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Dissertation

Transfer Operators and Zeta Functions for Spin Chains

Transferoperatoren und Zetafunktionen für Spinkettensysteme

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Introduction

Classical lattice spin systems have their origin in the description of the ferromagnetism of a solid. Empirical observations show that an electron is a spin-$\frac{1}{2}$ particle and that in a magnetic field its direction is quantised so that it orients either parallel or antiparallel to the field. One observes two phenomena: The interaction energy of parallel spins is smaller than the interaction energy of antiparallel ones. This makes the system tend towards a uniform parallel configuration. As temperature increases, the system tends to disorder. One observes a phase transition from ferromagnetism to paramagnetism: At low temperature the spins are parallel, at high temperature one has thermal noise. It was observed by E. Ising in his 1924 doctoral thesis [Is25] that a one-dimensional Ising chain with nearest neighbour interaction has no phase transition. Since that time this model attracts attention due to its applicability in many branches of science. It can be used to explain phenomena where individual elements (e.g., atoms, animals, protein folds, biological membrane, social behaviour, etc.) modify their behaviour so as to conform to the behaviour of other individuals in their vicinity. Abstractly speaking, a (classical) lattice spin system consists of a discrete space, the position space, where on each point a classical spin variable is attached. The spin variable can have very different interpretations, say as the charge of a particle, say as the number of particles present at the point, the species of particles, or as a classical spin variable with possible values "spin up" and "spin down". The position space may be interpreted as the locations of the atoms of a solid, with or even without a regular alignment structure. This setting allows to treat spin systems, lattice gas models, and alloy models from the same mathematical point of view. The particles interact with each other via a symmetric pair potential which is isotropic, i. e. it only depends on the distance of the particles, not on their absolute positions. If only the members of the same species interact, we have a Potts model [Po52]. A generalisation of the Ising model is the class of M-vector models which was introduced by H. E. Stanley in [St68a]. It allows to model one-component fluids, binary alloys, mixture processes, λ-transition in a Bose fluid, as well as ferromagnetism. An important parameter of an interaction is its range. While interactions with finite range are well-understood, even in one dimensional systems there are still open problems concerning long-range interactions. Roughly speaking, the faster the interaction decays, the easier is its mathematical treatment. However, many physically interesting interactions do not decay fast, for instance, the van der Waals potential or the Coulomb potential decay like the inverse of a polynomial. Exponentially decaying interactions are studied since M. Kac's 1966 paper [Ka66].

We interpret the sum of all interactions energies between a finite number of particles as the energy of a subconfiguration. After averaging over all possible configurations with the Boltzmann factor as weight we obtain the main object of statistical mechanics, the partition function. The partition function is an interesting object since many properties of the system, such as the total energy, free energy, entropy, and pressure, can be expressed in terms of the partition function and its derivatives. We are interested in the thermodynamic limit, i. e., we let tend the number of particles to infinity, and study the properties of the sequence of partition functions.

One method to investigate the physical system is the use of so called transfer operators. The transfer operator method consists in finding a linear operator, called the transfer operator, such that certain asymptotic properties of the partition functions can be expressed in terms of the spectrum of the operator. For interactions with finite range this method has been invented by H. Kramers and G.

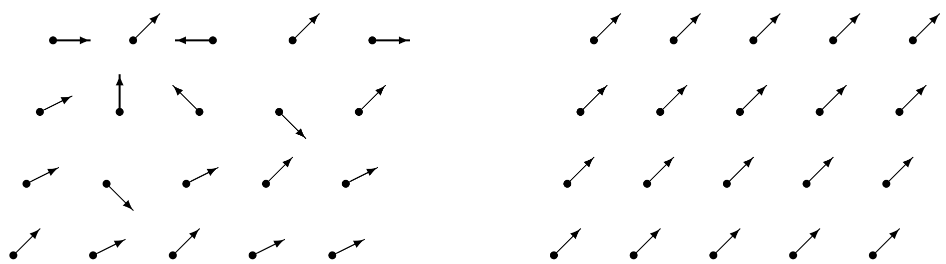


Figure 1: Phase transition: Disordered configuration (high temperature), ordered configuration (low temperature)

Wannier [KrWa41], E. Montroll [M41], and E. Ising [Is25], the operator is called a transfer matrix. For long range interactions D. Ruelle [Ru68], H. Araki [A69], G. Gallavotti and S. Miracle-Sole [GaMS70] introduced the transfer operator approach. The Ruelle-Perron-Frobenius theorem states that the Ruelle transfer operator has a positive leading eigenvalue, a corresponding positive eigenfunction, and a positive eigenmeasure. The leading eigenvalue has a physical interpretation, since its logarithm is closely related to the free energy. The leading eigenfunction of the Ruelle transfer operator together with the eigenmeasure determine the equilibrium state of the system.
In works of D. Mayer, K. Viswanathan, B. Moritz, and J. Hilgert [May76], [Vi76], [MayVi77], [ViMay77], [May80a], [Mo89], [HiMay02], [HiMay04], there are several examples of interactions known for which a so called dynamical trace formula holds, i. e., there exists a trace class operator, nowadays called the Ruelle-Mayer transfer operator, such that the partition functions can be expressed in terms of the traces of the powers of the transfer operator. As soon as an operator is found satisfying such a dynamical trace formula, the problem of computing the partition function is shifted to the functional analytic problem consisting in the determination of the spectrum.
The properties of the partition function are also studied by using a method from number theory. Putting the partition function as coefficients of a generating function one obtains a formal power series which converges under weak assumptions in a neighbourhood of zero. This function is called the dynamical zeta function and has been introduced by D. Ruelle [Ru76]. As in number theory one studies the analytic properties of zeta which imply, using Wiener-Tauber type arguments, conclusions on the mean behaviour of the coefficients. For this reasoning one needs the existence of a meromorphic continuation of zeta beyond the first pole. With methods similar to proof of the classical prime number theorem, W. Parry and M. Pollicott obtained prime orbit theorems on the distribution of prime orbits, see [PaPo90]. If there exists a transfer operator which satisfies a dynamical trace formula, then this leads to a representation of Ruelle's zeta as a quotient of Fredholm determinants of the transfer operator and hence to a meromorphic continuation of zeta to the entire complex plane. From this spectral interpretation of zeta J. Hilgert and D. Mayer obtained in [HiMay02] and [HiMay04] the existence of infinitely many equally spaced ("trivial") and infinitely many non-trivial zeros and poles along lines in the complex plane, which is a phenomenon also known from number theoretical zeta functions.
Spin chains, i. e., one-dimensional spin systems, with exponentially decaying Ising interaction have been firstly studied in [Ka66] by M. Kac via the transfer operator method. He, and in similar form also M. Gutzwiller [Gu82], introduced integral operators acting on the space of square-integrable functions on the real line. This Kac-Gutzwiller transfer operator satisfies a dynamical trace formula. D. Mayer derived in [May80a] his transfer operator for the same interaction. His transfer operator acts on a Banach space of holomorphic functions and also satisfies a (similar) dynamical trace formula. In [Mo89] it was shown that the spectra of both operators (almost) coincide. In [HiMay02] and [HiMay04] there was found a way how one explicitly relates the two operators. This construction uses essentially the Bargmann transform which provides a unitary isomorphism between the both spaces.
Motivated by these results we ask which is the class of interactions for which a dynamical trace formula holds. We analyse the known examples of Ruelle-Mayer transfer operators and determine what they have in common. We introduce a family of Ising type interactions which contains all the known Ising interactions with finite-range, superexponentially, exponentially, or polynomial-exponentially decaying distance function. We give some new examples, for instance Ruelle-Mayer transfer operators for M-vector models and Potts models, and new distance functions. We can formulate a general frame work for the construction of the Ruelle-Mayer transfer operator associated to interactions belonging to this class and prove a dynamical trace formula. For this class of interactions we investigate the dynamical zeta function and show its meromorphic continuation to the entire complex plane. Using the Bargmann transform we compute the Kac-Gutzwiller transfer operator for polynomial-exponentially decaying interactions and also for finite range interactions explicitly and study its properties.
The main contributions of this dissertation are

(i) A unification of all known examples of Ising interactions admitting a Ruelle-Mayer transfer operator which satisfies a dynamical trace formula,

(ii) Generalisations in view of the interaction: Ising type interactions, the possible spin values, and a (slightly) weaker decay of the distance function,

(iii) A concise treatment of the dynamical zeta function, in particular in the presence of a dynamical trace formula, including a proof of an Euler product expansion and meromorphic extension to the entire complex plane, and

(iv) A direct construction of the Kac-Gutzwiller transfer operator associated to a Ruelle-Mayer transfer operator, both for full shifts and matrix subshifts.

Our treatment is based on Hilbert space techniques and Schatten class operators, instead of Banach space techniques and nuclear operators used for instance in the work of D. Mayer. The Hilbert space techniques simplify many arguments concerning values of traces or determinants.
We will now explain our main results in detail and give an outline of this dissertation afterwards.

Theorem 2.13.8. *Let $F \subset \mathbb{C}$ be a bounded set, interpreted as spin values, and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift. Let ϕ be a two-body Ising type interaction with potential $q \in \mathcal{C}_b(F)$ and distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$, say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$, and interaction matrix $r \in \mathcal{C}_b(F \times F)$ with $r(x, y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$ with $s_i,\ t_j \in \mathcal{C}_b(F)$. Let $A_{(\phi)}$ be the standard observable. Then there exists an index $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geq n_0$ the Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}((\ell^2\mathbb{N})^M) \to \mathcal{F}((\ell^2\mathbb{N})^M)$ acting via*

$$(\mathcal{M}_\beta f)(z_1, \ldots, z_M) := \int_F \exp\Big(\beta q(\sigma) + \beta \sum_{l=1}^{M} s_l(\sigma)\, \langle z_l | w \rangle\Big)\, f\Big(t_1(\sigma) v + \mathbb{B} z_1, \ldots, t_M(\sigma) v + \mathbb{B} z_M\Big)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \mathbb{B}^n)^M \operatorname{trace}(\mathcal{M}_\beta)^n$. □

Before giving an outline of the proof we will explain this result and the notations. We begin with the prerequisites of the theorem, define the left hand side of the dynamical trace formula, i.e., the (dynamical) partition function $Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta)$ ($\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)})$, respectively), and show afterwards a generalisation of this result.
A one-sided one-dimensional shift is one of the most studied examples of lattice spin systems. It is defined as follows: Let F be a Hausdorff space F equipped with a finite Borel measure. We interpret F as the space of possible spin values. We consider as underlying position space the positive integers $\mathbb{N}$. Let $\tau : F^{\mathbb{N}} \to F^{\mathbb{N}},\ (\tau\xi)_i := \xi_{i+1}$ be the left shift on the space of F-valued sequences. Then $\mathbb{N}_0 \times F^{\mathbb{N}} \to F^{\mathbb{N}},\ n \cdot \xi := \tau^n \xi$ defines a semigroup action, i. e., a time-discrete dynamical system. If $\Omega \subset F^{\mathbb{N}}$ is a closed non-empty τ-invariant subspace, then Ω carries also an $\mathbb{N}_0$-action via this formula and is called a subshift of $F^{\mathbb{N}}$. We will first consider the full shift $F^{\mathbb{N}}$. Later we generalise our result to a family of subshifts, the matrix subshift, and prove a similar theorem for it.
The particles interact with each other via a two-body interaction $\phi = (\phi_\Lambda)$, indexed by all finite subsets $\Lambda \subset \mathbb{N}$, of the form

$$(1) \quad \phi_\Lambda : F^\Lambda \to \mathbb{C},\ \xi_\Lambda \mapsto \phi_\Lambda(\xi_\Lambda) := \begin{cases} -q(\xi_i) & , \text{ if } \Lambda = \{i\},\ \xi_\Lambda = (\xi_i), \\ -d(|i-j|)\, r(\xi_i, \xi_j) & , \text{ if } \Lambda = \{i, j\},\ \xi_\Lambda = (\xi_i, \xi_j),\ (i \neq j), \\ 0 & , \text{ otherwise,} \end{cases}$$

where $r : F \times F \to \mathbb{C}$ is a continuous bounded symmetric function, called the interaction matrix, $q \in \mathcal{C}_b(F)$ is a potential, and the distance function $d : \mathbb{N} \to \mathbb{C}$ belongs to (a subspace of) $\ell^1\mathbb{N}$.
A detailed analysis of the examples given in [May76], [Vi76], [MayVi77], [ViMay77], [May80a], [Mo89], [HiMay02], [HiMay04], i. e., all examples where a dynamical trace formula has been known before, yields that the distance function d can be written as

$$d(k) = \big\langle \mathbb{B}^{k-1} v \,\big|\, w \big\rangle_{\mathcal{H}_0}$$

for vectors v, w in a (separable) Hilbert space $(\mathcal{H}_0, \langle \cdot \,|\, \cdot \rangle)$ and a trace class operator $\mathbb{B} : \mathcal{H}_0 \to \mathcal{H}_0$ with operator norm $\|\mathbb{B}\| < 1$. This leads to our class of distance functions.

Definition 2.7.1. We define the subspaces $\mathcal{D}_1^{(p)} \subset \ell^1\mathbb{N}$ (for $p \in [1, \infty[$ as follows: $d \in \mathcal{D}_1^{(p)}$ if and only if there exist a bounded linear operator $\mathbb{B} : \mathcal{H}_0 \to \mathcal{H}_0$ on a Hilbert space $\mathcal{H}_0$ belonging to the Schatten class $\mathcal{S}_p(\mathcal{H}_0)$ with spectral radius $\rho_{\text{spec}}(\mathbb{B})$ less than one and vectors v, $w \in \mathcal{H}_0$ such that

$$d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \langle \mathbb{B}^{k-1} v | w \rangle_{\mathcal{H}_0}.$$

We call $(\mathbb{B}, v, w)$ a generating triple for d and $\mathbb{B}$ a generator. □

Our class includes the following subclasses of examples:

Example 2.7.7. (i) Finite range (Section 2.5): There exists $\rho_0 \in \mathbb{N}$, the range of d, such that

$$d(k) = 0 \text{ for all } k > \rho_0.$$

(ii) Superexponential (Section 2.9): Let $\gamma > 0$, $\delta > 1$ and

$$d : \mathbb{N} \to \mathbb{C},\ k \mapsto a(k) \exp(-\gamma\, k^{\delta}),$$

where $a : \mathbb{N} \to \mathbb{C}$ is of lower order[1] in the sense that $\lim_{k\to\infty} a(k) \exp(-\epsilon_1\, k^{\epsilon_2}) = 0$ for all $\epsilon_1,\ \epsilon_2 > 0$.

(iii) Polynomial-exponential (Section 2.11):

$$d : \mathbb{N} \to \mathbb{C},\ k \mapsto \lambda^k\, p(k),$$

where $p \in \mathbb{C}[z]$ is a polynomial and $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ is the decay rate.

(iv) Suitable infinite superpositions of exponentially decaying terms $\mathcal{D}_1^{(p),\Delta} \subset \mathcal{D}_1^{(p)}$ (Section 2.10):

$$d(k) = \sum_{i=1}^{\infty} c_i\, \lambda_i^k,$$

where $\lambda \in \ell^p\mathbb{N}$ and $c : \mathbb{N} \to \mathbb{C}$ such that $c\lambda : \mathbb{N} \to \mathbb{C}$, $n \mapsto c_n\, \lambda_n$ belongs to $\ell^1\mathbb{N}$.

We will see that (i) and (iii) correspond to generators $\mathbb{B}$ which are finite-rank operators, (i) and (ii) have nilpotent generators, and that the generators corresponding to (iii) are invertible. Since $\mathcal{D}_1^{(p)}$ is a complex vector space, every linear combination belongs again to $\mathcal{D}_1^{(p)}$. Every generator can be decomposed into the direct sum of its Jordan blocks. Since we require a generator to belong to the Schatten class $\mathcal{S}_p(\mathcal{H}_0)$, the occurring blocks belong either to (i), (ii), or to (iii). In the newly defined classes of part (iv) we collect the distance functions with an invertible semi-simple generator. □

We show that if a distance function belongs to $\mathcal{D}_1^{(p)}$, then it has at least exponential decay at infinity, i.e. $\limsup_{k\to\infty} \sqrt[k]{|d(k)|} < 1$, which for instance excludes the distance functions $d(k) = k^{-\alpha}$ (for some $\alpha > 0$), e.g. the Coulomb and the van der Waals potential.
We say that a symmetric function $r \in \mathcal{C}_b(F \times F)$ is of Ising type if it admits a decomposition

$$r(x, y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$$

for some continuous bounded functions $s_i,\ t_i : F \to \mathbb{C}$. In particular, we are interested in the case of F being a bounded subset of the complex numbers $\mathbb{C}$ and r of the form $r(x, y) = xy$ which is usually called the Ising model. Stanley's M-vector model [St68a] also belongs to this class of interaction matrices. If the space F of spin values is compact, one has an approximation property: The space of Ising type interaction matrices is dense. If F is a finite set, which is the most studied case, then every interaction matrix is of Ising type. Hence the famous M-states Potts model can be studied from the same mathematical point of view as the Ising model.
Next we explain the left hand side of the dynamical trace formula, the (dynamical) partition function, $Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)$ and $\widetilde{Z}^{b^{\mathbb{N}_0}}_n(\beta A_{(\phi)})$ respectively. We first introduce the dynamical partition function, which is of natural interest from the mathematical point of view. For any continuous bounded function $A : F^{\mathbb{N}} \to \mathbb{C}$ we define the dynamical partition function via

$$\widetilde{Z}^{b^{\mathbb{N}_0}}_n(A) := \int_{F^n} \exp\Big(\sum_{k=0}^{n-1} A(\tau^k(\overline{x_1 \ldots x_n})) \Big)\, d\nu(x_1) \ldots d\nu(x_n).$$

Notice that the argument $\sum_{k=0}^{n-1} A(\tau^k(\overline{x_1 \ldots x_n}))$ of the exponential in the integrand is (n-times) the average of the observable A along the closed τ-orbit through $\overline{x_1 \ldots x_n} := (x_1, \ldots, x_n, x_1, \ldots, x_n, \ldots)$.

[1] The decay estimate can be weakend, cp. Corollary 2.9.3.

In terms of statistical physics this sum is interpreted as the interaction energy of $x_1, \ldots, x_n$ and the dynamical partition function $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A)$ is the normalisation factor of the Gibbs-Boltzmann distribution $\exp\big(\sum_{k=0}^{n-1} A(\tau^k(\overline{x_1 \ldots x_n}))\big)\, d\nu(x_1) \ldots d\nu(x_n)$. Of particular interest is the so called standard observable

$$A_{(\phi)} : \Omega \to \mathbb{C},\ \xi \mapsto q(\xi_1) + \sum_{i=2}^{\infty} r(\xi_1, \xi_i)\, d(i-1), \tag{2}$$

where $r : F \times F \to \mathbb{C}$ is the interaction matrix, $q \in \mathcal{C}_b(F)$ is the potential, and $d : \mathbb{N} \to \mathbb{C}$ is the distance function of a two-body interaction ϕ (1). The observable $A_{(\phi)}$ is interpreted as the sum of two-body interactions of the particle at position one and the particles sitting at the rest of the half lattice. There is a second interpretation of the argument of the exponential in the integrand of the dynamical partition function $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A_{(\phi)})$: Think of n particles aligned on a circle and count the sum of all possible interactions. Take this sum as the energy of the n-particle configuration. In analogy to Boltzmann's distribution we include a parameter $\beta = 1/kT$, the so called inverse temperature, by replacing $A_{(\phi)}$ by $\beta\, A_{(\phi)}$. In (1.7.1) we define the (usual) physical partition function $Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)$ with periodic boundary condition and two-body interaction $\phi = (\phi_\Lambda)$ (1) to be the normalisation factor of the corresponding Boltzmann distribution. In Corollary 1.11.3 we show that for one-dimensional spin systems the both notions of partition function coincide, $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)$, which gives an interpretation from dynamical systems to the partition function and substantiates our interest in the observable $A_{(\phi)}$.

For every irreducible aperiodic measurable function $\mathbb{A} : F \times F \to \{0,1\}$, which we call the transition matrix, the set $\Omega_{\mathbb{A}} := \{\xi \in F^{\mathbb{N}} \,|\, \mathbb{A}(\xi_i, \xi_{i+1}) = 1\ \forall i \in \mathbb{N}\}$ is a closed shift-invariant subset of $F^{\mathbb{N}}$, we call a matrix subshift of $F^{\mathbb{N}}$. In the case of a finite alphabet, then $\mathbb{A}$ can be viewed as a matrix and $\Omega_{\mathbb{A}}$ is often called a subshift of finite type. One considers matrix subshifts if certain configurations are not permitted by nearest neighbour exclusion rules like "Particles of type X don't like to sit close to particles of type Y" or "Particles of type Z only occur separately". Matrix subshifts are used in coding: Suppose we are given a trajectory on some space which is partitioned into a family of pieces, each labelled by a different symbol. Following the orbit of a point one obtains a symbolic coding by writing down the symbols of the pieces the orbit meets. This process is meant to be some kind of data reduction. Then the following questions arise: Can one recover the original trajectories from the symbolic coding? Is there a description of the space of symbolic trajectories? For the last purpose one may use matrix subshifts. Note that a finer partitioning tends to result in less reduction of data and more information on the system, but needs more symbols which may cause mathematical problems. We obtain from Theorem 2.13.8 together with a certain tensoring trick the following dynamical trace formula.

Theorem 3.2.6. Let $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift. Let ϕ be a two-body Ising type interaction with potential $q \in \mathcal{C}_b(F)$ and distance function $d \in \mathcal{D}_1^{(p)}$ for some finite p, say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$, and interaction matrix $r \in \mathcal{C}_b(F \times F)$ with $r(x,y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$ with $s_i,\, t_j \in \mathcal{C}_b(F)$. Let $A_{(\phi)}$ be the standard observable (2). Then there exists an index $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geq n_0$ the iterates $\mathcal{M}_\beta^n \in \mathrm{End}(L^2(F,\nu) \hat{\otimes} \mathcal{F}((\ell^2\mathbb{N})^M))$ of the Ruelle-Mayer transfer operator

$$(\mathcal{M}_\beta f)(x; z_1, .., z_M) = \int_F \mathbb{A}_{\sigma,x} \exp\Big(\beta q(\sigma) + \beta \sum_{l=1}^{M} s_l(\sigma) \langle z_l | w \rangle\Big)\, f(\sigma; t_1(\sigma) v + \mathbb{B} z_1, .., t_M(\sigma) v + \mathbb{B} z_M)\, d\nu(\sigma)$$

satisfy the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta) = \det(1 - \mathbb{B}^n)^M\, \mathrm{trace}\, (\mathcal{M}_\beta)^n$. □

A consequence of the above Theorems 2.13.8 and 3.2.6 is the result 4.4.4 on the associated dynamical zeta function which we will explain after sketching their proofs.

The PROOF of Theorems 2.13.8 and 3.2.6 is done in steps: First we have to show that a certain power of the Ruelle-Mayer transfer operator is trace class, then we have to find an expression for the traces. Finally we compare both sides of the dynamical trace formula and show that they are equal.

For the first step we decompose the Ruelle-Mayer transfer operator as an integral of operators and show that each of them is trace class. Then we use the following folklore theorem for which we give a proof in Appendix A.

Theorem A.7.6. Let $(\mathcal{M}_y)_{y\in Y}$ be a measurable family of trace class operators on a separable Hilbert space $\mathcal{H}$ such that $\int_Y \|\mathcal{M}_y\|_{\mathcal{S}_1(\mathcal{H})}\, dy < \infty$. Then the linear operator $\mathcal{M} : \mathcal{H} \to \mathcal{H},\ \mathcal{M}f := \int_Y \mathcal{M}_y f\, dy$ is a trace class operator with

$$\operatorname{trace} \mathcal{M} = \int_Y \operatorname{trace} \mathcal{M}_y\, dy$$

□

Each summand in the decomposition of the Ruelle-Mayer transfer operator is a (generalised) composition operator, by which we mean an operator acting on a space of functions via

$$(Tf)(z) = \phi(z)\,(f \circ \psi)(z),$$

where ϕ is a scalar-valued map and ψ is a self-map of the domain where the functions are defined on. Composition operators and their spectral properties are well-studied in the literature, see for instance [Sh93]. A classical result is the Atiyah-Bott fixed point formula (see [AtBo67], [May80a]) which can be stated as follows:

Theorem 2.4.2. Let $U \subset \mathbb{C}^k$ be an open bounded complex domain. Let $\phi : U \to \mathbb{C}$ and $\psi : U \to U$ be holomorphic functions with continuous extensions to the closure $\overline{U}$ of U and, moreover, $\psi(\overline{U}) \subset U$. Then ψ has a unique fixed point $z^* \in U$ and the generalised composition operator

$$T : A^\infty(U) \to A^\infty(U),\ (Tf)(z) = \phi(z)\,(f \circ \psi)(z)$$

is nuclear of order zero with trace given by the Atiyah-Bott type fixed point formula

$$\operatorname{trace}_{A^\infty(U)} T = \frac{\phi(z^*)}{\det(1 - \psi'(z^*))}.$$

□

Here, $A^\infty(U)$ denotes the space of holomorphic functions on U which are continuous on $\overline{U}$, which is a Banach space with respect to the supremum norm. Mainly for two arguments we need to work with Hilbert space techniques and the Schatten ideals $\mathcal{S}_p(\mathcal{H}_0)$ instead of Banach spaces and nuclear operators. The first one is Theorem A.7.6 above which is not known to us in a Banach space setting. The other occurs in our treatment of dynamical zeta functions in Chapter 4.
For these reasons the Atiyah-Bott theorem has to be transfered to a Hilbert space setting. This will be done in Appendix B. It turns out that the Fock space $\mathcal{F}(\mathbb{C}^d)$ is suitable for our purposes. The Fock space is defined as the space of entire functions on $\mathbb{C}^d$ which are square-integrable with respect to a normalised Gaussian measure. We prove the following result which considers the special case of composing with an affine map.

Theorem B.3.4. Let $b \in \mathbb{C}^d$, $\mathbb{B} \in \mathrm{Gl}(d;\mathbb{C})$ with operator norm $\|\mathbb{B}\| < 1$, and $\phi : \mathbb{C}^d \to \mathbb{C}$ an entire function which can be estimated by $|\phi(z)| \le c\exp(a\,\|z\|)$ for all z. Let T be the composition operator acting via

$$(Tf)(z) = \phi(z)\, f(\mathbb{B}z + b)$$

both on the Fock space $\mathcal{F}(\mathbb{C}^d)$ and the spaces $A^\infty(B(0;r))$ for all balls $B(0;r) := \{z \in \mathbb{C}^d \mid \|z\| < r\}$ with sufficiently large radius $r > \frac{\|b\|}{1-q}$. Then $T : \mathcal{F}(\mathbb{C}^d) \to \mathcal{F}(\mathbb{C}^d)$ is a trace class operator with the Atiyah-Bott trace formula

$$\operatorname{trace}_{A^\infty(B(0;r))} T = \operatorname{trace}_{\mathcal{F}(\mathbb{C}^d)} T = \frac{\phi((1-\mathbb{B})^{-1}b)}{\det(1-\mathbb{B})}.$$

□

The previous result can be interpreted in such a way that all eigenfunctions of $T : A^\infty(B(0;r)) \to A^\infty(B(0;r))$ corresponding to non-zero eigenvalues belong to the smaller space $\mathcal{F}(\mathbb{C}^d)$, in particular the eigenfunctions extend to the entire space and fulfill a growth condition. This idea can be expanded to a proof of B.3.4.
In our case concerning the Ruelle-Mayer transfer operator the multiplication part of the composition operator is given by the function $\phi : \ell^2\mathbb{N} \to \mathbb{C},\ z \mapsto \exp(\beta\sigma\langle z|w\rangle)$. For this particular choice the previous theorem also holds true on the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ in infinitely many variables. This space can be characterised equivalently (Thm. A.4.8) either as the unique Hilbert space with reproducing

kernel $k(z,w) = \exp(\pi\langle z|w\rangle)$ or as the projective limit of the Hilbert spaces $\mathcal{F}(\mathbb{C}^d)$. It seems this equivalence has not been noticed before and will be proved in Appendix A which also contains a little introduction to reproducing kernel Hilbert spaces.
In Theorem 2.7.6 we consider the one-sided one-dimensional full shift with the pure two-body Ising interaction with the distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1) given as $d(k) = \langle \mathbb{B}^{k-1}v|w\rangle_{\ell^2\mathbb{N}}$. We will show that the Ruelle-Mayer transfer operator

$$\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\Big(\beta\sigma\langle z|w\rangle\Big)\, f(\sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = \det(1-\mathbb{B}^n)\,\mathrm{trace}\,(\mathcal{M}_\beta)^n$. Then the case of a general Ising type interaction is obtained by a superposition principle. By tensoring with $L^2(F,d\nu)$ our Theorem 2.13.8 will be extended to matrix subshifts.
To prove the fact that the trace of the Ruelle-Mayer transfer operator equals the partition function we use a general principle due to D. Mayer for which we give a representation theoretic interpretation. Let $\mathcal{B}$ be a Banach space and $\mathcal{B}'$ its dual, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$, and $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ a bounded linear operator with spectral radius $\rho_{\mathrm{spec}}(\mathbb{B}) < 1$. Define a distance function via $d(k) := \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$ for all $k \in \mathbb{N}$. Then d belongs to $\ell^1\mathbb{N}$ and

$$\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \mathcal{B},\ \xi \mapsto \sum_{k=1}^{\infty} \xi_k \mathbb{B}^{k-1} v$$

defines a bounded linear operator which satisfies $\pi_{\mathbb{B},v}(\sigma \vee \xi) = \sigma v + \mathbb{B}\pi_{\mathbb{B},v}(\xi)$ for all $\sigma \in \mathbb{C}$, $\xi \in \ell^\infty\mathbb{N}$. Here $(\sigma \vee \xi)$ denotes the sequence $(\sigma, \xi_1, \xi_2, \ldots)$. In representation theoretic terms the linear mapping $\pi_{\mathcal{B},v} : \ell^\infty\mathbb{N} \to \mathcal{B}$ intertwines the $\mathbb{N}_0$-representations

$$\alpha_1 : \mathbb{N}_0 \times \ell^\infty\mathbb{N} \to \ell^\infty\mathbb{N},\ \alpha_1(n,\xi) := (\tau')^n(\xi)$$

and

$$\alpha_2 : \mathbb{N}_0 \times \mathcal{B} \to \mathcal{B},\ \alpha_2(n,z) := \mathbb{B}^n z,$$

where $\tau' : \ell^\infty\mathbb{N} \to \ell^\infty\mathbb{N},\ \xi \mapsto (0 \vee \xi)$ is the dual operator of the left shift $\tau : \ell^1\mathbb{N} \to \ell^1\mathbb{N},\ (\tau\xi)_i := \xi_{i+1}$. Note that $\mathbb{B}$ generates the semigroup $(\mathbb{B}^n)_{n\in\mathbb{N}_0}$ which explains our notion "generating triple" in the context of our class of distance functions. Moreover, if the alphabet F is a bounded subset of $\mathbb{C}$, then the configuration space $F^{\mathbb{N}}$ is a bounded subset of $\ell^\infty\mathbb{N}$ and the standard observable $A_{(\phi)}$ (2) can be expressed as

$$A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sigma\langle \pi_{\mathbb{B},v}(\xi), w'\rangle_{\mathcal{B},\mathcal{B}'}.$$

We can show that under certain assumptions every distance function allowing this approach belongs to one of our classes $\mathcal{D}_1^{(p)}$. Let

$$\mathcal{L}_{\beta A_{(\phi)}} : \mathcal{C}_b(F^{\mathbb{N}}) \to \mathcal{C}_b(F^{\mathbb{N}}),\ (\mathcal{L}_{\beta A_{(\phi)}} f)(\xi) = \int_F \exp\big(\beta A_{(\phi)}(\sigma \vee \xi)\big)\, f(\sigma \vee \xi)\, d\nu(\sigma)$$

be the Ruelle transfer operator associated to the standard observable $\beta A_{(\phi)}$ and $C_{\pi_{\mathbb{B},v}} : \mathcal{C}_b(\mathcal{B}) \to \mathcal{C}_b(F^{\mathbb{N}}),\ f \mapsto f \circ \pi_{\mathbb{B},v}$ the composition operator associated to $\pi_{\mathbb{B},v}$. Then the previous considerations imply that (formally)

$$\mathcal{L}_{\beta A_{(\phi)}} \circ C_{\pi_{\mathbb{B},v}} = C_{\pi_{\mathbb{B},v}} \circ \mathcal{M}_\beta.$$

This shows the relation between the Ruelle-Mayer transfer operator $\mathcal{M}_\beta$ and the Ruelle operator.

The second main result of this dissertation concerns Ruelle's dynamical zeta function. It is defined as the formal power series in $z \in \mathbb{C}$

$$\zeta_R(z,\beta) := \exp\left(\sum_{n=1}^{\infty} Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta)\,\frac{z^n}{n}\right)$$

where $\beta \in \mathbb{C}$ is a parameter, i. e., zeta is the generating function of the partition functions. One easily shows that under weak assumptions zeta defines a holomorphic function in a neighbourhood of zero. A natural question concerns the existence of a meromorphic continuation discussed for instance in [MayVi77], [May80a]. We show the following result:

Corollary 4.4.4. Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift. Let ϕ be a two-body Ising interaction with distance function $d \in \mathcal{D}_1^{(1)}$ (2.7.1) of the form $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$, and potential $q \in \mathcal{C}_b(F)$. Let $\mathcal{M}_\beta : L^2(F,\nu)\hat{\otimes}\mathcal{F}(\ell^2\mathbb{N}) \to L^2(F,\nu)\hat{\otimes}\mathcal{F}(\ell^2\mathbb{N})$,

$$(\mathcal{M}_\beta f)(x,z) = \int_F \mathbb{A}_{\sigma,x}\, \exp\Big(\beta q(\sigma) + \beta\sigma\langle z|w\rangle\Big)\, f(\sigma, \sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

be the Ruelle-Mayer transfer operator defined in Theorem 3.2.6. Then there exists $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that the dynamical zeta function satisfies

$$\zeta_R(z,\beta) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n}\, \widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)})\Big) \lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M} \det_{n_0}(1 - z\underline{\lambda}^\alpha\, \mathcal{M}_\beta)^{(-1)^{|\alpha|+1}},$$

where $(\lambda_j)_{j\in\mathbb{N}}$ are the eigenvalues of $\mathbb{B}$ and $\underline{\lambda}^\alpha := \prod_{i=1}^M \lambda_i^{\alpha_i}$. The right hand side has an Euler product expansion and a meromorphic continuation to $\mathbb{C}$. □

Here, $\det_{n_0}(1 - \mathcal{M}_\beta)$ denotes the regularised determinant of order n_0, which exists, since by Theorem 3.2.6 the operator $(\mathcal{M}_\beta)^n$ is trace class for all $n \geq n_0$. Using the theory of regularised determinants, which we briefly recall in Section A.1, we can locate the poles and zeros of the dynamical zeta function.
Analogously, the same result holds for Ising type interactions (with rank M) when replacing the sequence $\underline{\lambda} = (\lambda_j)_{j\in\mathbb{N}}$ of eigenvalues of $\mathbb{B} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ by the sequence $\underline{\lambda}_M$ of eigenvalues of the M-fold direct sum $\mathbb{B}_M : (\ell^2\mathbb{N})^M \to (\ell^2\mathbb{N})^M$ of $\mathbb{B}$. Obviously, $\mathbb{B}_M$ has the same eigenvalues as $\mathbb{B}$, but with the M-fold multiplicity.
Concerning the PROOF of Corollary 4.4.4: Under our assumptions a dynamical trace formula holds. Hence we have to study generating functions of a special kind. For $u \in \mathbb{N}$, $z \in \mathbb{C}$, and G belonging to the Schatten class $\mathcal{S}_u(\mathcal{H})$ let

$$g_u(z,a,G) := \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, a_n \operatorname{trace} G^n\Big)$$

be the generating function associated to the sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. It is apparent, Prop. 4.2.1, that $g_u(\,\cdot\,,a,G)$ defines a holomorphic function in a neighbourhood of zero provided $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ is finite.
For the special choice $a_n = 1 - \lambda^n$ with $0 < |\lambda| < 1$ occurring for exponentially decaying interaction it was shown by Moritz [Mo89] that $g_1(\,\cdot\,,a,G)$ and hence zeta can be represented as a quotient of Fredholm determinants. This results easily extends to the case where $a_n = \det(1 - \Lambda^n)$ for a finite rank operator Λ. To handle the case that the dynamical trace formula only holds for almost all $n \in \mathbb{N}$ we have to use the theory of regularised determinants. We obtain the following generalisation of the result of Moritz:

Theorem 4.3.4. Let $\Lambda \in \mathcal{S}_1(\mathcal{H}_0)$ be a trace class operator[2] with $\rho_{\text{spec}}(\Lambda) < 1$ and $a_n := \det(1-\Lambda^n)$. Let $(\lambda_j)_{j\in\mathbb{N}}$ be the eigenvalues of Λ. Then for any $G \in \mathcal{S}_u(\mathcal{H})$

$$g_u(z,a,G) = \lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M} \det_u(1 - z\underline{\lambda}^\alpha\, G)^{(-1)^{|\alpha|+1}} = \frac{\lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M : |\alpha|\equiv 1(2)} \det_u(1 - z\underline{\lambda}^\alpha\, G)}{\lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M : |\alpha|\equiv 0(2)} \det_u(1 - z\underline{\lambda}^\alpha\, G)}.$$

In particular, the generating function $g_u(\,\cdot\,,a,G)$ extends to a meromorphic function on the entire $\mathbb{C}$-plane and has an Euler product expansion. □

As an immediate consequence we obtain the meromorphic continuation of the dynamical zeta function to the entire z-plane:

Corollary 4.4.2. Suppose there is a transfer operator $G \in \mathcal{S}_{n_0}(\mathcal{H})$ which satisfies the dynamical trace formula $Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi} = \det(1-\Lambda^n) \operatorname{trace} G^n$ for all $n \geq n_0$, where $\Lambda \in \mathcal{S}_1(\mathcal{H}_0)$ is a trace class operator with $\rho_{\text{spec}}(\Lambda) < 1$. Set $a_n := \det(1-\Lambda^n)$ and $a = (a_n)_{n\in\mathbb{N}}$. Then

$$\zeta_R(z) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n}\, Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}\Big)\, g_{n_0}(z,a,G)$$

[2] Later, I realised that there is no assumption needed about the spectral radius.

gives the meromorphic continuation to all of $\mathbb{C}$. □

As an immediate consequence of Theorem 4.3.4 and Corollary 4.4.2 we obtain Corollary 4.4.4.

The third main part of this dissertation concerns so called Kac-Gutzwiller transfer operators. Due to the correspondence between the Ruelle-Mayer transfer operator [May80a] and the Kac-Gutzwiller transfer operator [Gu82] for the one-dimensional Ising model with exponentially decaying interaction established in [HiMay02] and [HiMay04] we investigate the conjugates of Ruelle-Mayer transfer operators under the Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$ which provides a unitary isomorphism between the space $L^2(\mathbb{R}^n)$ of square-integrable functions and the Fock space $\mathcal{F}(\mathbb{C}^n)$. Given a Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ we call the conjugated integral operator $B^{-1} \circ \mathcal{M}_\beta \circ B$ on $L^2(\mathbb{R}^n)$ a Kac-Gutzwiller transfer operator. The Kac-Gutzwiller transfer operator for exponentially decaying Ising interactions has been constructed by M. Gutzwiller using ad hoc methods which have failed to work for other than this specific interaction. Our approach is applicable for all Ising type interactions with a finite-rank generator and works both for full and for matrix subshifts. We will compute the Kac-Gutzwiller transfer operator for polynomial-exponentially decaying interactions and also for finite range interactions explicitly and study its properties. As a case in point for Ising type interactions we consider the Potts model. We hope that our approach leads to better understanding of this kind of transfer operators.
In order to compute the Kac-Gutzwiller operator we use again the decomposition of the Ruelle-Araki-Mayer transfer operator into an integral over a family of generalised composition operators of the following type: For any $a,\, b \in \mathbb{C}^n,\ \Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with operator norm $\|\Lambda\| < 1$ we define a composition operator $\mathcal{L}_{a,b,\Lambda} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ acting via

$$(\mathcal{L}_{a,b,\Lambda} f)(z) = e^{(z|a)}\, f(\Lambda z + b).$$

It turns out that the composition operators of this kind belong to the extended Fock oscillator semigroup $E\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ which is well-understood due to its representation theoretic use. Via the Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$ the extended Fock oscillator semigroup is conjugate to the extended oscillator semigroup $E\Omega_n$ which consists of integral operators on $L^2(\mathbb{R}^n)$ with (not necessarily centered) Gaussians as kernel. Since every operator of type $\mathcal{L}_{a,b,\Lambda}$ can be written as a product of a composition operator $(C_\Lambda f)(z) = f(\Lambda z)$, a translation $(\tau_s f)(z) = f(z-s)$, and a multiplication by an exponential $(m_{\exp_s} f)(z) = e^{(z|s)}\, f(z)$, we study the conjugates of each constituent. [HiMay02] gives the conjugates of translations and of multiplication operators $B \circ \tau_r \circ B^{-1},\ B \circ m_{\exp_s} \circ B^{-1}$ both acting on $\mathcal{F}(\mathbb{C}^n)$. We also ask for the operators $B^{-1} \circ m_{\exp_a} \circ B,\ B^{-1} \circ \tau_{-b} \circ B$ both acting on $L^2(\mathbb{R}^n)$ and obtain a similar result in Proposition 5.3.6. These two results will lead to an explicit description of the extended Fock oscillator semigroup. This will be used to compute the Bargmann conjugate $B^{-1} \circ \mathcal{L}_{a,b,\Lambda} \circ B$ of $\mathcal{L}_{a,b,\Lambda}$ (Prop. 5.3.5). Then by general arguments we obtain the Kac-Gutzwiller transfer operator, first for Ising interactions, then for Ising type and, using results from Chapter 3, for matrix subshifts.

The OUTLINE of this dissertation is as follows:
In Chapter 1 we will provide some background material on the so called thermodynamical formalism for lattice spin systems. In particular we turn our attention to lattice spin systems with periodic boundary condition. For this we give a new definition which is independent of the particular semigroup which acts. The main object of interest is the partition function and its thermodynamic limit, i. e., as the number of particles tends to infinity. We will exploit all kinds of symmetries in order to simplify the computation of the partititon function. For one-dimensional lattice spin systems such considerations have led both to the invention of the Ruelle, the Ruelle-Mayer, and the Kac-Gutzwiller transfer operator. Besides establishing the connection between the dynamical and the (physical) partition function which is essentially needed for the construction of the Ruelle and the Ruelle-Mayer transfer operator our results are meant as a preparation for other types of future transfer operators.
In Chapter 2 we review the concept of a transfer operator. We introduce Ruelle's transfer operator and state some of its properties, for instance, the Ruelle-Perron-Frobenius theorem which relates the leading eigenvalue, the corresponding positive eigenfunction, and the positive eigenmeasure to physisal quantities. The Ruelle transfer operator is in general far from being a trace class operator. Since we are interested in dynamical trace formulas, we focus on the so called Ruelle-Mayer transfer operator. It occurs by restricting the Ruelle operator to a suitable invariant subspace of $\mathcal{C}_b(\Omega)$. We briefly recall

Ruelle's concept of counting traces which suggests that a larger part of the spectrum of the Ruelle transfer operator should be investigated than just its leading eigenvalue. We introduce the abstract Ruelle-Mayer transfer operator and prove its trace formula. Then we turn our attention to Ising two-body interactions and reformulate the known examples of Ruelle-Mayer transfer operators which satisfy a dynamical trace formula in our setting. The remaining part of this chapter is devoted to the proof of our main result Theorem 2.13.8.
In Chapter 3 we define the Ruelle-Mayer transfer operator for matrix subshifts. The operator was known in the case of a finite alphabet F. We eliminate this requirement and prove our second main result Theorem 3.2.6. For this purpose we investigate the behaviour of the trace formula and the spectral properties of the Ruelle-Mayer operator under a certain tensoring with the transition matrix which is needed for the dynamical trace formula for matrix subshifts.
The main concern of Chapter 4 is the study of the dynamical zeta function in the presence of a dynamical trace formula. We use the theory of regularised determinants for the study of the generating functions of type $g_u(z, a, G)$ introduced above and finally prove the main result on the meromorphic continuation of Ruelle's zeta (Corollary 4.4.4).
In Chapter 5 we give, based on [Fo89], an introduction to the extended oscillator semigroup and its Bargmann conjugate, the extended Fock oscillator semigroup. Given a Ruelle-Mayer transfer operator the Bargmann transform leads to a corresponding Kac-Gutzwiller operator. We make use of the fact that a Ruelle-Mayer transfer operator can be decomposed into an integral over a family of composition operators each of them of type $\mathcal{L}_{a,b,\Lambda}$ as above which are contained as a small subsemigroup in the extended Fock oscillator semigroup. We apply the conjugation formulas known in the literature to this type of composition operators and obtain the corresponding Kac-Gutzwiller operator. By choosing special generating triples for the distance function this integral operator on $L^2(\mathbb{R}^n)$ has a (relatively) simple integral kernel which can be used to investigate the spectrum of the transfer operator in detail.
Appendix A contains background material from functional analysis. The first three sections recall the definition of traces and (regularised) determinants based on [GoGoKr00]. First we give an introduction to the axiomatic approach, then we study the Hilbert space setting, i. e., the trace class and the Schatten classes as an example of embedded subalgebras. In Section A.2 we provide the (Hilbert space) theory of regularised determinants as needed for the investigation of the dynamical zeta function. In Section A.3 we briefly comment on the Banach space setting and mention exemplarily nuclear operators and the Grothendieck 2/3-theorem.
For the investigation of the Ruelle-Mayer transfer operator we use the fact that the Fock space is a reproducing kernel Hilbert space. In Section A.4 we give an introduction to reproducing kernel Hilbert spaces, discuss the main examples, and focus then on the classification of Fock spaces. We end this chapter with a proof of the folklore theorem A.7.6.
The main issue of Appendix B is the investigation of composition operators and their spectral properties. In particular we are interested in those cases in which the Atiyah-Bott fixed point formula 2.4.2 holds. This question is investigated both for finite-dimensional and infinite-dimensional complex domains. Thus both appendices provide essential tools for the proof of the dynamical trace formula and are of independent interest, too.

Future prospects:
In Chapter 1 we introduce lattice spin systems over (possibly) high dimensional lattices. However, the rest of this dissertation only concerns lattice dimension equal to one due to the fact that no reasonable candidate generalising the Ruelle transfer operator is known. In order to pave the way towards higher dimensional transfer operators we rewrite the partition function by emploiting symmetries which might be useful for the direct construction of future Kac-Gutzwiller type transfer operators. Another approach might be the investigation of the dynamical partition function which could be a suitable replacement for the partition function in higher dimensions.
In Chapters 2 and 3 we study Ruelle-Mayer transfer operators for classes of interactions which have a fast decay at infinity. It would be interesting to find other approaches which allow to treat slower decaying interactions (or to show that those are not accessible via the transfer operator method). We construct the Ruelle-Mayer transfer operator as an integral over a family of composition operators of a special type. As long as distance functions with finite rank generator are concerned, these composition operators $\mathcal{L}_{a,b,\Lambda}$ form a small subsemigroup inside the extended Fock oscillator semigroup which likewise consists of trace class operators. This motivates the hope that future transfer operators may be built up from (a larger part of) the extended (Fock) oscillator semigroup.

In Chapter 4 we prove the Euler product of the dynamical zeta function in the presence of a dynamical trace formula and hence obtain a spectral interpretation of zeta's zeros and poles. Using the Kac-Gutzwiller transfer operator this offers the possibility to study zero statistics of the dynamical zeta function as done for instance in [HiMay02], i. e., statistics like the average number of zeros in a certain interval or the average spacing of two consecutive zeros.
The methods of Chapter 5 are restricted to distance functions with finite rank generator. Using methods from probability theory one might be able to treat also arbitrary generators.
In Appendix B we enter the world of generalised composition operators and their spectral properties. For the purposes of this dissertation and in particular of the construction of the Ruelle-Mayer transfer operator it is sufficient to prove the trace formula (and the trace norm formula) for composition operators of type $\mathcal{L}_{a,b,\Lambda}$ (Thm. B.4.3). The natural question concerns the general setting for this theorem.
Some results of this dissertation will appear in a joint article with J. Hilgert under the title „Meromorphic continuation of dynamical zeta functions via transfer operators".

1 Thermodynamic formalism for lattice spin systems

The purpose of the thermodynamic formalism or statistical mechanics is the understanding of a dynamical system which consists of a huge number of similar subsystems ("particles"). The microscopic properties of the subsystems and their interactions determine the macroscopic properties of the system. Think for instance of a solid consisting of a huge number of atoms. Magnetism is a property which depends on the average properties of the magnetic momenta ("spins") of the particles. One observes a macroscopic ferromagnetism if the elementary (microscopic) magnetic momenta are aligned in such a way that they do not cancel each other. Physical observations show a phase transition from ferromagnetism to paramagnetism: At low temperature the spins are parallel leading to a ferromagnetic behaviour of the solid, at high temperature one has thermal noise, i. e., a disordered spin configuration and thus a paramagnetic behaviour. E. Ising proposed a model for describing ferromagnetism of a solid, where the spins of the electrons can only take values in a set with two elements, "spin up" or "spin down". The interaction energy of a pair of parallel spins is smaller than that of a pair of antiparallel ones. This makes the system tend towards a uniform parallel configuration. As temperature increases the system tends to disorder. Ising's original model considers nearest-neighbour interactions of particles aligned on a one-dimensional lattice. Although Ising's one-dimensional model does not exhibit a phase transition, his model and its generalisations (higher dimensional lattices, other types of imposed interaction) have been applied successfully in many branches of science to explain phenomena where individual elements (e.g., atoms, animals, protein folds, biological membrane, social behaviour, etc.) modify their behaviour so as to conform to the behaviour of other individuals in their vicinity. For some biological applications we refer to [Th72, Ch. 7]. More than 12 000 papers have been published between 1969 and 1997 using the Ising model.

We will now give an outline of this chapter. Roughly, a (classical)[3] lattice spin system consists of a configuration space equipped with the structure of a dynamical system. After introducing the notion of interaction and energy we will define the partition function which is the central object of this chapter.

In Section 1.1 we define the configuration space. Given a Hausdorff space F equipped with a finite Borel measure, the full configuration space is the space $F^{\mathbb{L}}$ of F-valued functions which assign to each lattice point $i \in \mathbb{L}$ in a fixed discrete space $\mathbb{L}$ a so called spin value $\xi(i) \in F$. The position space $\mathbb{L}$ is often interpreted as a crystal. Depending on the modelled physical situation the set F can be interpreted as charge, as classical spin values $F = \{\pm 1\}$ ("spin up", "spin down"), as occupation numbers $F = \{0, 1, \ldots, n\}$, or as species of particles present at a lattice point. Thus we can treat spin systems, lattice gas models, and alloy models from the same mathematical point of view. In the literature only the case of compact F or even finite is covered which for instance excludes to model a system where a particle can have arbitrarily large charge. A non-empty closed subspace $\Omega \subset F^{\mathbb{L}}$ is called a (restricted) configuration space. In Section 1.2 we provide the configuration space with the structure of a dynamical system. Given a left semigroup action $\Gamma \times \mathbb{L} \to \mathbb{L}$, $(\gamma, i) \mapsto \gamma \cdot i$ on the position space we define a right semigroup action on the full configuration space in the natural way, i. e.,

$$\tau : \Gamma \times F^{\mathbb{L}} \to F^{\mathbb{L}},\ \tau(\gamma, \xi)(i) := \xi(\gamma \cdot i)$$

for all $i \in \mathbb{L}$. The triple $(F^{\mathbb{L}}, \Gamma, \tau)$ is called a shift system. Now, a lattice spin system is a 5-tuple

[3] "Classical" in contrast to "quantum". We only consider classical systems without mentioning it in the future.

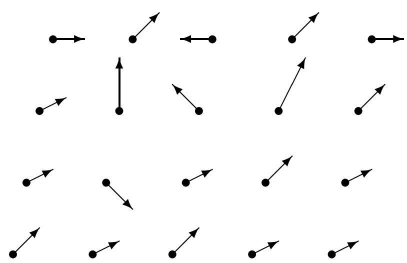


Figure 2: A two-dimensional lattice with attached spins.

$(\Omega, F, \mathbb{L}, \Gamma, \tau)$, where $(F^{\mathbb{L}}, \Gamma, \tau)$ is a shift system and $\Omega \subset F^{\mathbb{L}}$ is a closed τ-invariant subspace. An important and widely studied class of examples is the D-dimensional matrix subshift, see Def. 1.2.8. For every subset $\Lambda \subset \mathbb{L}$ one defines a restriction map $\rho_\Lambda : F^{\mathbb{L}} \to F^\Lambda,\ \xi = (\xi_i)_{i\in\mathbb{L}} \mapsto \xi|_\Lambda = (\xi_i)_{i\in\Lambda}$ and $\Omega_\Lambda := \rho_\Lambda(\Omega)$. The elements of Ω_Λ are called subconfigurations. An observable is a bounded continuous complex-valued map defined on the configuration space Ω. We call $f \in \mathcal{C}_b(\Omega)$ localised in the (finite) region $\Lambda \subset \mathbb{L}$ if $f = f_\Lambda \circ \rho_\Lambda$ for some $f_\Lambda \in \mathcal{C}_b(\Omega_\Lambda)$. In Section 1.4 we use the Stone-Weierstrass theorem to prove that the space of localised observables is dense provided the space F of spin values is compact. Many interesting observables, as for instance the energy, are built up from so called interactions. An interaction is a family of localised observables which we define in Section 1.5. An important role play interactions of the following type: Define $\phi = (\phi_\Lambda)$ via

$$\phi_\Lambda(\xi_\Lambda) = \begin{cases} -q(\xi_i) & , \text{ if } \Lambda = \{i\},\, \xi_\Lambda = (\xi_i), \\ -\tilde{d}(i,j)\, r(\xi_i, \xi_j) & , \text{ if } \Lambda = \{i,j\},\, \xi_\Lambda = (\xi_i, \xi_j),\ (i \neq j), \\ 0 & , \text{ otherwise,} \end{cases}$$

where $r : F \times F \to \mathbb{C}$ and $\tilde{d} : \mathbb{L} \times \mathbb{L} \to \mathbb{C}$ are symmetric functions, $q : F \to \mathbb{C}$, and $\xi_i,\ \xi_j \in F$, $\Lambda \subset \mathbb{L}$. An interaction of this form is called a two-body interaction with interaction matrix r, anisotropy matrix $\tilde{d}$ and potential q. From Section 1.8 on we will restrict our considerations to the case of two-body interactions. Two-body interactions are both simple to handle and still interesting for applications. Depending on the interaction matrix one has an Ising model, a Potts model, or one of Stanley's M-vector models.

Our approach to lattice spin systems is affected by the tradition of thermodynamics, i. e., we look at volume elements which contain a finite number of particles, investigate the localised observables, and then study the asymptotic behaviour of the localised observables as the volume tends to infinity. Average properties of certain observables can be encoded in the partition function which is of particular interest in thermodynamics. Its definition depends on a couple of things we will introduce first.

The way how a subconfiguration gets embedded into the configuration space is described by the so called boundary condition. Such an embedding should be a (partial) right inverse of the restriction map $\rho_\Lambda : \Omega \to \Omega_\Lambda$ and will be defined at least for certain subsets Λ of the position space $\mathbb{L}$. In Section 1.3 we introduce two types of such boundary conditions, the zero boundary condition as a particular example of an external field boundary condition, and for a particular class of semigroup actions the so called periodic boundary condition. In order to do this we use the orbit relation with respect to a semigroup action. The periodic boundary condition has been studied in the context of $\mathbb{Z}^D$-actions, whereas our generalisation will be applicable also in other situations, such as actions of an arbitrary abelian semigroup. We examplarily study the one-dimensional matrix subshift and give sufficient conditions on the transition matrix such that the associated matrix subshift allows a periodic boundary condition.

In Section 1.6 we define the energy of a subconfiguration $\xi_\Lambda \in \Omega_\Lambda$ if an interaction is fixed. The total energy $U_\Lambda^{b,\phi}$ consists of two parts. The inner energy comes from all interactions of subconfigurations inside this configuration and can be defined for any region $\Lambda \subset \mathbb{L}$ provided that the interaction satisfies a certain summability condition which we call compatibility. Given a boundary condition $b = (b_\Lambda : F^\Lambda \to F^{\mathbb{L}})_\Lambda$, a subconfiguration determines a configuration on the whole position space via the boundary condition. The outer part of the energy counts the interactions between the inside and its extension. We will give a sufficient growth condition imposed on the interaction which ensures the convergence of the possibly infinite sums. In the special case of zero boundary conditions the total energy is just the inner energy and the condition ensuring the absolute convergence can be weakened. For the periodic boundary condition we will find a weaker condition in Proposition 1.9.3.

Section 1.7 introduces the main object of this chapter, the so called partition function. The partition function depends on the (scaled inverse) temperature $\beta \in \mathbb{C}$, the volume element $\Lambda \subset \mathbb{L}$, and via the energy $U_\Lambda^{b,\phi}(\xi_\Lambda)$ on the microstates $\xi_\Lambda \in (b_\Lambda)^{-1}(\Omega)$ of a finite number $|\Lambda| < \infty$ of particles. The partition function is defined as follows: Let $\Lambda \subset \mathbb{L}$ be a finite subset of the position space, F a Hausdorff space with a finite Borel measure ν and let ν^Λ be the product measure on F^Λ. Given a configuration space $\Omega \subset F^{\mathbb{L}}$, an interaction ϕ which is compatible with the boundary condition b we define the partition function with boundary condition b as

$$Z_\Lambda^{b,\phi}(\beta) := \int_{b_\Lambda^{-1}(\Omega)} \exp\Big(-\beta\, U_\Lambda^{b,\phi}(\xi_\Lambda)\Big)\, d\nu^\Lambda(\xi_\Lambda).$$

The average properties of the microstates determine the partition function. Many of the thermodynamic variables of the system, such as the total energy, free energy, entropy, and pressure, can be expressed in terms of the partition function, its derivatives, and their asymptotic behaviour.
In Section 1.9 we restrict to the study of two-body interactions, see Def. 1.8.1. We determine the class of interactions which are compatible in the sense of Def. 1.6.4 with a periodic boundary condition. In Proposition 1.9.3 we formulate a sufficient condition ensuring compatibility which is easy to check, in particular if the two-body interaction is given via a distance function, an interaction matrix, and a potential. Afterwards we discuss some examples of such distance functions.
We would like to stress that the number of particles is typically thought as being huge (number of atoms in a piece of matter.) Hence all our efforts serve to simplify the computation of the partititon function. In this chapter we will see different approaches. They all consist in exploiting symmetries: Two-body interactions given via a distance function, an interaction matrix, and a potential have special properties which allow to simplify the integrand of the partition function further. In the case of an Ising spin system, see Ex. 1.8.3, with vanishing potential we obtain the representation

$$Z_{\Lambda}^{b^{\Gamma},\phi}(\beta) = \int_{(b_{\Lambda}^{\Gamma})^{-1}(\Omega)} \exp\Big(\beta \sum_{i,l\in\Lambda} \hat{t}_{i,l}^{\Gamma_{\alpha}}\, \xi_i\, \xi_l\Big)\, d\nu^{\Lambda}(\xi_{\Lambda})$$

of the partition function with periodic boundary condition $b^{\Gamma} = (b_{\Lambda}^{\Gamma})_{\Lambda}$, where $(\hat{t}_{i,l}^{\Gamma_{\alpha}})_{i,l\in\Lambda}$ is a symmetric quadratic matrix and hence the integrand should be viewed as the exponential of a quadratic form in the variables $\xi_i \in F \subset \mathbb{C}$ $(i \in \Lambda)$. This is a generalisation of the situation [HiMay02, p. 26] in the construction of the Kac-Gutzwiller integral operator. This representation only depends on the fact that we have a periodic boundary condition defined via an orbit relation. Specific information about the semigroup is not needed.
In the last two sections we specialise to one-sided $\mathbb{Z}^D$-subshifts. We use the specific semigroup structure of $\mathbb{N}_0^{\mathbb{D}}$ as a subsemigroup of $\mathbb{Z}^D$ and prove explicit formulas for the energy and the partition function. The considerations of Section 1.11 lead to the dynamical interpretation of the partition function in Corollary 1.11.3 which we explained in the introduction. We introduce the so called standard observable $A_{(\phi)}$ and show that for one-dimensional systems the energy with periodic boundary condition can be expressed in terms of $A_{(\phi)}$ and the $\mathbb{N}_0$-action. This is an essential idea in the construction both of the Ruelle transfer operator (2.1.3) and the Ruelle-Mayer transfer operator (2.3.7).
In the way of presentation we are inspired by the books [Ru78] and [May80a] which mainly deal with the lattice $\mathbb{L} \subseteq \mathbb{Z}^D$ and $\mathbb{Z}^D$-actions on it. We generalise in the following respects: We allow as position space a countable set equipped with a semigroup action. The spin variable can take values in a Hausdorff space with finite measure, the hitherto existing setting was a compact Hausdorff space. We decided to place more emphasis on the dynamical system and to give a mathematically satisfactory definition of a periodic boundary condition.

1.1 Lattice systems

A lattice spin system consists of an underlying fixed discrete space, the lattice $\mathbb{L}$, where on each lattice point $i \in \mathbb{L}$ a classical spin variable $\xi(i) \in F$ is attached. The set F can be interpreted as spin values for instance if $F = \{\pm 1\}$ has two states which are usually called "spin up" and "spin down". If one models a lattice gas the set $F = \{0, 1, \dots, n\}$ is interpreted as the number of particles present at a lattice point, so called occupation numbers. The alloy model thinks of $\xi(i) \in F$ as the species of the particle present at the lattice point $i \in \mathbb{L}$. We introduce the full and the restricted configuration space, define the restriction operation and with its help the notion of a subconfiguration.

Definition 1.1.1. (Configuration space)

(i) Let $\mathbb{L}$ be a countable set, called the *position space*, and F a Hausdorff space, called the *alphabet*. In many situations we will assume that F carries a finite Borel measure ν, called the *a priori measure*.

(ii) Let $F^{\mathbb{L}} := \prod_{i\in\mathbb{L}} F = \{f : \mathbb{L} \to F\}$ be equipped with the product topology. An element $\xi \in F^{\mathbb{L}}$ is a mapping which assigns to each lattice point $i \in \mathbb{L}$ a spin value $\xi_i = \xi(i) \in F$. We call $F^{\mathbb{L}}$ the *(full) configuration space* and its elements *configurations*. Let $f_0 \in F$ be a special element to denote an empty lattice point, i. e., $\xi(i) = f_0$ means that there is no spin attached to the point $i \in \mathbb{L}$.

(iii) Let $\emptyset \neq \Omega \subset F^{\mathbb{L}}$ be closed. We call Ω a *(restricted) configuration space* of the spin system. The elements of Ω are called *allowed* configurations. □

Think of F as the set of possible values of a classical spin variable and of $\mathbb{L}$ as the position space of a particle or as a crystal lattice. Important examples are the cases where F is compact, in particular if F is a finite set equipped with the discrete topology. In some examples the countable set $\mathbb{L}$ has a group structure which explains the notion "lattice". Think of the (restricted) configuration space Ω as defined by some constraints, e. g. if two spin values are not allowed at lattice points which are "close" to each other. In (1.2.8) we will introduce a class of non-trivial configuration spaces.
On the configuration space we have the restriction operation which we will define next. The purpose of these restriction mappings is that we want to define the total energy of a configuration as the sum of the energies coming from the interactions of all subconfigurations.

Definition 1.1.2. Let $F^{\mathbb{L}}$ be a full configuration space (1.1.1). For $\Lambda \subset M \subset \mathbb{L}$ we define *restriction maps*

$$\rho_\Lambda : F^{\mathbb{L}} \to F^\Lambda := \prod_{i\in\Lambda} F,\ \xi = (\xi_i)_{i\in\mathbb{L}} \mapsto \xi|_\Lambda = (\xi_i)_{i\in\Lambda}$$

and $\rho_{\Lambda,M} := \rho_\Lambda|_{F^M} : F^M \to F^\Lambda$. □

The image of a configuration under the restriction map ρ_Λ we call a *subconfiguration.*

Remark 1.1.3. (Subconfigurations) Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1), $\Lambda \in \mathcal{P}_f(\mathbb{L})$, where $\mathcal{P}_f(\mathbb{L}) := \{\Lambda \subset \mathbb{L};\ 0 < |\Lambda| < \infty\}$ is the set of non-empty finite subsets of $\mathbb{L}$, and ρ_Λ the restriction map (1.1.2).

(i) Let $\Omega_\Lambda := \rho_\Lambda(\Omega)$ for $\Lambda \in \mathcal{P}_f(\mathbb{L})$. We will call the elements ξ_Λ of Ω_Λ *subconfigurations.* Note that every subconfiguration $\xi_\Lambda \in \Omega_\Lambda$ can be extended to an allowed configuration: This means there exists a $\xi \in \Omega$ with $\rho_\Lambda(\xi) = \xi_\Lambda$.

(ii) The mappings ρ_Λ, $\rho_{\Lambda,M}$ are continuous and surjective and hence so are their restrictions $\rho_\Lambda|_\Omega$, $\rho_{\Lambda,M}|_{\Omega_M}$.

(iii) For $\Lambda \subset M \subset \mathbb{L}$ we clearly have $\rho_\Lambda = \rho_{\Lambda,M} \circ \rho_M$. □

We conclude this section with a little topological remark.

Remark 1.1.4. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1), $\Lambda \subset \mathbb{L}$.

(i) We have equipped $F^{\mathbb{L}}$ with the product topology. A basis of the topology are the so called cylinder sets. The topology is metrisable, see for instance [Ki98, p. 2, p. 226].

(ii) If F is compact, also the space $F^{\mathbb{L}}$ (by Tychonoff's theorem) and its images $F^\Lambda = \rho_\Lambda(F^{\mathbb{L}})$ and their closed subspaces Ω_Λ are compact. □

1.2 Shift operators

A dynamical system is a semigroup action $\tau : G \times M \to M$ where the semigroup G is interpreted as time. Thus typical examples are semigroup actions of the integers $\mathbb{Z}$, the non-negative integers $\mathbb{N}_0$, the real numbers $\mathbb{R}$, or the positive real numbers $\mathbb{R}_\geq$. Given a semigroup left action on the position space, this induces a right action on the configuration space, which equips the configuration space with the structure of a dynamical system. Such a dynamical system we call a shift system. We will discuss the question when the dynamical system leaves the restricted configuration space invariant and thus induces a dynamical system there. As an important class of examples we introduce the D-dimensional shift. A large family of non-trivial $\mathbb{Z}^D$-subshifts, the so called matrix subshifts, is given via a transition matrix $\mathbb{A} : F \times F \to \{0, 1\}$. We start with the definition of a semigroup action and a dynamical system.

Definition 1.2.1. A 3-tuple (M, G, τ) is called a *dynamical system*, if $\tau : G \times M \to M$ is a *right semigroup action* of the semigroup G on the set M, i. e., if

(i) $\tau(e, m) = m$ for all $m \in M$, if G contains a neutral element $e \in G$, and

(ii) $\tau(h, \tau(g, m)) = \tau(gh, m)$ for all $m \in M$, $g, h \in G$.

If $\tau(h, \tau(g, m)) = \tau(hg, m)$ for all $m \in M$, $g, h \in G$, then $\tau : G \times M \to M$ is a *left semigroup action.* □

In the following remark we give some examples of dynamical systems.

Remark 1.2.2. Let (M, G, τ) be a dynamical system (1.2.1).

(i) If G are the real numbers or the integers, then τ can be interpreted as the time evolution rule of M: Think of $\tau(g, m)$ being the actual state of $m \in M$ after "time $g \in G$".

(ii) As an example, let $T : M \to M$ be a map, denote by $T^n = \underbrace{T \circ \ldots \circ T}_{n-\text{times}}$ the n-th iterate of T. Then

$$\tau : \mathbb{N}_0 \times M \to M,\ (n, m) \mapsto \tau(n, m) := T^n(m) \tag{3}$$

defines a $\mathbb{N}_0$-action and $(M, \mathbb{N}_0, \tau)$ is a dynamical system. If T is invertible, then

$$\tau : \mathbb{Z} \times M \to M,\ (n, m) \mapsto \tau(n, m) := T^n(m) \tag{4}$$

defines a $\mathbb{Z}$-action. $(M, \mathbb{N}_0, \tau)$ and $(M, \mathbb{Z}, \tau)$ respectively, are called time-discrete dynamical systems induced by T, see Figure 3.

(iii) Every set M can be seen as a dynamical system: Take G to be the trivial group $\{e\}$ consisting of the neutral element. Then $\tau(e, m) := m$ for all $m \in M$ defines a G-action. □

We specialise to the case where M is a space of functions. In Remark 1.11.5 we will see that in turn every dynamical system induced by a map $T : M \to M$ can be written as an action on a space of functions.

Definition 1.2.3. (Shift) Let $F^{\mathbb{L}}$ be a full configuration space (1.1.1), Γ a semigroup, and $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ a left action of Γ on $\mathbb{L}$ (1.2.1).

(i) We have an induced right action on the space of F-valued functions on $\mathbb{L}$, also denoted by τ,

$$\tau : \Gamma \times F^{\mathbb{L}} \to F^{\mathbb{L}},\ \tau(\gamma, \xi)(i) := \xi(\gamma \cdot i)$$

for $i \in \mathbb{L}$, $\xi \in F^{\mathbb{L}}$, $\gamma \in \Gamma$, called the *shift action.* Hence $(F^{\mathbb{L}}, \Gamma, \tau)$ is a special dynamical system, called a *shift system.*

(ii) For fixed $\gamma \in \Gamma$ we have a continuous[4] map

$$\tau_\gamma : F^{\mathbb{L}} \to F^{\mathbb{L}},\ \tau_\gamma(\xi) = \tau(\gamma, \xi),$$

called the *shift operator* associated with γ.

[4]The cylinder sets are a basis of the topology, hence the continuity of τ_γ is easy to see, see [Kea91].

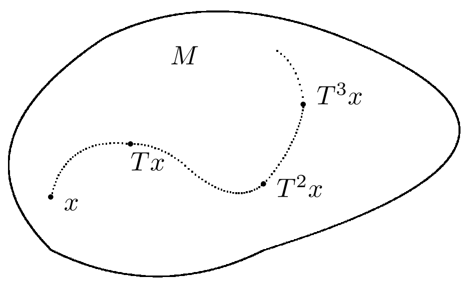


Figure 3: A time-discrete dynamical system generated by a self-map $T : M \to M$.

(iii) For fixed $\gamma \in \Gamma$ and $\Lambda \subset \mathbb{L}$ we have an induced operator $\tau_{\gamma;\gamma\cdot\Lambda} : F^{\gamma\cdot\Lambda} \to F^{\Lambda}$, such that the diagram

(5)
$$\begin{array}{ccc} F^{\mathbb{L}} & \xrightarrow{\tau_\gamma} & F^{\mathbb{L}} \\ \downarrow{\scriptstyle \rho_{\gamma\cdot\Lambda}} & & \downarrow{\scriptstyle \rho_\Lambda} \\ F^{\gamma\cdot\Lambda} & \xrightarrow{\tau_{\gamma;\gamma\cdot\Lambda}} & F^{\Lambda} \end{array}$$

commutes: For all $\xi \in F^{\mathbb{L}}$ with $\xi_{\gamma\cdot\Lambda} = \rho_{\gamma\cdot\Lambda}(\xi)$ let

$$\tau_{\gamma;\gamma\cdot\Lambda}\xi_{\gamma\cdot\Lambda} := (\rho_\Lambda \circ \tau_\gamma)(\xi).$$

This is well-defined: Given $\xi_{\gamma\cdot\Lambda} \in F^{\gamma\cdot\Lambda}$ and two $\rho_{\gamma\cdot\Lambda}$-preimages ξ, $\eta \in F^{\mathbb{L}}$, i. e., $\xi_{\gamma\cdot\Lambda} = \rho_{\gamma\cdot\Lambda}(\xi) = \rho_{\gamma\cdot\Lambda}(\eta)$, we have $(\rho_\Lambda \circ \tau_\gamma)(\xi) = (\xi(\gamma \cdot i))_{i\in\Lambda} = (\rho_\Lambda \circ \tau_\gamma)(\eta)$. □

If Γ is a group and τ is a group action, then the corresponding dynamical system $(F^{\mathbb{L}}, \Gamma, \tau)$ is invertible and hence in a certain sense deterministic. Nevertheless it can have non-invertible subsystems, a phenomenon which we will explain next.

Example 1.2.4. Let $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ be a group action, $H \leq \Gamma$ a subsemigroup, and $\Lambda \subset \mathbb{L}$. Then $\mathbb{L}_1 := H \cdot \Lambda$ is H-invariant and $\tau : H \times \mathbb{L}_1 \to \mathbb{L}_1$ is a semigroup action. We give some examples of semigroups, which are not groups.

(i) Let $\gamma \in \Gamma$ be an element of infinite order and $H = \{\gamma^n;\, n \in \mathbb{N}_0\}$.

(ii) Slightly more generally, take a finite number of commuting elements $\gamma_1, \dots, \gamma_n \in \Gamma$ of infinite order and let $H = \{\gamma_1^{\alpha_1} \dots \gamma_n^{\alpha_n} \mid \alpha_1, \dots, \alpha_n \in \mathbb{N}_0\}$.

(iii) As a concrete example let $\Gamma = \prod_{i=1}^D (n_i\mathbb{Z})$, where $n_i \geq 0$, which acts on $\mathbb{L} = \mathbb{Z}^D$ via left translation as usual. The semigroup $H = \prod_{i=1}^D (n_i\mathbb{N}_0)$ generated by the multiples $n_i\, e_i$ of the standard basis elements $e_i \in \mathbb{Z}^D$, has orbits of the form $k + H$, $k \in \mathbb{Z}^D$, and leaves invariant any set of the form $\mathbb{L}_1 = k + \mathbb{N}^D$, $k \in \mathbb{Z}^D$, i. e., a translate of the positive quadrant in $\mathbb{Z}^D$. □

An important class of lattice systems are the so called *matrix subshifts.* At first we will introduce the two- and the one-sided full shift and then their subshifts.

Example 1.2.5. (Full shift) Let $D \in \mathbb{N}$. The group $\mathbb{L} = \mathbb{Z}^D$, and hence all subsemigroups $\Gamma \leq \mathbb{Z}^D$, act on $\mathbb{L}$ by (left) translations

(6)
$$\tau : \Gamma \times \mathbb{Z}^D \to \mathbb{Z}^D,\ \tau(k, m) = k + m.$$

The induced action on the space of F-valued functions is the regular representation

(7)
$$\tau : \Gamma \times F^{\mathbb{Z}^D} \to F^{\mathbb{Z}^D},\ \tau(k, \xi)(m) = \xi(k + m)$$

for all $k, m \in \mathbb{Z}^D$. The dynamical system $(F^{\mathbb{Z}^D}, \mathbb{Z}^D, \tau)$ is called the *D-dimensional full shift.* □

An example for a non-invertible subsystem is the following, which has been mentioned already in Example 1.2.4 (iii).

