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# Maps between Tree and Band Modules

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The indecomposable modules over special biserial algebras are known to be of a specially simple form, so-called "tree" and "band" modules (cf. [6, 2]). Crawley-Boevey has shown how to describe the homomorphisms between tree modules (cf. [3]) and our objective in this paper is to extend this result to band modules.

Tree and band modules occur as indecomposable representations of arbitrary algebras. Nevertheless we follow Crawley-Boevey and consider zero-relation algebras as the appropriate context. Given two tree or band modules we first study certain quiver homomorphisms between their underlying trees or bands and it turns out that a map between the representations is completely described by $k$-linear maps which are associated to these quiver homomorphisms.

It is interesting to note that in general the classification of maps between representations of a special biserial algebra $A$ is a wild problem. In fact the maps are equivalent to the representations of $T_2(A)$. For instance take for $A$ the path algebra of an $A_n$, $n \geqslant 6$ then $T_2(A)$ is wild although the maps between indecomposable representations of $A$ are easily to describe.

Throughout, $k$ will be a fixed algebraically closed field. Maps are written on the left.

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## TREE AND BAND MODULES

Let $Q = (Q_0, Q_1)$ be a *quiver*, that is a locally finite oriented graph with vertices $Q_0$ and arrows $Q_1$. The *path algebra kQ* is the algebra which

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has as basis the paths in $Q$, i.e., formal compositions of arrows. A locally bounded factor $kQ/I$ (cf. [1]) is called a *zero-relation algebra* if $I$ is generated by a set of paths of length at least two.

We denote by $\mathrm{mod}(Q, I)$ the category of *finite dimensional $kQ/I$-modules.* An object $M$ in $\mathrm{mod}(Q, I)$ is given by finite dimensional spaces $M(x)$, one for each vertex $x$ in $Q$, with $M(x)=0$ for all but a finite number of vertices, together with linear maps $M(\alpha)$: $M(x) \to M(y)$ for each arrow $\alpha: x \to y$ in $Q$ satisfying $M(\alpha_n)\cdots M(\alpha_1)=0$ for each path $\alpha_n \cdots \alpha_1$ in $I$. A *map* $f: M \to N$ in $\mathrm{mod}(Q, I)$ is a family of linear maps $f(x)$: $M(x) \to N(x)$, $x \in Q_0$ satisfying $N(\alpha) f(x) = f(y)\, M(\alpha)$ for each arrow $\alpha: x \to y$ in $Q$.

We call a quiver a *tree* or *cyclic* if it is a finite quiver whose underlying graph is simply connected or an $\tilde{A}_n$. Now let $S$ be such a quiver. A quiver homomorphism $F: S \to Q$ which satisfies the following conditions (W1) and (W2) is called a *winding* of $S$ *on* $Q$:

(W1) $F$ is *injective on sinks and sources*, i.e., there is no subquiver of the form

$$\cdot \xleftarrow{\alpha} \cdot \xrightarrow{\beta} \cdot \qquad \text{or} \qquad \cdot \xrightarrow{\alpha} \cdot \xleftarrow{\beta} \cdot$$

in $S$ with $F(\alpha) = F(\beta)$.

(W2) If $S$ is cyclic, then $F$ is *non-periodic*, i.e., there is no automorphism $\sigma \neq id$ of $S$ with $F\sigma = F$.

If $S$ is cyclic, then we call a winding $F: S \to Q$ *cyclic* and we associate with $S$ a fixed arrow $\alpha_S$ in $S$. Morphisms between windings are commutative triangles, where $\varphi$ is a quiver homomorphism,

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "F"'] & & S' \arrow[dl, "F'"] \\ & Q & \end{tikzcd}$$

The notation $\varphi: S \to S'$ always implies $F = F'\varphi$. Observe that $\varphi$ is an isomorphism if $S$ is cyclic. Then we may assume that $\varphi(\alpha_S) = \alpha_{S'}$.

