Dissertation

Circular flows on signed graphs

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in partial fulfillment of the requirements for the degree of
doctor rerum naturalium (Dr. rer. nat.)

Paderborn 2018
Abstract

For planar graphs, integer nowhere-zero flows and face colorings are equivalent concepts. Considering embeddings in orientable surfaces every face coloring can be expressed as an integer nowhere-zero flow. However, not every integer nowhere-zero flow can be expressed as a face-coloring. Therefore, nowhere-zero flows generalize the concept of colorings.

In this thesis we study flows on signed graphs. Signed graphs \((G,\sigma)\) generalize ordinary unsigned graphs \(G\) in such a way that each edge is either positive or negative. While positive edges must be directed in the ordinary way, negative edges must be either oriented introverted or extroverted. Nowhere-zero flows on signed graphs differ essentially from nowhere-zero flows on unsigned graphs because negative edges define a set of edges that act like sources and sinks.

We will motivate the study of flows on signed graphs over colorings and tensions. A graph \(G\) has a nowhere-zero \(k\)-tension if and only if \(G\) has a \(k\)-coloring. For every graph \(G\) there exists one surface such that \(G\) can be embedded without crossing-edges and each circuit in this embedding is a contractible curve. Corresponding to one wisely chosen embedding, a face coloring of \(G\) always exist. We show that the corresponding tension leads in a natural way to an integer nowhere-zero flow on a certain dual signed graph.

Integer nowhere-zero flows on signed graphs establish a generalization for colorings. We also study a refinement, circular nowhere-zero \(r\)-flows where we allow edges to have real flow-values from the set \(\{x \in \mathbb{R} | 1 \leq |x| \leq r - 1\}\). We study the relation between the circular flow number \(F_c((G,\sigma))\) and the integer flow number \(F((G,\sigma))\) that are defined as the infimum over all \(r\) such that \((G,\sigma)\) admits a circular nowhere-zero \(r\)-flow or respectively an integer nowhere-zero \(r\)-flow. For flow-admissible graphs, Raspaud and Zhu proved that \(F((G,\sigma)) \leq 2\lfloor F_c((G,\sigma))\rfloor - 1\), and
they conjectured $F_c((G, \sigma)) > F((G, \sigma)) - 1$. We disprove this conjecture by showing that $\sup\{F((G, \sigma)) - F_c((G, \sigma))\} \geq 2$. Furthermore, we show some sufficient conditions for $[F_c((G, \sigma))] = F((G, \sigma))$.

The circular or integer flow spectrum ($\mathcal{S}(G)$ or $\bar{\mathcal{S}}(G)$, respectively) of a graph $G$ is the set of all possible circular respectively integer flow numbers given due to arbitrary flow-admissible signatures. We study integer and circular flow spectra on regular graphs. The circular flow number $2 + \frac{1}{t}$ is the smallest possible flow number in a regular graph. We characterize $(2t + 1)$-regular graphs whose flow spectrum contains $2 + \frac{1}{t}$. Furthermore, we analyze some cases for the flow spectrum of a graph $G$ if $G$ has a 1-factor. By introducing the concept of $r$-minimal sets we prove for a graph $G \neq K_2^3$ the following statements are equivalent:

(1) $G$ has a 1-factor. (2) $3 \in \mathcal{S}(G)$. (3) $3 \in \bar{\mathcal{S}}(G)$. (4) $4 \in \bar{\mathcal{S}}(G)$.

We find cubic graphs whose integer flow spectrum does not contain 5 or 6, and we construct an infinite family of bridgeless cubic graphs with integer flow spectrum $\{3, 4, 6\}$. We prove some sharp bounds for the cardinality of smallest 3-minimal and 4-minimal sets, respectively. If $G$ is not 3-edge-colorable, then these bounds are formulated in terms of its resistance and oddness.

A Kotzig graph is a cubic graph that has three 1-factors such that the union of any two of them induces a hamiltonian circuit. We give a proof of Bouchet’s conjecture for Kotzig-graphs.

Let $\mathcal{F}^c$ be the set of circular flow numbers that can be obtained by signed graphs and $\mathcal{F}^c_k$ be the set of circular flow numbers that can be obtained by $k$-regular signed graphs. We show that $\mathcal{F}^c_{2k+1} = (\mathcal{F}^c - [2; 2 + \frac{2}{2k-1}]) \cup \{2 + \frac{1}{k}\}$. As a corollary for unsigned graphs we entirely determine the set of flow numbers for regular graphs up to 5, which is best possible if Tutte’s 5-flow conjecture is true.
Lastly, we relate the problem of finding certain circular and integer nowhere-zero flows to the problem of finding a set of orientations with special properties. In this regard, we characterize all nowhere-zero flows on signed graphs.

Zusammenfassung


In dieser Arbeit beschäftigen wir uns mit signierten Graphen. Signierte Graphen \((G,\sigma)\) verallgemeinern gewöhnliche unsignierte Graphen \(G\), sodass jede Kante entweder eine positive oder negative Kante ist. Positive Kanten werden wie im gewöhnlichen Fall orientiert. Negative Kanten sind entweder introvertiert oder extrovertiert. Nirgends-null Flüsse auf signierten Graphen unterscheiden sich wesentlich von Flüssen auf unsignierten Graphen, da negative Kanten eine Menge von Kanten bilden, die als Quellen und Senken interpretiert werden können.

Wir motivieren das Studium von Flüssen auf signierten Graphen über Färbungen und Tensionen. Ein Graph \(G\) hat eine nirgends-null \(k\)-Tension genau dann, wenn \(G\) eine \(k\)-Färbung hat. Für jeden Graphen \(G\) existiert eine Fläche, in die \(G\) ohne Kreuzungskanten eingebettet werden kann und jeder Kreis in dieser Einbettung bildet eine Kurve, die sich auf einen Punkt zusammenziehen lässt. Für eine mit Bedacht gewählte Einbettung existiert eine Flächenfärbung. Wir zeigen, dass die zugehörige Tension
in natürlicher Art und Weise zu einem ganzzahligen nirgends-null Fluss auf einem dualen signierten Graphen führt.

Ganzzahlige nirgends-null Flüsse auf signierten Graphen verallgemeinern den Färbungsbegriff. Wir untersuchen neben ganzzahligen nirgends-null Flüssen zudem eine Verfeinerung: zirkuläre nirgends-null r-Flüsse, bei denen wir reelle Flusswerte aus der Menge \( \{ x \in \mathbb{R} | 1 \leq |x| \leq r - 1 \} \) erlauben. Wir betrachten die Beziehung zwischen der zirkulären Flusszahl \( F_c((G, \sigma)) \) und der ganzzahligen Flusszahl \( F((G, \sigma)) \). Für Graphen, die einen nirgends-null Fluss zulassen, bewiesen Raspaud and Zhu \( F((G, \sigma)) \leq 2[F_c((G, \sigma))] - 1 \) und stellten die Vermutung \( F_c((G, \sigma)) > F((G, \sigma)) - 1 \) auf. Wir widerlegen diese Vermutung, indem wir \( \sup\{F((G, \sigma)) - F_c((G, \sigma))\} \geq 2 \) zeigen. Desweiteren stellen wir einige hinreichende Bedingungen für \( [F_c((G, \sigma))] = F((G, \sigma)) \) auf.

Das zirkuläre bzw. das ganzzahlige Flusspektrum \( S(G) \) bzw. \( \mathcal{S}(G) \) eines Graphen \( G \) ist die Menge aller möglichen zirkulären bzw. ganzzahligen Flusszahlen, die durch beliebige Signaturen, welche einen nirgends-null Fluss zulassen, gegeben sind. Wir untersuchen ganzzahlige und zirkuläre Flusspektren auf regulären Graphen. Der Wert \( 2 + \frac{1}{t} \) ist die kleinste mögliche Flusszahl eines regulären Graphen. Wir charakterisieren \( (2t + 1) \)-reguläre Graphen, deren Flusspektrum \( 2 + \frac{1}{t} \) enthält. Desweiteren untersuchen wir einige Fälle des Flusspektrums eines Graphen \( G \), falls \( G \) einen 1-Faktor hat. Mithilfe des entwickelten Konzepts von \( r \)-minimalen Mengen beweisen wir, dass für einen Graphen \( G \neq K_2^3 \) folgende Aussagen equivalent sind:

1. \( G \) hat einen 1-Faktor.
2. \( 3 \in S(G) \).
3. \( 3 \in \mathcal{S}(G) \).
4. \( 4 \in \mathcal{S}(G) \).

Wir finden Graphen, deren ganzzahliges Flusspektrum nicht 5 oder 6 enthält und wir konstruieren eine unendliche Familie von brückenlosen kubischen Graphen mit ganzzahligem Flusspektrum \( \{3, 4, 6\} \). Wir beweisen einige scharfe Schranken für die Kardinalität von kleinsten 3-
minimalen bzw. 4-minimalen Mengen. Falls $G$ nicht 3-kantenfärbbar ist, sind diese Schranken über den Widerstand und die Ungradheit von $G$ definiert.

Ein Kotzig Graph ist ein kubischer Graph, der drei 1-Faktoren besitzt, sodass die Vereinigung zweier beliebiger 1-Faktoren einen Hamiltonkreis induziert. Wir beweisen Bouchets Vermutung für Kotzig Graphen.

Sei $F^c$ die Menge der zirkulären Flusszahlen, die durch signierte Graphen erhalten werden können und sei $F^c_k$ die Menge der zirkulären Flusszahlen, die durch $k$-reguläre signierte Graphen erhalten werden können. Wir zeigen $F^c_{2k+1} = (F^c - [2; 2 + \frac{2}{2k-1}]) \cup \{2 + \frac{1}{k}\}$. Für unsignierte Graphen ist $F^c_k$ damit vollständig bis 5 bestimmt. Dies ist bestmöglich, falls Tuttes 5-Fluss Vermutung wahr ist.

Abschließend überführen wir das Problem gewisse zirkuläre oder ganzzahlige nirgends-null Flüsse zu finden in das Problem eine Menge von Orientierungen mit speziellen Eigenschaften zu finden. In diesem Zusammenhang charakterisieren wir alle nirgends-null Flüsse.
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Chapter 1

Introduction

1.1 Preliminaries

One of the most famous issues in graph theory is the Four color theorem. Nearly one and a half century ago Francis Guthrie raised the conjecture that four colors are always sufficient to color a map in such a way, that adjacent regions receive different colors. Once the conjecture was stated, there appeared many proofs and counterexamples that emerge to be wrong until Appel and Haken finally proved the conjecture in 1976. The mathematical historical significance for that theorem established since this is one of the first major theorems which could just be prooven with the aid of a computer so far.

This thesis is about flows on signed graphs. On the one hand, flows can be interpreted as a generalization of map-colorings and on the other hand, signed graphs establish a more general concept for graphs. A more detailed explanation about the connection of colorings and flows is given in chapter 2. However, map-colorings are only defined for crossing-free embeddings of graphs. The fact that flows extend the intuitional concept
of map-colorings in the non-planar case made a great stir and caused flows to be a well-respected topic in graph theory.

In 1954, Tutte stated the conjecture that every bridgeless graph admits a nowhere-zero 5-flow. A nowhere-zero 5-flow of a graph is an assignment of a direction and a value from the set \{1, 2, 3, 4\} for each edge, in such a way that for each vertex the sum over all incoming values equals the sum over all outgoing values. This conjecture became one of the most famous in flow theory and until now it is unsolved. However, there are several approaches for that conjecture as for example Seymour’s 6-flow theorem, where the conjecture is prooven to be true by exchanging the set of numbers by \{1, 2, 3, 4, 5\}.

For signed graphs the concept of nowhere-zero flows is basically the same, except the fact that we allow graphs to have negative and positive edges. The orientation of a positive edge is always defined from one vertex towards another vertex, whereas the orientation of a negative edge can only be directed towards each incident vertex, or away from each incident vertex. Regarding that concept of flows on signed graphs, flows on unsigned graphs are a special case where all edges are positive.

Until now, no flow-admissible all-positive graph is known that does not have a nowhere-zero 5-flow. On the contrary, in the signed case infinitely many graphs exist that do not admit a signed nowhere-zero 5-flow.

In this context two famous conjectures remain unsolved, for all-positive graphs the above mentioned Tutte’s 5-flow-conjecture and for signed graphs the conjecture that every flow-admissible graph admits a nowhere-zero 6-flow stated by Bouchet in 1983.
1.2 Basic Definitions

A graph $G$ is an ordered pair $G = (V,E)$ where $V = V(G)$ is a set of vertices and $E = E(G)$ is a set of edges that are 2-element subsets of $V$. Two vertices $v, w \in V$ are neighbors and called adjacent if there exists an edge $e \in E$ with $e = (v,w)$ or short $vw$. We can construe and visualize graphs as vertices that are connected by edges. In this thesis we consider graphs to be finite.

For a vertex $v \in V(G)$ let $E(v)$ be the set of edges which are incident to $v$, and let $|E(v)|$ be the degree of $v$ which is denoted by $d_G(v)$. A graph $G$ is $d$-regular if all vertices of $G$ have the degree $d$. The maximum degree in a graph is denoted by $\Delta(G)$ and the minimum degree in a graph is denoted by $\delta(G)$.

A signed graph $(G,\sigma)$ consists of a graph $G$ and a function $\sigma$ from $E(G)$ into $\{-1,1\}$. The function $\sigma$ is called a signature. Let $e \in E(G)$ an edge. Either $\sigma(e) = 1$ in which case $e$ is called a positive edge, or $\sigma(e) = -1$ in which case $e$ is called a negative edge. For a graph $G$ the set which consists of all negative edges is denoted by $N_\sigma$. It is called the set of negative edges while $E(G) - N_\sigma$ is called the set of positive edges. Every subset $X$ of $E(G)$ defines a signature $\sigma$ of $G$ with $N_\sigma = X$. If all edges of $(G,\sigma)$ are positive, i. e. when $N_\sigma = \emptyset$, we will call $(G,\sigma)$ an all-positive signed graph. An unsigned graph is an all-positive signed graph. If we refer to the unsigned case we will denote an all-positive signed graph $(G,\sigma)$ simply by $G$ and if we refer to the signed case we will denote $(G,\sigma)$ by $(G,1)$.

Let $e \in E(G)$ be an edge which is incident to the vertices $u$ and $v$. We divide $e$ into two half-edges $h^+_e$ and $h^-_e$, one incident to $u$ and one incident to $v$. The set of the half-edges of $G$ is denoted by $H(G)$. For each half-edge $h \in H(G)$, the corresponding edge in $E(G)$ is denoted by $e_h$. For a
vertex $v$, $H(v)$ denotes the set of half-edges incident to $v$. An orientation of $(G, \sigma)$ is a function $\tau : H(G) \to \{\pm 1\}$ such that $\tau(h^u_v)\tau(h^v_u) = -\sigma(e)$ for each edge $e = uv$. The function $\tau$ can be interpreted as an assignment of a direction to each edge in the following way:

For a positive edge exactly one half-edge is incoming and the other one is outgoing. For a negative edge either both half-edges are incoming, in which case $e$ is an extroverted edge, or both half-edges are outgoing, in which case $e$ is an introverted edge. Taken together, a positive edge can be directed like or like and a negative edge can be directed like (extroverted edge) or like (introverted edge). An oriented signed graph is also called a bidirected graph.

For a vertex $v$ let $H^+(v)$ be the set of outgoing half-edges and $H^-(v)$ be the set of incoming half-edges which are incident to $v$.

Let $n \geq 1$ and $P = u_0u_1...u_n$ be a path. We say that $P$ is a $v$-$w$-path if $v = u_0$ and $w = u_n$. Let $(G, \sigma)$ be oriented. If a path $P$ of $G$ does not contain any negative edge and for every $i \in \{0, \ldots, n-1\}$ the edge $u_iu_{i+1}$ is directed from $u_i$ to $u_{i+1}$, then we say that $P$ is a directed $v$-$w$-path. If $P$ is not directed, then every edge of $P$ that is directed from $u_i$ to $u_{i+1}$ will be called forward-edge and every edge of $P$ that is directed from $u_{i+1}$ to $u_i$ will be called backward-edge.

Let $r, r', x$ be real numbers with $0 \leq r' < r$. We write $x \equiv r' \pmod{r}$ if there is an integer $t$ such that $x - r' = tr$.

Let $(G, \sigma)$ be a signed graph with orientation $\tau$. The boundary of the orientation $\tau$ is a function $\delta\tau : V(G) \to \mathbb{R}$ with $\delta\tau = \sum_{h \in H(v)} \tau(h)$. The boundary of a function $f : E(G) \to \mathbb{R}$ is a function $\delta f : V(G) \to \mathbb{R}$ with $\delta f(v) = \sum_{h \in H(v)} \tau(h)f(e_h)$.

The function $f$ is a (modular) $r$-flow on $(G, \sigma)$, if $|f(e)| \in \{x : 1 \leq x \leq r-1\} \cup \{0\}$ for every $e \in E(G)$ and $\delta f(v) = 0$ ($\delta f(v) \equiv 0 \pmod{r}$) for every $v \in V(G)$. An $r$-flow $f$ is also called an circular flow. If $f(e) \in \mathbb{Z}$
for all $e \in E(G)$ we also call $f$ an integer $r$-flow. Every flow $f$ has one underlying orientation $\tau_f$. For defining one flow we also use the tuple $(D, f)$ consisting of an orientation $D$ and a flow $f$.

The set $\{ e : f(e) \neq 0 \}$ is the support of $f$, and $f$ is a nowhere-zero (modular) $r$-flow on $(G, \sigma)$ if $E(G)$ is the support of $f$.

A signed graph $(G, \sigma)$ is flow-admissible, if there exists an orientation $\tau$ and a number $r \geq 2$ such that $(G, \sigma)$ has a nowhere-zero $r$-flow. The circular flow number of a flow-admissible signed graph $(G, \sigma)$ is

$$F_c((G, \sigma)) = \inf\{ r : (G, \sigma) \text{ admits a nowhere-zero } r\text{-flow} \}.$$ 

If we restrict our studies on flows that are integer-valued functions, the corresponding integer flow number is denoted by $F(G, \sigma)$.

Let $(D, \phi)$ be a nowhere-zero $k$-flow on $(G, \sigma)$. If we reverse the orientation of an edge $e$ (or of the two half-edges, respectively) and replace $\phi(e)$ by $-\phi(e)$, then we obtain another nowhere-zero $k$-flow $(D^*, \phi^*)$ on $(G, \sigma)$. Hence, if $(G, \sigma)$ is flow-admissible, then it has always a nowhere-zero flow with all flow values positive. This also shows that if $F_c(G, \sigma) = r$, then there is a nowhere-zero $r$-flow for any prescribed orientation of $(G, \sigma)$.

Let $(G, \sigma)$ be a signed graph. For $i \in \{1, 2\}$ let $\phi_i$ be a flow on $(G, \sigma)$ with underlying orientation $\tau_i$. Note that for each edge $e = uv$ either $\tau_1(h_e^u) = \tau_2(h_e^u)$ and $\tau_1(h_e^v) = \tau_2(h_e^v)$ or $\tau_1(h_e^u) \neq \tau_2(h_e^u)$ and $\tau_1(h_e^v) \neq \tau_2(h_e^v)$. The sum $\phi_1 + \phi_2$ is the function $\phi$ on $(G, \sigma)$ with orientation $\tau$, where $\tau = \tau_1\{h_{\phi_1(e_u) \geq \phi_2(e_u)}\} \cup \tau_2\{h_{\phi_2(e_u) > \phi_1(e_u)}\}$, and $\phi(e) = \phi_1(e) + \phi_2(e)$ if $e$ has the same direction in $\tau_1$ and $\tau_2$, and otherwise $\phi(e) = |\phi_1(e) - \phi_2(e)|$. Clearly, if $|\phi(e)| \geq 1$ for every edge with $\phi(e) \neq 0$, then $\phi$ is a flow.

Let $(G, \sigma)$ be a signed graph. A switching at $v$ defines a graph $(G, \sigma^*)$ with $\sigma^*(e) = -\sigma(e)$ if $e$ is incident to $v$, and $\sigma^*(e) = \sigma(e)$ otherwise. We say that signed graphs $(G, \sigma)$ and $(G, \sigma^*)$ are switching equivalent if they
can be obtained from each other by a sequence of switchings. We also say that $\sigma$ and $\sigma^*$ are *switching equivalent signatures* of $G$. If we consider a signed graph with an orientation $D$, then switching at $v$ is a change of the orientations of the half-edges that are incident with $v$. If $D^*$ is the resulting orientation, then we say that $D$ and $D^*$ are *switching equivalent orientations*.

A circuit in $(G, \sigma)$ is *balanced*, if it contains an even number of negative edges; otherwise it is *unbalanced*. Note that a circuit of $(G, \sigma)$ remains balanced (resp. unbalanced) after switching at any vertex of $(G, \sigma)$. The signed graph $(G, \sigma)$ is an *unbalanced graph*, if it contains an unbalanced circuit; otherwise $(G, \sigma)$ is a *balanced graph*.

A signed graph $(G, \sigma)$ is called a barbell if either $(G, \sigma)$ consists of two circuits that intersect in exactly one vertex or $(G, \sigma)$ consists of two disjoint circuits $C_1, C_2$ and a path $P$ that connects $C_1$ with $C_2$ and $V(C_1) \cap V(C_2) \cap P$.

A circuit cover of a graph $G$ is a set of circuits $C_1, C_2, ..., C_n$ in $G$ such that the union $\bigcup_{i=1}^{n} E(C_i) = E(G)$. A signed circuit is a graph that is either a balanced circuit or a barbell. A graph which has a circuit that visits each vertex precisely once is a *hamilton graph*. The corresponding circuit is called a *hamiltonian circuit*.

An *eulerian* graph is a graph, that has one circuit that contains each edge of the graph precisely once. Clearly, if $(G, \sigma)$ and $(G, \sigma')$ are equivalent and $H$ is an eulerian subgraph of $G$, then $|N_\sigma \cap E(H)|$ and $|N_{\sigma'} \cap E(H)|$ have the same parity. Hence, $H$ is unbalanced in $(G, \sigma)$ if and only if it is unbalanced in $(G, \sigma')$.

1.3 Contribution of the thesis

Some results of this thesis have already been published:
1.4 State of the art

- The results of section 3.1 (except 3.1.5 and 3.1.6) and of chapter 4 have been published in

- The results of the parts 5.1.1 - 5.1.3 and section 5.2 have been published for unsigned graphs in

1.4 State of the art

Certainly, for studying flows we are mainly interested in flow-admissible graphs. In general, one can always construct a nowhere-zero flow by the summation of proper flows if the support of those covers the whole edge set. The question arises, what subgraphs are the smallest possible in such a way that we can define flows on these subgraphs in order to cover each edge at least once. It turns out that flow-admissibility is based on the collection of signed circuits. Let \((G, \sigma)\) be a signed graph. \((G, \sigma)\) is flow-admissible if and only if there exists a set of signed circuits in \((G, \sigma)\) that covers \(E(G)\). For an unsigned graph \(G\) it follows that \(G\) is flow-admissible if and only if \(G\) has a circuit cover what is equivalent of being bridgeless. If \((G, \sigma)\) does not admit any nowhere-zero flow, we set \(F_c((G, \sigma)) = \infty\).

It is easy to see that two switching equivalent graphs \((G, \sigma)\) and \((G, \sigma')\) have the same (circular) flow numbers. Thus, switching is an equivalence relation. It is well known (see e.g. \[30\]) that \((G, \sigma)\) is balanced if and only if it is switching equivalent to \((G, 1)\). Bouchet \[3\] showed that \((G, \sigma)\) is not flow-admissible if and only if \((G, \sigma)\) is switching equivalent to \((G, \sigma')\) with \(|N_{\sigma'}| = 1\) or \(G\) has a bridge \(b\) and a component of \(G - b\) is balanced.
For an unsigned graph $G$, an orientation $D$, and a set $X \subseteq V(G)$ of $G$ the set $D^+(X)$ denotes the set of edges oriented from $V(G) - X$ towards $X$ and $D^-(X)$ denotes the set of edges oriented from $X$ towards $V(G) - X$. From a theorem of [11] it can be shown that an unsigned graph $G$ admits a circular nowhere-zero $r$-flow if and only if $G$ has one orientation such that for all $X \subseteq V(G)$, $r - 1 \geq \frac{|D^+(X)|}{|D^-(X)|}$ and $r - 1 \geq \frac{|D^-(X)|}{|D^+(X)|}$ [6]. Thus, for flow-admissible unsigned graphs it can be seen that the infimum $F_c(G)$ is a minimum and it is obtained at a rational number. For signed graphs the authors of [30] generalize the above mentioned characterization for circular flows and deduce that for every flow-admissible graph $(G,\sigma)$, $F_c((G,\sigma))$ is a minimum and it is obtained at a rational number.

In 1954, Tutte established the topic of flows. Flows on graphs constitute a very active field of research in graph theory and there are many difficult conjectures on circular flows on graphs, see [37] for a brief overview. The flow number of unsigned graphs has been studied intensively in the recent years. One of the most famous conjectures in this context is Tutte’s 5-flow conjecture [41].

**Conjecture 1.4.1.** [41] If $G$ is flow-admissible, then $G$ admits a nowhere-zero 5-flow.

The smallest counterexample for that an unsigned flow-admissible graph does not admit a nowhere-zero $k$-flow for $k \geq 2$ is cubic [33]. Hence, it is sufficient to proof Conjecture 1.4.1 for cubic graphs.

