An Application of Microlocal Analysis to the Representation Theory of Compact Lie Groups

Dissertation

An Application of Microlocal Analysis to the Representation Theory of Compact Lie Groups

vorgelegt von

Sameh Keliny

Universität Paderborn
Fakultät für Elektrotechnik, Informatik und Mathematik

March 19, 2009

Danksagung

Mit Dir kann ich Hindernisse überwinden.
Mit Dir springe ich über Mauern.

(Psalm 18, 30)

Der Glaube an Gott hat mir den Rücken gestärkt, Mut und Ausdauer verliehen.

Erstens möchte ich mich herzlich bei Prof. Dr. Sönke Hansen für seine Unterschützung im wissenschaftlichen Bereich sowie für seine Tür, die für mathematische Diskussionen und Ratschläge immer offen stand, bedanken. Besonders zu schätzen wusste ich die Freiheit, die er mir sowohl in meiner Forschung als auch in der Lehre gegeben hat. Zweitens gilt mein Dank Prof. Dr. Joachim Hilgert für die Überlassung dieses interessanten Dissertationsthemas, seine Hilfsbereitschaft und wichtigen Anregungen. Weiterhin danke ich meinen Arbeitskollegen sowohl im wissenschaftlichen als auch im nicht-wissenschaftlichen Bereich. Last but not least möchte ich mich bei meinen Eltern, die keine Kosten und Mühen gescheut haben, um mir die besten Voraussetzungen für das Leben zu schaffen, bedanken. Weiterhin bedanke ich mich bei meinen Freunden, insbesondere bei meiner Frau Heimke für ihre grenzenlose Geduld in Krisenzeiten.

Sameh Keliny

Ich danke den Gutachtern

Herrn Prof. Dr. Sönke Hansen,
Institut für Mathematik, Universität Paderborn

Herrn Prof. Dr. Joachim Hilgert,
Institut für Mathematik, Universität Paderborn

Contents

Zusammenfassung

Wir untersuchen Charaktere von reduziblen unitären Darstellungen einer kompakten zusammenhängenden halbeinfachen Liegruppe. Wir geben eine geometrische Bedingung, unter welcher diese Charaktere auf abgeschlossenen Untergruppen eingeschränkt werden können. Das wurde schon 1998 von T. Kobayashi untersucht, um ein Kriterium für die diskrete Zerlegbarkeit von Einschränkungen reduzibler untärer Darstellungen von reduktiven Liegruppen auf reduktive abgeschlossene Untergruppen zu erhalten. Der Schlüssel in seinem Beweis besteht darin, Charaktere von reduziblen unitären Darstellungen einer kompakten zusammenhängenden Liegruppe und ihre Einschränkungen auf abgeschlossene Untergruppen mit Hilfe von Methoden der mikrolokalen Analysis für Hyperfunktionen zu betrachten. In dieser Arbeit benutzen wir mikrolokale Analysis für Distributionen. Das Neue ist die Benutzung der Stetigkeit zwischen angepassten Distributionenräumen und der Einschränkung auf abgeschlossene Untermannigfaltigkeiten. Diese Stetigkeit ist im Falle der Hyperfunktionen nicht vorhanden.

Abstract

We consider characters of a reducible unitary representation of a compact connected semisimple Lie group. We provide a geometric condition under which these characters can be restricted to closed subgroups. This was already considered by T. Kobayashi in 1998 to establish a criterion for the discrete decomposability of restrictions of unitary representations of reductive Lie groups to reductive closed subgroups. The crucial point in his proof is to consider characters of a reducible unitary representation of a compact connected Lie group and their restriction to closed subgroups using microlocal analysis methods in the hyperfunctions setting. In this thesis we use microlocal analysis of distribution theory instead. The novelty consists in using the continuity between adapted spaces of distributions and the restriction to closed submanifolds. This continuity is not readily available in the hyperfunctions setting.

Introduction

The main motivation of this thesis is to help understanding how an irreducible representation of a real reductive linear Lie group decomposes when restricted to a subgroup. Following Kobayashi [Ko98] the main problem is studied by considering characters of unitary representation of a connected compact semisimple Lie group and the decomposition of its restriction to closed subgroups. Based on the the work of Kashiwara and Vergne [KV79] Kobayashi used microlocal analysis methods in the hyperfunctions setting. In this thesis we use microlocal analysis of distribution theory instead.

Chapter 1 provides notations, basic definitions, various theorems, and examples to be used.

In Chapter 2 we consider a connected, compact, and semisimple Lie group K with a maximal abelian torus T. $\mathfrak{k}$ denotes the semisimple Lie algebra of K, $\mathfrak{t}$ its subalgebra corresponding to T, and $\mathcal{D}'(K)$ the distributions defined on K. We identify the equivalence classes of irreducible representation $\widehat{K}$ with their highest weights, $\widehat{K} = L \cap \overline{C}$ ($\overline{C} \subset i\mathfrak{t}^*$, $i = \sqrt{-1}$, denotes the closure of the (dual) Weyl chamber and L the weight lattice in $i\mathfrak{t}^*$). Then due to Peter-Weyl Theorem, we get that for each $u \in \mathcal{D}'(K)$ we have the Fourier expansion

$$u = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda \,,$$

If the Fourier coefficients vanish outside a closed cone $\Omega \subset \overline{C} \setminus 0$, $\varphi_\lambda = 0$ if $\lambda \notin \Omega$, then the wave front set of u satisfies

$$\mathrm{WF}(u) \subset K \times \mathrm{Ad}^* K(-i\Omega) \subset T^*K.$$

Furthermore, the Fourier series of u converges to u in $\mathcal{D}'_\Gamma(K)$, where $\Gamma := K \times \mathrm{Ad}^* K(-i\Omega)$ is a closed cone in T^*K. Here $\mathcal{D}'_\Gamma(K)$ denotes the space of distributions having their wave front set in Γ. The convergence in $\mathcal{D}'_\Gamma(K)$ is an important tool in defining and analyzing the restriction of a distribution to a closed submanifold. We will also prove a more precise relationship between the wave front set of distribution and the asymptotic behavior of the L^2-norm of the Fourier coefficients (was first introduced by [KV79]).

In the first part of chapter 3 we introduce the K-character,

$$\Theta_\tau^K = \sum_{\lambda \in L \cap \overline{C}} m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda},$$

of a unitary representation τ of K with multiplicities $m_K(\pi_\lambda : \tau)$ which are polynomially bounded in $|\lambda|$. Here π_λ denotes an irreducible representation of K corresponding to the highest weight λ and χ_{π_λ} the character of π_λ. Θ_τ^K is a well-defined distribution on K.

Furthermore, the set of λ's with $m_K(\pi_\lambda : \tau) \neq 0$ is called the support of τ. The support is approximated by the asymptotic K-support of τ which is a closed conic subset $\mathrm{AS}_K(\tau) \subset \overline{C}$. We show that the wave front set of this distribution satisfies

$$\mathrm{WF}(\Theta_\tau^K) \subset K \times \mathrm{Ad}^* K(-i\,\mathrm{AS}_K(\tau)) \subset T^*K \setminus 0 \simeq K \times \mathfrak{k}^*.$$

The rest of chapter 3 is devoted to the restriction of Θ_τ^K to a closed subgroup. Here we consider a closed subgroup H of K and their Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, respectively. The space of conormals is denoted by $\mathfrak{h}^\perp \subset \mathfrak{k}^*$. If we assume that

$$\mathrm{AS}_K(\tau) \cap i\ \mathrm{Ad}^*\,K(\mathfrak{h}^\perp) = \emptyset,$$

then the restriction $\Theta_\tau^K|_H$ is a well-defined distribution on H. Moreover, the restricted distribution $\Theta_\tau^K|_H$ coincides with the distribution $\Theta_{\tau|_H}^H$, i.e.,

$$\Theta_{\tau|_H}^H = \Theta_\tau^K|_H.$$

This dissertation gives a simplified proof, in the framework of distributions, of a Theorem of Kobayashi [Ko98] on the restrictions of a K-character to closed subgroup. The novelty consists in using the continuity between adapted spaces of distributions and restriction to closed submanifolds. This continuity is not readily available in the hyperfunctions setting which is used by Kobayashi [Ko98].

1 Preliminaries

1.1 Roots and Root Spaces

We will give in this section a brief overview of some facts about root (weight) spaces.

Let K be a compact, connected, and semisimple Lie group with a maximal torus T. Let $\mathfrak{k}$ and $\mathfrak{t}$ be the semisimple Lie algebra of K and T. The algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ has the **root-space decomposition**

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Lambda} \mathfrak{k}_{\alpha} \tag{1.1}$$

where α is a linear form on $\mathfrak{t}_{\mathbb{C}}$, $\Lambda = \Lambda(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is called the set of **roots** and

$$\mathfrak{k}_{\alpha} := \{X \in \mathfrak{k}_{\mathbb{C}} \mid [Y, X] = \langle \alpha, Y\rangle X,\ \forall Y \in \mathfrak{t}_{\mathbb{C}}\}$$

is the corresponding **root space** with respect to $\mathfrak{t}_{\mathbb{C}}$, where $\langle \cdot, \cdot\rangle$ denotes the duality bracket $\mathfrak{t}_{\mathbb{C}}^* \times \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$, see [He01] p. 165.

An element $X \in \mathfrak{t}$ is said to be **regular** in $\mathfrak{t}$, if $\langle \alpha, X\rangle \neq 0$ for all roots $\alpha \in \Lambda$, $\mathfrak{t}^{\text{reg}} := \mathfrak{t} \setminus \bigcup_{\alpha\in\Lambda} \ker \alpha$. Let $\mathfrak{c}$ be a connected component of $\mathfrak{t}^{\text{reg}}$, it follows that for any $\alpha \in \Lambda$, $i^{-1}\langle \alpha, X\rangle$ either > 0 or < 0 on $\mathfrak{c}$. Hence,

$$\Lambda = \Lambda_+ \cup \Lambda_- \qquad \Lambda_+ \cap \Lambda_- = \emptyset, \tag{1.2}$$

if we take

$$\Lambda_+(\mathfrak{c}) = \Lambda_+ := \{\alpha \in \Lambda \mid i^{-1}\alpha > 0 \text{ on } \mathfrak{c}\}, \tag{1.3}$$

$\Lambda_- = \{-\alpha \mid \alpha \in \Lambda_+\}$. Conversely, if Λ_+ is a subset of Λ satisfying (1.2), then the set

$$\mathfrak{c}(\Lambda_+) := \{X \in \mathfrak{t} \mid i^{-1}\langle \alpha, X\rangle > 0,\ \ \forall \alpha \in \Lambda_+\} \tag{1.4}$$

is an open, convex polyhedral cone in $\mathfrak{t}$, contained in $\mathfrak{t}^{\text{reg}}$, and equal to a connected component of $\mathfrak{t}^{\text{reg}}$, if and only if, the convex cone in $i\mathfrak{t}^*$ generated by Λ_+ is *proper*, i.e., if Λ_+ satisfies the following condition:

$$\sum_{\alpha\in\Lambda_+} c_\alpha \alpha = 0, \quad c_\alpha \geq 0,\ \forall \alpha \in \Lambda_+ \Rightarrow c_\alpha = 0,\ \forall \alpha \in \Lambda_+. \tag{1.5}$$

Remark 1.1. A subset Λ_+ of Λ satisfying (1.2) and (1.5), is called *a choice of positive roots*. The connected components of $\mathfrak{t}^{\text{reg}}$ are called the *Weyl chambers* in $\mathfrak{t}$ (a equivalent definition of the Weyl chambers will be presented later on). Moreover, the relation (1.3) and (1.4) defines a bijection between the choice of a positive roots Λ_+ and the Weyl chambers $\mathfrak{c}$ (see [DK00], p.144-147).

The following definitions will be needed later on. For each $x \in K$ the map $\underline{\mathbf{Ad}}\, x : y \mapsto xyx^{-1}$ is the conjugation by x in the group K. Because $\underline{\mathbf{Ad}}\, x(e) = e$, the tangent mapping of $\underline{\mathbf{Ad}}\, x$ at e is a linear mapping

$$\text{Ad}\, x := T_e(\underline{\mathbf{Ad}}\, x) : \mathfrak{k} \longrightarrow \mathfrak{k}$$

called the adjoint mapping of x. That is, the mapping

$$\mathrm{Ad} : K \longrightarrow GL(\mathfrak{k})$$

is a homomorphism of groups and called the **adjoint representation** of K in $\mathfrak{k} = T_e K$. Accordingly, the linear map $\mathrm{ad} := T_e(\mathrm{Ad})$ is given by

$$\mathrm{ad} : \mathfrak{k} \longrightarrow L(\mathfrak{k}, \mathfrak{k}),$$

$\mathrm{ad}\, X(Y) := [X, Y]$, $X, Y \in \mathfrak{k}$ where $[\cdot, \cdot]$ is called the Lie bracket of X and Y (see [DK00], p. 3). The bilinear form on a Lie algebra is given by

$$B(X, Y) := \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y), \qquad X, Y \in \mathfrak{k}, \tag{1.6}$$

B is called the **Killing form** of $\mathfrak{k}$. This bilinear form is symmetric, because in general $\mathrm{tr}(A \circ B) = \mathrm{tr}(B \circ A)$, for linear endomorphisms A and B. Since the real trace of a linear mapping is equal to the complex trace of its complex linear extension, the Killing form extends to a complex bilinear form on $\mathfrak{k}_{\mathbb{C}}$ by $B(X, Y) = \mathrm{tr}_{\mathbb{C}}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$, where $X, Y \in \mathfrak{k}_{\mathbb{C}}$ and $\mathrm{ad}\, X \circ \mathrm{ad}\, Y$ is considered as an element of $L_{\mathbb{C}}(\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ (see [DK00], p. 148). Moreover, for each X in the Lie algebra $\mathfrak{k}$ of a compact Lie group, $\mathrm{ad}\, X \in L_{\mathbb{C}}(\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ is diagonalizable, with only purely imaginary eigenvalues (see [DK00], Lemma 3.5.1).

Remark 1.2. $B|_{\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}}$ is nondegenerate, consequently to each root α corresponds X_α in $\mathfrak{t}_{\mathbb{C}}$ with $\langle \alpha, X \rangle = B(X, X_\alpha)$ for all $X \in \mathfrak{t}_{\mathbb{C}}$ (see [Kn02], Proposition 2.17). On $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$ the Killing form is given by

$$B(X, X') = \sum_{\alpha \in \Lambda} \langle \alpha, X \rangle \langle \alpha, X' \rangle \tag{1.7}$$

see [Kn02], Corollary 2.24. Moreover, we transfer the restriction to $\mathfrak{t}_{\mathbb{C}}$ of the Killing form to a bilinear form on the dual $\mathfrak{t}_{\mathbb{C}}^*$ by the definition

$$\langle \alpha, \lambda \rangle = B(X_\alpha, X_\lambda) = \langle \alpha, X_\lambda \rangle = \langle \lambda, X_\alpha \rangle \tag{1.8}$$

for $\alpha, \lambda \in \mathfrak{t}_{\mathbb{C}}^*$ (see [Kn02], p. 144). Combining (1.7) and (1.8), we obtain

$$\langle \alpha, \lambda \rangle = B(X_\alpha, X_\lambda) = \sum_{\beta \in \Lambda} \langle \beta, X_\alpha \rangle \langle \beta, X_\lambda \rangle = \sum_{\beta \in \Lambda} \langle \beta, \alpha \rangle \langle \beta, \lambda \rangle. \tag{1.9}$$

The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{t}^* \times \mathfrak{t}^*$ is a positive definite inner product (see [Kn02], p. 147-149).

Definition 1.3. A **simple Lie algebra** is a non-abelian Lie algebra whose ideals are 0 or itself. A **semisimple Lie algebra** is a direct sum of simple Lie algebra. A Lie group called semisimple if its Lie algebra is semisimple. A **reductive Lie algebra** is a sum of abelian and semisimple lie algebra. A Lie group called reductive if its Lie algebra is reductive.

Definition 1.4. An **abstract root system**, in a finite-dimensional real inner product space V with inner product $\langle\cdot,\cdot\rangle$ which induces a norm $|\cdot|$, is a finite set Λ of non-zero elements of V such that;

(i) Λ spans V,

(ii) the orthogonal transformation $s_\alpha(\beta) = \beta - \frac{2\langle\alpha,\beta\rangle}{|\alpha|^2}\alpha$, for $\alpha \in \Lambda$, map Λ to itself,

(iii) $\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$ is an integer whenever α and β are in Λ.

An abstract root system is said to be **reduced system** if $\alpha \in \Lambda$ implies $2\alpha \notin \Lambda$. If α is a root and $1/2\,\alpha$ is not a root, we say that α is **reduced element** (see [Kn02], p. 152). We say that a member λ of V is **dominant** if $\langle\lambda,\alpha\rangle \geq 0$ for all $\alpha \in \Lambda_+$ (see [Kn02], p. 168).

The next theorem will give the relation between roots and abstract root systems.

Theorem 1.5. *The root system of a complex semisimple Lie algebra $\mathfrak{k}_\mathbb{C}$ with respect to a Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ forms a reduced abstract root system in $\mathfrak{t}^*$ with respect to the inner product $\langle\cdot,\cdot\rangle$ defined in 1.9 (see [Kn02], Proposition 2.42).*

In the following we will present some definitions and results which are needed to state *Theorem of The Highest Weight*.

The presence of the groups K and T gives us additional information about the root-space decomposition (1.1). In fact, $\mathrm{Ad}(T)$ acts by orthogonal transformations on $\mathfrak{k}$ relative to our given inner product (Killing form). If we extend this inner product on $\mathfrak{k}$ to a Hermitian inner product on $\mathfrak{k}_\mathbb{C}$, then $\mathrm{Ad}(T)$ acts on $\mathfrak{k}$ by a commuting family of unitary transformations. Such a family must have a eigenspace decomposition, that is (1.1). The action of $\mathrm{Ad}(T)$ on the 1-dimensional space $\mathfrak{k}_\alpha$ is a 1-dimensional representation of T, necessarily of the form

$$\mathrm{Ad}(t)X = \chi_\alpha(t)X \qquad \text{for } t \in T, \tag{1.10}$$

where $\chi_\alpha : T \to S^1$ is a continuous homomorphism of T into the group of complex numbers of modulus 1. We call χ_α a **multiplicative character** (see [Kn02], p. 254).

Proposition 1.6. *If $\lambda \in \mathfrak{t}^*$, then the following are equivalent:*

(i) Whenever $H \in \mathfrak{t}$ satisfies $\exp(H) = 1$, then $\langle\lambda, H\rangle$ is in $2\pi i\mathbb{Z}$.

(ii) There is a multiplicative character χ_λ of T with $\chi_\lambda(\exp(H)) = e^{\langle\lambda,H\rangle}$ for all $H \in \mathfrak{t}$

(see [Kn02], Proposition 4.58)

Remark 1.7. A linear form satisfying (i) and (ii) is called **analytically integral**.