If $F: S \to Q$ is a winding which satisfies

(W3) There is no path in $S$ whose image lies in $I$. Then $F$ induces a *push-down functor* $F_\lambda$: $\mathrm{mod}(S, 0) \to \mathrm{mod}(Q, I)$ defined by

$$F_\lambda M(a) = \bigoplus_{F(x)=a} M(x) \qquad \text{for } a \text{ either a vertex or an arrow in } Q.$$

If $S$ is a tree, we denote by $M_{S,1,1}$ the $kS$-module which is one-dimensional at each vertex and in which arrows are represented by the identity.

For cyclic $S$ denote by $M_{S,n,v}$, $n \in \{1, 2, 3, \ldots\}$ and $v \in k^*$, the $kS$-module which is $n$-dimensional at each vertex and in which the identity corresponds to every arrow except $\alpha_S$, where $M_{S,n,v}(\alpha_S)$ is the $n \times n$ Jordan block $J_{n,v}$ with eigenvalue $v$.

$$J_{n,v} = \begin{pmatrix} v & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & v \end{pmatrix}.$$

The modules of the form $F_\lambda M_{S,n,v}$ are called *tree modules* if $S$ is a tree (cf. [3]); otherwise we call them *band modules* (cf. [2]).

*Remarks.* (1) Tree and band modules are indecomposable. We obtain this from covering theory if we observe that the push-down $F_\lambda$ occurs as a restriction of $\bar{F}_\lambda : \operatorname{mod}(\bar{Q}, \bar{I}) \to \operatorname{mod}(Q, I)$ which is induced by a covering $\bar{F}: \bar{Q} \to Q$ extending $F$.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "F"'] & \bar{Q} \arrow[d, "\bar{F}"] \\ & Q \end{tikzcd} \qquad \begin{tikzcd} \operatorname{mod}(S, 0) \arrow[r, hook] \arrow[dr, "F_\lambda"'] & \operatorname{mod}(\bar{Q}, \bar{I}) \arrow[d, "\bar{F}_\lambda"] \\ & \operatorname{mod}(Q, I) \end{tikzcd}$$

Take for $\bar{F}$ the universal covering $\tilde{Q} \to Q$ if $S$ is a tree. Otherwise interpret $F$ as an element $w_F$ of the fundamental group $\Pi(Q, x)$, $x \in F(S)$ acting freely on the universal cover $\tilde{Q}$, and choose as covering $\bar{F}: \tilde{Q}/N \to Q$, where $N$ is the normal subgroup of $\Pi(Q, x)$ generated by $w_F$. The covering $\bar{F}$ is Galois (cf. [4]) and extends $F$ because $F$ is injective on sinks and sources. The induced push-down $\bar{F}_\lambda$ preserves indecomposables (cf. [4]).

(2) Tree and band modules may be defined over an arbitrary algebra $kQ/I$ if we replace (W3) by the following condition:

(W3′) If $v_1, \ldots, v_n$ are pairwise different paths in $Q$ with $\sum_{i=1}^{n} \alpha_i v_i \in I$ for some $\alpha_i \in k^*$, then there is no path $v$ in $S$ with $F(v) = v_i$ for any $i$.

In fact we may then regard $F_\lambda$ as the composition

$$F_\lambda : \operatorname{mod}(S, 0) \to \operatorname{mod}(Q, I') \to \operatorname{mod}(Q, I),$$

where $I' \supseteq I$ is generated by a suitable set of paths, and $\operatorname{mod}(Q, I') \to \operatorname{mod}(Q, I)$ is the canonical embedding. In this sense all results of this note remain true over an arbitrary algebra.

(3) An algebra $A$ is called *special biserial* (cf. [5]) if $A$ is isomorphic to a locally bounded factor $kQ/I$ which satisfies the following condition:

(S) If $F: S \to Q$ is a winding which satisfies (W3), then there is no subquiver of Dynkin class $D_4$ in $S$.

For a special biseral algebra tree and band modules correspond to the representations of the first kind and the representations of the second kind, respectively, and there are no more indecomposable modules except those projective-injective modules which are not uniserial (cf. [6, 2]).