There exist infinitely many graphs $G$ for which the circular flow number $F_c(G) = 5$ (see [18]). The first graph that was known in this regard is the Petersen graph. A proof that for the Petersen graph $P$ holds $F_c(P) = 5$ can be found in [35]. Thus, the bound in Tutte’s conjecture is sharp. So far, the closest approach for Tutte’s conjecture is Seymour’s 6-Flow Theorem.
1.4 State of the art

Theorem 1.4.2. [33] If $G$ is flow-admissible, then $G$ admits a nowhere-zero 6-flow.

In fact Tutte’s 5-flow Conjecture as well as Seymour’s 6-flow Theorem were stated for integer flows. Until now, it is still an open issue if there exists an $\epsilon \in (0, 1)$ such that every flow-admissible graph $G$ admits a nowhere-zero $(5 + \epsilon)$-flow. Furthermore, Tutte’s 5-flow conjecture and Seymour’s 6-flow Theorem were stated without the consideration of a graph to be signed, since that generalization was established later.

Naturally, the concept of nowhere-zero flows has been extended in several ways. In this thesis we study one generalization of these – nowhere-zero flows on signed graphs. Flows on signed graphs were introduced by Bouchet, who stated the following conjecture.

Conjecture 1.4.3. [3] Let $(G, \sigma)$ be a signed graph. If $(G, \sigma)$ is flow-admissible, then $(G, \sigma)$ admits a nowhere-zero 6-flow.

The bound on Bouchet’s conjecture is best possible, since the Petersen graph, equipped with a certain signature, has flow number 6. However, there exists no published proof, yet. To show that a flow number of a graph equals a specific value $k$ we need to show that there exist one nowhere-zero $k$-flow on the graph and there exist no nowhere-zero $k'$-flow with $k' < k$. The latter case usually is quite hard to show since there are many possibilities. In Appendix A we will prove that the Petersen graph, equipped with a certain signature, has flow number 6.

For cubic graphs, Bouchet’s conjecture 1.4.3 is as follows:

Conjecture 1.4.4. Let $(G, \sigma)$ be a signed cubic graph. If $(G, \sigma)$ is flow-admissible, then $(G, \sigma)$ admits a nowhere-zero 6-flow.

It is well-known that Bouchet’s conjecture is equivalent to its restriction on cubic graphs. Note, that for showing the equivalence for signed
graphs, a proof similar to that in [33] does not work. Since there seems to be no proof published so far, we will give one proof in the following.

**Theorem 1.4.5.** Conjecture 1.4.3 and Conjecture 1.4.4 are equivalent.

**Proof.** If Conjecture 1.4.3 is true, then obviously Conjecture 1.4.4 is true. Now, assume that Conjecture 1.4.4 is true. Let \((G, \sigma)\) be a non-cubic flow-admissible signed graph. We suppress all vertices of degree 2 and for each vertex \(v \in V(G)\) with \(d_G(v) \geq 4\) we subdivide each incident edge \(e_i^v (i \in \{1, \ldots, d_G(v)\})\) by adding a new vertex \(v_i\). Now, delete \(v\) and add the edges \((e_i^v, e_{i+1}^v)\). The addition of the indices is taken modulo \(d_G(v)\). The edges joining the circuit are defined to be \(f_i^v\), such that each \(v_i\) has the edges \(e_i^v, e_{i+1}^v, f_i^v\). We repeat this procedure for each vertex \(v \in V(G)\). We end up with a cubic graph \(G'\) and every vertex \(v \in V(G)\) corresponds to a circuit \(C_v\) of \(G'\). The edge set \(E(G)\) induces a 1-factor \(F_1\) in \(G'\), and the circuits in \(G'\) which correspond to the vertices in \(G\) form a 2-factor \(F_2\). Let \(\sigma' : E(G') \rightarrow \{\pm 1\}\) be the signature with \(\sigma'(e) = \sigma(e)\) for each \(e \in F_1\) and \(\sigma'(e) = 1\) for each \(e \in F_2\). We claim that \((G', \sigma')\) is flow-admissible. Construct a nowhere-zero flow \(\psi\) as follows: Let \(\phi\) be a nowhere-zero flow on an orientation \((G, \tau)\) of \((G, \sigma)\). We may assume that \(\phi(f) \geq 1\) for all \(f \in F_1\). As otherwise, we switch the direction of \(f\) and set the flow value to its negative inverse. For every circuit \(C_v\) let \(\delta_G^+(v)\) be the set of edges such that the corresponding half-edge is directed to the circuit. We define \(\psi : E(G') \rightarrow \mathbb{R}\) with \(\psi(e) = \phi(e)\) for \(f \in F_1\). For each circuit \(C_v\) in \(F_2\) we direct all edges in one direction along the circuit and set:

\[
\psi(e_1) = \sum_{u \in \delta_G^+(v)} \phi(u) + 1
\]

\[
\psi(e_{i+1}) = \begin{cases} 
\psi(e_i) + f_i, & \text{if } f_i \text{ is directed towards } C_v \\
\psi(e_i) - f_i, & \text{if } f_i \text{ is directed away from } C_v 
\end{cases}
\]
Hence, \((G', \sigma')\) is flow-admissible and due to Conjecture 1.4.4 \((G', \sigma')\) admits a nowhere-zero 6-flow. Now we contract the circuits given by \(F_2\) and insert the edges of degree 2 which were suppressed in \(G\) and by contraction we get \(F((G, \sigma)) \leq 6\).

\[\square\]

The best published approximation to Bouchet’s Conjecture 1.4.3 is proved by the following theorem of Zýka.

**Theorem 1.4.6.** [52] Let \((G, \sigma)\) be a signed graph. If \((G, \sigma)\) is flow-admissible, then \((G, \sigma)\) admits a nowhere-zero 30-flow.

Moreover, DeVos proved in one manuscript that every flow-admissible graph \((G, \sigma)\) admits a nowhere-zero 12-flow [5].

For 2-edge-connected signed graphs the following improvement was proved.

**Theorem 1.4.7.** [4] Let \((G, \sigma)\) be a 2-edge-connected signed graph. If \((G, \sigma)\) is flow-admissible, then \((G, \sigma)\) admits a nowhere-zero 11-flow.

Bouchet’s conjecture has been confirmed for particular classes of graphs [23, 21] and also for signed graphs with restricted edge-connectivity (for example [30]). By Theorem 1.4.2 it is also true for signed graphs with all edges positive, because it describes the special case for all-positive graphs. Furthermore, if for a graph \((G, \sigma)\) there is an edge-cut \(X \subseteq E(G)\) with \(X = N_{\sigma}\), then \((G, \sigma)\) is switching equivalent to \((G, 1)\) and it follows that their (circular) flow numbers coincide. This can be seen easily by switching all vertices of one side of the cut.

Another interesting fact about flows in unsigned graphs is, that whenever we consider one flow, we are able to scale the flow values in a specific way as shown by the following theorem.
Theorem 1.4.8. [35] Let $G$ be a graph and $F_c(G, \sigma) = \frac{p}{q} + 1$. Then there exists a nowhere-zero $(\frac{p}{q} + 1)$-flow $\phi$ on $G$, such that for each edge $e \in E(G)$ exists one $n_e \in \mathbb{Z}$ such that $\phi(e) = \frac{n_e}{q}$.

For signed graphs this theorem does not hold in general.
Chapter 2

Motivation

There exist several studies that connect signed graphs with interesting applications from other branches besides mathematics. As an example, for politics we would like to refer to a structural analysis of the situation in the middle east in 1956 [9] or a graph-theoretical approach for the analysis of international relations [10]. In physics signed graphs are often used to study spin glasses (see [39] for example). Moreover, in chemistry signed graphs are used to study molecular structures (see [8] for example). This is just a short overview about topics where signed graphs are used.

For more information in [49] a mathematical bibliography of signed and gain graphs and allied areas is given with more than 340 pages. Moreover, the author of [49] regularly updates a webpage which contains a list of hundreds of publications that fall in the area of signed graphs. For signed graphs pure mathematical concepts, as for example colorings, are also studied (see [47], [46], [48]). An interesting point is that there are even different concepts for colorings of signed graphs (see [19] and [16] for example). In this thesis we will focus on a pure mathematical analysis of flows of signed graphs. A survey article about flows on signed graphs is given in [15].
2.1 Flow and Coloring Duality

Throughout this chapter we consider flows to be integer flows. The motivation for studying flows emerges by the fact that flows and colorings are dual concepts. For unsigned graphs, this can be seen quite easily, while for signed graphs it turns out to be more complicated. However, the connection of colorings and flows, especially for signed graphs, establishes a great motivation. Therefore, we will give a brief overview about that topic.

Let $G$ be a graph. An embedding $M$ of $G$ into a surface $\Sigma$ is an injective function $\phi : V \to \Sigma$ together with a set of homomorphisms $\{\psi_e : e = (x, y) \in E\}$ such that:

1. $\psi_e : [0, 1] \to \Sigma$
2. $\{\psi_e(0), \psi_e(1)\} = \{\phi(x), \phi(y)\}$
3. $\psi_e((0, 1)) \cap (\{\phi(v) : v \in V\} \cup \{\psi_{e'} : e' \in E, e' \neq e, x \in [0, 1]\}) = \emptyset$

A crossing-free embedding is an embedding of a graph such that for any two edges there is no intersection. With respect to one surface $\Sigma$ we call a graph $G$ crossing-free if there exists a crossing-free embedding of $G$ into $\Sigma$. A graph $G$ is planar if there exists one crossing-free embedding of $G$ into a sphere.

The Jordan Curve Theorem allows us to define faces of a crossing-free embedding of a graph given by the connected components that are separated by edges which generate a Jordan curve.

**Theorem 2.1.1** (Jordan Curve Theorem [14]). Every simple closed curve $y$ separates the 2-sphere into two connected components of which $y$ is their common boundary.

The first proof of Theorem 2.1.1 was submitted in [14], however, some cases were missing. For a detailed proof we refer to [44].
Defining faces in one embedding is only meaningful if the embedding is crossing-free. However, for every graph $G$ exists a surface $\Sigma_G$ such that $G$ has a crossing-free embedding into $\Sigma$. For example, let $M$ be an embedding of a graph $G$ into $S^2$ with minimum number of crossing edges. For one intersection of two edges $e, f$ we add one handle $H$ to our surface, connecting the endvertices of $e$ and exchange the Jordan curve that represents $e$ by a Jordan curve that connects the end vertices of $e$ over $H$. By applying this procedure we end up with one surface $\Sigma^*$ that is orientable and one crossing-free embedding of $G$ into $\Sigma^*$.

However, this procedure might lead to the problem that not every face is bounded by one circuit of the graph, because not every closed Jordan Curve in the resulting surface is contractible. For defining face-colorings we need to consider embeddings where each face is bounded by one circuit. Thus, we can use the above described procedure and to avoid crossing edges we insert cross caps instead of adding handles. Eventually, we reach to one non-orientable surface $\Sigma^*$ and one crossing-free embedding of $G$ into $\Sigma^*$. Furthermore, each Jordan Curve is contractible in $\Sigma^*$. In summary, to ensure that for every graph a surface exists where we can define a face coloring on a proper crossing free embedding, we cannot rely on orientable surfaces. Jaeger proposed the following conjecture.

**Conjecture 2.1.2** (The strong embedding conjecture [13]). Every 2-connected graph $G$ has an embedding where each face boundary is a circuit in some orientable surface.

Thus, whenever we consider a face-coloring of a graph, we may require the corresponding embedding to be crossing-free and each face bounded by one circuit. For an embedding of a graph $G$ the set of faces is denoted by $F(G)$.

A *face-coloring* of an embedding $M$ of a graph $G$ is a mapping $c$ that assigns a color to every face of $G$ in $M$ in such a way that any two
adjacent faces receive different colors. Whenever not stated explicitly, for convenience we will use the set of integer numbers \(\{0, \ldots, k - 1\}\) to describe a set of \(k\) colors. A graph regarding to one embedding has a \(k\)-face-coloring, if it has a face-coloring which uses at most \(k\) colors. In a similar manner we define edge-colorings and vertex-colorings. A \(k\)-vertex-coloring of a graph is an assignment \(c : V(G) \rightarrow \{0, \ldots, k - 1\}\) such that adjacent vertices receive different colors. Moreover, a \(k\)-edge-coloring of a graph is an assignment \(c : E(G) \rightarrow \{0, \ldots, k - 1\}\) such that adjacent edges receive different colors. The smallest number of colors needed to edge-color or vertex-color \(G\) is the chromatic index or the chromatic number of \(G\), respectively.

Let \(G\) be a graph embedded in a surface \(\Sigma\) with faces \(F = F_1, \ldots, F_n\). Regarding one corresponding embedding \(M\) of \(G\), the dual graph \(G'\) is given by \(V(G) = \{v_1, \ldots, v_n\}\) and \(E(G) = \{(v_i, v_l) : F_i\text{ and }F_l\text{ are adjacent}\}\). By identifying each face of \(G\) as a vertex of \(G'\) and by connecting each pair of vertices if and only if the corresponding faces are adjacent we naturally get one embedding of \(G'\) in \(\Sigma\) as well. Clearly, \(G\) is \(k\)-face colorable if and only if \(G'\) is \(k\)-vertex colorable. For unsigned graphs Tutte proved the following.

**Theorem 2.1.3.** Let \(k \in \mathbb{Z}\). A graph \(G\) embedded in \(S^2\) has a \(k\)-face-coloring if and only if \(G\) admits a nowhere-zero \(k\)-flow.

Since the definition of nowhere-zero flows is independent of embeddings, flows on unsigned graphs establish a generalization for the concept of face-colorings.

To fathom a similar relation for signed graphs we need the concept of tensions. Let \(G\) be a graph and for every circuit \(C\) let \(D_C\) be an orientation of \(C\) such that each vertex has one incoming and one outgoing edge. For an orientation \(D\) of \(G\) we partition the set of edges of each circuit \(C\) into two sets \(C_1\) and \(C_2\). \(C_1\) contains all edges, that have
2.1 Flow and Coloring Duality

the same orientation in $D$ and $D_C$, and $C_2$ contains all edges that are oriented differently. $G$ has a $k$-tension if there exists an orientation and a function $\phi : E(G) \to \mathbb{R}$ such that for each circuit $C$ holds $\sum_{e \in C_1} \phi(e) = \sum_{e \in C_2} \phi(e)$. An embedded graph $G$ has a local $k$-tension if there exists an orientation and a function $\phi : E(G) \to \mathbb{R}$ such that for each circuit $C$, which is a contractible curve, holds $\sum_{e \in C_1} \phi(e) = \sum_{e \in C_2} \phi(e)$. A $k$-tension $\phi$ or a local $k$-tension $\phi$ where $1 \leq \phi(e) \leq k - 1$ is a nowhere-zero $k$-tension or nowhere-zero local $k$-tension, respectively. One equivalence between colorings and tensions is given by the following theorem. Note, that the following statements are well known. For the sake of motivating flows on signed graph, the proofs are given.

**Theorem 2.1.4.** Let $G$ be a graph and $k \in \mathbb{Z}$. $G$ has a nowhere-zero $k$-tension if and only if $G$ has a $k$-coloring.

**Proof.** We may assume that $G$ is connected. Let $f : E(G) \to \mathbb{Z}$ be a $k$-tension with an orientation $D$. We will define a $k$-coloring $c : E(G) \to \mathbb{Z}$ as follows: Choose a vertex $u \in V(G)$ and set $c(u) = 0$. To color the remaining vertices we successively apply the following rule: Let $x, y$ be two vertices that are connected by one edge $e = (x, y)$ such that $x$ is colored and $y$ is not colored. We set $c(y) \equiv c(x) + f(e) \pmod{k}$ if $e$ is directed towards $y$ and $c(y) \equiv c(x) - f(e) \pmod{k}$ if $e$ is directed towards $x$. Thus, for a vertex $z$ the value of $c(x)$ is the sum of $f(e)$ of forward edges $e$ minus the sum of $f(g)$ of backward edges $g$ on a $u$-$z$-path. Next, we have to show that the coloring which we get, is well-defined. Let $z$ be a vertex and $u_1$ and $u_2$ be two different neighbors of $z$. We assume that there is one $u$-$u_1$-path $W_1$ and one $u$-$u_2$-path $W_2$ in $G$, both paths do not contain $z$. Furthermore, we may assume that $W_1$ and $W_2$ only intersect in $u$ and $z$ and thus, we get a circuit $C$ consisting of $E(W_1) \cup E(W_2)$. If $W_1$ and $W_2$ would intersect in another vertex, then the last $v \in V(W_1) \cup V(W_2) \setminus \{u, z\}$ on the $u$-$u_1$-path would replace
u. We may color $z$ by a sequence of vertex colorings corresponding to $W_1$ or corresponding to $W_2$. We have to show, that, independent of the choice, $c(z)$ is uniquely defined. We define a direction $D_C$ of $C$ such that each vertex has one incoming and one outgoing edge and define $C_1 := \{ e \in C | D_C(e) = D(e) \}$ and $C_2 := C \setminus C_1$. Note that we may change the direction of one edge regarding to the tension and exchange its value in $f$ by its negative inverse and receive a valid tension again. Thus, we may assume that $C_1 = W_1 \cup \{ u_1 z \}$ and $C_2 = W_2 \cup \{ u_2 z \}$. Since $f$ is a tension it follows

$$\sum_{e \in C_1} f(e) - \sum_{e \in C_2} f(e) = 0. \quad (2.1)$$

For the color of $z$ regarding $W_1$ it is

$$c(z) \equiv c(u) + \sum_{e \in C_1} f(e) \pmod{k} \quad (2.2)$$

and with (2.1) it follows

$$c(z) \equiv c(u) + \sum_{e \in C_2} f(e) \pmod{k}. \quad (2.3)$$

Thus, independent of the choice, $c(z)$ is uniquely defined and therefore is $c$ well-defined.

Now, let $c : V(G) \rightarrow \mathbb{Z}$ be a $k$-coloring. We will define a nowhere-zero $k$-tension $f : E(G) \rightarrow \mathbb{Z}$ as follows. Let $e = uv$ be an edge. If $c(v) > c(u)$, we direct $e$ from $u$ towards $v$ and set $f(e) = c(v) - c(u)$ and if $c(u) > c(v)$, we direct $e$ from $v$ towards $u$ and set $f(e) = c(u) - c(v)$. Since $c$ is a proper $k$-coloring, $f$ is a proper $k$-tension.
In the following we observe the connection between colorings of embedded unsigned graphs and flows on the corresponding dual graphs. Therefore, we introduce a method for creating a bidirected dual signed graph from a directed unsigned graph embedded in one surface $\Sigma$. Let $G$ be a directed unsigned graph. For each cycle $C$ of $G$ we redefine one orientation $D_C$ such that each vertex of $C$ has one outgoing and one incoming edge. Let $G'$ be a dual graph of $G$ in regard to $\Sigma$. For a vertex $v'$ of $G'$ with incident edges $e'_1, \ldots, e'_{d_{G'}(v')}$ the corresponding edges $e_1, \ldots, e_{d_G(v)}$ in $G$ are given by the circuit $C_{v'}$ which bounds the corresponding face. We define a direction of the incident half-edges of $v'$ as follows: If $e_l$ ($l \in \{1, \ldots, d_G(v')\}$) has the same orientation in $G$ and $D_C$, then $h_{v}(e'_l)$ is incoming and if $e_l$ has a different orientation in $G$ and $D_C$, then $h_{v}(e'_l)$ is outgoing. Repeating this procedure for each vertex $u' \in G'$ leads to a bidirected dual graph with an induced signature. Let $\phi : E(G) \to \mathbb{R}$ be a mapping and let $\phi' : E(G') \to \mathbb{R}$ by $\phi'(e'_l) = \phi(e_l)$ for all $i \in \{1, \ldots, d_G(v)\}$. By this construction it is easy
to see that $\phi$ is a nowhere-zero $k$-flow if and only if $\phi'$ is a nowhere-zero local $k$-tension. Hence, a graph $G$ has a nowhere-zero $k$-flow if and only if the corresponding dual Graph $G'$ has a nowhere-zero local $k$-tension. In figure 2.1 one signed Petersen graph embedded in the projective plane and the corresponding dual graph is depicted with a nowhere-zero 6-flow and a nowhere-zero local 6-tension, respectively.
Chapter 3

Relations between the circular flow number and the integer flow number

The results of section 3.1, except 3.1.5 and 3.1.6, have already been published in [32]. In the previous chapter we concluded that, regarding certain embeddings, colorings and integer flows are dual concepts. The definition of a coloring leads to integer valued flows in a natural way. When we allow flows to be real-valued, we get a refinement of the concept of flows and therefore a refinement for colorings as well. However, the focus of this thesis is on flows. In this chapter we study the relation between the integer flow number and its refinement, the circular flow number. Clearly, for a signed graph \((G, \sigma)\) it holds \(F_c((G, \sigma)) \leq F((G, \sigma))\). For an unsigned graph \(G\), it is proven in [6] that \(F(G) = \lceil F_c(G) \rceil\).
3.1 The difference between $F_c((G, \sigma))$ and $F((G, \sigma))$

Raspaud and Zhu [30] proved that $F((G, \sigma)) \leq 2\lceil F_c((G, \sigma)) \rceil - 1$, and they conjectured the following.

**Conjecture 3.1.1.** [30] For every graph $(G, \sigma)$ holds $F_c((G, \sigma)) > F((G, \sigma)) - 1$.

We will show that this conjecture is not true. Let $\delta_F = \sup\{F((G, \sigma)) - F_c((G, \sigma)) : (G, \sigma) \text{ is flow-admissible}\}$. Let $t \geq 1$ be an integer and $H_t$ be the graph which is obtained from $2t + 1$ triangles $T_i$, one vertex $v$ and precisely one vertex of each triangle, say $v_i$, is adjacent to $v$. For $i \in \{1, \ldots, 2t + 1\}$ let $b_i = vv_i$. Clearly, each $b_i$ is a bridge and $H_t$ has no 1-factor. We define the signature $\sigma^*$ by the set of negative edges $N^\sigma$ which includes precisely each edge between the two bivalent vertices of each triangle.

**Theorem 3.1.2.** $F_c((H_t, \sigma^*)) = 3 + \frac{2}{t}$ and $F((H_t, \sigma^*)) = 5$, for each integer $t \geq 1$. Furthermore, $H_t$ has an integer nowhere-zero 5-flow $\phi$ such that $\phi(e) \in \{1, 2, 4\}$, and a $(3 + \frac{2}{t})$-flow with $\phi_c$ with $\phi_c(e) \in \{1, 1 + \frac{1}{t}, 2, 2 + \frac{2}{t}\}$ for all $e \in E(H_t)$.

**Proof.** We will construct an integer nowhere-zero 5-flow on $(H_t, \sigma^*)$.

Define an orientation $\tau$ on $H_t$ as follows: Let $e_i \in N^\tau \cap E(T_i)$ and let $e_1, \ldots, e_{t+1}$ be extroverted and $e_{t+2}, \ldots, e_{2t+1}$ be introverted. Orient the positive edges of the triangles such that the triangles are oriented like a "loop". For $i \in \{1, \ldots, t + 1\}$, $v$ is the terminal end of $b_i$, and for $j \in \{t + 1, \ldots, 2t + 1\}$, $v$ is the initial end of $b_j$. Let $\phi$ be the integer nowhere-zero 5-flow with $\phi(e) = 1$ if $e \in \bigcup_{i=1}^{2t} E(T_i)$, $\phi(e) = 2$, if $e \in \{b_1, \ldots, b_{2n}\} \cup E(T_{2t+1})$, and $\phi(e) = 4$, if $e = b_{2t+1}$. Hence, $F((H_t, \sigma^*)) \leq 5$. 
3.1 The difference between $F_c((G, \sigma))$ and $F((G, \sigma))$

Let $\psi$ be an integer nowhere-zero flow on $(H_t, \sigma^*)$. Let $E^+(v)\ (E^-(v))$ be the set of incoming (outgoing) edges at $v$. Assume that $|E^+(v)| \geq t + 1$. Since $\psi$ is an integer flow it follows that $\psi(b_i)$ is even for every bridge. Hence, $\sum_{b \in E^+(v)} \psi(b) \geq 2t + 2$. Since $|E^-(v)| \leq t$ and $\psi$ is an integer flow it follows that there is a bridge $b$ with $\psi(b) \geq 4$. Hence, $F((H_t, \sigma^*)) = 5$.

We construct a nowhere-zero $(3 + \frac{2}{t})$-flow $\phi_c$ on $(H_t, \sigma^*)$. Let $\tau$ be as above and $\phi_c(e) = \phi(e)$ if $e \in \bigcup_{i=1}^{t+1} E(T_i)$, $\phi_c(e) = 2$, if $e \in \{b_1, \ldots, b_{t+1}\}$, $\phi_c(e) = 2 + \frac{2}{t}$ if $e \in \{b_{t+2}, \ldots, b_{2t+1}\}$, and $\phi_c(e) = 1 + \frac{1}{t}$ if $e \in \bigcup_{i=t+2}^{2t+1} E(T_i)$. Hence, $F_c((H_t, \sigma^*)) \leq 3 + \frac{2}{t}$.

