A QUESTION ON INDUCTIVE LIMITS OF WEIGHTED LOCALLY CONVEX FUNCTION SPACES

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In the first part of this paper, we give the definition and a survey of some properties of the inductive limits of weighted spaces with which we are dealing here. The particular question on the inductive limit spaces mentioned in the title will be presented, together with a partial solution, in the second chapter.

1. Inductive limits of weighted spaces of continuous resp. holomorphic functions.

For proofs of the results surveyed here, we refer to the joint paper [2] with R. Meise. - We start, for simplicity, with a locally compact Hausdorff space X; a non-negative (real-valued) continuous function v on X is called a *weight* (on X).

1. *Definition* - We introduce here two types of *weighted* (or *Nachbin*) spaces of *continuous* functions connected with a weight v on X:

$Cv(X) := \{ f : X \to \mathbb{C}$ continuous; vf is bounded on $X\}$,

$Cv_o(X) := \{f\colon X \to \mathbb{C}$ continuous; vf vanishes at infinity$\}$.

Both spaces are endowed with the semi-norm $p_v(f) := \sup_{x\in X} v(x)|f(x$ such that $Cv_o(X)$ is a closed topological vector subspace of $Cv(X)$.

In the same way, if X is an open subset of $\mathbb{C}^N$ $(N \geq 1)$, two types of weighted spaces of <u>holomorphic</u> functions on X are defined by

$Hv(X) := \{f\colon X \to \mathbb{C}$ holomorphic; vf is bounded on $X\}$,

$Hv_o(X) := \{f\colon X \to \mathbb{C}$ holomorphic; vf vanishes at infinity$\}$.

From now on, we shall always <u>assume that</u> v <u>has no zeros on</u> X (we write $v > 0$). Then p_v is even a norm, $Hv(X)$ and $Hv_o(X)$ are closed subspaces of $Cv(X)$, too, and, in fact, all four types of weighted spaces in 1. are Banach spaces under p_v: p_v induces a topology which is stronger than the topology co of uniform convergence on compact subsets of the locally compact space X.

Let now $\mathcal{V} := \{v_n\}_{n\in\mathbb{N}}$ be a (countable) decreasing system of weights $v_n > 0$ on X, i.e. $v_1 \geq v_2 \geq v_3 \geq \dots$. The union $\mathcal{V}C(X) := \bigcup_{n\in\mathbb{N}} Cv_n(X)$ of the increasing sequence of Banach spaces $Cv_n(X)$ — with $Cv_n(X) \hookrightarrow Cv_{n+1}(X)$ continuous (indeed, of norm ≤ 1) — is endowed with the locally convex inductive topology with respect to all inclusions $i_n\colon Cv_n(X) \hookrightarrow \mathcal{V}C(X)$, i.e. with the strongest locally convex topology such that all i_n $(n \in \mathbb{N})$ are continuous. Of course, $\mathcal{V}C(X)$ with this topology is the (locally convex) inductive limit of the Banach spaces $Cv_n(X)$. — We can proceed similarly in the other three cases and hence come to:

2. <u>Definition</u> - $\mathcal{V}C(X) := \operatorname{ind}_{n\to} Cv_n(X)$, $\mathcal{V}_oC(X) := \operatorname{ind}_{n\to} C(v_n)_o(X)$,

$\mathcal{V}H(X) := \operatorname{ind}_{n\to} Hv_n(X)$, $\mathcal{V}_oH(X) := \operatorname{ind}_{n\to} H(v_n)_o(X)$.

As the inductive limit topologies are always stronger than co, they are Hausdorff, and so all inductive limits are ultrabornological (DF)-spaces.

In connection with inductive limits, several natural questions arise: E.g., is the inductive limit <u>regular</u> ? Is the inductive limit topology even <u>complete</u> ? — <u>Regularity</u> here means that, for instance, each bounded subset of $\mathcal{V}C(X)$ should be contained and bounded in some Banach space $Cv_n(X)$.

