An alternative dynamical description of Quantum Systems

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For the analysis of finite-dimensional classical completely integrable Hamiltonian systems the representation by action-angle-variables is an essential tool. This representation can be carried over to the infinite-dimensional situation by use of mastersymmetries, thus leading to a suitable Viasoro algebra in the vector fields. For the quantum case, however, such a structure cannot exist in the corresponding operator algebra, due to a classical theorem of Kaplansky, although the concept of mastersymmetries can be used to give a formal description of the infinite-dimensional symmetry group of those "nonlinear" Quantum systems which are accessible by the Quantum Inverse Scattering Transform. In order to make that precise and to transfer classical notions and methods to the quantum case an alternative dynamical concept for quantum systems is proposed. We give two examples, in the discrete case we consider spin chains, like the Heisenberg anisotropic spin chain, and in the continuous case, where additional difficulties arise, we consider the quantization of the KdV.

1 Introduction

The Viasoro algebra of mastersymmetries ([11], [21]) is an essential tool for completely integrable systems. These time-dependent symmetries are closely connected to recursion operators and to the action-angle representation of the dynamics [15].

We give the necessary definition in the abstract case of an arbitrary Lie algebra $\mathcal{L}$. Let $\mathcal{L}_1$ be a sub-Lie algebra of $\mathcal{L}$. Recall that a map $d : \mathcal{L}_1 \to \mathcal{L}$ is said to be a *derivation* (on $\mathcal{L}_1$) if

$$d[A, B] = [d(A), B] + [A, d(B)] \text{ for all } A, B \in \mathcal{L}_1. \tag{1.1}$$

Special derivations are given by the adjoint $\hat{G}$ of elements $G \in \mathcal{L}$ (i.e. $\hat{G}A := [G, A]$ for all $A \in \mathcal{L}_1$). These derivations are called *inner*. A derivation on $\mathcal{L}_1$ is said to be an $\mathcal{L}_1$-*mastersymmetry* if it maps $\mathcal{L}_1$ into $\mathcal{L}_1$. The mastersymmetries are a sub-Lie algebra.

We briefly explain the use of mastersymmetries. For example, if we fix $K \in \mathcal{L}$ and define $\mathcal{L}_1 = K^{\perp} = \{G \in \mathcal{L} \mid [G, K] = 0\}$ to be the commutant of K, then a $K^{\perp}$-mastersymmetry d has the property that

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$d(K), d^2(K), \ldots, d^n(K)$ are elements of $K^\perp$. Hence, we are able to generate in a recursive way out of K infinitely many elements of $K^\perp$, maybe even all of $K^\perp$.

It looks as if for the construction of those mastersymmetries d which can be used to generate elements of $K^\perp$ we have to check how d acts on *all* of $K^\perp$. Fortunately this is not really necessary when $K^\perp$ is abelian:

Observation 1.1: *Consider $\mathcal{L}_1 \subset \mathcal{B} \subset \mathcal{L}$, where $\mathcal{L}_1$ and $\mathcal{B}$ are sub-Lie algebras of $\mathcal{B}$ and $\mathcal{L}$, respectively. Fix $K \in \mathcal{L}_1$ such that $\mathcal{L}_1$ is equal to the commutant $K^\perp(\mathcal{B}) = \{A \in \mathcal{B} \mid [A, K] = 0\}$ of K in $\mathcal{B}$ (not in $\mathcal{L}$). Assume that $\mathcal{L}_1 = K^\perp(\mathcal{B})$ is abelian. Then an inner derivation $d : \mathcal{B} \to \mathcal{L}$ with $d(\mathcal{L}_1) \subset \mathcal{B}$ is a $K^\perp(\mathcal{B})$-mastersymmetry if and only if $d(K) \in K^\perp(\mathcal{B})$.*

Thus, in case of abelian structure we only have to try out how d acts on K, that means we only have to check if $d(K)$ commutes with K. The proof of this simple fact is mainly based on a successive application of the Jacobi identity, see [11]. Under very mild additional conditions, one can show that in this case the Lie algebra generated by K and d is a Viasoro algebra (see [15]).