## The Theorem

Let $kQ/I$ be a zero-relation algebra and let $X = F_\lambda M_{S,n,\nu}$ and $Y = G_\lambda M_{T,m,\mu}$ be tree or band modules. The theorem gives a description of $\operatorname{Hom}(X, Y)$ which is based on an analysis of maps between the underlying windings $F$ and $G$.

Consider *triples* $(U, \sigma, \tau)$ *connecting* $F$ *and* $G$, where $H: U \to Q$ is a winding and $\sigma: U \to S$ and $\tau: U \to T$ are winding morphisms. According to our convention they satisfy $F\sigma = H = G\tau$.

$$\begin{tikzcd} S \arrow[dr, "F"'] & U \arrow[l, "\sigma"'] \arrow[r, "\tau"] \arrow[d, "H"] & T \arrow[dl, "G"] \\ & Q & \end{tikzcd}$$

A morphism between triples $(U', \sigma', \tau')$ and $(U, \sigma, \tau)$ is given by a winding morphism $\iota: U' \to U$ which satisfies $\sigma' = \sigma\iota$ and $\tau' = \tau\iota$. This is an isomorphism if $\iota$ is an isomorphism and we define $(U'\sigma', \tau') \leqslant (U, \sigma, \tau)$ if $\iota$ is an inclusion. Now choose a complete set $\mathscr{U} = \mathscr{U}(F, G)$ of representatives of isomorphism classes of triples. The relation $\leqslant$ induces a partial order on $\mathscr{U}$ and therefore on any subset of $\mathscr{U}$. An element $(U, \sigma, \tau)$ is called *cyclic* if $U$ is cyclic. We call a triple $(U, \sigma, \tau)$ *admissable* if it satisfies the following conditions:

(A1) If $x$ is a vertex in $U$ and $\beta$ is an arrow in $S$ ending at $\sigma(x)$, then there is an arrow $\alpha$ ending at $x$ with $\sigma(\alpha) = \beta$.

(A2) If $x$ is a vertex in $U$ and $\beta$ is an arrow in $T$ starting at $\tau(x)$, then there is an arrow $\alpha$ starting at $x$ with $\tau(\alpha) = \beta$.

Lemma. *Let $i, j$ be a pair of vertices in $S$ and $T$, respectively. There is at most one admissible triple in*

$$\mathscr{U}_{ij} = \{(U, \sigma, \tau) \mid \textit{there is a vertex } x \textit{ in } U \textit{ with } \sigma(x) = i, \tau(x) = j\}.$$

*Proof.* Suppose first that there is no cyclic element in $\mathscr{U}_{ij}$. It is an immediate consequence of (A1) and (A2) that any admissable triple is maximal in $\mathscr{U}$. Therefore it is sufficient to prove that a maximal element

$(U, \sigma, \tau)$ in $\mathscr{U}_{ij}$ is uniquely determined. Suppose there is a second maximal element $(U', \sigma', \tau')$ in $\mathscr{U}_{ij}$. Consider a maximal triple $u = (\bar{U}, \bar{\sigma}, \bar{\tau})$ in $\mathscr{U}_{ij}$ which satisfies $u \leqslant (U, \sigma, \tau)$ and $u \leqslant (U', \sigma', \tau')$. We may identify $U$ with the corresponding isomorphic subquivers of $U$ and $U'$ respectively and our claim is $U = \bar{U} = U'$. Otherwise there is an arrow $\alpha$ in $U' \setminus \bar{U}$ starting or ending at a vertex $x$ in $\bar{U}$. We seek to contradict the maximality of $(U, \sigma, \tau)$. Therefore attach $\alpha$ to $U$ at $x$ and extend $\sigma$ and $\tau$ by $\sigma(\alpha) = \sigma'(\alpha)$ and $\tau(\alpha) = \tau'(\alpha)$ respectively. It remains to verify (W1) for the extended map $H = F\sigma = G\tau$. Assume there is an arrow $\beta$ in $U \setminus \bar{U}$ such that $\alpha$ and $\beta$ are both starting or ending at $x$. Since $\beta \notin \bar{U}_1$ we have $\sigma(\beta) \neq \sigma'(\alpha) = \sigma(\alpha)$ or $\tau(\beta) \neq \tau'(\alpha) = \tau(\alpha)$; otherwise $u$ would not be maximal. Then, applying (W1) for $F$ and $G$, it follows that $H(\beta) \neq H(\alpha)$, hence (W1) for $H$. Thus we have constructed a triple of $\mathscr{U}_{ij}$, which is strictly greater than $(U, \sigma, \tau)$. Contradiction.