3. <u>Proposition</u> - The inductive limits $\mathcal{V}C(X) = \operatorname{ind}_{n\to} Cv_n(X)$ and $\mathcal{V}H(X) = \operatorname{ind}_{n\to} Hv_n(X)$ are <u>always</u> regular. — On the other hand, in general e.g. $\mathcal{V}_oC(X) = \operatorname{ind}_{n\to} C(v_n)_o(X)$ is <u>not</u> regular (without further assumptions).

The proof of the first part of this proposition (see [2], 1.7) relies on a theorem of Grothendieck on bounded subsets of a courtable inductive limit of (DF)-spaces, while a counterexample to regularity of $\mathcal{V}_oC(X)$, similar to a counterexample of G.Köthe in the case of sequence spaces, can be found in [2], 2.3.

As the second part of proposition 3. (together with general theory) shows, $\mathcal{V}_oC(X)$ will in general <u>not even</u> be <u>sequentially complete</u>. No counterexample to completeness in the case of $\mathcal{V}C(X)$ is known, but to prove completeness of this space, a (sufficient) condition on $\mathcal{V}$ had to be added in [2].

4. <u>Definition</u> - $\mathcal{V}$ is said to satisfy <u>condition</u> (V), if for each $n \in \mathbb{N}$ there exists $m > n$ such that the quotient v_m/v_n vanishes at infinity.

If this condition is met, $\mathcal{V}C(X) = \mathcal{V}_oC(X)$ and $\mathcal{V}H(X) = \mathcal{V}_oH(X)$ holds true.

5. <u>Theorem</u> - If $\mathcal{V}$ satisfies condition (V), $\mathcal{V}C(X)$ and $\mathcal{V}H(X)$ are complete, and $\mathcal{V}H(X)$ is even a Silva (or (DFS)-space, i.e. strong dual of a Fréchet-Schwartz space (and hence separable, complete, and a Montel space).

In fact, if, for given $n \in \mathbb{N}$, $m > n$ is chosen according to condition (V), then,on any bounded subset B of $Cv_n(X)$ resp. $Hv_n(X)$, $Cv_m(X)$ induces the same topology as co (and hence the same as the inductive limit topology of $\mathcal{V}C(X)$ resp. $\mathcal{V}H(X)$).

The second part of this theorem can be verified easily (cf. [2], 1.6), and it follows that $\mathcal{V}C(X)$ and $\mathcal{V}H(X)$ are quasi-complete (that is, have all their closed bounded subsets complete) and hence complete, because both notions coincide in the case of (DF)-spaces. — The Semi-Montel property of the space $(H(X),co)$ of all holomorphic functions on X is then enough to conclude that $\mathcal{V}H(X)$ is a Silva space — a result that is, in fact, well-known. To state a (slightly) <u>weaker</u> sufficient condition for completeness of $\mathcal{V}C(X)$ resp. $\mathcal{V}H(X)$, we first have to give (resp. recall) the following definitions:

6. <u>Definition</u> - $\bar{V} := \{\bar{v}$ weight on X; $\frac{\bar{v}}{v_n}$ is bounded on X for each $n \in \mathbb{N}\}$

$= \{$all weights $\bar{v}$ on X majorized by some $\inf_{n \in \mathbb{N}} \lambda_n v_n$ with all $\lambda_n > 0\}$,

$C\bar{V}(X) := \{f: X \to \mathbb{C}$ continuous; $\bar{v}f$ bounded on X for every $\bar{v} \in \bar{V}\}$,

$C\bar{V}_o(X) := \{f: X \to \mathbb{C}$ continuous; $\bar{v}f$ vanishes at infinity for every $\bar{v} \in \bar{V}\}$. —

All the spaces $C\bar{V}(X)$, $C\bar{V}_o(X)$, $H\bar{V}(X)$, and $H\bar{V}_o(X)$ are complete (Hausdorff) locally convex spaces under the system $\{p_{\bar{v}}\}_{\bar{v} \in \bar{V}}$ of semi-norms, where $p_{\bar{v}}(f) := \sup_{x \in X} \bar{v}(x)|f(x)|$.