We illustrate this crucial notion in different situations. First we concentrate on the classical situation. Let $\mathcal{M}$ be a C^∞-manifold, denote the variable on $\mathcal{M}$ by u and consider an evolution equation $u_t = K(u)$ where K is a C^∞-vector field on $\mathcal{M}$. Recall that the C^∞-vector fields are endowed with a Lie-algebra structure, namely the infinitesimal structure of the group of C^∞-diffeomorphisms on $\mathcal{M}$. Therefore the construction of $K^\perp$ via mastersymmetries amounts to the construction of the infinitesimal generators of the one-parameter symmetry groups. This way of construction works for all the popular completely integrable systems like KdV, mKdV, SG, BO, KP etc. (see [11] or [21]). Complete integrability in all these cases implies that $K^\perp$ is abelian. In addition to that, the $K^\perp$-mastersymmetries have a direct meaning in terms of time-dependent symmetry groups. To see this, consider $G(t) := \exp(t\hat{K})G_0$ for a mastersymmetry G_0. Then, due to the mastersymmetry property, the Taylor series reduces to a polynomial of first order in t. Thus $G(t)$ is a time-dependent symmetry generator.

In case our system is hamiltonian, i.e. the vector field $K = \Theta\nabla H$ is the image of a gradient field ∇H (H the Hamiltonian) under an invertible implectic (inverse symplectic operator or Poisson operator) map Θ [10]. Then in the space of zero-forms (scalar quantities on $\mathcal{M}$) one has a canonical Lie-algebra structure $\{,\}_\Theta$ induced by Θ (*Poisson brackets* with respect to Θ). A zero form is a conserved quantity with respect to the flow if and only if it commutes with H in the Lie algebra of Poisson brackets. Hence, we are able to construct out of H further conserved scalar fields via commutation with $H^{\perp}$-mastersymmetries. As above, if γ_0 is a mastersymmetry for H in the Lie algebra of Poisson brackets, then $\gamma(t) = \exp(t\hat{H})\gamma_0$ is a time-dependent scalar field, invariant under the flow, and a polynomial of first order in t; hence an angle variable.

For quantum systems the Lie-algebra under consideration are the operators on a suitable Hilbert space. The Lie-product is given by the usual commutation of operators (denoted in the following by $[\![\ ]\!]$). Given an operator H then, via $H^{\perp}$-mastersymmetries, we would be able to construct in a recursive way operators commuting with H. This would be most interesting, because if H is normal, then knowing a maximal abelian subalgebra of $H^{\perp}$ is the same as knowing the spectral resolution of H. Hence finding $H^{\perp}$-mastersymmetries is a big step forward towards the diagonalization of H. But at this point one gets disillusioned since there is a well known theorem of Kaplansky [17] (see [20] for the unbounded case), stating that for every continuous derivation d on an operator algebra, fullfilling $[H, d(H)] = 0$, the spectral radius of $d(H)$ must be equal to zero. So, mastersymmetries cannot exist in the proper sense. The way out of this is to consider mastersymmetries of outer type which can be represented by operators which are "very unbounded". This actually will be our concept for spin chains.

2 Canonical formulation of quantum systems

We embed the usual formulation of quantum mechanics into the frame of classical hamiltonian systems. The manifold $\mathcal{M}$ under consideration is the space of selfadjoint operators on some Hilbert space H. Since $\mathcal{M}$ is a linear space we can identify $\mathcal{M}$ with the typical fiber of its tangent bundle. If some selfadjoint operator H is fixed, then a dynamic on $\mathcal{M}$ is described by the linear evolution equation

$$\dot{A}(t) = i[\![\ H, A(t)]\!] \quad . \tag{2.1}$$

Symmetries of this system are given in the well-known way: Consider another flow of this type $\dot{A}(t) = i[\![\ H_1, A]\!]$, then it commutes with (2.1) if and only if $[\![\ H_1, H]\!] = 0$.