Let $\psi_c$ be a nowhere-zero flow on $(H_t, \sigma^*)$. Assume that $|E^+(v)| \geq t + 1$. Since $\psi_c(b_i) \geq 2$ for every bridge it follows that $\sum_{b \in E^+(v)} \psi_c(b) \geq 2t + 2$. Hence, there is a bridge $b$ with $\psi(b) \geq 2 + \frac{2}{t}$. Therefore, $F((H_t, \sigma^*)) = 3 + \frac{2}{t}$.

For $t \geq 2$, the graphs $(H_t, \sigma^*)$ are counterexamples to the conjecture of Raspaud and Zhu. One interesting fact about $H_t$ is also, that $H_t$ stays to be a counterexample under any arbitrary flow-admissible signature $\sigma$. 
Lemma 3.1.3. For $t \geq 2$, let $(H_t, \sigma)$ be a flow admissible graph. Then, $F((H_t, \sigma)) = F((H_t, \sigma^*))$ and $F_c((H_t, \sigma)) = F_c((H_t, \sigma^*))$.

Proof. Let $(H_t, \sigma)$ be flow-admissible. We show that $(H_t, \sigma)$ is switching equivalent to $(H_t, \sigma^*)$ where precisely the edge between the two bivalent vertices of each triangle is negative. Switch, if necessary, at vertices $v_i$ to obtain a switching equivalent signature where all bridges are positive. Clearly, each triangle is unbalanced. Hence, if three edges of a triangle are negative, then switch at a bivalent vertex such that precisely one edge of that triangle is negative. Now, if necessary, switch at a bivalent vertex to obtain $(H_t, \sigma^*)$. Hence, $F((H_t, \sigma)) = F((H_t, \sigma^*))$ and $F_c((H_t, \sigma)) = F_c((H_t, \sigma^*))$.

We saw, that for the graph $H_t$ exists only one flow number over all signatures. In chapter 4 we will study possible flow numbers for fixed graphs and variable signatures, and will introduce the term spectrum. The observation of lemma 3.1.3 shows that, the circular flow spectrum and the integer flow spectrum of $H_t$ contains precisely one element which is a seemingly rare property.

Graph $H_2$ is shown in Figure 3.1. As a consequence of Theorem 3.1.2 we state:

Corollary 3.1.4. $\delta_F \geq 2$.

The statement of Theorem 3.1.2 holds also for graphs obtained from $H_t$ by replacing the triangles by (negative) loops. Clearly, the argumentation for the lower bounds of the flow numbers in the proof of Theorem 3.1.2 works also if replace the triangles of $H_t$ by any other unbalanced component and $v$ by a balanced component. After this result was published, Mácajová and Steffen 22 proved $\delta_F \geq 3$ by using a slightly
modified graph, where each negative loop is exchanged by $2t$ negative loops. Furthermore, when Bouchet’s Conjecture is true, then $\delta_F = 3$. The constructed graphs have a special property.

Let $K_{1,t}$ be a graph consisting of one vertex, which is connected to $t$ different vertices by a single edge.

**Definition 3.1.5.** A star-cut is an induced subgraph $S$ isomorphic to $K_{1,t}$ of $G$ such that every vertex and every edge of $H$ is a cut.

Up to today, all examples with the property that $[F_c((G,\sigma))] < F((G,\sigma))$ contain a star-cut. It becomes natural to ask whether for each 2-edge-connected signed graph $(G,\sigma)$ the numbers $[F_c((G,\sigma))]$ and $F((G,\sigma))$ are the same. We deny this question by giving a counterexample.

By $K_4$ we denote the complete graph on four vertices.

**Proposition 3.1.6.** Let $(G,\sigma)$ be the signed graph obtained from the all-positive $(K_4,1)$ by deleting an edge $v_1v_2$ and adding two negative loops, $l_1$ at $v_1$ and $l_2$ at $v_2$. It holds $F_c((G,\sigma)) = 3$ and $F((G,\sigma)) = 4$.

**Proof.** Claim that $(G,\sigma)$ admits a circular nowhere-zero 3-flow. The graph $(K_4,1)$ admits a nowhere-zero 4-flow $f$ with precisely one edge $v_1v_2$ of flow value 3. Hence, define $\phi$ of $G$ from $f$ assigning $1 + \frac{1}{2}$ to each loop. We claim that $(G,\sigma)$ does not admit an integer nowhere-zero 3-flow. Suppose that $\phi$ is one positive integer nowhere-zero 3-flow on $(G,\sigma)$. Since $(G,\sigma)$ has only two negative edges, both loops are oriented in opposite directions and get the same flow value $x$ with $x \in \{1,2\}$. An integer flow $f$ of $(K_4,1)$ can be obtained from $\phi$ on $G$ by assigning $2x + \phi(v_1v_2)$ to the edge $v_1v_2$. If $x = 1$, then $f$ itself is a nowhere-zero 3-flow of $(K_4,1)$, which forms a contradiction. If $x = 2$, then $f(v_1v_2) = 4$ and $f : E(K_4) \to \{\pm1,\pm2,4\}$. Hence, $f$ is a modulo 3-flow of $(K_4,1)$.
(here, \( f(v_1v_2) \equiv 1 \pmod{3} \)), thus, by Lemma 3.2 of [45] \((K_4, 1)\) has an integer nowhere-zero 3-flow, which forms a contraction.

\[\square\]

### 3.2 Some sufficient conditions for

\[\lceil F_c((G, \sigma)) \rceil = F((G, \sigma))\]

In 3.1.2 it was shown that the supremum of \( F((G, \sigma)) - F_c((G, \sigma)) \) is at least 2. In the following we give some sufficient conditions for \( \lceil F_c((G, \sigma)) \rceil = F((G, \sigma)) \).

A signed graph \((G, \sigma)\) has a circular-flow-\(\frac{1}{\mu k}\)-property if for each nowhere-zero \(\frac{n}{k}\)-flow \(\phi\) on \((G, \sigma)\) exists a nowhere-zero \(\frac{n}{k}\)-flow \(\phi'\) on \((G, \sigma)\) such that \(\mu k \phi'(e) \in \mathbb{Z}\) for each edge \(e\). The family of signed graphs that have a circular-flow-\(\frac{1}{\mu k}\)-property is denoted by \(G_{\mu}\). With this notion, we can restate Theorem 1.4.8 as follows: All unsigned flow-admissible graphs are members of \(G_1\). Now, we will show the corresponding result for general signed graphs that all signed graphs are members of \(G_2\). This is best possible since in general flow-values can not be scaled by multiples of \(\frac{1}{q}\), as for example figure 3.1 shows. It is easy to see that every circular nowhere-zero 4-flow on \((H_2, \sigma)\) must contain an edge with the flow value \(1 + \frac{1}{2}\).

For the next theorem we need the concept of pseudoflows:

Let \((G, \sigma)\) be a signed graph. A function \(f : E(G) \to \mathbb{R}\) with orientation \(\tau_f\) which fulfills \(0 = \sum_{h \in \mathcal{H}(v)} \tau(h) f(e_h)\) for each vertex \(v \in V(G)\) is a pseudo-\(r\)-flow if \(0 \leq |f(e)| \leq r - 1\) for each edge \(e \in E(G)\). Obviously, a pseudo-\(r\)-flow is a nowhere-zero \(r\)-flow if \(|f(e)| \geq 1\) for each \(e \in E(G)\).
3.2 Some sufficient conditions for $\lceil F_c((G,\sigma)) \rceil = F((G,\sigma))$

For two nowhere-zero flows $f$ and $g$ with orientations $\tau_f$ and $\tau_g$, the difference $f - g$ is defined for all edges $e \in E(G)$ as

- an orientation $\tau_h$ where $\tau_h(e) = \tau_f(e)$ if $f(e) \geq g(e)$ or $\tau_h(e) = \tau_g(e)$ if $f(e) < g(e)$ and

- a function $h : E(G) \to \mathbb{R}$ with $h(e) = f(e) - g(e)$ if $f(e) \geq g(e)$ or $h(e) = g(e) - f(e)$ if $f(e) < g(e)$.

For a signed graph $(G,\sigma)$ with two pseudo flows $f$ and $g$, it is easy to see that $f - g$ is also a pseudo flow.

**Theorem 3.2.1.** Let $(G,\sigma)$ be a graph and $F_c((G,\sigma)) = \frac{p}{q} + 1$. Then there exists a nowhere-zero $(\frac{p}{q} + 1)$-flow $\phi$ on $(G,\sigma)$, such that $\forall e \in E(G)$, $\exists n_e \in \mathbb{Z} : \phi(e) = \frac{n_e}{2q}$.

**Proof.** Given a nowhere-zero $(\frac{p}{q} + 1)$-flow $\psi$ let $F_\psi = \{e \in E(G) : 2q\psi(e) \notin \mathbb{Z}\}$. Choose a nowhere-zero $(\frac{p}{q} + 1)$-flow $\phi$ of $(G,\sigma)$ for which $F_\phi$ has minimum cardinality. We may assume that $\phi(e) \geq 1$ for each $e \in E(G)$. If $F_\phi = \emptyset$ then the flow value on each edge is a multiple of $\frac{1}{2q}$ and we are done. Thus, assume $F_\phi$ is not empty. Furthermore, each vertex of $G$ is incident to either zero or at least two edges of $F_\phi$.

**Claim 1:** There is no subset of $F_\phi$ which induces a signed circuit.

Suppose to the contrary that there is a subset $F' \subseteq F_\phi$ that induces a signed circuit $C$. First, we consider the case that $C$ is a balanced circuit or consists of two unbalanced circuits that intersect in exactly one vertex. Since $C$ admits a nowhere-zero 2-flow let $D_B$ be an orientation corresponding to a nowhere-zero 2-flow of $C$ in $G$. Let $\epsilon = \min_{e \in F_\phi} \{\frac{p}{q} - \phi(e), \phi(e) - 1\}$. Let $\phi_p$ be a pseudoflow on $G$ with orientation $D_{\phi_p} = D_B \cup D_\phi|_{E(G) - E(C)}$, $\phi_p(e) = \epsilon$ for $e \in C$, and $\phi_p(e) = 0$
for \( e \in E(G) \setminus C \). Then, the mappings \( \phi + \phi_p \) and \( \phi - \phi_p \) are nowhere-zero \( (\frac{p}{q} + 1) \)-flows and \( F_{\phi + \phi_p} \) or \( F_{\phi - \phi_p} \) is a proper subset of \( F_{\phi} \) contradicting the choice of \( \phi \). It remains to show that \( C \) is not a barbell. Suppose \( C \) consists of two unbalanced circuits \( C_1 \) and \( C_2 \) and a path \( P \) connecting them. Define a parameter \( \alpha_e \) with \( \alpha_e = \frac{1}{2} \) for \( e \in P \) and \( \alpha_e = 1 \) for \( e \notin P \). Since \( C \) admits a nowhere-zero 3-flow let \( DB \) be an orientation corresponding to a nowhere-zero 3-flow. Let \( \epsilon = \min_{e \in F_{\phi}} \{ \alpha_e (\frac{p}{q} - \phi(e)), \alpha_e (\phi(e) - 1) \} \). Let \( \phi_p \) be a pseudoflow on \( G \) with orientation \( D_{\phi_p} = DB \cup D_{\phi|E(G) - E(C)} \), \( \phi_p(e) = \epsilon \) for \( e \in C_1 \cup C_2 \), \( \phi_p(e) = 2\epsilon \) for \( e \in P \), and \( \phi_p(e) = 0 \) for \( e \in E(G) \setminus C \). Again, \( \phi + \phi_p \) and \( \phi - \phi_p \) are nowhere-zero \( (\frac{p}{q} + 1) \)-flows and \( F_{\phi + \phi_p} \) or \( F_{\phi - \phi_p} \) is a proper subset of \( F_{\phi} \) contradicting the choice of \( \phi \).

**Claim 2:** For each \( v \in V \), \( |\delta(v) \cap F_{\phi}| \in \{0, 2\} \). For sure, \( |\delta(v) \cap F_{\phi}| \neq 1 \). Assume to the contrary that for \( v_1 \in V(G) \) holds \( |\delta(v_1) \cap F_{\phi}| \geq 3 \).

If \( v_1 \) belongs to a path in \( G[F_{\phi}] \), that connects two circuits, then either one circuit is balanced or \( v_1 \) belongs to a barbell. That contradicts Claim 1. Thus, \( v_1 \) belongs to one unbalanced circuit \( C \) and any path from \( v_1 \) must end in \( C \). Let \( P = v_1, w_1, \ldots, w_l, v_k \) (\( k \in [2, n] \), \( l \in \mathbb{Z} \)) (the \( w_i \)'s and \( v_j \)'s are pairwise disjoint) be such a path and let \( C = v_1, \ldots, v_n \). However, either the circuit \( C_1 = v_1, \ldots, v_k, w_1, \ldots, w_l, v_1 \) or \( C_2 = v_k, \ldots, v_n, v_1, w_1, \ldots, w_l, v_k \) is balanced contradicting Claim 1.

To complete the proof it remains to show that there is no unbalanced circuit in \( F_{\phi} \). Assume \( C = v_1, \ldots, v_n \) (\( n \in \mathbb{Z} \)) is an unbalanced circuit in \( F_{\phi} \) with edges \( e_l = (v_l, v_{l+1}) \), \( l \mod n \). We may change the orientation of \( C \) such that each vertex but \( v_1 \) is incident to exactly one incoming and one outgoing half-edge. For each edge for which we change the direction we also exchange its flow-value by its negative inverse and \( \phi \) remain a proper nowhere-zero flow. Since a vertex in \( C \)
3.2 Some sufficient conditions for \( [F_c((G, \sigma))] = F((G, \sigma)) \)

has exactly two incident edges from \( F_\phi \) we get \( 2q(\phi(e_n) + \phi(e_1)) \in \mathbb{Z} \) and \( 2q(\phi(e_k) - \phi(e_{k-1})) \in \mathbb{Z} \) for \( k \in \{2, \ldots, n\} \). By induction we get \( 2q(\phi(e_n) - \phi(e_1)) \in \mathbb{Z} \) and hence, \( 4q\phi(e_n) \in \mathbb{Z} \) and it follows that each \( \phi(e_k) \) is an odd multiple of \( \frac{1}{4q} \) for \( k \in \{1, \ldots, n\} \). Since \( C \) has an odd number of negative edges the minimum cut \( X \) that separates \( C \) from \( G \) carries an excess \( \sum_{e \in X \cap C^+} \phi(e) - \sum_{e \in X \cap C^-} \phi(e) = \frac{2(2m+1)}{4q} = \frac{2m+1}{2q} \) (\( m \in \mathbb{Z} \)). Let \( F' = \{ e \in E(G) : \phi(e) = \frac{2k+1}{2q}, k \in \mathbb{Z} \} \).

We get \( |F' \cap X| \equiv 1 \mod 2 \) and for each \( v \in V(G) - V(G[F_\phi]) \), \( |\delta(v) \cap F'| \equiv 0 \mod 2 \). Therefore, not every path with edges of \( F' \) that begins with a vertex in \( C \) ends in \( C \) as well. There exists a barbell \( B \) consisting of a path \( P \subseteq F' \) that connects \( C \) with another unbalanced circuit \( C' \subseteq F_\phi \). Since \( B \) admits a nowhere-zero 3-flow let \( D_B \) be an orientation corresponding to a nowhere-zero 3-flow. Let \( \phi_p \) be a pseudoflow on \( G \) with orientation \( D_{\phi_p} = D_B \cup D_{\phi}|_{E(G)-E(B)} \), \( \phi_p(e) = \frac{1}{4q} \) for \( e \in C \cup C' \), \( \phi_p(e) = \frac{1}{2q} \) for \( e \in P \), and \( \phi_p(e) = 0 \) for \( e \in E(G) \setminus B \). Again, \( \phi + \phi_p \) and \( \phi - \phi_p \) are nowhere-zero \( \left( \frac{p}{q}+1 \right) \)-flows and \( F_{\phi+\phi_p} \) or \( F_{\phi-\phi_p} \) is a proper subset of \( F_\phi \) contradicting the choice of \( \phi \) which completes the proof.

\( \square \)

**Lemma 3.2.2.** Let \( (G, \sigma) \in \mathcal{G}_1 \). Then \( [F_c((G, \sigma))] = F((G, \sigma)) \).

**Proof.** Let \( (G, \sigma) \in \mathcal{G}_1 \) with a circular nowhere-zero \( \frac{p}{q} \)-flow \( f \) and let \( h = \lceil \frac{p}{q} \rceil \). Since \( f \) can also be considered as a circular nowhere-zero \( h \)-flow, \( (G, \sigma) \) admits a circular nowhere-zero \( h \)-flow \( f' \) with rational flow values from the set \( \{1, 1+\frac{1}{1}, 1+\frac{2}{1}, \ldots, h-1\} \). Obviously, \( f' \) is an integer valued nowhere-zero \( h \)-flow.

\( \square \)
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Lemma 3.2.3. If \((G, \sigma)\) has no long barbell, then \((G, \sigma) \in G_1\).

Proof. Given a nowhere-zero \((\frac{p}{q} + 1)\)-flow \(\psi\) let \(w(e) \equiv \phi(e) \pmod{1}\) and \(F_\psi = \{ e \in E(G) : w(e) \neq 0 \}\). Choose a nowhere-zero \((\frac{p}{q} + 1)\)-flow \(\phi\) of \((G, \sigma)\) for which \(F_\phi\) has minimum cardinality. We may assume that \(\phi(e) \geq 1\) for each \(e \in E(G)\). If \(F_\phi = \emptyset\) then we are done. Thus, we suppose to the contrary, that \(F_\phi \neq \emptyset\).

Claim 1: \(F_\phi\) contains neither a balanced circuit nor consists of two unbalanced circuits that intersect in exactly one vertex.

Suppose that there is a subset \(F'_\phi \subseteq F_\phi\) that induces a signed circuit \(C\) that is either a balanced circuit or consists of two unbalanced circuits that intersect in exactly one vertex. Since \(C\) admits a nowhere-zero 2-flow let \(D_B\) be an orientation corresponding to a nowhere-zero 2-flow of \(C\) in \(G\). Let \(\epsilon = \min_{e \in F_\phi} \{ \frac{p}{q} - \phi(e), \phi(e) - 1 \}\). Let \(\phi_p\) be a pseudoflow on \(G\) with orientation \(D_{\phi_p} = D_B \cup D_{\phi'}|_{E(G) - E(C)}\), \(\phi_p(e) = \epsilon\) for \(e \in C\), and \(\phi_p(e) = 0\) for \(e \in E(G) \setminus C\). Then, the mappings \(\phi + \phi_p\) and \(\phi - \phi_p\) are nowhere-zero \((\frac{p}{q} + 1)\)-flows and \(F_{\phi + \phi_p}\) or \(F_{\phi - \phi_p}\) is a proper subset of \(F_\phi\) contradicting the choice of \(\phi\).

Thus, \(F\) contains an unbalanced circuit \(C_1\). By switching, we may assume that in \(C_1\) there is precisely one negative edge \(e_1\). We may assume that \(e_1\) is extroverted. Since \(\phi\) is balanced at every vertex, the total inflow and outflow of all negative edges is zero.

\[
\sum_{e \in F_\phi \cap N_\sigma} 2\phi(e) = 0.
\]

That is,

\[
\sum_{e \in F_\phi \cap N_\sigma} \phi(e) = 0.
\]
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Thus, there must be another negative edge $e_2$ in $F_\phi$. The edges $e_1, e_2$ are contained in different unbalanced circuits $C_1$ and $C_2$ of $F_\phi$. Joining $C_1$ and $C_2$ by a path of $G$, we get a long barbell, forming a contradiction.

□

With lemma 3.2.2 and 3.2.3 it simply follows the next Corollary.

**Corollary 3.2.4.** If $(G, \sigma)$ has no long barbell, then

$$[F_c((G, \sigma))] = F((G, \sigma)).$$
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Chapter 4

Flows on signed regular graphs

4.1 Preface

The results of this chapter have been published in [32]. Let $G$ be a graph and $X \subseteq E(G)$. Let $\Sigma_X(G)$ be the set of signatures $\sigma$ of $G$, for which $(G,\sigma)$ is flow-admissible and $N_\sigma \subseteq X$. We define $S_X(G) = \{r : \text{there is a signature } \sigma \in \Sigma_X(G) \text{ such that } F_c((G,\sigma)) = r\}$ to be the $X$-flow spectrum of $G$. The $E(G)$-flow spectrum is the flow spectrum of $G$ and it is denoted by $S(G)$. If we consider integer-valued flows, then $S_X(G)$ denotes the integer $X$-flow spectrum of $G$.

Section 4.2 characterizes $(2t+1)$-regular graphs whose flow spectrum contains $2 + \frac{1}{t}$. Furthermore, if a $(2t+1)$-regular graph has a 1-factor, then its integer flow spectrum contains 3. However, for every $t \geq 2$, there is a signed $(2t+1)$-regular graph $(H,\sigma)$ with integer flow number 3 and $H$ does not have a 1-factor.

One of the earliest results on flows on graphs is Tutte’s characterization of bipartite cubic graphs [10]. His observation that a cubic graph
is bipartite if and only if it admits a nowhere-zero 3-flow motivated the following statement.

**Theorem 4.1.1.** Let $t \geq 1$ be an integer. A $(2t + 1)$-regular graph $G$ is bipartite if and only if $F_c((G, 1)) = 2 + \frac{1}{t}$. Furthermore, if $G$ is not bipartite, then $F_c((G, 1)) \geq 2 + \frac{2}{2t-1}$.

The situation does not change in the more general case of flow numbers on signed $(2t + 1)$-regular graphs. It is proven that if $r$ is an element of the flow spectrum of a $(2t + 1)$-regular graph, then $r = 2 + \frac{1}{t}$ or $r \geq 2 + \frac{2}{2t-1}$. In order to generalize the structural part of Theorem 4.1.1 we will need the following definition: Let $r \geq 2$ be a real number and $G$ be a graph. A set $X \subseteq E(G)$ is $r$-minimal if

1) there is a signature $\sigma$ of $G$ such that $F_c((G, \sigma)) = r$ and $N_\sigma = X$, and

2) $F_c((G, \sigma')) \neq r$ for every signature $\sigma'$ of $G$ with $N_{\sigma'} \subset X$.

In Section 4.3 we show that a set $X \subseteq E(G)$ is a minimal set such that $G - X$ is bipartite if and only if $X$ is $(2 + \frac{1}{t})$-minimal.

Since Bouchet’s conjecture is equivalent to its restriction on cubic graphs we study flows on signed cubic graphs in Section 4.4. Let $K_2^3$ be the unique cubic graph on two vertices which are connected by three edges. We study the relation between 3- and 4-minimal sets and deduce that if $G$ has a 1-factor and $G \neq K_2^3$, then $\{3, 4\}$ is a subset of its flow spectrum and of its integer flow spectrum. Furthermore, if $G \neq K_2^3$, then the following four statements are equivalent: (1) $G$ has a 1-factor. (2) $3 \in \mathcal{S}(G)$ (3) $3 \in \overline{\mathcal{S}}(G)$. (4) $4 \in \overline{\mathcal{S}}(G)$. There are cubic graphs whose integer flow spectrum does not contain 5 or 6, and we construct an infinite family of bridgeless cubic graphs with integer flow spectrum $\{3, 4, 6\}$. 
4.2 Smallest possible flow numbers of signed \((2t + 1)\)-regular graphs

We prove some sharp bounds for the cardinality of smallest 3-minimal and 4-minimal sets, respectively. If \(G\) is not 3-edge-colorable, then these bounds are formulated in terms of its resistance and oddness.

A *Kotzig graph* is a cubic graph that has three 1-factors such that the union of any two of them induces a hamiltonian circuit. The chapter concludes with a proof of Bouchet’s conjecture for Kotzig-graphs.

### 4.2 Smallest possible flow numbers of signed \((2t + 1)\)-regular graphs

This section characterizes \((2t + 1)\)-regular graphs whose flow spectrum contains \(2 + \frac{1}{t}\).

Let \((G, \sigma)\) be a signed graph that admits a (modular) \(r\)-flow \(\phi\). Let \(e\) be an edge of the support of \(\phi\). If we reverse the orientation of \(e\) and replace \(\phi(e)\) by \(-\phi(e)\) \((r - \phi(e))\) then we obtain another (modular) \(r\)-flow \(\phi'\) with the same support as \(\phi\). Hence, we can assume that a graph which has a (modular) \(r\)-flow \(\phi\) has a (modular) \(r\)-flow \(\phi'\) with the same support and \(\phi'(e) \geq 0\) for all \(e \in E(G)\).

**Lemma 4.2.1.** Let \(t \geq 1\) be an integer and \((G, \sigma)\) be a signed \((2t + 1)\)-regular graph. If \((G, \sigma)\) admits a modular nowhere-zero \((2 + \frac{1}{t})\)-flow \(\phi\), then \(|\phi(e)| \in \{1, 1 + \frac{1}{t}\}\) for every \(e \in E(G)\).

**Proof.** Let \(\phi\) be a modular nowhere-zero \((2 + \frac{1}{t})\)-flow on \((G, \sigma)\). Suppose to the contrary that there is an edge \(e'\) with \(|\phi(e')| \notin \{1, 1 + \frac{1}{t}\}\).