Example 1.2.6. (One-sided shift) The restriction of the $\mathbb{Z}^D$-action (6) to the semigroup $\mathbb{N}_0^D \leq \mathbb{Z}^D$ leaves invariant any set of the form $\mathbb{L}_1 = k + \mathbb{N}^D$, $k \in \mathbb{Z}^D$, hence we have an induced action on $\mathbb{N}_0^D$ via

(8)
$$\tau^{>} : \mathbb{N}_0^D \times \mathbb{N}^D \to \mathbb{N}^D,\ \tau(k, m) = k + m$$

and also on the space of functions

(9)
$$\tau^{>} : \mathbb{N}_0^D \times F^{\mathbb{N}^D} \to F^{\mathbb{N}^D},\ \tau(k, \xi)(m) = \xi(k + m).$$

The dynamical system $(F^{\mathbb{N}^D}, \mathbb{N}_0^D, \tau^{>})$ is called the *one-sided D-dimensional full shift*, in contrast to the *two-sided* full shift $(F^{\mathbb{Z}^D}, \mathbb{Z}^D, \tau)$. By abuse of notation we will sometimes write τ instead of $\tau^{>}$, if the acting semigroup is clear from the context. Figure 4 illustrates the surjective, non-injective mapping $\tau_{(1,1)} : F^{\mathbb{N}^2} \to F^{\mathbb{N}^2}$, which moves the configuration one step down and one step to the left. Look for instance at the motion of the block of spins inside the dotted frame. □

In (1.1.1) we defined full and restricted configuration spaces. In (1.2.3) we have equipped the full configuration space with the structure of a dynamical system. In order to get a dynamical system on a restricted configuration space we have to assume that it is invariant under the semigroup action.

Definition 1.2.7. Let $(F^{\mathbb{L}}, \Gamma, \tau)$ be a shift system (1.2.3). A subset $\Omega \subset F^{\mathbb{L}}$ is called *τ-invariant*, if Ω is τ_γ-invariant for all $\gamma \in \Gamma$. If $\Omega \subset F^{\mathbb{L}}$ is τ-invariant and closed, then the restriction of τ to Ω

$$\tau : \Gamma \times \Omega \to \Omega$$

defines a dynamical system (Ω, Γ, τ), called a *subshift* of $(F^{\mathbb{L}}, \Gamma, \tau)$. A *lattice spin system* is a 5-tuple $(\Omega, F, \mathbb{L}, \Gamma, \tau)$, where $(F^{\mathbb{L}}, \Gamma, \tau)$ is a shift system (1.2.3) and $\Omega \subset F^{\mathbb{L}}$ is a closed τ-invariant subspace. □

As an example we will now define a family of non-trivial subshifts of the shift systems $(F^{\mathbb{N}^D}, \mathbb{N}_0^D, \tau^>)$ and $(F^{\mathbb{Z}^D}, \mathbb{Z}^D, \tau)$ introduced in Example 1.2.6. We only use the property of $\mathbb{Z}^D$ that every point $i \in \mathbb{Z}^D$ has a finite number of direct neighbours, hence this definition can be extended to more general lattices. Given a function $\mathbb{A} : F \times F \to \{0, 1\}$ which assigns to a pair of spin values a "allowed" or a "not allowed", a configuration is allowed if all pairs of spins at adjacent positions are allowed. Think of $\mathbb{A}$ as a nearest-neighbour exclusion rule in the sense explained in the introduction.
Such so called matrix subshifts arise for instance when a dynamical system where not all transitions are allowed is encoded into a symbolic dynamical system. In Section 3.4 we will introduce a new modelling of the so called hard rods model as a matrix subshift. For other examples we refer to [Ki98, 1.2].

Definition 1.2.8. (Matrix subshift) In continuation of Example 1.2.5:

(i) A map $\mathbb{A} : F \times F \to \{0, 1\}$ is called a *transition matrix* or *transition rule*.

(ii) A configuration $\xi \in F^{(\mathbb{Z}^D)}$ is called *allowed* if

$$\mathbb{A}(\xi_i, \xi_{i+e}) = 1$$

for all $i \in \mathbb{Z}^D$, $e \in \{u \in \mathbb{N}^D; \|u\| = 1\}$, where $\|\cdot\|$ is the standard euclidean norm on $\mathbb{R}^D$. We denote the set of allowed configurations by $\Omega_{\mathbb{A}}$.

(iii) Clearly, $\Omega_{\mathbb{A}}$ is τ-invariant and closed in $F^{\mathbb{Z}^D}$. The dynamical system $(\Omega_{\mathbb{A}}, \mathbb{Z}^D, \tau)$ is called the *two-sided D-dimensional matrix subshift.* The set F is called the *alphabet* of the shift.

(iv) Similarly, let $\tau^>$ be as in (8) of (1.2.6). Then the restriction $\Omega_{\mathbb{A}}^> := \rho_{\mathbb{N}^D}(\Omega_{\mathbb{A}})$ is $\tau^>$-invariant and closed in $F^{\mathbb{N}^D}$, and $(\Omega_{\mathbb{A}}^>, \mathbb{N}_0^D, \tau^>)$ is called the *one-sided D-dimensional matrix subshift.* □

The following remark gives a visualisation of the set of allowed configurations of a matrix subshift as paths in a directed graph. This graph theoretic interpretation is helpful for many ergodic problems related to matrix subshifts.

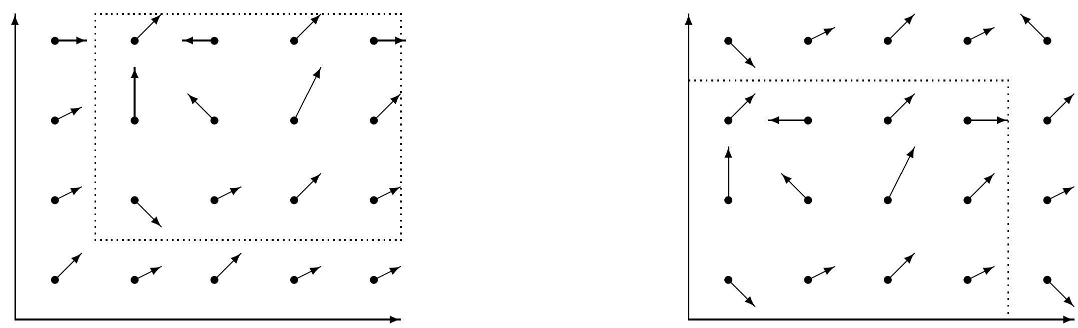
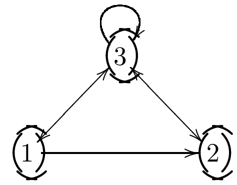


Figure 4: The one-sided two-dimensional shift (See 1.2.6).

Remark 1.2.9. (Matrix subshift) A path in a directed graph is a sequence of vertices which are linked by edges of the graph. The length $\ell(\gamma)$ of a path γ is the number of its edges. One can visualise the configuration space $\Omega_{\mathbb{A}}$ of the one-dimensional matrix subshift $(\Omega_{\mathbb{A}}, \mathbb{Z}, \tau)$ (1.2.8) as the set of (two-sided infinite) paths in the directed graph with vertices F and edges from x to y iff $\mathbb{A}(x, y) = 1$. Similarly, $\Omega_{\mathbb{A}}^{>}$ consists of the one-sided infinite paths in this graph. Consider for instance the following one-dimensional two-sided matrix subshift defined by the data $F = \{1, 2, 3\}$ and the transition matrix $\mathbb{A} = \Big(\mathbb{A}(i,j)\Big)_{i,j=1,2,3} = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ corresponding to

(10)



We will assume that the graph is strongly connected, which means by definition that for each pair of vertices one can find a path connecting them. In this case the transition matrix $\mathbb{A}$ is called *irreducible.* For each vertex $x \in F$ we define its period to be the greatest common divisor (gcd) of the length of closed paths through x:

$$\text{period}(x) := \gcd \{\ell(\gamma) \,|\, \gamma \text{ connects } x \text{ with itself}\}.$$

A vertex $x \in F$ is called aperiodic if $\text{period}(x) = 1$. A graph is called aperiodic if all its vertices are aperiodic. The example (10) shows an aperiodic graph. A transition matrix $\mathbb{A}$ is called *aperiodic* if its associated graph is aperiodic. □

Whereas for our main example, the matrix subshift, the shift invariance is obvious, the general situation is much more difficult. Let $(F^{\mathbb{L}}, \Gamma, \tau)$ be a shift system (1.2.3). We would like to determine which subspaces $\Omega \subset F^{\mathbb{L}}$ are τ-invariant. The τ-invariance of Ω clearly imposes some constraints on the "local" objects $\Omega_\Lambda = \rho_\Lambda(\Omega)$ $(\Lambda \subset \mathbb{L})$, where $(\rho_\Lambda)_{\Lambda \subset \mathbb{L}}$ is the family of restriction maps (1.1.2). For every $\Lambda \subset \mathbb{L}$ we have a diagram which is analogous to (5) in Definition 1.2.3

(11)
$$\begin{array}{ccccccc}
\Omega & \hookrightarrow & F^{\mathbb{L}} & \xrightarrow{\tau_\gamma} & F^{\mathbb{L}} & \hookleftarrow & \Omega \\
\downarrow{\scriptstyle \rho_{\gamma\cdot\Lambda}} & & \downarrow{\scriptstyle \rho_{\gamma\cdot\Lambda}} & & \downarrow{\scriptstyle \rho_\Lambda} & & \downarrow{\scriptstyle \rho_\Lambda} \\
\Omega_{\gamma\cdot\Lambda} & \hookrightarrow & F^{\gamma\cdot\Lambda} & \xrightarrow{\tau_{\gamma;\gamma\cdot\Lambda}} & F^{\Lambda} & \hookleftarrow & \Omega_\Lambda
\end{array}$$

where the middle square is commutative by the definition of $\tau_{\gamma;\gamma\cdot\Lambda}$, see Def. 1.2.3 (iii). As a necessary condition for the τ-invariance of Ω one has the commutativity of the diagram (11):

Proposition 1.2.10. *Let $(F^{\mathbb{L}}, \Gamma, \tau)$ be a shift system (1.2.3), $\gamma \in \Gamma$, $\tau_\gamma : F^{\mathbb{L}} \to F^{\mathbb{L}}$ and $\tau_{\gamma;\gamma\cdot\Lambda} : F^{\gamma\cdot\Lambda} \to F^{\Lambda}$ be as in Definition 1.2.3. Let $\Omega \subset F^{\mathbb{L}}$ be a non-empty subset. Then:*

(i) *If $\Omega \subset F^{\mathbb{L}}$ is τ_γ-invariant, i. e., $\tau_\gamma(\Omega) \subset \Omega$, then $\tau_{\gamma;\gamma\cdot\Lambda}(\Omega_{\gamma\cdot\Lambda}) \subset \Omega_\Lambda$ for all $\Lambda \subset \mathbb{L}$.*

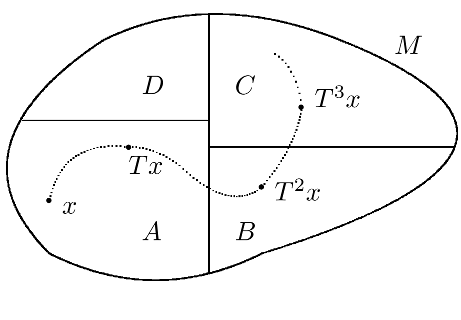


Figure 5: Coding a dynamical system: Given a map $T : M \to M$, and a partioning of $M = A \cup B \cup C \cup D$, then for example the sequence x, Tx, T^2x, T^3x gets encoded by $AABC$.

(ii) *Let $\tau_\gamma(\Omega) \subset \Omega$. The map $\tau_\gamma|_\Omega : \Omega \to \Omega$ is surjective if and only if $\tau_{\gamma;\gamma\cdot\Lambda}(\Omega_{\gamma\cdot\Lambda}) = \Omega_\Lambda$ for all $\Lambda \subset \mathbb{L}$.*

Proof. If $\Omega \subset F^{\mathbb{L}}$ is τ_γ-invariant, then by the definition of $\Omega_{\gamma\cdot\Lambda} = \rho_{\gamma\cdot\Lambda}(\Omega)$ in (1.1.3) and of $\tau_{\gamma;\gamma\cdot\Lambda}$ in (1.2.3) we have

$$\tau_{\gamma;\gamma\cdot\Lambda}(\Omega_{\gamma\cdot\Lambda}) = (\tau_{\gamma;\gamma\cdot\Lambda} \circ \rho_{\gamma\cdot\Lambda})(\Omega) = (\rho_\Lambda \circ \tau_\gamma)(\Omega) \subseteq \rho_\Lambda(\Omega) = \Omega_\Lambda.$$

The equality $\tau_{\gamma;\gamma\cdot\Lambda}(\Omega_{\gamma\cdot\Lambda}) = \Omega_\Lambda$ holds if $\tau_\gamma|_\Omega : \Omega \to \Omega$ is surjective. For the converse let $\Lambda = \mathbb{L}$. □

Now we consider the converse of Proposition 1.2.10, i. e., the problem whether the τ-invariance of Ω can be guaranteed by local constraints on the $\Omega_\Lambda \subset F^\Lambda$. Let $V_\Lambda \subset F^\Lambda$ be *any* family of closed subspaces parametrised by $\Lambda \in \mathcal{P}_f(\mathbb{L})$. We define

$$\tilde{\Omega}((V_\Lambda)_\Lambda) := \{\xi \in F^{\mathbb{L}} \mid \rho_\Lambda(\xi) \in V_\Lambda \ \forall \Lambda \in \mathcal{P}_f(\mathbb{L})\}. \tag{12}$$

Under some constraints on the "local" objects V_Λ, $\Lambda \subset \mathbb{L}$, the resulting $\tilde{\Omega}((V_\Lambda)_\Lambda) \subset F^{\mathbb{L}}$ is τ-invariant:

Proposition 1.2.11. *Let $(F^{\mathbb{L}}, \Gamma, \tau)$ be a shift system (1.2.3), $\gamma \in \Gamma$ and $\Omega \subset F^{\mathbb{L}}$ be a non-empty subset.*

(i) *If $\Omega \subset F^{\mathbb{L}}$ is τ-invariant and $\Omega_\Lambda := \rho_\Lambda(\Omega)$ for all finite $\Lambda \subset \mathbb{L}$, then $\Omega \subset \tilde{\Omega}((\Omega_\Lambda)_\Lambda)$, the latter defined in (12).*

(ii) *Let $(V_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ be any family of closed subspaces $V_\Lambda \subset F^\Lambda$ such that $\tau_{\gamma;\gamma\cdot\Lambda}(V_{\gamma\cdot\Lambda}) \subset V_\Lambda$ for all finite $\Lambda \subset \mathbb{L}$, then τ_γ leaves $\tilde{\Omega}((V_\Lambda)_\Lambda)$ invariant.*

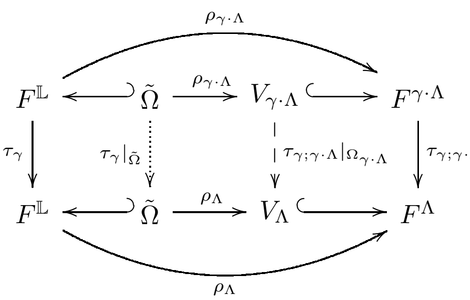
(iii) *In particular, if in addition $\Omega = \tilde{\Omega}((V_\Lambda)_\Lambda)$, then $\tau_\gamma|_\Omega : \Omega \to \Omega$ leaves Ω invariant.*

Proof. Let $\xi \in \tilde{\Omega}((V_\Lambda)_\Lambda)$. Then

$$\rho_\Lambda(\tau_\gamma(\xi)) = \tau_{\gamma;\gamma\cdot\Lambda}(\rho_{\gamma\cdot\Lambda}(\xi)) \in \tau_{\gamma;\gamma\cdot\Lambda}(V_{\gamma\cdot\Lambda}) \subset V_\Lambda$$

for all finite $\Lambda \subset \mathbb{L}$, hence $\tau_\gamma(\xi) \in \tilde{\Omega}((V_\Lambda)_\Lambda)$. □

The situation of Proposition 1.2.11 (ii) gives the commutativity of the following diagram



As remarked in [Ru78, p. 68], the hardest problem is indeed to show that $\tilde{\Omega}((V_\Lambda)_\Lambda)$ defined in (12) is non-empty. This can be undecidable in the sense of logic.

1.3 Boundary conditions

In thermodynamics one often studies a lattice spin system by looking at volume elements which contain a finite number of particles, investigating the localised observables, and then studying the asymptotic behaviour of the localised observables as the volume tends to infinity. Subconfigurations can be embedded into the configuration space in different ways. This process is described by the so called boundary condition. Since Ω_Λ was defined in (1.1.3) to be the image $\rho_\Lambda(\Omega)$ of the restriction map $\rho_\Lambda : \Omega \to \Omega_\Lambda$ (1.1.2), every $\xi_\Lambda \in \Omega_\Lambda$ has a preimage in Ω. We would like to have at least for certain finite $\Lambda \subset \mathbb{L}$ a (partial) right inverse of the restriction map. These admissible sets will be collected in the subset $\mathcal{P}$ of $\mathcal{P}_f(\mathbb{L}) = \{\Lambda \subset \mathbb{L};\ 0 < |\Lambda| < \infty\}$. We will introduce two types of such inverse maps which we call boundary conditions, the zero boundary condition as a particular example of an external field boundary condition, and for a particular class of semigroup actions the so called periodic boundary condition. The periodic boundary condition has been studied in the context of $\mathbb{Z}^D$-actions, whereas our generalisation will be applicable also in other situations, such as actions of an arbitrary abelian semigroup. We will give sufficient conditions on the transition matrix such that the associated one-dimensional matrix subshift allows a periodic boundary condition.

Definition 1.3.1. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1) and $(\rho_\Lambda : F^{\mathbb{L}} \to F^\Lambda)_{\Lambda\subset\mathbb{L}}$ the family of restriction mappings from (1.1.2). A *boundary extension* is a family $(b_\Lambda)_{\Lambda\in\mathcal{P}}$ of maps $b_\Lambda : F^\Lambda \to F^{\mathbb{L}}$ parametrised by some non-empty subset $\mathcal{P} \subset \mathcal{P}_f(\mathbb{L})$ such that $\rho_\Lambda \circ b_\Lambda = \mathrm{id}_{F^\Lambda}$ for all $\Lambda \in \mathcal{P}$. The subsets of the position space belonging to $\mathcal{P}$ are called *admissible* for the boundary extension. □

It is appearent that $b_\Lambda : F^\Lambda \to F^{\mathbb{L}}$ is a right inverse of $\rho_\Lambda : F^{\mathbb{L}} \to F^\Lambda$. We will now consider the restricted configuration space.

Remark 1.3.2. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1), $(b_\Lambda : F^\Lambda \to F^{\mathbb{L}})_{\Lambda\in\mathcal{P}}$ a boundary extension (1.3.1), and $(\rho_\Lambda : F^{\mathbb{L}} \to F^\Lambda)_{\Lambda\subset\mathbb{L}}$ the family of restriction mappings from (1.1.2). For $\Lambda \in \mathcal{P}$ let

$$\Omega'_\Lambda := (b_\Lambda)^{-1}(\Omega) = \{\xi_\Lambda \in F^\Lambda \mid b_\Lambda(\xi_\Lambda) \in \Omega\},$$

which is a subset of Ω_Λ, since for all $\xi_\Lambda \in \Omega'_\Lambda$ we have by definition of the boundary extension (1.3.1)

$$\xi_\Lambda = \rho_\Lambda(b_\Lambda(\xi_\Lambda)) \in \rho_\Lambda(\Omega) = \Omega_\Lambda,$$

where the latter identity holds by (1.1.3). Hence the restriction of b_Λ to Ω'_Λ defines a map $b_\Lambda|_{\Omega'_\Lambda} : \Omega'_\Lambda \to \Omega$, which is a partial right inverse of $\rho_\Lambda : \Omega \to \Omega_\Lambda$. By quite the same argument the following map is well-defined: $\rho_{\mathbb{L}\setminus\Lambda} \circ b_\Lambda|_{\Omega'_\Lambda} : \Omega'_\Lambda \to \Omega_{\mathbb{L}\setminus\Lambda}$, since for all $\xi_\Lambda \in \Omega'_\Lambda$ we have $\rho_{\mathbb{L}\setminus\Lambda}(b_\Lambda(\xi_\Lambda)) \in \rho_{\mathbb{L}\setminus\Lambda}(\Omega) = \Omega_{\mathbb{L}\setminus\Lambda}$. □

Definition 1.3.3. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space and $(b_\Lambda : F^\Lambda \to F^{\mathbb{L}})_{\Lambda\in\mathcal{P}}$ a boundary extension (1.3.1) parametrised by $\mathcal{P} \subset \mathcal{P}_f(\mathbb{L})$. If $\Omega'_\Lambda := (b_\Lambda)^{-1}(\Omega) \neq \emptyset$ for all $\Lambda \in \mathcal{P}$, the family $(b_\Lambda : \Omega'_\Lambda \to \Omega)_{\Lambda\in\mathcal{P}}$ is called a *boundary condition.* If $(b_\Lambda)^{-1}(\Omega) = \Omega_\Lambda$ for all $\Lambda \in \mathcal{P}_f(\mathbb{L})$, then we say that b is globally defined. □

By Remark 1.3.2 the map $b_\Lambda : \Omega'_\Lambda \to \Omega$ is well-defined. We review what we have achieved so far. On the full configuration space $F^{\mathbb{L}}$ the map $b_\Lambda : F^\Lambda \to F^{\mathbb{L}}$ is a right inverse of $\rho_\Lambda : F^{\mathbb{L}} \to F^\Lambda$, whereas on the restricted configuration space the picture is different: The map $b_\Lambda : \Omega'_\Lambda \to \Omega$ is only a partial right inverse of $\rho_\Lambda : \Omega \to \Omega_\Lambda$ on a (possibly strict) subset Ω'_Λ of Ω_Λ, as shown in Remark 1.3.2.

There are many ways to define extension maps. The simplest way is to choose the boundary extension (1.3.1) to be constant "outside" Λ. This leads to the so called external field extensions. In order to glue together both parts we use the following concatenation operator.

Definition 1.3.4. Let $F^{\mathbb{L}}$ be a full configuration space (1.1.1) and M, $N \subset \mathbb{L}$ non-empty disjoint sets. We define a *concatenation operator* $\oplus : F^M \times F^N \to F^{M\cup N}, (\xi_M, \eta_N) \mapsto \xi_M \oplus \eta_N$ via

$$(\xi_M \oplus \eta_N)(i) := \begin{cases} \xi_M(i) & , i \in M, \\ \eta_N(i) & , i \in N. \end{cases}$$

□

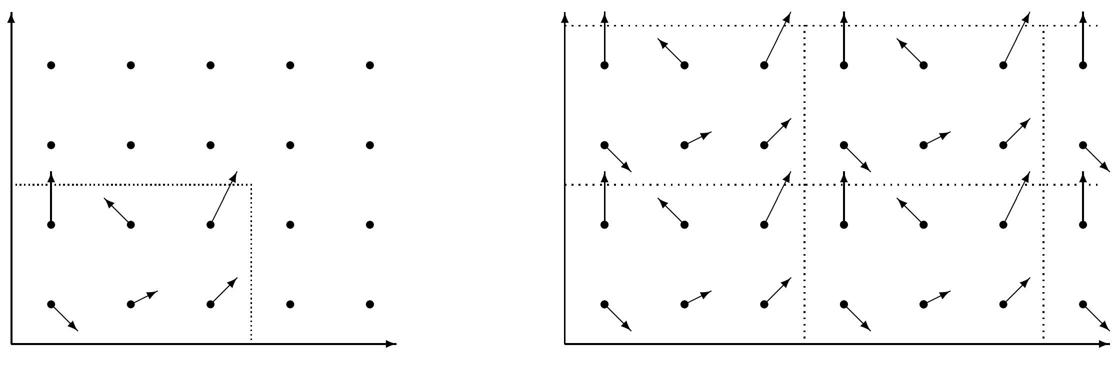


Figure 6: Zero boundary condition, periodic boundary condition.

Example 1.3.5. (External field extension) Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space and $\eta \in F^{\mathbb{L}}$ a fixed configuration, the "external field". The admissible family $\mathcal{P}_\eta$ for the external field extension consists of all finite subsets of $\mathbb{L}$, i. e., $\mathcal{P}_\eta = \mathcal{P}_f(\mathbb{L})$. The *external field extension* is the family

$$b^\eta_\Lambda : F^\Lambda \to F^{\mathbb{L}},\ \xi_\Lambda \mapsto \xi_\Lambda \oplus \rho_{\mathbb{L}\setminus\Lambda}(\eta)$$

for $\Lambda \in \mathcal{P}_\eta$. If $\Omega'_\Lambda := (b^\eta_\Lambda)^{-1}(\Omega) \neq \emptyset$ for all $\Lambda \in \mathcal{P}_\eta$, the family $(b^\eta_\Lambda : \Omega'_\Lambda \to \Omega)_{\Lambda\in\mathcal{P}_\eta}$ is called the *external field condition.* - If $\eta^0(i) = f_0$ for all $i \in \mathbb{L}$, where $f_0 \in F$ is the empty spin defined in (1.1.1), then the boundary extension is denoted by b^0_Λ and is called the *zero boundary extension*, respectively $(b^0_\Lambda : \Omega'_\Lambda \to \Omega)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ is called the *zero boundary condition* if it exists. □

In Remark 1.3.13 we will discuss a concrete example of a configuration space and a necessary and sufficient condition for the existence of the zero boundary condition.
Next we will define the periodic boundary extension. The idea is that certain subsets $\Lambda \subset \mathbb{L}$ give rise to a tiling of $\mathbb{L}$ such that the periodic boundary extension $b_\Lambda(\xi_\Lambda)$ restricted to a tile coincides with ξ_Λ. First we need some preparation concerning semigroup orbits.

Definition 1.3.6. Let $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ be a semigroup action (1.2.1). We define the *orbit relation* $\sim_\Gamma \subset \mathbb{L} \times \mathbb{L}$ via

$$i \sim_\Gamma j \ \text{ iff there exist } g,\, h \in \Gamma \text{ such that } \tau(g, i) = \tau(h, j).$$

□

Clearly the orbit relation with respect to a semigroup action is symmetric and reflexive. But, unlike group actions the orbit relation is (in general) not transitive, hence not an equivalence relation. A condition ensuring the transitivity is "commutator-free" which for instance holds if the semigroup is abelian.

Proposition 1.3.7. *Let $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ be a semigroup action of a semigroup Γ. The action is said to be commutator-free if $\tau(gh, \cdot) = \tau(hg, \cdot)$ for all $g,\, h \in \Gamma$. If τ is commutator-free, then the orbit relation $\sim_\Gamma\ \subset \mathbb{L} \times \mathbb{L}$ (1.3.6) with respect to Γ is an equivalence relation.*

Proof. We have to show that the relation $\sim_\Gamma \subset \mathbb{L} \times \mathbb{L}$ is transitive: Let $a \sim_\Gamma b$, $b \sim_\Gamma c$. Then there exist $f, g, h, i \in \Gamma$ such that $\tau(f, a) = \tau(g, b)$ and $\tau(h, b) = \tau(i, c)$. Hence $\tau(hf, a) = \tau(hg, b) = \tau(gh, b) = \tau(gi, c)$, which by definition means that $a \sim_\Gamma c$. □

In our applications the acting semigroups are abelian, but one could ask for other criteria forcing the orbit relation to be an equivalence relation.

Remark 1.3.8. Let $\Gamma \times \mathbb{L} \to \mathbb{L}$ be a semigroup action such that the orbit relation $\sim_\Gamma \subset \mathbb{L} \times \mathbb{L}$ from (1.3.6) is an equivalence relation.

(i) The position space $\mathbb{L}$ decomposes into a disjoint union of equivalence classes with respect to the orbit relation. The equivalence classes are called *(generalised)* Γ*-orbits.* We will write $\mathbb{L}/\Gamma := \mathbb{L}/\sim_\Gamma$ for the quotient space with respect to the orbit relation and call it the *(generalised)* Γ*-orbit space.* We call a set $\Lambda_\Gamma \subset \mathbb{L}$ of representatives of $\mathbb{L}/\Gamma$ a *fundamental domain* for the Γ-action.[5]

(ii) A fundamental domain $\Lambda \subset \mathbb{L}$ for the Γ-action is said to satisfy the *tiling condition*, if

$$\Gamma \cdot \Lambda := \big\{\gamma \cdot i \,|\, i \in \Lambda,\, \gamma \in \Gamma\big\} = \mathbb{L}.$$

(iii) If $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ is a group action, then our definition of the orbit relation coincides with the usual one

$$a \sim_\Gamma b \ :\Leftrightarrow\ (\exists \gamma \in \Gamma)\ \tau(\gamma, a) = b$$

and defines an equivalence relation. Every fundamental domain satisfies the tiling condition.

[5]Note that $\mathbb{L}$ is assumed to be countable.

(iv) As an example take the integers $\mathbb{L} := \mathbb{Z}$, $n \in \mathbb{N}$, and the usual translation action (6) of $n\mathbb{N}_0$ on $\mathbb{Z}$. Then the generalised orbits are of the form $i + n\mathbb{Z}$ $(i = 1, \dots, n)$, which differs from the pointwise orbit

$$\tau(n\mathbb{N}, i) := \{\tau(nm, i) = nm + i \in \mathbb{Z} \,|\, m \in \mathbb{N}_0\} = i + n\mathbb{N}_0$$

of $i \in \mathbb{Z}$. The (standard) fundamental domain for this action is the set $\{1, 2, \dots, n\}$. Note that also the $n\mathbb{Z}$-action on $\mathbb{Z}$ has this fundamental domain. □

Given a semigroup action $\Gamma \times \mathbb{L} \to \mathbb{L}$, $(\gamma, g) \mapsto \gamma \cdot g$ (1.2.1) and a family $(\Gamma_\alpha)_{\alpha \in A}$ of subsemigroups $\Gamma_\alpha \leq \Gamma$, we are now prepared to define a periodic boundary extension. Such a periodic boundary extension assigns to every function defined on a fundamental domain of Γ_α the periodic continuation with respect to the Γ_α-orbits.

Example 1.3.9. (Periodic boundary extension) Let Γ be a semigroup and $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$, $\tau(\gamma, g) = \gamma \cdot g$ a semigroup action (1.2.1). Let $\Gamma_\alpha \leq \Gamma$ be a family of subsemigroups of Γ (indexed by $\alpha \in A$) such that

(i) The orbit relation $\sim_\alpha := \sim_{\Gamma_\alpha}$ with respect to Γ_α is an equivalence relation for all $\alpha \in A$,

(ii) $\mathbb{L}/\Gamma_\alpha$ is finite for all $\alpha \in A$, and

(iii) There is no pair of distinct semigroups having the same set of representatives: For all $\alpha \neq \beta \in A$, for all sets Λ_α of representatives of $\mathbb{L}/\Gamma_\alpha$ and Λ_β of $\mathbb{L}/\Gamma_\beta$ we have $\Lambda_\alpha \neq \Lambda_\beta$.

We say that such a family $(\Gamma_\alpha)_{\alpha \in A}$ *defines a periodic boundary extension.* Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space. We define the admissible sets for the periodic boundary extension to be the family $\mathcal{P}_{(\Gamma_\bullet)}$ of sets of representatives of Γ_α-equivalence classes, i. e.,

$$\mathcal{P}_{(\Gamma_\bullet)} := \mathcal{P}_{(\Gamma_\alpha)_{\alpha \in A}} := \{\Lambda_\alpha \subset \mathbb{L} \,|\, \Lambda_\alpha \text{ is a set of representatives of } \mathbb{L}/\Gamma_\alpha,\ \alpha \in A\}. \tag{13}$$

Let $\Lambda_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$ be a set of representatives of $\mathbb{L}/\Gamma_\alpha$. We define the Γ_α-periodic continuation of $\xi_{\Lambda_\alpha} \in F^{\Lambda_\alpha}$ to be

$$r_{\Gamma_\alpha}(\xi_{\Lambda_\alpha}) : \mathbb{L} \to F,\ r_{\Gamma_\alpha}(\xi_{\Lambda_\alpha})(i) := \xi_{\Lambda_\alpha}(j)$$

where $i \in \mathbb{L}$ and j is the unique element $j \in \Lambda_\alpha$ with $i \sim_\alpha j$. This defines a map

$$r_{\Gamma_\alpha} : F^{\Lambda_\alpha} \to F^{\mathbb{L}},\ \xi_{\Lambda_\alpha} \mapsto r_{\Gamma_\alpha}(\xi_{\Lambda_\alpha}). \tag{14}$$

The *periodic boundary extension* associated to $(\Gamma_\alpha)_{\alpha \in A}$ is the family

$$b^{\Gamma}_{\Lambda_\alpha} := r_{\Gamma_\alpha} : F^{\Lambda_\alpha} \to F^{\mathbb{L}} \tag{15}$$

for all $\Lambda_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$. If $\Omega'_\Lambda := (b^\Gamma_\Lambda)^{-1}(\Omega) \neq \emptyset$ for all $\Lambda \in \mathcal{P}_{(\Gamma_\bullet)}$, then the family $(b^\Gamma_\Lambda : \Omega'_\Lambda \to \Omega)_{\Lambda \in \mathcal{P}_{(\Gamma_\bullet)}}$ is called the *periodic boundary condition* associated to the family $(\Gamma_\alpha)_{\alpha \in A}$. □

Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space. In order to investigate the question whether the periodic boundary extension gives rise to a periodic boundary condition, we have to determine $\Omega'_{\Lambda_\alpha} := (b^\Gamma_{\Lambda_\alpha})^{-1}(\Omega) \subset \Omega_{\Lambda_\alpha}$. It turns out, see Proposition 1.3.14, that Ω'_{Λ_α} is just the ρ_{Λ_α}-restriction of the Γ_α-periodic configurations.

Remark 1.3.10. Let $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ be a semigroup action of an abelian semigroup Γ. Then all subsemigroups $\Gamma_\alpha \leq \Gamma$ are abelian and hence define equivalence relations $\sim_\alpha$ by (1.3.7). Hence condition (i) in Example 1.3.9 is automatically satisfied.
It remains an open problem to decide under which assumptions the following holds: "Let $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ be a semigroup action, such that the orbit relation is an equivalence relation. Then the orbit relation with respect to a subsemigroup $H \leq \Gamma$ is an equivalence relation." □

The periodic boundary condition seems at first to be quite artificial. Apart from the fact that it allows a beautiful mathematical treatment as we will see, the derived physically interesting functions do not depend on the boundary condition at least in the thermodynamic limit which we define next.

Definition 1.3.11. (i) The name of *a thermodynamic limit* is given to a limit process in position space $\mathbb{L}$, when a sequence/a net of subsets $\Lambda \subset \mathbb{L}$ "tends to infinity", i. e., to $\mathbb{L}$. We write $\Lambda_n \to \mathbb{L}$ if $(\Lambda_n)_{n\in\mathbb{N}}$ is a sequence of subsets $\Lambda_n \subset \mathbb{L}$ such that for all finite $M \subset \mathbb{L}$ there exists an index $n_M \in \mathbb{N}$ such that $M \subset \Lambda_n$ for all $n \geq n_M$.

(ii) We say that a periodic boundary extension $(b^{\Gamma}_{\Lambda})_{\Lambda\in\mathcal{P}_{(\Gamma_\bullet)}}$ *allows a thermodynamic limit*, if there exists a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of subsets $\Lambda_n \in \mathcal{P}_{(\Gamma_\bullet)}$ with $\Lambda_n \to \mathbb{L}$. □

We will now give some examples for orbit relations. Part (i) corresponds to two-sided shifts, whereas (ii) corresponds to one-sided shifts.

Example 1.3.12. (i) We return to the situation of Example 1.2.5, where we let $\Gamma = \mathbb{Z}^D$ act on itself by left translations (6) via $\tau : \Gamma \times \mathbb{Z}^D \to \mathbb{Z}^D,\ \tau(k,m) = k + m$. The standard family of subgroups of Γ is given as the family of all $\Gamma_n = \prod_{i=1}^{D}(n_i\mathbb{Z})$, where $n = (n_1,\ldots,n_D)$ runs through $\mathbb{N}^D$. The group Γ is abelian and the family of admissible sets consists of the translates of the standard fundamental domains

$$\mathcal{P}_{(\Gamma_\bullet)} := \mathcal{P}_{(\Gamma_n)_{n\in\mathbb{N}^D}} = \big\{k + \prod_{i=1}^{D}\{1,\ldots,n_i\} \,|\, k \in \mathbb{Z}^D,\, n \in \mathbb{N}^D\big\}.$$

A second family of semigroups is $H_n = \prod_{i=1}^{D}(n_i\mathbb{N}_0)$, $(n \in \mathbb{N}^D)$, with the same family of admissible sets $\mathcal{P}_{(H_\bullet)} = \mathcal{P}_{(\Gamma_\bullet)}$. The advantage of (Γ_n) is that every $\Lambda \in \mathcal{P}_{(\Gamma_\bullet)}$ satisfies the tiling condition (1.3.8) with respect to the associated semigroup having Λ as fundamental domain, whereas no $\Lambda \in \mathcal{P}_{(H_\bullet)}$ satisfies a tiling condition with respect to (H_n).

(ii) The restriction of the $\mathbb{Z}^D$-action (6) to the semigroup $\mathbb{N}_0^D \leq \mathbb{Z}^D$ leaves $\mathbb{N}^D$ invariant, hence we have an induced action $\tau : \mathbb{N}_0^D \times \mathbb{N}^D \to \mathbb{N}^D,\ \tau(k,m) = k+m$. The standard family of semigroups consists of $H_n = \prod_{i=1}^{D}(n_i\mathbb{N}_0)$ parametrised by $n \in \mathbb{N}^D$ and

$$\mathcal{P}_{(H_\bullet)} = \big\{k + \prod_{i=1}^{D}\{1,\ldots,n_i\} \,|\, k,\, n \in \mathbb{N}^D\big\}$$

is the corresponding family of admissible sets. The elements $\Lambda \in \mathcal{P}_{(H_\bullet)}$ satisfying the tiling condition are precisely the sets

$$\mathcal{P}^{\text{tile}}_{(H_\bullet)} = \big\{\prod_{i=1}^{D}\{1,\ldots,n_i\} \,|\, n \in \mathbb{N}^D\big\}.$$

Note that in both cases our choice of the families of subsemigroups makes the fundamental domains finite and allows a thermodynamic limit. An example of a family $(\Gamma_n)_{n\in\mathbb{N}}$ of subsemigroups of $\mathbb{N}^2$ which does not allow a thermodynamic limit is the family $\Gamma_n := \mathbb{N} \times (n\mathbb{N})$ parametrised by $n \in \mathbb{N}$. □

In the following Remark 1.3.13 we will discuss a concrete example of a configuration space and different boundary extensions and conditions on it. In particular, we will see an example where $b^{-1}_{\Lambda}(\Omega) \subsetneq \Omega_\Lambda$. We study the one-dimensional matrix subshift (1.2.8) and use the visualisation technique from (1.2.9).

Remark 1.3.13. (Boundary conditions for the one-dimensional matrix subshift) Let $\mathbb{A}$ be an irreducible transition matrix (1.2.9) and $(\Omega_{\mathbb{A}}, \mathbb{Z}, \tau)$ be the associated one-dimensional matrix subshift as defined in Example 1.2.8. For all $\Lambda \subset \mathbb{L}$ let $\Omega_{\mathbb{A},\Lambda} := \rho_\Lambda(\Omega_{\mathbb{A}})$ be the image under the restriction map (1.1.2).

(i) If $\mathbb{A}$ has all entries equal to 1 so that $\Omega_{\mathbb{A}} = F^{\mathbb{Z}}$, then clearly all boundary extensions are boundary conditions and every subset $\Lambda \subset \mathbb{Z}$ is admissible for any boundary condition.

(ii) Let $f_0 \in F$ be the empty spin. The zero boundary extension $(b^0_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ (1.3.5) is globally defined in the sense of Def. 1.3.4, if and only if

$$\mathbb{A}(x, f_0) = 1 = \mathbb{A}(f_0, x)$$

for all $x \in F$. In fact: Suppose $(b^0_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ is globally defined. Let $\Lambda = \{i\} \subset \mathbb{L}$, then $\Omega_{\mathbb{A},\Lambda} = F^{\{i\}} \cong F$ since the graph is connected. By the definition of the zero boundary extension (1.3.5) the sequence $b^0_\Lambda(\xi_\Lambda) \in F^{\mathbb{Z}}$ has the entries

$$b^0_\Lambda(\xi_\Lambda)(j) = \begin{cases} \xi_\Lambda(i) & , \text{ if } j = i, \\ f_0 & , \text{ otherwise.} \end{cases}$$

Hence $b^0_\Lambda(\xi_\Lambda) \in \Omega_{\mathbb{A}}$ iff $\mathbb{A}(\xi_\Lambda(i), f_0) = 1 = \mathbb{A}(f_0, \xi_\Lambda(i))$ and $\mathbb{A}(f_0, f_0) = 1$. Vary over all $\xi_\Lambda \in \Omega_{\mathbb{A},\Lambda}$, i. e., over all $x \in F$, to get the assertion.

Conversely: If $\mathbb{A}(x, f_0) = 1 = \mathbb{A}(f_0, x)$ for all $x \in F$, let $\xi_\Lambda \in \Omega_{\mathbb{A},\Lambda}$ and $\eta := b^0_\Lambda(\xi_\Lambda) \in F^{\mathbb{L}}$. For $i,\ i+1 \in \Lambda$ we have $\mathbb{A}(\eta(i), \eta(i+1)) = 1$ by definition of $\Omega_{\mathbb{A},\Lambda}$. If $i \in \Lambda$, $i+1 \in \mathbb{L}\setminus\Lambda$ then $\mathbb{A}(\eta(i), \eta(i+1)) = \mathbb{A}(\eta(i), f_0) = 1$ by assumption, similarly the case $i \in \Lambda$, $i-1 \in \mathbb{L}\setminus\Lambda$. Hence $\eta \in \Omega_{\mathbb{A}}$ and the zero boundary extension is a (globally defined) boundary condition.

(iii) We consider the following one-dimensional two-sided matrix subshift defined by the data $F = \{1, 2, 3\}$ and the transition matrix $\mathbb{A} = \Big(\mathbb{A}(i,j)\Big)_{i,j=1,2,3} = \left(\begin{smallmatrix} 0&1&1\\ 0&0&1\\ 1&1&1 \end{smallmatrix}\right)$ for which we have drawn in (10) of (1.2.9) the corresponding directed graph. We choose the standard family of subgroups of $\mathbb{Z}$ consisting of $(n\mathbb{Z})_{n\in\mathbb{N}}$. We observe that the three-periodic sequence $\overline{1\,2\,3}$ belongs to $\Omega_{\mathbb{A}}$, hence $(1,2) \in \Omega_{\mathbb{A},\{1,2\}} := \rho_{\{1,2\}}(\Omega_{\mathbb{A}})$, but its $2\mathbb{Z}$-periodic extension $b^{2\mathbb{Z}}_{\{1,2\}}(1,2) = \overline{1\,2} \notin \Omega_{\mathbb{A}}$.

(iv) In example (iii) we can easily read off the fixed point set

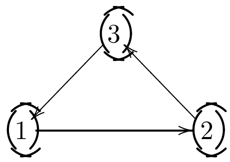
$$\bigcap_{\gamma\in 2\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega_{\mathbb{A}} \to \Omega_{\mathbb{A}}) = \{\overline{3} = \overline{3\,3},\ \overline{1\,3},\ \overline{2\,3}\},$$

hence for all $\Lambda = \{i, i+1\}$ one has

$$\Omega'_\Lambda := (b^\Gamma_\Lambda)^{-1}(\Omega) = \rho_\Lambda(\bigcap_{\gamma\in 2\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega_{\mathbb{A}} \to \Omega_{\mathbb{A}})) = \{(3,3),\ (3,1),\ (3,2),\ (1,3),\ (2,3)\} \subsetneq \Omega_\Lambda$$

and the periodic boundary extension (1.3.9) induces a boundary condition $b^\Gamma_\Lambda : \Omega'_\Lambda \to \Omega$ on a strict subset Ω'_Λ of Ω_Λ.

(v) We consider the following one-dimensional two-sided matrix subshift defined by the data $F = \{1, 2, 3\}$ and the transition matrix $\mathbb{A}' = \Big(\mathbb{A}'(i,j)\Big)_{i,j=1,2,3} = \left(\begin{smallmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{smallmatrix}\right)$ corresponding to



which is $\mathbb{A}$ from (iii) with some arrows removed. We choose the standard family of subgroups of $\mathbb{Z}$ consisting of $(n\mathbb{Z})_{n\in\mathbb{N}}$. We have

$$\bigcap_{\gamma\in n\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega_{\mathbb{A}'} \to \Omega_{\mathbb{A}'}) = \{\overline{1\,2\,3}\}$$

for all n congruent 0 modulo 3 and the empty set otherwise. □

We will now determine the domain $(b^\Gamma_\Lambda)^{-1}(\Omega) \subset \Omega_\Lambda$ of the periodic boundary condition. As suggested by Remark 1.3.13 (iv) the set $\Omega'_\Lambda := (b^\Gamma_\Lambda)^{-1}(\Omega)$ can be expressed in terms of joint fixed points.

Proposition 1.3.14. *Let $(\Omega, F, \mathbb{L}, \Gamma, \tau)$ be a lattice spin system (1.2.7), $(\Gamma_\alpha)_{\alpha\in A}$ a family of subsemigroups of Γ which defines a periodic boundary extension $(b^\Gamma_\Lambda)_{\Lambda\in\mathcal{P}_{(\Gamma_\bullet)}}$ (1.3.9). Let $N_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$ be a set of representatives of ${}^{\mathbb{L}}\!/_{\Gamma_\alpha}$, then*

$$\rho_{N_\alpha} : \bigcap_{\gamma\in\Gamma_\alpha} \mathrm{Fix}(\tau_\gamma : \Omega \to \Omega) \to (b^\Gamma_{N_\alpha})^{-1}(\Omega)$$

is a bijection with inverse $b^\Gamma_{N_\alpha}$, where $\mathrm{Fix}(\tau_\gamma : \Omega \to \Omega)$ is the set of fixed points of the map $\tau_\gamma : \Omega \to \Omega$.

Proof. Set $N := N_\alpha$ as an abbreviation. First observe that for all $\xi \in \bigcap_{\gamma\in\Gamma_\alpha} \mathrm{Fix}(\tau_\gamma) \subset \Omega$ we have $(b_N^\Gamma \circ \rho_N)(\xi) = \xi$ by the definition of the periodic boundary extension (1.3.9), hence

$$\rho_N : \bigcap_{\gamma\in\Gamma} \mathrm{Fix}(\tau_\gamma) \to (b_N^\Gamma)^{-1}(\Omega)$$

is well-defined. If one has $\xi, \eta \in \bigcap_{\gamma\in\Gamma_\alpha} \mathrm{Fix}(\tau_\gamma)$ with $\rho_N(\xi) = \rho_N(\eta)$, then $\xi(i) = \eta(i)$ for all $i \in N$. Let $j \in \mathbb{L}$, then there exists a Γ_α-equivalent point $j_\alpha \in N$, i. e., $j \sim_\alpha j_\alpha$. Hence there exist $g, h \in \Gamma_\alpha$ such that $g \cdot j = h \cdot j_\alpha$. Since ξ is Γ_α-periodic, we have

$$\xi(j) = \xi(g \cdot j) = \xi(h \cdot j_\alpha) = \xi(j_\alpha).$$

Similarly for η. By assumption $\xi(j_\alpha) = \eta(j_\alpha)$, hence $\xi(j) = \eta(j)$ for all $j \in \mathbb{L}$, hence $\xi = \eta$, and ρ_N is injective. Let $\xi_N \in \Omega'_N := (b_N^\Gamma)^{-1}(\Omega)$, then by definition of the periodic boundary extension (15) the element $\eta = b_N^\Gamma(\xi_N) \in \Omega$ is fixed by all maps $\tau_\gamma : \Omega \to \Omega$ for all $\gamma \in \Gamma_\alpha$:

$$\tau_\gamma(\eta)(i) = \eta(\gamma \cdot i) = \eta(i)$$

for all $i \in \mathbb{L}, \gamma \in \Gamma_\alpha$. Here we used the fact that $\gamma \cdot i \sim_\Gamma i$, which follows from $\gamma \cdot (\gamma \cdot i) = (\gamma^2) \cdot i$. Hence $b_N^\Gamma(\Omega'_N) \subset \bigcap_{\gamma\in\Gamma_\alpha} \mathrm{Fix}(\tau_\gamma)$, and hence the definition of a boundary extension gives

$$\Omega'_N = (\rho_N \circ b_N^\Gamma)(\Omega'_N) \subset \rho_N\big(\bigcap_{\gamma\in\Gamma_\alpha} \mathrm{Fix}(\tau_\gamma)\big)$$

and thus the surjectivity of $\rho_N : \bigcap_{\gamma\in\Gamma} \mathrm{Fix}(\tau_\gamma) \to (b_N^\Gamma)^{-1}(\Omega)$. □

We will use Proposition 1.3.14 to prove the following fact: If $\mathbb{A}$ is aperiodic (1.2.9), then the periodic boundary extension with respect to the family $(n\mathbb{Z})_{n\in\mathbb{N}}$ gives rise to a periodic boundary condition. By the previous proposition it suffices to show that $\bigcap_{\gamma\in n\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega_\mathbb{A} \to \Omega_\mathbb{A}) \neq \emptyset$ for almost all $n \in \mathbb{N}$. For this we need a number theoretic lemma:

Lemma 1.3.15. *Let $\lambda_1, \dots, \lambda_n \in \mathbb{N}$ with $\gcd(\lambda_1, \dots, \lambda_n) = 1$. Then there exists $N \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} \,|\, k \geq N\} \subset \Big\{ \sum_{j=1}^n \lambda_j\, n_j \,|\, n_j \in \mathbb{N}\Big\}.$$

Proof. Euclid's algorithm provides us with the existence of $\tilde{m}_j \in \mathbb{Z}$ such that $1 = \sum_{j=1}^n \lambda_j\, \tilde{m}_j$. By relabelling the λ_j's we can assume that there exists $1 \leq k \leq n$ and *positive* integers $m_1, \dots, m_n$ such that $1 = \sum_{j=1}^k \lambda_j\, m_j - \sum_{j=k+1}^n \lambda_j\, m_j$. Set $M := \sum_{j=k+1}^n \lambda_j\, m_j$. Write $k \in \mathbb{N}$ as $k = k_1 + k_2\, M$ with $1 \leq k_1 \leq M - 1$. Then

$$k = k_1 \cdot 1 + k_2\, M = k_1 \sum_{j=1}^k \lambda_j\, m_j - k_1 \sum_{j=k+1}^n \lambda_j\, m_j + k_2 \sum_{j=k+1}^n \lambda_j\, m_j = \sum_{j=1}^k \lambda_j\, k_1\, m_j + \sum_{j=k+1}^n \lambda_j\, m_j\, (k_2 - k_1).$$

Hence if $k \geq 1 + (M-1)\, M$, then $k_1 \leq k_2$ and $k \in \big\{ \sum_{j=1}^n \lambda_j\, n_j \,|\, n_j \in \mathbb{N}\big\}$. □

Corollary 1.3.16. *Let $(\Omega_\mathbb{A}, \mathbb{Z}, \tau)$ be a one-dimensional matrix subshift (1.2.8). If the transition matrix $\mathbb{A} : F \times F \to \{0, 1\}$ is aperiodic (1.2.9), then there exists $N_\mathbb{A} \in \mathbb{N}$ such that $\bigcap_{\gamma\in n\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega_\mathbb{A} \to \Omega_\mathbb{A}) \neq \emptyset$ for all $n \geq N_\mathbb{A}$. In other words, there exists $N_\mathbb{A} \in \mathbb{N}$ such that the periodic boundary extension with respect to the family $(n\mathbb{Z})_{n\in\mathbb{N}, n\geq N_\mathbb{A}}$ gives rise to a periodic boundary condition.*

Proof. Recall the graph theoretic interpretation of $\Omega_\mathbb{A}$ from Remark 1.2.9. If $\mathbb{A}$ is aperiodic, then for each vertex $x \in F$ there exist a finite number of paths $\gamma_1, \dots, \gamma_m$ through x, whose lengths don't have a common divisor. Hence by Lemma 1.3.15 for all $n \in \mathbb{N}$ sufficiently large there exists a concatenation $h := \gamma_1 \circ \dots \circ \gamma_1 \circ \dots \circ \gamma_m \circ \dots \circ \gamma_m$ such that the closed path h has length $\ell(h) = n$. Since every closed path in the graph gives rise to a periodic configuration, this shows that $\bigcap_{\gamma\in n\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega_\mathbb{A} \to \Omega_\mathbb{A}) \neq \emptyset$. □

In the following remark we will consider the one-sided one-dimensional matrix subshift and show the analogon of Corollary 1.3.16.

Remark 1.3.17. (i) Let $D \geq 1$ and $(\Omega, F, \mathbb{Z}^D, \Gamma = \mathbb{Z}^D, \tau)$ be a lattice spin system (1.2.7). Via restriction to the positive quadrant $\mathbb{N}_0^D$ we obtain the (so called one-sided) lattice spin system $(\Omega^> = \rho_{\mathbb{N}^D}(\Omega), F, \mathbb{N}^D, H = \mathbb{N}_0^D, \tau^>)$. For $n = (n_1, \ldots, n_D) \in \mathbb{N}^D$ let $\Gamma_n = \prod_{i=1}^D n_i\mathbb{Z}$ and $H_n = \prod_{i=1}^D n_i\mathbb{N}_0$ be the families as defined in Example 1.3.12 for a periodic boundary extension. Then

$$\rho_{\mathbb{N}^D} : \bigcap_{\gamma\in\Gamma_n} \mathrm{Fix}(\tau_\gamma : \Omega \to \Omega) \to \bigcap_{\gamma\in H_n} \mathrm{Fix}(\tau_\gamma^> : \Omega^> \to \Omega^>),\ \xi \mapsto \rho_{\mathbb{N}^D}(\xi)$$

is a bijection with inverse $r_{\Gamma_N} \circ \rho_{\Lambda_n}$, where $\Lambda_n = \prod_{i=1}^D \{1, \ldots, n_i\}$ and r_{Γ_n} are defined in (14). This is an immediate consequence of Proposition 1.3.14. We obtain, using $r_{\Gamma_N} = b_N^\Gamma$,

$$(b_N^\Gamma)^{-1}(\Omega) = (b_N^H)^{-1}(\Omega) = \rho_N\Big(\bigcap_{\gamma\in H_n} \mathrm{Fix}(\tau_\gamma^> : \Omega^> \to \Omega^>)\Big) = \rho_N\Big(\bigcap_{\gamma\in\Gamma_n} \mathrm{Fix}(\tau_\gamma : \Omega \to \Omega)\Big).$$

Hence by Proposition 1.3.14 the periodic boundary extension associated to the family $(\Gamma_n)_{n\in\mathbb{N}^D}$ gives rise to a periodic boundary condition if the same holds true for the one-sided system and the associated family $(H_n)_{n\in\mathbb{N}^D}$.

(ii) Let $(\Omega_{\mathbb{A}}^>, F, \mathbb{N}_0, \tau^>)$ be a one-sided one-dimensional matrix subshift (1.2.8) and $x_1, \ldots, x_n \in F$. Note that the sequence

$$\overline{x_1 \ldots x_n} := (\rho_{\mathbb{N}} \circ b_N^{n\mathbb{Z}})(x_1, \ldots, x_n) = r_{n\mathbb{N}_0}(x_1, \ldots, x_n) \in F^{\mathbb{N}} \tag{16}$$

belongs to the fixed point set $\mathrm{Fix}(\tau_n^> : \Omega_{\mathbb{A}}^> \to \Omega_{\mathbb{A}}^>)$ of the shift operator on the restricted configuration space if and only if $\mathbb{A}_{x_1,x_2} \cdot \ldots \cdot \mathbb{A}_{x_{n-1},x_n} \cdot \mathbb{A}_{x_n,x_1} = 1$. Hence by part (i) and Proposition 1.3.14 we have

$$(b_N^{\mathbb{N}_0})^{-1}(\Omega) = \big\{x = (x_1, \ldots, x_n) \in F^n \,|\, \mathbb{A}_{x_1,x_2} \cdot \ldots \cdot \mathbb{A}_{x_{n-1},x_n} \cdot \mathbb{A}_{x_n,x_1} = 1\big\}. \tag{17}$$

(iii) Let $(\Omega_{\mathbb{A}}^>, \mathbb{N}_0, \tau)$ be a one-dimensional one-sided matrix subshift as defined in Example 1.2.8 with an aperiodic transition matrix $\mathbb{A} : F \times F \to \{0, 1\}$ (1.2.9) and $(\Omega_{\mathbb{A}}, \mathbb{N}_0, \tau)$ the corresponding two-sided matrix subshift. By Corollary 1.3.16 there exists $N_{\mathbb{A}} \in \mathbb{N}$ such that $\bigcap_{\gamma\in n\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega_{\mathbb{A}} \to \Omega_{\mathbb{A}}) \neq \emptyset$ for all $n \geq N_{\mathbb{A}}$. Its $\rho_{\mathbb{N}}$-projection is non-empty and by part (1) precisely $(b_{n\mathbb{N}_0}^{\mathbb{N}_0})^{-1}(\Omega_{\mathbb{A}}^>)$, hence the periodic boundary extension with respect to the family $(n\mathbb{N}_0)_{n\in\mathbb{N}, n\geq N_{\mathbb{A}}}$ gives rise to a periodic boundary condition, i. e., the analogous result to Corollary 1.3.16. □

1.4 Observables

We introduce observables in the sense of classical mechanics as continuous functions on the configuration space. An observable is a quantity to be measured. Examples of such observables are the energy of a (sub-)configuration or the sum of interactions between two subconfigurations, which will be introduced in the next sections.

Definition 1.4.1. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1). A bounded continuous complex valued function $f : \Omega \to \mathbb{C}$ is called an *observable*[6]. The space $\mathcal{C}_b(\Omega) := \{f : \Omega \to \mathbb{C} \,|\, f \text{ continuous, bounded}\}$ of observables on Ω is a Banach space with respect to the supremum norm. If F is compact, then clearly $\mathcal{C}_b(\Omega) = \mathcal{C}(\Omega)$. □

Remark 1.4.2. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1) and $\Lambda \subset \mathbb{L}$. Since the restriction map $\rho_\Lambda : \Omega \to \Omega_\Lambda = \rho_\Lambda(\Omega)$ from (1.1.2) is by definition surjective, the map

$$\rho_\Lambda^\star : \mathcal{C}_b(\Omega_\Lambda) \to \mathcal{C}_b(\Omega),\ f_\Lambda \mapsto f_\Lambda \circ \rho_\Lambda$$

is an isometric linear mapping. In fact, $\|\rho_\Lambda^\star(f_\Lambda)\|_{\mathcal{C}(\Omega)} = \sup_{\xi\in\Omega} |(f_\Lambda) \circ \rho_\Lambda(\xi)| = \sup_{\xi\in\Omega} |f_\Lambda(\xi|_\Lambda)| = \|f_\Lambda\|_{\mathcal{C}_b(\Omega_\Lambda)}$. In particular, $\rho_\Lambda^\star$ is injective with $\|\rho_\Lambda^\star\| = 1$. Hence, we can regard $\mathcal{C}_b(\Omega_\Lambda)$ as a subspace of $\mathcal{C}_b(\Omega)$. □

[6]The usual notion from classical physics of an observable is a continuous real valued function on a compact space.

The image $\rho_\Lambda^\star(\mathcal{C}_b(\Omega_\Lambda)) = \{f \in \mathcal{C}_b(\Omega) \mid f \text{ constant along } \rho_\Lambda\text{-fibers}\} \subset \mathcal{C}_b(\Omega)$ is interpreted as the set of those physical quantities which are localised in the region Λ inside the position space $\mathbb{L}$. If F is compact, by the (complex) Stone-Weierstrass theorem the union of these images is dense in $\mathcal{C}(\Omega)$:

Proposition 1.4.3. *Let F be compact and $\Omega \subset F^{\mathbb{L}}$ be a configuration space, then*

$$A := \{f_\Lambda \circ \rho_\Lambda : \Omega \to \mathbb{C} \mid f_\Lambda \in \mathcal{C}(\Omega_\Lambda),\, \Lambda \in \mathcal{P}_f(\mathbb{L})\}$$

is dense in $\mathcal{C}(\Omega)$.

Proof. Recall the (complex) *Stone-Weierstrass theorem*, which can be stated as follows: Suppose K is a compact Hausdorff space and A is a subset of $\mathcal{C}(K)$ which separates points. Then the unital $*$-algebra generated by A is dense in $\mathcal{C}(K)$. We show that $A := \{f_\Lambda \circ \rho_\Lambda : \Omega \to \mathbb{C} \mid f_\Lambda \in \mathcal{C}(\Omega_\Lambda),\, \Lambda \in \mathcal{P}_f(\mathbb{L})\}$ is indeed a unital $*$-algebra: Let $f, g \in A$. Then there exist finite subsets Λ, M of $\mathbb{L}$ such that $f = f_\Lambda \circ \rho_\Lambda$ and $g = g_M \circ \rho_M$. Let $N = \Lambda \cup M$, then using Remark 1.1.3 we have the following representations $f = (f_\Lambda \circ \rho_{\Lambda,N}) \circ \rho_N$, $g = (g_M \circ \rho_{M,N}) \circ \rho_N$ where the functions $f_\Lambda \circ \rho_{\Lambda,N}$, $g_M \circ \rho_{M,N}$ belong to $\mathcal{C}(\Omega_N)$. For $\odot \in \{+, \cdot\}$ we have

$$f \odot g = (f_\Lambda \circ \rho_{\Lambda,N} \odot g_M \circ \rho_{M,N}) \circ \rho_N \in A.$$

The algebra A is closed under taking complex conjugates

$$\overline{f} = \overline{f_\Lambda \circ \rho_\Lambda} = \overline{f_\Lambda} \circ \rho_\Lambda \in A$$

and contains the constant function $1 \in \mathcal{C}(\Omega)$, since $1 = 1_\Lambda \circ \rho_\Lambda \in A$ for all $\Lambda \subset \mathbb{L}$. Hence A is a unital $*$-subalgebra of $\mathcal{C}(\Omega)$. Let $x \neq y \in \Omega$, then there exists a finite subset $\Lambda \subset \mathbb{L}$ such that $x_\Lambda := \rho_\Lambda(x) \neq \rho_\Lambda(y) =: y_\Lambda \in \Omega_\Lambda$. Since the Banach space $\mathcal{C}(\Omega_\Lambda)$ separates the points of Ω_Λ, there exists a function $f_\Lambda \in \mathcal{C}(\Omega_\Lambda)$ with $f_\Lambda(x_\Lambda) \neq f_\Lambda(y_\Lambda)$. Then $f_\Lambda \circ \rho_\Lambda \in A$ and $(f_\Lambda \circ \rho_\Lambda)(x) \neq (f_\Lambda \circ \rho_\Lambda)(y)$. □

1.5 Interactions

Most of the observables (energy, partition function) we are going to investigate later will be functions which depend on a given interaction. An interaction assigns to every subconfiguration (over a finite position region $\Lambda \in \mathcal{P}_f(\mathbb{L})$) a (complex or real) number, i. e., an interaction is a family of localised observables. In the physical interpretation this accounts to look at a finite number of particles and quantify their interactions.

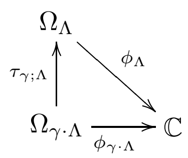
Definition 1.5.1. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1).

(i) An *interaction* is a family $(\phi_\Lambda)_{\Lambda \in \mathcal{P}_f(\mathbb{L})}$, indexed by all finite subsets $\Lambda \subset \mathbb{L}$, of continuous bounded mappings $\phi_\Lambda : \Omega_\Lambda \to \mathbb{C}$ with the property that $\phi_\Lambda(\xi_\Lambda) = 0$ if the empty spin $f_0 \in F$ from (1.1.1) belongs to the image of ξ_Λ.

(ii) Let $(\Omega, F, \mathbb{L}, \Gamma, \tau)$ be a lattice spin system (1.2.7). An interaction $\phi = (\phi_\Lambda)_{\Lambda \in \mathcal{P}_f(\mathbb{L})}$ is called Γ-*translation invariant*, if for all $\gamma \in \Gamma$ and all $\Lambda \in \mathcal{P}_f(\mathbb{L})$

$$\phi_{\gamma\cdot\Lambda} = \phi_\Lambda \circ \tau_{\gamma;\gamma\cdot\Lambda}$$

as functions $\Omega_{\gamma\cdot\Lambda} \to \mathbb{C}$, i. e. if the following diagram



commutes, where $\tau_{\gamma;\gamma\cdot\Lambda}$ is defined in (1.2.3).

(iii) An interaction $(\phi_\Lambda)_{\Lambda \in \mathcal{P}_f(\mathbb{L})}$ is called an *n-body interaction*, if n is the minimal integer such that $\phi_\Lambda = 0$ for all $\Lambda \in \mathcal{P}_f(\mathbb{L})$ with $|\Lambda| > n$. □

In Section 1.8 we will give some examples of two-body interactions. Interactions will occur as the "building blocks" of many interesting observables, e. g. the energy of a subconfiguration ξ_Λ over a finite region $\Lambda \subset \mathbb{L}$ is defined as the sum of all interactions of its subconfigurations $\rho_{N,\Lambda}(\xi_\Lambda)$ where $N \subset \Lambda$. When the physical observations of a given system suggest that the energy does not depend on the position of the region inside $\mathbb{L}$, but only on its volume, then one should choose a model with a Γ-translation invariant interaction, where Γ is a semigroup of translations on $\mathbb{L}$.

Remark 1.5.2. Interactions form a $\mathbb{C}$-vector space with respect to pointwise operations, which by definition means, given a complex number $c \in \mathbb{C}$ and two interactions $(\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ and $(\psi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$, we define

$$(\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})} + c\,(\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})} := (\phi_\Lambda + c\,\psi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}.$$

This leads to the interpretation that interactions form a subvector space of $\prod_{\Lambda\in\mathcal{P}_f(\mathbb{L})} \mathcal{C}_b(\Omega_\Lambda)$ defined via a certain vanishing condition. □

1.6 Energies

Given an interaction as introduced in Section 1.5, we define the energy of a subconfiguration. The total energy consists of two parts, the inner and the outer part. The *inner energy* comes from all interactions of subconfigurations inside this configuration - see Proposition 1.6.1. If we have fixed a boundary condition, observe that a subconfiguration determines a configuration on the whole position space via the boundary condition. The outer part of the energy counts the interactions between the inside and its extension, see Proposition 1.6.5. In order to make the infinite sums, which appear when one sums up all interactions, convergent, we have to introduce classes of interactions of certain suitable decay. We begin with the inner energy.

Proposition 1.6.1. *(Inner energy) Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1) and $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ an interaction (1.5.1) such that*

$$|\phi|_i := \sum_{\Lambda\in\mathcal{P}_f(\mathbb{L});\, \Lambda\ni i} \frac{1}{|\Lambda|} \sup_{\xi_\Lambda\in\Omega_\Lambda} |\phi_\Lambda(\xi_\Lambda)| < \infty \tag{18}$$

for all $i \in \mathbb{L}$. Then for all $\Lambda \in \mathcal{P}_f(\mathbb{L})$ the so called inner energy

$$\widetilde{U}^\phi_\Lambda : \Omega_\Lambda \to \mathbb{C},\ \xi_\Lambda \mapsto \sum_{\emptyset\neq M\subset\Lambda} \phi_M(\rho_{M,\Lambda}(\xi_\Lambda)) \tag{19}$$

is well-defined, depends linearly on the interaction, and $\|\widetilde{U}^\phi_\Lambda\|_{\mathcal{C}_b(\Omega_\Lambda)} = \sup_{\xi_\Lambda\in\Omega_\Lambda} |\widetilde{U}^\phi_\Lambda(\xi_\Lambda)| \leq \sum_{i\in\Lambda} |\phi|_i$.

Proof. For any $\Lambda \in \mathcal{P}_f(\mathbb{L})$ and any sequence $(a_M)_{M\subset\Lambda}$ we have

$$\sum_{\emptyset\neq M\subset\Lambda} a_M = \sum_{i\in\Lambda} \sum_{M\subset\Lambda; M\ni i} \frac{a_M}{|M|},$$

since every a_M is precisely counted $|M|$-times on the right hand side. Using this identity we get

$$\begin{aligned}\sup_{\xi_\Lambda\in\Omega_\Lambda} |\widetilde{U}^\phi_\Lambda(\xi_\Lambda)| &= \sup_{\xi_\Lambda\in\Omega_\Lambda} \Big|\sum_{i\in\Lambda} \sum_{M\subset\Lambda; M\ni i} \frac{1}{|M|}\phi_M(\rho_{M,\Lambda}(\xi_\Lambda))\Big| \\ &\leq \sum_{i\in\Lambda}\Big(\sum_{M\subset\Lambda; M\ni i} \frac{1}{|M|} \sup_{\xi_M\in\Omega_M} |\phi_M(\xi_M)|\Big) \leq \sum_{i\in\Lambda} |\phi|_i < \infty.\end{aligned}$$

The linearity then is obvious. In the following remark we comment on this peculiar upper bound. □

Remark 1.6.2. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space and $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ an interaction (1.5.1). Then the obvious bound ensuring the absolute convergence of the inner energy $\widetilde{U}^\phi_\Lambda : \Omega_\Lambda \to \mathbb{C}$ (19) is the following:

$$\sup_{\xi_\Lambda\in\Omega_\Lambda} |\widetilde{U}^\phi_\Lambda(\xi_\Lambda)| \leq \sum_{M\subset\Lambda} \sup_{\xi_M\in\Omega_M} |\phi_M(\xi_M)| \leq 2^{|\Lambda|} \max_{M\subset\Lambda} \sup_{\xi_M\in\Omega_M} |\phi_M(\xi_M)|,$$

since $\sum_{M\subset\Lambda} 1 = |\mathcal{P}(\Lambda)| = 2^{|\Lambda|}$. As far as the limit $|\Lambda| \to \infty$ is concerned, this bound seems to be worse than that from Proposition 1.6.1, which grows linearly, but the complexity is hidden in the computation of the seminorms $|\phi|_i$. □

Given two finite disjoint subsets Λ, $M \subset \mathbb{L}$ of the position space and an interaction ϕ, one can compare the inner energies $\widetilde{U}^{\phi}_{\Lambda\cup M}$, $\widetilde{U}^{\phi}_{\Lambda} + \widetilde{U}^{\phi}_{M}$ inside the regions Λ, M and $\Lambda \cup M$. The difference between these two terms was introduced by D. Ruelle [Ru78, 1.2].

Remark 1.6.3. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space, Λ, $M \subset \mathbb{L}$, $\Lambda \cap M = \emptyset$, and Λ finite. Let $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ be an interaction (1.5.1) such that

$$W_{\Lambda,M} : \Omega_{\Lambda\cup M} \to \mathbb{C},\ \xi \mapsto \sum_{N\in\mathcal{P}_f(\Lambda\cup M):N\cap\Lambda\neq\emptyset\neq N\cap M} \phi_N(\rho_{N,\Lambda\cup M}(\xi)) \tag{20}$$

converges absolutely. Note that $W_{\Lambda,M} : \Omega_{\Lambda\cup M} \to \mathbb{C}$ is well-defined, since by the definition of subconfigurations (1.1.2) and the restriction maps (1.1.3) we have the identity $\rho_{N,\Lambda\cup M}(\Omega_{\Lambda\cup M}) = (\rho_{N,\Lambda\cup M} \circ \rho_{\Lambda\cup M})(\Omega) = \rho_N(\Omega) = \Omega_N$. If the seminorms

$$\|\phi\|_i := \sum_{\Lambda\in\mathcal{P}_f(\mathbb{L});\,\Lambda\ni i} \sup_{\xi_\Lambda\in\Omega_\Lambda} |\phi_\Lambda(\xi_\Lambda)| \tag{21}$$

are finite for all $i \in \mathbb{L}$, then

$$|W_{\Lambda,M}(\xi)| \leq \sum_{N\in\mathcal{P}_f(\mathbb{L}):N\cap\Lambda\neq\emptyset} \sup_{\xi_N\in\Omega_N} |\phi_N(\xi_N)| \leq \sum_{i\in\Lambda} \sum_{N\in\mathcal{P}_f(\mathbb{L}):N\ni i} \sup_{\xi_N\in\Omega_N} |\phi_N(\xi_N)| = \sum_{i\in\Lambda} \|\phi\|_i.$$

If Λ, $M \subset \mathbb{L}$ are both finite, then one has

$$\widetilde{U}^{\phi}_{\Lambda\cup M}(\xi_{\Lambda\cup M}) = \widetilde{U}^{\phi}_{\Lambda}(\rho_{\Lambda,\Lambda\cup M}(\xi_{\Lambda\cup M})) + \widetilde{U}^{\phi}_{M}(\rho_{M,\Lambda\cup M}(\xi_{\Lambda\cup M})) + W_{\Lambda,M}(\xi_{\Lambda\cup M}).$$

□

The function $W_{\Lambda,M}$ will play a role in Remark 2.2.11 in the context of the leading eigenfunction of the Ruelle transfer operator. Given a boundary condition $(b_\Lambda)_{\Lambda\in\mathcal{P}}$ we will now define the total energy as the sum of the inner energy (19) and a term which depends on the boundary condition and the function $W_{\Lambda,M}$.

Definition 1.6.4. Given a configuration space $\Omega \subset F^{\mathbb{L}}$ (1.1.1), a boundary condition $(b_\Lambda : \Omega'_\Lambda \to \Omega)_{\Lambda\in\mathcal{P}}$ (1.3.3), and an interaction $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ (1.5.1) we define for $\Lambda \in \mathcal{P}$

$$U^{b,\phi}_\Lambda : b_\Lambda^{-1}(\Omega) \to \mathbb{C},\ U^{b,\phi}_\Lambda(\xi_\Lambda) := \widetilde{U}^{\phi}_\Lambda(\xi_\Lambda) + W_{\Lambda,\mathbb{L}\setminus\Lambda}(b_\Lambda(\xi_\Lambda)) \tag{22}$$

and call $U^{b,\phi}_\Lambda(\xi_\Lambda)$ the *total energy* of a subconfiguration $\xi_\Lambda \in b_\Lambda^{-1}(\Omega)$, provided the defining series converges. We say that an interaction ϕ is *compatible* with a boundary condition $(b_\Lambda)_{\Lambda\in\mathcal{P}}$ if $U^{b,\phi}_\Lambda(\xi_\Lambda)$ converges absolutely for all $\xi_\Lambda \in b_\Lambda^{-1}(\Omega)$ and $\Lambda \in \mathcal{P}$. □

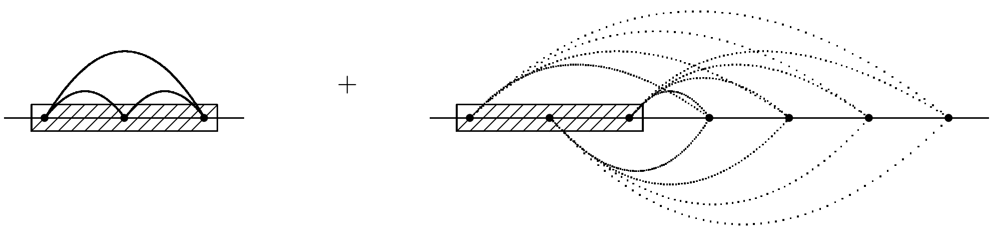


Figure 7: Total energy = inner energy (from interactions inside framed box, dots denote particles, arcs denote interacting pairs) + outer part (interactions between box and its outside, interaction strength decays with increasing distance).

The total energy is the sum of all interactions of spins over Λ with all other spins determined by the b_Λ-extension of ξ_Λ. By Remark 1.6.3 an interaction with the property that the seminorms $\|\phi\|_i$ (21) are finite for all $i \in \mathbb{L}$, is compatible with any boundary condition. For both types of boundary conditions introduced in (1.3.5) and (1.3.9), we will enlarge the class of compatible interactions, see Propositions 1.6.6 and 1.9.3.