Proposition 1.8. *If $\lambda \in \mathfrak{t}^*$ is analytically integral, then λ satisfies the following condition*

$$\lambda(\alpha)^\vee := \frac{\langle \lambda, \alpha \rangle}{|\alpha|^2} \quad \text{is in } \mathbb{Z} \text{ for each } \alpha \in \Lambda \tag{1.11}$$

(see [Kn02], Proposition 4.59)

Remark 1.9. A linear form satisfying condition (1.11) is called **algebraically integral**.

Proposition 1.10. *Let π be an irreducible finite-dimensional representation of K then we have:*

1. *If λ is the highest weight of π, then $\lambda(\alpha)^\vee \geq 0$ for all $\alpha \in \Lambda_+$.*
2. *For a weight λ of π, the following are equivalent:*
   (i) λ is a highest weight of π.
   (ii) If $\alpha \in \Lambda_+$, then $\alpha + \lambda$ is not a weight of π.
   (iii) For any weight μ of π, we have $\mu = \lambda - \sum_{\alpha \in \Lambda_+} n_\alpha \alpha$ for some $n_\alpha \in \mathbb{N}_0$

(see [DK00], Proposition 4.9.4)

Definition 1.11. One introduces a partial **ordering** $\leq$ by writting $\mu \leq \lambda$ if and only if $\mu = \lambda - \sum_{\alpha \in \Lambda_+} n_\alpha \alpha$ for some $n_\alpha \in \mathbb{N}_0$. The customary definition is to call a weight λ a highest weight of an irreducible representation π, if it is a maximal element of the set of weights of π, with respect to the partial ordering $\leq$; this is just condition (iii) in proposition (1.10) (2) (see [DK00], p. 260).

Definition 1.12. The restriction of the Killing form to $\mathfrak{t}_\mathbb{C}$ and to $\mathfrak{t}$ is defined in (1.8) and (1.9).

$$C \quad := \quad \{\lambda \in i\mathfrak{t}^* \mid \langle \lambda, \alpha \rangle > 0, \ \ \forall \alpha \in \Lambda_+\} \tag{1.12}$$

is called (dual) **Weyl chamber**. This definition is compatible with the definition of the Weyl chamber we gave in remark 1.1, since we have a bijection between the choice of a positive roots Λ_+ and the Weyl chamber $\mathfrak{c}(\Lambda_+)$ (see (1.4) and remark 1.1).

Definition 1.13. Suppose that $\Pi = \{\alpha_1, \ldots \alpha_l\}$ is any set of l independent reduced elements α_i (see definition 1.4), such that every expression of a member $\alpha \in \Lambda$ as $\alpha = \sum_i n_i \alpha_i$ has all non-zero n_i of the same sign. We call Π a **system of simple roots** (see [Kn02], p. 164).

The continuous representation $\sigma : K \to GL(U)$ and $\tau : K \to GL(V)$, where V and U are finite-dimensional vector spaces, respectively are said to be equivalent if there is a toplogical linear isomorphism L from U onto V, such that $L \circ \sigma(k) = \tau(k) \circ L$ for all $k \in K$. The set of equivalence classes of irreducible representations of K is called the dual $\widehat{K}$ of K (see [DK00], p. 210).

Let π be an irreducible representation of $\mathfrak{k}$ on a finite-dimensional complex vector space V and $V_\lambda := \{v \in V \mid \pi(H)v = \lambda(H)v, \ \forall H \in \mathfrak{t}\}$ be a weight space of V (eigenspace with respect to π). Due to proposition 1.10 and definition 1.11 the largest weight in the partial ordering is called the highest weight of π.

Theorem 1.14 (Theorem of The Highest Weight)**.** *Up to equivalence the irreducible finite-dimensional representations π of $\mathfrak{k}$ are in one-one correspondence with the dominant algebraically integral linear functionals on $\mathfrak{t}_{\mathbb{C}}$, the correspondence being that λ is the highest weight of π_λ. The highest weight λ of π_λ has the following additional properties:*

1. *λ depends only on the simple system Π and not on the ordering used to define Π.*
2. *The weight space V_λ for λ is 1-dimensional.*
3. *Each root vector E_α for an arbitrary $\alpha \in \Lambda_+$ annihilates the members of V_λ, and the elements of V_λ are the only vectors with this property.*
4. *Every weight of π_λ is of the form $\lambda - \sum_i n_i\alpha_i$ with $n_i \in \mathbb{N}_0$ and the α_i in Π.*

(see [Kn02], Theorem 5.5)

Definition 1.15. Let $L_T := \{X \in \mathfrak{t} \mid e^X = 1\}$. Then

$$L := \{\alpha \in i\mathfrak{t}^* \mid \langle \alpha, X\rangle \in 2\pi i\mathbb{Z}, \;\; \forall X \in L_T\}$$

is called the **weight lattice** in $i\mathfrak{t}^*$ (see [DK00]. p. 271).

Remark 1.16. Due to the Theorem of Highest Weight 1.14 we can identify $L \cap \overline{C}$ with $\widehat{K}$ where $\overline{C}$ the closure of the Weyl chamber and $\widehat{K}$ the set of equivalence classes of irreducible representations of K.

1.2 Analysis on Compact Groups

In this section we will present the Peter-Weyl Theorem.

Let K be a compact connected Lie group and π a continuous representation of K on the Hilbert space $\mathcal{H}$. Then π is said to be **unitary representation** if each $\pi(x)$, for $x \in K$, is a unitary transformation in $\mathcal{H}$, i.e.,

$$(\pi(x)v, \pi(x)w) = (v, w), \qquad \forall\, v, w \in \mathcal{H}.$$

Furthermore, since K is a compact group, then there exists a *Hermitian* inner product $\mathcal{H}$ (i.e., symmetric sesquilinear form) for which the representation π is unitary (see [DK00], Corollary 4.2.2.). Let $\widehat{K}$ denote the equivalent classes of unitary irreducible representation of K.

If π is a unitary representation of K on some Hilbert space $\mathcal{H}$, then the functions

$$\phi_{u,v}(x) = (\pi(x)u, v), \qquad u, v \in \mathcal{H}$$

are called **matrix elements** of π. If u and v are elements of an orthonormal basis $\{e_j\}$ for $\mathcal{H}$, then $\phi_{u,v}(x)$ is one of the entries of the matrix for $\pi(x)$ with respect to that basis, namely

$$\pi_{ij}(x) = \phi_{e_j,e_i}(x) = (\pi(x)e_j, e_i). \tag{1.13}$$

We denote the linear span of the matrix elements of π by E_π. E_π is a subspace of $C(K)$ and hence also of $L^p(K)$ for all p (see [Fo95a], p. 129).

Remark 1.17. We denote the character $\mathrm{tr}(\pi(x))$ of π by $\chi_\pi(x)$. The convolution is defined as follows

$$f * \chi_\pi(x) := \int_K f(xy^{-1})\chi_\pi(y)\ dy = \int_K f(y)\chi_\pi(y^{-1}x)\ dy$$

We note that the space $L^1(K)$ is an algebra with respect to convolution.

Proposition 1.18. *E_π depends only on the unitary equivalence class of π. It is invariant under left and right translation and is a two-sided ideal in $L^1(K)$. If $\dim \mathcal{H} = n < \infty$ then $\dim E_\pi \leq n^2$ (see [Fo95a], Proposition 5.6).*

We note that E which is given by

$$E = \text{ the linear span of } \bigcup_{\pi\in\widehat{K}} E_\pi,$$

is an algebra (see [Fo95a], Proposition 5.10).

Any unitary representation π of K on $\mathcal{H}$ determines another representation $\overline{\pi}$ on the dual space $\mathcal{H}'$ of $\mathcal{H}$, namely $\overline{\pi}(x) = \pi(x^{-1})^t$ where the t denotes the transpose. Here we identify a Hilbert space with its dual. Thus, if we choose an orthonormal basis for $\mathcal{H}$, so that $\pi(x)$ is represented by a matrix $M(x)$, then the matrix for $\overline{\pi}(x)$ is the inverse transpose of $M(x)$, and since π is unitary this is nothing but the complex conjugate of $M(x)$ and $\overline{\pi}$ is called the **contragredient** of π. We set $d_\pi = \dim \mathcal{H}$ ($d_\pi = \dim \mathcal{H} < \infty$ if π is irreducible see proposition 1.18).

Theorem 1.19 (Peter-Weyl Theorem)**.** *Let K be a compact group. Then E is uniformly dense in $C(K)$, $L^2(K) = \bigoplus_{\pi\in\widehat{K}} E_\pi$ (direct Hilbert sum). Let π_{ij} be defined as in (1.13), then*

$$\{\sqrt{d_\pi}\pi_{ij} \mid i,j = 1,\dots,d_\pi,\ \pi \in \widehat{K}\}$$

is an orthonormal basis for $L^2(K)$. Each $\pi \in \widehat{K}$ occurs in the right and left regular representations of K with multiplicity d_π. More precisely, for $i = 1,\dots,d_\pi$ the subspace of E_π (respectively $E_{\overline{\pi}}$) spanned by the i-th row (respectively the i-th column) of the matrix (π_{ij}) (respectively $(\overline{\pi}_{ij})$) is invariant under the right (respectively. left) regular representation, and the latter representation is equivalent to π there (see [Fo95a], Theorem 5.12).

Due to the Peter-Weyl theorem, if $f \in L^2(K)$ we have

$$f = \sum_{\pi\in\widehat{K}} \sum_{i,j=1}^{d_\pi} c^\pi_{ij}\pi_{ij}, \qquad c^\pi_{ij} = d_\pi \int_K f(x)\overline{\pi_{ij}(x)}\,dx. \tag{1.14}$$

Note that this decomposition of L^2 is dependent on the choice of an orthonormal basis for $\mathcal{H}$. However, it is possible to reformulate the equation (1.14) to avoid this dependency.

Definition 1.20. If $f \in L^1(K)$ and $\pi \in \widehat{K}$, we define the operator $\widehat{f}(\pi)$ on $\mathcal{H}$ by:

$$\widehat{f}(\pi) = \int_K f(x)\pi(x)^*\,dx = \int_K f(x)\pi(x^{-1})\;dx, \tag{1.15}$$

where dx denotes the Haar measure. This map $f \longmapsto \widehat{f}$ is called the **group Fourier transformation** of f at π (see [Fo95a], p.134). If we choose an orthonormal basis for $\mathcal{H}$ so that $\pi(x)$ is represented by the matrix $(\pi_{ij}(x))$, then $\widehat{f}(\pi)$ is given by the matrix

$$\widehat{f}(\pi)_{ij} = \int_K f(x)\overline{\pi_{ji}(x)}\,dx = \frac{1}{d_\pi}c^\pi_{ji}.$$

But then

$$\sum_{i,j} c^\pi_{ij}\pi_{ij}(x) = d_\pi \sum_{i,j} \widehat{f}(\pi)_{ji}\pi_{ij}(x) = d_\pi \operatorname{tr}\left[\widehat{f}(\pi)\pi(x)\right]$$

so that (1.14) becomes a Fourier inversion formula,

$$f(x) = \sum_{\pi\in\widehat{K}} d_\pi \operatorname{tr}\left[\widehat{f}(\pi)\pi(x)\right] \tag{1.16}$$

We get that

$$\begin{aligned}\operatorname{tr}\left[\widehat{f}(\pi)\pi(x)\right] &= \int_K f(y)\operatorname{tr}\left[\pi(y^{-1})\pi(x)\right]\,dy \\ &= \int_K f(y)\operatorname{tr}\left[\pi(y^{-1}x)\right]\,dy \\ &= f*\chi_\pi(x),\end{aligned}$$

so equation (1.16) becomes

$$f = \sum_{\pi\in\widehat{K}} d_\pi f*\chi_\pi = \sum_{\lambda\in L\cap\overline{C}} d_{\pi_\lambda} f*\chi_\pi. \tag{1.17}$$

In particular, $d_\pi f*\chi_\pi$ is the orthogonal projection of f onto E_π.

1.3 Tangent and Cotangent Bundles of a Lie Group

Let K be a real Lie group with Lie algebra $\mathfrak{k}$. Let TK and T^*K denote the tangent and contangent bundle of K. $\mathfrak{k}$ is defined as the tangent space T_eK at the identity element $e \in K$. The Lie bracket is given by the Lie bracket of the left invariant vector fields $v, w \in C^\infty(K, TK)$ as follows: $[X, Y] = [v, w]\,(e)$ if $X = v(e)$, $Y = w(e)$. Elements $X \in \mathfrak{k}$ are also viewed as generators of one-parameter group, $X = \frac{d}{dt}|_{t=0}\,e^{tX}$. Let $L_x : K \to K$, $y \mapsto xy$, denote left translation. Then we identify the tangent bundle TK with $K \times \mathfrak{k}$ using the bundle isomorphism

$$\begin{aligned} K\times\mathfrak{k} &\xrightarrow{\sim} TK \\ (x, X) &\longmapsto (dL_x)(X) = \frac{d}{dt}|_{t=0}\,x\,e^{tX}. \end{aligned}$$

This gives a global trivialization of the tangent bundle. Similarly, we identify the cotangent bundle T^*K with $K \times \mathfrak{k}^*$ as follows: Let $dL_x : T_eK \to T_xK$, $dL_x^* : T_x^*K \to T_e^*K$, and $(dL_x^*)^{-1} : T_e^*K \to T_x^*K$. The last map defines the identification

$$\begin{array}{rcl} K \times \mathfrak{k}^* & \xrightarrow{\sim} & T^*K \\ (x, \xi) & \longmapsto & (dL_x^*)^{-1}(\xi). \end{array}$$

Definition 1.21. The map $\mathrm{Ad}^* : K \to GL(\mathfrak{k}^*)$ is called the **co-adjoint representation** of K which is defined by: $\langle \mathrm{Ad}^* k(\lambda), X\rangle := \langle \lambda, \mathrm{Ad}\, k^{-1}(X)\rangle$ for all $\lambda \in \mathfrak{k}^*$, $k \in K$, and $X \in \mathfrak{k}$ where $\langle \cdot, \cdot \rangle$ denotes the duality bracket $\mathfrak{k}^* \times \mathfrak{k} \to \mathbb{C}$

Proposition 1.22. *Let $\phi : K \times K \to K$, $(x, y) \mapsto (yxy^{-1})$. Using the identification $T_xK = \mathfrak{k}$ via $X \mapsto \frac{d}{dt}|_{t=0}\, x\, e^{tX}$. Then the derivative of ϕ is given as follows:*

$$\begin{array}{rcl} d\phi(x, y) : \mathfrak{k} \times \mathfrak{k} & \longrightarrow & \mathfrak{k} \\ (X, Y) & \longmapsto & d\phi(x, y)(X, Y) = \mathrm{Ad}\, y \left(X - Y + \mathrm{Ad}\, x^{-1}(Y) \right). \end{array}$$

The adjoint map $d\phi(x, y)^* : \mathfrak{k}^* \longrightarrow \mathfrak{k}^* \times \mathfrak{k}^*$ is given by

$$d\phi(x, y)^* : \xi \longmapsto (\mathrm{Ad}^*\, y^{-1}(\xi), \mathrm{Ad}^*\, xy^{-1}(\xi) - \mathrm{Ad}^*\, y^{-1}(\xi))$$

Proof. We start first by fixing y, then we have

$$\begin{array}{rcl} d\phi(x) : \mathfrak{k} & \longrightarrow & \mathfrak{k} \\ \frac{d}{dt}|_{t=0}\, x\, e^{tX} & \longmapsto & \frac{d}{dt}_{|t=0}\, y\, x\, e^{tX}\, y^{-1}, \end{array}$$

and

$$\frac{d}{dt}|_{t=0} y\, x\, e^{tX}\, y^{-1} = \frac{d}{dt}_{|t=0}\, y\, x\, y^{-1} e^{t\, \mathrm{Ad}\, y(X)}.$$

Then we get

$$d\phi(x) : X \mapsto \mathrm{Ad}\, y(X).$$

Similarly, now fix x. Then we have

$$\begin{array}{rcl} d\phi(y) : \mathfrak{k} & \longrightarrow & \mathfrak{k} \\ \frac{d}{dt}|_{t=0}\, y\, e^{tY} & \longmapsto & \frac{d}{dt}|_{t=0}\, y\, e^{tY}\, x\, e^{-tY}\, y^{-1}. \end{array}$$

We compute

$$\begin{aligned} \frac{d}{dt}|_{t=0}\, y\, e^{tY}\, x\, e^{-tY}\, y^{-1} &= \frac{d}{dt}|_{t=0}\, y\, x e^{t\, \mathrm{Ad}\, x^{-1}(Y)}\, e^{-tY}\, y^{-1} \\ &= \frac{d}{dt}|_{t=0}\, y\, x\, e^{t\, \mathrm{Ad}\, x^{-1}(Y)}\, y^{-1}\, e^{-t\, \mathrm{Ad}\, y(Y)} \\ &= \frac{d}{dt}|_{t=0}\, y\, x\, y^{-1}\, e^{t\, \mathrm{Ad}\, yx^{-1}(Y)}\, e^{-t\, \mathrm{Ad}\, y(Y)} \\ &= \frac{d}{dt}|_{t=0} \left(y\, x\, y^{-1}\, e^{t\, \mathrm{Ad}\, yx^{-1}(Y) - t\, \mathrm{Ad}\, y(Y)} + O(t^2) \right) \\ &= \frac{d}{dt}|_{t=0} \left(y\, x\, y^{-1}\, e^{t\, \mathrm{Ad}\, y(\mathrm{Ad}\, x^{-1}(Y) - Y)} + O(t^2) \right). \end{aligned}$$

Then

$$d\phi(y) : Y \mapsto \operatorname{Ad} y(\operatorname{Ad} x^{-1}(Y) - Y).$$

Therefore,

$$\begin{array}{rcl} d\phi(x,y) : \mathfrak{k} \times \mathfrak{k} & \longrightarrow & \mathfrak{k} \\ (X,Y) & \mapsto & d\phi(x)(X) + d\phi(y)(Y) \\ & = & \operatorname{Ad} y(X) + \operatorname{Ad} y(\operatorname{Ad} x^{-1}(Y) - Y) \\ & = & \operatorname{Ad} y(X - Y + \operatorname{Ad} x^{-1}(Y)) \end{array}$$

For any $\xi \in \mathfrak{k}^*$ we get

$$\begin{array}{rcl} \langle \xi, d\phi(x,y)(X,Y)\rangle & = & \langle \xi, \operatorname{Ad} y(X + \operatorname{Ad} x^{-1}(Y) - Y)\rangle \\ & = & \langle \operatorname{Ad}^* y^{-1}(\xi), X + \operatorname{Ad} x^{-1}(Y) - Y\rangle. \end{array}$$

Then the adjoint map is given by

$$\begin{array}{rcl} d\phi^* : \mathfrak{k}^* & \longrightarrow & \mathfrak{k}^* \times \mathfrak{k}^* \\ \xi & \longmapsto & (\operatorname{Ad}^* y^{-1}(\xi), \operatorname{Ad}^* xy^{-1}(\xi) - \operatorname{Ad}^* y^{-1}(\xi)) \end{array}$$

□

We will need later on two more derivatives which we will compute in the following examples.