By the remark above there is modular nowhere-zero \((2 + \frac{1}{t})\)-flow \(\psi\) with \(\psi(e) > 0\) for every \(e \in E(G)\). Since \(-\phi(e') \notin \{1, 1 + \frac{1}{t}\}\), and \(2 + \frac{1}{t} - \phi(e') \notin \{1, 1 + \frac{1}{t}\}\) it follows that \(\psi(e') \notin \{1, 1 + \frac{1}{t}\}\).

Let \(e' \in E(v)\). We can assume that there is an orientation of the half-edges of \((G, \sigma)\) such that \(|H^+(v)| = 2t + 1\) and a modular nowhere-
zero \((2 + \frac{1}{t})\)-flow \(\psi'\) such that \(\psi'(e') \not\in \{1, 1 + \frac{1}{t}\}\) and \(\psi'(e) > 0\) for every \(e \in E(G)\). It follows that there is a positive integer \(k\) such that 
\[
\sum_{h \in H^+(v)} \psi'(e_h) = k(2 + \frac{1}{t}).
\]
We have \(\sum_{h \in H^+(v)} \psi'(e_h) > 2t+1\) and hence, \(k > t\). On the other side \(\sum_{h \in H^+(v)} \psi'(e_h) < 2t+3+\frac{1}{t}\) and hence, \(k < t+1\), a contradiction. Therefore, \(|\phi(e)| \in \{1, 1 + \frac{1}{t}\}\) for every \(e \in E(G)\).

\[\square\]

The next theorem is a generalization of Lemma 3.2 of \cite{paper45}.

**Theorem 4.2.2.** Let \(t \geq 1\) be an integer. A signed \((2t+1)\)-regular graph \((G, \sigma)\) admits a nowhere-zero \((2 + \frac{1}{t})\)-flow if and only if \((G, \sigma)\) admits a modular \((2 + \frac{1}{t})\)-flow and \(G\) has a \(t\)-factor.

**Proof.** Let \(\phi\) be a nowhere-zero \((2 + \frac{1}{t})\)-flow on \((G, \sigma)\) with \(\phi(e) > 0\) for every \(e \in E(G)\). Since every \((2 + \frac{1}{t})\)-flow is a modular \((2 + \frac{1}{t})\)-flow it follows with Lemma 4.2.1 that \(\phi(e) \in \{1, 1 + \frac{1}{t}\}\) for every \(e \in E(G)\). The set of edges with flow value \(1 + \frac{1}{t}\) induces a \(t\)-factor of \(G\), and \(\phi\) is a modular nowhere-zero \((2 + \frac{1}{t})\)-flow.

If \((G, \sigma)\) admits a modular nowhere-zero \((2 + \frac{1}{t})\)-flow, then it follows with Lemma 4.2.1 there is one, say \(\phi\), such that \(\phi(e) = 1\) for every \(e \in E(G)\). Since \(G\) is \((2t+1)\)-regular, it follows that every vertex of \(G\) is incident to either incoming or outgoing edges, only. Let \(F\) be a \(t\)-factor of \(G\). If we reverse the orientation of the edges of \(F\), then the function \(\phi'\) with \(\phi'(e) = 1\) if \(e \in E(G) - E(F)\) and \(\phi'(e) = 1 + \frac{1}{t}\) if \(e \in E(F)\) is a nowhere-zero \((2 + \frac{1}{t})\)-flow on \((G, \sigma)\).

\[\square\]

We will need the following result of Petersen.

**Theorem 4.2.3.** \cite{paper29} Let \(k\) be a positive integer and \(G\) a \(k\)-regular graph. If \(k\) is even, then \(G\) has a 2-factor.
4.2 Smallest possible flow numbers of signed \((2t + 1)\)-regular graphs

**Theorem 4.2.4.** Let \(t \geq 1\) be an integer.

1) A \((2t + 1)\)-regular graph \(G\) has a \(t\)-factor if and only if \(2 + \frac{1}{t} \in \mathcal{S}(G)\).

2) If \(G\) is \((2t + 1)\)-regular and has a 1-factor, then \(3 \in \overline{\mathcal{S}}(G)\).

Furthermore, for each \(t > 1\) there is a \((2t + 1)\)-regular graph \(G_t\) which has no 1-factor and \(3 \in \overline{\mathcal{S}}(G_t)\).

**Proof.** 1) Let \(G\) have a \(t\)-factor. Let \(\sigma\) be the signature of \(G\) with \(N_\sigma = E(G)\). The function \(\phi\) with \(\phi(e) = 1\) for every edge \(e\) is a modular nowhere-zero \((2 + \frac{1}{t})\)-flow on \(G\). It follows with Theorem 4.2.2 that \(2 + \frac{1}{t} \in \mathcal{S}(G)\). If \(2 + \frac{1}{t} \in \mathcal{S}(G)\), then it follows with Theorem 4.2.2 that \(G\) has a \(t\)-factor.

2) If \(t = 1\), then \(\phi'\) is an integer 3-flow and it follows from 1) that \(3 \in \overline{\mathcal{S}}(G)\). Let \(t \geq 2\) and \(F_1\) be a 1-factor of \(G\). By Theorem 4.2.3, \(G - F_1\) has a 2-factor \(F_2\). Hence, \(F_1 \cup F_2\) induces a spanning cubic subgraph \(H\) of \(G\) which has a 1-factor. By 1), \(H\) has a signature \(\sigma\) such that \((H, \sigma)\) has an integer nowhere-zero 3-flow. Furthermore, \(H' = G - E(H)\) is \((2t - 2)\)-regular and hence, \((H', 1)\) has a nowhere-zero 2-flow. Thus, \((G, \sigma)\) has an integer nowhere-zero 3-flow. Since a 3-flow is the smallest possible integer flow on a \((2t + 1)\)-regular graph it follows that \(3 \in \overline{\mathcal{S}}(G)\).

It remains to construct the graph \(G_t\) for \(t > 1\). Let \(T\) be a triangle where exactly two vertices are joined by two parallel edges, all other vertices are connected by a single edge. Take four copies \(T_1, \ldots, T_4\) of \(T\) and connect each bivalent vertex of \(T_1, T_2, T_3\) with the bivalent vertex \(y\) of \(T_4\) by an edge. Let \(H\) be this graph. Let \(\sigma\) be the signature of \(H\) where \(N_\sigma\) is the set of the parallel edges of \(T_1, \ldots, T_4\). Graph \((H, \sigma)\) with nowhere-zero 3-flow is shown in Figure 4.1. Since \(H\) has a vertex of degree 3, it follows that \(F((H, \sigma)) = 3\).

Let \(K'_{n,n}\) be the complete bipartite graph on \(2n\) vertices where one edge \(uv\) is replaced by a path \(uxv\). For \(t > 1\) and for each trivalent vertex \(z\) of \(H\) take \(t - 1\) copies of \(K'_{2t+1,2t+1}\) and identify \(z\) and the bivalent
vertices of the copies of $K'_{2t+1,2t+1}$. Do the same with $t - 2$ copies of $K'_{2t+1,2t+1}$ and $y$. The resulting graph $G_t$ is $(2t + 1)$-regular. We have $N_σ \subseteq E(G_t)$, and since $F((K'_{2t+1,2t+1}, 1)) = 3$ it follows that $3 \in \overline{S}(G_t)$.

Since $G_t - y$ has more than one odd component it follows that $G_t$ does not have a 1-factor.

\[\square\]

The second part of Theorem 4.1.1 can be generalized to signed graphs.

**Theorem 4.2.5.** [32] Let $t \geq 1$ be an integer and $(G, σ)$ be a signed $(2t + 1)$-regular graph. If $F_c((G, σ)) = r$, then $r = 2 + \frac{1}{t}$ or $r \geq 2 + \frac{2}{2t-1}$.

Theorems 4.2.4 and 4.2.5 imply the following corollary.

**Corollary 4.2.6.** Let $t \geq 1$ be an integer and $(G, σ)$ be a flow-admissible signed $(2t + 1)$-regular graph. If $G$ does not have a $t$-factor, then $F_c((G, σ)) \geq 2 + \frac{2}{2t-1}$.

### 4.3 $r$-minimal sets

This section studies the structural implications of the existence of a nowhere-zero $(2 + \frac{1}{t})$-flow on a signed $(2t + 1)$-regular graph. Hence, it extends the first part of Theorem 4.1.1 to signed graphs.
Proposition 4.3.1. Let $r \geq 2$ and $G$ be graph.

1) The empty set is $r$-minimal if and only if $F_c((G, 1)) = r$.
2) $F_c((G, 1)) \in S_X(G)$ for every $r$-minimal set $X$.

Let $G$ be a $(2t + 1)$-regular graph and $X \subseteq E(G)$. Let $H$ and $H'$ be two copies of $G - X$. For $v \in V(H)$ and $e \in E(H)$, let $v' (e')$ be the corresponding vertex (edge) in $H'$. For each edge $uv \in X$ add edges $uu'$ and $vv'$ (between $H$ and $H'$) to obtain a new $(2t + 1)$-regular graph $G^2_X$.

Let $E^2_X$ be the set of the added edges. Let $\sigma$ be a signature on $G$, and $\sigma|_{G - X}$ be the restriction of $\sigma$ on $G - X$. Let $\sigma^2_X$ be the signature on $G^2_X$ which is equal to $\sigma|_{G - X}$ on $H$ and $H'$ and all edges of $E^2_X$ are positive. Note that $|N_{\sigma^2_X}| = 2|N_{\sigma - X}|$. In particular, if $N_\sigma \subseteq X$, then $\sigma^2_X$ is the empty signature.

Lemma 4.3.2. Let $t \geq 1$ be an integer and $(G, \sigma)$ be a signed $(2t + 1)$-regular graph. Let $r \geq 2$ and $X \subseteq E(G)$. Every nowhere-zero $r$-flow on $(G, \sigma)$ induces a nowhere-zero $r$-flow on $(G^2_X, \sigma^2_X)$.

Proof. If $\tau$ is an orientation of the half-edges of $(G, \sigma)$, then $\overline{\tau}$ denotes the orientation of the half-edges of $(G, \sigma)$ which is obtained from $\tau$ by reversing the orientation of every half-edge. Now, if $\phi$ is a flow on $(G, \sigma)$ with orientation $\tau$, then $\phi$ is also a flow on $(G, \sigma)$ with orientation $\overline{\tau}$.

If $\phi$ is a nowhere-zero $r$-flow with orientation $\tau$, then define a nowhere-zero $r$-flow on $(G^2_X, \sigma^2_X)$ as follows. Let $\psi$ be the restriction of $\phi$ on $(H, \sigma)$ with orientation $\tau$ and $\psi'$ be the restriction of $\psi$ on $(H', \sigma')$ with orientation $\tau' = \overline{\tau}$. Extend these orientations to an orientation $\tau^2_X$ on $(G^2_X, \sigma^2_X)$ as follows. The orientation of the half-edges of $H$ or $H'$ is unchanged, and for an edge $e \in X$ with $e = uv$ orient the edges $uu'$ and $vv'$ of $E^2_X$ as follows: $\tau^2_X(h^u_{uw}) = \tau(h^u_v)$, $\tau^2_X(h^u_{uw'}) = \tau'(h^u_{v'})$, $\tau^2_X(h^v_{ww}) = \tau(h^v_u)$, and $\tau^2_X(h^v_{ww'}) = \tau'(h^v_{u'})$. To obtain a nowhere-zero $r$-flow $\phi^2_X$ on
(\(G^2_X, \sigma^2_X\)) let \(\psi\) and \(\psi'\) be unchanged and for \(uu', vv' \in E^2_X\) which are obtained from edge \(e \in X\) with \(e = uv\) let \(\phi^2_X(uu') = \phi^2_X(vv') = \phi(e)\).

\[\square\]

**Theorem 4.3.3.** Let \(t \geq 1\) be an integer and \(G\) be a \((2t + 1)\)-regular graph. A set \(X \subseteq E(G)\) is \((2 + \frac{1}{t})\)-minimal if and only if \(G\) has a \(t\)-factor and \(X\) is a minimal set such that \(G - X\) is bipartite.

**Proof.** Let \(X\) be \((2 + \frac{1}{t})\)-minimal. By definition, there is a signature \(\sigma\) of \(G\), such that \(F((G, \sigma)) = 2 + \frac{1}{t}\), \(N_\sigma = X\) and \(F((G, \sigma')) \neq 2 + \frac{1}{t}\) for every signature \(\sigma'\) of \(G\) with \(N_{\sigma'} \subset X\). By Theorem 4.2.4, \(G\) has a \(t\)-factor.

Let \(\phi\) be a nowhere-zero \((2 + \frac{1}{t})\)-flow on \((G, \sigma)\). By Lemma 4.3.2, \(\phi\) induces a nowhere-zero \((2 + \frac{1}{t})\)-flow on \((G^2_X, 1)\). By Theorem 4.1.1, \(G^2_X\) is bipartite and therefore \(G - X\) as well.

Now suppose to the contrary that there is a proper subset \(X^*\) of \(X\) such that \(G - X^*\) is bipartite.

\((*)\) It easily follows that \(G^2_{X^*}\) is bipartite and therefore, \((G^2_{X^*}, 1)\) has a nowhere-zero \((2 + \frac{1}{t})\)-flow by Theorem 4.1.1. This \((2 + \frac{1}{t})\)-flow can be modified to a modular nowhere-zero \((2 + \frac{1}{t})\)-flow \(\phi^*\) such that \(\phi^*(e) = 1\) for every edge \(e \in E^2_{X^*}\). Hence, if we reconstruct \(G\) from \(G^2_{X^*}\), by keeping the orientation of the half-edges appropriately, we obtain a signature \(\sigma^*\) of \(G\) with \(N_{\sigma^*} \subseteq X^*\) and a modular nowhere-zero \((2 + \frac{1}{t})\)-flow \(\phi^*\) on \(G\).

By Theorem 4.2.2, \((G, \sigma^*)\) has a nowhere-zero \((2 + \frac{1}{t})\)-flow. But \(N_{\sigma^*}\) is a proper subset of \(X\), contradicting the fact that \(X\) is \((2 + \frac{1}{t})\)-minimal.

Let \(X\) be a minimal set such that \(G - X\) is bipartite. Clearly, if \(X \neq \emptyset\), then \(|X| \geq 2\). As above \((*)\), we deduce that \(G\) has a signature \(\sigma\) with \(N_\sigma \subseteq X\) and a nowhere-zero \((2 + \frac{1}{t})\)-flow \(\phi\) on \((G, \sigma)\). Suppose to the contrary that there is an edge \(e \in X - N_\sigma\). By Lemma 4.3.2, \(\phi\) induces a nowhere-zero \((2 + \frac{1}{t})\)-flow on \((G^2_{X-e}, 1)\). Hence, \(G - (X - e)\) is bipartite,
contradicting the minimality of $X$. Therefore, $X$ is a $(2 + \frac{1}{t})$-minimal set.

\[\Box\]

An exhaustive survey on sufficient conditions for the existence of factors in regular graphs is given in [1].

4.4 The flow spectrum of cubic graphs

This section studies the flow spectrum and $r$-minimal sets of cubic graphs. We will construct some flows on cubic graphs.

**Theorem 4.4.1.** Let $G$ be a cubic graph. If $G$ is bipartite, then it has signature $\sigma$ with $|N_\sigma| = 2$ and $F_c((G, \sigma)) = F((G, \sigma)) = 4$.

**Proof.** By Theorem 4.1.1, $(G, 1)$ has a nowhere-zero 3-flow $\phi$. We can assume that $\phi(e) \geq 1$ for each $e \in E(G)$. The edges with flow value 1 induce a 2-factor $F$ of $G$. Let $C$ be a circuit of $F$. Any two adjacent edges of $C$ are both oriented towards the vertex they share or both away from it. Hence, $\phi$ can be modified to a nowhere-zero 3-flow $\phi'$, where the circuits of $F$ are directed circuits and the flow values on the edges are 1 and -1, alternately.

**Claim 4.4.2.** There is a circuit $C$ of $F$ and a path $P$ of three consecutive edges $e_1, e_2, e_3$ of $C$ such that $G - \{e_1, e_3\}$ is connected. Furthermore, $G$ contains a circuit $D$ such that $e_1 \in E(D)$ and $e_3 \notin E(D)$.

**Proof.** Every circuit of $F$ has a length of at least 4. Let $C$ be a circuit of $F$, and $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$ and $e_4 = v_4v_5$ ($v_1 = v_5$ is not excluded) four consecutive edges in $C$. Furthermore, $v_2$ has a neighbor $x \notin \{v_1, v_3, v_4\}$. If $\{e_1, e_3\}$ is a 2-edge-cut in $G$, then choose $P'$ with edges $e_2, e_3, e_4$. Suppose to the contrary that $\{e_2, e_4\}$ is an edge-cut. It
follows that \( e_2 \) is simple and \( v_2x \) is a bridge, contradicting the fact that \( G \) is bridgeless. Hence, there is a path as claimed. Furthermore, \( G - e_3 \) is bridgeless and hence, \( G - e_3 \) contains a circuit \( D \) with \( e_1 \in E(D) \).

\( \square \)

Let \( P \) be the path in \( C \) with three consecutive edges \( e_1, e_2, e_3 \) (\( e_i = v_i v_{i+1} \)) such that \( G - \{e_1, e_3\} \) is connected. We can assume that \( e_2 \) is directed from \( v_2 \) to \( v_3 \) and that \( \phi'(e_2) = 1 \). Let \( \tau \) be the orientation of \( H(G) \) which is obtained from the underlying orientation for \( \phi' \) by reversing the orientation of \( h_{e_1}^v \) and \( h_{e_3}^v \). Hence, we obtain a signature \( \sigma \) of \( G \) with \( N_{\sigma} = \{e_1, e_3\} \), where \( e_1 \) is extroverted and \( e_3 \) introverted.

Consider \( \phi' \) on \( (G, \sigma) \), then \( \delta \phi'(v_1) = -2 \), \( \delta \phi'(v_4) = 2 \), and \( \delta \phi'(v) = 0 \) if \( v \in V(G) - \{v_1, v_4\} \). Let \( E(\overline{P}) = E(C) - E(P) \). The function \( \psi : E(G) \to \{1, 2, 3\} \) with \( \psi(e) = \phi'(e) \) if \( e \in E(G) - E(\overline{P}) \) and \( \psi(e) = \phi'(e) + 2 \) if \( e \in E(\overline{P}) \) is a nowhere-zero 4-flow on \( (G, \sigma) \). Since \( \psi \) is an integer flow it follows that \( F((G, \sigma)) \leq 4. \)

By Claim 4.4.2 and Theorem 4.2.5 it follows that \( F_c((G, \sigma)) \geq 4 \) and hence, \( F_c((G, \sigma)) = F((G, \sigma)) = 4. \)

\( \square \)

We will use the strict form of Petersen’s Theorem on 1-factors in cubic graphs.

**Theorem 4.4.3** ([29]). Let \( G \) be a bridgeless cubic graph. For every \( e \in E(G) \) there is a 1-factor of \( G \) that contains \( e \).

The minimum number of odd circuits of a 2-factor of a cubic graph \( G \) is the *oddness* of \( G \) and it is denoted by \( \omega(G) \). A 2-factor that has precisely \( \omega(G) \) odd circuits is a minimum 2-factor of \( G \).
Theorem 4.4.4. Every non-bipartite cubic graph $G$ with a 1-factor has a signature $\sigma$ such that $|N_\sigma| = \omega(G)$ and $F_c((G, \sigma)) = F((G, \sigma)) = 4$. Furthermore, if $G$ is bridgeless, then for every 3-minimal set $X_3$ there is a 4-minimal set $X_4$ with $X_4 \subset X_3$ and $|X_4| \leq \omega(G) < |X_3|$.

Proof. If $G$ is 3-edge-colorable, then the empty set is 4-minimal and the statement follows.

Thus, we assume that $G$ is not 3-edge-colorable in the following. Let $\omega(G) = 2n$, $F_2$ be a 2-factor with odd circuits $C_1, \ldots, C_{2n}$, and $F_1$ be the complementary 1-factor. If $G$ has a bridge, then there is one, say $b$, such that one component of $G - b$ is bridgeless. Such components will be called end-components.

We first show that there is a signature $\sigma$ with $|N_\sigma| = \omega(G)$ and $F_c((G, \sigma)) = F((G, \sigma)) = 4$. For $i \in \{1, \ldots, 2n\}$ choose $f_i \in E(C_i)$ with the following restrictions if $G$ has bridges or if an odd circuit of $F_2$ contains a multi-edge.

1. One of the odd circuits of $F_2$, say $C_k$, has two vertices which are connected by two edges in $G$. Choose $f_k$ to be one of these two edges, and for $i \neq k$ choose $f_i \in E(C_i)$ arbitrarily.

2. All edges of the odd circuits of $F_2$ are simple in $G$ and $G$ has an end-component $K$ such that the bivalent vertex $x$ is contained in an odd circuit $C_k$ of $F_2$. Let $x_1$ and $x_2$ be the two neighbors of $x$ in $K$. Then choose $f_k$ to be an edge of $C_k$ which is incident to $x_1$ and different from $xx$. For $i \neq k$ choose $f_i \in E(C_i)$ arbitrarily.

In all other cases choose $f_i \in E(C_i)$ arbitrarily.

Subdivide $f_i$ by a vertex $u_i$ and add edges $e_k = u_{2k-1}u_{2k}$, for $k \in \{1, \ldots, n\}$. The resulting graph $G'$ is cubic. The set $F'_1 = F_1 \cup \{e_k : k = 1, \ldots, n\}$ is a 1-factor and $F'_2 = E(G') \setminus F'_1$ is an even 2-factor of $G'$. The odd circuits $C_i$ of $F_2$ are transformed into even circuits $C'_i$ of $F'_2$. Let $f'_i$ and $f''_i$ be the two edges of $C'_i$ which are incident to $u_i$. Let $c$
be a proper 3-edge-coloring of $G'$ that colors the edges of $F'_1$ with color 1, and the edges of the circuits of $F'_2$ with colors 2 and 3. Assume that $f'_i \in c^{-1}(2)$. Let $\phi_1$ be a nowhere-zero 2-flow on $G'[c^{-1}(1) \cup c^{-1}(2)]$. Let $\phi_2$ be a nowhere-zero 2-flow on $G[c^{-1}(2) \cup c^{-1}(3)]$, with the additional property that the orientation of $f'_i$ is different in $\phi_1$ and $\phi_2$. Flow $\phi_2$ exists, since for every $i \in \{1, \ldots, 2n\}$ there is precisely one edge $f'_i$ in $C'_i$.

Now, $2\phi_1 + \phi_2$ is a nowhere-zero 4-flow $\psi$ on $G'$ with the additional property that for each $i \in \{1, \ldots, 2n\}$ the vertex $u_i$ is either the terminal vertex of $f'_i$ and of $f''_i$ or it is the initial vertex of both of these edges. Furthermore, $\psi(f'_i) = \psi(f''_i) = 1$.

For each $i \in \{1, \ldots, n\}$, remove edge $e_i$, suppress vertices $u_{2i-1}, u_{2i}$ and consider $f'_i$ and $f''_i$ as two half-edges of $f_i$ to construct a signature $\sigma$ of $G$ with $N_\sigma = \{f_1, \ldots, f_{2n}\}$. Furthermore, $\psi$ induces an integer nowhere-zero 4-flow on $(G, \sigma)$. Hence, $F((G, \sigma)) \leq 4$.

It remains to show that $F_c((G, \sigma)) \geq 4$. Suppose to the contrary that $F_c((G, \sigma)) < 4$. Then, $F_c((G^2_{N_\sigma})) < 4$ and hence, $G - N_\sigma$ is bipartite by Theorem 4.1.1.

(1’) If there is $k \in \{1, \ldots, 2n\}$ such that $C_k$ has two vertices which are connected by two edges in $G$, then it follows from the construction of $\psi$, that $G - N_\sigma$ contains an odd circuit, a contradiction. Hence, $F_c((G, \sigma)) = F((G, \sigma)) = 4$.

We may assume that all edges of the odd circuits of $F_2$ are simple in $G$.

(2’) If $G$ has an end-component, then there is one, say $K$, which is respected in the construction of $\psi$. Let $F_2[E(K)]$ be the 2-factor of $K$ which is a subgraph of $F_2$. It follows with Theorem 4.4.3 (suppress the bivalent vertex), that there is a 2-factor $F'$ of $K$ that does not contain $f_k$. Since $K$ has odd order and $F$ contains at least as many odd circuits as $F_2[E(K)]$ it follows that $F'$ contains an odd circuit that does not contain...
any edge of $N_{\sigma}$. Thus, $G - N_{\sigma}$ contains an odd circuit, a contradiction. Hence, $F_c((G, \sigma)) = F((G, \sigma)) = 4$.

It remains to consider the case when $G$ is bridgeless. Let $X_3$ be a 3-minimal set. Since $X_3$ contains an edge of every odd circuit of $F_2$ it follows that $|X_3| \geq \omega(G)$. For $i \in \{1, \ldots, 2n\}$ let $f_i \in X_3 \cap E(C_i)$. Let $\sigma'$ be the signature on $G$ with $N_{\sigma'} = \{f_1, \ldots, f_{2n}\}$ and construct an integer nowhere-zero 4-flow on $(G, \sigma')$ as above. Hence, $X_3$ contains a 4-minimal set $X_4$ with $|X_4| \leq \omega(G)$.

