On Numerical Invariants in Algebraic Complexity Theory

Abstract

A common theme in mathematics is the classification of mathematical objects by assigning numerical invariants to them. There are two ways in which such numerical invariants can appear in relation to computational complexity. On the one hand, mathematical invariants are used in the context of proving *lower complexity bounds*: they serve as obstructions to the existence of fast algorithms for solving certain problems. On the other hand, it is the computational complexity of actually computing such invariants that is of interest. The first part of this thesis is concerned with lower bounds for the problems of computing linear and bilinear maps. The invariants used, namely the *mean square volume*, *singular values*, and *rigidity*, belong to linear algebra. One of the main results is a tight lower bound of order $\Omega(n \log n)$ for the problem of multiplying two polynomials, in the model of bounded coefficient circuits. This lower bound is extended to circuits for which a limited number of unbounded scalar multiplications (help gates) are allowed. The second part is concerned with the complexity of actually computing numerical invariants. The objects of study are two of the most prominent invariants in algebraic geometry and topology: the *Euler characteristic* and the *Hilbert polynomial* of complex projective varieties. These problems are studied within the framework of counting complexity classes. It is shown that the problem of computing the Euler characteristic of a complex projective variety is on essentially the same level of difficulty as the problem of counting the number of solutions of a system of polynomial equations. A similar result is proved for the Hilbert polynomials, when the input variety is assumed to be smooth and equidimensional.