On Some Basic Aspects of Transfer Operator Methods for Coupled Cell Systems

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Acknowledgement

This text is the result of almost twelve years of studying, learning, teaching and working in mathematics, and of at least three years work dedicated (more or less) expressly to the preparation of a doctoral thesis. I not only feel the duty, it is also my desire to say thanks to a number of people for a number of reasons that have accumulated over this time.

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It may not be appropriate to write this in this place, but my ultimate motiva-

[1] I have to be thankful for receiving German taxpayer's money, in particular through the DFG under grant De 448 / 8, and from the state of Nordrhein-Westfalen.

tion to pursue my research, and to bring it into the form of this thesis, did not come from the intrinsic interest and satisfaction that wrestling with mathematical problems has for me[2], or anything the like. It came from my family: my wife Carolina, my children Clemens, Margareta and Philipp, my parents. I am grateful for their love, their support, their encouragement.

[2]Although these (both interest and satisfaction) are considerable.

Abstract

Given a (measurable) space X and a measurable map $f : X \to X$, the transfer operator for f is a linear map $P_f : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by the prescription $P_f\mu(A) = \mu(f^{-1}(A))$ for all measurable sets $A \subset X$ and all $\mu \in \mathcal{M}(X)$. Here $\mathcal{M}(X)$ is a suitably chosen linear space of measures on X.
In this thesis, transfer operators are considered for a particular class of maps: those that describe a coupled cell system. A coupled cell system is a dynamical system admissible to a coupled cell network. This concept is used both in applications and in theoretical works to model dynamical systems that are built up from smaller parts (called cells), that influence each other in the temporal evolution of their internal state. Thus the state space X of a coupled cell system is the cartesian product of the state spaces X_c of the individual cells, and each component map f_c may depend on c and several other cells.
The structure of the network underlying a coupled cell system can be described in an algebraic way by means of its so-called symmetry groupoid. Admissibility of a map f on the network can then be expressed as equivariance of f with respect to a certain action of this groupoid. In the case of dynamical systems with "classical" symmetries, expressed by equivariance with respect to group actions, the results of linear representation theory have implications for a system that allow far-reaching characterisations of its dynamics. In particular, its transfer operator can be shown to possess invariant subspaces due to symmetry.
In view of these results, the aim of this thesis is to describe the structural implications that equivariance with respect to the action of the symmetry groupoid has for the transfer operator of a coupled cell system. How can the structural properties of the map f be translated into properties of the transfer operator P_f? To answer this question, this thesis shows that it is possible to decompose $\mathcal{M}(X)$ into the direct sum of subspaces $U_\mathcal{D}$ parametrized by the set of subsets $\mathcal{D}$ of the set of cells in such a way that the coupling structure of the network is reflected in the corresponding block decomposition of the transfer operator.
Furthermore, to analyse the structure of P_f due to symmetry properties of the network, a family $\Gamma_\mathcal{D}$ of symmetry groups for subsets of the set of cells is associated to the symmetry groupoid. These groups make it possible to use results from representation theory to determine further structural properties of the transfer operator.
The direct application of the theoretical results in a numerical scheme for the approximation of the transfer operator is prevented by the fact that standard methods for this task rely on the usage of bases of $\mathcal{M}(X)$ that are derived from discretizations of the state space X in a specific manner. These bases appear to be incompatible with the decomposition of $\mathcal{M}(X)$. The reasons underlying this problem are explained in detail, and an alternative method for the efficient approximation of P_f is sketched.

Zusammenfassung

Ist X ein Messraum und $f : X \to X$ eine messbare Abbildung, so ist der Transferoperator zu f eine lineare Abbildung $P_f : \mathcal{M}(X) \to \mathcal{M}(X)$, wobei $\mathcal{M}(X)$ ein geeignet gewählter Vektorraum von Maßen über X ist und P_f durch die Vorschrift $P_f\mu(A) = \mu(f^{-1}(A))$ für alle messbaren Mengen A und alle Maße $\mu \in \mathcal{M}(X)$ definiert wird.
In dieser Dissertation werden Transferoperatoren für eine spezielle Klasse von Abbildungen betrachtet, nämlich jene, die ein *coupled cell system*[3] beschreiben. Ein solches ist ein auf einem *coupled cell network* zulässiges dynamisches System. Dieses Konzept wird in anwendungsorientierten wie in theoretischen Arbeiten verwendet, um dynamische Systeme zu beschreiben, die aus einzelnen Teilen (genannt Zellen) aufgebaut sind, die sich gegenseitig in der zeitlichen Entwicklung ihrer inneren Zustände beeinflussen. Der Zustandsraum X des *coupled cell system* ist damit das kartesische Produkt der Zustandsräume X_c der einzelnen Zellen, und jede einzelne Komponentenabbildung f_c kann von c und mehreren anderen Zellen abhängen.
Die Struktur des einem *coupled cell system* unterliegenden Kopplungsnetzwerks kann mit Hilfe seines sogenannten Symmetriegruppoiden algebraisch beschrieben werden. Die Zulässigkeit einer Abbildung f auf einem Netzwerk lässt sich dann als Äquivarianz der Abbildung in Bezug auf eine bestimmte Wirkung des Gruppoiden ausdrücken. Im Falle eines dynamischen Systems mit "klassischen", durch die Äquivarianz in Bezug auf die Wirkung einer Gruppe ausgedrückten Symmetrien ziehen die Ergebnisse der linearen Darstellungstheorie Implikationen für ein System nach sich, die weit reichende Beschreibungen seiner Dynamik erlauben. Insbesondere lässt sich zeigen, dass der Transferoperator aufgrund der Symmetrie bestimmte Unterräume invariant lässt.
Im Hinblick auf diese Ergebnisse ist es das Ziel dieser Dissertation, die strukturellen Konsequenzen zu beschreiben, die die Äquivarianz in Bezug auf die Wirkung des Symmetriegruppoiden für den Transferoperator eines *coupled cell system* nach sich zieht. Wie lassen sich strukturelle Eigenschaften der Abbildung f in Eigenschaften von P_f übersetzen? Zur Beantwortung dieser Fragen wird eine Zerlegung des Maßraumes $\mathcal{M}(X)$ in eine direkte Summe von durch die Teilmengen $\mathcal{D}$ der Menge der Zellen parametrisierten Unterräumen $U_\mathcal{D}$ eingeführt, die es erlaubt, die Kopplungsstruktur des Netzes in der zugehörigen Blockzerlegung des Transferoperators wiederzufinden.
Weiterhin wird zur Analyse der Symmetriebeziehungen dem Symmetriegruppoiden des Netzwerks eine Familie von Symmetriegruppen $\Gamma_\mathcal{D}$ für Teilmengen von Zellen zugeordnet, die es erlauben, klassische Ergebnisse der Darstellungstheorie

[3] Der Autor hat auch nach längerer Suche keinen Beleg für eine Verwendung deutschsprachiger Äquivalente der Begriffe *coupled cell network* und *coupled cell system* in deutschsprachigen Texten gefunden, und zieht es daher vor, die in der Originalliteratur verwendeten Termini für die Zwecke dieser Zusammenfassung unübersetzt zu belassen.

von Gruppen zu nutzen, um weitere Strukturmerkmale des Transferoperators zu bestimmen.
Der direkten Verwendung der theoretischen Ergebnisse in einem numerischen Verfahren zur Berechnung einer Näherung an den Transferoperator steht die Tatsache entgegen, dass übliche Methoden für eine solche Berechnung die Verwendung von Basen von $\mathcal{M}(X)$ voraussetzen, die auf eine bestimmte Weise aus Diskretisierungen des Zustandsraumes X hervorgehen. Diese Basen scheinen unverträglich mit der Zerlegung von $\mathcal{M}(X)$ zu sein. Die diesem Problem zu Grunde liegenden Ursachen werden detailliert erklärt; weiterhin wird eine alternative Methode für die effiziente Berechnung von P_f skizziert.

Contents

1 Introduction

The title of this thesis combines two concepts – *transfer operators* and *coupled cell systems* – which both come from the same, rather rich and diverse area of mathematics, that of the study of *dynamical systems*, but which nevertheless can justly be described as being quite some distance apart from each other. It is the purpose of this work to start forming a connection between them which eventually may help reduce this distance, and to lay foundations that can be used for a more elaborated transfer operator theory for coupled cell systems. Before we can describe this aim in greater detail, we need to take closer looks at the two concepts involved.

Transfer operators

Dynamical Systems A *dynamical system* is, for the purposes of this thesis, a pair (X, f) where the *state space* X is a set of a suitable class: a topological space, a vector space, a (tangent bundle of) a manifold, or something similar; and the map $f : X \to X$ typically is in a suitable way compatible with the structure of X. In any case, for this thesis we will require that X is a measurable space, i. e. equipped with a σ-algebra, and that f is measurable with respect to this σ-algebra[1]. This small set of assumptions will already be enough to define the notion of the transfer operator in the way we will use it in this thesis.

When one considers a map $f : X \to X$ as a dynamical system, it is implied that the interest of one's study lies in *trajectories* of the map, i. e. in sequences in X of the form $x, f(x), f^2(x), f^3(x), \ldots$; and in describing the possible kinds of behaviour of these sequences[2]. In the mathematical theory of dynamical systems, this local, "pointwise" or "trajectory-wise" point of view is complemented by a global analytical approach that can be called "statistical" or "stochastic". The main object of study in this approach is the transfer operator that is associated with a dynamical system.

[1] See Section 2.3 for explanations of these terms.

[2] This is of course a rather short description of the term. A brief general introduction to the theory of dynamical systems can be found in [47], a detailed reference is [34].

Transfer operators In this thesis we will use the term "transfer operator" [3] for a given dynamical system to describe a linear operator on a space of (signed or complex) measures. Here the measures are seen as mathematical representatives of the notion of "ensembles" of initial conditions of the dynamical system. The dynamical system $f : X \to X$ determines the temporal evolution of both single initial conditions and ensembles of them, and the transfer operator P_f is the mathematical tool to describe the evolution of associated measures. It is thus defined through the requirement that

$$\int_X \varphi \, dP_f\mu = \int_X \varphi \circ f \, d\mu$$

for some measure μ and integrable function φ on X, or equivalently by

$$P_f\mu(A) = \mu(f^{-1}(A))$$

for measurable subsets A. The name "transfer operator" refers to its task of "transferring" probability density or mass density between regions of state space. Maybe the most concise description of the purposes transfer operators are used for in the analysis of dynamical systems is to say that it is often an appropriate tool when one wants to look at statistical, or stochastic, properties of a system. In particular, using the transfer operator formalism is a convenient choice when one wants to analyse systems subject to random disturbances. More concretely, functional analytic properties of the transfer operator (usually defined on suitable Banach spaces of densities of measures) can be linked to properties of the dynamical system such as ergodicity, mixing, or "chaoticity"; and vice versa. In certain settings, expansion rates of a system are connected with the *essential spectral radius* (see e. g. [43, 29, 35]) of the operator, and often isolated eigenvalues with modulus larger than that radius correspond to so-called *almost invariant sets* of the dynamical systems [10]. For a well readable introduction to the use of transfer operators in the theory of dynamical systems, we refer to [36].
On the side of applications, it is the possibility to interpret its action as describing "mass transport" that makes the transfer operator an interesting tool. It has proven to be useful for the analysis of transport phenomena in so different fields as ocean dynamics (see e. g. [21, 22]), solar system dynamics (e. g. [12, 13]) or theoretical chemistry (e. g. [7, 14, 15]). The main idea behind these applications is to use approximations of the transfer operator to compute almost invariant sets of the system, and also transport rates between these. For this purpose, one needs to be able to efficiently compute approximation to the transfer operator.

[3] The term "transfer operator" is today used for a variety of related, but distinct concepts in disciplines such as statistical mechanics, functional analysis, ergodic theory, or the theory of (abstract) dynamical systems. The meaning we will give it in the whole of this thesis is, to the authors knowledge, the most basic notion behind its usage in dynamical systems theory.

Numerics For the numerical approximation of the transfer operator, a simple recipe is often used that was probably first proposed by S. Ulam [46]: One chooses a sufficiently fine partition of the state space of the system and computes *transition rates* or *transition probabilities* between the sets in the partition. These probabilities, arranged as a matrix, can be understood as a finite-dimensional approximation of the transfer operator, where the underlying finite-dimensional space of measures is the one generated by the measures assigning the value one to precisely one of the sets of the partition, and zero to the others. Already this simple description of the procedure allows to get a feeling for the numerical effort necessary for its realisation: Roughly, it is the product of the number of sets in the partition and of the effort necessary for the estimation of transition probabilities from one box. Typically, the number of sets in a partition is proportional to δ^{-d}, where δ is a fineness parameter (e. g. the maximal diameter of a set), and d is the dimension of the state space[4].Even neglecting the fact that the costs for the estimation of transition probabilities from one box will normally depend on d, it becomes clear that the effort grows exponentially with d, so that numerical realisations normally are feasible only for few dimensions, e. g. $d \in \{1, 2, 3, 4\}$. This fact constitutes one half of the specific motivation for the research project behind this thesis.

Coupled cell systems

Networks are nearly ubiquitous in nature, and hence, in the natural sciences. On a "metalevel", their appearance is linked to the reductionist view science has on nature: Often one finds them while "zooming back out" after one "zoomed in" on some class of phenomena, studied their individual constituents in detail and is now trying to put together the parts to some larger "whole". Brain research can be cited as an (admittedly, rather ambitious) example here. What looks like some kind of homogeneous "grey matter" at first sight (upon opening a skull, say), is seen to consist of a large number of highly structured entities (neurons) on the microscopic level. Studying single neurons, one recognizes their electrochemical activity, the importance of electrical potential inside the neuron and of certain biological substances on its outside. One begins to describe their spiking behaviour and arrives at mathematical models for this behaviour like the Hodgkin-Huxley model, or the FitzHugh-Nagumo model. Zooming back out a little bit, one recognizes that in an indivuals' nervous system large numbers of neurons are connected with each other to form vast, overwhelmingly complex networks.
Similar phenomena can be observed not only in many biological systems, but

[4]This dimension is not necessarily equal to the number of variables used to describe the system. Systems with a relatively high-dimensional description can possess low-dimensional attractors, in which case an approximation of the transfer operator is possible, see [33, Section 3.5.3] for an example.

also in the man-made part of the world. Here it is not a reductionist but a constructivist principle that could be given as a cause: Engineers, in designing technical systems, tend to link together smaller technical systems (perhaps made by other engineers). In this way single transistors, reduced to nanometer size, are connected with each other to form logical units, say NAND-arrays, which are put together to form processors or memory chips. Several of these again form a computer, and many computers are connected with each other via the internet.
The mathematical abstraction from these and many more, very diverse examples is that of a *directed graph*: a set $\mathcal{C}$ of vertices, (sometimes called nodes, in this thesis usually called cells) together with a set $\mathcal{E} \subset \mathcal{C} \times \mathcal{C}$ of edges (which are here often called couplings). The graph alone, however, does not yet describe what we are interested in in this thesis. Many, though certainly not all networks in science and engineering exhibit some form of "dynamics": that is, the individual nodes are characterized by an internal "state" which evolves with time, and the edges between nodes represent influences that the states of the nodes exert on each others' evolution. To model this feature mathematically, one attaches a state space X_c to each cell c of the network, and describes the temporal evolution in a suitable manner, e. g. through differential or difference equations, Markov chains or some other form of stochastic process. (In this thesis, we will restrict our attention to the case of deterministic dynamical systems with discrete time.) The edges of the network are represented here through the fact that the equation for the evolution of the state of cell c depends only on states of cells from which an edge points to c.
For several years now, the term *coupled cell systems* has been used in the mathematical literature to describe a class of dynamical systems which are constructed from subsystems that are coupled together in some way. In particular, this name is used for a system, or a class of systems, when the interest of the analysis it is subjected to lies more in the *structure* of the system than in the particular set of equations that governs its evolution. The term "structure" here refers to both the *coupling* structure, i. e. to the answer to the question "Which cells' behaviour is influenced by which other cells?", and to what might be called *permutational* structure, i. e. to the answer to the question "Which permutations of cells are possible that result in the same (sub-)system?"
Questions of this kind have been analysed for a long time. In many cases, the permutational structure can be expressed as *equivariance* with respect to *permutation groups*. As an example, one can name the analysis of animal locomotion modes by Stewart et al. that is summarized e. g. in [25], see also references therein. There are a number of quite powerful tools from the theory of equivariant dynamical systems and representation theory of groups (see e. g. [26]) that can be employed for the analysis of such systems. However, being based on group theory, the applicability of these tools is limited to networks with *global* symmetry, that is, to networks that are invariant under permutations of *all* their components.

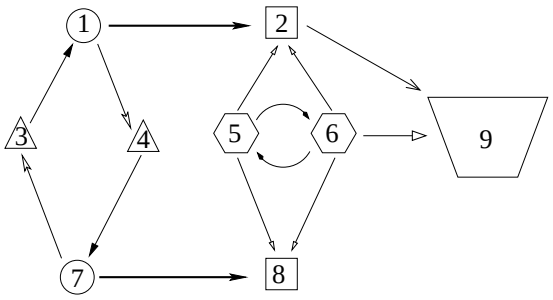

Figure 1.1: An example of a highly structured network without any global symmetry. If cell 9 and the arrows pointing to it are removed, the remaining graph admits a $S_2 \times S_2$ permutation symmetry.

It is not difficult to imagine cases for which this requirement seems a very strict, if not to say too strict one. There are many examples of networks as the one described in Figure 1.1 that posses a very high degree of *structure* that could easily be called symmetry, were it not for the fact that there is no permutation of *all* cells that leaves the *whole* network invariant.

As a remedy, Golubitsky, Stewart et al. developed a formalism that is closely related to the framework of group equivariance. (The earliest description of this formalism known to the author is given in [45], it was further developed in subsequent publications; an overview is contained in [25]. An equivalent alternative description that avoids much of the algebraic formalism was given by Field in [20].) In their framework, the place of the permutations – which are bijections on the set of all cells – is taken by so-called *input isomorphisms*, which are mappings defined only "locally", on certain subsets of the set of all cells. The greater flexibility that this change brings is paid for with a more complex algebraic structure. The set of all such input isomorphisms forms the so-called *symmetry groupoid* of the network. A groupoid is a generalization of a group; the most prominent difference between the two is that a group's composition is defined for arbitrary pairs of elements, while the composition of elements of the groupoid is defined only for particular pairs. It turns out (see e. g. [45]) that dynamical systems which one would intuitively consider to be admissible to a coupled cell network can be described as being equivariant to the symmetry groupoid of the network. (For details, see also Section 2.1.)

If by permutations of cells in a network one arrives at something that is regarded the same network, one naturally has to have cells in the network that are regarded to be of the same type. For two such cells it is quite natural to compare their states, or the sequence of their states over time. In particular it is an interesting situation when two such cells exhibit the *same* sequence of states. In such a case one says the two cells are *synchronous*.

In the case of symmetric networks (both of the classical group-theoretic kind and

the new, groupoid-equivariant kind), it turns out that synchrony often is not accidental. For such networks, it is possible to show that the network structure determines the presence of synchronicity patterns that are "robust" in the sense that they do not depend on the specific dynamical system, and hence do not disappear if the system is slightly changed. In [45] the term "robust polysynchronous subspace" is used for sets of such solutions; their presence is related to a certain combinatorial property of the network, namely the possibility of finding so-called "balanced equivalence relations" on the set of cells.

Transfer operators for coupled cell systems – combining both concepts

From the short descriptions we have given of the two concepts it becomes obvious that a problem must arise if one wants to use the transfer operator formalism for the (numerical) analysis of a coupled cell system: While the former is possible (or at least feasible) only in low dimensions, the latter almost necessarily implies a high-dimensional state space. So it is immediately clear that a naive, straightforward approach of simply using existing numerical solutions with a coupled cell system plugged in as the dynamical system to analyse will in most cases be too expensive to be of any practical use. However, this judgement does not take the additional structure of a coupled cell system into account. For a generic six-dimensional system a numerical computation of the transfer operator may not be feasible, but what if the system at hand consists of, for example, six cells with one-dimensional dynamics, three of one type, the other three of another type, each cell influenced only by one other cell (besides itself). Is it in this case still clear that it is practically impossible to compute its transfer operator?

To go towards an answer to questions like this one, clearly the influence of the system's structure on the structure of its transfer operator has to be explored. Here lies the starting point of this thesis' endeavours. The most basic rationale is that there must be a relationship of the sort "The more structured a system is, the more can be done to reduce numerical effort necessary for the computation of the transfer operator." To justify this statement (which at first was nothing more than a firm, unproven, but apparently plausible conviction), to explore and describe it, and finally to use it for algorithms for the numerical analysis of transfer operators – this is the research programme to which this thesis is devoted.

When the author started trying to realise this idea, his first approach was to follow a path on which other researchers had been successful before. The argumentation behind this path can be described as follows. We saw that coupled cell systems are similar to group-equivariant dynamical systems because the coupling structure can be expressed algebraically in such a way that the coupled cell systems appears to be equivariant with respect to the symmetry groupoid.

The transfer operator for an equivariant dynamical system is a linear operator that commutes with a particular linear representation of the underlying group. For linear mappings of this kind strong structural statements can be made on the basis of the classical linear representation theory for groups. It seemed plausible that there had to be some kind of generalisation of the representation theory for groups to the groupoid case, and it seemed promising to use this assumed generalisation in order to prove coupled cell system analogs of the theorems already established for "normal" dynamical systems. This idea did not occur exclusively to the author of this thesis. In particular, in [25], Golubitsky and Stewart express a similar line of thought. After explaining how representation theory can be usefully employed in the bifurcation analysis of equivariant dynamical systems depending on an additional bifurcation parameter, they go on asking

> Is there an analogous theory for network dynamics? Some features of the group-theoretic case carry over. For example, balanced polydiagonals play the same role as fixed-point subspaces. However, it is not so straightforward to find suitable analogs of irreducible representations and isotypic components. If we ask too much of these analogs, they do not exist; if we relax the conditions imposed on them, they may not be very useful.
>
> [25, Section 16, p. 359]

For the research programme this work is devoted to, these comments imply in particular that one cannot expect to find an off-the-shelf representation theory for groupoids which only had to be applied to coupled cell systems. On the contrary, fundamental questions for this approach are apparently wide open. It seems therefore advisable to look for a more fundamental approach to deal with a our research programme.
The main principle on which the results developed in this thesis are based is the observation that the transfer operator described above can be considered a special case of a more general idea in measure theory. If one has two spaces X and Y, each equipped with a σ-algebra, a measurable map $f : X \to Y$ and a measure μ on X, then an *image measure* ν of μ under f can be assigned to μ by the same formula $\nu(B) = \mu(f^{-1}(B))$ (for all measurable $B \subset Y$) which we already encountered above. This principle – the forming of image measures – describes a covariant functor that allows to transport all kinds of mappings – projections, embeddings, restrictions, group actions etc. – from the "state space level" to the "measure space level", and thus gives us a tool to describe the structure of a coupled cell network on the measure space level, that is in terms of transfer operators. It is the application of this principle upon which most of (the theoretical part of) this thesis rests.

This thesis

The overall structure of this thesis is as follows. In **Chapter 2**, the mathematical background material is presented that will be necessary for the development of the structural transfer operator theory in the following chapters. For most aspects, we concentrate on a brief presentation and provide references to more detailed expositions that can be found in the existing literature. However, we present in more detail some notions that will be used in this work and which the author could not find explicitly in the literature.[5] Finally, in Section 2.4.3 we present in the form of an "example theorem" an argument showing why a simpler path for the structural analysis of transfer operators of coupled cell systems, a path that might seem viable at first sight, leads in fact to a dead end.
In **Chapter 3**, the main matter of the thesis is reached. A structural theory of the transfer operator for a coupled cell system is developed that draws its results from two main principles. The first is briefly expressed by stating that the *independence* of a particular component map of the dynamical system from cells that are not in the input set of this cell must be reflected in the transfer operator in some way. The second principle could be called the *symmetry* principle, relating the (generalized) symmetries of a coupled cell network to structural properties of a corresponding transfer operator. The main technique that allows to derive the results is a method to decompose the domain of the transfer operator into subspaces adapted in certain ways to the structure of the underlying coupled cell network. Consequently, the results are expressed in terms of these decompositions.
This fact has an important consequence, the bottom line of which is that at present *the author of this thesis does not see a way to use the structural results on the transfer operator in numerical algorithms.* The core difficulty seems to lie in incompatibilities between a basis for the linear measure space prescribed by the structure-adapted decompositions on the one hand, and the basis implicitely assumed by apparently any numerical method that relies on the evaluation of the dynamical system at individual points on the other hand.
In the first part of **Chapter 4**, this problem is described in more detail. The only remedy this thesis has to offer fills the remainder of that chapter. Leaving the structural theory aside, the method for numerical computations that is described goes back to the definition of a coupled cell system, and provides a way to use the simple observation "A coupled cell system is completely defined by its component maps, and so must be its transfer operator." for a numerical algorithm. This algorithm describes a way in which a finite dimensional approximation of the transfer operator can be obtained using test points sampling the domains of the component maps only, in contrast to a test point method that aims at sampling the complete cartesian product of the state spaces.

[5] This does of course not mean that the material cannot be found at all; on the contrary, with some probability it will be there, possibly in much clearer form.

Depending on the connection graph, in particular on the sum of the dimensions of the domains of the component maps, the method can save considerable time for the evaluation of the transition matrix, assuming that a function evaluation is costly. It does not, however, reduce the dimension of the linear space in which the transfer operator acts, and therefore does not reduce the costs incurred by numerical linear algebra in standard uses for this matrix such as uncertainty propagation using direct evolution, or spectral analysis of the transfer operator to identify invariant or almost invariant measures. In an application where the major part of numerical effort is due to matrix-vector-products in linear algebra, the results of this thesis will not be able to offer much advantage. If however the simulation of the dynamics is the most costly part, and at the same time each individual component map has a low-dimensional domain, then a substantial reduction of computational cost is possible.

2 Mathematical background

2.1 Coupled Cell Systems

2.1.1 Coupled Cell Networks

In the following, we present the coupled cell formalism that will be the basis for the rest of this work. We will closely follow the works by Golubitsky, Stewart et al. that were already mentioned above, but we will modify the framework introduced there with respect to two issues. Firstly, we will not use the so-called *multi-arrow formalism* for a network that is used in the more recent publications [25, 24, 27], but instead use the earlier form of the concept presented e. g. in [45]. The reason for this is that it seems to the author that while the multi-arrow formalism has undeniable advantages when considering questions about synchrony phenomena in coupled cell networks and their related quotient networks, it also adds a notational overhead that seems to be of little use for the purposes for which the formalism will be used in this work.
The second difference concerns the definition of a dynamical system on a coupled cell network. It is somewhat more than just a formal difference. While in the literature time-continuous dynamical systems expressed by ordinary differential equations are considered, this work will concern itself with time-discrete systems given by maps on the state space. The reason for this is plain: we want to consider transfer operators, and transfer operators correspond to maps rather than to vector fields. The analogue of the transfer operator for a time-continuous system is the generator of an operator (semi-)group associated with the flow of the system. While it seems to the author that a corresponding theory could be developed, similar in spirit to the one for transfer operators that is used in this thesis, it would be much more demanding as far as technical aspects are concerned. To avoid that the main issues of this thesis are obfuscated by the resulting technical questions, we restrict our considerations to the case of transfer operators for maps, noting that a similar analysis for the case of flows should be possible, and probably far more difficult technically.
A coupled cell system is a dynamical system on a network of coupled cells. To define this concept formally, two main ingredients are needed which will be introduced one after the other. First comes the network, which defines how many cells of how many types there are, and how they are connected. Given a network, a dynamical system on it is defined by specifying state spaces and maps in a manner consistent with the network topology.

Coupled cell networks

We begin with the first ingredient. The concept for the network that will be used in the following is essentially that of a directed graph with labeled nodes. The following definition was introduced by Golubitsky, Stewart, and Pivato in [45] and will be used in the remainder of this work.

Definition 2.1
A coupled cell network *is given by a tuple* $G = (\mathcal{C}, \mathcal{E}, \sim_\mathcal{C}, \sim_\mathcal{E})$*, where*

- $\mathcal{C}$ *is a finite set, the elements of which are called* cells,
- $\mathcal{E} \subset \mathcal{C} \times \mathcal{C}$ *is a set of ordered pairs* (c, d) *called* directed edges *or* arrows,
- $\sim_\mathcal{C}$ *is an equivalence relation on* $\mathcal{C}$*,*
- $\sim_\mathcal{E}$ *is an equivalence relation on* $\mathcal{E}$ *satisfying*

$$(c_1, d_1) \sim_\mathcal{E} (c_2, d_2) \implies c_1 \sim_\mathcal{C} c_2 \ \wedge \ d_1 \sim_\mathcal{C} d_2.$$

For $e = (c, d) \in \mathcal{E}$*,* c *is called the* tail *of* e*, and* d *is called the* head *of* e*. An edge* (c, c) *with identical head and tail is called the* internal edge *of cell* c*. It is assumed that every cell* $c \in \mathcal{C}$ *has an internal edge, i. e. that* $(c, c) \in \mathcal{E}$ *for all* $c \in \mathcal{C}$*. Furthermore it is required that internal edges of equivalent cells are equivalent, and that internal and non-internal edges are never equivalent, i. e. that for all* $c, d, d' \in \mathcal{C}$ *the equivalence*

$$(c, c) \sim_\mathcal{E} (d, d') \iff d = d' \text{ and } c \sim_\mathcal{C} d$$

holds.

Less formally, a coupled cell network can be depicted as a directed graph with different kinds of symbols (such as triangles, circles, and so on) as nodes, and different kinds of arrows as edges. Figure 1.1 shows an example.

Definition 2.2
Let a coupled cell network $G = (\mathcal{C}, \mathcal{E}, \sim_\mathcal{C}, \sim_\mathcal{E})$ *be given. For each* $c \in \mathcal{C}$*, the set*

$$I(c) = \{i \in \mathcal{C} \mid (i, c) \in \mathcal{E}\}$$

is called the input set *of* c*. Two cells* c *and* d *are called* input-isomorphic *if there is a bijection* $\phi : I(c) \to I(d)$ *satisfying the requirements* $\phi(c) = d$ *and* $(i, c) \sim_\mathcal{E} (\phi(i), d)$ *for all* $i \in I(c)$*. Such a map is called an* input isomorphism*. We define* $B(c, d)$ *to be the set of all input isomorphisms between* c *and* d*. If* $B(c, d) \neq \emptyset$*, we call* c *and* d input equivalent*. Finally, for* $\mathcal{D} \subset \mathcal{C}$ *we write* $I(\mathcal{D}) = \cup_{c \in \mathcal{D}} I(c)$*.*

Golubitsky et al. note in [45] that the set of all input isomorphisms for a given coupled cell network, equipped with a "product" given by the composition of maps where defined, forms an algebraic structure called a *groupoid*.[1] We follow their example and introduce the following notation.

Definition 2.3
Let a coupled cell network $G = (\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}})$ be given. We define the symmetry groupoid *of G to be*

$$\mathcal{B}_G = \bigcup_{c,d \in \mathcal{C}} B(c,d).$$

For later reference, we now briefly paraphrase parts of the description of $\mathcal{B}_G$ given by Golubitsky et al.