We will show that $|X_3| > \omega(G)$. Suppose to the contrary that $|X_3| = \omega(G)$. Then $\{f_i\} = E(C_i) \cap X_3$ for each $i \in \{1, \ldots, 2n\}$. By Theorem 4.4.3 there is a 1-factor of $G$ that contains $f_1$. The complementary 2-factor $F'$ has at least $\omega(G)$ odd circuits. Thus, there is an odd circuit of $F'$ that does not contain an edge of $X_3$ which implies that $G - X_3$ is not bipartite, a contradiction. Thus, $|X_3| > \omega(G)$, $X_4 \subset X_3$, and $F_c((G, \sigma')) = F((G, \sigma')) = 4$.

\[\square\]

Every signature of $K^3_2$ is either equivalent to a signature with no negative edges or to a signature with precisely one negative edge. Hence, $\mathcal{S}(K^3_2) = \mathcal{S}(K^3_2) = \{3\}$. The following statement follows with Theorems 4.2.4, 4.4.1, and 4.4.4.

**Theorem 4.4.5.** Let $G$ be a cubic graph which has a 1-factor. If $G \neq K^3_2$, then $\{3, 4\} \subseteq \mathcal{S}(G) \cap \mathcal{S}(G)$.

**Theorem 4.4.6.** Let $G$ be a cubic graph. If $G \neq K^3_2$, then the following statements are equivalent.

1) $G$ has a 1-factor.
2) $3 \in \mathcal{S}(G)$.
3) $3 \in \mathcal{S}(G)$.
4) $4 \in \mathcal{S}(G)$.
Proof. Statement 1) implies 2) by Theorem 4.4.5. By Lemma 4.2.1 it follows that every circular nowhere-zero 3-flow on \( G \) is an integer nowhere-zero 3-flow. Hence, statement 2) implies statement 3), which implies statement 1 by Theorem 4.2.4.

We show the equivalence of statements 1) and 4). If \( G \) has a 1-factor, then \( 4 \in \mathcal{S}(G) \) by Theorems 4.4.1 and 4.4.4. If \( 4 \in \mathcal{S}(G) \), then \( G \) has an integer nowhere-zero 4-flow \( \phi \) with \( \phi(e) > 0 \) for each edge \( e \). It is easy to see that \( F = \{ e : \phi(e) = 2 \} \) is a 1-factor of \( G \).

\[ \square \]

Theorem 4.4.5 says that if \( G \) has a 1-factor, then \( 4 \in \mathcal{S}(G) \). However, the other direction is not true.

Proposition 4.4.7. There is a cubic graph \( H \) which has no 1-factor and \( 4 \in \mathcal{S}(H) \).

Proof. The graph \( H \) in Figure 4.2 has a nowhere-zero 4-flow. It has no 1-factor since \( H - v \) has more than one odd component. By Theorem 4.4.6 it has no nowhere-zero 3-flow. Hence, \( 4 \in \mathcal{S}(G) \) by Theorem 4.2.5.

\[ \square \]
Corollary 4.4.8. Let $G$ be a cubic graph that does not have a 1-factor. If $k \in \mathfrak{F}(G)$, then $k \geq 5$.

Smallest $r$-minimal sets

Let $r \geq 2$. An $r$-minimal set $X$ is a smallest $r$-minimal set of $G$ if $|X| \leq |X'|$ for every $r$-minimal set $X'$ of $G$.

Proposition 4.4.9. Let $t \geq 1$ be an integer, $G$ a $(2t + 1)$-regular graph and $r \geq 2$. If $X \subseteq E(G)$ is a smallest $r$-minimal set, then $\Delta(G[X]) \leq t$.

Proof. Suppose to the contrary that $\Delta(G[X]) > t$. Then there is $v \in V(G)$ such that $d_{G[X]}(v) > t$. If we switch at $v$, then we obtain an equivalent graph $(G,\sigma)$ with $|N_\sigma| < |X|$ and $F_e((G,\sigma)) = r$. But $N_\sigma$ contains an $r$-minimal set $X'$, contradicting the fact that $X$ is a smallest $r$-minimal set.

We will prove some bounds for the cardinality of smallest $r$-minimal sets. The independence number of $G$ is denoted by $\alpha(G)$.

Proposition 4.4.10. Let $t \geq 1$ be an integer and $G$ be a $(2t + 1)$-regular graph. If $X \subseteq E(G)$ is a smallest $(2 + \frac{1}{t})$-minimal set, then $|X| \leq \min\{(\frac{1}{2}|V(G)| - \alpha(G))(2t + 1), \frac{2}{t}|V(G)|\}$.

Proof. Since $G$ has a $(2 + \frac{1}{t})$-minimal set, it follows by Theorem 4.2.4 that $G$ has a $t$-factor. Let $(G,\sigma)$ be the graph with $N_\sigma = E(G)$ and $V \subseteq V(G)$ be an independent set with $|V| = \alpha(G)$. The function $\phi : E(G) \rightarrow \{1\}$ is a modular nowhere-zero $(2 + \frac{1}{t})$-flow on $(G,\sigma)$. Switch at every vertex of $V$ to obtain a switching equivalent graph $(G,\sigma')$ with a modular nowhere-zero $(2 + \frac{1}{t})$-flow $\phi'$ and $|N_{\sigma'}| \leq |E(G)| - \alpha(G)(2t + 1) = \frac{2}{t}|V(G)|$. The proof is completed.
It follows with Theorem 4.2.2 that \((G, \sigma')\) has a nowhere-zero \((2 + \frac{1}{2})\)-flow. Therefore, \(|X| \leq (\frac{1}{2}|V(G)| - \alpha(G))(2t + 1)\). Proposition 4.4.9 implies that \(|X| \leq \frac{1}{2}|V(G)|\).

\[
\frac{1}{2}|V(G)| - \alpha(G))(2t + 1). \text{ It follows with Theorem 4.2.2 that } (G, \sigma') \text{ has a nowhere-zero } (2 + \frac{1}{2})\text{-flow. Therefore, } |X| \leq (\frac{1}{2}|V(G)| - \alpha(G))(2t + 1). \text{ Proposition 4.4.9 implies that } |X| \leq \frac{1}{2}|V(G)|. \]

\[
\frac{1}{2}|V(G)| - \alpha(G))(2t + 1). \text{ It follows with Theorem 4.2.2 that } (G, \sigma') \text{ has a nowhere-zero } (2 + \frac{1}{2})\text{-flow. Therefore, } |X| \leq (\frac{1}{2}|V(G)| - \alpha(G))(2t + 1). \text{ Proposition 4.4.9 implies that } |X| \leq \frac{1}{2}|V(G)|. \]

Let \(G\) be a bridgeless cubic graph. The resistance \(r(G)\) is the cardinality of a minimum color class, where the minimum is taken over all proper 4-edge-colorings of \(G\). It is easy to see that \(r(G) \leq \omega(G)\) and if \(r(G) \neq 0\), then \(r(G) \geq 2\) (see [34]). A bridgeless cubic graph which is not 3-edge-colorable is called a snark.

Theorem 4.4.11. Let \(G\) be a cubic graph which has a 1-factor and \(G \neq K^3_3\). For each \(i \in \{3, 4\}\) there is a smallest \(i\)-minimal set \(X_i\) in \(G\), and

1) if \(G\) is bipartite, then \(|X_3| = 0\) and \(|X_4| = 2\).

2) if \(G\) is 3-edge-colorable and not bipartite, then \(2 \leq |X_3| \leq 3(\frac{1}{2}|V(G)| - \alpha(G))\) and \(|X_4| = 0\).

3.1) if \(G\) is not 3-edge-colorable, then \(r(G) \leq |X_4| \leq \omega(G) \leq |X_3| \leq \min\{3(\frac{1}{2}|V(G)| - \alpha(G)), \frac{1}{2}|V(G)|\}\).

3.2) if \(G\) is a snark, then \(r(G) \leq |X_4| \leq \omega(G) < |X_3| \leq \min\{3(\frac{1}{2}|V(G)| - \alpha(G)), \frac{1}{2}|V(G)|\}\).

Proof. It follows from Theorem 4.4.5 that there is a smallest \(i\)-minimal set \(X_i\) in \(G\) for each \(i \in \{3, 4\}\). By Proposition 4.4.10 \(|X_3| \leq 3(\frac{1}{2}|V(G)| - \alpha(G))\).

1) If \(G\) is bipartite, then \(|X_3| = 0\) and it follows with Theorem 4.4.1 that \(G\) has a signature \(\sigma\) with \(|N_\sigma| = 2\) and \(F((G, \sigma)) = 4\). Since every signature of a flow-admissible graph has at least two edges it follows that \(|X_4| = 2\).

2) If \(G\) is not bipartite and 3-edge-colorable, then \(|X_4| = 0\), and as above we get that \(|X_3| \geq 2\).
3) Let $G$ be not 3-edge-colorable. Suppose to the contrary that there is a smallest 4-minimal set $X$ such that $|X| < r(G)$. By Lemma 4.3.2, $(G_X^2, 1)$ has a nowhere-zero 4-flow and hence, it is 3-edge-colorable. By Proposition 4.4.9, $X$ is an independent set. Hence, a 3-edge-coloring of $G_X^2$ induces a proper 4-edge-coloring of $G$ which has a minimal color class $c$ with $|c| \leq |N_\sigma| < r(G)$, contradiction. Thus, $r(G) \leq |X_4|$. The other inequalities follow by Theorem 4.4.4.

The bounds of Theorem 4.4.11 are sharp for the Petersen graph. In [36] it is shown that for every positive integer $k$ there is a cubic graph $G$ such that $\omega(G) - r(G) \geq k$ and that there is cyclically 5-edge-connected cubic graph $H$ and $r(H) \geq k$. If $G$ is not 3-edge-colorable and $r(G) > 2$, then there is a signature $\sigma$ such that $(G, \sigma)$ is flow-admissible and $2 \leq |N_\sigma| < r(G)$. Theorem 4.4.11 implies $F((G, \sigma)) > 4$. Hence, we obtain the following corollary which is similar to Corollary 4.4.8.

**Corollary 4.4.12.** Let $G$ be a cubic graph. If $\omega(G) > 2$, then for every $k$ with $2 \leq k < r(G)$ there is a signature $\sigma$ such that $|N_\sigma| = k$ such that $F_c((G, \sigma)) > 4$ and $F((G, \sigma)) \geq 5$.

**The integer flow spectrum of a class of cubic graphs**

It is quite difficult to determine the flow spectrum of a graph. Indeed even for the integer flow spectrum it is difficult. So far the integer flow spectrum has been determined only for eulerian graphs in [23], and for complete and complete bipartite graphs in [20]. For instance, it is known that $\overline{S}(G) = \{3, 4, 5, 6\}$, if $G$ is the Petersen graph.

For $n \geq 1$, let $G_n$ be the cubic graph which is obtained from a circuit of length $2n$, where every second edge is replaced by two parallel edges.
Theorem 4.4.13. If $n = 1$, then $\mathcal{S}(G_n) = \{3\}$. If $n = 2$, then $\mathcal{S}(G_n) = \{3, 4\}$, and if $n \geq 3$, then $\mathcal{S}(G_n) = \{3, 4, 6\}$.

Proof. If $n = 1$, then $G_1 = K_3^2$. Hence, $\mathcal{S}(G_1) = \{3\}$.

Let $n \geq 2$. Theorem 4.4.5 implies that $\{3, 4\} \subseteq \mathcal{S}(G_n)$. Let $V(G_n) = \{v_0, \ldots, v_{2n-1}\}$. For $i \in \{0, \ldots, n-1\}$ the vertices $v_{2i}, v_{2i+1}$ are connected by two parallel edges and the vertices $v_{2i+1}, v_{2i+2}$ are connected by a simple edge (indices are added modulo $2n$). Every signature of $G_n$ is equivalent to a signature $\sigma$ where for each $i \in \{0, \ldots, n-1\}$ at most one edge between $v_{2i}, v_{2i+1}$ is negative and all other edges are positive. We call $\sigma$ a normal signature of $G_n$. We say that $\sigma$ is odd or even, depending on whether $|N_\sigma|$ is odd or even. Hence, we have only to consider the two cases, whether $\sigma$ is even or odd.

If $\sigma$ is even, then $G_n$ is the union of two balanced eulerian graphs and hence, $F((G, \sigma)) \leq 4$ by Lemma 4.4.14. Hence, $\mathcal{S}(G_2) = \{3, 4\}$.

It remains to consider the case when $n \geq 3$ and $\sigma$ is odd. Then $|N_\sigma| \geq 3$. Let $e_1, e_2 \in N_\sigma$. There are hamiltonian circuits $C_1, C_2$ such that $E(C_i) \cap N_\sigma = N_\sigma - \{e_i\}$. Both circuits are balanced and hence, there are nowhere-zero 2-flows $\phi_i$ on $C_i$. Let $e'_1$ be the positive edges which is parallel to $e_1$. Then $\psi = 2\phi_1 + \phi_2$ is a 4-flow on $G_n$, and $\psi(e) \neq 0$ if $e \in N_\sigma, \psi(e) \in \{0, 1, 3\}$ if $e \in E(G_n) - (N_\sigma \cup \{e'_1\})$, and $\psi(e'_1) = 2$. Let $\tau_\psi$ be the underlying orientation of $H(G)$ for $\psi$. There is a hamiltonian circuit $C$ that consists only of positive edges, that contains $e'_1$ and all edges $e$ with $\psi(e) = 0$. Let $\tau$ be an orientation of $H(C)$ such that $C$ is a directed circuit and $\tau$ and $\tau_\psi$ coincide on $e'_1$. Let $\psi'$ be a nowhere-zero 2-flow on $C$ with orientation $\tau$. Then, $\psi + 2\psi'$ is a nowhere-zero 6-flow on $G_n$.

Suppose to the contrary that $F((G_n, \sigma)) < 6$. Let $k < 6$ and $\psi_n$ be nowhere-zero $k$-flow on $G_n$. Without loss of generality we assume in the following that all flow values are positive.
We first show that if \( n > 3 \), then \( F((G_n, \sigma)) < 6 \) implies that there is a \( m < n \) such that \( F((G_m, \sigma_m)) < 6 \), where \( \sigma_m \) is normal and odd.

Let \( n > 3 \). It is easy to see that if one edge of the two edges between \( v_{2i} \) and \( v_{2i+1} \) is negative, then \( \psi_n(v_{2i-1}v_{2i}) \neq \psi_n(v_{2i+1}v_{2i+2}) \). If there are \( i, j \in \{0, \ldots, n-1\} \) with \( i < j \) and \( \psi_n(v_{2i+1}v_{2i+2}) = \psi_n(v_{2j+1}v_{2j+2}) \), then remove these two edges and add edges \( v_{2i+2}v_{2j+1} \) and \( v_{2i+1}v_{2j+2} \) to obtain two graphs \( G_{n_1} \) and \( G_{n_2} \) with nowhere-zero \( k \)-flows \( \psi_{n_1} \) and \( \psi_{n_2} \), respectively. Depending on the orientation of the half-edges of \( v_{2i+1}v_{2i+2} \) and of \( v_{2j+1}v_{2j+2} \) the new edges might be negative. For one of these two graphs, say \( G_{n_1} \), \( \psi_{n_1} \) is equivalent to a \( k \)-flow \( \psi'_{n_1} \) on an odd normal signature \( \sigma'_{n_1} \) of \( G_{n_1} \), since for otherwise \( G_n \) and an even normal signature which is equivalent to \( \sigma_n \) could be reconstructed from \( (G_{n_1}, \sigma_{n_1}) \) and \( (G_{n_2}, \sigma_{n_2}) \). Hence, \( |N_{\sigma_{n_1}}| \geq 3 \). Thus, we can assume that \( G_n \) has an odd normal signature \( \sigma_n \) with \( |N_{\sigma}| = n \). In particular, \( n \) is odd.

Since \( k < 6 \), it follows that if \( G_n \) is not reducible to a smaller graph \( G_m \), then \( n = 3 \). Consider \( (G_3, \sigma_3) \). Since \( \sigma_3 \) is normal it follows that the difference between the flow values of any two simple edges is at least 2. Since \( \psi_3(e) \geq 1 \) for every edge \( e \in G_3 \), it follows that there is a simple edge with \( \psi_3(e) \geq 5 \), contradicting our assumption that \( k < 6 \).

Therefore, \( F((G_n, \sigma)) = 6 \) and \( \mathfrak{S}(G_n) = \{3, 4, 6\} \) if \( n > 2 \).

\( \square \)

Every graph \( G_n \) is bipartite. Hence, the empty set is a smallest 3-minimal set for all \( n \geq 1 \). If \( n \geq 2 \), then a smallest 4-minimal set contains precisely two edges, and if \( n \geq 3 \), then a smallest 6-minimal set consists of three edges.
Bouchet’s conjecture

To prove Bouchet’s conjecture it is sufficient to prove it for cubic graphs (see Theorem 1.4.5). Hence, we are in a similar situation as for Tutte’s 5-flow conjecture. However, as Section 3.1 shows Bouchet’s conjecture has to be proven for integer flows explicitly.

Lemma 4.4.14. Let \((G, \sigma)\) be a signed graph. If \((G, \sigma)\) is the union of two balanced eulerian graphs, then \(F((G, \sigma)) \leq 4\).

Proof. Let \(H_1\) and \(H_2\) be eulerian graphs such that \(E(H_1) \cup E(H_2) = E(G)\). For \(i \in \{1, 2\}\), let \(\sigma_i\) be the restriction of \(\sigma\) to \(H_i\). Since \(H_i\) is balanced it follows that there is a nowhere-zero 2-flow \(\phi_i\) on \((H_i, \sigma_i)\). Hence, \(\phi_1 + 2\phi_2\) is a nowhere-zero 4-flow on \((G, \sigma)\).

\[\square\]

Theorem 4.4.15. Let \((G, \sigma)\) be a flow-admissible signed cubic graph. If \(G\) is a Kotzig-graph, then \(F((G, \sigma)) \leq 6\).

Proof. Since \(G\) is a Kotzig-graph, \(G\) has three 1-factors \(M_1, M_2, M_3\) such that the union of any two of them induces a hamiltonian circuit of \(G\). It follows that there are two, say \(M_1\) and \(M_2\), such that \(|N_\sigma \cap M_1|\) and \(|N_\sigma \cap M_2|\) have the same parity. Hence, \(|N_\sigma \cap M_1| + |N_\sigma \cap M_2|\) is even, and \((G[M_1 \cup M_2], \sigma_{1,2})\) with \(\sigma_{1,2} = \sigma|_{M_1 \cup M_2}\) is balanced. Clearly, \((G, \sigma)\) is equivalent to \((G, \sigma')\) with \(N_{\sigma'} \cap (M_1 \cup M_2) = \emptyset\).

If \(|M_3 \cap N_{\sigma'}|\) is even, then \((G[M_1 \cup M_3], \sigma_{1,3})\) with signature \(\sigma_{1,3} = \sigma'|_{M_1 \cup M_3}\) is balanced. Hence, \(F(G, \sigma) \leq 4\) by Lemma 4.4.14.

If \(|M_3 \cap N_{\sigma'}|\) is odd, then \(|N_{\sigma'}| \geq 3\). Let \(e = xy\) be an extroverted edge of \((G, \sigma')\), and let \((G, \sigma^*)\) be the graph which is obtained from \((G, \sigma')\) by changing the direction of \(h_\pi^\xi\). Then \(e\) is a positive edge which is directed from \(x\) to \(y\) in \((G, \sigma^*)\). It follows as above, that \((G, \sigma^*)\) has a nowhere-zero 4-flow \(\phi\). Without loss of generality we can assume that all flow
values are positive, all edges of $M_3$ have flow value 1, and $M_1 \cup M_2$ is a directed circuit $C$. For two vertices $a, b$, let $P(a, b)$ denote the directed path from $a$ to $b$ in $C$.

If we consider $\phi$ on $(G, \sigma')$, then $\delta \phi(x) = 2$, and $\delta \phi(v) = 0$ for all $v \in V(G) \setminus \{x\}$. Since all flow values are positive, it follows that there is an introverted edge $f = uw$. Let $x, u, w$ be the sequent order of these three vertices in $C$. We define a nowhere-zero 6-flow $\phi^*$ on $(G, \sigma')$ as follows: $\phi^*(e) = \phi(e)$, if $e \in E(G) - (E(P(x, w)) \cup \{f\})$, $\phi^*(e) = \phi(e) + 2$, if $e \in E(P(x, u))$, $\phi^*(e) = \phi(e) + 1$, if $e \in E(P(u, w))$, and $\phi^*(f) = 2$.

Greenwell and Kronk [7] proved that every uniquely 3-edge-colorable cubic graph has precisely three hamiltonian circuits, which are induced by the three color classes. Hence, we obtain the following corollary.

**Corollary 4.4.16.** Let $(G, \sigma)$ be a flow-admissible signed cubic graph. If $G$ is uniquely 3-edge-colorable, then $F((G, \sigma)) \leq 6$.

Further results and references on Kotzig-graphs and uniquely 3-edge-colorable graphs can be found in [51].

Remark: For $t \geq 2$ it can be proven by a slight modification of the proof of Theorem 4.4.15 that $F((G, \sigma)) \leq 4$ for every flow-admissible $(2t + 1)$-regular Kotzig-graph $(G, \sigma)$. However, this is also a simple consequence of a result of Raspaud and Zhu [30] that every flow-admissible 4-edge-connected graph has a nowhere-zero 4-flow.
Chapter 5

The set of circular flow numbers of regular graphs

5.1 Circular flow numbers of $d$-regular graphs

The parts 5.1.1 - 5.1.3 and section 5.2 have been published for unsigned graphs in [31]. In [31], the graphs are considered to be unsigned; this section is the generalization of that study for signed graphs.

In this chapter we consider graphs to be loopless. The constructions that we will use require multiedges, so-called $t$-edges. For $t \geq 1$, a $t$-edge between two vertices $u$ and $v$ is a set of $t$ (parallel) edges between $u$ and $v$; that is, an edge is a 1-edge.

If we assign a direction to each edge of $(G, \sigma)$, then we obtain a directed graph $D((G, \sigma))$. For $k \geq 1$, a $k$-edge $e$ in a directed graph is denoted by $(u, v)_k$, if all $k$ edges of $e$ are directed from $u$ to $v$. The shorthand $(u, v)$ is used for a 1-edge $(u, v)_1$. We say that $u$ is the initial vertex and $v$ is the end vertex of $e$. 

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A network is an ordered pair \(((G, \sigma), U)\) consisting of a signed graph \((G, \sigma)\) and a subset \(U \subseteq V(G)\) whose elements are called terminals. A nowhere-zero \(r\)-flow \(\phi\) on a network \(((G, \sigma), U)\) is an assignment of a direction and a real flow value \(\phi(e)\) to each edge \(e \in E(G)\) such that \(1 \leq \phi(e) \leq r - 1\) and the excess flow \(\delta\phi(v)\) is equal to 0 for every \(v \in V(G) - U\), where \(\delta\phi(v) = \sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h)\). Considering flows on terminals we will use the more intuitive term excess flow instead of boundary. Clearly, if \(\phi\) is a nowhere-zero \(r\)-flow on a network \(((G, \sigma), U)\), then \(\sum_{v \in U} \delta\phi(v) = 0\).

We say that an edge \(e\) has flow value \(x\) in \(((G, \sigma), U)\) if \(\phi(e) = x\). Moreover, we say that a \(k\)-edge \((u, v)_k\) has flow value \(x\) if each of its \(k\) edges has flow value \(x\). In this chapter terminals will always have degree 1. The set of terminals \(U\) is partitioned into two sets \(U^+\) and \(U^-\), where \(U^+\) denotes the set of terminals which are end vertices and \(U^-\) denotes the set of terminals which are initial vertices of an edge. Note that a signed graph \((G, \sigma)\) is a network \(((G, \sigma), \emptyset)\).

For a set \(A\), let \(A_\mathbb{Q} = A \cap \mathbb{Q}\) and

\[ F^c = \{ r : \text{there is a signed graph } (G, \sigma) \text{ with } F_c(G, \sigma) = r \}. \]

If we restrict our studies on all-positive graphs and include Seymour’s 6-flow Theorem 1.4.2 then \(F^c \subseteq [2; 6]\). In general, using the 12-flow theorem of DeVos [5] we get for signed graphs \(F^c \subseteq [2; 12]\).

**Theorem 5.1.1.** [23] For every \(r \in [2; 5]_\mathbb{Q}\), there is an all-positive graph \((G, 1)\) with \(F_c(G, 1) = r\).

If Bouchet’s 6-flow conjecture is true, then \(F^c \subseteq [2; 6]_\mathbb{Q}\).

Many flow conjectures are equivalent to their restrictions to regular graphs. For instance, Tutte’s 5-flow conjecture and Bouchet’s 6-flow conjecture are equivalent to its restrictions to cubic graphs (see Theorem...
5.1 Circular flow numbers of $d$-regular graphs

and the 3-flow conjecture \cite{3} is equivalent to its restriction to 5-
regular graphs. However, the set of flow values of graphs with a specific
odd regularity is not a closed interval, as Theorem \cite{4} shows.

In this chapter we determine the set of flow numbers $F_d^c$ of $d$-regular
graphs for odd $d$. Clearly, 1-regular graphs do not have any nowhere-zero
flows.