Remark that the functions $\bar{v} \in \bar{V}$ <u>may</u> have zeros on X. Obviously each non-negative continuous function with compact support belongs to $\bar{V}$. — $\bar{V}$ is called <u>the maximal Nachbin family associated with</u> $\mathfrak{V}$, and we have:

$$\mathfrak{V}C(X) \subset C\bar{V}(X)\ ,\ \mathfrak{V}_o C(X) \subset C\bar{V}_o(X)\ ,\ \mathfrak{V}H(X) \subset H\bar{V}(X)\ ,\ \mathfrak{V}_o H(X) \subset H\bar{V}_o(X)$$

with continuous injections.

7. <u>Definition</u> - $\mathfrak{V}$ is said to satisfy <u>condition</u> (wV), if for each $n \in \mathbb{N}$ there exists $m > n$ such that for each $\epsilon > 0$ there exists some $\bar{v} \in \bar{V}$ with:

$$\bar{v}(x) < v_m(x) \quad \text{implies} \quad \frac{v_m(x)}{v_n(x)} \leq \epsilon .$$

8. <u>Theorem</u> - ([2], Anhang, 1.6') - If $\mathfrak{V}$ satisfies condition (wV), $\mathfrak{V}C(X)$ and $\mathfrak{V}H(X)$ are complete.

In fact, if, for given $n \in \mathbb{N}$, $m > n$ is chosen according to condition (wV), then, on any bounded subset B of $Cv_n(X)$ resp. $Hv_n(X)$, $Cv_m(X)$ induces the same topology as $C\bar{V}(X)$ (and hence the same as the inductive limit topology of $\mathcal{V}C(X)$ resp. $\mathcal{V}H(X)$).

Again, the second part of this theorem may be verified (easily) by a small calculation, and this implies, together with proposition 3., a strong regularity condition on the inductive limits in question (that we were used to call "<u>strongly boundedly retractive</u>") which, in turn, leads to quasi-completeness and hence completeness.

However, in the case of theorem 8., $\mathcal{V}C(X) = \mathcal{V}_oC(X)$ does <u>not</u> follow, and $\mathcal{V}H(X)$ will in general no longer be a Silva space: In fact, (wV) is certainly <u>always</u> satisfied, if $v_n \equiv$ a <u>fixed</u> $v > 0$ on X for all $n \in \mathbb{N}$ (whence $\bar{V} = \{\bar{v}$ weight on X; $\bar{v} \leq Cv$ for some $C > 0\}$).

2. <u>A weighted projective description of inductive limits of weighted spaces</u>.

We can now specify the main question we are dealing with here:

9. <u>Problem</u> - Under which conditions do the following equations hold <u>algebraically</u> and <u>topologically</u>:

$$\mathcal{V}C(X) = C\bar{V}(X) \quad , \quad \mathcal{V}_oC(X) = C\bar{V}_o(X)$$

$$\mathcal{V}H(X) = H\bar{V}(X) \quad , \quad \mathcal{V}_oH(X) = H\bar{V}_o(X) \ ?$$

So we are asking for a "nice" description of the inductive limits of weighted spaces (as introduced in section 1.) again as — projectively defined — weighted (or Nachbin) spaces. — Of course, at least e.g. in the case of $\mathcal{V}_o C(X)$, the topological vector space equality with $C\bar{V}_o(X)$ cannot hold in full generality, because $C\bar{V}_o(X)$ is always complete under our assumptions, whereas the counterexample mentioned under 3. above shows that $\mathcal{V}_o C(X)$ may even fail to be sequentially complete. (We could still ask whether $C\bar{V}_o(X)$ induces on $\mathcal{V}_o C(X)$ the inductive limit topology or whether $C\bar{V}_o(X)$ is just the completion of $\mathcal{V}_o C(X)$.)