For the moment we restrict our attention to the special case of finite dimensional Hilbert space in order to avoid convergence difficulties. In that case we have a well defined *duality* on the tangent bundle given by $< A, B >:= \mathrm{trace}\{AB\}$. Via this duality we can identify tangent space $T\mathcal{M}$

and cotangent space $T^*\mathcal{M}$ and we can compute gradients. For example if $P(A) = (1/2) < A, A >$ then $\nabla P(A)$ is given by

$$< \nabla P(A), B >= \frac{1}{2}\frac{\partial}{\partial\epsilon}_{|\epsilon=0} < A + \epsilon B, A + \epsilon B > = < A, B > .$$

Hence $\nabla P(A) = A$. It is easily seen that whenever a selfadjoint $H_1 \in E$ commutes with H, then $F(A) =< A, H_1 A >= trace\{AH_1A\}$ is a conservation law for (2.1). So, from this aspect, for quantum mechanical systems there is no essential difference between symmetries and conserved quantities, or, in other words, all symmetries are hamiltonian.

We can use Noethers theorem (mapping conserved quantities to symmetry generators) to construct dynamical laws which are not of the type (2.1). We introduce a map $\Theta : T\mathcal{M}^* \to T\mathcal{M}$ by $\Theta(A) = i[\![H, A]\!]$. It is easily verified that Θ is skew-symmetric with respect to the duality introduced before. So Θ must be implectic because it is independent of the particular manifold point. The system (2.1) is hamiltonian in the classical sense because it can be rewritten as $\dot{A}(t) = \Theta \circ \nabla P(A)$. One should remark that this dynamical law is truly linear whatever the form of the hamiltonian is. This observation is puzzling insofar as, by the technique of the quantum inverse scattering transform, *nonlinear* methods really yield relevant results for certain quantum systems: Take some H_1 with $[\![H, H_1]\!] = 0$ then using the image (under Θ) of ∇F we find that the flow $\dot{A}(t) = i[\![H, AH_1 + H_1 A]\!]$ commutes with (2.1).

So, on the first view, there are additional symmetries for quantum systems. However, some of these additional symmetries may not really be relevant because either they destroy the commutation relations between canonical variables or they do not lead to new information. To some extent, this is due to the linear nature of the dynamics given by (2.1). Therefore, one of the aims of this paper is to see how some quantum systems can be considered as truly *nonlinear* systems. We illustrate this new viewpoint first in case of spin chains.

3 Hamiltonian Mastersymmetries for spin chains

We present a direct method for the computation of the commutants of the hamiltonians of the XYZ-model in ferromagnets. For details concerning the importance and the physical relevance of these spin chains we refer to the literature ([3]-[5], [1], [7], see also [2] where similar results can be found for the XYh-model).

At each point n of the lattice $\mathbb{Z}$ a three-component spin operator $\vec{\mathbf{S}}_n = (S_n^X, S_n^Y, S_n^Z)$ is given. We assume spin-1/2 operators. i.e.

$$S_n^j S_n^k = \delta_{jk} + i \sum_l \varepsilon^{jkl} S_n^l \tag{3.1}$$

where ε^{jkl} is the cyclic totally antisymmetric tensor with $\varepsilon^{XYZ} = 1$. We either consider the unbounded case, where no periodicity of the lattice is assumed, i.e. where all spin operators at different places commute $[S_n^j, S_m^k] = 0$ for $n \neq m$, or we consider the periodic case where some N is given such that $S_n^k = \pm S_{n+N}^k$ and where the spin operators only commute for those $n \neq m$ which are different modulo N. The manifold of polynomials in the S_n^k fulfilling the constraints (3.1) we denote by $\mathcal{M}(1/2)$.

The Hamiltonian of the XYZ-model is

$$H_{XYZ} = \sum_{n,k} J_k S_n^k S_{n-1}^k \tag{3.2}$$

where the sum for k goes over $\{X, Y, Z\}$ and for n it either goes over all $n \in \mathbb{Z}$ (unbounded case) or from 1 to N (periodic case with periodicity N). The equation of motion is $\vec{\mathbf{S}}_t = i[\![\ H, \vec{\mathbf{S}}]\!]$ or explicitly:

$$\dot{S}_n^k = -2 \sum_{l,r} \varepsilon^{klr} J_r S_n^l (S_{n-1}^r + S_{n+1}^r) \ . \tag{3.3}$$

To describe the commutants of this hamiltonian we look for hamiltonian mastersymmetries. Here, hamiltonian mastersymmetries are operators (in some extended operator algebra) such that if they are commuted with the operator H, then we obtain again operators commuting with H.