The initial motivation of the study in this chapter, which has its origin
in \cite{3}, is to estimate the set of flow numbers for regular all-positive
graphs. Nonetheless, in case for graphs of odd regularities these results
can be obtained for signed graphs as well.

If $d$ is even, then $F_c(G,1) = 2$ for every $d$-regular graph $G$.

Therefore, we focus on the case when $d$ is odd and $d \geq 3$. Let $k \geq 1$
and $F_{2k+1}^c = \{r : \text{there is a signed } (2k+1)\text{-regular graph } (G,\sigma) \text{ with } F_c(G,\sigma) = r\}$. In \cite{17} it is proven that for every $r \in [4;5]_Q$ there is
a cubic graph $(G,1)$ with $F_c(G,1) = r$. Indeed there is the additional
constraint, that these graphs are cyclically 4-edge connected and that
they have girth at least 5. We prove a similar result for $(2k+1)$-regular
signed graphs, and we show that in case that Tutte’s 5-flow conjecture
is false, then gaps for circular flow numbers in the interval $[5;6]$ are due
for all graphs and not just for regular graphs.

**Theorem 5.1.2.** For all $k \geq 1 : F_{2k+1}^c = (F^c - [2; 2 + \frac{2}{2k-1}]) \cup \{2 + \frac{1}{k}\}$.

Theorems 5.1.1 and 5.1.2 imply the following result.

**Corollary 5.1.3.** For every integer $k \geq 1$ and every rational number
$r \in \{2 + \frac{1}{k}\} \cup [2 + \frac{2}{2k-1};5]$, there exists a $(2k+1)$-regular graph $(G,\sigma)$
with $F_c(G,\sigma) = r$.

In \cite{35}, Theorem 2.6, it is proven that if there is a graph $(G,1)$ with
$F_c(G,1) = r$, then there is a simple graph $G'$ with $F_c(G',1) = r$. The
construction of \[35\] that originates from \[27\] preserves \(d\)-regularity. However, that proof uses a complicated method that is just applicable for all-positive graphs. Here, we will give an easier proof for the general case.

Let \((G, \sigma)\) be a signed graph and \(v\) be a vertex with \(\delta(v) = n\). Let \(v_1, \ldots, v_n\) be the neighbors of \(v\) that are not necessarily distinct. Further, let \((H, 1)\) be the complete all-positive bipartite graph \(K_{n,n-1}\) and \(u_1, \ldots, u_n\) be the vertices of degree \(n - 1\). The graph \((G^n, \sigma^n)\) obtained from \((G - v, \sigma)\) and \((H, 1)\) by adding the edges \(v_iu_i\) with signature \(\sigma(v_iu_i) = \sigma(v_iv)\) for \(i \in \{1, \ldots, n\}\) is called a Meredith-extension of \(G\).

**Theorem 5.1.4.** If \((G^n, \sigma^n)\) is a Meredith-extension of a signed graph \((G, \sigma)\) then \(F_c((G, \sigma)) = F_c((G^n, \sigma^n))\).

**Proof.** Let \(v\) be a vertex with \(\delta(v) = n, v_1, \ldots, v_n\) be the neighbors of \(v\), \((H, 1)\) be the complete all-positive bipartite graph \(K_{n,n-1}\), \(u_1, \ldots, u_n\) be the vertices of degree \(n - 1\) and \(x_1, \ldots, x_{n-1}\) be the vertices of degree \(n\). \((G^n, \sigma^n)\) is obtained from \((G - v, \sigma)\) and \((H, 1)\) by adding the edges \(v_iu_i\) with signature \(\sigma(v_iu_i) = \sigma(v_iv)\) for \(i \in \{1, \ldots, n\}\).

Suppose \(\phi\) is a nowhere-zero \(r\)-flow on \((G^n, \sigma^n)\). Since \((H, 1)\) is an all-positive graph, we get a nowhere-zero \(r\)-flow \(\phi'\) on \((G, \sigma)\) by defining \(\phi'(v_i) = \phi(v_iu_i)\) and \(\phi'(e) = \phi(e)\) if \(e \neq (v_iu_i)\) and corresponding orientation \(\tau_{\phi'}(v_i) = \tau_{\phi}(v_iu_i)\) and \(\tau_{\phi'}(e) = \tau_{\phi'}(v_i)\) if \(e \neq (v_iu_i)\) for \(i \in \{1, \ldots, n\}\). Hence, \(F_c((G, \sigma)) \leq F_c((G^n, \sigma^n))\).

To show the other direction we will construct a nowhere-zero \(r\)-flow on \(F_c((G^n, \sigma^n))\) from a given nowhere-zero \(r\)-flow \(\phi\) on \((G, \sigma)\). Define a nowhere-zero \(r\)-flow on \(F_c((G^n, \sigma^n))\) as follows: For \(i \in \{1, \ldots, n\}\) we define the corresponding orientation \(\tau_{\phi}(v_iu_i) = \tau_{\phi'}(v_i)\), \(\tau_{\phi'}(u_i, x_i) = \tau_{\phi'}(v_{i+1}, v)\) (the summation of the indices is taken modulo \(n\)), and \(\tau_{\phi'}(e) = \tau_{\phi}(e)\) for \(e \notin \{v_iu_i : i \in \{1, \ldots, n\}\} \cup E(H)\). For \(i \in \{1, \ldots, n\}\) let \(\phi'(v_iu_i) = \phi(v_i)\), \(\phi'(u_i, x_i) = \phi(v_{i+1}, v)\) (the summation of the indices is
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We show that $\phi'$ is a proper flow. Kirchoff’s law is obviously fulfilled at all vertices $v \in V(G^M) - V(H)$. For every $u_i$ we get

$$
\sum_{e \in H(v)} \tau(e) \phi'(e) = \sum_{h \in H(v)} \tau(e_h) \phi(e_h) = 0
$$

Similarly we get $\sum_{h \in H(x_i)} \tau'(e_h) \phi'(e_h) = \sum_{h \in H(v)} \tau(e_h) \phi(e_h) = 0$. Hence, $\phi'$ is a proper nowhere-zero $r$-flow on $(G^M, \sigma^M)$ and $F_c((G, \sigma)) \geq F_c((G^M, \sigma^M))$.

In this regard it is interesting to note that a more general concept of the Meredith-extension as given in [38] and [2] will not keep the flow number in general.

Let $(G, \sigma)$ be a signed graph, $v$ be a vertex with $\delta(v) = n$ and $\Delta = \Delta(G)$ be the maximum degree of $G$. Let $v_1, \ldots, v_n$ be the neighbors of $v$ that are not necessarily distinct. Further, let $(H, 1)$ be the complete all-positive bipartite graph $K_{\Delta, \Delta-1}$ and $u_1, \ldots, u_n$ be the vertices of degree $n - 1$. The graph $(G^{M'}, \sigma^{M'})$ optained from $(G - v, \sigma)$ and $(H, 1)$ by adding the edges $v_i u_i$ with signature $\sigma(v_i u_i) = \sigma(v_i v)$ for $i \in \{1, \ldots, n\}$ is called a general $\Delta$-Meredith-extension of $G$.

Let $(J, 1)$ be the all-positive graph given by $K_{3,3}$ with one edge subdi-vided by a vertex $v$. Since $F_c(K_{3,3}, 1) = 3$ [35], we get $F_c(J, 1) = 3$. Now, let $(H, 1)$ be the graph obtained by applying a $\Delta$-Meredith-extension on $v$ in $(J, 1)$. We will show that $F_c(H, 1) \geq 4$. After contraction of $(K_{3,3}, 1)$ in $(J, 1)$ and suppression of two vertices of degree two we get the complete
The set of circular flow numbers of regular graphs

graph on four vertices \((K_4, 1)\). Since we consider all-positive graphs, it follows \(F_c(H, 1) \geq F_c(K_4, 1) = 4\).

Hence, the Meredith-extension keeps the flow-value for every graph, whereas there exist graphs where a \(\Delta\)-Meredith-extension does not keep the flow value.

Thus, Theorem 5.1.1, Theorem 5.1.2 and Corollary 5.1.3 are true for the restriction on simple graphs as well.

5.2 Proof of Theorem 5.1.2

Let \(k \geq 1\) be an integer and \((G, \sigma)\) be a signed \((2k + 1)\)-regular graph. If \(F_c(G, \sigma) = r\), then \(r = 2 + \frac{1}{t}\) or \(r \geq 2 + \frac{2}{2k-1}\). Hence, we need to consider the case that \(r \geq 2 + \frac{2}{2k-1}\). Let \((G, \sigma)\) be a graph with \(F_c(G, \sigma) = r\). We have to show that there is a \((2k + 1)\)-regular graph \((H, \tau)\) with \(F_c(H, \tau) = r\).

**Lemma 5.2.1.** For every \(r \in F^c\), there is a graph \((H, \tau)\) with \(F_c(H, \tau) = r\) and a nowhere-zero \(r\)-flow \(\phi\) such that the following holds: If \(v \in V(H)\) and \(d_H(v) \notin \{2, 3\}\), then \(\phi(e) < 2\) for every edge \(e\) which is incident to \(v\).

**Proof.** If \(r < 3\), then there is nothing to prove. Let \(r \in F^c \cap [3; 6]\), \((G, \sigma)\) be a graph with \(F_c(G, \sigma) = r\), and \(\phi\) be a nowhere-zero \(r\)-flow on \((G, \sigma)\). There is an edge \(e\) with \(\phi(e) \geq 2\). Let \(t_{(G, \sigma)}(v) = \sum_{e \in E(v)} (|\phi(e)| - 1), V^-(G) = \{v : v \in V(G)\) and \(d_G(v) \notin \{2, 3\}\}\), and \(t((G, \sigma)) = \sum_{v \in V^-(G)} t_{(G, \sigma)}(v)\).

Among all graphs with circular flow number \(r\) let \(H\) be a graph with \(t((H, \tau))\) minimum. We claim that \(t((H, \tau)) = 0\) and hence \((H, \tau)\) has the desired property.

Suppose to the contrary that \(t((H, \tau)) > 0\). Then there is \(v \in V^-(H)\) with \(t_{(H, \tau)}(v) > 0\). Let \(e \in E(v)\) and \(\phi(e) \geq 2\).
We assume that \( v \) is the end vertex of \( e \). Subdivide \( e \) by a vertex \( w \) and add an positive edge \( wv \) to obtain a new graph \( (H', \tau') \). Then \( wv \) is a 2-edge, say \((w, v)_2 = \{e_1, e_2\} \) and both edges, \( e_1 \) and \( e_2 \), are directed from \( w \) to \( v \). Extend \( \phi \) to a nowhere-zero \( r \)-flow \( \phi' \) on \( (H', \tau') \) with \( \phi'(e_1) = 1 \) and \( \phi'(e_2) = \phi(e) - 1 \). Since \((H, \tau)\) can be obtained from \((H', \tau')\) by contraction of an all-positive subgraph, it follows that \( F_r(H', \tau') = r \). Furthermore, \( t_{(H, \tau)}(v) = t_{(H', \tau')}(v) - 1 \) and hence, \( t((H', \tau')) < t((H, \tau)) \), which contradicts our choice of \((H, \tau)\).

\[ \square \]

Let \(((G, \sigma), U)\) be a network with a nowhere-zero \( r \)-flow \( \phi \). Let \( v \in V(G) - U \), and \( E(v) = \{e_1, \ldots, e_n\} \). For each \( i \in \{1, \ldots, n\} \), subdivide \( e_i \) by a new vertex \( v_i \) and delete \( v \), to obtain a network \(((G', \sigma'), U')\). Let \( e_i' \) be the edge incident to \( v_i \) in \(((G', \sigma'), U')\). We extend the direction of \(((G, \sigma), U)\) to \(((G', \sigma'), U')\) by letting the direction of all edges which are not in \( \{e_1', \ldots, e_n'\} \) unchanged and giving \( e_i' \) the same direction as \( e_i \), that is, if \( v \) is the initial (end) vertex of \( e_i \), then \( v_i \) is the initial (end) vertex of \( e_i' \). Furthermore, \( \phi \) is extended to a nowhere-zero \( r \)-flow \( \phi' \) on \(((G', \sigma'), U')\) as follows: Let \( \phi'(e) = \phi(e) \) if \( e \notin \{e_1', \ldots, e_n'\} \) and \( \phi'(e_i') = \phi(e_i) \), for \( i \in \{1, \ldots, n\} \). If \( v_i \) is an end vertex, then \( \delta \phi'(v_i) = \phi'(e_i') \) and otherwise \( \delta \phi'(v_i) = -\phi'(e_i') \). Therefore, \( \sum_{i=1}^{n} \delta \phi'(v_i) = \sum_{e \in E^+(v)} \phi(e) - \sum_{e \in E^-(v)} \phi(e) = 0 \). We say that the network \(((G', \sigma'), U')\) is obtained from \(((G, \sigma), U)\) by splitting up \( v \) into terminals.

**Construction 5.2.2.** For \( k \geq 1 \), let \((T_k, 1)\) be the triangle where any two vertices are connected by a \( k \)-edge, let \((G, \tau)\) be a signed graph, and let \( v \) be a trivalent vertex of \( G \). Let \( V(T_k) = \{w_1, w_2, w_3\} \). Split up \( v \) into terminals \( v_1, v_2, v_3 \) and identify \( v_i \) and \( w_i \) for \( i \in \{1, 2, 3\} \) to obtain the graph \((G^k_v, \sigma)\).
Note that if \( k \geq 2 \), then \((G^*_v, \sigma)\) has one trivalent vertex less than \((G, \tau)\). Furthermore, all vertices of \( T_k \) have degree \( 2k + 1 \) in \( G^*_v \).

**Lemma 5.2.3.** Let \( k \geq 2 \), \((G, \tau)\) be a graph with \( F_c(G, \tau) = r \), and let \( v \in V(G) \). If \( d_G(v) = 3 \), then \( F_c(G, \tau) = F_c(G^*_v, \sigma) \).

**Proof.** Since \( G \) has a vertex \( v \) of degree 3, it follows that \( r \geq 3 \). We first consider the case when \( k = 2 \). Let \( E(v) = \{e_1, e_2, e_3\} \), without loss of generality we assume that \( v \) is the end vertex of \( e_1 \) and \( e_2 \) and the initial vertex of \( e_3 \).

Consider \((G_v^2, \sigma)\), and let \( v_1, v_2, v_3 \) be the vertices of \( T_2 \). We assume that \( v_1 \) and \( v_2 \) are end vertices of \( e_1' \) and \( e_2' \), respectively, and that \( v_3 \) is the initial vertex of \( e_3' \). Furthermore, denote the two edges between \( v_i \) and \( v_j \) \((i \neq j)\) by \( e_{i,j} \) and \( f_{i,j} \). Let us orient the new edges as follows (note that all of those are positive): Let \( e_{1,3} = (v_1, v_3) \), \( f_{1,3} = (v_1, v_3) \), \( e_{1,2} = (v_2, v_1) \), \( f_{1,2} = (v_2, v_1) \), \( e_{2,3} = (v_2, v_3) \) and \( f_{2,3} = (v_3, v_2) \). Extend \( \phi \) to a nowhere-zero \( r \)-flow \( \phi_v \) on \((G_v^2, \sigma)\) with \( \phi_v(e_{1,3}) = \phi(e_1) \), \( \phi_v(f_{1,3}) = \phi_v(f_{2,3}) = 2 \), \( \phi_v(e_{2,3}) = \phi(e_2) \) and \( \phi_v(e_{1,2}) = \phi_v(f_{1,2}) = 1 \). Clearly, \( \phi_v \) is a nowhere-zero \( r \)-flow on \((G_v^2, \sigma)\).

If \( k > 2 \), then first construct \((G_v^2, \sigma)\) as above and then add \( k - 2 \) directed (all-positive) triangles with a nowhere-zero 2-flow to the triangle \( T_2 \) of \((G_v^2, \sigma)\) to obtain a graph \((G^*_v, \sigma')\). Clearly, \( F_v(G^*_v, \sigma') = r \), for all \( k \geq 2 \). The replacement of \( v \) by \( T_k \) is shown in Figure 5.1.

\[ \square \]

We first consider the interval \([4; 6]_Q\). Let \(((G, \sigma), U)\) be a network, \( k \geq 1 \) be an integer, and \( V'(G) = \{v : v \in V(G) - U \text{ and } d_G(v) \notin \{2, 2k + 1\}\} \).

**Lemma 5.2.4.** For all \( k \geq 1 : \mathcal{F}^c \cap [4; 6] \subseteq \mathcal{F}^c_{2k+1} \)
5.2 Proof of Theorem 5.1.2

Proof. We first prove the statement for \( k = 1 \). Let \( r \in F^c \cap [4; 12] \), \((G, \sigma)\) be a signed graph with \( F_c(G, \sigma) = r \), and let \( \phi \) be a nowhere-zero \( r \)-flow on \((G, \sigma)\). We assume that \((G, \sigma)\) has no bivalent vertices and that it satisfies the conditions of Lemma 5.2.1. If \( V'(G) = \emptyset \), then we are done.

If there is a vertex \( v \in V'(G) \), then let \( ((G_1, \sigma_1), U_1) \) be the network which is obtained from \((G, \sigma)\) by splitting up \( v \) into terminals \( v_1, \ldots, v_l \). For \( i \in \{1, \ldots, l\} \) let \( E(v_i) = \{e'_i\} \), and let \( \phi_1 \) be the corresponding nowhere-zero \( r \)-flow on \((G_1, U_1)\). Note, that \( |V'(G_1)| = |V'(G)| - 1 \).

**Step 1:** Consider terminal \( v_i \). We assume that \( v_i \) is the end vertex of \( e'_i \). The argumentation in the other case is similar. If \( \delta \phi_1(v_i) > \frac{3}{2} \), then add six vertices \( x_1, x_2, u_1, u_2, u_3, u_4 \) and the positive edges \((v_i, x_1), (x_1, x_2), (u_1, v_i), (x_1, u_2), (x_2, u_3), (x_2, u_4)\). Extend \( \phi_1 \) to a nowhere-zero \( r \)-flow \( \phi_2 \) on the new network \(((G_2, \sigma_2), U_2)\), where \( \phi_2((x_1, u_2)) = \phi_2((x_2, u_3)) = \phi_2((x_2, u_4)) = 1, \phi_2((v_i, x_1)) = 3, \phi_2((x_1, x_2)) = 2 \) and \( \phi_2((u_1, v_i)) = 3 - \delta \phi_1(v_i) \). Then \( U_2 = (U_1 - \{v_i\}) \cup \).
\{u_1, \ldots, u_4\}, \delta \phi_2(u_i) \leq \frac{3}{2} \text{ and } \delta \phi_2(u_j) = 1 \text{ for } j \in \{2, 3, 4\}. \text{ By Lemma 5.2.1 we have } 1 < \phi_2((u_1, v_i)) < \frac{3}{2}. \text{ The construction is depicted in Figure 5.2.}

Repeat Step 1 until we obtain a network \((G_3, \sigma_3, U_3)\) with a nowhere-zero flow \(\phi_3\) where the absolute value of the excess flow of all terminals is at most \(\frac{3}{2}\). Note that all vertices of \((G_3, \sigma_3, U_4)\) have degree 1 or 3, and that the number of terminals with excess flow different from 1 is unchanged.

\textbf{Step 2:} If there are two terminals \(w_1, w_2\) of \((G_3, U_3)\) with an excess flow greater than 1, then identify \(w_1\) and \(w_2\) to a single vertex \(y_1\) and add vertices \(y_2, y_3, y_4\) and positive edges \((y_1, y_2), (y_2, y_3), (y_2, y_4)\). Extend the nowhere-zero \(r\)-flow on \((G_3, U_3)\) to a nowhere-zero \(r\)-flow \(\phi_4\) on the new network \((G_4, \sigma_4, U_4)\), where \(\phi_4((y_1, y_2)) = \delta \phi_3(w_1) + \delta \phi_3(w_2), \phi_4((y_2, y_3)) = 1\) and \(\phi_4((y_2, y_4)) = \delta \phi_3(w_1) + \delta \phi_3(w_2) - 1\). The construction is depicted in Figure 5.3.

If \(\delta \phi_4(y_4) = 2\), then add two further vertices \(y'_4, y''_4\) and positive edges \((y_4, y'_4), (y_4, y''_4)\) and define the flow value on these two edges to be 1. Then we have \(U_4 = (U_3 - \{w_1, w_2\}) \cup \{y_3, y'_4, y''_4\}\) and all terminals have excess flow 1. Hence, the number of terminals with an excess flow different from 1 is reduced by 2.

If \(\frac{3}{2} < \delta \phi_4(y_4) < 2\), then we have \(U_4 = (U_3 - \{w_1, w_2\}) \cup \{y_3, y_4\}\), and the number of terminals with excess flow different from 1 is reduced by 1. Apply Step 1 again to obtain a new network with the same number of
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Figure 5.3: Construction for two terminals with an excess flow greater than 1

terminals with excess flow different from 1 and where all terminals have an excess flow of at most $\frac{3}{2}$.

Since the number of terminals with excess flow different from 1 is unchanged in Step 1 and reduced in Step 2 we eventually obtain (after a finite number of applications of these two steps) a network $((G_5, \sigma_5), U_5)$ which has a nowhere-zero $r$-flow $\phi_5$ and the following property: Let $U_5^+ = \{z_1, \ldots, z_n\}$, and $U_5^- = \{z'_1, \ldots, z'_m\}$. At most one terminal of $U_5^+$, say $z_1$, has an excess flow greater than 1 and all the other terminals have an excess flow of 1. By our construction we have that $\delta \phi_5(z_1) < 2$.

Since $\phi_5$ is a nowhere-zero $r$-flow on $((G_5, \sigma_5), U_5)$ which is successively constructed from the nowhere-zero $r$-flow $\phi$ on $(G, \sigma)$, the same holds true for $U_5^-$. Hence, $n = m$ and $\delta \phi_5(z_1) = \delta \phi_5(z'_1)$. Identify terminals $z_i$ and $z'_i$, to obtain a cubic graph $H$ and a nowhere-zero $r$-flow on $H$. Since $(G, \sigma)$ can be obtained from $(H, \tau)$ by contracting an all-positive subgraph of $H$ it follows that $F_c(H, \tau) = r$.

For $k \geq 2$ apply Construction 5.2.2 to every vertex of $(H, \tau)$ to obtain a $(2k + 1)$-regular graph, which has circular flow number $r$ by Lemma 5.2.3.
Since an all-positive bipartite cubic graph has circular flow number 3 and by Lemma 5.2.4 there is a cubic graph with circular flow number \( r \) for every \( r \in \mathcal{F}_c \cap [4; 6] \), it follows that Theorem 5.1.2 is proven for \( k = 1 \).

We now consider the interval \([2 + \frac{2}{2k-1}; 4)\) for \( k \geq 2 \). Clearly, Construction 5.2.2 and Lemma 5.2.3 allow us to eliminate trivalent vertices. The following lemma describes a "target substructure" in a graph \((G, \sigma)\), which allows to replace a specific vertex \( v \) with \( d_G(v) \neq 2k+1 \) by a graph \((H_v, \tau_v)\), such that the new graph \((H, \tau)\) has the same circular flow number as \((G, \sigma)\) and \(|V'(H)| = |V'(G)| - 1\).

**Lemma 5.2.5.** Let \( n \geq 1 \) and \( k \geq 2 \) be integers, \( r \in [2 + \frac{2}{2k-1}; 4) \) and \((G, \sigma)\) be a graph with \( F_c((G, \sigma)) = r \). If there is a nowhere-zero \( r \)-flow \( \phi \) of \((G, \sigma)\) and a vertex \( v \in V'(G) \) with \(|E^+(v)| = n(2k-1)\), \(|E^-(v)| = 2kn\) and \( \phi(e) = 1 + \frac{1}{2k-1} \) if \( e \in E^+(v) \) and \( \phi(e) = 1 \) if \( e \in E^-(v) \), then there is a graph \((H, \rho)\) with \( F_c((H, \rho)) = F_c((G, \sigma)) \) and \(|V'(H)| = |V'(G)| - 1\).

**Proof.** Let \( v \) be a vertex of a graph \( G \) with \( d_G(v) = n(4k - 1) \), and \( E(v) = \{e_1, \ldots, e_{n(4k-1)}\} \). Assume that the directions and the flow values on the edges of \( E(v) \) are as stated above.

Let \(((G_1, \sigma_1), U_1)\) be the network which is obtained from \((G, \sigma)\) by splitting up \( v \) into terminals \( v_1, \ldots, v_{n(4k-1)} \). Let \( E(v_i) = \{e'_i\} \) and \( \phi_1 \) be the corresponding nowhere-zero \( r \)-flow on \(((G_1, \sigma_1), U_1)\). Note that \(|U_1^+| = n(2k - 1)\) and \(|U_1^-| = 2kn|\).

We transform \(((G_1, \sigma_1), U_1)\) into a network \(((G_2, \sigma_2), U_2)\) with \(|U_2^+| = (n-1)(2k-1)|\) and \(|U_2^-| = 2k(n-1)\). Let \( X^+ \) be a set of \( 2k - 1 \) elements of \( U_1^+ \) and \( X^- \) be a set of \( 2k \) elements of \( U_1^- \).