Example 1.23. Let $\omega : K \times K \to K$, $(x, y) \mapsto x$. The derivative of this function is denoted by $d\omega$. First we fix y, then we get

$$\begin{array}{rcl} d\omega : \mathfrak{k} \times \mathfrak{k} & \longrightarrow & \mathfrak{k} \\ (X,Y) & \longmapsto & X. \end{array}$$

Then the adjoint map of is given by

$$\begin{array}{rcl} d\omega^* : \mathfrak{k}^* & \longrightarrow & \mathfrak{k}^* \times \mathfrak{k}^* \\ \xi & \longmapsto & (\xi, 0) \end{array}$$

Example 1.24. Let $\Phi : K \times K \to K$, $(x, y) \mapsto xy$. The derivative of this function is denoted by $d\Phi$. We start first by fixing y, then we have

$$\begin{array}{rcl} d\Phi(x) : \mathfrak{k} & \longrightarrow & \mathfrak{k} \\ \frac{d}{dt}|_{t=0}\, x\, e^{tX} & \longmapsto & \frac{d}{dt}|_{t=0}\, x\, e^{tX}\, y \end{array}$$

where

$$\frac{d}{dt}|_{t=0}\, x\, e^{tX}\, y = \frac{d}{dt}|_{t=0}\, x\, y\, e^{t \operatorname{Ad} y^{-1}(X)}.$$

Then

$$d\Phi(x) : X \mapsto \operatorname{Ad} y^{-1}(X).$$

Similarly, we will fix x, then we have

$$\begin{aligned} d\Phi(y) : \mathfrak{k} &\longrightarrow \mathfrak{k} \\ \frac{d}{dt}|_{t=0}\, y\, e^{tY} &\longmapsto \frac{d}{dt}|_{t=0}\, x\, y\, e^{tY} \end{aligned}$$

Then

$$d\Phi(y) : Y \mapsto Y.$$

Therefore,

$$\begin{aligned} d\Phi(x,y) : \mathfrak{k} \times \mathfrak{k} &\longrightarrow \mathfrak{k} \\ (X, Y) &\mapsto d\Phi(x)(X) + d\Phi(y)(Y) \\ &= \operatorname{Ad} y^{-1}(X) + Y \end{aligned}$$

Then the adjoint map of is given by

$$\begin{aligned} d\Phi^* : \mathfrak{k}^* &\longrightarrow \mathfrak{k}^* \times \mathfrak{k}^* \\ \xi &\longmapsto (\operatorname{Ad}^* y(\xi), \xi) \end{aligned}$$

1.4 Distributions

Definition 1.25. Let X be an open set in $\mathbb{R}^n$. A **distribution** u in X is a linear form on $C_0^\infty(X)$ such that for every compact set $B \subset X$ there exist a constant C and integer k such that:

$$|\langle u|\varphi\rangle| \leq C \sum_{|\alpha|\leq k} \sup_B |\partial^\alpha \varphi|, \qquad \forall \varphi \in C_0^\infty(B). \tag{1.18}$$

The set of all distribution in X is denoted by $\mathcal{D}'(X)$ (see [Ho83], Definition 2.1.1).

Distributions can be restricted to open subsets. Let $u \in \mathcal{D}'(X)$, then the support of u, denoted by $\operatorname{supp} u$, is the set of points in X having no open neighborhood to which the restriction of u is 0.

Definition 1.26. If $u \in \mathcal{D}'(X)$, then the singular support of u, denoted by $\operatorname{sing\,supp} u$, is the set of points in X having no open neighborhood to which the restriction of u is a smooth function.

If $u \in \mathcal{D}'(X)$ has a compact support, then $\langle u|\varphi\rangle$ can be defined for all $\varphi \in C^\infty(X)$. Let $\psi \in C_0^\infty(X)$ and $\psi = 1$ in a neighborhood of $\operatorname{supp} u$, so we define $\langle u|\varphi\rangle := \langle u|\psi\varphi\rangle$. This definition does not depend on the choice of ψ. It follows from (1.18) and the product rule that

$$|\langle u|\varphi\rangle| \leq \sum_{|\alpha|\leq k} \sup_B |\partial^\alpha \varphi|, \qquad \forall \varphi \in C^\infty(X),$$

where $B = \operatorname{supp}\varphi$. Conversely, suppose that we have a linear form v on $C^\infty(X)$ such that for some constant C, integer k, and some compact set $L \subset X$

$$|\langle v|\varphi\rangle| \leq \sum_{|\alpha|\leq k} \sup_L |\partial^\alpha \varphi|, \qquad \forall \varphi \in C^\infty(X).$$

Then the restriction of v to $C_0^\infty(X)$ is a distribution with support contained in L. We denote the space of **distribution with compact support** $\mathcal{E}'(X)$ (see [Ho83], p. 44).

Remark 1.27. The set $\mathcal{E}'(X)$ can be identified with the set of distributions in $\mathcal{E}'(\mathbb{R}^n)$ with supports contained in X (see [Ho83], Theorem 2.3.1).

Theorem 1.28. *The Fourier transformation of a distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ is the function*

$$\widehat{u}(\xi) = \langle u_x | e^{-i\langle \xi | x\rangle}\rangle. \tag{1.19}$$

The right-hand side is also defined for every complex vector $\xi \in \mathbb{C}^n$ and is entire analytic function of ξ, called the ***Fourier-Laplace transformation*** *of u (see [Ho83], Theorem 7.1.14).*

Remark 1.29. If B is a compact set in $\mathbb{R}^n$, $u \in \mathcal{E}'(\mathbb{R}^n)$ is a linear form on $C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B$, such that, if Ω is a neighborhood of B,

$$|\langle u|\varphi\rangle| \leq \sum_{|\alpha|\leq k} \sup_\Omega |D^\alpha \varphi|, \qquad \forall \varphi \in C^\infty(\mathbb{R}^n).$$

where $D^\alpha := (1/i)\,\partial^\alpha$. One can extend $\langle u|\varphi\rangle$ by continuity to all $\varphi \in C^\infty(\Omega)$.

The derivatives of an analytic function can be estimated in a compact set by the maximum of its absolute value in a neighborhood. Therefore the following definition makes sense.

Definition 1.30. Let $B \subset \mathbb{C}^n$ be a compact set, then $\mathcal{A}'(B)$, the set of **analytic functionals** carried by B, is the space of linear forms u on the space $\mathcal{A}$ of entire analytic functions in $\mathbb{C}^n$, such that for every neighborhood Ω of B

$$|\langle u|\varphi\rangle| \leq C_\Omega \sup_\Omega |\varphi|, \qquad \forall \varphi \in \mathcal{A}.$$

Definition 1.31. Let X be an open and bounded set in $\mathbb{R}^n$. The spaces of **hyperfunctions** which is denoted by $\mathcal{B}(X)$ can be defined as follows:

$$\mathcal{B}(X) := \mathcal{A}'(\overline{X})/\mathcal{A}'(\partial X)$$

(see [Ho83], Definition 9.2.1).

The following theorem is a special case of Theorem 1.56 below.

Theorem 1.32. *Let $U_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$, be an open sets, and $f : U_1 \to U_2$ a C^∞ map such that $f'(x)$ is surjective for all $x \in U_1$. Then there is a unique continuous linear map $f^* : \mathcal{D}'(U_2) \to \mathcal{D}'(U_1)$ such that $f^* u = u \circ f$ when $u \in C(U_2)$. One calls $f^* u$ the pull-back of u by f (see [Ho83], Theorem 6.1.2).*

Definition 1.33 (**Distributions on a Manifold**)**.** Let M be a smooth manifold. Assume that to every coordinate system $\kappa : U_\kappa \subset M \to V_\kappa \subset \mathbb{R}^n$ in M we have a distribution $u_\kappa \in \mathcal{D}'(V_\kappa)$ such that

$$u_{\widetilde{\kappa}} = (\kappa \circ \widetilde{\kappa}^{-1})^* u_\kappa \qquad \text{in } \widetilde{\kappa}(U_\kappa \cap U_{\widetilde{\kappa}}). \tag{1.20}$$

Here $(\cdot)^*$ denotes the pull-back of the map $\kappa \circ \widetilde{\kappa}^{-1}$. We call the system u_κ a distribution u in M. The set of all distributions in M is denoted by $\mathcal{D}'(M)$.

Theorem 1.34. *Let $\mathcal{F}$ be an atlas for M. If for every $\kappa \in \mathcal{F}$ we have a distribution $u_\kappa \in \mathcal{D}'(V_\kappa)$, and (1.20) is valid when κ and $\widetilde{\kappa}$ belongs to $\mathcal{F}$, it follows that there exists one and only one distribution $u \in \mathcal{D}'(M)$ such that $u \circ \kappa^{-1} = u_\kappa$ for every $\kappa \in \mathcal{F}$ (see [Ho83], Theorem 6.3.4).*

In our case the smooth manifold is a compact Lie group K. We can define the space of distributions as the dual space of $C_0^\infty(K)$. Then we can identify functions with distributions via the Haar measure dk:

$$L^2(K) \to \mathcal{D}'(K), \qquad f \mapsto (\varphi \mapsto \int_K f(k)\varphi(k)\, dk)$$

Theorem 1.35. *Let K be a a connected compact Lie group and u be a function on K which has the Fourier expansion $u = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda$ due to Peter-Weyl Theorem (compare (1.17)), then*

1. *$u = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda$ is a C^∞-function on K if and only if for any $m \in \mathbb{N}$, there exists a positive number L_m such that*

$$\|\varphi_\lambda\|_{L^2} \leq L_m \, (1 + |\lambda|)^{-m}.$$

2. *$u = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda$ is a distribution on K if and only if there are positive numbers m and L such that*

$$\|\varphi_\lambda\|_{L^2} \leq L \, (1 + |\lambda|)^{m}.$$

(see [Se65])

1.5 Microlocal Analysis

We will give the definitions of the wave front set of a distribution, the pull-back, and the push-forward. Let $\Gamma \subseteq \mathbb{R}^n \setminus \{0\}$ **conic** set (i.e., if $\xi \in \Gamma$, $t > 0 \Rightarrow t\xi \in \Gamma$, $\forall t \in \mathbb{R}$)

Definition 1.36. Let u be a distribution in an open subset X of $\mathbb{R}^n$. The **wave front set** of u is the subset $\mathrm{WF}(u) \subseteq X \times (\mathbb{R}^n \setminus \{0\})$ defined as follows: $(x_0, \xi_0) \notin \mathrm{WF}(u)$ iff there exists a conic neighborhood Γ of ξ_0 and $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ such that:

$$|\widehat{\varphi u}(\xi)| \leq C_N (1 + |\xi|)^{-N}, \qquad N = 1, 2, \ldots, \xi \in \Gamma, \tag{1.21}$$

(see [Ho83], p. 252). Notice there is an equivalent statement which often used

$$\widehat{\varphi u}(\xi) = O(|\xi|)^{-N}, \qquad \text{as } \Gamma \ni \xi \longrightarrow \infty \tag{1.22}$$

for all N. Another equivalent formulation is: There is a neighborhood U of x such that (1.22) holds for every $\varphi \in C_0^\infty(U)$. Now 1.22 is equivalent to

$$\langle u | e^{-it\langle \xi | \cdot \rangle} \varphi \rangle = O(t^{-N}) \qquad \text{for } t \longrightarrow \infty \tag{1.23}$$

uniformly in $|\xi| = 1$ where $\xi \in \Gamma$, for all N where $\mathbb{R} \ni t \geq 1$. Here we tested the distribution u with oscillatory test function $e^{-it\langle \xi | \cdot \rangle} \varphi(x)$ and then investigated the asymptotic behavior letting the frequency variable t go to ∞ (see [Du96], p.15).

Remark 1.37. For any linear differential operator with C^∞-coefficients P we have

$$\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$$

(see [Ho83], p. 256).

Definition 1.38. Let X be an open set in $\mathbb{R}^n$ and Γ be a closed cone in $X \times \mathbb{R}^n \setminus \{0\}$, then we define the following:

$$\mathcal{D}'_\Gamma(X) := \left\{ u \in \mathcal{D}'(X) \mid \mathrm{WF}(u) \subset \Gamma \right\}$$

Lemma 1.39. *A distribution $u \in \mathcal{D}'(X)$ is in $\mathcal{D}'_\Gamma(X)$ if and only if for every $\phi \in C_0^\infty(X)$ and every closed cone in $V \subset \mathbb{R}^n$ with*

$$\Gamma \cap (\mathrm{supp}(\phi) \times V) = \emptyset \tag{1.24}$$

we have;

$$\sup_{\xi \in V} |\xi|^N \, |\widehat{\phi u}(\xi)| < \infty \qquad N \in \mathbb{N}$$

Proof. See [Ho83], Lemma 8.2.1. □

Definition 1.40. For a sequence $u_j \in \mathcal{D}'_\Gamma(X)$ and $u \in \mathcal{D}'_\Gamma(X)$ we say that $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$ if

$$u_j \to u \qquad \text{in } \mathcal{D}'(X)(\text{weakly}) \tag{1.25}$$

$$\sup_{\xi \in V} |\xi|^N \, |\widehat{\phi u}(\xi) - \widehat{\phi u_j}(\xi)| \to 0, \qquad j \to \infty, \tag{1.26}$$

for $N \in \mathbb{N}$ if $\phi \in C_0^\infty(X)$ and V is a closed cone in $\mathbb{R}^n$ such that (1.24) is valid. Since (1.25) implies that $\widehat{\phi u_j} \to \widehat{\phi u}$ uniformly on every compact set and N is arbitrary in (1.26), we can replace (1.26) by

$$\sup_j \sup_{\xi \in V} |\xi|^N \, |\widehat{\phi u_j}(\xi)| < \infty \qquad N \in \mathbb{N} \tag{1.27}$$

Theorem 1.41. *For every* $u \in \mathcal{D}'_\Gamma(X)$ *there is a sequence* $u_j \in C_0^\infty(X)$ *such that* $u_j \to u$ *in* $\mathcal{D}'_\Gamma(X)$.

Proof. See [Ho83], Theorem 8.2.3. □

We need more definitions. Let P be a differential operator of order m with C^∞-coefficients defined on a manifold X. In local coordinates we have

$$P = P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha.$$

The **principal symbol** P_m is invariantly defined on $T^*X \setminus 0$

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

The **characteristic variety** (set) $\operatorname{Char} P$ is defined by

$$\operatorname{Char} P := \{(x, \xi) \in T^*X \setminus 0 \mid P_m(x, \xi) = 0\}. \tag{1.28}$$

(See [Ho83], p. 271.)

Theorem 1.42. *If* P *is a differential operator of order* m *with* C^∞*-coefficients on a manifold* X*, then*

$$\operatorname{WF}(u) \subset \mathit{Char}\, P \cup \operatorname{WF}(Pu), \qquad u \in \mathcal{D}'(X),$$

(see [Ho83], Theorem 8.3.1).

If P is elliptic, that is, $P_m(x, \xi) \neq 0$ in $T^*X \setminus 0$, then

$$\operatorname{WF}(u) = \operatorname{WF}(Pu), \qquad u \in \mathcal{D}'(X).$$

(see [Ho83], Corollary 8.3.2)

Proposition 1.43. *Let* X *be a manifold and* Y *a submanifold with co-normal bundle*

$$N^*(Y) := \left\{(y, \xi) \mid y \in Y,\, \xi \in T^*_{\iota(y)}X,\, \langle \xi, T_y Y\rangle = 0\right\},$$

where $\iota : Y \hookrightarrow X$ *denotes the inclusion. For every distribution* u *in* X *with*

$$\operatorname{WF}(u) \cap N^*(Y) = \emptyset$$

the restriction $u_{|_Y}$ *to* Y *is well defined distribution on* Y*, the pull-back by the inclusion.*

Proof. see [Ho83], Corollary 8.2.7. □

Remark 1.44. Let X be a smooth manifold. We will give an equivalent definition of $\mathcal{D}'_\Gamma(X)$ using pseudo-differential operator instead of Fourier transformation. First we will give the definition of a pseudo-differential operator.

Definition 1.45. Let $r \in \mathbb{R}$. $a(x,\eta) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ is a **symbol** of order $\leq r$ iff

$$|\partial_x^\beta \partial_\eta^\alpha a(x,\eta)| \leq C_{\alpha,\beta}(1+|\eta|)^{r-|\alpha|} \qquad (\forall \alpha,\ \beta \in \mathbb{N}_0^N). \tag{1.29}$$

The lowest upper bounds of the constants in (1.29) are seminorms on the symbol space S^r turning it into a Fréchet space. $S^\infty = \cup_r S^r$ and $S^{-\infty} = \cap_r S^r$ (see[Ho85], Definition 18.1.1).

Let $\mathcal{S}$ denotes the Schwartz class and $\mathcal{S}'$ the space of tempered distributions.

Theorem 1.46. *Let $a \in S^m$ and $u \in \mathcal{S}$. Then*

$$Au(x) = a(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x|\xi\rangle} a(x,\xi)\widehat{u}(\xi)\, d\xi \tag{1.30}$$

defines a function $a(x,D)u \in \mathcal{S}$, and the bilinear map $(a,u) \mapsto a(x,D)u$ is continuous. The commutator with D_j and the multiplication by x_j are

$$[a(x,D)u, D_j] = i(\partial_{x_j} a(x,D)) \tag{1.31}$$
$$[a(x,D), x_j] = -i(\partial_{\xi_j} a(x,D)). \tag{1.32}$$

*One calls $A = a(x,D)$ a **pseudo-differential operator** of order m and denoted by $A \in \Psi^m$ (see [Ho85], Theorem 18.1.6).*

Remark 1.47. Due to the definition of $\widehat{u}$ in (1.30), it follows that the **Schwartz kernel** $\mathcal{K}$ of A is given by

$$\mathcal{K}(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y|\xi\rangle} a(x,\xi)\, d\xi, \tag{1.33}$$

which is a partial Fourier transformation of a (see [Ho85], p. 69).

Theorem 1.48. *Let $a \in S^m$. We denote by $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ the Schwartz Kernel of A defined by (1.33). Then $\mathcal{K} \in C^j(\mathbb{R}^n \times \mathbb{R}^n)$ if $m+j+n<0$, and $\mathcal{K} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle)$ for any m if $\triangle$ is the diagonal $\{(x,x), x \in \mathbb{R}^n\}$. More precisely,*

$$\mathrm{WF}(\mathcal{K}) \subset \{(x,x,\eta,-\eta) \mid x,\eta \in \mathbb{R}^n\} \tag{1.34}$$

which is the the co-normal bundle of $\triangle$. We have, for every $u \in \mathcal{S}'$,

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u), \tag{1.35}$$
$$\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u. \tag{1.36}$$

If $a \in S^{-\infty}$, then $Au \subset C^\infty$ (see [Ho85], Theorem 18.1.16).