Proposition 2.4 (Cf. [45, p. 620])

1. *If two cells c and d are not input equivalent, then $B(c,d) = \emptyset$.*
2. *For any cell $c \in \mathcal{C}$, the input set $I(c)$ can be written as $I(c) = I_1 \dot\cup \ldots \dot\cup I_k$, where every I_j is a (maximal) set of cells coupled to c by equivalent edges. Let $d_j = |I_j|$. Then the set $B(c,c)$ is isomorphic to the group $S_{d_1} \times \ldots \times {S_{d_k}}^2$, where the elements of S_{d_j} act as permutations of I_j.*
3. *If $c \neq d$ are input equivalent, then we have for any $\beta \in B(c,d)$ that*

$$\beta \circ B(c,c) = B(c,d) = B(d,d) \circ \beta.$$

We stress in particular that for any cell c, the set $B(c,c)$ is a group. From now on we will call it the *vertex group* of the cell c.

2.1.2 From Networks to Systems

With a coupled cell network, one has, so to speak, the skeleton in ones hands of a dynamical system to be described. To put the flesh on the bones, one has to define what makes a dynamical system *admissible* to a given network. This will be the main concern of the following section. In the definition, we will again essentially follow [45], with the exception that we consider maps instead of vector

[1] The notion of a groupoid is a generalization of the concept of a group and was introduced by Brandt (who was using the term "*Gruppoid*") in [4]. It has since been applied in several fields to describe notions of *symmetry* more general than those given by classical groups, see e. g. [5] for a survey, or [48] for a short "tour through some examples". The main difference to groups is that in groupoids the product of two elements may not be defined for arbitrary pairs. For example, in our case the composition of $\phi_1 \in B(c,d)$ with $\phi_2 \in B(e,f)$ is defined if and only if $d = e$. To avoid confusion, the reader should note that what is called a "groupoid" here is often called a "Brandt groupoid" in the modern algebraic literature, as the simple term has been given a different meaning in that field.

[2] See also Section Section 2.2.1

fields. The restriction on the choice of state space as compact subsets of $\mathbb{R}^d$ that we make here is essentially arbitrary and serves just to simplify the treatment of technical issues related to the choice of measure spaces later on.

Definition 2.5
Let a coupled cell network $G = (\mathcal{C}, \mathcal{E}, \sim_\mathcal{C}, \sim_\mathcal{E})$ be given. For each cell c let X_c be a compact subset of $\mathbb{R}^{d_c}$. The collection $(X_c)_{c\in\mathcal{C}}$ is admissible *to G iff $X_c = X_d$ whenever $c \sim_\mathcal{C} d$. In that case, X_c is called the* state space *for cell c, and*

$$X = \prod_{c\in\mathcal{C}} X_c$$

is called the total state space.

If $\mathcal{D} \subset \mathcal{C}$ is some subset of the cells, and an admissible collection of state spaces X_c is given, we also use the notation

$$X_\mathcal{D} = \prod_{c\in\mathcal{D}} X_c$$

for the partial state spaces, and

$$\pi_\mathcal{D} : X \to X_\mathcal{D}$$

for the canonical projection from X onto $X_\mathcal{D}$. In the case $\mathcal{D} = \{c\}$, we will omit the braces and write π_c. Also we write $x_\mathcal{D} = \pi_\mathcal{D}(x)$ for all $x \in X$.
If $\phi : I(c) \to I(d)$ is an input isomorphism and $(X_c)_{c\in\mathcal{C}}$ is an admissible collection, then ϕ can be used to define a map between the corresponding partial state spaces. Having done so, we denote the result also by ϕ, and thus have a map $\phi : X_{I(c)} \to X_{I(d)}$ given by

$$\pi_{\phi(i)}(\phi(x)) = \pi_i(x)$$

for all $x \in X_{I(c)}$ and $i \in I(c)$.[3]

Definition 2.6
Let a coupled cell network G and an admissible state space $X = \prod_{c\in\mathcal{C}} X_c$ be given. A map $f : X \to X$ consisting of component maps $f_c : X \to X_c$ is admissible to G iff there is a collection of maps $\hat{f}_c : X_{I(c)} \to X_c$ such that for each $c \in \mathcal{C}$, $f_c(x) = \hat{f}_c(\pi_{I(c)}(x))$ holds for all $x \in X$, and for each input isomorphism $\phi : I(c) \to I(d)$, $\hat{f}_c(x') = \hat{f}_d(\phi(x'))$ holds for all $x' \in X_{I(c)}$.

[3] The equality sign here expresses equality, but not 'sameness'. This is not the only place where clarity suffers somewhat from that fact that common mathematical notation makes it difficult to distinguish the statement that two objects are the *same* from the statement that they are *equal*. Here it is meant that $\pi_{\phi(i)}(\phi(x)) \in X_{\phi(i)}$ and $\pi_i(x) \in X_i$ are equal although the two spaces are being thought of as distinct sets.

Remark 2.7
The above requirement can be understood as a generalisation of the *equivariance* of the map f with respect to the action of some permutation group. To see this, imagine that a map $g : \mathbb{R}^n \to \mathbb{R}^n$ is equivariant with respect to the action of S_n on $\mathbb{R}^n$ given by $\gamma.(x_1, \ldots, x_n)^T = (x_{\gamma^{-1}(1)}, \ldots, x_{\gamma^{-1}(n)})^T$. If g has components g_i, $i = 1, \ldots, n$, then equivariance of g just means that

$$g_{\gamma^{-1}(i)} = g_i \circ \gamma$$

for all $i = 1, \ldots, n$ and all $\gamma \in S_n$. Now, identifying the spaces X_c and X_d, the defining requirement on the component maps of coupled cell systems can also be written as

$$\hat{f}_c = \hat{f}_d \circ \phi$$

for all $c \in \mathcal{C}$ and all input isomorphisms $\phi : I(c) \to I(d)$, which in turn just means that

$$\hat{f}_{\phi^{-1}(d)} = \hat{f}_d \circ \phi.$$

This form of the admissibility condition for a coupled cell system resembles closely the equivariance condition, the difference being that while the set of all γ above forms a group, the set of mappings ϕ considered here only forms a groupoid. An admissible map thus is "equivariant" with respect to the action of the groupoid of input isomorphisms. This is the reason why Golubitsky, Stewart et al. introduce the term "symmetry groupoid" for the set of input isomorphisms.

Later on, it will sometimes be convenient to refer to "partial maps" that are composed from several component maps $\hat{f}_c$. So if $\mathcal{D} \subset \mathcal{C}$, we denote by $\hat{f}_\mathcal{D} : X_{I(\mathcal{D})} \to X_\mathcal{D}$ the map defined by $\pi_\mathcal{D}(f(x)) = \hat{f}_\mathcal{D}(\pi_{I(\mathcal{D})}(x))$ for all $x \in X$.

2.2 Prerequisites from the Representation Theory for Groups

One of the mathematical tools that will be used in the following is the classical linear representation theory for groups. In this short section, the necessary terms and results will be presented. Depending upon the base field that is used for vector spaces, the theory exists in two variants with slightly differing results. The complex variant often seems to be preferable to the algebraist, while the use of real vector spaces might be more appropriate for certain applications. We will leave this question open as often as possible, and merely assume, unless otherwise stated, that any vector space is built over either the reals or the complex numbers[4].Furthermore, we here restrict our attention to the case of *finite* groups, although many of the results can be generalised to more general cases. More

[4]We can do so with some right, as for the case of the symmetric groups S_d, the theory is rather independent of the choice of base fields. Their irreducible representations over complex

complete accounts of the topics touched here can be found in a large number of textbooks, e. g. in the ones by Serre [44], or by Fulton and Harris [23], and also in Chapter XII of [26]. These texts were used for the preparation of this thesis.

2.2.1 Basics

While we can state most of the theory which we are going to present in terms of general finite groups, we are in fact mostly interested in a particular class of these groups, and in particular subgroups of the members of this class. More concretely, we will be looking at *permutation groups* S_d (for $d \in \mathbb{N}$). S_d is the group of permutations of the set $\mathbb{N}_d := \{1, \ldots, d\}$. Under any such permutation π, $\mathbb{N}_d$ decomposes into invariant subsets. The minimal (with respect to the partial order defined by set inclusion) invariant subsets are called the *cycles* of π, and they suffice to completely define the permutation. If necessary, we will therefore denote a particular element of S_d in cycle notation, that is, we list its cycles, excluding those of minimal period one (the fixed elements).
A finite sequence $d_1, \ldots, d_k$ of natural numbers with sum d defines a partition $\mathbb{N}_d = \{1, \ldots, d_1\} \cup \{d_1 + 1, \ldots, d_1 + d_2\} \cup \ldots \cup \{\sum_{i=1}^{k-1} d_i, \ldots, d\}$. The elements of S_d that leave this partition invariant form a subgroup of S_d which is isomorphic to the product group $S_{d_1} \times \ldots \times S_{d_k}$. We have already seen in Proposition 2.4 that subgroups of this kind arise naturally as the vertex groups $B(c, c)$ of a coupled cell network, consisting of permutations of the input sets $I(c)$ of a cell c that respect cell and edge equivalence. For this reason we pay special attention to them.

Definition 2.8
Let G be a finite group. Let V be a vector space. A linear representation *of G in V is a group homomorphism $\theta : G \to GL(V)$. The dimension of V is called the* degree *of θ. A representation θ is* reducible *if there is a non-zero proper subspace of $U \subset V$ that is invariant under $\theta(g)$ for every $g \in G$. If a representation is not reducible, it is called* irreducible*. Two representations θ_1 in V_1 and θ_2 in V_2 of the same group G are isomorphic if there is a vector space isomorphism $T : V_1 \to V_2$ such that $T \circ \theta_1(g) = \theta_2(g) \circ T$ for all $g \in G$.*

---

vector spaces are precisely the ones over real vector spaces. This statement can be justified as follows (see the text below for an explanation of the terms): The *number* of irreducible representations of S_d is just the number of conjugacy classes of S_d which is independent of the choice of base field. So it just needs to be decided whether a given complex irreducible representation of S_d is *realizable*, i. e. whether it can be written as a representation on a real vector space. Now from the general representation theory, it is known that the question whether a given complex representation $\theta : G \to GL(V)$ is realizable, is equivalent to the question whether there is a G-invariant non-degenerate symmetric bilinear form on V (see e. g. [23, Theorem 3.37]). The complex irreducible representations of S_d are given as certain G-invariant subspaces of the group algebra $\mathbb{C}S_d$, which is naturally equipped with such a bilinear form. Its restriction to the representation space thus provides the necessary criterion that allows to say that any complex irreducible representation of S_d is realizable.

Any finite group can only have a finite number of distinct (i. e. non-isomorphic) irreducible representations.
It can be shown that for every representation θ of a finite group G in a vector space V, a scalar product can be defined on V such that every $\theta(g)$ is orthogonal (in the real case) or unitary (in the complex case)[5]. If necessary, we will in the following assume that this is done.
An important consequence of this is the fact that if $\theta : G \to GL(V)$ is a representation, V can be written as the direct sum $V = \bigoplus_{\iota \in I} U_\iota$ of irreducible subspaces U_ι[6]. In general, this decomposition is not unique. However, it can be shown that any decomposition of V into irreducible subspaces leads to a unique decomposition of V when these subspaces are grouped together by isomorphy of representations, i. e. if one forms subspaces $W_1 = \bigoplus_{\iota \in I_1} U_\iota, \ldots, W_k = \bigoplus_{\iota \in I_k} U_\iota$ with pairwise disjoint subsets I_j satisfying $\cup_{j=1}^{k} I_j = I$ and such that for every $\iota_1 \in I_i$ and $\iota_2 \in I_j$, the representations in U_{ι_1} and U_{ι_2} are isomorphic whenever $i = j$, and are not isomorphic whenever $i \neq j$.

Definition 2.9
The decomposition $V = W_1 \oplus \ldots \oplus W_k$ described above is called the isotypic *decomposition of V with respect to θ. A single W_j is called an* isotypic component *of V with respect to θ.*

Every direct sum decomposition of a vector space comes with a set of projections onto the summands of the decomposition. In the case of the isotypic decomposition, the projections can be described using the *character* of the decomposition.

Definition 2.10
Let $\theta : G \to GL(V)$ be a representation. The character *of θ is the map $\chi_\theta : G \to \mathbb{C}$ given by $\chi_\theta(g) = \mathrm{Tr}(\theta(g))$, where $\mathrm{Tr}(\theta(g))$ denotes the trace[7] of the endomorphism $\theta(g)$.*

Lemma 2.11
In the situation of Definition 2.9, let n_j be the degree and χ_j be the character of the j^{th} irreducible representation. Then a projection onto the isotypic component W_j is given by the map

$$p_j := \frac{n_j}{|G|} \sum_{g \in G} \overline{\chi_j(g)} \theta(g).$$

[5] This is done essentially through Haar integration over an arbitrary scalar product. For finite groups, Haar integration is nothing else but averaging over the group elements.
[6] This is sometimes also expressed by saying that θ is a *completely reducible* or *semisimple* representation.
[7] If an endomorphism $A : V \to V$ is represented through a matrix (a_{ij}), then $\mathrm{Tr}(A) = \sum a_{ii}$. The trace can also be defined in a basis-independent way as the contraction of A viewed as a $(1,1)$-tensor on V.

The main purpose of this section is to describe the structural implications *equivariance* has for a linear map. In the following we define this term and briefly sketch the way that leads to the main structural result.

Definition 2.12
Let G be a finite group, let $\theta_1 : G \to GL(V_1)$ and $\theta_2 : G \to GL(V_2)$ be representations of G, and let $A : V_1 \to V_2$ be a linear map. A is called equivariant *with respect to θ_1 and θ_2 if $A \circ \theta_1(g) = \theta_2(g) \circ A$ holds for all $g \in G$.*

The main step towards the basic structure theorem is made with the following famous lemma.

Lemma 2.13 (Schur's Lemma)
Let $\theta_1 : G \to GL(V_1)$ and $\theta_2 : G \to GL(V_2)$ be irreducible representations. Let a linear map $A : V_1 \to V_2$ be equivariant with respect to θ_1 and θ_2. Then either $A = 0$ or A is invertible.

There are two important immediate consequences of this lemma. Firstly, if θ_1 and θ_2 are not isomorphic, then A has to be zero. Secondly, if $A \neq 0$, then θ_1 and θ_2 are isomorphic. Together with the decomposition of V_1 and V_2 into isotypic components, these findings yield the following theorem, which describes the general structure of an equivariant linear mapping.

Theorem 2.14
Let G be a finite group, let $\theta_1 : G \to GL(V_1)$ and $\theta_2 : G \to GL(V_2)$ be representations of G. Let $A : V_1 \to V_2$ be equivariant with respect to θ_1 and θ_2. Then the block matrix representation of A with respect to the isotypic decompositions of V_1 and V_2 is a diagonal matrix.

We should not forget to mention that the use of so-called *symmetry-adapted* bases allows more refined statements about the structure of an equivariant mapping. These are bases for V_1 and V_2 with respect to which every $\theta_i(g)$ has block diagonal structure, where the diagonal blocks contain only irreducible representations. For more details on this see e. g. [44, p. 23 ff] for a complex, or [49, Theorem 2.3] for a real version.
In Chapter 3 we will apply Theorem 2.14 to (parts of) the transfer operator of a coupled cell system. We do this as we wish to determine as far as possible the consequences which the structure of the coupling network has for the transfer operator. In order to carry out this programme, we will need two more prerequisites: Firstly, we will need to be able to tell, for a given group, which irreducible representations it has, and secondly, we will need to be able to compute an isotypic decomposition of a given representation. The first point is, in group theoretic terms, the problem of enumerating the different irreducible representations. In the classical representation theory this problem has been solved for many kinds of groups, e. g. for the case of the permutation groups S_d by

using so-called *Young diagrams*. We will briefly sketch this solution below. To deal with the second problem, one can in principle use Lemma 2.11. However, for concrete computations in specific examples, the projection formula given often needs to be analyzed further in order to obtain more concrete expressions for the isotypic components. For the case of products of symmetric groups acting on tensor products of vector spaces, this programme has been carried out. To give a (superficial) impression of the nature of the result, we sketch it in Section 2.2.3 below. We are going to use it in Example 3.28 to give at least a rough estimate of the savings in numerical effort that can possibly result from using the structural theory we are developing in this thesis.
Finally, we have to note that for practical purposes theoretical answers of this kind may be of limited value. For this reason, we want to point out that also algorithmic approaches are possible that compute for a given representation of some (finite) group on some vector space the complete isotypic decomposition. A simple example for an algorithm of this kind is given in [16].

2.2.2 Description of the irreducible representations of S_d

Young diagrams and irreducible representations In the following, we sketch the theory presented in [23, Chapter 4] as briefly as possible. Our aim is to construct all irreducible representations of the permutation group S_d acting on d elements. First we introduce the *group algebra* $\mathbb{C}S_d$ as the vector space of formal linear combinations of the elements of S_d: $\mathbb{C}S_d := \{\sum_{i=1}^{l} \alpha_i\gamma_i \mid l \in \mathbb{N}, \alpha_i \in \mathbb{C}, \gamma_i \in S_d\}$, and observe that the simple "linear extension" of the group multiplication equips $\mathbb{C}S_d$ with a multiplication, thus making it an algebra. Further we observe that the elements of $\mathbb{C}S_d$ act as linear operators on $\mathbb{C}S_d$ by right multiplication. More generally, any representation $\theta : S_d \to GL(V)$ can be extended linearly to an algebra homomorphism $\theta : \mathbb{C}S_d \to \mathrm{End}(V)$. We refer to this extension when we speak of "the action of c on V" for some $c \in \mathbb{C}S_d$.
Now we start with an arbitrary size-ordered partition λ of d, that is, a finite sequence $\lambda_1 \geq \ldots \geq \lambda_k \geq 1$ of natural numbers with $\sum_{i=1}^{k} \lambda_i = d$. We associate to such a partition a so-called *Young diagram*, that is, a staircase-like two-dimensional structure with k rows containing λ_i blocks in the i-th row, numbered top-down from left to right (see Figure 2.1 for an example).

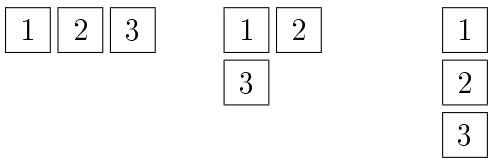

Figure 2.1: Young diagrams for the three partitions of the number three.

Next we define two subgroups of S_d, namely

$$P_\lambda := \{\gamma \in S_d \mid \gamma \text{ leaves the rows invariant}\} \text{ and}$$
$$Q_\lambda := \{\gamma \in S_d \mid \gamma \text{ leaves the columns invariant.}\}$$

These subgroups are now used to define three elements of the group algebra,

$$\begin{aligned} a_\lambda &:= \sum_{\gamma \in P_\lambda} \gamma, \\ b_\lambda &:= \sum_{\gamma \in Q_\lambda} \operatorname{sgn}(\gamma)\gamma, \quad \text{and} \\ c_\lambda &:= a_\lambda \cdot b_\lambda, \end{aligned}$$

where $\operatorname{sgn}(\gamma)$ denotes the sign of the permutation γ. The element c_λ is called the *Young symmetrizer* associated to λ. In the following, we cite a fundamental result on representations of symmetric groups from Lecture 4 in [23].

Theorem 2.15
Consider $c_\lambda : \mathbb{C}S_d \to \mathbb{C}S_d$ as a linear map and define

$$V_\lambda := \operatorname{im} c_\lambda.$$

Then V_λ is invariant under the representation of S_d on $\mathbb{C}S_d$, and the restriction of this representation to V_λ is an irreducible representation of S_d. Every irreducible representation of S_d can be obtained in this way for a unique partition λ.

Example 2.16
As an example, we show how to obtain the irreducible representations for S_3 from the three Young diagrams shown in Figure 2.1. We denote the elements of S_3 in cycle notation, e. g. the notation (123) is used for the permutation that moves each element "one right", and (12) for the permutation that fixes 3 and interchanges 1 and 2.
For the first diagram, we have $P_\lambda = S_3$ and $Q_\lambda = \{\mathrm{Id}\}$, and hence obtain

$$\begin{aligned} c_\lambda \left(\sum_{i=1}^{l} \alpha_i \gamma_i \right) &= \left(\sum_{i=1}^{l} \alpha_i \gamma_i \right) \cdot \left(\sum_{\gamma \in S_3} \gamma \right) \\ &= \left(\sum_{i=1}^{l} \alpha_i \left(\sum_{\gamma \in S_3} \gamma \right) \right) \\ &= \sum_{i=1}^{l} \alpha_i \sum_{\gamma \in S_3} \gamma. \end{aligned}$$

Thus the image of c_λ is one-dimensional, and we obtain the trivial representation from this diagram.

For the second diagram, we have $P_\lambda = \{\mathrm{Id}, (12)\}$ and $Q_\lambda = \{\mathrm{Id}, (13)\}$, hence $a_\lambda = \mathrm{Id} + (12)$ and $b_\lambda = \mathrm{Id} - (13)$, and finally $c_\lambda = \mathrm{Id} + (12) - (13) - (123)$. A direct computation shows that the image of c_λ is spanned e. g. by c_λ itself and by $(13) \cdot c_\lambda$. Therefore this diagram generates a two-dimensional irreducible representation, which is called the *standard* representation of S_3. It can be visualised as a representation on $\mathbb{R}^2$ by choosing a straight line (e. g. the x-axis) through the origin and assigning to (13) the reflection on this line, and to (123) the clockwise rotation by an angle of $\frac{2\pi}{3}$ around the origin.
In the third diagram, P_λ and Q_λ are just interchanged when compared with the first diagram, and so one obtains analogously that

$$c_\lambda \left(\sum_{i=1}^{l} \alpha_i \gamma_i \right) = \sum_{i=1}^{l} \mathrm{sgn}(\gamma_i) \alpha_i \sum_{\gamma \in S_3} \gamma.$$

So this diagram also yields a one-dimensional representation, the so-called *alternating* representation of S_3 that is most simply described by saying that every group element is mapped to its sign.

Products of groups We have just seen how the irreducible representations of symmetric groups $S_d, d \in \mathbb{N}$, can be constructed. However, most of the groups with which we will have to deal in Chapter 3 are *sub*groups of symmetric groups. While there is in general no simple way to derive the irreducible representations of a subgroup from those of the larger group, we are in the fortunate situation that it is indeed possible to construct the irreducible representations of the groups we are interested in from those of the full symmetric groups S_d. The key for this construction is the fact that we can write the groups of our interest as *product groups* with factors of the form S_d, $d \in \mathbb{N}$. Fortunately, there is a simple rule for the construction of irreducible representations of product groups from those of the factors. The basic notion we need for this is that of the tensor product of representations.

Definition 2.17
Let G_1 and G_2 be finite groups, let $\theta_1 : G_1 \to GL(V_1)$ and $\theta_2 : G_2 \to GL(V_2)$ be representations thereof. The representation $\theta_1 \otimes \theta_2 : G_1 \times G_2 \to GL(V_1 \otimes V_2)$ defined by $\theta_1 \otimes \theta_2(\gamma_1, \gamma_2)(v_1 \otimes v_2) = \theta_1(\gamma_1)(v_1) \otimes \theta_2(\gamma_2)(v_2)$ is called the tensor product *of the representations θ_1 and θ_2.*

Now the matter is as simple as:

Theorem 2.18 ([44])
If in the setting of Definition 2.17 θ_1 and θ_2 are irreducible, then $\theta_1 \otimes \theta_2$ is an irreducible representation of $G_1 \times G_2$. Each irreducible representation of $G_1 \times G_2$ is isomorphic to a representation of this form.

2.2.3 Description of isotypic decompositions of $V^{\otimes d}$

In the previous section, we have seen how the irreducible representations of S_d can be described, and Theorem 2.18 tells us how the irreducible representations of products of the form $S_{d_1} \times \ldots \times S_{d_k}$ can be constructed from them. In our application, the equivariance theory for transfer operators of coupled cell systems, we will be dealing with a particular kind of representations, and what is missing until know is a description of the isotypic decomposition of these representations. We will describe their construction in the following.
Concretely, we will have to deal with vector spaces

$$V^{\otimes d} = \underbrace{V \otimes \ldots \otimes V}_{d \text{ factors}}$$

that are "tensor powers" of some vector space V, and with the representation of S_d on it that is defined via "index permutation", that is the representation that assigns to $\gamma \in S_d$ the endomorphism on $V^{\otimes d}$ given through[8]

$$v_1 \otimes v_2 \otimes \ldots \otimes v_d \mapsto v_{\gamma^{-1}(1)} \otimes v_{\gamma^{-1}(2)} \otimes \ldots \otimes v_{\gamma^{-1}(d)}.$$

Surprisingly, this representation seems not to have been given a name in representation theory. For ease of further descriptions, we will call it the *factor permutation representation* of S_d in the rest of this thesis.
The construction we are going to sketch in the following is due to Hermann Weyl and presented in a large number of textbooks, e. g. in [23, Chapter 6]. We already know one of its ingredients, namely the irreducible representations V_λ of S_d, parametrized by the partitions λ of d. These spaces were constructed as images of the Young symmetrizers c_λ, defined as linear mappings on the group algebra $\mathbb{C}S_d$. For the second main ingredient, we consider the action of c_λ on $V^{\otimes d}$ (see p. 19) and define the *Weyl module* corresponding to λ as

$$\mathbb{S}_\lambda V := \operatorname{im}\left(c_\lambda : V^{\otimes d} \to V^{\otimes d}\right).$$

Having defined these terms, we can now formulate the result that is of interest to us.

Theorem 2.19 (Exercise 6.30 in [23])
Let V be a vector space. The isotypic decomposition of the factor permutation representation of S_d on $V^{\otimes d}$ is given by

$$V^{\otimes d} = \bigoplus_\lambda \left(\mathbb{S}_\lambda V \otimes V_\lambda\right),$$

where the summation is over the partitions λ of d.

[8] This statement suffices to define the automorphisms as the elementary tensors span the whole tensor power $V^{\otimes d}$.

In order to be able to estimate the amount of potential savings achievable by using a block-diagonalization based on the isotypic decomposition, it will be helpful to know about the dimensions of the parts in this decomposition. Fortunately, expressions for the dimensions both of V_λ and $\mathbb{S}_\lambda V$ are available, e. g. in [23, p. 50 (Equation (2.1)) and p. 78 (Equation (2.2))]. Not surprisingly, they depend on three factors, namely d, the partition λ, and the dimension k of V. More precisely, the partition λ determines a set of d so-called *hook lengths* h_{ij}, the product of which enters the dimension formulas. These are defined as follows: The box (i, j) located in row i, column j of the Young diagram corresponding to λ determines a *hook* whose corner it is. This hook is the set of boxes that are either below or to the right of the box (i, j), together with the box (i, j) itself. Now the hook length h_{ij} is simply the number of boxes in the hook corresponding to box (i, j). With this notation, we can define

$$H_\lambda := \prod_{(i,j)} h_{ij},$$

where the product is defined over the boxes in the Young diagram associated to λ. Now we have the following formulas.

$$\dim V_\lambda = \frac{d!}{H_\lambda} \tag{2.1}$$

$$\dim \mathbb{S}_\lambda V = \frac{1}{H_\lambda} \prod_{(i,j)} (k - i + j) \tag{2.2}$$

We notice that for the case $k >> d$, which will be of principal interest for our application, $\dim \mathbb{S}_\lambda V \approx \frac{k^d}{H_\lambda}$, so that H_λ roughly determines the potential savings. As an example, and for later reference, in Table 2.1 the hook lengths and dimensions for the cases $d \in \{2, \ldots, 5\}$ are listed.

2.3 Measure Theoretical Prerequisites

In the following section measure theoretical prerequisites that will be needed in the rest of this work will be presented. Its main purpose is to fix notation. Everything that is presented in the section titled 'Basic concepts' can be found in virtually every textbook on measure theory, although the specific manner of the presentation often differs substantially from the choice we made for our exposition. For the preparation of this work, the textbooks by H. Bauer [2] and J. Elstrodt [19], and especially J. L. Doob's conception of measure theory [18] were helpful sources. The section titled 'Specific concepts' presents the specific notation we will adopt in this work to deal with measure spaces for coupled cell networks. Most of its contents is not taken from existing literature, but has been developed by the author of this thesis during his efforts to understand transfer operators for coupled cell systems.

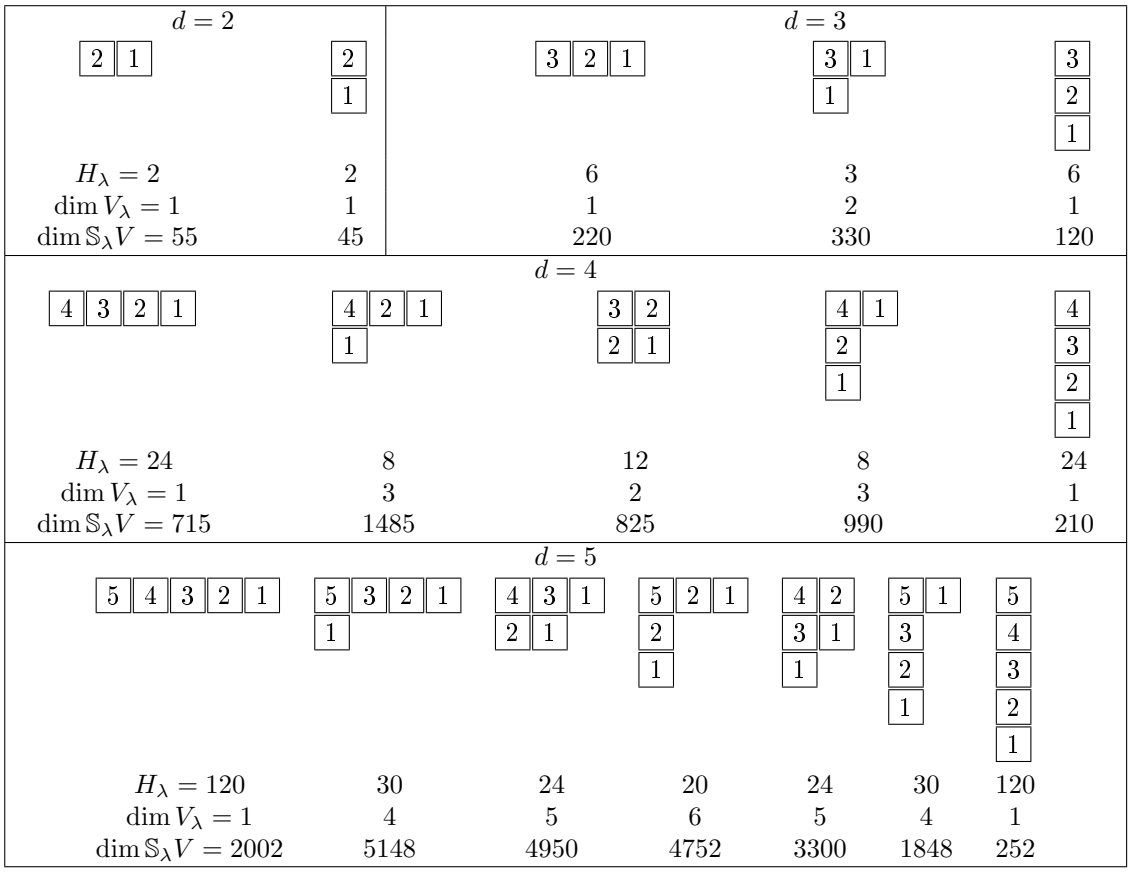

Table 2.1: Young diagrams for the irreducible representations of S_2, S_3, S_4, and S_5. The boxes in the Young diagrams are filled with their hook lengths, the lines below the diagrams give the hook length products, the dimension of the irreducible representation and, as an example, the dimension of the Weyl module for a vector space V with dimension $k = 10$.