**Step 1:** For \( u_1, u_2 \in X^+ \), we add six vertices \( v_1, v_2, v_3, v_4, u_3, u_4 \). We add the positive edges \((v_2, v_1), (v_4, v_3), (v_1, v_4)_{k+1}, (v_4, v_1)_{k-2}, (v_2, v_3)_{k-1}, \)
(v₃, v₂), (u₁, v₁), (u₂, v₂), (v₃, u₃) and (v₄, u₄). The construction is depicted in Figure 5.4. We extend φ₁ to a nowhere-zero r-flow φ₂ on ((G₂, σ₂), U₂) as follows:

Let φ₂((v₃, u₃)) = φ₂((v₃, v₂)ₖ) = φ₂((v₃, v₂)ₖ₊₁) = 1, φ₂((u₁, v₁)) = φ₂((u₂, v₂)) = φ₂((v₄, v₃)) = 1 + 1/₂k−₁, and φ₂((v₂, v₁)) = φ₂((v₂, v₁)ₖ−₂) = φ₂((v₂, v₃)ₖ−₁) = φ₂((v₄, u₄)) = 1 + 2/₂k−₁. Note that we have changed two terminals of X⁺ with excess flow 1 + 1/₂k−₁ into two terminals, where one has excess flow 1 and the other has excess flow 1 + 2/₂k−₁. The successive application of Step 1 on k − 1 pairs of terminals of X⁺ transfer 2k − 2 terminals with excess flow 1 + 1/₂k−₁ into k − 1 terminals with excess flow 1 and k − 1 terminals with excess flow 1 + 2/₂k−₁. Let X⁺₁ be the set of terminals with excess flow 1 and X⁺₂ be the set of terminals with excess flow 1 + 2/₂k−₁. We continue with Step 2.
Step 2: First, identify each of the \( k - 1 \) elements of \( X^+_1 \) with precisely one terminal of \( X^- \), and identify the \( k + 1 \) remaining elements of \( X^- \) to a single vertex \( v^- \). Then identify the elements of \( X^+_2 \) and the remaining terminal of \( X^+ \) with excess flow \( 1 + \frac{2}{2k-1} \) to a single vertex \( v^+ \). Finally identify \( v^+ \) and \( v^- \) to a single vertex \( v \) to obtain the desired network \(((G_2, \sigma_2), U_2)\) (see Figure 5.5).

Repeat Step 1 and Step 2 until we get a graph \( H \) with \( F_c(H, \rho) = r \), and \( |V'(H)| = |V'(G)| - 1 \).

\[\square\]

**Lemma 5.2.6.** For all \( k \geq 2 \): \( F^c \cap [2 + \frac{2}{2k-1}; 4] \subseteq F^c_{2k+1} \)

**Proof.** Let \( k \geq 2 \), \( r \in F^c \cap [2 + \frac{2}{2k-1}; 4] \) and \( (G, \sigma) \) be a graph with \( F_c((G, \sigma)) = r \) and let \( \phi \) be a nowhere-zero \( r \)-flow on \( (G, \sigma) \). Due to Lemmas 5.2.1, 5.2.3 and Construction 5.2.2 we can assume that if \( e = uv \) is an edge and \( \phi(e) > 2 \), then \( d_G(v) = d_G(u) = 2k + 1 \). If \( V'(G) = \emptyset \), then we are done. If \( V'(G) \neq \emptyset \), then we proceed as follows:

**Step 1:** Let \( v \in V'(G) \), and \(((G_1, \sigma_1), U_1)\) be the network which is obtained from \( (G, \sigma) \) by splitting up \( v \) into terminals \( v_1, \ldots, v_n \). Let \( E(v_i) = \)
{e'_i} and φ₁ be the corresponding nowhere-zero r-flow on (G₁, U₁). Note that |V'(G₁)| = |V'(G)| − 1.

**Step 2:** We are now going to transfer ((G₁, σ₁), U₁) stepwise into a network ((G₄, σ₄), U₄) with |V'(G₁)| = |V'(G₄)| and a nowhere-zero r-flow φ₄ with |δφ₄(u)| ≤ 1 + \frac{1}{2k-1} for every terminal u. For the constructions we only consider positive terminals, for negative terminals the constructions work analogously.

**Step 2.1:** If there is a terminal v in ((G₁, σ₁), U₁) with δφ₁(v) ≥ 1 + \frac{2}{2k-1}, then add 2k new vertices and connect each new vertex with v by a single positive edge. We extend the nowhere-zero r-flow as follows: Exactly k + 1 of these edges have v as initial vertex and they have flow value 1 and exactly k − 1 edges have v as end vertex and they have flow value \frac{k+1-δφ₁(v)}{k-1}. Since 1 + \frac{1}{2k-1} < δφ₁(v) < 2, it follows that 1 < \frac{k+1-δφ₁(v)}{k-1} < 1 + \frac{1}{2k-1} ≤ r − 1. The construction is shown in Figure 5.6. We apply this construction to each terminal u of ((G₁, σ₁), U₁) with δφ₁(u) ≥ 1 + \frac{2}{2k-1} to obtain a network (G₂, σ₂), U₂) with a nowhere-zero r-flow φ₂ such that the excess flow is smaller than 1 + \frac{1}{2k-1} for every terminal. Note that |V'(G₂)| = |V'(G)|. If δφ₂(v) ≤ 1 + \frac{1}{2k-1}, for all v ∈ U₂, then set ((G₄, σ₄), U₄) = ((G₂, σ₂), U₂).
Figure 5.7: Construction for $v$ if $1 + \frac{1}{2k-1} < \delta \phi_2(v) < 1 + \frac{2}{2k-1}$.

**Step 2.2:** If there is a terminal $v \in U_2$ with $1 + \frac{1}{2k-1} < \delta \phi_2(v) < 1 + \frac{2}{2k-1}$, then construct a new network $((G_3, \sigma_3), U_3)$ by adding seven new vertices $v_1, v_2, v_3, v_4, u_1, u_2, u_3$. We add the positive edges $(v_1, v_2)_k$, $(v_2, v_3)_{k-1}$, $(v_3, v_4)_{k-1}$, $(v_4, v_1)_k$, $(v_2, u_1)$, $(v_3, v_2)$, $(v_3, u_2)$, $(v_4, v_3)$, $(u_3, v_4)$ and $(v_1, v_4)$. The construction is shown in Figure 5.7. Extend $\phi_2$ to a nowhere-zero $r$-flow $\phi_3$ on $((G_3, \sigma_3), U_3)$ as follows: Let $\phi_3((v, v_1)) = \delta \phi_2(v)$, $\phi_3((u_3, v_4)) = 2 + \frac{2}{2k-1} - \delta \phi_2(v)$, $\phi_3((v_1, v_4)) = \delta \phi_2(v) - \frac{1}{2k-1}$, $\phi_3((v_2, v_3)_{k-1}) = \phi_3((v_4, v_1)_k) = \phi_3((v_3, v_4)_{k-1}) = 1 + \frac{2}{2k-1}$, $\phi_3((v_2, u_1)) = \phi_3((v_3, u_2)) = 1 + \frac{1}{2k-1}$, and $\phi_3((v_3, v_2)) = \phi_3((v_1, v_2)_k) = \phi_3((v_3, v_4)_{k-1}) = 1$. Terminal $v$ is replaced by three new terminals $u_1, u_2, u_3$ and all other vertices of $((G_2, \sigma_2), U_2)$ are unchanged. The absolute value of the excess flow at two terminals is $1 + \frac{1}{2k-1}$ and the other one is $2 + \frac{2}{2k-1} - \delta \phi_2(v)$, which is at least 1 and and smaller than $1 + \frac{1}{2k-1}$. Repeat this construction to obtain a
5.2 Proof of Theorem 5.1.2

\[ 2 + \frac{1}{2k-1} < \delta \phi_4(u_1) + \delta \phi_4(u_2) < 2 + \frac{2}{2k-1} \]

Figure 5.8: Construction if \( 2 + \frac{1}{2k-1} < \delta \phi_4(u_1) + \delta \phi_4(u_2) < 2 + \frac{2}{2k-1} \)

network \( ((G_4, \sigma_4), U_4) \), where the excess flow of every terminal is at most \( 1 + \frac{1}{2k-1} \).

**Step 3:** Now we are going to construct a new network where all but at most two (one positive and one negative) terminals have excess flow in \( \{ \pm 1, \pm (1 + \frac{1}{2k-1}) \} \). If one of the aforementioned networks has this property, then we are done. Again, for the constructions we only consider positive terminals, for negative terminals it works analogously.

Suppose that \( ((G_4, \sigma_4), U_4) \) does not have this property. Then there are two (positive) terminals \( u_1 \) and \( u_2 \) with \( \delta \phi_4(u_1), \delta \phi_4(u_2) \not\in \{ 1, 1 + \frac{1}{2k-1} \} \).

**Step 3.1:** If \( 2 + \frac{1}{2k-1} < \delta \phi_4(u_1) + \delta \phi_4(u_2) < 2 + \frac{2}{2k-1} \), then add six new vertices \( v_1, v_2, v_3, v_4, u_3, u_4 \) to obtain a new network \( ((G_5, \sigma_5), U_5) \). We add the positive edges \( (v_2, v_1), (v_4, v_3), (v_3, v_2), (v_2, v_3)_{k-1}, (v_3, v_2)_{k-1}, (v_4, v_1)_{k-1}, (v_1, v_4)_{k}, (u_1, v_1), (v_2, u_3), (u_2, v_3) \) and \( (v_4, u_4) \). The construction is depicted in Figure 5.8.

Extend \( \phi_4 \) to a nowhere-zero \( r \)-flow \( \phi_5 \) on \( ((G_5, \sigma_5), U_5) \) as follows:
Let $\phi_5((u_1, v_1)) = \delta \phi_4(u_1)$, $\phi_5((v_2, u_3)) = \delta \phi_4(u_1) + \delta \phi_4(u_2) - (1 + \frac{1}{2k-1})$, $\phi_5((u_2, v_3)) = \delta \phi_4(u_2)$, $\phi_5((v_2, v_1)) = 2 + \frac{1}{2k-1} - \delta \phi_4(u_1)$, $\phi_5((v_3, v_2)) = \delta \phi_4(u_2) + \frac{1}{2k-1}$, $\phi_5((v_4, u_4)) = 1 + \frac{1}{2k-1}$, $\phi_5((v_1, v_4)_k) = \phi_5((v_3, v_2)_{k-1}) = 1 + \frac{2}{2k-1}$, and $\phi_5((v_4, v_3)) = \phi_5((v_4, v_1)_{k-1}) = \phi_5((v_2, v_3)_{k-1}) = 1$.

**Step 3.2:** If $2 < \delta \phi_4(u_1) + \delta \phi_4(u_2) \leq 2 + \frac{1}{2k-1}$, then add four vertices $v_1$, $v_2$, $u_3$, $u_4$ and the following edges to obtain a new network $((G_6, \sigma_6), U_6)$: We add the positive edges $(u_1, v_1)$, $(u_2, v_1)$, $(v_2, u_3)$, $(v_2, v_1)_k$ and $(v_1, v_2)_k$. The construction is shown in Figure 5.9.

Extend $\phi_4$ to a nowhere-zero $r$-flow $\phi_6$ on $((G_6, \sigma_6), U_6)$ as follows: Let $\phi_6((u_1, v_1)) = \delta \phi_4(u_1)$, $\phi_6((u_2, v_1)) = \delta \phi_4(u_2)$, $\phi_6((v_2, u_4)) = \delta \phi_4(u_1) + \delta \phi_4(u_2) - 1$, $\phi_6((v_1, v_2)_k) = \frac{\delta \phi_4(u_1) + \delta \phi_4(u_2) - k + 1}{2k-1}$, and $\phi_6((v_2, v_1)_{k-1}) = \phi_6((v_2, u_3)) = 1$. Then $u_3$ and $u_4$ are terminals in $(G_6, U_6)$ and their excess flow is at most $1 + \frac{1}{2k-1}$.

Repeat these two constructions until we get a network $(G_7, U_7)$ where all terminals but at most two (one positive and one negative) terminals have excess flow in $\{\pm 1, \pm (1 + \frac{1}{2k-1})\}$. 

![Figure 5.9: Construction if $2 < \delta \phi_4(u_1) + \delta \phi_4(u_2) \leq 2 + \frac{1}{2k-1}$](image-url)
Now we are going to reduce the number of terminals so that we eventually get a signed graph \((H^*, \sigma^*)\) with \(F_c((H^*, \sigma^*)) = r\) and \(|V'(H^*)| = |V'(G)| - 1\).

Let \(((G_7, \sigma_7), U_7)\) be the network where the only possible excess flow values are \(\pm 1\) or \(\pm(1 + \frac{1}{2k-1})\) except for at most two terminals (one with positive and one with negative excess flow) \(x\) and \(y\) with \(|\delta \phi_7(x)|, |\delta \phi_7(y)| < 1 + \frac{1}{2k-1}\). Let \(aU^+_7\) be the set of terminals in \(U^+_7\) which have excess flow 1, and \(bU^-_7\) be the set of terminals in \(U^-_7\) which have excess flow \(1 + \frac{1}{2k-1}\); \(aU^-_7\) and \(bU^-_7\) are defined analogously.

If only one of \(x\) and \(y\) exists, say \(x\), then, since \(\phi_7\) is a nowhere-zero \(r\)-flow on \(((G_7, \sigma_7), U_7)\), \(x\) is a multiple of \(\frac{1}{2k-1}\), contradicting the fact that \(|\delta \phi_7(x)| \in (1; 1 + \frac{1}{2k-1})\).

If \(x\) and \(y\) exist then it follows that
\[
|bU^+_7|(1 + \frac{1}{2k-1}) + |aU^+_7| + |\delta \phi_7(x)| = |bU^-_7|(1 + \frac{1}{2k-1}) + |aU^-_7| + |\delta \phi_7(y)|
\]
and therefore
\[
|\delta \phi_7(y)| - |\delta \phi_7(x)| = (|bU^+_7| - |bU^-_7|)(1 + \frac{1}{2k-1}) + |aU^+_7| - |aU^-_7|.
\]

Hence, \(|\delta \phi_7(y)| - |\delta \phi_7(x)|\) is a multiple of \(\frac{1}{2k-1}\). Since \(|\delta \phi_7(x)|, |\delta \phi_7(y)| \in (1; 1 + \frac{1}{2k-1})\) it follows that \(|\delta \phi_7(y)| - |\delta \phi_7(x)| \in (-\frac{1}{2k-1}; \frac{1}{2k-1})\), and therefore \(|\delta \phi_7(y)| = |\delta \phi_7(x)|\).

Without loss of generality we assume that \(|aU^+_7| \leq |aU^-_7|\). Since \(((G_7, \sigma_7), U_7)\) has a nowhere-zero \(r\)-flow, it follows that \(|bU^+_7| \geq |bU^-_7|\).

**Step 4:** Identify \(x\) and \(y\). Then identify every terminal of \(aU^+_7\) with precisely one terminal of \(aU^-_7\), and every terminal of \(bU^-_7\) with precisely one terminal of \(bU^+_7\) to obtain a network \(((G_8, \sigma_8), U_8)\) where all positive terminals have excess flow \(1 + \frac{1}{2k-1}\) and all negative terminals have excess
flow \(-1\). Then identify these remaining terminals in one vertex \(v\) and suppress the bivalent vertices to obtain a signed graph \((G_9, \sigma_9)\) with 
\[F_c((G_9, \sigma_9)) = r.\] If \(|V'(G_9)| = |V'(G)| - 1\), then \((G_9, \sigma_9) = (H^*, \sigma^*)\) and we are done. Otherwise, \(|V'(G_9)| = |V'(G)|\) and there is an integer \(n \geq 1\) such that 
\[d_G(v) = n(4k - 1), \quad |E^+(v)| = n(2k - 1), \quad |E^-(v)| = 2kn,\]
and if \(e \in E^+(v)\), then \(e\) has flow value \(1 + \frac{1}{2k-1}\), and if \(e \in E^-(v)\), then \(e\) has flow value \(1\). By Lemma 5.2.5 there is a graph \((H^*, \sigma^*)\) with 
\[F_c((H^*, \sigma^*)) = F_c((G_9, \sigma_9)) = r \quad \text{and} \quad |V'(H^*)| = |V(G_9)| - 1 = |V(G)| - 1.\]

Repeat Step 1 to Step 4 until we get a graph \((H, \tau)\) with \(V'(H) = \emptyset\); i.e. \(H\) is \((2k + 1)\)-regular and 
\[F_c((H, \tau)) = r.\]

\[\square\]

Theorem 5.1.2 follows from Lemmas 5.2.4 and 5.2.6.
Chapter 6

Flows and orientations

6.1 Flows and signed circuits

Let $G$ be a graph and $e_1, e_2 \in E(G)$ be two edges incident with a vertex $v_1$. We say, under an orientation $D$ of $G$, $e_1, e_2$ are consistent at $v_1$ if either

- $h_{e_1}^{v_1}$ is directed towards $v_1$ and $h_{e_2}^{v_1}$ is directed away from $v_1$, or

- $h_{e_1}^{v_1}$ is directed away from $v_1$ and $h_{e_2}^{v_1}$ is directed towards $v_1$.

We also say that $v_1$ is consistently balanced. A trail or a balanced cycle $S$ is consistent, if $S$ is consistently balanced in each vertex. An odd cycle $C$ is consistent if $C$ is consistently balanced at each vertex but one.

We say that a nowhere-zero flow $\phi$ on a graph $(G, \sigma)$ is $A$-decomposable, if there exist a set $A$ of subgraphs of $G$ endowed with certain nowhere-zero flows, such that

$$\phi = \sum_{S \in A} n_S \phi_S$$
where \( \phi_S \) is the corresponding nowhere-zero flow on \( S \) and \( n_S \) is a positive integer.

For a graph \((G, \sigma)\) let \( \mathcal{F}(G, \sigma) \) be the set of signed circuits in \((G, \sigma)\).

For a signed graph \((G, \sigma)\) with a positive integer nowhere-zero flow \( \phi \) let \( F^i_{(G, \sigma)} \in \mathcal{F}(G, \sigma) \) \((i \in \mathbb{Q})\) be the set of signed circuits that are endowed with pseudo flows with the following properties:

- if \( S \in F^i_{(G, \sigma)} \) is a barbell, then all edges in the unbalanced circuits receive the value \( i \) and all edges on the path (possibly zero edges) that connects these circuits receive the value \( 2i \).

- if \( S \in F^i_{(G, \sigma)} \) is a balanced circuit, then all edges receive value 1

Mácajová and Škoviera proved the following theorem:

**Theorem 6.1.1.** [25] Let \( \phi \) be a positive integer nowhere-zero flow on a signed graph \((G, \sigma)\). Then \( \phi \) is \( F^{\frac{1}{2}}_{(G, \sigma)} \)-decomposable such that all graphs in \( F^{\frac{1}{2}}_{(G, \sigma)} \) have the same orientation as in \((G, \sigma)\).

The issue that orientations are kept seems to be a strong condition. However, here, we decompose a nowhere-zero flow into non-proper nowhere-zero flows. The question at hand is, whether the use of non-proper nowhere-zero flows with fractional values is necessary. As an example for the necessity the authors of [25] gave the following example.

Let \((K^*_2, \sigma)\) be the graph consisting of two vertices connected by two parallel positive edges, where a negative loop is attached to each vertex. It is easy to see that \((K^*_2, \sigma)\) admits a nowhere-zero 2-flow, but each decomposition of this flow into flows of signed circuits contains two distinct barbells that share both negative loops.

Thus, in this setting we do not get rid of non-proper flows with fractional flow values. It seems natural to express a nowhere-zero flow of a graph as the sum of nowhere-zero flows of signed circuits. For this
6.1 Flows and signed circuits

purpose, in the further study of decompositions of a signed graph \((G, \sigma)\) we do not want to fix the orientation of the signed circuits in \(F^{i}_{(G, \sigma)}\) corresponding to \((G, \sigma)\). It turns out that this will give us the flexibility to express a nowhere-zero flow by the sum of proper nowhere-zero flows of signed circuits.

**Theorem 6.1.2.** Let \((G, \sigma)\) be a flow-admissible signed graph and \(\phi\) be a positive integer nowhere-zero flow of \((G, \sigma)\). Then \(\phi\) is \(F^{1}_{(G, \sigma)}\)-decomposable.

**Proof.** Let \(\phi\) be a positive integer nowhere-zero flow on \((G, \sigma)\).

We will prove the statement by induction on \(\sum_{e \in E(G)} \phi(e)\). Therefore, in each step we are going to find a directed signed circuit in \((G, \sigma)\) and choose one corresponding \(C \in \mathcal{F}(G, \sigma)\) together with a nowhere-zero flow \(\phi_{C}\) such that \(\tau_{\phi_{C}}(e) = \tau_{\phi}(e)\) for each \(e \in E(G)\). Next, we consider the integer nowhere-zero flow \(\phi' = \phi - \phi_{C}\) on the directed graph \((G', \sigma')\) with \(E(G') = \{e \in E(G) | \phi(e) - \phi_{C}(e) \neq 0\}\) and \(\sigma' = \sigma|_{E(G')}\). If \(C\) is a balanced circuit or a short barbell, then

\[
\sum_{e \in E(G')} \phi'(e) = \sum_{e \in E(G)} \phi(e) - |E(C)|.
\]

Assume \(C\) is a barbell consisting of two disjoint unbalanced circuits \(C_1\) and \(C_2\) and a path \(T\) connecting \(C_1\) and \(C_2\). It follows

\[
\sum_{e \in E(G')} \phi'(e) = \sum_{e \in E(G)} \phi(e) - |E(C_1) \cup E(C_2)| - |\{e \in E(T) : \phi(e) \neq 1\}|.
\]

Note, that in \((G', \sigma')\) edges in \(T\) with flow value 1 reversed their direction.

It remains to show that in each step we can find a signed circuit \(C \in \mathcal{F}(G, \sigma)\) as a subgraph. Suppose after a series of steps we get one graph \((G', \sigma')\) with a pseudoflow \(\phi'\) and there are no signed circuits \(C \in \mathcal{F}(G, \sigma)\)
as subgraphs in \((G',\sigma')\). Let \(v \in V(G')\). Note, since \(\phi'\) is a positive nowhere-zero flow, for each vertex \(v\) exist two edges that are consistent in \(v\). Beginning with a vertex \(v\) and an outgoing half-edge we successively add consistent vertices and construct a maximum consistent trail \(P_1\). The trail \(P_1\) must contain at least one vertex, say \(u_1\) twice. The corresponding circuit \(C_1\) must be unbalanced.

Next, beginning with \(v\) and an incoming half-edge we successively add consistent vertices and construct a maximum consistent trail \(P_2\). Similarly \(P_2\) must contain at least one vertex, say \(w_1\) twice, forming an unbalanced circuit \(C_2\). If \(C_1\) and \(C_2\) are pairwise disjoint or intersect in precisely one vertex, then \((G',\sigma')\) contains a barbell, which contradicts our assumption. Thus, there are two circuits \(C_u = u_1, \ldots, u_p, \ldots, u_n, u_1\) and \(C_w = w_1, \ldots, w_p, \ldots, w_m, w_1\) \((p, n, m \in \mathbb{Z})\) and \(u_i = w_i\) for \(i \in \{1, \ldots, p\}\). The circuit \(C_t\) formed by the the trails \(P_1 = u_p, u_{p+1}, \ldots, u_n\) and \(P_2 = w_p, w_{p+1}, \ldots, w_m\) must also be unbalanced. Therefore, precisely one trail is balanced. Suppose \(P_1\) is balanced and \(P_2\) is unbalanced. However, since \(C_1\) and \(C_2\) are unbalanced, the trail \(u_1, \ldots, u_p\) cannot be balanced, nor unbalanced, a contradiction.

\(\square\)

6.2 Characterization of flows

As seen in Theorem 6.1.2 orientations play an important role of the way we are able to decompose a flow. Just by allowing a higher flexibility we are able to prove a decomposition into proper nowhere-zero flows. In this section, for a signed graph \((G,\sigma)\) we characterize a nowhere-zero flow \(\phi\) by finding a set \(Q\) of orientations of \((G,\sigma)\). In this regard for the orientations in \(Q\) there are no constraints. An orientation \(D \in Q\) even
does not have to admit a positive nowhere-zero flow. We will relate the problem of finding a nowhere-zero $k$-flow to the problem of finding a set of orientations with the property that a linear combination with given coefficients over the boundary of these so called reorientations equals zero. Let $(G, \sigma)$ be an directed signed graph with orientation $D$. A reorientation of $(G, \sigma)$ is an assignment $f_D : E(G) \to \{-1, +1\}$. We interpret $f_D$ as a new orientation of $(G, \sigma)$ itself. If $f_D(e) = 1$ for an edge $e$, then each half-edge of $e$ is oriented as in $D$ and if $f(e) = -1$, then each half-edge of $e$ is oriented in the opposite direction. We may view each orientation as an reorientation of one fixed orientation. Here, we use the expression $\partial f$ to denote the corresponding boundary of an orientation $f$.

The following theorem characterizes unsigned graphs that admit a nowhere-zero 5-flow by orientations.

**Theorem 6.2.1.** [12] Let $G$ be a directed unsigned graph. $G$ admits a nowhere-zero 5-flow if and only if $G$ has three reorientations $f_1, f_2, f_3$, such that

$$\partial f_1 + 2\partial f_2 + 5\partial f_3 = 0$$

In the following we will give a similar characterization which is more general for unsigned graphs that admit a nowhere-zero $(\frac{p}{q} + 1)$-flow.

