Hence $I_m({}^tQ)$ is the intersection of weakly closed sets and hence is weakly closed. Now suppose ${}^tQ(v) = 0$. Let $Q_v \in PD_{S\theta}(E)$ such that $\beta(Q_v) = v$. For each $x \in E$ and $P \in \mathcal{P}_{b\theta}(E)$ we have $[Q_v * Q](P)(x) =$ $= [(Q_v * Q)(\tau_{-x}P)](0) = [Q_v(Q(\tau_{-x}P))](0) = \beta(Q_v)[Q(\tau_{-x}P)] =$ $= v[Q(\tau_{-x}P)] = 0$. Hence $Q_v * Q = 0$ and then $\widehat{\beta(Q_v)}\widehat{\beta(Q)} = 0$. Since $Q \neq 0$ and $\widehat{\beta(Q_v)}, \widehat{\beta(Q)} \in \mathcal{H}(E^*)$ we have $\widehat{\beta(Q_v)} = \hat{v} = 0$. Hence ${}^tQ$ is one to one.

<u>(5.19) Corollary</u>. If E has a countable basis for the elements of $\mathcal{B}_E$ for the boundeds sets and $Q \in PD_{S\theta}(E)$ with $Q \neq 0$, then $Q(\mathfrak{F}\,\mathcal{P}_{b\theta}(E)) \subset \mathfrak{F}\,\mathcal{P}_{b\theta}(E)$.

<u>Proof</u>. In the conditions of stated we have that $\mathfrak{F}\,\mathcal{P}_{b\theta}(E)$ is a Fréchet space. By the Dieudonne-Schwartz theorem, to show that $Q \neq 0$ is onto is suffices to show $Im({}^tQ)$ is closed for the weak topology on $\mathfrak{F}\,\mathcal{P}_{b\theta}(E)'$ defined by $\mathfrak{F}\,\mathcal{P}_{b\theta}(E)$. But this fact we have proved.

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THE APPROXIMATION-THEORETIC LOCALIZATION OF SCHWARTZ'S APPROXIMATION PROPERTY FOR WEIGHTED LOCALLY CONVEX FUNCTION SPACES AND SOME EXAMPLES

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INTRODUCTION

The method of an approximation-theoretic localization of the locally convex approximation property (a.p.) for weighted function spaces (or rather modules) was first explained in [4]. Its usefulness was then demonstrated in the last section of [11] in connection with investigations on spaces of functions of "mixed dependence" on subsets of products.

At the time when [11] was written the present author "computed" a number of concrete examples in order to get a better idea how far the applications of the fundamental localization theorem went and where it had its limitations. Only the examples which had direct connections with product sheaves could be included in [11].

In 1977, we came back to the remaining examples, and, based on the results of our paper [7], we have then been able to construct some new ones, too. When we looked once more at the localization theorem in the light of recent work in vector-valued approximation theory by Machado-Prolla [22] and in connection with Prolla's paper [25], it turned out that it was now possible to remove the completeness assumptions, which were needed in [11], completely. That no change of the general method was actually necessary, but that the research of Machado-Prolla on Nachbin spaces of vector fibrations in the non self-adjoint case, appropriately reinterpreted and used, enabled us to generalize our former results considerably, came as some surprise to this author's attention. It demonstrates that the method of dealing with with spaces of vector-valued functions by aid of L. Schwartz's ϵ-product is not really limited to complete spaces, but applies in full generality. So we have decided to give the main idea of the approximation-theoretic localization of the a.p. and the general version of the fundamental localization theorem in the first part of this article (sections 1. and 2.) while the examples and applications form the second part (sections 3. and 4.).

Let us now review the contents briefly: In section 1, we present the two methods how one can apply approximation-theoretic results to proofs of the a.p. of a function space: a "direct" one which sets out to interpret the space of continuous linear operators on a locally convex function space as a space of vector-valued functions (cf. Prolla [25]) and the one using the ϵ-product (cf. [4], [11]). It is rather obvious that the two methods are not essentially different, and we discuss their relation. For the rest of the paper we follow the second method because it takes the generality of the problem into account from the very start and because ϵ-product representations are readily available in the applications. - Since some technical problems and results complicate the general proof of the main localization theorem and since it is quite probable that the method applies in some other cases, too, section 1 does not go into details and may also serve as an introduction.

The first part of section 2 collects all the notation and all "ingredients" for the proof of the fundamental localization Theorem 16 of Schwartz's a.p. for subspaces of weighted spaces of type $CV_0(X)$ (i.e. Nachbin spaces) [which are modules over an algebra such that the weighted (Bernstein-Nachbin) approximation problem is localizable]. Lemma 11 is the crucial point which makes it possible

to do without the former completeness assumptions: Here we show that the space of (only weakly continuous) F-valued functions which corresponds with the ϵ-product $CV_0(X) \epsilon F$ is still a Nachbin space $L\tilde{V}_0$ of cross-sections in the sense of (say) [22]. In fact, whenever the topology of the function space is stronger than the compact-open topology, it is enough to assume only hypocontinuity of the functions here, and the localization theorem also applies in such a situation (as we point out in Theorem 18).

Section 3 is devoted to (more) examples for the localization of the a.p. among weighted spaces with "mixed dependence" on subsets Λ of products. In [11], we had dealt with a (rather) simple case already; now we construct concrete examples for the "regularity assumptions" on Λ which are needed if the "localized spaces" are not subspaces of nuclear spaces. We are mainly interested in the two cases where the "slices" Λ_t of Λ are open or compact. In the last case we make use of the results of [7] to get interesting new examples (see e.g. 29 and 30); we also include (in 33 and 34) "density theorems" completely analogous to [11], 4.11 and 4.12.

-At the end of the section, we look at the general setting for all examples in section 3 and point out that they have a natural interpretation as Nachbin spaces of cross-sections. (Let us note that 11, 18, and the remarks at the end of section 3 yield a number of interesting new examples of Nachbin spaces of cross-sections which arise quite naturally in the applications and which apparently have not been mentioned in the literature so far.)

Section 4 concludes the article with other examples (a different kind of "mixed dependence") and some applications to vector-valued approximation theory: In connection with the paper [26] of Prolla-Machado we consider Weierstrass-Stone, Kakutani-Stone, and Grothendieck subspaces of $CV_0(X,F)$ and generalize Blatter's method of [13] to weighted spaces.

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As I have mentioned before, part of the results in the last two sections of this article date back to the time when the joint publication [11] was prepared. I would like to thank B. Gramsch and R. Meise for many helpful conversations and remarks in this connection. - During the Campinas Conference on Approximation Theory in 1977 I had the opportunity to speak on the results (of [7] and) of (mainly the second part of) this article (part of which, in some sense, is a sequel to [7]). The author gratefully acknowledges support under the GMD/CNPq - agreement during his stay at UNICAMP July-September 1977 without which it would not have been possible to attend the Conference in Campinas. I would also like to thank J.B. Prolla for some conversations on his papers [22] - [26] which we had at this occasion.

1. THE GENERAL IDEA

Let E be an arbitrary (always Hausdorff) locally convex (l.c.) space over $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$). The following definition is taken from Grothendieck [16] (resp. Schwartz [29]):

<u>1 Definition</u>. E is said to have the approximation property (a.p.) (resp. Schwartz's a.p.) if the identity id_E of E can be approximated uniformly on every precompact (resp. every absolutely convex and compact) subset of E by continuous linear operators from E into E of finite rank (i.e. with finite dimensional range).

For two l.c. spaces E and F, let $\mathcal{L}(E,F)$ denote the space of all continuous linear operators from E into F, and put $\mathcal{L}(E) := \mathcal{L}(E,E)$. If $\mathcal{L}(E,F)$ is endowed with the topology of uniform convergence on all precompact (resp. absolutely convex compact) subsets of E, we write $\mathcal{L}_c(E,F)$ (resp. $\mathcal{L}_{cc}(E,F)$); so $\mathcal{L}_c(E)$ and $\mathcal{L}_{cc}(E)$ denote $\mathcal{L}_c(E,E)$, and $\mathcal{L}_{cc}(E,E)$, respectively. As, under the natural identification, the space of all continuous linear operators of finite rank from E into F corresponds with the tensor product

$E' \otimes F$, we get easily:

<u>2 Proposition</u> (Grothendieck [16], Proposition 35, p. 164-165). The following assertions are equivalent:

(1) E has the a.p. (resp. Schwartz's a.p.),

(2) $id_E \in \overline{E' \otimes E}^{\mathcal{L}_c(E)}$ (resp. $id_E \in \overline{E' \otimes E}^{\mathcal{L}_{cc}(E)}$),

(3) $E' \otimes E$ is dense in $\mathcal{L}_c(E)$ (resp. $\mathcal{L}_{cc}(E)$),

(4) for each l.c. space F, $E' \otimes F$ is dense in $\mathcal{L}_c(E,F)$ (resp. $\mathcal{L}_{cc}(E,F)$),

(5) for each l.c. space F, $F' \otimes E$ is dense in $\mathcal{L}_c(F,E)$ (resp. $\mathcal{L}_{cc}(F,E)$).

A counterexample of Enflo (1972), with subsequent refinements due to (among others) Davie and Szankowski, shows that there are even closed subspaces of the sequence spaces l^p <u>without a.p.</u> for each p, $1 \leq p \leq \infty$, $p \neq 2$. And recently Szankowski [30] proved (surprisingly) that the C^*-algebra $\mathcal{L}(H)$ of all continuous linear operators on an infinite dimensional (separable) Hilbert space H (under its canonical operator norm) does <u>not</u> have the a.p.

So, in view of the many interesting applications of the a.p. in the theory of topological tensor products (cf. Grothendieck [16]), it is reasonable to ask for methods to establish the a.p. for "concrete" l.c. spaces. One such method which applies to <u>a general class</u> of l.c. function spaces is discussed here. - Usually, a proof of the a.p. of a function space immediately implies useful results on (the approximation of) vector-valued functions and on ("slice products" of) functions of several variables (cf. e.g. [2], [3], [4]), but our general idea here is, conversely, to apply known theorems on the approximation of vector-valued functions to prove the a.p. of spaces of scalar functions.

<u>3 Remark</u>. The a.p. of a complete space $\mathfrak{F}$ of scalar functions, an ϵ-tensor product representation $\mathfrak{F}_F = \mathfrak{F} \check{\otimes}_\epsilon F$ of a space $\mathfrak{F}_F$ of

F-valued functions, and the known fact (cf. Schwartz [29], Proposition 11, Corollaire 2, p. 48) that the ϵ-tensor product of two complete l.c. spaces with a.p. again has the a.p. together imply that $\mathfrak{F}_F$ has the a.p. for each complete l.c. space F with a.p., too. - Hence we may restrict our attention to spaces of scalar functions here without any real loss of generality (for most purposes).

Now the most powerful tools of approximation theory in spaces (rather, algebras or, more generally, modules) $\mathfrak{F} = \mathfrak{F}(X)$ of continuous functions on a (Hausdorff) topological space X, like generalizations of the Stone-Weierstrass theorem or of Bishop's theorem on uniform algebras, yield a covering $X = \bigcup_{\alpha \in A} X_\alpha$ of X by (pairwise disjoint closed) sets X_α such that approximation can be "localized" to the sets X_α ($\alpha \in A$) in a natural way. Correspondingly, one may try to prove that (under suitable conditions) the a.p. of $\mathfrak{F}$ can be "localized" to the sets X_α in the following sense: If, for each $\alpha \in A$, $\mathfrak{F}|_{X_\alpha}$ has the a.p., then $\mathfrak{F}$ must have the a.p., too. - As many examples of spaces of continuous functions with a.p. are known (since Grothendieck and Schwartz), such a localization principle for the a.p. will yield a number of new concrete examples (say, among spaces of functions with "mixed dependence" on subsets of products, where Schwartz's result on the a.p. of ϵ-tensor products of complete l.c. spaces with a.p. can only be applied on full product sets.) The theorem is, in some sense, an interesting and very useful approximation-theoretic "permanence property" of the a.p. in spaces of continuous functions.

However, the definition of the a.p. (see Proposition 2) does not allow a direct application of approximation theorems for vector-valued continuous functions on X to our situation of approximating $id_{\mathfrak{F}(X)}$ by elements of $\mathfrak{F}(X)' \otimes \mathfrak{F}(X)$ in $\mathfrak{L}_c(\mathfrak{F}(X))$.

There are essentially two ways (which, as a careful analysis shows, are closely related and, in fact, equivalent) to overcome this

difficulty. - If $(X,(F_x)_{x\in X})$ is a "vector fibration" over a Hausdorff topological space X with "fibers" F_x ($x \in X$) and if $L = LV_o$ is a "Nachbin space of cross-sections" (see below) with $L(x) := \{f(x);\ f \in L\} = F_x$ for each $x \in X$ such that L is a module over a selfadjoint and separating subalgebra of $CB(X)$ (= continuous and bounded scalar functions on X), Prolla [25] (modifying a previous idea of G. Gierz) has recently shown that the a.p. of all spaces F_x (under the semi-norms $(v(x))_{v\in V}$) implies the a.p. of L. In his proof he represents the space $\mathfrak{L}(L)$ as a Nachbin space of cross-sections over X with fibers $\mathfrak{L}(L,F_x)$ and then applies the solution of the "weighted approximation problem" (for Nachbin spaces of cross-sections) in the separating and selfadjoint bounded case. - The corresponding representation is given as follows: Let, for each $x \in X$, $\delta_x\colon L \to F_x$ be the point evaluation at x ($\delta_x(f) := f(x)$ for all $f \in L$). For $T \in \mathfrak{L}(L)$, $\delta_x \circ T$ belongs to $\mathfrak{L}(L,F_x)$, and $R\colon T \to \hat{T} := (\delta_x \circ T)_{x\in X}$ represents $\mathfrak{L}(L)$ as a Nachbin space of cross-sections over X with fibers $\mathfrak{L}(L,F_x)$.

In our general case, it is clear that $\mathfrak{L}(\mathfrak{F}(X))$ can similarly be represented as a space of functions $\hat{T}$ on X with values in $\mathfrak{F}(X)'$, if the topology of $\mathfrak{F}(X)$ is stronger than pointwise convergence on X. As $\mathfrak{F}(X)$ consists of continuous functions, any such function $\hat{T}\colon x \to (f \to Tf(x))$ is continuous from X into $\mathfrak{F}(X)'\ [\sigma(\mathfrak{F}(X)', \mathfrak{F}(X))]$ and continuous into $\mathfrak{F}(X)'_c := \mathfrak{L}_c(\mathfrak{F}(X),\mathbb{K})$ (resp. $\mathfrak{F}(X)'_{cc} := \mathfrak{L}_{cc}(\mathfrak{F}(X),\mathbb{K})$) if and only if, for each precompact (resp. absolutely convex and compact) set K in $\mathfrak{F}(X)$, the image $T(K)$ under T is equicontinuous on X (which certainly holds for each $T \in \mathfrak{L}(\mathfrak{F}(X))$ if each precompact [resp. absolutely convex compact] subset of $\mathfrak{F}(X)$ is equicontinuous on X). - Now the topology of $\mathfrak{L}_c(\mathfrak{F}(X))$ [resp. $\mathfrak{L}_{cc}(\mathfrak{F}(X))$] is given by the set $\{q_{K,p};$ p continuous seminorm on $\mathfrak{F}(X)$, K precompact (resp. absolutely convex compact) in $\mathfrak{F}(X)\}$ of seminorms $q_{K,p}(T) = \sup_{f\in K} p(Tf)$ for all

$T \in \mathfrak{L}(\mathfrak{F}(X))$. For a large class of function spaces $\mathfrak{F}(X)$, it turns out that this topology corresponds with a (it is enough: topology weaker than a) "natural" topology on the space $\widehat{\mathfrak{L}(\mathfrak{F}(X))}$ of all functions $\hat{T}: X \to \mathfrak{F}(X)'_c$ (resp. $\mathfrak{F}(X)'_{cc}$) in the sense that vector-valued approximation theorems apply to approximation of the "evaluation mapping" $\widehat{id}_{\mathfrak{F}(X)}: x \to \delta_x$ by elements from $\widehat{\mathfrak{F}(X)' \otimes \mathfrak{F}(X)}$ (which are functions on X with values in finite dimensional subspaces of $\mathfrak{F}(X)'$ obviously).

In general, if there are vector-valued approximation theorems "only" for spaces of continuous functions, then this fact clearly restricts the class of spaces $\mathfrak{F}(X)$ to which our method applies. (Compare the equivalent condition for continuity of $\hat{T}: X \to \mathfrak{F}(X)'_c$ [resp. $\mathfrak{F}(X)'_{cc}$] which we have mentioned above; in the applications, this usually amounts to a completeness type assumption on $\mathfrak{F}(X)$.) In Prolla's case, however, we realize that the notion of Nachbin space of cross-sections (and the corresponding generalization of the Stone-Weierstrass theorem) is already flexible enough to conclude without any restriction. - Thus we can avoid completeness assumptions in our fundamental theorem below. On the other hand, such an assumption is quite natural, and most of the function spaces which occur in the applications are complete. So there is no great loss of generality e.g. in dealing mainly with Schwartz's a.p. throughout this paper.

Let us remark at this point (cf. Schwartz [28]) that the equivalence (2) ⇔ (5) of Proposition 2 implies, after a similar representation of $\mathfrak{L}(F,\mathfrak{F}(X))$ as a space of F'-valued functions on X for arbitrary l.c. F, that the approximation of the (single) evaluation mapping $\widehat{id}_{\mathfrak{F}(X)}$ by elements of $\widehat{\mathfrak{F}(X)' \otimes \mathfrak{F}(X)}$ in the space $\widehat{\mathfrak{L}(\mathfrak{F}(X))}$ (under a suitable topology) is really equivalent to approximation of all functions in a certain space $\widehat{\mathfrak{L}(F,\mathfrak{F}(X))}$ of F'-valued functions on X (under a suitable topology) by elements of

$\widehat{F' \otimes \mathfrak{F}(X)}$ (which are functions on X with values in finite dimensional subspaces of F') for arbitrary l.c. spaces F. - This remark leads us immediately to the other method to allow application of vector-valued approximation theorems to a proof of the a.p. for spaces of scalar functions. This (in some sense less direct) method which takes into account the generality of the problem right from the start was already presented in [4] and [11] and relies on a useful equivalence of the a.p. due to Schwartz [29] (cf. also [2] and [8]):

4 Definition. For a l.c. space F, let $F'_{cc} = \mathfrak{L}_{cc}(F,\mathbb{K})$ denote the dual of F with the topology of uniform convergence on all absolutely convex compact subsets of F. Schwartz's ε-product of E and F is defined by $E\varepsilon F := \mathfrak{L}_e(F'_{cc},E)$, where the subscript e indicates the topology of uniform convergence on the equicontinuous subsets of F'.