Here we are only going to look at the cases of $\mathcal{V}C(X)$ and $\mathcal{V}H(X)$. In fact, the question $\mathcal{V}H(X) \stackrel{?}{=} H\bar{V}(X)$ is of particular interest in connection with Ehrenpreis's notion of analytically uniform space: The Paley-Wiener-Schwartz theorem shows that the Fourier transform is a topological isomorphism of $\mathcal{E}' = \mathcal{E}'(\mathbb{R}^N)$ $(N \geq 1)$, the space of distributions with compact support under its usual topology, onto $\mathcal{V}H(\mathbb{C}^N)$, where $\mathcal{V} = \{v_n\}_{n\in\mathbb{N}}$ with:

$$v_n(z) = \prod_{j=1}^{N} \frac{\exp(-n|\operatorname{Im}(z_j)|)}{(1+|z_j|)^n}, \quad z = (z_1, \ldots, z_N) \in \mathbb{C}^N.$$

This $\mathcal{V}$ clearly satisfies condition (V). — Now Ehrenpreis proved that $\mathcal{E}$ has "an analytically uniform structure", i.e. that $\mathcal{V}H(\mathbb{C}^N) = H\tilde{V}(\mathbb{C}^N)$ for some Nachbin family $\tilde{V}$ of positive continuous functions (cf. [1], p. 63-64). $\tilde{V}$ is not uniquely determined, but the maximal Nachbin family $\bar{V}$ associated with $\mathcal{V}$ is one such system $\tilde{V}$. — Similarly, if one wants to prove that certain other spaces, say, of ultradistributions are analytically uniform, known Paley-Wiener type theorems reduce the problem to the proof of an equality like $\mathcal{V}H(\mathbb{C}^N) = H\bar{V}(\mathbb{C}^N)$ for a certain system $\mathcal{V}$.

Starting from this observation, B.A. Taylor [4] treated the question whether $H\bar{V}(\mathbb{C}^N)$ induces on $\mathcal{V}H(\mathbb{C}^N)$ the inductive limit topology for systems $\mathcal{V} = \{v_n\}_{n\in\mathbb{N}}$, where $v_n(z) = \exp(-\Phi_n(z))$ with Φ_n plurisubharmonic for each $n \in \mathbb{N}$. He was able to give an affirmative answer under certain conditions that imply (V). Taylor's proof is rather complicated and uses techniques of Hörmander concerning L^2-estimates for solutions of the $\bar{\partial}$-equation.

A similar, more general theorem about inductive limit topologies on spaces of entire functions has recently been stated by C. Servien [3]. There is a gap in his proof, however, and hence it is not clear whether his main theorem holds in full generality. — We propose here to treat the case of $\mathcal{V}C(X)$ (in a simple way) with a partition of unity argument <u>first</u> and <u>then</u> to conclude in the holomorphic case by aid of an open mapping lemma due to A. Baernstein. — Let us start, however, with a proposition (already contained in [2], 2.8) the proof of which is but a slight refinement of an argument due to B.A. Taylor:

10. <u>Proposition</u> - Let X be locally compact <u>and σ-compact</u> (in the continuous case). Then we have $\mathcal{V}C(X) = C\bar{V}(X)$ and $\mathcal{V}H(X) = H\bar{V}(X)$ **<u>algebraically</u>. — In fact, the weighted topology of $C\bar{V}(X)$ resp. $H\bar{V}(X)$ has also the <u>same</u> bounded sets as the inductive limit topology of $\mathcal{V}C(X)$ resp. $\mathcal{V}H(X)$.**

After these preparations we can state our main theorem:

11. Theorem - Let X be locally compact and σ-compact and assume that $\mathcal{V}$ satisfies condition (V). Then $\mathcal{V}C(X) = C\bar{V}(X)$ holds even topologically.