For the two Systems (XYZ and XYh) such hamiltonian mastersymmetries can be found in the literature ([13], [2]). Indeed, by a computer algebra package [14], developed for this purpose, we were able to find such operators systematically.

For the XYZ-model we obtain that $M_0 = \sum_{n,k} n J_k S_n^k S_{n-1}^k$ is such a hamiltonian mastersymmetry. Indeed, commuting this operator formally with H_{XYZ} we find the well known operator $H_1 = \sum_{k,l,r} J_l J_r \varepsilon^{lkr} S_{n+1}^l S_n^k S_{n-1}^r$ (see [18], [13]), as first symmetry (or conservation law, if one likes). This process can be continued indefinitely and leads to an infinite sequence of hamiltonians which commute with H_{XYZ}. Of course, the quantity M_0 is not really an operator, it only defines an outer derivative.

One observes that M_0 does not give a translation invariant operator, so it is not compatible with the reductions leading to periodic lattices. Nevertheless one can also use this mastersymmetry in case of periodic lattices because a successive application of the commutator given by M_0 to the original H_{XYZ} always leads to translation invariant operators.

We did not find higher order mastersymmetries for the XYZ-model (contrary to the situation encountered for the XYh-model), although the one mastersymmetry already allows a simple construction of commuting hamiltonians. However, the XYZ-model is considered to be completely integrable, and in case of classical complete integrability we always can find a complete set of action-angle variables, and only the action variables are found, since these correspond to the commuting hamiltonians. The angle variables should correspond to mastersymmetries, and since these do not exist, some doubts seem to be cast upon the complete integrability of this system.

These doubts are not really justified, because the quantum mechanical formulation of the system only allows for hamiltonian vector fields, since all dynamical laws have to be given by operators, thus leading to hamiltonian structure. Even in the classical case of complete integrability, angle variables on infinite dimensional manifolds do not always exist, only nonhamiltonian mastersymmetries exist in these cases, which then yield the angle variables if finite dimensional reductions are taken. The point is, that the dynamical laws given by the quantum mechanical formulation are not a rich enough structure to allow for nonhamiltonian quantities. So, we have to look for an extension of the dynamics in such a way that for hamiltonian quantities the usual quantum mechanical dynamic prevails and nevertheless nonhamiltonian dynamical laws are possible.

4 An alternative description of the dynamics

Consider a vector operator $\mathbf{S}_j = (S_j^X, S_j^Y, S_j^Z)$ associated with every lattice point j. Let P(**S**) be the polynomials in S_j^n, where $n = X, Y, Z$ and $j \in \mathbb{Z}$. Define the space of *densities* (see [13] or [2]) to be the quotient of P(**S**) with respect to $Q(\mathbf{S}) :=$ linear span $\{AB - BA \mid A, B \in P(\mathbf{S})\}$. Equivalence classes will be denoted by $[\ ]$, and the equivalence by $\equiv$. The construction of density space is done in such a way that $Q(\mathbf{S})$ can be understood as the kernel of a tracelike operation; it is exactly that in case the operators are Hilbert-Schmidt.

Let $\mathbf{A}$ and $\mathbf{B}$ be three-component operator-valued vectors whose components are in P(**S**). Define for $\mathbf{A}, \mathbf{B} \in P(\mathbf{S})$ the inner product by

$$(\mathbf{A}, \mathbf{B}) :\equiv [\sum A_j^n B_j^n] \equiv \text{equivalence class of } \sum A_j^n B_j^n \ . \qquad (4.1)$$

Define the *directional derivative* of a density F in the direction of $\mathbf{B}$ by

$$F'[\mathbf{B}] = \frac{\partial}{\partial\epsilon}_{|\epsilon=0} [F(\mathbf{S}+\epsilon\mathbf{B})] \ . \tag{4.2}$$