Now suppose there exists a cyclic element in $\mathscr{U}_{ij}$. A cyclic triple $(U, \sigma, \tau)$ is unique because $\sigma$ and $\tau$ are isomorphisms by a previous remark. Any non-cyclic element $(U, \sigma, \tau) \in \mathscr{U}_{ij}$ consists of windings of the linear quiver $U$ round $S$ and $T$ by $\sigma$ and $\tau$, respectively. A simple argument shows that they do not satisfy (A1) and (A2).

We associate to each admissible triple $a = (U, \sigma, \tau)$ a space $H_a$ of $k$-linear maps. Define

$$H_a = \begin{cases} \operatorname{Hom}_k(k^n, k^m), & \text{if } a \text{ is non-cyclic;} \\ \{f \in \operatorname{Hom}_k(k^n, k^m) \mid f = (*)\}, & \text{if } a \text{ is cyclic and } \nu = \mu; \\ 0, & \text{if } a \text{ is cyclic and } \nu \neq \mu, \end{cases}$$

where $(*)$ is of form

$$(*) = \begin{pmatrix} 0 & \cdots & 0 & \kappa_r & \cdots & \kappa_2 & \kappa_1 \\ \vdots & & & \ddots & \ddots & & \kappa_2 \\ \vdots & & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \kappa_r \end{pmatrix}$$

or

$$(*) = \begin{pmatrix} \kappa_r & \cdots & \kappa_2 & \kappa_1 \\ 0 & \ddots & & \kappa_2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \kappa_r \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

with $\kappa_t \in k$ and $r = \min(n, m)$.

THEOREM. *Let $kQ/I$ be a zero-relation algebra and let $X = F_\lambda M_{S,n,\nu}$ and $Y = G_\lambda M_{T,m,\mu}$ be tree or band modules. If $\mathscr{A} = \mathscr{A}(F, G)$ denotes the set of all admissable triples in $\mathscr{U}(F, G)$, then there is an isomorphism between* $\mathrm{Hom}(X, Y)$ *and* $\bigoplus_{a \in \mathscr{A}} H_a$.

The proof provides the construction of $k$-linear maps $\Phi$: $\mathrm{Hom}(X, Y) \to \bigoplus_{a \in \mathscr{A}} H_a$ and $\Psi$: $\bigoplus_{a \in \mathscr{A}} H_a \to \mathrm{Hom}(X, Y)$ which are inverse to each other. This requires the choice of a single vertex $a_0$ in $U$ for each $a = (U, \sigma, \tau)$ in $\mathscr{A}$ which is arbitrary. A homomorphism $f$: $F_\lambda M_{S,n,\nu} \to G_\lambda M_{T,m,\mu}$ in $\mathrm{mod}(Q, I)$ is given by a collection of maps

$$f(x) = \bigoplus f_{ij}\colon \bigoplus_{F(i)=x} M_{S,n,\nu}(i) \to \bigoplus_{G(j)=x} M_{T,m,\mu}(j), \qquad x \in Q_0$$

and we may alternatively use the matrix $f = (f_{ij})_{i \in S_0, j \in T_0}$ of $k$-linear maps. Moreover to each arrow $\xi: s \to t$ in $Q$ there corresponds a commutative square