(Originally, Schwartz's ε-product was defined in [29] in a different way, but [29], Proposition 4, Corollaire 2, p.34 shows that our definition is equivalent up to topological isomorphism.) - Then $E\varepsilon F \cong F\varepsilon E$ holds, and the ε-product of two complete spaces is complete. Moreover, the tensor product $E \otimes F$ with the ε-topology of A. Grothendieck [16], i.e. $E \otimes_\varepsilon F$, is (canonically identified with) a topological subspace of $E\varepsilon F$, namely just the space of all (even $\mathfrak{L}(F',F)$-) continuous linear operators of finite rank in $E\varepsilon F$. Let $E \check{\otimes}_\varepsilon F$ denote the ε-tensor product of E and F, i.e. the completion of $E \otimes_\varepsilon F$. Then Schwartz's criterion for the a.p. (with some refinements) can be formulated as follows:

5 Theorem (Schwartz [29], Proposition 11, p. 46-47). The following assertions for a l.c. space E are equivalent:

(1) E has Schwartz's a.p.,

(2) $id_E \in \overline{E \otimes E'}^{E \varepsilon E'_{cc}}$,

(3) $E \otimes E'$ is dense in $E \varepsilon E'_{cc}$,

(4) for each l.c. space F, $E \otimes F$ is dense in $E \varepsilon F$,

(5) for each Banach space F, $E \otimes F$ is dense in $E \varepsilon F$.

If, additionally, E is <u>complete</u>, the a.p. of E (i.e. (1)-(5) above) is also equivalent to:

(6) for each complete l.c. (or each Banach) space F, $E \check{\otimes}_{\varepsilon} F = E \,\varepsilon\, F$ holds.

Here the proof of (2) ⇔ (1) follows from the fact (Schwartz [29], Proposition 5, Corollaire, p. 36-37) that, for two l.c. spaces E and F, $\mathfrak{L}_{cc}(E,F)$ is always a topological linear subspace of $F \,\varepsilon\, E'_{cc}$.

Now, in many cases, for a space $\mathfrak{F}(X)$ of continuous functions on X the ε-product $F \varepsilon \mathfrak{F}(X)$ for any (say) complete l.c. space F is known to be (up to the topological isomorphism $u \to (x \to u(\delta_x))$ nothing but "<u>the space of</u> F-<u>valued functions of type</u> $\mathfrak{F}(X)$", and $F \otimes \mathfrak{F}(X)$ corresponds with the subspace of all functions which take their values in finite dimensional subspaces of F. In view of this, it is immediate (and much more natural than we may have realized up to this point) to apply vector-valued approximation theorems to a proof of the a.p. of $\mathfrak{F}(X)$ (in the form of equivalence (5) of Theorem 5). (And, conversely, the approximation of vector-valued functions [by functions with values in finite dimensional subspaces] which is implied by the a.p. of a function space is also more interesting than we may have been tempted to think after a short glance at the first method.)

Furthermore, equivalence (2) of Theorem 5 shows then that we have to deal with the approximation of the <u>single</u> $\mathfrak{F}(X)'_{cc}$-valued function (evaluation mapping) $x \to \delta_x$ <u>only</u> (if we prefer). This remark links the first method with this one, and, since $\mathfrak{L}_{cc}(E,F)$ is always a topological linear subspace of $F \,\varepsilon\, E'_{cc}$, it turns out that both methods are essentially <u>equivalent</u> (except for technical details). Undoubtedly, the first method is more direct, but, in each case, it requires (new) investigations on a topological isomorphism of $\mathfrak{L}_c(\mathfrak{F}(X))$ with the space $\widehat{\mathfrak{L}(\mathfrak{F}(X))}$ of $\mathfrak{F}(X)'_c$-valued functions on X

under a "natural" topology, whereas one advantage of the second method comes from the fact that <u>known</u> results (needed for other applications) can be used in a natural way. Also, while it is useful to know that the "test space" $F = E'_{cc}$ suffices in equivalence (4) of Theorem 5, it is sometimes much easier to work only with Banach spaces F as in equivalences (5) and (6) above, because the topological vector space structure of E'_{cc} may be quite complicated even for "good" E.

Thus, in accordance with our interest in l.c. function spaces (and not in Nachbin spaces of cross-sections, where similar results hold), we will only discuss the second method <u>in more detail</u> from now on. - As pointed out above, to derive a localization theorem for the a.p. (in the sense described before) by the second method, we need <u>two types of results</u>: ε-product representations $\mathfrak{F}(X)\varepsilon F = \mathfrak{F}_F(X)$ and localization theorems for (vector-valued) approximation in $\mathfrak{F}_F(X)$ by elements of $\mathfrak{F}(X) \otimes F$. So let us now list some of the known results in this direction:

(I) ε-<u>product-representations</u>:

If F is a (quasi-) complete l.c. space, a representation of $\mathfrak{F}(X)\varepsilon F$ as a "natural" space $\mathfrak{F}_F(X)$ of F-valued functions on X can be found for weighted Nachbin spaces (of continuous functions) $\mathfrak{F}(X) = CV_o(X)$ resp. $CV(X)$ (with $\mathfrak{F}_F(X) = CV_o(X,F)$ resp. $CV^p(X,F)$ if X is a V_R-space [see below]) in [2], II, Theorem 4 or [4], for "weighted spaces of differentiable functions" in the thesis [1] of B. Baumgarten (cf. also L. Schwartz [28] and Garnir-de Wilde-Schmets [14]), for spaces of functions satisfying general Lipschitz conditions and spaces of continuously differentiable functions with Hölder conditions in W. Kaballo [18], 3 and 4 or [19], 3a). From these results and some general theorems on the ε-product of inductive limits, ε-product representations can also be derived for certain inductive limits of the spaces mentioned before. For inductive limits of weighted spaces e.g., this was done in [9]. (Recently some of the open problems here have been solved; see R. Hollstein [17], the author's paper [6] and [12], where it is shown that, in many interesting cases, inductive limits of weighted spaces are again [topologically] weighted spaces.) - At this point, it should be remarked that the ε-product obviously "preserves topological linear subspaces", and hence ε-product representations as above lead to analogous theorems for all topological subspaces, too. In fact, most of the spaces mentioned above are already known to have the a.p. (this usually follows from the ε-product representation theorem and results on the approximation of vector-valued functions, too), but not many results are available on the a.p. of topological subspaces, and here localization theorems might apply. (In the case of inductive limits, we still have the so-called "subspace problem" whether subspaces G of inductive limits $E = \operatorname{ind}_{\alpha\to} E_\alpha$ inherit the natural inductive limit topology $\operatorname{ind}_{\alpha\to}(G \cap E_\alpha)$, but in some applications this can be deduced from a lemma of A. Baernstein, cf. [9], 1 Satz 14).

(II) <u>Localization theorems for vector-valued approximation</u>: Localization theorems for approximation by elements of modules W in weighted Nachbin spaces $CV_o(X,F)$ of continuous F-valued functions were first given in the selfadjoint case by Nachbin-Machado-Prolla [23]. For modules over non selfadjoint algebras, vector-valued localization was obtained in the complex (restricted) bounded case of the weighted (Bernstein-Nachbin) approximation problem by G. Kleinstück [21] (generalizing a previous results of J.B. Prolla). The general complex case has recently been treated by Machado-Prolla [22] (even in the setting of Nachbin spaces of cross-sections) with a reduction to the finite dimensional Bernstein approximation problem on $\mathbb{R}^n$ $(n \geq 1)$ and yielding, as corollaries, the analytic and quasi-analytic criteria; see also Prolla's book [24], Chapter 5, §2-3 for the case of vector-valued functions. Of course, for the spaces

$CV^p(X,F)$ instead of $CV_o(X,F)$, localization results can only hold when, for the functions f in $CV^p(X,F)$ and the weights $v \in V$, vf is considered (by "canonical" extension) as function on a compactification $\hat{X}$ of X (or even on $\beta\tilde{X}$, the Stone-Čech compactification of $\tilde{X} := X$ under an appropriate finer topology), and approximation is then localized to certain subsets of $\hat{X}$ (or $\beta\tilde{X}$) rather. Such localization theorems for approximation from modules W in $CV^p(X,F)$ were proved in the (complex) "bounded case" by G. Kleinstück [21], 2, Theorems 9 and 11. - No (general) localization theorems for approximation in spaces of continuously differentiable functions or spaces of Lipschitz functions are known. (But, by the above - mentioned theorem of [6] resp. [12] that, at least in the case of Banach space-valued functions, inductive limits of weighted spaces may again be weighted spaces [topologically], localization theorems are also available for most of the interesting "inductively weighted spaces". - In general, for a proof of the a.p. of a function space with an inductive limit topology, it is important that one may restrict the attention to <u>Banach</u> "test spaces" in equivalence (5) of Theorem 5. On the other hand, the a.p. is inherited, say, under quasi-complete "<u>compactly regular</u>" inductive limits, cf. [8], and hence, in many interesting cases, the a.p. of $\operatorname{ind}_{\alpha\to} E_\alpha$ follows from the a.p. for all the spaces E_α already.)

2. THE FUNDAMENTAL LOCALIZATION THEOREM FOR THE a.p.

Let us now turn to the statement (and proof) of the fundamental localization theorem for the a.p. of weighted function spaces, which follows from some known results listed in (I) and (II) at the end of section 1. First we need some notation (and a number of definitions).

<u>6 Definitions</u>. From now on, let X always denote a completely regular Hausdorff space. A non-negative upper semicontinuous function v on X is called a <u>weight</u> (on X). A family $V \neq \emptyset$ of weights on X which is directed in the sense that for all $v_1, v_2 \in V$ and any

$\lambda \geq 0$ there exists $v \in V$ with $\lambda v_1, \lambda v_2 \leq v$ (pointwise on X) is said to be a Nachbin family (on X). We will always assume that $V > 0$, i.e. that for each point $x \in X$ there is $v \in V$ with $v(x) > 0$.

Let F be an arbitrary l.c. space. Two weighted (Nachbin) spaces of continuous F-valued functions on X with respect to the Nachbin family V (on X) are introduced as follows:

$CV^p(X,F) := \{f: X \to F$ continuous; $(vf)(X) := \{v(x)f(x);\ x \in X\}$ precompact in F for each $v \in V\}$,

$CV_o(X,F) := \{f: X \to F$ continuous; $vf: x \to v(x)f(x)$ vanishes at infinity on X (i.e., for each continuous semi-norm p on F and each $\varepsilon > 0$ there is a compact subset K of X with $p((vf)(x)) < \varepsilon$ for all $x \in X \setminus K$) for each $v \in V\}$,

both endowed with the l.c. topology generated by the system $\{b_{v,p};\ v \in V,\ p$ continuous semi-norm on $F\}$ of semi-norms

$$b_{v,p}(f) := \sup_{x\in X} v(x)p(f(x)) \quad \text{for all } f \in CV^p(X,F).$$

We put $CV(X) := CV^p(X,\mathbb{K}) = \{f: X \to \mathbb{K}$ continuous; vf bounded on X for each $v \in V\}$ and $CV_o(X) := CV_o(X,\mathbb{K})$. – Since all $v \in V$ are upper semicontinuous (and hence bounded on compact subsets of X), $CV_o(X,F)$ is a closed subspace of $CV^p(X,F)$, and the assumption $V > 0$ implies that the topology of $CV^p(X,F)$ (resp. $CV_o(X,F)$) is stronger than pointwise convergence on X (and hence Hausdorff).

The spaces CV(X) and $CV_o(X)$ were introduced by L. Nachbin (in connection with the weighted approximation problem). For more information on the weighted spaces $CV^p(X,F)$ and $CV_o(X,F)$ and some examples see e.g. [2] and [4]. The following is a sufficient condition for completeness in weighted spaces ([4], Proposition 22, p.38):

7 Proposition. For a given Nachbin family $V > 0$ on X, let X be a $V_{\mathbb{R}}$-space, i.e.: A function $f: X \to \mathbb{R}$ (or, equivalently, $f: X \to Y$, Y any completely regular [Hausdorff] space) is continuous if (and

always only if) $f|_{\{x\in X;\ v(x)\geq 1\}}$ is continuous for each $v \in V$. Then $CV^p(X,F)$ and $CV_o(X,F)$ are complete for each complete l.c. space F.

As usual, a space X is called a $k_{\mathbb{R}}$-space, if a function $f: X \to \mathbb{R}$ is continuous if (and only if) $f|_K$ is continuous for each compact set $K \subset X$. (All locally compact or metrizable spaces, and more generally the k-spaces of Kelley, are $k_{\mathbb{R}}$-spaces.) Then, if $W \leq V$ holds, i.e. if for each compact subset K of X we can find a weight $v \in V$ with $\inf_{x\in X} v(x) > 0$ (which implies that the topology of $CV^p(X,F)$ resp. $CV_o(X,F)$ is stronger than uniform convergence on compact subsets of X), then any $k_{\mathbb{R}}$-space X is a fortiori a $V_{\mathbb{R}}$-space.

In our proof below, we need an ε-product representation theorem (see [2], II, 2.1. (4) and 3.1.(1) resp. [4], Theorem 24, p.39) which requires the following definition:

8 Definition. $CV^{\sigma,c}(X,F) := \{f: X \to F[\sigma(F,F')]$ continuous; $\Gamma((vf)(X))$ [:= absolutely convex hull of $(vf)(X)$] relatively compact in F for each $v \in V\}$,

$CV_o^{\sigma,c}(X,F) := \{f \in CV^{\sigma,c}(X,F);\ vf$ vanishes at infinity (as a function from X into F) for each $v \in V\}$;

on these spaces, the semi-norms $b_{v,p}$ as in 6 are still well-defined, and we equip the spaces with the corresponding l.c. topology (such that $CV^p(X,F) \subset CV^{\sigma,c}(X,F)$ and $CV_o(X,F) \subset CV_o^{\sigma,c}(X,F)$ topologically).

Since on a relatively compact subset of F the topology of F coincides with $\sigma(F,F')$, it is easy to see (cf.[4], Prop.23, p.39) that any function in $CV^{\sigma,c}(X,F)$ is already continuous from X into F if X is a $V_{\mathbb{R}}$-space, and hence we obtain $CV^{\sigma,c}(X,F) = CV^p(X,F)$ and $CV_o^{\sigma,c}(X,F) = CV_o(X,F)$ if X is a $V_{\mathbb{R}}$-space and F quasi-complete.

9 Theorem. (1) $F\varepsilon CV(X) \cong CV(X)\varepsilon F \cong CV^{\sigma,c}(X,F)$ and $F\varepsilon CV_o(X) \cong CV_o(X)\varepsilon F \cong CV_o^{\sigma,c}(X,F)$ (up to the following canonical topological isomorphisms: $u \to (x \to u(\delta_x))$ of $F\varepsilon CV(X)$ [resp.

$F \epsilon CV_o(X)]$ onto $CV^{\sigma,c}(X,F)$ [resp. $CV_o^{\sigma,c}(X,F)$] and $f \to (f' \to f' \circ f)$ of $CV^{\sigma,c}(X,F)$ [resp. $CV_o^{\sigma,c}(X,F)$] onto $CV(X)\epsilon F$ [resp. $CV_o(X)\epsilon F$]).

(2) Hence $CV(X)\epsilon F \cong CV^p(X,F)$ and $CV_o(X)\epsilon F \cong CV_o(X,F)$ hold if X is a $V_{\mathbb{R}}$-space and F quasi-complete.

(3) Moreover, for any topological linear subspace E of $CV(X)$ resp. $CV_o(X)$, we obtain:

$E\epsilon F \cong \{f \in CV^{\sigma,c}(X,F)$ (resp. $CV_o^{\sigma,c}(X,F)$); $f' \circ f : x \to f'(f(x))$ belongs to E for each $f' \in F'\}$ (with the induced topology), which under the conditions of (2) becomes (more simply):

$E\epsilon F \cong \{f \in CV^p(X,F)$ (resp. $CV_o(X,F)$); $f' \circ f \in E$ for each $f' \in F'\}$.

Since we must make use of the solution of the weighted (Bernstein-Nachbin) approximation problem <u>for Nachbin spaces of cross-sections</u> later on, we recall the necessary definitions and prove an important lemma next.

<u>10 Definitions</u> (cf. [22], [23]). A <u>vector fibration</u> over X is a pair $(X,(F_x)_{x\in X})$, where each F_x is a vector space over the field $\mathbb{K}$. A <u>cross-section</u> is then any element of $\prod_{x\in X} F_x$, i.e., $f = (f(x))_{x\in X}$. A <u>"weight"</u> v on X is a function v on X such that $v(x)$ is a semi-norm on F_x for each $x \in X$. A <u>Nachbin space</u> LV_o is a vector space of cross-sections f such that the mapping $x \to v(x)[f(x)]$ is upper semicontinuous on X and vanishes at infinity for each "weight" $v \in V$, equipped with the l.c. topology defined by the family $\{\|\cdot\|_v\}_{v\in V}$ of semi-norms $\|f\|_v := \sup_{x\in X} v(x)[f(x)]$.

Of course, $CV_o(X,F)$ is certainly a Nachbin space $L\tilde{V}_o$ of cross-sections $f = (f(x))_{x\in X}$ with respect to the vector fibration $(X,(F_x)_{x\in X})$, where $F_x := F$ for each $x \in X$, and to the set $\tilde{V} := \{\tilde{v}_{v,p};\ v \in V,\ p$ continuous semi-norm on $F\}$ of "weights" on X, defined by

$$\tilde{v}_{v,p}(x)[e] := v(x)p(e) \quad \text{for each } x \in X \text{ and } e \in F.$$

However, we observe:

<u>11 Lemma</u>. (1) Let $v \in V$, p a continuous semi-norm on F, and $f \in CV_o^{\sigma,c}(X,F)$ be arbitrary. Then the function $vp\circ f: x \to v(x)p(f(x))$ is <u>upper semicontinuous</u> on X.

(2) In the same way as described above for $CV_o(X,F)$, $CV_o^{\sigma,c}(X,F)$ is also <u>a Nachbin space</u> $L\tilde{V}_o$ <u>of cross-sections</u>.

<u>Proof</u>. (1): Let $\epsilon > 0$ and fix an arbitrary point $x \in X$.

<u>First case</u>: $v(x) \neq 0$. Let $0 < \delta \leq 1$ satisfy $\delta p(f(x)) < \frac{\epsilon}{2}$ and put $\alpha := \frac{v(x)+1}{v(x)} > 1$.

Since $C := \alpha\Gamma((vf)(X))$ is relatively compact (and absolutely convex) in F, the (uniform structure resp.) topology of F coincides with $\sigma(F,F')$ on this set, and so p is uniformly continuous on C with respect to $\sigma(F,F')$. Hence, there exists a balanced neighbourhood V of 0 in $\sigma(F,F')$ such that $e_1, e_2 \in C$ and $e_1 - e_2 \in V$ imply $|p(e_1) - p(e_2)| < \frac{\epsilon}{2}$. As v is upper semicontinuous and $f: X \to F\ [\sigma(F,F')]$ continuous, we can find a neighbourhood $U(x)$ of x in S such that $v(y) < v(x) + \delta$ and $f(y) - f(x) \in \frac{1}{v(x)+1} V$ for all $y \in U(x)$. Then for any such $y \in U(x)$ we have:

$v(y)f(y) - v(y)f(x) \in \frac{v(y)}{v(x)+1} V \subset V$ with $v(y)f(y) \in (vf)(X) \subset C$ and $v(y)f(x) = \frac{v(y)}{v(x)} v(x)f(x) \in \alpha C((vf)(X)) \subset C$ (where C indicates the balanced hull), hence $|p(v(y)f(y)) - p(v(y)f(x))| < \frac{\epsilon}{2}$. It follows $p(v(y)f(y)) < p(v(y)f(x)) + \frac{\epsilon}{2} \leq p(v(x)f(x)) + \delta p(f(x)) + \frac{\epsilon}{2} < p(v(x)f(x)) + \epsilon$, that is, $vp\circ f$ is upper semicontinuous at x.