Since cyclic permutations of factors are allowed in densities we obtain the result ([13] or [16]) that there is a unique operator ∇F in $P(\mathbf{S})$ such that $F'[\mathbf{B}] = (\nabla F, \mathbf{B})$. The operator ∇F is defined to be the *gradient* of F. For example, one obtains the gradient of $[H_{XYZ}]$, the equivalence class given by H_{XYZ}, to be $\nabla[H_{XYZ}]_j = (J_X(S^X_{j+1} + S^X_{j-1}), J_Y(S^Y_{j+1} + S^Y_{j-1}), J_Z(S^Z_{j+1} + S^Z_{j-1}))$ or for example $(\nabla S^l_m)^k_n = \delta_{lk}\delta_{mn}$. Next one introduces a vector product, whose k-th component on the n-th place is given as

$$(\mathbf{B}\times\mathbf{A})^k_n = \frac{1}{2}\sum_{rs}\varepsilon^{rsk}(B^r_n A^s_n - A^r_n B^s_n) \ . \tag{4.3}$$

For example $\mathbf{S}\times\mathbf{S} = 0$. Now equation (3.3) can be written as

$$\dot{\mathbf{S}} = -2(\mathbf{S}\times\nabla[H]), \tag{4.4}$$

where $H = H_{XYZ}$. This suggests to consider this dynamical formulation as a flow on $\mathcal{M}(1/2)$. Observe that all flows of the form (4.4) leave this manifold $\mathcal{M}(1/2)$ invariant. We define a Lie algebra structure (Poisson brackets) by

$$\{[G],[H]\}_\Theta \equiv -2(\nabla[G], \mathbf{S}\times\nabla[H]) \tag{4.5}$$

which fulfills the Jacobi identity on this special manifold. A density G is *invariant* under the flow (4.4) if and only if $G_t + \{G,H\}_\Theta \equiv 0$.

Therefore, also in this formulation, H, or rather the density given by it, is said to be the *hamiltonian* of (4.4). Furthermore, the map $G \longrightarrow -2\mathbf{S}\times\nabla G$ is as usual a Lie algebra homomorphism from the Poisson brackets into the vector fields. Hence $\Theta = -2\mathbf{S}\times$ constitutes an implectic operator, i.e. an operator which can be used to define Poisson brackets not only for scalar fields but as well for covector fields. This yields in addition that for the system (4.4) Θ maps invariant covector fields onto invariant vector fields. These statements hold true for any flow of the type (4.4).

Observe, that we found a new hamiltonian structure for the given dynamical systems which drastically differs on the structural level from the one we had before. The main differences are:

•The manifold under consideration is not anymore the manifold of all selfadjoint operators but rather the manifold $\mathcal{M}(1/2)$ of suitable functions in the spin variable $\mathbf{S}$. Thus we have reduced the dynamics to a manifold which is considerably smaller.

• The dynamical system now truly is a nonlinear one, whereas in its canonical formulation if was linear.

This new approach, which completely fits into the classical formulation of hamiltonian systems, now allows us to look for flows on the new manifold which are not anymore of hamiltonian nature. It turns out that, at least for reductions to the periodic case, action angle variables can be found (a result which also follows from [7] or from Baxters work [3]-[5]).

5 Quantization of KdV

Recall that for the KdV $u_t = u_{xxx} + 6uu_x$ the Poisson bracket structure for scalar fields is defined by

$$\{F_1(u), F_2(u)\} := \int_{-\infty}^{+\infty} (\nabla F_1(u))(\nabla F_2(u))_x dx \tag{5.1}$$

where ∇ denotes the usual gradient. We shall follow the rule that quantum brackets are operator generalizations of the classical Poisson brackets. First we rewrite (5.1) for the case $F_i(u) = \int_{-\infty}^{+\infty} \varphi_i(x)u(x)dx$ where $\varphi_i(x)$ are suitable test functions. For these special fields we find $\{F_1, F_2\} = \int_{-\infty}^{+\infty} \varphi_i(x)\varphi_2(x)_x dx$. Now, taking limits such that $\varphi_1(x) \to \delta(\hat{x})$ and $\varphi_2(x) \to \delta(x - \hat{x})$, then we obtain $\{u(\hat{x}), u(\tilde{x})\} = \delta_{\hat{x}}(\hat{x} - \tilde{x})$ and the Poisson bracket between field variables at different points is a derivative of the δ-distribution. So quantization of the KdV-field must lead to

$$[u(x), u(\tilde{x})] = i\delta_x(x - \tilde{x}) \ . \tag{5.2}$$

Serious difficulties are to overcome in order to make this heuristic approach precise. To show that an algebra, fulfilling this relation, exists at all, we have to give an interpretation of terms like $(u(x)u(\tilde{x}) - u(\tilde{x})u(x))^2$ which would be equal to $\delta_x(x - \tilde{x})^2$, a quantity not yet defined. So, we first have to make some remarks about distribution multiplication. We follow closely the concept introduced in [12] and [8].