$$\begin{array}{ccc} X(s) = \bigoplus\limits_{F(i)=s} M_{S,n,\nu}(i) & \xrightarrow{\oplus f_{ij}} & \bigoplus\limits_{G(j)=s} M_{T,m,\mu}(j) = Y(s) \\ \Big\downarrow {\scriptstyle \bigoplus_{F(\alpha)=\xi} M_{S,n,\nu}(\alpha)} & & \Big\downarrow {\scriptstyle \bigoplus_{G(\beta)=\xi} M_{T,m,\mu}(\beta)} \\ X(t) = \bigoplus\limits_{F(p)=t} M_{S,n,\nu}(p) & \xrightarrow{\oplus f_{pq}} & \bigoplus\limits_{G(q)=t} M_{T,m,\mu}(q) = Y(t) \end{array} \qquad (**)$$

since $Y(\xi) f(s) = f(t) X(\xi)$. In particular one obtains from $(**)$ the following commutative square for a pair of arrows $\alpha$: $i \to p$ in $S$ and $\beta$: $j \to q$ in $T$ with $F(\alpha) = G(\beta)$:

$$\begin{array}{ccc} M_{S,n,\nu}(i) & \xrightarrow{f_{ij}} & M_{T,m,\mu}(j) \\ \Big\downarrow {\scriptstyle M_{S,n,\nu}(\alpha)} & & \Big\downarrow {\scriptstyle M_{T,m,\mu}(\beta)} \\ M_{S,n,\nu}(p) & \xrightarrow{f_{pq}} & M_{T,m,\mu}(q) \end{array} \qquad (***)$$

This decomposition of $(**)$ follows from (W1) for $F$ and $G$.

*First Step.* We describe the map $\Phi: f \mapsto (f_a)_{a \in \mathscr{A}}$. For each $a = (U, \sigma, \tau) \in \mathscr{A}$ put $f_a = f_{ij}$, where $i = \sigma(a_0)$ and $j = \tau(a_0)$. Clearly $f_a \in \mathrm{Hom}_k(k^n, k^m)$. In order to show that $f_a \in H_a$ we have to verify an additional property if $a$ is cyclic. To this end we may arrange the squares of form $(***)$, where the pair $(\alpha, \beta)$ runs through $\{(\sigma(\gamma), \tau(\gamma)) \mid \gamma \in U_1\}$ in a sequence as follows, using the fact that $\sigma$ and $\tau$ are isomorphisms between cyclic quivers:

$$\begin{tikzcd}
M_{S,n,v}(i) \arrow[r, "M_{S,n,v}(\alpha_S)"] \arrow[d, "f_{ij}"'] & M_{S,n,v}(i') \arrow[d] \cdots & M_{S,n,v}(i) \arrow[d, "f_{ij}"] \\
M_{T,m,\mu}(j) \arrow[r, "M_{T,m,\mu}(\alpha_T)"] & M_{T,m,\mu}(j') \cdots & M_{T,m,\mu}(j)
\end{tikzcd}$$

For convenience we have choosen $a_0$ as the starting vertex of $\alpha_U$. The dots indicate the maps $M_{S,n,v}(\alpha)$ and $M_{T,m,\mu}(\alpha)$, where $\alpha$ runs through all arrows of $S$ and $T$, respectively, except $\alpha_S$ and $\alpha_T$. The vertical maps are the corresponding summands of $f$. Since the horizontal arrows, except the first, in each row represent the identity of $k^n$ and $k^m$ respectively we obtain $f_{ij}J_{n,v} = J_{m,\mu}f_{ij}$. This implies $f_{ij}=0$ if $v \neq \mu$ and, if $v=\mu$, then $f_{ij}$ if of form $(*)$ as in the definition of $H_a$.