Proposition 1.6.5. *Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space, $(b_\Lambda : \Omega'_\Lambda \to \Omega)_{\Lambda\in\mathcal{P}}$ a boundary condition (1.3.3), and $(\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ an interaction such that the seminorms $\|\phi\|_i$ (21) are finite for all $i \in \mathbb{L}$. Then*

$$U_\Lambda^{b,\phi} : b_\Lambda^{-1}(\Omega) \to \mathbb{C},\ U_\Lambda^{b,\phi}(\xi_\Lambda) = \sum_{M\in\mathcal{P}_f(\mathbb{L}), M\cap\Lambda\neq\emptyset} \phi_M(\rho_M \circ b_\Lambda(\xi_\Lambda)) \tag{23}$$

Proof. At first note that for all $\xi_\Lambda \in b_\Lambda^{-1}(\Omega)$ we have $(\rho_M \circ b_\Lambda)(\xi_\Lambda) \in \rho_M(\Omega) = \Omega_M$, hence the interaction $\phi_M(\rho_M \circ b_\Lambda(\xi_\Lambda))$ is defined. Consider the summands in (23) with $M \subset \Lambda$. Note that b_Λ is a (partial) right inverse of ρ_Λ, i. e., $(\rho_\Lambda \circ b_\Lambda)(\xi_\Lambda) = \xi_\Lambda$ for all $\xi_\Lambda \in b_\Lambda^{-1}(\Omega)$. The sum of those terms gives exactly $\widetilde{U}_\Lambda^\phi$. The condition $\|\phi\|_i < \infty$ ensures the absolute convergence, hence we can rearrange terms to get $U_\Lambda^{b,\phi} = \widetilde{U}_\Lambda^\phi + W_{\Lambda,\mathbb{L}\setminus\Lambda} \circ b_\Lambda$. □

In the special case of a zero boundary condition (1.3.5) the total energy (22) is just the inner energy (19) and the condition ensuring the absolute convergence of its defining sum can be weakened.

Proposition 1.6.6. *(Inner energy) Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space admitting the zero boundary condition $(b_\Lambda^0)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ (1.3.5) and $(\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{L})}$ an interaction such that the seminorms $|\phi|_i$ (18) are finite for all $i \in \mathbb{L}$. Then for all $\Lambda \in \mathcal{P}_f(\mathbb{L})$ the inner energy coincides with the total enery for the zero boundary condition, i. e., $U_\Lambda^{b^0,\phi} = \widetilde{U}_\Lambda^\phi$ as functions on $(b_\Lambda^0)^{-1}(\Omega)$.*

Proof. We use the definition (1.3.5) of the zero boundary condition and the fact that an interaction vanishes by definition on subconfigurations which contain the empty spin $f_0 \in F$, hence the difference term vanishes, $(W_{\Lambda,\mathbb{L}\setminus\Lambda} \circ b_\Lambda^0)(\xi_\Lambda) = 0$ for all $\xi_\Lambda \in (b_\Lambda^0)^{-1}(\Omega)$. □

Remark 1.6.7. (i) Let $\{\|\cdot\|_i;\ i \in \mathbb{L}\}$, $\{|\cdot|_i;\ i \in \mathbb{L}\}$ be the families of seminorms introduced in (18) and (21), respectively. Set

$$\mathcal{A} := \{\phi \in \prod_{\Lambda\in\mathcal{P}_f(\mathbb{L})} \mathcal{C}_b(\Omega_\Lambda);\ \forall i \in \mathbb{L} : \|\phi\|_i < \infty\} \text{ and } \mathcal{A}^0 := \{\phi \in \prod_{\Lambda\in\mathcal{P}_f(\mathbb{L})} \mathcal{C}_b(\Omega_\Lambda);\ \forall i \in \mathbb{L} : |\phi|_i < \infty\}.$$

(ii) By Propositions 1.6.1 and 1.6.5 the total energy for a general boundary condition is well-defined and bounded if the interaction belongs to $\mathcal{A}$; for the zero boundary condition $\mathcal{A}^0$ suffices.

(iii) If ϕ is a Γ-translation invariant interaction, then the seminorms $\|\cdot\|_i$ and $\|\cdot\|_{\gamma\cdot i}$ are equivalent for an arbitrary point $i \in \mathbb{L}$ and $\gamma \in \Gamma$, similarly $|\cdot|_i$ and $|\cdot|_{\gamma\cdot i}$. □

From Section 1.8 on we will focus mainly on two-body interactions. In Section 1.9 we will give (necessary and sufficient) conditions on two-body interaction such that the total energy converges absolutely. This will be applied in Example 1.9.7 where we give some examples of interactions belonging to the class $\mathcal{A}$ defined in (1.6.7).

1.7 Partition functions

We will now introduce the main object of this chapter, the so called partition function, which encodes many statistical properties of a system. The partition function depends on the temperature, the volume, and the microstates of a finite number of particles. Many of the thermodynamic variables of the system, such as the total energy, free energy, entropy, and pressure, can be expressed in terms of the partition function, its derivatives, and their asymptotic behaviour.

Definition 1.7.1. Let F be a Hausdorff space with a finite Borel measure ν, called the *a priori measure*. For $\Lambda \in \mathcal{P}_f(\mathbb{L})$ let ν^Λ be the product measure on F^Λ. Given a configuration space $\Omega \subset F^{\mathbb{L}}$ (1.1.1), an interaction ϕ which is compatible with the boundary condition $b = (b_\Lambda)_{\Lambda\in\mathcal{P}}$ (1.6.4) such that the total energy $U_\Lambda^{b,\phi}$ belongs to $L^\infty(b_\Lambda^{-1}(\Omega), \nu^\Lambda)$ for all $\Lambda \in \mathcal{P}$, we define the *partition function with boundary condition b* as

$$Z_\Lambda^{b,\phi}(\beta) := \int_{b_\Lambda^{-1}(\Omega)} \exp\big(-\beta\, U_\Lambda^{b,\phi}(\xi_\Lambda)\big)\, d\nu^\Lambda(\xi_\Lambda),$$

where $\beta \in \mathbb{C}$ is a parameter, called the (scaled) *inverse temperature.* □

In this picture the microstates are just the subconfigurations $\xi_\Lambda \in (b_\Lambda)^{-1}(\Omega)$. Every microstate has a certain energy from which by integration the partition function is obtained. The partition function in turn allows to determine other statistical properties of the system.

Remark 1.7.2. Since we suppose in Def. 1.7.1 the total energy to be bounded, the partition function $Z_\Lambda^{b,\phi}$ exists as an integral over a ν^Λ-almost everywhere bounded function with respect to a finite measure. In particular, for interactions belonging to the classes $\mathcal{A}$ and $\mathcal{A}_0$ as defined in (1.6.7) the energy functions $U_\Lambda^{b,\phi}$ are continuous and (everywhere) bounded as we have seen in Remark 1.6.7. Note that for the parameter β equal to zero, the partition function is nothing but the volume $Z_\Lambda^{b,\phi}(0) = \nu^\Lambda(b_\Lambda^{-1}(\Omega))$. □

Our definition 1.7.1 of the partition function is motivated by the so called canonical ensemble. In statistical thermodynamics an ensemble is the collection of all configurations of a fixed system, e. g. the canonical ensemble is the collection of all configurations with constant number of particles, constant volume and constant temperature. If there is a unique probability measure on the ensemble, this is often called ensemble also.

Example 1.7.3. (Gibbs measure) Let F be a Hausdorff space with a finite Borel measure ν. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space and ϕ an interaction which is compatible (1.6.4) with the zero boundary condition $b^0 = (b_\Lambda^0)_{\Lambda \in \mathcal{P}_f(\mathbb{L})}$ (1.3.5). For fixed positive real β the *Gibbs measure* or *canonical ensemble* for a finite region $\Lambda \subset \mathbb{L}$ is the probability measure on $(b_\Lambda^0)^{-1}(\Omega)$ which has the density

$$\mu_\Lambda(\xi_\Lambda) := (Z_\Lambda^{b^0,\phi})^{-1}\ \exp\big(-\beta\, U_\Lambda^{b^0,\phi}(\xi_\Lambda)\big)$$

with respect to ν^Λ. The parameter $\beta = 1/kT$ is interpreted as Boltzmann's factor, k the Boltzmann constant, and T the absolute temperature. This explains the notion of (scaled) inverse temperature for β. The physical model behind it is the following: Given a system with possible states $\xi_\Lambda \in (b_\Lambda^0)^{-1}(\Omega)$ such that a microstate ξ_Λ has the energy $U_\Lambda^{b^0,\phi}(\xi_\Lambda)$ and an exterior large source of heat which is at temperature T, then it is a physical fact, see [Bo75, p. 4], that $\mu_\Lambda(\xi_\Lambda)$ is the probability to find the system in the state ξ_Λ (after long time). □

Since we know the domain of integration in the case of the periodic boundary condition by Proposition 1.3.14, we can rewrite the partition function as follows.

Corollary 1.7.4. *Let F be a Hausdorff space with a finite Borel measure ν, $(\Omega, F, \mathbb{L}, \Gamma, \tau)$ a lattice spin system (1.2.7), let $b^\Gamma = (b_\Lambda^\Gamma)_{\Lambda \in \mathcal{P}_{(\Gamma_\bullet)}}$ be the periodic boundary condition (1.3.9) associated with the family $(\Gamma_\alpha)_{\alpha \in A}$ of subsemigroups of Γ, and ϕ a compatible interaction (1.6.4) such that $U_\Lambda^{b,\phi} \in L^\infty(b_\Lambda^{-1}(\Omega), \nu^\Lambda)$. Let $N_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$ be a fundamental domain of Γ_α, then*

$$Z_{N_\alpha}^{b^\Gamma,\phi}(\beta) = \int_{\rho_{N_\alpha}(\bigcap_{\gamma \in \Gamma_\alpha} \mathrm{Fix}(\tau_\gamma))} \exp\big(-\beta\, U_{N_\alpha}^{b^\Gamma,\phi}(\xi_{N_\alpha})\big)\, d\nu^{N_\alpha}(\xi_{N_\alpha}). \tag{24}$$

In particular, if F is finite and ν is the counting measure, then

$$Z_{N_\alpha}^{b^\Gamma,\phi}(\beta) = \sum_{\xi \in \bigcap_{\gamma \in \Gamma_\alpha} \mathrm{Fix}(\tau_\gamma)} \exp\big(-\beta\, U_{N_\alpha}^{b^\Gamma,\phi}(\rho_{N_\alpha}(\xi))\big). \tag{25}$$

Proof. Combine the definition of the partition function (1.7.1) with Proposition 1.3.14. □

Proposition 1.3.14 and its Corollary 1.7.4 show that partition functions can be expressed as sums over fixed points. This leads to a dynamical interpretation of the partition function and will be used together with Section 1.11 in the construction of the Ruelle type transfer operator in Chapter 2.

1.8 Two-body interactions

In the following we will restrict our considerations to the case of two-body interactions and give some examples. Recall from (1.5.1) that an interaction $\phi = (\phi_\Lambda)_{\Lambda \in \mathcal{P}_f(\mathbb{L})}$ is called two-body if $\phi_\Lambda = 0$ for all finite subsets $\Lambda \subset \mathbb{L}$ with cardinality $|\Lambda| > 2$. Two-body interactions occur where interaction forces superpose without further interference, i. e., the energy of a three particle configuration is the sum of the interactions of all possible pairs and singletons. Of particular interest are those interactions which are given via an interaction matrix, an anisotropy matrix (respectively, a distance function), and a potential. Among them are the following physical models: Ising model, Potts model, and Stanley's M-vector model. We introduce the new class of Ising type interactions which contains both Ising model and Stanley's M-vector model. We will show later that for the special case of a finite alphabet F every interaction matrix is of Ising type.
We start with the normal form of a two-body interaction and define a special type of two-body interactions, which will be of interest later on.

Definition 1.8.1. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1) and $\phi = (\phi_\Lambda)_{\Lambda \in \mathcal{P}_f(\mathbb{L})}$ be a two-body interaction (1.5.1). We can always write ϕ_Λ as

$$\phi_\Lambda : \Omega_\Lambda \to \mathbb{C},\ \xi_\Lambda \mapsto \phi_\Lambda(\xi_\Lambda) := \begin{cases} -\varphi_1(i;\xi_i) & \text{, if } \Lambda = \{i\},\ \xi_\Lambda = (\xi_i), \\ -\varphi_2(i,j;\xi_i,\xi_j) & \text{, if } \Lambda = \{i,j\},\ \xi_\Lambda = (\xi_i,\xi_j),\ (i \neq j), \\ 0 & \text{, otherwise,} \end{cases} \tag{26}$$

where $\xi_i, \xi_j \in F$, $\Lambda \in \mathcal{P}_f(\mathbb{L})$ and $\varphi_1 : \mathbb{L} \times F \to \mathbb{C}$, $\varphi_2 : \mathbb{L}^2 \times F^2 \to \mathbb{C}$ are some functions[7].

(i) If $\varphi_2(i,j;x,y) = \varphi_2(j,i;y,x) \quad \forall i,j \in \mathbb{L},\ x,y \in F$, then ϕ is called *symmetric.*

(ii) ϕ is called a *pure two-body interaction* if $\varphi_1 = 0$.

(iii) If ϕ is of the form

$$\phi_\Lambda(\xi_\Lambda) = \begin{cases} -q(\xi_i) & \text{, if } \Lambda = \{i\},\ \xi_\Lambda = (\xi_i), \\ -\tilde{d}(i,j)\, r(\xi_i,\xi_j) & \text{, if } \Lambda = \{i,j\},\ \xi_\Lambda = (\xi_i,\xi_j),\ (i \neq j), \\ 0 & \text{, otherwise,} \end{cases}$$

where $r : F \times F \to \mathbb{C}$ and $\tilde{d} : \mathbb{L} \times \mathbb{L} \to \mathbb{C}$ are symmetric functions[8], $q : F \to \mathbb{C}$, and $\xi_i, \xi_j \in F$, $\Lambda \subset \mathbb{L}$, then ϕ is called a *two-body interaction* with *interaction matrix* r, *anisotropy matrix* $\tilde{d}$ and *potential* q. Denote by $\phi^r_{\tilde{d},q}$ the two-body interaction with anisotropy matrix $\tilde{d}$, potential q, and interaction matrix r. Such interactions are automatically symmetric. □

In (3.4.4) we will introduce another type of two-body interactions, the so called hard rods interaction. The restriction to two-body interactions simplifies many arguments, for instance, the energy can be calculated quite explicitly. Before doing this we give examples how an anisotropy matrix or an interaction matrix can look like, see Remark 1.8.2 and Example 1.8.3.

Remark 1.8.2. Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1).

(i) Let ϕ be a two-body interaction on Ω of the form (26). Then ϕ is Γ-translation invariant (1.5.1) iff φ_1 and φ_2 are Γ-invariant in the following sense: $\varphi_1(\gamma \cdot i; x) = \varphi_1(i;x)$ and $\varphi_2(\gamma \cdot i, \gamma \cdot j; x, y) = \varphi_2(i,j;x,y)$ for all $i,j \in \mathbb{L}, \gamma \in \Gamma, x,y \in F$. In fact: Let $\Lambda = \{i\} \subset \mathbb{L}$, $\xi_\Lambda : i \mapsto x \in F$, $\gamma \in \Gamma$, then $-\phi_{\gamma \cdot \Lambda}(\xi_\Lambda) = \varphi_1(\gamma \cdot i; x) = \varphi_1(i;x) = -\phi_\Lambda \circ \tau_{\gamma;\gamma\cdot\Lambda}(\xi_\Lambda)$. Similarly, let $\Lambda = \{i,j\} \subset \mathbb{L}$, $\xi_\Lambda = (\xi_i, \xi_j)$, $\gamma \in \Gamma$, then $-\phi_{\gamma\cdot\Lambda}(\xi_\Lambda) = \varphi_2(\gamma \cdot i, \gamma \cdot j; \xi_i, \xi_j) = \varphi_2(i,j;\xi_i,\xi_j) = -\phi_\Lambda \circ \tau_{\gamma;\gamma\cdot\Lambda}(\xi_\Lambda)$.

(ii) If $\mathbb{L}$ is a group and $d : \mathbb{L} \to \mathbb{C}$ is a function satisfying the symmetry condition $d(i) = d(i^{-1})$, then

$$\tilde{d} : \mathbb{L} \times \mathbb{L} \to \mathbb{C},\ \tilde{d}(i,j) := d(i^{-1}j)$$

defines an anisotropy matrix, i. e., $\tilde{d}(i,j) = \tilde{d}(j,i)$. Such a function $d : \mathbb{L} \to \mathbb{C}$ is called a *distance function.*

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[7] $\varphi_2(i,i;x,x)$ can be chosen arbitrarily.

[8] A function $f : X \times X \to Y$ is symmetric if $f(x_1,x_2) = f(x_2,x_1)$ for all $x_1, x_2 \in X$. We denote the space of symmetric functions by $\{f : X \times X \to Y\}^{\mathbb{Z}_2}$.

(iii) If $\mathbb{L}$ is a group, $\Gamma \leq \mathbb{L}$ a subgroup of $\mathbb{L}$, which acts by left translation on $\mathbb{L}$, i. e., $\Gamma \times \mathbb{L} \to \mathbb{L}$, $(\gamma, g) \mapsto \gamma \cdot g := \gamma g$, and $\tilde{d}$ is defined as in part (ii) via a distance function, then every two-body interaction with anisotropy matrix $\tilde{d}$, potential q, and interaction matrix r is Γ-invariant, since $(\gamma i)^{-1}(\gamma j) = i^{-1}\gamma^{-1}\gamma j = i^{-1}j$.

(iv) As a partial converse of part (iii): Let $\Gamma = \mathbb{Z}$ act on $\mathbb{L} = \mathbb{Z}$ by left translation, then

$$\tilde{d} \mapsto \big(n \mapsto d(n) := \tilde{d}(0, n)\big)$$

defines a $\mathbb{C}$-linear isomorphism between the space of $\mathbb{Z}$-invariant anisotropy matrices and the even functions $d : \mathbb{Z} \to \mathbb{C}$. In fact: Using the invariance and the symmetry of d we obtain

$$d(-n) = \tilde{d}(0, -n) = \tilde{d}(n, n - n) = \tilde{d}(n, 0) = \tilde{d}(0, n) = d(n),$$

hence the map is well-defined. Its linearity and injectivity are obvious, it is surjective by (iii). □

The next examples introduce the widely studied physical models, namely the Ising model, the Potts model, and Stanley's M-vector model.

Example 1.8.3. (Physical models)

(i) Ising model: Let $F \subset \mathbb{C}$ be a bounded set and $r(x, y) = xy$. In Ising's original model he took $F = \{\pm 1\}$, the so called spin-$\frac{1}{2}$ model, in order to describe ferromagnetism of a solid, where the spins of the electrons can only take values in a set with two elements, "spin up" or "spin down".

(ii) Potts model: Let F be any set and $r(x, y) = \delta(x, y)$, where $\delta : F \times F \to \mathbb{C}$ is Kronecker's delta on F. This model is due to R. Potts [Po52] and describes the situation where only electrons having the same spin (members of the same species) interact.

(iii) An interaction matrix r is called *of Ising type* if

$$r(x, y) = \sum_{k=1}^{l} a_i(x)\, b_i(y)$$

for some functions $a_i, b_i : F \to \mathbb{C}$. The minimal number l is called the *rank* of r. □

Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space , $\tilde{d}$ an anisotropy matrix $\tilde{d}$, and q be a potential q. We call Ω when equipped with the two-body interaction $\phi^{r}_{\tilde{d},q}$ (1.8.1) an *Ising model*, *Potts model*, or *Ising type model*, if the interaction matrix r is of the form (i), (ii), or (iii) respectively.

With respect to the construction of transfer operators, Ising type interactions can be treated by a superposition principle which we will describe in Section 2.13. This has been observed for instance in [May80a, p. 98]. Note: If F is finite, then *every* interaction matrix is of Ising type as we will explain in Section 2.13. The prototype of an Ising type model is Stanley's M-vector model (see [St68a], [St74]), which we discuss next.

Example 1.8.4. Let $s > 0$, $M \in \mathbb{N}$, $\mathbb{L}$ a countable set, and $\tilde{d} : \mathbb{L} \times \mathbb{L} \to \mathbb{C}$ be a symmetric function. The (generalised[9]) Stanley M-vector model is defined by the following data: The spins take values in the $(M - 1)$-sphere with radius s, i. e.,

$$F := S_{M-1}(s) := \big\{\sigma = (\sigma^{(1)}, \dots, \sigma^{(M)}) \in \mathbb{R}^M \;:\; \sum_{i=1}^{M} |\sigma^{(i)}|^2 = s^2\big\},$$

equipped with the (normalised) surface measure ν on F, and the configuration space is $\Omega := F^{\mathbb{L}}$. The interaction is the two-body interaction with anisotropy matrix $\tilde{d}$ and interaction matrix $r(x, y) = \big(x \,\big|\, y\big) := \sum_{i=1}^{M} x_i\, y_i$, i. e.,

$$\phi_\Lambda(\xi_\Lambda) = \begin{cases} -q(\xi_i) & \text{, if } \Lambda = \{i\},\ \xi_\Lambda = (\xi_i), \\ -\frac{J}{2}\, \tilde{d}(i, j)\ \big(\xi_i \,\big|\, \xi_j\big) & \text{, if } \Lambda = \{i, j\},\ \xi_\Lambda = (\xi_i, \xi_j), \\ 0 & \text{, otherwise,} \end{cases}$$

[9]Stanley has considered these models only for finite range interactions.

where $J \in \mathbb{C}$ is the energy of a pair of parallel spins and $q : F \to \mathbb{C}$ is a potential. Note that r has rank M (1.8.3).
The following table gives a list of physical models which can be seen as applications of Stanley's M-vector model. Depending on the parameter M these models have special names.

Rank	Special name	System
1	Ising model	one-component fluid, binary alloy, mixture
2	Planar model/Vaks-Larkin model	λ-transition in a Bose fluid
3	Heisenberg model	(anti-)ferromagnetism
$M > 3$	general M-vector model	no physical system discovered yet
∞	Stanley spherical model	no physical system discovered yet

This table has appeared in [St74, p. 488] with lots of references therein to the physical models. Note that the rank 1 case gives $F = S_0(1) = \{\pm 1\}$ and hence the spin-$\frac{1}{2}$ Ising model, see (1.8.3). The rank ∞ case is treated in [St68b]. □

Another non-trivial example is the following: Let $F \subseteq \mathrm{Mat}(D, D; \mathbb{C})$ be a bounded set, for example a compact matrix group, and the interaction matrix r be defined by $r(x, y) := \text{trace}\ (xy)$. Then r is symmetric and of Ising type, since it can be written as

$$r(x, y) = \text{trace}\ (xy) = \sum_{i=1}^{D} (xy)_{i,i} = \sum_{i,j=1}^{D} x_{i,j}\, y_{j,i},$$

for $x = (x_{i,j})_{i,j=1,\ldots,D}$ and $y = (y_{i,j})_{i,j=1,\ldots,D}$. As a generalisation of this example any $\mathbb{R}$-bilinear symmetric form on a finite-dimensional normed vector space V in which F is contained as a bounded set defines an Ising type interaction matrix.

1.9 Energy and partition function in the case of two-body interactions

We will now determine the class of two-body interactions (1.8.1) which are compatible in the sense of (1.6.4) with a periodic boundary condition. For this purpose we decompose the total energy into a pure one-body term and a pure two-body interaction term and discuss the necessary decay conditions. Let Γ be a semigroup acting on the position space $\mathbb{L}$ and $(\Gamma_\alpha)_{\alpha \in A}$ a family of subsemigroups of Γ. A key step is Proposition 1.9.3 which states that a symmetric two-body interaction ϕ is compatible (1.6.4) with the periodic boundary condition $(b^{\Gamma}_{\Lambda})_{\Lambda \in \mathcal{P}_{(\Gamma_\bullet)}}$ associated to $(\Gamma_\alpha)_{\alpha \in A}$ iff $\sum_{\gamma \in \Gamma_\alpha} \varphi_2(i, \gamma \cdot l; x, y)$ converges absolutely for all $i, l \in \Lambda$, $x, y \in F$, $\alpha \in A$. This condition can be checked in particular easily if the two-body interaction is given via a distance function, an interaction matrix, and a potential. We discuss some examples of such distance functions in Example 1.9.7. Then we compute the partition function in the case of a two-body interaction and discuss special situations where the integrand can be simplified further. In the case of an Ising spin system (1.8.3) with vanishing potential we finally obtain a representation

$$Z^{b^\Gamma, \phi}_{\Lambda}(\beta) = \int_{(b^{\Gamma}_{\Lambda})^{-1}(\Omega)} \exp\Big(\beta \sum_{i,l \in \Lambda} \hat{t}^{\Gamma_\alpha}_{i,l}\, \xi_i\, \xi_l\Big)\, d\nu^{\Lambda}(\xi_\Lambda),$$

where $(\hat{t}^{\Gamma_\alpha}_{i,l})_{i,l}$ is a symmetric quadratic matrix and hence the integrand should be viewed as the exponential of a quadratic form in $(\xi_1, \ldots, \xi_n) \in F^n$. This is a generalisation of the situation [HiMay02, p. 26] in the construction of the Kac-Gutzwiller integral operator. We would like to stress that this representation only depends on the fact that we have a periodic boundary condition defined via an orbit relation. We start with the decomposition of the energy $U^{b,\phi}_{\Lambda}$ (1.6.4) into a potential term and a pure two-body interaction term.

Proposition 1.9.1. *Let $\Omega \subset F^{\mathbb{L}}$ be a configuration space (1.1.1), ϕ be a symmetric two-body interaction (1.8.1) compatible with the boundary condition $b = (b_\Lambda)_{\Lambda \in \mathcal{P}}$ (1.6.4), and $\Lambda \in \mathcal{P}$. Then*

$$U^{b,\phi}_{\Lambda} = U^{b,\phi}_{\Lambda,1} + U^{b,\phi}_{\Lambda,2},$$

where

$$U^{b,\phi}_{\Lambda,1} : b_\Lambda^{-1}(\Omega) \to \mathbb{C},\ \xi_\Lambda \mapsto -\sum_{i\in\Lambda} \varphi_1(i;\xi_\Lambda(i))$$

is the so called magnetic potential term and $U^{b,\phi}_{\Lambda,2} : b_\Lambda^{-1}(\Omega) \to \mathbb{C}$ *defined via*

$$U^{b,\phi}_{\Lambda,2}(\xi_\Lambda) = -\frac{1}{2}\sum_{i\in\Lambda}\sum_{k\in\Lambda\setminus\{i\}} \varphi_2\Big(i,k;b_\Lambda(\xi_\Lambda)(i),b_\Lambda(\xi_\Lambda)(k)\Big) - \sum_{i\in\Lambda}\sum_{k\in\mathbb{L}\setminus\Lambda} \varphi_2\Big(i,k;b_\Lambda(\xi_\Lambda)(i),b_\Lambda(\xi_\Lambda)(k)\Big)$$

is the so called pure two-body interaction term.

Proof. Let $\xi_\Lambda \in b_\Lambda^{-1}(\Omega)$. By our assumptions the series defining $U^{b,\phi}_\Lambda(\xi_\Lambda)$ is absolutely convergent. We use the explicit form of the total energy given in Proposition 1.6.5 and change the order of summation. We collect the first summands where $M \in \mathcal{P}_f(\mathbb{L})$ with $M\cap\Lambda \neq \emptyset$ is a singleton, and then the summands where M consists of two points. We distinguish further whether both points belong to Λ or not.

$$\begin{aligned}
U^{b,\phi}_\Lambda(\xi_\Lambda) &= \sum_{M\in\mathcal{P}_f(\mathbb{L}),M\cap\Lambda\neq\emptyset} \phi_M(\rho_M\circ b_\Lambda(\xi_\Lambda)) \\
&= \sum_{M\in\mathcal{P}_f(\mathbb{L}),M\cap\Lambda\neq\emptyset,|M|=1} \phi_M(\rho_M\circ b_\Lambda(\xi_\Lambda)) + \sum_{M\in\mathcal{P}_f(\mathbb{L}),M\cap\Lambda\neq\emptyset,|M|=2} \phi_M(\rho_M\circ b_\Lambda(\xi_\Lambda)) \qquad (27)\\
&= -\sum_{i\in\Lambda}\varphi_1(i;\xi_\Lambda(i)) - \frac{1}{2}\sum_{i\in\Lambda}\sum_{k\in\Lambda\setminus\{i\}} \varphi_2\Big(i,k;b_\Lambda(\xi_\Lambda)(i),b_\Lambda(\xi_\Lambda)(k)\Big) \\
&\quad -\sum_{i\in\Lambda}\sum_{k\in\mathbb{L}\setminus\Lambda} \varphi_2\Big(i,k;b_\Lambda(\xi_\Lambda)(i),b_\Lambda(\xi_\Lambda)(k)\Big) \qquad (28)\\
&= U^{\phi}_{\Lambda,1}(\xi_\Lambda) + U^{b,\phi}_{\Lambda,2}(\xi_\Lambda),
\end{aligned}$$

since $b_\Lambda(\xi_\Lambda)(k) = \xi_\Lambda(k)$ for all $k \in \Lambda$ by definition of the boundary extension (1.3.1). □

The only influence of the boundary condition on the magnetic potential term is the domain $b_\Lambda^{-1}(\Omega)$ where it is defined on. In any case, its defining sum is a finite sum since $\mathcal{P} \subset \mathcal{P}_f(\mathbb{L})$ and hence does not influence the convergence of the total energy. If b is the zero boundary condition, then all sums in Proposition 1.9.1 have only finitely many non-zero summands and thus there is no convergence problem. Another set of examples are the so called finite-range interactions:

Example 1.9.2. (Finite range interaction) Let $(\mathbb{L},\rho)$ be a countable metric space and $\Omega \subset F^{\mathbb{L}}$ a configuration space. A two-body interaction ϕ has so called *finite range* ρ_0 if and only if

$$\varphi_2(i,j;x,y) = 0$$

for all $x, y \in F$ whenever $\rho(i,j) > \rho_0$ and ρ_0 is minimal with this property. For any boundary condition $(b_N)_{N\in\mathcal{P}}$ and $N \in \mathcal{P}$ one has the representation

$$U^{b,\phi}_{N,2}(\xi_N) = \sum_{i\in N}\sum_{k\in\mathbb{L}:0<\rho(i,k)\leq\rho_0} \varphi_2\Big(i,k;b_N(\xi_N)(i),b_N(\xi_N)(k)\Big)$$

of the pure two-body interaction term (1.9.1) which is a finite sum. □

Proposition 1.9.3. *Let* $(\Omega, F, \mathbb{L}, \Gamma, \tau)$ *be a lattice spin system (1.2.7) and* $(b^\Gamma_\Lambda)_{\Lambda\in\mathcal{P}_{(\Gamma_\bullet)}}$ *a periodic boundary condition associated to the family* $(\Gamma_\alpha)_{\alpha\in A}$ *of subsemigroups of* Γ *such that every* $\Lambda_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$ *satisfies the tiling condition (1.3.8). Then a symmetric two-body interaction* ϕ *is compatible (1.6.4) with the boundary condition iff* $\sum_{\gamma\in\Gamma_\alpha}\varphi_2(i,\gamma\cdot l;x,y)$ *converges absolutely for all* $i, l \in \Lambda$, $x, y \in F$, $\alpha \in A$.

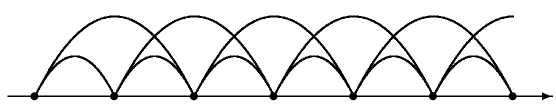

Figure 8: A (pure) two-body interaction with range two: Arcs denote interacting particles.

Proof. Let $\Lambda := \Lambda_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$ be a fundamental domain of the subsemigroup $\Gamma_\alpha \leq \Gamma$ and let $\xi_\Lambda \in \Omega'_\Lambda := (b^\Gamma_\Lambda)^{-1}(\Omega)$. The absolute convergence of the total energy $U^{b^\Gamma,\phi}_\Lambda(\xi_\Lambda)$ is equivalent to the absolute convergence of $\sum_{i\in\Lambda}\sum_{k\in\mathbb{L}}\varphi_2\big(i,k;b^\Gamma_\Lambda(\xi_\Lambda)(i),b^\Gamma_\Lambda(\xi_\Lambda)(k)\big)$, since we can disregard a finite number of summands (for instance the magnetic potential term and the first term of the pure two-body interaction term (1.9.1) in equation (28)). Since $\Lambda \subset \mathbb{L}$ is finite, this (double-) series converges absolutely, if and only if the inner series $\sum_{k\in\mathbb{L}}\varphi_2\big(i,k;b^\Gamma_\Lambda(\xi_\Lambda)(i),b^\Gamma_\Lambda(\xi_\Lambda)(k)\big)$ converges absolutely for all $i \in \Lambda$. We now use the definition of the periodic boundary extension (1.3.9) and the tiling condition $\Gamma_\alpha \cdot \Lambda = \mathbb{L}$. If the convergence is absolute, we can rearrange terms in the following way

$$\begin{aligned}\sum_{k\in\mathbb{L}}\varphi_2\big(i,k;b^\Gamma_\Lambda(\xi_\Lambda)(i),b^\Gamma_\Lambda(\xi_\Lambda)(k)\big) &= \sum_{k\in\Gamma_\alpha\cdot\Lambda}\varphi_2\big(i,k;b^\Gamma_\Lambda(\xi_\Lambda)(i),b^\Gamma_\Lambda(\xi_\Lambda)(k)\big)\\ &= \sum_{l\in\Lambda}\sum_{\gamma\in\Gamma_\alpha}\varphi_2\big(i,\gamma\cdot l;b^\Gamma_\Lambda(\xi_\Lambda)(i),b^\Gamma_\Lambda(\xi_\Lambda)(\gamma\cdot l)\big)\\ &= \sum_{l\in\Lambda}\sum_{\gamma\in\Gamma_\alpha}\varphi_2\big(i,\gamma\cdot l;b^\Gamma_\Lambda(\xi_\Lambda)(i),b^\Gamma_\Lambda(\xi_\Lambda)(l)\big)\\ &= \sum_{l\in\Lambda}\sum_{\gamma\in\Gamma_\alpha}\varphi_2\big(i,\gamma\cdot l;\xi_\Lambda(i),\xi_\Lambda(l)\big) \qquad (29)\end{aligned}$$

since $b^\Gamma_\Lambda(\xi_\Lambda)(k) = \xi_\Lambda(k)$ for all $k \in \Lambda$ by definition of the boundary extension. Hence the absolute convergence of $U^{b^\Gamma,\phi}_\Lambda(\xi_\Lambda)$ is equivalent to the absolute convergence of $\sum_{\gamma\in\Gamma_\alpha}\varphi_2(i,\gamma\cdot l;x,y)$ for all $i,l \in \Lambda$, $x,y \in F$. □

Remark 1.9.4. In the case of a two-body interaction with interaction matrix r and anisotropy matrix $\tilde{d}$ expression (29) reduces to $\sum_{k\in\Lambda} r\big(\xi_\Lambda(i),\xi_\Lambda(k)\big)\sum_{\gamma\in\Gamma_\alpha}\tilde{d}(i,\gamma\cdot k)$. Hence the absolute convergence of the total energy is equivalent to the absolute convergence of $\sum_{\gamma\in\Gamma_\alpha}\tilde{d}(i,\gamma\cdot l)$ for all $i,l \in \Lambda$. Since $\Gamma_\alpha \subset \Gamma$, this can be bounded by $\sum_{\gamma\in\Gamma}|\varphi_2(i,\gamma\cdot l;x,y)|$, respectively by $\sum_{\gamma\in\Gamma}|\tilde{d}(i,\gamma\cdot l)|$. □

Corollary 1.9.5. *Let $(\Omega,F,\mathbb{L},\Gamma,\tau)$ be a lattice spin system (1.2.7). Let $b^\Gamma = (b^\Gamma_\Lambda)_{\Lambda\in\mathcal{P}_{(\Gamma_\bullet)}}$ be a periodic boundary condition associated to the family $(\Gamma_\alpha)_{\alpha\in A}$ of subsemigroups of Γ such that every $\Lambda_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$ satisfies the tiling condition (1.3.8). Let $\phi^r_{\tilde{d},q}$ be the two-body interaction (1.8.1) with anisotropy matrix $\tilde{d}$, potential q, and interaction matrix $r \in \mathcal{C}_b(F\times F)^{\mathbb{Z}_2}$. Then $\phi^r_{\tilde{d},q}$ is compatible with the boundary condition b^Γ in the sense of (1.6.4) if $\sum_{\gamma\in\Gamma}|\tilde{d}(i,\gamma\cdot l)| < \infty$ for all $i,l \in \Lambda$. In this case the map*

$$\mathcal{C}_b(F\times F)^{\mathbb{Z}_2} \to \mathcal{C}_b((b^\Gamma_\Lambda)^{-1}(\Omega)),\ r \mapsto U^{b^\Gamma,\phi^r_{\tilde{d},q}}_{\Lambda,2}$$

is linear and continuous.

Proof. The compatibility of the interaction was shown in Remark 1.9.4. The linearity of the mapping $\mathcal{C}_b(F\times F)^{\mathbb{Z}_2} \to \mathcal{C}_b(\Omega'_\Lambda)$, $r \mapsto U^{b^\Gamma,\phi^r_{\tilde{d},q}}_{\Lambda,2}$ is obvious by the definition of the pure two-body interaction term in (1.8.1) and (1.9.1). Concerning the continuity observe that

$$\sum_{i\in\Lambda}\sum_{k\in\mathbb{L}}|\tilde{d}(k,i)| \leq \sum_{i\in\Lambda}\sum_{k\in\Lambda}\sum_{\gamma\in\Gamma}|\tilde{d}(\gamma\cdot k,i)| \leq |\Lambda|^2\sup_{k,i\in\Lambda}\sum_{\gamma\in\Gamma}|\tilde{d}(\gamma\cdot k,i)|,$$

hence, setting $\|r\|_{\mathcal{C}_b(F\times F)} := \sup_{x,y\in F}|r(x,y)|$, we obtain

$$\sup_{\xi_\Lambda\in\Omega'_\Lambda}|U^{b^\Gamma,\phi^r_{\tilde{d},q}}_{\Lambda,2}(\xi_\Lambda)| \leq \sum_{i\in\Lambda}\sum_{k\in\mathbb{L}}|\tilde{d}(k,i)|\,\|r\|_{\mathcal{C}_b(F\times F)} \leq |\Lambda|^2\,\|r\|_{\mathcal{C}_b(F\times F)}\sup_{k,i\in\Lambda}\sum_{\gamma\in\Gamma}|\tilde{d}(\gamma\cdot k,i)|. \qquad (30)$$

□

Proposition 1.9.6. *Let Γ be a subsemigroup of $\mathbb{Z}$ acting by translation on $\mathbb{L} = \mathbb{Z}$ and $(\Omega,F,\mathbb{L},\Gamma,\tau)$ a lattice spin system (1.2.7). Let $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{Z})}$ be a pure two-body interaction of the type*

$$\phi_\Lambda(\xi_\Lambda) = -d(i-j)\,r(\xi_i,\xi_j),\ \text{ if } \Lambda = \{i,j\},\ \xi_\Lambda = (\xi_i,\xi_j),$$

where $d : \mathbb{Z} \to \mathbb{C}$ is an even function with $\sum_{n=0}^\infty |d(n)| < \infty$ and $r \in \mathcal{C}_b(F\times F)^{\mathbb{Z}_2}$. Then the interaction ϕ belongs to the class $\mathcal{A}$ as defined in (1.6.7).

Proof. The interaction ϕ is translation invariant by Remark 1.8.2 (i), hence by Remark 1.6.7 it suffices to show $\|\phi\|_i < \infty$ for an arbitrary point $i \in \mathbb{L} = \mathbb{Z}$. The subsets of $\mathbb{L}$, which contribute to the seminorm $\|\phi\|_i$ (21), are of the form $\{i, j\}$ with $j \in \mathbb{L} \setminus \{i\}$. Then we have

$$\|\phi\|_i = \sum_{\Lambda \in \mathcal{P}_f(\mathbb{L});\, \Lambda \ni i} \sup_{\xi_\Lambda \in \Omega_\Lambda} |\phi_\Lambda(\xi_\Lambda)| = \sum_{j \in \mathbb{L}, j \neq i} \sup_{\xi_{\{i,j\}} \in \Omega_{\{i,j\}}} |\varphi_2(\xi_{\{i,j\}})| \leq \sup_{x,y \in F} |r(x,y)| \sum_{j \in \mathbb{Z}} |d(i-j)| < \infty,$$

since $\sum_{j\in\mathbb{Z}} |d(i-j)| = \sum_{\gamma \in \mathbb{Z}} |d(\gamma)| < 2 \sum_{n=0}^{\infty} |d(n)| < \infty$. □

Example 1.9.7. The following distance functions satisfy the condition $\sum_{n=0}^{\infty} |d(n)| < \infty$ in Proposition 1.9.6

(i) Exponentially decaying interactions $d(k) = \lambda^{|k|}$ for $0 < |\lambda| < 1$,

(ii) Polynomially decaying interactions $d(k) = |k|^{-s}$ for $\mathbf{Re}(s) > 1$,

(iii) Logarithmic interaction $d(k) = \log(1 - c\lambda^{|k|})$ for $0 < |c|, |\lambda| < 1$,

(iv) Plummer potential: $d(k) = (\epsilon + |k|^2)^{-\alpha/2}$ for $\epsilon > 0$, $\alpha > 1$, and

(v) Finite range interactions $d(k) = 0$ for all $|k| > r_0$ for some $r_0 \in \mathbb{N}$.

For the proof of (i) use the geometric series and for (ii) Riemann's zeta function. Concerning (iii) we use $|\log(1-z)| \leq -\log(1-|z|)$ for $|z| < 1$ and conclude from that

$$\sum_{k=0}^{\infty} |\log(1 - c\lambda^k)| \leq -\sum_{k=0}^{\infty} \log(1 - |c|\,|\lambda|^k) = -\log\Big(\prod_{k=0}^{\infty}(1 - |c|\,|\lambda|^k)\Big),$$

which converges since $\sum_{k=0}^{\infty} |\lambda|^k < \infty$ by the standard criterion for the convergence of infinite products. For case (iv) we observe that $(\epsilon + k^2)^{-\alpha/2} \leq k^{-\alpha}$ since $\epsilon + k^2 \geq k^2$. Hence $\sum_{k=0}^{\infty}(\epsilon + k^2)^{-\alpha/2} \leq \epsilon^{-\alpha/2} + \sum_{k=1}^{\infty} k^{-\alpha}$. The case (v) is trivial. □

We will focus on *long range interactions*, which by definition means a nowhere vanishing anisotropy matrix (respectively, non-vanishing distance function). The associated models are sometimes called *Kac model*, whereas in other references this name is reserved to the special case of exponentially decaying interactions from Example 1.9.7 (i) after M. Kac's article [Ka66].
In Proposition 1.9.1 we have seen that the total energy $U^{b,\phi}_{\Lambda_\alpha}$ of a finite region $\Lambda \subset \mathbb{L}$ amounts to evaluating the interaction at an infinite number of pairs of points. This shows that the computation of the partition function (1.7.1) via its definition as the integral of $\exp(-\beta\, U^{b,\phi}_{\Lambda_\alpha})$ is ineffective. In the case of a periodic boundary condition the spin values over the complement of a fundamental domain Λ_α are determined by the spin values of their Γ_α-equivalent points. Using this idea we can transport back all computations on Λ_α by integrating a new function depending on the interaction and the semigroup Γ_α.

Theorem 1.9.8. *Let $(\Omega, F, \mathbb{L}, \Gamma, \tau)$ be a lattice spin system (1.2.7), $(\Gamma_\alpha)_{\alpha \in A}$ a family of subsemigroups of Γ defining a periodic boundary condition $(b^\Gamma_\Lambda)_{\Lambda \in \mathcal{P}_{(\Gamma_\bullet)}}$ and ϕ a symmetric compatible two-body interaction (1.6.4) such that the Γ_α-averaged interaction function*

$$\widehat{t}_{\Gamma_\alpha}(i, l; x, y) := -\frac{1}{2}\varphi_2(i, l; x, y) + \frac{1}{2} \sum_{\gamma \in \Gamma_\alpha} \Big(\varphi_2(i, \gamma \cdot l; x, y) + \varphi_2(l, \gamma \cdot i; y, x)\Big) \tag{31}$$

converges absolutely for all $i, l \in \mathbb{L}$, $x, y \in F$, $\alpha \in A$ and $\sup_{x,y\in F} |\widehat{t}_{\Gamma_\alpha}(i, l; x, y)| < \infty$ for all $i, l \in \mathbb{L}$, $\alpha \in A$. Suppose that $\Lambda_\alpha \in \mathcal{P}_{(\Gamma_\bullet)}$ satisfies the tiling condition $\mathbb{L} = \Gamma_\alpha \cdot \Lambda_\alpha$ (1.3.8). Then the partition function defined in (1.7.1) can be expressed as

$$Z^{b^\Gamma,\phi}_{\Lambda_\alpha}(\beta) = \int_{(b^\Gamma_{\Lambda_\alpha})^{-1}(\Omega)} \exp\Big(\beta \sum_{i,l \in \Lambda_\alpha} \widehat{t}_{\Gamma_\alpha}(i, l; \xi_i, \xi_l) + \beta \sum_{i \in \Lambda_\alpha} \varphi_1(i; \xi_i) - \frac{\beta}{2} \sum_{i \in \Lambda_\alpha} \varphi_2(i, i; \xi_i, \xi_i)\Big)\, d\nu^{\Lambda_\alpha}(\xi_{\Lambda_\alpha}).$$

In particular, denote by $\phi^r_{\tilde{d},q}$ the two-body interaction (1.8.1) with anisotropy matrix $\tilde{d}$, potential q, and interaction matrix r, then the map $\mathcal{C}_b(F \times F)^{\mathbb{Z}_2} \to \mathbb{C}$, $r \mapsto Z_\Lambda^{b^\Gamma, \phi^r_{\tilde{d},q}}(\beta)$ is continuous.

Proof. Set

$$t_{\Gamma_\alpha}(i,l;x,y) := \frac{1}{2}\sum_{\gamma\in\Gamma_\alpha}\Big(\varphi_2(i,\gamma\cdot l;x,y)+\varphi_2(l,\gamma\cdot i;y,x)\Big) \tag{32}$$

for $i,l\in\Lambda_\alpha$, $x,y\in F$, $\alpha\in A$. Obviously, the function t_Γ has the following symmetry: $t_\Gamma(i,l;x,y) = t_\Gamma(l,i;y,x)$. Set $\eta := (y_i)_{i\in\mathbb{L}} = b^\Gamma_{\Lambda_\alpha}(\xi_{\Lambda_\alpha}) \in \bigcap_{\gamma\in\Gamma_\alpha}\mathrm{Fix}(\tau_\gamma)$. By Proposition 1.9.1 we know that

$$U^{b^\Gamma,\phi}_{\Lambda_\alpha}(\xi_{\Lambda_\alpha}) = -\sum_{i\in\Lambda_\alpha}\varphi_1(i;\xi_{\Lambda_\alpha}(i)) - \frac{1}{2}\sum_{i\in\Lambda_\alpha}\sum_{k\in\Lambda_\alpha\setminus\{i\}}\varphi_2(i,k;y_i,y_k) - \sum_{i\in\Lambda_\alpha}\sum_{k\in\mathbb{L}\setminus\Lambda_\alpha}\varphi_2(i,k;y_i,y_k).$$

By the tiling condition $\mathbb{L}=\Gamma_\alpha\cdot\Lambda_\alpha$ we have

$$\begin{aligned}\sum_{i\in\Lambda_\alpha}\sum_{k\in\mathbb{L}}\varphi_2(i,k;y_i,y_k) &= \sum_{i\in\Lambda_\alpha}\sum_{k\in\Gamma_\alpha\cdot\Lambda_\alpha}\varphi_2(i,k;y_i,y_k)\\ &= \sum_{i\in\Lambda_\alpha}\sum_{l\in\Lambda_\alpha}\sum_{\gamma\in\Gamma_\alpha}\varphi_2(i,\gamma\cdot l;y_i,y_l)\\ &= \sum_{i\in\Lambda_\alpha}\sum_{l\in\Lambda_\alpha}t_{\Gamma_\alpha}(i,l;y_i,y_l),\end{aligned} \tag{33}$$

since by assumption the last sum converges absolutely and, by relabelling,

$$\sum_{i\in\Lambda_\alpha}\sum_{l\in\Lambda_\alpha}\varphi_2(i,\gamma\cdot l;y_i,y_l) = \sum_{i\in\Lambda_\alpha}\sum_{l\in\Lambda_\alpha}\varphi_2(l,\gamma\cdot i;y_l,y_i).$$

Therefore

$$\begin{aligned}-U^{b^\Gamma,\phi}_{\Lambda_\alpha}(\xi_{\Lambda_\alpha}) &= \sum_{i\in\Lambda_\alpha}\varphi_1(i;\xi_{\Lambda_\alpha}(i)) + \frac{1}{2}\Big(\sum_{i,k\in\Lambda_\alpha}\varphi_2(i,k;y_i,y_k) - \sum_{i\in\Lambda_\alpha}\varphi_2(i,i;y_i,y_i)\Big)\\ &\quad+\Big(\sum_{i\in\Lambda_\alpha}\sum_{k\in\mathbb{L}}\varphi_2(i,k;y_i,y_k) - \sum_{i,k\in\Lambda_\alpha}\varphi_2(i,k;y_i,y_k)\Big)\\ &= \sum_{i\in\Lambda_\alpha}\varphi_1(i;y_i) + \sum_{i,l\in\Lambda_\alpha}t_{\Gamma_\alpha}(l,i;y_l,y_i) - \frac{1}{2}\sum_{i,l\in\Lambda_\alpha}\varphi_2(i,l;y_i,y_l) - \frac{1}{2}\sum_{i\in\Lambda_\alpha}\varphi_2(i,i;y_i,y_i).\end{aligned} \tag{34}$$

By our assumptions this is a finite sum of bounded functions, hence integrable with respect to the finite measure ν^Λ on $(b^\Gamma_\Lambda)^{-1}(\Omega)\subset\Omega_\Lambda$. This proves the first claim. Note that for any pair of measurable functions f, g we have

$$\Big|\int e^f - e^g\Big| = \Big|\int e^f(1-e^{g-f})\Big| \le \|1-e^{g-f}\|_\infty \int |e^f| \le (e^{\|g-f\|_\infty}-1)\int |e^f|.$$

Recalling the notation $\phi^r_{\tilde d,q}$ from (1.8.1) and the definition (1.9.1) of the pure two-body interaction term, we get for any pair of continuous bounded symmetric functions $r,s: F^2\to\mathbb{C}$ the identity

$$U^{b^\Gamma,\phi^r_{\tilde d,q}}_\Lambda - U^{b^\Gamma,\phi^s_{\tilde d,q}}_\Lambda = U^{b^\Gamma,\phi^r_{\tilde d,q}}_{\Lambda,2} - U^{b^\Gamma,\phi^s_{\tilde d,q}}_{\Lambda,2}.$$

Hence, by the proof of Corollary 1.9.5 (30)

$$\begin{aligned}\Big|Z^{b^\Gamma,\phi^r_{\tilde d,q}}_\Lambda(\beta) - Z^{b^\Gamma,\phi^s_{\tilde d,q}}_\Lambda(\beta)\Big| &= \Big|\int_{\Omega_\Lambda}\exp\Big(-\beta U^{b^\Gamma,\phi^r_{\tilde d,q}}_\Lambda(\xi_\Lambda)\Big) - \exp\Big(-\beta U^{b^\Gamma,\phi^s_{\tilde d,q}}_\Lambda(\xi_\Lambda)\Big)\,d\nu^\Lambda(\xi_\Lambda)\Big|\\ &\le (\exp\Big(|\beta|\|U^{b^\Gamma,\phi^r_{\tilde d,q}}_{\Lambda,2} - U^{b^\Gamma,\phi^s_{\tilde d,q}}_{\Lambda,2}\|_{\mathcal{C}_b(\Omega_\Lambda)}\Big)-1)\int_{\Omega_\Lambda}\Big|\exp\Big(-\beta U^{b^\Gamma,\phi^r_{\tilde d,q}}_\Lambda(\xi_\Lambda)\Big)\Big|\,d\nu^\Lambda(\xi_\Lambda)\\ &\le (\exp\Big(|\beta|\,|\Lambda|^2\,c(\tilde d)\,\|r-s\|_{\mathcal{C}_b(F^2)}\Big)-1)\int_{\Omega_\Lambda}\Big|\exp\Big(-\beta U^{b^\Gamma,\phi^r_{\tilde d,q}}_\Lambda(\xi_\Lambda)\Big)\Big|\,d\nu^\Lambda(\xi_\Lambda)\end{aligned}$$

for some constant $c(\tilde d)>0$ depending on the anisotropy matrix $\tilde d$. Since the integral is finite, the difference $\Big|Z^{b^\Gamma,\phi^r_{\tilde d,q}}_\Lambda(\beta) - Z^{b^\Gamma,\phi^s_{\tilde d,q}}_\Lambda(\beta)\Big|$ tends to zero as $\|r-s\|_{\mathcal{C}_b(F^2)}$ tends to zero. □

The function $t_{\Gamma_\alpha} : \mathbb{L} \times \mathbb{L} \times F \times F \to \mathbb{C},\ (i,l,x,y) \mapsto t_{\Gamma_\alpha}(i,l;x,y)$ as defined in (32) in the proof of Theorem 1.9.8 has an additional symmetry if $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ is a group action:

Proposition 1.9.9. *Let $\tau : \Gamma \times \mathbb{L} \to \mathbb{L}$ be a group action and $f : \mathbb{L}^2 \times F^2 \to \mathbb{C}$ a Γ-invariant function with the symmetry condition $f(i,j;x,y) = f(j,i;y,x)$, then*

$$\frac{1}{2} \sum_{\gamma \in \Gamma} \Big(f(i, \gamma \cdot l; x, y) + f(l, \gamma \cdot i; y, x) \Big) = \sum_{\gamma \in \Gamma} f(i, \gamma \cdot l; x, y)$$

for all $i, l \in \mathbb{L}$, $x, y \in F$.

Proof. Use the symmetry, the Γ-invariance, and summation over a group to calculate

$$\sum_{\gamma \in \Gamma} f(l, \gamma \cdot i; y, x) = \sum_{\gamma \in \Gamma} f(\gamma \cdot i, l; x, y) = \sum_{\gamma \in \Gamma} f(\gamma^{-1} \cdot (\gamma \cdot i), \gamma^{-1} \cdot l; x, y) = \sum_{\gamma \in \Gamma} f(i, \gamma \cdot l; x, y).$$

□

We end this section by a further specialisation. Let $\mathbb{L}$ be a group, e its identity element. Set $\tilde{d}(i,j) := d(i^{-1}j)$, where $d : \mathbb{L} \to \mathbb{C}$ is a distance function (1.8.2), i. e., a function with the symmetry condition $d(i) = d(i^{-1})$. Then the map $\Lambda \to \mathbb{C},\ i \mapsto \tilde{d}(i,i) = d(i^{-1}i) = d(e)$ is constant and every two-body interaction $\phi^r_{\tilde{d},q}$ with this anisotropy matrix $\tilde{d}$ has the properties that the maps $\Lambda \to \mathbb{C},\ i \mapsto \varphi_1(i;x) = q(x)$ and $i \mapsto \varphi_2(i,i;x,x) = r(x,x)\, d(e)$ are constant. This motivates the assumptions of the following corollary:

Corollary 1.9.10. *Suppose in addition to the hypotheses of Theorem 1.9.8 that the maps $\Lambda \to \mathbb{C},\ i \mapsto \varphi_1(i;x)$ and $i \mapsto \varphi_2(i,i;x,x)$ are constant for all $x \in F$. Set $p : F \to \mathbb{C},\ p(x) := \varphi_1(i;x) - \frac{1}{2}\varphi_2(i,i;x,x)$ and let $\widehat{t}_\Gamma$ be as in (31). Then the partition function defined in (1.7.1) can be expressed as*

$$Z_\Lambda^{b^\Gamma,\phi}(\beta) = \int_{(b_\Lambda^\Gamma)^{-1}(\Omega)} \exp\Big(\beta \sum_{i,l \in \Lambda} \widehat{t}_\Gamma(i,l;\xi_i,\xi_l) + \beta \sum_{i \in \Lambda} p(\xi_i) \Big)\, d\nu^\Lambda(\xi_\Lambda).$$

□

Corollary 1.9.10 implies that the integrand of the partition function for an Ising interaction can be written as the exponential of a symmetric quadratic form as we will show next.

Remark 1.9.11. Suppose the same hypotheses as in Corollary 1.9.10. For any subsemigroup Γ_α of Γ with fundamental domain $\Lambda_\alpha \subset \mathbb{L}$ we define two symmetric matrices $\widehat{T}_{\Gamma_\alpha} = \big(\hat{t}^{\Gamma_\alpha}_{i,l}\big)_{i,l \in \Lambda_\alpha}$ and $T_{\Gamma_\alpha} = \big(t^{\Gamma_\alpha}_{i,l}\big)_{i,l \in \Lambda_\alpha}$ via $t^{\Gamma_\alpha}_{i,l} := \sum_{\gamma \in \Gamma_\alpha} \tilde{d}(i, \gamma \cdot l)$, $\hat{t}^{\Gamma_\alpha}_{i,l} := -\frac{1}{2}\tilde{d}(i,l) + \sum_{\gamma \in \Gamma_\alpha} \tilde{d}(i, \gamma \cdot l)$ for all $i, l \in \Lambda_\alpha$. Then the averaged interaction functions defined (31) and (32) can be simplified to

$$\widehat{t}_{\Gamma_\alpha}(i,l;x,y) = -\frac{1}{2}\tilde{d}(i,l)\, r(x,y) + t^{\Gamma_\alpha}_{i,l}\, r(x,y), \quad t_{\Gamma_\alpha}(i,l;x,y) = t^{\Gamma_\alpha}_{i,l}\, r(x,y).$$

We combine these considerations with Corollary 1.9.10 and obtain

$$Z_\Lambda^{b^\Gamma,\phi}(\beta) = \int_{(b_\Lambda^\Gamma)^{-1}(\Omega)} \exp\Big(\beta \sum_{i,l \in \Lambda} \hat{t}^{\Gamma_\alpha}_{i,l}\, r(\xi_i,\xi_l) + \beta \sum_{i \in \Lambda} p(\xi_i) \Big)\, d\nu^\Lambda(\xi_\Lambda).$$

In particular, in the case of an Ising spin system (1.8.3) with vanishing potential we have

$$Z_\Lambda^{b^\Gamma,\phi}(\beta) = \int_{(b_\Lambda^\Gamma)^{-1}(\Omega)} \exp\Big(\beta \sum_{i,l \in \Lambda} \hat{t}^{\Gamma_\alpha}_{i,l}\, \xi_i\, \xi_l \Big)\, d\nu^\Lambda(\xi_\Lambda),$$

where the sum in the exponential should be viewed as a quadratic form in $(\xi_1, \ldots, \xi_n) \in F^n \subset \mathbb{C}^n$. □

Note that for $\delta \gg 0$ the matrices $\widehat{T}_\Gamma + \delta, T_\Gamma + \delta$ in Remark 1.9.11 are positive definite such that one may try to use the Kac-Gutzwiller trick for the evaluation of the partition function. This trick is due to M. Kac [Ka66] and has been adopted by M. Gutzwiller [Gu82] to periodic boundary conditions. This trick uses the well-known identity on Gaussian integrals

$$e^{\pi(z|\alpha z)} = (\det\alpha)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\pi(x|\alpha^{-1}x)-2\pi(z|x)}\,dx$$

for $\alpha \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\alpha = \alpha^\top$ and $\mathbf{Re}(\alpha)$ positive definite and any $z \in \mathbb{C}^n$. For one-dimensional one-sided shifts with exponentially decaying interaction this idea was an essential step in the construction of the Kac-Gutzwiller transfer operator. For details we refer to [Gu82], [Hel02], [HiMay02, p. 26], or [Ri03]. In Chapter 5 we will construct Kac-Gutzwiller transfer operators also for polynomial-exponential and finite-range interactions using a different approach.

1.10 One-sided $\mathbb{Z}^D$-subshifts: explicit formulas

By the results of Section 1.9 the partition function with periodic boundary condition both for the one-sided and two-sided shift, as defined in Example 1.2.6, can be expressed as an integral over a "semigroup averaged interaction". If this average is taken over a group (as for instance for the two-sided shift), this yields additional symmetry properties of the averaged interaction. In the special situation of one-dimensional systems one can easily relate the pure two-body interaction terms (1.9.1) for the one- and the two-sided system, Prop. 1.10.1, since $\mathbb{N}_0^D$ sits inside $\mathbb{Z}^D$ in a special way. This result will be generalised to higher dimensions and will be used to prove Theorem 1.10.3, which is quite similar to Theorem 1.9.8.

Proposition 1.10.1. *Let $(\Omega, F, \mathbb{Z}, \mathbb{Z}, \tau)$ be a one-dimensional two-sided subshift (1.2.5) and denote by $b^{\mathbb{Z}} = (b^{\mathbb{Z}}_\Lambda)_{\Lambda\in\mathcal{P}_{(n\mathbb{Z})}}$ the periodic boundary condition on Ω associated to the family $(n\mathbb{Z})_{n\in\mathbb{N}}$ (1.3.9). Let $\Omega_> = \rho_{\mathbb{N}}(\Omega)$ (1.1.2) and $(\Omega_>, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be its one-sided subshift (1.2.6) and $b^{\mathbb{N}_0} = (b^{\mathbb{N}_0}_\Lambda)_{\Lambda\in\mathcal{P}_{(n\mathbb{N})}}$ be the periodic boundary condition on $\Omega_>$ associated to the family $(n\mathbb{N}_0)_{n\in\mathbb{N}}$, and ϕ a $\mathbb{Z}$-invariant compatible (1.6.4) symmetric two-body interaction on Ω (and hence on $\Omega_>$). Let $\Lambda = \{1,\dots,n\}$. Then on $\rho_{\mathbb{N}}\big(\bigcap_{\gamma\in n\mathbb{Z}} \mathrm{Fix}(\tau_\gamma : \Omega\to\Omega)\big)$ one has*

$$U^{b^{\mathbb{Z}},\phi}_{\Lambda,2}(\xi_\Lambda) = 2U^{b^{\mathbb{N}_0},\phi}_{\Lambda,2}(\xi_\Lambda) + \frac{1}{2}\sum_{i,l\in\Lambda}\varphi_2(i,l;\xi_\Lambda(i),\xi_\Lambda(l)) - \frac{1}{2}\sum_{i\in\Lambda}\varphi_2(i,i;\xi_\Lambda(i),\xi_\Lambda(i)).$$

Proof. Let $\xi_\Lambda \in \rho_\Lambda(\bigcap_{\gamma\in n\mathbb{Z}}\mathrm{Fix}(\tau_\gamma:\Omega\to\Omega))$ and $(y_i)_{i\in\mathbb{Z}} := b^{\mathbb{Z}}_\Lambda(\xi_\Lambda)$. Using the tiling property (1.3.8) $n\mathbb{N}_0\cdot\Lambda = \mathbb{N}$ and the $n\mathbb{Z}$-periodicity of the sequence $(y_i)_{i\in\mathbb{Z}}$ we have

$$\sum_{i\in\Lambda}\sum_{k\in\mathbb{N}\setminus\Lambda}\varphi_2(i,k;y_i,y_k) = \sum_{i\in\Lambda}\sum_{k\in\Lambda}\sum_{\gamma\in n\mathbb{N}}\varphi_2(i,\gamma\cdot k;y_i,y_{\gamma\cdot k}) = \sum_{i\in\Lambda}\sum_{k\in\Lambda}\sum_{\gamma\in n\mathbb{N}}\varphi_2(i,\gamma\cdot k;y_i,y_k).$$

Because of the $\mathbb{Z}$-invariance and the symmetry one has

$$\begin{aligned}
\sum_{i\in\Lambda}\sum_{k\in\Lambda}\sum_{\gamma\in n\mathbb{N}}\varphi_2(i,\gamma\cdot k;y_i,y_k) &= \sum_{i\in\Lambda}\sum_{k\in\Lambda}\sum_{\gamma\in n\mathbb{N}}\varphi_2(\gamma^{-1}\cdot i,k;y_i,y_k)\\
&= \sum_{i\in\Lambda}\sum_{k\in\Lambda}\sum_{\gamma\in -n\mathbb{N}}\varphi_2(\gamma\cdot i,k;y_{\gamma\cdot i},y_k)\\
&= \sum_{i\in\Lambda}\sum_{k\in-\mathbb{N}_0}\varphi_2(k,i;y_k,y_i)\\
&= \sum_{i\in\Lambda}\sum_{k\in-\mathbb{N}_0}\varphi_2(i,k;y_i,y_k).
\end{aligned}$$

We will use this relation in $(\star)$. Starting with the representation (1.9.1) of $U^{b^{\mathbb{N}_0},\phi}_{\Lambda,2}$ we calculate using the relations found during the proof of (1.9.8)

$$\begin{aligned}
-2U_{\Lambda,2}^{b^{\mathbb{N}_0},\phi}(\xi_\Lambda)+\sum_{i\in\Lambda}\varphi_2(i,i;y_i,y_i) &= 2\cdot\frac{1}{2}\sum_{i\in\Lambda}\sum_{k\in\Lambda\setminus\{i\}}\varphi_2(i,k;y_i,y_k)+2\sum_{i\in\Lambda}\sum_{k\in\mathbb{N}\setminus\Lambda}\varphi_2(i,k;y_i,y_k)\\
&\quad+\sum_{i\in\Lambda}\varphi_2(i,i;y_i,y_i)\\
&\overset{(\star)}{=} \sum_{i\in\Lambda}\Big(\sum_{k\in\mathbb{N}\setminus\Lambda}+\sum_{k\in\Lambda}+\sum_{k\in-\mathbb{N}_0}\Big)\varphi_2(i,k;y_i,y_k)\\
&= \sum_{i\in\Lambda}\sum_{k\in\mathbb{Z}}\varphi_2(i,k;y_i,y_k)\overset{(33)}{=}\sum_{i,l=1}^{n}t_{n\mathbb{Z}}(i,l;y_i,y_l)\\
&\overset{(34)}{=} -U_{\Lambda,2}^{b_0^{\mathbb{N}},\phi}(\xi_\Lambda)+\frac{1}{2}\sum_{i,l\in\Lambda}\varphi_2(i,l;y_i,y_l)+\frac{1}{2}\sum_{i\in\Lambda}\varphi_2(i,i;y_i,y_i),
\end{aligned}$$

where the function $t_{n\mathbb{Z}}:\mathbb{L}^2\times F^2\to\mathbb{C},\ (i,l,x,y)\mapsto t_{n\mathbb{Z}}(i,l;x,y)$ was defined in (32) in the proof of Theorem 1.9.8. ☐

The proof essentially is based on summation techniques used in [Gu82] and [HiMay02]. We will now generalise Proposition 1.10.1 to higher dimensional systems. This makes the resulting expression for the pure two-body interaction term $U_{\Lambda_n,2}^{b_0^{\mathbb{N}^D},\phi}$ much more complicated. We will use Proposition 1.10.2 for the proof of Theorem 1.10.3.

Proposition 1.10.2. *Let $(\Omega,F,\mathbb{Z}^D,\mathbb{Z}^D,\tau)$ be a D-dimensional two-sided subshift (1.2.5) and $b^{\mathbb{Z}^D}=(b_\Lambda^{\mathbb{Z}^D})_{\Lambda\in\mathcal{P}_{(\Gamma_\bullet)}}$ the periodic boundary condition on Ω associated to the family $(\Gamma_n:=\prod_{i=1}^D n_i\mathbb{Z})_{n\in\mathbb{N}^D}$. Let $(\Omega_>=\rho_{\mathbb{N}^D}(\Omega),F,\mathbb{N}^D,\mathbb{N}_0^D,\tau)$ be its one-sided subshift (1.2.6) and $b^{\mathbb{N}_0^D}=(b_\Lambda^{\mathbb{N}_0^D})_{\Lambda\in\mathcal{P}_{(H_\bullet)}}$ be the periodic boundary condition (1.3.9) on $\Omega_>$ associated to the family $(H_n:=\prod_{i=1}^D n_i\mathbb{N}_0)_{n\in\mathbb{N}^D}$. Let ϕ be a $\mathbb{Z}^D$-invariant[10] symmetric two-body interaction such that $t_{\Gamma_n}(i,l;x,y)=\sum_{\gamma\in\Gamma_n}\varphi_2(\gamma\cdot i,l;x,y)$ converges absolutely for all $i,l\in\mathbb{Z}^D$, $x,y\in F$, $n\in\mathbb{N}_0^D$. For $k\in\mathbb{N}$ let $A(k,\epsilon_i):=\begin{cases}\{1,\dots,k\} & ,\epsilon_i=1\\ \mathbb{Z} & ,\epsilon_i=-1\end{cases}$ and for $\epsilon=(\epsilon_1,\dots,\epsilon_D)\in\{\pm1\}^D$, $n\in\mathbb{N}^D$ let $A_n(\epsilon):=\prod_{i=1}^D A(n_i,\epsilon_i)$. Then for $\Lambda_n:=\prod_{i=1}^D\{1,\dots,n_i\}$ and for all $\xi_\Lambda\in\rho_\Lambda(\bigcap_{\gamma\in\Gamma_n}\mathrm{Fix}(\tau_\gamma:\Omega\to\Omega))$*

$$\begin{aligned}
U_{\Lambda_n,2}^{b_0^{\mathbb{N}^D},\phi}(\xi_{\Lambda_n}) &= -2^{-D}\sum_{j\in\Lambda_n}\sum_{\epsilon\in\{\pm1\}^D}\sum_{i\in A(n,\epsilon)}\varphi_2(i,j;\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(j))\\
&\quad+\frac{1}{2}\sum_{i,j\in\Lambda_n}\varphi_2(i,j;\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(j))-\frac{1}{2}\sum_{i\in\Lambda_n}\varphi_2(i,i;\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(i)).
\end{aligned}$$

Proof. Let $\Lambda:=\Lambda_n$, $\xi_\Lambda\in\rho_\Lambda(\bigcap_{\gamma\in\Gamma_n}\mathrm{Fix}(\tau_\gamma:\Omega\to\Omega))$ and $y:=b_\Lambda^{\mathbb{Z}^D}(\xi_\Lambda)$. Let

$$\Phi_\Lambda:(b_\Lambda^{\mathbb{Z}^D})^{-1}(\Omega)\to\mathbb{C},\ \Phi_\Lambda(\xi_\Lambda):=\sum_{i\in\Lambda}\sum_{j\in\mathbb{N}^D}\varphi_2(i,j;y_i,y_j)$$

be the sum of interactions between Λ and the positive quadrant. Using the definition of $U_{\Lambda,2}^{b_0^{\mathbb{N}^D},\phi}$ in Proposition 1.9.1 one easily confirms that

$$-U_{\Lambda,2}^{b_0^{\mathbb{N}^D},\phi}(\xi_\Lambda)=\Phi_\Lambda(\xi_\Lambda)-\frac{1}{2}\sum_{i,j\in\Lambda}\varphi_2(i,j;y_i,y_j)-\frac{1}{2}\sum_{i\in\Lambda}\varphi_2(i,i;y_i,y_i).$$

We show by induction (over the dimension D) that

$$2^D\,\Phi_\Lambda(\xi_\Lambda)=\sum_{j\in\Lambda}\sum_{\epsilon\in\{\pm1\}^D}\sum_{i\in A_n(\epsilon)}\varphi_2(i,j;y_i,y_j) \tag{35}$$

[10] We can apply Prop. 1.9.9, hence the definition of $t_{\Gamma_n}(i,l;x,y)$ confirms well with (32) in the proof of Theorem 1.9.8

which gives the assertion. This identity has a geometric interpretation, namely it yields a representation of the positive quadrant as a signed sum of boxes whose sides are either finite or all of $\mathbb{Z}$.
Let $D = 1$: By equation $(\star)$ in the proof of Proposition 1.10.1 we have

$$\begin{aligned}\sum_{i\in\Lambda}\sum_{j\in\mathbb{Z}}\varphi_2(i,j;y_i,y_j) &= \sum_{i\in\Lambda}\Big(\sum_{j\in\mathbb{N}\setminus\Lambda}+\sum_{j\in\Lambda}+\sum_{j\in-\mathbb{N}_0}\Big)\varphi_2(i,j;y_i,y_j)\\ &= 2\sum_{i\in\Lambda}\sum_{j\in\mathbb{N}\setminus\Lambda}\varphi_2(i,j;y_i,y_j)+\sum_{i\in\Lambda}\sum_{j\in\Lambda}\varphi_2(i,j;y_i,y_j).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i\in\Lambda}\sum_{j\in\mathbb{Z}}\varphi_2(i,j;y_i,y_j)+\sum_{i\in\Lambda}\sum_{j\in\Lambda}\varphi_2(i,j;y_i,y_j) &= 2\sum_{i\in\Lambda}\sum_{j\in\mathbb{N}\setminus\Lambda}\varphi_2(i,j;y_i,y_j)+2\sum_{i\in\Lambda}\sum_{j\in\Lambda}\varphi_2(i,j;y_i,y_j)\\ &= 2\sum_{i\in\Lambda}\sum_{j\in\mathbb{N}}\varphi_2(i,j;y_i,y_j)=2\Phi_\Lambda(\xi_\Lambda)\end{aligned}$$

The induction step is a straight forward calculation: Let $i',j',\epsilon'\in\mathbb{Z}^{D-1}$, $\Lambda'=\prod_{l=1}^{D-1}\{1,\dots,n_l\}$, $\Lambda_D=\{1,\dots,n_D\}$, $i=(i',i_D), j=(j',j_D), \Lambda=\Lambda'\times\Lambda_D$. Then

$$\begin{aligned}\text{RHS of (35)} &= \sum_{i'\in\Lambda'}\sum_{i_D\in\Lambda_D}\sum_{\epsilon'\in\{\pm1\}^{D-1}}\sum_{\epsilon_D=\pm1}\sum_{j'\in A_{n'}(\epsilon')}\sum_{j_D\in A(n_D,\epsilon_D)}\varphi_2(i,j;y_i,y_j)\\ &= \sum_{i_D\in\Lambda_D}\sum_{\epsilon_D=\pm1}\sum_{j_D\in A(n_D,\epsilon_D)}\sum_{i'\in\Lambda'}\sum_{\epsilon'\in\{\pm1\}^{D-1}}\sum_{j'\in A_{n'}(\epsilon')}\varphi_2(i,j;y_i,y_j)\\ &\overset{\text{ind.}}{=} \sum_{i_D\in\Lambda_D}\sum_{\epsilon_D=\pm1}\sum_{j_D\in A(n_D,\epsilon_D)}2^{D-1}\sum_{i'\in\Lambda'}\sum_{j'\in\mathbb{N}^{D-1}}\varphi_2(i,j;y_i,y_j)\\ &= 2^{D-1}\sum_{i'\in\Lambda'}\sum_{j'\in\mathbb{N}^{D-1}}\Big(\sum_{i_D\in\Lambda_D}\sum_{j_D\in\Lambda_D}\varphi_2(i,j;y_i,y_j)+\sum_{i_D\in\Lambda_D}\sum_{j_D\in\mathbb{Z}}\varphi_2(i,j;y_i,y_j)\Big)\\ &= 2^{D-1}\sum_{i'\in\Lambda'}\sum_{j'\in\mathbb{N}^{D-1}}\Big(2\sum_{i_D\in\Lambda_D}\sum_{j_D\in\mathbb{N}}\varphi_2(i,j;y_i,y_j)\Big)=\text{LHS of (35)}\end{aligned}$$

□

An immediate consequence of Proposition 1.10.2 is the following analogue of Theorem 1.9.8.