2.3.1 Basic concepts

Measurable maps and measures. To begin, we remember a number of basic concepts. A *σ-algebra* Σ on a set X is a collection of subsets of X with the properties that $\emptyset \in \Sigma$, that $A \in \Sigma$ implies $X \setminus A \in \Sigma$, and that for any infinite sequence $A_1, A_2, \ldots \in \Sigma$ also $\cup_{i=1}^{\infty} A_i \in \Sigma$ holds.
A *measurable space* is a pair (X, Σ), where X is a set and Σ is a σ-algebra on X. The cases most important to this work are those of topological spaces X, for which there is a natural choice of algebra, namely the *Borel* σ-algebra on X, which is defined to be the smallest σ-algebra on X that contains all open subsets of X. We note that an equivalence relation on X is defined by regarding two elements $x, y \in X$ as equivalent whenever $\chi_A(x) = \chi_A(y)$ holds for the characteristic functions χ_A of all $A \in \Sigma$. The equivalence classes for this relation

are called the *atoms* of Σ. Atoms are the "smallest" entities in Σ in the sense that no non-empty proper subset of an atom is contained in Σ. Given two measurable spaces (X, Σ_X) and (Y, Σ_Y), a map $F : X \to Y$ is called *measurable* if $F^{-1}(A) \in \Sigma_X$ for all $A \in \Sigma_Y$.
A *signed measure* on a measurable space (X, Σ) is a function $\mu : \Sigma \to \mathbb{R} \cup \{-\infty, \infty\}$ with the properties that $\mu(\emptyset) = 0$, that μ is countably additive, which means that for a sequence of pairwise disjoint subsets $A_i \in \Sigma$ the equality

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

holds, and that at least one of the values $-\infty, \infty$ is not taken. The signed measure μ is called *finite* if $\mu(X)$ is finite, and σ-*finite* if X is a countable union of sets with finite μ-measure. A measure is *non-negative* if it takes only non-negative values. A non-negative measure μ on a space X with $\mu(X) = 1$ is called a *probability measure*. Given a non-negative measure μ and a signed or complex measure ν over the same σ-algebra Σ, ν is called *absolutely continuous* with respect to μ if for all sets $A \in \Sigma$, $\mu(A) = 0$ already implies $\nu(A) = 0$. At the other end of the spectrum are the δ-measures which concentrate all weight in one point: $\delta_x(A) = 1$ if $x \in A$, otherwise $\delta_x(A) = 0$.
Normally, in the literature a distinction is made between complex measures, signed measures and ("true") measures, the latter being signed measures which only take non-negative values. As the measures this work will be concerned with will always be thought of as being elements of vector spaces, we will not make this distinction, and mean "complex measures" or "signed measures" whenever we speak of "measures" unless otherwise stated. Furthermore, for the same reason we will often assume that all measures that are considered are finite. Finally, in the literature, the term "measure space" is usually applied to a tuple (X, Σ, μ) fixing a specific measure on a specific σ-algebra over a specific space. For this work, it will be more convenient to use that term as a short version of the somewhat longish "linear space of measures". So in the following that term is to be understood with this meaning, unless it is explicitely said or otherwise clear that the standard meaning is intended.

Integrals. Measures are there, of course, to integrate real- or complex-valued functions. So to complete this brief exposition, we sketch a path for the definition of integrable functions with respect to a given measure space (X, Σ, μ). This path has three stages: measurable step functions, limits of non-negative measurable step functions, and finally linear combinations of those. Here we will only describe the steps one goes towards the stages, and we refer the reader to the literature for all questions about correctness, well-definedness, or alternative constructions. Step functions are functions $\varphi : X \to \mathbb{R}$ that take only finitely many values $y_1, \ldots, y_k \in \mathbb{R}$. If such a function is measurable, its integral with

respect to the measure μ can quite simply be defined as

$$\int_X \varphi \, d\mu = \sum_{j=1}^{k} y_j \mu(\varphi^{-1}(y_j)).$$

The basis for the next step is the fact that every non-negative measurable function can be obtained as the pointwise limit of a monotone sequence of non-negative measurable step functions. Also, the sequence of integrals of functions in such a sequence is convergent, and this allows to define the integral of non-negative measurable functions as the limit of the sequence of integrals. Thus the second stage is reached, and an integral is defined for all non-negative step functions, which (at this stage) is allowed to take the value infinity. For the last step, one notices that complex-valued measurable functions can be written as (complex) linear combinations of two real-valued measurable functions (namely their real and imaginary parts), and that real-valued measurable functions can be written as a linear combination (in fact, as the difference) of two non-negative measurable functions. Thus it is obvious that the integral of arbitrary measurable functions should be defined as the corresponding linear combination of the integrals of its constituents. To do this, one has to take care, however, that in the final linear combination no expression of the form "$\infty - \infty$" can occur, and therefore one performs the final step only for those real- or complex-valued measurable functions whose positive and negative parts of their real and imaginary parts have all finite integrals. These functions are called μ-integrable functions.

Image measures. If μ is a measure on (X, Σ_X), a measurable map $F : X \to Y$ induces a measure ν on (Y, Σ_Y) by means of the formula

$$\nu(A) = \mu(F^{-1}(A)).$$

The measure ν is also called the *image measure* of μ under the map F. The effect of image measures on the integration of functions is given by the fact that if ϕ is a ν-integrable function on Y, then $\phi \circ F$ is a μ-integrable function on X, and

$$\int_X \phi \circ F \, d\mu = \int_Y \phi \, d\nu. \tag{2.3}$$

Remark 2.20
We remark that the construction of image measures can be repeated: Let, in addition to the above situation, a map $G : Y \to Z$ be given that is measurable with respect to the σ-algebras Σ_Y on Y and Σ_Z on Z. If we write $\nu = F(\mu)$ for the image measure of μ under F, and $G(\nu)$ for the image measure of ν under G, then we have

$$G(F(\mu)) = (G \circ F)(\mu).$$

A short way of saying this is to state that forming an image measure measure represents a *covariant functor* from the category of measurable spaces into a suitably chosen category of spaces of measures.

Restrictions of measures If μ is a measure on (X, Σ_X) and $A \in \Sigma_X$, then the *restriction* $\mu_{|A}$ of μ to A is defined by $\mu_{|A}(B) = \mu(A \cap B)$ for all $B \in \Sigma_X$.

Product measures. Let arbitrary measure spaces (X, Σ_X, μ) and (Y, Σ_Y, ν) be given, and define a σ-algebra $\Sigma_{X\times Y}$ as the smallest σ-algebra that contains all sets of the form $A \times B$ with $A \in \Sigma_X$ and $B \in \Sigma_Y$. Then it can be shown that there is a unique measure $\mu \otimes \nu$ on $X \times Y$ satisfying $\mu \otimes \nu(A \times B) = \mu(A) \cdot \nu(B)$ for all $A \in \Sigma_X$ and $B \in \Sigma_Y$. This measure is called the *product measure* of μ and ν. By iterating this construction, product measures with an arbitrary finite number of factors can be defined.

Remark 2.21
In the notation $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_N(A_1 \times A_2 \times \cdots \times A_N)$ that is used for product measures, a specific order for both the factor measures and for the factors of the cartesian product is implicitely assumed. It is worth to note that this requirement indeed purely stems from the particular *notation* and is in no way intrinsic to the *notion* itself. This is helpful for the cases in which product spaces and measures will be used in this thesis, as the index set that is typically used is the set of all cells $\mathcal{C}$, or some subset thereof. These sets are not naturally equipped with any order, and we do not need to impose one. In particular, we remark that when we write down expressions like

$$\left[\bigotimes_{c\in\mathcal{C}} \mu_c\right] \left(\prod_{c\in\mathcal{C}} A_c\right),$$

we assume that "every term finds its correct counterpart" even when ambiguous choices due to identical copies of a space in the product would be possible.

For later use, we briefly remind of the Radon-Nikodým theorem which states that a measure that is absolutely continuous with respect to some other measure can be expressed as an integral with respect to this measure by means of a *density*.

Theorem 2.22 (Radon-Nikodým, cited from [19])
Let μ be a σ-finite measure on a space X, and let ν be a signed measure that is absolutely continuous with respect to μ. Then there is a measurable function $h : X \to \mathbb{R} \cup \{\infty\}$ such that $\nu(A) = \int_A h \, d\mu$ for every measurable set A. The function h is μ-integrable if and only if ν is finite, and h is real-valued if and only if ν is σ-finite.

2.3.2 Specific concepts

Building upon the basic ideas just described, we now introduce concepts specific to this thesis.
Dealing with transfer operators and with coupled cell systems, we will have to consider measure spaces over the cartesian product of some or all of the state spaces for the cells, and linear mappings between those measure spaces. Given a coupled cell system with a collection of state spaces, there may be different choices for these collections of measure spaces, each of which can be deemed interesting in its own right and each of which may be suitable for its own kind of problems. Instead of developing separate theories for each of these settings, we prefer taking a more abstract point of view, formulating once the requirements we have for a particular setting, and providing examples that meet these requirements afterwards.

Suitable settings

In the following, we assume we are given a coupled cell system with a set of cells $\mathcal{C}$ and with a state space X_c for each cell c. We assume that each X_c is equipped with a σ-algebra Σ_c. For each subset of cells $\mathcal{D} \subset \mathcal{C}$ we equip the partial state space $X_\mathcal{D}$ with the product σ-algebra $\Sigma_\mathcal{D}$ formed from the σ-algebras Σ_c, $c \in \mathcal{D}$. Note that this implies that all canonical projections $\pi_\mathcal{D} : X \to X_\mathcal{D}$ are measurable maps.

Definition 2.23
In the context described above, a pair $(\{\mathcal{M}(X_\mathcal{D}) \mid \mathcal{D} \subset \mathcal{C}\}, \{\mathcal{F}(X_\mathcal{D}) \mid \mathcal{D} \subset \mathcal{C}\})$, *where each* $\mathcal{M}(X_\mathcal{D})$ *is a subspace of the linear space of complex*[9] *measures on* $(X_\mathcal{D}, \Sigma_\mathcal{D})$, *and each* $\mathcal{F}(X_\mathcal{D})$ *is a vector space of measurable functions on* $X_\mathcal{D}$, *is called a* suitable setting, *if the following properties are satisfied.*

1. *Whenever* $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{C}$ *with* $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, $\mu \in \mathcal{M}(X_{\mathcal{D}_1})$ *and* $\nu \in \mathcal{M}(X_{\mathcal{D}_2})$, *one has* $\mu \otimes \nu \in \mathcal{M}(X_{\mathcal{D}_1 \cup \mathcal{D}_2})$.

2. *For each* $\mathcal{D} \subset \mathcal{C}$ *and each measure* $\mu \in \mathcal{M}(X)$, *the image measure of* μ *under the canonical projection* $\pi_\mathcal{D}$ *must be contained in* $\mathcal{M}(X_\mathcal{D})$.

3. *Each* $\phi \in \mathcal{F}(X_\mathcal{D})$ *is integrable with respect to every element of* $\mathcal{M}(X_\mathcal{D})$.

4. *The weak topology on* $\mathcal{M}(X_\mathcal{D})$ *with respect to* $\mathcal{F}(X_\mathcal{D})$ *is Hausdorff, i. e. for any* $\mu \in \mathcal{M}(X_\mathcal{D})$,

$$\forall \phi \in \mathcal{F}(X_\mathcal{D}) : \quad \int_{X_\mathcal{D}} \phi \, d\mu = 0$$

implies that $\mu = 0$.

[9] In this place one could equally well write "signed measures" to consider real vector spaces.

5. *Conversely, $\mathcal{M}(X_\mathcal{D})$ separates the elements of $\mathcal{F}(X_\mathcal{D})$, i. e. for any $\phi \in \mathcal{F}(X_\mathcal{D})$,*

$$\forall \mu \in \mathcal{M}(X_\mathcal{D}): \quad \int_{X_\mathcal{D}} \phi \, d\mu = 0$$

implies that $\phi = 0$.

Furthermore, for any $\mathcal{D} \subset \mathcal{C}$, we define

$$\mathcal{F}_\mathcal{D}(X) = \{\phi \in \mathcal{F}(X) \mid \exists \hat{\phi} : X_\mathcal{D} \to \mathbb{R} : \phi = \hat{\phi} \circ \pi_\mathcal{D}\}$$

as the subspace of functions depending only on cells in $\mathcal{D}$.

Definition 2.24
Condition 2 from Definition 2.23 implies that the process of forming the images of measures in $\mathcal{M}(X)$ under the canonical projections $\pi_\mathcal{D} : X \to X_\mathcal{D}$ defines a linear map from $\mathcal{M}(X)$ to $\mathcal{M}(X_\mathcal{D})$. From here on, the notation $\pi_\mathcal{D}$ will also be used to denote this map, that is

$$\begin{aligned} \pi_\mathcal{D} : \mathcal{M}(X) &\to \mathcal{M}(X_\mathcal{D}) \\ \mu &\mapsto \mu \circ \pi_\mathcal{D}^{-1}. \end{aligned}$$

The measure $\pi_\mathcal{D}\mu$ is called the marginal measure *of μ with respect to the space $X_\mathcal{D}$.*

Example 2.25
The effect of applying $\pi_\mathcal{D}$ can be described as "integrating out" the factors in the product space that are not in $\mathcal{D}$. To see what this means, we consider the case of a product measure on X. More concretely, let $\mu \in \mathcal{M}(X_\mathcal{D})$ and $\nu \in \mathcal{M}(X_{\mathcal{C}\setminus\mathcal{D}})$. Then $\mu \otimes \nu \in \mathcal{M}(X)$, and for $A \subset X_\mathcal{D}$ measurable we have

$$\begin{aligned} \pi_\mathcal{D}(\mu \otimes \nu)(A) &= (\mu \otimes \nu)(A \times X_{\mathcal{C}\setminus\mathcal{D}}) \\ &= \mu(A) \cdot \nu(X_{\mathcal{C}\setminus\mathcal{D}}), \end{aligned}$$

so that $\pi_\mathcal{D}(\mu \otimes \nu) = \nu(X_{\mathcal{C}\setminus\mathcal{D}}) \cdot \mu$.

In the following, we will present examples for suitable settings $\mathcal{M}(X)$ together with corresponding function spaces $\mathcal{F}(X)$.

Example 2.26
The probably simplest example is that of finite-dimensional measure spaces. Let a coupled cell system be given with a set $\mathcal{C}$ of cells and state spaces X_c with σ-algebras Σ_c. Assume that each σ-algebra Σ_c has a finite number n_c of atoms. (The simplest example is the case where each X_c has only finitely many elements and Σ_c is its power set.) Let $A_c \subset X_c$ be an atom for each $c \in \mathcal{C}$. Then it is easy to see that for any $\mathcal{D} \subset \mathcal{C}$, the cartesian product $\prod_{c\in\mathcal{D}} A_c$ is an atom of $\Sigma_\mathcal{D}$,

and that all atoms of $\Sigma_{\mathcal{D}}$ can be obtained in this way. Therefore the number of atoms in $\Sigma_{\mathcal{D}}$ is also finite and given by the product of the numbers of atoms in the X_c, i. e. $\Sigma_{\mathcal{D}}$ has $\prod_{c\in\mathcal{D}} n_c$ atoms. A suitable setting for this situation is given as follows.
For any $c \in \mathcal{C}$, we take $\mathcal{F}(X_c)$ to be space of complex valued Σ_c-measurable functions. As these functions have to be constant on the atoms, each $\mathcal{F}(X_c)$ is isomorphic to $\mathbb{C}^{n_c}$. Likewise, for any $\mathcal{D} \subset \mathcal{C}$ we define $\mathcal{F}(X_{\mathcal{D}})$ to be the space of $\Sigma_{\mathcal{D}}$-measurable complex valued functions, which can be written as the tensor product

$$\mathcal{F}(X_{\mathcal{D}}) = \bigotimes_{c\in\mathcal{D}} \mathcal{F}(X_c) \cong \bigotimes_{c\in\mathcal{D}} \mathbb{C}^{n_c} \cong \mathbb{C}^{\prod_{c\in\mathcal{D}} n_c}.$$

Now we can simply define the measure space $\mathcal{M}(X_{\mathcal{D}})$ as the linear algebraic dual spaces of their function space counterparts, that is, we define for any $\mathcal{D} \subset \mathcal{C}$

$$\mathcal{M}(X_{\mathcal{D}}) := (\mathcal{F}(X_{\mathcal{D}}))^* \cong \mathbb{C}^{\prod_{c\in\mathcal{D}} n_c}.$$

Example 2.27
Let the state spaces X_c be compact subsets of $\mathbb{R}^{d_c}$ for some finite d_c, equipped with the Borel σ-algebra. Then we take as measures in $\mathcal{M}(X_c)$ all finite measures that are absolutely continuous with respect to the Lebesgue measure m_c on X_c. By the Radon-Nikodým theorem there is a density in $L^1(X_c, m_c)$ for every one of these measures. Conversely, each function in $L^1(X_c, m_c)$ defines a finite signed Lebesgue-absolutely continuous measure, and therefore the above described $\mathcal{M}(X_c)$ can be identified with $L^1(X_c, m_c)$. For the spaces $\mathcal{M}(X_{\mathcal{D}})$, $\mathcal{D} \subset \mathcal{C}$, we take the finite measures on $X_{\mathcal{D}}$ that are absolutely continuous with respect to the product $\otimes_{c\in\mathcal{D}} m_c$ of the individual Lebesgue measures, and note that these contain the products of measures from the individual measure spaces $\mathcal{M}(X_c)$. (The density of the product measure is given by the product of the densities, which by the Fubini theorem is in L^1 if the individual densities are in L^1.)
A suitable choice for a test function space $\mathcal{F}(X)$ in this case is given by the continuous functions on X_c, as the weak topology on $\mathcal{M}(X)$ with respect to this $\mathcal{F}(X)$ is Hausdorff (cf. e. g. [19, Ch. VIII.4.2]).

Example 2.28
For a third example, let the state spaces be as in the previous setting, and again define $\mathcal{F}(X_{\mathcal{D}})$ to be the space of continous functions. We note that these spaces, equipped with the $\|\cdot\|_\infty$-norm, are Banach spaces, and take their dual spaces as the spaces $\mathcal{M}(X_{\mathcal{D}})$, which can be identified with the space of (signed or complex) measures. We note that in this example, $\mathcal{M}(X)$ is larger than in the previous example and contains e. g. the δ-measures.

Remark 2.29
At least two of our examples share the property that the measure space over the cartesian product of a collection of state spaces can be understood as the

tensor product of the measure spaces over the individual state spaces. We explicitely described this for the finite-dimensional setting in Example 2.26. The analogue property for the L^1-setting of Example 2.27 was shown by Alexandre Grothendieck[10] in his doctoral thesis. While this property is not crucial for the theory we are going to develop below (for our purposes, the first condition in Definition 2.23 is sufficient), it is probably instructive for the reader to add it to the mental picture about the relationships between the measure spaces.

Structure of $\mathcal{M}(X)$

In the rest of this section, we begin to explore the structure of suitable settings in the just defined sense. First we show how elements of the several spaces are linked with each other by means of "projections"[11]and their right inverses, then we characterise the kernels of these projections.

Lemma 2.31
Let a coupled cell system and a suitable setting for it be given. For all $c \in \mathcal{C}$ choose $\lambda_c \in \mathcal{M}(X_c)$ with $\lambda_c(X_c) = 1$ and $\lambda_c = \lambda_d$ whenever $c \sim_{\mathcal{C}} d$. Then for $\mathcal{D} \subset \mathcal{C}$, the map

$$\mathcal{M}(X_{\mathcal{D}}) \to \mathcal{M}(X) \qquad \mu_{\mathcal{D}} \mapsto \left(\bigotimes_{c \notin \mathcal{D}} \lambda_c \right) \otimes \mu_{\mathcal{D}}$$

is a right inverse to $\pi_{\mathcal{D}}$ and denoted by $\pi_{\mathcal{D}}^{-1}$.

[10]In essence, this is the question for the relation between the tensor product $L^1(\mu) \otimes L^1(\nu)$ and $L^1(\mu \otimes \nu)$. It has been given a rather general answer by Alexandre Grothendieck, who in his doctoral thesis [28] considers topological tensor products of vector spaces. If one considers tensor products of topological vector spaces E and F, it turns out that in general there is more than one sensible possibility for a topology on the tensor product $E \otimes F$. One reasonable choice is the so-called π-topology which is defined as the finest topology on $E \otimes F$ such that the natural bilinear map $E \times F \to E \otimes F$ is continuous. If one considers the tensor product with this topology, it is written $E \otimes_\pi F$. In the following theorem by Grothendieck, which is cited here in English translation (from the original French) and with mildly adapted notation, the completion of $L^1(\mu) \otimes_\pi L^1(\nu)$ with respect to this topology is shown to be isomorphically homeomorphic to $L^1(\mu \otimes \nu)$.

Theorem 2.30 (Grothendieck, [28, p. 61f, Corollaire 4])
Let X and Y be locally compact spaces equipped with positive measures μ and ν, respectively. The space $\overline{L^1(\mu) \otimes_\pi L^1(\nu)}$ can be identified with the space $L^1(\mu \otimes \nu)$.

[11]This term is used here not in the sense of projection operators, i. e. of idempotent linear endomorphisms, but in the sense borrowed from the geometry of $\mathbb{R}^n$, where "picking" certain components of an n-vector can be understood as projection onto a certain coordinate plane.

Proof: A very simple computation suffices. Let $\mu_{\mathcal{D}} \in \mathcal{M}(X_{\mathcal{D}})$. Then

$$\begin{aligned} \pi_{\mathcal{D}} \circ \pi_{\mathcal{D}}^{-1}(\mu) &= \pi_{\mathcal{D}} \left[\left(\bigotimes_{c \notin \mathcal{D}} \lambda_c \right) \otimes \mu_{\mathcal{D}} \right] \\ &= \left(\prod_{c \notin \mathcal{D}} \lambda_c(X_c) \right) \cdot \mu_{\mathcal{D}} \\ &= \mu_{\mathcal{D}}. \end{aligned}$$

□

With this lemma, we obtain the possibility to embed the spaces $\mathcal{M}(X_{\mathcal{D}})$ (for $\mathcal{D} \subset \mathcal{C}$) into the measure space $\mathcal{M}(X)$, which will be extensively used in the developments in the next chapters.

Definition 2.32
For $\mathcal{D} \subset \mathcal{C}$, the map $\pi_{\mathcal{D}}^{-1}$ (and sometimes its image) is called the embedding *of $\mathcal{M}(X_{\mathcal{D}})$ into $\mathcal{M}(X)$.*

Example 2.33
In the finite dimensional setting (Example 2.26), the map $\pi_{\mathcal{D}}$ is given by summation over the indices belonging to cells not in $\mathcal{D}$. More concretely, we assume that the cells are enumerated in such a way that $\mathcal{C} = \{1, \ldots, N\}$ and $\mathcal{D} = \{K+1, \ldots, N\}$ for some $K < N$. Then a measure $\mu \in \mathcal{M}(X)$ can be written as a tensor in $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_N}$, that is

$$\mu = \left(\mu^{i_1,\ldots,i_N}\right)_{1 \le i_1 \le n_1, \ldots, 1 \le i_N \le n_N},$$

and one readily sees that

$$\pi_{\mathcal{D}}(\mu) = \left(\sum_{i_1=1}^{n_1} \cdots \sum_{i_K=1}^{n_K} \mu^{i_1,\ldots,i_N} \right)_{1 \le i_{K+1} \le n_{K+1}, \ldots, 1 \le i_N \le n_N}.$$

With measures $\lambda_c \in \mathbb{C}^{n_c}$ chosen such that $\sum_{i=1}^{n_c} \lambda_c^i = 1$, the right inverse $\pi_{\mathcal{D}}^{-1}$ takes the form

$$\pi_{\mathcal{D}}^{-1}(\tilde{\mu}) = \left(\tilde{\mu}^{i_{K+1},\ldots,i_N} \cdot \lambda_1^{i_1} \cdot \ldots \cdot \lambda_K^{i_K}\right)$$

for $\tilde{\mu} \in \mathcal{M}(X_{\mathcal{D}})$.

Remark 2.34
1. To simplify notation, we will in the following sometimes write $\lambda_{\mathcal{C}\setminus\mathcal{D}}$ instead of $\bigotimes_{c \notin \mathcal{D}} \lambda_c$.
2. For further use, we remark that for any $\mathcal{D} \subset \mathcal{C}$, the composition $\pi_{\mathcal{D}} \circ \pi_{\mathcal{D}}^{-1}$ is the identity mapping on $\mathcal{M}(X_{\mathcal{D}})$, while the composition $\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}$ is a projection onto $\operatorname{im} \pi_{\mathcal{D}}^{-1} \subset \mathcal{M}(X)$:

$$(\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}})^2 = \pi_{\mathcal{D}}^{-1} \circ (\pi_{\mathcal{D}} \circ \pi_{\mathcal{D}}^{-1}) \circ \pi_{\mathcal{D}} = \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}.$$

3. Furthermore, we remark that the measures λ_c can be used for a similar construction on the side of the test functions. Using these measures, a linear map $\mathcal{F}(X) \to \mathcal{F}(X_\mathcal{D})$ can be defined by assigning to a function ϕ on X the function $\phi_\mathcal{D}$ on $X_\mathcal{D}$ obtained from "integrating out" the factors not in $\mathcal{D}$:

$$\phi_\mathcal{D}(\xi_\mathcal{D}) = \int_{X_{\mathcal{C}\setminus\mathcal{D}}} \phi(\xi_\mathcal{D}, \eta)\, d\lambda_{\mathcal{C}\setminus\mathcal{D}}(\eta)$$

Using this map, we can express the right inverse $\pi_\mathcal{D}^{-1}$ through its behaviour on integrals. If $\phi \in \mathcal{F}(X)$ and $\mu_\mathcal{D} \in \mathcal{M}(X_\mathcal{D})$, then

$$\int_{X_\mathcal{D}} \phi_\mathcal{D}\, d\mu_\mathcal{D} = \int_X \phi\, d(\pi_\mathcal{D}^{-1}\mu_\mathcal{D}),$$

which is in essence again an application of the Fubini theorem.

Lemma 2.35
In a suitable setting, the kernels of the linear maps $\pi_\mathcal{D}$, $\mathcal{D} \subset \mathcal{C}$, have the following characterisation:

$$\mu \in \ker \pi_\mathcal{D} \iff \int_X \phi\, d\mu = 0 \quad \forall \phi \in \mathcal{F}_\mathcal{D}(X).$$

Proof:

$\Rightarrow$ Let $\mu \in \mathcal{M}(X)$ with $\pi_\mathcal{D}(\mu) = 0$, and let $\phi \in \mathcal{F}_\mathcal{D}(X)$. Then there is $\tilde{\phi} \in \mathcal{F}(X_\mathcal{D})$ such that $\phi = \tilde{\phi} \circ \pi_\mathcal{D}$, and then

$$\int_X \phi\, d\mu = \int_X \tilde{\phi} \circ \pi_\mathcal{D}\, d\mu = \int_{X_\mathcal{D}} \tilde{\phi}\, d\pi_\mathcal{D}(\mu) = 0.$$

$\Leftarrow$ Let $\int_X \phi\, d\mu = 0$ for all $\phi \in \mathcal{F}_\mathcal{D}(X)$. Using the above equation, this implies

$$\int_{X_\mathcal{D}} \tilde{\phi}\, d\pi_\mathcal{D}(\mu) = 0 \text{ for all } \tilde{\phi} \in \mathcal{F}(X_\mathcal{D}),$$

and thus $\pi_\mathcal{D}(\mu) = 0$ by Condition 4 from Definition 2.23.

□

Corollary 2.36
If $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{C}$, then $\ker \pi_{\mathcal{D}_2} \subset \ker \pi_{\mathcal{D}_1}$.

Proof: This follows from $\mathcal{F}_{\mathcal{D}_1}(X) \subset \mathcal{F}_{\mathcal{D}_2}(X)$ and Lemma 2.35. □

The next lemma can be seen as the analog of Lemma 2.35 on the test function side.

Lemma 2.37
In a suitable setting, the spaces of functions depending only on a subset $\mathcal{D} \subset \mathcal{C}$ of all cells have the following characterisation:

$$\phi \in \mathcal{F}_{\mathcal{D}}(X) \iff \int_X \phi \, d\mu = 0 \quad \forall \mu \in \ker \pi_{\mathcal{D}}.$$

Proof:

$\Leftarrow$ Let $\phi \in \mathcal{F}(X)$ be such that $\int_X \phi \, d\mu = 0$ for all $\mu \in \ker \pi_{\mathcal{D}}$. Define $\psi = \phi - (\phi_{\mathcal{D}} \circ \pi_{\mathcal{D}})$. For any $\mu \in \mathcal{M}(X)$ define $\mu_1 = \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}(\mu)$ and $\mu_0 = \mu - \mu_1$, so that $\mu_0 \in \ker \pi_{\mathcal{D}}$. Then we have

$$\begin{aligned}
\int \psi \, d\mu &= \int \psi \, d\mu_0 + \int \psi \, d\mu_1 \\
&= \int \phi - (\phi_{\mathcal{D}} \circ \pi_{\mathcal{D}}) \, d\mu_0 + \int \phi - (\phi_{\mathcal{D}} \circ \pi_{\mathcal{D}}) \, d\mu_1 \\
&= \underbrace{\int \phi \, d\mu_0}_{=0 \text{ by assumption}} - \underbrace{\int \phi_{\mathcal{D}} \, d\pi_{\mathcal{D}}\mu_0}_{=0, \text{ as } \mu_0 \in \ker \pi_{\mathcal{D}}} + \int \phi \, d\mu_1 - \int \phi_{\mathcal{D}} \circ \pi_{\mathcal{D}} \, d\mu_1 \\
&= \int \phi \, d\mu_1 - \int \phi_{\mathcal{D}} \, d\pi_{\mathcal{D}}\mu_1 \\
&= \int \phi \, d\mu_1 - \int \phi \, d \underbrace{\pi_{\mathcal{D}}^{-1}\pi_{\mathcal{D}}\mu_1}_{=\mu_1} \\
&= 0.
\end{aligned}$$

But then, using Condition 5 from Definition 2.23, $\psi = 0$, and this means nothing else than $\phi \in \mathcal{F}_{\mathcal{D}}(X)$.