<u>Second case</u>: $v(x) = 0$. Since vf vanishes at infinity (as a function from X into F), there exists a compact subset K of X such that $p(v(y)f(y)) < \epsilon$ for all $y \in X\setminus K$. Since $f: X \to F\ [\sigma(F,F')]$ is continuous, $f(K)$ is $\sigma(F,F')$-compact and hence bounded in F; let $M > 0$ satisfy $p(f(y)) \leq M$ for all $y \in K$. Since v is upper semicontinuous and $v(x) = 0$, there exists a neighbourhood $U(x)$ of x in X such that $v(y) < \frac{\epsilon}{M+1}$ for any $y \in U(x)$. Then for any such $y \in U(x)$ we have $p(v(y)f(y)) < \epsilon = p(v(x)f(x)) + \epsilon$, since $y \notin K$ certainly implies $p(v(y)f(y)) < \epsilon$ while $p(v(y)f(y)) \leq \frac{\epsilon}{M+1} p(f(y)) < \epsilon$ for any $y \in U(x) \cap K$. So $vp\circ f$ is again upper semicontinuous at x.

(2): By definition, any function $v p\circ f: x \to \tilde{v}_{v,p}(x)[f(x)] = v(x)p(f(x))$ with $f \in CV_o^{\sigma,c}(X,F)$ vanishes at infinity, and by (1) this function is also upper semicontinuous on X (for arbitrary $v \in V$ and p = continuous semi-norm on F). Also, the topology of $CV_o^{\sigma,c}(X,F)$ is given by the directed family $\{\|\cdot\|_{\tilde{v}_{v,p}}\}_{v,p}$ of semi-norms

$$\|f\|_{\tilde{v}_{v,p}} = \sup_{x\in X} \tilde{v}_{v,p}(x)[f(x)] = \sup_{x\in X} v(x)\ p(f(x)). \quad \square$$

The following are the definitions and results we need from approximation theory (cf. [23]):

12 Definitions. Let A be a subalgebra of $C(X)$ (= continuous scalar functions on X), let LV_o be a Nachbin space of cross-sections over X and let Z denote a vector subspace of LV_o which is an A-module (with respect to pointwise multiplication, i.e., $a \in A$ and $z = (z(x))_{x\in X} \in Z$ imply $az = (a(x)z(x))_{x\in X} \in Z$, too). In this context, the weighted (Bernstein-Nachbin) approximation problem asks for a description of the closure of Z in LV_o.

Let $\mathfrak{X}$ be a covering of X by pairwise disjoint closed subsets. Z is said to be $\mathfrak{X}$-localizable in LV_o if $f \in LV_o$ belongs to $\bar{Z}^{LV_o}$ if (and always only if), given any $K \in \mathfrak{X}$, any $v \in V$, and any $\varepsilon > 0$, there is some $z \in Z$ such that $v(x)[f(x)-z(x)] < \varepsilon$ for all $x \in K$.

$\mathfrak{X}_A$ denotes the system of all maximal A-antisymmetric subsets of X: A subset K of X is called A-antisymmetric if $f \in A$, $f|_K$ real-valued always imply $f|_K$ constant. Each A-antisymmetric set is contained in a (with respect to inclusion) maximal A-antisymmetric subset, and hence the system $\mathfrak{X}_A$ of all such sets is a covering of X by pairwise disjoint closed sets. - If A is selfadjoint, $\mathfrak{X}_A$ coincides with the set of equivalence classes with respect to the equivalence relation $\underset{A}{\sim}$ on X: $x \underset{A}{\sim} y$ if and only if $a(x) = a(y)$ for all $a \in A$.

For simplicity, let us agree to say that Z is localizable under A in LV_o if Z is $\mathfrak{X}_A$-localizable in LV_o.

Sufficient conditions for an A-module $Z \subset LV_o$ to be localizable under A in LV_o were derived by Machado-Prolla [22]. (Remark that "Z sharply localizable under A in LV_o" in the terminology of [22] implies "Z localizable under A in LV_o" in our notation.) In Theorem 14 resp. 15 of [22], Machado-Prolla reduce the search for sufficient conditions for localizability to the n-dimensional resp. one-dimensional Bernstein approximation problem (on fundamental weights on $\mathbb{R}^n$ resp. $\mathbb{R}$), generalizing previous results of Nachbin-Machado-Prolla [23] in the selfadjoint case. As corollaries (Theorems 16, 17, 18 of [22]), they derive the analytic resp. quasi-analytic criterion of localizability and localizability in the so-called "bounded case" of the weighted approximation problem. - It would take us too far to state all these results here, and so we confine ourselves to a specialization of Machado-Prolla's "bounded case" (which, however, is essentially enough for all the examples we have in mind).

13 Definition. Let $Z \subset LV_o$ be an A-module. We say that we are in the bounded case (of the weighted approximation problem for Z) if, given any $v \in V$, $a \in A$, and $z \in Z$, the function a is bounded on the support of $v\circ z: x \to v(x)[z(x)]$. This is certainly true if, given any $v \in V$ and $a \in A$, a is bounded on $\operatorname{supp} v := \overline{\{x \in X;\ v(x) \neq 0\}}$. When the latter condition holds, we say that we are in the restricted bounded case.

So the restricted bounded case occurs for instance, if all the functions $a \in A$ are bounded or if each $v \in V$ has compact support.

14 Theorem (Machado-Prolla [22], Theorem 18). In the bounded case, Z is always localizable under A in LV_o.

In fact, as Machado-Prolla [22], Theorem 18 shows, it is sufficient that

(*) $$a|_{\operatorname{supp}(v\circ z)} \text{ is bounded}$$

for all $v \in V$, $a \in A$, and $z \in G(Z)$, a "set of generators" for Z, that is, the A-module of Z generated by $G(Z)$ is dense in Z for the topology of LV_o. Similarly, the class of functions $a \in A$ for which condition (*) must be "tested" may be restricted to $G(A)$, a so-called "strong set of generators" for A. (If A has a set $G(A)$ of real-valued functions such that the subalgebra of A generated by $G(A)$ is dense in A for the topology of uniform convergence on the compact subsets of X, then $G(A)$ is a strong set of generators; and the whole algebra itself is always such a set.) - On the other hand, Kleinstück's previous result in [21] even assumed the restricted bounded case.

In (the proof of) our fundamental theorem, we assume that E is a topological vector subspace of $CV_o(X)$ which is an A-module for a subalgebra A of $C(X)$ and apply, for arbitrary l.c. space F, solutions of the weighted approximation problem to $Z := E \otimes F$ in $CV_o^{\sigma,c}(X,F) = L\tilde{V}_o$ (cf. 11 (2)). Now Z is clearly an A-module, too, and if $G(E)$ is a set of generators for E (in $CV_o(X)$), then $G(E) \otimes F$ is a set of generators for Z in $CV_o^{\sigma,c}(X,F)$, because $CV_o^{\sigma,c}(X,F)$ (or, equivalently, $CV_o(X,F)$) always induces the ε-topology on the tensor product $CV_o(X) \otimes F$ (and hence also on $E \otimes F = Z$), cf. Theorem 9 (1) above (and the remarks after Definition 4). - Moreover, any "reasonable" sufficient condition for localizability applies to $Z = E \otimes F$ (and the set of generators $G(E) \otimes F$) in $L\tilde{V}_o = CV_o^{\sigma,c}(X,F)$ if it applies to E (and the set of generators $G(E)$) in $CV_o(X)$. This is true e.g. for the conditions of Theorems 14 and 15 of [22] (reduction to the n - resp. one-dimensional Bernstein approximation problem) as well as for the analytic and quasi-analytic criterion (say, in the form of [22], Theorems 16 and 17). Again, we state only:

15 Remark. If the A-module $E \subset CV_o(X)$ satisfies (e.g. for a set of generators $G(E)$) the conditions of the bounded [resp. restricted bounded] case, then the A-module $Z = E \otimes F$ satisfies (for the set $G(E) \otimes F$ of generators) the conditions of the bounded [resp. restricted bounded] case in $CV_o^{\sigma,c}(X,F) = L\tilde{V}_o$ (F any l.c. space).

After all these preparations, we can finally state the fundamental localization theorem for the a.p. of modules in weighted spaces $CV_o(X)$:

16 Theorem. Let X be a completely regular Hausdorff space and $V > 0$ a Nachbin family on X. Let A denote a subalgebra of $C(X)$ and E a topological linear subspace of $CV_o(X)$ which is an A-module. Let $\mathcal{K}$ be a covering of X by pairwise disjoint closed subsets such that, for an arbitrary l.c. space F, (the A-module) $Z := E \otimes F$ is $\mathcal{K}$-localizable in the space $L\tilde{V}_o = CV_o^{\sigma,c}(X,F)$. [E.g. let $\mathcal{K} = \mathcal{K}_A$ and assume that $E \otimes F$ is localizable under A in LV_o.] Then, if $E|_K = \{f|_K;\ f \in E\}$, as a topological linear subspace of $C(V|_K)_o(K)$, has Schwartz's a.p. for each $K \in \mathcal{K}$, the space E has Schwartz's a.p., too.

For instance, if under the assumptions of the first paragraph of the theorem E satisfies the condition of reduction to the n- resp. one-dimensional Bernstein approximation problem ([22], Theorems 14 and 15) or the condition in the analytic or quasi-analytic criterion ([22], Theorems 16 and 17), then by the remarks before 15 (and the results of Machado-Prolla [22] mentioned before 13) $E \otimes F$ is always localizable under A in $L\tilde{V}_o$, and 16 applies. Especially:

17 Corollary. Let X, V, A, and E be as in Theorem 16. Assume that we are in the bounded case of the weighted approximation problem for E in $CV_o(X)$. Then, if $E|_K$ (as a topological linear subspace of $C(V|_K)_o(K)$) has Schwartz's a.p. for each maximal A-antisymmetric subset K of X, the space E has Schwartz's a.p. (and hence the a.p., if E is quasi-complete).

For the restricted bounded case, this generalizes Satz 4.5 of [11], where a completeness assumption (X V_R-space) for $CV_o(X)$ was necessary. - The method of proof of Theorem 16 (see below) is the same as for our previous result, but instead of using Kleinstück's solution of the weighted approximation problem for vector-valued continuous functions in the restricted bounded case, we have preferred here to formulate the result in full generality and to apply the recent approximation theorems of Machado-Prolla for Nachbin spaces of cross-sections. This not only allows to relax the approximation-theoretic conditions in the theorem, but, as we have already seen in section 1, the use of Nachbin spaces of cross-sections (inspired by the results of Prolla's paper [25]) in this context makes the previous completeness assumptions superfluous.

Proof of Theorem 16. By Theorem 5 (4), it a enough to show that, for an arbitrary l.c. space F, $E \otimes F$ is dense in $E \varepsilon F$. Since $E\varepsilon F \cong \{f \in CV_o^{\sigma,c}(X,F);\ f' \circ f \in E \text{ for all } f' \in F'\}$ by Theorem 9 (3), we have to prove: Each function $f \in CV_o^{\sigma,c}(X,F)$ with $f' \circ f \in E$ for each $f' \in F'$ belongs to the closure of $E \otimes F$ in ($E\varepsilon F$ or, equivalently) $CV_o^{\sigma,c}(X,F)$. As $CV_o^{\sigma,c}(X,F)$ is a Nachbin space $L\tilde{V}_o$ of cross-sections (cf. 11 (2)) in which (by assumption) $Z := E \otimes F$ is $\mathcal{K}$-localizable, it suffices to verify that: Given any $K \in \mathcal{K}$, $f|_K$ belongs to the closure of $Z|_K = E|_K \otimes F$ in $L\tilde{V}_o|_K = CV_o^{\sigma,c}(X,F)|_K$ (under the weighted topology generated by the Nachbin family $V|_K$ on K).

But $f|_K \in CV_o^{\sigma,c}(X,F)|_K \subset C(V|_K)_o^{\sigma,c}(K,F)$ (as any $K \in \mathcal{K}$ is closed in X) satisfies $f' \circ (f|_K) = (f' \circ f)|_K \in E|_K \subset CV_o(X)|_K \subset C(V|_K)_o(K)$ for each $f' \in F'$, and by the description $(E|_K)\varepsilon F = \{g \in C(V|_K)_o^{\sigma,c}(K,F);\ f' \circ g \in E|_K \text{ for each } f' \in F'\}$ (which follows again from Theorem 9 (3)), we get $f|_K \in (E|_K)\varepsilon F$. By assumption $E|_K \subset C(V|_K)_o(K)$ has Schwartz's a.p., and so (once more) Theorem 5 (4) implies $f|_K \in \overline{E|_K \otimes F}^{(E|_K)\varepsilon F} \subset \overline{Z|_K}^{C(V|_K)_o^{\sigma,c}(K,F)}$ or $f|_K \in \overline{Z|_K}^{CV_o^{\sigma}(X,F)|_K}$, which is just what we had left to verify. □

Remark. It is not clear whether a converse of 16 holds in general, that is, whether Schwartz's a.p. for E also implies Schwartz's a.p.

for the topological subspaces $E|_K$ of $C(V|_K)_o(K)$, $K \in \mathcal{K}$. - In our scheme, this question is of course related to the problem whether, for l.c. (or only Banach) spaces F, $(E\varepsilon F)|_K = (E|_K)\varepsilon F$ holds algebraically. (The inclusion $(E\varepsilon F)|_K \subset (E|_K)\varepsilon F$ is obvious, and the topologies agree. - Remark also that $(E\otimes F)|_K = (E|_K) \otimes F$ is certainly always true; hence $(E|_K)\varepsilon F \subset (E\varepsilon F)|_K$ holds whenever both $(E|_K) \otimes F$ is dense in $(E|_K)\varepsilon F$ and $(E\varepsilon F)|_K$ is a closed subspace.) In fact, if $(E\varepsilon F)|_K = (E|_K)\varepsilon F$ holds for all Banach spaces F, the a.p. of E (by density of $E \otimes F$ in $E\varepsilon F$) yields density of $(E|_K) \otimes F = (E \otimes F)|_K$ in $(E|_K)\varepsilon F = (E\varepsilon F)|_K$ such that $E|_K$ has the a.p. by 5 (5).

In the situation of 16 and for Banach spaces F, we are thus led to ask (and this may be of independent interest): Does a function $f \in C(V|_K)_o^{\sigma,c}(K,F)$ which "extends to E weakly", i.e. satisfies $f' \circ f \in E|_K$ for each $f' \in F'$, extend to an element $g \in E\varepsilon F$, i.e. satisfy $f = g|_K$ for some $g \in CV_o^{\sigma,c}(X,F)$ such that $f' \circ g \in E$ for each $f' \in F'$? - Here the methods of Gramsch [15] (above all, cf. 2.5.-2.8.) can be applied and yield (at least) an idea how one might proceed: Fix $K \in \mathcal{K}$ and a Banach space F. Let E_o be the (by $V > 0$) closed linear subspace $\{e \in E;\ e|_K = 0\}$ of E. Then any $f \in (E|_K)\varepsilon F$ induces a canonical linear mapping $\tilde{f}: F' \to E/E_o$ (say, $\tilde{f}(f')$ = "extension" of $f' \circ f \in E|_K$ to E, modulo E_o) which, as one can immediately verify, is closed for the weak topologies $\sigma(F',F)$ and $\sigma(E/E_o, (E/E_o)')$ [and hence for all stronger topologies]. In many cases, $\tilde{f}$ must then already be continuous from F'_b into E/E_o: This follows sometimes directly from (general) closed graph theorems. Furthermore, it may also turn out that E/E_o carries (it is enough: a topology weaker than) the projective topology of a system $(X_\alpha)_{\alpha \in A}$ of Banach spaces with respect to linear mappings $\pi_\alpha: E/E_o \to X_\alpha$ such that all the compositions $\pi_\alpha \circ \tilde{f}$ are closed linear mappings; it is then sufficient to apply the classical closed graph theorem to get continuity of $\pi_\alpha \circ \tilde{f}$ for all α and hence continuity of $\tilde{f}: F'_b \to E/E_o$

(cf. Gramsch [15], 2.13). Of course, we would like to prove continuity of $\tilde{f}: F'_{cc} \to E/E_o$ rather and then to get a continuous linear "lifting" $\tilde{\tilde{f}}: F'_{cc} \to E$ of $\tilde{f}$ (i.e., for the quotient map $\pi: E \to E/E_o$, $\tilde{f} = \pi \circ \tilde{\tilde{f}}$ holds). In this case (after the canonical identification of $\tilde{\tilde{f}}$ with an element of $CV_o^{\sigma,c}(X,F)$) clearly $\tilde{\tilde{f}}|_K = f$, i.e. $f \in (E\varepsilon F)|_K$. (For the existence of liftings in concrete cases see e.g. Kaballo [18] - [20].)

Naturally, from the point of view of applications, a converse of 16 is of secondary importance anyway: It is much more interesting to derive the a.p. of the "complicated" space E from the a.p. of the "simpler" spaces $E|_K$ than conversely. - In fact, the smaller the sets $K \in \mathcal{K}$ in 16 are, the simpler the spaces $E|_K$ will become. Then the chances are better that the a.p. of all $E|_K$ is already known and that 16 can be applied to prove the a.p. of E.

This is perhaps a good point to remark that, instead of restricting our attention to spaces E of continuous functions as in 16, we could also have started with weighted spaces E of scalar functions on X of which only the restrictions to certain "characteristic" subsets of X are continuous: For a given completely regular space X, a Nachbin family $V > 0$ on X and a l.c. space F let, as in [4], p.39, $\mathfrak{F}_V$ denote the system of all sets $F_v := \{x \in X;\ v(x) \geq 1\}$, $v \in V$, and $RV_o(X,F) := \{f: X \to F;$ $f|_S$ continuous for each $S \in \mathfrak{F}_V$, $(vf)(X)$ precompact in F and vf vanishes at infinity for each $v \in V\}$, equipped with the natural l.c. topology given by the system $\{b_{v,p}\}$ of semi-norms as defined in 6. Again put $RV_o(X) := RV_o(X,\mathbb{K})$. $RV_o(X,F)$ is complete whenever F is.

We will assume from now on that $W \leq V$. Then each function $f \in RV_o(X,F)$ is a fortiori hypocontinuous, i.e. the restriction of f to each compact subset of X is continuous, and so clearly $(vf)(X)$ is precompact in F whenever vf vanishes at infinity. This yields, more simply, $RV_o(X,F) = \{f: X \to F;\ f|_{\{x \in X;\ v(x) \geq 1\}}$

is continuous and vf vanishes at infinity for each $v \in V\}$.