A distribution $\phi(x)$ is said to be *almost-bounded* if, for every $n \in \mathbf{N}$, its n-th derivative is of the form $\phi^{(n)}(x) = b(x) + \Delta(x)$ where b is a locally bounded function and where Δ is a distribution with discrete support without accumulation point.

A fundamental observation is that in the space of almost-bounded distributions, there is a canonical algebraic structure fulfilling associativity, product-rule of differentiaton, translation invariance with respect to x, and having the property that it extends the usual pointwise algebra of functions. The algebra is non-commutative, hence there must be two different algebras (interchange of order of factors). In that canonical algebra the product of two distributions with discrete support vanishes. The two product realizations are given by $\phi(x)\phi(\tilde{x}) = \lim_{\epsilon\downarrow 0}\phi(x+\epsilon)\tilde{\phi}(x)$ or $\phi(x)\phi(\tilde{x}) = \lim_{\epsilon\downarrow 0}\phi(x)\tilde{\phi}(x+\epsilon)$. To make the following considerations consistent we choose one of these realizations, say the last.

Now we denote by $u(x)$ a variable in the space of real almost-bounded distributions of degree 3 (third order derivatives of continuous functions). We define $\mathcal{F}(x)$ to be the algebra, generated in the space of almost-bounded distributions, by $u(x)$, all its translations $u(x+\tilde{x}), \tilde{x} \in \mathbb{R}$ and by the almost-bounded distributions. Observe that this is a non-commutative algebra. By $\otimes\mathcal{F}(x)$ we define the algebra of arbitrary tensor products. We will realize an algebra fulfilling (5.2) by taking suitable congruence classes in that algebra. Consider the ideal J generated in $\otimes\mathcal{F}(x)$ by the following relations $\simeq$:

$$\begin{aligned}
\phi_1(x)\otimes\phi_2(\tilde{x}) &\simeq \phi_1(x)\phi_2(\tilde{x})\otimes 1 \simeq 1\otimes\phi_1(x)\phi_2(\tilde{x})\\
u(x)\otimes\phi_1(\tilde{x}) &\simeq u(x)\phi_1(\tilde{x})\otimes 1 \simeq 1\otimes u(x)\phi_1(\tilde{x})\\
\phi_1(\tilde{x})\otimes u(x) &\simeq \phi_1(\tilde{x})u(x)\otimes 1 \simeq 1\otimes\phi_1(\tilde{x})u(x)\\
u(\hat{x})\otimes u(\tilde{x}) - u(\tilde{x})\otimes u(\hat{x}) &\simeq i\delta_x(\hat{x}-\tilde{x})\otimes 1\\
A\otimes 1 \simeq 1\otimes A &\simeq A
\end{aligned} \tag{5.3}$$

for ϕ_1, ϕ_2 arbitrary almost-bounded distributions, and $A \in \otimes\mathcal{F}(x)$.

Taking now the quotient $QF(x) = \otimes\mathcal{F}(x)/J$ of $Q\mathcal{F}(x)$ with respect to the ideal J we have found our quantum realization (named $QF(x)$, the *quantum fields* generated by $\mathcal{F}(x)$). In this new algebraic structure we then have $u(x)\cdot u(\tilde{x}) - u(\tilde{x})\cdot u(x) = i\delta_x(x-\tilde{x})$. Since elements of $QF(x)$ may be considered as operators (by multiplication) on $QF(x)$ itself, we have found the required operator representation of the Poisson structure of the KdV. Now, we have the prerequisites to define the time evolution for quantum systems by taking suitable Hamiltonian operators. For example, taking