$\Rightarrow$ Let $\phi \in \mathcal{F}_{\mathcal{D}}(X)$ be given and a function $\hat{\phi}$ on $X_{\mathcal{D}}$ such that $\phi = \hat{\phi} \circ \pi_{\mathcal{D}}$. Let $\mu \in \ker \pi_{\mathcal{D}}$. Then

$$\int_X \phi \, d\mu = \int_X \hat{\phi} \circ \pi_{\mathcal{D}} \, d\mu = \int_{X_{\mathcal{D}}} \hat{\phi} \, d\pi_{\mathcal{D}}\mu = 0.$$

□

Lifts of permutations

We now begin to combine the measure-theoretical concepts that were just introduced with the concepts describing "symmetry" in coupled cell networks. In principle we are going to describe a straightforward construction, that of the

"lift" of permutations of cells to state spaces and to measure spaces. Although these constructions are not particularly difficult or mathematically demanding, we perform them in detail, as we think this will contribute a good deal to the clarity of the whole thesis.
We assume that we are working in a suitable setting in the above defined sense. We are going to look at permutations of cells, and how these permutations can be made "act" on state spaces $X_{\mathcal{D}}$, on measure spaces $\mathcal{M}(X_{\mathcal{D}})$, and on the embeddings of these into $\mathcal{M}(X)$. More precisely, let us assume that $\mathcal{D} \subset \mathcal{C}$ and that $\varphi : \mathcal{D} \to \mathcal{C}$ is an injective mapping respecting the cell types, i. e. such that $c \sim_{\mathcal{C}} \varphi(c)$ for each $c \in \mathcal{D}$. Such a mapping can be visualised as "picking" the cells from $\mathcal{D}$ and putting them into the places $\varphi(c)$. If one additionally visualises the state spaces X_c as "attached" to the cells, this immediately defines mappings between the state spaces, and also between their cartesian products.
To express this in a more precise way, we note that as φ respects the cell types, X_c is identical to $X_{\varphi(c)}$ for each $c \in \mathcal{D}$. Therefore φ defines a map $\hat{\varphi} : X_{\mathcal{D}} \to X_{\varphi(\mathcal{D})}$ by

$$(\hat{\varphi}(x))_{c'} = x_{\varphi^{-1}(c')} \quad \forall c' \in \varphi(\mathcal{D}) \quad \forall x \in X_{\mathcal{D}},$$

that is by assigning to the c'-component of $\hat{\varphi}(x)$ the value of the component "where c' was before application of φ ".[12] Now this map defines a linear map $\bar{\varphi} : \mathcal{M}(X_{\mathcal{D}}) \to \mathcal{M}(X_{\varphi(\mathcal{D})})$ through the assignment of image measures, that is through the prescription

$$\bar{\varphi}(\mu)(A) = \mu(\hat{\varphi}^{-1}(A)) \quad \forall A \subset X_{\varphi(\mathcal{D})} \text{ measurable} \quad \forall \mu \in \mathcal{M}(X_{\mathcal{D}}).$$

Finally, the mapping $\bar{\varphi}$ can be "embedded" into $\mathcal{M}(X)$ via the maps $\pi_{\mathcal{D}}$ and $\pi^{-1}_{\varphi(\mathcal{D})}$ to obtain a map between subspaces of $\mathcal{M}(X)$. With the following definition we subsume these developments.

Definition 2.38
We call $\bar{\varphi}$ the lift of φ to $\mathcal{M}(X_{\mathcal{D}})$. *Furthermore, we call the map*

$$\tilde{\varphi} : \mathcal{M}(X) \supset \operatorname{im} \pi^{-1}_{\mathcal{D}} \to \operatorname{im} \pi^{-1}_{\varphi(\mathcal{D})} \subset \mathcal{M}(X)$$

defined by

$$\tilde{\varphi} = \pi^{-1}_{\varphi(\mathcal{D})} \circ \bar{\varphi} \circ \pi_{\mathcal{D}}$$

the lift of φ to $\mathcal{M}(X)$.

2.3.3 Probabilistic concepts

While there is no reason not to see probability theory simply as a part of measure theory, there is also plenty of reason to see it as a part of mathematics of its

[12] We implicitely already encountered this construction in the definition of admissibility of a map on a coupled cell network.

own right, not the least of which are its history, the large number of genuinely probabilistic (or stochastic) concepts which often stand quite apart from the rest of measure theoretic notions, and its seemingly infinite applicability to real world problems. This is the reason why we introduce the small set of probabilistic terms which we will need in the later chapters into a section of its own. Basically, the aim of this section is to introduce the notion of a *Markov chain*. In order to keep up some consistence of this thesis, we will start in the base camp to which the section on basic concepts from measure theory brought us, and from there begin to climb a steep mountain, from the ridges of which we will be able to see, but not to climb, its higher peaks. (That is to say, we will sketch the definition of a Markov chain, but we will not say many words about the vast theory that has been developed around the concept).
Again, the material presented here is standard knowledge contained in virtually any textbook on probability theory from the graduate level onwards. To mention a reference, we mention two and refer the reader to the classical textbook [17] by J. L. Doob, or the more modern, extensive account [41] by S. P. Meyn and R. L. Tweedie.

Random variables and conditional probabilities Let (Ω, Σ, P) be a measurable space with a probability measure P. If $B \in \Sigma$ with $P(B) > 0$, then B defines another probability measure $P(\cdot \mid B)$ on Ω via

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{for all } A \in \Sigma.$$

This measure is called the *conditional probability* under the condition B. Let (Y, Σ_Y) be any other measurable space. A *random variable* Z with values in Y is a measurable map $Z : \Omega \to Y$. For some set $A \in \Sigma_Y$ one uses the notation $P(Z \in A)$ for $P(Z^{-1}(A))$ and reads this as "the probability that Z is in A". Likewise, if X is another random variable on Ω with values in Y and $B \in \Sigma_Y$ with $P(X \in B) > 0$, then $P(Z \in A \mid X \in B)$ denotes the probability that Z is in A under the condition that X is in B. Analogously one defines conditional probabilities for conditions consisting of more than one term, i. e. conditional probabilities of the form $P(Z \in A \mid X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n)$. To shorten notation, one often leaves out explicit mention of the sets $A, B_1, \ldots, B_n$ and simply writes $P(Z \mid X_1, \ldots, X_n)$, as we shall do from here on. In this notation, the set Ω has completely vanished, and often the measure $P(Z \in \cdot \mid X_1, \ldots, X_n)$ is seen as a probability measure on Y, that is, it is identified with its image measure under the map Z.

Stochastic processes and Markov chains A (discrete-time) stochastic process with values in Y is a sequence $(Z_t)_{t \in \mathbb{N}}$ of random variables with values in Y. The space Y is also called the *state space* of the process, especially in the case

of Markov chains to which we will come in a moment. This is a rather general concept, and in a moment, we will add some important conditions to this definition in order to define the notion of a Markov chain. However, before we do this, we acknowledge that it is time now to explain, rather than only describe, what is going on.

A stochastic process as just defined should be thought of as a probabilistic version of a dynamical system. One could think of trajectories in an ω-wise sense, that is, as sequences of points $(Z_t(\omega))_{t\in\mathbb{N}}$ with $\omega \in \Omega$. These sequences are called the *sample paths* of the process. While there is sometimes a good reason to speak of a stochastic process in terms of its sample paths[13], often this point of view is not appropriate. A more probabilistic view is better suited, and thus one often speaks about a stochastic process in terms of its probability measures and their evolution. As an example, the sequence $(P(Z_t \in \cdot \mid Z_0 = y_0))_{t\in\mathbb{N}}$ of probability measures is used to describe the process under the condition that it starts in $y_0 \in Y$.

Amongst all stochastic processes, Markov chains[14]are those processes that share an important feature with a deterministic dynamical system: the property that it is "memoryless". This means that in a deterministic dynamical system, the position of a particle at time $t+1$ depends only on its position at time t, but not on its "history", i. e. the sequence of positions at times $0, \ldots, t$. Translated into the probabilistic setting, this property defines a Markov process. More precisely, a stochastic process $(Z_t)_t \in \mathbb{N}$ is a Markov chain if it satisfies

$$P(Z_{t+1} \mid Z_0, \ldots, Z_t) = P(Z_{t+1} \mid Z_t),$$

that is, if the conditional probability of Z_{t+1} under the condition of the complete history is equal to the conditional probability under the condition of the last time step only. A chain for which the probability $P(Z_{t+1} \mid Z_t)$ does not depend on t, i. e. for which

$$P(Z_{t+1} \mid Z_t) = P(Z_2 \mid Z_1) \quad \text{for all } t \in \mathbb{N}$$

holds, is called a *stationary* Markov chain.

We will now specialize our considerations to the case of a (finite or infinite) *countable* state space $Y = (y_1, y_2, \ldots)$ with the power set as σ-algebra. A probability measure on Y can then be written as a sequence (or vector) $p_1, p_2, \ldots$ of non-negative numbers with $\sum_{i=1}^{\infty} p_i = 1$. Then it is an easy consequence of Bayes' theorem that

$$P(Z_{t+1} = y_i) = \sum_{j=1}^{\infty} P(Z_{t+1} = y_i \mid Z_t = y_j) P(Z_t = y_j).$$

[13]For example, one may think of the Wiener process whose sample paths are almost all continuous with bounded second variation.

[14]The term "chain" signifies that a discrete time process is described. The analogous concept for a continuous time variable $t \in \mathbb{R}_{\geq 0}$ is usually called a Markov process.

This means that the probability measure for Z_{t+1} is derived from the one for Z_t by means of a simple "matrix-vector-product". If $(Z_t)_{t\in\mathbb{N}}$ is a stationary Markov chain, the t-independent matrix $P_{ij} = P(Z_{t+1} = y_i \mid Z_t = y_j)$ appearing therein is called the *transition matrix* of the chain. The transition matrix of a stationary Markov chain is the stochastic counterpart of the transfer operator of a deterministic autonomous dynamical system, which will be the topic of the following section.

2.4 Transfer Operators

2.4.1 Transfer Operators for Dynamical Systems

As we already saw in Chapter 1, the basic idea behind the notion of the transfer operator of a dynamical system is the observation that, when performed on a linear space of measures, the process of assigning to a measure its image measure under a measurable map defines a linear operator. We also mentioned that this operator is a useful tool for different kinds of analysis of dynamical systems, and that there is a wide range of possible applications of this tool, from purely mathematical ergodic theory to the analysis of uncertainty propagation for "noisy" systems.
In this section, we will introduce the formal mathematical definition of the transfer operator for a dynamical system that we will be using in the rest of this thesis. While our focus will be on the case of (deterministic) maps as dynamical systems, we will also mention the stochastic counterpart of the transfer operator, the *transition matrix* of a Markov chain. Our reason for doing so is twofold: on the one hand, most of the following developments could be applied to "coupled cell Markov chains"[15]as well, and we will have to resist the temptation to elaborate this idea in this thesis in order not to loose the focus on the consequences of the network structure. On the other hand, whenever we will be looking at numerical approximations of a transfer operator, we will in fact be looking at the transition matrix of a suitably defined Markov chain.
The definition we give now is somewhat more general than what is found in the literature, in that it refers to maps between two spaces X and Y. The reason

[15]This term is used here to describe a Markov chain version of a coupled cell system. It is different from and should not be confused with notions of "coupled", "uncoupled" or "almost uncoupled" Markov chains in the literature, see e. g. [38, 40], or [39] for an overview. The main difference lies in the fact that the concept considered in these publications assumes that the state space of a Markov chain is given as the *disjoint union* of smaller spaces, such that each of these is viewed as the state space of a smaller Markov chain on its own, which is coupled (in a suitable sense) to the other ones. The concept of "coupledness" that is considered in this thesis, in contrast, assumes that the state space is given as a *cartesian product* of spaces. The former concept is contained as a special case in our concept, and one could formulate a Markov chain version of Theorem 2.42 that explains the relations between the concepts.

for this is that in the context of coupled cell systems, in addition to the "full" dynamical system $f : X \to X$, we will need to consider the constituent maps $\hat{f}_c : X_{I(c)} \to X_c$, and the projections $\pi_{\mathcal{D}} : X \to X_{\mathcal{D}}$ between total and partial state spaces. It will be convenient to have the notion of the transfer operator also for these maps.

Definition 2.39
Let measurable spaces X, Y and a measurable map $f : X \to Y$ be given. Let $\mathcal{M}(X)$ be a subspace of the vector space of complex-valued measures on X, and let $\mathcal{M}(Y)$ be a subspace of the vector space of complex-valued measures on Y, such that the image measure under f of any measure in $\mathcal{M}(X)$ is contained in $\mathcal{M}(Y)$. Then the transfer operator $P_f : \mathcal{M}(X) \to \mathcal{M}(Y)$ *associated with f is defined by the equation*

$$P_f\mu(A) = \mu(f^{-1}(A))$$

for every measurable $A \subset Y$.

Example 2.40
Let $f : X \to X$ be a dynamical system. If the measure space $\mathcal{M}(X)$ contains the δ-measures, then it is possible to fully recover the map f from its transfer operator P_f. Indeed, for $x \in X$ one has

$$P_f\delta_x(A) = \delta(f^{-1}(A)) = \delta_{f(x)}(A)$$

for every measurable A, so that $P_f\delta_x = \delta_{f(x)}$.

Remark 2.41
1. For a usual dynamical system $f : X \to X$ one normally requires that both subspaces of measures are identical. Then the condition on f that $\mathcal{M}(X)$ is invariant under the forming of image measures takes different concrete realisations for different kinds of spaces $\mathcal{M}(X)$.
The probably most classical setting is the case where there is a distinguished measure m on X, such as Lebesgue measure on compact subsets of $\mathbb{R}^d$, and one takes as $\mathcal{M}(X)$ the space of m-absolutely continuous finite measures. By the Radon-Nikodým theorem this space can by be identified with $L^1(X, m)$. Then the condition on f is that f is *non-singular*, which means that whenever $m(A) = 0$, also $m(f^{-1}(A)) = 0$ has to hold.
In the literature devoted to the study of functional analytic properties of the transfer operator, very often spaces are considered that are tailor-made for the system at hand. These spaces are usually Banach spaces obtained either as the closure of a specific subset of $L^1(X, m)$ with respect to specific norms, or as dual spaces of equally carefully crafted spaces of functions. Their construction may depend on the dimension d of X, on whether f is e. g. expanding, contracting or otherwise hyperbolic, and on the degree of smoothness of f. Also, in these settings one often finds that, although f is a map on one space X, the

transfer operator is defined as an operator between two different Banach spaces. By means of so-called *Lasota-Yorke* inequalities it can then be shown that the transfer operator is quasi-compact, from which further conclusions can be drawn on ergodic properties of the dynamical system. For a well-readable introduction into this field see [37], a more detailed reference is e. g. [1].
2. In order to employ the finite-dimensional setting, one has to equip X with a σ-algebra Σ_N generated by finitely many atoms $A_1, \ldots, A_N$. (For example, one can think of the atoms as disjoint boxes covering X.) The only requirement which f has to fulfill is that of measurability with respect to such a σ-algebra. However, in this setting, the transfer operator does not fully resolve the dynamics of f. There may be different measurable maps $f, g : X \to X$ which satisfy $f^{-1}(A_i) = g^{-1}(A_i)$ for all $i = 1, \ldots, N$, and thus have the same transfer operator. (For an example, see Figure 2.2.) Thus the transfer operator does not represent the map in a sensible way. One can, however, use any of these maps to define a Markov chain whose transition matrix is essentially the transfer operator of f with respect to the σ-algebra Σ_N. To see this, equip the measurable space (X, Σ_N) with a probability measure P, take as state space the set $Y = \{1, \ldots, N\}$ with the power set as σ-algebra, and define the Markov chain as the sequence of random variables $(F_k)_{k\in\mathbb{N}}$ with $F_k : X \to Y, x \mapsto i(f^k(x))$ where $i(x)$ is the unique index such that $x \in A_{i(x)}$. (These maps are measurable as f is Σ_N-measurable.) One easily sees that this defines a stationary Markov chain with transition matrix given as

$$p_{ij} = P(F_{t+1} = i \mid F_t = j) = \frac{P(A_j \cap f^{-1}(A_i))}{P(A_j)} \quad \text{for } i, j \in Y. \tag{2.4}$$

We will shortly see that this Markov chain plays a prominent role in the numerical approximation of the transfer operator for a dynamical system.

2.4.2 Numerical Approximation of the Transfer Operator

In this section, we will briefly discuss a popular method for the numerical approximation of transfer operators, in order to provide the foundations for the discussion of numerical questions in Chapter 4. Besides, this method gives the reason for which we also consider the setting of finite-dimensional measure spaces.

Ulam's method The probably most popular, and probably also oldest suggestion for a numerical approximation of a transfer operator of a dynamical system was made by S. Ulam in [46], where he suggests to approximate an invariant density of a map f on the open interval $(0, 1)$ by a sequence of eigenvectors for the eigenvalue one of matrices that are obtained as follows: For a partition $\{A_1, \ldots, A_n\}$ of the interval into n parts, the matrix (a_{ij}) is defined by $a_{ij} = \frac{m(f^{-1}(A_i)\cap A_j)}{m(A_j)}$, where m is the Lebesgue measure on $(0, 1)$. These matrices

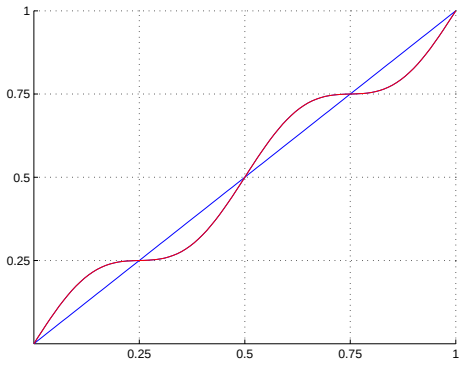

Figure 2.2: Two maps on the unit interval which are measurable with respect to the partition shown by the dotted lines. Although the maps are clearly different, they have the same transfer operator with respect to the σ-algebra generated by that partition.

are easily seen to all have the row sum one, and therefore they all have an eigenvalue one, together with a corresponding left eigenvector. In a by now famous conjecture on this subject, Ulam asks when, assuming that an invariant density of f exists, the left eigenvectors, viewed as step functions on the interval, converge (for increasingly fine partitions) to the invariant density. While this question has been answered positively in a number of settings[16], the extended question of convergence of a sequence of Ulam matrices to the corresponding transfer operator has only been given partial answers. However, in this thesis we will not pursue this question further and shall be content with a short exposition of a concrete implementation of a numerical scheme going back to Ulam's proposal.

Numerical realizations of Ulam's method For completeness' sake, we are now going to sketch a numerical realization of Ulam's approach for the approximation of the transfer operator of a dynamical system. We have seen in equation (2.4) how matrix entries can be defined that form the transition matrix of a Markov chain which approximates a given dynamical system on a given σ-algebra with finitely many atoms. For a concrete implementation of this idea, three questions have to answered: How to obtain the σ-algebra, which probability measure P to take, and how to approximate the right hand side of (2.4).
A common answer to the first question is the use of *box coverings* of X obtained e. g. from schemes that start with one box and iteratively subdivide and select boxes until a sufficiently fine covering of the state space X is achieved. Variations of this approach are known in the literature under the names *cell mapping*

[16]See e. g. [37] for an overview, or [42] for a detailed exposition of this topic.

methods (see e. g. [31, 30, 32]), *subdivision schemes* (e. g. [8, 9]) or more generally *set-oriented numerics* (e. g. [33, 6, 11]). For want of better possibilities, the second question is often answered by using (normalized) Lebesgue measure m on the (compact) state space as the probability measure. The third question, how to approximate the value of $m(A_j \cap f^{-1}(A_i))$ for two sets A_i and A_j from the box collection, normally finds an answer as it is e. g. described in [33] or [6]. The basic idea is to use a sufficiently large number of test points in A_j, and to form the fraction of the number of test points that are mapped by f into A_i with the total number of points. We will consider this approach again in Chapter 4.

2.4.3 A Structural Theorem for Transfer Operators of Coupled Cell Systems

In the final section of this preparatory chapter, we will take a first closer look at the transfer operator for a coupled cell system. We will present a theorem the consequences of which help explain why the undertaking of the following chapters is necessary. The basic thought that leads to this theorem is as follows. When starting freshly to think about how structural properties of a transfer operator P_f depend on the structure of the underlying dynamical system $f : X \to X$ with component maps $f_c : X_{I(c)} \to X_c$, one might first be guided by the observation that as f is determined by its components, so must $P_f : \mathcal{M}(X) \to \mathcal{M}(X)$ be determined by the component transfer operators $P_{f_c} : \mathcal{M}(X_{I(c)}) \to \mathcal{M}(X_c)$. Further, as one is considering a problem where all involved mappings are linear mappings, one might be trapped into thinking that this is equivalent to saying that any image vector $P_f\mu$ must be determined "in a linear way" from vectors of the form $P_{f_c}\pi_{I(c)}\mu \in \mathcal{M}(X_c)$. This amounts to saying that there should be linear mappings between $\oplus_{c\in\mathcal{C}}\mathcal{M}(X_c)$ and $\mathcal{M}(X)$ and a mapping $\tilde{P}$ on $\oplus_{c\in\mathcal{C}}\mathcal{M}(X_c)$ constructed from the P_{f_c} such that the diagram

$$\begin{array}{ccc} \mathcal{M}(X) & \xrightarrow{P_f} & \mathcal{M}(X) \\ \downarrow & & \uparrow \\ \bigoplus_{c\in\mathcal{C}} \mathcal{M}(X_c) & \xrightarrow[\tilde{P}]{} & \bigoplus_{c\in\mathcal{C}} \mathcal{M}(X_c) \end{array}$$

commutes. Written in this way, it becomes clear that this conjecture is too ambitious. As the space $\mathcal{M}(X)$ is, intuitively put, "much larger" than the space $\oplus_{c\in\mathcal{C}}\mathcal{M}(X_c)$, the mapping on the right side of this diagram can at best be injective, but cannot be expected to be surjective. Consequently, the image of P_f had to be restricted to the image of that mapping. However, e. g. for invertible mappings f, P_f must be invertible also, which leads to a contradiction. At this point, one is forced to weaken the conjecture. If one cannot find a $\tilde{P}$ that could fully replace the operator, maybe one can find one that reproduces, in some way,

the "essential" properties of P_f? For example an operator $\tilde{P}$ that is contained as a factor in P_f, such that this diagram commutes.

$$\begin{array}{ccc} \mathcal{M}(X) & \xrightarrow{P_f} & \mathcal{M}(X) \\ {\scriptstyle \oplus\pi_c}\downarrow & & \downarrow{\scriptstyle \oplus\pi_c} \\ \bigoplus_{c\in\mathcal{C}} \mathcal{M}(X_c) & \xrightarrow[\tilde{P}]{} & \bigoplus_{c\in\mathcal{C}} \mathcal{M}(X_c) \end{array}$$

For this to be true it is necessary that the kernel of the map $\oplus_{c\in\mathcal{C}}\pi_c$ is invariant under the action of P_f. It turns out that this is a very special property that is essentially the result of *independence* of maps, and that the above idea is wrong for most "proper" coupled cell systems. We will see this in the following from a simplified example[17]. To bring forth the argument, we consider a very simple coupled cell system that is given by two *independent* dynamical systems $f : X \to X$ and $g : Y \to Y$ which are simply *viewed* as one coupled system - with trivial (i. e. non-existent) coupling. Now while formally the transfer operator of this coupled system is defined on the space $\mathcal{M}(X \times Y)$ it is immediately clear that it is not necessary to consider an operator acting on so large a space in order to capture the dynamics of the two systems. In fact, by just taking the two individual transfer operators P_f and P_g one can define a linear operator $\tilde{P}$ on the much smaller space $\mathcal{M}(X) \oplus \mathcal{M}(Y)$ simply by prescribing that $\tilde{P}(\mu_X, \mu_Y) = (P_f\mu_X, P_g\mu_Y)$, i. e. by viewing the individual transfer operators as diagonal blocks of $\tilde{P}$. This operator is able to reproduce everything the formal transfer operator on $\mathcal{M}(X \times Y)$ can describe, while it is at the same time not only defined on a smaller space, but even comes in a block-diagonalised form.
This "result" of course does not come as a surprise. Block-diagonalisability of a linear map is equivalent to the existence of invariant subspaces on each of which the map is independent from the other ones, and thus with *reducibility* of a system into smaller parts. In our example the reducibility of the operator is a direct consequence of the reducibility of the dynamical system. Therefore, one cannot expect to find a block-diagonal form of the transfer operator for interconnected, truly *coupled* cell systems. As we argued above, one might however suspect that it is possible to define an operator on the direct sum, instead of the tensor product, of the constituent measure spaces that is able to describe at least an "essential" part (defined in a suitable manner) of the dynamics of the system.
With the following result we show that this hope has to be frustrated. It states that whenever the full transfer operator for a coupled cell system has a factor that is an operator defined on the direct sum of the constituent spaces, then the system is of a very special nature, consisting of two parts that are either indeed

[17] As the result that is exemplified is a negative result, just showing which way one *cannot* go, the author deemed it acceptable for this thesis to refrain from formulating a more general version, assuming that the fundamental idea necessary to convince the reader is carried across best by simplicity.

uncoupled, or ignorant of their own state and governed only by the other cell[18], or of a master-slave nature where one cell's evolution depends on its own state only, while it at the same time completely determines the other cell's evolution. We will formulate the theorem only for two cells, but it is easy to see that it can be generalised to any coupled cell network by successive application. Similarly, while the theorem is formulated in a specific setting of operators between L^1-spaces, it should not be difficult to generalise it to the other settings considered in this thesis, and probably to more general ones.

Theorem 2.42
Let compact state spaces $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ with non-empty interior be given. Denote by λ the Lebesgue measure on X, on Y and on $X \times Y$. Let $\mathcal{M}(X) = L^1(X, \lambda)$, $\mathcal{M}(Y) = L^1(Y, \lambda)$ and $\mathcal{M}(X \times Y) = L^1(X \times Y, \lambda)$. Let $f : X \times Y \to X \times Y$ be a continuous non-singular map such that $P_f : \mathcal{M}(X \times Y) \to \mathcal{M}(X \times Y)$ is its transfer operator. Let $\pi : \mathcal{M}(X \times Y) \to \mathcal{M}(X) \oplus \mathcal{M}(Y)$ be defined by $\pi\mu = (\pi_X\mu, \pi_Y\mu)$. If there is a linear operator $\tilde{P} : \mathcal{M}(X) \oplus \mathcal{M}(Y) \to \mathcal{M}(X) \oplus \mathcal{M}(Y)$ such that $\pi \circ P_f = \tilde{P} \circ \pi$, then f is of the form

$$f(x,y) = \begin{cases} (f_X(x), f_Y(y)) & \text{or} \\ (f_X(y), f_Y(x)) & \text{or} \\ (f_X(x), f_Y(x)) & \text{or} \\ (f_X(y), f_Y(y)). & \end{cases}$$

For the proof of this statement we will need the following lemma. As a side remark we want to note that it is related to Lemma 2.35, in that it draws a similar, but weaker conclusion from similar, but weaker assumptions. The proof given here, however, is quite different from the proof used for Lemma 2.35, and it would be interesting to see whether there is a similar proof, or whether Lemma 2.43 can be proved from Lemma 2.35.

Lemma 2.43
If in the setting of Theorem 2.42 there is a closed set $A \subset X \times Y$ with non-empty interior for which $\mu(A) = 0$ holds for all measures $\mu \in \ker \pi$, then there are closed sets $A_x \subset X$ and $A_y \subset Y$ with positive Lebesgue measure such that one of the statements

$$A \subset X \times A_y \text{ and } \lambda(A) = \lambda(X \times A_y)$$

or

$$A \subset A_x \times Y \text{ and } \lambda(A) = \lambda(A_x \times Y)$$

is true.

[18] In this case, the dynamical system obtained by looking only at every second step is uncoupled again.

Proof: Before we start, we note the following fact. Whenever we consider an open non-empty set in $X \times Y$, we can assume that this open set contains a cartesian product $A \times B$ of open sets $A \subset X$, $B \subset Y$. To see this is true, remember that all norms on $\mathbb{R}^{n+m}$ are equivalent and thus generate the same topology. So we can safely assume that any open set contains an open ball with respect to the $\|\cdot\|_\infty$-norm. But the open balls with respect to this norm are just cartesian products as required.
To start with the proof of the lemma, define $A_x = \pi_X(A)$ and $A_y = \pi_Y(A)$, and note that these sets are closed, with non-empty interior. From $\lambda(A) > 0$ it follows that $\lambda(A_x) > 0$ and $\lambda(A_y) > 0$. We will now show that one of these sets has to have full measure. To see this, assume that this is not case, i. e. that both $\lambda(A_x) < \lambda(X)$ and $\lambda(A_y) < \lambda(Y)$. Then define $B_x := X \setminus A_x$ and $B_y := Y \setminus A_y$ which are both open and have positive Lebesgue measure. Define now

$$\mu^x := \frac{1}{\lambda(A_x)} \cdot \lambda_{|A_x}, \quad \mu^y := \frac{1}{\lambda(A_y)} \cdot \lambda_{|A_y}, \quad \nu^x := \frac{1}{\lambda(B_x)} \cdot \lambda_{|B_x}, \text{ and } \nu^y := \frac{1}{\lambda(B_y)} \cdot \lambda_{|B_y}.$$

With this, define $\mu := \mu^x \otimes \mu^y + \nu^x \otimes \nu^y - \mu^x \otimes \nu^y - \nu^x \otimes \mu^y$. Now $\pi_X(\mu) = \mu^x \cdot 1 + \nu^x \cdot 1 - \mu^x \cdot 1 - \nu^x \cdot 1 = 0$ and similarly $\pi_Y(\mu) = 0$, so that $\mu \in \ker \pi$. But

$$\mu(A) = \underbrace{\mu^x \otimes \mu^y(A)}_{=1} + \underbrace{\nu^x \otimes \nu^y(A)}_{=0} - \underbrace{\mu^x \otimes \nu^y(A)}_{=0} - \underbrace{\nu^x \otimes \mu^y(A)}_{=0} = 1,$$

which is a contradiction to the assumptions of the lemma. So for the following, we assume without loss of generality that $\lambda(A_x) = \lambda(X)$.
Now we wish to show that $\lambda(A) = \lambda(X \times A_y)$. Assume again that this is not the case, i. e. that $\lambda(A) < \lambda(X \times A_y)$. Then we define $C := (X \times A_y^\circ) \setminus A$ and see that it is open in $X \times Y$ with positive Lebesgue measure. Remembering the initial remark, we see that there are open sets $\tilde{C}_x \subset X$ and $\tilde{C}_y \subset A_y$ such that $\tilde{C}_x \times \tilde{C}_y \subset C$. As $\pi_Y^{-1}(\tilde{C}_y) \cap A^\circ$ is open, it contains a cartesian product $D_x \times C_y$ of open sets in X and C_y, respectively. Similarly we assert the existence of open sets $C_x \subset \tilde{C}_x$ and $D_y \subset A_y$ such that $C_x \times D_y \subset A$. Altogether, we have $C_x \times C_y \subset C$, $C_x \times D_y \subset A$ and $D_x \times C_y \subset A$. (About the location of $D_x \times D_y$ not more can be said than that it may be contained in A, in C, or may intersect both.) Now we proceed in a similar way as before. We define

$$\mu^x = \frac{1}{\lambda(C_x)} \cdot \lambda_{|C_x}, \quad \mu^y = \frac{1}{\lambda(C_y)} \cdot \lambda_{|C_y}, \quad \nu^x = \frac{1}{\lambda(D_x)} \cdot \lambda_{|D_x}, \text{ and } \nu^y = \frac{1}{\lambda(D_y)} \cdot \lambda_{|D_y},$$

and μ just as above. In the same way as above we can conclude that $\mu \in \ker \pi$, but

$$\mu(A) = \underbrace{\mu^x \otimes \mu^y(A)}_{=0} + \underbrace{\nu^x \otimes \nu^y(A)}_{\in[0,1]} - \underbrace{\mu^x \otimes \nu^y(A)}_{=1} - \underbrace{\nu^x \otimes \mu^y(A)}_{=1} = \nu^x \otimes \nu^y(A) - 2 < 0,$$

as $0 \leq \nu^x \otimes \nu^y(A) \leq 1$. So we have again found a contradiction, which leaves us to conclude that the statement is true. □

Proof of Theorem 2.42: Write $\tilde{P}$ as a block matrix

$$\tilde{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

Pick an arbitrary $\mu \in \ker \pi = (\ker \pi_X \cap \ker \pi_Y) \subset \mathcal{M}(X_1 \times X_2)$. Then $\pi \circ P_f = \tilde{P} \circ \pi$ implies $\pi_1 P_f \mu = P_{11}\pi_1\mu + P_{22}\pi_2\mu = 0$ and likewise $\pi_2 P_f \mu = 0$, i. e. $P_f\mu \in \ker \pi$. To show now that the X-component of f is of the form stated in the theorem, let $A \subset X$ be an arbitrary open set. Then the preimage $f_X^{-1}(\bar{A})$ of the closure $\bar{A}$ is closed, as f is continuous. Both sets have positive Lebesgue measure, as they contains open sets. Let $1_{\bar{A}\times Y}$ be the characteristic function of $\bar{A} \times Y$. Then $1_{\bar{A}\times Y} \in \mathcal{F}_X(X \times Y)$, and with Lemma 2.35 we can conclude that

$$0 = P_f\mu(\bar{A} \times Y) = \mu(f^{-1}(\bar{A} \times Y)) = \mu(f_X^{-1}(\bar{A})).$$

So we have established that whenever $\mu \in \ker \pi$, then $\mu(f_X^{-1}(\bar{A})) = 0$ for $A \subset X$ open. By Lemma 2.43 this means that the closed set $f_X^{-1}(\bar{A})$ is either of the form $\tilde{A} \times Y$ with $\tilde{A} \subset X$ or of the form $X \times \tilde{B}$ with $\tilde{B} \subset Y$. But this implies that f_X depends either only on X or only on Y, just as stated. To prove the statement about f_Y we can repeat this proof with the roles of X and Y interchanged. □

3 Transfer Operators for Coupled Cell Systems

In this chapter, transfer operators associated with admissible maps on a coupled cell system are considered in detail. As was described in Section 2.1, the fact that a mapping $f : X = \prod_{c\in\mathcal{C}} X_c \to X$ is admissible to a given coupled cell system has implications on the internal structure of the map in terms of (in-) dependencies of the component maps f_c on other components. The aim in this chapter will be to derive consequences of the structure of f for the structure of P_f. To this end, the following standing assumptions will be made throughout this chapter.