Theorem 1.10.3. *Let* $(\Omega,F,\mathbb{N}^D,\mathbb{N}_0^D,\tau)$ *be a one-sided subshift (1.2.6), let* $b^{\mathbb{N}_0^D}=(b_\Lambda^{\mathbb{N}_0^D})_{\Lambda\in\mathcal{P}_{(H_\bullet)}}$ *be the periodic boundary condition (1.3.9) on* Ω *associated to the family* $(H_n:=\prod_{i=1}^D n_i\mathbb{N}_0)_{n\in\mathbb{N}^D}$*, and* ϕ *a compatible* $\mathbb{Z}^D$*-invariant two-body interaction (1.8.1) with interaction matrix* $0\neq r\in\mathcal{C}_b(F\times F)$*, potential* $q\in\mathcal{C}_b(F)$*, and anisotropy matrix* $\tilde d$ *of the form* $\tilde d(i,j)=\prod_{l=1}^D\tilde d^{(l)}(i_l,j_l)$*. For* $i,j\in\mathbb{Z}$ *set*

$$t_{i,j}^{n_l\mathbb{Z},(l)}:=\sum_{k\in\mathbb{Z}}\tilde d^{(l)}(i,j+kn_le_l),$$

where $e_l\in\mathbb{Z}^D$ *denotes the* l*-th standard unit vector. Set* $q':\mathbb{N}^D\times F\to\mathbb{C}$, $(i,x)\mapsto q(x)-\frac12\tilde d(i,i)\,r(x,x)$*. Then for* $\Lambda_n:=\prod_{i=1}^D\{1,\dots,n_i\}$ *the partition function (1.7.1) is given as*

$$\begin{aligned}Z_{\Lambda_n}^{b^{\mathbb{N}_0^D},\phi}(\beta) &= \int_{\Omega'_{\Lambda_n}}\exp\Big(\frac{\beta}{2^D}\sum_{i,j\in\Lambda_n}\sum_{\epsilon\in\{\pm1\}^D}\big(\prod_{l:\epsilon_l=1}\tilde d^{(l)}(i_l,j_l)\prod_{l:\epsilon_l=-1}t_{i_l,j_l}^{n_l\mathbb{Z},(l)}\big)\,r(\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(j))\Big)\\ &\qquad\exp\Big(-\frac{\beta}{2}\sum_{i,j\in\Lambda_n}\tilde d(i,j)\,r(\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(j))+\beta\sum_{i\in\Lambda_n}q'(i,\xi_{\Lambda_n}(i))\Big)\,d\nu^{\Lambda_n}(\xi_{\Lambda_n}),\end{aligned}$$

where $\Omega'_{\Lambda_n}:=(b_{\Lambda_n}^{\mathbb{N}_0^D})^{-1}(\Omega)$*; in particular for* $D=1$, $\Lambda_n=\{1,\dots,n\}$

$$Z_{\Lambda_n}^{b^{\mathbb{N}_0},\phi}(\beta)=\int_{(b_{\Lambda_n}^{n\mathbb{Z}})^{-1}(\Omega)}\exp\Big(\frac{\beta}{2}\sum_{i,l=1}^n t_{i,l}^{n\mathbb{Z}}\,r(x_i,x_l)+\beta\sum_{i=1}^n q(x_i)-\frac{\beta}{2}\sum_{i=1}^n\tilde d(i,i)\,r(x_i,x_i)\Big)\,d\nu^{\Lambda_n}(x_1,\dots,x_n).$$

Proof. By the compatibility assumption (1.6.4) the defining series for $t_{i,j}^{n_l\mathbb{Z},(l)}$ converges absolutely (1.9.3). Then apply Proposition 1.10.2 and write the energy as

$$\begin{aligned} U^{b^{\mathbb{N}_0^D},\phi}_{\Lambda_n,2}(\xi_{\Lambda_n}) &= -2^{-D}\sum_{i,j\in\Lambda_n}\sum_{\epsilon\in\{\pm1\}^D}\prod_{l:\epsilon_l=1}\tilde{d}^{(l)}_{i_l,j_l}\prod_{l:\epsilon_l=-1}t^{n_l\mathbb{Z},(l)}_{i_l,j_l}\, r(\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(j)) \\ &\quad +\frac{1}{2}\sum_{i,j\in\Lambda_n}\tilde{d}(i,j)\, r(\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(j)) - \frac{1}{2}\sum_{i\in\Lambda_n}\tilde{d}(i,i)\, r(\xi_{\Lambda_n}(i),\xi_{\Lambda_n}(i)). \end{aligned}$$

By the compatibility assumption (1.6.4) and the boundedness of the interaction matrix r and the potential q, the integrand is a bounded function. Hence the integral converges. □

Theorems 1.9.8 and its Corollary 1.9.10 as well as Theorem 1.10.3 show that partition functions can be expressed as integrals over exponentials of symmetric quadratic forms. These results are generalisations of ideas appearing in [Gu82], [HiMay02, p. 26], or [Ri03]. We hope that this idea can be applied in a direct construction of future Kac-Gutzwiller type transfer operators.

1.11 One-sided $\mathbb{Z}$-subshift

In this section we restrict our considerations to the case of one-sided one-dimensional subshifts. We will introduce the so called standard observable $A_{(\phi)}$. It depends on the sum of interactions between the spin at position 1 and the rest of the half line. Given a one-dimensional system with periodic boundary condition, the energy can be expressed in terms of $A_{(\phi)}$ and the $\mathbb{N}_0$-action

$$\tau : \mathbb{N}_0 \times F^{\mathbb{N}} \to F^{\mathbb{N}},\ (n,\xi) \mapsto \tau_n(\xi)(m) = \xi(n+m).$$

This leads to a dynamical interpretation of the partition function in Corollary 1.11.3 and allows a higher dimensional generalisation which we call the dynamical partition function. The dynamical interpretation will be important for the construction of the Ruelle transfer operator in (2.1.3) in the next chapter and is also the link between the thermodynamic formalism for lattice systems and the thermodynamic formalism for expanding maps. Whereas the Ruelle transfer operator (which we will define in the next chapter) was invented in order to describe the partition function for one-dimensional lattice spin systems, for higher dimensional systems there is up to now no reasonable transfer operator available. We think that it might be easier to find a generalisation of Ruelle's transfer operator which describes the dynamical partition function (also in higher dimension) than one for the ordinary partition function, since a fixed point interpretation is built in by definition.

Definition 1.11.1. Let $(\Omega, F, \mathbb{N}^D, \mathbb{N}_0^D, \tau)$ be a D-dimensional one-sided subshift (1.2.6), and ϕ a two-body interaction (1.8.1) on Ω such that $\sum_{i\in\mathbb{N}^D}\varphi_2(\underline{1}, i; x, \xi_i)$ is absolutely convergent for all $\xi\in\Omega$, $x\in F$, where $\underline{1} = (1,\ldots,1)\in\mathbb{N}^D$. Set

$$A_{(\phi)} : \Omega \to \mathbb{C},\ A_{(\phi)}(\xi) := \varphi_1(\underline{1};\xi_{\underline{1}}) + \sum_{i\in\mathbb{N}^D\setminus\{\underline{1}\}}\varphi_2(\underline{1}, i; \xi_{\underline{1}}, \xi_i).$$

We call the function $A_{(\phi)}$ the *standard observable*. It expresses the energy coming from all interactions between the spin at position $\underline{1}$ and all the spins over the rest of the positive quadrant $\mathbb{N}^D$, in fact:

$$A_{(\phi)}(\xi) = \tilde{U}^{\phi}_{\{\underline{1}\}}(\xi_{\underline{1}}) + W_{\{\underline{1}\},\mathbb{N}^D\setminus\{\underline{1}\}}(\xi),$$

where $\tilde{U}^{\phi}_{\Lambda} : \Omega_\Lambda \to \mathbb{C}$ is the inner energy (1.6.1) and $W_{\{\underline{1}\},\mathbb{N}^D\setminus\{\underline{1}\}} : \Omega \to \mathbb{C}$ is Ruelle's difference term from (1.6.3). □

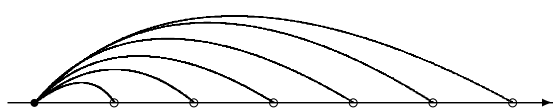

Figure 9: The standard observable counts the pair interactions between the first particle and the halfline.

The standard observable $A_{(\phi)}$ is an interesting object since for one-dimensional systems it has a special property (Prop. 1.11.2), which will play an essential role in the construction of the Ruelle transfer operator for lattice spin systems and which is due to Ruelle [Ru78].

Proposition 1.11.2. *Let* $(\Omega, F, \mathbb{N}, \mathbb{N}_0, \tau)$ *be a one-sided one-dim. subshift (1.2.6),* $(b_N^{\mathbb{N}})_{N \in \mathcal{P}_{(n\mathbb{N}_0)}}$ *the periodic boundary condition (1.3.9) associated to the family* $(n\mathbb{N}_0)_{n\in\mathbb{N}}$ *of subsemigroups of* $\mathbb{N}_0$*, and* ϕ *a compatible two-body interaction (1.6.4). Let* $N = \{1, \dots, n\}$*. Then for all* $\xi_N = (x_1, \dots, x_n) \in \rho_{\mathbb{N}}(\mathrm{Fix}(\tau_n : \Omega \to \Omega))$ *one has*

$$U_N^{b^{\mathbb{N}_0},\phi}(\xi_N) = \sum_{k=0}^{n-1} A_{(\phi)}(\tau_k(\overline{x_1 \dots x_n})),$$

where $\overline{x_1 \dots x_n} = (\rho_{\mathbb{N}} \circ r_{n\mathbb{Z}})(x_1, \dots, x_n)$ *is the periodic extension of the subconfiguration* $(x_1, \dots, x_n)$ *to the half lattice* $\mathbb{N}$*.*

Proof. By the compatibility assumption we can rearrange terms. Using the representation of the energy $U_N^{b,\phi}$ given in the proof of Proposition 1.9.1 (27) we obtain

$$\begin{aligned}
-U_N^{b^{\mathbb{N}_0},\phi}(\xi_N) &= \sum_{i=1}^{n} \varphi_1(i; x_i) + \sum_{i=1}^{n} \sum_{j>i} \varphi_2(i, j; x_i, x_j) \\
&= \sum_{k=0}^{n-1} \varphi_1(1+k; x_{1+k}) + \sum_{k=0}^{n-1} \sum_{l=1}^{\infty} \varphi_2(1+k, 1+k+l; x_{1+k}, x_{1+k+l}) \\
&= \sum_{k=0}^{n-1} A_{(\phi)}(\tau_k(\overline{x_1 \dots x_n}))
\end{aligned}$$

by definition of $A_{(\phi)}$ (1.11.1). □

We would like to underline that the expression on the right hand side of Proposition 1.11.2 can be seen as n-times the orbit mean of the observable $A_{(\phi)}$ along the closed $n\mathbb{N}_0$-orbit through $\overline{x_1 \dots x_n}$. We will return to this point of view in Remark 1.11.5. We add the remark that Proposition 1.11.2 (i. e. expressing the total energy $U_N^{b^{\mathbb{N}_0},\phi}(\xi_N)$ with periodic boundary condition as the sum of the values of one fixed function evaluated at translates of ξ_N where the translations are parametrised by N) seems to be limited to one-dimensional systems. All our attempts to find a higher dimensional analogue failed.
In the following we will assume that ν is a finite Borel measure on F and that the transition matrix $\mathbb{A} : F \times F \to \{0, 1\}$ is a $\nu \otimes \nu$-measurable function and irreducible aperiodic (1.2.9). The latter assumption guarantees by (1.3.16) that the matrix subshift $\Omega_{\mathbb{A}}$ (1.2.8) admits a periodic boundary condition with respect to the standard family of subsemigroups of $\mathbb{Z}$.

Corollary 1.11.3. *Let* $(\Omega_{\mathbb{A}}^{>}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ *be a one-sided one-dimensional matrix subshift as defined in (1.2.8). Let* $(b_N^{\mathbb{N}})_{N \in \mathcal{P}_{(n\mathbb{N}_0)}}$ *be the periodic boundary condition (1.3.9) associated to the family* $(n\mathbb{N}_0)_{n\in\mathbb{N}}$ *of subsemigroups of* $\mathbb{N}_0$*, and* ϕ *a compatible two-body interaction (1.6.4). Then for all* $n \in \mathbb{N}$ *the partition function (1.7.1) can be expressed as*

$$Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \int_{F^n} \prod_{i=1}^{n} \mathbb{A}_{x_i, x_{i+1}} \exp\Big(\beta \sum_{k=0}^{n-1} A_{(\phi)}(\tau_k(\overline{x_1 \dots x_n}))\Big)\, d\nu^n(x_1, \dots, x_n),$$

where $\overline{x_1 \dots x_n} = (\rho_{\mathbb{N}} \circ r_{n\mathbb{Z}})(x_1, \dots, x_n)$ *and* $x_{n+1} = x_1$*.*

Proof. Remark 1.3.17 gives a characterisation of the periodic sequences belonging to the configuration space $\Omega_{\mathbb{A}}^{>}$, hence a reformulation of the domain of integration. Then use Proposition 1.11.2 to see that the integrand has the stated form. □

One can interpret the above formula as the average of the observable $\exp(\beta\, A)$ over (a parametrisation of) the joined fixed point set of the shift operators $\tau_m : \Omega_{\mathbb{A}}^{>} \to \Omega_{\mathbb{A}}^{>}$ $(m \in n\mathbb{N}_0)$. This suggests the following generalisation, the dynamical partition function, where we replace the standard observable

$A_{(\phi)}$ (1.11.1) by arbitrary observables $A \in \mathcal{C}_b(\Omega)$. Dynamical partition functions have been introduced by Ruelle, [Ru78, 3.3], in a similar form without using this name.
In Chapters 2 and 3 we will construct (for certain interactions ϕ) a so called transfer operator such that the trace of its iterates allows to express the partition function. Therefore the interpretation of the above formula for $Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)$ as a fixed point formula will be used.

Definition 1.11.4. Let F be a Hausdorff space with a finite Borel measure ν, $(\Omega, F, \mathbb{N}^D, \mathbb{N}_0^D, \tau)$ a one-sided D-dimensional subshift (1.2.6). For $n \in \mathbb{N}^D$ let $H := \mathbb{N}_0^D$, $H_n := \prod_{i=1}^D (n_i\mathbb{N}_0)_{n\in\mathbb{N}^D}$, $N_n = \prod_{i=1}^D \{1,\ldots,n_i\} \subset \mathbb{N}^D$, and $M_n := \prod_{i=1}^D \{0,\ldots,n_i-1\} \subset \mathbb{N}_0^D$. Let $b^{\mathbb{N}_0^D} = (b^H_N)_{N\in\mathcal{P}_{(h_n)}}$ be the periodic boundary condition (1.3.9) associated to the family $(H_n)_{n\in\mathbb{N}^D}$. Let $A \in \mathcal{C}_b(\Omega)$. We define the *dynamical partition function* to be

$$\widetilde{Z}^{b^{\mathbb{N}_0^D}}_n(A) := \int_{\rho_{N_n}(\bigcap_{\gamma\in H_n}\mathrm{Fix}(\tau_\gamma:\Omega\to\Omega))} \exp\Big(\sum_{m\in M_n} A(\tau_m b^H_{N_n}\xi_{N_n})\Big)\, d\nu^{N_n}(\xi_{N_n}).$$

□

We note that we will always consider the dynamical partition function coming from the periodic boundary condition $b^{\mathbb{N}_0^D} = (b^H_N)_{N\in\mathcal{P}_{(h_n)}}$ associated to the family $(H_n)_{n\in\mathbb{N}^D}$.
We will now give examples for the application of the dynamical partition function which show the connection between the thermodynamic formalism for lattice spin systems (which is the main topic of this dissertation) and the thermodynamic formalism for expanding maps (see [Ru78, 7.26 ff.], [May91, 7.3]) which we only touch occasionally.

Remark 1.11.5. Typical choices of $A \in \mathcal{C}_b(\Omega)$ are the following:

(i) Let $D = 1$ and $A_{(\phi)}$ be the standard observable (1.11.1), then by Corollary 1.11.3

$$\widetilde{Z}^{b^{\mathbb{N}_0}}_n(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta),$$

since Remark 1.3.17 provides an alternative characterisation of the domain of integration, i.e., of the joint fixed points of τ_γ ($\gamma \in H_n = n\mathbb{N}_0$), in terms of the transition matrix and the integrands of both partition functions coincide. We raise the question whether the thermodynamic limit of the dynamical partition function behaves like the limit of the (ordinary) partition function which seems to be likely in view of [Ru78, 3.3].

(ii) (In the notation of 1.11.4) The trivial choice $A = 0$ leads to a measurement of the number of the joint fixed points of τ_n ($n \in H_n$) in Ω, since $\widetilde{Z}^{b^H}_n(0) = \nu^{N_n}(\rho_{N_n}(\bigcap_{\gamma\in H_n}\mathrm{Fix}(\tau_\gamma : \Omega \to \Omega)))$.

(iii) Let X be a set and $T : X \to X$ a map such that for all $n \in \mathbb{N}$ the n-th iterate $T^n : X \to X$ of T has only finitely many fixed points[11]. Define an $\mathbb{N}_0$-action (1.2.2) via

$$\mathbb{N}_0 \times X \to X,\ (k, x) \mapsto T^k(x).$$

Define $\Omega_T := \{\xi \in X^{\mathbb{N}} \,|\, T(\xi_i) = \xi_{i+1}\}$, which is invariant under the shift action

$$\tau : \mathbb{N}_0 \times \Omega_T \to \Omega_T,\ \tau(n, \xi)_i = \xi_{i+n}.$$

The elements of Ω_T are (in bijection to) T-orbits: Let $\xi = (\xi_i)_{i\in\mathbb{N}} \in \Omega_T$, then $\xi_{n+1} = T\xi_n = \ldots = T^n\xi_1$. Hence the map T is encoded in the sequence space Ω_T and we can apply our technics for lattice spin systems. Let X be equipped with the counting measure and $A : X \to \mathbb{C}$ be an observable. Then we obtain the following expression for the dynamical partition function (1.11.4)

$$\widetilde{Z}^{b^{\mathbb{N}_0}}_n(A) = \sum_{x\in\mathrm{Fix}(T^n:X\to X)} \exp\Big(\sum_{k=0}^{n-1} A(T^k x)\Big).$$

Note that the term in the exponential is the *orbit mean* of A along the closed T-orbit of x.

[11] Provided a summability/integrability condition this assumption can be weakened.

(iv) The trivial choice $A = 0$ of the observable in (iii) leads to $Z_n^{b^{\mathbb{N}_0}}(0) = |\mathrm{Fix}(T^n : X \to X)|$, which is the number of the fixed points of T^n in X.

(v) Let $X = I$ be a bounded domain in $\mathbb{R}^n$ or $\mathbb{C}^n$ and $T : I \to I$ a piecewiece continuously differentiable map such that $T^n : I \to I$ has only finitely many fixed points for all $n \in \mathbb{N}$. For the particular choice of the almost everywhere defined function $A(x) = -\log|\det(T'(x))|$ in (iii) one obtains

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}\big(-\beta \log|\det(T'(x))|\big) = \sum_{x \in \mathrm{Fix}(T^n : I \to I)} \prod_{k=0}^{n-1} \frac{1}{|\det(T'(T^k x))|^{\beta}},$$

which is the standard notion of the partition function in the context of expanding maps. □

2 Transfer operators for the full shift

The Ruelle transfer operator is an important tool for the investigation of dynamical systems, statistical mechanics, quantum chaos, and fractals. The idea is to encode the dynamical information into an operator, study its spectral properties, and to deduce back from these some dynamical properties. The transfer operator is defined as follows: Given a fixed self map $T : I \to I$ of a set I, this gives rise to an $\mathbb{N}_0$-action, i. e., a time-discrete dynamical system, by Remark 1.2.2. If every point $x \in I$ has only finitely many T-preimages, the formula

$$(\mathcal{L}_A f)(x) = \sum_{y \in T^{-1}(x)} \exp(A(y))\, f(y)$$

defines the Ruelle transfer operator acting on complex-valued functions $f : I \to \mathbb{C}$. Here the function $\exp \circ A : I \to \mathbb{C}$ is the weight corresponding to the observable A. The Ruelle transfer operator is a (sum of) composition operator(s) acting on a huge function space. Nevertheless, one can show that (in many situations) this operator has a spectral gap, i. e., there exists a leading eigenvalue λ_1 and the rest of the spectrum is contained in a disk with radius strictly smaller than $|\lambda_1|$. The Ruelle-Perron-Frobenius Theorem 2.1.4 uses these analytic properties and concludes from them certain dynamical properties, such as the existence of an equilibrium state. One technique for the investigation of the spectrum is the restriction of the transfer operator to an invariant subspace which is easier to analyse. For instance, if one can transport the (restriction of the) transfer operator to one of the commonly used function spaces from functional analysis, then many results on composition operators are available. The smaller the subspace, the more "improve" the spectral properties of the (restricted) operator (for instance the operator becomes bounded, compact, Schatten class). But of course in this process information about the physical properties is lost. The choice of a suitable subspace or a direct construction of a good transfer operator is a hard problem and in many cases depends on a skilled view.

In this chapter we consider the one-dimensional one-sided (full) shift endowed with a Ising two-body interaction ϕ (1.8.3) given via a distance function $d \in \ell^1\mathbb{N}$ and spin values in a bounded subset $F \subset \mathbb{C}$. In this case the self map T from above is the shift operator acting on the configuration space $\Omega \subset F^{\mathbb{N}}$ and the observable A is the standard observable $A_{(\phi)}$ from (1.11.1) corresponding to the interaction ϕ. In several papers ([May76], [Vi76], [MayVi77], [ViMay77], [May80a], [Mo89], [HiMay02], [HiMay04]) examples of interactions have been found for which one could identify a certain subspace which is invariant under the Ruelle transfer operator yielding the so called Ruelle-Mayer transfer operator which - via a so called dynamical trace formula - gives a complete description of the physical system, i. e., the sequence of partition functions can be expressed in terms of the spectrum of the transfer operator. The thermodynamic formalism shows that many properties of the dynamical system depend as functions on the partition function. Motivated by these examples we ask for the class of interactions for which the dynamical system can be completely described by a transfer operator. We can find a class of interactions in which all the above examples are contained and give some new examples. The main Theorems 2.7.6 and 2.13.8 of this chapter will be explained during the following outline of the chapter.

In Section 2.1 we introduce the Ruelle transfer operator $\mathcal{L}$ and formulate the Ruelle-Perron-Frobenius Theorem 2.1.4. We introduce the dual shift τ' and show that the leading eigenfunction of the Ruelle transfer operator factors through a family of continuous linear functionals. This observation is the key idea in finding the suitable $\mathcal{L}$-invariant subspace in Section 2.6. In Section 2.2 we discuss some ways to find a natural Hilbert space $\mathcal{H}$ which contains a preimage of the leading eigenvector of the Ruelle transfer operator. For our constructions we will need a certain suitable stronger decay of the distance function d and hence define subspaces of $\ell^1\mathbb{N}$. By Cauchy's root test the space $\ell^1\mathbb{N}$ of distance functions splits into three parts, namely $d \in \ell^1\mathbb{N}$ either satisfies

$$\limsup_{k\to\infty} \sqrt[k]{|d(k)|} = \begin{cases} 0, & (d \text{ has } \textit{faster than exponential decay}), \\ q \text{ with } 0 < q < 1, & (d \text{ has } \textit{exponential decay at infinity), or} \\ 1, & (d \text{ has } \textit{subexponential decay}). \end{cases}$$

Our methods are limited to cover (parts of) the first two cases as we will see in Remark 2.6.8. In Section 2.3 we give another motivation for the study of the Ruelle transfer operator. We recall the definition of the so called counting trace of a composition operator. We show that the counting trace of the iterates of the Ruelle transfer operator is precisely the dynamical partition function (1.11.4). By virtue of Ansatz 2.3.3 we can transport the dynamical system on Ω to a dynamical system on another topological space E, such that the Ruelle transfer operator induces a composition operator, the Ruelle-Mayer (RM) operator $\mathcal{M}_\beta : \mathcal{C}_b(E) \to \mathcal{C}_b(E)$. For a certain class of examples this operator looks like

$$(\mathcal{M}_\beta f)(z) = \int_F \exp\left(\beta x \langle z|v\rangle\right) \, f(xw + \mathbb{B}z) \, d\nu(x),$$

where v, $w \in E := \ell^2\mathbb{N}$, $\mathbb{B}$ is a trace class operator on $\ell^2\mathbb{N}$, and $f : \ell^2\mathbb{N} \to \mathbb{C}$. It is an interesting observation that the dynamical partition function can be expressed as the counting trace of this induced operator.

In Section 2.4 we will assume that the Ruelle-Mayer transfer $\mathcal{M}_\beta$ operator is trace class. We calculate the spectral traces of its iterates and relate this to the dynamical partition function (1.11.4). We show that

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = \det(1 - \mathbb{B}^n) \, \text{trace} \, (\mathcal{M}_\beta)^n$$

which we call a dynamical trace formula. In Section 2.5 we investigate arbitrary finite range interactions, construct the Ruelle-Mayer transfer operator, and prove its trace formula with the methods of Section 2.4. For the rest of this chapter we will restrict to Ising (type) interactions. In Section 2.6 we will introduce a general method how one can choose a projection map in order to construct a Ruelle-Mayer transfer operator which works for a large class of (longe range) distance functions. The main idea is to find a family of linear continuous maps from $\ell^\infty\mathbb{N}$ (which contains our configuration space Ω) into $\mathbb{C}$, which translates the shift action on $\ell^\infty\mathbb{N}$ into affine maps on some complex vector space. In Section 2.6 we investigate the Banach space situation, whereas in Section 2.7 the Hilbert space case is concerned. Our approach is new compared to the existing literature and allows to treat the following classes of distance functions from a unified point of view: finite range interactions in Section 2.8, superexponentially decaying interactions in Sections 2.9, polynomial-exponential decaying interactions in 2.11, and their superpositions. These distance functions have in common that they can be written as

$$d : \mathbb{N} \to \mathbb{C}, \; k \mapsto d(k) := \langle \mathbb{B}^{k-1} v | w \rangle_{\mathcal{H}}$$

where $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$ with spectral radius $\rho_{\text{spec}}(\mathbb{B}) < 1$, and two fixed vectors v, $w \in \mathcal{H}$. Abstractly speaking, the restriction of the shift operator τ to the subspace of $\ell^1\mathbb{N}$ generated by the functions $\tau^k d$ ($k \in \mathbb{N}_0$) is contractive. This point of view allows a classification of this type of distance functions and implies that the above list is almost complete.

In (2.7.1) we define the classes of distance functions $\mathcal{D}_1^{(p)} \subset \ell^1\mathbb{N}$ (for $p < \infty$) via $d \in \mathcal{D}_1^{(p)}$ iff d has a generating triple $(\mathbb{B}, v, w)$, i. e., admits a representation $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$, where $\mathbb{B}$ belongs to the Schatten class $\mathcal{S}_p(\ell^2\mathbb{N})$ and $\rho_{\text{spec}}(\mathbb{B}) < 1$. The following main theorem of this chapter states that for all Ising interactions with distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ a dynamical trace formula holds at least for almost all $n \in \mathbb{N}$:

Theorem 2.7.6. Let $F \subset \mathbb{C}$ be a bounded set and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1), say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$. Then there exists an index $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geq n_0$ the Ruelle-Mayer transfer operator

$$\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}), \; (\mathcal{M}_\beta f)(z) = \int_F \exp\Big(\beta q(\sigma) + \beta \sigma \langle z|w\rangle\Big) \, f(\sigma \, v + \mathbb{B} \, z) \, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \mathbb{B}^n) \, \text{trace} \, (\mathcal{M}_\beta)^n$. □

In Section 2.12 we will make some comments on the classification of this class of distance functions. In the following sections we will construct Ruelle-Mayer transfer operators for finite range interactions (Subs. 2.8), superexponentially decaying Ising interactions (Subs. 2.9), polynomial-exponentially decaying Ising interactions (Subs. 2.11), and $\mathcal{D}_1^{(p),\Delta}$ Ising interactions (Subs. 2.10). The latter are distance functions which are suitable infinite superpositions of exponentially decaying interactions and have not been studied before.

We end this chapter by transferring the result 2.7.6 to Ising type interactions. Recall that an interaction matrix is called of Ising type if it can be written as a finite sum of Ising interactions. A typical example of an Ising type interaction is Stanley's M-vector model [St68a]. If the alphabet F is a finite set, which is often the case, then every interaction matrix is of Ising type. This allows to study the M-states Potts model from the same mathematical point of view as for instance the Ising model. Our Theorem 2.13.8 will be obtained by taking the Ruelle-Mayer transfer operator for each summand and form (mainly) their tensor power.
In Chapter 3 we will generalise the results of this chapter to one-dimensional one-sided matrix subshifts and obtain a similar dynamical trace formula.

2.1 Ruelle transfer operator

In this section we will explain the concept of a transfer operator. We start with the original definition by D. Ruelle and discuss some of its properties. The Ruelle transfer operator is a composition operator acting on the space of continuous bounded functions on the configuration space. Under additional assumptions one can show that this operator has a spectral gap, i. e., there exists a leading eigenvalue λ_1 and the rest of the spectrum is contained in a disk with radius strictly smaller than $|\lambda_1|$. The Ruelle-Perron-Frobenius Theorem 2.1.4 exploits this fact, deduces further properties of the Ruelle transfer operator, and relates these analytic facts to certain properties of the dynamical system.

Remark 2.1.1. We return to the situation of Remark 1.11.5 (v): Let I be a bounded domain in $\mathbb{R}^n$ or $\mathbb{C}^n$ and $T : I \to I$ a continuous map, such that every point $x \in I$ has only finitely many preimages. Let $A \in \mathcal{C}_b(I)$ and define the Ruelle transfer operator

$$\mathcal{L}_A : \mathcal{C}_b(I) \to \mathcal{C}_b(I),\ (\mathcal{L}_A f)(x) = \sum_{y \in T^{-1}(x)} \exp\big(A(y)\big)\, f(y). \tag{36}$$

There is a closely related operator: The *Perron-Frobenius operator* $\mathcal{L}$ is defined on $L^1(I)$ via

$$\int_I (\mathcal{L}f)(x)\, g(x)\, dx = \int_I f(x)\, (g \circ T)(x)\, dx$$

for all $f \in L^1(I)$, $g \in L^\infty(I)$. If T is piecewiece continuously differentiable, then the function $A(x) = -\log|\det(T'(x))|$ is almost everywhere defined and the Ruelle operator associated with this observable coincides with the restriction of the Perron-Frobenius operator to the space $\mathcal{C}_b(I)$ of continuous bounded functions on I. □

In the following we will only deal with the one-dimensional matrix subshift for which we can determine the preimages explicitly:

Example 2.1.2. Let $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6). The shift operator is the surjective, non-injective map

$$\tau := \tau_1 : F^{\mathbb{N}} \to F^{\mathbb{N}},\ (\tau\xi)(i) = \xi(i+1).$$

We give a description of the preimages of a configuration $\xi \in F^{\mathbb{N}}$ under the shift operator: Recall the bijective right shift $\tau_{-1;\mathbb{N}} : F^{\mathbb{N}} \to F^{\mathbb{N}_{\geq 2}}$ defined in (1.2.3). For any $\sigma \in F$ the element $(\sigma \vee \xi) = (\sigma, \xi_1, \xi_2, \ldots) = (\sigma \oplus \tau_{-1;\mathbb{N}}(\xi)) \in F^{\mathbb{N}}$ defined via

$$(\sigma \vee \xi)_i := \begin{cases} \sigma & , i = 1, \\ \xi_{i-1} & , i \geq 2 \end{cases}$$

is a preimage of ξ. For $n \in \mathbb{N}$, $\sigma_1, \ldots, \sigma_n \in F$, $\xi \in F^n$ we define

$$\sigma_n \vee \ldots \vee \sigma_2 \vee \sigma_1 \vee \xi := \sigma_n \vee (\sigma_{n-1} \ldots \vee (\sigma_2 \vee (\sigma_1 \vee \xi)) \ldots). \tag{37}$$

Let $(\Omega_{\mathbb{A}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift (1.2.8) of $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$. The preimages of $\xi \in \Omega_{\mathbb{A}}$ are precisely those configurations $(\sigma \vee \xi) \in F^{\mathbb{N}}$ with $\mathbb{A}_{\sigma,\xi_1} = 1$. □

This leads to the following definition:

Definition 2.1.3. Let F be a Hausdorff space equipped with a finite Borel measure ν, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6), and $A \in \mathcal{C}_b(\Omega)$ an observable, then the associated *Ruelle transfer operator* $\mathcal{L}_A : \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega)$ is defined as

$$(\mathcal{L}_A f)(\xi) := \int_F \exp\big(A(\sigma \vee \xi)\big)\, f(\sigma \vee \xi)\, d\nu(\sigma).$$

□

As one easily checks, the Ruelle transfer operator is a bounded linear operator on $\mathcal{C}_b(\Omega)$. Moreover cp. [May91, p. 181], [Ru78], if F is compact and A is real-valued, the operator $\mathcal{L}_A : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ is a positive operator and hence has a positive, separated leading eigenvalue, which implies the following theorem:

Theorem 2.1.4. *(Ruelle-Perron-Frobenius Theorem*[12]*) Let F be compact, $(\Omega, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a lattice spin system (1.2.7), and $A : \Omega \to \mathbb{R}$ a Hölder continuous function. Then:*

(i) *There exists $h_1 \in \mathcal{C}(\Omega)$, $h_1 > 0$, $\lambda_1 > 0$ such that $\mathcal{L}_A h_1 = \lambda_1 h_1$.*

(ii) *There exists a probability measure $\nu_1 \in \mathcal{C}(\Omega)'$, $\nu_1 \geq 0$, $\nu_1(h_1) = 1$ and $(\mathcal{L}_A)'\nu_1 = \lambda_1 \nu_1$.*

(iii) *For any $f \in \mathcal{C}(\Omega)$*

$$\lim_{n\to\infty} \|\lambda_1^{-n} (\mathcal{L}_A)^n f - \nu_1(f) h_1\|_{\mathcal{C}(\Omega)} = 0.$$

(iv) *The following formula holds for the topological pressure*[13]

$$P(A) := \lim_{n\to\infty} \frac{1}{n} \log \widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \log \lambda_1. \tag{38}$$

(v) *The probability measure $\mu_1 := h_1\, \nu_1$ is shift invariant and is a Gibbs state*[14]. □

The spectral properties of the transfer operator (quasi-compact, compact, Schatten class) determine the limit behaviour of the dynamical system, see for instance the book [Ba00]. For this purpose it is helpful to study the Ruelle transfer operator on subspaces of $\mathcal{C}_b(\Omega)$ which are easier to treat. To preserve the main (spectral) information about the original operator, such a subspace should contain the constant function $1 \in \mathcal{C}_b(\Omega)$ and its image under the iterates of $\mathcal{L}$, i. e., the space V defined as

$$V := \overline{\operatorname{span}\{\mathcal{L}^n 1 \,|\, n \in \mathbb{N}_0\}}^{\mathcal{C}_b(\Omega)}, \tag{39}$$

since by the Ruelle-Perron-Frobenius Theorem 2.1.4 the eigenvector h_1 of $\mathcal{L}$ corresponding to the leading eigenvalue λ_1 can be approximated by the normalised $\mathcal{L}_A$-iterates of the constant function 1 due to the fact that

$$\lim_{n\to\infty} \|\lambda_1^{-n} \mathcal{L}^n 1 - h_1\|_{\mathcal{C}(\Omega)} = 0.$$

In this dissertation we restrict our attention to Ising type interactions which contain many physically interesting systems. For the mathematical treatment it is often expedient to consider first the Ising model and the transfer operator $\mathcal{L}_{A_{(\phi)}}$, see Example 1.8.3, with spin values in a bounded subset $F \subset \mathbb{C}$, and then to generalise to Ising type models. If $F \subset \mathbb{C}$ is bounded, then the configuration space $\Omega = F^{\mathbb{N}}$ is a bounded subset of $\ell^\infty\mathbb{N}$.

The previous considerations motivate the investigation of the image of the constant function 1 under the iterates of $\mathcal{L}$ in order to obtain an explicit description of the space V (39). It turns out, see Proposition 2.1.8, that the functions $(\mathcal{L}_{A_{(\phi)}})^n 1$ depend on a family of functionals which are defined via the distance function and the shift. These functionals will be introduced in Remark 2.1.6.

[12] see for instance [Bo75] or [PaPo90].

[13] The topological pressure can equivalently be characterised by a variational principle, which can be stated as: $P(A) = \sup\{h_\mu(\tau) + \int_\Omega A\, d\mu \,|\, \mu\; \tau\text{-invariant probability measure on } \Omega\}$, where $h_\mu(\tau)$ is the entropy of τ with respect to μ.

[14] Gibbs states can be equivalently characterised as solutions of the Dobrushin-Lanford-Ruelle equations and also by certain conditional probabilities, see [Ru78].

Remark 2.1.5. The (left) shift operator $\tau : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}},\ (\tau\xi)_i = \xi_{i+1}$ leaves invariant all the spaces $\ell^p\mathbb{N}$ for $1 \le p \le \infty$ and defines continuous linear operators on these spaces:

$$\|\tau(\xi)\|_{\ell^p(\mathbb{N})} \le \|\xi\|_{\ell^p\mathbb{N}}.$$

With respect to the usual bilinear pairing $\langle \xi, \eta\rangle_{\ell^p\mathbb{N},\ell^q\mathbb{N}} = \sum_{i=1}^{\infty} \xi_i\,\eta_i$ (where p, q are dual exponents, i. e., $\frac{1}{p} + \frac{1}{q} = 1$), the dual operator of $\tau : \ell^p\mathbb{N} \to \ell^p\mathbb{N}$ is $\tau' : \ell^q\mathbb{N} \to \ell^q\mathbb{N}$ acting via $\tau'(\xi) = 0 \vee \xi$ (2.1.2) defined via

$$\tau' : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}},\ (\tau'\xi)_i = \begin{cases} 0, & i = 1, \\ \xi_{i-1}, & i \ge 2. \end{cases}$$

In particular, $(\tau\tau')(\xi) = \tau(0 \vee \xi) = \xi$ and $(\tau'\tau)(\sigma \vee \xi) = \tau'(\xi) = 0 \vee \xi$. □

Remark 2.1.6. Let $\tau : \ell^1\mathbb{N} \to \ell^1\mathbb{N}$ be the shift operator (2.1.5). With respect to the usual bilinear pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\ell^p\mathbb{N},\ell^q\mathbb{N}}$ a sequence $d \in \ell^1\mathbb{N}$ gives rise to a family of continuous linear functionals $\pi_k^d \in (\ell^\infty\mathbb{N})'$ (indexed by $k \in \mathbb{N}$) defined via

(40) $$\pi_k^d : \ell^\infty(\mathbb{N}) \to \mathbb{C},\ \xi \mapsto \langle \xi, \tau^{k-1} d\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}} = \sum_{i=1}^{\infty} \xi_i\, d(k+i-1).$$

Obviously,

$$|\pi_k^d(\xi)| \le \sup_{i\in\mathbb{N}} |\xi_i|\, \|\tau^{k-1} d\|_{\ell^1\mathbb{N}} \le \|\xi\|_{\ell^\infty\mathbb{N}}\, \|d\|_{\ell^1\mathbb{N}}.$$

For any finite set $J \subset \mathbb{N}$ set $\pi_J^d := (\pi_j^d)_{j\in J} : \ell^\infty(\mathbb{N}) \to \mathbb{C}^{|J|}$. Then π_J^d is a continuous linear operator. □

The functionals $\pi_k^d \in \ell^\infty(\mathbb{N})'$ are interesting objects due to their relation to the standard observable (1.11.1) and the following proposition.

Remark 2.1.7. Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a lattice spin system (1.2.7), then $\Omega \subset F^{\mathbb{N}} \subset \ell^\infty(\mathbb{N})$ is a bounded set and the observable $A_{(\phi)}$ (1.11.1) associated to the Ising interaction (1.8.3) with distance function $d \in \ell^1\mathbb{N}$ and potential $q \in \mathcal{C}_b(F)$ is given as

(41) $$A_{(\phi)} : \Omega \to \mathbb{C},\ A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sigma\, \pi_1^d(\xi).$$

In this situation, we call $A_{(\phi)}$ the *standard Ising observable.* By Remark 2.1.6 we have $A_{(\phi)} \in \mathcal{C}_b(\Omega)$. If q extends to a continuous map $\mathbb{C} \to \mathbb{C}$, then also $A_{(\phi)}$ extends to a continuous map via

$$A_{(\phi)} : \ell^\infty\mathbb{N} \to \mathbb{C},\ A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sigma\, \pi_1^d(\xi).$$

In fact: Let $\eta = \sigma \vee \xi \in \Omega$, then using (1.8.1), (1.8.2), and (1.11.1)

$$A_{(\phi)}(\eta) = \varphi_1(1;\eta_1) + \sum_{i>1} \varphi_2(1, i; \eta_1, \eta_i) = q(\eta_1) + \sum_{i>1} d(i-1)\,\eta_1\,\eta_i = q(\sigma) + \sigma \sum_{i=1}^{\infty} d(i)\,\xi_i = q(\sigma) + \sigma\, \pi_1^d(\xi),$$

which proves (41). By the continuity of π_1^d (2.1.6) and q also $A_{(\phi)}$ is continuous:

$$\begin{aligned} |A_{(\phi)}(\sigma \vee \xi) - A_{(\phi)}(\sigma' \vee \xi')| &\le |q(\sigma) - q(\sigma')| + |\sigma\, \pi_1^d(\xi) - \sigma'\, \pi_1^d(\xi')| \\ &\le |q(\sigma) - q(\sigma')| + |\sigma|\, \|d\|_{\ell^1\mathbb{N}}\, \|\xi - \xi'\|_{\ell^\infty\mathbb{N}} + |\sigma - \sigma'|\, \|d\|_{\ell^1\mathbb{N}}\, \|\xi'\|_{\ell^\infty\mathbb{N}}. \end{aligned}$$

□

In Proposition 2.6.6 we will return to the idea that the standard Ising observable can be expressed with a certain linear functional. The next proposition says that the spectral information about the leading eigenvalue of the Ruelle transfer operator is concentrated on a subspace of $\mathcal{C}_b(\Omega)$ which is characterised by the functionals $\pi_k^d \in \ell^\infty(\mathbb{N})'$ from Remark 2.1.6. This observation will be essential for the construction of the Ruelle-Mayer transfer operator.

Proposition 2.1.8. *Let $F \subset \mathbb{C}$ be a bounded set, $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided full shift (1.2.6), ϕ a two-body Ising interaction (1.8.3) with distance function $d \in \ell^1\mathbb{N}$, $A_{(\phi)}$ the standard Ising observable (2.1.7), and $\mathcal{L}_{A_{(\phi)}}$ be the associated Ruelle transfer operator (2.1.3), then $((\mathcal{L}_{A_{(\phi)}})^n 1)(\xi)$ depends on ξ via the functions $\pi_k^d|_\Omega$ (40) for $k = 1, \ldots, n$:*

$$(\mathcal{L}_{\beta A_{(\phi)}} 1)(\xi) = \int_F \exp\Big(\beta q(\sigma) + \beta\sigma\pi_1^d(\xi)\Big)\, d\nu(\sigma),$$

$$((\mathcal{L}_{\beta A_{(\phi)}})^n 1)(\xi) = \int_{F^n} \exp\Big(\beta\sum_{k=1}^n q(\sigma_k) + \beta\sum_{k=1}^n \sigma_k \sum_{i=1}^{k-1}\sigma_i\, d(k-i) + \beta\sum_{k=2}^n \sigma_k\pi_k^d(\xi)\Big) d\nu(\sigma_n)\ldots d\nu(\sigma_1).$$

Proof. Let $A \in \mathcal{C}_b(\Omega)$ and $\mathcal{L}_A$ be the associated Ruelle transfer operator (2.1.3). We write this operator as an integral over a family of composition operators as

$$(\mathcal{L}_A f)(\xi) = \int_F \exp\big(A_\sigma(\xi)\big)\,(f\circ\psi_\sigma)(\xi)\, d\nu(\sigma) = \int_F L_x\, d\nu(x),$$

where for any $\sigma \in F$ we set $\psi_\sigma : F^{\mathbb{N}} \to F^{\mathbb{N}},\ \xi \mapsto \sigma \vee \xi$ (2.1.2), $A_\sigma : F^{\mathbb{N}} \to \mathbb{C},\ \xi \mapsto A(\sigma\vee\xi)$, and $L_\sigma : \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega)$, $(L_\sigma f)(\xi) = \exp\big(A_\sigma(\xi)\big)\,(f\circ\psi_\sigma)(\xi)$. We compute the n-th iterate of $\mathcal{L}_A = \int_F L_x\, d\nu(x)$ with Corollaries A.7.7 and B.1.3 which yields for $n \geq 2$

$$\begin{aligned}(\mathcal{L}_A^n f)(\xi) &= \int_{F^n} (L_{x_n}\circ\ldots\circ L_{x_1} f)(\xi)\, d\nu(x_1)\ldots d\nu(x_n)\\ &= \int_{F^n} \exp\Big(\sum_{k=1}^n A(x_k\vee\ldots\vee x_n\vee\xi)\Big) f(x_1\vee\ldots\vee x_n\vee\xi)\, d\nu(x_1)\ldots d\nu(x_n)\end{aligned} \tag{42}$$

using the definition of $\vee$ (37) given in (2.1.2) For the particular choices $A = \beta A_{(\phi)}$ (1.11.1) and f being the constant function 1 we obtain (for $n \geq 2$)

$$\begin{aligned}((\mathcal{L}_{\beta A_{(\phi)}})^n 1)(\xi) &= \int_{F^n} \exp\Big(\beta\sum_{k=1}^n A_{(\phi)}(x_k\vee\ldots\vee x_n\vee\xi)\Big)\, d\nu(x_1)\ldots d\nu(x_n)\\ &= \int_{F^n} \exp\Big(\beta\sum_{k=1}^n A_{(\phi)}(x_k\vee\ldots\vee x_1\vee\xi)\Big)\, d\nu(x_1)\ldots d\nu(x_n)\\ &= \int_{F^n} \exp\left(\beta\sum_{k=1}^n \varphi_1(k,\sigma_k) + \beta\sum_{k=1}^n\Big(\sum_{i=2}^k \varphi_2(1,i;\sigma_k,\sigma_{k+1-i}) + \sum_{i=1}^\infty \varphi_2(k+i,1;\xi_i,\sigma_k)\Big)\right) d\nu^n(\sigma_1,..,\sigma_n)\end{aligned}$$

which gives in the case ϕ being an Ising two-body interaction (1.8.3)

$$((\mathcal{L}_{\beta A_{(\phi)}})^n 1)(\xi) = \int_{F^n} \exp\left(\beta\sum_{k=1}^n q(\sigma_k) + \beta\sum_{k=1}^n \sigma_k\Big(\sum_{i=1}^{k-1}\sigma_i\, d(k-i) + \sum_{i=1}^\infty \xi_i\, d(k+i-1)\Big)\right) d\nu^n(\sigma_1,..,\sigma_n).$$

□

We will make a first attempt in finding a subspace of $\mathcal{C}_b(\Omega)$ which is invariant under the Ruelle transfer operator $\mathcal{L}_{A_{(\phi)}}$, contains the leading eigenfunction, and can be described without knowing the eigenfunctions of $\mathcal{L}_{A_{(\phi)}}$.

Remark 2.1.9. Let $F \subset \mathbb{C}$ be a bounded set, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided full shift (1.2.6), ϕ a two-body Ising interaction (1.8.3) with distance function $d \in \ell^1\mathbb{N}$, $A_{(\phi)}$ the standard Ising observable (2.1.7), and $\mathcal{L}_{A_{(\phi)}}$ be the associated Ruelle transfer operator (2.1.3), then in view of Proposition 2.1.8 a reasonable candidate for the investigation of spectral properties of $\mathcal{L}_{A_{(\phi)}}$ is

$$W := \overline{\operatorname{span}\{f \in \mathcal{C}_b(\Omega)\,|\,\exists J \subset \mathbb{N}_0 \text{ finite}, g \in \mathcal{C}_b(U_J) :\ f = g\circ\pi_J^d\}}^{\mathcal{C}_b(\Omega)},$$

where $U_J \subset \mathbb{C}^J$ is some (connected) neighbourhood of the image $\pi_J^d(\Omega) \subset \mathbb{C}^J$ of π_J^d defined in (2.1.6). The space V from (39) is contained in W. A priori it is not clear whether V and W are non-trivial subspaces of $\mathcal{C}_b(\Omega)$, whether $V = W$, and whether W is $\mathcal{L}_{A_{(\phi)}}$-invariant. Concerning the latter: Let $g_J \in \mathcal{C}_b(U_J)$, $f = g_J \circ \pi_J^d \in W$, then

$$(\mathcal{L}_{A_{(\phi)}} f)(\xi) = \int_F \exp\bigl(\beta\sigma\pi_1^d(\xi)\bigr)\,(g_J \circ \pi_J^d)(\sigma \vee \xi)\,d\nu(\sigma) = \int_F \exp\bigl(\beta\sigma\pi_1(\xi)\bigr)\,g_J(\sigma d_J + \pi_{J+1}^d(\xi))\,d\nu(\sigma),$$

where $d_J := \bigl(d(j)\bigr)_{j\in J} \in \mathbb{C}^J$ and $J+1 := \{j+1; j \in J\}$. Hence the image depends on the functionals π_k^d (2.1.6) for $k \in \{1\} \cup J+1$. □

2.2 The leading eigenfunction of the Ruelle transfer operator

Let ϕ be a pure Ising two-body interaction (1.8.3) with distance function $d \in \ell^1\mathbb{N}$ and spin values in a bounded subset $F \subset \mathbb{C}$. We look for a small subspace of $\mathcal{C}_b(\Omega)$ on which the spectral information of the Ruelle transfer operator is concentrated. In Proposition 2.1.8 we have seen that the leading eigenvector of the Ruelle transfer operator $\mathcal{L}_{A_{(\phi)}}$ (2.1.3) depends on all the functionals $\pi_k^d : \ell^\infty\mathbb{N} \to \mathbb{C}$ $(k \in \mathbb{N})$ as defined in equation (40) of Remark 2.1.6. This observation suggests to consider $\pi^d =$"$\lim_{J\to\mathbb{N}} \pi^d$" of π_J^d and to consider the action of the Ruelle transfer operator on functions $f = g \circ \pi^d$. Hence one has to estimate the size of $\pi_J^d(\xi)$. First we will use the $\ell^2\mathbb{N}$-norm, which leads to the class $\mathcal{D}_2$ of distance functions such that π^d is continuous. Later, we will investigate distance functions belonging to subspaces of $\mathcal{D}_2$ characterised by a stronger decay condition. The potential q does not play a role for these considerations and will be included only later.

Definition 2.2.1. Let $\mathcal{D}_2$ be the subspace of $\ell^1\mathbb{N}$ consisting of all sequences $d : \mathbb{N} \to \mathbb{C}$ such that

$$\|d\|_{\mathcal{D}_2}^2 := \sum_{j=1}^\infty \Bigl(\sum_{k=1}^\infty |d(k+j-1)|\Bigr)^2 = \sum_{j=1}^\infty \Bigl(\sum_{k=j}^\infty |d(k)|\Bigr)^2 < \infty.$$

□

The requirement that a distance function d belongs to $\mathcal{D}_2$ is the natural condition which guarantees the continuity of the linear map $\pi^d : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N}$ defined next.

Proposition 2.2.2. *Let $d \in \mathcal{D}_2$ (2.2.1). Then*

$$\pi^d : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N},\ \pi^d(\xi)_j := \pi_j^d(\xi) = \langle \xi, \tau^{j-1} d\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}} = \sum_{k=1}^\infty \xi_k\, d(k+j-1) \tag{43}$$

is a continuous linear map with $\|\pi^d\| \le \|d\|_{\mathcal{D}_2}$ and

$$\pi^d(\sigma \vee \xi) = \sigma\, d + \tau\pi^d(\xi) \tag{44}$$

for all $\sigma \in \mathbb{C}$, $\xi \in \ell^\infty\mathbb{N}$, where $\tau : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $\tau(\xi)_i := \xi_{i+1}$ is the shift defined in Remark 2.1.5.

Proof. Let $\xi \in \ell^\infty\mathbb{N}$. Then by Remark 2.1.6

$$\|\pi^d(\xi)\|^2 = \sum_{j=1}^\infty |\pi_j^d(\xi)|^2 \le \|\xi\|_{\ell^\infty\mathbb{N}}^2 \sum_{j=1}^\infty \|\tau^{j-1} d\|_{\ell^1\mathbb{N}}^2 = \|\xi\|_{\ell^\infty\mathbb{N}}^2 \sum_{j=1}^\infty \Bigl(\sum_{k=1}^\infty |d(k+j-1)|\Bigr)^2.$$

Let $j \in \mathbb{N}$, $\sigma \in F$, $\xi \in \ell^\infty\mathbb{N}$, then

$$\pi^d(\sigma \vee \xi)_j = \pi_j^d(\sigma \vee \xi) = \langle \sigma \vee \xi, \tau^{j-1} d\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}} = \sigma\, d(j) + \sum_{k=1}^\infty \xi_k\, d(k+j) = \sigma\, d(j) + \pi_{j+1}(\xi).$$

□

Example 2.2.3. Let $d \in \mathcal{D}_2$ (2.2.1). Cauchy-Schwarz's inequality yields a majorant for $\|d\|_{\mathcal{D}_2}^2$ via

$$\|d\|_{\mathcal{D}_2}^2 = \sum_{j=1}^\infty \Bigl(\sum_{k=1}^\infty |d(k+j-1)|\Bigr)^2 \le \|d\|_{\ell^\infty\mathbb{N}} \sum_{j=1}^\infty \sum_{k=1}^\infty |d(k+j-1)| = \|d\|_{\ell^\infty\mathbb{N}} \sum_{j=1}^\infty j|d(j)|, \tag{45}$$

where the last equality can be shown by counting the number of equal terms in the double series. This upper bound will reappear in Definition 2.2.4. Examples of distance functions belonging to $\mathcal{D}_2$ will be given in (2.2.5). □

In the following we will require a suitable stronger decay of the distance function d such that the image of $\ell^\infty\mathbb{N}$ under π^d (43) lies in $\ell^1\mathbb{N}$. This is also the natural condition for the absolute convergence of the function $W_{\mathbb{Z}_\leq,\mathbb{Z}_>}$ from Remark 1.6.3. The interest in the function $W_{\mathbb{Z}_\leq,\mathbb{Z}_>}$ is its close relation to the leading eigenfunction of the Ruelle transfer operator. This we will show in Remark 2.2.11. We will prove a useful representation of $W_{\mathbb{Z}_\leq,\mathbb{Z}_>}$ with the help of a continuous trilinear mapping W, as we will see in Remark 2.2.10. This will allow us to investigate whether the leading eigenvector h_1 of the Ruelle operator $\mathcal{L}_{A_{(\phi)}}$ (2.1.3) has a preimage $h_1 = \tilde{h}_1 \circ \pi^d$ under composition with π^d.

Definition 2.2.4. Let $\mathcal{D}_1$ be the subspace of $\ell^1\mathbb{N}$ consisting of all sequences $d : \mathbb{N} \to \mathbb{C}$ with

$$\|d\|_{\mathcal{D}_1} := \sum_{i=1}^\infty \sum_{j=1}^\infty |d(i+j-1)| = \sum_{i=1}^\infty i\,|d(i)| < \infty.$$

□

By Example 2.2.3 we have $\mathcal{D}_1 \subset \mathcal{D}_2$, and obviously $\|d\|_{\ell^1\mathbb{N}} \leq \|d\|_{\mathcal{D}_1}$ holds. We give some examples of physically relevant distance functions belonging to this class.

Example 2.2.5. (i) Let $\alpha > 1$ and $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto k^{-\alpha}$. It belongs to $\mathcal{D}_1$ (2.2.4), iff $\alpha > 2$, since

$$\sum_{\nu=1}^\infty \nu\,|d(\nu)| = \sum_{\nu=1}^\infty \nu^{-1-\alpha} = \zeta(\alpha+1),$$

where ζ denotes the Riemann zeta function. This class of distance functions contains the *van der Waals potential* ($\alpha = 6$) of particle physics, but not the *Coulomb potential* ($\alpha = 2$) of electrostatics.

(ii) Plummer potential: Let $\epsilon > 0$, $\alpha > 1$ and $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto (\epsilon + |k|^2)^{-\alpha/2}$, which is an approximation of $d'(k) = k^{-\alpha}$. In fact: One can choose $c_\epsilon > 0$ such that $c_\epsilon\, k^{-\alpha} \leq d(k) \leq k^{-\alpha}$. Hence it belongs to $\mathcal{D}_1$, iff $\alpha > 2$.

(iii) Let γ, $\delta > 0$, $n \in \mathbb{N}_0$, and $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto k^n \exp(-\gamma k^\delta)$, which appears for instance in [May80a, p. 100] in the case $n = 0$. It belongs to $\mathcal{D}_1$ by Proposition 2.2.6.

(iv) Let α, $\gamma > 0$ and $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto \int_0^1 t^\alpha \exp(-\gamma t k)\,dt$ which for large k behaves like $k^{-\alpha+1}$. This distance appears in [May80a, p. 109]. It belongs to $\mathcal{D}_1$ for all $\gamma > 0$, $\alpha > 2$: Note that the function $[0, \infty[\to \mathbb{R}$, $t \mapsto t^\alpha \exp(-\gamma t)$ attains its maximum at γ/α, hence $d(k) \leq (\alpha/\gamma)^\alpha\, k^{-\alpha} \exp(-\alpha)$ and $\sum_{\nu=1}^\infty \nu\,|d(\nu)| \leq (\alpha/\gamma)^\alpha \exp(-\alpha) \sum_{k=1}^\infty k^{1-\alpha}$, which converges for $\alpha > 2$. □

The following auxiliary proposition is left from Example 2.2.5.

Proposition 2.2.6. *Let ϵ_1, $\epsilon_2 > 0$, $n \in \mathbb{N}_0$, then $\sum_{k=1}^\infty k^n \exp(-\epsilon_1\, k^{\epsilon_2}) < \infty$.*

Proof. The series converges if and only if the condensed series

$$\sum_{k=1}^\infty 2^k\,(2^k)^n \exp(-\epsilon_1\, 2^{k\epsilon_2}) = \sum_{k=1}^\infty (2^{n+1})^k \exp(-\epsilon_1\, 2^{k\epsilon_2})$$

converges. By the root test this holds true, since

$$\lim_{k\to\infty} \sqrt[k]{(2^{n+1})^k \exp(-\epsilon_1\, 2^{k\epsilon_2})} = 2^{n+1} \lim_{k\to\infty} \exp\big(-\frac{\epsilon_1}{k}\, 2^{k\epsilon_2}\big) = 0.$$

□

Remark 2.2.7. Let $d \in \ell^1\mathbb{N}$ be a given distance function. Cauchy's root test implies that there are only three possible cases

$$\limsup_{k\to\infty} \sqrt[k]{|d(k)|} = \begin{cases} 0, & d \text{ has } \textit{faster than exponential decay}; \\ 1, & d \text{ has } \textit{subexponential decay}; \\ q \text{ with } 0 < q < 1, & d \text{ has } \textit{exponential decay at infinity}. \end{cases}$$

□

In Example 2.6.9 we will investigate this limit behaviour measured by 2.2.7 for the distance functions introduced in Example 2.2.5. We will now show that the distance function with at least exponential decay at infinity belong to the class $\mathcal{D}_1$.

Proposition 2.2.8. *Let $d : \mathbb{N} \to \mathbb{C}$ be a function with $\limsup_{k\to\infty} \sqrt[k]{|d(k)|} < 1$, then $d \in \mathcal{D}_1$ (2.2.4)* .

Proof. The root test implies that $\|d\|_{\mathcal{D}_1} = \sum_{k=1}^{\infty} k\,|d(k)|$ is finite. □

The converse of Proposition 2.2.8 is not true: Consider for instance $d(k) = k^{-3}$ (2.2.5) (i).
We will construct trace class Ruelle-Mayer (RM) transfer operators for some classes of interactions which decay fast enough. It is an open problem to find trace class RM transfer operators for interactions with slower than exponential decay, or to prove that those cannot exist.
The requirement $d \in \mathcal{D}_1$ is the natural assumption which guarantees the continuity of the map W, which we define next.

Proposition 2.2.9. *For $d \in \mathcal{D}_1$ (2.2.4) let $\pi^d : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N}$ be the linear map defined in (2.2.2). Then the image of $\ell^\infty\mathbb{N}$ under π^d is contained in $\ell^1\mathbb{N}$ and the linear map $\pi^d : \ell^\infty\mathbb{N} \to \ell^1\mathbb{N}$ is continuous with $\|\pi^d(x)\|_{\ell^1\mathbb{N}} \le \|d\|_{\mathcal{D}_1}\,\|x\|_{\ell^\infty\mathbb{N}}$. Set*

$$W : \ell^\infty\mathbb{N} \times \ell^\infty\mathbb{N} \times \mathcal{D}_1 \to \mathbb{C},\ (x, y, d) \mapsto W(x, y; d) := -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i\, y_j\, d(i + j - 1).$$

Then W is a continuous trilinear map, $|W(x, y; d)| \le \|d\|_{\mathcal{D}_1}\,\|x\|_{\ell^\infty\mathbb{N}}\,\|y\|_{\ell^\infty(\mathbb{N})}$, with

$$W(x, y; d) = W(y, x; d) = -\langle x, \pi^d(y)\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}} = -\langle \pi^d(x), y\rangle_{\ell^1\mathbb{N},\ell^\infty\mathbb{N}}. \tag{46}$$

Proof. To prove the continuity of π^d we calculate for $x,\ y \in \ell^\infty\mathbb{N}$, $d \in \mathcal{D}_1$

$$\|\pi^d(x)\|_{\ell^1\mathbb{N}} = \sum_{j=1}^{\infty} |\pi^d_j(x)| = \sum_{j=1}^{\infty} \Big|\sum_{k=1}^{\infty} x_k\, d(j + k - 1)\Big| \le \|x\|_{\ell^\infty\mathbb{N}} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |d(j + k - 1)|,$$

which is finite by definition of $\mathcal{D}_1$ (2.2.4). Rearranging terms gives

$$-W(x, y; d) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i\, y_j\, d(i + j - 1) = \sum_{i=1}^{\infty} x_i \sum_{j=1}^{\infty} y_j\, d(i + j - 1) = \sum_{i=1}^{\infty} x_i\, \pi^d_i(y) = \langle x, \pi^d(y)\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}},$$

which is linear and continuous in $x,\ y \in \ell^\infty\mathbb{N}$, $d \in \mathcal{D}_1$. The symmetry in the first two arguments of W concludes the proof. □

The interest in the function W from Proposition 2.2.9 is the following connection with the function $W_{\mathbb{Z}_\le,\mathbb{Z}_>}$ from Remark 1.6.3 and a theorem of D. Ruelle which we recall in Remark 2.2.11.

Remark 2.2.10. Let $F \subset \mathbb{C}$ be a bounded set and $(F^{\mathbb{Z}}, \mathbb{Z}, \tau)$ a one-dimensional two-sided full shift (1.2.5). Let ϕ be a pure two-body Ising interaction (1.8.3) with distance function $d \in \mathcal{D}_1$ (2.2.4). Set $\mathbb{Z}_\le := -\mathbb{N}_0$ and $\mathbb{Z}_> := \mathbb{N}$. Then the function

$$W_{\mathbb{Z}_\le,\mathbb{Z}_>} : F^{-\mathbb{N}_0} \times F^{\mathbb{N}} \to \mathbb{C},\ (\eta, \xi) \mapsto W_{\mathbb{Z}_\le,\mathbb{Z}_>}(\eta \oplus \xi) = -\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \varphi_2(-i, j; \eta_{-i}, \xi_j)$$

from Remark 1.6.3 can be written as

$$W_{\mathbb{Z}_\le,\mathbb{Z}_>}(\eta \oplus \xi) = W(S(\overleftarrow{\eta}), \xi; d). \tag{47}$$

where W is the continuous trilinear map from Proposition 2.2.9, $S : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}}$, $(Sx)_i = x_{i-1}$ is the bijective right shift[15], $\overleftarrow{\ }$ is the *inversion map* $\overleftarrow{\ } : \mathbb{C}^{-\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, $\overleftarrow{\eta}_i := \eta_{-i}$, and $\oplus$ is the concatenation operator (Def. 1.3.4). $W_{\mathbb{Z}_\le,\mathbb{Z}_>}$ extends to a bilinear continuous map

$$W_{\mathbb{Z}_\le,\mathbb{Z}_>} : \ell^\infty(\mathbb{Z}_\le) \times \ell^\infty(\mathbb{Z}_>) \to \mathbb{C},\ (\eta, \xi) \mapsto W_{\mathbb{Z}_\le,\mathbb{Z}_>}(\eta \oplus \xi) = W(S(\overleftarrow{\eta}), \xi; d).$$

[15] S was denoted by $S = \tau_{-1;\mathbb{N}_0}$ in the notation of Definition 1.2.3.

In fact: Let $\eta \in \ell^\infty(\mathbb{Z}_\leq)$, $\xi \in \ell^\infty(\mathbb{Z}_>)$, then

$$W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi) = -\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \eta_{-i}\,\xi_j\, d(i+j) = -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \eta_{1-i}\,\xi_j\, d(i+j-1) = W(S(\overleftarrow{\eta}), \xi; d).$$

□

We are now prepared to formulate a result due to D. Ruelle which gives an explicit description of the leading eigenfunction of the Ruelle transfer operator.

Remark 2.2.11. Let $F \subset \mathbb{R}$ be a bounded set, $(F^{\mathbb{Z}}, \mathbb{Z}, \tau)$ a one-dimensional two-sided full shift (1.2.5), and $(F^{\mathbb{Z}_\leq}, \mathbb{N}_0, \overline{\tau})$ be the one-sided subshift on the negative half axis $\mathbb{Z}_\leq$ endowed with the semigroup action $\overline{\tau} : \mathbb{N}_0 \times F^{\mathbb{Z}_\leq} \to F^{\mathbb{Z}_\leq}$, $\overline{\tau}(n,\xi)(i) := \xi(i-n)$. Let ϕ be a two-body Ising interaction (1.8.3) with real-valued distance function $d \in \mathcal{D}_1$ (2.2.4) and potential $q \in \mathcal{C}_b(F)$. Furthermore, let $A_{(\phi)} \in \mathcal{C}_b(F^{\mathbb{N}})$ be the standard Ising observable (2.1.7). Let $W_{\mathbb{Z}_\leq,\mathbb{Z}_>}$ be as in Remark 1.6.3 and $\beta > 0$ the thermodynamic constant from (1.7.1). D. Ruelle [Ru78, 5.12] has shown that the eigenspace corresponding to the leading eigenvalue of the Ruelle transfer operator $\mathcal{L}_{\beta A_{(\phi)}} : \mathcal{C}_b(F^{\mathbb{N}}) \to \mathcal{C}_b(F^{\mathbb{N}})$ (2.1.3) is spanned by

(48) $$h_1 : F^{\mathbb{N}} \to \mathbb{C},\ \xi \mapsto \int_{F^{\mathbb{Z}_\leq}} \exp(-\beta\, W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi))\, d\mu_\leq(\eta).$$

Here $\mu_\leq$ denotes the (unique) Gibbs measure on $F^{\mathbb{Z}_\leq}$ for the interaction ϕ. The existence and uniqueness is of this finite measure is shown in [Ru78, 5.9]. Usually, one is interested in the unique positive, normalised eigenfunction $\frac{1}{K}\, h_1 > 0$, where $K := \int_{F^{\mathbb{N}}} h_1(\xi)\, d\mu_\leq(\xi)$ is a known constant [Ru78, 5.9, 5.12], which is of independent interest. Note that h_1 is independent of the potential $q \in \mathcal{C}_b(F)$. □

We look for a (small) Hilbert subspace $\mathcal{H}$ of $\mathcal{C}_b(F^{\mathbb{N}})$ which is invariant under the Ruelle transfer operator and still contains the main spectral information. In view of the Ruelle-Perron-Frobenius theorem we require that this subspace contains a preimage of the leading eigenvector of the Ruelle transfer operator. A starting point for the idenfication of a suitable Hilbert space $\mathcal{H}$ is obtained as a combination of Remarks 2.2.10 and 2.2.11. More precisely, we look for a Hilbert space such that the composition operator

$$C_{\pi^d} : \mathcal{H} \to \mathcal{C}_b(F^{\mathbb{N}}),\ f \mapsto f \circ \pi^d$$

is continuous and $C_{\pi^d}(\mathcal{H})$ contains the spectral information of the Ruelle transfer operator $\mathcal{L}_{A_{(\phi)}}$.

Remark 2.2.12. Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{Z}}, \mathbb{Z}, \tau)$ a one-dimensional two-sided full shift (1.2.5), $\Omega_\leq = \rho_{-\mathbb{N}_0}(\Omega)$, and $\Omega_> = \rho_{\mathbb{N}}(\Omega)$. Let ϕ be a two-body Ising interaction (1.8.3) with distance function $d \in \mathcal{D}_1$ (2.2.4). Let $\pi^d : \ell^\infty\mathbb{N} \to \ell^1\mathbb{N}$ be defined by formula (43), $\overleftarrow{\ } : \mathbb{C}^{-\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ the inversion map, and $S = \tau_{-1;\mathbb{N}_0} : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}}$ the bijective right shift. Then by Proposition 2.2.9 and Remark 2.2.10 we have

$$W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi) = W(S(\overleftarrow{\eta}), \xi; d) = -\langle S(\overleftarrow{\eta}), \pi^d(\xi)\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}}.$$

Hence by Remark 2.2.11 the leading eigenvector of the Ruelle transfer operator $\mathcal{L}_{\beta A_{(\phi)}}$ (2.1.3) is given as

$$h_1(\xi) = \int_{\Omega_\leq} \exp(\beta\, \langle S(\overleftarrow{\eta}), \pi^d(\xi)\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}})\, d\mu_\leq(\eta).$$

This, together with Proposition 2.1.8, suggests to look for a Hilbert space $\mathcal{H}$, which can be embedded into $\mathcal{C}(\Omega_>)$ via the composition operator $C_{\pi^d} : \mathcal{H} \to \mathcal{C}(\Omega_>)$, $f \mapsto f \circ \pi^d$. If

$$\tilde{h}_1 : \ell^1\mathbb{N} \to \mathbb{C},\ z \mapsto \int_{\Omega_\leq} \exp\Big(\beta\, \langle S(\overleftarrow{\eta}), z\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}}\Big)\, d\mu_\leq(\eta)$$

belongs to $\mathcal{H}$, then $h_1 = \tilde{h}_1 \circ \pi^d \in C_{\pi^d}(\mathcal{H}) \subset \mathcal{C}(\Omega_>)$. We were not able to implement this idea, maybe because we tried to find a reproducing kernel Hilbert space[16]. It remains an open problem to find a (reproducing kernel) Hilbert space $\mathcal{H} \subset \mathcal{C}(\ell^1\mathbb{N})$ such that $\mathcal{H}$ contains the vector $\tilde{h}_1$: As a first step

[16]Reproducing kernel Hilbert spaces (rkhs) are introduced in Appendix A.4.

one could look for a rkhs such that for any $z_0 \in \ell^\infty\mathbb{N}$ the map $g_{z_0} : \ell^1\mathbb{N} \to \mathbb{C},\ z \mapsto \exp(\langle z_0, z\rangle_{\ell^\infty\mathbb{N},\ell^1\mathbb{N}})$ belongs to that rkhs and in a second step show that the $\Omega_{\leq}$-averaged function $\tilde{h}_1$ belongs to it.
Note that in general z_0 does not belong to the subspace $\ell^2\mathbb{N}$ of $\ell^\infty\mathbb{N}$, hence g_{z_0} and $\tilde{h}_1$ are undefined on $\ell^2\mathbb{N}$, in particular they do not belong to the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ defined in (A.4.6). The reproducing kernel k of $\mathcal{F}(\ell^2\mathbb{N})$ and its restriction to $\ell^1\mathbb{N} \times \ell^1\mathbb{N}$ are functions of positive type, hence there exists an rkhs $H(\ell^1\mathbb{N}, k|_{\ell^1\mathbb{N}\times\ell^1\mathbb{N}})$. By [Ar50, I.5.1] this space is given as

$$H(\ell^1\mathbb{N}, k|_{\ell^1\mathbb{N}\times\ell^1\mathbb{N}}) = \{f : \ell^1\mathbb{N} \to \mathbb{C} \,|\, \exists F \in \mathcal{F}(\ell^2\mathbb{N}) :\ F|_{\ell^1\mathbb{N}} = f\}.$$

If $z_0 \in \ell^\infty\mathbb{N} \setminus \ell^2\mathbb{N}$, then in view of Theorem A.4.8 it is impossible to find $F \in \mathcal{F}(\ell^2\mathbb{N})$ such that $F|_{\ell^1\mathbb{N}} = g_{z_0}$. □

In Section 2.7 we introduce a class of examples of distance functions for which we are able to find a reproducing kernel Hilbert space which contains a preimage of the leading eigenvector h_1 of the Ruelle operator $\mathcal{L}_{A_{(\phi)}} : \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega)$ and investigate the associated transfer operators. Our class $\mathcal{D}_1^{(\infty)}$ of distance functions is a proper subclass of $\mathcal{D}_1$ (Def. 2.2.4) and contains a large family of distance functions which are characterised by a certain exponential decay at infinity. Among them are the known examples of finite range interactions, polynomial-exponentially decaying interactions, and a class of superexponentially decaying interactions. Moreover, for this class of interactions we will find a subspace of $\mathcal{C}_b(\Omega)$ which is invariant under the Ruelle operator and induces the so called Ruelle-Mayer transfer operator. A positive answer to the question raised in the previous Remark 2.2.12 might lead to a larger class of distance functions, which can be treated with our method.

2.3 The counting trace

Besides the Ruelle-Perron-Frobenius Theorem 2.1.4 which relates the leading eigenvalue of the Ruelle transfer operator (2.1.3) to physical quantities, we will give a second motivation for the study of this operator: The counting trace of its iterates is precisely the dynamical partition function (1.11.4). By virtue of Ansatz 2.3.3 we can translate the dynamical system on Ω to a dynamical system on another topological space E, such that the Ruelle transfer operator induces a composition operator, the Ruelle-Mayer operator, acting on $\mathcal{C}_b(E)$ and the dynamical partition function can be expressed as the counting trace of this induced operator. Remark 1.11.5 will lead us to a representation of the partition function with periodic boundary condition. At several points in this section we will assume the set F of spin values to be finite (and still write an integral sign). We think that the results still hold true for general F, but this would complicate the arguments. Our main intention in this section is to give a motivation for the Ruelle-Mayer transfer operator.

Definition 2.3.1. Let E be a topological space. Let Trace^c be the so called *counting trace* defined as the linear extension of

$$\mathrm{Trace}^c\, T = \sum_{\psi x = x} \phi(x)$$

to the algebra of composition operators generated by simple composition operators

$$T : \mathcal{C}_b(E) \to \mathcal{C}_b(E),\ (Tf)(z) = \phi(z)\,(f \circ \psi)(z)$$

where $\phi : E \to \mathbb{C}$ and $\psi : E \to E$ are continuous functions and ψ has only finitely many fixed points. □

The counting trace was first introduced by D. Ruelle, see [Ru02], since the counting trace of the Ruelle transfer operator gives the dynamical partition function: Let F be a finite set, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6), $A \in \mathcal{C}_b(\Omega)$ an observable, and

$$\mathcal{L}_A : \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega),\ (\mathcal{L}_A f)(\xi) = \int_F \exp\bigl(A(x \vee \xi)\bigr) f(x \vee \xi)\, d\nu(x) \tag{49}$$

be the corresponding Ruelle transfer operator (2.1.3). For every $x \in F$ the map $\xi \mapsto x \vee \xi$ has the unique fixed point $\overline{x} = (x, x, \ldots)$ and hence

$$\mathrm{Trace}^c\, \mathcal{L}_A = \int_F \exp\bigl(A(\overline{x})\bigr)\, d\nu(x) = \widetilde{Z}_1^{b^{\mathbb{N}_0}}(A).$$

The following proposition shows that this equality also holds for the higher iterates of $\mathcal{L}_A$. This suggests that not only the first eigenvalue of the Ruelle transfer operator is interesting, see Theorem 2.1.4, but also certain tracial functionals evaluated on its powers.