For later use we observe that, given a pair $i, j \in S_0 \times T_0$ such that $a \in \mathscr{U}_{ij} \cap \mathscr{A}$, the maps $f_{ij}$ and $f_a$ are equal up to certain isomorphisms which depend on the choice of $\alpha_S$, $\alpha_T$ and $a_0$:

$$f_{ij} = J_{m,\mu}{}^t f_a J_{n,v}{}^s \qquad \text{for some integers } s \text{ and } t.$$

*Second Step.* We shall construct maps $\Psi_a: H_a \to \mathrm{Hom}(X, Y)$ which extend to a map $\Psi: \bigoplus_{a \in \mathscr{A}} H_a \to \mathrm{Hom}(X, Y)$. Let $a = (U, \sigma, \tau) \in \mathscr{A}$ and $f \in H_a$. Define $\Psi_a(f) = (f_{ij})$ by $f_{ij} = f$ if $i = \sigma(a_0)$ and $j = \tau(a_0)$, and by claiming commutativity of the diagrams $(***)$, where the pair $(\alpha, \beta)$ runs through $\{(\sigma(\gamma), \tau(\gamma)) \mid \gamma \in U_1\}$. Finally put $f_{ij} = 0$ if there is no vertex $x$ in $U$ with $i = \sigma(x)$ and $j = \tau(x)$. If we summarize the commutativity of all single squares, then we obtain the commutativity of the square $(**)$ for each arrow $\xi$ in $Q$ because $a$ is an admissable triple and $f \in H_a$. Hence $\Psi_a(f)$ is a homomorphism in $\mathrm{mod}(Q, I)$.

*Third Step.* To conclude the proof of the theorem we verify that the maps $\Psi$ and $\Phi$ which we have constructed are inverse to each other. The key for this is the lemma. Let $f = (f_a)_{a \in \mathscr{A}}$ be an element of $\bigoplus_{a \in \mathscr{A}} H_a$ and let $a' \in \mathscr{A}$ be a fixed triple. The equality $\Phi\Psi_{a'}(f_{a'})_{a'} = f_{a'}$ is trivial, and the fact $\Phi\Psi_a(f_a)_{a'} = 0$ for all $a \neq a'$ is a consequence of the lemma. Thus $\Phi\Psi(f)_{a'} = f_{a'}$ for any $f$ and any $a'$, hence $\Phi\Psi = \mathrm{id}$.

We now show that $\Psi\Phi(f) = f$ for a map $f: X \to Y$. Fix a pair $i, j \in S_0 \times T_0$. Suppose first that there is no admissable element in $\mathscr{U}_{ij}$. Then $g_{ij} = 0$ for every $g \in \mathrm{Hom}(X, Y)$ follows from the commutativity of the diagram $(**)$ (after replacing $f$ by $g$). This proves $\Psi\Phi(f)_{ij} = 0 = f_{ij}$. Suppose now that $\mathscr{U}_{ij} \cap \mathscr{A} \neq \varnothing$. Then there is a unique admissible $a' \in \mathscr{U}_{ij}$. If we write $\Phi(f) = (f_a)$, then $f_{a'}$ and $f_{ij}$ are related by certain isomorphisms which were discussed in the first step. The definition of $\Psi_{a'}$, evaluated at $i, j$, involves the inverse of these relating isomorphisms and gives $\Psi_{a'}(f_{a'})_{ij} = f_{ij}$; the lemma provides $\Psi_a(f_a)_{ij} = 0$ for all $a \neq a'$ and therefore

$\Psi\Phi(f)_{ij} = f_{ij}$. Hence $\Psi\Phi = \mathrm{id}$ holds and the proof of the theorem is complete.

*Remark.* If we restrict $\Psi\colon \bigoplus_{a \in \mathscr{A}} H_a \to \mathrm{Hom}(X, Y)$ to a fixed summand $H_a$, then we may describe $f = \Psi(f_a)\colon X \to Y$, where $0 \neq f_a \in H_a$, by a natural factorisation, whose form depends on $a = (U, \sigma, \tau)$ as follows:

$$f\colon X \xrightarrow{f_\sigma} Z \xrightarrow{f_\tau} Y, \; Z = \begin{cases} (H_\lambda M_{U,1,1})^d, & \text{if } U \text{ is not cyclic;} \\ H_\lambda M_{U,d,v}, & \text{if } U \text{ is cyclic.} \end{cases}$$

Here $H$ denotes the winding $F\sigma = G\tau$ and $d = \dim_k \mathrm{Im}\, f_a$. If $\sigma$ is monic then $f_\sigma$ is an epimorphism and if $\tau$ is monic then $f_\tau$ is a monomorphism.