1. A fixed coupled cell system $(\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}})$ is given.
2. A choice of state spaces X_c has been made.
3. A consistent choice of spaces $\mathcal{M}(X_c)$ and $\mathcal{M}(X)$ has been made.
4. An admissible map $f : X \to X$ is given such that $P_f : \mathcal{M}(X) \to \mathcal{M}(X)$ is the transfer operator to be analysed.
5. A consistent choice of measures $\lambda_c \in \mathcal{M}(X_c)$ has been made, which defines right inverses $\pi_{\mathcal{D}}^{-1}$ as in Lemma 2.31.

3.1 Consequences of Independence – Block decompositions of P_f

In our effort to analyse the structure of the transfer operator P_f resulting from the structure of a coupled cell network on which the map f is admissible, we want to understand the operator as something that is build up from smaller parts. More precisely, we want to find a way to write P_f as a block matrix, preferably with blocks that are easier to describe (or to compute) than the whole operator itself. Writing a linear operator as a block matrix is tantamount to finding a direct sum decomposition of the underlying linear space, so our results about the transfer operator go hand in hand with suitable decompositions of $\mathcal{M}(X)$. In particular, the second part of this section will be devoted to the description of a "universal" decomposition of $\mathcal{M}(X)$ that will enable us to describe all structural features of the transfer operator that result from *independence* (as described in the Introduction), and many of the features that result from *symmetry*.

3.1.1 Coarse Block Decompositions of P_f

Our analysis of the structure of P_f starts with the following simple observation.

Lemma 3.1
For each $c \in \mathcal{C}$,

$$P_f\left(\ker \pi_{I(c)}\right) \subset \ker \pi_c.$$

Proof: Let $\mu \in \ker \pi_{I(c)}$. As for all $\phi \in \mathcal{F}_c(X)$ one has $\phi \circ f \in \mathcal{F}_{I(c)}(X)$ and therefore

$$\int_X \phi \, dP_f\mu = \int_X \phi \circ f \, d\mu = 0,$$

Lemma 2.35 implies that $P_f\mu \in \ker \pi_c$. □

Despite its simplicity, this observation is of some importance for the subsequent developments. First of all, note that by Corollary 2.36, $\ker \pi_c \subset \ker \pi_{I(c)}$. The fact that a subspace of $\mathcal{M}(X)$ is mapped into a space containing this subspace suggests to consider direct sum decompositions of $\mathcal{M}(X)$ into three subspaces, so that the operator P_f is represented by a block matrix structure.

Lemma 3.2
For $c \in \mathcal{C}$ define

1. $N_1(c) = \ker \pi_{I(c)}$,
2. $N_2(c) = \ker \pi_c \cap \operatorname{im} \pi_{I(c)}^{-1}$, *and*
3. $N_3(c) = \operatorname{im} \pi_c^{-1}$.

Then $\mathcal{M}(X) = N_1(c) \oplus N_2(c) \oplus N_3(c)$.

Proof:

1. To see that $N_1(c) + N_2(c) + N_3(c) = \mathcal{M}(X)$, observe that

   $$\mathcal{M}(X) = \ker \pi_c + \operatorname{im} \pi_c^{-1} = \ker \pi_c + N_3(c),$$

   as π_c^{-1} is an injective mapping. Therefore it suffices to show that $N_1(c) + N_2(c) = \ker \pi_c$. As $\ker \pi_{I(c)} \subset \ker \pi_c$, one has trivially that $N_1(c) + N_2(c) \subset \ker \pi_c$. To prove the inclusion $N_1(c) + N_2(c) \supset \ker \pi_c$, let $\mu \in \ker \pi_c$ and define $\mu_1 = \pi_{I(c)}^{-1} \circ \pi_{I(c)}(\mu)$ and $\mu_2 = \mu - \mu_1$. Then $\mu_1 \in N_2(c)$, as $\pi_{I(c)}^{-1} \circ \pi_{I(c)}$ is a projection onto $\operatorname{im} \pi_{I(c)}^{-1}$, and $\mu_2 \in N_1(c)$, as

   $$\begin{aligned} \pi_{I(c)}(\mu_2) = \pi_{I(c)}(\mu - \mu_1) &= \pi_{I(c)}(\mu) - \pi_{I(c)}(\pi_{I(c)}^{-1} \circ \pi_{I(c)}(\mu)) \\ &= \pi_{I(c)}(\mu) - \pi_{I(c)}(\mu) = 0. \end{aligned}$$

   Therefore $\mu = \mu_1 + \mu_2 \in N_1(c) + N_2(c)$.

2. To assert that $N_i(c) \cap N_j(c) = 0$ for $i, j \in \{1,2,3\}$ with $i \neq j$, consider first the cases $\{i,j\} = \{1,2\}$ and $\{i,j\} = \{2,3\}$, and remember that the images of the right inverses have to have zero intersection with their respective kernels, for otherwise they were no right inverses. For the case $\{i,j\} = \{1,3\}$ remember $N_1(c) \subset \ker \pi_c$ and apply the same argument. □

Corollary 3.3
With respect to the decomposition $\mathcal{M}(X) = N_1(c) \oplus N_2(c) \oplus N_3(c)$, P_f has the block matrix structure

$$P_f = \left(\begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ \hline 0 & * & * \end{array}\right).$$

Proof: It suffices to show that if $\mu \in N_1(c)$, then $P_f\mu \in N_1(c) \oplus N_2(c)$. By Lemma 3.1, $P_f\mu \in \ker \pi_c = N_1(c) \oplus N_2(c)$ which can be seen from the proof of Lemma 3.2. □

While the introduction of the direct sum decomposition in Lemma 3.2 was motivated from the statement of Lemma 3.1, it has relevance going beyond that relation. The subspaces $N_1(c), N_2(c)$ and $N_3(c)$ allow to write down embeddings of the measure spaces $\mathcal{M}(X_{I(c)})$ and $\mathcal{M}(X_c)$ into $\mathcal{M}(X)$. These spaces are domain and image space of the transfer operator of the component map $\hat{f}_c$. The next proposition presents consequences of this fact.

Proposition 3.4
Let $\mathcal{M}(X) = N_1(c) \oplus N_2(c) \oplus N_3(c)$ as in Corollary 3.3 be given, let $P_{\hat{f}_c} : \mathcal{M}(X_{I(c)}) \to \mathcal{M}(X_c)$ be the transfer operator for $\hat{f}_c$, and let $\widetilde{P_{\hat{f}_c}} : N_2(c) \oplus N_3(c) \to N_3(c)$ be given by

$$\widetilde{P_{\hat{f}_c}} = \pi_c^{-1} \circ P_{\hat{f}_c} \circ \pi_{I(c)}.$$

With respect to this decomposition, the transfer operator P_f has the block matrix structure

$$P_f = \left(\begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ \hline 0 & \multicolumn{2}{c}{\widetilde{P_{\hat{f}_c}}} \end{array}\right).$$

Proof: From $\pi_c \circ f = f_c = \hat{f}_c \circ \pi_{I(c)}$ one obtains $\pi_c \circ P_f = P_{\hat{f}_c} \circ \pi_{I(c)}$. For $\widetilde{P_{\hat{f}_c}}$ as above this implies

$$\begin{aligned} \widetilde{P_{\hat{f}_c}} &= \pi_c^{-1} \circ P_{\hat{f}_c} \circ \pi_{I(c)} \\ &= (\pi_c^{-1} \circ \pi_c) \circ P_f. \end{aligned}$$

The left factor of this factorization is just the projection upon the third summand $N_3(c)$ of the direct sum decomposition of $\mathcal{M}(X)$. □

On the one hand, this proposition can be deemed quite satisfactory. As far as the component maps $\hat{f}_c$ are considered as building blocks of the coupled cell system f, it is natural to find the corresponding transfer operators $P_{\hat{f}_c}$ as building blocks of P_f, and no further refinement can be expected in this direction.
On the other hand, the situation as it stands now is quite unsatisfactory. Each $P_{\hat{f}_c}$ can be 'seen' only with respect to a direct sum decomposition specific to a particular cell c. Not only is this more than unfortunate for practical purposes, but also from a structural (mathematical) point of view the question is immediate whether these different pieces of information on P_f can be combined into one view that shows everything 'at once'. In the following section we will see that this is indeed possible.

3.1.2 A Finer Block Decomposition of P_f

In the previous section, for every cell $c \in \mathcal{C}$ the vector space decomposition $\mathcal{M}(X) = N_1(c) \oplus N_2(c) \oplus N_3(c)$ was introduced as a decomposition with respect to which the transfer operator corresponding to the map f_c can be identified within P_f. To combine the pieces of information obtained from each single decomposition, it is desirable to have a *joint refinement* of these decompositions. In this section, it is shown that for the cases of interest in this thesis, joint refinements always do exist, and how they can be constructed.

Definition 3.5
Let V be a vector space, let $\mathcal{C}$ be a finite set, let $k \in \mathbb{N}$, and for each $c \in \mathcal{C}$, let

$$V = N_1(c) \oplus \ldots \oplus N_k(c)$$

be a given direct sum decomposition of V. A decomposition

$$V = \bigoplus_{j=1}^{n} U_j$$

is called a joint refinement *of the given decompositions if for each $c \in \mathcal{C}$ and for each $i \in \{1, \ldots, k\}$ there is $J(c,i) \subset \{1, \ldots, n\}$ such that*

$$N_i(c) = \bigoplus_{j \in J(c,i)} U_j.$$

Example 3.6
For general vector space decompositions, joint refinements need not exist even in the simplest cases. For the standard basis $e_1, \ldots, e_5$ of $\mathbb{R}^5$ consider the two decompositions

$$\begin{aligned}\mathbb{R}^5 &= < e_1, e_2 > \oplus < e_3, e_4 > \oplus < e_5 > \\ &= < e_1 + e_3, e_2 + e_4 > \oplus < e_1 - e_3, e_2 - e_4 > \oplus < e_5 > .\end{aligned}$$

There is no joint refinement, although the dimensions of the first two subspaces would allow for a further refinement of the decompositions.

In the following we will prove that for the decompositions of $\mathcal{M}(X)$ obtained for a coupled cell system, a joint refinement always exists. In our development, we are going to deal with subsets $\mathcal{D} \subset \mathcal{C}$. To simplify the presentation and avoid a lot of special considerations for the case $\mathcal{D} = \emptyset$, we want to introduce the following

Notation: We define

$$\operatorname{im} \pi_\emptyset^{-1} := \bigcap_{\emptyset \neq \mathcal{D} \subset \mathcal{C}} \operatorname{im} \pi_\mathcal{D}^{-1} =< \otimes_{c \in \mathcal{C}} \lambda_c >$$

and

$$\ker \pi_\emptyset := \sum_{\emptyset \neq \mathcal{D} \subset \mathcal{C}} \ker \pi_\mathcal{D}.$$

(Note that in the latter definition, the sign $\sum$ signifies normal vector space addition and *not* the direct sum.) With these definitions, we again have that $\mathcal{M}(X) = \ker \pi_\emptyset \oplus \operatorname{im} \pi_\emptyset^{-1}$, as $\operatorname{im} \pi_\emptyset^{-1}$ has trivial intersection with every $\ker \pi_\mathcal{D}$. With the following lemma, we make the first step towards the proof of the existence of a joint refinement of the decompositions of $\mathcal{M}(X)$.

Lemma 3.7
Let $\mathcal{D}_1, \ldots, \mathcal{D}_k \subset \mathcal{C}$ be given, and let the maps

$$q_i = \pi_{\mathcal{D}_i}^{-1} \circ \pi_{\mathcal{D}_i} : \mathcal{M}(X) \to \mathcal{M}(X) \text{ and } p_i = \mathrm{Id} - q_i$$

be the projections onto $\operatorname{im} \pi_{\mathcal{D}_i}^{-1}$ and $\ker \pi_{\mathcal{D}_i}$ respectively. Then arbitrary compositions of these projections are also projections.

Proof: At the heart of the proof lies the fact that all these projections commute. Once this is established, it is easily shown that arbitrary compositions are projections. The commutativity is shown by means of the following computation, for which it suffices to consider the q_i only, because $p_i = \mathrm{Id} - q_i$.
Let $\mu \in \mathcal{M}(X)$. To show that the measures $(q_i \circ q_j)(\mu)$ and $(q_j \circ q_i)(\mu)$ are equal it suffices to show they are equal on sets of the form $A = \prod_{c \in \mathcal{C}} A_c$, with $A_c \subset X_c$ measurable. First, observe that

$$q_i \mu = (\pi_{\mathcal{D}_i}^{-1} \circ \pi_{\mathcal{D}_i})\mu = \lambda_{\mathcal{C} \setminus \mathcal{D}_i} \otimes \pi_{\mathcal{D}_i} \mu \tag{3.1}$$

and therefore

$$\begin{aligned}
(q_i \circ q_j)(\mu)(A) &= \lambda_{\mathcal{C} \setminus \mathcal{D}_i} \otimes \pi_{\mathcal{D}_i}(q_j \mu)(A) \\
&= \lambda_{\mathcal{C} \setminus \mathcal{D}_i} \otimes \pi_{\mathcal{D}_i}(\lambda_{\mathcal{C} \setminus \mathcal{D}_j} \otimes \pi_{\mathcal{D}_j} \mu)(A) \\
&= \lambda_{\mathcal{C} \setminus \mathcal{D}_i} \left(\prod_{c \notin \mathcal{D}_i} A_c \right) \cdot \underbrace{\pi_{\mathcal{D}_i}(\lambda_{\mathcal{C} \setminus \mathcal{D}_j} \otimes \pi_{\mathcal{D}_j} \mu) \left(\prod_{c \in \mathcal{D}_i} A_c \right)}_{(*)}
\end{aligned}$$

We take a close look at the expression $(*)$ and see that

$$\begin{aligned}
(*) &= \lambda_{\mathcal{C}\setminus\mathcal{D}_j} \otimes \pi_{\mathcal{D}_j}\mu \left(\prod_{c\in\mathcal{D}_i} A_c \times \prod_{c\notin\mathcal{D}_i} X_c \right) \\
&= \lambda_{\mathcal{C}\setminus\mathcal{D}_j} \left(\prod_{c\in\mathcal{D}_i\setminus\mathcal{D}_j} A_c \times \prod_{c\notin(\mathcal{D}_i\cup\mathcal{D}_j)} X_c \right) \cdot \pi_{\mathcal{D}_j}\mu \left(\prod_{c\in\mathcal{D}_i\cap\mathcal{D}_j} A_c \times \prod_{c\in\mathcal{D}_j\setminus\mathcal{D}_i} X_c \right) \\
&= \lambda_{\mathcal{C}\setminus\mathcal{D}_j} \left(\prod_{c\in\mathcal{D}_i\setminus\mathcal{D}_j} A_c \times \prod_{c\notin(\mathcal{D}_i\cup\mathcal{D}_j)} X_c \right) \cdot \mu \left(\prod_{c\in\mathcal{D}_i\cap\mathcal{D}_j} A_c \times \prod_{c\in\mathcal{D}_j\setminus\mathcal{D}_i} X_c \times \prod_{c\notin\mathcal{D}_j} X_c \right) \\
&= \lambda_{\mathcal{C}\setminus\mathcal{D}_j} \left(\prod_{c\in\mathcal{D}_i\setminus\mathcal{D}_j} A_c \times \prod_{c\notin(\mathcal{D}_i\cup\mathcal{D}_j)} X_c \right) \cdot \mu \left(\prod_{c\in(\mathcal{D}_i\cap\mathcal{D}_j)} A_c \times \prod_{c\notin(\mathcal{D}_i\cap\mathcal{D}_j)} X_c \right).
\end{aligned}$$

Altogether, this gives us

$$\begin{aligned}
(q_i \circ q_j)(\mu)(A) &= \prod_{c\notin\mathcal{D}_i} \lambda_c(A_c) \cdot \prod_{c\in\mathcal{D}_i\setminus\mathcal{D}_j} \lambda_c(A_c) \cdot \prod_{c\notin(\mathcal{D}_i\cup\mathcal{D}_j)} \lambda_c(X_c) \cdot \mu \left(\prod_{c\in(\mathcal{D}_i\cap\mathcal{D}_j)} A_c \times \prod_{c\notin(\mathcal{D}_i\cap\mathcal{D}_j)} X_c \right) \\
&= \prod_{c\notin(\mathcal{D}_i\cap\mathcal{D}_j)} \lambda_c(A_c) \cdot \prod_{c\notin(\mathcal{D}_i\cup\mathcal{D}_j)} \lambda_c(X_c) \cdot \mu \left(\prod_{c\in(\mathcal{D}_i\cap\mathcal{D}_j)} A_c \times \prod_{c\notin(\mathcal{D}_i\cap\mathcal{D}_j)} X_c \right).
\end{aligned}$$

A close inspection of the last line reveals that it remains invariant under exchange of i and j, which shows that $(q_i \circ q_j) = (q_j \circ q_i)$.
We now show that arbitrary compositions of these projections are also projections. The proof is performed by induction on the number j of maps in the composition. For $j = 2$, from the above argument we have directly

$$\begin{aligned}
(q_{i_1} \circ q_{i_2})^2 &= (q_{i_1} \circ q_{i_2}) \circ (q_{i_1} \circ q_{i_2}) \\
&= (q_{i_1} \circ q_{i_2}) \circ (q_{i_2} \circ q_{i_1}) \\
&= q_{i_1} \circ q_{i_2}^2 \circ q_{i_1} \\
&= q_{i_1} \circ q_{i_2} \circ q_{i_1} \\
&= q_{i_1} \circ q_{i_1} \circ q_{i_2} = q_{i_1} \circ q_{i_2}.
\end{aligned}$$

Now if a composition of any $j-1$ of the projections is a projection, then we have for compositions of j projections:

$$\begin{aligned}
(q_{i_1} \circ \ldots \circ q_{i_j})^2 &= (q_{i_1} \circ \ldots \circ q_{i_j}) \circ (q_{i_j} \circ q_{i_1} \circ q_{i_{j-1}}) \\
&= (q_{i_1} \circ \ldots \circ q_{i_{j-1}}) \circ q_{i_j}^2 \circ (q_{i_1} \circ \ldots \circ q_{i_{j-1}}) \\
&= (q_{i_1} \circ \ldots \circ q_{i_{j-1}}) \circ q_{i_j} \circ (q_{i_1} \circ \ldots \circ q_{i_{j-1}}) \\
&= (q_{i_1} \circ \ldots \circ q_{i_{j-1}})^2 \circ q_{i_j} \\
&= q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j},
\end{aligned}$$

so that the claim is proved. □

Remark 3.8
For the following considerations it should be noted that the range of the composition of commuting projections is just the intersection of the ranges of the projections: Let q_1, q_2 be projections as above, let $Q_1 = \operatorname{im} q_1$ and $Q_2 = \operatorname{im} q_2$, then $q_1 \circ q_2 = q_2 \circ q_1$ implies $\operatorname{im}(q_1 \circ q_2) \subset Q_1 \cap Q_2$. On the other hand, one has $q_1 \circ q_2(\operatorname{im} q_1 \cap \operatorname{im} q_2) = q_1(\operatorname{im} q_1 \cap \operatorname{im} q_2) = \operatorname{im} q_1 \cap \operatorname{im} q_2$, which implies $Q_1 \cap Q_2 \subset \operatorname{im}(q_1 \circ q_2)$.

We are now in the position to prove the existence of a joint refinement of *all* decompositions of the form $\mathcal{M}(X) = \ker \pi_{\mathcal{D}} \oplus \operatorname{im} \pi_{\mathcal{D}}^{-1}$ for $\mathcal{D} \subset \mathcal{C}$.

Theorem 3.9
Let $\mathcal{P} = \mathcal{P}(\mathcal{C}) = \{D_1, \ldots, D_{2^N}\}$ be the power set of the set $\mathcal{C}$ of cells of a coupled cell system. For each $\mathcal{D} \in \mathcal{P}$ define

$$\begin{aligned} p_{1,\mathcal{D}} &= \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}, & N_1(\mathcal{D}) &= \operatorname{im} p_{1,\mathcal{D}} = \operatorname{im} \pi_{\mathcal{D}}^{-1}, \\ & \text{and} & & \\ p_{2,\mathcal{D}} &= \mathrm{Id} - p_{1,\mathcal{D}}, & N_2(\mathcal{D}) &= \operatorname{im} p_{2,\mathcal{D}} = \ker \pi_{\mathcal{D}}, \end{aligned}$$

and finally for every $\vartheta : \mathcal{P} \to \{1, 2\}$ define

$$p_\vartheta = p_{\vartheta(\mathcal{D}_1),\mathcal{D}_1} \circ \cdots \circ p_{\vartheta(\mathcal{D}_{2^N}),\mathcal{D}_{2^N}}. \tag{3.2}$$

A joint refinement of all decompositions $\mathcal{M}(X) = \ker \pi_{\mathcal{D}} \oplus \operatorname{im} \pi_{\mathcal{D}}^{-1}$, $\mathcal{D} \in \mathcal{P}$ exists and is given by the non-trivial elements of the collection of spaces $\operatorname{im} p_\vartheta, \vartheta \in \{1, 2\}^{\mathcal{P}}$, i. e. by the spaces

$$\operatorname{im} p_\vartheta = \bigcap_{\mathcal{D} \in \mathcal{P}} N_{\vartheta(\mathcal{D})}(\mathcal{D}).$$

Before we come to the proof of the theorem, we insert a quick note on our terminology. While the formulation "joint refinement of the decompositions $\mathcal{M}(X) = \ker \pi_{\mathcal{D}} \oplus \operatorname{im} \pi_{\mathcal{D}}^{-1}$ " is technically correct, it is also terribly longwinded. To simplify the language, we will in the following often speak of the "joint decomposition" of $\mathcal{M}(X)$.
Proof: For the proof of the statement, two things need to be shown.

1. The intersection of any two different subspaces is trivial. To see this, let $\vartheta_1, \vartheta_2 \in \{1, 2\}^{\mathcal{P}}$ with $\vartheta_1 \neq \vartheta_2$ be given. Then there is $\mathcal{D} \in \mathcal{P}$ with $\vartheta_1(\mathcal{D}) \neq \vartheta_2(\mathcal{D})$. Without loss of generality we assume $\vartheta_1(\mathcal{D}) = 1$ and $\vartheta_2(\mathcal{D}) = 2$. By Lemma 3.7, one then has $\operatorname{im} p_{\vartheta_1} \subset N_1(\mathcal{D})$ and $\operatorname{im} p_{\vartheta_2} \subset N_2(\mathcal{D})$ (see also Remark 3.8). This implies $\operatorname{im} p_{\vartheta_1} \cap \operatorname{im} p_{\vartheta_2} \subset N_1(\mathcal{D}) \cap N_2(\mathcal{D}) = 0$.

2. All subspaces of the form $\ker \pi_{\mathcal{D}}$ and $\operatorname{im} \pi_{\mathcal{D}}^{-1}$ can be written as sums of some of the subspaces $\operatorname{im} p_\vartheta$. To see this for $\operatorname{im} \pi_{\mathcal{D}}^{-1}$, define $I = \{\vartheta \in \{1,2\}^{\mathcal{P}} \mid \vartheta(\mathcal{D}) = 1\}$. We will then show that

$$\sum_{\vartheta \in I} p_\vartheta = \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}},$$

which implies the statement. To see this, we remember from the proof of Lemma 3.7 that the factors in p_ϑ commute, and extract the common factor $p_{1,\mathcal{D}}$ out of the sum, so that

$$\sum_{\vartheta \in I} p_\vartheta = p_{1,\mathcal{D}} \circ \sum_{\vartheta \in I} \tilde{p}_\vartheta,$$

where $\tilde{p}_\vartheta$ is a composition of projections defined just as p_ϑ, execpt that the factor $p_{1,\mathcal{D}}$ is left out. As for each $\mathcal{D} \neq \mathcal{D}' \in \mathcal{P}$, there is an element $\vartheta \in I$ with $\vartheta(\mathcal{D}') = 1$ and also an element $\vartheta' \in I$ with $\vartheta'(\mathcal{D}') = 2$, the sum can be factored out to give

$$\begin{aligned} \sum_{\vartheta \in I} \tilde{p}_\vartheta &= (p_{1,\mathcal{D}_1} + p_{2,\mathcal{D}_1}) \circ \ldots \circ (p_{1,\mathcal{D}_{2^N}} + p_{2,\mathcal{D}_{2^N}}) \\ &= \mathrm{Id}, \end{aligned}$$

where the factor corresponding to $\mathcal{D}$ is left out in the upper line. Remembering the definition of $p_{1,\mathcal{D}}$, we see that the proof is finished. To prove that $\ker \pi_{\mathcal{D}}$ can be constructed from the subspaces $\operatorname{im} p_\vartheta$, one proceeds analogously, collecting in I those ϑ with $\vartheta(\mathcal{D}) = 2$.

□

Remark 3.10
A particular problem for the description of the joint decomposition is the question how one denotes the individual elements of the decomposition. The description of the joint decomposition that is used in the preceding theorem is quite ineffective in the sense that a large number of the mappings p_ϑ are in fact zero mappings. For example, if for some $\vartheta \in \{1,2\}^{\mathcal{P}}$ there are three distinct subsets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \in \mathcal{P}$ with $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ such that $\vartheta(\mathcal{D}_1) = 1 = \vartheta(\mathcal{D}_2)$ and $\vartheta(\mathcal{D}_3) = 2$, then $\operatorname{im} p_\vartheta = 0$. To see this, note that $\operatorname{im} p_\vartheta \subset \operatorname{im} \pi_{\mathcal{D}_1}^{-1} \cap \operatorname{im} \pi_{\mathcal{D}_2}^{-1} \cap \ker_{\mathcal{D}_3}$. However, as $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, one easily sees that $\operatorname{im} \pi_{\mathcal{D}_1}^{-1} \cap \operatorname{im} \pi_{\mathcal{D}_2}^{-1}$ is the one-dimensional subspace spanned by the measure $\otimes_{c \in \mathcal{C}} \lambda_c$. This measure is not contained in the kernel of any of the maps $\pi_{\mathcal{D}}$, so that the intersection of its span with $\ker \pi_{\mathcal{D}_3}$ is the trivial space.

In the following we propose a description of the elements of the joint decomposition that appears to be considerably more lucid than the one given by the projections p_ϑ which we just used.

The idea behind it is quite simple to describe. We make use of the facts that on the one hand *every* subspace $\operatorname{im}\pi_{\mathcal{D}}^{-1}$ can be written as the direct sum of some elements of the joint decomposition and on the other hand for $\mathcal{D}' \subset \mathcal{D}$ we also have that $\operatorname{im}\pi_{\mathcal{D}'}^{-1}$ is a subspace of $\operatorname{im}\pi_{\mathcal{D}}^{-1}$. This allows to assign to every subset $\mathcal{D} \subset \mathcal{C}$ the sum of those elements of the joint decomposition that are contained in $\operatorname{im}\pi_{\mathcal{D}}^{-1}$, but not contained in any of the subspaces $\operatorname{im}\pi_{\mathcal{D}'}^{-1}$ for proper subsets $\mathcal{D}' \subsetneq \mathcal{D}$. With the proposition following the formal definition of this construction, we describe the relation of these subspaces to the elements of the joint decomposition.

Definition 3.11
For all sets $\mathcal{D} \in \mathcal{P}(\mathcal{C})$ we define a subspace $U_{\mathcal{D}} \subset \mathcal{M}(X)$ as follows. For $\mathcal{D} = \emptyset$ we set $U_\emptyset =< \otimes_{c\in\mathcal{C}}\lambda_c >$, and for $\emptyset \neq \mathcal{D} \in \mathcal{P}(\mathcal{C})$ we define $U_{\mathcal{D}}$ to be the sum of those elements of the joint decomposition of $\mathcal{M}(X)$ that are contained in $\operatorname{im}\pi_{\mathcal{D}}^{-1}$, but not in $\operatorname{im}\pi_{\mathcal{D}'}^{-1}$ for any $\mathcal{D}' \subsetneq \mathcal{D}$.