From [4], Theorem 27, p. 40, we know that for quasi-complete F

$$RV_o(X)\varepsilon F = RV_o(X,F)$$

holds, and so for topological linear subspaces E of $RV_o(X)$ we get again: $E\varepsilon F \cong \{f \in RV_o(X,F);\ f' \circ f \in E$ for all $f' \in F'\}$.

Similarly as in Lemma 11 above, let us now prove that for arbitrary $v \in V$, p continuous semi-norm on F and $f: X \to F$ hypocontinuous with vf vanishing at infinity the function $vp \circ f: x \to v(x)\ p(f(x))$ is still upper semicontinuous on X:

To do so, take $\varepsilon > 0$ and fix $x \in X$. Since vf vanishes at infinity there exists a compact subset K of X such that $p(v(y)f(y)) < p(v(x)f(x)) + \varepsilon$ for all $y \in X\setminus K$. If $x \notin K$, $X\setminus K$ is an open neighbourhood $U(x)$ of x in X such that $p(v(y)f(y)) < p(v(x)f(x)) + \varepsilon$ for all $y \in U(x)$. Now let $x \in K$. Since f is hypocontinuous, $f|_K$ is continuous, and so ($v \in V$ being upper semicontinuous) there exists a neighbourhood $U(x)$ of x in X such that, with $\delta := \min(1, \frac{\varepsilon}{3(p(f(x))+1)})$ and $\tilde{\delta} := \min(\frac{\varepsilon}{3}, \frac{\varepsilon}{3(v(x)+1)})$, $v(y) < v(x) + \delta$ for all $y \in U(x)$ and $p(f(y)-f(x)) < \tilde{\delta}$ for all $y \in U(x) \cap K$. Then we get again $p(v(y)f(y)) < p(v(x)f(x)) + \varepsilon$ for all $y \in U(x)$ since this is true whenever $y \in X\setminus K$ while for $y \in U(x) \cap K$:

$$v(y)p(f(y)) < (v(x)+\delta)(p(f(x))+\tilde{\delta}) = v(x)p(f(x)) + \delta p(f(x)) + \tilde{\delta}v(x) + \delta\tilde{\delta}$$

$$< v(x)p(f(x)) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = v(x)p(f(x)) + \varepsilon.$$

So f is upper semicontinuous at x.

It follows that even $RV_o(X,F)$ is a Nachbin space $L\tilde{V}_o$ of cross-sections in the same way as before (cf. 11 (2))! An inspection of the proof of 16 (and use of, say, 5 (5) instead of 5 (4)) now shows:

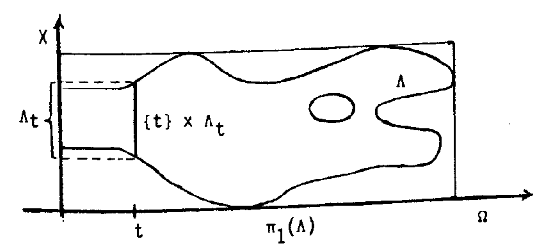
18 Theorem. If $W \leq V$, then 16 and 17 hold also for subspaces E of $RV_o(X)$ (and $L\tilde{V}_o = RV_o(X,F)$, F quasi-complete or Banach, as well as $R(V|_K)_o(K)$, $K \in \mathcal{K}$) [instead of $CV_o(X)$, $L\tilde{V}_o = CV_o^{\sigma,c}(X,F)$, and $C(V|_K)_o(K)$, respectively]. - Remark that E then even has the a.p. whenever it is closed in $RV_o(X)$.

Finally, let us note that our method of proof (as outlined in section 1) can also be applied, mutatis mutandis, to A-modules $E \subset CV(X)$ (instead of $CV_o(X)$): Here the ϵ-product representation of $E \epsilon F$ (F an arbitrary l.c. space) as a space of F-valued functions on X (in fact, a topological linear subspace of $CV^p(X,F)$ for $V_{\mathbb{R}}$-spaces X and quasi-complete F) is contained in Theorem 9, too, and, say, Theorem 9 (resp. 11) of section 2 of Kleinstück [21] yields "localization" of approximation from A-modules in $CV^p(X,F)$ in the (restricted) "bounded ($\hat{X}$-) case" (if $CV(X)$ satisfies condition (*) of [21], p.11). Since Kleinstück's approximation theorems work "only" in spaces of continuous functions, the (completeness type) assumption X $V_{\mathbb{R}}$-space is needed. Moreover, the results necessarily involve compactifications $\hat{X}$ of X (and extensions $(vf)^\wedge$ to $\hat{X}$ of the functions vf for arbitrary $v \in V$ and $f \in CV^p(X,F)$). As no "splitting" $(vf)^\wedge = v^\wedge f^\wedge$ is possible in general, we must rather suppose in this case that, for arbitrary complete l.c. (or Banach) space F and arbitrary $v \in V$, the F-valued functions $(vf)^\wedge$ with $f \in E \epsilon F$ can be approximated, uniformly on maximal A_V-antisymmetric subsets K_{A_V} of $\hat{X}_V$, by functions $(vz)^\wedge$ with $z \in E \otimes F$ (instead of assuming Schwartz's a.p. for restrictions $E|_K$ of E to maximal A-antisymmetric subsets K of X as in 17). - The corresponding proposition is (necessarily) technically involved, and as we do not want to repeat the notation and the definitions of [21] here, we leave its exact formulation to the interested reader. Because of the technical difficulties, this proposition is not as useful as 17, but Kleinstück [21], section 3, Theorem 5 and Korollar 6, has still been able to derive at least the a.p. of $CV(X)$ on $V_{\mathbb{R}}$-spaces X (and a slightly stronger result) in this way.

3. EXAMPLES FOR THE LOCALIZATION OF THE a.p. AMONG WEIGHTED SPACES WITH MIXED DEPENDENCE ON SUBSETS OF A PRODUCT

We turn to a number of examples for the localization of the a.p. which follow from Corollary 17 and which illustrate the application of the fundamental localization Theorem 16 to one concrete situation, namely to the case of weighted spaces "with mixed dependence" on subsets of a topological product (cf. [11]).

The general setting of these examples is as follows: Let Ω and X be completely regular (Hausdorff) spaces, $\Lambda \subset \Omega \times X$ a topological subspace, $V > 0$ a Nachbin family on Λ and Y a topological linear subspace of $CV_o(\Lambda)$. The canonical projection $\Omega \times X \to \Omega$ is denoted by π_1. For $t \in \pi_1(\Lambda)$ we identify $\Lambda_t := \{x \in X;\ (t,x) \in \Lambda\}$ with the "slice" $\{t\} \times \Lambda_t$. Correspondingly, we identify $Y_t := Y|_{\{t\} \times \Lambda_t}$ with a function space on $\Lambda_t \subset X$ and $V_t := V|_{\{t\} \times \Lambda_t}$ with a Nachbin family on Λ_t, too. Y_t is then a topological linear subspace of $C(V_t)_o(\Lambda_t)$.



19 Proposition. Let Y be a module over a subalgebra A of $C(\Lambda)$ and assume that we are in the bounded case of the weighted approximation problem for Y in $CV_o(\Lambda)$ (e.g., let $a|_{\text{supp } v}$ be bounded for each $a \in A$ and each $v \in V$). We suppose that, for an appropriate subset T of $\pi_1(\Lambda)$, $\mathcal{K}_A = \{\{t\} \times \Lambda_t;\ t \in T\} \cup \{\{p\};\ p \in \Lambda \setminus (\bigcup_{t \in T} \{t\} \times \Lambda_t)\}$. Then if the completion $\hat{Y}_t$ of Y_t has (Schwartz's) a.p. for each $t \in T$, Y has Schwartz's a.p., too.

Proof. We have only to check that $Y|_K$ has Schwartz's a.p. for each $K \in \mathcal{K}_A$ and may then apply Corollary 17. But in case $K \in \mathcal{K}_A$ is the one point set $\{p\}$, $p \in \Lambda \setminus (\bigcup_{t\in T} \{t\}\times\Lambda_t)$, $Y|_K$ trivially has the a.p., whereas for each $t \in T$, Schwartz's a.p. for $Y|_{\{t\}\times\Lambda_t} \cong Y_t$ follows from the a.p. of the completion $\hat{Y}_t$ (cf. Séminaire Schwartz [27], Exposé nº 15, Théorème 7). □

To demonstrate the rôle of the set $T \subset \pi_1(\Lambda)$ in 19, we consider just one simple example:

20 Example. Let $X = \mathbb{C}^N$ $(N \geq 1)$ [or, more generally, a quasi-complete dual-nuclear locally convex space]. Let Λ be a subspace of $\Omega \times X$ and T a closed subset of $\pi_1(\Lambda) \subset \Omega$ such that Λ_t is open in X for each $t \in T$ (e.g. $\Lambda \subset \Omega\times X$ open). $C\mathcal{O}_T(\Lambda) \subset \Omega$ denotes the space of all continuous functions f on Λ such that $f(t,\cdot)$ is holomorphic [i.e. continuous and G-analytic] on Λ_t for each $t \in T$, endowed with the topology of uniform convergence on all compact subsets of Λ. Then $C\mathcal{O}_T(\Lambda)$ has Schwartz's a.p.

Proof. $Y := C\mathcal{O}_T(\Lambda)$ is a module over the algebra $A := \{f \in CB(\Lambda)$ (i.e. continuous and bounded on Λ); $f|_{\{t\}\times\Lambda_t}$ constant for each $t \in T\}$.

We are in the restricted bounded case of the weighted approximation problem for $C\mathcal{O}_T(\Lambda)$ in $CV_o(\Lambda)$, where $V = W = \{\lambda\chi_K;\ \lambda > 0,\ K$ compact in $X\}$ (and $\chi_K :=$ characteristic function of K). Since A is selfadjoint, $\mathcal{K}_A$ coincides with the system of maximal subsets of Λ on which all functions in A are constant. Now (real-valued) bounded continuous functions on $\pi_1(\Lambda)$ separate points, and by constant "extension" along X it follows that each set $K \in \mathcal{K}_A$ is contained in a slice $\{t\}\times\Lambda_t$, $t \in \pi_1(\Lambda)$. Since T is closed, each slice $\{t_o\} \times \Lambda_{t_o}$ with $t_o \notin T$ clearly "splits up" in one point sets. (There exists a [real-valued] $\varphi \in CB(\pi_1(\Lambda))$ with $\varphi(t_o) = 1$, $\varphi|_T = 0$; then the functions $\varphi \otimes g|_\Lambda : \lambda = (t,x) \to \varphi(t)g(x)$, g [real-valued] $\in CB(X)$, belong to A and separate points of $\{t_o\} \times \Lambda_{t_o}$.) Hence $\mathcal{K}_A = \{\{t\}\times\Lambda_t;\ t \in T\} \cup \{\{p\};\ p \in \Lambda \setminus (\bigcup_{t\in T} \{t\} \times \Lambda_t)\}$, and 19 applies: For each $t \in T$, Y_t is a topological subspace of $(\mathcal{O}(\Lambda_t),co)$, the space of holomorphic functions on Λ_t, endowed with the topology of uniform convergence on all compact subsets of Λ_t. Since $(\mathcal{O}(\Lambda_t),co)$ is nuclear [in the infinite dimensional case, this follows from a recent theorem of Boland and Waelbroeck], Y_t (and, a fortiori, $\hat{Y}_t$) is nuclear, too, and hence has the a.p.. □

For all other examples from now on, we will no longer be interested in this (full) generality of 19, thus we will assume that $T = \pi_1(\Lambda)$, i.e. $\mathcal{K}_A = \{\{t\}\times\Lambda_t;\ t \in \pi_1(\Lambda)\}$, holds. This is certainly <u>the most interesting case</u>.

Let us introduce some notation at this point: $\mathfrak{F}$ will always denote a (topological) <u>sub-(pre-)sheaf</u> of the sheaf C_X of continuous functions on X, that is, for each open subset U of X, $\mathfrak{F}(U)$ denotes a (topological) linear subspace of $(C(U),co)$, the space of all continuous (scalar) functions on U (endowed with the compact-open topology co of uniform convergence on all compact subsets of U).
- The first case in which we are interested is in some sense similar to Example 20 and requires $\Lambda \subset \Omega\times X$ to be a subspace with the property that Λ_t is <u>open</u> in X for each $t \in \pi_1(\Lambda)$. Then we can define:

21 Definition. $C\mathfrak{F}(\Lambda) := \{f$ continuous on Λ; $f(t,\cdot) \in \mathfrak{F}(\Lambda_t)$ for each $t \in \pi_1(\Lambda)\}$, endowed with the topology co of uniform convergence on all compact subsets of Λ; $C\mathfrak{F}V_o(\Lambda) := C\mathfrak{F}(\Lambda) \cap CV_o(\Lambda)$ with the topology induced by $CV_o(\Lambda)$, and, similarly, for $t \in \pi_1(\Lambda)$: $\mathfrak{F}(V_t)_o(\Lambda_t) := \mathfrak{F}(\Lambda_t) \cap C(V_t)_o(\Lambda_t)$ (with the weighted topology from the Nachbin family V_t).

22 Proposition. Let $Y := C\mathfrak{F}V_o(\Lambda)$ and assume that one of the following conditions (a) or (b) is satisfied for each $t \in \pi_1(\Lambda)$:

(a) $\mathfrak{F}(V_t)_o(\Lambda_t)$ is nuclear,

(b) Y_t is dense in $\mathfrak{F}(V_t)_o(\Lambda_t)$, and $\mathfrak{F}(V_t)_o(\Lambda_t)$ has Schwartz's a.p. Then $C\mathfrak{F}V_o(\Lambda)$ has Schwartz's a.p., too.

Proof. Take A as in Example 20 (with $T = \pi_1(\Lambda)$), so $\mathfrak{X}_A = \{[\{t\}\times$ ...; $t \in \pi_1(\Lambda)\}$ holds. Again Y is quite obviously a module over this algebra. Then for each $t \in \pi_1(\Lambda)$ Schwartz's a.p. for $Y|_{\{t\}\times\Lambda_t} \cong Y_t$ follows from our conditions: In case (a), Y_t is a subspace of the nuclear space $\mathfrak{F}(V_t)_o(\Lambda_t)$ and hence nuclear, too, while in case (b), Y_t inherits Schwartz's a.p. from $\mathfrak{F}(V_t)_o(\Lambda_t)$ of which it is a dense topological linear subspace. □

Under additional completeness assumptions, Theorem 19 and Proposition 22 had already been given as 4.7 and 4.8 in [11]. See ([10] and) [11] for examples of concrete sheaves $\mathfrak{F}$, for some concrete examples following from 22 and for applications to vector-valued functions and "density theorems" (which we will not repeat here). - 4.9 and 4.10 of [11] considered only the compact-open topology, and 4.10 (2) is the only example for an application of case (b) in Proposition 22 which was mentioned there. (More examples were promised in [11], and we are now ready to keep this promise.)

The density condition in 22 (b) leads to a "regularity" assumption on the set Λ, as we shall see in a moment (cf. also [11], 2): We will state a sufficient condition which implies the required density in our next remark. - For the rest of our discussion of this first case, we will assume that Λ is an open subset of $\Omega\times X$: this is certainly satisfied in the most interesting examples based on 22.

23 Remark. Fix $t_o \in \pi_1(\Lambda)$. Let $\mathfrak{F}_{t_o}V$ denote the set of all functions g as follows:

We suppose that there is an open neighbourhood $U \subset \pi_1(\Lambda)$ of t_o in Ω and a function $\varphi \in CB(\Omega)$ with $\varphi(t_o) \neq 0$, but $\varphi \equiv 0$ outside of U. Let Λ_U be the open topological subspace $\bigcup_{t\in U} \Lambda_t$ of X.

Then $g \in \mathfrak{F}(\Lambda_U)$, and the function $\varphi_g \in C\mathfrak{F}(\Lambda)$, defined by

$$\varphi_g(t,x) = \begin{cases} \varphi(t)g(x), & t \in \bar{U} \text{ and } x \in \Lambda_t \\ 0, & \text{elsewhere on } \Lambda \end{cases},$$

belongs to $C\mathfrak{F}V_o(\Lambda)$ (i.e. satisfies the weight conditions induced by V). If $\mathfrak{F}_{t_o}V|_{\Lambda_{t_o}}$ is dense in $\mathfrak{F}(V_{t_o})_o(\Lambda_{t_o})$, then density of Y_{t_o} (for $Y = C\mathfrak{F}V_o(\Lambda)$) in $\mathfrak{F}(V_{t_o})_o(\Lambda_{t_o})$ holds, as required in 22 (b).

Proof. Let U, φ and g be arbitrary as in the definition of $\mathfrak{F}_{t_o}V$. Let $(t,x) \in \Lambda$ satisfy $t \in \bar{U}\setminus U$. Since Λ is open, there exist neighbourhoods U_1 of t in Ω and U_2 of x in X such that $U_1\times U_2 \subset \Lambda$. Because of $t \in \bar{U}$ we can find $t_1 \in U_1 \cap U$. Then $(t_1,x) \in U_1\times U_2 \subset \Lambda$, and hence $x \in \Lambda_{t_1} \subset \Lambda_U$ holds. It is now clear that φ_g is well-defined and continuous on Λ and so belongs to $C\mathfrak{F}(\Lambda)$.

After this remark, it suffices to point out that for any $g \in \mathfrak{F}_{t_o}V$ obviously

$$g|_{\Lambda_{t_o}} = \frac{1}{\varphi(t_o)}\varphi_g(t_o,\cdot) \in Y_{t_o}$$

holds and that therefore density of $\mathfrak{F}_{t_o}V|_{\Lambda_{t_o}}$ obviously implies $Y_{t_o} \subset \mathfrak{F}(V_{t_o})_o(\Lambda_{t_o})$ dense. □

We note that the definition of $\mathfrak{F}_{t_o}V$ in 23 involves two different restrictions (as compared with elements of $\mathfrak{F}(V_{t_o})_o(\Lambda_{t_o})$): The functions $g \in \mathfrak{F}_{t_o}V$ are defined and belong to the sheaf $\mathfrak{F}$ on some open set Λ_U which may be strictly larger than Λ_{t_o}, and their constant "extensions" along Ω, multiplied by a suitable "cut-off" function φ, must satisfy the weight conditions given by V on $\bigcup_{t\in U}\{t\}\times\Lambda_t \subset \Lambda$. The cutting-off process was introduced here in order to let the elements by which we approximate in $\mathfrak{F}(V_{t_o})_o(\Lambda_{t_o})$ belong to the sheaf $\mathfrak{F}$ only on some Λ_U (and not on all of $\bigcup_{t\in\pi_1(\Lambda)}\Lambda_t$) and, on the other hand, to take care of the growth conditions "in Ω-direction". (If Ω is locally compact, U may be chosen relatively compact, then φ has compact support in Ω.) -

The regularity assumption on Λ which we have mentioned before 23 comes - even if weight conditions do not exist, cf. [11], 4.10 (2) - from the approximation in $\mathfrak{F}(V_{t_o})_o(\Lambda_{t_o})$ by functions extending to elements of $\mathfrak{F}$ on some open set $\Lambda_U \supset \Lambda_{t_o}$.