We can define pseudo-differential operators on a manifold as follows

Definition 1.49. Let X smooth manifold. A pseudo-differential operator of order m on X is a continuous linear map $A : C_0^\infty(X) \to C^\infty(X)$ such that for every local coordinate patch $U_\kappa \subset X$ with coordinates $U_\kappa \ni x \to \kappa(x) \in V_\kappa \subset \mathbb{R}^n$ and all $\phi, \psi \in C_0^\infty(V_\kappa)$ the map

$$\mathcal{S}^{'}(\mathbb{R}^n) \ni u \longrightarrow \phi(\kappa^{-1})^* A \kappa^*(\psi u) \tag{1.37}$$

is in $\Psi^m(X)$. We can extend A, by continuity, to a map $\mathcal{E}^{'}(X) \to \mathcal{D}^{'}(X)$. (see [Ho85], Definition 18.1.20)

Definition 1.50. The (pseudo-differential) operator A in X is said to be **properly supported** if both projections from the support of the kernel in $X \times X$ to X are proper maps, that is, for every compact subset $B \subset X$ there is a compact set $B^{'} \subset X$ such that

$$\operatorname{supp} u \subset B \Rightarrow \operatorname{supp} Au \subset B^{'}. \tag{1.38}$$

(see [Ho85], Definition 18.1.21)

Definition 1.51. If $a \in S^m(T^*X)$ is a principal symbol of $A \in \Psi^m(X)$ then A is said to be non-characteristic at $(x_0, \xi_0) \in T^*X \setminus 0$ if $ab - 1 \in S^{-1}$ in a conic neighborhood of (x_0, ξ_0) for some $b \in S^{-m}(T^*X)$. The set of characteristic points is denoted by $\operatorname{Char} A$. The operator is said to be elliptic at a non-characteristic point (see [Ho85], Definition 18.1.25). This definition is independent of the choice of a. The proof of Theorem 18.1.9 in [Ho85] shows that in local coordinates an equivalent condition is that; A is elliptic at (x_0, ξ_0) if and only if there exists a neighborhood U of x_0, a conic neighborhood V of ξ_0, and a constant $C > 0$ such that $|a(y, \eta)| \geq |\eta|^n / C$ for $y \in U$, $\eta \in V$, $n \in \mathbb{R}$, $|\eta| > C$. If A has a homogeneous principal symbol a, the condition is equivalent to $a(x_0, \xi_0) \neq 0$, then the last definition of $\operatorname{Char} A$ coincides with (1.28) for differential operator.

Since $\mathrm{WF}(\mathcal{K})$, $\mathcal{K}$ denotes the kernel of the pseudo-differential operator A, is contained in the diagonal of $T^*X \setminus 0 \times T^*X \setminus 0$ it is natural to identify it with a conic subset of $T^*X \setminus 0$. We shall write

$$\mathrm{WF}(A) = \{\gamma \in T^*X \setminus 0 \mid (\gamma, \gamma) \in \mathrm{WF}(\mathcal{K})\} \tag{1.39}$$

Theorem 1.52. *If $u \in \mathcal{D}^{'}(X)$ we have for every $m \in \mathbb{R}$*

$$\mathrm{WF}(u) = \bigcap \mathit{Char}\, A \tag{1.40}$$

where the intersection is taken over all properly supported $A \in \Psi^m(X)$ such that $Au \in C^\infty(X)$ (see [Ho85], Theorem 18.1.27).

Definition 1.53. Let X be a smooth manifold and Γ be a closed cone in $T^*X \setminus 0$. The convergence of a sequence, $u_j \to u$ in $\mathcal{D}^{'}_\Gamma(X)$ is equivalent to

$$u_j \to u \qquad \text{in } \mathcal{D}^{'}(X)(\text{weakly}) \tag{1.41}$$

and that there exists for every $(x, \xi) \in (T^*X \setminus 0) \setminus \Gamma$, a pseudo-differential operator $A \in \Psi^m(X)$ such that $(x, \xi) \notin \operatorname{Char} A$, $\mathrm{WF}(A) \cap \Gamma = \emptyset$, and $Au_j \to Au \in C^\infty(X)$. (see [Ho85], p. 89 remark following Theorem 18.1.28)

Remark 1.54. To prove that $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$ it suffices to show that for every closed cone $\Gamma_1 \subset T^*X \setminus 0$ such that $\Gamma \subset \overset{\circ}{\Gamma}_1$ the following holds

$$u_j \to u \qquad \text{in } \mathcal{D}'_{\Gamma_1}(X). \tag{1.42}$$

This is clear from the last definition since for a given $(x,\xi) \in (T^*X \setminus 0) \setminus \Gamma$ we can find Γ_1 such that $(x,\xi) \notin \Gamma_1$, a pseudo-differential operator $A \in \Psi^m(X)$ such that $(x,\xi) \notin \operatorname{Char} A$, $\operatorname{WF}(A) \cap \Gamma_1 = \emptyset$, and $Au_j \to Au \in C^\infty(X)$.

Proposition 1.55. *Let K be a Lie group and Γ be a closed cone in $T^*K \setminus 0$. Let (u_j) be a sequence in $\mathcal{D}'_\Gamma(K)$ with $u_j \to u$ in $\mathcal{D}'(K)$. Assume that for every $(x_0,\xi_0) \in (T^*K \setminus 0) \setminus \Gamma$, there exist an open neighborhood $U \subset K$ of x_0, a real-valued function $\varphi \in C^\infty(U \times \mathfrak{k}^*)$, linear in the second variable, $\eta_0 \in \mathfrak{k}^*$ where $\xi_0 = \varphi'_x(x_0,\eta_0)$, and an open conic neighborhood $W_0 \subset \mathfrak{k}^* \setminus 0$ of η_0 where $\det \varphi''_{x\eta}(x,\eta) \neq 0$ for $(x,\eta) \in (U \times W_0)$, such that*

$$\sup_j \sup_{\eta \in W_0} |\eta|^N |\langle u_j | e^{-i\varphi(\cdot,\eta)}\psi\rangle| < \infty \qquad \forall N \in \mathbb{N}, \tag{1.43}$$

and all $\psi \in C_0^\infty(U)$. Then u_j converges to u in $\mathcal{D}'_\Gamma(K)$.

Proof. Let $(x,\xi) \in (T^*K \setminus 0) \setminus \Gamma$. Choose U, W_0, and φ as in the hypothesis, with U a coordinate neighborhood, and $\varphi''_{x\eta} \neq 0$ on $U \times W_0$. Let $\psi \in C_0^\infty(U)$ and $a \in S^0(\mathfrak{k}^*)$ is elliptic at η_0 with $\operatorname{supp}(a) \Subset W_0$. This implies that $(x_0,\xi_0) \notin \operatorname{Char}(A)$. Consider the operator $A : C^\infty(K) \to \mathcal{D}'(K)$ with Schwartz Kernel compactly supported in $U \times U$. The operator A is, in terms of local coordinates, given by

$$\begin{aligned} Au(y') &= \int_{W_0}\int_U e^{i\varphi(y',\eta) - i\varphi(y,\eta)}\psi(y')a(\eta)\psi(y)u(y)\,dyd\eta \\ &= \int_{W_0} e^{i\varphi(y',\eta)}\psi(y')a(\eta)\langle v|e^{-i\varphi(\cdot,\eta)}\psi\rangle\,d\eta, \end{aligned} \tag{1.44}$$

where $y', y \in U$. Observe that the singular support of the Schwartz Kernel of A is contained in the diagonal (compare theorem 1.48) and K can be covered with open sets $U \subset K$. Let $u \in \mathcal{D}'(K)$. Using theorem 1.52 we obtain that $\operatorname{WF}(u) = \bigcap \operatorname{Char} A$, where the intersection is taken over all properly supported $A \in \Psi^0(K)$ such that $Au \in C^\infty(K)$. The distributions space $\mathcal{D}'_\Gamma(K)$ is equipped with a local convex topology (see definition 1.24). Moreover, for every $u \in \mathcal{D}'_\Gamma(K)$ there exists a sequence $u_j \in C_0^\infty(X)$ such that $u_j \to u \in \mathcal{D}'_\Gamma(K)$ (see theorem 1.41). From definition 1.53 we obtain that a sequence $u_j \to u \in \mathcal{D}'_\Gamma(K)$ is equivalent to $u_j \to u \in \mathcal{D}'(K)$ (weakly) and the existence of, for every $(x,\xi) \in (T^*K \setminus 0) \setminus \Gamma$, a pseudo-differential operator $A \in \Psi^0(K)$ such that $(x,\xi) \notin \operatorname{Char} A$, and $Au_j \to Au \in C^\infty(K)$.

We want to show that (Au_j) is bounded sequence in $C^\infty(K)$. Since $A \in \Psi^0(K)$, then we can estimate the integral in (1.44) using (1.43),

$$|Au_j(y')| \leq \int_{W_0} |e^{i\varphi(y',\eta)}\psi(y')a(\eta)|\,d\eta, \leq c\int_{W_0} |\eta|^{-N}\,d\eta \leq C \tag{1.45}$$

where the constants $c, C > 0$ are independent of j. This implies that (Au_j) and $(D^\alpha Au_j)$ are bounded sequences in $L^\infty(K)$ for some $\alpha \in \mathbb{N}_0^N$. This implies that (Au_j) is bounded sequence in $C^\infty(K)$. Using Ascoli's theorem and the continuity of A on $\mathcal{D}'(K)$ we conclude that $Au_j \to Au \in C^\infty(K)$. Since we can choose a and ψ such that $\mathrm{WF}(A) \cap \Gamma = \emptyset$, then u_j converges to u in $\mathcal{D}'_\Gamma(K)$. □

Let X and Y be an open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and $f : X \to Y$ be a smooth map. The map $f^* : C^\infty(Y) \to C^\infty(X)$, $u \mapsto u \circ f$, have a unique continuous linear $f^* : \mathcal{D}'(Y) \to \mathcal{D}'(X)$ if f' is surjective (see Theorem 1.32) or under some conditions which will be presented in theorem 1.56. First we recall the definition of the pull-back of a distribution.

The pull-back of distribution is define as follows: Let X and Y be an open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and $f : X \to Y$ be a smooth map. For $u \in C_0^\infty(Y)$ and $\varphi \in C_0^\infty(X)$ we have

$$\langle f^*u|\varphi\rangle = (2\pi)^{-n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} e^{i\langle \eta, f(x)\rangle} \widehat{u}(\eta)\varphi(x)\, d\eta dx \tag{1.46}$$

We will rewrite this equality to generalize the pull-back operator to distributions which have their wavefront sets in suitable position. The assumption on the wave front set arises from the geometry. Let K be compact subset of X which denotes the support of φ and V be closed cone in $\mathbb{R}^n \setminus \{0\}$. Moreover, we assume

$$x \in K,\ \eta \in V \Rightarrow (f'(x))^t \eta \neq 0.$$

Choose $\Phi \in S^0$ such that $\mathrm{supp}(\Phi) \subset V$ and $|(f'(x))^t\eta| \geq c|\eta|$ when $x \in K$ and $\Phi(\eta) \neq 0$. We note that

$$\frac{\partial}{\partial_{x_j}} e^{i\langle \eta | f(x)\rangle} = i\langle \eta | \frac{\partial f(x)}{\partial_{x_j}}\rangle e^{i\langle \eta | f(x)\rangle}.$$

Therefore we can find a differential operator $L = \sum_j a_j(x,\eta)\partial/\partial_{x_j}$ with coefficients $a_j(x,\eta) \in S^{-1}$ such that

$$\Phi(\eta) L e^{i\langle \eta | f(x)\rangle} = \Phi(\eta) e^{i\langle \eta | f(x)\rangle}.$$

Now let $u \in \mathcal{E}'(Y)$, $\mathrm{WF}(u) \subseteq Y \times V$. Using partial integrations, we rewrite equation (1.46) as follows:

$$\begin{aligned}\langle f^*u|\varphi\rangle &= (2\pi)^{-n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} e^{i\langle \eta, f(x)\rangle} \Phi(\eta)\widehat{u}(\eta)(L^t)^N \varphi(x)\, d\eta dx \\ &\quad + (2\pi)^{-n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} e^{i\langle \eta, f(x)\rangle} (1-\Phi(\eta))\widehat{u}(\eta)\varphi(x)\, d\eta dx\end{aligned}$$

When $N \in \mathbb{N}$ is sufficiently large the integrals exist and we take this equation as the definition of the **pull-back of distribution** u (see [Du96], p. 19). More precisely, the following is true.

Theorem 1.56. *Let X and Y be an open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $f : X \to Y$ be a smooth map. Denote the set of the normals of the map by*

$$N_f = \{(f(x), \eta) \in Y \times \mathbb{R}^n \mid (f'(x))^t\eta = 0\}.$$

where $(f'(x))^t$ is the transpose of $f'(x)$. Then the pull-back f^*u can be defined in one and only way for all $u \in \mathcal{D}'(Y)$ with

$$N_f \cap \mathrm{WF}(u) = \emptyset \tag{1.47}$$

so that $f^*u = u \circ f$ when $u \in C^\infty$ and for any closed conic subset $\Gamma \subset Y \times \mathbb{R}^n \setminus \{0\}$ with $\Gamma \cap N_f = \emptyset$ we have a continuous map $f^* : \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'_{f^*\Gamma}(X)$, where

$$f^*\Gamma = \{\left(x, (f'(x))^t\eta\right) \mid (f(x), \eta) \in \Gamma\}.$$

In particular we have for every $u \in \mathcal{D}'(Y)$ satisfying (1.47)

$$\mathrm{WF}(f^*u) \subset f^* \mathrm{WF}(u).$$

(see [Ho83], Theorem 8.2.4)

Remark 1.57. It is useful to introduce the following sets

$$C_f := \left\{(x, y; \xi, \eta) \mid y = f(x),\ (f'(x))^t\eta + \xi = 0\right\} \tag{1.48}$$

and

$$C'_f := \{(x, y; \xi, \eta) \mid (x, y; -\xi, \eta) \in C_f\} \tag{1.49}$$

Then

$$N_f = C'_f \circ \{(x, \xi) \mid \xi = 0\} \tag{1.50}$$

and

$$f^*\Gamma = C'_f \circ \Gamma. \tag{1.51}$$

Remark 1.58. If X is a smooth manifold and $u \in \mathcal{D}'(X)$ we can now define $\mathrm{WF}(u) \subset T^*X \setminus \{0\}$ so that the restriction to a coordinate patch X_κ is equal to $\kappa^* \mathrm{WF}(u \circ \kappa^{-1})$. In fact when f is a diffeomorphism between open set in $\mathbb{R}^n$ it follows from last theorem 1.56 that $\mathrm{WF}(f^*u)$ is the pull-back of $\mathrm{WF}(u)$ considered as a subset of the cotangent bundle. Hence the definition of the pull-back is independent of the choice of local coordinates. Moreover, $\mathrm{WF}(u)$ is a closed subset of $T^*X \setminus \{0\}$ which is conic in the sense that the intersection with the vector space T^*_xX is a cone for every $x \in X$ (see [Ho83], p. 265).

The following remark is a supplement to proposition 1.43 using theorem 1.56.

Remark 1.59. Let X be a manifold and Y a submanifold with the inclusion $\iota : Y \hookrightarrow X$. From proposition 1.43 we obtain that, for every distribution u defined on X with $\mathrm{WF}(u) \cap N^*(Y) = \emptyset$, the restriction $u_{|Y}$ is a well-defined distribution on Y and $u_{|Y}$ is the pull-back of u by the inclusion ι. From theorem 1.56 and remark 1.57 we obtain the following

$$C_\iota := \left\{ (y, x; \eta, \xi) \mid x = \iota(y),\, (\iota^{'}(y))^t \xi + \eta = 0 \right\}, \tag{1.52}$$

here $(\iota^{'}(y))^t : T^*_{\iota(y)} X \to T^*_y Y$ and

$$C_\iota^{'} := \{ (y, x; \eta, \xi) \mid (y, x; -\eta, \xi) \in C_\iota \} \tag{1.53}$$

Then

$$N_\iota = C_\iota^{'} \circ \{ (y, \eta) \mid \eta = 0 \} = N^*(Y), \tag{1.54}$$

and

$$C_\iota^{'} \circ \mathrm{WF}(u) = \{ (y, \eta) \in T^*Y \mid \exists (x, \xi) \in \mathrm{WF}(u), x = \iota(y), (\iota^{'}(y))^t \xi = \eta \}. \tag{1.55}$$

We can define the tensor product $u \otimes v \in \mathcal{D}^{'}(X \times X)$ of two distributions $u, v \in \mathcal{D}^{'}(X)$. The corresponding bilinear map is separately sequentially continuous. In case $u, v \in C(\mathbb{R}^n)$ we have $u \otimes v(x, y) := u(x)v(y)$.

Theorem 1.60. *Let X, Y be smooth manifolds and $u \in \mathcal{D}^{'}(X)$, $v \in \mathcal{D}^{'}(Y)$, then*

$$\begin{aligned} \mathrm{WF}(u \otimes v) \subset (\mathrm{WF}(u) \times \mathrm{WF}(v)) \quad & \cup \quad ((\operatorname{supp} u \times \{0\}) \times \mathrm{WF}(v)) \\ & \cup \quad (\mathrm{WF}(u) \times (\operatorname{supp} v \times \{0\})) \end{aligned} \tag{1.56}$$

Proof. See [Ho83], Theorem 8.2.9. □

Proposition 1.61. *Let X and Y be a smooth manifolds. Let $\Phi : X \to Y$ be a proper map. The dual map $\Phi_* = (\Phi^*)^t : \mathcal{D}^{'}(X) \to \mathcal{D}^{'}(Y)$ is defined and continuous. It is called the push-forward by Φ. Moreover, for a given closed cone $\Gamma \subset T^*X \setminus 0$ the restriction $\Phi_* : \mathcal{D}^{'}_\Gamma(X) \to \mathcal{D}^{'}_{\Phi_*\Gamma}(Y)$ is sequentially continuous. Here we get that the set $\Phi_*\Gamma$ is contained in*

$$\begin{aligned} & \{ (y, \eta) \in T^*Y \setminus \{0\} \mid y = \Phi(x) \,\wedge\, (x, (d_x\Phi(x))^t \eta) \in \Gamma, x \in X \} \\ \cup \quad & \{ (y, \eta) \in T^*Y \setminus \{0\} \mid y = \Phi(x) \,\wedge\, (d_x\Phi(x))^t \eta = 0 \} \end{aligned} \tag{1.57}$$

where $d\Phi(x) : T^*_x X \to T^*_{\Phi(x)} Y$.

Proof. See [Du96], Proposition 1.3.4. and [FJ98], Proposition 11.3.3. □

2 Wave Front Set and Fourier Coefficients

In this chapter we are going to calculate the wave front set of a distribution defined on connected, compact, and semisimple Lie groups.