Proposition 3.12
For given $\emptyset \neq \mathcal{D} \subset \mathcal{C}$ let $\vartheta_{\mathcal{D}} : \mathcal{P} \to \{1, 2\}$ be defined through

$$\vartheta_{\mathcal{D}}(\mathcal{D}') = \begin{cases} 1 & \text{for } \mathcal{D} \subset \mathcal{D}' \\ 2 & \text{else} \end{cases}$$

Let $p_{\vartheta_{\mathcal{D}}}$ be defined as in Theorem 3.9. Then $U_{\mathcal{D}} = \operatorname{im} p_{\vartheta_{\mathcal{D}}}$.

Proof: We perform the proof in three steps. First we will show that $\operatorname{im} p_{\vartheta_{\mathcal{D}}}$ has a trivial intersection with $\operatorname{im}\pi_{\mathcal{D}'}^{-1}$ for any $\mathcal{D}' \subsetneq \mathcal{D}$. After that, we are going to show that $\operatorname{im} p_{\vartheta_{\mathcal{D}}}$ has the form

$$\operatorname{im} p_{\vartheta_{\mathcal{D}}} = \operatorname{im}\pi_{\mathcal{D}}^{-1} \cap \bigcap_{c\in\mathcal{D}} \ker \pi_{\mathcal{D}\setminus\{c\}}. \tag{3.3}$$

Using this description, we then complete the proof by showing that $\operatorname{im} p_{\vartheta_{\mathcal{D}}} + \sum_{\mathcal{D}'\subsetneq\mathcal{D}} \operatorname{im}\pi_{\mathcal{D}'}^{-1} = \operatorname{im}\pi_{\mathcal{D}}^{-1}$.

1. Let $\mathcal{D}' \subsetneq \mathcal{D}$. Then $\vartheta(\mathcal{D}') = 2$, and thus $\operatorname{im} p_{\vartheta_{\mathcal{D}}} \subset \ker \pi_{\mathcal{D}'}$, which implies $\operatorname{im} p_{\vartheta_{\mathcal{D}}} \cap \operatorname{im}\pi_{\mathcal{D}'}^{-1} = 0$.

2. The key for the proof of the second part is the commutativity of the projections that was established in Lemma 3.7, together with Remark 3.8 (in which we showed that the range of the composition of these projections is the intersection of their ranges).

   Using the commutativity, we can write $p_{\vartheta_{\mathcal{D}}}$ as the composition of three maps, whose effects we can analyse separately. To do so, we divide the subsets of $\mathcal{C}$ into the three groups

   $$\begin{aligned} \mathcal{P}_0 &= \{\mathcal{D}' \mid \mathcal{D} \subseteq \mathcal{D}'\}, \\ \mathcal{P}_1 &= \{\mathcal{D}' \mid \mathcal{D}' \subsetneq \mathcal{D}\}, \text{ and} \\ \mathcal{P}_2 &= \{\mathcal{D}' \mid \mathcal{D} \not\subset \mathcal{D}' \ \wedge\ \mathcal{D}' \not\subset \mathcal{D}\}, \end{aligned}$$

and for $i \in \{0,1,2\}$ define p_i as the composition of those terms from equation (3.2) belonging to sets in $\mathcal{P}_i$. Then $p_{\vartheta_\mathcal{D}} = p_0 \circ p_1 \circ p_2$, and $\operatorname{im} p_{\vartheta_\mathcal{D}} = \operatorname{im} p_0 \cap \operatorname{im} p_1 \cap \operatorname{im} p_2$. (Note that p_0 is the composition of projections onto the ranges of right inverses $\pi_{\mathcal{D}'}^{-1}$, while p_1 and p_2 are projections onto the kernels of the maps $\pi_{\mathcal{D}'}$.) From $\operatorname{im} \pi_\mathcal{D}^{-1} \subset \operatorname{im} \pi_{\mathcal{D}'}^{-1}$ for every $\mathcal{D}' \in \mathcal{P}_0$ we see that

$$\operatorname{im} p_0 = \operatorname{im} \pi_\mathcal{D}^{-1}.$$

As for every $\mathcal{D}' \in \mathcal{P}_1$ there is a $c \in \mathcal{D}$ such that $\mathcal{D}' \subset \mathcal{D} \setminus \{c\}$, we have $\ker \pi_{\mathcal{D}\setminus\{c\}} \subset \ker \pi_{\mathcal{D}'}$ for every $\mathcal{D}' \in \mathcal{P}_1$ and the corresponding c. Combining these relations for all $c \in \mathcal{C}$, we obtain

$$\operatorname{im} p_1 = \bigcap_{c\in\mathcal{D}} \ker \pi_{\mathcal{D}\setminus\{c\}}.$$

We now want to prove that $p_2(\operatorname{im}(p_0 \circ p_1)) = \operatorname{im}(p_0 \circ p_1)$, that is, that an application of p_2 remains without effect on the intersection of the ranges of p_0 and p_1. To see that this is the case, let $\mathcal{D}' \in \mathcal{P}_2$, and observe that

$$\begin{aligned} p_0 \circ p_1 &= ((\mathrm{Id} - \pi_{\mathcal{D}'}^{-1}\pi_{\mathcal{D}'}) + \pi_{\mathcal{D}'}^{-1}\pi_{\mathcal{D}'}) \circ (p_0 \circ p_1) \\ &= (\mathrm{Id} - \pi_{\mathcal{D}'}^{-1}\pi_{\mathcal{D}'}) \circ (p_0 \circ p_1) + \pi_{\mathcal{D}'}^{-1}\pi_{\mathcal{D}'} \circ (p_0 \circ p_1). \end{aligned}$$

Note that the first summand is of the form occuring in the composition $p_2 \circ p_0 \circ p_1$. Accordingly, we have to show that the second summand is zero. To see that this is the case, assume first that $\emptyset \neq \mathcal{D} \cap \mathcal{D}'$ and observe that

$$\begin{aligned} \operatorname{im}\left(\pi_{\mathcal{D}'}^{-1}\pi_{\mathcal{D}'} \circ p_0 \circ p_1\right) &= \operatorname{im} \pi_{\mathcal{D}'}^{-1} \cap \operatorname{im} p_0 \cap \operatorname{im} p_1 \\ &= \underbrace{\operatorname{im} \pi_{\mathcal{D}'}^{-1} \cap \operatorname{im} \pi_\mathcal{D}^{-1}}_{=\operatorname{im} \pi_{\mathcal{D}\cap\mathcal{D}'}^{-1}} \cap \bigcap_{c\in\mathcal{D}} \ker \pi_{\mathcal{D}\setminus\{c\}} \\ &= 0, \end{aligned}$$

as $\mathcal{D} \cap \mathcal{D}' \subset \mathcal{D} \setminus \{c\}$ for a c. In case that $\emptyset = \mathcal{D} \cap \mathcal{D}'$, the same argument holds, with $\operatorname{im} \pi_{\mathcal{D}\cap\mathcal{D}'}^{-1}$ replaced by $U_\emptyset$.

3. For the next step, we first note that whenever P is an idempotent linear endomorphism on $\mathcal{M}(X)$ with $\operatorname{im} P \subset \operatorname{im} \pi_\mathcal{D}^{-1}$, we have

$$(\operatorname{im} \pi_\mathcal{D}^{-1} \cap \ker P) + \operatorname{im} P = \operatorname{im} \pi_\mathcal{D}^{-1}. \tag{3.4}$$

To see this, note that the inclusion "$\subset$" is trivially true, and that the inclusion "$\supset$" follows from the fact that any $\mu \in \operatorname{im} \pi_\mathcal{D}^{-1}$ can be written as $P\mu + (\mu - P\mu)$, with the first part from $\operatorname{im} P$ and the second part from $\operatorname{im} \pi_\mathcal{D}^{-1} \cap \ker P$. Noting that $\operatorname{im} \pi_{\mathcal{D}\setminus\{c\}} \subset \operatorname{im} \pi_\mathcal{D}$ for every $c \in \mathcal{D}$, we see that

we can apply equation (3.4) to equation (3.3) with $P = \pi^{-1}_{\mathcal{D}\setminus\{c\}} \circ \pi_{\mathcal{D}\setminus\{c\}}$ for every $c \in \mathcal{D}$ to obtain

$$\operatorname{im} p_{\vartheta_{\mathcal{D}}} + \sum_{c\in\mathcal{D}} \operatorname{im} \pi^{-1}_{\mathcal{D}\setminus\{c\}} = \operatorname{im} \pi^{-1}_{\mathcal{D}},$$

which implies the statement.

□

Corollary 3.13
The joint decomposition of $\mathcal{M}(X)$ constructed in Theorem 3.9 consists precisely of the sets $U_{\mathcal{D}}$ from Definition 3.11.

Proof: By definition, $\sum_{\mathcal{D}\in\mathcal{P}(\mathcal{C})} U_{\mathcal{D}} = \mathcal{M}(X)$, so every subspace in the joint decomposition is contained in $\{U_{\mathcal{D}} \mid \mathcal{D} \in \mathcal{P}(\mathcal{C})\}$. On the other hand, by Proposition 3.12 every $U_{\mathcal{D}}$ consists of precisely one element of the joint decomposition. □

Example 3.14
In the finite-dimensional setting (see Example 2.26), a basis for $\mathcal{M}(X)$ can be obtained by forming all the tensor products of the basis elements of the individual $\mathcal{M}(X_c)$. If we consider a basis of the form $\{\mu^{(c)}_j | j = 1, \ldots, n_c\}$ for each $\mathcal{M}(X_c)$ with $\mu^{(c)}_1 = \lambda_c$ and $\mu^{(c)}_j(X_c) = 0$ for $j = 2, \ldots, n_c$, then $U_{\mathcal{D}}$ is spanned by the basis vectors of the form $\otimes_{c\in\mathcal{C}} \mu^{(c)}_{j_c}$ for which $j_c = 1$ if and only if $c \notin \mathcal{D}$.
To see this, we use the characterization of $U_{\mathcal{D}}$ given by equation (3.3). On the one hand, a measure of the form $\otimes_{c\in\mathcal{C}} \mu^{(c)}_{j_c}$ is in $\operatorname{im} \pi^{-1}_{\mathcal{D}}$ if and only if $j_c = 1$ for $c \notin \mathcal{D}$. On the other hand, we have

$$\pi_{\mathcal{D}\setminus\{c\}}(\otimes_{c'\in\mathcal{C}} \mu^{(c')}_{j_{c'}}) = \prod_{c'\notin\mathcal{D}} \lambda_{c'}(X_{c'}) \cdot \mu^{(c)}_{j_c}(X_c) \cdot \bigotimes_{c\neq c'\in\mathcal{D}} \mu^{(c')}_{j_{c'}},$$

and therefore $\otimes_{c\in\mathcal{C}} \mu^{(c)}_{j_c} \in \ker \pi_{\mathcal{D}\setminus\{c\}}$ if and only if $\mu^{(c)}_{j_c}(X_c) = 0$ for $c \in \mathcal{D}$, that is, if $j_c \neq 1$ for $c \in \mathcal{D}$.

The motivation for the construction of the joint decomposition came from Proposition 3.4, which gave us a cell-dependent decomposition of P_f. Accordingly, the first result we get using the joint decomposition is a "simultaneous" version of that proposition, combining the decompositions of P_f not only for all cells, but even for all (non-empty) subsets $\mathcal{D} \subset \mathcal{C}$. For its statement we need a little bit of further notation as follows. We know that the restrictions of the maps $\pi_{\mathcal{D}}$ to the images of their right inverses are vector space isomorphisms. As for two subsets $\mathcal{D}' \subset \mathcal{D} \subset \mathcal{C}$ we have $U_{\mathcal{D}'} \subset \operatorname{im} \pi^{-1}_{\mathcal{D}}$, we see that $\pi_{\mathcal{D}}$ maps $U_{\mathcal{D}'}$ isomorphically

onto its image $\pi_{\mathcal{D}}(U_{\mathcal{D}'}) =: U^{\mathcal{D}}_{\mathcal{D}'} \subset \mathcal{M}(X_{\mathcal{D}})$. In this way we obtain a decomposition of $\mathcal{M}(X_{\mathcal{D}})$ that is completely analogous to the joint decomposition of $\mathcal{M}(X)$. It is easy to check that a projection in $\mathcal{M}(X_{\mathcal{D}})$ onto $U^{\mathcal{D}}_{\mathcal{D}'}$ is given by the map $\pi_{\mathcal{D}} \circ p_{\vartheta_{\mathcal{D}'}} \circ \pi_{\mathcal{D}}^{-1}$.

Theorem 3.15
For $\emptyset \neq \mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{C}$ let $P_{\mathcal{D}_1,\mathcal{D}_2} : U_{\mathcal{D}_2} \to U_{\mathcal{D}_1}$ be an entry of the block matrix decomposition of P_f with respect to the joint decomposition of $\mathcal{M}(X)$. If $\mathcal{D}_2 \not\subset I(\mathcal{D}_1)$, then $P_{\mathcal{D}_1,\mathcal{D}_2} = 0$. If $\mathcal{D}_2 \subset I(\mathcal{D}_1)$, then $P_{\mathcal{D}_1,\mathcal{D}_2}$ can be identified with a block of the transfer operator $P_{\hat{f}_{\mathcal{D}_1}} : \mathcal{M}(X_{I(\mathcal{D}_1)}) \to \mathcal{M}(X_{\mathcal{D}_1})$ with respect to the decompositions $\mathcal{M}(X_{\mathcal{D}_1}) = \oplus_{\mathcal{D}' \subset \mathcal{D}_1} U^{\mathcal{D}_1}_{\mathcal{D}'}$ and $\mathcal{M}(X_{I(\mathcal{D}_1)}) = \oplus_{\mathcal{D}' \subset I(\mathcal{D}_1)} U^{I(\mathcal{D}_1)}_{\mathcal{D}'}$.

Proof: The proof can be performed analogously to those of Lemma 3.1 and of Proposition 3.4. More concretely, in the same way as in Lemma 3.1, we see that

$$P_f(\ker \pi_{I(\mathcal{D}_1)}) \subseteq \ker \pi_{\mathcal{D}_1}. \tag{3.5}$$

For the rest, we can conveniently use the machinery of projections in $\mathcal{M}(X)$ we have developed. The block $P_{\mathcal{D}_1,\mathcal{D}_2}$ can be expressed through the mapping $p_{\vartheta_{\mathcal{D}_1}} \circ P_f \circ p_{\vartheta_{\mathcal{D}_2}}$. If $\mathcal{D}_2 \not\subset I(\mathcal{D}_1)$, we have that $\vartheta_{\mathcal{D}_2}(I(\mathcal{D}_1)) = 2$, and thus $U_{\mathcal{D}_2} \subset \ker \pi_{I(\mathcal{D}_1)}$. But then we have

$$\begin{aligned}
\operatorname{im}(p_{\vartheta_{\mathcal{D}_1}} \circ P_f \circ p_{\vartheta_{\mathcal{D}_2}}) &\subseteq \operatorname{im}(\pi_{\mathcal{D}_1}^{-1}\pi_{\mathcal{D}_1} \circ P_f \circ p_{\vartheta_{\mathcal{D}_2}}) \\
&\subseteq \operatorname{im}(\pi_{\mathcal{D}_1}^{-1}\pi_{\mathcal{D}_1} \circ P_f(\ker \pi_{I(\mathcal{D}_1)})) \\
&\subseteq \operatorname{im}(\pi_{\mathcal{D}_1}^{-1}\pi_{\mathcal{D}_1}(\ker \pi_{\mathcal{D}_1})) \\
&= 0,
\end{aligned}$$

which proves the first claim. For the second claim, we first use that $\pi_{\mathcal{D}_1} \circ f = f_{\mathcal{D}_1} = \hat{f}_{\mathcal{D}_1} \circ \pi_{I(\mathcal{D}_1)}$ to obtain $\pi_{\mathcal{D}_1} \circ P_f = P_{\hat{f}_{\mathcal{D}_1}} \circ \pi_{I(\mathcal{D}_1)}$. We then note that as $\operatorname{im} p_{\vartheta_{\mathcal{D}_1}} \subset \operatorname{im} \pi_{\mathcal{D}_1}^{-1}$ we can write $p_{\vartheta_{\mathcal{D}_1}} = p_{\vartheta_{\mathcal{D}_1}} \circ \pi_{\mathcal{D}_1}^{-1}\pi_{\mathcal{D}_1}$ and analogously also $p_{\vartheta_{\mathcal{D}_2}} = \pi_{I(\mathcal{D}_1)}^{-1}\pi_{I(\mathcal{D}_1)} \circ p_{\vartheta_{\mathcal{D}_2}}$. This gives us

$$\begin{aligned}
p_{\vartheta_{\mathcal{D}_1}} \circ P_f \circ p_{\vartheta_{\mathcal{D}_2}} &= p_{\vartheta_{\mathcal{D}_1}} \circ \pi_{\mathcal{D}_1}^{-1}\pi_{\mathcal{D}_1} \circ P_f \circ \pi_{I(\mathcal{D}_1)}^{-1}\pi_{I(\mathcal{D}_1)} \circ p_{\vartheta_{\mathcal{D}_2}} \\
&= p_{\vartheta_{\mathcal{D}_1}} \circ \pi_{\mathcal{D}_1}^{-1} \circ P_{\hat{f}_{\mathcal{D}_1}} \circ \pi_{I(\mathcal{D}_1)} \circ p_{\vartheta_{\mathcal{D}_2}},
\end{aligned}$$

and thus

$$\pi_{\mathcal{D}_1} \circ P_{\mathcal{D}_1,\mathcal{D}_2} \circ \pi_{I(\mathcal{D}_1)}^{-1} = (\pi_{\mathcal{D}_1} \circ p_{\vartheta_{\mathcal{D}_1}} \circ \pi_{\mathcal{D}_1}^{-1}) \circ P_{\hat{f}_{\mathcal{D}_1}} \circ (\pi_{I(\mathcal{D}_1)} \circ p_{\vartheta_{\mathcal{D}_2}} \circ \pi_{I(\mathcal{D}_1)}^{-1}).$$

As the terms to the left and to the right of $P_{\hat{f}_{\mathcal{D}_1}}$ are the projections onto $U^{\mathcal{D}_1}_{\mathcal{D}_1}$ and $U^{I(\mathcal{D}_1)}_{\mathcal{D}_2}$, we see that the right hand side of this equation describes a block of $P_{\hat{f}_{\mathcal{D}_1}}$ with respect to the corresponding joint decompositions. The identification between blocks is thus obtained via the maps $\pi_{\mathcal{D}_1}$ and $\pi_{I(\mathcal{D}_1)}^{-1}$.

□

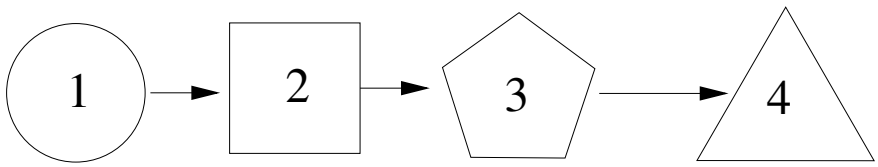

Figure 3.1: A four cell linear chain considered in Example 3.16.

Example 3.16
We are now going to illustrate the statement of Theorem 3.15 using the example of a very simple network: Four cells arranged in a linear chain which feed input forward into the next (see Figure 3.1.) For this network we have $I(1) = \{1\}$ and $I(c) = \{c-1, c\}$ for $c = 2, 3, 4$. With the power set of $\{1, 2, 3, 4\}$ given the order

$$\begin{array}{cccccccc} \{1,2,3,4\}, & \{1,2,3\}, & \{1,2,4\}, & \{1,3,4\}, & \{2,3,4\}, & \{1,2\}, & \{1,3\}, & \{1,4\}, \\ \{2,3\}, & \{2,4\}, & \{3,4\}, & \{1\}, & \{2\}, & \{3\}, & \{4\}, & \emptyset, \end{array}$$

applying Theorem 3.15 to this network leads to the following form of the transfer operator.

$$P_f = \left(\begin{array}{c|c|c|c|c||c|c|c|c|c|c||c|c|c|c||c} * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ \hline 0 & * & 0 & 0 & 0 & * & * & 0 & * & 0 & 0 & * & * & * & 0 & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & * & * & 0 & * & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & * & 0 & 0 & * & * & 0 & 0 & * & * & 0 & * & * & * \\ 0 & * & 0 & 0 & 0 & * & * & 0 & * & 0 & 0 & * & * & * & 0 & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & * & * & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & * & * & * \\ \hline * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \end{array}\right)$$

(Here the double lines are used to mark the blocks belonging to subsets of $\{1, 2, 3, 4\}$ with four, three, two, one and zero elements, respectively.) It is worthwhile to point out that this description is misleading in the way that in this matrix the same space is allocated for each block, while the actual dimensions (in the finite-dimensional case) will differ over a wide range, from $1 = \dim U_\emptyset$ to $O(n^4) = \dim U_{\mathcal{C}}$, the dimensions of the individual measure spaces $\mathcal{M}(X_c)$ being assumed to be proportional to n. (Cf. also Examples 2.26 and 3.14.)

We have now reached a position where it seems appropriate to recapitulate what we have achieved in this chapter up to this point. We started with a very simple observation about the images of certain subspaces under the transfer operator of a coupled cell system, and used this observation to obtain a first, very rough block decomposition of the transfer operator. Having to construct this decomposition separately for each cell c, we immediately asked ourselves whether one could not combine all of these decompositions into a unique, finer decomposition. To do this, we would have had to construct a joint refinement of all decompositions of the form $\mathcal{M}(X) = \ker \pi_c \oplus \operatorname{im} \pi_c^{-1}$ and of the form $\mathcal{M}(X) = \ker \pi_{I(c)} \oplus \operatorname{im} \pi_{I(c)}^{-1}$, with $c \in \mathcal{C}$. During the preparation of this thesis, it turned out that instead of doing this, one could equally well construct a joint refinement of all decompositions of the form $\mathcal{M}(X) = \ker \pi_{\mathcal{D}} \oplus \operatorname{im} \pi_{\mathcal{D}}^{-1}$, with $\mathcal{D} \subset \mathcal{C}$. Consequently, we have done this, and furthermore we have found an efficient way of writing down the joint decomposition in form of the subspaces $U_{\mathcal{D}}$. The first reward we received for our efforts is Theorem 3.15, which can be seen as the fulfillment of the programme we formulated after Proposition 3.4. It seems to the author that with this theorem we have completed the analysis of the consequences of the *independence* relations determined by the coupled cell system. In the next section, we will see how far we can get by exploiting the *symmetry* relations, that is, the equivariance of the dynamical system with respect to the groupoid associated with a coupled cell network.

3.2 Consequences of Groupoid Equivariance for the Transfer Operator

In Chapter 2 we saw that a coupled cell network is given by a set $\mathcal{C}$ of cells together with an equivalence relation, and by a set of couplings $\mathcal{E}$ together with their equivalence relation. Both pieces of information about the network are used to form its symmetry groupoid. In the previous sections, we mostly used information about the cells in the network, and almost completely disregarded its coupling structure. This was naturally the case as our interest was focussed on a particular structuring of the measure space $\mathcal{M}(X)$, which only depends on the state spaces X_c and thus only on the cells. In the following we will turn our attention more towards the coupling structure, and in particular to the symmetry groupoid $\mathcal{B}_G$ which was used to describe the structure in an algebraic way. We saw in Section 2.1.2 that admissibility of a map f on a coupled cell network is tantamount to f being equivariant with respect to this groupoid. We are now going to study the consequences of this equivariance for the transfer operator. An important tool for our study will be the decomposition $\mathcal{M}(X) = \oplus_{\mathcal{D} \subset \mathcal{C}} U_{\mathcal{D}}$ which was constructed in the previous section.

3.2.1 Symmetry in coupled cell networks

As we already mentioned in the Introduction, the concept of coupled cell systems spans an arc from highly structured systems that are symmetric in the classical sense (i. e. invariant with respect to a global permutation symmetry group) to practically unstructured general "coupled systems". Quite generally the question comes up how to systematically describe the position of a given coupled cell system in this wide field, that is, how to describe which "symmetric structure" it actually has. In the following we propose to describe the system by associating to it a collection of groups, which will be used to obtain structural results for the transfer operator. More specific, we propose to analyse the "symmetry structure" by looking at each subset of the set of cells and determining an appropriate symmetry group for this subset.

Definition 3.17
Let a coupled cell network $(\mathcal{C}, \mathcal{E})$ be given. For any non-empty $\mathcal{D} \subset \mathcal{C}$, we call a bijective mapping $\varphi : I(\mathcal{D}) \to I(\mathcal{D})$ a $\mathcal{D}$-relative symmetry *of the cell network, if $\varphi(\mathcal{D}) = \mathcal{D}$ and $\varphi_{|I(c)}$ is an input isomorphism for every $c \in \mathcal{D}$. Further we define $\Gamma_{\mathcal{D}}$ to be the set of all $\mathcal{D}$-relative symmetries, and call it the* $\mathcal{D}$-relative symmetry group *of the coupled cell network.*

To justify this name, we show that $\Gamma_{\mathcal{D}}$ forms indeed a group.

Lemma 3.18
For each $\mathcal{D}$, $\Gamma_{\mathcal{D}}$ is a group.

Proof: We have to show that $\Gamma_{\mathcal{D}}$ is closed under composition of maps. If $\varphi_1 \in \Gamma_{\mathcal{D}}$ and $\varphi_2 \in \Gamma_{\mathcal{D}}$, their composition is also a bijection, and $\varphi_1 \circ \varphi_2(\mathcal{D}) = \mathcal{D}$ is inherited from the corresponding properties of φ_1 and φ_2. Now let $c \in \mathcal{D}$. Then $(\varphi_1 \circ \varphi_2)_{|I(c)} = \varphi_{1|I(\varphi_2(c))} \circ \varphi_{2|I(c)}$ is the composition of two input isomorphisms, hence it is one itself. This concludes the proof. □

Remark 3.19
Golubitsky, Stewart et al. consider a very similar notion which they call *interior symmetry* of a network. (Cf. e. g. [24] or [25].) For a subset $\mathcal{D} \subset \mathcal{C}$ they define an interior symmetry on $\mathcal{D}$ to be an "input equivalence preserving permutation" $\sigma : \mathcal{C} \to \mathcal{C}$ "that is the identity on the complement of $\mathcal{D}$" (quotations from [24, Definition 7][1]), and use the concept of the interior symmetry group on $\mathcal{D}$ to obtain coupled cell network analogues of classical results in equivariant bifurcation

[1] As in [24] the "multi-arrow formalism" is used, the actual definition given there is a little bit more complicated. In particular, the notation $I(\mathcal{D})$ there refers to a collection of input *edges* which we do not need to consider at all. Quoting selectively, we here rephrase the definition to fit into the "classical" coupled cell network formalism we are working with.

theory[2].
Requiring the permutation to be defined on the whole set $\mathcal{C}$ *and* to be the identity on the complement of $\mathcal{D}$, this notion is considerably more restrictive than the concept of $\mathcal{D}$-relative symmetries we propose here.

Example 3.20
To illustrate our definition, we consider some special cases. Our intention is in particular to persuade the reader that non-trivial $\mathcal{D}$-relative symmetry groups are nothing exceptional in networks possessing some "visual symmetry".

1. Let $\mathcal{D} = \{c\}$ for some $c \in \mathcal{C}$. Then $\Gamma_{\mathcal{D}} = B(c,c)$. This follows directly from the definition.

2. If the group $\Gamma_{\mathcal{C}}$ is non-trivial, the full network has $\Gamma_{\mathcal{C}}$ as (classical) symmetry group, and any admissible map f is (classically) equivariant with respect to the action of this group on X. (Cf. Lemma 3.23 below.) Again, this can be seen directly from the definition.

3. If $\mathcal{D} = \{c,d\}$ for two cells $c \neq d$ and the set $B(c,d)$ of input isomorphisms between $I(c)$ and $I(d)$ is non-empty, then we call a pair (φ_1, φ_2) with either $\varphi_1 \in B(c,d)$ and $\varphi_2 \in B(d,c)$ or $\varphi_1 \in B(c,c)$ and $\varphi_2 \in B(d,d)$ a *non-conflicting pair* if $\varphi_{1|I(c)\cap I(d)} = \varphi_{2|I(c)\cap I(d)}$. For such a pair, we write $\binom{\varphi_1}{\varphi_2}$ for the map $I(c) \cup I(d) \to I(c) \cup I(d)$ that coincides with φ_1 and with φ_2 on their respective domains. Using this notation, we have

$$\Gamma_{\mathcal{D}} = \left\{ \binom{\varphi_1}{\varphi_2} \,\middle|\, (\varphi_1, \varphi_2) \text{ is non-conflicting} \right\},$$

   i. e. $\Gamma_{\mathcal{D}}$ consists of "pairings" of input isomorphisms which either permute $I(c)$ and $I(d)$ separately or which "swap" these two input sets and permute them.

4. The construction used in the previous example can be generalized to subsets $\mathcal{D}$ with more than two elements. The key requirement always is that the tuples of input isomorphisms that are to be used to form an element of $\Gamma_{\mathcal{D}}$ have to be pairwise non-conflicting.

Remark 3.21
Considering there is always the whole collection $\{\Gamma_{\mathcal{D}} \mid \mathcal{D} \subset \mathcal{C}\}$ of relative symmetry groups associated with a coupled cell network, one is lead to ask whether there might be reasonably simple, general relationships between these groups, e. g. between $\Gamma_{\mathcal{D}_1}$ and $\Gamma_{\mathcal{D}_2}$ for the case $\mathcal{D}_1 \subset \mathcal{D}_2$, or more specific for the case $\mathcal{D}_2 = \mathcal{D}_1 \cup \{c\}$. However, at second sight one recognizes that there might be

[2] More precisely, they formulate analogues to the Equivariant Branching Lemma and the Equivariant Hopf Theorem, see e. g. [26, Ch. XIII and Ch. XVI].

no such simple relationships, and if there are, that they might not be easy to discover. In fact, the simple operation of adding one cell to the subset (that is, of passing from $\mathcal{D}_1$ to $\mathcal{D}_2$ in the latter example) can dramatically change the nature of the associated relative symmetry group, both enlarging or reducing it. As an example, consider a subset Γ_1 of N identical cells with identical coupling between any two of them. Then the relative symmetry group will be isomorphic to the full symmetric group S_N. If one now considers one more cell which is coupled in a different way to each of the others, the relative symmetry group will be trivial. An opposite effect can be observed if $\mathcal{D}_1$ is taken to be a chain of N identical cells, each of which receives input only from its immediate predecessor, except the first one. In this situation, $\Gamma_{\mathcal{D}_1}$ is trivial. If only one cell is introduced with edges that "close the loop" to make $\mathcal{D}_2$ a "ring", then $\Gamma_{\mathcal{D}_2}$ is the group $\mathbb{Z}_{N+1}$. These examples illustrate that little, if anything, can be said for the general situation. Therefore one will have to consider each particular network separately, determine its collection of relative symmetry groups and apply the theorems we are going to present in the next section to the individual network in order to gain insight into its transfer operator.