## An Example

Given two windings $F\colon S \to Q$ and $G\colon T \to Q$, calculation of the elements $(U, \sigma, \tau) \in \mathscr{A} = \mathscr{A}(F, G)$ is fairly simple. In fact the injectivity on sinks and sources of $\sigma$ implies that $U$ may be identified with a connected subquiver $S'$ of the universal cover $\tilde{S}$ of $S$. Thus $(U, \sigma, \tau)$ is described by a homomorphism $\varphi\colon S' \to T$. We give a precise alternative description of $\mathscr{A}$.

Let $\pi\colon \tilde{S} \to S$ be the universal covering of the quiver $S$ and denote by $\mathscr{A}'$ the set of maps of form $\varphi\colon S' \to T$ which satisfy the following conditions:

(H1) $S'$ is a finite connected subquiver of $\tilde{S}$.

(H2) $\varphi$ is a quiver homomorphism with $F\pi = G\varphi$.

(H3) If $x$ is a vertex in $S'$ and $\alpha$ is an arrow in $\tilde{S}$ ending at $x$ then $\alpha$ lies in $S'$.

(H4) If $x$ is a vertex in $S'$ and $\beta$ is an arrow in $T$ starting at $\varphi(x)$ then there is an arrow $\alpha$ in $S'$ starting at $x$ with $\varphi(\alpha) = \beta$.

Proposition. *The map $\varphi \mapsto (S', \pi\iota, \varphi)$, where $\varphi$ starts at $S'$ and $\iota$ is the inclusion $S' \subsetneq \tilde{S}$, defines a bijection between $\mathscr{A}'$ and the non-cyclic triples of $\mathscr{A}$. There is a cyclic triple in $\mathscr{A}$, which is necessarily unique, if and only if $F$ and $G$ are isomorphic cyclic windings.*

Note that according to the lemma the cardinality of $\mathscr{A}$ is bounded by

$$\mathrm{card}\, \mathscr{A} \leqslant \mathrm{card}\{(i, j) \in S_0 \times T_0 \mid F(i) = G(j)\}$$

although $\tilde{S}$ may be infinite.

As an example we consider the following quiver with relations:

$$Q\colon \alpha \circlearrowleft \cdot \circlearrowright \beta \qquad \beta\alpha = \alpha\beta = \alpha^3 = \beta^3 = 0.$$

The cyclic quivers $S$ and $T$ in this example are obtained by identifying the ends of the following linear quivers. Their arrows are labeled with their images under $F$ and $G$ respectively.

$$S: \quad 1 \xleftarrow{\beta} 2 \xrightarrow{\alpha} 3 \xleftarrow{\beta} 4 \xleftarrow{\beta} 5 \xrightarrow{\alpha} 6 \xrightarrow{\alpha} 7 \xleftarrow{\beta} 8 \xrightarrow{\alpha} 1$$

$$T: \quad 1 \xleftarrow{\beta} 2 \xleftarrow{\beta} 3 \xrightarrow{\alpha} 4 \xrightarrow{\alpha} 1$$

The following table shows the maps in $\mathscr{A}'$. A map $\varphi_p: S_p \to T$ in $\mathscr{A}'$ is given by their images $\varphi_p(q)$ which are listed horizontally. Here $q$ runs through the vertices of $S_p$.

| $q =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\varphi_1$ | — | 4 | 1 | 2 | 3 | 4 | 1 | 2 |
| $\varphi_2$ | 1 | 2 | — | — | — | — | — | 4 |
| $\varphi_3$ | — | — | — | 1 | 2 | — | — | — |
| $\varphi_4$ | — | — | — | — | 4 | 1 | — | — |
| $\varphi_5$ | — | 1 | — | — | — | — | — | — |
| $\varphi_6$ | — | — | — | — | 1 | — | — | — |
| $\varphi_7$ | — | — | — | — | — | — | — | 1 |

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