Let K be a connected, semisimple, and compact Lie group and π an irreducible representation of K on $L^2(K)$. We fix a Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ of $\mathfrak{k}_\mathbb{C}$ and a positive system of roots $\Lambda_+ = \Lambda_+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. We denote by $\overline{C} \subset i\mathfrak{t}^*$ the closure of the Weyl chamber (see definition 1.12) and $L \subset i\mathfrak{t}^*$ the weight lattice (see definition 1.15). Recall that $\widehat{K}$ denotes the set of equivalence classes of irreducible representations of K. We identify $\widehat{K}$ with $\lambda \in L \cap \overline{C}$ using the theorem of Highest Weights 1.14. Due to the Peter-Weyl Theorem 1.19 we have

$$L^2(K) = \bigoplus_{\lambda \in L \cap \overline{C}} E_\lambda. \tag{2.1}$$

For $u \in L^2(K)$ we have the Fourier series

$$u = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda$$

where φ_λ is an element of E_λ (compare theorem 1.19 $E_\lambda = E_{\pi_\lambda}$). Here $\varphi_\lambda = d_{\pi_\lambda} u * \chi_{\pi_\lambda}$, d_{π_λ} is the dimension of the representation $\pi \in \widehat{K}$, χ_{π_λ} is the character of π. If $u \in \mathcal{D}'(K)$ then we have the same Fourier series because; $\chi_{\pi_\lambda} \in C_0^\infty(K)$, $u * \chi_{\pi_\lambda} \in E_\lambda \subset C^\infty(K)$, and the convolution of a distribution with a $C_0^\infty(K)$-function is defined by $u * \chi_{\pi_\lambda}(x) = \int_k u(xy^{-1})\chi_{\pi_\lambda}(y)dy$, where the duality bracket written as an integral. Hence, the Fourier series u converges to u in $\mathcal{D}'(K)$.

In the following Ω will always denote a closed cone contained in $\overline{C} \setminus 0 \subset i\mathfrak{t}^*$.

2.1 Wave Front Set of Truncated Distributions

In this section we aim to calculate the wave front set of a truncated distribution $u_\Omega = \sum_{\lambda \in L \cap \Omega} \varphi_\lambda$.

Remark 2.1. Let $X \in \mathfrak{k}$ and $\widetilde{X}$ be the cooresponding left invariant vector field (see section 1.3). Let $(x, \xi) \in K \times \mathfrak{k}^* \simeq T^*K$. We want to show that the principal symbol of $\widetilde{X}$, denoted by $\sigma(\widetilde{X})$, at the point (x, ξ), equals $i\langle \xi, X\rangle$ where $\langle \cdot, \cdot \rangle$ denotes the duality brackets $\mathfrak{k}^* \times \mathfrak{k} \to \mathbb{C}$. Choose $\psi \in C^\infty(K)$ with $d\psi(e) = \xi$, where e is the identity element of K. Set $\varphi := (L_{x^{-1}})^*\psi$ where $(L_{x^{-1}})^*$ denotes the pull-back of ψ by the left translation $L_{x^{-1}}$ for $x \in K$. Then $d\psi(x) = (x, \xi)$ under the identification $T_x^*K \simeq \mathfrak{k}^*$. Then the principal symbol of $\widetilde{X}$, in terms of local coordinates, can be define as follows, for $0 < t \in \mathbb{R}$ and $x \in K$,

$$\begin{aligned}
\sigma(\widetilde{X})(d\psi(x)) &= \lim_{t\to\infty} \frac{1}{t} e^{-it\varphi(x)} \left(\widetilde{X} e^{it\varphi(x)}\right)(x) \\
&= \lim_{t\to\infty} \frac{1}{t} e^{-it(L_{x^{-1}})^*\psi(x)} \left(\widetilde{X} e^{it(L_{x^{-1}})^*\psi(x)}\right)(x) \\
&= \lim_{t\to\infty} \frac{1}{t} (L_{x^{-1}})^* e^{-it\psi(x)} (L_{x^{-1}})^* \left(\widetilde{X} e^{it\psi(x)}\right)(x) \\
&= \lim_{t\to\infty} \frac{1}{t} (L_{x^{-1}})^* \left(e^{-it\psi(x)} \left(\widetilde{X} e^{it\psi(x)}\right)\right)(x) \\
&= \lim_{t\to\infty} \frac{1}{t} \left(e^{-it\psi(x)} \left(\widetilde{X} e^{it\psi(x)}\right)\right)(e) \\
&= \lim_{t\to\infty} \frac{1}{t} e^{-it\psi(e)} e^{it\psi(e)} it \left(\widetilde{X}\psi(x)\right)(e) \\
&= i\langle d\psi(e), \widetilde{X}(e)\rangle \\
&= i\langle \xi, X\rangle.
\end{aligned}$$

This imply that $\sigma(\widetilde{X})(x,\xi) = i\langle \xi, X\rangle$.

Proposition 2.2. *Let $u_\Omega = \sum_{\lambda\in L\cap\Omega}\varphi_\lambda$ be a distribution on K and all φ_λ are highest weight vectors with respect to the left action. Then u_Ω converges in $\mathcal{D}'_\Gamma(K)$ where $\Gamma := K \times (-i)\Omega$ is a closed cone in T^*K. In particular* $\mathrm{WF}(u_\Omega) \subset \Gamma$.

Proof. We assume also that Ω is convex. Let each $\varphi_\lambda \in E_\lambda$ be a highest weight vector of the highest weight λ with respect to the left action on $L^2(K)$. We denote by $\langle\cdot,\cdot\rangle$ denotes the duality bracket $\mathfrak{t}^*_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$. Let $\mathfrak{n} := \bigoplus_{\alpha\in\Lambda_+}\mathfrak{k}_\alpha$, then we have

$$X\cdot\varphi_\lambda = 0 \qquad \text{for } X \in \mathfrak{n} \tag{2.2}$$

$$X\cdot\varphi_\lambda = \langle\lambda, X\rangle\,\varphi_\lambda \qquad \text{for } X \in \mathfrak{t} \tag{2.3}$$

for the left action. Since $X \in \mathfrak{k}$ is not only an element of the Lie algebra but acts as a differential operator of first order with a smooth coefficients which we denote by a tilde $\widetilde{X}$ to distinguish from the Lie algebra element, then $X\cdot\varphi_\lambda = \widetilde{X}\varphi_\lambda$.

From (2.2) and because $\widetilde{X}$ is a continuous operator we get

$$\widetilde{X}\sum_{\lambda\in L\cap\Omega}\varphi_\lambda = \sum_{\lambda\in L\cap\Omega}\widetilde{X}\varphi_\lambda = 0$$

Furthermore, due to theorem 1.42 we get

$$\mathrm{WF}(u_\Omega) \subset \mathrm{Char}\widetilde{X}, \qquad \text{for } X \in \mathfrak{n}$$

where $\mathrm{Char}\widetilde{X}$ denotes the characteristic variety of the operator $\widetilde{X}$ (see (1.28)). Moreover, using remark 2.1, we get that

$$\mathrm{Char}\widetilde{X} = K \times X^\perp \qquad \text{for } X \in \mathfrak{n}.$$

Hence,

$$\mathrm{WF}(u_\Omega) \subset \bigcap_{X \in \mathfrak{n}} K \times X^\perp = K \times \mathfrak{t}^*.$$

Recall the identification of the cotangent bundle $T^*K = K \times \mathfrak{k}^*$ (see 1.3).

Now we consider equation (2.3). Let U_0 be a neighborhood of $0 \in \mathfrak{k}$ and V_0^x be a neighborhood of $x \in K$ defined by $V_0^x = L_x \circ \exp(U_0)$, where L_x denotes the left translation on K. Let $\kappa : V_0^x \subset K \to U_0 \subset \mathfrak{k}$, $y \mapsto \exp^{-1}(x^{-1} \cdot y)$ be a chart ($y \in V_0^x$). Then κ is a diffeomorphism: $V_0^x \to U_0$ (see [DK00], Theorem 1.6.3).

Let $\widetilde{X}$ and $\approx\!\! X$ ($\overset{\approx}{X}$) be differential operators of first order with real smooth coefficients such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}'(V_0^x) & \xrightarrow{\widetilde{X}} & \mathcal{D}'(V_0^x) \\ {\scriptstyle (\kappa^{-1})^*}\big\downarrow & & \big\downarrow{\scriptstyle (\kappa^{-1})^*} \\ \mathcal{D}'(U_0) & \xrightarrow{\overset{\approx}{X}} & \mathcal{D}'(U_0) \end{array}$$

where $(\kappa^{-1})^*$ denotes the pull-back operator (see theorem 1.56). In local coordinates $\overset{\approx}{X}$ is given by $\sum_j c_j(y)\, \partial/\partial y_j$.

The pull-back of Haar measure is $(\kappa^{-1})^* dy = J_\kappa(Y) dY$, $J_\kappa(Y)$ denotes the Jacobian of κ^{-1}. The function $J_\kappa(Y)$ is smooth and will be calculated explicitly in remark 2.5.

To characterize the wave front set we use definition 1.53 and proposition 1.55. Let $\vartheta \in C^\infty(U_0 \times \mathfrak{t}^*)$ be a real-valued function satisfying the conditions in proposition 1.55. $\langle\cdot|\cdot\rangle$ denotes the duality brackets between $\mathcal{D}'(K) \times C_0^\infty(K)$ and $\overset{\approx}{X}{}^t$ denotes the transpose of $\overset{\approx}{X}$. We choose a suitable localization function $\psi \in C_0^\infty(\mathfrak{k})$. Then, for $Y \in U_0 \subset \mathfrak{k}$, $X \in \mathfrak{t}$, $\xi \in \mathfrak{k}^*$, we have

$$\begin{aligned} \langle \lambda, X\rangle \left\langle (\kappa^{-1})^*(\varphi_\lambda) \,|\, \psi e^{-i\vartheta(\cdot,\xi)} \right\rangle &= \int_{\mathfrak{k}} \langle \lambda, X\rangle \varphi_\lambda(\kappa^{-1}(Y))\psi(Y)\, e^{-i\vartheta(Y,\xi)}\, J_\kappa(Y) dY \\ &\overset{(2.3)}{=} \int_{\mathfrak{k}} (\widetilde{X}\varphi_\lambda)(\kappa^{-1}(Y))\psi(Y) e^{-i\vartheta(Y,\xi)}\, J_\kappa(Y) dY \\ &= \int_{\mathfrak{k}} \overset{\approx}{X}(\varphi_\lambda(\kappa^{-1}(Y)))\psi(Y) e^{-i\vartheta(Y,\xi)}\, J_\kappa(Y) dY \\ &= -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y)) \overset{\approx}{X}{}^t \left(\psi(Y) e^{-i\vartheta(Y,\xi)} J_\kappa(Y)\right) dY \\ &= -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y)) \overset{\approx}{X}{}^t \left(\psi(Y) J_\kappa(Y)\right) e^{-i\vartheta(Y,\xi)} dY \\ &\quad -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y)) \psi(Y) J_\kappa(Y) \overset{\approx}{X}{}^t \left(e^{-i\vartheta(Y,\xi)}\right) dY \end{aligned}$$

Recall that $\widetilde{\widetilde{X}} = \sum_j c_j(Y)\, \partial/\partial y_j$ in local coordinates. Then

$$\begin{aligned}
\widetilde{\widetilde{X}}^{\,t}\left(e^{-i\vartheta(Y,\xi)}\right) &= \sum_j^n \frac{\partial}{\partial y_j}\left(c_j(Y)\, e^{-i\vartheta(Y,\xi)}\right) \\
&= \sum_j^n c_j(Y)\, \frac{\partial}{\partial y_j}\, e^{-i\vartheta(Y,\xi)} + \sum_j^n \left(\frac{\partial}{\partial y_j} c_j(Y)\right)\, e^{-it\vartheta(Y,\xi)} \\
&= \widetilde{\widetilde{X}}\left(e^{-i\vartheta(Y,\xi)}\right) + \underbrace{\sum_j^n \left(\frac{\partial}{\partial y_j} c_j(Y)\right)}_{=:Z(Y)}\, e^{-it\vartheta(Y,\xi)} \qquad (2.4)
\end{aligned}$$

Claim 2.3. Let $\widetilde{\widetilde{X}}$ be a vector field with a real smooth coefficients. Then due to the method of characteristics we obtain

$$\widetilde{\widetilde{X}}(e^{-i\vartheta(Y,\xi)}) = -i\langle \xi, X\rangle e^{-i\vartheta(Y,\xi)}, \qquad (2.5)$$

where $\vartheta \in C^\infty(U_0 \times \mathfrak{t}^*)$ is a real-valued function satisfying the conditions in proposition 1.55.

Proof of claim 2.3. It is sufficient to consider the equation

$$\begin{aligned}
\widetilde{\widetilde{X}}\,\vartheta(Y,\xi) &= \langle \xi, X\rangle \\
\sum_j^n c_j(y)\, \frac{\partial\vartheta}{\partial y_j}(Y,\xi) &= \langle \xi, X\rangle \qquad (2.6)
\end{aligned}$$

This is a linear partial differential equation of first order. Assume, without loss of generality, $c_n(y) \neq 0$, then due to the method of characteristics (see [Fo95], p. 34-38) we can find a solution ϑ. It satisfies along any characteristic curve, i.e., a solution of

$$\begin{aligned}
\frac{dy_j}{dt} &= c_j(Y) \\
y_j(0) &= (y_0)_j,
\end{aligned}$$

the differential equation,

$$\begin{aligned}
\langle \xi, X\rangle &= \frac{d\vartheta}{dt}(Y,\xi) = \sum_j^n \frac{dy_j}{dt}\frac{\partial\vartheta}{\partial y_j}(Y,\xi) = \sum_j^n c_j(Y)\frac{\partial\vartheta}{\partial y_j}(Y,\xi) \\
\vartheta(Y,\xi) &= \langle \xi, Y\rangle \qquad \text{on} \quad y_n = 0.
\end{aligned}$$

Observe that in local coordinates we can give a basis $(y_1, \cdots, y_n) = Y \in U_0 \subset \mathfrak{k}$, and $(\xi_1, \cdots, \xi_n) = \xi \in \mathfrak{k}^*$, respectively. $\square$

Due to claim 2.3 and (2.4) we get that

$$\begin{aligned}\langle\lambda, X\rangle \left\langle(\kappa^{-1})^*(\varphi_\lambda)\,|\,\psi e^{-i\vartheta(\cdot,\xi)}\right\rangle &= -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y))\overset{\approx}{X}{}^t(\psi(Y)J_\kappa(Y))e^{-i\vartheta(Y,\xi)}dY \\ &\quad -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y))\psi(Y)J_\kappa(Y)\overset{\approx}{X}{}^t(e^{-i\vartheta(Y,\xi)})dY \\ &= -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y))\overset{\approx}{X}{}^t(\psi(Y)J_\kappa(Y))e^{-i\vartheta(Y,\xi)}dY \\ &\quad +\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y))\psi(Y)J_\kappa(Y)\langle i\xi, X\rangle e^{-i\vartheta(Y,\xi)}dY \\ &\quad -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y))\psi(Y)J_\kappa(Y)Z(Y)e^{-i\vartheta(Y,\xi)}dY\end{aligned}$$

Then we have the following

$$\begin{aligned}\int_{\mathfrak{k}}\langle\lambda - i\xi, X\rangle\varphi_\lambda(\kappa^{-1}(Y))\psi(Y)e^{-i\vartheta(Y,\xi)}\, J_\kappa(Y)dY = \\ -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y))\overset{\approx}{X}{}^t(\psi(Y)J_\kappa(Y))e^{-i\vartheta(Y,\xi)}dY \\ -\int_{\mathfrak{k}} \varphi_\lambda(\kappa^{-1}(Y))\psi(Y)J_\kappa(Y)Z(Y)e^{-i\vartheta(Y,\xi)}dY.\end{aligned}$$

The term on the left hand side equals to $\langle\lambda - i\xi, X\rangle \left\langle(\kappa^{-1})^*(\varphi_\lambda)\,|\,\psi e^{-i\vartheta(\cdot,\xi)}\right\rangle$. The terms on the right hand side, using Cauchy-Schwarz inequality, are bounded by a constant C_1 times $\|\varphi_\lambda\|$. Iterating $2N$ times we obtain

$$|\langle\lambda - i\xi, X\rangle^{2N} \left\langle(\kappa^{-1})^*(\varphi_\lambda)\,|\,\psi e^{-i\vartheta(\cdot,\xi)}\right\rangle| \le C_1\,\|\varphi_\lambda\|,$$

where C_1 is independent of $\lambda \in L\cap\Omega$ and $\xi \in \mathfrak{k}^*$. Since the L^2-norms of the Fourier coefficients are polynomially bounded we obtain, for some $N_0 \in \mathbb{N}$

$$|\left\langle(\kappa^{-1})^*(\varphi_\lambda)\,|\,\psi e^{-i\vartheta(\cdot,\xi)}\right\rangle| \le C_1\,|\langle\lambda - i\xi, X\rangle|^{-2N}|\lambda|^{N_0}. \tag{2.7}$$

Claim 2.4. Let Ω be a non-empty convex cone and V a closed cone contained in $i\mathfrak{k}^*$ such that $\Omega\cap V = \emptyset$. Then there exists $C_2 := C_{\Omega,V} > 0$ and $X \in \mathfrak{t}$ such that the following estimate holds true:

$$\min_{\substack{|\lambda|+|\xi|=1 \\ i\xi\in V,\, \lambda\in\Omega}} |\langle\lambda - i\xi, X\rangle| \ge C_2. \tag{2.8}$$

In particular, we get

$$|\langle\lambda - i\xi, X\rangle| \ge C_2(|\lambda| + |\xi|) \qquad \forall\lambda\in\Omega,\ \forall i\xi\in V \tag{2.9}$$

Proof of claim 2.4. Due to the condition above we get from Hahn-Banach separation theorem (see [We05], Theorem III.2.5): There exists an $X \in \mathfrak{t}$ $\forall \lambda \in \Omega$, $\forall i\xi \in V$ such that $\langle \lambda - i\xi, X\rangle \neq 0$. The function $|\langle \lambda - i\xi, X\rangle|$ is continuous and defined on a compact set where $|\lambda| + |\xi| = 1$, then it will take its minimum on this set which proves assertion (2.8). Moreover, due to the homogeneity of λ, ξ, X we get assertion (2.9). □

Therefore, we obtain from claim 2.4 and estimate (2.7), for $i\xi \in V$

$$|\left\langle (\kappa^{-1})^*(\varphi_\lambda)\,|\,\psi e^{-i\vartheta(\cdot,\xi)}\right\rangle| \leq C_1\,|\langle \lambda - i\xi, X\rangle|^{-2N}|\lambda|^{N_0} \leq C_1 C_2^{-2N}\,|\lambda|^{-N+N_0}|\xi|^{-N}. \quad (2.10)$$

Hence, for N sufficiently large we obtain

$$\sup_{\lambda\in\Omega}\sup_{\xi\in V}|\xi|^N|\left\langle (\kappa^{-1})^*(\varphi_\lambda)\,|\,\psi e^{-i\vartheta(\cdot,\xi)}\right\rangle| < \infty \qquad \forall N \in \mathbb{N}. \quad (2.11)$$