In our next corollary, we will put much stronger regularity conditions on Λ than an application of 23 would really require. We will also restrict our attention to the case that $\mathfrak{F} = \mathfrak{O}$, the sheaf of holomorphic functions on $\mathbb{C}^N$ $(N \geq 1)$ here.

24 Corollary. Let $X = \mathbb{C}^N$ $(N \geq 1)$, and let Λ be an open subset of $\Omega \times X$. Then $C\mathfrak{O}V_o(\Lambda)$ has Schwartz's a.p. if for each $t \in \pi_1(\Lambda)$ the following conditions hold:

(i) $\mathfrak{O}(V_t)_o(\Lambda_t)$ has Schwartz's a.p.,

(ii) for each polynomial p (on $\mathbb{C}^N$), constant "extension" along Ω leads, after multiplication by a suitable cut-off function $\varphi \in CB(\Omega)$ with $\varphi(t) \neq 0$, to a function in $C\mathfrak{O}V_o(\Lambda)$ (i.e. the function φ_p, defined as in 23, satisfies the weight conditions given by V),

(iii) (restrictions of) polynomials are dense in $\mathfrak{O}(V_t)_o(\Lambda_t)$.

It should be clear by now that 24 is immediate from 22 (and 23). - Condition (ii) of 24 already implies that $Y_t \subset \mathfrak{O}(V_t)_o(\Lambda_t)$ (for $Y = C\mathfrak{O}V_o(\Lambda)$) contains all (restriction of) polynomials. Together with (iii), this is certainly a rather crude way to ensure that Y_t is dense in $\mathfrak{O}(V_t)_o(\Lambda_t)$! However, as on the question when (iii) is satisfied much information can be found in the literature, Corollary 24 makes it easy to construct concrete examples. - Roughly spoken, by condition (ii), V may impose rather arbitrary growth conditions in Ω-direction (above all if Ω is locally compact), but V (or rather the Nachbin families V_t on Λ_t for $t \in \pi_1(\Lambda)$) must allow polynomial growth in X-direction. (This is indeed fulfilled if the sets

Λ_t are relatively compact and if all the weights in V_t vanish at infinity on Λ_t.)

Not too many general results on the a.p. of the spaces $\mathfrak{O}V_o(G)$ on open sets $G \subset \mathbb{C}^N$ are known, however. All these results require a very special form of G and / or very special Nachbin families V. Let us mention the following:

(1) On the open unit disk $D \subset \mathbb{C}$, $\mathfrak{O}V_o(D)$ has the a.p. if V consists of "normal" weights v only (i.e. v is positive, continuous, and radial, and there exist $0 < \varepsilon < k$ and $r_o < 1$ such that for all $r \geq r_o$ as $r \to 1-$: $\frac{v(r)}{(1-r)^\varepsilon} \searrow 0$ and $\frac{v(r)}{(1-r)^k} \nearrow \infty$). - This follows from a theorem of Shields-Williams, see [8], §2, where we have also pointed out that then the a.p. of $\mathfrak{O}V_o(G)$ follows for (certain) simply connected regions G in $\mathbb{C}$ and some product regions $G \subset \mathbb{C}^N$ (if V is of a special form).

(2) Kaballo carried the methods of (1) over to more general domains in $\mathbb{C}^N$ (and developed new methods) to prove analogues of the Shields-Williams theorem, and to get the a.p., say, for $\mathfrak{O}v_o(G)$ $(= \mathfrak{O}(V_v)_o(G)$ with $V_v := \{\lambda v;\ \lambda > 0\})$

- on the unit ball G (or a polydisc G) $\subset \mathbb{C}^N$ if the weight v satisfies only a weaker condition than normality ([19], 3.12), or

- on a bounded strictly pseudoconvex region G with C^∞-boundary if the weight is normal in a restricted sense ([20], 2.7).

- Moreover, $\mathfrak{O}V_o(G)$ also has the a.p. on a bounded region G which is "approximable from the exterior" if V is a countable Nachbin family of "admissible" weights which satisfies a certain "compactness" condition ([18], 6.6).

(3) Some work has also been done in the case $V = C_o^+(G)$ (= all non-negative continuous functions which vanish at infinity on G), when $\mathfrak{O}V_o(G) = (H^\infty(G), \beta)$ is the space of all bounded holomorphic functions with the strict topology β: In [2] and [3], we established the a.p.

of this space for arbitrary simply connected regions $G \subset \mathbb{C}$ and for products of such regions. Recently Kaballo ([18], 6.6 and 6.9) proved the a.p. of $(H^\infty(G),\beta)$ also for strictly pseudoconvex regions $G \subset \mathbb{C}^N$ with C^4-boundary and for bounded regions which are approximable from the exterior.

Normal weights clearly vanish at infinity. Remark also that (say, by the results of Shields-Williams) polynomials are dense in $\mathfrak{H}V_o(D)$ for Nachbin families V of normal weights on the open unit disk. Hence we can easily construct examples of sets Λ and of Nachbin families V, such that each V_t consists of normal weights on Λ_t $(t \in \pi_1(\Lambda))$, and such that $C\mathfrak{H}V_o(\Lambda)$ has Schwartz's a.p. by 24 (and the preceding remarks).

We leave the formulation of a general theorem of this type to the reader and note only the following (somewhat "curious") situation: Even on product sets, say, on $\Lambda = (0,1) \times D$, it is possible to exhibit Nachbin families V of continuous functions, with V_t consisting only of normal weights on D for each $t \in (0,1)$, such that $C\mathfrak{H}V_o(\Lambda)$ (is necessarily complete and) has the a.p. by 24, but such that no ε-tensor product "decomposition" of $C\mathfrak{H}V_o(\Lambda)$ holds (which would allow an easier proof of the a.p.)!

If the Nachbin family V is of the very special form $V_1 \otimes V_2$ (with Nachbin families V_1 on Ω and V_2 on X), however, general "slice product theorems" (cf. [3]) will usually give

$$C\mathfrak{F}(V_1 \times V_2)_o(\Omega \times X) = C(V_1)_o(\Omega)\, \check{\otimes}_\varepsilon\, \mathfrak{F}(V_2)_o(X),$$

and then the a.p. of $C\mathfrak{F}V_o(\Omega \times X)$ may also follow from the fact (Schwartz [29], p.48) that $E \varepsilon F$ inherits Schwartz's a.p. from the (quasi-complete) spaces E and F. (Because of this, we had only been interested in subsets of products and not in product sets in [11].)

Let us finish the first case with another example which is based on the results recalled in (3) above:

25 Example. Let $X = \mathbb{C}$, let Ω be locally compact and $\Lambda \subset \Omega \times \mathbb{C}$ open. We asssume that V is a Nachbin family of continuous weights on Λ such that for each $t \in \pi_1(\Lambda)$:

(i) Λ_t is a bounded region in $\mathbb{C}$ with $\Lambda_t = \mathring{\overline{\Lambda}}_t$ and $\mathbb{C} \setminus \overline{\Lambda}_t$ connected,

(ii) $V_t = C_o^+(\Lambda_t)$ (up to "equivalence"),

(iii) each polynomial belongs to Y_t (for $Y = C\mathfrak{H}V_o(\Lambda)$).

Then $C\mathfrak{H}V_o(\Lambda)$ has the a.p.

Proof. Local compactness of Λ, continuity of the weights and (ii) combine to yield $CV_o(\Lambda)$ complete; then its closed subspace $C\mathfrak{H}V_o(\Lambda)$ is complete, too. - Now fix $t \in \pi_1(\Lambda)$. Since $\Lambda_t = \mathring{\overline{\Lambda}}_t$ holds, $\mathbb{C} \setminus \Lambda_t$ is the closure of the connected set $\mathbb{C} \setminus \overline{\Lambda}_t$ by (i) and hence itself connected so that Λ_t is simply connected. Clearly (ii) gives $\mathfrak{H}(V_t)_o(\Lambda_t) = (H^\infty(\Lambda_t),\beta)$ which then has the a.p. by (3) above. Moreover, (i) allows to apply the well-known theorem of Farrel on "pointwise bounded" approximation by polynomials to get density of polynomials in $(H^\infty(\Lambda_t),\beta)$. Hence the proof is finished by 24. □

Several different sufficient conditions which imply 25 (iii) are conceivable, but instead of discussing possible special cases of 25, we turn to the second general class of weighted spaces with mixed dependence in which we are interested and where the localization method can be applied to prove the a.p.

We start by introducing (resp. recalling, cf. [7]) some notation: Ω and X are like at the beginning of this paragraph, and $\mathfrak{F}$ again denotes a (pre-) sheaf (of continuous functions) on X. We assume now that the topological subspace $\Lambda \subset \Omega \times X$ has the property that Λ_t is compact in X for each $t \in \pi_1(\Lambda)$. Let V be a Nachbin family on Λ which satisfies $W \leq V$ (such that the weighted topology of $CV_o(\Lambda)$ is stronger than uniform convergence on compact subsets of Λ, and hence $CV_o(\Lambda)$ is complete whenever Λ is a

$k_{\mathbb{R}}$-space, see 7 above). Then for any topological linear subspace Y of $CV_o(\Lambda)$, the space Y_t is obviously (topologically isomorphic to) a <u>normed</u> subspace of $C(\Lambda_t)$ (= Banach space of all continuous functions on Λ_t under its canonical sup-norm) for each $t \in \pi_1(\Lambda)$, because upper semicontinuous functions are bounded on compact subsets. (<u>So there are no weight conditions whatsoever in</u> X-<u>direction</u> here!) Hence a condition like the one in 22(a) is of no use in this case, but under the assumptions of 19 the space Y has Schwartz's a.p. if for each $t \in T$ (here again $= \pi_1(\Lambda)$ for simplicity) the closure $\overline{Y_t}$ of Y_t in $C(\Lambda_t)$ has the a.p.

For a compact set K in X and a l.c. space F we define (with $C(K,F)$ = space of all continuous F-valued functions on K under the topology of uniform convergence on K):

$A_{\mathfrak{Z}}(K,F) := \{f \in C(K,F);\ f' \circ f|_{\overset{\circ}{K}} \in \mathfrak{Z}(\overset{\circ}{K})$ for each $f' \in F'\}$, and

$H_{\mathfrak{Z}}(K,F)$:= the closure in $C(K,F)$ of

$\{f \in C(K,F)$; there exists an open neighbourhood U of K (depending on f) and a function $g: U \to F$ continuous with $f' \circ g \in \mathfrak{Z}(U)$ for any $f' \in F'$ such that $g|_K = f\}$.

If $\mathfrak{Z}$ is a closed locally convex sub- (pre-) sheaf of C_X, i.e. if for each open subset U of X the topological linear subspace $\mathfrak{Z}(U)$ of $(C(U),co)$ is closed - <u>which we will assume from now on</u> -, $A_{\mathfrak{Z}}(K,F)$ is a <u>closed</u> subspace of $C(K,F)$ and hence $H_{\mathfrak{Z}}(K,F) \subset A_{\mathfrak{Z}}(K,F)$ holds. Both spaces are endowed with the topology induced by $C(K,F)$. If $F = \mathbb{K}$, we omit this symbol and so have introduced the Banach spaces $A_{\mathfrak{Z}}(K)$ and $H_{\mathfrak{Z}}(K)$.

<u>26 Definition</u>. $CA_{\mathfrak{Z}}(\Lambda,F)$ [resp. $CH_{\mathfrak{Z}}(\Lambda,F)$] $:= \{f: \Lambda \to F$ continuous; $f(t,\cdot) \in A_{\mathfrak{Z}}(\Lambda_t,F)$ [resp. $H_{\mathfrak{Z}}(\Lambda_t,F)$] for each $t \in \pi_1(\Lambda)\}$, endowed with the topology co of uniform convergence on all compact subsets of Λ;

and
$$CA_{\mathfrak{Z}}V_o(\Lambda,F) := CA_{\mathfrak{Z}}(\Lambda,F) \cap CV_o(\Lambda,F)$$

$$CH_{\mathfrak{Z}}V_o(\Lambda,F) := CH_{\mathfrak{Z}}(\Lambda,F) \cap CV_o(\Lambda,F),$$

both endowed with the weighted topology induced by $CV_o(\Lambda,F)$. - If $F = \mathbb{K}$, we again omit this symbol in each case.

Under our general assumptions ($\mathfrak{Z}$ closed and $W \leq V$), $CH_{\mathfrak{Z}}V_o(\Lambda,F) \subset CA_{\mathfrak{Z}}V_o(\Lambda,F)$ are closed subspaces of $CV_o(\Lambda,F)$ and hence complete whenever Λ is a $k_{\mathbb{R}}$-space and F complete. - The following proposition is for the present case what 22(b) was for the first case:

<u>27 Proposition</u>. Let $Y := CA_{\mathfrak{Z}}V_o(\Lambda)$ [resp. $CH_{\mathfrak{Z}}V_o(\Lambda)$]. Assume that for each $t \in \pi_1(\Lambda)$ the following two conditions hold:

(a) Y_t is <u>dense</u> in $A_{\mathfrak{Z}}(\Lambda_t)$ [resp. $H_{\mathfrak{Z}}(\Lambda_t)$],

(b) $A_{\mathfrak{Z}}(\Lambda_t)$ [resp. $H_{\mathfrak{Z}}(\Lambda_t)$] has the a.p.

Then Y has Schwartz's a.p. (and hence the a.p. if Λ is a $k_{\mathbb{R}}$-space).

<u>Proof</u>. After our preceding remarks we have only to observe that Y is again quite obviously a module over the algebra A, defined as in Example 20 (with $T = \pi_1(\Lambda)$).

Conditions (a) and (b) imply that Y_t has the a.p. as dense subspace of a Banach space with a.p. for each $t \in \pi_1(\Lambda)$, so we may apply 19. □

As before, the density assumption in 27 (a) leads to regularity restrictions on Λ. In fact:

<u>28 Remark</u>. Fix $t_o \in \pi_1(\Lambda)$. With $Y = CA_{\mathfrak{Z}}V_o(\Lambda)$, the following are two examples of <u>sufficient</u> conditions for density of Y_{t_o} in $A_{\mathfrak{Z}}(\Lambda_{t_o})$, as required in 27 (a):

(i) Let $A_{\mathfrak{Z}}^{t_o}V$ denote the set of all functions g as follows: We suppose that there is a neighbourhood U of t_o in $\pi_1(\Lambda)$ with $\Lambda_U := \bigcup_{t \in U} \Lambda_t$ <u>relatively compact</u> in X and a function $\varphi \in CB(\pi_1(\Lambda))$ with $\varphi(t_o) \neq 0$, but $\varphi \equiv 0$ off U. Then $g \in A_{\mathfrak{Z}}(\overline{\Lambda_U})$, and the function $\varphi_g \in CA_{\mathfrak{Z}}(\Lambda)$, defined by:

$$\varphi_g(t,x) := \begin{cases} \varphi(t)g(x), & t \in U \text{ and } x \in \Lambda_t \\ 0 \quad, & \text{elsewhere on } \Lambda \end{cases},$$

belongs to $CA_{\mathfrak{F}}V_o(\Lambda)$ (i.e. satisfies the weight conditions induced by V). - Under these assumptions, our condition reads: $A_{\mathfrak{F}}^{t_o}V|_{\Lambda_{t_o}}$ is <u>dense</u> in $A_{\mathfrak{F}}(\Lambda_{t_o})$.

(ii) There is a neighbourhood U of t_o in $\pi_1(\Lambda)$ such that $\overline{\Lambda_U} = \overline{\bigcup_{t\in U} \Lambda_t}$ is <u>compact in</u> X and $\overline{\Lambda \cap (U\times X)} = \overline{\bigcup_{t\in U} \{t\} \times \Lambda_t}$ compact in Λ and such that we have $A_{\mathfrak{F}}(\overline{\Lambda_U})|_{\Lambda_{t_o}} \subset A_{\mathfrak{F}}(\Lambda_{t_o})$ <u>dense</u>.

<u>Proof</u>. Since any function $g \in A_{\mathfrak{F}}(\overline{\Lambda_U})$ is uniformly bounded, it is easy to see that φ_g (defined as above) is continuous on Λ and hence belongs to $CA_{\mathfrak{F}}(\Lambda)$, as claimed. Then the proof of (i) follows (cf. 23) from the fact that for any function $g \in A_{\mathfrak{F}}^{t_o}V$, $g|_{\Lambda_{t_o}} \in Y_{t_o}$. - For (ii) remark that the weight conditions given by V are certainly satisfied by all functions which vanish off the compact subset $\overline{\Lambda \cap (U\times X)}$ of Λ. Hence we may apply the method of (i) even with the <u>fixed</u> neighbourhood U of t_o alone (and arbitrary cut-off function φ). □

Similarly, with $Y = CH_{\mathfrak{F}}V_o(\Lambda)$, the following is a <u>sufficient</u> condition for density of Y_{t_o} in $H_{\mathfrak{F}}(Y_{t_o})$: $H_{\mathfrak{F}}^{t_o}V|_{\Lambda_{t_o}}$ <u>dense</u> in $H_{\mathfrak{F}}(\Lambda_{t_o})$, where $H_{\mathfrak{F}}^{t_o}V$ is the set of all functions g as follows: We suppose that there is a neighbourhood U of t_o in $\pi_1(\Lambda)$ and a function $\varphi \in CB(\pi_1(\Lambda))$ with $\varphi(t_o) \neq 0$, but $\varphi \equiv 0$ off U. Let N denote an open neighbourhood of $\Lambda_{\overline{U}} := \bigcup_{t\in\overline{U}} \Lambda_t$. Then $g \in \mathfrak{F}(N)$ and the function $\varphi_g \in CH_{\mathfrak{F}}(\Lambda)$, defined as in 28(i), belongs to $CH_{\mathfrak{F}}V_o(\Lambda)$ (i.e. satisfies the weight conditions induced by V).

However, if we are willing to assume <u>a stronger regularity condition for</u> Λ, the density assumption 27(a) becomes <u>superfluous</u> for $Y = CH_{\mathfrak{F}}V_o(\Lambda)$:

<u>29 Proposition</u>. Assume that Λ is <u>regular</u> in the following sense: For each $t_o \in \pi_1(\Lambda)$ and arbitrary open neighbourhood N of Λ_{t_o} in X, there exists a neighbourhood U of t_o in $\pi_1(\Lambda)$ such that $\overline{\Lambda_U} = \overline{\bigcup_{t\in U} \Lambda_t}$ is <u>compact and contained in</u> N and $\overline{\Lambda \cap (U\times X)} = \overline{\bigcup_{t\in U} \{t\} \times \Lambda_t}$ compact in Λ. Then Y_{t_o} (with $Y = CH_{\mathfrak{F}}V_o(\Lambda)$) is <u>dense</u> in $H_{\mathfrak{F}}(\Lambda_{t_o})$ for each $t_o \in \pi_1(\Lambda)$, and hence $CH_{\mathfrak{F}}V_o(\Lambda)$ has Schwartz's a.p. whenever $H_{\mathfrak{F}}(\Lambda_t)$ has the a.p. for each $t \in \pi_1(\Lambda)$.