We include a full proof of this result to demonstrate the ideas involved: First (by renumbering the sequence $\{v_n\}_{n\in\mathbb{N}}$) in view of condition (V), we can assume without loss of generality that v_{n+1}/v_n vanishes at infinity for each $n \in \mathbb{N}$. Furthermore, as a (general) neighborhood U of 0 in the inductive limit $\mathcal{V}C(X)$, we may take $U = \overline{\Gamma(\bigcup_{n=1}^{\infty} U_n)}$, where the closure is with respect to the inductive limit topology, where Γ indicates the absolutely convex hull, and where

$$U_n := \{f \in Cv_n(X) \; ; \; \|f\|_n := p_{v_n}(f) = \sup_{x\in X} v_n(x)|f(x)| < \rho_n\},$$

$\rho_n > 0$ monotonically decreasing.

As X is locally compact and σ-compact, there is a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of X with $K_n \subset \mathring{K}_{n+1}$ $(n = 1,2,\ldots)$ and $X = \bigcup_{n\in\mathbb{N}} K_n$. Making use of condition (V) in the form stated above, we have without loss of generality - renumbering the sequence $\{K_n\}$, if necessary -

$$\frac{v_{n+1}}{v_n} < \frac{\rho_{n+1}}{2^{n+1}} \quad \text{off } K_n \quad (n = 1, 2, \ldots).$$

We now take a continuous partition of unity $\{\varphi_n\}_{n\in N}$ subordinated to the locally finite open covering

$$X = \bigcup_{n=1}^{\infty} (\mathring{K}_{n+1} \setminus K_{n-1}) \qquad (\text{where } K_o := \emptyset),$$

i.e. φ_n continuous on X, $0 \leq \varphi_n \leq 1$, $\varphi_n = 0$ off $\mathring{K}_{n+1} \setminus K_{n-1}$, and $\sum_{n=1}^{\infty} \varphi_n \equiv 1$ on X.

Put $\bar{v} := \sum_{k=1}^{\infty} v_{k-1} \varphi_k$, where $v_o :\equiv 1$. By definition, the sum exists in the space $(C(X),co)$ of all continuous functions on X. It is not hard to see that, in fact, $\bar{v} \in \bar{V}$. For $x \notin K_{n+1}$, we have $\varphi_k(x) = 0$ for all $k < n+1$, hence for all such x:

$$\frac{\bar{v}(x)}{v_n(x)} = \sum_{k=n+1}^{\infty} \varphi_k(x) \frac{v_{k-1}(x)}{v_n(x)} \leq \sum_{k=n+1}^{\infty} \varphi_k(x) \leq 1.$$

But on K_{n+1}, the continuous function $\frac{\bar{v}}{v_n}$ is bounded, hence $\frac{\bar{v}}{v_n}$ is bounded on all of X. As $n \in \mathbb{N}$ was arbitrary, $\bar{v} \in \bar{V}$ follows.

Now we claim that the neighborhood $\bar{U}$ of 0 in $C\bar{V}(X)$ defined as follows:

$$\bar{U} := \{f \in C\bar{V}(X) \ ; \ p_{\bar{v}}(f) = \sup_{x \in X} \bar{v}(x)|f(x)| < \epsilon := \min(1, \frac{\rho_1}{2 \sup_{x \in K_2} v_1(x)})\},$$

satisfies $\bar{U} \subset U$. If we have established this claim, the proof is of course finished. However, to prove our claim, we first remark that any $f \in C(X)$ can be represented in $(C(X),co)$ in the following way:

$$f = \sum_{n=1}^{\infty} \varphi_n f = \sum_{n=1}^{\infty} \frac{1}{2^n} 2^n \varphi_n f \quad , \quad \text{or} \quad : \quad f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$$

with $f_n := 2^n \varphi_n f$ continuous and with compact support.