3.2.2 Equivariance

How can we use the $\mathcal{D}$-relative symmetry groups at all? To answer this question, we begin with two lemmas that prepare the stage for our later developments. We first look at the interaction of one of the groups $\Gamma_{\mathcal{D}}$ with an admissible map. For this we first note that $\Gamma_{\mathcal{D}}$ acts on the spaces $X_{\mathcal{D}}$ and $X_{I(\mathcal{D})}$.

Definition 3.22
We call the action of $\Gamma_{\mathcal{D}}$ on $X_{\mathcal{D}}$ given by the lift of the elements of $\Gamma_{\mathcal{D}}$ to $X_{\mathcal{D}}$ (see Definition 2.38) the output action *of $\Gamma_{\mathcal{D}}$, and we call the corresponding action on $X_{I(\mathcal{D})}$ the* input action *of $\Gamma_{\mathcal{D}}$.*

Lemma 3.23
If $f : X \to X$ is an admissible map, then the partial map $\hat{f}_{\mathcal{D}} : X_{I(\mathcal{D})} \to X_{\mathcal{D}}$ commutes with the input and output action of $\Gamma_{\mathcal{D}}$:

$$\hat{\varphi} \circ \hat{f}_{\mathcal{D}} = \hat{f}_{\mathcal{D}} \circ \hat{\varphi} \quad \forall \varphi \in \Gamma_{\mathcal{D}}.$$

Proof: We have to show that for every $c \in \mathcal{D}$ and every $x \in X_{I(\mathcal{D})}$ the $\varphi(c)$-components of $(\hat{\varphi} \circ \hat{f}_{\mathcal{D}})(x)$ and of $(\hat{f}_{\mathcal{D}} \circ \hat{\varphi})(x)$ are the same. But on the one hand,

$$[(\hat{\varphi} \circ \hat{f}_{\mathcal{D}})(x)]_{\varphi(c)} = [\hat{f}_{\mathcal{D}}(x)]_c = \hat{f}_c(\pi_{I(c)}(x)),$$

by definition, and on the other hand

$$[\hat{f}_{\mathcal{D}}(\hat{\varphi}(x))]_{\varphi(c)} = \hat{f}_{\varphi(c)}(\pi_{I(\varphi(c))}(x)) = \hat{f}_{\varphi(c)}(\pi_{\varphi(I(c))}(x)) = \hat{f}_c(\pi_{I(c)}(x)),$$

as $\varphi_{|I(c)}$ is an input isomorphism and f is admissible. □

Now we analyse the action of elements of $\Gamma_{\mathcal{D}}$ on the decomposition of $\mathcal{M}(X)$. Here we can slightly relax the restrictions on the mappings φ and consider a more general situation.

Lemma 3.24
Let $\mathcal{D} \subset \mathcal{C}$, and let $\varphi : \mathcal{D} \to \mathcal{C}$ be an injective map such that $c \sim_{\mathcal{C}} \varphi(c)$ for all $c \in \mathcal{D}$. Let $\tilde{\varphi} : \operatorname{im} \pi_{\mathcal{D}}^{-1} \to \operatorname{im} \pi_{\varphi(\mathcal{D})}^{-1}$ be its lift to $\mathcal{M}(X)$ (see Definition 2.38). Then for every $\mathcal{D}' \subset \mathcal{D}$, $\tilde{\varphi}(U_{\mathcal{D}'}) = U_{\varphi(\mathcal{D}')}$.

Proof: First of all notice that if one denotes the restriction $\varphi_{|\mathcal{D}'}$ by ψ, then its lift $\tilde{\psi}$ to $\mathcal{M}(X)$ is just the restriction of $\tilde{\varphi}$ to $\operatorname{im} \pi_{\mathcal{D}'}^{-1}$. Thus we see that for every $\mathcal{D}' \subset \mathcal{D}$, $\tilde{\varphi}(\operatorname{im} \pi_{\mathcal{D}'}^{-1}) = \operatorname{im} \pi_{\varphi(\mathcal{D}')}^{-1}$. Bearing this in mind, one sees that the statement follows directly from the definition of $U_{\mathcal{D}'}$ and $U_{\varphi(\mathcal{D}')}$. □

Now we are ready to present the main result about the structure of the transfer operator of a coupled cell system. The overall line of argument is simple. From Lemma 3.24 we can see that for every $\mathcal{D} \subset \mathcal{C}$, the lift of each element of $\Gamma_{\mathcal{D}}$ leaves the subspaces $U_{\mathcal{D}}$ and $\operatorname{im} \pi_{I(\mathcal{D})}^{-1}$ invariant. This means that the lift defines two representations of the group $\Gamma_{\mathcal{D}}$ on these spaces. It is not difficult to see that the composition of P_f with the projection to $U_{\mathcal{D}}$ (which can be visualized as the "row" of P_f belonging to the subspace $U_{\mathcal{D}}$) is equivariant (in the classical sense!) with respect to these representations. This puts us in a position to use Theorem 2.14 to derive statements about a block diagonal structure of the $U_{\mathcal{D}}$-row of P_f. In particular, one obtains statements about zero blocks in a "block matrix representation" of P_f that go beyond those obtained from the "independence" relation that were exploited in Proposition 3.4.
We first take a close look at the two (or rather three) linear representations that are involved in this analysis. The first representation is easily described. Following Definition 2.38, we see that every element $\varphi \in \Gamma_{\mathcal{D}}$ can be lifted to an invertible map $\tilde{\varphi}$ defined on the subspace $\operatorname{im} \pi_{I(\mathcal{D})}^{-1}$ of $\mathcal{M}(X)$. This is the first of the representations we are interested in. As we require that $\mathcal{D}$ is invariant under the elements of $\Gamma_{\mathcal{D}}$, we can restrict them to this set before lifting them to $\mathcal{M}(X)$. This leads us to a group of invertible linear mappings $\widetilde{\varphi_{|\mathcal{D}}} : \operatorname{im} \pi_{\mathcal{D}}^{-1} \to \operatorname{im} \pi_{\mathcal{D}}^{-1}$ which defines the second representation. Finally, Lemma 3.24 tells us that these mappings leave the subspace $U_{\mathcal{D}}$ of $\operatorname{im} \pi_{\mathcal{D}}^{-1}$ invariant, which allows us to restrict them to it. This gives the third representation.

Definition 3.25
The representation of $\Gamma_{\mathcal{D}}$ defined by

$$\varphi \mapsto (\tilde{\varphi} : \operatorname{im} \pi_{I(\mathcal{D})}^{-1} \to \operatorname{im} \pi_{I(\mathcal{D})}^{-1})$$

is called the input representation. *The representation defined by*

$$\varphi \mapsto (\widetilde{\varphi_{|\mathcal{D}}} : \operatorname{im} \pi_{\mathcal{D}}^{-1} \to \operatorname{im} \pi_{\mathcal{D}}^{-1})$$

is called the output representation *of* $\Gamma_{\mathcal{D}}$*, and its restriction to* $U_{\mathcal{D}}$ *is called the* restricted output representation.

Theorem 3.26
For $\mathcal{D} \subset \mathcal{C}$ *let* $p_{\vartheta_{\mathcal{D}}}$ *be the projection to* $U_{\mathcal{D}}$ *as in Proposition 3.12. The operator* $\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f : \operatorname{im} \pi_{I(\mathcal{D})}^{-1} \to U_{\mathcal{D}}$ *is equivariant with respect to the input representation and the output representation of* $\Gamma_{\mathcal{D}}$*. The operator* $p_{\vartheta_{\mathcal{D}}} \circ P_f : \operatorname{im} \pi_{I(\mathcal{D})}^{-1} \to U_{\mathcal{D}}$ *is equivariant with respect to the input representation and the restricted output representation of* $\Gamma_{\mathcal{D}}$*.*

Proof: We first show that the transfer operator for the map $\hat{f}_{\mathcal{D}}$ is equivariant with respect to the lifts of $\Gamma_{\mathcal{D}}$ to $\mathcal{M}(X_{\mathcal{D}})$ and to $\mathcal{M}(X_{I(\mathcal{D})})$. After that, we show how this result can be "embedded" into $\mathcal{M}(X)$ to obtain an analogous statement for the input representation and the output representation. This proves the statement for the output representation. Finally, invariance of $U_{\mathcal{D}}$ will allow us to prove the statement of the theorem for the restricted output representation.
To see that the transfer operator for the map $\hat{f}_{\mathcal{D}}$ is equivariant with respect to the lifts of $\Gamma_{\mathcal{D}}$ to $\mathcal{M}(X_{\mathcal{D}})$ and to $\mathcal{M}(X_{I(\mathcal{D})})$, we can simply go back to the definitions of the different terms. Let $\varphi \in \Gamma_{\mathcal{D}}$, let $\bar{\mu} \in \mathcal{M}(X_{I(\mathcal{D})})$, let $A \in X_{\mathcal{D}}$ be measurable. Then

$$(P_{\hat{f}_{\mathcal{D}}} \circ \bar{\varphi})\mu(A) = \mu(\hat{\varphi}^{-1}(\hat{f}_{\mathcal{D}}^{-1}(A))) = \mu(\hat{f}_{\mathcal{D}}^{-1}(\hat{\varphi}^{-1}(A))) = (\overline{\varphi_{|\mathcal{D}}} \circ P_{\hat{f}_{\mathcal{D}}})\mu(A),$$

where the innermost equation is due to Lemma 3.23.
The next step consists of a sequence of transformations, most of which do not involve much more than plugging in definitions. We show that on the subspace $\operatorname{im} \pi_{I(\mathcal{D})}^{-1}$ the map $(\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}) \circ P_f$, i. e. the transfer operator followed by the projection onto $\operatorname{im} \pi_{\mathcal{D}}^{-1}$, commutes with the representations of $\Gamma_{\mathcal{D}}$.

$$\begin{aligned}
(\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}) \circ P_f \circ \tilde{\varphi}(\pi_{I(\mathcal{D})}^{-1}\mu) &= \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f \circ \pi_{I(\mathcal{D})}^{-1} \circ \bar{\varphi} \circ \pi_{I(\mathcal{D})}(\pi_{I(\mathcal{D})}^{-1}\mu) \\
&= \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f \circ \pi_{I(\mathcal{D})}^{-1} \circ \bar{\varphi}(\mu) \\
&= \pi_{\mathcal{D}}^{-1} \circ P_{\hat{f}_{\mathcal{D}}} \circ \pi_{I(\mathcal{D})} \circ \pi_{I(\mathcal{D})}^{-1} \circ \bar{\varphi}(\mu) \\
&= \pi_{\mathcal{D}}^{-1} \circ P_{\hat{f}_{\mathcal{D}}} \circ \bar{\varphi}(\mu) \\
&= \pi_{\mathcal{D}}^{-1} \circ \overline{\varphi_{|\mathcal{D}}} \circ P_{\hat{f}_{\mathcal{D}}}(\mu) \\
&= \pi_{\mathcal{D}}^{-1} \circ \overline{\varphi_{|\mathcal{D}}} \circ P_{\hat{f}_{\mathcal{D}}} \circ \pi_{I(\mathcal{D})} \circ \pi_{I(\mathcal{D})}^{-1}(\mu) \\
&= \pi_{\mathcal{D}}^{-1} \circ \overline{\varphi_{|\mathcal{D}}} \circ \pi_{\mathcal{D}} \circ P_f \circ \pi_{I(\mathcal{D})}^{-1}(\mu) \\
&= (\pi_{\mathcal{D}}^{-1} \circ \overline{\varphi_{|\mathcal{D}}} \circ \pi_{\mathcal{D}}) \circ (\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}) \circ P_f \circ \pi_{I(\mathcal{D})}^{-1}(\mu) \\
&= \widetilde{\varphi_{|\mathcal{D}}} \circ (\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}) \circ P_f(\pi_{I(\mathcal{D})}^{-1}\mu)
\end{aligned}$$

This proves the statement for the output representation. To finish the proof for the restricted output representation, we first note that invariance of $U_{\mathcal{D}}$ with respect to the output action (Lemma 3.24) implies that the projection $p_{\vartheta_{\mathcal{D}}}$ onto $U_{\mathcal{D}}$ commutes with the action. As furthermore $U_{\mathcal{D}} \subset \operatorname{im} \pi_{\mathcal{D}}^{-1}$, we have $p_{\vartheta_{\mathcal{D}}} \circ (\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}) = p_{\vartheta_{\mathcal{D}}}$, and we can obtain the desired result as follows:

$$\begin{aligned}(p_{\vartheta_{\mathcal{D}}} \circ P_f) \circ \tilde{\varphi}(\pi_{I(\mathcal{D})}^{-1}\mu) &= p_{\vartheta_{\mathcal{D}}} \circ (\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}) \circ P_f \circ \tilde{\varphi}(\pi_{I(\mathcal{D})}^{-1}\mu) \\ &= p_{\vartheta_{\mathcal{D}}} \circ \widetilde{\varphi_{|\mathcal{D}}} \circ (\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}}) \circ P_f(\pi_{I(\mathcal{D})}^{-1}\mu) \\ &= \widetilde{\varphi_{|\mathcal{D}}} \circ (p_{\vartheta_{\mathcal{D}}} \circ P_f)(\pi_{I(\mathcal{D})}^{-1}\mu)\end{aligned}$$

□

Corollary 3.27
If $\Gamma_{\mathcal{D}}$ has k irreducible representations, let $U_{\mathcal{D}} = U_{\mathcal{D}}^1 \oplus U_{\mathcal{D}}^2 \oplus \cdots \oplus U_{\mathcal{D}}^k$ be the isotypic decomposition of $U_{\mathcal{D}}$ with respect to the restricted output representation of $\Gamma_{\mathcal{D}}$, let $\operatorname{im} \pi_{\mathcal{D}}^{-1} = V^1 \oplus V^2 \oplus \cdots \oplus V^k$ be the isotypic decomposition with respect to the output representation, and let $\operatorname{im} \pi_{I(\mathcal{D})}^{-1} = W^1 \oplus W^2 \oplus \cdots \oplus W^k$ be the isotypic decomposition of $\operatorname{im} \pi_{I(\mathcal{D})}^{-1}$ with respect to the input representation of $\Gamma_{\mathcal{D}}$. Then $p_{\vartheta_{\mathcal{D}}} \circ P_f$ and $\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f$ respectively have block diagonal structure with respect to these decompositions.

Proof: These statements follow directly from the linear representation theory for groups (see Theorem 2.14). □

To assess the meaning of these statements, one should first note that in practice, the block diagonal structure means that the off-diagonal blocks are zero blocks, which means that one does not need to take them into account for computations. Secondly, it is important to realise that the operators $p_{\vartheta_{\mathcal{D}}} \circ P_f : \mathcal{M}(X) \to U_{\mathcal{D}}$ and $\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f : \mathcal{M}(X) \to \operatorname{im} \pi_{\mathcal{D}}^{-1}$ correspond to "rows" (or "multirows", respectively) of the block matrix representation of P_f with respect to the joint decomposition. That means that this theorem allows to decompose P_f "block-row-wise", where "block rows" can either correspond to a single $U_{\mathcal{D}}$ or a whole subspace $\operatorname{im} \pi_{\mathcal{D}}^{-1}$. This distinction is of importance e. g. in the case where for two subsets $\mathcal{D}_1 \subset \mathcal{D}_2$ the relative symmetry group $\Gamma_{\mathcal{D}_1}$ is larger than the symmetry group $\Gamma_{\mathcal{D}_2}$, so that the block diagonalization with respect to the representations of $\Gamma_{\mathcal{D}_2}$ brings less information ("fewer zeroes") than with respect to those of $\Gamma_{\mathcal{D}_1}$. In this case it is advantageous to decompose the row corresponding to $\operatorname{im} \pi_{\mathcal{D}_1}$ with respect to $\Gamma_{\mathcal{D}_1}$, and only the row corresponding to $U_{\mathcal{D}_2}$ with respect to $\Gamma_{\mathcal{D}_2}$. On the other hand, when $\Gamma_{\mathcal{D}_2}$ has more irreducible representations than $\Gamma_{\mathcal{D}_1}$ for all subsets $\mathcal{D}_1 \subset \mathcal{D}_2$, it is of course better to decompose the whole row corresponding to $\operatorname{im} \pi_{\mathcal{D}_2}^{-1}$ than only the one corresponding to $U_{\mathcal{D}_2}$.

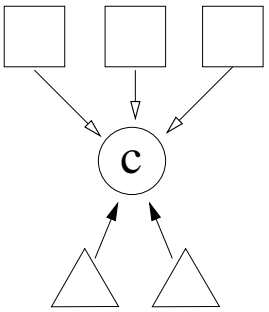

Figure 3.2: Network considered in Example 3.28, number 1.

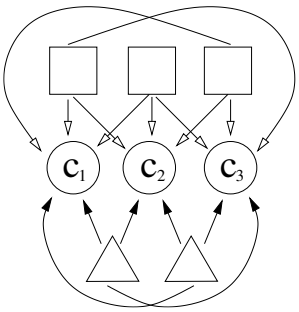

Figure 3.3: Network considered in Example 3.28, number 2

Example 3.28
In the following we consider two examples by which we want to illustrate the considerations just stated. Although the networks we present in the figures appear to be closed networks, standing for themselves, they can always be considered as forming parts of larger networks, so that the figures depict only that part of the network given by the sets $\mathcal{D}$ and $I(\mathcal{D})$.

1. As a first example, we consider the case $\mathcal{D} = \{c\}$. In this case the output-representation is trivial, and thus $\operatorname{im} \pi_c^{-1}$ has only one isotypic component. But we know from Proposition 2.4 that $\Gamma_{\{c\}} = B(c,c)$ is of the form $S_{d_1} \times \ldots \times S_{d_k}$, and in Section 2.2.2 and Section 2.2.3 we saw how the irreducible representations of groups of this kind can be determined. To be more concrete, we consider the case where a cell c receives input from three cells of one kind and two cells of another kind, in each case via identical couplings. (See Figure 3.2.) Then by Theorem 2.18 the relative symmetry group $\Gamma_{\{c\}} = S_3 \times S_2$ has six irreducible representations, two of which are two-dimensional (the tensor products of the standard representation of S_3 with the trivial and the alternating representation of S_2) and four of which are one-dimensional. If we write U, V, W for the measure spaces over the state spaces for the "circle", "triangle" and "square" cell type, respectively, we can represent $\operatorname{im} \pi_{I(c)}^{-1}$ as $U \otimes V^{\otimes 2} \otimes W^{\otimes 3}$. Here Corollary 3.27 implies that the "row" of P_f corresponding to $\operatorname{im} \pi_c^{-1}$ can be divided into six blocks, of which only one (for the trivial representation on $\operatorname{im} \pi_{I(c)}^{-1}$) is a non-zero block. In the finite-dimensional setting, we can use equations (2.1) and (2.2) to determine the dimensions of the blocks. As an example, we can read off Table 2.1 that in the case $\dim V = \dim W = 10$, the trivial isotypic component of $U \otimes V^{\otimes 2} \otimes W^{\otimes 3}$ has the dimension $\dim U \cdot 55 \cdot 220 = 12100 \cdot \dim U$. In contrast, the whole space has dimension $\dim U \cdot 100000$, so that exploitation of the symmetry would reduce the size of the block that is to be computed to roughly one eighth.

2. As a second example, we extend the situation we just considered in such a

way that not one, but three cells of type "circle" receive inputs in identical manners from the same five cells as before. (See Figure 3.3.) If we now take $\mathcal{D} = \{c_1, c_2, c_3\}$, we get $\Gamma_\mathcal{D} = S_3 \times S_3 \times S_2$, which has eighteen irreducible representations, namely the tensor products of the three representations of S_3 with themselves and the trivial and the alternating representation of S_2. In contrast to the previous situation, now the output action is not trivial, but instead given by the lifts of the action of S_3 on $\{c_1, c_2, c_3\}$. This means that the output representation will contain only the three irreducible representations of S_3, and only 3 of the $3 \cdot 18$ blocks of the representation of $\pi_\mathcal{D}^{-1}\pi_\mathcal{D} \circ P_f$ with respect to the isotypic decompositions are non-zero. Again, the dimensions of the isotypic decomposition can be determined using equations (2.1) and (2.2).

3.2.3 Combining Independence and Equivariance Structure

In Section 3.1 we have analysed which consequences it has for the transfer operator of a map f admissible to a coupled cell network that each component map f_c is independent from cells not in the input set $I(c)$; with Theorem 3.15 we have obtained a satisfying description of the implications of independence relations for the structure of the transfer operator. In the current section we have looked at the symmetry properties of an admissible map, as they are expressed by its equivariance with respect to the symmetry groupoid of the network. These properties imply equivariance with respect to certain groups, and we have described their implications with Theorem 3.26 and Corollary 3.27. However, as it stands, the two results make quite independent statements. On the one hand, we learn in Theorem 3.15 about the structure of the block decomposition with respect to the *joint decomposition* of $\mathcal{M}(X)$. On the other hand, we are told that the block decomposition of the "non-zero parts" of the transfer operator with respect to an *isotypic decomposition* of the involved subspaces has in fact a diagonal structure. In this situation, it is natural to ask how these two statements can be combined. More concretely, one would like to know what the block diagonalisations due to symmetry imply for the block decomposition with respect to the joint decomposition. This question is related to the question how, for given $\mathcal{D}$, the joint decomposition and the isotypic decompositions of $\operatorname{im}\pi_{I(\mathcal{D})}$ and of $\operatorname{im}\pi_\mathcal{D}$ are related to each other. To be more specific, we quickly recollect the situation to be regarded. For each subset $\mathcal{D} \subset \mathcal{C}$ of cells, we have the input representation of the group $\Gamma_\mathcal{D}$ on $\operatorname{im}\pi_{I(\mathcal{D})}^{-1}$. The question is how the isotypic decomposition of this representation can be expressed in terms of the collection of subspaces $U_{\mathcal{D}'}$ with $\mathcal{D}' \subset I(\mathcal{D})$, and vice versa. Furthermore, the corresponding questions for the output representation has to be answered.
We can see from the projection formula (Lemma 2.11) for the isotypic components and from Lemma 3.24 that in general isotypic components will not coincide with subspaces $U_{\mathcal{D}'}$. One could, in principle, use the projection formula to obtain

linear dependencies between isotypic components and pieces of the joint decomposition, and one could also go on and invert these relations in order to obtain linear equations involving the blocks of P_f with respect to the joint decompositions which describe which linear combinations of blocks sum up to zero – because they correspond to non-diagonal blocks of P_f with respect to an isotypic decomposition – and which ones sum up to diagonal blocks.
However, it seems to the author that in general the results of these calculations will be of limited use only. We demonstrate the intended procedure on the simplest example. We assume that the group $\Gamma_{\mathcal{D}}$ which we consider is such that the output representation is trivial, and we write $p^{(i)}, i = 1, \ldots, k$ for the projections onto the isotypic decomposition of $\operatorname{im} \pi_{I(\mathcal{D})}^{-1}$ with respect to the input representation, where $p^{(1)}$ is the projection onto the component associated with the trivial representation. We then have $\mathrm{Id} = \sum_{i=1}^{k} p^{(i)}$ on $\operatorname{im} \pi_{I(\mathcal{D})}^{-1}$ and hence

$$\begin{aligned} \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f &= \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f \circ \sum_{i=1}^{k} p^{(i)} \\ &= \pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f \circ p^{(1)}, \end{aligned} \tag{3.6}$$

as we know from Corollary3.27 that the blocks $\pi_{\mathcal{D}}^{-1} \circ \pi_{\mathcal{D}} \circ P_f \circ p^{(i)}$ are zero for $i \neq 1$. Combining equation (3.6) with the simple observation that $p^{(1)}(\mu) = p^{(1)}(\tilde{\varphi}(\mu))$ for any $\mu \in \operatorname{im} \pi_{I(\mathcal{D})}^{-1}$ and any $\varphi \in \Gamma_{\mathcal{D}}$, we now see that the equivariance of $\pi_{\mathcal{D}}^{-1} \pi_{\mathcal{D}} \circ P_f$ implies that $\pi_{\mathcal{D}}^{-1} \pi_{\mathcal{D}} \circ P_f(\mu) = \pi_{\mathcal{D}}^{-1} \pi_{\mathcal{D}} \circ P_f(\tilde{\varphi}(\mu))$. This in particular means that two blocks $P_{\mathcal{D}_1, \mathcal{D}_2}$ and $P_{\mathcal{D}_1, \mathcal{D}_2'}$ of P_f with respect to the joint decomposition are identical whenever $\mathcal{D}_2' = \varphi(\mathcal{D}_2)$ for some $\varphi \in \Gamma_{\mathcal{D}}$ with $\mathcal{D}_1 \subset \mathcal{D}$ and $\mathcal{D}_2 \subset I(\mathcal{D}), \mathcal{D}_2' \subset I(\mathcal{D})$.

4 Numerical Approximations of Coupled Cell Transfer Operators

> ...far more ability and sophistication is required to obtain a numerical solution than to establish the usual existence and uniqueness theorems. It is far more difficult to obtain an effective algorithm than one that stops with a demonstration of validity. A final goal of any scientific theory must be the derivation of numbers. Theories stand or fall, ultimately, upon numbers.

This passage from Richard E. Bellman's autobiography ([3, p.185]) found an unexpected but nevertheless striking verification in the preparation of this chapter. Aiming to comply to Bellman's statement about the "final goal of theories", the author was initially motivated to analyse the structure of the transfer operator of a coupled cell system in order to reduce the computational effort necessary for an approximation of the transfer operator of a coupled cell system. That is, the intention of this thesis was to narrow the gap between systems that are small enough to be accessible to a numerical analysis using transfer operator techniques on the one hand and systems that are sufficiently large to be interesting from a theoretical coupled cell systems point of view, or even for applications. As a proof that a good step has been made in that direction, one has good reason to expect a numerical example.

Unfortunately, this aim appears to be unreachable for this thesis. It turned out that there is an obstacle between the structural results that were obtained in the preceding chapter and their implementation into an algorithm for the approximation of the transfer operator which the author of this thesis was unable to circumvent.

In the following we will first describe this obstacle, and then, for lack of better ideas, leave the theory developed so far aside, go back to the beginning and present an alternative approach for the numerical approximation of the transfer operator that tries to exploit the structural features of the coupled cell network.

In everything that is presented in this chapter, it is assumed that we are working in the finite-dimensional setting (cf. Example 2.26). Furthermore, for simplicity we assume all vector spaces that we are considering to be real vector spaces. Most of what is said, however, could equally well be applied to complex vector spaces.

4.1 What prevents an algorithmic use of the structure results

Any numerical approximation of the transfer operator will essentially be given in the form of a matrix. This matrix is the representation of a linear endomorphism on a finite-dimensional subspace $\mathcal{M}(X)$ (of some other, infinite-dimensional measure space) that approximates P_f. This implies that the concrete form the matrix will take depends firstly on the choice of subspace, secondly on the choice of the approximating endomorphism and finally also on the choice of a basis for the finite-dimensional subspace. With the first choice, the designer of an algorithm can influence the precision of the approximation that is achieved. (For example, she can adjust the granularity of the discretization of X, or construct partitions specially adapted to the dynamical system to be analysed.) The approximating endomorphism is usually constructed as it is described in Section 2.4.2. The third point, finally, is usually decided upon in view of implementational criteria, i. e. in such a way that the evaluation of the resulting formulas is easy to implement. It is this question from which the problem results.

4.1.1 The dependence on the choice of basis

$U_\mathcal{D}$-adapted bases The structural description of the transfer operator in the preceding chapter was given in terms of blocks $P_{\mathcal{D}_1,\mathcal{D}_2}$ of P_f with respect to the decomposition $\mathcal{M}(X) = \oplus_{\mathcal{D}\subset\mathcal{C}} U_\mathcal{D}$. So in order to use it for a numerical algorithm we need to use a basis of $\mathcal{M}(X)$ with the property that the spaces $U_\mathcal{D}$ are linear spans of certain subsets of this basis. Furthermore, in our specific case, we not only have the space $\mathcal{M}(X)$ to consider, but also the spaces $\mathcal{M}(X_c)$ for all cells c, as well as the spaces $\mathcal{M}(X_\mathcal{D})$ for subsets $\mathcal{D} \subset \mathcal{C}$. As in all suitable settings $\mathcal{M}(X)$ is essentially the tensor product of the spaces $\mathcal{M}(X_c)$, it seems appropriate to use a basis for $\mathcal{M}(X)$ that consists of tensor products of bases of $\mathcal{M}(X_c)$. We have seen in Example 3.14 that such a basis is obtained if one takes tensor products of bases of the spaces $\mathcal{M}(X_c)$ with the property that one (in ordered bases, without loss of generality the first) basis element is λ_c and that the other basis elements are measures that assign the value zero to the whole space X_c. Such a basis will be called a $U_\mathcal{D}$-adapted basis in the following.

Matrix dependence on the basis To make a step towards a description of the "obstacle", we remind ourselves how a matrix representation of the transfer operator $P_f : \mathcal{M}(X) \to \mathcal{M}(X)$ with respect to an arbitrary basis is constructed. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of $\mathcal{M}(X)$. $\mathcal{B}$ defines a canonical linear isomorphism

$B : \mathbb{R}^n \to \mathcal{M}(X)$ via

$$B \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i b_i.$$

The single component mappings of the inverse $B^{-1} : \mathcal{M}(X) \to \mathbb{R}^n$ define elements b_i^* of the dual space $\mathcal{M}(X)^*$, which together form the *dual basis* $\mathcal{B}^*$. Now the entry at position (i, j) of the matrix of P_f with respect to a basis $\mathcal{B}$ is given simply through the formula $m_{ij} = b_i^* \; P_f \; b_j$. This means that, having decided upon a particular basis, to compute matrix entries one needs to solve two computational tasks: Firstly, to compute the image $P_f b_j$ for all matrix elements, and secondly, to evaluate the dual basis on these images.[1]
The core of the problem is now that it is not clear how either of these tasks can be performed for $U_{\mathcal{D}}$-adapted bases *except* by resorting to the "box basis" used by standard algorithms for an approximation of a transfer operator. To make clear what this means, we take a look at the standard procedure that is implemented e. g. in GAIO [6].

4.1.2 The standard procedure and why it cannot be used efficiently for coupled cell systems

GAIO The software package GAIO combines efficient algorithms for the creation of a *box covering* of "interesting parts" of the state space of a general dynamical system (e. g. fixed points, (relative) global attractors, or invariant manifolds) with an algorithm that approximates the transfer operator of the system, based on a previously generated covering. The coverings consist of disjoint (up to boundaries of Lebesgue zero measure) *coordinate boxes*[2]. These boxes take the place of the atoms of the σ-algebra from Example 2.26. A measure in $\mathcal{M}(X)$ is associated to each box B that assigns to any function in $\mathcal{F}(X)$ the value it takes on B. These measures determine a basis for $\mathcal{M}(X)$, which is from here on called the *box basis*. Among others, this basis has the particular advantage that its dual basis can easily be described. More concretely, if μ_i is an element of the box basis that is associated with the box B_i, then let $\chi_i \in \mathcal{F}(X)$ be the characteristic function on B_i. The dual basis element corresponding to μ_i is then "integration of χ_i", that means that for $\mu \in \mathcal{M}(X)$ the value $\int_X \chi_i \, d\mu$ is the coefficient of μ_i in the box basis expansion of μ. It turns out that in the

[1] An alternative approach would be to compute expressions for the maps $bi^* P_f$ first, and to evaluate these on the basis elements. Such a procedure could be termed "row-wise" computation, as every expression $b_i^* P_f$ corresponds to one row of the matrix. Some more thoughts in this direction can be found in Section 5.2.3.