Set $\Gamma = K \times (-i)\Omega$. Then due to definition 1.53 and proposition 1.55 we get that the series u_Ω converges to u_Ω in $\mathcal{D}'_\Gamma(K)$. Hence,

$$\mathrm{WF}(u_\Omega) \subset K \times (-i)\Omega \subset T^*K \simeq K \times \mathfrak{k}^*$$

We made an additional assumption at the beginning of the proof that Ω is convex. If this were not true; let $(\Omega_j)_{1\leq j\leq N}$ be a family of closed convex cones such that $\Omega \subset \cup\,\Omega_j$. Choose $A_j \subset \Omega_j$, $A_j \cap A_k = \emptyset$ for $j \neq k$, such that $u = \sum_j u_j$, $u_j = \sum_{\lambda\in A_j}\varphi_\lambda$. Then $\mathrm{WF}(u_j) \subset K \times (-iA_j)$, hence $\mathrm{WF}(u) \subset K \times \cup\,(-i\Omega_j)$. □

Remark 2.5. The pull-back of the Haar measure is $(\kappa^{-1})^*dy = J_\kappa(Y)dY$. We will determine the Jacobian of κ^{-1} which we denote by $J_\kappa(Y) = \det((\kappa^{-1})')$ we first have to calculate the derivative of κ^{-1} at $Y \in \mathfrak{k}$

$$(L_x \circ \exp)'(Y) = dL_x(\exp(Y)) \circ d\exp(Y) = dL_x(\exp(Y)) \circ dL_{\exp(Y)}(e) \circ \frac{1 - e^{-\operatorname{ad} Y}}{\operatorname{ad} Y},$$

the second equality follows from Theorem 1.7 in [He01]. Since the Haar measure is left invariant, it is sufficient to consider only the term $\dfrac{1 - e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}$. Then we get, for $Y \in \mathfrak{t}$

$$\det\left(\frac{1 - e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}\right) = \det\left(\frac{e^{\operatorname{ad} Y/2} - e^{-\operatorname{ad} Y/2}}{\operatorname{ad} Y/2}\right) = \prod_{\alpha\in\Lambda}\frac{e^{\langle\alpha,Y\rangle/2} - e^{-\langle\alpha,Y\rangle/2}}{\langle\alpha, Y\rangle} =: J_\kappa(Y).$$

The first equality follows from Corollary 5.5 [BGV92] and the second equality follows from [BGV92], p. 253. In general, for each $Y \in \mathfrak{k}$ there exists a $k \in K$ such that $\operatorname{Ad} k(Y) \in \mathfrak{t}$, then $\operatorname{ad} Y = \operatorname{Ad} k^{-1} \circ \operatorname{ad}(\operatorname{Ad} k(Y)) \circ \operatorname{Ad} k$. Since the trace is invariant under the adjoint action then the second equality is valid for all $Y \in \mathfrak{k}$. Observe that the enumerator and the denominator of $J_\kappa(Y)$ has the same simple zero where $\langle\alpha, Y\rangle = 0$.

Remark 2.6. Note that in the last proposition the elements φ_λ are assumed to be highest weight vectors in E_λ (see (1.13)). Thus very special elements.

2.1.1 Dirac-Distributions

Let us consider the **Dirac-distribution** δ which is supported at e (the identity element of the group) and defined by

$$\langle \delta \mid \psi \rangle = \psi(e),$$

where $\psi \in C_0^\infty(K)$. We note that the characters $\{\chi_{\pi_\lambda}\}$, $\chi_{\pi_\lambda} = \mathrm{trace}(\pi_\lambda)$, form an orthonormal basis for the space of $\mathrm{Ad}\,K$-invariant L^2-functions on K. χ_{π_λ} is the unique $\mathrm{Ad}\,K$-invariant function in E_λ. We obtain

$$\delta = \sum_{\lambda \in L \cap \overline{C}} d_{\pi_\lambda} \chi_{\pi_\lambda},$$

where $d_{\pi_\lambda} = \dim \pi_\lambda$. Note that δ is the identity of the convolution, i.e., $u * \delta = u$ (see also [DK00], p. 237).

2.1.2 Truncated Dirac-Distributions

In proposition 2.2 we assumed that the element φ_λ in the Fourier expansion $\sum_{\lambda \in L \cap \Omega} \varphi_\lambda$ are highest weight vectors.

Let π_λ be a continuous irreducible representation of K on the Hilbert space $\mathcal{H}_\pi$ and u_λ be a highest weight vector of π_λ, provided with a K-invariant hermitian inner product $(\cdot, \cdot)$. We set

$$\Psi_\lambda(y) = (\pi_\lambda(y^{-1})u_\lambda, u_\lambda)$$

where $y \in K$. Since the Haar measure is an invariant measure ([DK00], p. 184) we obtain

$$\int\limits_K \Psi_\lambda(yxy^{-1})dy = \int\limits_K \Psi_\lambda(ykx(yk)^{-1})dy = \int\limits_K \Psi_\lambda(ykxk^{-1}y^{-1})dy.$$

Then $x \mapsto \int_K \Psi_\lambda(yxy^{-1})\, dy$ is an $\mathrm{Ad}\,K$-invariant function in E_λ, hence proportional to $\overline{\chi_{\pi_\lambda}}$ and we obtain

$$\int_K \Psi_\lambda(yxy^{-1})\, dy = \frac{1}{d_{\pi_\lambda}} \overline{\chi_{\pi_\lambda}}(x), \tag{2.12}$$

(see [KV79] for more details). We define

$$\Psi_\Omega := \sum_{\lambda \in L \cap \Omega} d_{\pi_\lambda}^2 \Psi_\lambda \in \mathcal{D}'(K),$$

(see remark 2.7). We note that

1. The pull-back of Ψ_λ by $\phi : K \times K \to K$, $(x, y) \mapsto yxy^{-1}$, is

$$(\phi^* \Psi_\lambda)(x, y) = \Psi_\lambda(\phi(x, y)) = \Psi_\lambda(yxy^{-1})$$

2. The push-forward of Ψ_λ by $\omega : K \times K \to K$, $(x, y) \mapsto x$, is

$$\omega_* \Psi_\lambda(x) = \int_K \Psi_\lambda(x, y)\, dy,$$

Then

$$\omega_*\phi^*\Psi_\lambda(x) = \int_K \phi^*\Psi_\lambda(x,y)\,dy = \int_K \Psi_\lambda(\phi(x,y))\,dy = \int_K \Psi_\lambda(yxy^{-1})\,dy$$

and we get from (2.12)

$$d_{\pi_\lambda}^2 \int_K \Psi_\lambda(yxy^{-1})\,dy = d_{\pi_\lambda}\,\overline{\chi_{\pi_\lambda}}(x).$$

The **truncated Dirac-Distribution** δ_Ω is defined by the push-forward and pull-back of the distribution Ψ_Ω. Since the push-forward and pull-back are linear continuous operators we obtain

$$\delta_\Omega := \omega_*\phi^*\Psi_\Omega = \sum_{\lambda\in L\cap\Omega} d_{\pi_\lambda}\overline{\chi_{\pi_\lambda}}.$$

Remark 2.7. $d_{\pi_\lambda} = \prod_{\mu\in\Lambda_+}\frac{\langle\mu\,,\,\lambda+\rho\rangle}{\langle\mu\,,\,\rho\rangle}$, where ρ is the half sum of the positive roots (it is called **Weyl dimension formula** which holds for connected and compact group (see [DK00], Theorem 4.9.2)). Then due to the characterization theorem 1.35 we get that Ψ_Ω and δ_Ω are distribution on K.

Furthermore, due to Theorem 5.9 in [Fo95a] we have bijection between the matrix element $\Psi_\lambda \in E_\lambda$ and any vector $u \in \mathcal{H}_\pi$, in particular the highest weight vector u_λ. Hence, Ψ_λ satisfies equation (2.2) and (2.3). From proposition 2.2 we obtain that $\Psi_\Omega = \sum_{\lambda\in L\cap\Omega} d_{\pi_\lambda}^2\,\Psi_\lambda$ converges in $\mathcal{D}'_\Gamma(K)$ where $\Gamma = K\times(-i)\Omega$. In particular

$$\mathrm{WF}(\Psi_\Omega) \subset K\times(-i)\Omega$$

In the following we will study the effect of the pull-back and the push-forward of Ψ_Ω respectively, on the wave front set.

Lemma 2.8. *Let ϕ be defined as above and $u \in \mathcal{D}'_{K\times(-i\Omega)}(K)$. Then*

$$\mathrm{WF}(\phi^*u) \subset \phi^*\,\mathrm{WF}(u) \subset T^*K\times T^*K.$$

More precisely, $\phi^*\,\mathrm{WF}(u)$ is contained in the set $\widetilde{\Gamma}$ which is defined by

$$\{(x,y;\xi,\eta)\mid \exists\zeta : (yxy^{-1},\zeta)\in \mathrm{WF}(\Psi_\Omega), \xi = \mathrm{Ad}^*\,y^{-1}(\zeta), \eta = (\mathrm{Ad}^*\,x - I)\,\mathrm{Ad}^*\,y^{-1}(\zeta)\}$$

Proof. Following theorem 1.56 and using proposition 1.22 we get

$$\begin{array}{rcl} d\phi(x,y) : \mathfrak{k}\times\mathfrak{k} & \longrightarrow & \mathfrak{k} \\ (X,Y) & \mapsto & Z = d\phi(x,y)(X,Y) = \mathrm{Ad}\,y(X - Y + \mathrm{Ad}\,x^{-1}(Y)) \end{array}$$

Moreover, the graph of ϕ is given by

$$G(\phi) = \{(x,y,z)\mid z = yxy^{-1} = \phi(x,y)\}.$$

and the tangent space of this graph at (x, y, z) is given by

$$T_{x,y,z}G(\phi) = \{(X, Y, Z) \mid Z = d\phi(x, y)(X, Y) = \operatorname{Ad} y(X - Y + \operatorname{Ad} x^{-1}(Y))\}.$$

We will denote by $\xi, \eta, \zeta \in \mathfrak{k}^*$ the dual variable corresponding to $X, Y, Z \in \mathfrak{k}$. Let N_ϕ be the the normals of the map ϕ (compare theorem 1.56 and remark 1.57)

$$N_\phi = \{(z, \zeta) \mid z = yxy^{-1},\ \langle \zeta, d\phi(x, y)(X, Y)\rangle = 0,\ \forall X, Y \in \mathfrak{k}\},$$

The set C_ϕ equals

$$C_\phi = \{(x, y, z; \xi, \eta, \zeta) \mid \forall (X, Y, Z) \in T_{x,y,z}G(\phi),\ \langle \zeta, Z\rangle + \langle \xi, X\rangle + \langle \eta, Y\rangle = 0\},$$

Using the formula for $d\phi(x, y)(X, Y)$ we obtain

$$C_\phi \;=\; \left\{(x, y, z; \xi, \eta, \zeta) \mid \xi = -\operatorname{Ad}^* y^{-1}(\zeta), \eta = -\operatorname{Ad}^* xy^{-1}(\zeta) + \operatorname{Ad}^* y^{-1}(\zeta)\right\}.$$

Hence,

$$C'_\phi = \{(x, y, yxy^{-1}; \xi, \eta, \zeta) \mid \xi = \operatorname{Ad}^* y^{-1}(\zeta), \eta = (\operatorname{Ad}^* x - I) \operatorname{Ad}^* y^{-1}(\zeta)\}.$$

Then we get that $\mathrm{WF}(\phi^* u) \subset \phi^* \mathrm{WF}(u) = C'_\phi \circ \mathrm{WF}(u)$ equals

$$\{(x, y; \xi, \eta) \mid \exists \zeta : (yxy^{-1}, \zeta) \in \mathrm{WF}(u), \xi = \operatorname{Ad}^* y^{-1}(\zeta), \eta = (\operatorname{Ad}^* x - I) \operatorname{Ad}^* y^{-1}(\zeta)\}.$$

and

$$\mathrm{WF}(\phi^* u) \subset \phi^* \mathrm{WF}(u) \subset T^*K \times T^*K.$$

□

Proposition 2.9. *Set $\widetilde{\Gamma}$ as in lemma 2.8. Let ω be defined as above $u \in \mathcal{D}'_{\widetilde{\Gamma}}(K \times K)$. Then we get that*

$$\mathrm{WF}(\omega_* u) \subset K \times \operatorname{Ad}^* K(-i\Omega). \tag{2.13}$$

Proof. Following proposition 1.61 and using the calculation in example 1.23 we get

$$\begin{array}{rcl} d\omega^* : \mathfrak{k}^* & \longrightarrow & \mathfrak{k}^* \times \mathfrak{k}^* \\ \xi & \longmapsto & (\xi, 0) \end{array}$$

It follows from proposition 1.61

$$\mathrm{WF}(\omega_* u) \subset \{(x, \xi) \mid \exists y :\ (x, y, \xi, 0) \in \widetilde{\Gamma}\}$$

Hence, by definition of $\widetilde{\Gamma}$,

$$\mathrm{WF}(\omega_* u) \subset K \times \operatorname{Ad}^* K(-i\Omega)$$

□

Proposition 2.10. *The wave front set of the truncated Dirac-Distributions is contained in* $\Gamma := K \times \mathrm{Ad}^* K(-i\Omega)$ *and*

$$\delta_\Omega = \sum_{\lambda \in L \cap \Omega} d_{\pi_\lambda} \overline{\chi_{\pi_\lambda}} \qquad \textit{converges in } \mathcal{D}'_\Gamma(K). \tag{2.14}$$

Proof. Set $A := \omega_* \circ \phi^*$ and $\Psi_\Omega \in \mathcal{D}'_{(K \times -i\Omega)}(K)$. Then

$$A : \mathcal{D}'_{(K \times -i\Omega)}(K) \longrightarrow \mathcal{D}'_{(K \times \mathrm{Ad}^* K(-i\Omega))}(K)$$

is linear and continuous. Hence, (2.14) converges in $\mathcal{D}'_\Gamma(K)$. □

2.2 Wave Front Set of Convolution

The goal of this section is to show that $\mathrm{WF}(u * \delta_\Omega) \subset K \times \mathrm{Ad}^* K(-i\Omega)$. Observe that $\mathcal{D}'(K) = \mathcal{E}'(K)$.

Fact 2.11. Let $u, v \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support. Then $\mathrm{WF}(u*v)$ is contained in

$$\{(x+y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \mid (x, \xi) \in \mathrm{WF}(u),\ (y, \xi) \in \mathrm{WF}(v)\} \tag{2.15}$$

Proof. See [FJ98], p. 158 . □

Remark 2.12. The convolution in fact 2.11 is defined for distributions on $\mathbb{R}^n$. For distribution defined on K we will first express the convolution via tensor product. Next we will calculate the wave front set of the tensor product.

Let $\Phi : K \times K \longrightarrow K$ denote the smooth map $(x, y) \mapsto x \cdot y$. Then we write the convolution of $\delta_\Omega, u \in \mathcal{E}'(K)$, with the duality brackets written as integrals, as follows:

$$\begin{aligned}
\langle u * \delta_\Omega \mid \psi \rangle &= \int_K \int_K u(xy^{-1}) \delta_\Omega(y) \psi(x)\, dy dx \\
&\overset{x = zy}{=} \int_K \int_K u(z) \delta_\Omega(y) \psi(zy)\, dy dz \\
&= \int_K \int_K u(z) \delta_\Omega(y) \Phi^* \psi(z, y)\, dy dz \\
&= \langle u \otimes \delta_\Omega \mid \Phi^* \psi \rangle \\
&= \langle \Phi_*(u \otimes \delta_\Omega) \mid \psi \rangle
\end{aligned} \tag{2.16}$$

where $\psi \in C_0^\infty$, Φ^* is the pull-back of ψ by Φ, and Φ_* is the *push-forward* of $u \otimes \delta_\Omega$ by Φ. It follows that $u * \delta_\Omega = \Phi_*(u \otimes \delta_\Omega)$.

Proposition 2.13. *Let* $\delta_\Omega, u \in \mathcal{E}'(K)$ *and* $\Phi : K \times K \to K$, $(x, y) \mapsto x \cdot y$. *Then we obtain*

$$\mathrm{WF}(\Phi_*(u \otimes \delta_\Omega)) \subset K \times \mathrm{Ad}^* K(-i\Omega)$$

Proof. From proposition 1.60 we obtain that the wave front set of the tensor product satisfies

$$\begin{aligned}\mathrm{WF}(u\otimes\delta_\Omega)\subset(\mathrm{WF}(u)\times\mathrm{WF}(\delta_\Omega)) &\cup ((\mathrm{supp}(u)\times\{0\})\times\mathrm{WF}(\delta_\Omega))\\ &\cup (\mathrm{WF}(u)\times(\mathrm{supp}(\delta_\Omega)\times\{0\}))\,.\end{aligned}$$

Following proposition 1.61 and using the calculation in example 1.24 we get

$$\begin{aligned} d\Phi^*:\mathfrak{k}^* &\longrightarrow \mathfrak{k}^*\times\mathfrak{k}^*\\ \xi &\longmapsto (\mathrm{Ad}^*y(\xi),\xi).\end{aligned}$$

Hence, the wave front set of $\Phi_*(u\otimes\delta_\Omega)$ is contained in

$$\{(z,\zeta)\mid \exists x,y,\xi,\eta: z=xy=\Phi(x,y),(x,y;\xi,\eta)\in\mathrm{WF}(u\otimes\delta_\Omega),\xi=\mathrm{Ad}^*y(\zeta),\eta=\zeta\}$$

In particular, $\mathrm{WF}(\Phi_*(u\otimes\delta_\Omega))\subset K\times\mathrm{Ad}^*K(-i\Omega)$. □

Because of the relation between Fourier transformation and convolution, for a distribution $u=\sum_{\lambda\in L\cap\overline{C}}\varphi_\lambda$ on K we get

$$u*\delta_\Omega=\sum_{\lambda\in L\cap\Omega}\varphi_\lambda=u_\Omega$$

Proposition 2.14. *Let $\Gamma:=K\times\mathrm{Ad}^*K(-i\Omega)$ be a closed cone in T^*K. Then the convolution*

$$u*\delta_\Omega\qquad \textit{converges in } \mathcal{D}'_\Gamma(K)$$

In particular

$$\mathrm{WF}(u_\Omega)=\mathrm{WF}(u*\delta_\Omega)\subset K\times\mathrm{Ad}^*K(-i\Omega)$$

Proof. Due to proposition 2.10 we get that δ_Ω converges in $\mathcal{D}'_\Gamma(K)$. Moreover, we get from proposition 2.13 that $\mathrm{WF}(u*\delta_\Omega)\subset K\times\mathrm{Ad}^*K(-i\Omega)$. Finally, due the separately continuity of the map $*:\mathcal{D}'(K)\times\mathcal{D}'_\Gamma(K)\to\mathcal{D}'_\Gamma(K)$, $(u,\delta_\Omega)\mapsto\delta_\Omega*u$ (by fixing the first component) and the properties of the convolution we get that $u*\delta_\Omega$ converges in $\mathcal{D}'_\Gamma(K)$. □

2.3 Characterization of The Wave Front Set of Distributions

Finally we gather the results from the preceding discussion in the last two sections to state the following characterization theorem. The next theorem was introduced by [KV79] in the hyperfunction setting.