If f is even a member of $\mathcal{V}C(X)$, i.e. belongs to some $Cv_n(X)$, then the partial sums $\{\sum_{k=1}^{N} \varphi_k f\}_{N\in \mathbb{N}}$ of the series $\sum_{k=1}^{\infty} \frac{1}{2^k} f_k$ are all bounded in $\|\cdot\|_n$ by $\|f\|_n$. Therefore, in this case, the series not only converges uniformly on each compact subset of X, but also with respect to the inductive limit topology of $\mathcal{V}C(X)$ by theorem 5. Keeping this in mind, we see that our claim is established, if we only have the implication:

$$f \in \bar{U} \Rightarrow f_n \in U_n \quad \text{for all} \quad n \in \mathbb{N}.$$

But this is easy. If $f \in \bar{U}$,

$$\min\Big(1 , \frac{\rho_1}{2 \sup_{x\in K_2} v_1(x)}\Big) = \varepsilon > \sup_{x\in X} \sum_{k=1}^{\infty} v_{k-1}(x)\, \varphi_k(x) |f(x)| \geq$$

$$\geq \sup_{x\in X} v_n(x)\, \varphi_{n+1}(x) |f(x)| =$$

$$= \frac{1}{2^{n+1}} \sup_{x\in X} v_n(x) |f_{n+1}(x)|$$

resp. $\geq \sup_{x\in X} \varphi_1(x)|f(x)| = \frac{1}{2} \sup_{x\in X} |f_1(x)|$,

and hence $\sup_{x\in X} |f_1(x)| < \frac{\rho_1}{\sup_{x\in K_2} v_1(x)}$, $\sup_{x\in X} v_n(x)|f_{n+1}(x)| < 2^{n+1}$,

$n = 1 , 2 , \ldots$

So finally $f \in \bar{U}$ implies:

$$\sup_{x\in X} v_1(x)|f_1(x)| = \sup_{x\in K_2} v_1(x)|f_1(x)| < \rho_1 ,$$

i.e. $f_1 \in U_1$, and

$$\sup_{x\in X} v_{n+1}(x)|f_{n+1}(x)| = \sup_{x\in X\setminus K_n} \frac{v_{n+1}(x)}{v_n(x)} v_n(x)|f_{n+1}(x)| < \frac{\rho_{n+1}}{2^{n+1}} 2^{n+1} =$$

$$= \rho_{n+1} ,$$

that is, $f_{n+1} \in U_{n+1}$ for each $n \in \mathbb{N}$. □

The application to the problem $\mathcal{V}H(X) = H\bar{V}(X)$ is then based on:

12. Proposition - If $\mathcal{V}$ satisfies condition (V), $\mathcal{V}H(X)$ is a topological subspace of $\mathcal{V}C(X)$.

The proof of 12. is easy (cf. [2], 1.14): By 5., condition (V) implies that $\mathcal{V}H(X)$ is a Semi-Montel space (among other things), and by 3., we may then apply an open mapping lemma due to A. Baernstein to the canonical (continuous) injection $\mathcal{V}H(X) \to \mathcal{V}C(X)$ to obtain the assertion of the proposition.

The following corollary is now a direct consequence of 10., 11., and 12.:

13. Corollary - Assume that $\mathcal{V}$ satisfies condition (V). Then $\mathcal{V}H(X) = H\bar{V}(X)$ holds topologically, too.

Remark that we have used continuous partitions of unity in the proof of theorem 11. An analogous direct proof of corollary 13. would therefore not have been possible. — In fact, we realize now that the use of partitions of unity made the continuous case much easier than the holomorphic one, but, fortunately, Baernstein's open mapping lemma helped to deduce the holomorphic result as a corollary.

To finish, let me point out that the results in this section are only *a beginning*; much *more* can be done: Bill Summers has recently dealt with the case of $\mathcal{V}_o C(X)$ (using duality theory and a characterization of dual spaces of weighted spaces). And R. Meise proposed to even *eliminate continuity* from the argument — so he in fact introduced *a third case*. Furthermore, it is possible to treat *vector-valued* functions, i.e. functions with values in a Banach space or a special type of (LB)-space, in a similar manner. — We will not give more details here, but refer to a forthcoming joint article of the author with R. Meise and W.H. Summers.

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