[2] Coordinate boxes are sets of points $(x_1, \ldots, x_k)^T \in \mathbb{R}^k$ satisfying inequalities of the form $|x_i - c_i| \leq r_i$ for $i \in \{1, \ldots, k\}$. Here the vector $c \in \mathbb{R}^k$ is the *center* and the vector $r_i \in \mathbb{R}^k_+$ is the *radius* of the box.

computation of an approximation of P_f, this term can be evaluated efficiently, by simply counting points in boxes.
To see this, we have to consider how the image measures $P_f\mu_i$ are approximated for the elements μ_i of the box basis. For this purpose, the measure μ_i is approximated by a sum of δ-measures in N "evenly distributed[3]" points:

$$\mu_i \approx \frac{1}{N}\sum_{l=1}^{N}\delta_{x_{i,l}} \quad \text{with } \{x_{i,1},\dots,x_{i,N}\} \subset B_i.$$

We have seen in Example 2.40 that the images of δ-measures can be determined simply be determining the images of the corresponding points, so that one has

$$P_f\mu_i \approx \frac{1}{N}\sum_{l=1}^{N}\delta_{f(x_{i,l})}.$$

Thus in order to expand $P_f\mu_i$ with respect to the box basis, one simply has to count the number of point images $f(x_{i,l})$ within the box B_j:

$$\int_X \chi_j \, dP_f\mu_i \approx \frac{1}{N}\sum_{l=1}^{N}\chi_j(f(x_{i,l})) = \frac{1}{N}|\{l|f(x_{i,l}) \in B_j, 1 \le l \le N\}|,$$

which is precisely what the algorithm contained in GAIO does.

Computing matrix entries for $U_{\mathcal{D}}$-adapted bases It turns out that it is not easily possible to perform the two tasks "compute images of basis measures under P_f" and "evaluate the dual basis on them" for $U_{\mathcal{D}}$-adapted bases. First of all, assuming that the map f can be evaluated on single points only (which appears to be a reasonable assumption) it seems clear that all methods to compute images of measures have to use in some way or another the method that was described in the preceding paragraph: they will have to approximate basis measures by linear combinations of δ-measures, as these are the only ones, it appears, whose image under P_f can be directly computed. But this means that we do not loose generality when we decide to describe our $U_{\mathcal{D}}$-adapted basis of choice in terms of a box basis. Doing this not only means that we have *in principle* a way of computing images of basis measures (namely, computing the images of box basis measures and forming the appropriate linear combination), it also brings the additional benefit that – again, *in principle* – we know how to evaluate the dual

[3]How this idea is realized depends in particular on the numerical effort one is willing to pay. The convenient choice of choosing test points on a fine rectangular grid within B_i will become prohibitively expensive in higher dimensions, in which case one may be better off with choosing test points at random from a uniform distribution on B_i. Note that in this account of the procedure we completely leave out questions concerning the approximation quality.

basis. The reason for this lies basic linear algebra, more precisely in the fact that if M is a change-of-basis matrix between the box and the $U_{\mathcal{D}}$-adapted basis, i. e. if the j-th column of M describes which linear combination of the elements of the box basis is equal to the j-th element of the $U_{\mathcal{D}}$-adapted basis, then the matrix M^{-1} describes the dual of the $U_{\mathcal{D}}$-adapted basis in terms of the dual of the box basis.
To be more precise, let $\{\mu_1^1, \ldots, \mu_N^1\}$ be the box basis, and let $\{\mu_1^2, \ldots, \mu_N^2\}$ be the $U_{\mathcal{D}}$-adapted basis. These bases define linear isomorphisms $B_l : \mathbb{R}^N \to \mathcal{M}(X)$ via $B_l(e_j) = \mu_j^l$ (for $l = 1, 2$), where $\{e_1, \ldots, e_N\}$ is the standard basis of $\mathbb{R}^N$. Then $P_1 := B_1^{-1} \circ P_f \circ B_1$ is the representation of P_f with respect to the box basis, and likewise $P_2 := B_2^{-1} \circ P_f \circ B_2$ its representation with respect to the $U_{\mathcal{D}}$-adapted basis. Using this terminology, the change-of-basis matrix described above is given through $M = B_1^{-1} \circ B_2$, as we have $Me_j = B_1^{-1}\mu_j^2$ and thus $B_1 M e_j = \mu_j^2$. With this matrix, we have $P_2 = M^{-1} P_1 M$. This means in particular that we obtain a "formula" for the (i, j)-th entry of P_2, namely

$$(i,j)\text{-th entry of } P_2 = (i\text{-th row of } M^{-1}) \cdot P_1 \cdot (j\text{-th column of } M). \tag{4.1}$$

If M and M^{-1} were known the be sparse matrices, this expression could be reduced to a linear combination of "a few" entries of P_1. The problem now lies in the fact that the author of this thesis was unable to find a $U_{\mathcal{D}}$-adapted basis for which both M and M^{-1} are sparsely populated matrices. This, however, seems to be a necessary requirement if one aims at finding an *efficient* algorithm for the computation of P_2, for the efficiency of such an algorithm would primarily have to be measured by the number of test points that are necessary to compute the entries of P_2. The best the author has to offer is a $U_{\mathcal{D}}$-adapted basis described below in Example 4.1, for which M is relatively sparse, but M^{-1} is fully populated.
In the second-last sentence, we purposefully said "compute the entries" and not "compute one entry". It is not only possible, but likely that the effort necessary for the computation of the whole matrix is much less than the number of entries times the effort for one entry. It may well be that intermediate results from the computation of one entry can be reused for other entries. In fact, we encounter precisely this phenomenon in the "standard algorithm" (which would be used for the computation of P_1) that was described above. There, the computation of the images of test points from one box suffices to determine the matrix entries for one complete column of the matrix.
On the one hand, this is good news: if, as in Example 4.1, M is relatively sparse and M^{-1} is not, then the second and third term of the right hand side of Equation (4.1) can be combined to a linear combination of some (few) columns of P_1. Then fullness of M^{-1} means nothing else than that for the evaluation of Equation (4.1) one indeed *needs* full columns of P_1.

On the other hand, it seems this is also bad news, as it suggests that the information about zero blocks that one can obtain from the structural theory is effectively useless. In general, the block decomposition of P_f with respect to the joint decomposition will not contain full zero columns – observe e. g. that for the block row corresponding with $\mathcal{D} = \mathcal{C}$ there is no zero block due to independence. This means that usually one will have to compute entries *from each column* of the $U_{\mathcal{D}}$-adapted matrix representation. But this, in turn, makes it necessary, unless alternative means are found to obtain matrix entries for the $U_{\mathcal{D}}$-adapted representation, to compute *every full column* of the box basis representation – which means that there are no savings as long as one has to return to the box basis for actual computations. We saw above that it appears plausible that this restriction can hardly be relaxed – at least under the paradigm that the dynamical system is numerically accessible only through evaluation of the map f on single points.

Example 4.1
For each cell c, let $\{\mu_1^c, \dots, \mu_{n_c}^c\}$ be the box basis of $\mathcal{M}(X_c)$. Define $\tilde{\mu}_1^c = \frac{1}{n_c}\sum_{j=1}^{n_c} \mu_j^c$, and for $1 < i \leq n_c$ define $\tilde{\mu}_i^c = \mu_i^c - \mu_{i-1}^c$. Then $\{\tilde{\mu}_1^c, \dots, \tilde{\mu}_{n_c}^c\}$ is a basis of $\mathcal{M}(X_c)$ as described in Example 3.14, and we thus obtain a $U_{\mathcal{D}}$-adapted by forming all tensor products of all these basis elements.
We see from the above definitions that the change-of-basis matrix M^c that describes the $\tilde{\mu}_i^c$ in terms of the box basis is not only lower triangular, but also relatively sparse: there are non-zero entries in the first column, in the diagonal and in the first lower secondary diagonal, but nowhere else. If we now do not use the numbers 1 to $\prod_{c\in\mathcal{C}} n_c$ as indices for the change-of-basis matrix between the box basis for $\mathcal{M}(X)$ and the $U_{\mathcal{D}}$-adapted bases, but instead tuples $(i_{c_1}, \dots, i_{c_N})$ with $1 \leq i_c \leq n_c$, then it is easy to see that the entries of that change-of-basis matrix are given by the formula

$$m_{(i_{c_1},\dots,i_{c_N}),(i_{c_1},\dots,i_{c_N})} = \prod_{c\in\mathcal{C}} m^c_{i_c,j_c},$$

where the $m^c_{i_c,j_c}$ are entries of the individual change-of-basis matrices. We thus see that indeed this matrix M is a sparse matrix. Direct computations (performed in Matlab) however showed that at least for a number of examples, the inverse M^{-1} is a fully populated matrix. The author of this thesis was not able to find a $U_{\mathcal{D}}$-adapted basis which did not share this property.

With the argumentation laid out in this section, we cannot claim to have formally proved that it is fundamentally impossible to save effort due to the structural information one may have about P_2 from the structure of a coupled cell network to which it belongs. We made assumptions that may seem plausible, but were not proved, and at several points we glossed over complicated matters saying "it appears that" or "it seems that". Nevertheless, the author of this thesis is

convinced that the problems described are substantial, and he hopes that the above argumentation also convinces the reader.

4.1.3 Possible ways out

Although the argument presented in the previous section makes the author of this thesis generally pessimistic as far as the possibility of generic efficient numerical approximation of transfer operators of coupled cell systems is concerned, it shall not be denied that there might be solutions to the problems we encountered. Presently, the author sees some approaches which for completeness we shortly present here.

Orthogonal bases In order to facilitate the problems that stem from a fully populated change-of-basis matrix, it would be best to construct the $U_{\mathcal{D}}$-adapted basis in such a way that the resulting change-of-basis matrix is both sparse and orthogonal. This can be achieved in particular by choosing bases for each $\mathcal{M}(X_c)$ in such a way that the matrices M^c are sparse and orthogonal, which in turn means that the individual basis measures should have "small" (in terms of the number of boxes) supports (this implies sparsity) and that they should be pairwise orthogonal with respect to the scalar product on $\mathcal{M}(X_c)$ induced from the standard scalar product on $\mathbb{R}^{n_c}$ via the box basis. However, it is not clear to the author how one could start to construct such a basis.

Row-wise computation But even if M could be chosen sparse and orthogonal, one still had the problem that using that standard method, one cannot do any less than computing whole columns of P_1 at a time. This would be different if one found a different way of evaluating the dynamical system, which allows "row-wise" computations. Indeed it seems that such an approach is possible after a certain "paradigm shift": If one assumes that *pre-images* of subsets of X under f can be approximated, then it seems that it is possible to "transpose" the standard algorithm. This is explained in more detail in Section Section 5.2.3.

Number of test points An aspect that has hitherto not been touched, but which might lead to some savings, is the number of test points that is necessary to guarantee a certain quality of the approximation of the image of a box. If one takes into account the structural information on P_2, then after multiplication with M and M^{-1} one obtains linear equations as constraints for the entries of P_1. (If one succeeded to construct a basis in such a way that both M and M^{-1} are sparse (or at least of a simple structure), then these constraints could even be accessible analytically.) It might be that the number of test points that is necessary to obtain a certain quality of approximation for the entries of P_1 can be reduced due to these constraints, e. g. through an error-correction scheme based

on projections onto the constraint manifold. However, it appears that generally little is known about the relationship between the number of test points and the quality of approximation, although it seems that it should be possible to derive statements about this relationship from general statements about the Monte-Carlo approximation of higher-dimensional integrals.

4.2 Efficient Numerical Approximation

In this section we are going to describe another approach that allows to save numerical effort in the computation of the transfer operator of a coupled cell systems. This approach is rather unrelated to the developments presented so far. With the problem of incompatible bases necessary for the numerical computation on the one hand and the structural theory presented in the earlier chapters on the other hand, we find ourselves in what seems to be a dead end if we try to design algorithms for the numerical computation of the transfer operator on the basis of evaluations of the dynamical system. For this reason, we go back to the definition of a coupled cell system, remind ourselves of the standard Ulam approach for the approximation of the transfer operator described in Section 2.4.2, and see how this approach can be adapted to the coupled cell setting in such a way that it becomes unnecessary to sample the complete, high-dimensional state space with test points which is the prohibitive restriction for a straightforward application of the standard approach.
The basic idea behind the method is the following. We have seen that the standard approach to compute an Ulam approximation of the transfer operator for a map $f : X \to X$ estimates the ratios of measures of sets of the form

$$\frac{m(f^{-1}(A) \cap B)}{m(B)}$$

for certain sets (boxes) $A, B \subset X$ by computing the images of a sufficiently large number of test points in B, counting the number of images that are contained in A and taking the ratio of these numbers as an estimate for the ratio of the two measures. The naive use of this method for the transfer operator of coupled cell systems becomes prohibitively large for two reasons, both of which can be reduced to the high dimension of the state space X: On the one hand, the number of sets A necessarily grows exponentially in the dimension of X; on the other hand, the number of test points in each of these sets that is necessary to estimate the measure of $f^{-1}(A) \cap B$ grows exponentially as well. As we will see in the following example, it is possible to use smaller numbers of test points. Furthermore, if one considers the task of computing not only one of the sets $f^{-1}(A) \cap B$, but *all* sets $f^{-1}(B_i) \cap B_j$ for a box collection $\{B_i\}$, further savings are possible.

Example 4.2
As an example we consider a map f is admissible on the three cell network shown in Figure 4.1, defined on the cartesian product $X \times Y \times Z$ of three spaces. Then f has the three component maps f_1, f_2, f_3 such that $f_1(x, y, z) = \hat{f}_1(x, y)$,

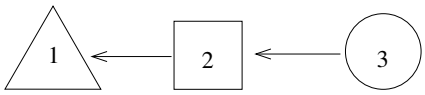

Figure 4.1: Three cell example network.

$f_2(x, y, z) = \hat{f}_2(y, z)$ and $f_3(x, y, z) = \hat{f}_3(z)$.For simplicity, we assume that the set B (e. g. a coordinate box in $X \times Y \times Z$) is approximated by a fine mesh of points, i. e. a collection of test points of the form $P = P_x \times P_y \times P_z$, where $P_x \subset X$, $P_y \subset Y$ and $P_z \subset Z$ are finite collections of points. Furthermore, we assume that the set A is of the form $A = A_1 \times A_2 \times A_3$ (an assumption that does not imply much loss of generality for box discretization schemes). Then one can rewrite the expression $f^{-1}(A) \cap B$ by noting that

$$f^{-1}(A) = \hat{f}_1^{-1}(A_1) \times Z \cap X \times \hat{f}_2^{-1}(A_2) \cap X \times Y \times \hat{f}_3^{-1}(A_3).$$

The interesting property of the right hand side is that its non-trivial parts are subsets of the 'low-dimensional' partial cartesian products $X \times Y$, $Y \times Z$ and Z alone, respectively. To approximate these sets, one needs only the relatively low-dimensional meshes $P_x \times P_y$, $P_y \times P_z$ and P_z, which are together of cardinality much smaller than P.
A rough description of an algorithm for our example building upon this observation could be as follows. Given boxes $A, B \subset X \times Y \times Z$ as above, choose point collections P_x, P_y, P_z so that P is a sufficiently fine approximation of B. Let $|P_x| = n_x$, $|P_y| = n_y$, and $|P_z| = n_z$. Next compute the image sets $\hat{f}_1(P_x \times P_y)$, $\hat{f}_2(P_y \times P_z)$ and $\hat{f}_3(P_z)$, and form $P_1 = (P_x \times P_y) \cap \hat{f}_1^{-1}(A_1)$, $P_2 = (P_y \times P_z) \cap \hat{f}_2^{-1}(A_2)$ and $P_3 = P_z \cap \hat{f}_3^{-1}(A_3)$ by checking whether an image point is in the correspondig set A_i. Finally count the number of points in $P_1 \times P_z \cap P_x \times P_2 \cap P_x \times P_y \times P_3$ simply by matching lists of points. The ratio of this number to $n_x \cdot n_y \cdot n_z$ is an approximation of the ratio $\frac{m(f^{-1}(A) \cap B)}{m(B)}$. Its computation needed $n_x \cdot n_y + n_y \cdot n_z + n_z$ evaluations of 'one-dimensional'[4]maps, as opposed to the $n_x \cdot n_y \cdot n_z$ evaluations of the multi-dimensional f needed in the standard approach.
Now imagine that not only the two sets A and B are considered, but instead box collections $\{B_i^X \mid i \in I_X\}$, $\{B_i^Y \mid i \in I_Y\}$ and $\{B_i^Z \mid i \in I_Z\}$ are given that cover the spaces X, Y and Z, respectively. Then the collection $\mathcal{B} = \{B_i \times B_j \times B_k \mid i \in$

[4]Taking the three spaces X, Y and Z as one-dimensional, which is of course an unnecessary restriction if taken literally.

$I_X, j \in I_Y, k \in I_Z\}$ covers $X \times Y \times Z$. Assume that all sets of the form $f^{-1}(B_i) \cap B_j$ for boxes $B_i, B_j \in \mathcal{B}$ are to be computed. Then of course one could use the procedure just described on every pair of boxes individually. However, this would be rather inefficient, as the images of the same sets of test points would be computed several times. To see this, consider test points in the two distinct boxes $B_1 = B_1^X \times B_1^Y \times B^Z$ and $B_2 = B_2^X \times B_2^Y \times B^Z$. For the computation of the z-components of their images only the identical test points from the z-component B^Z of these boxes are needed, which would be done twice if one just went through the procedure described above for B_1 and B_2 independently.
Instead, one should simply compute all needed test point images in advance, store them in some form of table and simply look up the specific test point images needed to compute an approximation for a specific set $f^{-1}(B_i) \cap B_j$.

In the following, we generalize and formalize the procedure described in this example. It should be noted that the algorithm described here can be viewed as a mere adaptation of the standard algorithm used for the computation of the transition matrix in GAIO [6], where the computation of test point images is 'outsourced' to table lookups, with the tables being prepared in a kind of pre-processing step.
The algorithm uses an enumeration of the box collection covering X that we describe in the following. Let box coverings $\mathcal{B}_c$ of X_c be given with $|\mathcal{B}_c| = n_c$ and arbitrary but fixed orderings $\mathcal{B}_c = \{B_c^i \mid 0 \leq i \leq n_c - 1\}$, together with an arbitrary but fixed order $\mathcal{C} = \{c_1, \ldots, c_N\}$ of the set of cells. Then the total number of boxes in $\mathcal{B}_X = \{\prod_{c \in \mathcal{C}} B_c \mid B_c \in \mathcal{B}_c\}$ is $\prod_{j=1}^N n_{c_j}$, and we order them such that $B_k = B_{c_1}^{i_1} \times \ldots \times B_{c_N}^{i_N}$ with

$$k = \sum_{j=1}^{N} i_j \cdot \prod_{l=1}^{j-1} n_{c_l}.$$

(Figuratively speaking, one counts the boxes in $\mathcal{B}_X$ first along the axis associated with c_1, then along the axis associated with c_2 and so on.)

Algorithm 4.3 (Computation of a transition matrix)

Input:
- Coupled Cell network: Set of cells $\mathcal{C}$, input sets $I(c)$, maps $\hat{f}_c$.
- Box discretizations $\mathcal{B}_c$ of the individual state spaces X_c.
- Sets of test points P_c^B for each box $B \in \mathcal{B}_c$.

Part 1: **Prepare test point image tables** For each cell $c \in \mathcal{C}$ perform the following three steps.

1. Form the box collection $\mathcal{B}_{I(c)} = \{\prod_{c' \in I(c)} B_{c'} \mid B_{c'} \in \mathcal{B}_c\}$.

2. For each box $B = \prod_{c' \in I(c)} B_{c'} \in \mathcal{B}_{I(c)}$ form the corresponding set of domain test points:

$$P^B_{I(c)} = \prod_{c' \in I(c)} P^{B_{c'}}_{c'}.$$

3. Compute $I_B = \hat{f}_c(P^B_{I(c)})$, and find out to which box in $\mathcal{B}_c$ each element of I_B belongs.

Part 2: **Compute the transition matrix** For each $0 \leq s \leq \prod_{j=1}^{N} n_{c_j}$ compute the intersections of $f(B_s)$ with all boxes $B_t \in \mathcal{B}_X$ in the following way.

1. Initialise a list $T(p)$ indexed by the test points $p \in B_s$ with zero, and for each $j \in 1, \ldots, N$
   a) look up in which box $B^{i_j(p)}_{c_j} \in \mathcal{B}_{c_j}$ the images $f_{c_j}(p)$ of the test points are contained, and
   b) add the number $i_j(p) \cdot \prod_{l=1}^{j-1} n_{c_l}$ to $T(p)$.
2. For each $0 \leq t \leq \prod_{j=1}^{N} n_{c_j}$ determine $n(t) = |\{p \mid T(p) = t\}|$. Store the quotient $n(t)/m(s)$ as matrix entry p_{ts}.

Output: The transition matrix for $f : X \to X$ with respect to $\mathcal{B}_X$.

Remark 4.4
A computer implementation of this algorithm does not need to handle actual boxes from $\mathcal{B}_X$ as full-dimensional object, as the second part of the algorithm uses B only for indexing purposes. While it is therefore not necessary to have a representation of $\mathcal{B}_X$ in memory, it is not in principle avoidable to store the transition matrix, which may severely limit the practical usability of the algorithm for already moderately sized systems. However, in many situations the matrix can be expected to be sparse, e. g. for systems that are time-T maps of differential equations with sufficiently small T. In these cases the usability limit can be reached at substantially larger systems.

5 Conclusion

This chapter concludes the thesis with a review of what has been reached, describing in brief what can be concluded from the main material of this thesis; and with a listing of some possibilities for further research which I recognized while writing. For different reasons, both parts are kept rather short. While in the review I try to bring this thesis' endeavours to a point, I feel obliged to restrict myself to short sketches of the "further possibilities".

5.1 What can be concluded

Let us briefly review the contents of this work. Its objective was to analyze and understand structural features of the transfer operator for a coupled cell system that originate from the underlying network structure. As the state space of a coupled cell system has a natural description as the cartesian product of the individual cell state spaces X_c, the domain of the transfer operator has a natural description as the tensor product $\mathcal{M}(X) = \bigotimes_{c\in\mathcal{C}} \mathcal{M}(X_c)$ of the measure spaces over the individual state spaces. Having understood (by Theorem 2.42) that generically it cannot be expected that the tensor product could be replaced by a direct sum, we looked for alternative ways to decompose $\mathcal{M}(X)$, and constructed a direct sum decomposition of $\mathcal{M}(X) = \bigoplus_{\mathcal{D}\subset\mathcal{C}} U_{\mathcal{D}}$ as a joint refinement of decompositions suggested from Lemma 3.1. Based on this decomposition, we were able to identify transfer operators for component maps f_c or partial maps $f_{\mathcal{D}}$ within P_f as "rows" of the block matrix description of P_f. We defined the $\mathcal{D}$-relative symmetry groups associated with a coupled cell network, and found that results from classical representation theory can be used to obtain block diagonalisations of these rows with respect to the isotypic decomposition associated to the actions of the $\mathcal{D}$-relative symmetry groups.
The relative success we could enjoy in describing the implications of the network structure on the structure of the operator is balanced by the relative failure we had to accept when we tried to make the structural information useful for a numerical approximation of the transfer operator. The author was not able to devise a method for the approximation that allows to use the structural information, and had to accept the difficulties that prevented him from finding such a method. The core of the problem is that – assuming that the map f is numerically accessible by point evaluations only – it seems one cannot circumvent using the basic idea employed in the standard algorithm for approximating the transfer operator; that is, it seems one has to approximate images of boxes. This

implies that one cannot make use of information about blocks of P_f with respect to the joint decomposition, as in any case one has to compute full columns of the operator. We were able to describe an algorithm that is able to exploit the structure of the coupled cell network for the purpose of reducing the number of test points (and therefore: of evaluations of f) necessary for a computation of the transfer operator, but which does not take into account the structural properties of the transfer operator and which therefore cannot offer an efficient numerical representation of the transfer operator. In the end, it seems we have to accept that the additional structure that comes from a coupled cell network does not provide a liberating spell that could be used to counteract the curse of dimension.
Summing up, the following points can be considered to be the conclusions one can draw out of this work.

1. The decomposition $\mathcal{M}(X) = \bigoplus_{\mathcal{D}\subset\mathcal{C}} U_{\mathcal{D}}$ allows to describe implications of the network structure for the transfer operator in terms of a block matrix decomposition.

2. Independence of the evolution of one cell from other cells is reflected by zero blocks with respect to this decomposition.

3. Also non-classical forms of symmetry in a network can be described using the notion of relative symmetry groups.

4. Unless alternative ways of evaluating the map f are given, there seem to be serious problems preventing the usage of the structural information for an efficient numerical representation of the transfer operator.

5. It is possible to adapt the standard algorithm for the approximation of the transfer operator to make use of a coupled cell network structure. This reduces the number of evaluations of f, but not the size of the representation of P_f.

5.2 What else could be done

The aim of this section is to collect some of the questions that arose and remained open during the preparation of the thesis.

5.2.1 Relative symmetry groups and the groupoid

Having introduced the notion of $\mathcal{D}$-relative symmetry groups, it was the initial aim of the author to show that knowledge of all the groups $\Gamma_{\mathcal{D}}$ is equivalent

to knowledge of the symmetry groupoid $\mathcal{B}_G$, i. e. that the groupoid can be reconstructed from the groups. While in view of the structure of $\mathcal{B}_G$ (see Proposition 2.4) this seemed an easy task at first sight, the author did not succeed in proving this relationship. The main problem is to guarantee that for each non-empty $B(c,d)$ one can derive an element of $B(c,d)$ from the collection of the groups $\Gamma_{\mathcal{D}}$.

5.2.2 Symmetry-adapted bases

Here we describe a possible approach for dealing with the problem of how to combine the $U_{\mathcal{D}}$-adapted basis, the box basis and the isotypic decompositions of subspaces of $\mathcal{M}(X)$ with respect to actions of the relative symmetry groups $\Gamma_{\mathcal{D}}$. In Section 2.2.1 we already briefly mentioned the notion of a symmetry-adapted basis. If $\theta : G \to GL(V)$ is a representation, this is a basis of V with the property that every θ-equivariant endomorphism in V is represented by a matrix with block diagonal structure. This structure encompasses the diagonalisation with respect to the isotypic decomposition, but in general it can be even finer. At least theoretically, such bases can be constructed using projector formulas similar to the one presented in Lemma 2.11.
In the classical applications of representation theory in dynamical systems theory (see e. g. [26]) the representation space is typically the state space of the system to be considered, that is, the state space carries a vector space structure and can thus be identified with its own tangent spaces (alternatively, the tangent bundle can be written as $V \times V$). This means that given a symmetry-adapted basis, one only has to find an expression for the dynamical system, which, if not analytically, can at least be dealt with numerically using change-of-basis matrices.
In our case, however, the situation is somewhat more complicated. We need a symmetry adapted-basis not for the state space of our dynamical system, but for measure spaces $\mathcal{M}(X_{\mathcal{D}})$ over partial state spaces. Ideally, one would like to construct a basis that combines three properties at once, a basis that is a symmetry-adapted $U_{\mathcal{D}}$-adapted box basis. Probably, this is too ambitious. To relax our requirements, we can e. g. drop the $U_{\mathcal{D}}$-adaptedness. Thus, it appears whorthwhile to investigate the following question: Is it possible to find a coordinate system on $X = \prod_{c \in \mathcal{C}} X_c$ such that a box basis with respect to this coordinate system generates a symmetry-adapted basis on $\mathcal{M}(X)$? It seems it is difficult to answer this question.

5.2.3 Row-wise approximation of the transfer operator

In Chapter 4 we discussed the difficulties arising from the fact that conventional numerical schemes for the approximation of the transfer operator operate "column-wise". Due to these problems, it appears reasonable to search for other schemes that do not have this property, and it turns out that in fact this seems

not impossible to achieve. All possibilities the author of this thesis has found arise from the thought that one seeks to "transpose" the mode of operation (in a very vague sense), and that the transposition of the transition matrix of a Markov chain is related to time-reversal. Computationally, time-reversal is essentially the computation of pre-images instead of images, and thus at the heart of these methods lies the assumption that one can somehow compute pre-images of points under f.
We already saw that the key task in computing the transition matrix of a Markov chain approximating a transfer operator is to determine expressions of the type $m(f^{-1}(B_i) \cap B_j)$, where f is a map, m is some measure on the state space and B_i are elements of a partition of the state space. When going forward in time, one chooses test points in B_j and approximates the set $f^{-1}(B_i) \cap B_j$ by those points which have images in B_i. Thus by computing the images of all the test points in B_j, one obtains approximations of the matrix entries $p_{ij} = \frac{m(f^{-1}(B_i) \cap B_j)}{m(B_j)}$ for all i at once, that is, one obtains an approximation of the j-th column.
To compute the i-th row, in contrast, one has to fix the box B_i in the expression for the transition probabilities, and to approximate $f^{-1}(B_i) \cap B_j$ for all j. Assuming one is capable of computing pre-images, this means that one chooses N test points in B_i and computes their pre-images. From this point on, there are different possibilities:

1. In cases where knowledge of the pre-images of test points in B_i is sufficient to estimate $m(f^{-1}(B_i) \cap B_j)$, one can do precisely that. This will usually imply that the "pre-image point cloud" gives a good geometric description of $f^{-1}(B_i)$. Depending on the specific situation, this can in particular be the case when further information on f, such as Lipschitz bounds or convexity, is given.

2. Otherwise, the following strategy could be pursued. For each box B_j, one counts the number k_j of pre-image points contained in B_j. Going back to equation (2.4), (and using the terminology used there,) one then observes

$$\begin{aligned} p_{ij} &= P(f(x) \in B_i \mid x \in B_j) \\ &= \frac{P(x \in B_j \ \wedge \ f(x) \in B_i)}{P(x \in B_j)} \\ &= \frac{P(f(x) \in B_i)}{P(x \in B_j)} \cdot \frac{P(x \in B_j \ \wedge \ f(x) \in B_i)}{P(f(x) \in B_i)} \\ &= \frac{P(f(x) \in B_i)}{P(x \in B_j)} \cdot P(x \in B_j \mid f(x) \in B_i) \\ &\approx \frac{P(f(x) \in B_i)}{P(x \in B_j)} \cdot \frac{k_j}{N}, \end{aligned}$$

as the fraction $\frac{k_j}{N}$ can be interpreted as an approximation of the conditional probability that $f^{-1}(x) \in B_j$ given that $x \in B_i$. This shifts the difficulty from estimating $m(f^{-1}(B_i) \cap B_j)$ to estimating $P(f(x) \in B_i)$. Although one might object that this in general assumes that the transfer operator is already known, one can imagine situations in which this problem is easier to handle. Most notably, when there is an invariant measure that is known *a-priori* (e. g. Liouville volume for the case of mechanical systems) the problem is reduced to evaluating that measure.

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