Proposition 2.3.2. *Let F be a finite set, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dim. full shift (1.2.6), and $A \in \mathcal{C}_b(\Omega)$. Then the dynamical partition function $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A)$ (1.11.4) is given as*

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \mathrm{Trace}^c\,(\mathcal{L}_A)^n,$$

where $\mathcal{L}_A : \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega)$ is the Ruelle transfer operator (49).

Proof. For fixed $n \in \mathbb{N}$, $x_1, \ldots, x_n \in F$ the map $\Omega \to \Omega$, $\xi \mapsto x_1 \ldots x_n \vee \xi = (x_1, \ldots, x_n, \xi_1, \ldots)$ defined via (37) has the unique fixed point $\overline{x_1 \ldots x_n} \in \Omega$. For $k \in \mathbb{N}_0$ let $\tau_k : F^{\mathbb{N}} \to F^{\mathbb{N}}$, $\tau_k(\xi)(n) = \xi(k+n)$ be the k-th iterate of the shift (1.2.3). For all $n \in \mathbb{N}$, $0 \leq k < n$, $x_1, \ldots, x_n \in F$, $\xi \in F^{\mathbb{N}}$ one has

$$\tau_k(x_1 \vee \ldots \vee x_n \vee \xi) = x_{k+1} \vee \ldots \vee x_n \vee \xi. \tag{50}$$

In fact: Let $n = 1$, hence $k = 0$ and $\tau_0(x \vee \xi) = x \vee \xi$. Induction step $n \to n+1$: Again $k = 0$ is trivial.

$$\tau_{k+1}(x_1 \vee \ldots \vee x_{n+1} \vee \xi) = \tau_k(x_2 \vee \ldots \vee x_{n+1} \vee \xi) = x_{k+1} \vee \ldots \vee x_{n+1} \vee \xi.$$

Then for $\xi = \overline{x_1 \ldots x_n} = x_1 \vee \ldots \vee x_n \vee \overline{x_1 \ldots x_n}$ we have

$$\tau_{k-1}(\overline{x_1 \ldots x_n}) = \tau_{k-1}(x_1 \vee \ldots \vee x_n \vee \overline{x_1 \ldots x_n}) \overset{(50)}{=} x_k \vee \ldots \vee x_n \vee \overline{x_1 \ldots x_n}.$$

Let $f \in \mathcal{C}_b(\Omega)$. An explicit formula for $(\mathcal{L}_A)^n$ is given in formula (42) in the proof of Proposition 2.1.8:

$$((\mathcal{L}_A)^n f)(\xi) = \int_{F^n} \exp\Big(\sum_{k=1}^{n} A(x_k \vee \ldots \vee x_n \vee \xi)\Big)\, f(x_1 \vee \ldots \vee x_n \vee \xi)\, d\nu(x_1) \ldots d\nu(x_n).$$

Hence the counting trace of $(\mathcal{L}_A)^n$ coincides with the dynamical partition function (1.11.4). □

We will now assume that the observable A factors through a so called projection map. Together with a second assumption on the projection map, this will allow us to transfer the dynamical system on Ω to another dynamical system on a topological space E. Via this transfer we obtain the Ruelle-Mayer transfer operator associated to a Ruelle transfer operator. We remark that a similar set of axioms has been proposed by D. Mayer in [May91, p. 192].

Ansatz 2.3.3. Let $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6), $A \in \mathcal{C}_b(\Omega)$, and $\pi : \Omega \to E$ a continuous map into a topological space E. Consider the following properties.

(S1) There exist continuous maps $\psi_\sigma : E \to E$, such that for all $\sigma \in F, \xi \in \Omega$

$$(\psi_\sigma \circ \pi)(\xi) = \pi(\sigma \vee \xi).$$

(S2) There exist continuous bounded functions $A_\sigma : E \to \mathbb{C}$ such that for all $\sigma \in F, \xi \in \Omega$

$$A(\sigma \vee \xi) = (A_\sigma \circ \pi)(\xi).$$

(S3) The families $(\psi_x)_{x \in F}$ and $(A_x)_{x \in F}$ are measurable with respect to the parameter $x \in F$ and the a priori measure ν on F.

We call the map π a *projection map* and ψ_σ a *linking map.* □

If the projection map π is the identity on Ω, then trivially every observable A possesses properties (S1) - (S3) by setting $A_x(\xi) := A(x \vee \xi)$ and $\psi_x(\xi) = x \vee \xi$. We look for projection maps with values in a space E with nicer properties such that additional structures can be used for the investigation and description of the dynamical system.
The main example we have in mind for the projection map π is the linear map $\pi^d : F^{\mathbb{N}} \subset \ell^\infty\mathbb{N} \to \ell^2\mathbb{N}$ from Proposition 2.2.2, hence the image is a subset of a Hilbert space.
Assuming conditions (S1) - (S3) we will rewrite our partition function in Proposition 2.3.5. We begin with an induction argument.

Proposition 2.3.4. *Let $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6) and $\pi : F^{\mathbb{N}} \to E$ a continuous map into a topological space E with property (S1). Then for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in F$, $\xi \in F^{\mathbb{N}}$*

$$(\psi_{x_1} \circ \ldots \circ \psi_{x_n})(\pi(\xi)) = \pi(x_1 \vee \ldots \vee x_n \vee \xi). \tag{51}$$

Proof. By induction: $\psi_{x_1}(\pi(\xi)) = \pi(x_1 \vee \xi)$. Let $x_1, \ldots, x_{n+1} \in F$, $\xi \in F^{\mathbb{N}}$. Then

$$(\psi_{x_1} \circ \ldots \circ \psi_{x_n})(\pi(\xi)) = \psi_{x_1}(\psi_{x_2} \circ \ldots \circ \psi_{x_n})(\pi(\xi)) = \psi_{x_1}(\pi(x_2 \vee \ldots \vee x_n \vee \xi)) = \pi(x_1 \vee x_2 \vee \ldots x_n \vee \xi).$$

□

Proposition 2.3.5. *Let $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6), $A \in \mathcal{C}_b(F^{\mathbb{N}})$ and $\pi : F^{\mathbb{N}} \to E$ a continuous map into a topological space E with properties (S1) - (S3) (2.3.3). Let $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift (1.2.8). Then the dynamical partition function $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A)$ (1.11.4) is given as*

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \int_{F^n} \mathbb{A}_{x_1,x_2} \cdot \ldots \cdot \mathbb{A}_{x_{n-1},x_n} \cdot \mathbb{A}_{x_n,x_1} \exp\Big(\sum_{k=1}^{n} (A_{x_k} \circ \psi_{x_{k+1}} \circ \ldots \circ \psi_{x_n})(z^{\star}_{x_1,\ldots,x_n})\Big) d\nu(x_1) \ldots d\nu(x_n),$$

where $z^{\star}_{x_1,\ldots,x_n} := \pi(\overline{x_1 \ldots x_n}) \in E$.

Proof. By Corollary 1.11.3 it suffices to show that for all $\xi = \overline{x_1 \ldots x_n} \in \mathrm{Fix}(\tau_n)$ and $k = 0, \ldots, n-1$

$$A(\tau_k \xi) = (A_{x_{k+1}} \circ \psi_{x_{k+2}} \circ \ldots \circ \psi_{x_n})(z^{\star}_{x_1,\ldots,x_n}).$$

This follows from properties (S1), (S2), and Proposition 2.3.4:

$$A(\tau_k \xi) \overset{(50)}{=} A(x_{k+1} \vee \ldots \vee x_n \vee \xi) = (A_{x_{k+1}} \circ \pi)(x_{k+2} \vee \ldots \vee x_n \vee \xi) \overset{(51)}{=} (A_{x_{k+1}} \circ \psi_{x_{k+2}} \circ \ldots \psi_{x_n} \circ \pi)(\xi).$$

□

The assumptions (S1) - (S3) from (2.3.3) lead to a factorisation of the Ruelle transfer operator. The resulting operator is the so called Ruelle-Mayer transfer operator:

Remark 2.3.6. Let F be a finite set, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6), $A \in \mathcal{C}_b(\Omega)$ an observable, and $\pi : \Omega \to E$ a continuous map into a topological space E with properties (S1) - (S3) (2.3.3). Let $C_\pi : \mathcal{C}_b(E) \to \mathcal{C}_b(\Omega)$, $f \mapsto f \circ \pi$ be the associated composition operator and $\mathcal{L}_A : \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega)$ the Ruelle operator (2.1.3). Let $f \in \mathcal{C}_b(E)$ and $g = C_\pi(f) = f \circ \pi \in \mathcal{C}_b(\Omega)$, then

$$(\mathcal{L}_A g)(\xi) = \sum_{\sigma \in F} \exp\big(A(\sigma \vee \xi)\big)\,(f \circ \pi)(\sigma \vee \xi) = \sum_{\sigma \in F} \exp\big(A_\sigma \circ \pi(\xi)\big)\,(f \circ \psi_\sigma \circ \pi)(\xi),$$

i. e. $\mathcal{L}_A$ leaves the image $C_\pi(\mathcal{C}_b(E))$ of C_π invariant. Thus the operator

$$\mathcal{M} : \mathcal{C}_b(E) \to \mathcal{C}_b(E),\ (\mathcal{M} f(z) = \sum_{x \in F} \exp\big(A_x(z)\big)\, f(\psi_x(z))$$

makes the following diagram commutative:

$$\begin{array}{ccccc}
\mathcal{C}_b(E) & \xrightarrow{C_\pi} & C_\pi(\mathcal{C}_b(E)) & \hookrightarrow & \mathcal{C}_b(\Omega) \\
\downarrow{\scriptstyle \mathcal{M}} & & \downarrow{\scriptstyle \mathcal{L}_A} & & \downarrow{\scriptstyle \mathcal{L}_A} \\
\mathcal{C}_b(E) & \xrightarrow{C_\pi} & C_\pi(\mathcal{C}_b(E)) & \hookrightarrow & \mathcal{C}_b(\Omega)
\end{array}$$

□

Remark 2.3.6 motivates the following definition:

Definition 2.3.7. Let $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6), $A \in \mathcal{C}_b(\Omega)$ an observable, and $\pi : \Omega \to E$ a continuous map into a topological space E with properties (S1) - (S3) (2.3.3). The (possibly unbounded) operator

$$\mathcal{M} : \mathcal{C}_b(E) \to \mathcal{C}_b(E),\ (\mathcal{M}f)(z) = \int_F \exp(A_x(z))\ f(\psi_x(z))\, d\nu(x)$$

is called the (formal) *Ruelle-Mayer (RM) transfer operator.* □

If the projection map π is the identity on Ω, then trivially the Ruelle-Mayer operator is just the Ruelle operator and we do not gain any new insights. Both the Ruelle and the Ruelle-Mayer operator are generalised composition operators[17]. If for instance the projection map π takes values in a bounded complex domain, then more techniques are available to determine the spectral properties of $\mathcal{M}$, since the Ruelle-Mayer transfer operator looks like the Ruelle transfer operator for an expanding map. In this interpretation, the properties (S1) - (S3) provide a link between the two types of operators. Depending on the functions ψ_x and A_x, the Ruelle-Mayer operator preserves smoothness and integrability on bounded sets. It turns out that under additional assumptions this operator is trace class with a nice trace formula directly linked to the partition function. Before proving this result, known as the *transfer operator method*, we will compute the counting trace of its iterates. For the one-dimensional one-sided full shift we have an analogous result to Proposition 2.3.2. It uses the general idea that the counting trace remains unchanged under every factorisation which preserves fixed points.

Proposition 2.3.8. *Let F be a finite set, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dim. full shift (1.2.6), $A \in \mathcal{C}_b(\Omega)$ an observable, and $\pi : \Omega \to E$ a continuous map into a topological space E with properties (S1) - (S3) 2.3.3. Suppose that for all $x_1, \ldots, x_n \in F$ the map $\psi_{x_1} \circ \ldots \circ \psi_{x_n} : E \to E$ has a unique fixed point. Then the Ruelle-Mayer transfer operator*

$$\mathcal{M} : \mathcal{C}_b(E) \to \mathcal{C}_b(E),\ (\mathcal{M}f(z) = \int_F \exp\big(A_x(z)\big)\ f(\psi_x(z))\, d\nu(x)$$

satisfies the (pre-) trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \mathrm{Trace}^c\, \mathcal{M}^n$.

Proof. By Propositions 2.3.2 and 2.3.5 it suffices to show that $z^\star_{x_1,\ldots,x_n} := \pi(\overline{x_1 \ldots x_n})$ is a fixed point of the map $\psi_{x_1} \circ \ldots \circ \psi_{x_n} : E \to E$. Apply (51) from Proposition 2.3.4 for $\xi = \overline{x_1 \ldots x_n}$

$$(\psi_{x_1} \circ \ldots \circ \psi_{x_n})(\pi(\overline{x_1 \ldots x_n})) \overset{(51)}{=} \pi(x_1 \vee x_2 \vee \ldots x_n \vee \overline{x_1 \ldots x_n}) = \pi(\overline{x_1 \ldots x_n}).$$

□

We end this section with a superposition principle: Given a finite number of observables $A^{(\alpha)} \in \mathcal{C}_b(\Omega)$ $(\alpha = 1, \ldots, l)$ with properties (S1) - (S3), then also the observable $A := \sum_{\alpha=1}^{l} A^{(\alpha)} : \Omega \to \mathbb{C}$ admits a factorisation with these properties.

Proposition 2.3.9. *Let $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6). For $\alpha = 1, \ldots, l$ let $A^{(\alpha)} \in \mathcal{C}_b(\Omega)$ be observables, and $\pi^{(\alpha)} : \Omega \to E^{(\alpha)}$ continuous maps into topological spaces $E^{(\alpha)}$ with properties (S1) - (S3) (2.3.3). Set $E := \prod_{\alpha=1}^{l} E^{(\alpha)}$. For all $\sigma \in F$ we define the maps $\psi_\sigma := (\psi_\sigma^{(1)}, \ldots, \psi_\sigma^{(l)}) : E \to E$ and $A_\sigma := \sum_{\alpha=1}^{l} A_\sigma^{(\alpha)} : E \to \mathbb{C}$. Then $\pi := (\pi^{(1)}, \ldots, \pi^{(l)}) : \Omega \to E$ satisfies $\pi(\sigma \vee \xi) = (\psi_\sigma \circ \pi)(\xi)$ and $A(\sigma \vee \xi) = (A_\sigma \circ \pi)(\xi)$ for all $\sigma \in F$ and $\xi \in F^{\mathbb{N}}$.*

Proof. For all $\sigma \in F$ and $\xi \in F^{\mathbb{N}}$ we have

$$\pi(\sigma \vee \xi) = \Big(\pi^{(\alpha)}(\sigma \vee \xi)\Big)_{\alpha=1,\ldots,l} = \Big((\psi_\sigma^{(\alpha)} \circ \pi^{(\alpha)})(\xi)\Big)_{\alpha=1,\ldots,l} = (\psi_\sigma \circ \pi)(\xi)$$

and

$$A(\sigma \vee \xi) = \sum_{\alpha=1}^{l} A^{(\alpha)}(\sigma \vee \xi) = \sum_{\alpha=1}^{l} (A_\sigma^{(\alpha)} \circ \pi^{(\alpha)})(\xi) = (A_\sigma \circ \pi)(\xi).$$

□

[17]The reader finds an introduction to composition operators in Appendix B.

Moreover, for any $x \in F$ the fixed points of ψ_x are precisely the products of the fixed points of the $\psi_x^{(\alpha)}$, since

$$(\psi_{\xi_1} \circ \ldots \circ \psi_{\xi_n})(z) = \left(\psi_{\xi_1}^{(\alpha)} \circ \ldots \circ \psi_{\xi_n}^{(\alpha)})(z^{(\alpha)})\right)_{\alpha=1,\ldots,l}.$$

Proposition 2.3.9 can be used to construct a (family of) Ruelle-Mayer transfer operator(s) corresponding to the observable A in the case that Ruelle-Mayer transfer operators are known for each of the observables $A^{(\alpha)}$. This idea will be used for the construction of Ruelle-Mayer transfer operators for Ising type interactions in Section 2.13.

2.4 The Ruelle-Mayer transfer operator and the dynamical trace formula

In (2.3.7) we have given the general definition of a Ruelle-Mayer (RM) transfer operator. We will now assume that we are in a situation where the RM transfer operator is trace class. This depends on the data A_x, ψ_x, and the space on which the operator acts via this formula. We adress this question in the subsequent sections. In this section we show, based on results of D. Mayer, that the Ruelle-Mayer transfer satisfies a dynamical trace formula, i. e., the (spectral) traces of its iterates determine the (dynamical) partition functions.
In order to prove the dynamical trace formula we proceed in steps. We suppose that the base space E is a topological vector space and that the Ruelle-Mayer transfer operator leaves a Hilbert subspace $\mathcal{H} \subset \mathcal{C}(E)$ invariant. We write the Ruelle-Mayer transfer operator (2.3.7) as an integral $(\mathcal{M}g)(z) := \int_F (\mathcal{M}_x g)(z)\, d\nu(x)$ over a family of composition operators

$$\mathcal{M}_x : \mathcal{H} \to \mathcal{H},\ (\mathcal{M}_x f)(z) = \exp(A_x(z))\,(f \circ \psi_x)(z).$$

For each of them we will apply a generalisation of the so called Atiyah-Bott formula, which expresses the trace of a composition operator as a fixed point formula. Then we will compare the traces of the iterates of the Ruelle-Mayer transfer operator with the dynamical trace formula. The next lemma, which we will prove in Appendix A.7.7, provides an abstract trace formula for operators of the type $\mathcal{M} = \int_F \mathcal{M}_x\, d\nu(x)$.

Lemma 2.4.1. *Let ν be a Borel measure on F and $(T_x)_{x\in F}$ a measurable family of trace class*[18] *operators on a Hilbert space $\mathcal{H}$ with $\int_F \|T_x\|_{\mathcal{S}_1(\mathcal{H})}\, d\nu(x) < \infty$. Then $T : \mathcal{H} \to \mathcal{H},\ Tg := \int_F T_x g\, d\nu(x)$ is a trace class operator with*

$$T^n f = \int_{F^n} T_{x_n} \circ \ldots \circ T_{x_1} f\, d\nu(x_1) \ldots d\nu(x_n)$$

and

$$\text{trace } T^n = \int_{F^n} \text{trace } (T_{x_n} \circ \ldots \circ T_{x_1})\, d\nu(x_1) \ldots d\nu(x_n).$$

□

We add the remark that in case the set F is finite, then Lemma 2.4.1 simply states the linearity of the trace functional.
For each of the composition operators $\mathcal{M}_x$ we would like to apply an Atiyah-Bott type fixed point formula. The classical formulation of this theorem, which we sketch in B.2.4, is the following: Let $U \subset \mathbb{C}^k$ be an open bounded complex domain. We denote by $A^\infty(U)$ the space of holomorphic functions on U, which are continuous up to the closure $\overline{U}$ of U. The space $A^\infty(U)$ is a Banach space with respect to the supremum norm.

Theorem 2.4.2. *(Atiyah-Bott type fixed point formula) Let $U \subset \mathbb{C}^k$ be an open bounded complex domain. Let $\phi \in A^\infty(U)$ and $\psi : U \to U$ be a holomorphic function with continuous extension to $\overline{U}$ and $\psi(\overline{U}) \subset U$, i. e., ψ is strictly contractive. Then ψ has a unique fixed point $z^* \in U$ and the generalised composition operator*

$$T : A^\infty(U) \to A^\infty(U),\ (Tf)(z) = \phi(z)\,(f \circ \psi)(z)$$

[18] $\|\cdot\|_{\mathcal{S}_1(\mathcal{H})}$ denotes the trace norm as defined in (A.2.2).

is nuclear of order zero[19] *with trace given by*

$$\operatorname{trace}_{A^\infty(U)} T = \frac{\phi(z^*)}{\det(1-\psi'(z^*))}. \tag{52}$$

□

In Appendix B we will discuss the question whether the trace formula (52) holds on other function spaces and also over infinite-dimensional domains. In the following we will take this trace formula for granted and investigate how the trace of the iterates of the Ruelle-Mayer transfer operator and the partition function are related.

Proposition 2.4.3. *Let E be a topological vector space. For $\sigma \in F$ let $\psi_\sigma : E \to E$ and $A_\sigma : E \to \mathbb{C}$ be continuous maps with the following property: Suppose that for all $\sigma_1, \ldots, \sigma_n \in F$ the map $\psi_{\sigma_1} \circ \ldots \circ \psi_{\sigma_n} : E \to E$ has a unique fixed point, denoted by $z^\star_{\sigma_1,\ldots,\sigma_n}$, and that the linear map $\psi'_\sigma(z^*_\sigma) \in \operatorname{End}(E)$ admits a Fredholm determinant. Suppose that the algebra generated by the composition operators*

$$\mathcal{M}_\sigma : \mathcal{H} \to \mathcal{H}, \ (\mathcal{M}_\sigma f)(z) = \exp\bigl(A_\sigma(z)\bigr)\,(f \circ \psi_\sigma)(z)$$

consists of trace class operators on a Hilbert space $\mathcal{H} \subset \mathcal{C}(E)$ and satisfies the trace formula

$$\operatorname{trace} \mathcal{M}_\sigma = \frac{\exp(A_\sigma(z^*_\sigma))}{\det(1-\psi'_\sigma(z^*_\sigma))}. \tag{53}$$

Let ν be a Borel measure on F such that $\int_F \|\mathcal{M}_\sigma\|_{\mathcal{S}_1(\mathcal{H})}\, d\nu(\sigma) < \infty$. Then the operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$, $(\mathcal{M}g)(z) := \int_F (\mathcal{M}_\sigma g)(z)\, d\nu(\sigma)$ is trace class with

$$\operatorname{trace} \mathcal{M}^n = \int_{F^n} \frac{\exp\bigl(\sum_{k=1}^n (A_{\sigma_k} \circ \psi_{\sigma_{k+1}} \circ \ldots \circ \psi_{\sigma_n})(z^\star_{\sigma_1,\ldots,\sigma_n})\bigr)}{\det(1-(\psi_{\sigma_1} \circ \ldots \circ \psi_{\sigma_n})'(z^\star_{\sigma_1,\ldots,\sigma_n}))}\, d\nu(\sigma_1)\ldots d\nu(\sigma_n).$$

Proof. By Corollary B.1.3 we have

$$(\mathcal{M}_{x_n} \circ \ldots \circ \mathcal{M}_{x_1} f)(z) = \exp\Bigl(\sum_{k=1}^n (A_{x_k} \circ \psi_{x_{k+1}} \circ \ldots \circ \psi_{x_n})(z)\Bigr)\,(f \circ \psi_{x_1} \circ \ldots \circ \psi_{x_n})(z).$$

We apply the trace formula to $\mathcal{M}_{x_n} \circ \ldots \circ \mathcal{M}_{x_1}$ and use Lemma 2.4.1. □

We comment on the determinant condition in (A.1.1) and Appendix A.2: It is satisfied for instance for trace class maps on a Hilbert space and for the Grothendieck class of maps on a Banach space which are nuclear of order 2/3 (A.3.1). Proposition 2.4.3 can also be stated if E is a finite dimensional manifold. In this case $\psi'_\sigma(z^*_\sigma) \in \operatorname{End}(T_{z^*_\sigma}E) \cong \operatorname{End}(\mathbb{C}^{\dim E})$ is automatically trace class and hence admits a Fredholm determinant. Comparing the trace of the iterates of $\mathcal{M}$ and the partition function given in (2.3.5) leads to the following theorem in the finite dimensional setting which is due to D. Mayer [May91].

Theorem 2.4.4. *Let F be a finite set, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dim. full shift (1.2.6), $A \in \mathcal{C}_b(\Omega)$ an observable, and $\pi : \Omega \to U \subset \mathbb{C}^k$ a continuous map into a bounded complex domain with properties (S1) - (S3) (2.3.3). Suppose that the maps $\psi_x : U \to U$ are holomorphic and strictly contractive, i. e., $\psi_x(\overline{U}) \subset U$, and $A_x : U \to \mathbb{C}$ is holomorphic. Then the dynamical partition function (1.11.4) can be expressed as*

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \sum_{r=0}^k (-1)^r \operatorname{trace}\,(\mathcal{M}^{(r)})^n$$

for all $n \in \mathbb{N}$, where for $r = 0, \ldots, k$ the operator $\mathcal{M}^{(r)}$ is defined on $\bigwedge^r A^\infty(U)$ via

$$(\mathcal{M}^{(r)} w_r)(z) = \int_F \exp(A_x(z)) \wedge^r D\psi_x(z)(w_r)(\psi_x(z))\, d\nu(x),$$

if $w(z) = \sum_{1 \le i_1 < \ldots < i_r \le k} w_{i_1,\ldots,i_r}(z)\, dz_{i_1} \wedge \ldots \wedge dz_{i_r}$ with $w_{i_1,\ldots,i_r} \in A^\infty(U)$.

[19] For a definition of a nuclear operator see for instance (A.3.1) or [May80a, Appendix A].

Proof. The trace of $\mathcal{M}^{(r)}$ is given by the formula

$$\operatorname{trace} \mathcal{M}^{(r)} = \int_F \exp(A_x(z_x^*)) \frac{\wedge^r D\psi_x(z_x^*)}{\det(1 - \wedge^r D\psi_x(z_x^*))} \, d\nu(x)$$

for all $r = 0, \dots, k$. Use the following Lemma 2.4.5 and compare with the expressions for the dynamical partition function $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A)$ given in Propositions 2.3.5 and 2.3.8. □

We note that this theorem works for more general linking maps ψ_x, but requires F to be finite. We pose the question in which setting this theorem is true for an arbitrary alphabet F. For the proof one either requires a Banach space version of Lemma 2.4.1 or an identification of suitable Hilbert spaces replacing $\bigwedge^r A^\infty(U)$.
The following identity (2.4.5) from (advanced) multilinear algebra (see for instance [Si77]) concludes the proof of Theorem 2.4.4.

Lemma 2.4.5. *Let $A : \mathcal{H} \to \mathcal{H}$ be a trace class operator on a Hilbert space $\mathcal{H}$. Then*

$$\det(1 - A) = \sum_{r=0}^{\dim \mathcal{H}} (-1)^r \operatorname{trace} \wedge^r A,$$

where $\wedge^r A : \wedge^r \mathcal{H} \to \wedge^r \mathcal{H}$, $\wedge^r A(e_1 \wedge \ldots \wedge e_r) := Ae_1 \wedge \ldots \wedge Ae_r$ is the r-fold exterior product of A. □

An important special case of Theorem 2.4.4 is the following, where the linking maps are affine. This happens for all known examples of Ruelle-Mayer transfer operators for one-dimensional spin systems. In Section 2.7 we will explain how the affine linking maps arise in our context.

Theorem 2.4.6. *Let $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6), $A \in \mathcal{C}_b(\Omega)$ an observable, and $\pi : \Omega \to E$ a continuous map into a Banach space E with properties (S1) - (S3) (2.3.3). Assume that the maps $\psi_x : E \to E$ are affine and of the form $\psi_x : E \to E$, $z \mapsto \psi_x(z) := a_x + \mathbb{B}z$ for some fixed map $\mathbb{B} \in \operatorname{End}(E)$ which admits a Fredholm determinant and has operator norm $\|\mathbb{B}\|_{\text{op}} < 1$. Suppose that the algebra generated by the composition operators*

$$\mathcal{M}_x : \mathcal{H} \to \mathcal{H}, \ (\mathcal{M}_x f)(z) = \exp(A_x(z)) \, (f \circ \psi_x)(z)$$

consists of trace class operators on a Hilbert space $\mathcal{H} \subset \mathcal{C}(E)$ and satisfies the trace formula (53). Let ν be a Borel measure on F such that $\int_F \|\mathcal{M}_x\|_{\mathcal{S}_1(\mathcal{H})} \, d\nu(x) < \infty$. Then for all $n \in \mathbb{N}$ the Ruelle-Mayer transfer operator

$$\mathcal{M} : \mathcal{H} \to \mathcal{H}, \ (\mathcal{M}f)(z) = \int_F \exp(A_x(z)) \, (f \circ \psi_x)(z) \, d\nu(x)$$

satisfies $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \det(1 - \mathbb{B}^n) \operatorname{trace} \mathcal{M}^n$.

Proof. Using von Neumann's series the operator $1 - \mathbb{B}$ is invertible and hence ψ_x has precisely one fixed point in E. Compare the expression for the trace of $\mathcal{M}^n$ given in Proposition 2.4.3 with Propositions 2.3.5 and 2.3.8. By our assumption on the special form of the linking maps ψ_x, the determinant is independent of the integration variable, hence it can be pulled out. □

We call a formula of the type $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \det(1 - \mathbb{B}^n) \operatorname{trace} \mathcal{M}^n$ a *dynamical trace formula.* We have shown in the abstract setting of Theorem 2.4.6 that the Ruelle-Mayer transfer operator satisfies a dynamical trace formula. In the rest of this chapter we investigate classes of observables and - by $Z_{\{1,\dots,m\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \widetilde{Z}_m^{b^{\mathbb{N}_0}}(\beta A_{(\phi)})$ (1.11.5) - of interactions for which the hypotheses of the previous theorems are fulfilled. This means one has to identify the projection map and to solve several analytic problems, for instance find the suitable Hilbert spaces.

2.5 Transfer operators for finite range spin systems

In this section we review the transfer operator method as developed in [May80a, II.2.] for finite range interactions. Recall: Let $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6), then a two-body translation invariant interaction ϕ (1.8.1) has finite range ρ_0, if $\varphi_2(i, j; x, y) = 0$ for all $x, y \in F$ whenever $|i - j| > \rho_0$ (1.9.2). We denote by $q \in \mathcal{C}_b(F)$ the potential term of ϕ. We will construct the

Ruelle-Mayer transfer operator and prove its trace formula using the methods of Section 2.4. Finite range interactions are the simplest case where the dynamical trace formula (2.4.4) is valid, since the higher iterates of the Ruelle-Mayer operator are superpositions of degenerate composition operators (B.1.4). We will point out the connection between the Ruelle transfer operator, the Ruelle-Mayer transfer operator, and the Kramers-Wannier transfer matrix.
For a finite range interaction ϕ the standard observable $A_{(\phi)} \in \mathcal{C}_b(\Omega)$ (1.11.1) reduces to a finite sum,

$$A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sum_{k=1}^{\rho_0} \varphi_2(k+1, 1; \xi_k, \sigma).$$

Let $\mathcal{L}_{A_{(\phi)}}$ be the corresponding Ruelle transfer operator (2.1.3). The observation that $A_{(\phi)}(\sigma \vee \xi)$ depends on ξ only via the first ρ_0 entries $\xi_1, \dots, \xi_{\rho_0}$ leads to an $\mathcal{L}_{A_{(\phi)}}$-invariant subspace of $\mathcal{C}_b(\Omega)$: For $\sigma \in F$ set

$$A_\sigma : F^{\rho_0} \to \mathbb{C},\ z = (z_1, \dots, z_{\rho_0}) \mapsto q(\sigma) + \sum_{k=1}^{\rho_0} \varphi_2(k+1, 1; z_k, \sigma)$$

and $\psi_\sigma : F^{\rho_0} \to F^{\rho_0},\ z = (z_1, \dots, z_{\rho_0}) \mapsto (\sigma, z_1, \dots, z_{\rho_0 - 1})$. Let $\mathrm{pr} : \Omega \to F^{\rho_0},\ \xi \mapsto (\xi_1, \dots, \xi_{\rho_0})$ be the projection onto the first ρ_0 components. We will show that the choice of the maps $\pi = \mathrm{pr}$, ψ_σ, and A_σ satisfies properties (S1) - (S3) as defined in (2.3.3): For all $\xi \in \Omega$, $\sigma \in F$ we have $A_{(\phi)}(\sigma \vee \xi) = (A_\sigma \circ \mathrm{pr})(\xi)$ and $\mathrm{pr}(\sigma \vee \xi) = (\sigma, \xi_1, \dots, \xi_{\rho_0-1}) = (\psi_\sigma \circ \mathrm{pr})(\xi)$, i. e., (S1) and (S2) from (2.3.3). Thus by Definition 2.3.7 we have a Ruelle-Mayer transfer operator.

Proposition 2.5.1. *Let $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body translation invariant interaction with finite range ρ_0 and potential $q \in \mathcal{C}_b(F)$. Then for all $m \geq \rho_0$ the Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{C}_b(F^{\rho_0}) \to \mathcal{C}_b(F^{\rho_0})$*

$$(\mathcal{M}_\beta f)(z_1, \dots, z_{\rho_0}) = \int_F \exp\Big(\beta q(\sigma) + \beta \sum_{k=1}^{\rho_0} \varphi_2(k+1, 1; z_k, \sigma)\Big) f(\sigma, z_1, \dots, z_{\rho_0 - 1})\, d\nu(\sigma).$$

satisfies the dynamical trace formula $\widetilde{Z}_m^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\dots,m\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \mathrm{trace}\ (\mathcal{M}_\beta)^m$.

Proof. For $m \geq \rho_0$ and any choice of $\sigma_1, \dots, \sigma_m \in F$ the maps $\psi_{\sigma_1} \circ \dots \circ \psi_{\sigma_m} : F^{\rho_0} \to F^{\rho_0},\ z \mapsto (\sigma_1, \dots, \sigma_{\rho_0})$ are constant. By Corollary B.1.5 the higher mixed iterates $S_{x_1} \circ \dots \circ S_{x_m}$ of the composition operators $S_\sigma : \mathcal{C}_b(F^{\rho_0}) \to \mathcal{C}_b(F^{\rho_0}),\ (S_\sigma f)(z) = \exp(\beta A_\sigma(z))\, (f \circ \psi_\sigma)(z)$ are (for $m \geq \rho_0$) rank one operators and hence their nuclear norm $\|\cdot\|_{L^1(\mathcal{C}_b(F^{\rho_0}))}$ (see Def. A.3.1) can be bounded by

$$\|S_{x_1} \circ \dots \circ S_{x_m}\|_{L^1(\mathcal{C}_b(F^{\rho_0}))} \leq \Big(\sup_{\sigma \in F} \|\exp(\beta A_\sigma(\,\cdot\,))\|_{\mathcal{C}_b(F^{\rho_0})}\Big)^m \leq \exp(m|\beta|\, \|A\|_{\mathcal{C}_b(\Omega)}),$$

hence $\mathcal{M}_\beta^m$ is a trace class operator. By Corollary B.1.5 the operators $S_{x_1} \circ \dots \circ S_{x_m}$ satisfy the fixed point trace formula (53) provided $m \geq \rho_0$, hence Theorem 2.4.4 gives the assertion. □

Let $\mathrm{pr} : \Omega \to F^{\rho_0},\ \xi \mapsto (\xi_1, \dots, \xi_{\rho_0})$ be the projection and $C_{\mathrm{pr}} : \mathcal{C}_b(F^{\rho_0}) \to \mathcal{C}_b(\Omega),\ f \mapsto f \circ \mathrm{pr}$ the associated composition operator. It was the observation that

$$C_{\mathrm{pr}} \circ \mathcal{M}_\beta = \mathcal{L}_{\beta A_{(\phi)}} \circ C_{\mathrm{pr}}$$

which was historically the starting point for the theory of the (nowadays called) Ruelle-Mayer transfer operators in [May80a].

Remark 2.5.2. Let $\mathcal{M}_\beta : \mathcal{C}_b(F^{\rho_0}) \to \mathcal{C}_b(F^{\rho_0})$ be the Ruelle-Mayer transfer operator for finite range interactions, which we introduced in Proposition 2.5.1 as an integral over a family of generalised composition operators. We get from Corollary B.1.3 for $m \geq \rho_0$

$$\begin{aligned}(\mathcal{M}_\beta^m f)(z_1, \dots, z_{\rho_0}) &= \int_{F^m} \exp\Big(\beta \sum_{k=1}^m (A_{x_k} \circ \psi_{x_{k+1}} \circ \dots \circ \psi_{x_m})(z)\Big)\, (f \circ \psi_{x_1} \circ \dots \circ \psi_{x_m})(z)\, d\nu^{\rho_0}(x) \\ &= \int_{F^m} \exp\Big(\beta \sum_{k=1}^m A_{x_k}(x_{k+1}, \dots, x_m, z_1, \dots, z_{\rho_0 - m + k})\Big)\, f(x_1, \dots, x_{\rho_0})\, d\nu(x_1) \dots d\nu(x_m),\end{aligned}$$

which opens an alternative view on $\mathcal{M}_\beta^m$, namely views it as an integral operator on $\mathcal{C}_b(F^{\rho_0})$. □

Our Proposition 2.5.1 is a minor extension of known results:

Remark 2.5.3. Let F be a finite set, $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6), and ϕ a pure two-body translation invariant interaction with finite range ρ_0. Let $\mathrm{pr} : \Omega \to F^{\rho_0}$, $\xi \mapsto (\xi_1, \ldots, \xi_{\rho_0})$ be the projection and $C_{\mathrm{pr}} : \mathcal{C}_b(F^{\rho_0}) \to \mathcal{C}_b(\Omega)$ the associated composition operator. The image $C_{\mathrm{pr}}(\mathcal{C}_b(F^{\rho_0})) \subset \mathcal{C}_b(\Omega)$ of C_{pr} is a finite (in fact $|F|^{\rho_0}$-) dimensional complex vector space, hence every linear operator is trace class and $\mathcal{M}_\beta$ can be written as

$$(\mathcal{M}_\beta f)(\xi_1, \ldots, \xi_{\rho_0}) = \sum_{\underline{\sigma} \in F^{\rho_0}} \mathbb{M}_{\underline{\sigma},(\xi_1,\ldots,\xi_{\rho_0})}\, f(\underline{\sigma}),$$

where

$$\begin{aligned} \mathbb{M}_{(\sigma_1,\ldots,\sigma_{\rho_0}),(\xi_1,\ldots,\xi_{\rho_0})} &= \exp\Big(\beta q(\sigma_1) + \beta \sum_{k=1}^{\rho_0} \varphi_2(k+1, 1; \xi_k, \sigma_1)\Big) \prod_{k=2}^{\rho_0} \delta_{\xi_{k-1},\sigma_k} \\ &= \begin{cases} \exp\Big(\beta (A_{(\phi)} \circ \mathrm{pr})(\sigma_1, \xi_1, \ldots, \xi_{\rho_0 - 1})\Big), & \text{if } (\sigma_2, \ldots, \sigma_{\rho_0}) = (\xi_1, \ldots, \xi_{\rho_0-1}), \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

By direct computation(see [May80a, II.2.1]) D. Mayer was able to show that $\mathbb{M}^{\rho_0}$ is exactly the transfer matrix $\mathbb{K}$ used by H. Kramers and G. Wannier in [KrWa41], hence

$$\widetilde{Z}^{b^{\mathbb{N}_0}}_{n\rho_0}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\rho_0\}}(\beta) = \mathrm{trace}\ \mathbb{K}^n = \mathrm{trace}\ \mathbb{M}^{n\rho_0} = \mathrm{trace}\ (\mathcal{M}_\beta)^{n\rho_0}$$

for all $n \in \mathbb{N}$. Later we will extend this idea to handle the case where $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ is a matrix subshift (1.2.8), simply by replacing the matrix $\mathbb{M}$ by $\mathbb{M}^{\mathbb{A}}_{(\sigma_1,\ldots,\sigma_{\rho_0}),(\xi_1,\ldots,\xi_{\rho_0})} := \mathbb{A}_{\sigma,\xi_1}\, \mathbb{M}_{(\sigma_1,\ldots,\sigma_{\rho_0}),(\xi_1,\ldots,\xi_{\rho_0})}$ □

2.6 Linear models and $\mathbb{N}_0$-representations

In Section 2.5 we have seen an example how one can choose a projection map in order to construct a Ruelle-Mayer transfer operator. We will now specialise to Ising interactions and show that there is a general method how to choose the projection map which works for a large class of longe range interactions. This method explains the results [May76], [Vi76], [MayVi77], [ViMay77], [May80a], [Mo89], [HiMay02], [HiMay04] from the same point of view and also allows to handle new distance functions. In particular our method will be applied for the following classes of distance functions: For finite range interactions in Section 2.8, for superexponentially decaying interactions in Sections 2.9 and for polynomial-exponential decaying interactions in 2.11.
We have the following setting: Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with distance function $d : \mathbb{N} \to \mathbb{C}$ and potential $q \in \mathcal{C}_b(F)$. The boundedness of $F \subset \mathbb{C}$ implies that the configuration space Ω is a bounded subset of $\ell^\infty\mathbb{N}$.
The main idea is to find a family of linear continuous maps from $\ell^\infty\mathbb{N}$ (which contains our configuration space Ω) into $\mathbb{C}$, which translates the shift action on $\ell^\infty\mathbb{N}$ into affine maps on some vector space. In this section we investigate the Banach space situation and explain the main ideas, whereas in the next Section 2.7 the Hilbert space case is treated and the trace formula is proven using the additional Hilbert space structure.
Recall the Definition 2.3.3 of the properties (S1) - (S3). We will restrict to the case where the projection map π extends to a continuous linear map $\ell^\infty\mathbb{N} \to \mathcal{B}$, where $\mathcal{B}$ is a complex Banach space. Recall that the family of linking maps is defined as a family $\psi_\sigma : \mathcal{B} \to \mathcal{B}$ indexed by $\sigma \in F$ with the property that $\psi_\sigma(\pi(\xi)) = \pi(\sigma \vee \xi)$. Under this assumption the linking maps are affine and their linear part has special properties:

Proposition 2.6.1. *Let $\pi : \ell^\infty\mathbb{N} \to \mathcal{B}$ be a continuous linear map, which is the projection map of a (S1) - (S3) Ansatz (2.3.3). Then for any $\sigma \in F$ the linking map $\psi_\sigma : \mathcal{B} \to \mathcal{B}$ is affine and continuous on $\pi(\ell^\infty\mathbb{N})$. Moreover, there exists a continuous linear map $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ such that $\psi_\sigma(z) = \sigma\, \pi(e_1) + \mathbb{B}z$ on $\pi(\ell^\infty\mathbb{N}) \subset \mathcal{B}$. Let $\tau' : \ell^\infty\mathbb{N} \to \ell^\infty\mathbb{N}$, $\tau'(\xi) := 0 \vee \xi$ be the dual shift (2.1.5), then for all $\xi \in \ell^\infty\mathbb{N}$*

$$(\pi \circ \tau')(\xi) = (\mathbb{B} \circ \pi)(\xi). \tag{54}$$

where $e_1 = (1, 0, \ldots) \in \ell^\infty\mathbb{N}$ is the first standard basis vector.

Proof. Let $\sigma \in F$. At first we determine the constant part of ψ_σ, which is

$$\psi_\sigma(0) = (\psi_\sigma \circ \pi)(0) = \pi(\sigma \vee 0) = \sigma\, \pi(e_1).$$

The norm of the constant part of ψ_σ can be estimated by $\|\sigma\, \pi(e_1)\|_{\mathcal{B}} \leq |\sigma|\, \|\pi\|\, \|e_1\|_{\ell^\infty \mathbb{N}} = |\sigma|\, \|\pi\|$. Using property (S1), the linearity of π, and the definition of τ' we obtain

$$\psi_\sigma(\pi(\xi)) = \pi(\sigma \vee \xi) = \sigma\, \pi(e_1) + \pi(0 \vee \xi) = \sigma\, \pi(e_1) + (\pi \circ \tau')(\xi).$$

Observe that by Remark 2.1.5 the map $\tau' : \ell^\infty \mathbb{N} \to \ell^\infty \mathbb{N}$ is linear and continuous, hence the linking map ψ_σ is affine and continuous on the closed span $\tilde{\mathcal{B}} := \overline{\mathrm{span}}\, \pi(\ell^\infty \mathbb{N}) \subset \mathcal{B}$ of the image of π. Without loss of generality we can assume that $\tilde{\mathcal{B}} = \mathcal{B}$. Note that we have not yet identified the linear part, but the previous argument also shows that the linear part of ψ_σ is independent of σ and will be denoted by $\mathbb{B} : \mathcal{B} \to \mathcal{B}$. By property (S1) it has to satisfy equation (54). □

In order to get a trace formula for the transfer operator we will have to assume that the linear part of this affine map is contractive in a strong sense which will be specified later, see (2.7.1). Proposition 2.6.1 has a representation theoretic interpretation:

Remark 2.6.2. (i) Rephrasing equation (54) in other words, the projection map π is an intertwining map for the following representations a_1 and a_2 of the semigroup $\mathbb{N}_0$ defined by

$$a_1 : \mathbb{N}_0 \times \ell^\infty \mathbb{N} \to \ell^\infty \mathbb{N},\ (n, \xi) \mapsto a_1(n, \xi) := (\tau')^n(\xi)$$

and

$$a_2 : \mathbb{N}_0 \times \mathcal{B} \to \mathcal{B},\ (n, z) \mapsto a_2(n, z) := \mathbb{B}^n z.$$

Intertwining means that $\pi(a_1(n, \xi)) = a_2(n, \pi(\xi))$ for all $\xi \in \ell^\infty \mathbb{N}$ and $n \in \mathbb{N}_0$.

(ii) The representation a_1 and the representation

$$a_3 : \mathbb{N}_0 \times \ell^1 \mathbb{N} \to \ell^1 \mathbb{N},\ (n, d) \mapsto a_3(n, d) := \tau^n d,$$

defined by $a_3(n, d)(k) := (\tau^n d)(k) = d(n + k)$ are dual to each other in the following way: $a_1 = a_3'$, i.e., $a_1(n, \cdot) = a_3(n, \cdot)'$ for all $n \in \mathbb{N}_0$, and $a_1'\big|_{\mathbb{N}_0 \times \ell^1 \mathbb{N}} = a_3$. This is due to the facts that $(\ell^1 \mathbb{N})' = \ell^\infty \mathbb{N}$,

$$\langle a_1(n, \xi), d \rangle_{\ell^\infty \mathbb{N}, \ell^1 \mathbb{N}} = \langle (\tau')^n(\xi), d \rangle_{\ell^\infty \mathbb{N}, \ell^1 \mathbb{N}} = \langle \xi, \tau^n d \rangle_{\ell^\infty \mathbb{N}, \ell^1 \mathbb{N}} = \langle \xi, a_3(n, d) \rangle_{\ell^\infty \mathbb{N}, \ell^1 \mathbb{N}},$$

but $\ell^1 \mathbb{N} \subsetneq (\ell^\infty \mathbb{N})'$. To see that $a_1'\big|_{\mathbb{N}_0 \times \ell^1 \mathbb{N}} = a_3$, let $d \in \ell^1 \mathbb{N}$ and calculate as follows

$$a_1'(n, d)(k) = \langle e_k, a_1'(n, d) \rangle = \langle a_1(n, e_k), d \rangle = \langle e_{k+n}, d \rangle = d(k + n).$$

(iii) The dual representation of a_2 is given on the strong dual $\mathcal{B}'$ of $\mathcal{B}$ as

$$a_2' : \mathbb{N}_0 \times \mathcal{B}' \to \mathcal{B}',\ (n, z') \mapsto a_2'(n, z') := (\mathbb{B}')^n z'$$

where $\mathbb{B}' : \mathcal{B}' \to \mathcal{B}'$ is the dual map of $\mathbb{B} : \mathcal{B} \to \mathcal{B}$. With the identification $\tau'' = \tau$ and $(\pi \circ \tau')' = \tau'' \circ \pi' = \tau \circ \pi'$ equation (54) transforms into

$$(\tau \circ \pi')(d) = (\pi \circ \tau')'(d) = (\mathbb{B} \circ \pi)'(d) = (\pi' \circ \mathbb{B}')(d) \tag{55}$$

for all $d \in (\ell^\infty \mathbb{N})'$, which implies that π' intertwines a_1' and a_2', i. e., $\pi'(a_2'(n, z')) = a_1'(n, \pi' z')$ for all $n \in \mathbb{N}_0$, $z' \in \mathcal{B}'$, since for all $z' \in \mathcal{B}'$, $\xi \in \ell^\infty \mathbb{N}$ we have

$$\begin{aligned} \langle \xi, \pi'(a_2'(n, z')) \rangle_{\mathcal{B}, \mathcal{B}'} &= \langle \xi, \pi'((\mathbb{B}')^n z') \rangle_{\mathcal{B}, \mathcal{B}'} = \langle \mathbb{B}^n \pi(\xi), z' \rangle_{\mathcal{B}, \mathcal{B}'} \\ &= \langle \pi(\tau')^n(\xi), z' \rangle_{\mathcal{B}, \mathcal{B}'} = \langle \xi, \tau^n \pi' z' \rangle_{\mathcal{B}, \mathcal{B}'} \\ &= a_1'(n, \pi' z')(\xi). \end{aligned}$$

□

Proposition 2.6.1 and Remark 2.6.2 say that in order to fulfill the condition (S1) one has to look for intertwining maps for certain representations of the semigroup $\mathbb{N}_0$. We will assume in addition to the hypotheses of (2.6.1) that there is a connection between the intertwining map of the $\mathbb{N}_0$-representations a_1 and a_2 from (2.6.2) and the standard Ising observable $A_{(\phi)}$ (2.1.7). Then the distance function is of the form $d(k) = \langle \mathbb{B}^{k-1} v, w' \rangle_{\mathcal{B},\mathcal{B}'}$ for some linear operator $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ and $v \in \mathcal{B}$, $w' \in \mathcal{B}'$:

Proposition 2.6.3. *Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator and $\alpha : \ell^\infty\mathbb{N} \to \mathcal{B}$ be a continuous linear map such that $\alpha(0 \vee \xi) = \mathbb{B}\alpha(\xi)$ for all $\xi \in \ell^\infty\mathbb{N}$, i. e., $\alpha \circ \tau' = \mathbb{B} \circ \alpha$, hence α is an intertwiner of a_1 and a_2 from (2.6.2). Let $d \in \ell^1\mathbb{N}$. Suppose there exists $\alpha' \in \mathcal{B}'$ such that*

$$\langle \alpha(\xi), \alpha' \rangle_{\mathcal{B},\mathcal{B}'} = \sum_{k=1}^{\infty} \xi_k \, d(k) \tag{56}$$

for all $\xi \in \ell^\infty\mathbb{N}$. Then $d(k) = \langle \mathbb{B}^{k-1}\alpha(e_1), \alpha' \rangle_{\mathcal{B},\mathcal{B}'}$ for all $k \in \mathbb{N}$.

Proof. We set $\beta : \ell^\infty\mathbb{N} \to \mathbb{C}$, $\beta(\xi) := \sum_{k=1}^\infty \xi_k \, d(k)$ and evaluate β at the standard basis elements $e_j := (\delta_{j,k})_{k\in\mathbb{N}} \in \ell^\infty\mathbb{N}$. This gives $\beta(e_j) = \sum_{k=1}^\infty \delta_{j,k} \, d(k) = d(j)$. We show by induction that $\alpha(e_k) = \mathbb{B}^{k-1}\alpha(e_1)$. The case $k = 1$ is trivial. Since $0 \vee e_k = e_{k+1}$, we have

$$\alpha(e_{k+1}) = \alpha(0 \vee e_k) = \mathbb{B}\alpha(e_k) \overset{(I)}{=} \mathbb{B}\mathbb{B}^{k-1}\alpha(e_1) = \mathbb{B}^k\alpha(e_1)$$

using at (I) the induction hypothesis. Hence $d(k) = \beta(e_k) \overset{(56)}{=} \langle \alpha(e_k), \alpha' \rangle_{\mathcal{B},\mathcal{B}'} = \langle \mathbb{B}^{k-1}\alpha(e_1), \alpha' \rangle_{\mathcal{B},\mathcal{B}'}$. □

Proposition 2.6.3 shows that a distance function $d \in \ell^1\mathbb{N}$ with the property (56) is necessarily of the form $d(k) = \langle \mathbb{B}^{k-1}\alpha(e_1), \alpha' \rangle_{\mathcal{B},\mathcal{B}'}$, hence $\sum_{k=1}^\infty |\langle \mathbb{B}^{k-1}\alpha(e_1), \alpha' \rangle_{\mathcal{B},\mathcal{B}'}|$ has to be finite. Whereas the upper bound $|\langle \mathbb{B}^{k-1}\alpha(e_1), \alpha' \rangle_{\mathcal{B},\mathcal{B}'}| \le \|\mathbb{B}^{k-1}\alpha(e_1)\|_{\mathcal{B}} \, \|\alpha'\|_{\mathcal{B}'}$ is straight forward, a sharp lower bound is missing. A natural sufficient condition[20] which ensures the convergence of $\sum_{k=1}^\infty \|\mathbb{B}^{k-1}\alpha(e_1)\|_{\mathcal{B}}$ is that the spectral radius of $\mathbb{B}$ is less than one. We briefly remind the reader of the notion of the spectral radius of a linear operator acting on a Banach space and recall some of its properties. In Proposition 2.6.5 we will show that each bounded linear operator with spectral radius less than one gives rise to a distance function which decays exponentially and has the desired intertwining properties.

Remark 2.6.4. Recall (for instance [We00, p. 231]) the definition of the *spectral radius* of a bounded linear operator $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ on a Banach space $\mathcal{B}$ as

$$\rho_{\text{spec}}(\mathbb{B}) := \sup\big\{|z| \in \mathbb{R} \mid z \in \text{spec}(\mathbb{B})\big\}.$$

(i) The spectral radius can be characterised via

$$\rho_{\text{spec}}(\mathbb{B}) = \max\big\{|z| \in \mathbb{R} \mid z \in \text{spec}(\mathbb{B})\big\} = \lim_{k\to\infty} \sqrt[k]{\|\mathbb{B}^k\|} = \inf_{k\to\infty} \sqrt[k]{\|\mathbb{B}^k\|}.$$

From this formula it is obvious, that $\rho_{\text{spec}}(\mathbb{B}) \le \|\mathbb{B}\|$.

(ii) Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a linear operator with $\|\mathbb{B}^{k_0}\| < 1$ for some $k_0 \in \mathbb{N}$. Then $\rho_{\text{spec}}(\mathbb{B}) < 1$, since

$$\rho_{\text{spec}}(\mathbb{B}) = \lim_{k\to\infty} \sqrt[k]{\|\mathbb{B}^k\|} = \lim_{n\to\infty} \sqrt[nk_0]{\|\mathbb{B}^{nk_0}\|} \le \lim_{n\to\infty} \sqrt[nk_0]{\|\mathbb{B}^{k_0}\|^n} = \sqrt[k_0]{\|\mathbb{B}^{k_0}\|} < 1.$$

(iii) Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, then there exists $k_0 \in \mathbb{N}$ such that $\|\mathbb{B}^k\| < 1$ for all $k \ge k_0$. In fact: For all $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ one has $|\rho_{\text{spec}}(\mathbb{B}) - \|\mathbb{B}^k\|^{1/k}| < \epsilon$. Hence $\|\mathbb{B}^k\|^{1/k} \le \rho_{\text{spec}}(\mathbb{B}) + \epsilon$, which gives the assertion if we let $\epsilon < 1 - \rho_{\text{spec}}(\mathbb{B})$. □

Proposition 2.6.5. *Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$. Set $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto d(k) := \langle \mathbb{B}^{k-1} v, w' \rangle_{\mathcal{B},\mathcal{B}'}$. Then d belongs to the space $\mathcal{D}_1$ of distance functions defined in (2.2.4) and $\limsup_{k\to\infty} \sqrt[k]{|d(k)|} \le \rho_{\text{spec}}(\mathbb{B}) < 1$, which by definition means that d is (at least) exponentially decreasing at infinity.*

[20] It would be sufficient that the restriction of $\mathbb{B}$ to the subspace $\overline{\text{span}}\{\mathbb{B}^k\alpha(e_1) \in \mathcal{B} \mid k \in \mathbb{N}_0\} \subset \mathcal{B}$ generated by $\mathbb{B}$ and $\alpha(e_1)$ has spectral radius less than one, so one might assume that $\alpha(e_1)$ is a cyclic vector.

Proof. We have the trivial estimate $|d(k)| = |\langle \mathbb{B}^{k-1}v, w'\rangle| \leq \|\mathbb{B}^{k-1}\| \|v\| \, \|w'\|$ and hence

$$\limsup_{k\to\infty} \sqrt[k]{|d(k)|} \leq \lim_{k\to\infty} \sqrt[k]{\|\mathbb{B}^{k-1}\| \|v\| \, \|w'\|} = \lim_{k\to\infty} \sqrt[k]{\|\mathbb{B}^{k-1}\|} \overset{(*)}{=} \lim_{k\to\infty} \sqrt[k]{\|\mathbb{B}^{k}\|} = \rho_{\text{spec}}(\mathbb{B}) < 1.$$

Hence by Proposition 2.2.8 the distance function d belongs to $\mathcal{D}_1$ (2.2.4). It remains to show the stated equality $(*)$: We use that $\rho_{\text{spec}}(\mathbb{B}) = \lim_{k\to\infty} \|\mathbb{B}^k\|^{1/k}$. Hence for all $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ one has the estimate $|\rho_{\text{spec}}(\mathbb{B}) - \|\mathbb{B}^k\|^{1/k}| < \epsilon$. Hence

$$(\rho_{\text{spec}}(\mathbb{B}) + \epsilon)^{-1/(k+1)} \leq (\|\mathbb{B}^k\|^{1/k})^{-1/(k+1)} \leq (\rho_{\text{spec}}(\mathbb{B}) - \epsilon)^{-1/(k+1)},$$

thus $\lim_{k\to\infty}(\|\mathbb{B}^k\|^{1/k})^{-1/(k+1)} = 1$ and

$$\lim_{k\to\infty} \sqrt[k+1]{\|\mathbb{B}^k\|} = \lim_{k\to\infty} \|\mathbb{B}^k\|^{1/k} \, \|\mathbb{B}^k\|^{-1/k(k+1)} = \lim_{k\to\infty} \|\mathbb{B}^k\|^{1/k} \lim_{k\to\infty} \|\mathbb{B}^k\|^{-1/k(k+1)} = \lim_{k\to\infty} \|\mathbb{B}^k\|^{1/k}.$$

□

In the remark following Proposition 2.6.3 we asked the following question: Given $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$, find a lower asymptotic bound for $\sqrt[k]{|d(k)|} = \sqrt[k]{|\langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}|}$ depending on the spectral properties of $\mathbb{B}$. This seems to be a harder problem than the upper bound. In particular this limit behaviour will (in general) also depend on the data $v \in \mathcal{B}$, $w' \in \mathcal{B}'$. Note that from $\limsup_{k\to\infty} \sqrt[k]{|\langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}|} \leq 1$ one cannot conclude that the spectral radius of $\mathbb{B}$ is less than one. As an example take $\mathbb{B} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ which has $\rho_{\text{spec}}(\mathbb{B}) = 2$, but $d(k) = \langle \mathbb{B}^{k-1}e_1|e_1\rangle$ is a non-zero finite range interaction: $d(1) = 1$ and $d(k) = 0$ for all $k \geq 2$. In order to tackle the above problem it might be a good idea to consider the restriction $\mathbb{B}\big|_{\tilde{\mathcal{B}}}$ of $\mathbb{B}$ to the subspace $\tilde{\mathcal{B}} := \overline{\text{span}}\{\mathbb{B}^k v \,|\, k \in \mathbb{N}_0\} \subset \mathcal{B}$. In the above example one has $\tilde{\mathcal{B}} = \mathbb{C}e_1$ and $\rho_{\text{spec}}(\mathbb{B}\big|_{\tilde{\mathcal{B}}}) = 0$.
With the preparation of Remark 2.6.4 and Proposition 2.6.5 we can now give a construction scheme for linear maps which intertwine the $\mathbb{N}_0$-representations a_1 and a_2 from Remark 2.6.2 if the distance function $d : \mathbb{N} \to \mathbb{C}$ is given as $d(k) = \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$.

Proposition 2.6.6. *Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$.*

(i) *Set $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \mathcal{B}, \pi_{\mathbb{B},v}(\xi) := \sum_{k=1}^\infty \xi_k \, \mathbb{B}^{k-1}v$. Then $\pi_{\mathbb{B},v}$ is linear, continuous, and satisfies $\pi_{\mathbb{B},v}(e_1) = v$ and $\pi_{\mathbb{B},v}(0 \vee \xi) = \mathbb{B}\, \pi_{\mathbb{B},v}(\xi)$.*

(ii) *Let $\mathbb{B}' : \mathcal{B}' \to \mathcal{B}'$ be the dual map of $\mathbb{B}$ on the (strong) dual $\mathcal{B}'$ of $\mathcal{B}$. Set $\pi_{\mathbb{B}',w'} : \ell^\infty\mathbb{N} \to \mathcal{B}'$, $\pi_{\mathbb{B}',w'}(\xi) := \sum_{k=1}^\infty \xi_k \, (\mathbb{B}')^{k-1}w'$. Then $\pi_{\mathbb{B}',w'}$ is linear, continuous, and satisfies $\pi_{\mathbb{B}',w'}(e_1) = w'$ and $\pi_{\mathbb{B}',w'}(0 \vee \xi) = \mathbb{B}'\, \pi_{\mathbb{B}',w'}(\xi)$.*

(iii) *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential q and distance function $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto d(k) := \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$. Then for all $\sigma \vee \xi \in \Omega$ we can express the standard Ising observable (2.1.7) as*

$$A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sigma\langle \pi_{\mathbb{B},v}(\xi), w'\rangle_{\mathcal{B},\mathcal{B}'} = q(\sigma) + \sigma\langle v, \pi_{\mathbb{B}',w'}(\xi)\rangle_{\mathcal{B},\mathcal{B}'}.$$

Proof. Let $\xi \in \ell^\infty\mathbb{N}$, then

$$\|\pi_{\mathbb{B},v}(\xi)\| \leq \sum_{k=1}^\infty |\xi_k| \, \|\mathbb{B}^{k-1}v\| \leq \|\xi\|_{\ell^\infty\mathbb{N}} \sum_{k=1}^\infty \|\mathbb{B}^{k-1}v\| \leq \|\xi\|_{\ell^\infty\mathbb{N}} \|v\| \sum_{k=1}^\infty \|\mathbb{B}^{k-1}\|.$$

The latter series converges by the root test and our assumptions on the spectral radius of $\mathbb{B}$. This shows the boundedness of $\pi_{\mathbb{B},v}$ and $\|\pi_{\mathbb{B},v}\| \leq \sum_{k=1}^\infty \|\mathbb{B}^{k-1}v\|$. Recall from Example 2.1.2 that $0 \vee \xi = (0, \xi_1, \xi_2, \ldots)$ is the insertion of 0 at the first position and a simultaneous right shift, hence

$$\pi_{\mathbb{B},v}(0 \vee \xi) = \sum_{k=1}^\infty (0 \vee \xi)_k \, \mathbb{B}^{k-1}v = \sum_{k=1}^\infty \xi_k \, \mathbb{B}^k v = \mathbb{B}\, \pi_{\mathbb{B},v}(\xi).$$

Similarly one proceeds for $\pi_{\mathbb{B}',w'}$ noting that $\|\mathbb{B}\| = \|\mathbb{B}'\|$. The standard Ising observable is given as $A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sigma \sum_{k=1}^\infty \xi_k \, d(k)$ for all $\sigma \vee \xi \in \Omega$. Concerning the second term we note that

$$\sum_{k=1}^\infty \xi_k \, d(k) = \sum_{k=1}^\infty \xi_k \, \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'} = \langle \sum_{k=1}^\infty \xi_k \, \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'} = \langle \pi_{\mathbb{B},v}(\xi), w'\rangle_{\mathcal{B},\mathcal{B}'}.$$

□

The first part of Proposition 2.6.6 states that the map $\pi_{\mathbb{B},v}$ intertwines the representations a_1 and a_2. Part (iii) shows that the standard observable admits an (S1) - (S3) Ansatz (2.3.3).
Next, we recall that every distance function $d \in \ell^1\mathbb{N}$ has a representation $d(k) = \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$ for some $v \in \mathcal{B}$, $w' \in \mathcal{B}', \mathbb{B} \in \mathrm{End}(\mathcal{B})$ using the shift (2.1.5). We determine its spectral properties which - unfortunately - will not be sufficient for our purposes.

Remark 2.6.7. Let $\tau : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}},\ (\tau\xi)_n := \xi_{n+1}$ be the shift from Remark 2.1.5. Note that

$$d(k) = \langle \tau^{k-1}d, e_1\rangle_{\ell^1\mathbb{N},\ell^\infty\mathbb{N}} \tag{57}$$

for *all* $d \in \ell^1\mathbb{N}$ and $k \in \mathbb{N}$ where $e_1 = (1, 0, \ldots) \in \ell^\infty\mathbb{N}$. We call (57) the trivial representation of d, since τ and e_1 do not depend on d. All powers τ^m $(m \in \mathbb{N})$ have operator norm equal to one on all the sequence spaces $\ell^p\mathbb{N}$ $(1 \le p \le \infty)$, hence $\rho_{\text{spec}}(\tau) = 1$. Similarly to Remark 2.1.5 we get $\tau^m(\tau')^m = \mathrm{id}$, hence no power of τ is trace class. □

We want to understand which distance functions d can be represented as $d(k) = \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$ with $\rho_{\text{spec}}(\mathbb{B}) < 1$. Such a representation can be viewed as a subrepresentation of $a_3 : \mathbb{N}_0 \times \ell^1\mathbb{N} \to \ell^1\mathbb{N}$ defined by $a_3(n, d)(k) := (\tau^n d)(k) = d(n + k)$ in (2.6.2). In Example 2.7.7 we will give a couple of examples of distance functions which have such a representation and give a (partial) classification in (2.12.3). In particular, we show that there are two types of finite dimensional representations, which either come from finite range or from polynomial-exponential distance functions (See Sections 2.8 and 2.11, respectively).
The following remark is just the contraposition of one of the statements from Proposition 2.6.5, but it provides a test which is both simple to handle and applicable in important situations.

Remark 2.6.8. Let $d \in \ell^1\mathbb{N}$. If $\lim_{k\to\infty} \sqrt[k]{|d(k)|} = 1$, then by Proposition 2.6.5 there is no representation $d(k) = \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$ for a linear operator $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ with $\rho_{\text{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$. □

Remark 2.6.8 will imply that, unfortunately, some physically relevant distance functions, which we introduced in Example 2.2.5, cannot be treated with our method.

Example 2.6.9. We recall that the distance functions introduced in (2.2.5) (i) - (iv) belong to $\mathcal{D}_1$ (2.2.4). By Remark 2.6.8 the Examples (i) and (ii) do not have a representation $d(k) = \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$, where $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ is a bounded linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$.

(i) Let $\alpha > 1$ and $d(k) := k^{-\alpha}$. Then $\lim_{k\to\infty} \sqrt[k]{|d(k)|} = (\lim_{k\to\infty} \sqrt[k]{k})^{-\alpha} = 1$.

(ii) Plummer potential: Let $\epsilon > 0$, $\alpha > 1$ and $d(k) := (\epsilon + |k|^2)^{-\alpha/2}$. The Plummer potential satifies $c_\epsilon\, k^{-\alpha} \le d(k) \le k^{-\alpha}$ for some $c_\epsilon > 0$ as stated in 2.2.5 (ii), hence we have $\lim_{k\to\infty} \sqrt[k]{|d(k)|} = (\lim_{k\to\infty} \sqrt[k]{k})^{-\alpha} = 1$.

(iii) Let γ, $\delta > 0$, and $d(k) := \exp(-\gamma k^\delta)$. Then

$$\lim_{k\to\infty} \sqrt[k]{|d(k)|} = \lim_{k\to\infty} \exp(-\gamma k^{\delta-1}) = \begin{cases} 0, & \text{if } \delta > 1, \\ 1, & \text{if } \delta < 1, \\ \exp(-\gamma), & \text{if } \delta = 1. \end{cases}$$

The exponential case $\delta = 1$ can be treated with our method, see Example 2.10.6, the subexponential case $\delta < 1$ cannot be treated by Remark 2.6.8. The case $\delta > 1$ of superexponential decay was first solved by D. Mayer via a similar approach for an arbitrary interaction matrix, but finite alphabet F. We will study this in Section 2.9.

(iv) Let α, $\gamma > 0$ and $d(k) := \int_0^1 t^\alpha \exp(-\gamma t k)\, dt$. The estimate given in Example 2.2.5 implies that $\lim_{k\to\infty} \sqrt[k]{|d(k)|} \le 1$; a lower asymptotic bound remains open. □

We will now review some results from Section 2.2 and explain them with the methods from this section. As shown in Remark 2.6.7, the shift $\tau : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}},\ (\tau\xi)_n := \xi_{n+1}$ (2.1.5) yields the trivial representation $d(k) = \langle \tau^{k-1}d, e_1\rangle_{\ell^1\mathbb{N},\ell^\infty\mathbb{N}}$ for all $d \in \ell^1\mathbb{N}$ and $k \in \mathbb{N}$. Besides the projection map

$$\pi_{\tau,d} : \ell^\infty\mathbb{N} \to \ell^1\mathbb{N},\ \xi \mapsto \sum_{j=1}^{\infty} \xi_j\, \tau^{j-1} d$$

associated to the trivial representation via Proposition 2.6.6, we have the projection map π^d defined in Remark 2.2.2. By looking at their components $\langle \pi_{\tau,d}(\xi), e_i\rangle_{\ell^1\mathbb{N},\ell^\infty\mathbb{N}} = \sum_{j=1}^{\infty} \xi_j\, d(j+i-1) = \pi^d(\xi)_i$ we see that they coincide. The projection map π^d appeared in the continuous bilinear extension of the map $W_{\mathbb{Z}_\leq,\mathbb{Z}_>} : \Omega_\leq \times \Omega_> \to \mathbb{C}$ (1.6.3) to $\ell^\infty(\mathbb{Z}_\leq) \times \ell^\infty(\mathbb{N}) \to \mathbb{C}$ via

$$W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi) = -\langle \pi^d(\xi), S(\overleftarrow{\eta})\rangle_{\ell^1\mathbb{N},\ell^\infty\mathbb{N}} = -\langle \pi^d(\xi), S(\overleftarrow{\eta})\rangle_{\ell^1\mathbb{N},\ell^\infty\mathbb{N}},$$

which is a pairing between $\ell^1\mathbb{N}$ and $\ell^\infty\mathbb{N}$. We will now give a generalisation of this pairing situation and use its connection with Ruelle's representation of the leading eigenfunction of the Ruelle transfer operator. We recall the definition of the inversion map $\overleftarrow{\ } : \mathbb{C}^{-\mathbb{N}} \to \mathbb{C}^{\mathbb{N}},\ \overleftarrow{\eta}_i = \eta_{-i}$ and the bijective right shift $S = \tau_{-1;\mathbb{N}_0} : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}},\ (Sx)_i = x_{i-1}$.

Proposition 2.6.10. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{Z}}, \mathbb{Z}, \tau)$ a two-sided one-dimensional full shift (1.2.5), $\Omega_\leq = \rho_{-\mathbb{N}_0}(\Omega)$, and $\Omega_> = \rho_{\mathbb{N}}(\Omega)$. Let ϕ be a two-body Ising interaction (1.8.3) with distance function $d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$ where $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ is a bounded linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$. Let $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \mathcal{B}$ and $\pi_{\mathbb{B}',w'} : \ell^\infty\mathbb{N} \to \mathcal{B}'$ be defined as in Prop. 2.6.6. Then the map $W_{\mathbb{Z}_\leq,\mathbb{Z}_>} : \Omega_< \times \Omega_\geq \to \mathbb{C}$ (1.6.3) has a continuous bilinear extension $W_{\mathbb{Z}_\leq,\mathbb{Z}_>} : \ell^\infty(\mathbb{Z}_\leq) \times \ell^\infty(\mathbb{Z}_>) \to \mathbb{C}$ via*

$$W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi) = -\langle \pi_{\mathbb{B},v}(\xi), (\pi_{\mathbb{B}',w'} \circ S)(\overleftarrow{\eta})\rangle_{\mathcal{B},\mathcal{B}'} = -\langle (\pi_{\mathbb{B},v} \circ S)(\overleftarrow{\eta}), \pi_{\mathbb{B}',w'}(\xi)\rangle_{\mathcal{B},\mathcal{B}'}.$$

Let h_1 be the leading eigenfunction of the Ruelle transfer operator $\mathcal{L}_{\beta A_{(\phi)}} : \mathcal{C}(\Omega_>) \to \mathcal{C}(\Omega_>)$ (2.1.3) and

$$\tilde{h}_1 : \mathcal{B} \to \mathbb{C},\ z \mapsto \int_{\Omega_\leq} \exp\Big(\beta\, \langle z, (\pi_{\mathbb{B}',w'} \circ S)(\overleftarrow{\eta})\rangle_{\mathcal{B},\mathcal{B}'}\Big)\, d\mu_\leq(\eta),$$

then $\tilde{h}_1$ belongs to $\mathcal{C}(\mathcal{B})$ with $h_1 = \tilde{h}_1 \circ \pi_{\mathbb{B},v}$.

Proof. Let $\xi \in \ell^\infty\mathbb{N}$, $\eta \in \ell^\infty(\mathbb{Z}_\leq)$. By Proposition 2.6.5 the distance function d belongs to $\mathcal{D}_1$, hence by Remark 2.2.10 the series defining $W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi)$ converges absolutely and is bounded by

$$|W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi)| \leq \|d\|_{\mathcal{D}_1}\, \|\eta\|_{\ell^\infty(\mathbb{Z}_\leq)}\, \|\xi\|_{\ell^\infty\mathbb{N}}.$$

For this reason the following sums can be interchanged

$$\begin{aligned}
-W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi) &= \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} \eta_{-k}\, \xi_j\, d(k+j) \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \eta_{1-k}\, \xi_j\, \langle \mathbb{B}^{k+j-2}v, w'\rangle_{\mathcal{B},\mathcal{B}'} \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \eta_{1-k}\, \xi_j\, \langle \mathbb{B}^{j-1}v, (\mathbb{B}')^{k-1}w'\rangle_{\mathcal{B},\mathcal{B}'} \\
&= \langle \sum_{j=1}^{\infty} \xi_j \mathbb{B}^{j-1}v, \sum_{k=1}^{\infty} \eta_{1-k}(\mathbb{B}')^{k-1}w'\rangle_{\mathcal{B},\mathcal{B}'} = \langle \pi_{\mathbb{B},v}(\xi), (\pi_{\mathbb{B}',w'} \circ S)(\overleftarrow{\eta})\rangle_{\mathcal{B},\mathcal{B}'}
\end{aligned}$$

Proposition 2.6.6 shows that the latter expression is well-defined. Similarly one obtains

$$-W_{\mathbb{Z}_\leq,\mathbb{Z}_>}(\eta \oplus \xi) = \langle (\pi_{\mathbb{B},v} \circ S)(\overleftarrow{\eta}), \pi_{\mathbb{B}',w'}(\xi)\rangle_{\mathcal{B},\mathcal{B}'}.$$

The continuity of $\tilde{h}_1$ is straight forward, the representation $h_1(\xi) = \tilde{h}_1(\pi_{\mathbb{B},v}(\xi))$ follows from Ruelle's result (48) in Remark 2.2.11. □

In the special case that $\mathcal{B} = \ell^2\mathbb{N}$ we will show that the leading eigenfunction of the Ruelle transfer operator has a preimage in the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ under the composition operator $C_{\pi_{\mathbb{B},v}} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{C}_b(\Omega)$. This answers a question raised in Remark 2.2.12.