**Theorem 6.2.2.** A directed unsigned graph $G$ admits a nowhere-zero $(\frac{p}{q} + 1)$-flow if and only if $G$ has reorientations $f_1, \ldots, f_{p-q+1}$, such that

$$\sum_{i=1}^{p-q+1} \alpha_i \partial f_i = 0 \text{ with } \alpha_1 = \frac{q+p}{2q} \text{ and } \alpha_l = \frac{1}{2q} \text{ for } l \in \{2, \ldots, p-q+1\}$$

**Proof.** Let $\phi$ be a nowhere-zero $(\frac{p}{q} + 1)$-flow on $G$ which is chosen according to Theorem [1.4.8] such that for each edge $e$, $\phi(e)$ is a multiple of $\frac{1}{q}$ and let $D$ be the underlying orientation. We define the required reorientations in the following way:

For $e \in E(G)$ and $i \in \{1, \ldots, p-q+1\}$ let $f_i(e) = 1$ if $1 + \frac{i-1}{q} \leq \phi(e)$ or $-1 - \frac{i-1}{q} < \phi(e) \leq -1$ and $f_i(e) = -1$ otherwise. Next, we consider the vector consisting of all possible flow values of $\phi$. We are going to express
each flow value of an edge $e$ by one unique set of reorientations regarding $e$. We will determine the coefficients $\alpha_i$ of the equation $\sum_{i=1}^{p-q+1} \alpha_i f_i(e) = \phi(e)$. For reasons of symmetry we can restrict the linear system to only positive flow values.

\[
\begin{pmatrix}
1 & -1 & \cdots & -1 \\
1 & 1 & -1 & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{p-q+1} \\
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{q} \\
\frac{q+1}{q} \\
\vdots \\
\frac{p}{q} \\
\end{pmatrix}. 
\tag{6.1}
\]

The unique solution is given by $\alpha_1 = \frac{q+p}{2q}$ and $\alpha_l = \frac{1}{2q}$ for $l \in \{2, \ldots, p-q+1\}$. Hence,

\[
\sum_{i=1}^{p-q+1} \alpha_i \partial f_i(v) = \sum_{i=1}^{p-q+1} \alpha_i \left( \sum_{h \in H(v)} f_i(h_e) \right)
= \sum_{i=1}^{p-q+1} \sum_{h \in H^+(v)} \alpha_i f_i(h_e) - \sum_{i=1}^{p-q+1} \sum_{h \in H^-(v)} \alpha_i f_i(h_e)
= \sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0
\tag{6.2}
\]

To prove the other direction, let $\alpha_1 = \frac{q+p}{2q}$ and $\alpha_l = \frac{1}{2q}$ for $l \in \{2, \ldots, p-q+1\}$ and $f_1, \ldots, f_{p-q+1}$ be reorientations of $(G, \sigma)$ such that $\sum_{i=1}^{p-q+1} \alpha_i \partial f_i = 0$. Let $\phi : E(G) \to \mathbb{R}$ be an assignment with $\phi(e) = \sum_{i=1}^{p-q+1} \alpha_i f_i(e)$. We need to show, that $\phi$ is a nowhere-zero flow. Due to the choice of $\phi$ we get the same linear system as before. We get $\sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0$ for every $v \in V(G)$ and from equation 6.1 follows $\phi^{-1}(E) \subseteq \{\pm \frac{2}{q}, \pm \frac{q+1}{q}, \ldots, \pm \frac{p}{q}\}$. Thus, $\phi$ is a proper nowhere-zero $(\frac{p}{q}+1)$-flow. \qed
In a similar way we can find a characterization for general signed graphs. For flows on signed graphs we need more possible flow values. That will result in an increasing number of reorientations.

**Theorem 6.2.3.** A directed signed graph $(G, \sigma)$ admits a nowhere-zero $(\frac{p}{q} + 1)$-flow if and only if $(G, \sigma)$ has reorientations $f_1, \ldots, f_{2p-2q+1}$, such that $\sum_{i=1}^{2p-2q+1} \alpha_i \partial f_i = 0$ with $\alpha_1 = \frac{p+q}{2q}$ and $\alpha_l = \frac{1}{4q}$ for $l \in \{2, \ldots, 2p-2q+1\}$.

**Proof.** Let $\phi$ be a nowhere-zero $(\frac{p}{q} + 1)$-flow of $(G, \sigma)$ which is chosen according to Theorem 3.2.1 such that for each edge $e$, $\phi(e)$ is a multiple of $\frac{1}{2q}$ and let $D$ be the underlying orientation. We define the required reorientations in the following way:

For $e \in E(G)$ and $i \in \{1, \ldots, 2p-2q+1\}$ let $f_i(e) = 1$ if $1 + \frac{i-1}{2q} \leq \phi(e)$ or $-1 - \frac{i-1}{2q} < \phi(e) \leq -1$ and $f_i(e) = -1$ otherwise. Next, we consider the vector consisting of all possible flow values of $\phi$. We are going to express each flow value of an edge $e$ by one unique set of reorientations regarding $e$.

We will determine the coefficients $\alpha_i$ of the equation $\sum_{i=1}^{2p-2q+1} \alpha_i f_i(e) = \phi(e)$. For reasons of symmetry we can restrict the linear system to only positive flow values.

\[
\begin{pmatrix}
1 & -1 & \ldots & -1 \\
1 & 1 & -1 & \vdots \\
\vdots & \ddots & 1 & -1 \\
1 & \ldots & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{2p-2q+1}
\end{pmatrix}
= \begin{pmatrix}
2q \\
2q+1 \\
\vdots \\
2p
\end{pmatrix}.
\] (6.3)

The unique solution is given by $\alpha_1 = \frac{1}{2q}(p + q)$ and $\alpha_l = \frac{1}{4q}$ for $l \in \{2, \ldots, 2(p - q) + 1\}$. Hence,
\[
\sum_{i=1}^{2p-2q+1} \alpha_i \partial f_i(v) = \sum_{i=1}^{2p-2q+1} \alpha_i \left( \sum_{h \in H(v)} f_i(h_e) \right) \\
= \sum_{i=1}^{2p-2q+1} \sum_{h \in H^+(v)} \alpha_i f_i(h_e) - \sum_{i=1}^{2p-2q+1} \sum_{h \in H^-(v)} \alpha_i f_i(h_e) \\
= \sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0
\]

(6.4)

To prove the other direction, let \( \alpha_1 = \frac{p+q}{2q} \) and \( \alpha_l = \frac{1}{4q} \) for \( l \in \{2, \ldots, 2p-2q+1\} \) and \( f_1, \ldots, f_{2p-2q+1} \) be reorientations of \((G, \sigma)\) such that \( \sum_{i=1}^{2p-2q+1} \alpha_i \partial f_i = 0 \). Let \( \phi : E(G) \to \mathbb{R} \) be an assignment with \( \phi(e) = \sum_{i=1}^{2p-2q+1} \alpha_i f_i(e) \). We need to show, that \( \phi \) is a nowhere-zero flow. Due to the choice of \( \phi \) we get the same linear system as before. We get \( \sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0 \) for every \( v \in V(G) \) and from equation 6.3 it follows \( \phi^{-1}(E) \subseteq \{\pm \frac{2q}{2q}, \pm \frac{2q+1}{2q}, \ldots, \pm \frac{2p}{2q}\} \). Thus, \( \phi \) is a proper nowhere-zero \( \left(\frac{p}{q} + 1\right)\)-flow.

\( \square \)
Chapter 7

Conclusion

Nowhere-zero flows on signed graphs generalize the coloring concept in a natural way. Recall that $\delta_F = \sup \{ F((G, \sigma)) - F_c((G, \sigma)) : (G, \sigma) \text{ is flow-admissible} \}$. In Chapter 3 we showed $\delta_F \geq 2$. The strengthening in [22] cannot be improved anymore in the case if Bouchet’s Conjecture 1.4.3 is true. We also showed that not every counterexample to Conjecture 3.1.1 which states $F_c((G, \sigma)) > F((G, \sigma)) - 1$ has a star-cut. However, up to now the only graphs that are known to be counterexamples to Conjecture 3.1.1 contain a vertex cut consisting of precisely one vertex (we may interprete a vertex with a negative loop as such a vertex cut). In this context it is interesting to study whether these graphs are the only counterexamples or not.

In Chapter 4 we studied the flow spectrum of graphs and established the concept of $r$-minimal sets. If a graph $H$ has an $r$-minimal set $X$ of cardinality 2, then $S_X(H) = \{ r, F_c((H, 1)) \}$. Theorem 4.4.11 implies that every 3-minimal set of a bridgeless non-3-edge-colorable cubic graph contains at least three edges. The Petersen graph $P$ has a 3-minimal set $X$ with $|X| = 3$. Hence, $S_X(P) = \{3, 4, 5\}$. On the other hand, $P$ has a 6-minimal set $X'$ with $|X'| = 3$. It follows that $3 \not\in S_{X'}(P)$; indeed
The dashed lines represent set $X$ on the left and $X'$ on the right on the Petersen graph $S_X(P) = \{4, 5, 6\}$. The sets $X$ and $X'$ are indicated in Figure 7.1. Two switches at $v$ and $w$ yield a signature $\sigma$ of $P$ such that $(P, \sigma)$ and $(P, X)$ are equivalent. But $X' \subset N_\sigma$ and therefore, $S_X(P) \neq S_{N_\sigma}(P)$.

Some further problems are:

- Let $r \geq 2$, $(G, \sigma)$ be a flow-admissible signed graph and $X$ a (non-empty) $r$-minimal set. Determine the (integer) $X$-flow spectrum of $G$.

- Let $r \geq 2$ and $G$ be a graph. Let $(G, \sigma)$ and $(G, \sigma')$ be flow-admissible. Is it true that if $N_\sigma$ and $N_{\sigma'}$ are (both) smallest $r$-minimal sets, then $S_{N_\sigma}(G) = S_{N_{\sigma'}}(G)$ (or $\overline{S}_{N_\sigma}(G) = \overline{S}_{N_{\sigma'}}(G)$, if $r$ is an integer)?

- Let $G$ be a snark and $X_3$ a 3-minimal set. Is it true that there exists a 4-minimal set $X_4$ and a 5-minimal set $X_5$ such that $X_5 \subset X_4 \subset X_3$?

According to Theorem 4.4.11 for snarks every 3-minimal set contains a proper 4-minimal set. If Tutte’s 5-flow conjecture is true, then the last problem has an affirmative answer.
In Chapter 5 we determined the set of circular flow numbers for \((2k + 1)\)-regular signed graphs of odd regularity. An eulerian signed graph \((G, \sigma)\) with an even number of negative edges has \(F_c((G, \sigma)) = 2\). Let \((H, \tau)\) be the signed graph consisting of two vertices that are connected by \(2k + 1\) positive parallel edges and \(2k + 1\) negative parallel edges. \(H\) is eulerian and it is easy to see that \(F_c((H, \tau)) = 2 + \frac{1}{k}\). It is an interesting question whether for each \(r \in [2, 6]\) there exist one eulerian graph \((G, \sigma)\) with \(F_c((G, \sigma)) = r\). Until now, only the integer flow numbers of eulerian graphs have been characterized [24], [26].

In Chapter 6 we showed that a positive flow \(\phi\) on a signed graph \((G, \sigma)\) is \(F^1_{(G, \sigma)}\)-decomposable. In our concept the orientation of the edges in the decomposition are free. Therefore, we are able to decompose \((G, \sigma)\) into proper nowhere-zero flows. The authors of [25] proved that if a signed graph \(G\) admits an integer nowhere-zero \(k\)-flow, then it has a signed circuit cover of total length at least \(2(k - 1)|E(G)|\).

Suppose \(k \geq 3\). It would be interesting to analyse if we can lower the total length of a signed circuit cover with the new concept.
Appendix A

Details

A.1 A graph \((G, \sigma)\) with \(F((G, \sigma)) = 6\)

In the following we will show that the Petersen graph equipped with a special signature has integer flow number 6. The proof via case differentiation is quite technical. Considering, that we prove that a certain highly symmetric graph with ten vertices and 15 edges has a specific flow number the afford for proving turns out to be huge. Moreover, in the proof we use the fact that the Petersen graph is highly symmetric in order to shorten the proof. This gives a feeling of how hard it is in general to determine the flow number for a given graph. Up to switching isomorphism, there are six ways to put signs on the edges of the Petersen graph \([50]\), thus, the spectrum of the Petersen graph \(P\) consists at most of five elements, since at least one switching equivalence class contains no flow-admissible graph. Indeed, it is well known, that \(\mathfrak{S}(P) = \{3, 4, 5, 6\}\), and in the following we determine one element of the spectrum.

In the following the summation of the indices is taken modulo 5. Let \((P, \sigma^*)\) be the Petersen graph consisting of two circuits \(C_1 = u_1, u_2, u_3, u_4, u_5\) and \(C_2 = v_1, v_2, v_3, v_4, v_5\) that are connected
A.1 A graph \((G, \sigma)\) with \(F((G, \sigma)) = 6\)

The signed Petersen graph \((P, \sigma^*)\) is depicted in figure A.1.

**Lemma A.1.1.** The signed Petersen graph \((P, \sigma^*)\) has flow number \(F((P, \sigma^*)) = 6\).

**Proof.** Suppose \((P, \sigma^*)\) has a nowhere zero \(k\)-flow \(\phi\) and \(k \geq 6\). Let \(N_{ex}\) and \(N_{in}\) be the sets of extroverted and introverted edges of \(\phi\), respectively. Without loss of generality we may assume that either \(|N_{ex}| = 4\) and \(|N_{in}| = 1\) or \(|N_{ex}| = 3\) and \(|N_{in}| = 2\). First, we suppose \(|N_{ex}| = 4\) and \(|N_{in}| = 1\). Due to symmetry we may assume \(N_{ex} = \{e_1, e_2, e_3, e_4\}\) and \(N_{in} = \{e_5\}\). Since \(4 \leq \sum_{e \in N_{ex}} \phi(e) = \sum_{e \in N_{in}} \phi(e) \leq 4\), it follows \(\phi(e_i) = 1\) for \(i = 1, 4\) and \(\phi(e_5) = 4\). It follows that \(g_1\) and \(g_5\) are directed towards \(C_1\) and \(g_2, g_3, g_4\) are directed towards \(C_2\) with \(\phi(g_1) = \phi(g_5) = 3\) and \(\phi(g_2) = \phi(g_3) = \phi(g_4) = 2\). Due to symmetry we may assume that \(f_i\)
is directed towards $v_4$ and hence $f_3$ is directed towards $v_3$. If $\phi(f_4) = 1$, it follows $\phi(f_3) = 3$ and $\phi(f_2) = 0$. If $\phi(f_4) = 2$ we get $\phi(f_5) = 0$. Finally, if $\phi(f_4) \geq 3$, it follows $\phi(f_3) \geq 5$. In all cases we obtain a contradiction to the fact, that $\phi$ is a proper nowhere zero $k$-flow and $k \geq 6$. Hence, $|N_{ex}| = 3$ and $|N_{in}| = 2$. Let $(a, b, c, d, e)$ be a 5-tuple for $\phi$ in such a way that $a, b$ and $c$ represent the flow values for the extroverted edges and $d$ and $e$ represent the flow values of introverted edges. Taken into account that $a + b + c = \sum_{e \in N_{ex}} \phi(e) = \sum_{e \in N_{in}} \phi(e) = d + e \leq 8$ and $\phi(e) \leq 4$ there are without regarding permutations 19 possible 5 tuples: $(1, 1, 1, 1, 2), (1, 1, 2, 1, 3), (1, 1, 2, 2, 2), (1, 1, 3, 1, 4), (1, 1, 3, 2, 3), (1, 1, 4, 2, 4), (1, 1, 4, 3, 3), (1, 2, 2, 1, 4), (1, 2, 2, 2, 3), (1, 2, 3, 2, 4), (1, 2, 3, 3, 3), (1, 2, 4, 3, 4), (1, 3, 3, 3, 4), (1, 3, 4, 4, 4), (2, 2, 2, 2, 4), (2, 2, 2, 3, 3), (2, 2, 3, 3, 4), (2, 2, 4, 4, 4), (2, 3, 3, 4, 4)$. Next, we will diminish that list of possible flow values for negative edges. If one flow value occurs more than 3 time in the 5-tuple, then there must be one introverted edge incident to one extroverted edge with the same flow value, but that would force another incident edge to have no proper flow value and forms a contradiction. Thus, we get rid of $(1, 1, 1, 1, 2)$ and $(2, 2, 2, 2, 4)$. If $d + e \geq 5$, the introverted edges must not be adjacent, since the value on the third incident edge would be $\geq 5$. If additionally the value $d$ or $e$ occurs at least three times in the 5-tuple, there will always be one incident edge forced to get flow value zero, which generate a contradiction. Thus, we get rid of $(1, 1, 3, 1, 4), (1, 2, 2, 2, 3), (1, 2, 3, 3, 3), (1, 3, 4, 4, 4), (1, 3, 3, 3, 4), (2, 2, 4, 4, 4)$. We can also exclude the tuples $(2, 3, 3, 4, 4), (1, 1, 4, 2, 4)$ because at least two extroverted edges are adjacent, but the sum of flow values for any two extroverted edges is $\geq 5$.

Thus, there are seven tuples left, where exists a possible distribution for the flow values in $C_1$, $(1, 1, 2, 1, 3), (1, 1, 2, 2, 2), (1, 1, 3, 2, 3), (1, 1, 4, 3, 3), (1, 2, 3, 2, 4), (1, 2, 3, 3, 3), (1, 2, 4, 3, 4)$.
A.1 A graph $(G, \sigma)$ with $F((G, \sigma)) = 6$

$(1,1,4,3,3), (1,2,2,1,4), (1,2,3,2,4), (2,2,2,3,3)$. We will show that for every tuple up to symmetry just exists one possible distribution in the circuit. We consider the following cases:

(i) Suppose $(a,b,c,d,e) = (1,1,2,1,3)$. The introverted edge with with value 1, say $e_1$ must not be adjacent to any introverted edge with value 1, hence $e_3$ and $e_4$ are extroverted with $\phi(e_3) = \phi(e_4) = 1$. Due to symmetry we may assume $e_2$ is introverted, $e_5$ is extroverted, $\phi(e_2) = 3$, and $\phi(e_5) = 2$.

(ii) Suppose $(a,b,c,d,e) = (1,1,2,2,2)$. The introverted edges with value 2 must be adjacent, otherwise, no extroverted edge could receive value 2. Hence, $e_1$ and $e_2$ are introverted, $e_3, e_4$ and $e_5$ are extroverted, and $\phi(e_1) = \phi(e_2) = \phi(e_4) = 2$ and $\phi(e_3) = \phi(e_5) = 1$.

In the following we may consider $(1,2,2,1,4)$ represented by $(2,2,1,4,1)$. For all remaining tuples $(a,b,c,d,e) \in \{(1,1,3,2,3), (1,1,4,3,3), (2,2,1,4,1), (1,2,3,2,4), (2,2,2,3,3)\}$ we have $d + e \geq 5$. Thus, the introverted edges cannot be incident. Let $e_1$ and $e_3$ be the introverted edges with $\phi(e_1) = d$ and $\phi(e_3) = e$ and $e_2, e_4$ and $e_5$ are extroverted.

(iii) Suppose $(a,b,c,d,e) = (1,1,3,2,3)$ or $(a,b,c,d,e) = (2,2,1,4,1))$. Since $c = e$ we get $\phi(e_5) = c$ and because of $a = b$, $\phi(e_2) = \phi(e_4) = a$.

(iv) Suppose $(a,b,c,d,e) = (1,1,4,3,3)$. It follows $\phi(e_2) = 4$ and $\phi(e_4) = \phi(e_5) = 1$ since the edge with value 4 must not be adjacent to any other extroverted edge.

(v) and (vi) Suppose $(a,b,c,d,e) = (1,2,3,2,4)$. Since the extroverted edges with values 2 and 3 must not be adjacent and the extroverted
(vii) Suppose \((a, b, c, d, e) = (2, 2, 2, 3, 3)\). We get \(\phi(e_2) = \phi(e_4) = \phi(e_5) = 2\).

In figure A.2 the possibilities for \(C_1\) with corresponding edges \(g_i \ (i = 1..5)\) are depicted.

Note, for further consideration, that whenever there is one incoming and one outgoing edge adjacent with same the flow value, we may exclude that case, since the flow cannot be extended properly.
A.1 A graph \((G, \sigma)\) with \(F((G, \sigma)) = 6\)

(i) \(E^+(C_1) = \{e_1, e_4, e_5\}\) and \(E^-(C_1) = \{e_2, e_3\}\) and \(\phi(g_1) = 1, \phi(g_2) = 4, \phi(g_3) = \phi(g_4) = 2, \phi(g_5) = 3\): The edges \(f_3\) and \(f_4\) must be directed towards \(v_4\). It follows \(\phi(f_3) \notin \{3, 4\}\) and \(f_2\) is directed towards \(v_2\). If \(\phi(f_3) = 1\), then \(\phi(f_2) = 2\), and if \(\phi(f_3) = 2\), then \(\phi(f_4) = 2\), which forms a contradiction.

(ii) \(E^+(C_1) = \{e_4, e_5\}\) and \(E^-(C_1) = \{e_1, e_2, e_3\}\) and \(\phi(g_1) = \phi(g_3) = 1, \phi(g_2) = 4, \phi(g_4) = \phi(g_5) = 3\): The edges \(f_3\) and \(f_4\) must be directed towards \(v_4\). It follows \(\phi(f_3) \notin \{3, 4\}\) and \(f_2\) is directed towards \(v_2\). If \(\phi(f_3) = 1\), then \(\phi(f_4) = 3\), and if \(\phi(f_3) = 2\), then \(\phi(f_2) = 1\), which forms a contradiction.

(iii) \(E^+(C_1) = \{e_1, e_5\}\) and \(E^-(C_1) = \{e_2, e_3, e_4\}\) and \(\phi(g_1) = \phi(g_2) = 1, \phi(g_3) = \phi(g_4) = 4, \phi(g_5) = 3\): The edges \(f_2\) and \(f_3\) must be directed away from \(v_3\). It follows \(\phi(f_3) \notin \{1, 4\}\) and \(f_4\) is directed towards \(v_5\). If \(\phi(f_3) = 2\), then \(\phi(f_2) = 2\), and if \(\phi(f_3) = 3\), then \(\phi(f_4) = 2\), which forms a contradiction.

(iv) \(E^+(C_1) = \{e_2, e_3, e_5\}\) and \(E^-(C_1) = \{e_1, e_4\}\) and \(\phi(g_1) = \phi(g_4) = \phi(g_5) = 2, \phi(g_2) = \phi(g_3) = 1\): Due to symmetry we can assume that \(f_5\) is directed towards \(v_1\). It follows that \(\phi(f_5) = 1\), and hence \(f_1\) is directed towards \(v_1\) with \(\phi(f_1) = 1\), which forms a contradiction.

(v) \(E^+(C_1) = \{e_3, e_4, e_5\}\) and \(E^-(C_1) = \{e_1, e_2\}\) and \(\phi(g_1) = \phi(g_3) = 1, \phi(g_2) = 2, \phi(g_3) = \phi(g_4) = 3\): The edge \(f_2\) must be directed towards \(v_2\) and \(\phi(f_2) \notin \{3, 4\}\). If \(\phi(f_2) = 1\) then \(f_3\) must be directed towards \(v_4\) with \(\phi(f_3) = 2\), a contradiction. If \(\phi(f_2) = 2\) then \(f_1\) must be directed towards \(v_1\) with \(\phi(f_1) = 3\), that forms a contradiction.

(vi) \(E^+(C_1) = \{e_1, e_3, e_4\}\) and \(E^-(C_1) = \{e_2, e_5\}\) and \(\phi(g_1) = \phi(g_2) = \phi(g_3) = 1, \phi(g_4) = 2, \phi(g_5) = 3\): The edge \(f_3\) must be directed
towards $v_4$ and $\phi(f_3) \notin \{3, 4\}$. If $\phi(f_3) = 1$ then $f_4$ must be directed towards $v_5$ with $\phi(f_4) = 2$, a contradiction. If $\phi(f_3) = 2$ then $f_2$ must be directed towards $v_2$ with $\phi(f_2) = 1$, a contradiction.

(vii) $E^+(C_1) = \{e_1, e_2, e_3, e_4\}$ and $E^-(C_1) = \{e_5\}$ and $\phi(g_1) = \phi(g_2) = \phi(g_3) = \phi(g_4) = 1, \phi(g_5) = 4$: The edge $f_3$ must be directed towards $v_4$, hence $f_4$ must be directed towards $v_5$, $f_1$ must be directed towards $v_1$ and $\phi(f_3) \notin \{1, 4\}$. If $\phi(f_3) = 2$ then $\phi(f_4) = 1$, a contradiction. If $\phi(f_3) = 3$ then $f_2$ must be directed towards $v_2$ with $\phi(f_2) = 1$, a contradiction.

To show that $F((P, \sigma^*)) \leq 6$ it is sufficient to find a nowhere-zero 6-flow as depicted in figure A.3.

□
A.1 A graph \((G, \sigma)\) with \(F((G, \sigma)) = 6\)

Figure A.3: A nowhere-zero 6 flow on \((P, \sigma^*)\)
Bibliography