$$H = \int_{-\infty}^{+\infty}\left\{\frac{1}{2}u_\xi(\xi)u_\xi(\xi) + u(\xi)u(\xi)u(\xi)\right\}d\xi \tag{5.4}$$

and defining the action of a commutator on an integral, as integral (in convolution sense) over the commutator with its integrand, then we find

$$u(x)_t = i[H, u(x)] = u_{xxx}(x) + 3u_x(x)u(x) + 3u(x)u_x(x)\ . \tag{5.5}$$

This is the quantum version of the KdV, it leaves the crucial relation (5.2) invariant. The main problem is to prove that this equation is completely integrable in the usual sense, i.e. that it has infinitely many commuting symmetry groups (or conserved quantities which are in involution).

6 Densities

In order to give a recursive description of the symmetries and the conserved quantities of the evolution (5.5), as before an alternative representation of its dynamics is introduced.

Define the space of *densities* to be the quotient of $QF(x)$, first with respect to $\mathcal{L}_1 =$ linear span $\{AB-BA \mid A, B \in QF(x)\}$ and then with respect to $\mathcal{L}_2 =$ linear span $\{DA - AD \mid A \in QF(x)\}$, where D denotes differentiation with respect to the variable x. The equivalence relation coming from the successive spaces $\mathcal{L}_1$ and $\mathcal{L}_2$ will be denoted by $\equiv$ and the class of A by $[A]$. The construction of density space is done in such a way that the factor spaces can be understood as the kernel of a trace operation such that formal integrals (from $-\infty$ to $+\infty$) over total derivatives vanish. Indeed, it were exactly that kernel, in case our operators were Hilbert-Schmidt operators vanishing rapidly at $x = \pm\infty$.

Let $\mathbf{A}$ and $\mathbf{B}$ elements in $QF(x)$. Define for $\mathbf{A}, \mathbf{B} \in QF(x)$ an *inner product* by

$$< \mathbf{AB} >:\equiv \text{equivalence class of } \int_{-\infty}^{+\infty} \mathbf{A}(x)\mathbf{B}(x)dx \ . \tag{6.1}$$

Observe that, due to $\mathcal{L}_2$, the differential operator is antisymmetric with respect to that density-valued inner product. Let $F = F(u)$ be a density depending in some way on the field variable u, define the *directional derivative* of F in the direction of $\mathbf{B}$ by $F'[\mathbf{B}] := \partial/\partial\epsilon_{|\epsilon=0} F(u + \epsilon\mathbf{B})$. The equivalence relation yields the result that there is a unique operator ∇, mapping densities into density-valued linear functionals on $QF(x)$ such that $F'[\mathbf{B}] =< \nabla F, \mathbf{B} >$ for all $\mathbf{B} \in QF(x)$. The quantity ∇F is said to be the *gradient* of F. In case that the densities can be understood as kernels of traces, then the gradient defined this way is indeed the classical gradient of the corresponding scalar quantity given by the integral over the trace. For example, one obtains the gradient of $[H_1]$, where

$$H_1 = \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} u_\xi(\xi) u_\xi(\xi) + u(\xi)u(\xi)u(\xi) \right\} d\xi \tag{6.2}$$

as $u_{xxx}(x)+3u_x(x)u(x)+3u(x)u_x(x)$. The evolution equation (5.5) therefore can be rewritten as

$$u_t = D\nabla[H_1] \tag{6.3}$$

where H_1 is given above, and where D denotes the operator of taking the derivative with respect to x. This suggests, to consider this alternative dynamical formulation of the system (5.5). This is a flow not on all of $QF(x)$ but rather on the manifold given by those $u(x)$ which are realizations of (5.2). However, this now is a hamiltonian system in the classical sense since D is an implectic operator.

Given this implectic operator Θ, define a Lie algebra structure in the space of densities (*Poisson brackets*) by $\{G,H\}_\Theta :\equiv < \nabla G, \Theta\nabla H >$. These fulfill the Jacobi identity. A density $G(u,t)$ is then *invariant* under the flow $u_t = \Theta\nabla H$ if and only if $G_t + \{G,H\}_\Theta \equiv 0$. If G does not depend explicitly on t then G is invariant if and only if its Poisson bracket with H vanishes. Therefore, H is said to be the *classical hamiltonian* of $u_t = \Theta\nabla H$. The map $G \longrightarrow \Theta\nabla G$ is a Lie algebra homomorphism from the Poisson brackets into the vector fields.