Corollary 2.6.11. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{Z}}, \mathbb{Z}, \tau)$ a two-sided one-dimensional full shift (1.2.5), $\Omega_{\leq} = \rho_{-\mathbb{N}_0}(\Omega)$ and $\Omega_{>} = \rho_{\mathbb{N}}(\Omega)$. Let ϕ be a two-body Ising interaction (1.8.3) with distance function $d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \big(\mathbb{B}^{k-1}v \,\big|\, w\big)_{\ell^2\mathbb{N}}$ where $v,\, w \in \ell^2\mathbb{N}$ and $\mathbb{B} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ is a bounded linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$. Let $\mathbb{B}^{\perp} = \mathbb{B}' : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ be its dual. Let $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N}$ and $\pi_{\mathbb{B}^\top,w} : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N}$ be the linear maps defined in (2.6.6). Let h_1 be the leading eigenfunction of the Ruelle transfer operator $\mathcal{L}_{\beta A_{(\phi)}} : \mathcal{C}(\Omega_>) \to \mathcal{C}(\Omega_>)$ (2.1.3) and*

$$\tilde{h}_1 : \ell^2\mathbb{N} \to \mathbb{C},\ z \mapsto \int_{\Omega_{\leq}} \exp\Big(\beta\, (z \,|\, (\pi_{\mathbb{B}^\top,w} \circ S)(\overleftarrow{\eta}))\Big)\, d\mu_{\leq}(\eta).$$

Then $\tilde{h}_1$ belongs to $\mathcal{F}(\ell^2\mathbb{N})$ with $h_1 = \tilde{h}_1 \circ \pi_{\mathbb{B},v}$, where $\overleftarrow{\cdot}: \mathbb{C}^{-\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is the inversion map and $S = \tau_{-1;\mathbb{N}_0} : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}}$ the bijective right shift.

Proof. For all $\eta \in \Omega_{\leq}$ the vector $(\pi_{\mathbb{B}^\top,w} \circ S)(\overleftarrow{\eta})$ belongs to $\ell^2\mathbb{N}$. By Proposition A.4.9 we have $\tilde{h}_1 \in \mathcal{F}(\ell^2\mathbb{N})$. Hence the assertion follows from Proposition 2.6.10. □

Our next goal is the construction of a Ruelle-Mayer transfer operator $\mathcal{M}$ such that for all $k \geq k_0$ the operator $\mathcal{M}^k$ is trace class and a dynamical trace formula holds. By Proposition 2.6.6 the standard Ising observable admits an (S1) - (S3) Ansatz (2.3.3) using the linear map $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \mathcal{B}$ from Proposition 2.6.6 as a projection map. Definition 2.3.7 directly yields the corresponding (formal) Ruelle-Mayer operator

$$\text{(58)}\qquad \mathcal{M}_\beta : \mathcal{C}_b(\mathcal{B}) \to \mathcal{C}_b(\mathcal{B})\ (\mathcal{M}_\beta f)(z) := \int_F \exp\big(\beta q(\sigma) + \beta\sigma\langle z, w'\rangle_{\mathcal{B},\mathcal{B}'}\big)\, f(\sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma),$$

which formally satisfies $\mathcal{L}_{\beta A_{(\phi)}} \circ C_{\pi_{\mathbb{B},v}} = C_{\pi_{\mathbb{B},v}} \circ \mathcal{M}_\beta$, where $C_{\pi_{\mathbb{B},v}} : \mathcal{C}_b(\mathcal{B}) \to \mathcal{C}_b(\Omega),\ g \mapsto g \circ \pi_{\mathbb{B},v}$ is the composition operator associated to $\pi_{\mathbb{B},v}$. In order to obtain a bounded Ruelle-Mayer transfer operator one has to identify a suitable small space of functions which is invariant under the operator. The Ruelle-Mayer transfer operator is a superposition of the generalised composition operators[21]

$$\mathcal{M}_{\beta,\sigma} : \mathcal{C}_b(\mathcal{B}) \to \mathcal{C}_b(\mathcal{B})\ (\mathcal{M}_{\beta,\sigma} f)(z) := \exp\big(\beta q(\sigma) + \beta\sigma\langle z, w'\rangle_{\mathcal{B},\mathcal{B}'}\big)\, f(\sigma\, v + \mathbb{B}\, z).$$

The spectral properties of the operators $\mathcal{M}_{\beta,\sigma}$ and hence of $\mathcal{M}_\beta$ depend on the space on which the operator acts. Appendix B addresses this problem. The next Remark 2.6.13 investigates the structure of the algebra generated by the composition operators $\mathcal{M}_{(\sigma)}$ from which the operator $\mathcal{M}$ is built up. We will need a preparatory proposition on the compositions of a special type of affine maps which arise as linking maps as we have seen in Proposition 2.6.1.

Proposition 2.6.12. *Let V be a complex vector space, $\mathbb{B} : V \to V$ a linear operator, and $a \in V$. For $x \in \mathbb{C}$ put $\psi_x : V \to V,\ z \mapsto xa + \mathbb{B}z$. Then for all $k \in \mathbb{N},\ x_1, \ldots, x_k \in \mathbb{C},\ z \in V$ one has*

$$(\psi_{x_1} \circ \ldots \circ \psi_{x_k})(z) = \mathbb{B}^k z + \sum_{j=0}^{k-1} x_{j+1}\mathbb{B}^j a.$$

Proof. By induction: Let $x_1, \ldots, x_{k+1} \in F$, then $\psi_{x_1}(z) = x_1\mathbb{B}^0 a + \mathbb{B}^1 z$ and

$$(\psi_{x_1} \circ \ldots \circ \psi_{x_{k+1}})(z) = \mathbb{B}^k \psi_{x_{k+1}}(z) + \sum_{j=0}^{k-1} x_{j+1}\mathbb{B}^j a = \mathbb{B}^{k+1} z + x_{k+1}\mathbb{B}^k a + \sum_{j=0}^{k-1} x_{j+1}\mathbb{B}^j a.$$

□

Remark 2.6.13. Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a bounded operator, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$, $F \subset \mathbb{C}$, and $q \in \mathcal{C}(F)$. For all $\sigma \in F$ we define an unbounded operator

$$\mathcal{M}_{(\sigma)} : \mathcal{C}_b(\mathcal{B}) \to \mathcal{C}_b(\mathcal{B}),\ (\mathcal{M}_{(\sigma)} g)(z) = \exp\big(q(\sigma) + \sigma\langle z, w'\rangle_{\mathcal{B},\mathcal{B}'}\big)\, g(\sigma\, v + \mathbb{B}z).$$

Then for all $g \in \mathcal{C}_b(\mathcal{B})$, $n \in \mathbb{N}$ and all choices $\sigma_1, \ldots, \sigma_n \in F$ one has by Corollary B.1.3 and Proposition 2.6.12

$$(\mathcal{M}_{(\sigma_1)} \circ \ldots \circ \mathcal{M}_{(\sigma_n)} g)(z) = \exp\Big(\sum_{k=1}^n q(\sigma_k) + \sum_{k=1}^n \sigma_k \langle \mathbb{B}^{n-k} z + \sum_{j=1}^{n-k} \sigma_{k+j}\mathbb{B}^{j-1}v, w'\rangle\Big)\, g(\mathbb{B}^n z + \sum_{k=1}^n \sigma_k \mathbb{B}^{k-1} v).$$

□

[21] (Generalised) composition operators, their spectral properties and trace formulas will be discussed in Appendix B.

In addition to the hypotheses of Remark 2.6.13 we will now assume that the linear operator $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ is contractive: If the operator norm of $\mathbb{B}$ is strictly less than one, then Proposition 2.6.14 shows that any affine map of the form $\mathcal{B} \to \mathcal{B},\ z \mapsto \mathbb{B}z + b$ is strictly contractive on suitable large balls $B(0;\rho)_{\mathcal{B}} := \{z \in \mathcal{B}; \|z\|_{\mathcal{B}} < \rho\}$. More generally, if the spectral radius of $\mathbb{B}$ is strictly less than one, then at least certain (mixed) iterates of affine maps are contractive:

Proposition 2.6.14. *Let $F \subset \mathbb{C}$ be a bounded set, $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, and $v \in \mathcal{B}$. For $x \in F$ let $\psi_x : \mathcal{B} \to \mathcal{B},\ z \mapsto xv + \mathbb{B}z$. Then there exists $\rho > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $x_1, \ldots, x_k \in F$*

$$(\psi_{x_1} \circ \ldots \circ \psi_{x_k})(\overline{B(0;\rho)_{\mathcal{B}}}) \subset B(0;\rho)_{\mathcal{B}}.$$

If $\|\mathbb{B}\| < 1$, then this holds for all $k \in \mathbb{N}$.

Proof. We start with the special case $\|\mathbb{B}\| < 1$. Put $c_F := \sup_{x\in F} |x|$ and let $\rho > \frac{c_F\,\|v\|}{1-\|B\|}$. Then for all $z \in \mathcal{B}$ with $\|z\| \leq \rho$ we have

$$\|\psi_x(z)\| \leq \|xv\| + \|\mathbb{B}\|\,\|z\| \leq c_F\,\|v\| + \|\mathbb{B}\|\,\rho < \rho.$$

Concerning the general case: By Remark 2.6.4 (iii) there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\|\mathbb{B}^k\| < 1$. Let $c_k := \sum_{j=1}^{k} \|\mathbb{B}^{j-1}v\|$ and $\rho_k > \frac{c_F\,c_k}{1-\|B^k\|}$ for all $k \in \mathbb{N}$. Then for all $k \geq k_0$, $z \in \overline{B(0;\rho_k)_{\mathcal{B}}}$, and $x_1, \ldots, x_k \in F$ we have by the previous Proposition 2.6.12

$$\|(\psi_{x_1} \circ \ldots \circ \psi_{x_k})(z)\| \leq \|\sum_{j=0}^{k-1} x_{j+1}\mathbb{B}^j v\| + \|\mathbb{B}^k\|\,\|z\| \leq c_F\,c_k + \|\mathbb{B}^k\|\,\rho_k < \rho_k.$$

By induction $(\psi_{x_1} \circ \ldots \circ \psi_{x_{nk_0}})(\overline{B(0;\rho_{k_0})_{\mathcal{B}}}) \subset B(0;\rho_{k_0})_{\mathcal{B}}$ for all $n \in \mathbb{N}$. Then the assertion follows with $\rho := \max_{j=k_0,\ldots,2k_0-1} \rho_j$. □

Using the previous propositions we can identify a domain on which the formal Ruelle-Mayer transfer operator (58) is bounded.

Corollary 2.6.15. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential q and distance function $d : \mathbb{N} \to \mathbb{C}$ given as $d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \langle \mathbb{B}^{k-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}$, where $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ is a linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$. Then there are $\rho > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the higher iterates $(\mathcal{M}_\beta)^k : \mathcal{C}_b(B(0;\rho)_{\mathcal{B}}) \to \mathcal{C}_b(B(0;\rho)_{\mathcal{B}})$ of the Ruelle-Mayer transfer operator $\mathcal{M}_\beta$ defined by*

$$(\mathcal{M}_\beta f)(z) := \int_F \exp\big(\beta q(\sigma) + \beta\sigma\langle z, w'\rangle_{\mathcal{B},\mathcal{B}'}\big)\, f(\sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

are bounded operators with $(\mathcal{L}_{\beta A_{(\phi)}})^k \circ C_{\pi_{\mathbb{B},v}} = C_{\pi_{\mathbb{B},v}} \circ (\mathcal{M}_\beta)^k$.

Proof. By Proposition 2.6.6 the image $\pi_{\mathbb{B},v}(\Omega)$ is a bounded subset of $\mathcal{B}$. In fact, it is contained in the ball $B(0;\rho_d)_{\mathcal{B}}$ with radius $\rho_d := \|\pi_{\mathbb{B},v}\| \sup_{x\in F} |x|$. By Proposition 2.6.14 we can find an index $k_0 \in \mathbb{N}$ and a radius $\rho > 0$ such that for all $k \geq k_0$ the operator $(\mathcal{M}_\beta)^k$ leaves $\mathcal{C}_b(B(0;\rho)_{\mathcal{B}})$ invariant, since

$$\begin{aligned}
\sup_{z\in B(0;\rho)_{\mathcal{B}}} \big|((\mathcal{M}_\beta)^k f)(z)\big| &= \sup_{z\in B(0;\rho)_{\mathcal{B}}} \Big| \int_{F^n} (\mathcal{M}_{(\sigma_1)} \circ \ldots \circ \mathcal{M}_{(\sigma_n)} f)(z)\big|\, d\nu^n(\sigma_1, \ldots, \sigma_n)\Big| \\
&\leq \int_{F^n} \sup_{z\in B(0;\rho)_{\mathcal{B}}} \Big|\exp\Big(\sum_{k=1}^{n} q(\sigma_k) + \sum_{k=1}^{n} \sigma_k \langle \mathbb{B}^{n-k} z + \sum_{j=1}^{n-k} \sigma_{k+j}\mathbb{B}^{j-1}v, w'\rangle_{\mathcal{B},\mathcal{B}'}\Big)\Big| \\
&\qquad \sup_{z\in B(0;\rho)_{\mathcal{B}}} \Big| f(\mathbb{B}^n z + \sum_{k=1}^{n} \sigma_k \mathbb{B}^{k-1} v)\Big|\, d\nu^n(\sigma_1, \ldots, \sigma_n) \\
&\leq \int_F \exp\Big(\Big|\sum_{k=1}^{n} q(\sigma_k)\Big| + \sum_{k=1}^{n} |\sigma_k|\,\big(\|\mathbb{B}^{n-k}\|\rho + \sum_{j=1}^{n-k} \|\sigma_{k+j}\mathbb{B}^{j-1}v\|\,\|w'\|\Big)\, d\nu^n(\sigma_1, \ldots, \sigma_n)\,\|f\|_{\mathcal{C}_b(B(0;\rho)_{\mathcal{B}})}.
\end{aligned}$$

The property $(\mathcal{L}_{\beta A_{(\phi)}})^k \circ C_{\pi_{\mathbb{B},v}} = C_{\pi_{\mathbb{B},v}} \circ (\mathcal{M}_\beta)^k$ is now an immediate consequence of Proposition 2.6.6 and an adapted version of Remark 2.3.6. □

One could now proceed as in [May80a] and study the spectral properties of the Ruelle-Mayer operator on certain invariant subspaces consisting of analytic functions on (bounded) domains in the complex Banach space $\mathcal{B}$. For our purpose it is sufficient to study the Hilbert space setting where we can find an invariant Hilbert space of holomorphic functions on which the Ruelle-Mayer operator is trace class. All examples of Ising interactions for which a Ruelle-Mayer transfer operator is known can also be treated within such a setting.

2.7 Linear models: The Hilbert space setting

In Section 2.6 we have seen a general method how to choose the projection map for Ising interactions with a special type of distance function. We will now assume that the distance function is of the form

$$d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \langle \mathbb{B}^{k-1} v \,|\, w \rangle_{\mathcal{H}}$$

where $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a (separable) Hilbert space $(\mathcal{H}, \langle \cdot \,|\, \cdot \rangle_{\mathcal{H}})$ with spectral radius $\rho_{\text{spec}}(\mathbb{B}) < 1$, and $v, w \in \mathcal{H}$. Hence the projection map $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \mathcal{H}$ (Prop. 2.6.6) takes values in a Hilbert space. The corresponding Ruelle-Mayer transfer operator (58) is given as

$$(\mathcal{M}_\beta f)(z) = \int_F \exp\bigl(\beta q(\sigma) + \beta\sigma\langle z|w\rangle\bigr)\, f(\sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma). \tag{59}$$

In this section (Theorem 2.7.6) we will show that for a large class of distance functions a dynamical trace formula holds. The Ruelle-Mayer transfer operator $\mathcal{M}_\beta$ (59) viewed as operator acting on the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ satisfies the trace formula

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \mathbb{B}^n)\,\text{trace}\,(\mathcal{M}_\beta)^n$$

for almost all $n \in \mathbb{N}$. This is one of the main results of this dissertation and shows that the spectrum of the Ruelle-Mayer transfer operator gives a complete description of the sequence of partition functions and thus of many properties of the dynamical system. In Chapter 4 we will use this result to show that the associated dynamical zeta function has a meromorphic continuation to the entire complex plane and an Euler product.
We will now aim for conditions on the distance function ensuring that at least a certain power of the Ruelle-Mayer transfer operator is trace class. We introduce the following classes of distance functions.

Definition 2.7.1. Given a bounded linear operator $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ on a (separable) Hilbert space $\mathcal{H}$, $v, w \in \mathcal{H}$, we define a function $d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \langle \mathbb{B}^{k-1} v \,|\, w \rangle_{\mathcal{H}}$. We define the subspaces $\mathcal{D}_1^{(p)} \subset \ell^1\mathbb{N}$ (for $p \in [1, \infty]$) via

(i) $d \in \mathcal{D}_1^{(p)}$ for $p < \infty$ iff $\rho_{\text{spec}}(\mathbb{B}) < 1$ and $\mathbb{B}$ belongs to the Schatten[22] class $\mathcal{S}_p(\mathcal{H})$,

(ii) $d \in \mathcal{D}_1^{(\infty)}$ iff $\rho_{\text{spec}}(\mathbb{B}) < 1$.

We call $(\mathbb{B}, v, w)$ a *generating triple* for d and $\mathbb{B}$ a *generator*. □

In Example 2.7.7 we will give a list of examples of distance functions belonging to these spaces. These will be investigated in forthcoming sections.
We would like to point out that $\mathcal{D}_1^{(p)}$ (for each $p \leq \infty$) is a complex vector space: Let $(\mathbb{B}_i, v_i, w_i)$ be generating triples for $d_i \in \mathcal{D}_1^{(p)}$ $(i = 1, 2)$, then for any $c \in \mathbb{C}$ the distance function $d_1 + c\,d_2$ has a representation in the Hilbert space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ via $d_1(k) + c\,d_2(k) = \langle \mathbb{B}^{k-1} v \,|\, w \rangle_{\mathcal{H}}$ with $\mathbb{B} := \left(\begin{smallmatrix} \mathbb{B}_1 & \\ & \mathbb{B}_2 \end{smallmatrix}\right)$, $v := \left(\begin{smallmatrix} v_1 \\ cv_2 \end{smallmatrix}\right)$, and $w := \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$. By Proposition 2.3.9 it suffices to construct Ruelle-Mayer transfer operators for each distance function d_i, then by tensorising we obtain a Ruelle-Mayer operator for $d_1 + c\,d_2$.
The spaces $\mathcal{D}_1^{(p)}$ for $p < \infty$ will lead to Schatten class Ruelle-Mayer transfer operators, whereas this fails for $\mathcal{D}_1^{(\infty)}$. This is caused by the qualitative difference of the corresponding operators: Note that an operator $\mathbb{B}$ corresponding to a distance function $d \in \mathcal{D}_1^{(\infty)}$ need not be compact. In Example 2.10.4 we show that there are compact operators such that no power is trace class. On the other hand, let

[22] For the definition of the Schatten classes see Appendix A.2.

$\mathbb{B}$ be an operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$ and $\mathbb{B} \in \mathcal{S}_p(\mathcal{H})$. By the theory of Schatten classes the operator $\mathbb{B}^n$ belongs to $\mathcal{S}_{\max(1,p/n)}(\mathcal{H})$ and by Remark 2.6.4 (iii) we find an index n_0 depending on $\mathbb{B}$ with the property that $\mathbb{B}^n$ is trace class and has operator norm strictly smaller than one for all $n \geq n_0$. This last property is essential for the proof of the dynamical trace formula. For other goals a detailed investigation of (subspaces of) $\mathcal{D}_1^{(\infty)}$ might be advisable.

Proposition 2.7.2. *For all $1 \leq p \leq q \leq \infty$ we have $\mathcal{D}_1^{(p)} \subset \mathcal{D}_1^{(q)} \subset \mathcal{D}_1$, the latter defined in (2.2.4). Moreover, for all $d \in \mathcal{D}_1^{(\infty)}$ we have $\limsup_{k\to\infty} \sqrt[k]{|d(k)|} < 1$.*

Proof. Proposition 2.6.5 implies that $\limsup_{k\to\infty} \sqrt[k]{|d(k)|} < 1$ for all $d \in \mathcal{D}_1^{(\infty)}$, the rest is obvious. □

Remark 2.7.3. Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with distance function $d \in \mathcal{D}_1^{(\infty)}$ (2.7.1). Since we always assume a Hilbert space to be separable, we can assume that $\mathcal{H} = \ell^2\mathbb{N}$ or at least $\mathcal{H} \subseteq \ell^2\mathbb{N}$. Let $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \mathcal{H}$ be as in Proposition 2.6.6. Hence by Corollary 2.6.11 the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ contains a preimage under the composition operator $C_{\pi_{\mathbb{B},v}} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{C}(\Omega),\ f \mapsto f \circ \pi_{\mathbb{B},v}$ of the leading eigenvector of the Ruelle transfer operator $\mathcal{L}_{A_{(\phi)}} : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ defined in (2.1.3). This observation motivates the study of the Ruelle-Mayer transfer operator as an operator acting on the Fock space. □

We will now prove the dynamical trace formula for the Ruelle-Mayer transfer operator $\mathcal{M}$ (59). Remark 2.6.13 gives an explicit formula for the mixed iterates $\mathcal{M}_{(\sigma_1)} \circ \ldots \circ \mathcal{M}_{(\sigma_n)}$ from which $(\mathcal{M}_\beta)^n$ is built up. It implies that it suffices to prove an analogue of the Atiyah-Bott fixed point formula only for the case that the linear map $\mathbb{B} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ is a trace class operator with $\|\mathbb{B}\| < 1$. This will be done in (2.7.4).
The following theorem, which we prove in Appendix B.4.3, will imply that certain powers of the Ruelle-Mayer operator $\mathcal{M} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ (59) are trace class and satisfy a dynamical trace formula.

Theorem 2.7.4. *Let $a,\, b \in \ell^2\mathbb{N}$ and $\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}\| < 1$. Let*

$$T : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (Tf)(z) = e^{\pi\langle z|a\rangle} f(\mathbb{B}z + b)$$

be the corresponding composition operator. Then the trace norm (A.2.2) of T is equal to

$$\|T\|_{\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N}))} = \frac{\exp\Big(\frac{\pi}{2}\|a\|^2 + \frac{\pi}{2}\|(1 - \mathbb{B}\mathbb{B}^\star)^{-1/2}(\mathbb{B}a + b)\|^2\Big)}{\det(1 - |\mathbb{B}|)}$$

and T is trace class with the Atiyah-Bott fixed point formula

$$\operatorname{trace} T = \frac{\exp\big(\pi\langle(1-\mathbb{B})^{-1}b|a\rangle\big)}{\det(1-\mathbb{B})}.$$

□

This theorem together with Remark 2.6.4 enables us to weaken the spectral conditions on the linear map $\mathbb{B}$ such that for all $d \in \mathcal{D}_1^{(p)}$ with $p < \infty$ we will get a Ruelle-Mayer operator for which a dynamical trace formula holds at least for almost all powers.

Lemma 2.7.5. *Let $F \subset \mathbb{C}$ be a bounded set with a finite Borel measure ν, $q \in \mathcal{C}_b(F)$, $\mathbb{B} \in \mathcal{S}_p(\ell^2\mathbb{N})$ for some $1 \leq p < \infty$ and $\rho_{\text{spec}}(\mathbb{B}) < 1$, and $v,\, w \in \ell^2\mathbb{N}$. Let*

$$\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\big(\beta q(\sigma) + \beta\sigma\langle z|w\rangle\big)\, f(\sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

be the Ruelle-Mayer operator (59). Then there exists $n \in \mathbb{N}$ such that the operator $(\mathcal{M}_\beta)^n$ is trace class.

Proof. For all $\sigma \in F$ we set

$$\mathcal{M}_{\beta,\sigma} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_{\beta,\sigma} g)(z) := \exp\big(\beta q(\sigma) + \beta\sigma\langle z|w\rangle\big)\, g(\sigma\, v + \mathbb{B}z).$$

For a moment we assume that $\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}\| < 1$. Then, by Theorem 2.7.4 the composition operator $\mathcal{M}_{\beta,\sigma}$ is trace class with trace norm given as

$$\|\mathcal{M}_{\beta,\sigma}\|_{\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N}))} = \frac{\exp\Big(\mathbf{Re}(\beta q(\sigma)) + \frac{\pi}{2}\|\frac{\overline{\sigma\beta}}{\pi}w\|^2 + \frac{\pi}{2}\|(1-\mathbb{B}\mathbb{B}^\star)^{-1/2}(\frac{\overline{\sigma\beta}}{\pi}\mathbb{B}w + \sigma v)\|^2\Big)}{\det(1-|\mathbb{B}|)} \tag{60}$$

and satisfies the Atiyah-Bott fixed point formula (53). The function $F \to \mathbb{R},\ \sigma \mapsto \|\mathcal{M}_{\beta,\sigma}\|_{\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N}))}$ is a bounded function, hence integrable with respect to a finite measure. By Theorem A.7.6 the operator $\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ is trace class. Now we return to the general case: By Remark 2.6.4 there exists $k_0 \in \mathbb{N}$ such that $\|\mathbb{B}^k\| < 1$ for all $k \geq k_0$. For all $k \geq p$ we have $\mathbb{B}^k \in \mathcal{S}_{\max(1,p/k)}(\ell^2\mathbb{N}) = \mathcal{S}_1(\ell^2\mathbb{N})$. We let $n = \max(\lceil p \rceil, k_0)$. Then for all choices of $\sigma_1, \ldots, \sigma_n \in F$ the operator $\mathcal{M}_{\beta,\sigma_1} \circ \ldots \circ \mathcal{M}_{\beta,\sigma_n} \in \mathrm{End}(\mathcal{F}(\ell^2\mathbb{N}))$ acts by Remark 2.6.13 via

$$(\mathcal{M}_{\beta,\sigma_1}\circ..\circ\mathcal{M}_{\beta,\sigma_n} g)(z) = \exp\Big(\beta\sum_{k=1}^{n} q(\sigma_k) + \beta\sum_{k=1}^{n}\sigma_k\langle\mathbb{B}^{n-k}z + \sum_{j=1}^{n-k}\sigma_{k+j}\mathbb{B}^{j-1}v|w\rangle\Big) g(\mathbb{B}^n z + \sum_{k=1}^{n}\sigma_k\mathbb{B}^{k-1}v).$$

The above argument shows that $\mathcal{M}_{\beta,\sigma_1} \circ \ldots \circ \mathcal{M}_{\beta,\sigma_n} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ is trace class and hence by Theorem A.7.6 the operator $(\mathcal{M}_\beta)^n$ is trace class. □

Since $\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N})) \subset \mathrm{End}(\mathcal{F}(\ell^2\mathbb{N}))$ is an operator ideal, all higher iterates $(\mathcal{M}_\beta)^m$ for $m \geq n$ are trace class.
The exact formula (60) for the trace norm of the operators $\mathcal{M}_{\beta,\sigma}$ allows to weaken the condition on the measure ν and on the boundedness of $F \subset \mathbb{C}$. This will be used in Proposition 2.10.7.
We now can easily prove our main theorem of this section which states that for all Ising interactions with distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ a dynamical trace formula holds at least for almost all $n \in \mathbb{N}$. In the following sections we will apply this theorem for instance to finite range interactions, superexponentially decaying interactions, and polynomial-exponentially decaying interactions.

Theorem 2.7.6. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1), say $d(k) = \langle\mathbb{B}^{k-1}v|w\rangle_{\ell^2\mathbb{N}}$. Then there exists an index $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geq n_0$ the Ruelle-Mayer transfer operator*

$$\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\big(\beta q(\sigma) + \beta\sigma\langle z|w\rangle\big)\, f(\sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1-\mathbb{B}^n)\,\mathrm{trace}\,(\mathcal{M}_\beta)^n.$

Proof. Set $\pi_{\mathbb{B},v} : \Omega \to \ell^2\mathbb{N}$ (Prop. 2.6.6), $A_\sigma : \ell^2\mathbb{N} \to \mathbb{C},\ z \mapsto \beta\, q(\sigma) + \sigma\,\beta\langle z|w\rangle$ and $\psi_\sigma : \ell^2\mathbb{N} \to \ell^2\mathbb{N},\ z \mapsto \sigma\, v + \mathbb{B}z$. This gives an (S1) - (S3) Ansatz (2.3.3) by Proposition 2.6.6. By Lemma 2.7.5 there exists $n_0 \in \mathbb{N}$ such that $(\mathcal{M}_\beta)^n$ is trace class for all $n \geq n_0$. The operator $(\mathcal{M}_\beta)^n$ is an n-fold integral over the family of composition operators $\mathcal{M}_{\beta,\sigma_1} \circ \ldots \circ \mathcal{M}_{\beta,\sigma_n} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$. By Remark 2.6.13 and Theorem 2.7.4 they satisfy the Atiyah-Bott fixed point formula (53). The trace formula for $(\mathcal{M}_\beta)^n$ now follows from Theorem 2.4.6 and Remark 1.11.5. □

In particular, given a generating triple $(\mathbb{B}, v, w)$, Theorem 2.7.6 directly constructs the corresponding Ruelle-Mayer transfer operator. In the next section we will discuss how transfer operators corresponding to different generating triples are related. The following example gives a list of the types of distance functions which can be treated with our method. By the remark following Definition 2.7.1 all finite superpositions of distance functions from these classes lead to Ruelle-Mayer transfer operators with dynamical trace formula.

Example 2.7.7. (i) Finite range (Section 2.8): Let $d : \mathbb{N} \to \mathbb{C}$ be a distance function with finite range ρ_0, i.e. $d(k) = 0$ for all $k > \rho_0$. Then[23]

$$d(k) = \left\langle (\mathbb{S}_{\rho_0})^{k-1} v^d \,\middle|\, e_1 \right\rangle,$$

where $\mathbb{S}_{\rho_0} := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathrm{Mat}(\rho_0, \rho_0; \mathbb{Z})$ is the standard ρ_0-step nilpotent matrix and $v^d = (d(1), \dots, d(\rho_0))^\top \in \mathbb{C}^{\rho_0}$.

(ii) Superexponential (Section 2.9): Let $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto a(k) \exp(-\gamma\, k^\delta)$, where $\gamma > 0$, $\delta > 1$ and $a : \mathbb{N} \to \mathbb{C}$ is a lower order term[24] in the sense that $\lim_{k\to\infty} a(k) \exp(-\epsilon_1\, k^{\epsilon_2}) = 0$ for all ϵ_1, $\epsilon_2 > 0$. Then

$$d(k) = \left\langle \mathbb{S}^{k-1} \tilde{v}^d \,\middle|\, e_1 \right\rangle,$$

where $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $(\mathbb{S}z)_k := \frac{\exp(\gamma (k-1)^\delta)}{\exp(\gamma\, k^\delta)}\, z_{k+1}$, and $\tilde{v}^d : \mathbb{N} \to \mathbb{C}$, $\tilde{v}^d_k := \frac{\exp(\gamma (k-1)^\delta)}{\exp(\gamma\, k^\delta)}\, a(k)$.

(iii) The classes $\mathcal{D}_1^{(p),\Delta} \subset \mathcal{D}_1^{(p)}$ (Section 2.10): Let $\lambda \in \ell^p\mathbb{N}$ for some $1 \le p < \infty$ with $\|\lambda\|_{\ell^\infty\mathbb{N}} < 1$ and $c : \mathbb{N} \to \mathbb{C}$ such that $\sqrt{c\lambda} : \mathbb{N} \to \mathbb{C}$, $n \mapsto (c_n\, \lambda_n)^{1/2}$ belongs to $\ell^2\mathbb{N}$. Set

$$d(k) = \sum_{i=1}^{\infty} c_i\, \lambda_i^k = \left\langle\, \mathrm{diag}(\lambda)^{k-1} \sqrt{c\lambda} \mid \overline{\sqrt{c\lambda}}\, \right\rangle..$$

(iv) Polynomial-exponential (Section 2.11): Let $c = (c_0, \dots, c_p) \in \mathbb{C}^{p+1}$, $0 < |\lambda| < 1$ and $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto \lambda^k \sum_{i=0}^{p} c_i\, k^i$. Then

$$d(k) = \left\langle \lambda (\lambda \mathbb{B}^{(p+1)})^{k-1} \underline{1} \,\middle|\, \overline{c} \right\rangle,$$

where the matrix $\mathbb{B}^{(p+1)} \in \mathrm{Gl}(p+1; \mathbb{C})$ is given in Remark 2.11.1 and $\underline{1} : \{0, \dots, p\} \to \mathbb{C}$ is the constant function one. □

Given a generating triple $(\mathbb{B}, v, w)$, Proposition 2.6.6 shows that $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N}$, $\xi \mapsto \sum_{j=1}^\infty \xi_j \mathbb{B}^{j-1} v$ is a suitable projection map for a (S1) - (S3) Ansatz (2.3.3). For the sake of completeness and in order to simplify a comparison with the literature we list the corresponding projection maps for the known examples and and compute their coefficients explicitly. Hence the reader who is familiar with the literature can find a new interpretation of the old results.

Corollary 2.7.8. *Let $d : \mathbb{N} \to \mathbb{C}$ be a finite range distance function, say $d(k) = 0$ for all $k > \rho_0$, and $v^d := (d(1), \dots, d(\rho_0))^\top \in \mathbb{C}^{\rho_0}$. Let $\mathbb{S}_{\rho_0} \in \mathrm{Mat}(\rho_0, \rho_0; \mathbb{Z})$ be the standard ρ_0-step nilpotent matrix from Example 2.7.7 (i). Then*

$$\pi_{\mathbb{S}_{\rho_0}, v^d} : \ell^\infty\mathbb{N} \to \mathbb{C}^{\rho_0},\ \xi \mapsto \sum_{j=1}^{\infty} \xi_j\, \mathbb{S}_{\rho_0}^{j-1} v^d = \begin{pmatrix} \langle \xi, \tau^0 d\rangle \\ \vdots \\ \langle \xi, \tau^{\rho_0 - 1} d\rangle \end{pmatrix}.$$

Proof. For all $1 \le k < \rho_0$ we have $(\mathbb{S}_{\rho_0})^\top e_k = e_{k+1}$ and hence, by iteration, $e_k = (\mathbb{S}_{\rho_0}^\top)^{k-1} e_1 = (\mathbb{S}_{\rho_0}^{k-1})^\top e_1$. We compute the coefficients of the projection map $\pi_{\mathbb{S}_{\rho_0}, v^d}(\xi) = \sum_{j=1}^\infty \xi_j\, \mathbb{S}_{\rho_0}^{j-1} v^d$ as

$$\begin{aligned} \langle \pi_{\mathbb{S}_{\rho_0}, v^d}(\xi) | e_k \rangle &= \sum_{j=1}^{\infty} \xi_j\, \langle \mathbb{S}_{\rho_0}^{j-1} v^d \mid e_k \rangle = \sum_{j=1}^{\infty} \xi_j\, \langle \mathbb{S}_{\rho_0}^{j-1} v^d \mid (\mathbb{S}_{\rho_0}^{k-1})^\top e_1 \rangle = \sum_{j=1}^{\infty} \xi_j\, \langle \mathbb{S}_{\rho_0}^{j+k-2} v^d \mid e_k \rangle \\ &= \sum_{j=1}^{\infty} \xi_j\, d(k+j-1), \end{aligned}$$

which coincides with $\langle \pi_{\tau,d}(\xi) \mid e_k \rangle = \langle \pi^d(\xi) \mid e_k \rangle = \pi^d_k(\xi)$ from Remark 2.1.6. □

[23] In Section 2.8 we will find a better representation of finite-range distance functions which uses a generator of smaller operator norm.

[24] The assumptions on the lower order term can be weakened.

The following result is an immediate consequence of the definitions.

Proposition 2.7.9. *Let $d \in \mathcal{D}_1^{(\infty),\Delta}$ (2.10.1), say $d(j) = \sum_{i=1}^{\infty} c_i \, \lambda_i^j$ for all $j \in \mathbb{N}$. Then*

$$\pi_{\operatorname{diag}(\lambda),\sqrt{c\lambda}} : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N},\ \xi \mapsto \sum_{j=1}^{\infty} \operatorname{diag}(\lambda)^{j-1}\sqrt{c\lambda}$$

is a continuous linear map with components $\langle \pi_{\operatorname{diag}(\lambda),\sqrt{c\lambda}}(\xi) \,|\, e_i\rangle = c_i^{1/2} \sum_{j=1}^{\infty} \xi_j \, \lambda_i^{j-1/2}$. □

For the superexponentially decaying interactions we have to cite some results from Section 2.9.

Proposition 2.7.10. *Let $d : \mathbb{N} \to \mathbb{C}$, $\tilde{v}^d : \mathbb{N} \to \mathbb{C}$, and $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ be as in Example 2.9.4. Then*

$$\pi_{\mathbb{S},\tilde{v}^d} : \ell^\infty\mathbb{N} \to \ell^2\mathbb{N}, \quad \xi \mapsto \sum_{k=1}^{\infty} \xi_k \, \mathbb{S}^{k-1}\tilde{v}^d$$

has the components $\langle \pi_{\mathbb{S},\tilde{v}^d}(\xi) \,|\, e_k\rangle = \exp(\gamma(k-1)^\delta) \sum_{j=1}^{\infty} \xi_j \, a(j+k-1) \exp(-\gamma(j+k-1)^\delta)$.

Proof. By Proposition 2.9.2 we can compute the coefficients of $\pi_{\mathbb{S},\tilde{v}^d}$ explicitly as

$$\langle \pi_{\mathbb{S},\tilde{v}^d}(\xi) \,|\, e_j\rangle = \sum_{k=1}^{\infty} \xi_k \, \langle \mathbb{S}^{k-1}\tilde{v}^d \,|\, e_j\rangle = \sum_{k=1}^{\infty} \xi_k \, \frac{\exp(\gamma(j-1)^\delta)}{\exp(\gamma\,(j+k-1)^\delta)} \, a(j+k-1).$$

□

In the following we will construct trace class Ruelle-Mayer transfer operators for finite range interactions (2.8) and superexponentially decaying Ising interactions (2.9), $\mathcal{D}_1^{(p),\Delta}$ Ising interactions 2.10, and polynomial-exponentially decaying Ising interactions 2.11. In view of Theorem 2.7.6 it suffices to find a representation via a generating triple $(\mathbb{B}, v, w)$, i. e., to prove Example 2.7.7. In doing so we will investigate the limitations of the methods used for each type of distance function.

2.8 Ruelle-Mayer transfer operators for finite range Ising interactions

In this section we return to finite range interactions, for which we gave in Section 2.5 a full description. We will now specialise to the case of Ising interactions, which can also be treated with the methods from Section 2.7. Let $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with a finite range distance function $d : \mathbb{N} \to \mathbb{C}$, say $d(k) = 0$ for all $k > \rho_0$, and potential $q \in \mathcal{C}_b(F)$. We will construct a trace class Ruelle-Mayer transfer operator such that the dynamical trace formula holds for all $n \in \mathbb{N}$.
We state some obvious facts as a proposition without proof.

Proposition 2.8.1. *For $\rho_0 \in \mathbb{N}_{>1}$ let*

$$\mathbb{S}_{\rho_0} := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \operatorname{Mat}(\rho_0, \rho_0; \mathbb{Z}). \tag{61}$$

(i) *Then $\mathbb{S}_{\rho_0}$ is a ρ_0-step nilpotent matrix, i. e., $(\mathbb{S}_{\rho_0})^{\rho_0-1} \neq 0$ and $(\mathbb{S}_{\rho_0})^{\rho_0} = 0$. In particular, the continuous linear map $\mathbb{S}_{\rho_0} : \mathbb{C}^{\rho_0} \to \mathbb{C}^{\rho_0}$ has spectral radius $\rho_{\text{spec}}(\mathbb{S}_{\rho_0}) = 0$.*

(ii) *Let $d : \mathbb{N} \to \mathbb{C}$ be a finite range distance function, say $d(k) = 0$ for all $k > \rho_0$, and $v^d := (d(1), \ldots, d(\rho_0))^\top \in \mathbb{C}^{\rho_0}$. Then $d(k) = \big\langle (\mathbb{S}_{\rho_0})^{k-1} v^d \,\big|\, e_1 \big\rangle$ for all $k \in \mathbb{N}$.* □

We will call $\mathbb{S}_{\rho_0}$ the *standard ρ_0-step nilpotent matrix*. The disadvantage of the matrix $\mathbb{S}_{\rho_0}$ is that the matrices $(\mathbb{S}_{\rho_0})^k$ (for $k = 0, \ldots, \rho_0 - 1$) have operator norm equal to one. Hence the dynamical trace formula for the Ruelle-Mayer transfer operator built from $\mathbb{S}_{\rho_0}$

$$\mathcal{M}_\beta : \mathcal{C}_b(\mathbb{C}^{\rho_0}) \to \mathcal{C}_b(\mathbb{C}^{\rho_0}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\big(\beta q(x) + \beta\, z_1\big)\, f(x\, v^d + \mathbb{S}_{\rho_0} z)\, d\nu(x) \tag{62}$$

only holds for all $n \geq \rho_0$. In this finite-dimensional setting we can avoid this by the following trick.

Proposition 2.8.2. *Let $0 < \lambda < 1$ and $\mathbb{S}_{\rho_0}$ be the standard ρ_0-step nilpotent matrix from (61). Let $d : \mathbb{N} \to \mathbb{C}$ be a finite range distance function, say $d(k) = 0$ for all $k > \rho_0$, and $w^d \in \mathbb{C}^{\rho_0}$ with entries $w^d(k) = \lambda^{1-k}\, d(k)$. Then $d(k) = \langle (\lambda \mathbb{S}_{\rho_0})^{k-1} w^d \,|\, e_1 \rangle$ for all $k \in \mathbb{N}$.* □

An immediate consequence of Proposition 2.8.2 and Theorem 2.7.6 is the following trace formula:

Corollary 2.8.3. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with finite range distance function $d : \mathbb{N} \to \mathbb{C}$, say $d(k) = 0$ for all $k > \rho_0$, and potential $q \in \mathcal{C}_b(F)$. Let $0 < \lambda < 1$, $w^d \in \mathbb{C}^{\rho_0}$ with entries $w^d(k) = \lambda^{1-k}\, d(k)$, and $\mathbb{S}_{\rho_0} \in \mathrm{Mat}(\rho_0, \rho_0; \mathbb{Z})$ be as in (61). Then for all $m \in \mathbb{N}$ the Ruelle-Mayer transfer operator*

$$\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^{\rho_0}) \to \mathcal{F}(\mathbb{C}^{\rho_0}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\bigl(\beta q(x) + \beta\, z_1\bigr)\, f(x\, w^d + \lambda \mathbb{S}_{\rho_0} z)\, d\nu(x)$$

satisfies the dynamical trace formula $\widetilde{Z}_m^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,m\}}(\beta) = \mathrm{trace}\ (\mathcal{M}_\beta)^m$.

Proof. By Proposition 2.8.1 we can write $d(k) = \langle (\lambda \mathbb{S}_{\rho_0})^{k-1} w^d \,|\, e_1 \rangle$ for all $k \in \mathbb{N}$. The linear map $\lambda \mathbb{S}_{\rho_0}$ defined on the finite dimensional space $\mathbb{C}^{\rho_0}$ is automatically trace class and has operator norm equal to λ. Hence the assertion follows from Theorem 2.7.6. The determinant factor in the dynamical trace formula obviously vanishes. □

The naive generalisation of this result to the case of long range interactions fails, since the shift matrix $\mathbb{S}_{\rho_0}$ is trace class precisely if the interaction range ρ_0 is finite. The infinite analogue of $\mathbb{S}_{\rho_0}$ is the shift map $\tau : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $(\tau\xi)_n := \xi_{n+1}$ from Remark 2.1.5, which has operator norm equal to one and is not trace class. In the next section we will replace the shift by a so called weighted shift which allows us to treat a certain class of long range interactions, namely the superexponentially decaying ones.

2.9 Ruelle-Mayer transfer operators for superexponentially decaying Ising interactions

In this section we will study long range Ising two-body interactions with superexponentially decaying distance function of a special type. Our class contains in particular distance functions of the following form:

$$d : \mathbb{N} \to \mathbb{C},\ k \mapsto a(k)\, \exp(-\gamma\, k^\delta),$$

where $\gamma > 0$, $\delta > 1$ and $a : \mathbb{N} \to \mathbb{C}$ is a lower order term, in the sense that $\lim_{k\to\infty} a(k)\, \exp(-\epsilon_1\, k^{\epsilon_2}) = 0$ for all ϵ_1, $\epsilon_2 > 0$. This interaction has been investigated by D. Mayer in [May80a, p. 100]. He worked with Banach space techniques and nuclear operators, whereas we will use the methods from Section 2.7 and Hilbert space techniques. The essential step in our approach is the finding of a suitable generating triple for a given distance function. Then by Theorem 2.7.6, a possibly large power (which depends on the spectral properties of the generator) of the Ruelle-Mayer transfer operator is trace class and satisfies the dynamical trace formula. We will examine the generating triple in detail which leads to a slightly larger class of distance functions which can be represented via a so called weighted shift. In particular for distance functions of the Mayer type introduced above, our transfer operator is trace class. In this section we restrict to Ising interactions, but allow spin values in a bounded subset $F \subset \mathbb{C}$. By Section 2.13 our results will extend to the case of arbitrary Ising type interactions. If the alphabet F is finite, then every interaction matrix is of Ising type, and hence we can reproduce D. Mayer's result who worked with a finite alphabet.
The operator $\mathbb{S}$ introduced next will serve as a generator for superexponentially decaying distance functions d. In Proposition 2.9.2 we show a representation $d(k) = \langle \mathbb{S}^{k-1} v | w \rangle$. Using the methods of Section 2.7 we obtain the dynamical trace formula in Corollary 2.9.3.

Proposition 2.9.1. *Let $g : \mathbb{N} \to \mathbb{C}\backslash\{0\}$ with $\sum_{k=1}^\infty \bigl|\frac{g(k)}{g(k+1)}\bigr|^p < \infty$. We define $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $(\mathbb{S}z)_k := \frac{g(k)}{g(k+1)}\, z_{k+1}$ and call it the weighted shift operator. Then:*

(i) *$\mathbb{S}$ leaves invariant the spaces $\ell^q\mathbb{N}$ for $1 \le q < \infty$ and defines continuous operators on these spaces with $\|\mathbb{S}\|_{\ell^q\mathbb{N} \to \ell^q\mathbb{N}} \le \sup_{k\in\mathbb{N}} \bigl|\frac{g(k)}{g(k+1)}\bigr|$ for all $1 \le q \le \infty$,*

(ii) *For all* $z \in \ell^p\mathbb{N}$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, *we have* $(\mathbb{S}^n z)_k = \frac{g(k)}{g(k+n)}\, z_{k+n}$,

(iii) $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ *belongs to the Schatten class* $\mathcal{S}_p(\ell^2\mathbb{N})$, *is not normal, and* $\rho_{\mathrm{spec}}(\mathbb{S}) < 1$.

Proof. Let $1 \le q < \infty$ and $z \in \ell^q\mathbb{N}$, then

$$\|\mathbb{S}z\|^q_{\ell^q\mathbb{N}} = \sum_{k=1}^{\infty} \left| \frac{g(k)}{g(k+1)}\, z_{k+1} \right|^q \le \sup_{k\in\mathbb{N}} \left| \frac{g(k)}{g(k+1)} \right|^q \|z\|^q_{\ell^q\mathbb{N}}.$$

We show assertion (ii) by induction. The case $n = 0$ is trivial. For the induction step observe that

$$(\mathbb{S}^{n+1}z)_k = \frac{g(k)}{g(k+1)}\,(\mathbb{S}^n z)_{k+1} = \frac{g(k)}{g(k+1)}\,\frac{g(k+1)}{g(k+n+1)}\, z_{k+n+1} = \frac{g(k)}{g(k+n+1)}\, z_{k+n+1}.$$

From

$$\langle Sz \,|\, w\rangle = \sum_{k=1}^{\infty} \frac{g(k)}{g(k+1)}\, z_{k+1}\, \overline{w_k} = z_1 \cdot 0 + \sum_{k=1}^{\infty} z_k\, \overline{\frac{g(k-1)}{g(k)}\, w_{k-1}} = \langle z \,|\, S^* w\rangle$$

one obtains the $\ell^2\mathbb{N}$-adjoint $\mathbb{S}^*$ of $\mathbb{S}$ as

$$\mathbb{S}^* : \ell^2\mathbb{N} \to \ell^2\mathbb{N},\ (\mathbb{S}^*\xi)_i = \begin{cases} 0, & i = 1, \\ \frac{g(i-1)}{g(i)}\, \xi_{i-1}, & i \ge 2. \end{cases}$$

In particular, $((\mathbb{S}\mathbb{S}^*)(\xi))_k = \left|\frac{g(k)}{g(k+1)}\right|^2 \xi_k$, which shows that $\mathbb{S}\mathbb{S}^*$ is diagonal with respect to the standard basis. We can read off the singular numbers of $\mathbb{S}$ being the square roots of the diagonal entries. By assumption they belong to $\ell^p\mathbb{N}$. On the other hand

$$((\mathbb{S}^*\mathbb{S})(\xi))_k = \begin{cases} 0, & i = 1, \\ \left|\frac{g(i-1)}{g(i)}\right|^2 \xi_i, & i \ge 2. \end{cases}$$

The operator norm of $\mathbb{S}^n$ is bounded by $\sup_{k\in\mathbb{N}} \left|\frac{g(k)}{g(k+n)}\right|$. The sequence $k \mapsto \left|\frac{g(k)}{g(k+1)}\right|$ tends to zero, hence one can find $k_0 \in \mathbb{N}$ such that $\left|\frac{g(k)}{g(k+1)}\right| < \frac{1}{2}$ for all $k \ge k_0$. Let $C = \max_{k=1,\dots,k_0} \left|\frac{g(k)}{g(k+1)}\right|$. Then for all $k \in \mathbb{N}$ one has

$$\left|\frac{g(k)}{g(k+n)}\right| = \left|\frac{g(k)}{g(k+1)}\frac{g(k+1)}{g(k+2)} \cdots \frac{g(k_0)}{g(k_0+1)}\frac{g(k_0+1)}{g(k_0+2)} \cdots \frac{g(k+n-1)}{g(k+n)}\right| \le \left(\frac{1}{2}\right)^{n-k_0+k} C^{k_0-1},$$

which tends to zero as $n \to \infty$. Hence we can find $n \in \mathbb{N}$ such that $\|\mathbb{S}^n\| < 1$ and hence the spectral radius $\rho_{\mathrm{spec}}(\mathbb{S})$ of $\mathbb{S}$ is less than one. □

For any non-vanishing sequence $s \in \ell^p\mathbb{N}$ one gets by setting $g : \mathbb{N} \to \mathbb{C},\ g(k) := \left(\prod_{l=1}^{k-1} s(l)\right)^{-1}$ a function g as required in Proposition 2.9.1. In particular, $s(k) = \frac{g(k)}{g(k+1)}$ and $|s| : \mathbb{N} \to \mathbb{C},\ n \mapsto |s(n)|$ is the sequence of singular numbers of the corresponding weighted shift operator. A typical function $g : \mathbb{N} \to \mathbb{C}$ satisfying the summability condition $\sum_{k=1}^{\infty} \left|\frac{g(k)}{g(k+1)}\right|^p < \infty$ is, for instance, $g(k) = \exp(\gamma k^\delta)$ with $\gamma > 0$, $\delta > 1$. In Remark 2.9.5 we will explain the notion of a weighted shift operator.
The interest in the map $\mathbb{S}$ is that it allows to express the superexponentially decreasing distance function d in such a way that we can apply the general theory from Section 2.7.

Proposition 2.9.2. *Let* $g : \mathbb{N} \to \mathbb{C} \setminus \{0\}$ *and* $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ *be as in Proposition 2.9.1 and* $d : \mathbb{N} \to \mathbb{C}$ *such that* $\tilde{v}^d : \mathbb{N} \to \mathbb{C},\ \tilde{v}^d_k := g(k)\, d(k)$ *belongs to* $\ell^2\mathbb{N}$. *Then*

$$d(k) = \frac{1}{g(1)} \left\langle \mathbb{S}^{k-1}\tilde{v}^d \,\middle|\, e_1 \right\rangle_{\ell^2\mathbb{N}}.$$

Proof. As a consequence of Proposition 2.9.1 (ii) we have

$$(\mathbb{S}^n \tilde{v}^d)_l = \frac{g(k)}{g(l+n)}\, \tilde{v}^d_{l+n} = \frac{g(l)}{g(l+n)}\, g(l+n)\, d(l+n)$$

for all $n \in \mathbb{N}_0$, $l \in \mathbb{N}$, which immediately implies that $\left\langle \mathbb{S}^{k-1}\tilde{v}^d \,\middle|\, e_1 \right\rangle = (\mathbb{S}^{k-1}\tilde{v}^d)_1 = g(1)\, d(k)$. □

From the above propositions and Theorem 2.7.6 we obtain the dynamical trace formula for the Ruelle-Mayer transfer operator.

Corollary 2.9.3. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function $d : \mathbb{N} \to \mathbb{C}$ given as follows: There exists $g : \mathbb{N} \to \mathbb{C} \setminus \{0\}$ with $\sum_{k=1}^{\infty} \big|\frac{g(k)}{g(k+1)}\big|^p < \infty$ and $g(1) = 1$ such that $\tilde{v}^d : \mathbb{N} \to \mathbb{C}$, $\tilde{v}^d_k := g(k)\, d(k)$ belongs to $\ell^2\mathbb{N}$. Let $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $(\mathbb{S}z)_k := \frac{g(k)}{g(k+1)}\, z_{k+1}$. Then there exists an index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the Ruelle-Mayer transfer operator*

$$\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\big(\beta\, q(\sigma) + \beta\, \sigma\, z_1\big)\, f(\sigma\, \tilde{v}^d + \mathbb{S}z)\, d\nu(\sigma) \tag{63}$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \mathrm{trace}\,(\mathcal{M}_\beta)^n$.

Proof. By Proposition 2.9.2 we can write $d(k) = \big\langle \mathbb{S}^{k-1}\tilde{v}^d \,\big|\, e_1 \big\rangle$ for all $k \in \mathbb{N}$, where $\mathbb{S} \in \mathcal{S}_p(\ell^2\mathbb{N})$ with $\rho_{\mathrm{spec}}(\mathbb{S}) < 1$ by Proposition 2.9.1. With respect to the standard basis of $\ell^2\mathbb{N}$ the operator $\mathbb{S}$ is an upper triangular matrix with zeros along the diagonal, hence $\det(1 - \mathbb{S}^n) = 1$ for all $n \geq n_0$. The assertion follows from Theorem 2.7.6. ☐

This result is similar to Corollary 2.8.3 for a finite range interaction, since there is no determinant factor in the trace formula, which appears for instance in Corollary 2.10.5 for exponentially decaying interactions. At this point our Hilbert space approach seems to be more effective than the Banach space approach of D. Mayer [May80a, p. 106], since we can easily see that the determinant factor is equal to one. This will lead in Section 4.2 to a simpler form of the corresponding dynamical zeta function.
We will now investigate which distance functions $d : \mathbb{N} \to \mathbb{C}$ satisfy the assumptions of Corollary 2.9.3 and give an example first which is due to D. Mayer [May80a, p. 100].

Example 2.9.4. Consider the distance function d given as $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto a(k)\, \exp(-\gamma\, k^\delta)$, where $\gamma > 0$, $\delta > 1$ and $a : \mathbb{N} \to \mathbb{C}$ is a lower order term, in the sense that $\lim_{k\to\infty} a(k)\, \exp(-\epsilon_1\, k^{\epsilon_2}) = 0$ for all $\epsilon_1, \epsilon_2 > 0$. We show that d has a representation as required in Corollary 2.9.3. Let $g : \mathbb{N} \to \mathbb{C}$, $k \mapsto \exp(\gamma(k-1)^\delta)$. Then $g(1) = 1$ and g satisfies the summability condition: For $\delta > 1$, $j, k \geq 1$ we have $(j+k)^\delta - k^\delta = (j+k)\,(j+k)^{\delta-1} - k\, k^{\delta-1} \geq (j+k-k)\, k^{\delta-1} = j\, k^{\delta-1}$. Hence

$$\sum_{k=1}^{\infty} \Big|\frac{\exp(\gamma(k-1)^\delta)}{\exp(\gamma\, k^\delta)}\Big|^p = \sum_{k=1}^{\infty} \exp(-\gamma\, p\, (k^\delta - (k-1)^\delta)) \leq \sum_{k=0}^{\infty} \exp(-\gamma\, p\, k^{\delta-1}), \tag{64}$$

which is finite by Proposition 2.2.6 for all $p > 0$. Hence the corresponding weighted shift operator $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ is trace class. Moreover, $\mathbb{S}$ has operator norm bounded by $\exp(-\gamma) < 1$, hence by Theorem 2.7.4 the Ruelle-Mayer transfer operator is trace class. It remains to show that $\tilde{v}^d \in \ell^2\mathbb{N}$. We proceed similar to the previous estimate (64). For $0 < \epsilon_1 < \gamma$, $0 < \epsilon_2 \leq \delta - 1$, by our assumptions on the lower order term a we can find a constant $C > 0$ such that

$$\|\tilde{v}^d\|_{\ell^1\mathbb{N}} = \sum_{k=1}^{\infty} \exp(-\gamma(k^\delta - (k-1)^\delta))\, |a(k)| \leq C \sum_{k=1}^{\infty} \exp(-\gamma k^{\delta-1} + \epsilon_1 k^{\epsilon_2}) \leq C \sum_{k=1}^{\infty} \exp(-(\gamma - \epsilon_1) k^{\delta-1}),$$

which is finite by Proposition 2.2.6. Hence $\tilde{v}^d \in \ell^1\mathbb{N} \subset \ell^2\mathbb{N}$. In order to apply the methods of Section 2.7 it would be sufficient that $\tilde{v}^d \in \ell^2\mathbb{N}$ and (64) for some $p < \infty$. These observations allow to weaken the conditions on the lower order term. For instance the sequence a might grow like $k \mapsto \exp(\gamma\, k^{\delta-1-\epsilon})$ for all $\epsilon > 0$. ☐

Next we explain the notion of a weighted shift operator and discuss the possible weights g (and the possible distance functions d) for which the Ruelle-Mayer transfer operator satisfies a dynamical trace formula. It turns out that this approach is limited to superexponentially decaying interactions.

Remark 2.9.5. Let $g : \mathbb{N} \to \mathbb{C} \setminus \{0\}$ and $\mathbb{S} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $(\mathbb{S}z)_k := \frac{g(k)}{g(k+1)}\, z_{k+1}$. The operator $\mathbb{S}$ is a weighted shift operator, in the sense that it acts as a left shift composed with a diagonal operator.

Formal conjugation by the (possibly unbounded) operator $\operatorname{diag}(g) : \ell^2\mathbb{N} \to \ell^2\mathbb{N},\ (\operatorname{diag}(g)z)_k := g(k)\, z_k$ gives

$$(\operatorname{diag}(g)^{-1}\, \mathbb{S} \operatorname{diag}(g)z)_k = g(k)^{-1}\, \frac{g(k)}{g(k+1)}\, g(k+1)\, z_{k+1} = z_{k+1}.$$

Hence the weighted shift $\mathbb{S}$ is conjugate to the (unweighted) shift $\tau : \ell^2\mathbb{N} \to \ell^2\mathbb{N},\ (\tau z)_k := z_{k+1}$ from Remark 2.1.5. We recall from Remark 2.6.7 the trivial representation $d(k) = \langle \tau^{k-1} d, e_1\rangle_{\ell^1\mathbb{N},\ell^\infty\mathbb{N}} = \langle \tau^{k-1} d | e_1\rangle_{\ell^2\mathbb{N}}$. By formal calculation we obtain

$$d(k) = \left\langle \tau^{k-1} d \,\middle|\, e_1 \right\rangle_{\ell^2\mathbb{N}} = \left\langle \operatorname{diag}(g)^{-1}\, \mathbb{S}^{k-1} \operatorname{diag}(g) d \,\middle|\, e_1 \right\rangle_{\ell^2\mathbb{N}} = \left\langle \mathbb{S}^{k-1} \operatorname{diag}(g) d \,\middle|\, \operatorname{diag}(g)^{-1} e_1 \right\rangle.$$

In order to fulfill the premisses of Theorem 2.7.6 we have to investigate the spectral properties of $\mathbb{S}$ depending on g: By Proposition 2.9.1 the operator $\mathbb{S}$ belongs to the Schatten class $\mathcal{S}_p(\ell^2\mathbb{N})$ iff

$$\sum_{k=1}^{\infty} \left| \frac{g(k)}{g(k+1)} \right|^p < \infty. \tag{65}$$

In this case the spectral radius of $\mathbb{S}$ is automatically less than one. In order to satisfy (65) the sequence g must increase more than exponentially fast. It remains to investigate for which growth rates of d and g the vector $\tilde{v}^d := \operatorname{diag}(g) d : \mathbb{N} \to \mathbb{C},\ \tilde{v}^d_k := g(k)\, d(k)$ defines an $\ell^2\mathbb{N}$-sequence. Thus, given d with superexponential decay, g must not increase too much in comparison to d. A candidate for the auxiliary sequence g is the following: Set $d(0) = 1$ and $g : \mathbb{N} \to \mathbb{C}$ with $g(k) \approx \frac{1}{d(k-1)}$. The problem with this point of view are the possible zeros of d. Vice versa, given a weight g, the distance function d must decay more than exponentially. We can use (65) in two different ways. First, we write

$$\|\tilde{v}^d\|^2_{\ell^2\mathbb{N}} = \sum_{k=1}^{\infty} |g(k)|^2\, |d(k)|^2 = \sum_{k=1}^{\infty} \left| \frac{g(k)}{g(k+1)} \right|^p |g(k)^{2-p}\, g(k+1)^p\, d(k)^2|.$$

For $\tilde{v}^d \in \ell^2\mathbb{N}$ it suffices by Cauchy-Schwarz inequality and (65) that the sequence

$$\mathbb{N} \to \mathbb{C},\ k \mapsto g(k)^{2-p}\, g(k+1)^p\, d(k)^2$$

is bounded. This happens for instance if d can be written as $d(k) = \frac{b_1(k)}{\sqrt{|g(k)^{2-p}\, g(k+1)^p|}}$ with $b_1 \in \ell^\infty\mathbb{N}$. The second approach, which coincides with the first for $p = 1$, is to write

$$\|\tilde{v}^d\|^2_{\ell^2\mathbb{N}} = \sum_{k=1}^{\infty} |g(k)|^2\, |d(k)|^2 = \sum_{k=1}^{\infty} \frac{|g(k)|}{|g(k+1)|}\, |g(k)\, g(k+1)\, d(k)^2|.$$

By Cauchy-Schwarz inequality and (65) we need for $\tilde{v}^d \in \ell^2\mathbb{N}$ that $\sum_{k=1}^{\infty} |g(k)\, g(k+1)\, d(k)^2|^q < \infty$ for $q \le \frac{p}{1-p}$. This happens for instance if d can be written as $d(k) = \frac{b_2(k)}{\sqrt{|g(k)\, g(k+1)|}}$ with $b_2 \in \ell^q\mathbb{N}$. ☐

2.10 Ruelle-Mayer transfer operators for a special class of Ising interactions

We will now investigate a new class of distance functions consisting of suitable superpositions of infinitely many exponentially decaying Ising interactions $d : \mathbb{N} \to \mathbb{C},\ d(j) := \sum_{i=1}^{\infty} c_i\, \lambda_i^j$ for which we can apply the methods from Section 2.7. Besides finite superpositions of exponentially decaying Ising interactions as studied in [May80a], [HiMay02], [HiMay04], this class contains for instance the following distance functions: Let $0 < |\lambda| < 1$ and $d(k) := \frac{\lambda^k}{1-\lambda^k}$, the logarithmic interaction $d(k) := -\log(1-\lambda^k)$ from Example 1.9.7, $d(k) := \frac{\lambda^k}{(1-\lambda^k)^2}$, and $d(k) := e^{(\lambda^k)} - 1$. Due to the special form the limit behaviour of these distance functions can be analysed in detail. It turns out that these distance functions are characterised by an exponential decay at infinity.

We define the following operations on sequences: For any two sequences $a, b : \mathbb{N} \to \mathbb{C}$ we define their pointwise product $ab : \mathbb{N} \to \mathbb{C},\ (ab)_i := a_i b_i$.

We fix the branch of the complex square root which is positive on the positive real line. For any complex sequence $a : \mathbb{N} \to \mathbb{C}$ we define its pointwise square root $\sqrt{a} : \mathbb{N} \to \mathbb{C},\ (\sqrt{a})_i := \sqrt{a_i}$.

Definition 2.10.1. Given $\lambda : \mathbb{N} \to \mathbb{D} := B(0;1)_{\mathbb{C}} := \{z \in \mathbb{C} \,|\, |z| < 1\}$ and $c : \mathbb{N} \to \mathbb{C}$ such that $\sqrt{c\lambda} \in \ell^2\mathbb{N}$, we define a distance function $d : \mathbb{N} \to \mathbb{C}$, $d(j) := \sum_{i=1}^{\infty} c_i\, \lambda_i^j = \big\langle \operatorname{diag}(\lambda)^{j-1}\sqrt{c\lambda} \mid \overline{\sqrt{c\lambda}} \big\rangle_{\ell^2\mathbb{N}}$ for all $j \in \mathbb{N}$. We define the subspaces $\mathcal{D}_1^{(p),\Delta} \subset \ell^1\mathbb{N}$ (for $p \in [1,\infty]$) via

(i) $d \in \mathcal{D}_1^{(p),\Delta}$ for $p < \infty$ iff $\|\lambda\|_{\ell^p\mathbb{N}} < \infty$,

(ii) $d \in \mathcal{D}_1^{(\infty),\Delta}$ iff $\|\lambda\|_{\ell^\infty\mathbb{N}} < 1$. □

In Example 2.10.10 we will show that there are many distance functions belonging to $\mathcal{D}_1^{(\infty),\Delta}$ (2.10.1) and its subspaces. The following Proposition 2.10.2 explains the notation Δ in the definition of the spaces $\mathcal{D}_1^{(p),\Delta}$ (2.10.1) as those distance functions coming from diagonal matrices.

Proposition 2.10.2. *A distance function $d : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{D}_1^{(p),\Delta}$ (2.10.1) for $p < \infty$ if and only if there exists a generating triple $(\mathbb{B}, v, w)$ where $\in \ell^2\mathbb{N}$, $\mathbb{B} \in \mathcal{S}_p(\ell^2\mathbb{N})$ with $\mathbb{B} = \mathbb{B}^\top$ and $\rho_{\text{spec}}(\mathbb{B}) < 1$, such that $d(k) = \langle \mathbb{B}^{k-1}v \,|\, w\rangle$.*

Proof. Let $U : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ be a unitary operator such that $U\mathbb{B}U^\star = \Delta = \operatorname{diag}(\lambda)$ is diagonal with respect to the standard basis of $\ell^2\mathbb{N}$. Set $v' = Uv$ and $w' = Uw$ which both belong to $\ell^2\mathbb{N}$. Then

$$d(k) = \langle \mathbb{B}^{k-1}v \,|\, w\rangle = \langle U^\star\Delta^{k-1}Uv \,|\, w\rangle = \langle \operatorname{diag}(\lambda)^{k-1}v' \,|\, w'\rangle = \sum_{i=1}^{\infty} \lambda_i^{k-1} v_i'\overline{w_i'} = \langle \operatorname{diag}(\lambda)^{k-1}p \,|\, p\rangle\,,$$

where $p := \sqrt{v'\overline{w'}} \in \ell^2\mathbb{N}$, since $\|p\|_{\ell^2\mathbb{N}}^2 = \sum_{i=1}^\infty |p_i|^2 = \sum_{i=1}^\infty |v_i'w_i'| \le \|v'\|_{\ell^2\mathbb{N}}\|w'\|_{\ell^2\mathbb{N}} = \|v\|_{\ell^2\mathbb{N}}\|w\|_{\ell^2\mathbb{N}}$. The spectral radius of a diagonal operator is its operator norm which is the supremum norm of the diagonal entries. The converse is obvious. □

We will now investigate the inclusion relations of the classes $\mathcal{D}_1^{(q),\Delta}$. In particular, we will see that the distance functions belonging to $\mathcal{D}_1^{(q),\Delta}$ decay exponentially at infinity. The second part states that a generator $\mathbb{B}$ of $d \in \mathcal{D}_1^{(p),\Delta}$ has necessarily operator norm strictly less than one, hence $\mathcal{D}_1^{(p),\Delta} \subset \mathcal{D}_1^{(\infty),\Delta}$.

Proposition 2.10.3. (i) *For all $1 \le p \le q \le \infty$ we have $\mathcal{D}_1^{(p),\Delta} \subsetneq \mathcal{D}_1^{(q),\Delta} \subset \mathcal{D}_1^{(q)}$, the latter defined in (2.7.1). In particular, $\limsup_{k\to\infty} \sqrt[k]{|d(k)|} < 1$ for all $d \in \mathcal{D}_1^{(\infty),\Delta}$.*

(ii) *Let $p < \infty$ and $d \in \mathcal{D}_1^{(p),\Delta}$, say $d(j) = \sum_{i=1}^\infty c_i\,\lambda_i^j$. Then $\|\lambda\|_{\ell^\infty\mathbb{N}} = \max_{i\in\mathbb{N}}|\lambda_i| < 1$.*

Proof. We begin with the second assertion: Let $d(j) = \sum_{i=1}^\infty c_i\,\lambda_i^j$ for all $j \in \mathbb{N}$. If $p < \infty$ and $d \in \mathcal{D}_1^{(p),\Delta}$ with $\lambda \in \ell^p\mathbb{N}$, then the sequence λ_i tends to zero as $i \to \infty$, hence the maximum $\max_{i\in\mathbb{N}}|\lambda_i|$ exists and is strictly less than one. Concerning part (i) observe that $\|\lambda\|_{\ell^\infty\mathbb{N}} = \|\operatorname{diag}(\lambda)\|_{\ell^2\mathbb{N}\to\ell^2\mathbb{N}} < 1$ and use Propositions 2.7.2 and 2.10.2. It remains to show that for $p < q < \infty$ the inclusion $\mathcal{D}_1^{(p),\Delta} \subsetneq \mathcal{D}_1^{(q),\Delta}$ is strict. For instance consider the sequence $\lambda : \mathbb{N} \to \mathbb{C}$, $k \mapsto k^{-1/p}$ belonging to $\ell^q\mathbb{N} \setminus \ell^p\mathbb{N}$. In (2.10.4) we give an example of a sequence which vanishes at infinity, but does not belong to any $\ell^p\mathbb{N}$ for $p < \infty$. □

We will now give an example of a sequence which converges to zero, but does not belong to any sequence space $\ell^p\mathbb{N}$ for a finite p. This will complete the proof of the preceding proposition.

Example 2.10.4. We show that

$$\bigcup_{p<\infty} \ell^p\mathbb{N} \subsetneq c_0,$$

where c_0 as usually denotes the space of complex-valued sequences $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = 0$. The stated inclusion is obvious. We have to show that it is strict. The idea is to define a sequence which looks like a flight of stairs where the length of the stairs increases faster than their height decays. Let $(y_n)_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of non-negative real numbers (the height of the stairs), which will be specified later. We define $(x_n)_{n\in\mathbb{N}}$ depending on $(y_n)_{n\in\mathbb{N}}$ via

$$x_{2^{n!}} = \ldots = x_{2^{(n+1)!}-1} = y_n$$

for all $n \in \mathbb{N}$. Let $p \geq 1$. Then counting the number of equal terms yields

$$\sum_x x_n^p = \sum_n (2^{(n+1)!} - 2^{n!})\, y_n^p = \sum_n 2^{n!}\,(2^{n+1} - 1)\, y_n^p.$$

We set $z_n := 2^{n!}\,(2^{n+1} - 1)\, y_n^p$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence tending to zero with $\lim_{n\to\infty} \sqrt[n]{y_n} = 1$, take for instance $y_n = 1/n$. Then the root test applied to the z_n implies that $(x_n)_{n\in\mathbb{N}} \notin \ell^p\mathbb{N}$, since

$$\sqrt[n]{z_n} = 2^{(n-1)!}\,\sqrt[n]{2^{n+1} - 1}\,(\sqrt[n]{y_n})^p \simeq 2^{(n-1)!} \cdot 2 \cdot 1$$

as $n \to \infty$, independently of p. □

Viewing a sequence as an diagonal operator on $\ell^2\mathbb{N}$, the last example states that there are compact symmetric operators such that no power is trace class.
We will now apply Theorem 2.7.6 from Section 2.7 to the Ruelle-Mayer transfer operator for the one-dimensional one-sided full shift (1.2.6) with Ising spin interactions (1.8.3) and distance function $d \in \mathcal{D}_1^{(p),\Delta}$. By Proposition 2.10.3 (ii) there is nothing to prove.

Corollary 2.10.5. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with distance function $d \in \mathcal{D}_1^{(p),\Delta}$ (2.10.1), say $d(j) = \sum_{i=1}^{\infty} c_i\, \lambda_i^j$, and potential q. Let $A_{(\phi)}$ be the standard Ising observable (2.1.7). Then for all $n \geq \lceil p \rceil$ the Ruelle-Mayer transfer operator*

$$(66)\quad \mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\Big(\beta q(\sigma) + \beta\sigma\big\langle z\big|\overline{\sqrt{c\lambda}}\,\big\rangle\Big)\, f\big(\sigma\sqrt{c\lambda} + \operatorname{diag}(\lambda) z\big)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \operatorname{diag}(\lambda)^n)\operatorname{trace}(\mathcal{M}_\beta)^n$. □

In particular Corollary 2.10.5 includes Ising interactions whose distance function is a superposition of *finitely* many exponentially decaying terms, which is a setting which has been investigated for instance by D. Mayer and J. Hilgert in [May80a], [HiMay02], and [HiMay04].

Example 2.10.6. (Finite superpositions) Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function d being a superposition of finitely many exponentially decaying interactions, say $d(k) = \sum_{i=1}^{n} c_i\, \lambda_i^k$ with $0 < |\lambda_i| < 1$. The distance function d belongs to $\mathcal{D}_1^{(1),\Delta}$ (2.10.1) and the corresponding Ruelle-Mayer transfer operator is given as

$$\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\Big(\beta q(\sigma) + \beta\sigma\big\langle z\big|\overline{\sqrt{c\lambda}}\,\big\rangle\Big)\, f\big(\sigma\sqrt{c\lambda} + \operatorname{diag}(\lambda) z\big)\, d\nu(\sigma).$$

Its conjugate $\mathcal{L}_\beta := C^{-1}_{\operatorname{diag}(\sqrt{\lambda/c})} \circ \mathcal{M}_\beta \circ C_{\operatorname{diag}(\sqrt{\lambda/c})} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ acts via

$$(\mathcal{L}_\beta f)(z) = \int_F \exp\Big(\beta q(\sigma) + \beta\sigma\langle z|\overline{c}\rangle\Big)\, f(\sigma\,\lambda + \operatorname{diag}(\lambda) z)\, d\nu(\sigma),$$

which is precisely the operator discussed[25] in [HiMay02]. By Corollary 2.10.5 it satisfies the trace formula

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \operatorname{diag}(\lambda)^n)\operatorname{trace}(\mathcal{L}_\beta)^n.$$

□

We write the Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ defined in (66) as an integral over a family of Schatten class operators. Then the explicit trace norm formula (60) of Lemma 2.7.5 allows to weaken the condition on the measure ν and on the boundedness of $F \subset \mathbb{C}$ imposed in Corollary 2.10.5.

[25] There only the finite alphabet $F = \{\pm 1\}$ has been considered.

Proposition 2.10.7. *Let $F \subset \mathbb{R}$ be a ν-measurable set and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with real-valued distance function $d \in \mathcal{D}_1^{(1),\Delta}$ (2.10.1), say $d(j) = \sum_{i=1}^{\infty} c_i \lambda_i^j$, and potential q. Let $m := \max_{i\in\mathbb{N}:\, c_i\lambda_i\neq 0} |\lambda_i| < 1$. If the map*

$$F \to \mathbb{C},\ \sigma \mapsto \exp\Big(\mathbf{Re}(\beta q(\sigma)) + \frac{\pi}{2}\big(\frac{\beta^2}{\pi^2} + \frac{4}{1-m^2}\big(\frac{\beta}{\pi}+1\big)^2\big)\,\sigma^2\,\|\sqrt{c\lambda}\|^2_{\ell^2\mathbb{N}}\Big)$$

is ν-integrable, then the Ruelle-Mayer operator $\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ defined in (66) is trace class for $\beta \in \mathbb{R}$.