<u>Proof</u>. Fix $f \in H_{\mathfrak{F}}(\Lambda_{t_o})$ and $\varepsilon > 0$. By definition there exists an open neighbourhood N of Λ_{t_o} and a function $g \in \mathfrak{F}(N)$ such that $\sup_{x\in\Lambda_{t_o}} |f(x)-g(x)| < \varepsilon$. By the assumed <u>regularity of</u> Λ, we can find a neighbourhood U of t_o in $\pi_1(\Lambda)$ such that $\overline{\Lambda_U}$ is compact $\subset N$ and $\overline{\Lambda \cap (U\times X)}$ compact in Λ. The function φ_g, defined as in 28(i), is continuous on Λ (since $g|_{\Lambda_U}$ must be bounded) and clearly belongs to $CH_{\mathfrak{F}}V_o(\Lambda)$ because it vanishes off the compact subset $\overline{\Lambda \cap (U\times X)}$ of Λ. But since $g|_{\Lambda_{t_o}} = \frac{1}{\varphi(t_o)} \varphi_g|_{\{t_o\}\times\Lambda_{t_o}}$, it turns out that $g|_{\Lambda_{t_o}} \in Y_{t_o}$, and as $\varepsilon > 0$ was arbitrary, f may be approximated by elements of Y_{t_o}, which proves our claim. □

We have treated the general problem of the a.p. for $A_{\mathfrak{F}}(K)$ and $H_{\mathfrak{F}}(K)$ in [7], and surveyed the known theorems in the cases $\mathfrak{F} = \mathfrak{O}$, the sheaf of <u>holomorphic</u> functions on $\mathbb{C}^N$ ($N \geq 1$), and, say, $\mathfrak{F} = \mathfrak{K}$, the sheaf of <u>harmonic</u> functions on $\mathbb{R}^N$ ($N \geq 2$) [resp. (some) sheaves of harmonic functions in <u>axiomatic potential theory</u>], in the last sections of that paper. We will not repeat the results of [7] here, but shall now note some of the examples of spaces $CH_{\mathfrak{F}}V_o(\Lambda)$ with a.p. which immediately follow from these results (and from 29 above):

<u>30 Proposition</u>. Assume that Λ is <u>regular</u> in the sense of 29. Then $Y = CH_{\mathfrak{F}}V_o(\Lambda)$ has Schwartz's a.p. in each of the following cases:

(a) $X = \mathbb{C}$, $\mathfrak{F} = \mathfrak{O}$;

(b) $X = \mathbb{C}^N$ ($N > 1$); $\mathfrak{F} = \mathfrak{O}$, and for each $t \in \pi_1(\Lambda)$ the compact set Λ_t = the closure of a strictly pseudoconvex region with sufficiently smooth (say, C^3-) boundary or the closure of a regular Weil polyeder; more generally:

(c) $X = \mathbb{C}^N$ $(N>1)$, $\mathfrak{Z} = \mathcal{O}$, and for each $t \in \pi_1(\Lambda)$, Λ_t = a product $\Lambda_t^1 \times \ldots \times \Lambda_t^k$, where each compact set Λ_t^j $(j=1,\ldots,k)$ is either a subset of $\mathbb{C}$ or the closure of a strictly pseudoconvex region with sufficiently smooth boundary or the closure of a regular Weil polyeder;

(d) $X = \mathbb{R}^N$ $(N\geq 2)$ [or X the space of definition of a (suitable) harmonic sheaf of axiomatic potential theory], $\mathfrak{Z} = \mathcal{H}$ (and for each $t \in \pi_1(\Lambda)$ the compact set Λ_t = the closure of an open subset of X).

When $A_{\mathfrak{Z}}(\Lambda_t) = H_{\mathfrak{Z}}(\Lambda_t)$ holds for all $t \in \pi_1(\Lambda)$, $CH_{\mathfrak{Z}}V_o(\Lambda)$ is equal to $CA_{\mathfrak{Z}}V_o(\Lambda)$, and hence 30. gives some information on the a.p. of $CA_{\mathfrak{Z}}V_o(\Lambda)$, too. It is interesting, however, to presente some simple concrete examples of spaces of type $CA_{\mathfrak{Z}}V_o(\Lambda)$ with a.p. which follow from the results mentioned in [7] and from 27, because we will not require Λ to be regular in the sense of 29 here. For simplicity let us consider the case $V = W$ (of the compact-open topology on Λ) only and so restrict our attention to $CA_{\mathfrak{Z}}(\Lambda)$ (instead of $CA_{\mathfrak{Z}}V_o(\Lambda)$ for general Nachbin families V on Λ).

31 Example. Let Ω be locally compact, $X = \mathbb{C}^N$, and Λ a closed subset of $\Omega \times X$ (with Λ_t compact for each $t \in \pi_1(\Lambda)$). Then $Y = CA_{\mathfrak{Z}}(\Lambda)$ has the a.p. in each of the following cases:

(a) $N = 1$, $\mathfrak{Z} = \mathcal{O}$ or $\mathcal{H}$, and for each $t \in \pi_1(\Lambda)$, the set $\mathbb{C}\setminus\Lambda_t$ is connected; (b) $N > 1$, $\mathfrak{Z} = \mathcal{O}$, and for each $t \in \pi_1(\Lambda)$, Λ_t is polynomially convex, has the so-called "segment property" and is a product $\Lambda_t^1 \times \ldots \times \Lambda_t^k$, where each compact set Λ_t^j $(j=1,\ldots,k)$ is either a "fat" compact subset of $\mathbb{C}$ with $\mathbb{C} \setminus \Lambda_t^j$ connected or the closure of a strictly pseudoconvex region with sufficiently smooth boundary.

Proof. We remark that Λ is locally compact as a closed subset of $\Omega \times \mathbb{C}^N$. Moreover, under the conditions above, polynomials [resp. real parts of complex polynomials] are dense in $A_{\mathcal{O}}(\Lambda_t)$ [resp. $A_{\mathcal{H}}(\Lambda_t)$]

for each $t \in \pi_1(\Lambda)$. (In case (a), this is Mergelyan's theorem resp. the Walsh-Lebesgue theorem, for (b) use e.g. [5], Theorem 5.2.) Therefore the density assumption 27(a) is certainly satisfied. For the a.p. of the spaces $A_{\mathfrak{Z}}(\Lambda_t)$, $t \in \pi_1(\Lambda)$, which is needed in 27(b), we refer to [7]. □

Let us now have a look at the spaces of vector-valued functions introduced in 26 and derive a "density theorem" (similar to [11], 4.11 resp. 4.12 for $C\mathfrak{Z}V_o(\Lambda,F)$ resp. $C\mathfrak{Z}(\Lambda,F)$).

32 Proposition. Let F be quasi-complete and Λ a $k_{\mathbb{R}}$-space.

(1) Then $CA_{\mathfrak{Z}}V_o(\Lambda,F) = CA_{\mathfrak{Z}}V_o(\Lambda)\varepsilon F$ holds.

(2) Hence we have $CA_{\mathfrak{Z}}V_o(\Lambda,F) = CA_{\mathfrak{Z}}V_o(\Lambda) \check{\otimes}_\varepsilon F$ whenever F is even complete and $CA_{\mathfrak{Z}}V_o(\Lambda)$ (or F) has the a.p.

Let now F be complete (and Λ a $k_{\mathbb{R}}$-space).

(3) Then we have the following inclusions of topological linear subspaces of $CV_o(\Lambda,F)$:

$$CH_{\mathfrak{Z}}V_o(\Lambda) \check{\otimes}_\varepsilon F \subset CH_{\mathfrak{Z}}V_o(\Lambda,F) \subset CH_{\mathfrak{Z}}V_o(\Lambda)\varepsilon F.$$

(4) Hence we have $CH_{\mathfrak{Z}}V_o(\Lambda,F) = CH_{\mathfrak{Z}}V_o(\Lambda)\varepsilon F = CH_{\mathfrak{Z}}V_o(\Lambda) \check{\otimes}_\varepsilon F$ whenever $CH_{\mathfrak{Z}}V_o(\Lambda)$ (or F) has the a.p..

Proof. (1) follows directly from the ε-product representation Theorem 9(3) (with $E = CA_{\mathfrak{Z}}V_o(\Lambda)$). [Remark that for each $t \in \pi_1(\Lambda)$ the space $A_{\mathfrak{Z}}(\Lambda_t,F)$ is clearly nothing but $A_{\mathfrak{Z}}(\Lambda_t)\varepsilon F$ by our definition and compare [7], 3(1).]

(2) is then an obvious consequence of Schwartz's Theorem 5. - Since one can easily show that $CH_{\mathfrak{Z}}V_o(\Lambda) \otimes_\varepsilon F$ is a (topological linear) subspace of $CH_{\mathfrak{Z}}V_o(\Lambda,F)$ and since $CH_{\mathfrak{Z}}V_o(\Lambda,F)$ is complete under our assumptions, the first inclusion of (3) follows. But $CH_{\mathfrak{Z}}V_o(\Lambda,F) \subset CH_{\mathfrak{Z}}V_o(\Lambda)\varepsilon F$ can again be deduced from 9(3). (For each $t \in \pi_1(\Lambda)$ we have $H_{\mathfrak{Z}}(\Lambda_t) \check{\otimes}_\varepsilon F \subset H_{\mathfrak{Z}}(\Lambda_t,F) \subset H_{\mathfrak{Z}}(\Lambda_t)\varepsilon F$, cf. [7], Theorem 4 and the following remark.) □

The next propositions are again formulated for the case of the compact-open topology (i.e. $V = W$) only although suitable analogues hold in the general weighted case, too.

33 Proposition. (1) If, for each $t \in \pi_1(\Lambda)$, $\mathfrak{Z}(X)|_{\Lambda_t}$ is dense in $H_{\mathfrak{Z}}(\Lambda_t)$, $C(\Omega) \otimes \mathfrak{Z}(X)|_{\Lambda}$ is dense in $CH_{\mathfrak{Z}}(\Lambda)$. (If we assume that $\mathfrak{Z}(X)|_{\Lambda_t}$ is even dense in $A_{\mathfrak{Z}}(\Lambda_t)$ for each $t \in \pi_1(\Lambda)$, this clearly implies $A_{\mathfrak{Z}}(\Lambda_t) = H_{\mathfrak{Z}}(\Lambda_t)$ for all t and hence $CA_{\mathfrak{Z}}(\Lambda) = CH_{\mathfrak{Z}}(\Lambda)$ such that then $C(\Omega) \otimes \mathfrak{Z}(X)|_{\Lambda}$ is also dense in $CA_{\mathfrak{Z}}(\Lambda)$.)

(2) Let $\Lambda_\Omega := \overline{\bigcup_{t\in\pi_1(\Lambda)} \Lambda_t}$ be compact in X. If, for each $t \in \pi_1(\Lambda)$, $H_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda_t}$ [resp. $A_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda_t}$] is dense in $H_{\mathfrak{Z}}(\Lambda_t)$ [resp. $A_{\mathfrak{Z}}(\Lambda_t)$], then $C(\Omega) \otimes H_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda}$ [resp. $C(\Omega) \otimes A_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda}$] is dense in $CH_{\mathfrak{Z}}(\Lambda)$ [resp. $CA_{\mathfrak{Z}}(\Lambda)$].

Proof. Apply (the scalar version of) 14 to the module
$Z := C(\Omega) \otimes \mathfrak{Z}(X)|_{\Lambda}$ (or $Z := C(\Omega) \otimes H_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda}$ resp.
$Z := C(\Omega) \otimes A_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda_B}$) over the selfadjoint algebra
$A := CB(\Omega) \otimes \mathbb{K}|_{\Lambda} = CB(\Omega) \otimes \{\text{constants on } X\}|_{\Lambda}$ (in the space $L\tilde{V}_o = (C(\Lambda),co)$): Since $K_A = \{\{t\}\times\Lambda_t;\ t \in \pi_1(\Lambda)\}$, $f \in C(\Lambda)$ belongs to $\tilde{Z}^{(C(\Lambda),co)}$ if (and only if) for each $t \in \pi_1(\Lambda)$ the restriction $f|_{\{t\}\times\Lambda_t}$ is an element of

$$\overline{Z|_{\{t\}\times\Lambda_t}}^{C(\{t\}\times\Lambda_t)}$$

or, equivalently, if, for each $t\in\pi_1(\Lambda)$,

$$f(t,.) \in \overline{\mathfrak{Z}(X)|_{\Lambda_t}}^{C(\Lambda_t)} \quad (\text{or } \overline{H_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda_t}}^{C(\Lambda_t)} \text{ resp. } \overline{A_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda_t}}^{C(\Lambda_t)}).$$

But this is satisfied for $f \in CH_{\mathfrak{Z}}(\Lambda)$ resp. $CA_{\mathfrak{Z}}(\Lambda)$ under our respective assumptions. □

It is of course possible to combine 32 and 33 to derive density of $C(\Omega) \otimes \mathfrak{Z}(X) \otimes F|_{\Lambda}$ (or $C(\Omega) \otimes H_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda}$ resp. $C(\Omega) \otimes A_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda}$) in $CH_{\mathfrak{Z}}(\Lambda,F)$ resp. $CA_{\mathfrak{Z}}(\Lambda,F)$ for a l.c. space F, but the corresponding result can also be obtained directly as follows:

By applying the vector-valued version of 14 to the module

$Z := C(\Omega) \otimes \mathfrak{Z}(X) \otimes F|_{\Lambda}$ (or $Z := C(\Omega) \otimes H_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda}$ resp. $Z := C(\Omega) \otimes A_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda}$) over the algebra $A := CB(\Lambda) \otimes \mathbb{K}|_{\Lambda}$ (in the space $L\tilde{V}_o = (C(\Lambda,F),co)$), we get similarly as in the proof of 33:

Let F be an arbitrary l.c. space.

(1) If, for each $t \in \pi_1(\Lambda)$, $\mathfrak{Z}(X) \otimes F|_{\Lambda_t}$ is dense in $H_{\mathfrak{Z}}(\Lambda_t,F)$, $C(\Omega) \otimes \mathfrak{Z}(X) \otimes F|_{\Lambda}$ is dense in $CH_{\mathfrak{Z}}(\Lambda,F)$. (If we assume that $\mathfrak{Z}(X) \otimes F|_{\Lambda_t}$ is even dense in $A_{\mathfrak{Z}}(\Lambda_t,F)$ for each $t \in \pi_1(\Lambda)$, this clearly implies $A_{\mathfrak{Z}}(\Lambda_t,F) = H_{\mathfrak{Z}}(\Lambda_t,F)$ for all t and hence $CA_{\mathfrak{Z}}(\Lambda,F) = CH_{\mathfrak{Z}}(\Lambda,F)$ such that then $C(\Omega) \otimes \mathfrak{Z}(X) \otimes F|_{\Lambda}$ is also dense in $CA_{\mathfrak{Z}}(\Lambda,F)$.)

(2) Let $\Lambda_\Omega := \overline{\bigcup_{t\in\pi_1(\Lambda)} \Lambda_t}$ be compact in X. If, for each $t\in\pi_1(\Lambda)$, $H_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda_t}$ [resp. $A_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda_t}$] is dense in $H_{\mathfrak{Z}}(\Lambda_t,F)$ [resp. $A_{\mathfrak{Z}}(\Lambda_t,F)$], then $C(\Omega) \otimes H_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda}$ [resp. $C(\Omega) \otimes A_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda}$] is dense in $CH_{\mathfrak{Z}}(\Lambda,F)$ [resp. $CA_{\mathfrak{Z}}(\Lambda,F)$].

Let now $\hat{F}$ denote the completion of F and fix $t \in \pi_1(\Lambda)$. Since $A_{\mathfrak{Z}}(\Lambda_t,F)$ is a topological linear subspace of $A_{\mathfrak{Z}}(\Lambda_t,\hat{F}) = A_{\mathfrak{Z}}(\Lambda_t)\varepsilon\hat{F}$ (cf. [7], 3.1) in which $A_{\mathfrak{Z}}(\Lambda_t) \otimes \hat{F}$ is dense if $A_{\mathfrak{Z}}(\Lambda_t)$ or $\hat{F}$ has the a.p. (cf. 5), we get a fortiori density of $[\mathfrak{Z}(X) \otimes F|_{\Lambda_t} = \mathfrak{Z}(X)|_{\Lambda_t} \otimes F$ resp.] $A_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda_t} = A_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda_t} \otimes F$ in $A_{\mathfrak{Z}}(\Lambda_t,F)$ whenever $[\mathfrak{Z}(X)|_{\Lambda_t}$ resp.] $A_{\mathfrak{Z}}(\Lambda_\Omega)|_{\Lambda_t}$ is dense in $A_{\mathfrak{Z}}(\Lambda_t)$ and one of the spaces $A_{\mathfrak{Z}}(\Lambda_t)$ or $\hat{F}$ has the a.p.

Similarly $H_{\mathfrak{Z}}(\Lambda_t,F)$ is a topological linear subspace of $H_{\mathfrak{Z}}(\Lambda_t,\hat{F})$, and $H_{\mathfrak{Z}}(\Lambda_t,\hat{F})$ equals $H_{\mathfrak{Z}}(\Lambda_t) \check{\otimes}_\varepsilon \hat{F}$ if $H_{\mathfrak{Z}}(\Lambda_t)$ of $\hat{F}$ has a.p. or if, for some basis $\mathfrak{G}$ of open neighbourhoods of Λ_t, $\mathfrak{Z}(U)$ has Schwartz's a.p. for each $U \in \mathfrak{G}$. (See [7], Theorem 4 (1) and the following remark. - Note also that $(C(U,\hat{F}),co)$ is always a topological linear subspace of $(C(U),co)\varepsilon\hat{F}$ by 9 (1), and thus the assumption $X = k_{\mathbb{R}}$-space which was made in [7] is not needed under our present definitions.) Hence density of $\mathfrak{Z}(X) \otimes F|_{\Lambda_t}$ resp. $H_{\mathfrak{Z}}(\Lambda_\Omega) \otimes F|_{\Lambda_t}$ in $H_{\mathfrak{Z}}(\Lambda_t,F)$ follows whenever $\mathfrak{Z}(X)|_{\Lambda_t}$ resp.

$H_{\mathfrak{F}}(\Lambda_\Omega)|_{\Lambda_t}$ is dense in $H_{\mathfrak{F}}(\Lambda_t)$ and (i) $H_{\mathfrak{F}}(\Lambda_t)$ or (ii) $\hat{F}$ has the a.p. or (iii) $\mathfrak{F}$ is a sheaf with Schwartz's a.p. (i.e. $\mathfrak{F}(U)$ has Schwartz's a.p. for each open $U \subset X$, e.g. $\mathfrak{F}$ nuclear).