Theorem 2.15. *Let $u=\sum_{\lambda\in L\cap\overline{C}}\varphi_\lambda$ be a distribution on K and Ω be a closed cone contained in $\overline{C}\setminus 0$. Then the following assertions are equivalent:*

(i) $\mathrm{WF}(u)\subset K\times\mathrm{Ad}^*K(-i\Omega)$.

(ii) For every closed cone $\widetilde{\Omega}$ in $\overline{C}$ such that $\widetilde{\Omega} \cap \Omega = \emptyset$ and every $N \in \mathbb{N}$, there exists a constant $L_N > 0$ such that

$$\|\varphi_\lambda\|_{L^2} \leq L_N \,(1 + |\lambda|)^{-N}, \qquad \textit{for } \lambda \in \widetilde{\Omega} \tag{2.17}$$

Proof. First we are going to prove that (ii) implies (i). Choose $\widetilde{\Omega}$ as in (ii), then we write

$$u = u * \delta_{\widetilde{\Omega}} + u * \delta_{\overline{C} \setminus \widetilde{\Omega}},$$

set $u_{\widetilde{\Omega}} = u * \delta_{\widetilde{\Omega}} = \sum_{\lambda \in L \cap \widetilde{\Omega}} \varphi_\lambda$ and $u_{\overline{C} \setminus \widetilde{\Omega}} = u * \delta_{\overline{C} \setminus \widetilde{\Omega}} = \sum_{\lambda \in L \cap \overline{C} \setminus \widetilde{\Omega}} \varphi_\lambda$. Due to the characterization theorem 1.35 the assertion (ii) is equivalent to $u_{\widetilde{\Omega}} \in C^\infty(K)$ which implies that $\mathrm{WF}(u_{\widetilde{\Omega}}) = \emptyset$. Moreover, we obtain from proposition 2.14 that the $\mathrm{WF}(u) = \mathrm{WF}(u_{\overline{C} \setminus \widetilde{\Omega}}) \subset K \times \mathrm{Ad}^* \, K(\overline{\overline{C} \setminus \widetilde{\Omega}})$. Since we can choose $(\overline{\overline{C} \setminus \widetilde{\Omega}})$ as close to Ω as we like, then we obtain assertion (i).

On the other hand, if assertion (i) is satisfied, then the $\mathrm{WF}(u_{\widetilde{\Omega}}) = \emptyset$, hence $u_{\widetilde{\Omega}} \in C^\infty(K)$. Then assertion (ii) follows from Theorem 1.35. □

3 Restriction of Characters

In the last chapter we presented the wave front set of a distribution defined on connected, semisimple, and compact Lie groups K. In this chapter we are going to apply the results from the last chapter to achieve the restriction of characters of an irreducible unitary representation of K to a closed subgroup H.

Let K be a connected, compact, and semisimple Lie group. We denote by $\overline{C} \subset i\mathfrak{t}^*$ the closure of the dominant Weyl chamber (see definition 1.12) and $L \subset i\mathfrak{t}^*$ the weight lattice (see definition 1.15). Let λ be the highest weight of an irreducible representation π of K.

Definition 3.1. Let S be subset of a real vector space $\mathbb{R}^n$. We define a closed cone S_∞ in $\mathbb{R}^n \setminus \{0\}$ by

$$S_\infty := \left\{ y \in \mathbb{R}^n \setminus \{0\} \mid \exists\, (y_n, t_n) \subset S \times \mathbb{R}_+ \text{ such that } \lim_{n\to\infty} t_n y_n = y \text{ and } \lim_{n\to\infty} t_n = 0 \right\}.$$

The following lemma is without proof in [Ko98], Lemma 2.5.

Lemma 3.2. *Let S be a subset of a real vector space $\mathbb{R}^n$, and Y a closed cone in $\mathbb{R}^n$. Then the following two condition are equivalent.*

1. *$S_\infty \cap Y = \emptyset$.*
2. *There exists an open cone V containing Y such that $S \cap V$ is relatively compact.*

Proof. First we are going to prove that (1) implies (2). Let $Y \subset V \subset \overline{V}$ and $S_\infty \cap \overline{V} = \emptyset$. It is sufficient to show that $S \cap V$ is bounded. Assume that $S \cap V$ is unbounded, then exits a sequence $(z_n) \subset S \cap V$ such that $|z_n| \to \infty$. Consider the sequence $(z_n/|z_n|) \subset \overline{V}$, $(z_{n_j}/|z_{n_j}|) \to y \in \overline{V}$. Since $S_\infty \cap \overline{V} = \emptyset$, then $y \notin S_\infty$. On the other hand $y = \lim_{j\to\infty} t_{n_j} z_{n_j}$ where $t_{n_j} = 1/|z_{n_j}|$, then $y \in S_\infty$. Hence, $S \cap V$ is bounded.

Second we are going to prove that (2) implies (1). Due to the boundedness of $S \cap V$ we get that $S_\infty \cap Y = (S \setminus (S \cap V))_\infty \cap Y = (S \setminus V)_\infty \cap Y$. Since $(S \setminus V) \cap Y = \emptyset$, it remains to show that $(S \setminus V)_\infty \cap Y = \emptyset$. Assume that $y \in (S \setminus V)_\infty \cap Y$, it follows that $\exists (y_n, t_n) \subset (S \setminus V) \times \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n y_n = y \in Y$ and $\lim_{n\to\infty} t_n = 0$. Moreover, we get that $t_n y_n \in V$ for $n \in \mathbb{N}$ large enough. On the other hand, we have that $(y_n) \subset S \setminus V \Rightarrow (y_n) \not\subset V$, then $(t_n y_n) \not\subset V$ because V is an open cone. Hence, $(t_n y_n) \subset S \setminus V$ but this contradicts to $t_n y_n \in V$ when $n \in \mathbb{N}$ is large enough. $\square$

Definition 3.3. Given $u = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda \in \mathcal{D}'(K)$ where $\varphi_\lambda \in E_\lambda$ ($E_\lambda = E_{\pi_\lambda}$ see theorem 1.19), we define

$$\operatorname{fsupp}(u) := \left\{ \lambda \in L \cap \overline{C} \mid \varphi_\lambda \neq 0 \right\} \subset i\mathfrak{t}^*. \tag{3.1}$$

Analogous to Lemma 2.6 in [Ko98], we will prove the following lemma using wave front set of distribution instead of singular spectrum of hyperfunction.

Lemma 3.4. *Using definition 3.3, we have*

1. $\operatorname{WF}(u) \subset K \times \operatorname{Ad}^* K(-i \operatorname{fsupp}(u)_\infty) \subset T^*K \setminus 0$

2. *Assume* $\exists\, M > 0\ \forall \lambda \in \operatorname{fsupp}(u)$: $\|\varphi_\lambda\|_{L^2(K)} \geq (1+|\lambda|)^{-M}/M$. *Assume there exists a closed cone* $W \subset \overline{C} \setminus 0$ *such that* $\operatorname{WF}(u) \subset K \times \operatorname{Ad}^* K(-iW)$. *Then* $\operatorname{fsupp}(u)_\infty \subset W$.

Proof. 1) We define a closed cone $S := \operatorname{fsupp}(u)_\infty \subset i\mathfrak{t}^*$. Let Ω be an arbitrary closed cone in $i\mathfrak{t}^*$ such that $\Omega \cap S = \emptyset$. By lemma 3.2 $1 \Rightarrow 2$ we can find an open cone V containing Ω such that $V \cap \operatorname{fsupp}(u)$ is relatively compact in $i\mathfrak{t}^*$. In particular, $\Omega \cap \operatorname{fsupp}(u)$ is a finite set because $\operatorname{fsupp}(u) \subset L \cap \overline{C}$ is discrete. Hence, for every $N \in \mathbb{N}$, there exists a constant $L_N > 0$ such that

$$\|\varphi_\lambda\|_{L^2(K)} \leq L_N (1+|\lambda|)^{-N} \qquad \forall\, \lambda \in \Omega \cap (L \cap \overline{C}), \tag{3.2}$$

here $L_N := \max_{\lambda\in\Omega}(1+|\lambda|)^N \|\varphi_\lambda\|_{L^2(K)} < +\infty$. It follows from theorem 2.15 $(ii) \Rightarrow (i)$ that $\operatorname{WF}(u) \subset K \times \operatorname{Ad}^* K(-i \operatorname{fsupp}(u)_\infty)$.

2) Suppose $\operatorname{fsupp}(u)_\infty \not\subset W$, then we can find a closed cones S' and S'' and an open cone V such that

$$\begin{aligned} \emptyset \neq S'' &\subset V \subset S' \subset \overline{C}, \\ S'' &\subset \operatorname{fsupp}(u)_\infty, \qquad (3.3) \\ S' &\cap W = \emptyset. \qquad (3.4) \end{aligned}$$

By assumption,

$$\operatorname{WF}(u) \subset K \times \operatorname{Ad}^* K(-iW). \tag{3.5}$$

From (3.4) and (3.5) we obtain using theorem 2.15 $(i) \Rightarrow (ii)$ that; for every $N \in \mathbb{N}$, there exists a constant L_N such that

$$\|\varphi_\lambda\|_{L^2(K)} \leq L_N\, (1+|\lambda|)^{-N} \qquad \text{for any } \lambda \in S' \cap (L \cap \overline{C}). \tag{3.6}$$

Because $\exists\, M > 0\ \forall \lambda \in \operatorname{fsupp}(u)$ such that $\|\varphi_\lambda\|_{L^2(K)} \geq (1+|\lambda|)^{-M}/M$, we have that $\operatorname{fsupp}(u) \cap S'$ is bounded. In particular,

$$\#(\operatorname{fsupp}(u) \cap S') < \infty. \tag{3.7}$$

Hence, $\operatorname{fsupp}(u) \cap V$ is relatively compact. It follows from lemma 3.2 $(2) \Rightarrow (1)$ that $S'' \cap \operatorname{fsupp}(u)_\infty = \emptyset$. This contradicts to (3.3) and $S'' \neq \emptyset$. Hence, we get that $\operatorname{fsupp}(u)_\infty \subset W$. □

3.1 K-Characters

Let τ be a representation of K on the Hilbert space $\mathcal{H}$. The K-**multiplicity** is defined by

$$m_K(\cdot : \tau) : \widehat{K} \to \mathbb{N}_0 \cup \{\infty\}, \qquad \pi \mapsto m_K(\pi : \tau) := \dim \operatorname{Hom}_K(\pi, \tau).$$

The **asymptotic K-support** of τ, denoted by $\mathrm{AS}_K(\tau)$, is defined by

$$\begin{aligned} \mathrm{supp}_K(\tau) &:= \left\{\lambda \in L \cap \overline{C} \mid m_K(\pi_\lambda : \tau) \neq 0\right\}, && (3.8)\\ \mathrm{AS}_K(\tau) &:= \mathrm{supp}_K(\tau)_\infty. && (3.9)\end{aligned}$$

$\mathrm{AS}_K(\tau)$ is a closed cone contained in $\overline{C} \subset i\mathfrak{t}^*$ because $\mathrm{supp}_K(\tau) \subset (L \cap \overline{C})$.

We say that $m_K(\cdot : \tau)$ is of **polynomial growth**, if there exist constants C and N such that

$$m_K(\pi_\lambda : \tau) \leq C\,(1 + |\lambda|)^N \qquad \text{for } \lambda \in L \cap \overline{C}. \tag{3.10}$$

We denote by χ_π the (trace) character of the irreducible representation π of K.

Lemma 3.5. *Suppose that $m_K(\cdot : \tau)$ is of polynomial growth. Then the K-character Θ_τ^K of τ,*

$$\Theta_\tau^K = \sum_{\lambda \in L \cap \overline{C}} m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda}, \tag{3.11}$$

is well-defined as a distribution on K.

Proof. Due to characterization theorem 1.35, Θ_τ^K is a distribution if and only if there exists a positive numbers M and a constant L such that

$$\|m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda}\|_{L^2} \leq L\,(1 + |\lambda|)^M. \tag{3.12}$$

Since characters form an orthonormal basis for $L^2(K)$ (see [DK00], Theorem 4.3.4) and because of the polynomial growth of $m_K(\cdot : \tau)$ we get that

$$\begin{aligned} \|m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda}\|_{L^2}^2 &= (m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda}, m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda})_{L^2}\\ &= (m_K(\pi_\lambda : \tau))^2(\chi_{\pi_\lambda}, \chi_{\pi_\lambda})_{L^2}\\ &\overset{(3.10)}{\leq} C^2\,(1 + |\lambda|)^{2N}. \end{aligned}$$

This proves inequality (3.12) which proves that Θ_τ^K is a distribution on K. Since Θ_τ^K depends only on the equivalent class of τ, then Θ_τ^K is well-defined. □

Analogous to Proposition 2.7. in [Ko98], we will prove the following proposition using wave front set of distribution instead of singular spectrum of hyperfunction.

Proposition 3.6. *Suppose that $m_K(\cdot : \tau)$ is of polynomial growth. Then*

1. *The wave front set of the K-character Θ_τ^K of τ satisfies*

$$\mathrm{WF}(\Theta_\tau^K) \subset K \times \mathrm{Ad}^*\,K(-i\,\mathrm{AS}_K(\tau)) \subset T^*K \setminus 0 \simeq K \times \mathfrak{k}^*. \tag{3.13}$$

2. *Conversely, if $W \subset \overline{C} \setminus 0$ is a closed cone such that*

$$\mathrm{WF}(\Theta_\tau^K) \subset K \times \mathrm{Ad}^*\,K(-iW).$$

Then

$$\mathrm{AS}_K(\tau) \subset W. \tag{3.14}$$

Proof. By definition $\Theta_\tau^K = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda$, $\varphi_\lambda = m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda}$. It follows from (3.1) and (3.8), that

$$\operatorname{fsupp}(\Theta_\tau^K) = \operatorname{supp}_K(\tau). \tag{3.15}$$

In particular, $\operatorname{fsupp}(\Theta_\tau^K)_\infty = \operatorname{supp}_K(\tau)_\infty = \mathrm{AS}_K(\tau)$. We apply now lemma 3.4 (1) with $u = \Theta_\tau^K$

$$\mathrm{WF}(\Theta_\tau^K) \subset K \times \mathrm{Ad}^* K(-i \operatorname{fsupp}(\Theta_\tau^K)_\infty) = K \times \mathrm{Ad}^* K(-i\, \mathrm{AS}_K(\tau)). \tag{3.16}$$

2) Observe that $\|m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda}\|_{L^2(K)} \geq 1$ if $\lambda \in \operatorname{supp}_K(\tau)$. Using lemma 3.4 (2) and the assumption $\mathrm{WF}(\Theta_\tau^K) \subset K \times \mathrm{Ad}^* K(-iW)$ we obtain for $u = \Theta_\tau^K$ that

$$\mathrm{AS}_K(\tau) = \operatorname{fsupp}(u)_\infty \subset W. \tag{3.17}$$

This complete the proof of the proposition. □

3.2 Restriction of Characters to a Closed Subgroup

Let H be a closed subgroup of K. We write $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}} : \mathfrak{k}^* \to \mathfrak{h}^*$ for the projection dual to the inclusion of the Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{k}$. Let $\mathfrak{h}^\perp := \ker\left(\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}\right)$.

Let $\mu \in \widehat{H}$, define the set

$$\triangle(\mu) := \left\{\lambda \in L \cap \overline{C} \mid [\pi_\lambda|_H : \mu] \neq 0\right\} \tag{3.18}$$

where $[\pi_\lambda|_H : \mu]$ is the multiplicity of μ in $\pi_\lambda|_H$ (see the following remark 3.7).

Remark 3.7. Let σ be a representation of a compact group H on a finite dimensional Hilbert space V.Then σ is completely reducible, i.e., V can be written as a direct sum of $\sigma(H)$-invariant subspaces V_j, such that $\sigma|_{V_j}$ is irreducible for each j (see [DK00], Corollary 4.2.2). Furthermore,

$$\widetilde{\chi}_\sigma = \sum_{\mu \in \widehat{H}} [\sigma : \mu]\, \widetilde{\chi}_\mu,$$

is a finite sum, where $\widetilde{\chi}_\mu$ is the trace of μ, $\widetilde{\chi}_\sigma$ is the trace of σ, and $[\sigma : \mu]$ is a number which denote the multiplicity of μ in σ (see [DK00], Corollary 4.3.5).

Definition 3.8. The embedding $\iota : H \hookrightarrow K$ defines the co-normal bundle $N^*(H) := \ker\rho \simeq H \times \mathfrak{h}^\perp$, here $\rho : T^*K|_H \simeq H \times \mathfrak{k}^* \to T^*H \simeq H \times \mathfrak{h}^*$ is the natural projection $(h, \alpha) \mapsto (h, \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}(\alpha))$. The projection $(\iota')^t$ in remark 1.59 agrees with ρ in this case.

Lemma 3.9. *Let H be a closed subgroup of K and τ a representation of K in $\mathcal{H}$. Suppose that $m_K(\cdot : \tau)$ is of polynomial growth. Define the closed cone $\Gamma := K \times \mathrm{Ad}^* K(-i\,\mathrm{AS}_K(\tau))$ in $T^*K \setminus 0$. Then we obtain the following:*

1. *The K-character of τ*

$$\Theta_\tau^K = \sum_{\lambda \in L \cap \overline{C}} m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda} \qquad \textit{converges in } \mathcal{D}'_\Gamma(K) \tag{3.19}$$

2. *Assume*

$$\Gamma \cap N^*(H) = \emptyset. \tag{3.20}$$

Then $\Theta_\tau^K|_H$ *is a well-defined distribution on* H. *Moreover,*

$$\Theta_\tau^K|_H = \sum_{\lambda \in L \cap \overline{C}} m_K(\pi_\lambda : \tau)(\chi_{\pi_\lambda})|_H \qquad \textit{converges in } \mathcal{D}'(H), \tag{3.21}$$

and the wave front set of $\Theta_\tau^K|_H$ *satisfies,*

$$\mathrm{WF}(\Theta_\tau^K|_H) \subset H \times \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}(\mathrm{Ad}^* K(-i\, \mathrm{AS}_K(\tau))) \tag{3.22}$$

Proof. Let Y be an arbitrary closed cone in $i\mathfrak{t}^*$ such that $\mathrm{AS}_K(\tau) \cap Y = \emptyset$. By lemma 3.2 1 $\Rightarrow$ 2, there exists an open cone V containing Y such that $\mathrm{supp}_K(\tau) \cap V$ is relative compact in $i\mathfrak{t}^*$. In particular, $\mathrm{supp}_K(\tau) \cap V$ is finite set because $\mathrm{supp}_K(\tau)$ is discrete. This implies that, for every closed cone $W \subset \overline{C} \setminus 0$ such that $\mathrm{AS}_K(\tau) \subset \overset{\circ}{W}$, the series $\Theta_\tau^K * \delta_{\overline{C}\setminus W}$ converges in $C^\infty(K)$. Using proposition 2.14 we deduce that the series $\Theta_\tau^K * \delta_W$ converges in $\mathcal{D}'_{\Gamma_1}(K)$ where $\Gamma_1 := K \times \mathrm{Ad}^* K(-iW)$. It follows that the series $\Theta_\tau^K = \Theta_\tau^K * \delta_{\overline{C}\setminus W} + \Theta_\tau^K * \delta_W$ converges in $\mathcal{D}'_{\Gamma_1}(K)$. The assertion (3.19) follows from remark 1.54.