Proof. We write the Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ (66) as an integral over a family of trace class operators

$$\mathcal{M}_{\beta,\sigma} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_{\beta,\sigma} f)(z) := \exp\big(\beta q(\sigma) + \beta\sigma\langle z|w\rangle\big)\, f(\sigma\, v + \mathbb{B}z)$$

with $v = w = \sqrt{c\lambda} \in \ell^2\mathbb{N}$ and $\mathbb{B} := \operatorname{diag}(\lambda)$. Then the trace norm formula (60) gives

$$\|\mathcal{M}_{\beta,\sigma}\|_{\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N}))} = \frac{\exp\Big(\mathbf{Re}(\beta q(\sigma)) + \frac{\pi\sigma^2}{2}\frac{\beta^2}{\pi^2}\|\sqrt{c\lambda}\|^2 + \frac{\pi\sigma^2}{2}\big(\frac{\beta}{\pi}+1\big)^2\|(1-\mathbb{B}\mathbb{B}^\star)^{-1/2}(1+\mathbb{B})\sqrt{c\lambda}\|^2\Big)}{\det(1-|\mathbb{B}|)}.$$

Suppose that $\|(1-\mathbb{B}\mathbb{B}^\star)^{-1/2}(1+\mathbb{B})\sqrt{c\lambda}\|^2_{\ell^2\mathbb{N}} \leq \dfrac{4}{1-m^2}\,\|\sqrt{c\lambda}\|^2_{\ell^2\mathbb{N}}$. Then the integrability assumption shows that $\int_F \|\mathcal{M}_{\beta,\sigma}\|_{\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N}))}\, d\nu(\sigma) < \infty$ and hence by Theorem A.7.6 the Ruelle-Mayer operator is trace class. Concerning the needed estimate we observe that

$$(1-\mathbb{B}\mathbb{B}^\star)^{-1/2} = (1-\operatorname{diag}(\lambda)\operatorname{diag}(\overline{\lambda}))^{-1/2} = \operatorname{diag}(\hat{\lambda}),$$

where $\hat{\lambda}_i := (1-|\lambda_i|^2)^{-1/2}$. Then $|1+\lambda_i| < 2$ for $|\lambda_i| < 1$ gives the stated bound:

$$\begin{aligned}
\|(1-\mathbb{B}\mathbb{B}^\star)^{-1/2}(1+\mathbb{B})\sqrt{c\lambda}\|^2_{\ell^2\mathbb{N}} &= \|\operatorname{diag}(\hat{\lambda})\operatorname{diag}(1+\lambda)\sqrt{c\lambda}\|^2_{\ell^2\mathbb{N}} \\
&= \sum_{k=1}^{\infty} \big|(1-|\lambda_i|^2)^{-1/2}(1+\lambda_i)(c_i\,\lambda_i)^{1/2}\big|^2 \\
&= \sum_{k=1}^{\infty} \frac{|1+\lambda_i|^2 |c_i\,\lambda_i|}{1-|\lambda_i|^2} \\
&\leq \frac{4}{1-m^2}\sum_{k=1}^{\infty}|c_i\,\lambda_i| = \frac{4}{1-m^2}\,\|\sqrt{c\lambda}\|^2_{\ell^2\mathbb{N}}
\end{aligned}$$

where $m := \max_{i\in\mathbb{N}:\, c_i\lambda_i\neq 0} |\lambda_i| < 1$ which exists by Proposition 2.10.3. □

We will now give criteria ensuring that a distance function belongs to $\mathcal{D}_1^{(1),\Delta}$ defined in (2.10.1) and give some examples. Let $d(k) = \sum_{i=1}^{\infty} c_i\,\lambda_i^k$. First, Prop. 2.10.8, we choose the sequence $\lambda : \mathbb{N} \to \mathbb{C}$ to be an exponentially decaying sequence times a lower order term. Then we specialise further and look in Corollary 2.10.9 at purely exponentially decreasing sequences λ.

Proposition 2.10.8. *Let $c,\ g : \mathbb{N} \to \mathbb{C}$ be complex-valued sequences with $r_c := \limsup_{\nu\to\infty} \sqrt[\nu]{|c_\nu|} < \infty$ and $r_g := \limsup_{\nu\to\infty} \sqrt[\nu]{|g_\nu|} < \infty$. For any $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \min(\frac{1}{r_c\, r_g}, \frac{1}{r_g})$ the distance function $d : \mathbb{N} \to \mathbb{C},\ k \mapsto \sum_{i=1}^{\infty} c_i\, g_i^k\, \lambda^{ik}$ belongs to $\mathcal{D}_1^{(1),\Delta}$ (2.10.1).*

Proof. At first observe that $\lim_{k\to\infty} \lambda^k\, g_k = 0$, since the series $\sum_{k=1}^{\infty} |\lambda^k\, g_k|$ converges by the root test:

$$\limsup_{k\to\infty} \sqrt[k]{|\lambda^k\, g_k|} = |\lambda|\, r_g < 1.$$

Set $\tilde{\lambda} : \mathbb{N} \to \mathbb{C},\ \tilde{\lambda}_i := \lambda^i\, g_i$ for all $i \in \mathbb{N}$, then for $l = 0, 1$ we have

$$\limsup_{i\to\infty} \sqrt[i]{|c_i^l\,\tilde{\lambda}_i|} \leq |\lambda|\ \limsup_{i\to\infty} \sqrt[i]{|c_i|^l}\ \limsup_{i\to\infty} \sqrt[i]{|g_i|} \leq |\lambda|\, r_c^l\, r_g.$$

Our assumptions on the range of λ imply that $|\lambda|\, r_c^l\, r_g < 1$. Now the root test implies that $\tilde{\lambda} \in \ell^1\mathbb{N}$ (for $l = 0$) and $\sqrt{c\tilde{\lambda}} \in \ell^2\mathbb{N}$ (for $l = 1$). □

Another subclass of the space $\mathcal{D}_1^{(1),\Delta}$ (2.10.1) consists of those distance functions which come from evaluating analytic functions in a neighbourhood of a zero point. Without loss of generality let f be an analytic function in a neighbourhood of zero with $f(0) = 0$ and λ small enough. Then $d(k) := f(\lambda^k)$ belongs to $\mathcal{D}_1^{(1),\Delta}$ as we will show next. For this phenomenon we will give some examples in (2.10.10).

Corollary 2.10.9. *Let f be an analytic function, whose Taylor expansion at zero has the radius of convergence $0 < r_f \leq \infty$, and $f(0) = 0$. Then for every $0 < |\lambda| < 1$ the distance function $d : \mathbb{N} \to \mathbb{C}$, $d(k) := f(\lambda^k)$ belongs to $\mathcal{D}_1^{(1),\Delta}$ (2.10.1), as long as $|\lambda| < \min(1, r_f)$.*

Proof. Let f be given as $f(z) = \sum_{i=1}^{\infty} c_i\, z^i$. Recall that the radius of convergence of the analytic function f satisfies $\frac{1}{r_f} = \limsup_{i\to\infty} \sqrt[i]{|c_i|} = r_c$ in the notation of Proposition 2.10.8. Set $g \equiv 1$ and apply (2.10.8). □

We denote the space of holomorphic functions on the unit disk $\mathbb{D} := B(0;1)_{\mathbb{C}}$ by $\mathcal{O}(\mathbb{D})$. A whole zoo of distance functions of the type described in Corollary 2.10.9 can be obtained by evaluating suitable analytic functions at points λ^k. We will now give some examples.

Example 2.10.10. Some examples for distance functions obtained via Corollary 2.10.9 are for instance the following (we retain the wording from there):

(i) Let $f \in \mathcal{O}(\mathbb{D})$ be defined by $f(z) := \frac{z}{1-z} = \sum_{i=1}^{\infty} z^i$. We have $c_i \equiv 1$ and $r_f = 1$. Hence $d(k) := f(\lambda^k) = \frac{\lambda^k}{1-\lambda^k}$ belongs to $\mathcal{D}_1^{(1),\Delta}$ (2.10.1) for all $\lambda \in \mathbb{D}$ by Corollary 2.10.9.

(ii) Let $f \in \mathcal{O}(\mathbb{D})$ be defined by $f(z) := \log(1-z) = \sum_{i=1}^{\infty} \frac{z^i}{i}$. We have $c_i = \frac{1}{i}$ and $r_f = 1$. Hence $d(k) := f(\lambda^k) = -\log(1-\lambda^k)$ belongs to $\mathcal{D}_1^{(1),\Delta}$ for all $\lambda \in \mathbb{D}$. This logarithmic interaction was introduced in Example 1.9.7.

(iii) Let $f \in \mathcal{O}(\mathbb{D})$ be defined by $f(z) := \frac{z}{(1-z)^2} = \sum_{i=1}^{\infty} i\, z^i$. We have $c_i = i$ and $r_f = 1$. Hence $d(k) := f(\lambda^k) = \frac{\lambda^k}{(1-\lambda^k)^2}$ belongs to $\mathcal{D}_1^{(1),\Delta}$ for all $\lambda \in \mathbb{D}$.

(iv) Let $f \in \mathcal{O}(\mathbb{C})$ be defined by $f(z) := e^z - 1 = \sum_{i=1}^{\infty} \frac{z^i}{i!}$. We have $c_i = \frac{1}{i!}$ and $r_f = \infty$. Hence $d(k) := f(\lambda^k) = e^{(\lambda^k)} - 1 = \sum_{i=1}^{\infty} \frac{\lambda^{ik}}{i!}$ belongs to $\mathcal{D}_1^{(1),\Delta}$ for all $\lambda \in \mathbb{D}$. □

Remark 2.10.11. Given a given distance function $d \in \mathcal{D}_1$ (Def. 2.2.4), we would like to know if d can be constructed via Corollary 2.10.9. In order to decide this one has to identify the corresponding holomorphic map f such that $d(k) = f(\lambda^k)$. Since $k = \frac{\log \lambda^k}{\log \lambda}$, the naive approach is to set $f(z) = d\big(\frac{\log z}{\log \lambda}\big)$, which satisfies $f(\lambda^k) = d\big(\frac{\log \lambda^k}{\log \lambda}\big) = d(k)$. Then one has to investigate whether this defines a holomorphic function f in a neighbourhood of zero (possibly there are restrictions on the choice of λ). For any $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ the set $\{\lambda^k \in \mathbb{C} \,|\, k \in \mathbb{N}\}$ has the accumulation point zero. Hence (for every fixed λ) a holomorphic function f, which is defined in a neighbourhood of zero and satisfies $f(\lambda^k) = d(k)$ for all $k \in \mathbb{N}$, is uniquely determined. Since $\lim_{k\to\infty} d(k) = 0$, the function f necessarily belongs to the ideal of holomorphic functions vanishing at zero. □

In the rest of this section we will derive necessary conditions on distance functions belonging to the spaces $\mathcal{D}_1^{(p),\Delta}$ (Def. 2.10.1). By Proposition 2.10.3 (ii) we know that every distance function $d \in \mathcal{D}_1^{(p),\Delta}$ has at least exponential decay at infinity, i. e., $\limsup_{k\to\infty} \sqrt[k]{|d(k)|} < 1$. We will show that the non-trivial distance functions belonging to $\mathcal{D}_1^{(p),\Delta}$ have *precisely* exponential decay at infinity which is surprising, since $\mathcal{D}_1^{(p),\Delta}$ is defined by inequalities. The following proposition is of preparatory nature.

Proposition 2.10.12. *Let $\lambda : \mathbb{N} \to \mathbb{C}$ be bounded and $c : \mathbb{N} \to \mathbb{C}$ such that $\sum_{i=1}^{\infty} c_i\, \lambda_i^k$ converges absolutely for all $k \in \mathbb{N}$. Then*

$$\lim_{k\to\infty} \sqrt[k]{\sum_{i=1}^{\infty} |c_i\, \lambda_i^k|} = \sup_{i\in\mathbb{N}:\, c_i\neq 0} |\lambda_i|.$$

Proof. We set $\lambda := \sup_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|$. By our assumptions the series $\sum_{i=1}^{\infty} |c_i\,\lambda_i|$ converges. Hence for all $k \in \mathbb{N}$ we have

$$\sqrt[k]{\Big|\sum_{i=1}^{\infty} c_i\,\lambda_i^k\Big|} \le \sqrt[k]{\lambda^k \sum_{i=1}^{\infty} |c_i|\,\Big|\frac{\lambda_i}{\lambda}\Big|^k} \le \sqrt[k]{\lambda^k \sum_{i=1}^{\infty} |c_i|\,\Big|\frac{\lambda_i}{\lambda}\Big|} = \lambda\,\sqrt[k]{\sum_{i=1}^{\infty} |c_i|\,\Big|\frac{\lambda_i}{\lambda}\Big|},$$

which tends to λ as $k \to \infty$. For all $\epsilon \in]0,\lambda[$ there exists $n \in \mathbb{N}$ such that $c_n \neq 0$ and $|\lambda_n| > \lambda - \epsilon$. Hence

$$\sqrt[k]{\sum_{i=1}^{\infty} |c_i\,\lambda_i^k|} \ge \sqrt[k]{|c_n|\,|\lambda_n|^k} \ge \sqrt[k]{|c_n|\,(\lambda-\epsilon)^k} = \sqrt[k]{|c_n|}\,(\lambda-\epsilon),$$

which tends to $\lambda - \epsilon$ as $k \to \infty$. Since $\epsilon > 0$ was arbitrary, we have $\lim_{k\to\infty} \sqrt[k]{\sum_{i=1}^{\infty} |c_i\,\lambda_i^k|} = \lambda$. ☐

If the supremum $\sup_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|$ in Proposition 2.10.12 is indeed a maximum, then we can determine the limit $\lim_{k\to\infty} \sqrt[k]{|\sum_{i=1}^{\infty} c_i\,\lambda_i^k|}$ which we interpret as $\lim_{k\to\infty} \sqrt[k]{|d(k)|}$ for a distance function $d \in \mathcal{D}_1^{(p),\Delta}$. The maximum is attained for instance if the sequence $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ converges to zero, which is equivalent to say that $\operatorname{diag}(\lambda) : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ is a compact operator.

Proposition 2.10.13. *Let $\lambda : \mathbb{N} \to \mathbb{C}$ be bounded and $c : \mathbb{N} \to \mathbb{C}$ such that $\max_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|$ exists and $\sum_{i=1}^{\infty} c_i\,\lambda_i^k$ converges absolutely for all $k \in \mathbb{N}$. Then*

$$\lim_{k\to\infty} \sqrt[k]{\Big|\sum_{i=1}^{\infty} c_i\,\lambda_i^k\Big|} = \max_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|.$$

Proof. We set $\lambda := \max_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|$. By relabelling we can assume that the sequence $(|\lambda_n|)_{n\in\mathbb{N}}$ attains its maximum only once (change the coefficients $(c_n)_{n\in\mathbb{N}}$ otherwise), without loss of generality $|\lambda_1| = \lambda$. Let $\epsilon > 0$. Since $\sqrt{c\lambda}$ belongs to $\ell^2\mathbb{N}$, there exists an index $n_0 \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} |c_i\,\lambda_i| < \epsilon\lambda$ for all $n \ge n_0$. Since $\lambda > |\lambda_i|$ for all $i \ge 2$, there exists an index $k_0 \in \mathbb{N}$ such that $\big|\sum_{i=2}^{n} c_i \left(\frac{\lambda_i}{\lambda}\right)^k\big| < \epsilon$ for all $k \ge k_0$. Then

$$\Big|\sum_{i=2}^{\infty} c_i \Big(\frac{\lambda_i}{\lambda}\Big)^k\Big| \le \Big|\sum_{i=2}^{n} c_i \Big(\frac{\lambda_i}{\lambda}\Big)^k\Big| + \sum_{i=n+1}^{\infty} |c_i|\,\Big|\frac{\lambda_i}{\lambda}\Big|^k \le \Big|\sum_{i=2}^{n} c_i \Big(\frac{\lambda_i}{\lambda}\Big)^k\Big| + \sum_{i=n+1}^{\infty} \Big|c_i\,\frac{\lambda_i}{\lambda}\Big| < 2\epsilon. \tag{67}$$

Hence for $\epsilon < |c_1|/2$ we have, using $|\lambda_1| = \lambda$,

$$\Big|\sum_{i=1}^{\infty} c_i\,\lambda_i^k\Big| \ge \Big||c_1|\,\lambda^k - \Big|\sum_{i=2}^{\infty} c_i\,\lambda_i^k\Big|\Big| = |c_1|\,\lambda^k\Big(1 - \Big|\sum_{i=2}^{\infty} \frac{c_i}{|c_1|}\Big(\frac{\lambda_i}{\lambda}\Big)^k\Big|\Big) \ge |c_1|\,\lambda^k\left(1 - \frac{2}{|c_1|}\epsilon\right).$$

This implies that

$$\sqrt[k]{\Big|\sum_{i=1}^{\infty} c_i\,\lambda_i^k\Big|} \ge \sqrt[k]{|c_1|\,\lambda^k\left(1 - \frac{2}{|c_1|}\epsilon\right)},$$

which tends to λ as $k \to \infty$. The obvious upper bound $\sqrt[k]{|\sum_{i=1}^{\infty} c_i\,\lambda_i^k|} \le \sqrt[k]{\sum_{i=1}^{\infty} |c_i\,\lambda_i^k|}$ also tends to $|\lambda_1|$ as $k \to \infty$ by the preceding Proposition 2.10.12. Hence the limit exists and has the stated value. ☐

Of course, we would like to prove[26]

Conjecture 2.10.14. Let $\lambda : \mathbb{N} \to \mathbb{C}$ be bounded and $c : \mathbb{N} \to \mathbb{C}$ such that $\sum_{i=1}^{\infty} c_i\,\lambda_i^k$ converges absolutely for all $k \in \mathbb{N}$. Then $\lim_{k\to\infty} \sqrt[k]{|\sum_{i=1}^{\infty} c_i\,\lambda_i^k|} = \sup_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|$. ☐

[26] By Propositions 2.10.12 and 2.10.13 it remains to show the lower bound $\lim_{k\to\infty} \sqrt[k]{|\sum_{i=1}^{\infty} c_i\,\lambda_i^k|} \le \sup_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|$ in the case the maximum $\max_{i\in\mathbb{N}\colon c_i\neq 0} |\lambda_i|$ is not attained.

We summarise the previous results and obtain the following description of the spaces $\mathcal{D}_1^{(p),\Delta}$ (2.10.1). It states that the non-trivial elements of $\mathcal{D}_1^{(p),\Delta}$ have precisely exponential decay at infinity.

Proposition 2.10.15. *Let $d \in \mathcal{D}_1$ (Def. 2.2.4) be a distance function.*

(i) *If $d \in \mathcal{D}_1^{(p),\Delta} \setminus \{0\}$ for some $p < \infty$ (2.10.1), then* $0 < \lim_{k\to\infty} \sqrt[k]{|d(k)|} < 1$.

(ii) *If $d \in \mathcal{D}_1^{(p),\Delta}$ for some $p < \infty$ and $\lim_{k\to\infty} \sqrt[k]{|d(k)|} = 0$, then $d = 0$.*

Proof. Let $d \in \mathcal{D}_1^{(p),\Delta}$ for some $p < \infty$, say $d(k) = \sum_{=1}^{\infty} c_i \lambda_i^k$. Since $\mathcal{D}_1^{(p),\Delta} \subset \mathcal{D}_1^{(p)}$, we know by Proposition 2.7.2 that $\lim_{k\to\infty} \sqrt[k]{|d(k)|} < 1$. By Proposition 2.10.3 the sequence $(|\lambda_i|)_{i\in\mathbb{N}}$ attains its maximum $\max_{i\in\mathbb{N}:\, c_i\neq 0} |\lambda_i| < 1$. On the other hand, the first assertion of Proposition 2.10.13 shows that $\lim_{k\to\infty} \sqrt[k]{|d(k)|} = \max_{i\in\mathbb{N}:\, c_i\neq 0} |\lambda_i|$. Hence, if $\lim_{k\to\infty} \sqrt[k]{|d(k)|} = 0$, then $\max_{i\in\mathbb{N}:\, c_i\neq 0} |\lambda_i| = 0$ and the distance function is the constant function zero. □

The converse of Proposition 2.10.15 (i) is not true in general, as we will see next.

Remark 2.10.16. Let $d \in \mathcal{D}_1$ (Def. 2.2.4) be a distance function with $0 < \lim_{k\to\infty} \sqrt[k]{|d(k)|} < 1$. Then d does not necessarily belong to some $\mathcal{D}_1^{(p),\Delta}$ with $p \leq \infty$. One is tempted to think that there should be a development of d as a sum of exponentially decaying functions by an iterative process. This however fails: Think of $d(k) := \lambda^k + \exp(-\gamma k^\delta)$ with $1 > \lambda > 0$, $\gamma > 0$, $\delta > 1$. Then $\lambda^k \leq d(k) \leq C\,\lambda^k$ for some constant $C > 0$ and hence $\lim_{k\to\infty} \sqrt[k]{|d(k)|} = \lambda$, but the next order term $\exp(-\gamma k^\delta)$ has no such expansion since $\lim_{k\to\infty} \sqrt[k]{|\exp(-\gamma k^\delta)|} = \lim_{k\to\infty} \exp(-\gamma k^{\delta-1}) = 0$. □

In this section we introduced a scale of new classes of distance functions which consist of suitable infinite superpositions of exponentially decaying terms. Due to the special shape of these distance functions their asymptotic behaviour can be well analysed. We showed that the (non-zero) distance functions belonging to these classes have exponential decay at infinity.

2.11 Ruelle-Mayer transfer operators for polynomial-exponential decaying interactions

In this section we construct the Ruelle-Mayer transfer operator for the one-sided one-dimensional full shift with polynomial-exponentially decaying Ising interaction, i.e., we consider the distance functions of the form $d : \mathbb{N} \to \mathbb{C},\ k \mapsto \lambda^k \sum_{i=0}^{p} c_i\, k^i$, where $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$, $c_i \in \mathbb{C}$. We determine a Ruelle-Mayer transfer operator and prove a dynamical trace formula. In view of Theorem 2.7.6 it remains to determine a generating triple for d. This directly reproduces the results of D. Mayer and J. Hilgert [May80a], [HiMay02], [HiMay04] on exponentially decaying distance functions and K. Viswanathan's result [Vi76] on polynomial-exponentially decaying interactions.

Remark 2.11.1. Let $\lambda \in \mathbb{C}^\times$, $x \geq 0$, and $p \in \mathbb{N}_0$. A standard basis for the $(p+1)$-dimensional vector space spanned by τ-images of the distance function $d : \mathbb{N} \to \mathbb{C},\ k \mapsto (k+x)^p \lambda^k$ are the functions $\widetilde{e}_l : \mathbb{N} \to \mathbb{C},\ \widetilde{e}_i(k) := (k+x)^i \lambda^k$ for $i = 0, \dots, p$. Because of the binomial formula they satisfy

$$\begin{pmatrix} \lambda^{k+1} \\ (k+1+x)\lambda^{k+1} \\ \vdots \\ (k+1+x)^{p-1}\lambda^{k+1} \\ (k+1+x)^p\lambda^{k+1} \end{pmatrix} = \lambda \mathbb{B}^{(p+1)} \begin{pmatrix} \lambda^k \\ (k+x)\lambda^k \\ \vdots \\ (k+x)^{p-1}\lambda^k \\ (k+x)^p\lambda^k \end{pmatrix}$$

for all $k \in \mathbb{N}$, where $\mathbb{B}^{(p+1)} \in \mathrm{Mat}(p+1, p+1; \mathbb{R})$ is the unipotent (lower) triangular matrix with entries

$$(\mathbb{B}^{(p+1)})_{i,j} = \begin{cases} \binom{i}{j} & , j \leq i, \\ 0 & , \text{otherwise.} \end{cases}$$

Obviously, $\det \mathbb{B}^{(p+1)} = 1$. Let $x = 0$, then $v_e := (\widetilde{e}_0(1), \ldots, \widetilde{e}_p(1))^\top = (\lambda, \ldots, \lambda) = \lambda \underline{1} \in \mathbb{C}^{p+1}$. By induction we obtain

$$\begin{pmatrix} \lambda^k \\ k\lambda^k \\ \vdots \\ k^p \lambda^k \end{pmatrix} = \lambda^k (\mathbb{B}^{(p+1)})^{k-1} \underline{1}$$

and hence we have found a generating triple for

$$d(k) := \lambda^k \sum_{i=0}^{p} c_i \, k^i = \big\langle \lambda^k (\mathbb{B}^{(p+1)})^{k-1} \underline{1} \, \big| \, \overline{c} \big\rangle$$

for all $c = (c_0, \ldots, c_p) \in \mathbb{C}^{p+1}$. □

As an immediate consequence of Remark 2.11.1 and Theorem 2.7.6 we obtain the dynamical trace formula for polynomial-exponentially decaying Ising interactions, a result which has been observed by K. Viswanathan [Vi76] in the case of a finite alphabet.

Corollary 2.11.2. *Let $F \subset \mathbb{C}$ be a bounded set and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto \lambda^k \sum_{i=0}^{p} c_i \, k^i$, where $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$, $c_i \in \mathbb{C}$. Let $\mathbb{B}^{(p+1)} \in \mathrm{Gl}(p+1; \mathbb{C})$ be the matrix given in Remark 2.11.1 and $\underline{1} : \{0, \ldots, p\} \to \mathbb{C}$ the constant function one. Then the Ruelle-Mayer transfer operator*

$$\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^{p+1}) \to \mathcal{F}(\mathbb{C}^{p+1}), \; (\mathcal{M}_\beta f)(z) = \int_F \exp\big(\beta q(x) + \beta \, x \, \langle z | \overline{c} \rangle\big) \, f\big(\lambda \, x \, \underline{1} + \lambda \mathbb{B}^{(p+1)} z\big) \, d\nu(x)$$

satisfies the dynamical trace formula $Z^{b^{\mathbb{N}_0}, \phi}_{\{1,\ldots,n\}}(\beta) = \widetilde{Z}^{b^{\mathbb{N}_0}}_n(\beta A_{(\phi)}) = (1 - \lambda^n)^{p+1} \, \mathrm{trace} \, (\mathcal{M}_\beta)^n$. □

2.12 Classification

Given a distance function $d \in \mathcal{D}_1^{(\infty)}$ (2.7.1), then there are many triples $(\mathbb{B}, v, w)$ which generate d. For instance, say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\mathcal{H}}$, then d can also be represented as $d(k) = \langle v | (\mathbb{B}^*)^{k-1} w \rangle_{\mathcal{H}} = \langle (\mathbb{B}^\top)^{k-1} \overline{w} | \overline{v} \rangle_{\mathcal{H}}$. Secondly, we can change v into $v + v'$ with $v' \in \ker \mathbb{B}$ and, similarly, w into $w + w'$ with $w' \in \ker \mathbb{B}^* = \mathbb{B}\mathcal{H}$. Thirdly, for every $S \in \mathrm{Gl}(\mathcal{H})$ one has

$$d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\mathcal{H}} = \langle S^{-1} (S \mathbb{B} S^{-1})^{k-1} S v | w \rangle_{\mathcal{H}} = \langle (S \mathbb{B} S^{-1})^{k-1} S v | S^{-\star} w \rangle_{\mathcal{H}}.$$

Given two different representations $d(k) = \langle (\mathbb{B}_i)^{k-1} v_i | w_i \rangle$ for $i = 1, 2$, then the corresponding Ruelle-Mayer transfer operators

$$\mathcal{M}_{\beta,(i)} : \mathcal{F}(\mathcal{H}_i) \to \mathcal{F}(\mathcal{H}_i), \; (\mathcal{M}_{\beta,(i)} f)(z) = \int_F \exp\big(\beta q(\sigma) + \beta \sigma \langle z | w_i \rangle\big) \, f(\sigma \, v_i + \mathbb{B}_i z) \, d\nu(\sigma)$$

are not conjugate in general, even if $\mathcal{H}_1 \cong \mathcal{H}_2$ because of possibly occurring kernels of $\mathbb{B}_i$ and of $\mathbb{B}_i^\star$. In this section we ask for normal forms. First we will deal with those generators which cannot be decomposed.

Definition 2.12.1. Let $\mathcal{H}$ be a Hilbert space. We call a linear map $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ *irreducible* if there is no closed subspace V of $\mathcal{H}$ different from $\{0\}$ and $\mathcal{H}$ such that both V and $V^\perp$ are $\mathbb{B}$-invariant. □

Remark 2.12.2. Let $d \in \mathcal{D}_1^{(\infty)}$ (2.7.1), say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\mathcal{H}}$, such that $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ is compact. This, for instance, happens if $d \in \mathcal{D}_1^{(p)}$ (2.7.1) for some $p < \infty$. Hence both the sequence of eigenvalues and the sequence of singular numbers of $\mathbb{B}$ converge to zero. Moreover, for all $\lambda \neq 0$ the generalised eigenspaces $E_\lambda := \{v \in \mathcal{H} \,|\, (\exists n \in \mathbb{N}) \; (\mathbb{B} - \lambda)^n v = 0\}$ are finite dimensional. For $\lambda \in \mathbb{C}$, $j \in \mathbb{N}$ we call

$$J_{(\lambda,j)} := \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix} \in \mathrm{Mat}(j, j; \mathbb{C}) \tag{68}$$

the *standard Jordan block* of size j and eigenvalue λ. A Jordan block is irreducible (in the sense of Def. 2.12.1) and by the Jordan decomposition theorem the Jordan blocks are (up to conjugation) the only irreducible maps on a finite dimensional complex vector space. Hence the only other irreducible maps are congruent to "infinite Jordan blocks" with eigenvalue zero. □

The operators $\mathbb{B} = \mathrm{diag}(\lambda) : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $(\mathbb{B}x)(i) = \lambda_i\, x_i$ corresponding to distance functions belonging to the class $\mathcal{D}_1^{(p),\Delta}$(see Example 2.7.7 (iii) and Section 2.10), are examples of highly non irreducible maps. - We make some attempts to give a classification of those generators which can be treated with our method.

Remark 2.12.3. Given $d \in \mathcal{D}_1^{(\infty)}$ (2.7.1), say $d(k) = \langle \mathbb{B}^{k-1} v | w\rangle_{\mathcal{H}}$, with $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ compact and irreducible (2.12.1). Using Remark 2.12.2 one obtains the following classification:

1. $\mathcal{H}$ finite dimensional: $\mathcal{H} \cong \mathbb{C}^j$. The irreducible maps are the Jordan blocks $J_{(\lambda,j)}$ (2.12.2).
   - 1.a Let $J_{(\lambda,j)} \in \mathrm{Mat}(j, j; \mathbb{C})$ be the standard Jordan block with $\lambda = 0$, i. e., $J_{(\lambda,j)} = \mathbb{S}_j$ is the standard j-step nilpotent matrix. Hence $d(k) = \langle (J_{(0,j)})^{k-1} v|w\rangle$ is a finite-range distance function with finite range j. See Section 2.8.
   - 1.b Let $J_{(\lambda,j)} \in \mathrm{Mat}(j, j; \mathbb{C})$ be the standard Jordan block with $\lambda \neq 0$. The matrix $J_{(\lambda,j)}$ is invertible and d is a polynomial-exponential distance function $d(k) = p_j(k)\, \lambda^k$ with a polynomial p_j in the variable k of degree $\deg p_j = j - 1$. See Section 2.11.
2. $\mathcal{H}$ infinite dimensional: $\mathcal{H} \cong \ell^2\mathbb{N}$. Necessarily spectrum of the irreducible maps consists of $\{0\}$. Up to now we know only one example, namely the superexponential distance functions, see Section 2.9.

Since the generators from (1.a) and (1.b) act on finite dimensional spaces, they are trace class and hence the corresponding distance functions belong to $\mathcal{D}_1^{(1)}$ (2.7.1). □

A possible normal form of a generating triple $(\mathbb{B}, v, w)$ consists of an operator $\mathbb{B}$ written as the direct sum of its Jordan blocks.
In the following we will give a characterisation of the generators acting on finite dimensional spaces via the study of shift invariant subspaces of distance functions.

Definition 2.12.4. Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a bounded operator on a Banach space $\mathcal{B}$. A vector $v \in \mathcal{B}$ is called $\mathbb{B}$-cyclic if the space spanned by the $\mathbb{B}$-iterates $\mathbb{B}^k v$ ($k \in \mathbb{N}_0$) is dense in $\mathcal{B}$. □

Proposition 2.12.5. *Let $d : \mathbb{N} \to \mathbb{C}$. The following are equivalent:*

(i) *There is a linear map $\mathbb{B}$ satisfying*

$$\mathbb{B} \begin{pmatrix} d(k) \\ \vdots \\ d(k+n-1) \end{pmatrix} = \begin{pmatrix} d(k+1) \\ \vdots \\ d(k+n) \end{pmatrix} \tag{69}$$

for all $k \in \mathbb{N}$ and n is minimal with this property.

(ii) *The shift operator $\tau : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, $(\tau f)(k) = f(k+1)$ (2.1.5) applied to d generates an n-dimensional complex vector space.*

(iii) *The functions $\tau^l d : \mathbb{N} \to \mathbb{C}$ $(l = 1, \ldots, n)$ are a basis of* $\mathrm{span}\{\tau^l d \mid l \in \mathbb{N}_0\}$.

(iv) *d solves an n-th order homogeneous linear difference equation with constant coefficients.*

(v) *There exists $\mathbb{B} \in \mathrm{Mat}(n, n; \mathbb{C})$ and a cyclic vector $v \in \mathbb{C}^n$ such that $d(k) = \left\langle \mathbb{B}^{k-1} v \,\middle|\, e_1 \right\rangle$ for all $k \in \mathbb{N}$.*

Proof. Let $\mathbb{B} = (\mathbb{B}_{i,j})_{i,j=1,\ldots,n}$.

(ii ⇒ iii) Look at the chain of vector spaces[27]

$$0 \to \operatorname{span}\{d\} \xrightarrow{\tau} \operatorname{span}\{d,\, \tau d\} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \operatorname{span}\{d, \ldots, \tau^{k-1} d\} \xrightarrow{\tau} \operatorname{span}\{d, \ldots, \tau^{k} d\} \xrightarrow{\tau} \ldots .$$

By assumption this chain is eventually constant. If $\tau(\operatorname{span}\{d, \ldots, \tau^{k-1}d\}) \subset \operatorname{span}\{d, \ldots, \tau^k d\}$ for some $k \in \mathbb{N}$, then all higher iterates $\tau^l d$ also belong to this space, hence $\operatorname{span}\{\tau^l d \mid l \in \mathbb{N}_0\} = \operatorname{span}\{d, \ldots, \tau^{k-1} d\}$ and $k = n$. The argument moreover shows that $\dim \operatorname{span}\{d, \ldots, \tau^l d\} = l+1$ for all $0 \le l \le k-1$, hence $d,\, \tau d, \ldots, \tau^{n-1} d$ are a basis in $\operatorname{span}\{\tau^l d \mid l \in \mathbb{N}_0\}$.

(iii ⇒ ii) Obvious.

(iii ⇒ iv) Since $d, \ldots, \tau^{n-1} d$ form a basis in $\operatorname{span}\{d, \tau d, \ldots, \tau^{n-1} d\}$, there are coefficients such that $\tau^n d = \sum_{l=0}^{n-1} \mathbb{B}_{n,l+1}\, \tau^l d$, i.e., $(\tau^n d)(k) = d(n+k) = \sum_{l=0}^{n-1} \mathbb{B}_{n,l+1}\, (\tau^l d)(k) = \sum_{l=1}^{n} \mathbb{B}_{n,l}\, d(k+l-1)$ for all $k \in \mathbb{N}$. Due to the special structure of the basis one has

$$\begin{pmatrix} (\tau d)(k) \\ \vdots \\ \vdots \\ (\tau^n d)(k) \end{pmatrix} = \begin{pmatrix} d(k+1) \\ \vdots \\ \vdots \\ d(k+n) \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \mathbb{B}_{n,1} & \cdots & \cdots & \mathbb{B}_{n,n} \end{pmatrix} \begin{pmatrix} d(k) \\ \vdots \\ \vdots \\ d(k+n-1) \end{pmatrix}. \tag{70}$$

By looking on the structure of the equations it suffices to solve the last one, i. e.,

$$d(j+n) = \sum_{k=1}^{n} \mathbb{B}_{n,k}\, d(j+k-1)$$

for all $j \in \mathbb{N}$. Introducing $c_k := -\mathbb{B}_{n,k+1}$ $(k = 0, \ldots, n-1)$, the last equation can be rewritten as

$$d(j+n) + \sum_{k=0}^{n-1} c_k\, d(j+k) = 0 \tag{71}$$

for all $j \in \mathbb{N}$, being the standard form of a homogeneous linear difference equation of n-th order with constant coefficients.

(iv ⇒ i) Write (71) as (70).

(i ⇒ v) By induction we obtain the representation

$$\begin{pmatrix} d(k+1) \\ \vdots \\ d(k+n) \end{pmatrix} = \mathbb{B}^{k-1} \begin{pmatrix} d(1) \\ \vdots \\ d(n) \end{pmatrix}$$

for all $k \in \mathbb{N}$. This implies that $(\tau^{i-1} d)(k) = \langle \mathbb{B}^{k-1} v^d \mid e_i \rangle$ for all $k \in \mathbb{N}$, $i = 1, \ldots, n$, where $v^d = \big(d(1), \ldots, d(n)\big)^\top$ and $e_i \in \mathbb{C}^n$ is the i-th standard unit vector. If v^d is not cyclic, then there is a non-zero vector $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ such that $\sum_{i=1}^n w_i\, \tau^{i-1} d = 0$, i. e.,

$$0 = \sum_{i=1}^{n} w_i\, (\tau^{i-1} d)(k) = \sum_{i=1}^{n} w_i\, \langle \mathbb{B}^{k-1} v^d \mid e_i \rangle = \langle \mathbb{B}^{k-1} v^d \mid \overline{w} \rangle$$

for all $k \in \mathbb{N}$. Hence the vectors $d, \ldots, \tau^{n-1} d$ are linearly dependent contradicting (iii).

(v ⇒ iii) Conversely, if there exists a vector $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ such that $\sum_{i=1}^n w_i\, \tau^{i-1} d = 0$, then $\langle \mathbb{B}^k v \mid \overline{w} \rangle$ for all $k \in \mathbb{N}$. Hence w is orthogonal to the span of the $\mathbb{B}^k v$, which by assumption is the whole space $\mathbb{C}^n$, hence $w = 0$ and the $d, \ldots, \tau^{n-1} d$ are linearly independent. Obviously, they are spanning. □

[27]The existence of such a "cyclic" basis does not depend on the specific choice of the shift operator.

Using the notation of the proof one calls

$$p(X) = X^n + \sum_{k=0}^{n-1} c_k\, X^k = X^n - \sum_{k=0}^{n-1} \mathbb{B}_{n,k+1}\, X^k$$

the *characteristic polynomial of the difference equation.* By induction one can show that $p(X) = \det(\mathbb{B} - X)$ is the characteristic polynomial of $\mathbb{B}$. Obviously, the matrix $\mathbb{B}$ is invertible if and only if $c_0 = -\mathbb{B}_{n,1} \neq 0$ if and only if $p(0) \neq 0$. The latter we call *non-degenerate.*
Next we show that non-degenerate difference equations are in correspondence with polynomial- exponential functions, i. e., pointwise products of a polynomial and an exponential function, and that degenerate difference equations correspond to finite range distance functions.

Proposition 2.12.6. *Let $d : \mathbb{N} \to \mathbb{C}$. The following are equivalent:*

(i) *There is a linear isomorphism $\mathbb{B}$ satisfying (69) for all $k \in \mathbb{N}$ and n is minimal with this property.*

(ii) $M_d := \operatorname{span}\{\tau^k d : \mathbb{N} \to \mathbb{C} \mid k \in \mathbb{N}_0\}$ *is a τ-invariant complex vector space and the restriction $\tau|_{M_d} : M_d \to M_d$ of the shift is a linear isomorphism.*

(iii) *There exists $\mathbb{B} \in \mathrm{Gl}(n;\mathbb{C})$ and a cyclic vector $v \in \mathbb{C}^n$ such that $d(k) = \langle \mathbb{B}^k v \,|\, e_1 \rangle$ for all $k \in \mathbb{N}_0$.*

(iv) *d solves a non-degenerate n-th order homogeneous linear difference equation with constant coefficients.*

(v) *There exist $\lambda_i \in \mathbb{C} \setminus \{0\}$ $(i = 1,\dots,l)$ and multiplicities $h_1,\dots,h_l \in \mathbb{N}$ with $\sum_{i=1}^{l} h_i = n$ and coefficients $c_{i,m} \in \mathbb{C}$ $(i = 1,\dots,l;\ m = 0,\dots,h_i - 1)$ such that for all $k \in \mathbb{N}$*

$$d(k) = \sum_{i=1}^{l} \sum_{m=0}^{h_i-1} c_{i,m}\, k^m\, \lambda_i^k.$$

Proof. By Proposition 2.12.5 the equivalences between (i), (iii), (iv) are obvious.

(i ⇔ ii) Note that $\mathbb{B}$ is the representing matrix of $\tau|_{M_d} : M_d \to M_d$ with respect to a special basis.

(iv ⇔ v) Let λ_i be the roots of the characteristic polynomial $p(X) = \det(\mathbb{B} - X)$ with multiplicity h_i, $i = 1,\dots,l$, then by the non-degeneracy the λ_i are non-zero complex numbers and the theory of difference equations [Mi90, p. 127] yields that

$$f_{i,m}(k) = k^m \lambda_i^k \quad (m = 0,\dots,h_i - 1,\ i = 1,\dots,l)$$

is a fundamental system of solutions, i. e., the n functions $f_{i,m}$ $(m = 0,\dots,h_i - 1; i = 1,\dots,l)$ are linearly independent and their span is τ-invariant. □

The following corollary considers the irreducible finite dimensional shift invariant subspaces and shows that they correspond to polynomial-exponential functions.

Corollary 2.12.7. *Let $d : \mathbb{N} \to \mathbb{C}$. The following are equivalent:*

(i) *There exist $\lambda \in \mathbb{C} \setminus \{0\}$ and coefficients $c_i \in \mathbb{C}$ $(i = 0,\dots,n-1)$ such that for all $k \in \mathbb{N}$*

$$d(k) = \lambda^k \sum_{i=0}^{n-1} c_i\, k^i.$$

(ii) $M_d := \operatorname{span}\{\tau^k d : \mathbb{N} \to \mathbb{C} \mid k \in \mathbb{N}_0\}$ *is an n-dimensional τ-invariant complex vector space and the restriction $\tau|_{M_d} : M_d \to M_d$, $(\tau g)(k) = g(k+1)$ of the shift is a linear, bijective and irreducible map (in the sense of Def. 2.12.1).*

(iii) *There is an irreducible linear isomorphism $\mathbb{B} \in \mathrm{Gl}(n;\mathbb{C})$ satisfying (69).*

Proof. If $\mathbb{B}$ is an irreducible isomorphism, then it is conjugate to a Jordan block $J_{(\lambda,n)} \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\lambda \neq 0$. Hence the roots of the characteristic polynomial are all equal to λ and hence d has the stated form by Proposition 2.12.6. If d is of this form, then $\lambda\mathbb{B}^{(n)}$ is a representing matrix of the shift operator restricted to M_d as shown in Example 2.11.1. The matrices $\lambda\mathbb{B}^{(n)}$ and $J_{(\lambda,n)}$ are both irreducible and have the same eigenvalues, hence are conjugate showing the equivalence of (i) and (iii). The implication (ii$\Rightarrow$iii) follows from looking at a representing matrix of $\tau|_{M_d} : M_d \to M_d$. It remains to show that given a polynomial $f \in \mathbb{C}[X]$ and $\lambda \in \mathbb{C}^\times$, the distance function $d : \mathbb{N} \to \mathbb{C}$, $d(n) := \lambda^n f(n)$ satisfies (ii). First consider the case $\lambda = 1$, i. e., $f = d$. For any monomial $m_k(X) := X^k \in \mathbb{C}[X]$ we have

$$(\tau m_k)(X) = m_k(X+1) = (X+1)^k = \sum_{l=0}^{k} \binom{k}{l} X^l \in \mathrm{span}\{m_0, \dots m_k\},$$

hence M_f is contained in the finite dimensional τ-invariant space $\mathrm{span}\{m_0, \dots m_{1+\deg f}\}$. The restriction of the shift $\tau|_{M_f} : M_f \to M_f$ is linear and injective, since $h \in M_f$ satisfies $(\tau h)(X) = h(X+1) = 0$ for all $X \in \mathbb{N}$ if and only if $h = 0$. Hence $\tau|_{M_f} : M_f \to M_f$ is bijective. The space M_f is τ-irreducible, since for every $h \in M_f$ of degree $\deg h$ the span of the iterates of h

$$M_h = \mathrm{span}\{\tau^k h : \mathbb{N} \to \mathbb{C} \,|\, k \in \mathbb{N}_0\} = \mathrm{span}\{m_0, \dots m_{\deg h}\}$$

is invariant, but the complement $M_f \setminus M_h$ is not invariant.
For arbitrary $\lambda \in \mathbb{C}^\times$ set $C_\lambda : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$, $(C_\lambda h)(n) := \lambda^n\, h(n)$. Then C_λ is a bijective linear map which almost (up to a constant scalar) commutes with the shift τ, since

$$(\tau C_\lambda h)(n) = \lambda^{n+1}\, h(n+1) = \lambda\, (C_\lambda \tau h)(n)$$

for any $h \in \mathbb{C}^\mathbb{N}$ and $n \in \mathbb{N}$. Hence the claim follows from the first case. ◻

In other words, Remark 2.11.1 computes the representing matrix of the restriction $\tau|_{M_d} : M_d \to M_d$ of the shift operator with respect to a suitable basis where $d(k) = k^p\, \lambda^k$.
In the same spirit of Proposition 2.12.6 one easily shows its analogon for finite range distance functions.

Proposition 2.12.8. *Let $d : \mathbb{N} \to \mathbb{C}$. The following are equivalent:*

(i) *There is a nilpotent linear map $\mathbb{B}$ satisfying (69) for all $k \in \mathbb{N}$ and n is minimal with this property.*

(ii) *$M_d := \mathrm{span}\{\tau^k d : \mathbb{N} \to \mathbb{C} \,|\, k \in \mathbb{N}_0\}$ is a τ-invariant complex vector space and the restriction $\tau|_{M_d} : M_d \to M_d$ of the shift is nilpotent.*

(iii) *There exists a nilpotent $\mathbb{B} \in \mathrm{Mat}(n,n;\mathbb{C})$ and a cyclic vector $v \in \mathbb{C}^n$ such that $d(k) = \langle \mathbb{B}^k v \,|\, e_1 \rangle$ for all $k \in \mathbb{N}_0$.*

(iv) *d solves a degenerate n-th order homogeneous linear difference equation with constant coefficients.*

(v) *d is a distance function with finite range n.* ◻

2.13 Ising type interaction

In Sections 2.7, 2.8 - 2.11, we have investigated lattice spin systems with Ising interaction and have constructed transfer operators for them. We will now use these results to find Ruelle-Mayer type transfer operators for one-sided one-dimensional full shifts with Ising type interaction. In our next main Theorem 2.13.8 we will prove the dynamical trace formula which generalises Theorem 2.7.6 to Ising type interactions. Recall from Remark 1.8.3 that an interaction matrix $r : F \times F \to \mathbb{C}$ is called *of Ising type* if

$$r(x,y) = \sum_{k=1}^{l} a_i(x)\, b_i(y)$$

for some functions a_i, $b_i : F \to \mathbb{C}$. As mentioned in Example 1.8.4 many physically relevant interaction matrices belong to this class, for instance Stanley's vector models and the finite state Potts model.
We start with some observations on the algebraic and analytic properties of the set of Ising type interaction matrices.

Remark 2.13.1. Let V be a subvector space of $F^{\mathbb{C}} = \{g : F \to \mathbb{C}\}$. A function $f : F \times F \to \mathbb{C}$ is called *decomposable* in V, if there exist functions $s_i,\, t_j \in V$ such that for all $x,\, y \in F$

$$f(x,y) = \sum_{i=1}^{M} s_i(x)\, t_i(y),, \tag{72}$$

where $s_{(M)} := (s_1,\dots,s_M)$, $t_{(M)} := (t_1,\dots,t_M) : F \to \mathbb{C}^M$. The minimal number $M \in \mathbb{N}$ is called the *rank* of f. If the rank is one, i. e., $f(x,y) = s(x)\,t(y)$, we call f *simple.*

(i) The space of decomposable functions is a subvector space of all functions $F \times F \to \mathbb{C}$.

(ii) If F is finite, then every function $f : F \times F \to \mathbb{C}$ is decomposable: Using Kronecker's delta function δ, we have the so called *trivial decomposition*

$$f(x,y) = \sum_{z\in F} f_z(x)\,\delta_z(y),$$

where $f_z(x) := f(x,z)$ and $\delta_z(y) = \delta_{z,y}$ for all $x,\, y,\, z \in F$.

(iii) Let F be compact. Note that $\mathcal{C}(F)\times\mathcal{C}(F)$ is a total subset in $\mathcal{C}(F\times F)$, hence every $r \in \mathcal{C}(F\times F)$ can be approximated by a sequence $(r_{(M)})_{M\in\mathbb{N}}$ in $\mathcal{C}(F\times F)$ such that each $r_{(M)}$ is decomposable with rank M.

(iv) Let F be a Hausdorff space. It is known that $\mathcal{C}_b(F)\times\mathcal{C}_b(F)$ is in general *not* a total subset in $\mathcal{C}_b(F\times F)$. □

Remark 2.13.2. (i) The space of Ising type interaction matrices is a subvector space of all symmetric functions $F \times F \to \mathbb{C}$.

(ii) Let F be compact. Note that $(\mathcal{C}(F)\times\mathcal{C}(F))^{\mathbb{Z}_2}$ is a total subset in $\mathcal{C}(F\times F)^{\mathbb{Z}_2}$. In fact, let $r \in \mathcal{C}(F\times F)^{\mathbb{Z}_2}$ and $(r_{(M)})_{M\in\mathbb{N}}$ be an approximating sequence in $\mathcal{C}(F\times F)$. Set $\tilde r_{(M)} : F\times F \to \mathbb{C}$, $\tilde r_{(M)}(x,y) := \frac{1}{2}\left(r_{(M)}(x,y) + r_{(M)}(y,x)\right)$ which is symmetric. Then $\tilde r_{(M)} \to r$ in $\mathcal{C}(F\times F)$, since

$$\sup_{x,y\in F}|r(x,y) - \tilde r_{(M)}(x,y)| \le \frac{1}{2}\sup_{x,y\in F}|r(x,y) - r_{(M)}(x,y)| + \frac{1}{2}\sup_{x,y\in F}|r(y,x) - r_{(M)}(y,x)|$$

which tends to zero as $M \to \infty$. □

Example 2.13.3. Let F be a finite set. The Potts model (see Example 1.8.3) with alphabet F has the interaction matrix $r_{\text{Potts}}(x,y) = \delta_{x,y} = \sum_{z\in F}\delta_{x,z}\,\delta_{z,y}$, where δ is the Kronecker delta function on $F\times F$. The Potts model interaction matrix has rank equal to $|F|$. In this example the trivial decomposition is also symmetric. □

If F is finite, then every interaction matrix has symmetric decompositions:

Proposition 2.13.4. *Let* $R \in \mathrm{Mat}(N,N;\mathbb{R})$ *be symmetric and* $0 \le \mathrm{rank}(R) \le N$ *the rank of the associated bilinear form*

$$\beta_R : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, (x,y) \mapsto \langle x|Ry\rangle.$$

(i) *For each* $M \ge \mathrm{rank}(R)$ *one has factorisations* $R = A_M A_M^\top$ *with* $A_M \in \mathrm{Mat}(N,M;\mathbb{C})$.

(ii) *There exists* $A_N = A_N^\top \in \mathrm{Mat}(N;N;\mathbb{C})$ *with* $R = A_N^2$.

Proof. The matrices $A_M \in \mathrm{Mat}(N,M;\mathbb{C})$ will be constructed using the matrices $\iota_M \in \mathrm{Mat}(N,M;\mathbb{C})$ which we define first. For $M \le N$ set

$$\iota_M := \begin{pmatrix} \mathrm{id}_M \\ O_{N-M,M} \end{pmatrix} \in \mathrm{Mat}(N,M;\mathbb{C})$$

and for $M \ge N$

$$\iota_M := \begin{pmatrix} \mathrm{id}_N & O_{N,M-N} \end{pmatrix} \in \mathrm{Mat}(N,M;\mathbb{C}).$$

where $O_{k,l}$ is the zero matrix in $\mathrm{Mat}(k,l;\mathbb{C})$. Depending on whether M or N is the larger number, we determine $\iota_M\iota_M^\top$. For $M \le N$ we have $\iota_M\iota_M^\top = \begin{pmatrix} \mathrm{id}_M & O_{M,N-M} \\ O_{N-M,M} & O_{N-M,N-M}\end{pmatrix} \in \mathrm{Mat}(N,N;\mathbb{C})$ and for $M \ge N$ we have $\iota_M\iota_M^\top = \mathrm{id}_N$.

(i) Denote by $\rho = \operatorname{rank}(R)$ the rank of R. There exists $T \in O(N)$ such that $R = TDT^\top$ where $D = \operatorname{diag}(d_1, \dots, d_\rho, 0, \dots, 0) \in \operatorname{Mat}(N, N; \mathbb{R})$. Let $\sqrt{\ }$ be the square root which is positive on the positive real line. For $M \geq \operatorname{rank}(R)$ set

$$E_M := \sqrt{D}\,\iota_M = \begin{pmatrix} \sqrt{\operatorname{diag}(d)} & O_{\rho,M-\rho} \\ O_{N-\rho,\rho} & O_{N-\rho,M-\rho} \end{pmatrix} \in \operatorname{Mat}(N, M; \mathbb{C}),$$

where $\sqrt{\operatorname{diag}(d)} = \operatorname{diag}(\sqrt{d_1}, \dots, \sqrt{d_\rho}) \in \operatorname{Mat}(\rho, \rho; \mathbb{C})$. By the previous considerations we have $E_M E_M^\top = \sqrt{D}\,\iota_M \iota_M^\top \sqrt{D} = D$. Set $A_M := TE_M \in \operatorname{Mat}(N, M; \mathbb{C})$, then

$$A_M A_M^\top = TE_M E_M^\top T^\top = TDT^\top = R.$$

(ii) Set $A_N = T\sqrt{D}T^\top$, then $A_N^\top = A_N$ and $A_N^2 = TDT^\top = R$. □

The following proposition will be the first step towards Ising type interactions. According to the decomposition of an Ising type interaction matrix one obtains a decomposition of the corresponding standard observable. Each summand in this decomposition can almost be represented as in Remark 2.1.7, but decorated with a homomorphism which we will introduce in Remark 2.13.5. In the second step, Prop. 2.13.7, this will lead to an (S1) - (S3) Ansatz (2.3.3) for Ising type interactions provided the distance function is of a special shape.

Remark 2.13.5. For any function $f : F \to \mathbb{C}$ we set $\underline{f} : F^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, $\underline{f}(\xi)_i := f(\xi_i)$. The map $\underline{f}$ is not linear, but a homomorphism in the following way: For all $\sigma \in F$, $\xi \in F^{\mathbb{N}}$ we have

$$\underline{f}(\sigma \vee \xi) = f(\sigma) \vee \underline{f}(\xi) = f(\sigma)e_1 + \big(0 \vee \underline{f}(\xi)\big),$$

where $e_1 = (1, 0, \dots) \in \mathbb{C}^{\mathbb{N}}$ is the first standard unit vector. □

Proposition 2.13.6. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising type interaction (1.8.3) with potential q, distance function $d \in \ell^1\mathbb{N}$, and interaction matrix $r \in \mathcal{C}_b(F \times F)$ given as $r(x, y) = \sum_{i=1}^M s_i(x)\, t_i(y)$ with s_i, $t_j \in \mathcal{C}_b(F)$. Then for all $\sigma \vee \xi \in \Omega$ we can express the standard observable (1.11.1) by dint of the linear map $\pi_1^d : \ell^\infty\mathbb{N} \to \mathbb{C}$, $\pi_1^d(\xi) = \sum_{k=1}^\infty \xi_k\, d(k)$ from Remark 2.1.6 as*

$$A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sum_{l=1}^M s_l(\sigma)\, \pi_1^d(\underline{t_l}(\xi)).$$

Proof. For all $\sigma \in F$, $\xi \in F^{\mathbb{N}}$ we have

$$\sum_{k=1}^\infty r(\sigma, \xi_k)\, d(k) = \sum_{k=1}^\infty \sum_{l=1}^M s_l(\sigma)\, t_l(\xi_k)\, d(k) = \sum_{l=1}^M s_l(\sigma) \sum_{k=1}^\infty t_l(\xi_k)\, d(k) = \sum_{l=1}^M s_l(\sigma)\, \pi_1^d(\underline{t_l}(\xi)).$$

□

We specialise to Ising type interactions with distance function of the type investigated in Section 2.6 and obtain the following result which is analogous to Proposition 2.6.6, i. e., the map $\pi_{\mathbb{B},v}^{(t)}$ is the projection map of an (S1) - (S3) Ansatz (2.3.3) for Ising type interactions.

Proposition 2.13.7. *Let $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator with $\rho_{\text{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$.*

(i) *Let $t = (t_1, \dots, t_M) : \mathbb{C} \to \mathbb{C}^M$ be bounded and $\pi_{\mathbb{B},v} : \ell^\infty\mathbb{N} \to \mathcal{B}$, $\pi_{\mathbb{B},v}(\xi) := \sum_{k=1}^\infty \xi_k\, \mathbb{B}^{k-1}v$ as in Proposition 2.6.6. Set*

$$\pi_{\mathbb{B},v}^{(t)} : \mathbb{C}^{\mathbb{N}} \to \mathcal{B}^M,\ \xi \mapsto \pi_{\mathbb{B},v}^{(t)}(\xi) := \Big(\pi_{\mathbb{B},v}(\underline{t_1}(\xi)), \dots, \pi_{\mathbb{B},v}(\underline{t_M}(\xi))\Big)$$

and $\underline{\mathbb{B}} : \mathcal{B}^M \to \mathcal{B}^M$, $\underline{\mathbb{B}}(z_1, \dots, z_M) := (\mathbb{B}z_1, \dots, \mathbb{B}z_M)$. Then for all $\sigma \vee \xi \in \mathbb{C}^{\mathbb{N}}$

$$\pi_{\mathbb{B},v}^{(t)}(\sigma \vee \xi) = \big(t_1(\sigma)v, \dots, t_M(\sigma)v\big) + \underline{\mathbb{B}}\pi_{\mathbb{B},v}^{(t)}(\xi).$$

(ii) *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising type interaction (1.8.3) with potential q, distance function $d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \langle \mathbb{B}^{k-1}v, w' \rangle_{\mathcal{B},\mathcal{B}'}$, and interaction matrix $r(x,y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$ with $s_i,\ t_j \in \mathcal{C}_b(F)$. Then for all $\sigma \vee \xi \in \Omega$ we can express the standard observable (1.11.1) as*

$$A_{(\phi)}(\sigma \vee \xi) = q(\sigma) + \sum_{l=1}^{M} s_l(\sigma)\, \langle \pi_{\mathbb{B},v}(\underline{t_l}(\xi)), w' \rangle_{\mathcal{B},\mathcal{B}'} = q(\sigma) + \langle \pi^{(t)}_{\mathbb{B},v}(\xi), \big(s_1(\sigma)w', \ldots, s_M(\sigma)w'\big) \rangle_{\mathcal{B}^M,(\mathcal{B}')^M}.$$

Proof. The first assertion follows from Propositions 2.6.6 (ii) and 2.13.6. Using the linearity of $\pi_{\mathbb{B},v}$ and the properties $\pi_{\mathbb{B},v}(e_1) = v$ and $\pi_{\mathbb{B},v}(0 \vee \xi) = \mathbb{B}\pi_{\mathbb{B},v}(\xi)$ as shown in Proposition 2.6.6 (i), we obtain

$$\begin{aligned} \pi^{(t)}_{\mathbb{B},v}(\sigma \vee \xi) &= \big(\pi_{\mathbb{B},v}(\underline{t_1}(\sigma \vee \xi)), \ldots, \pi_{\mathbb{B},v}(\underline{t_M}(\sigma \vee \xi))\big) \\ &= \big(\pi_{\mathbb{B},v}(t_1(\sigma) + 0 \vee \underline{t_1}(\xi)), \ldots, \pi_{\mathbb{B},v}(t_M(\sigma) + 0 \vee \underline{t_M}(\xi))\big) \\ &= \big(t_1(\sigma)v + \mathbb{B}\pi_{\mathbb{B},v}(\underline{t_1}(\xi)), \ldots, t_M(\sigma)v + \mathbb{B}\pi_{\mathbb{B},v}(\underline{t_M}(\xi))\big). \end{aligned}$$

□

Since $\pi^{(t)}_{\mathbb{B},v}$ is a projection map of an (S1) - (S3) Ansatz (2.3.3) for Ising type interactions, Definition 2.3.7 directly yields the (formal) Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{C}_b(\mathcal{B}^M) \to \mathcal{C}_b(\mathcal{B}^M)$,

$$(\mathcal{M}_\beta f)(z_1, \ldots, z_M) := \int_F \exp\Big(\beta q(\sigma) + \beta \sum_{l=1}^{M} s_l(\sigma)\, \langle z_l, w' \rangle_{\mathcal{B},\mathcal{B}'}\Big)\, f\big((t_1(\sigma)v, \ldots, t_M(\sigma)v) + \underline{\mathbb{B}}z\big)\, d\nu(\sigma).$$

Now we specialise to the Hilbert space setting introduced in Section 2.7 and prove the generalisation of Theorem 2.7.6 to Ising type interactions: For any Ising type two-body interaction with distance function belonging to $\mathcal{D}_1^{(p)}$ (2.7.1) we define a Ruelle-Mayer transfer operator which satisfies a dynamical trace formula.

Theorem 2.13.8. *Let $F \subset \mathbb{C}$ be a bounded set and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising type interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$, interaction matrix $r \in \mathcal{C}_b(F \times F)$ given as $r(x,y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$ with $s_i,\ t_j \in \mathcal{C}_b(F)$, and distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1), say $d(k) = \langle \mathbb{B}^{k-1}v | w \rangle_{\ell^2\mathbb{N}}$. Let $A_{(\phi)}$ be the standard observable (1.11.1). Then there exists an index $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geq n_0$ the Ruelle-Mayer transfer operator $\mathcal{M}_\beta^{(M)} : \mathcal{F}((\ell^2\mathbb{N})^M) \to \mathcal{F}((\ell^2\mathbb{N})^M)$,*

$$(73)\quad (\mathcal{M}_\beta^{(M)} f)(z_1, \ldots, z_M) := \int_F \exp\Big(\beta q(\sigma) + \beta \sum_{l=1}^{M} s_l(\sigma)\, \langle z_l | w \rangle\Big)\, f\Big((t_1(\sigma)v, \ldots, t_M(\sigma)v) + \underline{\mathbb{B}}z\Big)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \mathbb{B}^n)^M\ \mathrm{trace}\ (\mathcal{M}_\beta^{(M)})^n$.

Proof. We use the map $\pi^{(t)}_{\mathbb{B},v} : \Omega := F^{\mathbb{N}} \to \ell^2\mathbb{N}$ from Proposition 2.13.7 as a projection map of an (S1) - (S3) Ansatz (2.3.3). For all $\sigma \in F$ we define $A_\sigma : (\ell^2\mathbb{N})^M \to \mathbb{C},\ z \mapsto \beta\, q(\sigma) + \beta \sum_{l=1}^{M} s_l(\sigma)\, \langle z_l | w \rangle$ and the linking maps $\psi_\sigma : (\ell^2\mathbb{N})^M \to (\ell^2\mathbb{N})^M,\ (t_1(\sigma)v, \ldots, t_M(\sigma)v) + \underline{\mathbb{B}}z$. By Proposition 2.13.7 this gives an (S1) - (S3) Ansatz. The linking maps ψ_σ are affine and have the linear part $\underline{\mathbb{B}} : (\ell^2\mathbb{N})^M \to (\ell^2\mathbb{N})^M,\ \underline{(z_1, \ldots, z_M) \mapsto (\mathbb{B}z_1, \ldots, \mathbb{B}z_M)}$ in common. Note that $\det(1 - \underline{\mathbb{B}}^n) = \det(1 - \mathbb{B}^n)^M$. Apply Corollary B.4.5 for $\mathbb{B}_i = \mathbb{B}$, $a_i(\sigma) = \frac{\overline{\beta}}{\pi}\, s_i(\sigma)\, w$, and $b_i(\sigma) = t_i(\sigma)\, v$, which shows that

$$\|\mathcal{M}_{\beta,\sigma}\|_{\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N}^M))} = \frac{\exp\Big(\mathbf{Re}(\beta q(\sigma)) + \frac{\pi}{2} \sum_{i=1}^{M} (\|a_i\|^2 + \|(1 - \mathbb{B}_i\mathbb{B}_i^\star)^{-1/2}(\mathbb{B}_i a_i + b_i)\|^2)\Big)}{\prod_{i=1}^{M} \det(1 - |\mathbb{B}_i|)}$$

$$(74) = \frac{\exp\Big(\mathbf{Re}(\beta q(\sigma)) + \frac{|\beta|^2}{2\pi}\|w\|^2 \sum_{i=1}^{M} |s_i(\sigma)|^2 + \frac{\pi}{2} \sum_{i=1}^{M} \|(1 - \mathbb{B}\mathbb{B}^\star)^{-1/2}(\overline{\tfrac{\beta}{\pi} s_i(\sigma)}\mathbb{B}w + t_i(\sigma)v)\|^2\Big)}{\det(1 - |\mathbb{B}|)^M}$$

which is by our assumptions a bounded function in $\sigma \in F$. Then the assertion follows from Proposition 2.3.9 and Theorems 2.4.4, 2.4.6, and 2.7.6. □

In the following remark we explain which parameters of the interaction effect the Ruelle-Mayer operator and its spectral properties. Furthermore we comment on the approximation of interaction matrices.

Remark 2.13.9. (i) Let ϕ be a two-body Ising type interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1), say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$, and interaction matrix $r \in \mathcal{C}_b(F \times F)$ given as $r(x,y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$ with $s_i,\, t_j \in \mathcal{C}_b(F)$. The generator $\mathbb{B} \in \mathcal{S}_p(\mathcal{H}_0)$ of the distance function $d \in \mathcal{D}_1^{(p)}$ determines the space on which the Ruelle-Mayer transfer operator acts, namely on $\mathcal{F}(\mathcal{H}_0)^{\otimes M} \cong \mathcal{F}(\mathcal{H}_0^M)$, where M is the rank of the interaction matrix r. The spectral properties of the generator deeply influence the spectral properties of the RM operator in contrast to the vectors $v,\, w \in \mathcal{H}_0$. The potential q appears as the ν-density $\exp(\beta\, q)$. Together with the functions $s_i,\, t_j \in \mathcal{C}_b(F)$ its growth effects the finiteness of $\int_F \|\mathcal{M}_{\beta,\sigma}^{(M)}\|_{\mathcal{S}_1(\mathcal{H}_0^M)}\, d\nu(\sigma)$ and hence the question whether the Ruelle-Mayer operator is trace class.

(ii) Formula (74) moreover shows that given an arbitrary interaction matrix $r \in \mathcal{C}_b(F \times F)$ which can be approximated by a sequence $r_{(M)}$ of interaction matrices of rank M, the sequence of the corresponding Ruelle-Mayer transfer operators $\mathcal{M}_\beta^{(M)} : \mathcal{F}((\ell^2\mathbb{N})^M) \to \mathcal{F}((\ell^2\mathbb{N})^M)$ (73) is a Cauchy sequence with respect to the trace norm if and only if the Fredholm determinant of the generator is equal to one, i. e., for finite range and superexponentially decaying distance functions. This is due to the fact that a well approximating sequence of interaction matrices makes the argument of the exponential in (74) converge, but gives no control on the determinant factor, unless the latter vanishes. □

We end this chapter by returning to the main examples and commenting on the literature.

Example 2.13.10. Let $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1), say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$, be the distance function for the following one-sided one-dimensional full shifts (1.2.6).

(i) The Ising model (see Example 1.8.3) has the interaction matrix $r_{\text{Ising}}(x,y) = xy$. It has rank equal to one. The corresponding Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ is

$$(\mathcal{M}_{\beta,[\text{Ising}]} f)(z) = \int_F \exp\big(\beta q(x) + \beta x \,\langle z | w \rangle\big)\, f\big(xv + \mathbb{B}z\big)\, d\nu(x).$$

(ii) Let $F = \{1, \ldots, N\}$ be finite and the measure ν on F be identified with its distribution vector. The Potts model (see Example 1.8.3) has the interaction matrix $r_{\text{Potts}} : F \times F \to \{0,1\}$, $(x,y) \mapsto \delta_{x,y} = \sum_{z \in F} \delta_{x,z}\, \delta_{z,y}$, where δ is Kronecker's delta on $F \times F$. It has rank equal to $|F| = N$. The corresponding Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}((\ell^2\mathbb{N})^N) \to \mathcal{F}((\ell^2\mathbb{N})^N)$ is

$$(\mathcal{M}_{\beta,[\text{Potts}]} f)(z_1, \ldots, z_N) = \sum_{i=1}^{N} \nu_i \, \exp\big(\beta q_i + \beta \,\langle z_i | w \rangle\big)\, f\big((\delta_{i,m} v + \mathbb{B} z_m)_{m=1,\ldots,N}\big).$$

□

Remark 2.13.11. In the literature ([Vi76], [May76], [May80a],[HiMay02],[HiMay04]) mostly the Ising model (rank one case) for finite F is considered. [May76] shows the generalisation to arbitrary interaction matrices over a finite alphabet F. The approach to consider decomposable interaction matrices in the case of compact F was proposed in [May80a]. □

3 Transfer operators for the matrix subshift

Up to now we have constructed and investigated transfer operators for the one-sided one-dimensional full shift only. However, for applications this is an inadequate restriction, since the configuration space is often a strict subset of the full configuration space as we explained in the introduction. In the following we will treat the case of a one-dimensional matrix subshift as defined in (1.2.8). As in Section 1.11 we assume the space F of spin values to be a Hausdorff space endowed with a finite Borel measure ν and the transition matrix $\mathbb{A} : F \times F \to \{0, 1\}$ to be aperiodic (1.2.9) and $\nu \otimes \nu$-measurable. First we will define the Ruelle transfer operator and then the Ruelle-Mayer transfer operator. The main idea is to view the transition matrix as a Hilbert-Schmidt operator on $L^2(F, d\nu)$ and to investigate its tensor product with the Ruelle-Mayer operator for the full shift on the tensor product of $L^2(F, d\nu)$ with the Hilbert space on which the RM operator acts. In Section 3.3 we will provide the background on the spectral properties of a special kind of operators acting on tensor products of Hilbert spaces. In Theorem 3.2.6 we show that given a Ruelle-Mayer transfer operator for a full shift which satisfies a dynamical trace formula we can find a new transfer operator for the matrix subshift which satisfies a similar dynamical trace formula. In particular we obtain a generalisation of our main Theorem 2.4.6 on Ruelle-Mayer transfer operators for Ising type interactions with distance function belonging to $\mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1). As a second application we consider the hard rod model which we model as a matrix subshift. We will apply our techniques from Theorem 3.2.5 and construct a Schatten class Ruelle-Mayer transfer operator for polynomial-exponential interactions and prove a dynamical trace formula for it.

3.1 Ruelle transfer operator

In this section we will define Ruelle transfer operators for one-dimensional matrix subshifts (1.2.8). In the first step we assume that the alphabet F is finite.

Remark 3.1.1. Let F be a finite alphabet, $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional matrix subshift (1.2.8), and $A \in \mathcal{C}(\Omega_{\mathbb{A}})$ an observable. By Example 2.1.2 the preimage of $\xi \in \Omega_{\mathbb{A}}$ under the shift $\tau : \Omega_{\mathbb{A}} \to \Omega_{\mathbb{A}}$ consists precisely of those sequences $(\sigma \vee \xi) \in F^{\mathbb{N}}$ which fulfill $\mathbb{A}_{\sigma,\xi_1} = 1$. Then Remark 2.1.1 leads to a provisional definition of the Ruelle transfer operator $\mathcal{L}_A : \mathcal{C}(\Omega_{\mathbb{A}}) \to \mathcal{C}(\Omega_{\mathbb{A}})$ associated with A via

$$(\mathcal{L}_A f)(\xi) = \sum_{\eta \in (\tau_\gamma)^{-1}(\xi)} \exp(A(\eta))\, f(\eta) = \sum_{\sigma \in F} \mathbb{A}_{\sigma,\xi_1} \exp(A(\sigma \vee \xi))\; f(\sigma \vee \xi),$$

hence $\mathcal{L}_A f$ is a function which depends on ξ and in a special way on its first entry ξ_1. □

In [May91] D. Mayer considers the Ruelle (and the Ruelle-Mayer) transfer operator as operators acting on a direct sum of vector spaces which is indexed by the alphabet. This introductory section shall give a motivation for the right generalisation of the Ruelle and the Ruelle-Mayer transfer operator to matrix subshifts, also in the general case. We suggest to replace this direct sum by tensor products. We assume $\mathbb{A} : F \times F \to \{0, 1\}$ to be $\nu \otimes \nu$-measurable and put the following definition which shall be compared to Definition 2.1.3 for the full shift. Our proof of the continuity requires that $\mathcal{C}(F) \times \mathcal{C}(F^{\mathbb{N}})$ is total in $\mathcal{C}(F^{\mathbb{N}})$, which for instance happens if F is compact. Nevertheless, this Ruelle operator will lead to the right Ruelle-Mayer transfer operator in the next section.

Proposition 3.1.2. *Consider a Hausdorff space F equipped with a finite Borel measure ν and $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-dimensional one-dimensional matrix subshift (1.2.8). The Ruelle transfer operator*

$$\tilde{\mathcal{L}}_A : L^p(F, \nu) \hat{\otimes}_\pi \mathcal{C}_b(\Omega_{\mathbb{A}}) \to L^p(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(\Omega_{\mathbb{A}}),\ (\tilde{\mathcal{L}}_A f)(x, \xi) = \int_F \mathbb{A}_{\sigma,x} \exp(A(\sigma \vee \xi))\; f(\sigma, \sigma \vee \xi)\, d\nu(\sigma)$$

associated to the observable $A \in \mathcal{C}_b(F)$ is a bounded linear operator, $1 \leq p \leq \infty$.