So we have proved:

34 Proposition. Let F be an arbitrary l.c. space.

(1) Suppose that:

(a) $\mathfrak{F}(X)|_{\Lambda_t}$ is dense in $H_{\mathfrak{F}}(\Lambda_t)$ for each $t \in \pi_1(\Lambda)$, and:

(b) (i) $\hat{F}$ has the a.p. or (ii) $H_{\mathfrak{F}}(\Lambda_t)$ has the a.p. for each $t \in \pi_1(\Lambda)$ or (iii) $\mathfrak{F}$ is a sheaf with Schwartz's a.p.

Then $C(\Omega) \otimes \mathfrak{F}(X) \otimes F|_\Lambda$ is dense in $CH_{\mathfrak{F}}(\Lambda,F)$.

(If we assume instead of (a) that $\mathfrak{F}(X)|_{\Lambda_t}$ is even dense in $A_{\mathfrak{F}}(\Lambda_t)$ for each $t \in \pi_1(\Lambda)$, then $A_{\mathfrak{F}}(\Lambda_t) = H_{\mathfrak{F}}(\Lambda_t)$ for all t and hence the a.p. of $\hat{F}$ or the a.p. of $A_{\mathfrak{F}}(\Lambda_t)$ for each $t \in \pi_1(\Lambda)$ implies $A_{\mathfrak{F}}(\Lambda_t,F) = H_{\mathfrak{F}}(\Lambda_t,F)$ for all t such that also $CA_{\mathfrak{F}}(\Lambda,F) = CH_{\mathfrak{F}}(\Lambda,F)$, and $C(\Omega) \otimes \mathfrak{F}(X) \otimes F|_\Lambda$ is then dense in $CA_{\mathfrak{F}}(\Lambda,F)$.)

(2) Let $\Lambda_\Omega := \overline{\bigcup_{t\in\pi_1(\Lambda)} \Lambda_t}$ be compact in X. Suppose that:

(a) $H_{\mathfrak{F}}(\Lambda_\Omega)|_{\Lambda_t}$ [resp. $A_{\mathfrak{F}}(\Lambda_\Omega)|_{\Lambda_t}$] is dense in $H_{\mathfrak{F}}(\Lambda_t)$ [resp. $A_{\mathfrak{F}}(\Lambda_t)$] for each $t \in \pi_1(\Lambda)$, and

(b) (i) $\hat{F}$ has the a.p. or (ii) $H_{\mathfrak{F}}(\Lambda_t)$ has the a.p. for each $t \in \pi_1(\Lambda)$ or (iii) $\mathfrak{F}$ is a sheaf with Schwartz's a.p. [resp. (i) $\hat{F}$ has the a.p. or (ii) $A_{\mathfrak{F}}(\Lambda_t)$ has the a.p. for each $t \in \pi_1(\Lambda)$].

Then $C(\Omega) \otimes H_{\mathfrak{F}}(\Lambda_\Omega) \otimes F|_\Lambda$ [resp. $C(\Omega) \otimes A_{\mathfrak{F}}(\Lambda_\Omega) \otimes F|_\Lambda$] is dense in $CH_{\mathfrak{F}}(\Lambda,F)$ [resp. $CA_{\mathfrak{F}}(\Lambda,F)$].

To finish let us point out that the abstract setting for all the examples in this section, as outlined before 19, allows also examples of a more general kind than we have considered so far:

Let Λ be a topological subspace of the product $\Omega \times X$ (of completely regular spaces) and let $V > 0$ be a Nachbin family on Λ. For each $t \in \pi_1(\Lambda)$ identify $V|_{\{t\}\times\Lambda_t}$ with the Nachbin family V_t on Λ_t. Now take a topological linear subspace $\mathfrak{F}_t$ of the weighted

space $C(V_t)_o(\Lambda_t)$ for each $t \in \pi_1(\Lambda)$ and put (topologically, with the induced weighted topology)

$$Y := \{f \in CV_o(\Lambda);\ f(t,\cdot) \in \mathfrak{F}_t \text{ for each } t \in \pi_1(\Lambda)\}.$$

Then Y is clearly a module over the selfadjoint algebra

$$A := \{f \in CB(\Lambda);\ f(t,\cdot) \text{ constant on } \Lambda_t \text{ for arbitrary } t\in\pi_1(\Lambda)\},$$

we are in the bounded case of the weighted approximation problem for Y in $CV_o(\Lambda)$, and $K_A = \{\{t\}\times\Lambda_t;\ t \in \pi_1(\Lambda)\}$. Hence by the localization theorem Y has Schwartz's a.p. whenever for each $t \in \pi_1(\Lambda)$ the space $\mathfrak{F}_t$ is nuclear or $Y_t = Y|_{\{t\}\times\Lambda_t}$, identified with a space of functions on Λ_t, is dense in $\mathfrak{F}_t$ and $\mathfrak{F}_t$ has Schwartz's a.p..

In all our examples, except 20, the spaces $\mathfrak{F}_t$ have been "of the same type" for each $t \in \pi_1(\Lambda)$, e.g. $\mathfrak{F}_t = \mathfrak{F}(V_t)_o(\Lambda_t)$ for Λ_t open and a fixed sheaf $\mathfrak{F}$ on X or $\mathfrak{F}_t = A_{\mathfrak{F}}(\Lambda_t)$ resp. $H_{\mathfrak{F}}(\Lambda_t)$ for Λ_t compact, $W \leq V$, and again a fixed sheaf $\mathfrak{F}$ on X. In Example 20 we took $\mathfrak{F}_t = \mathcal{O}(\Lambda_t)$ for all $t \in T$ closed $\subset \pi_1(\Lambda)$ (with Λ_t open for each such t), but $\mathfrak{F}_t = (C(\Lambda_t),co)$ for each $t \in \pi_1(\Lambda)\setminus T$. More generally, it is of course possible to construct examples where the "type" of the spaces $\mathfrak{F}_t$ changes with $t \in \pi_1(\Lambda)$: For instance, let T_1 and T_2 be two closed disjoint subsets of $\pi_1(\Lambda)$, let Λ_t be open for all $t \in T_1 \cup T_2$ and take $\mathfrak{F}_t = \mathfrak{F}_i(\Lambda_t)$ for $t \in T_i$, $i = 1,2$, and $\mathfrak{F}_t = (C(\Lambda_t),co)$ for all $t \in \pi_1(\Lambda)\setminus(T_1 \cup T_2)$ with two different sheaves $\mathfrak{F}_1$ and $\mathfrak{F}_2$ on X. Or even, which is much more interesting, let all Λ_t be open [resp. compact] and put $\mathfrak{F}_t = \mathfrak{F}(\Lambda_t)$ [resp. $A_{\mathfrak{F}_t}(\Lambda_t)$ or $H_{\mathfrak{F}_t}(\Lambda_t)$] where the sheaves $\mathfrak{F}_t$ on X depend on the parameter $t \in \pi_1(\Lambda)$ (e.g. sheaves of [null-] solutions of hypoelliptic partial differential operators $P(x,D,t)$), etc.

Finally, we should perhaps point out that each space Y of scalar functions as above has a natural interpretation as a vector space of cross-sections with respect to the "vector-valued" vector fibration $(\pi_1(\Lambda), (\mathfrak{F}_t)_{t\in\pi_1(\Lambda)})$ by taking $f = (f(t,\cdot))_t$. The topology of Y is also given by the family $\tilde{V} = \{\tilde{v};\ v \in V\}$ of "weights"

$\tilde{v}$ on $\pi_1(\Lambda)$, defined by $\tilde{v}(t)[g] := \sup_{x\in\Lambda_t} v(t,x)|g(x)|$ for all $g \in \mathfrak{F}_t$, $t \in \pi_1(\Lambda)$ and $v \in V$. Moreover, for arbitrary $v \in V$ and $f \in Y$ the mapping $s: t \to \tilde{v}(t)[f(t,\cdot)] = \sup_{x\in\Lambda_t} v(t,x)|f(t,x)|$ is <u>upper semicontinuous on</u> $\pi_1(\Lambda)$ and <u>vanishes at infinity</u>:

To prove this, fix $\epsilon > 0$. Since $f \in CV_o(\Lambda)$, there exists a compact subset K of Λ with $v(\lambda)\,|f(\lambda)| < \epsilon$ for all $\lambda \in \Lambda\setminus K$. $\pi_1(K)$ is a compact subset of $\pi_1(\Lambda)$, and for each point $t \notin \pi_1(\Lambda)\setminus\pi_1(K)$ we get $(t,x) \notin K$ and hence $v(t,x)\,|f(t,x)| < \epsilon$ for all $x \in \Lambda_t$ which implies $s(t) = \sup_{x\in\Lambda_t} v(t,x)\,|f(t,x)| \le \epsilon$, i.e. s vanishes at infinity. - Now we show upper semicontinuity of s at $t_o \in \pi_1(\Lambda)$: Let $S := s(t_o) = \sup_{x\in\Lambda_{t_o}} v(t_o,x)\,|f(t_o,x)|$. Since $v|f|: \lambda \to v(\lambda)|f(\lambda)|$ is upper semi continuous on Λ and vanishes at infinity, the set $K := \{\lambda \in \Lambda: v(\lambda)\,|f(\lambda)| \ge S + \frac{\epsilon}{2}\}$ is compact. Let $\mathfrak{G}$ be the system of closed neighbourhoods of t_o in $\pi_1(\Lambda)$, and for each $U \in \mathfrak{G}$ let $F_U := \bigcup_{t\in U}\{t\}\times\Lambda_t$ $(= \{\lambda \in \Lambda;\ \pi_1(\lambda) \in U\})$. Then F_U is a <u>closed</u> subset of Λ and

$$\bigcap_{U\in\mathfrak{G}} (F_U \cap K) = \Big(\bigcap_{U\in\mathfrak{G}} F_U\Big) \cap K = (\{t_o\}\times\Lambda_{t_o}) \cap K = \emptyset$$

(Ω is [completely] regular). By the finite intersection property of compact sets we get a (closed) neighbourhood U of t_o in $\pi_1(\Lambda)$ such that $F_U \cap K = \emptyset$, i.e. $v(t,x)\,|f(t,x)| < S + \frac{\epsilon}{2}$ for all $t \in U$ and $x \in \Lambda_t$ which implies $s(t) = \sup_{x\in\Lambda_t} v(t,x)|f(t,x)| \le S + \frac{\epsilon}{2} < s(t_o)+\epsilon$ for all $t \in U$.

After what we have just proved, the canonical identification of Y with a vector space of cross-sections [with respect to the vector fibration $(\pi_1(\Lambda), (\mathfrak{F}_t)_{t\in\pi_1(\Lambda)})$] <u>yields a Nachbin space</u> $L\tilde{V}_o$. In fact, as such a vector space of cross-sections, Y is a module over $CB(\pi_1(\Lambda))$, and as we are in the selfadjoint case, the localization of the a.p. of Y to $Y|_{\{t\}\times\Lambda_t} = Y_t \subset \mathfrak{F}_t$ can also be deduced from Prolla's main theorem in [25]. - However, with this identification we cannot get localization to smaller sets than whole "slices"

$\{t\}\times\Lambda_t$ whereas considering Y as a space of functions on Λ has the advantage that a "<u>finer</u>" localization is possible whenever Y is a module over an "essentially larger" algebra than A above (cf. 20, but one can easily find more striking examples).

4. OTHER EXAMPLES

Another (obvious) case where the localization of the a.p. applies involves <u>a different kind of</u> "<u>mixed dependence</u>" (cf. already [4], Corollary 15, p. 13/14 for a very simple example):

Let X be an arbitrary completely regular space, $V > 0$ a Nachbin family on X, Λ an arbitrary topological subspace of X (with closure $\bar{\Lambda}$ in X), and $\mathfrak{F} = \mathfrak{F}(\Lambda)$ a topological linear subspace of $(C(\Lambda), co)$. Now define $CV_o(X;\mathfrak{F}) := \{f \in CV_o(X);\ f|_\Lambda \in \mathfrak{F}(\Lambda)\}$ with the induced weighted topology. (E.g. let Λ be open in X and $\mathfrak{F}$ a [pre-] sheaf of continuous functions on X as in the preceding section.) This space is a module over the algebra $A := \{f \in CB(X);\ f|_\Lambda \text{ constant}\}$ with $K_A = \bar{\Lambda} \cup \{\{x\};\ x \in X\setminus\bar{\Lambda}\}$. Since the restriction of $CV_o(X;\mathfrak{F})$ to each one point set certainly has the a.p., the localization Theorem 17 <u>reduces</u> Schwartz's a.p. of $CV_o(X;\mathfrak{F})$ to the question whether $CV_o(X;\mathfrak{F})|_{\bar{\Lambda}} \subset C(V|_{\bar{\Lambda}})_o(\bar{\Lambda};\mathfrak{F}) = \{f \in C(V|_{\bar{\Lambda}})_o(\bar{\Lambda});\ f|_\Lambda \in \mathfrak{F}(\Lambda)\}$ (with the restricted weighted topology) has Schwartz's a.p..

<u>35 Proposition</u>. $CV_o(X;\mathfrak{F})$ has Schwartz's a.p. whenever $CV_o(X;\mathfrak{F})|_\Lambda$ has.

<u>36 Remark</u>. If X is a V_R-space and F a quasi-complete l.c. space, 9 (3) yields: $CV_o(X;\mathfrak{F})\,\epsilon\, F = \{f \in CV_o(X,F);\ f'\circ f|_\Lambda \in \mathfrak{F}(\Lambda)$ for each $f' \in F'\}$ with the weighted topology of $CV_o(X,F)$, and hence then $\{f \in CV_o(X,F);\ f'\circ f|_\Lambda \in \mathfrak{F}(\Lambda)$ for each $f' \in F'\} = CV_o(X;\mathfrak{F})\,\check{\otimes}_\epsilon\, F$ holds whenever F is even complete and $CV_o(X;\mathfrak{F})|_{\bar{\Lambda}}$ (or F) has Schwartz's a.p..

In Proposition 35 one would sometimes like to <u>replace</u> $CV_o(X;\mathfrak{F})|_{\bar{\Lambda}}$ by $C(V|_{\bar{\Lambda}})_o(\bar{\Lambda};\mathfrak{F})$. Similarly as before this is possible whenever density of $CV_o(X;\mathfrak{F})|_{\bar{\Lambda}}$ in $C(V|_{\bar{\Lambda}})_o(\bar{\Lambda};\mathfrak{F})$ is known, that is, if the elements $f \in C(V|_{\bar{\Lambda}})_o(\bar{\Lambda};\mathfrak{F})$ which extend to functions in $CV_o(X)$ form a dense subset.

37 Remark. (a) (Even) $CV_o(X;\mathfrak{F})|_{\bar{\Lambda}} = C(V|_{\bar{\Lambda}})_o(\bar{\Lambda};\mathfrak{F})$ holds if we have $CV_o(X)|_{\bar{\Lambda}} \supset C(V|_{\bar{\Lambda}})_o(\bar{\Lambda};\mathfrak{F})$, e.g. if (*) $CV_o(X)|_{\bar{\Lambda}}$ equals $C(V|_{\bar{\Lambda}})_o(\bar{\Lambda})$.

(b) Condition (*) of (a) is satisfied for instance in the following cases:

(i) $V = W$ (if suffices that each function $v \in V$ has compact support which implies $CV_o(X) = C(X)$ algebraically) and $\Lambda \subset X$ relatively compact or X normal, or

(ii) V = positive constants on X (hence $CV_o(X) = C_o(X)$ and $C(V|_{\bar{\Lambda}})_o(\bar{\Lambda}) = C_o(\bar{\Lambda})$) and X locally compact or

(iii) X locally compact and $\Lambda \subset X$ relatively compact (but $V>0$ arbitrary).

In case (iii) of (b), each function $f \in C(\bar{\Lambda})$ clearly extends to a function in $C(X)$ which has compact support and thus satisfies arbitrary weight conditions. - At this point we should perhaps also observe that, if X is a topological subspace of (the completely regular space) X_o, e.g. $\mathbb{C}^N$ or $\mathbb{R}^N$, the closure of $\Lambda \subset X$ with respect to X_o need not coincide with $\bar{\Lambda} \subset X$ in general, but $\bar{\Lambda}$ is the intersection of the closure in X_o with X.

We note some interesting examples that follow from 35, 37 (b) (iii) and from the results in [7] (already used in the last section) on the a.p. of $A_{\mathfrak{F}}(K)$:

38 Example. Let X be locally compact, V a Nachbin family on X with $W \le V$, $\mathfrak{F}$ a closed l.c. sub-(pre-) sheaf of C_X, and Λ an open relatively compact subset of X with $\Lambda = \mathring{\bar{\Lambda}}$. Then $CV_o(X;\mathfrak{F}(\Lambda))$ has the a.p. whenever (in the notation of the preceding section) the Banach space $A_{\mathfrak{F}}(\bar{\Lambda})$ has.

E.g., let X be a locally compact subspace of $\mathbb{C}^N$ $(N \ge 1)$ resp. $\mathbb{R}^N$ $(N \ge 2)$ [or of the space of definition of a (suitable) harmonic sheaf $\mathcal{H}$ of axiomatic potential theory], $W \le V$, $\mathfrak{F} = \mathfrak{G}$ resp. $\mathcal{H}$, and Λ an open subset of $\mathbb{C}^N$ resp. $\mathbb{R}^N$ such that its closure $\bar{\Lambda}$ is a compact subset of X with $\mathring{\bar{\Lambda}} = \Lambda$.

Then $CV_o(X;\mathfrak{F}(\Lambda))$ has the a.p. provided $\mathfrak{F} = \mathcal{H}$ or $\mathfrak{F} = \mathfrak{G}$ and $N = 1$ or $\mathfrak{F} = \mathfrak{G}$, $N > 1$, and Λ is (a strictly pseudoconvex region with sufficiently smooth boundary or a regular Weil polyeder or) a product $\Lambda_1 \times \dots \times \Lambda_k$ where each open (relatively compact) set Λ_j (j=1,...,k) is either contained in $\mathbb{C}$ or a strictly pseudoconvex region with sufficiently smooth boundary or a regular Weil polyeder.

Proof. Since X is locally compact and $W \le V$, the topology of $CV_o(X)$ is complete and stronger than co. Since $\mathfrak{F}(\Lambda)$ is a closed topological subspace of $(C(\Lambda),co)$, $CV_o(X;\mathfrak{F}(\Lambda))$ is clearly a closed subspace of $CV_o(X)$ and hence complete, too. $\bar{\Lambda} \subset X$ compact and $\Lambda = \mathring{\bar{\Lambda}}$ imply $C(V|_{\bar{\Lambda}})_o(\bar{\Lambda};\mathfrak{F}) = A_{\mathfrak{F}}(\bar{\Lambda})$. Now the first part of 38 follows from 35 and 36 (b) (iii) while the second part is a consequence of the results surveyed in the third section of [7]. □

Let us now take X and V as before, but assume that there is a whole (finite or infinite) system $(\Lambda_\alpha)_\alpha$ of (disjoint) topological subspaces of X and a corresponding system $\mathfrak{F}_\alpha = \mathfrak{F}_\alpha(\Lambda_\alpha)$ of topological linear subspaces of $(C(\Lambda_\alpha),co)$. We look at the space $CV_o(X;(\mathfrak{F}_\alpha)) := \{f \in CV_o(X);\ f|_{\Lambda_\alpha} \in \mathfrak{F}_\alpha(\Lambda_\alpha) \text{ for each } \alpha\}$ with the induced weighted topology.