From assumption (3.20) and proposition 1.43 we obtain that $\Theta_\tau^K|_H$ is a well-defined distribution on H. Moreover, $\Theta_\tau^K|_H$ is the pull-back of Θ_τ^K by the inclusion $\iota : H \hookrightarrow K$. Using theorem 1.56 and remark 1.59 we obtain that $\iota^* : \mathcal{D}'_\Gamma(K) \to \mathcal{D}'_{\iota^*\Gamma}(H)$ is continuous. Hence, (3.21) holds and the wave front set of $\Theta_\tau^K|_H$ is contained in the image of the projection ρ (see definition 3.8), i.e., $\mathrm{WF}(\Theta_\tau^K|_H) \subset H \times \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}(\mathrm{Ad}^* K(-i\, \mathrm{AS}_K(\tau)))$, where $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}} : \mathfrak{k}^* \to \mathfrak{h}^*$. □

Remark 3.10. The following theorem is introduced in [Ko98], Theorem 2.8 using hyperfunctions. It was remarked in [Ko98] (remark following Theorem 2.8) that for a convergent sequence of analytic functionals (hyperfunctions) restricted to a submanifold may not converge in general. This limitation can be avoided when working, as in this thesis, with the distribution spaces $\mathcal{D}'_\Gamma(K)$.

Theorem 3.11. *Let* τ *be a representation of* K *and* H *be a closed subgroup of* K *with the inclusion* $H \hookrightarrow K$. *We assume*

$$\mathrm{AS}_K(\tau) \cap i\, \mathrm{Ad}^* K(\mathfrak{h}^\perp) = \emptyset. \tag{3.23}$$

Then we obtain the following:

1.

$$\#\,(\mathrm{supp}_K(\tau) \,\cap\, \triangle(\mu)) < \infty \tag{3.24}$$

for each $\mu \in \widehat{H}$. *If the* K*-multiplicity function* $m_K(\pi : \tau) < \infty$ *for any* $\pi \in \widehat{K}$. *Then the* H*-multiplicity function* $m_H(\mu : \tau|_H) := \mathrm{Hom}_H(\mu, \tau|_H) < \infty$ *for any* $\mu \in \widehat{H}$.

2. *If the K-multiplicity function $m_K(\cdot : \tau) : \widehat{K} \to \mathbb{N}_0$ is of polynomial growth, then so is the H-multiplicity function $m_H(\cdot : \tau|_H) : \widehat{H} \to \mathbb{N}_0$. Furthermore, the restriction of Θ^K_τ to the submanifold H is well-defined as a distribution and the resulting distribution $\Theta^K_\tau|_H$ coincides with*

$$\Theta^H_{\tau|_H} := \sum_{\mu \in \widehat{H}} m_H(\mu : \tau|_H)\widetilde{\chi}_\mu.$$

In particular, $\Theta^H_{\tau|_H}$ has the wave front set given by:

$$\mathrm{WF}\left(\Theta^H_{\tau|_H}\right) \subset H \times \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}\left(\mathrm{Ad}^* K(-i\,\mathrm{AS}_K(\tau))\right)$$

Proof. (1) Let $\mu \in \widehat{H}$. We fix a representation space $\mathcal{H}_\mu$(finite dimension). Let $\sigma = \mathrm{ind}^K_H(\mu)$ be the unitary representation of K induced by μ. We consider the representation space $\mathcal{H}_\sigma$ of σ as a subrepresentation of $L^2(K, \mathcal{H}_\mu)$ defined by,

$$u(kh) = \mu(h^{-1})u(k), \tag{3.25}$$

where $h \in H$ and $k \in K$.

Using (3.25) we obtain, for $Y \in \mathfrak{h}$,

$$\frac{d}{dt}|_{t=0}\left(u(ke^{tY})\right) = \frac{d}{dt}|_{t=0}\left(\mu(e^{-tY})u(k)\right) = -\mu_*(Y)u(k),$$

where $\mu_* : \mathfrak{h} \to \mathrm{End}(\mathcal{H}_\mu)$ is the Lie algebra representation induced by μ. Here,

$$\frac{d}{dt}|_{t=0}\left(u(ke^{tY})\right) = (k \cdot Y)u(k) =: \widetilde{Y}u(k).$$

Hence, $u \in L^2(K, \mathcal{H}_\mu)$ belongs to $\mathcal{H}_\sigma$, if and only if it satisfies, in the sense of distributions, the differential equations of first order,

$$\widetilde{Y}u + \mu_*(Y)u = 0.$$

This implies, using theorem 1.42,

$$\mathrm{WF}(u) \subset \mathrm{Char}(\widetilde{Y} + \mu_*(Y)) = \mathrm{Char}(\widetilde{Y}), \qquad \text{for } Y \in \mathfrak{h}$$

where $\mathrm{Char}(\widetilde{Y})$ denotes the characteristic variety of an operator $\widetilde{Y}$ (see (1.28)). Moreover, using remark 2.1, we get that

$$\mathrm{Char}(\widetilde{Y}) \subset K \times Y^\perp.$$

Hence,

$$\mathrm{WF}(u) \subset \bigcap_{Y\in\mathfrak{h}} \mathrm{Char}(\widetilde{Y}) \subset K \times \mathrm{Ad}^* K(\mathfrak{h}^\perp). \tag{3.26}$$

Claim 3.12. $\mathrm{AS}_K(\sigma) \subset i\,\mathrm{Ad}^* K(\mathfrak{h}^\perp)$.

Proof. Assume, to the contrary, $\mathrm{AS}_K(\sigma) \not\subset i\,\mathrm{Ad}^*K(\mathfrak{h}^\perp)$. Then there exists a closed cone $\Omega \subset \overline{C} \setminus 0$ such that $\Omega \cap i\,\mathrm{Ad}^*K(\mathfrak{h}^\perp) = \emptyset$ and $\mathrm{AS}_K(\sigma) \cap \mathring{\Omega} \neq \emptyset$. Choose $\varphi_\lambda \in E_\lambda$, $\lambda \in L \cap \Omega$, such that $\sum_{\lambda \in L\cap\Omega} \|\varphi_\lambda\|^2_{L^2(K,\mathcal{H}_\mu)} < +\infty$. Then $u := \sum_{\lambda\in L\cap\Omega} \varphi_\lambda$ converges in $L^2(K,\mathcal{H}_\mu)$, and since $\varphi_\lambda \in E_\lambda$ then $u \in \mathcal{H}_\sigma$. From (3.26) we obtain that

$$\mathrm{WF}(u) \subset K \times \mathrm{Ad}^*K(\mathfrak{h}^\perp). \tag{3.27}$$

In addition, we can choose φ_λ's such that for some $M > 0$,

$$\forall \lambda \in L \cap \Omega : \|\varphi_\lambda\|_{L^2(K)} \geq \frac{1}{M}(1+|\lambda|)^{-M}. \tag{3.28}$$

Then $L \cap \Omega \subset \mathrm{fsupp}(u)$. It follows that $\mathring{\Omega} \subset \mathrm{fsupp}(u)_\infty$. Therefore,

$$\mathrm{fsupp}(u)_\infty \not\subset i\,\mathrm{Ad}^*K(\mathfrak{h}^\perp).$$

On the other hand (3.27), (3.28), and lemma 3.4 (2) lead to the contradiction $\mathrm{fsupp}(u)_\infty \subset i\,\mathrm{Ad}^*K(\mathfrak{h}^\perp)$. □

Since $\mathrm{AS}_K(\sigma) \subset i\,\mathrm{Ad}^*K(\mathfrak{h}^\perp)$ then we obtain, using assumption (3.23), that $\mathrm{AS}_K(\sigma) \cap \mathrm{AS}_K(\tau) = \emptyset$. Therefore, using lemma 3.2 ($1 \Rightarrow 2$), $\mathrm{supp}_K(\sigma) \cap \mathrm{supp}_K(\tau)$ is relatively compact, hence finite. Using Frobenius reciprocity Theorem (see [Fo95a], Theorem 6.10) we obtain that $\mathrm{supp}_K(\sigma) = \triangle(\mu)$. This implies that $\#(\triangle(\mu) \cap \mathrm{supp}_K(\tau)) < \infty$. Moreover, we obtain

$$\begin{aligned} m_H(\mu : \tau|_H) &= \sum_{\lambda \in L\cap\overline{C}} m_K(\pi_\lambda : \tau)\,[\pi_\lambda|_H : \mu] \\ &= \sum_{\lambda \in \mathrm{supp}_K(\tau)\cap\triangle(\mu)} m_K(\pi_\lambda : \tau)\,[\pi_\lambda|_H : \mu]\,, \end{aligned} \tag{3.29}$$

is a finite sum. Since the K-multiplicity function $m_K(\pi : \tau) < \infty$ for every $\pi \in \widehat{K}$, then the H-multiplicity function $m_H(\mu : \tau|_H) < \infty$ for every $\mu \in \widehat{H}$ which proves assertion (1).

(2) Because of proposition 3.6 and (3.23) we get

$$\mathrm{WF}(\Theta^K_\tau) \cap N^*(H) \subset K \times \mathrm{Ad}^*K(-i\,\mathrm{AS}_K(\tau)) \cap \big(H \times \mathfrak{h}^\perp\big) = \emptyset.$$

Therefore, it follows from lemma 3.9 that the restriction $\Theta^K_\tau|_H$ is well-defined as a distribution on H and its wave front set satisfies

$$\mathrm{WF}(\Theta^K_\tau|_H) \subset H \times \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}(\mathrm{Ad}^*K(-i\,\mathrm{AS}_K(\tau))) \subset T^*H \setminus 0.$$

Set $\Gamma := K \times \mathrm{Ad}^*K(-i\,\mathrm{AS}_K(\tau))$ which is a closed cone in $T^*K \setminus 0$. It follows from lemma 3.9 (1) that

$$\Theta^K_\tau = \sum_{\lambda\in L\cap\overline{C}} m_K(\pi_\lambda : \tau)\chi_{\pi_\lambda} \quad \text{converges in } \mathcal{D}'_\Gamma(K).$$

Furthermore, from (3.23) and lemma 3.9 (2) we obtain

$$\Theta^K_\tau|_H \;=\; \sum_{\lambda\in L\cap\overline{C}} m_K(\pi_\lambda:\tau)(\chi_{\pi_\lambda})|_H \qquad \in \mathcal{D}'(H).$$

On the other hand we obtain

$$\Theta^K_\tau|_H \;=\; \sum_{\nu\in\widehat{H}} c_\nu\widetilde{\chi}_\nu \qquad \in \mathcal{D}'(H),$$

where $\widetilde{\chi}_\nu$ denotes the trace of $\nu\in\widehat{H}$. Observe that c_ν is of polynomial growth because $\Theta^K_\tau|_H$ is a distribution on H. We want to show: $c_\mu = m_H(\mu:\tau|_H)$ for each $\mu\in\widehat{H}$. Then the H-characters,

$$\Theta^H_{\tau|_H} = \sum_{\mu\in\widehat{H}} m_H(\mu:\tau|_H)\widetilde{\chi}_\mu \qquad \in \mathcal{D}'(H).$$

and $\Theta^K_\tau|_H = \Theta^H_{\tau|_H}$. This means we have to show

$$(\Theta^K_\tau|_H\,,\,\widetilde{\chi}_\mu) = m_H(\mu:\tau|_H) \tag{3.30}$$

where $(\cdot,\cdot)$ denotes the anti-duality brackets, anti-linear in the second variable, between $\mathcal{D}'(H)\times C^\infty_0(H)$ which is induced by the L^2 scalar product. Now

$$\begin{aligned}(\Theta^K_\tau|_H\,,\,\widetilde{\chi}_\mu) &= \sum_{\pi_\lambda\in\widehat{K}} m_K(\pi_\lambda:\tau)\,((\chi_{\pi_\lambda})|_H\,,\,\widetilde{\chi}_\mu)\\ &\overset{\text{remark 3.7}}{=} \sum_{\pi_\lambda\in\widehat{K}}\sum_{\nu\in\widehat{H}} m_K(\pi_\lambda:\tau)\,[\pi_\lambda|_H:\nu]\,(\widetilde{\chi}_\nu\,,\,\widetilde{\chi}_\mu).\end{aligned}$$

Since the characters are orthonormal basis for $L^2(H)$, then $(\widetilde{\chi}_\nu,\widetilde{\chi}_\mu)=\delta^\nu_\mu$. Furthermore,

$$\begin{aligned}(\Theta^K_\tau|_H\,,\,\widetilde{\chi}_\mu) &= \sum_{\pi_\lambda\in\widehat{K}}\sum_{\nu\in\widehat{H}} m_K(\pi_\lambda:\tau)\,[\pi_\lambda|_H:\nu]\,(\widetilde{\chi}_\nu,\widetilde{\chi}_\mu)\\ &= \sum_{\pi_\lambda\in\widehat{K}}\sum_{\nu\in\widehat{H}} m_K(\pi_\lambda:\tau)\,[\pi_\lambda|_H:\nu]\,\delta^\nu_\mu\\ &= \sum_{\pi_\lambda\in\widehat{K}} m_K(\pi_\lambda:\tau)\,[\pi_\lambda|_H:\mu]\,.\end{aligned}$$

This and (3.29) imply (3.30). Therefore $c_\mu = m_H(\mu:\tau|_H)$. Moreover we conclude that the H-multiplicity function $m_H(\cdot:\tau|_H)$ in (3.29) is of polynomial growth. Using lemma 3.9 we obtain

$$\mathrm{WF}\left(\Theta^H_{\tau|_H}\right) = \mathrm{WF}\left(\Theta^K_\tau|_H\right) \subset H\times \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}\left(\mathrm{Ad}^*K(-i\,\mathrm{AS}_K(\tau))\right)$$

this proves (2). □

References

[Be77] G. Bengel, *Wave Front Sets and Singular Supports of Convolutions*, Math. Ann., Springer-Verlag (1977), vol. 226, no. 3, p. 247-252.

[BGV92] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag (1992), Berlin Heidelberg.

[BD85] T. Bröcker and T.T. Dieck, *Representations of Compact Lie Groups*, Springer-Verlag (1985), New York Berlin Heidelberg Tokyo.

[CR69] A. Cerezo et F. Rouvier, *Solution elementaire d' un Operateur Differentiel Lineaire Invariant a Gauche sur un Groupe Lie Riel Compact et sur un Espace Homogene Reductif Compact* , Ann. Sci. E.N.S. 4 (1969), p. 561-581.

[Du96] J.J. Duistermaat, *Fourier Integral Operators*, Birkhäuser (1996), Bosten Basel Berlin.

[DK00] J.J. Duistermaat and J.A.C. Kolk, *Lie Groups*, Springer-Verlag (2000), New York Berlin Heidelberg.

[Fo89] G.B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press (1989), Princeton New Jersey.

[Fo95] G.B. Folland, *Introduction to Partial Differential Equations*, 2nd ed., Princeton University Press (1995), Princeton New Jersey.

[Fo95a] G.B. Folland, *A Course in Abstract Harmonic Analysis* , University of Washington CRC Press (1995), Boca Raton Ann Arbor London Tokyo.

[FJ98] G. Friedlander and M. Joshi, *Introduction to the Theory of Distributions*, 2nd ed., Cambridge University Press (1998).

[FH04] W. Fulton and J. Harris, *Representation theory*, A First Course, Springer-Verlag (2004).

[Hl06] B.C. Hall, *Lie Groups, Lie Algebras, and Representations, An Elementary Introduction*, 1st edition, Springer-Verlag (2006), New York Berlin Heidelberg.

[HC54] Harish-Chandra, *Representations of Semisimple Lie Groups III*, Transactions of the American Mathematical Society (1954), vol. 76, no. 2, pp. 234-253.

[He01] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, American Mathematical Society (2001).

[Ho83] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. I, Springer-Verlag (1983), reprint of the 2nd edition 1990, Berlin Heidelberg New York.

[Ho85] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. III, Springer-Verlag (1985), Berlin Heidelberg New York.

[KV79] M. Kashiwara and M. Vergne, *K-types and singular spectrum*, Lect. Notes in Math. (1979), vol. 728, Springer-Verlag, p. 177-200.

[Ki68] A.A. Kirillov, *Method of Orbits in The Theory of Unitary Representation of Lie Groups*, Functional Anal. i Prilozhen. 2 (1968), no. 1, p. 96-98.

[Ki04] A.A. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Mathematics, AMS (2004), vol. 64, Rhode Island.

[Kn86] A.W. Knapp, *Representation Theory of Semisimple Groups*, Princeton University Press (1986), Princeton New Jersey.

[Kn02] A.W. Knapp, *Lie Groups Beyond an Introduction*, 2nd edition, Progress in Mathematics, vol. 140, Birkhäuser-Verlag (2002), Boston Basel Berlin.

[Ko98] T. Kobayashi, *Discrete Decomposability of the Restriction of $A_q(\lambda)$ with Respect to Reductive Subgroups II: Micro-local Analysis and Asymptotic K-Support*, The Annals of Mathematics, 2nd Ser. (May 1998), vol. 147, no. 3, p. 709-729.

[Le97] J.M. Lee, *Riemannian Manifolds An Introduction to Curvature*, Springer-Verlag (1997), New York.

[Se65] R.T. Seeley, *Integro-Differential Operators on Vector Bundles*, Transactions of the American Mathematical Society, vol. 117 (May, 1965), p. 167-204.

[Ta86] M.E. Taylor, *Noncommutative Harmonic Analysis*, The American Mathematical Society (1986), Princeton New Jersey.

[Ta96] M.E. Taylor, *Partial Differential Equations I*, Springer-Verlag (1996), New York Berlin Heidelberg.

[Tr00] V.V. Trofimov, *Finite Subgroups of Compact Lie Groups and Maximal Exponents of Lie Algebras*, Journal of Mathematical Sciences, vol. 102, no. 6, (2000).

[We05] D. Werner, *Funktionalanalysis*, 5th edition, Springer-Verlag (2005), Berlin Heidelberg New York.

[ZE03] M. Zworski and L.C. Evans, *Lecture Notes for an Advanced Graduate Course on Semiclassical Analysis*, /math.berkeley.edu/∼zworski.

Index