Observe, that we found a new hamiltonian structure, for the given quantum system, which drastically differs from the one we had before in (5.5). The main differences are:

- The manifold under consideration is not anymore the manifold of all elements of $QF(x)$ but rather the manifold of all $u(x)$ fulfilling (5.2). Thus we have reduced the dynamics to a manifold which is considerably smaller.
- The dynamical system now is a truly nonlinear one, whereas in its canonical formulation $A_t = i[H,A]$, $A \in QF(x)$ it was a linear one.

This new approach, which completely fits into the classical formulation of hamiltonian systems, now allows to look for other implectic operators which generate the same dynamics. So, we may use the bi-hamiltonian formulation, given by that, to construct the recursion operator in the usual way.

7 Recursion operator and mastersymmetries

We are now able to obtain the second hamiltonian formulation of the quantum KdV. Denote by u the field variable and introduce $L(u)A := uA$ $R(u)A := Au$ where $a \in QF(x)$. These are the operators of multiplication with u from the left and from the right, respectively. Then set:

$$\Theta = D^3+DL(u)+DR(u)+R(u)D+L(u)D+(L(u)-R(u))D^{-1}(L(u)-R(u))$$

which gives an operator being antisymmetric with respect to the inner product defined in the last section. We claim Θ satisfies, for arbitrary $A, B, C \in QF(x)$, the *implectic condition*: $< A, \Theta'[\Theta B]C > +$ cyclic permutations $= 0$. Hence we can conclude that Θ is an implectic operator and provides therefore the second hamiltonian formulation of (6.3). The actual verification of the implectic condition is tedious and elaborate. However, the operator Θ is formally, up to a change of sign, the same as the one considered in the general KP-theory presented in [19] or [6]. So, one may apply formally the structural arguments found in these papers, although the operator space considered there is quite different from the one considered here.

Since the operator Θ satisfies the implectic condition, we may now consider the conserved quantity: $H_0 := \frac{1}{2}\int_{-\infty}^{+\infty} u(\xi)u(\xi)d\xi$. Then from the rules laid down in the previous section we get $\nabla[H_0] = u$, whence, $u_t = \Theta\nabla[H_0]$ again results in the flow (5.5).

Thus we have two classical hamiltonian formulations for this quantum flow. This allows to apply the usual theory of hereditary operators ([9] or [10] in order to have a recursive generation of conserved densities and vector fields. We observe that replacing $u(x)$ by $u(x) + \alpha$ in the operator Θ preserves the implectic character of that operator (trivial Bäcklund transformation). But this transformation now yields the operator $\Theta + 4\alpha D$. Hence, Θ and D are compatible implectic operators, so $\Phi := \Theta D^{-1}$ is hereditary and generates out of the vector field, given by the right side of (5.5), a hierarchy of commuting flows. All these flows then constitute symmetry group generators for the quantum KdV, since that equation is among the members of the hierarchy. On the other hand, we may consider the density $[H_0]$ as a conserved quantity for (5.5), then recursive application of Φ^+ (transpose) yields other conserved densities. This is an immediate consequence of the fact that Φ^+ generates, because of its hereditaryness, a sequence of elements whose Poisson brackets commute.

The complete set of mastersymmetries is now easily found. Define, as in the commutative situation, a scaling symmetry by $\tau_0 := xu_x + 2u$. Then, if $K(u)$ is the right side of the quantum KdV, we find for the Lie derivatives $L_{\tau_0}K(u) = 3K(u)$ and $L_{\tau_0}\Phi = 2\Phi$. Using now the hereditary structure of Φ we find that the $\{\tau_n \mid \tau_n := \Phi^n\tau_0,\ n \in \mathbb{N}\}$ is the set of mastersymmetries forming a Viasoro algebra together with the set of symmetries $\{K(u)_n \mid K(u)_n := \Phi^n K(u)_0,\ n \in \mathbb{N}\}$.

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