For instance, the sets Λ_α may be open in X, and $(\mathfrak{F}_\alpha)_\alpha$ may denote different (pre-)sheaves of continuous functions on X. It is also quite interesting to take $\mathfrak{F}_\alpha$ = the same sheaf $\mathfrak{F}$ for all α and to let $(\Lambda_\alpha)_\alpha$ denote the system of (connected) components of an open set $\Lambda = \bigcup_\alpha \Lambda_\alpha$ in a locally connected space X. (The sets Λ_α are then open, too.)

Clearly $CV_o(X;(\mathfrak{F}_\alpha))$ is a module over the selfadjoint algebra

$A := \{f \in CB(X); f|_{\Lambda_\alpha}$ constant for each $\alpha\}$, and all the sets $\bar{\Lambda}_\alpha$ are contained in maximal A-antisymmetric sets. But, even for only two sets Λ_1, Λ_2, it may happen that $\Lambda_1 \cap \Lambda_2 = \emptyset$, but that $\bar{\Lambda}_1 \cap \bar{\Lambda}_2$ is non-void; in this case $\bar{\Lambda}_1 \cup \bar{\Lambda}_2$ is a closed subset of X on which all functions in A are constant. So in general the localization of the a.p. of $CV_o(X;(\mathfrak{F}_\alpha))$ to K_A, i.e. to maximal sets of constancy of A, will not lead to a complete "splitting" of the different spaces $\mathfrak{F}_\alpha$. In fact, for a space X on which the system $(\Lambda_\alpha)_\alpha$ induces a kind of "swiss cheese" structure, it may be quite complicated to "compute" K_A explicitly, and a number of different situations occur. - We will not deal with such a topological problem here any more, but mention only a very simple case in which the localization theorem is useful (and which fornunately arises sometimes in concrete applications):

39 Proposition. Assume that the algebra A (as above) separates the sets $(\bar{\Lambda}_\alpha)_\alpha$ and points in $X \setminus \bigcup_\alpha \bar{\Lambda}_\alpha$ from $\bigcup_\alpha \bar{\Lambda}_\alpha$, i.e. $K_A = \{\bar{\Lambda}_\alpha;\alpha\} \cup \{\{x\}; x \in X \setminus \bigcup_\alpha \bar{\Lambda}_\alpha\}$. (This is certainly the case if (*) X is normal, the sets $\bar{\Lambda}_\alpha$ are disjoint, and if $\bigcup_{\alpha \neq \alpha_o} \bar{\Lambda}_\alpha$ is closed for each α_o.)
Then $CV_o(X;(\mathfrak{F}_\alpha))$ has Schwartz's a.p. whenever all the spaces $CV_o(X;(\mathfrak{F}_\alpha))|_{\bar{\Lambda}_\alpha}$ have.

Here $CV_o(X;(\mathfrak{F}_\alpha))|_{\bar{\Lambda}_\alpha} \subset C(V|_{\bar{\Lambda}_\alpha})_o(\bar{\Lambda}_\alpha;\mathfrak{F}_\alpha) = \{f \in C(V|_{\bar{\Lambda}_\alpha})_o(\bar{\Lambda}_\alpha);\ f|_{\Lambda_\alpha} \in \mathfrak{F}_\alpha(\Lambda_\alpha)\}$ (with the restricted weighted topology), and similarly as in 37 one can find conditions such that (even) $CV_o(X;(\mathfrak{F}_\alpha))|_{\bar{\Lambda}_\alpha} = C(V|_{\bar{\Lambda}_\alpha})_o(\bar{\Lambda}_\alpha;\mathfrak{F}_\alpha)$ holds: For instance this is the case if, additionally to the assumption (*) of 39, we have $V = W$ or: X locally compact and $\bar{\Lambda}_\alpha \subset X$ compact.

40 Example. Let X_o be a completely regular space and $\mathfrak{F}$ a closed l.c. subsheaf of C_{X_o}. Let then K denote a compact subset of X_o which is the union $\bigcup_\alpha K_\alpha$ of (disjoint) compact sets K_α such that $\mathring{K} = \bigcup_\alpha \mathring{K}_\alpha$ holds and the algebra $A := \{f \in C(K); f|_{K_\alpha}$ constant for each $\alpha\}$ separates the sets K_α. Then $A_{\mathfrak{F}}(K)$ has the a.p. if $A_{\mathfrak{F}}(K)|_{K_\alpha} \subset A_{\mathfrak{F}}(K_\alpha)$ has the a.p. for each α.

Proof. Take $X = K$, $\Lambda_\alpha = K_\alpha$, and $\mathfrak{F}_\alpha(\Lambda_\alpha) = A_{\mathfrak{F}}(K_\alpha)$ for each α in 39. With, say, V = positive constants on K we get $CV_o(X;(\mathfrak{F}_\alpha)) = \{f \in C(K);\ f|_{\mathring{K}_\alpha} \in \mathfrak{F}(\mathring{K}_\alpha)$ for each $\alpha\}$ which by $\mathring{K} = \bigcup_\alpha \mathring{K}_\alpha$ and the sheaf property of $\mathfrak{F}$ equals $A_{\mathfrak{F}}(K)$. - Of course we need these facts also to show that $A_{\mathfrak{F}}(K)$ is a module over the algebra A above. □

It is easy to construct (non-trivial) examples of sets (say) $K = \bigcup_{n \in \mathbb{N}} K_n$ as in 40 and even to arrange these examples in order to get $A_{\mathfrak{F}}(K)|_{K_n} = A_{\mathfrak{F}}(K_n)$ for each $n \in \mathbb{N}$. - In many cases density of $A_{\mathfrak{F}}(K)|_{K_\alpha}$ in $A_{\mathfrak{F}}(K_\alpha)$ will hold anyway, and this already suffices to replace by $A_{\mathfrak{F}}(K_\alpha)$ in 40. - It is of course possible to combine the two kinds of "mixtures" and to use spaces of type $CV_o(X;(\mathfrak{F}_\alpha))$ as $\mathfrak{F}_t$ in the general scheme given at the end of the preceding section. - We prefer, however, to conclude with applications of the localization theorem (and of the ε-product) in the vector-valued weighted approximation theory of continuous functions. These applications were mentioned very briefly (and without proofs) in Remark 4.6 d) of [11]; they are connected with the paper [26] by Prolla-Machado and generalize B Blatter's method from [13], Theorem 1.10 to arbitrary weighted spaces.

Let us start by introducing resp. recalling (cf. [26]) some notation: Let X be completely regular, $V > 0$ a Nachbin family on $X, F \neq \{0\}$ a locally convex space, and $Y = Y_F$ a topological linear subspace of $CV_o(X,F)$. The set G_Y of all pairs $(x,y) \in X \times X$ such that (with $\alpha\delta_x: f \to \alpha f(x)$ for all $f \in CV_o(X,F)$, arbitrary $x \in X$ and $\alpha \in \mathbb{K}$) either $\delta_x|_Y = \delta_y|_Y = 0$ or there exists $t \in \mathbb{K}$, $t \neq 0$, with $\delta_x|_Y = t\,\delta_y|_Y \neq 0$ yields an equivalence relation on X. Define $\varepsilon = \varepsilon_Y: G_Y \to \mathbb{K}$ by $\varepsilon(x,y) = 0$ if $0 = \delta_x|_Y = \delta_y|_Y$ and $\varepsilon(x,y) = t$ if $0 \neq \delta_x|_Y = t\,\delta_y|_Y$.

Similarly the subsets $KS_Y := \{(x,y) \in G_Y;\ g(x,y) \geq 0\}$ and $WS_Y := \{(x,y) \in G_Y;\ g(x,y) \in \{0,1\}\}$ yield equivalence relations on X. Now consider a "symbol" $\Delta \in \{G, KS, WS\}$. The closed (topological linear) subspace $\Delta(Y) := \{f \in CV_o(X,F);\ f(x) = g(x,y)f(y)$ for all $(x,y) \in \Delta_Y\} \supset Y$ of $CV_o(X,F)$ is called the Δ-hull of Y in $CV_o(X,F)$. Y is said to be a Δ-subspace of $CV_o(X,F)$ if $\Delta(Y)$ is just the closure of Y in $CV_o(X,F)$. (The letters G, KS, WS stand for Grothendieck, Kakutani-Stone, and Weierstrass-Stone, respectively.)

41 Proposition. Let Y $(= Y_{\mathbb{K}})$ be a topological linear subspace of $CV_o(X)$.

(a) Then $WS(Y)$ has Schwartz's a.p.

(b) Let $\Delta = G$ or KS. If $A(\Delta) := \{f \in CB(X);\ f$ is constant on each equivalence class modulo $\Delta_Y\}$ separates the equivalence classes mod Δ_Y, $\Delta(Y)$ has Schwartz's a.p..

Proof. Let $\Delta = G$, KS, or WS. Since $CV_o(X)$ is a module over $CB(X)$, $\Delta(Y)$ is a module over

$A(\Delta) = \{f \in CB(X);\ f$ is constant on all equivalence classes mod $\Delta_Y\}$.

We are in the bounded case of the weighted approximation problem for $\Delta(Y)$ in $CV_o(X)$, and A is selfadjoint. - Let us prove that $A(\Delta)$ always separates the equivalence classes mod Δ_Y in the case $\Delta = WS$ (a fact which is mentioned [for real scalars] in [26], p. 248): Take $x_1, x_2 \in X$ which belong to different equivalence classes mod WS_Y, i.e. there exists $h \in Y \subset WS(Y)$ such that $h(x_1) \neq h(x_2)$. $WS(Y)$ is clearly selfadjoint, hence we can find a real-valued $g \in WS(Y)$ such that $g(x_1) < g(x_2)$. But for real-valued functions $g_1, g_2 \in WS(Y)$ also $\sup(g_1,g_2) \in WS(Y)$. So without loss of generality we may assume $g \geq 0$. The function $f := \inf(g, g(x_2))$ is continuous, by $0 \leq f \leq g$ belongs to $CV_o(X)$, and it is easily checked that then even $f \in WS(Y)$, too. So f must be constant on the equivalence classes mod WS_Y, and we have constructed a real-valued function $f \in A(WS)$ which separates x_1 and x_2.

For the cases $\Delta = G$ resp. KS, we assume in (b) this separation property of $A(\Delta)$, and hence we have in all three cases that $\mathcal{K}_A$ is nothing but the system of equivalence classes mod Δ_Y. By the localization Theorem 17, $\Delta(Y)$ now has Schwartz's a.p. if $\Delta(Y)|_K$ has Schwartz's a.p. for each equivalence class K modulo Δ_Y.

In the case $\Delta = WS$ all functions $f \in \Delta(Y)$ are constant on each equivalence class K mod Δ_Y, that is $WS(Y)|_K = \{0\}$ or $\mathbb{K}$ which clearly has the a.p.. In the other cases $\Delta = KS$ or G it is easy to see that the values of an arbitrary function $f \in \Delta(Y)$ on K are completely determined by the value at one single point $x_o \in K$, and hence $\Delta(Y)|_K$ is again at most one-dimensional. □

For the assumption in 41 (b) see [26], 3.15. - From 41 we get by density of Δ-subspaces Y in $\Delta(Y)$, $\Delta = G$, KS or WS:

42 Corollary. Each WS-subspace $Y \subset CV_o(X)$ has Schwartz's a.p., and for $\Delta = G$ or KS any Δ-subspace Y of $CV_o(X)$ for which $A(\Delta)$ as in 41 separates the equivalence classes mod Δ_Y has Schwartz's a.p., too.

Let us now turn to the vector-valued case where (up to the a.p. of arbitrary closed KS- and G-subspaces of $CV_o(X)$ which followed in Blatter's case [X locally compact and V = positive constants on X] from a theorem of Lindenstrauss) we will generalize Theorem 1.10 of Blatter [13] below.

Let Y_F denote a topological linear subspace of $CV_o(X,F)$. Define $Y_o := \{f' \circ f;\ f' \in F',\ f \in Y_F\} \subset CV_o(X)$ with the weighted topology. We will always assume that

(*) $Y_o \times F := \{g \otimes e;\ g \in Y_o,\ e \in F\}$ is contained in Y_F.

Then obviously Y_o is a linear subspace of $CV_o(X)$, and Y_o is closed in $CV_o(X)$ whenever Y_F is closed in $CV_o(X,F)$. (See [2], II, 3.5 and compare [13], Remark 1.15 (i) as well as [26], Lemma 1.1)

43 Lemma. Let $\Delta \in \{G, KS, WS\}$. Then $\Delta_{Y_F} = \Delta_{Y_o}$, and $g_{Y_F}(x,y) = g_{Y_o}(x,y)$ for all $(x,y) \in \Delta_{Y_o} = \Delta_{Y_F}$.

__Proof.__ Using the Hahn-Banach theorem this is easily checked. □

__44 Proposition.__ (a) For an arbitrary topological linear subspace Y $(= Y_{\mathbb{K}})$ of $CV_o(X)$ we have

$\Delta(Y)\varepsilon F = \{f \in CV_o^{\sigma,c}(X,F);\ f(x) = g_Y(x,y)f(y)$ for all $(x,y) \in \Delta_Y\}$.

(b) Let Y_F be a topological linear subspace of $CV_o(X,F)$ with (

Then we get

$$\Delta(Y_F) = \Delta(Y_o)\varepsilon F$$

whenever F is quasi-complete and X a $V_{\mathbb{R}}$-space.

__Proof.__ (a) is immediate from 9 (3) and the Hahn-Banach theorem. Under the assumptions of (b) we have $CV_o^{\sigma,c}(X,F) = CV_o(X,F)$, and hence (b) follows from (a) and Lemma 43. □

Without any assumptions on F and X we get in case (b): $\Delta(Y_F) = \Delta(Y_o)\varepsilon F \cap CV_o(X,F)$.

__45 Proposition.__ An arbitrary __closed__ linear subspace Y_F of $CV_o(X,F)$ is a Δ-__subspace__ if and only if $Y_o := \{f'\circ f;\ f' \in F',\ f \in Y_F\}$ is a closed Δ-subspace of $CV_o(X)$ satisfying $Y_F = Y_o\varepsilon F \cap CV_o(X,F)$. (The intersection of $Y_o\varepsilon F$ with $CV_o(X,F)$ is __not necessary__ if X is a $V_{\mathbb{R}}$-space and F quasi-complete.)

__Proof.__ 1. Let Y_o be a closed Δ-subspace of $CV_o(X)$, i.e. $Y_o = \Delta(Y_o)$. Then $Y_F := Y_o\varepsilon F \cap CV_o(X,F)$ clearly contains $Y_o \times F$, and it is immediate that $\{f'\circ f;\ f' \in F',\ f \in Y_F\} = Y_o$ (F being $\neq \{0\}$). Hence we may apply 44 to get

$\Delta(Y_F) = \Delta(Y_o)\varepsilon F \cap CV_o(X,F) = Y_o\varepsilon F \cap CV_o(X,F) = Y_F$ which proves that Y_F is a closed Δ-subspace of $CV_o(X,F)$.

2. Let now Y_F be a closed Δ-subspace of $CV_o(X,F)$. Then one can verify __directly__ that $Y_o := \{f'\circ f;\ f' \in F',\ f \in Y_F\}$ satisfies $Y_o \times F \subset \Delta(Y_F) = Y_F$. By 44 we get consequently:

$Y_F = \Delta(Y_F) = \Delta(Y_o)\varepsilon F \cap CV_o(X,F)$, and it remains to show $\Delta(Y_o) = Y_o$.

But for $h \in \Delta(Y_o)$ and $e \in F$, $e \neq 0$, we have

$f := h \otimes e \in \Delta(Y_o) \times F \subset \Delta(Y_o) \otimes F \subset \Delta(Y_o)\varepsilon F \cap CV_o(X,F) = Y_F$. Choosing

$f' \in F'$ with $f'(e) = 1$ we finally obtain $f'\circ f = h \in Y_o$, i.e. $Y_o = \Delta(Y_o)$. □

__46 Corollary.__ Let X be a $V_{\mathbb{R}}$-space, F a complete l.c. space, and Y_F a closed Δ-subspace of $CV_o(X,F)$. If $\Delta = WS$ or if $\Delta = G$ or KS and $A(\Delta)$ in 41 separates the equivalence classes mod Δ_Y (or if F has the a.p.), we obtain:

$$Y_F = Y_o \check{\otimes}_{\varepsilon} F.$$

__Proof.__ Combine 41 and 45. □

The ε-product characterization 45 of closed Δ-subspaces Y_F of $CV_o(X,F)$ (in some sense) __reduces__ the study of Δ-subspaces in the vector-valued case to __scalar__ functions. (For some characterizations for Δ-subspaces $Y_o \subset CV_o(X)$ see [26].)

Finally let us note that the above results lead to new proofs of some propositions in [26] and yield better insight. We illustrate this with two examples (compare [26], 3.19):

In view of Lemma 2.2 and Remark 2.5 of [26], Theorem 2.9 of [26] can be __rephrased__ as follows:

Let Y_F be a topological linear subspace of $CV_o(X,F)$ such that Y_o satisfies $Y_o \times F \subset Y_F$ and is a WS-subspace of $CV_o(X)$. Then Y_F is a WS-subspace of $CV_o(X,F)$.

A proof based on the preceding results runs as follows: We have $Y_F \supset Y_o \otimes_{\varepsilon} F$ and a fortiori: $\bar{Y}_F \supset \overline{Y_o \otimes_{\varepsilon} F}^{CV_o(X,F)}$ which clearly contains $\overline{\bar{Y}_o \otimes_{\varepsilon} F}^{CV_o(X,F)}$ and hence $\bar{Y}_o\varepsilon F \cap CV_o(X,F)$ since $\bar{Y}_o \otimes F$ is dense in $\bar{Y}_o\varepsilon F$ by 5 because $\bar{Y}_o = WS(Y_o)$ has Schwartz's a.p. by Proposition 41 (a). On the other hand 44 yields:

$$WS(Y_F) = WS(Y_o)\varepsilon F \cap CV_o(X,F) = \bar{Y}_o\varepsilon F \cap CV_o(X,F),$$

and it follows that $WS(Y_F) \subset \bar{Y}_F$ which implies $\bar{Y}_F = WS(Y_F)$, i.e. Y_F is a WS-subspace of $CV_o(X,F)$.

And in view of [26], Proposition 3.11, Theorem 3.14 of [26] reads:

Let Y_F be a topological linear subspace of $CV_o(X,F)$ such that Y_o satisfies $Y_o \times F \subset Y_F$ and is a Δ-subspace of $CV_o(X)$ for Δ = KS or G. Assume that $A(\Delta)$ separates the equivalence classes modulo Δ_{Y_F}. Then Y_F is a Δ-subspace of $CV_o(X,F)$.

- A proof can be given exactly as before, using Proposition 41 (b) (and Lemma 43) this time.

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