Proof. We use the fact, see for instance [Scha50], that the projective tensor product can be characterised by the property that every bilinear, continuous map $T : X \times Y \to Z$ can be uniquely and continuously extended to a linear mapping $\overline{T} : X \hat{\otimes}_\pi Y \to Z$ with $\|T\| = \|\overline{T}\|$. Our operator $\tilde{\mathcal{L}}_A$ can be written as $\tilde{\mathcal{L}}_A = T_1 \circ T_2 \circ T_3$ with $T_i : L^1(F,\nu)\hat{\otimes}_\pi \mathcal{C}_b(\Omega_\mathbb{A}) \to L^1(F,d\nu)\hat{\otimes}_\pi \mathcal{C}_b(\Omega_\mathbb{A})$ uniquely defined on elementary tensors as

$$\begin{aligned}(T_1(f_1 \otimes f_2))(x,\xi) &= \int_F \mathbb{A}_{\sigma,x}\, f_1(\sigma)\, d\nu(\sigma)\, f_2(\xi),\\ (T_2(f_1 \otimes f_2))(x,\xi) &= f_1(x)\, f_2(x \vee \xi),\\ (T_3(f_1 \otimes f_2))(x,\xi) &= \exp(A(\xi))\, f_1(x)\, f_2(\xi).\end{aligned}$$

We will show that $\|T_1\| \le \nu(F)$, $\|T_2\| \le 1$, and $\|T_3\| \le \exp(\|A\|_{\mathcal{C}_b(\Omega_\mathbb{A})})$. We denote the dual exponent of p by q, defined by $\frac{1}{p}+\frac{1}{q}=1$. First note that $T_1 = T_{1,1} \otimes \mathrm{id}$ with

$$\begin{aligned}\|T_{1,1} f_1\|^p &= \int_F \Big| \int_F \mathbb{A}_{\sigma,x}\, f_1(\sigma)\, d\nu(\sigma)\Big|^p d\nu(x)\\ &\le \int_F \Big(\int_F 1\, d\nu(\sigma)\Big)^{p/q} \int_F |f_1(\sigma)|^p\, d\nu(\sigma)\, d\nu(x) = \nu(F)^p\, \|f_1\|^p.\end{aligned}$$

For the norm estimate of T_2 we use that every $f_2 \in \mathcal{C}_b(\Omega_\mathbb{A})$ can be approximated by a series $\sum_k g_1^{(k)} \otimes g_2^{(k)}$ with $g_1^{(k)} \in \mathcal{C}(F)$, $g_2^{(k)} \in \mathcal{C}(F^\mathbb{N})$. On elementary tensors $f_2 = g_1 \otimes g_2$ with $g_1 \in \mathcal{C}(F)$, $g_2 \in \mathcal{C}(F^\mathbb{N})$ we have

$$\begin{aligned}\|T_2(f_1 \otimes g_1 \otimes g_2)\| &= \Big(\int_F |f_1(\sigma)\, g_1(\sigma)|^p\, d\nu(\sigma)\Big)^{1/p} \sup_{\xi \in F^\mathbb{N}} |g_2(\xi)|\\ &\le \Big(\int_F |f_1(\sigma)|^p\, d\nu(\sigma)\Big)^{1/p} \sup_{\xi \in F^\mathbb{N}} |g_1(\xi)| \sup_{\xi \in F^\mathbb{N}} |g_2(\xi)|\\ &= \|f_1\|_{L^p(F,\nu)}\, \|g_1\|_{\mathcal{C}(F^\mathbb{N})}\, \|g_2\|_{\mathcal{C}(F^\mathbb{N})}.\end{aligned}$$

Let $T_{3,2} : \mathcal{C}_b(\Omega_\mathbb{A}) \to \mathcal{C}_b(\Omega_\mathbb{A})$ be the multiplication operator $(T_{3,2}f)(\xi) := \exp(A(\xi))\, f(\xi)$ which obviously satisfies $\|T_{3,2}\| \le \exp(\|A\|_{\mathcal{C}_b(\Omega_\mathbb{A})})$. Since $T_3 = \mathrm{id} \otimes T_{3,2}$, this concludes the proof. ☐

3.2 The Ruelle-Mayer transfer operator

Given an observable A which has properties (S1) - (S3) as defined in (2.3.3), we will define the (formal) Ruelle-Mayer transfer operator in a way which is similar to (2.3.7). We will transfer the ideas of Section 2.4 to the matrix subshift case such that for each transfer operator for a full shift we obtain a transfer operator for the matrix subshift with (quite) the same analytic properties. In particular we obtain the generalisation of Theorems 2.7.6 and 2.13.8 to matrix subshifts: For all Ising type interactions with distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ a dynamical trace formula holds at least for almost all $n \in \mathbb{N}$. This will be mainly accomplished by tensorising in a clever way with the space $L^2(F,d\nu)$ of square-integrable functions on the space F of spin values. Note that we always assume that the set F has finite measure with respect to the (a priori) Borel measure ν, hence by Cauchy-Schwarz's inequality a square-integrable function is absolutely integrable, i. e., $L^2(F,d\nu) \subset L^1(F,d\nu)$.

Remark 3.2.1. Let F be a Hausdorff space carrying a finite Borel measure ν and $(\Omega_\mathbb{A}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-dimensional matrix subshift (1.2.8). Let

$$\tilde{\mathcal{L}}_A : L^p(F,\nu)\hat{\otimes}_\pi \mathcal{C}_b(\Omega_\mathbb{A}) \to L^p(F,d\nu)\hat{\otimes}_\pi \mathcal{C}_b(\Omega_\mathbb{A}), (\tilde{\mathcal{L}}_A f)(x,\xi) = \int_F \mathbb{A}_{\sigma,x} \exp\big(A(\sigma \vee \xi)\big)\, f(\sigma, \sigma \vee \xi)\, d\nu(\sigma)$$

be the Ruelle operator (3.1.2) associated to an observable $A \in \mathcal{C}_b(\Omega_\mathbb{A})$ which has properties (S1) - (S3) as defined in (2.3.3). Let $\pi : \Omega_\mathbb{A} \to E$ be the corresponding projection map[28] into a topological space E, $\psi_\sigma : E \to E$ the linking maps ($\sigma \in F$), and $A_\sigma : E \to \mathbb{C}$ be the family of new observables. Set $\mathrm{id} \otimes \pi : F \times \Omega_\mathbb{A} \to F \times E$, $(x,\xi) \mapsto (x, \pi(\xi))$ and $\mathrm{id} \otimes C_\pi : \mathcal{C}_b(E) \to L^1(F,d\nu)\hat{\otimes}_\pi \mathcal{C}_b(E) \to L^p(F,d\nu)\hat{\otimes}_\pi \mathcal{C}_b(\Omega_\mathbb{A})$, $f_1 \otimes f_1 \mapsto f_1 \otimes f_2 \circ \pi$ the corresponding composition operator. As in the construction (2.3.6) of the Ruelle-Mayer operator for the full shift, we assume for a moment F to

[28]The projective = topological π-tensor product $X\hat{\otimes}_\pi Y$ has nothing to do with this map π.

be finite and apply the Ruelle transfer operator to $g := f \circ (\mathrm{id} \otimes \pi) \in L^p(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(\Omega_{\mathbb{A}})$ with $f \in L^p(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(E)$.

$$\begin{aligned}(\tilde{\mathcal{L}}_A g)(x, \xi) &= \sum_{\sigma \in F} \mathbb{A}_{\sigma,x} \exp\bigl(A(\sigma \vee \xi)\bigr)\,(f \circ (\mathrm{id} \otimes \pi))(\sigma, \sigma \vee \xi) \\ &= \sum_{\sigma \in F} \mathbb{A}_{\sigma,x} \exp\bigl(A_\sigma \circ \pi(\xi)\bigr)\, f(\sigma, \psi_\sigma(\pi(\xi))),\end{aligned}$$

i. e. the image of $\mathrm{id} \otimes C_\pi$ is $\tilde{\mathcal{L}}_A$-invariant and $\tilde{\mathcal{L}}_A \circ (\mathrm{id} \otimes C_\pi) = (\mathrm{id} \otimes C_\pi) \circ \tilde{\mathcal{M}}$, where we define the (formal) Ruelle-Mayer operator $\tilde{\mathcal{M}} : L^1(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(E) \to L^1(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(E)$ via

$$(\tilde{\mathcal{M}} f)(x, z) := \sum_{\sigma \in F} \mathbb{A}_{\sigma,x} \exp\bigl(A_\sigma(z)\bigr)\, f(\sigma, \psi_\sigma(z)).$$

□

Returning to the general case of an arbitrary alphabet F the previous considerations motivate the following definition which is related to Definition 2.3.7.

Definition 3.2.2. Let F be a Hausdorff space carrying a finite Borel measure ν, $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-dimensional matrix subshift (1.2.8), $A \in \mathcal{C}_b(\Omega_{\mathbb{A}})$ an observable, and $\pi : \Omega_{\mathbb{A}} \to E$ a continuous map into a topological space E with properties (S1) - (S3) (2.3.3). The operator

$$\tilde{\mathcal{M}} : L^1(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(E) \to L^1(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(E),\ (\tilde{\mathcal{M}} f)(x, z) := \int_F \mathbb{A}_{\sigma,x} \exp\bigl(A_\sigma(z)\bigr)\, f(\sigma, \psi_\sigma(z))\, d\nu(\sigma)$$

is called the (formal) *Ruelle-Mayer (RM) transfer operator* for the matrix subshift. □

Hence given a Ruelle-Mayer transfer operator for the full shift, we obtain by Definition 3.2.2 an associated RM operator for a matrix subshift. For example, if A is the standard observable of an Ising spin system with distance function $d \in \mathcal{D}_1^{(\infty)}$ (2.7.1), we will find such an operator, see Example 3.2.3. Another example will be discussed in Section 3.5 in which the hard rod model is concerned.

Example 3.2.3. (Cp. Remark 2.6.15) Let $F \subset \mathbb{C}$ be a bounded set equipped with a finite Borel measure ν and $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-dimensional matrix subshift (1.2.8). Let ϕ be a two-body Ising interaction (1.8.3) with potential q and distance function $d : \mathbb{N} \to \mathbb{C}$ given as $d : \mathbb{N} \to \mathbb{C},\ k \mapsto d(k) := \langle \mathbb{B}^{k-1} v, w' \rangle_{\mathcal{B},\mathcal{B}'}$, where $\mathbb{B} : \mathcal{B} \to \mathcal{B}$ is a linear operator with $\rho_{\mathrm{spec}}(\mathbb{B}) < 1$, $v \in \mathcal{B}$, $w' \in \mathcal{B}'$. Then by Proposition 2.6.6 the map $\pi_{\mathbb{B},v} : \ell^\infty \mathbb{N} \to \mathcal{B}, \pi_{\mathbb{B},v}(\xi) := \sum_{k=1}^\infty \xi_k\, \mathbb{B}^{k-1} v$ is a projection map of a (S1) - (S3) Ansatz (2.3.3). Let $\mathcal{M}_\beta : L^p(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(\mathcal{B}) \to L^p(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(\mathcal{B})$ be the Ruelle-Mayer transfer operator defined via

$$(\mathcal{M}_\beta f)(x, z) := \int_F \mathbb{A}_{\sigma,x}\, \exp\Bigl(\beta q(\sigma) + \beta \sigma \langle z, w' \rangle_{\mathcal{B},\mathcal{B}'}\Bigr)\, f(\sigma, \sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

and $\tilde{\mathcal{L}}_{A_{(\phi)}} : L^p(F, \nu) \hat{\otimes}_\pi \mathcal{C}_b(\Omega_{\mathbb{A}}) \to L^p(F, d\nu) \hat{\otimes}_\pi \mathcal{C}_b(\Omega_{\mathbb{A}})$ be the Ruelle operator (3.1.2) associated to the standard Ising observable $A_{(\phi)} \in \mathcal{C}_b(\Omega_{\mathbb{A}})$, then $\tilde{\mathcal{L}}_{\beta A_{(\phi)}} \circ (\mathrm{id} \otimes C_{\pi_{\mathbb{B},v}}) = (\mathrm{id} \otimes C_{\pi_{\mathbb{B},v}}) \circ \mathcal{M}_\beta$. □

Also in this context our intention to work with Hilbert spaces simplifies the arguments. Having identified a suitable Hilbert space $\mathcal{H}$ where the Ruelle-Mayer operator for the full shift acts, the corresponding RM for the matrix subshift acts on $L^2(F, d\nu) \hat{\otimes} \mathcal{H}$. Using the canonical isomorphisms $L^2(F, d\nu) \hat{\otimes} \mathcal{H} \cong L^2(F, \nu; \mathcal{H}) \cong \int_F^\oplus \mathcal{H}\, d\nu$ the reader can choose its preferred way of thinking.
We remark that in the widely considered case of a finite alphabet F we have $L^2(F, d\nu) \cong \mathbb{C}^{|F|}$ canonically, hence $L^2(F, d\nu) \hat{\otimes} \mathcal{H} \cong L^2(F, \nu; \mathcal{H}) \cong \int_F^\oplus \mathcal{H}\, d\nu \cong \mathcal{H}^{|F|}$.
Our next aim is to investigate the spectral properties of the Ruelle-Mayer transfer operator for the matrix subshift and to prove a dynamical trace formula for it. For this we need the following lemma which is an immediate consequence of Lemma 3.3.1 proved in the following Section 3.3.

Lemma 3.2.4. *Let ν be a finite measure on F and $\mathbb{A} : F \times F \to \{0, 1\}$ a $\nu \otimes \nu$-measurable transition matrix. Assume that $(S_x)_{x \in F}$ is a measurable family of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$ with $\int_F \|S_x\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(x) < \infty$. Then*

$$(\widetilde{T}(f_1 \otimes f_2))(\sigma) := \int_F \mathbb{A}_{x,\sigma}\, f_1(x)\, S_x f_2\, d\nu(x)$$

defines a Hilbert-Schmidt operator on $L^2(F, d\nu)\hat{\otimes}\mathcal{H}$ *with*

$$(\widetilde{T}^n(f_1 \otimes f_2))(\sigma) = \int_{F^n} \mathbb{A}_{x_1,x_2} \dots \mathbb{A}_{x_{n-1},x_n} \mathbb{A}_{x_n,\sigma} \,(S_{x_n} \circ \dots \circ S_{x_1} f_2)\, f_1(x_1)\, d\nu(x_1) \dots d\nu(x_n)$$

and (for $n \geq 2$*)*

$$\operatorname{trace} \widetilde{T}^n = \int_{F^n} \mathbb{A}_{x_1,x_2} \dots \mathbb{A}_{x_{n-1},x_n} \mathbb{A}_{x_n,x_1} \operatorname{trace}\,(S_{x_n} \circ \dots \circ S_{x_1})\, d\nu(x_1) \dots d\nu(x_n).$$

□

A tensorised version of Lemma 3.2.4 in a Banach space setting besides the case of a finite alphabet F is not known to us. Such a result could be used to obtain an analogon of Theorem 2.4.4 in the matrix subshift setting. For our purpose it is sufficient to transfer Theorem 2.4.6 to matrix subshifts.

Theorem 3.2.5. *Let* $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ *be a one-sided one-dimensional matrix subshift (1.2.8),* $A \in \mathcal{C}_b(\Omega_{\mathbb{A}})$ *an observable, and* $\pi : \Omega_{\mathbb{A}} \to E$ *a continuous map into a Banach space* E *with properties (S1) - (S3) (2.3.3). Assume that the maps* $\psi_x : E \to E$ *are affine and of the form* $\psi_x : E \to E,\ z \mapsto \psi_x(z) := a_x + \mathbb{B}z$ *for some fixed map* $\mathbb{B} \in \operatorname{End}(E)$ *which admits a Fredholm determinant and has operator norm* $\|\mathbb{B}\|_{\mathrm{op}} < 1$*. Suppose that the algebra generated by the composition operators*

$$\mathcal{M}_x : \mathcal{H} \to \mathcal{H},\ (\mathcal{M}_x f)(z) = \exp\big(A_x(z)\big)\,(f \circ \psi_x)(z)$$

consists of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H} \subset \mathcal{C}(E)$ *and satisfies the trace formula (53). Let* ν *be a finite Borel measure on* F *such that* $\int_F \|\mathcal{M}_x\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(x) < \infty$*. Then the dynamical partition function (1.11.4) can be expressed as*

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \det(1 - \mathbb{B}^n) \operatorname{trace} \tilde{\mathcal{M}}^n$$

for all $n \in \mathbb{N}_{\geq 2}$*, where* $\tilde{\mathcal{M}}$ *is the Ruelle-Mayer transfer operator*

$$\tilde{\mathcal{M}} : L^2(F, d\nu)\hat{\otimes}\mathcal{H} \to L^2(F, d\nu)\hat{\otimes}\mathcal{H},\ (\tilde{\mathcal{M}}f)(\sigma, z) = \int_F \mathbb{A}_{x,\sigma}\, \exp\big(A_x(z)\big)\, f(x, \psi_x(z))\, d\nu(x).$$

Proof. By Corollary B.1.3 and by the assumed trace formula we have

$$\operatorname{trace}\,(\mathcal{M}_{x_n} \circ \dots \circ \mathcal{M}_{x_1}) = \exp\Big(\sum_{k=1}^{n} (A_{x_k} \circ \psi_{x_{k+1}} \circ \dots \circ \psi_{x_n})(z^\star_{x_1,\dots,x_n})\Big)$$

for all $x_1, \dots, x_n \in F$. Hence the assertion follows from comparing the trace given by Lemma 3.2.4

$$\operatorname{trace} \tilde{\mathcal{M}}^n = \int_{F^n} \mathbb{A}_{x_1,x_2} \cdot \ldots \cdot \mathbb{A}_{x_{n-1},x_n} \cdot \mathbb{A}_{x_n,x_1} \operatorname{trace}\,(\mathcal{M}_{x_n} \circ \dots \circ \mathcal{M}_{x_1})\, d\nu(x_1) \dots d\nu(x_n)$$

with the expression for the partition function for the one-sided one-dimensional matrix subshift

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(A) = \int_{F^n} \mathbb{A}_{x_1,x_2} \cdot \ldots \cdot \mathbb{A}_{x_{n-1},x_n} \cdot \mathbb{A}_{x_n,x_1}\, \exp\Big(\sum_{k=1}^{n} (A_{x_k} \circ \psi_{x_{k+1}} \circ \dots \circ \psi_{x_n})(z^\star_{x_1,\dots,x_n})\Big) d\nu(x_1) \dots d\nu(x_n).$$

given in Proposition 2.3.5. □

An immediate consequence of Theorem 3.2.5 is our following result which generalises both Theorems 2.7.6 and 2.13.8 to matrix subshifts: For all Ising type interactions with distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ a dynamical trace formula holds at least for almost all $n \in \mathbb{N}$.

Theorem 3.2.6. *Let* $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ *be a one-sided one-dimensional matrix subshift (1.2.8). Let* ϕ *be a two-body Ising type interaction (1.8.3) with potential* $q \in \mathcal{C}_b(F)$ *and distance function* $d \in \mathcal{D}_1^{(p)}$ *for some* $p < \infty$ *(2.7.1), say* $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$*, and interaction matrix* $r \in \mathcal{C}_b(F \times F)$ *with* $r(x, y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$ *with* $s_i,\, t_j \in \mathcal{C}_b(F)$*. Let* $A_{(\phi)}$ *be the standard observable (1.11.1). Then* *there exists an index $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geq n_0$ the Ruelle-Mayer transfer operator $\mathcal{M}_\beta : L^2(F, d\nu)\hat{\otimes}\mathcal{F}((\ell^2\mathbb{N})^M) \to L^2(F, d\nu)\hat{\otimes}\mathcal{F}((\ell^2\mathbb{N})^M)$,*

$$(\mathcal{M}_\beta f)(x; z_1, .., z_M) = \int_F \mathbb{A}_{\sigma,x} \exp\Big(\beta q(\sigma) + \beta \sum_{l=1}^{M} s_l(\sigma)\langle z_l|w\rangle\Big)\, f(\sigma; t_1(\sigma)v + \mathbb{B}z_1, .., t_M(\sigma)v + \mathbb{B}z_M)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \mathbb{B}^n)^M \operatorname{trace}(\mathcal{M}_\beta)^n$.
In particular, if ϕ is a two-body Ising interaction (1.8.3), then for all $n \geq n_0'$ the Ruelle-Mayer transfer operator $\mathcal{M}_\beta : L^2(F, d\nu)\hat{\otimes}\mathcal{F}(\ell^2\mathbb{N}) \to L^2(F, d\nu)\hat{\otimes}\mathcal{F}(\ell^2\mathbb{N})$,

$$(\mathcal{M}_\beta f)(x, z) = \int_F \mathbb{A}_{\sigma,x} \exp\Big(\beta q(\sigma) + \beta\sigma\langle z|w\rangle\Big)\, f(\sigma, \sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1 - \mathbb{B}^n) \operatorname{trace}(\mathcal{M}_\beta)^n$.

Proof. By the superposition principle it suffices to prove the Ising case. In order to apply Theorem 3.2.5 we have to show that $\int_F \|\mathcal{M}_{\beta,(x))}\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(x) < \infty$, where $\mathcal{M}_{\beta,(x)} := e^{\beta q(x)}\, \mathcal{K}_{\overline{\frac{\beta}{\pi}xw},xv,\mathbb{B}}$ in the notation of Corollary B.4.4. We have

$$\|\mathcal{L}_{\overline{\frac{\beta}{\pi}xw},xv,\mathbb{B}}\|^2_{\mathcal{S}_2(\mathcal{F}(\ell^2\mathbb{N}))} = \frac{\exp\Big(2\mathbf{Re}(\beta q(x)) + \pi\|\overline{\frac{\beta}{\pi}xw}\|^2 + \pi\langle(1 - \mathbb{B}\mathbb{B}^*)^{-1}(\mathbb{B}\overline{\frac{\beta}{\pi}xw} + xv)|\mathbb{B}\overline{\frac{\beta}{\pi}xw} + xv\rangle\Big)}{\det(1 - \mathbb{B}\mathbb{B}^*)}.$$

Since $F \subset \mathbb{C}$ is bounded by assumption, the function $F \to \mathbb{R},\ x \mapsto \|\mathcal{K}_{\overline{\frac{\beta}{\pi}xw},xv,\mathbb{B}}\|^2_{\mathcal{S}_2(\mathcal{F}(\ell^2\mathbb{N}))}$ is bounded, hence ν-integrable. □

To emphasise the importance of the previous theorem we refer to Example 2.7.7 which gives a list of the classes of distance functions belonging to $\mathcal{D}_1^{(p)}$ defined in (2.7.1). The class of Ising type interactions, introduced in (1.8.3) and discussed in Section 2.13, contains many physically relevant interaction matrices such as Stanley's vector models (see Example 1.8.4) and the finite state Potts model.
The following corollary concerns the non-interacting case $\beta = 0$. The transfer operator is given as (the transpose of) the transition matrix interpreted as an integral operator on $L^2(F, d\nu)$. This was known for the special case of a finite alphabet F.

Corollary 3.2.7. *Let $(\Omega_\mathbb{A}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift (1.2.8) and ν a finite measure on F. Then for $n \geq 2$ the integral operator*

$$\mathcal{G}_\mathbb{A} : L^2(F, d\nu) \to L^2(F, d\nu),\ (\mathcal{G}_\mathbb{A} f)(x) = \int_F \mathbb{A}_{\sigma,x}\, f(\sigma)\, d\nu(\sigma)$$

associated to the transition matrix $\mathbb{A}$ satisfies the dynamical trace formula

$$\widetilde{Z}_n^{b^{\mathbb{N}_0}}(0) = Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},0} = \nu^n(\rho_{\{1,\dots,n\}}(\mathrm{Fix}(\tau^n : \Omega_\mathbb{A} \to \Omega_\mathbb{A}))) = \operatorname{trace}(\mathcal{G}_\mathbb{A})^n.$$

Proof. The operator $\mathcal{G}_\mathbb{A}$ is obviously Hilbert-Schmidt and can be seen as $\tilde{T}$ in Lemma 3.3.1 where all the operators $S_x = \mathrm{id} : \mathbb{C} \to \mathbb{C}$ are trivial. The trace of its iterates is given by Lemma 3.3.1 and 3.3.3 and coincides with the (dynamical) partition function given in Proposition 1.11.3. Setting $\beta = 0$ in Remark 1.7.2 which together with Proposition 1.3.14 concludes the proof. □

We end this section with the Ruelle-Mayer transfer operator for the special case of a finite alphabet.

Remark 3.2.8. Let $F = \{1, \dots, K\}$ be a finite alphabet and $(\Omega_\mathbb{A}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift (1.2.8). Let ϕ be a two-body Ising type interaction (1.8.3) with potential $q : F \to \mathbb{C}$ and distance function $d \in \mathcal{D}_1^{(p)}$ for some $p < \infty$ (2.7.1), say $d(k) = \langle\mathbb{B}^{k-1}v|w\rangle_{\ell^2\mathbb{N}}$, and interaction matrix $r \in \mathbb{C}^{F\times F}$ with $r(x, y) = \sum_{i=1}^{M} s_i(x)\, t_i(y)$ with $s_i,\, t_j : F \to \mathbb{C}$. Then the Hilbert space $L^2(F, d\nu)\hat{\otimes}\mathcal{F}((\ell^2\mathbb{N})^M)$ on which the Ruelle-Mayer transfer operator acts can be identified with $\mathcal{F}((\ell^2\mathbb{N})^M)^K$ and the operator can be written in compenents $(l = 1, \ldots, K)$

$$\begin{aligned}&(\mathcal{M}_\beta(f_1, \ldots, f_K)(z_1, \ldots, z_M))_l \\ &\quad = \sum_{k=1}^{K} \mathbb{A}_{k,l} \exp\Big(\beta\, q(k) + \beta \sum_{j=1}^{M} s_j(k)\, \langle z_j | w \rangle\Big)\, f_k(t_1(k)v + \mathbb{B}z_1, \ldots, t_M(k)v + \mathbb{B}z_M).\end{aligned}$$

We introduce the projections $\mathrm{pr}_l : \mathcal{F}((\ell^2\mathbb{N})^M)^K \to \mathcal{F}((\ell^2\mathbb{N})^M)$ onto the l-th component. Hence in short notation the transfer operator is characterised by

$$\mathrm{pr}_l \circ \mathcal{M}_\beta = \sum_{k=1}^{K} \mathbb{A}_{k,l}\, e^{\beta\, q_k} \bigotimes_{j=1}^{M} \mathcal{L}_{\beta s_j(k)\overline{w}, t_j(k)v, \mathbb{B}} \circ \mathrm{pr}_k,$$

where for any $a,\, b \in \ell^2\mathbb{N}$ and $\mathbb{B} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ the composition operator $\mathcal{L}_{a,b,\mathbb{B}} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$ acts via $(\mathcal{L}_{a,b,\mathbb{B}} f)(z) = e^{(z|a)}\, f(b + \mathbb{B}z)$. □

3.3 Tensor products

In this section we give a proof of Lemma 3.3.1 which we used in the previous section to prove the dynamical trace formula for the Ruelle-Mayer transfer operator associated to a matrix subshift. For this we need a small excursus on a special type of operators defined on tensor products of Hilbert spaces and their trace formulas. Namely, given a family $(S_x)_{x\in F}$ of operators on a Hilbert space $\mathcal{H}$ and a measurable function $g : F \times F \to \mathbb{C}$, we define

$$(\widetilde{T}(f_1 \otimes f_2))(\sigma) := \int_F g(x, \sigma)\, f_1(x)\, S_x f_2\, d\nu(x)$$

on $L^2(F, d\nu)\hat{\otimes}\mathcal{H}$. For the dynamical trace formula one puts $g(x, y) = \mathbb{A}_{x,y}$, where $\mathbb{A}$ is the transition matrix. In our result 3.3.1 we give an explicit formula for the Hilbert-Schmidt norm of the operator $\widetilde{T}$. We did not succeed to show that it is trace class which seems to be quite likely. We comment briefly on the occurring problems and cite some known results. We also investigate the behaviour of this kind of operators under unitary isomorphisms in the $\mathcal{H}$-variable which we will later apply in order to compute the Bargmann conjugate of the Ruelle-Mayer transfer operator.
First we give a formula for the iterates of $\widetilde{T}$. A simple induction argument shows that

$$(\widetilde{T}^n(f_1 \otimes f_2))(\sigma) = \int_{F^n} g(x_1, x_2) \ldots g(x_{n-1}, x_n)\, g(x_n, \sigma)\, f_1(x_1)\, S_{x_n} \circ \ldots \circ S_{x_1} f_2\, d\nu(x_1) \ldots d\nu(x_n).$$

In fact,

$$\begin{aligned}\widetilde{T}(\widetilde{T}^n(f_1 \otimes f_2))(\sigma) &= \int_F g(x, \sigma)\, S_x\Big((\widetilde{T}^n(f_1 \otimes f_2))(x)\Big)\, d\nu(x) \\ &= \int_F g(x, \sigma)\, S_x \int_{F^n} g(x_1, x_2) \ldots g(x_{n-1}, x_n)\, g(x_n, x)\, (S_{x_n} \circ \ldots \circ S_{x_1} f_2)\, f_1(x_1)\, d\nu^n(x_1, \ldots, x_n)\, d\nu(x) \\ &= \int_{F^{n+1}} g(x_1, x_2) \ldots g(x_{n-1}, x_n)\, g(x_n, x)\, g(x, \sigma)\, S_x(S_{x_n} \circ \ldots \circ S_{x_1} f_2)\, f_1(x_1)\, d\nu(x_1) \ldots d\nu(x_n)\, d\nu(x).\end{aligned}$$

If $\widetilde{T}$ is trace class, then one is tempted to think that

$$\text{trace}\ \widetilde{T}^n = \int_{F^n} g(x_1, x_2) \ldots g(x_{n-1}, x_n)\, g(x_n, x_1)\, \text{trace}\ (S_{x_n} \circ \ldots \circ S_{x_1})\, d\nu(x_1) \ldots d\nu(x_n) \tag{75}$$

for all $n \in \mathbb{N}$. This formula holds trivially if F is a finite set by the linearity of the trace, an idea which has been used for instance in [May91]. We will now extend this idea to arbitrary F. We can prove the following slightly weaker result which shows that the desired trace formula holds one step later. If one applies this lemma in the transfer operator setting in order to prove the meromorphic extension of the zeta function, then this result is sufficient. We add a comment to the trace class situation after the proof.

We note that our strategy is quite different from the proof of Theorem A.7.6 which we used for the dynamical trace formula for the full shift. There we have written the Ruelle-Mayer operator as an integral over a family of trace class operators, $Tf = \int_F S_x f\, d\nu(x)$ and showed that trace $T = \int_F \operatorname{trace} S_x\, d\nu(x)$. In the matrix subshift setting one should think of the operators S_x as integral operators on some $L^2(Z)$, say $(S_x f)(z) = \int_Z s_x(z,w)\, f(w)\, dw$. Then write

$$(\tilde{T}f)(\sigma, z) = \int_F g(x,\sigma)\, (S_x f(x,\cdot))(z)\, d\nu(x) = \int_F \int_Z g(x,\sigma)\, s_x(z,w)\, f(x,w)\, dw\, d\nu(x)$$

and investigate the integral kernel $\tilde{t}(\sigma, z; x, w) = g(x,\sigma)\, s_x(z,w)$ of $\tilde{T}$. This idea yields:

Lemma 3.3.1. *Let (F,ν) be a measure space, $g : F \times F \to \mathbb{C}$ a measurable function, and $(S_x)_{x\in F}$ a measurable family of operators on a separable Hilbert space $\mathcal{H}$. The formula*

$$(T(f_1 \otimes f_2))(\sigma) := \int_F g(x,\sigma)\, f_1(x)\, S_x f_2\, d\nu(x) \tag{76}$$

defines a Hilbert-Schmidt operator $T : L^2(F, d\nu)\hat{\otimes}\mathcal{H} \to L^2(F, d\nu)\hat{\otimes}\mathcal{H}$ if and only if

$$\int_F \int_F |g(x,\sigma)|^2\, d\nu(\sigma)\, \|S_x\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(x) < \infty. \tag{77}$$

In this case T satisfies

$$\|T\|^2_{\mathcal{S}_2(L^2(F,d\nu)\hat{\otimes}\mathcal{H})} = \int_F \int_F |g(x,\sigma)|^2\, d\nu(\sigma)\, \|S_x\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(x)$$

and

$$\operatorname{trace} T^n = \int_{F^n} \Big(\prod_{j=1}^{n-1} g(x_j, x_{j+1}) \Big)\, g(x_n, x_1)\, \operatorname{trace}\, (S_{x_n} \circ \ldots \circ S_{x_1})\, d\nu^n(x_1, \ldots, x_n)$$

for all $n \geq 2$. Moreover, for these n we have

$$\|T^n\|^2_{\mathcal{S}_2(L^2(F,d\nu)\hat{\otimes}\mathcal{H})} = \int_F \int_{F^n} \Big| \Big(\prod_{j=1}^{n-1} g(x_j, x_{j+1}) \Big)\, g(x_n, \sigma) \Big|^2 \|S_{x_n} \circ \ldots \circ S_{x_1}\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu^n(x_1, \ldots, x_n)\, d\nu(\sigma).$$

Proof. Suppose first that (76) defines a Hilbert-Schmidt operator. Fix orthonormal bases $(e_i)_{i\in\mathbb{N}}$, $(f_j)_{j\in\mathbb{N}}$ for $L^2(F, d\nu)$ and $\mathcal{H}$, respectively. Then, by Parseval's identity, one has

$$\begin{aligned}
\|T\|^2_{\mathcal{S}_2(L^2(F,d\nu)\hat{\otimes}\mathcal{H})} &= \sum_{i,j=1}^{\infty} \|T(e_i \otimes f_j)\|^2 \\
&= \sum_{i,j,k,l=1}^{\infty} \Big| \big\langle T(e_i \otimes f_j) \,\big|\, e_k \otimes f_l \big\rangle \Big|^2 \\
&= \sum_{i,j,k,l=1}^{\infty} \Big| \int_F \int_F g(x,\sigma)\, e_i(x)\, \langle S_x f_j \,|\, f_l \rangle\, d\nu(x)\, \overline{e_k(\sigma)}\, d\nu(\sigma) \Big|^2 \\
&= \sum_{j,l=1}^{\infty} \int_F \int_F \Big| g(x,\sigma)\, \langle S_x f_j \,|\, f_l \rangle \Big|^2 d\nu(x)\, d\nu(\sigma) \\
&= \int_F \int_F |g(x,\sigma)|^2 \sum_{j,l=1}^{\infty} \Big| \big\langle S_x f_j \,\big|\, f_l \big\rangle \Big|^2 d\nu(x)\, d\nu(\sigma) \\
&= \int_F \int_F |g(x,\sigma)|^2\, d\nu(\sigma)\, \|S_x\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(x).
\end{aligned}$$

Conversely, if (77) holds, we reverse this calculation and conclude that not only the integral (76) converges for almost all σ, but also that it defines a Hilbert-Schmidt operator on $L^2(F, d\nu)\hat{\otimes}\mathcal{H}$.

Now assume that T is Hilbert-Schmidt. Then for $n \geq 2$ the operator T^n is trace class. By the first part of the proof the S_{x_j} are Hilbert-Schmidt (for almost all x_j), hence the compositions $S_{x_n} \circ \ldots \circ S_{x_1}$ are trace class. Now the trace of T^n can be calculated as follows

$$\begin{aligned}
\text{trace } T^n &= \sum_{i,j=1}^{\infty} \langle T^n(e_i \otimes f_j) \,|\, e_i \otimes f_j \rangle \\
&= \sum_{i,j=1}^{\infty} \int_F \int_{F^n} g(x_1, x_2) \cdots g(x_{n-1}, x_n)\, g(x_n, \sigma) \times \\
&\qquad \times \langle S_{x_n} \circ \ldots \circ S_{x_1} f_j \,|\, f_j \rangle\, e_i(x_1)\, d\nu^n(x_1, \ldots, x_n)\, \overline{e_i(\sigma)}\, d\nu(\sigma) \\
&= \sum_{i=1}^{\infty} \int_F \int_{F^n} g(x_1, x_2) \cdots g(x_{n-1}, x_n)\, g(x_n, \sigma) \times \\
&\qquad \times \text{ trace } (S_{x_n} \circ \ldots \circ S_{x_1})\, e_i(x_1)\, d\nu^n(x_1, \ldots, x_n)\, \overline{e_i(\sigma)}\, d\nu(\sigma).
\end{aligned}$$

We claim that trace T^n can be rewritten as $\sum_{i=1}^{\infty} \langle \mathcal{G}_n e_i \,|\, e_i \rangle = \text{trace } \mathcal{G}_n$ with

$$(\mathcal{G}_n f)(\sigma) := \int_{F^n} \Big(\prod_{j=1}^{n-1} g(x_j, x_{j+1}) \Big)\, g(x_n, \sigma) \text{ trace } (S_{x_n} \circ \ldots \circ S_{x_1})\, f(x_1)\, d\nu^n(x_1, \ldots, x_n).$$

Note that (by Fourier expansion and induction)

$$\text{trace } (S_n \circ \ldots \circ S_1) = \sum_{i_1, \ldots, i_n = 1}^{\infty} \Big(\prod_{j=1}^{n-1} \langle S_j h_{i_j} \,|\, h_{i_{j+1}} \rangle \Big) \langle S_n h_{i_n} \,|\, h_{i_1} \rangle$$

for any orthonormal basis $(h_i)_{i \in \mathbb{N}}$ for $\mathcal{H}$ and Hilbert-Schmidt operators S_i on $\mathcal{H}$. Setting

$$(\mathcal{G}_{i,j} f)(\sigma) := \int_F g(x, \sigma)\, \langle S_x h_i \,|\, h_j \rangle\, f(x)\, d\nu(x)$$

for $i,\, j \in \mathbb{N}$, we can rewrite $\mathcal{G}_n$ as

$$\begin{aligned}
(\mathcal{G}_n f)(\sigma) &= \int_{F^n} \Big(\prod_{j=1}^{n-1} g(x_j, x_{j+1}) \Big)\, g(x_n, \sigma) \times \\
&\qquad \times \sum_{i_1, \ldots, i_n = 1}^{\infty} \prod_{j=1}^{n} \langle S_{x_i} h_{i_j} \,|\, h_{i_{j+1}} \rangle\, f(x_1)\, d\nu^n(x_1, \ldots, x_n) \\
&= \sum_{i_1, \ldots, i_n = 1}^{\infty} (\mathcal{G}_{i_n, i_1} \circ \mathcal{G}_{i_{n-1}, i_n} \circ \ldots \circ \mathcal{G}_{i_1, i_2} f)(\sigma).
\end{aligned}$$

The identity

$$\begin{aligned}
\sum_{i,i=1}^{\infty} \|\mathcal{G}_{i,j}\|^2_{\mathcal{S}_2(L^2(F, d\nu))} &= \sum_{i,j=1}^{\infty} \int_{F^2} |g(x,y)|^2\, |\langle S_x h_i \,|\, h_j \rangle|^2\, d\nu(x)\, d\nu(y) \\
&= \int_{F^2} |g(x,y)|^2 \sum_{i,j=1}^{\infty} |\langle S_x h_i \,|\, h_j \rangle|^2\, d\nu(x)\, d\nu(y) \\
&= \int_{F^2} |g(x,y)|^2\, \|S_x\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(x)\, d\nu(y)
\end{aligned} \tag{78}$$

implies that the $\mathcal{G}_{i,j}$ are Hilbert-Schmidt operators on $L^2(F, d\nu)$. Therefore, for each $(i_1, \ldots, i_n) \in \mathbb{N}^n$ the integral operator $\mathcal{G}_{i_n, i_1} \circ \mathcal{G}_{i_{n-1}, i_n} \circ \ldots \circ \mathcal{G}_{i_1, i_2}$ is trace class and by [Ka66, Ex. X. 1.18] its trace can be obtained by integrating the integral kernel along the diagonal. If $\mathcal{G}_n$ is trace class, we have

$$\begin{aligned} \text{trace } \mathcal{G}_n &= \sum_{i_1,\ldots,i_n=1}^{\infty} \text{trace } (\mathcal{G}_{i_n,i_1} \circ \mathcal{G}_{i_{n-1},i_n} \circ \ldots \circ \mathcal{G}_{i_1,i_2}) \\ &= \int_{F^n} \Big(\prod_{j=1}^{n-1} g(x_j, x_{j+1}) \Big) g(x_n, x_1) \, \text{trace } (S_{x_n} \circ \ldots \circ S_{x_1}) \, d\nu^n(x_1, \ldots, x_n) \\ &= \text{trace } T^n. \end{aligned}$$

Thus, to prove the claim it suffices to show that $\sum_{i_1,\ldots,i_n=1}^{\infty} \mathcal{G}_{i_n,i_1} \circ \ldots \circ \mathcal{G}_{i_1,i_2}$ converges in $\mathcal{S}_1\big(L^2(F, d\nu)\big)$. Using the technical Lemma 3.3.2 below, we obtain the estimate

$$\begin{aligned} \|\mathcal{G}_n\|_{\mathcal{S}_1\left(L^2(F,d\nu)\right)} &\leq \sum_{i_1,\ldots,i_n=1}^{\infty} \prod_{j=1}^{n} \|\mathcal{G}_{i_j,i_{j+1}}\|_{\mathcal{S}_2\left(L^2(F,d\nu)\right)} \\ &\leq \Big(\sum_{i,j=1}^{\infty} \|\mathcal{G}_{i_j,i_{j+1}}\|^2_{\mathcal{S}_2\left(L^2(F,d\nu)\right)} \Big)^{n/2} \\ &\overset{(78)}{=} \Big(\int_F \int_F |g(x,y)|^2 \, \|S_x\|^2_{\mathcal{S}_2(\mathcal{H})} \, d\nu(x) \, d\nu(y) \Big)^{n/2}, \end{aligned}$$

which proves the claim. To conclude the proof of the lemma one verifies the formula for Hilbert-Schmidt norm of T^n for $n \geq 2$, which can be done similarly as in the case $n = 1$. □

If ν is a finite measure on F, $g : F^2 \to \mathbb{C}$ is bounded, and $\int_F \|S_x\|^2_{\mathcal{S}_2(\mathcal{H})} \, d\nu(x)$ is finite, Lemma 3.3.1 shows that the associated operator T is Hilbert-Schmidt.

Lemma 3.3.2. *Let $n \geq 2$ and suppose that the functions $a_k : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ satisfy $\sum_{i,j=1}^{\infty} |a_k(i,j)|^2 < \infty$ for $k = 1, \ldots, n$. Then*

$$\Big| \sum_{i_1,\ldots,i_n=1}^{\infty} \prod_{k=1}^{n} a_k(i_k, i_{k+1}) \Big| \leq \prod_{k=1}^{n} \Big(\sum_{i,j=1}^{\infty} |a_k(i,j)|^2 \Big)^{1/2}$$

using the convention that $i_{n+1} = i_1$.

Proof. We proceed by induction. The case $n = 2$ follows from the estimate

$$\begin{aligned} \Big| \sum_{i_1,i_2=1}^{\infty} a_1(i_1,i_2) \, a_2(i_2,i_1) \Big| &\leq \sum_{i_1=1}^{\infty} \Big(\sum_{i_2=1}^{\infty} |a_1(i_1,i_2)|^2 \Big)^{1/2} \Big(\sum_{i_2=1}^{\infty} |a_2(i_2,i_1)|^2 \Big)^{1/2} \\ &\leq \prod_{k=1}^{2} \Big(\sum_{i,j=1}^{\infty} |a_k(i,j)|^2 \Big)^{1/2}. \end{aligned}$$

To do the induction step consider $\tilde{a}_n(i,j) := \sum_{m=1}^{\infty} |a_n(i,m) \, a_{n+1}(m,j)|$. Then

$$\begin{aligned} \sum_{i,j=1}^{\infty} |\tilde{a}_n(i,j)|^2 &= \sum_{i,j=1}^{\infty} \Big(\sum_{m=1}^{\infty} |a_n(i,m) \, a_{n+1}(m,j)| \Big)^2 \\ &\leq \Big(\sum_{i,m=1}^{\infty} |a_n(i,m)|^2 \Big) \Big(\sum_{m,j=1}^{\infty} |a_{n+1}(m,j)|^2 \Big), \end{aligned}$$

and induction yields

$$\begin{aligned} \Big| \sum_{i_1,\ldots,i_{n+1}=1}^{\infty} \prod_{k=1}^{n+1} a_k(i_k, i_{k+1}) \Big| &\leq \sum_{i_1,\ldots,i_n=1}^{\infty} \Big| \prod_{k=1}^{n-1} a_k(i_k, i_{k+1}) \Big| \, \tilde{a}_n(i_n, i_1) \\ &\leq \prod_{k=1}^{n-1} \Big(\sum_{i,j=1}^{\infty} |a_k(i,j)|^2 \Big)^{1/2} \Big(\sum_{i,j=1}^{\infty} |\tilde{a}_n(i,j)|^2 \Big)^{1/2} \\ &\leq \prod_{k=1}^{n+1} \Big(\sum_{i,j=1}^{\infty} |a_k(i,j)|^2 \Big)^{1/2}. \end{aligned}$$

□

The following remark concerns the question whether the last lemma can be adapted to a trace class setting and discusses trace formulas for integral operators.

Remark 3.3.3. Let ν be a finite measure on F and $g : F \times F \to \mathbb{C}$ a bounded measurable function. Assume that $(S_x)_{x\in F}$ is a measurable family of trace class operators on $\mathcal{H}$ with $\int_F \|S_x\|_{\mathcal{S}_1(\mathcal{H})}\, d\nu(x) < \infty$. One expects that the operator $\widetilde{T}$ (76) is trace class and satisfies the trace formula. But one encounters the following problem which arises from taking the trace on $L^2(F, d\nu)$: Recall, that if $K : L^2(F, d\nu) \to K : L^2(F, d\nu)$, $(Kf)(x) = \int_F k(x, y)\, f(y)\, d\nu(y)$ is a trace class integral operator, then the manifest trace formula

$$\text{trace } K = \int_F k(x, x)\, d\nu(x), \tag{79}$$

only holds under additional assumptions, say continuity of the kernel [GoGoKr00, Thm. 8.1]. The reason for this is that the diagonal in $F \times F$ is a set of measure zero, hence the kernel can be changed arbitrarily without influence on K, but with influence on the value of the trace integral. A limitation to continuous transition functions is not appropriate, since continuity forces the function to be constant on connected components. Formula (79) holds if one finds a (regularised) representant of the kernel. We present some ideas in this direction: In [K66, Ex. X. 1.18] appears the following: Since every trace class operator $K \in \mathcal{S}_1(\mathcal{H})$ has a representation as a product $K = AB$ of two Hilbert-Schmidt operators A, $B \in \mathcal{S}_2(\mathcal{H})$, the formula

$$\text{trace } K = \langle A \,|\, B^* \rangle = \sum_{n=1}^{\infty} \langle Ae_n \,|\, B^* e_n \rangle$$

for an arbitrary Hilbert basis $(e_n)_{n\in\mathbb{N}}$ is an equivalent definition for the trace of K. If K, A, B happen to be integral operators with integrals denoted by k, a, and $b \in L^2(F^2, d\nu^2)$ respectively, then by polarising the well-known formula (see for instance [We00])

$$\|K\|^2_{\mathcal{S}_2(L^2(F,d\nu))} = \|k\|^2_{L^2(F^2,d\nu^2)} = \int_F \int_F |k(x, y)|^2\, d\nu(x)\, d\nu(y)$$

one obtains

$$\text{trace } K = \int_F \int_F a(x, z)\, b(z, x)\, d\nu(z)\, d\nu(x).$$

In other words,

$$k(x, y) = \int_F a(x, z)\, b(z, y)\, d\nu(z)$$

defines a representant of k which fulfills formula (79). Another technique is Stekov's smoothing operator S_h, see for instance [GoGoKr00, p. 75]. One has also to mention the works of C. Brislawn [Br91] using the Hardy-Littlewood maximal operator. A more subtle problem is to find conditions on the integral kernel which guarantee that the corresponding integral operator is trace class. We mention that there is a well-developed theory for non-negative Hermitian operators. For instance, if ν is a finite measure on F and $K : L^2(F, d\nu) \to K : L^2(F, d\nu)$, $(Kf)(x) = \int_F k(x, y)\, f(y)\, d\nu(y)$ is an integral operator with a non-negative Hermitian bounded kernel, then K is trace class, [GoGoKr00, Cor. IV. 8.5]. □

The following proposition will be used for the computation of the Bargmann conjugate of the Ruelle-Mayer transfer operator in Chapter 5. Concerning the vector-valued integration we refer to Appendix A.7.

Proposition 3.3.4. *Let $(\mathcal{L}_y)_{y\in Y}$ be an integrable family of bounded operators on a separable Hilbert space $\mathcal{H}_1$, i. e., $\int_Y \|\mathcal{L}_y\|\, dy < \infty$. Let $\mathcal{L}f = \int_Y \mathcal{L}_y f\, dy : \mathcal{H}_1 \to \mathcal{H}_1$ be the bounded operator defined by Prop. A.7.3. Let $\mathbb{A} \in L^2(F^2, d\nu^2)$. Let $B : \mathcal{H}_2 \to \mathcal{H}_1$ be an isomorphism of Hilbert spaces and denote by $T_x := B^{-1} \circ L_x \circ B : \mathcal{H}_2 \to \mathcal{H}_2$. We define the operators $\widetilde{\mathcal{L}} : L^2(F, d\nu)\hat{\otimes}\mathcal{H}_1 \to L^2(F, d\nu)\hat{\otimes}\mathcal{H}_1$ via*

$$(\widetilde{\mathcal{L}}(f_1 \otimes f_2))(y) := \int_F \mathbb{A}_{x,y}\, f_1(x)\, L_x f_2\, d\nu(x)$$

and $\tilde{B} : L^2(F, d\nu)\hat{\otimes}\mathcal{H}_2 \to L^2(F, d\nu)\hat{\otimes}\mathcal{H}_1$ *via* $\tilde{B}(f_1 \otimes f_2) := f_1 \otimes (Bf_2)$*. Then* $B^{-1} \circ \mathcal{L} \circ B = \int_F T_x \, d\nu(x)$ *and* $\tilde{T} := \tilde{B}^{-1} \circ \tilde{\mathcal{L}} \circ \tilde{B} : L^2(F, d\nu)\hat{\otimes}\mathcal{H}_2 \to L^2(F, d\nu)\hat{\otimes}\mathcal{H}_2$ *acts via*

$$(\tilde{T}g)(y) = \int_F \mathbb{A}_{x,y} \, T_x g(x) \, d\nu(x).$$

Proof. For all f, $g \in \mathcal{H}_1$ one has

$$\left\langle \mathcal{L}f \,\middle|\, g \right\rangle_{\mathcal{H}_1} = \int_Y \left\langle L_x f \,\middle|\, g \right\rangle_{\mathcal{H}_1} dx = \int_Y \left\langle T_x B^{-1} f \,\middle|\, B^{-1} g \right\rangle_{\mathcal{H}_2} dx = \left\langle T B^{-1} f \,\middle|\, B^{-1} g \right\rangle_{\mathcal{H}_2},$$

hence $B^{-1} \circ \mathcal{L} \circ B = \int_F T_x \, d\nu(x)$. For all f_1, $g_1 \in \mathcal{H}_1$ f_2, $g_2 \in \mathcal{H}_2$

$$\begin{aligned}
\left\langle \tilde{\mathcal{L}}(f_1 \otimes f_2) \,\middle|\, g_1 \otimes g_2 \right\rangle_{L^2(F,d\nu)\hat{\otimes}\mathcal{H}_1} &= \int_Y \int_Y \mathbb{A}_{x,y} \, f_1(x) \, \left\langle L_x f_2 \,\middle|\, g_2 \right\rangle_{\mathcal{H}_1} dx \, \overline{g_1(y)} \, dy \\
&= \int_Y \int_Y \mathbb{A}_{x,y} \, f_1(x) \, \left\langle T_x B^{-1} f_2 \,\middle|\, B^{-1} g_2 \right\rangle_{\mathcal{H}_1} dx \, \overline{g_1(y)} \, dy \\
&= \left\langle \tilde{T} \tilde{B}^{-1}(f_1 \otimes f_2) \,\middle|\, \tilde{B}^{-1}(g_1 \otimes g_2) \right\rangle_{L^2(F,d\nu)\hat{\otimes}\mathcal{H}_2},
\end{aligned}$$

which shows the second assertion. □

Remark 3.3.5. One way of thinking of $L^2(F, d\nu)\hat{\otimes}\mathcal{H}$ is the so called constant field $\int_F^{\oplus} \mathcal{H} \, d\nu(\sigma)$ over F which is a very special direct integral of Hilbert spaces. See for instance [Ne00, A.II] for the definition and basic properties of the direct integral $\int_F^{\oplus} \mathcal{H}_x \, d\nu(\sigma)$ of Hilbert spaces $\mathcal{H}_x$. In our case the fibers $\mathcal{H}_x$ are constant, but for future transfer operators a non-constant direct integral could be useful. In particular we have in mind the formal Ruelle-Mayer transfer operator from Def. 3.2.2

$$(\tilde{\mathcal{M}} f)(x, z) := \int_F \mathbb{A}_{\sigma,x} \exp(A_\sigma(z)) \; f(\sigma, \psi_\sigma(z)) \, d\nu(\sigma)$$

which has been defined formally on $L^1(F, d\nu)\hat{\otimes}\, \mathcal{C}_b(E)$. If the nature of A_σ and ψ_σ depends on the parameter $\sigma \in F$ drastically, then a direct integral might be an appropriate choice when looking for a Hilbert space where the operator $\tilde{\mathcal{M}}$ acts on. □

3.4 Hard rods model

In this section we introduce a new approach to the so called hard rod model. This will enable us to use the methods from the previous sections of this chapter to find a Ruelle-Mayer transfer operator which satisfies a dynamical trace formula as we will show in Section 3.5.

The lattice spin systems as introduced in (1.2.7) consists of a fixed discrete lattice $\mathbb{L}$, where on each lattice point $i \in \mathbb{L}$ a spin value $\xi(i) \in F$ is attached. We have a semigroup action on the lattice inducing an action on the configuration space, see (1.2.3). This allows the spins to "move" on the (discrete) lattice.

Continuous models allow the spins to move in a non-discrete set. An important example of a continuous model is the one-dimensional *hard rod model* which models the situation of one-dimensional particles (the rods) with a finite positive length moving on the real line. The rods are solid (hard), i. e., they cannot intersect each other. This model has been firstly investigated by M. Kac, G. E. Uhlenbeck, and P. C. Hemmer in their series of joint papers from 1963, [KaUhHe63].

Our general notion of spin values (1.1.1) allows us to mimic this continuous model by a one-dimensional lattice spin system. Think of an "initial configuration" or "zero temperature configuration" in which the (left edge of the) i-th rod is at position $i \in \mathbb{L}$ (with $\mathbb{L} = \mathbb{N}$ or $\mathbb{Z}$), i. e., a uniform configuration. As the temperature increases, the particles start to move a little bit around their initial positions. Having this in mind we give the set $F \subset \mathbb{R}$ of spin values the interpretation as the set of possible movements. Mathematically, we will introduce a map which assigns to the spin value ξ_i of the i-th particle the real number $p_i = i + \xi_i$ interpreted as the position of the i-th particle. Since the particles shall not intersect each other, not all configurations are allowed. We use a matrix subshift to exclude intersections.

In this section we will describe the hard rod model, in the next section we will construct a Ruelle-Mayer type transfer operator for a specific choice of an interaction and prove that it satisfies a dynamical trace formula.

Example 3.4.1. (Hard rods model)

(i) Let $a > 0$ be a constant which will be interpreted as the length of the hard rods. Let $F \subset \mathbb{R}$ be an interval containing zero and $\mathbb{L} = \mathbb{Z}$. The *hard rod model* or *hard rod subshift* is defined as the two-sided one-dimensional matrix subshift $(\Omega_{\mathbb{A}_a}, F, \mathbb{Z}, \mathbb{Z}, \tau)$ (1.2.8), with the transition matrix

$$\mathbb{A}_a : F \times F \to \{0,1\},\ (x,y) \mapsto \mathbb{A}_a(x,y) = \begin{cases} 1, & \text{if } y+1-x \geq a, \\ 0, & \text{otherwise} \end{cases}$$

and the usual shift action $\tau : \mathbb{Z} \times \Omega_{\mathbb{A}_a} \to \Omega_{\mathbb{A}_a}$ (1.2.5). Similarly, one defines the *one-sided hard rod model* $(\Omega^{>}_{\mathbb{A}_a}, F, \mathbb{N}, \mathbb{N}_0, \tau^{>})$ to be the one-sided matrix subshift (1.2.8) with this transition matrix.

(ii) With a configuration $(\xi_i)_{i\in\mathbb{Z}} \in F^{\mathbb{Z}}$ we associate via the injective map

$$p : F^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}},\ (\xi_i)_{i\in\mathbb{Z}} \mapsto (\lambda_i)_{i\in\mathbb{Z}} := (i+\xi_i)_{i\in\mathbb{Z}} \tag{80}$$

the (which we call) *absolute position vector* $(\lambda_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$. The configuration $\eta^0 \in F^{\mathbb{Z}}$ defined via $\eta^0(i) = 0$ for all $i \in \mathbb{Z}$ corresponds to an "initial configuration" in which the (left edge of the) i-th rod is at position i. Interprete ξ_i as the motion of the i-th rod relative to its base point i, such that $\lambda_i = \xi_i + i$ is the absolute position of the (left edge of the) i-th rod.

(iii) The *classical hard rod model* allows the particles to move freely, which corresponds in our modelling to the choice $F = \mathbb{R}$. Hence we have translated a continuous spin into a discrete model at the (mathematical) "cost" of an unbounded set of spin values.

(iv) Let $\mathbb{L} = \mathbb{Z}$, $\delta_1 < 0 < \delta_2$, $F := [\delta_1, \delta_2]$ and $0 < a < 1$ be the length of the hard rods. We call the lattice spin system $(\Omega_{\mathbb{A}_a}, F, \mathbb{Z}, \mathbb{Z}, \tau)$ with this data a *mock hard rod model.* We will take F quite large to imitate the classical hard rod situation. This allows the i-th rod to move in the compact set $F + i$ so that the methods for discrete lattice systems with values in a compact set can be applied. □

Proposition 3.4.2 and the following remark will deepen the understanding of the absolute position map $p : F^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}},\ (\xi_i)_{i\in\mathbb{Z}} \mapsto (\lambda_i)_{i\in\mathbb{Z}} := (i+\xi_i)_{i\in\mathbb{Z}}$, which we introduced in 3.4.1 (ii).

Proposition 3.4.2. *Let $(\Omega_{\mathbb{A}_a}, F, \mathbb{Z}, \mathbb{Z}, \tau)$ be a hard rod subshift (3.4.1) with hard rod length $a > 0$ and position map $p : F^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ (80). Then:*

(i) *Let $\xi \in F^{\mathbb{Z}}$, then ξ belongs to $\Omega_{\mathbb{A}_a}$ iff $p(\xi) \in \mathbb{P}_{\mathbb{A}'_a}$, where $\mathbb{P}_{\mathbb{A}'_a} \subset \mathbb{R}^{\mathbb{Z}}$ is configuration space of the matrix subshift with alphabet $F' = \mathbb{R}$ and transition matrix*

$$\mathbb{A}'_a : F' \times F' \to \{0,1\},\ (x,y) \mapsto \mathbb{A}'_a(x,y) = \begin{cases} 1, & \text{if } y-x \geq a \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *Let $\xi \in \Omega_{\mathbb{A}_a}$ and $i \in \mathbb{N}$, then $\xi_i \geq \xi_1 + (i-1)(a-1)$.*

(iii) *If $a > 1$ and F is bounded from above, then $\Omega_{\mathbb{A}_a} = \emptyset$.*

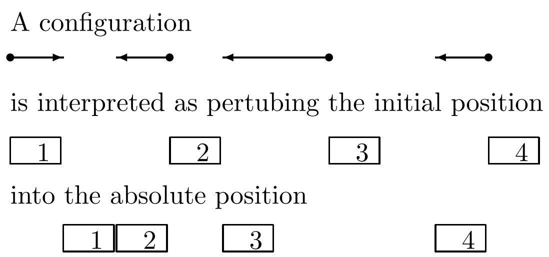

Figure 10: The hard rod model: Configurations and absolute positions.

Proof. For the first assertion observe that $\xi \in \Omega_{\mathbb{A}_a}$ by definition iff $\mathbb{A}_a(\xi_{i+1}, \xi_i) = 1 \ \forall i$, i. e., iff

$$\xi_{i+1} - \xi_i + 1 = i + 1 + \xi_{i+1} - i - \xi_i = (i + 1 + \xi_{i+1}) - (i + \xi_i) \geq a,$$

i. e. iff $p(\xi) \in \mathbb{P}_{\mathbb{A}'_a}$. The second assertion is easily done by induction: Let $\xi \in \Omega_{\mathbb{A}_a}$, then $\xi_1 \geq \xi_1$ and $\xi_{i+1} \geq (a-1) + \xi_i \geq (a-1) + \xi_1 + (i - 1(a-1))\xi_1 + i(a-1)$. The third is an immediate consequence of the second. □

The first assertion of Proposition 3.4.2 can be interpreted in such a way that the $(i+1)$-th rod and the i-th rod do not intersect: Let $\xi \in \Omega_{\mathbb{A}_a}$, then $p(\xi) \in \mathbb{P}_{\mathbb{A}'_a}$, and hence by definition $p(\xi)_{i+1} - p(\xi)_i \geq a$. The i-th rod has its left edge at $p(\xi)_i$, its right edge at $p(\xi)_i + a$ which is a position left to the left edge of the $(i+1)$-th rod at position $p(\xi)_{i+1}$. In our model the rods are not allowed to change their ordering. If one wants to allow this, one would have to take in account some combinatorics.
The action $\tau : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as defined in Example 1.2.5 (6) induces via formula (7) $\mathbb{Z}$-actions not only on the configuration space $F^{\mathbb{Z}}$, but also on $\mathbb{R}^{\mathbb{Z}}$. Hence we can compare the actions on the configuration spaces. It turns out that they almost commute.

Proposition 3.4.3. *Let $(\Omega_{\mathbb{A}_a}, F, \mathbb{Z}, \mathbb{Z}, \tau)$ be a hard rod subshift (3.4.1) with hard rod length $0 < a < 1$. Let $\tau_{(1)}$ be the shift on the configuration space $F^{\mathbb{Z}}$, $\tau_{(2)}$ the shift on $\mathbb{R}^{\mathbb{Z}}$ via formula (7), and $p : F^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ be the position map as in (80). Then one has on $F^{\mathbb{Z}}$ (and hence on $\Omega_{\mathbb{A}_a}$)*

$$\tau_{(2)} \circ p = p \circ \tau_{(1)} + 1.$$

Proof.

$$((\tau_{(2)} \circ p)(\xi))_i = (p(\xi))_{i+1} = \xi_{i+1} + i + 1 = ((p \circ \tau_{(1)})(\xi))_i + 1,$$

since $((p \circ \tau_{(1)})(\xi))_i = i + (\tau_{(1)}(\xi))_i = i + \xi_{i+1}$. □

We will now define a family of two-body interactions for the hard rod model. Having the interpretation via the absolute position map p (80) in mind, we will define the hard rod interaction of a subconfiguration $(\xi_i, \xi_j) \in F^{\{i,j\}}$ as a function which depends on the difference of the absolute positions.

Definition 3.4.4. (Hard rods interaction) Let Γ be a subsemigroup of $\mathbb{Z}$ which acts by translation on itself, $F \subset \mathbb{R}$ an interval containing zero, and $(\Omega, F, \mathbb{Z}, \Gamma, \tau)$ a lattice spin system (1.2.7). We will study pure two-body interactions of the type

$$\phi_\Lambda(\xi_\Lambda) = \varphi_2(i, j; \xi_i, \xi_j) = -\Delta(i + \xi_i - j - \xi_j)$$

where $\Lambda = \{i, j\}$ with $i \neq j$, $\xi_\Lambda = (\xi_i, \xi_j)$, and $\Delta : \mathbb{R} \to \mathbb{R}$ is some even function with a certain decay at infinity which we will specify later. Such an interaction we call a *hard rod two-body interaction with distance function* Δ. According to Remark 1.8.2 a hard rod interaction is translation invariant with respect to the shift action. □

We mention that one might consider the line $\mathbb{Z}$ as $\mathbb{Z} \times \{0\} \subset \mathbb{R}^n$ and spin values $F \subset \mathbb{R}^n$. Then one can model a system where particles are allowed to move in a tubular neighbourhood of the line $\mathbb{Z} \times \{0\} \subset \mathbb{R}^n$. Then define a hard rod interaction via $\varphi_2(i, j; \xi_i, \xi_j) = -\Delta(\|i + \xi_i - j - \xi_j\|)$ for an appropriate norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ and a function $\Delta : \mathbb{R} \to \mathbb{R}$ with certain decay.
We will now determine the hard rod interactions which are compatible with the periodic boundary condition. The analogue of Proposition 1.9.6 for the hard rod interactions is the following:

Proposition 3.4.5. *Let $\mathbb{Z}$ act by translation on itself, $F \subset \mathbb{R}$ be an interval containing zero, and $(\Omega, F, \mathbb{Z}, \mathbb{Z}, \tau)$ a lattice spin system (1.2.7). Let $b^{\mathbb{Z}} = (b^{\mathbb{Z}}_\Lambda)_{\Lambda \in \mathcal{P}_{(n\mathbb{Z})}}$ be the periodic boundary condition (1.3.9) associated to the family $(n\mathbb{Z})_{n \in \mathbb{N}}$ of subsemigroups of $\mathbb{Z}$. Let ϕ be a hard rod two-body interaction*

$$\varphi_2(i, j; \xi_i, \xi_j) = -\Delta(i + \xi_i - j - \xi_j),$$

where $\Delta : \mathbb{R} \to \mathbb{R}$ is even with $\sup_{z \in [0, \frac{1}{2}]} \sum_{\gamma \in \mathbb{Z}} |\Delta(\gamma + z)| < \infty$. Then the total energy $U^{b^{\mathbb{Z}}, \phi}_\Lambda$ defined in (1.6.4) converges absolutely, i. e., the boundary condition and the interaction are compatible.

Proof. By Proposition 1.9.3 it is sufficient to show (the last inequality of)

$$\sum_{\gamma\in\mathbb{Z}} |\varphi_2(i,\gamma\cdot l;x,y)| = \sum_{\gamma\in\mathbb{Z}} |\Delta(i+x-\gamma-l-y)| = \sum_{\gamma\in\mathbb{Z}} |\Delta(\gamma+x-y)| \le \sup_{z\in[0,\frac{1}{2}]} \sum_{\gamma\in\mathbb{Z}} |\Delta(\gamma+z)| < \infty.$$

Here we made substitutions $\gamma' = \gamma + l - i$, $\gamma' = \gamma + \lceil x - y \rceil$, and possibly $\gamma' = -\gamma$ to ensure that z can be chosen in the interval $[0, \frac{1}{2}]$. □

Remark 3.4.6. The term $\sup_{z\in[0,\frac{1}{2}]} \sum_{\gamma\in\mathbb{Z}} |\Delta(\gamma+z)|$ in our sufficient condition in Proposition 3.4.5 can be bounded from above by

$$\sup_{z\in[0,1]} \sum_{\gamma\in\mathbb{Z}} |\Delta(\gamma+z)| = \sup_{z\in[0,1]} \sum_{\gamma=0}^{\infty} \Big(|\Delta(\gamma+z)| + |\Delta(\gamma+1-z)|\Big) \le 2 \sup_{z\in[0,1]} \sum_{k=0}^{\infty} |\Delta(\gamma+z)|,$$

which is twice the bound for the corresponding one-sided system. □

Similarly to Example 1.9.7 we get the following examples of hard rod interactions which satisfy the summability condition from Proposition 3.4.5 and hence are compatible with the periodic boundary condition.

Example 3.4.7. The following distance functions $\Delta : \mathbb{R} \to \mathbb{R}$ satisfy the summability condition from Proposition 3.4.5, i. e., $\sup_{z\in[0,\frac{1}{2}]} \sum_{\gamma\in\mathbb{Z}} |\Delta(\gamma+z)| < \infty$.

(i) Exponentially decaying interactions $\Delta(z) = \lambda^{|z|}$ for $0 < |\lambda| < 1$.

(ii) Mock polynomially decaying interactions $\Delta(z) = (\epsilon + |z|)^{-s}$ for $\mathbf{Re}(s) > 1, \epsilon > 0$.

(iii) Logarithmic interaction $\Delta(z) = \log(1 - c\lambda^{|z|})$ for $0 < |c|, |\lambda| < 1$.

In fact:

(i) For $z \in [0,1]$ we have $\sum_{k=0}^{\infty} |\lambda|^{|k+z|} = |\lambda|^z \sum_{k=0}^{\infty} |\lambda|^k = \frac{|\lambda|^z}{1-|\lambda|} \le \frac{1}{1-|\lambda|} < \infty$.

(ii) For $z \in [0,1]$ we have

$$\sum_{k=0}^{\infty} |(\epsilon + |k+z|)^{-s}| = (\epsilon+z)^{-\mathbf{Re}(s)} + \sum_{k=1}^{\infty} (\epsilon+z+k)^{-\mathbf{Re}(s)} \le \epsilon^{-\mathbf{Re}(s)} + \sum_{k=1}^{\infty} k^{-\mathbf{Re}(s)} < \infty.$$

(iii) Using $|\log(1-z)| \le -\log(1-|z|)$ for $|z| < 1$ we get

$$\sum_{k=0}^{\infty} |\log(1-c\lambda^{k+z})| \le -\sum_{k=0}^{\infty} \log(1-|c|\,|\lambda|^{k+z}) = -\log\Big(\prod_{k=0}^{\infty}(1-|c|\,|\lambda|^{k+z})\Big),$$

which converges since $\sum_{k=0}^{\infty} |\lambda|^{k+z} < \infty$. □

3.5 Transfer operators for the hard rod model

In this section we derive the Ruelle-Mayer transfer operator for the one-sided one-dimensional hard rod model. By the previous Section 3.4 we have a representation of the hard rod model as a matrix subshift which makes the methods from Section 3.3 available. We will exemplarily deal with the polynomial-exponentially decaying hard rod interaction for which we can find a Ruelle-Mayer transfer operator. This RM operator satisfies a dynamical trace formula as we will prove in Corollary 3.5.2. Whereas for a mock hard rod model, i. e. the set F of spin values is a bounded subset of $\mathbb{R}$, the only condition on the a priori measure ν on F is its finiteness, for unbounded F we have to require a strong decay at infinity. We mention that in [MayVi77] there has been found a Ruelle-Mayer type transfer operator for exponentially decaying interactions starting from a different approach. The proof of the dynamical trace formula presented there is a long and technical computation, whereas our approach will directly yield the desired formula.
Let $a > 0$ be a positive number interpreted as the length of the hard rods, $F \subset \mathbb{R}$ an interval containing zero equipped with a finite Borel measure ν, and $(\Omega_{\mathbb{A}_a}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-sided hard rod

subshift (3.4.1). Let ϕ be a pure two-body hard rod interaction (3.4.4) with distance function Δ, i. e., ϕ is of the form

$$\phi_\Lambda(\xi_\Lambda) = -\Delta(i + \xi_i - j - \xi_j), \text{ if } \Lambda = \{i, j\},\ \xi_\Lambda = (\xi_i, \xi_j),$$

where $\Delta : \mathbb{R} \to \mathbb{R}$ is an even function. The *standard hard rod observable* is given as

$$A_{(\phi)} : \Omega_{\mathbb{A}_a} \to \mathbb{R},\ \xi \mapsto \sum_{i=2}^{\infty} \Delta(\xi_i - \xi_1 + i - 1). \tag{81}$$

In order to make this well-defined, one needs certain decay estimates for the distance function Δ, see for instance Proposition 3.4.6. Exemplarily we discuss the case of polynomial-exponentially decaying interactions, i. e., $\Delta : \mathbb{R} \to \mathbb{C},\ x \mapsto \lambda^{|x|} \sum_{i=0}^{m-1} c_i\, |x|^i$, where $0 < \lambda < 1$, and $c_i \in \mathbb{C}$ $(i = 0, \ldots, m-1)$. In Proposition 3.4.2 (ii) we showed that for all $\xi \in \Omega_{\mathbb{A}_a}$ and $i \in \mathbb{N}$ the estimate $\xi_i \geq \xi_1 + (i-1)(a-1)$ holds true. Hence

$$\xi_i + i - \xi_1 - 1 \geq (i-1)(a-1) + i - 1 = (i-1)a \geq 0.$$

For all $0 < \epsilon < 1 - \lambda$ one can find $c > 0$ such that $\left|\lambda^{|x|} \sum_{i=0}^{m-1} c_i\, |x|^i\right| \leq c\,(\lambda + \epsilon)^x$ for all $x \in \mathbb{R}$. This implies

$$|A_{(\Delta)}(\xi)| \leq c \sum_{i=2}^{\infty} (\lambda + \epsilon)^{|\xi_i - \xi_1 + i - 1|} \leq c \sum_{i=2}^{\infty} (\lambda + \epsilon)^{(i-1)a} < \infty,$$

i. e., $A_{(\phi)} : \Omega_{\mathbb{A}_a} \to \mathbb{C}$ is bounded by a constant which does not depend on F.

According to Proposition 3.1.2 we define the Ruelle transfer operator $\tilde{\mathcal{L}}_{A_{(\phi)}} \in \operatorname{End}\big(L^2(F, \nu) \hat{\otimes}_\pi \mathcal{C}_b(\Omega)\big)$ associated to the observable $A_{(\phi)} \in \mathcal{C}_b(F)$ via

$$(\tilde{\mathcal{L}}_{A_{(\phi)}} f)(x, \xi) = \int_F \mathbb{A}_{\sigma,x} \exp\big(A_{(\phi)}(\sigma \vee \xi)\big)\, f(\sigma, \sigma \vee \xi)\, d\nu(\sigma).$$

The following proposition is the key step for the construction of the Ruelle-Mayer transfer operator. It states that π is a projection map leading to an (S1) - (S3) Ansatz 2.3.3 and thus provide a factorisation of the Ruelle transfer operator which we need in order to construct the Ruelle-Mayer transfer operator.

Proposition 3.5.1. *Let $a > 0$ be the length of the hard rods, $F \subset \mathbb{R}$ an interval containing zero equipped with a finite Borel measure ν, and $(\Omega_{\mathbb{A}_a}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-sided hard rod subshift (3.4.1). Let ϕ be a two-body hard rod interaction (3.4.4) with distance function $\Delta : \mathbb{R} \to \mathbb{C},\ x \mapsto \lambda^{|x|} \sum_{i=0}^{m-1} c_i\, |x|^i$, where $0 < \lambda < 1$ and $c_i \in \mathbb{C}$ $(i = 0, \ldots, m-1)$. Let*

$$\pi = (\pi_0, \ldots, \pi_{m-1}) : \Omega \to \mathbb{R}^m,\ \xi \mapsto \Big(\sum_{j=1}^{\infty} (\xi_j + j)^k\, \lambda^{\xi_j + j}\Big)_{k=0,\ldots,m-1}$$

and $\mathbb{D}_x^{(m)} \in \operatorname{Mat}(m, m; \mathbb{R})$ be the lower triangular matrix defined via

$$(\mathbb{D}_x^{(m)})_{i,j} := \begin{cases} x^{i-j} \binom{i}{j} & , j \leq i, \\ 0 & , \text{otherwise}, \end{cases}$$

for $x \in \mathbb{R}$ and $i, j = 0, \ldots, m-1$. Then $\pi(\sigma \vee \xi) = \Big((\sigma+1)^k \lambda^{\sigma+1}\Big)_{k=0,\ldots,m-1} + \lambda \mathbb{D}_0^{(m)} \pi(\xi)$ and the standard observable $A_{(\phi)} : \Omega \to \mathbb{R}$ (81) can be expressed as[29]

$$A_{(\phi)}(\sigma \vee \xi) = \lambda^{-\sigma} \left(\mathbb{D}_{-\sigma}^{(m)} \pi(\xi) \mid c\right).$$

Proof. Note that $\mathbb{D}_0^{(m)} = \mathbb{B}^{(m)} \in \operatorname{Mat}(m, m; \mathbb{R})$ as defined in Remark 2.11.1. For all $k = 0, \ldots, m-1$ and $\sigma \vee \xi \in \Omega_{\mathbb{A}_a}$ we compute

$$\begin{aligned} \pi_k(\sigma \vee \xi) &= (\sigma+1)^k \lambda^{\sigma+1} + \sum_{j=1}^{\infty} (\xi_j + j + 1)^k \lambda^{\xi_j + j + 1} \\ &= (\sigma+1)^k \lambda^{\sigma+1} + \lambda \sum_{l=0}^{k} \binom{k}{l} \sum_{j=1}^{\infty} (\xi_j + j)^l \lambda^{\xi_j + j} \\ &= (\sigma+1)^k \lambda^{\sigma+1} + \lambda \sum_{l=0}^{k} \binom{k}{l} \pi_l(\xi), \end{aligned}$$

[29] $(\cdot \mid \cdot)$ denotes the $\mathbb{C}$-bilinear extension of the euclidean scalar product, i. e., $(x \mid y) = \sum_i x_i\, y_i$.

which shows the first assertion. Concerning the second we note that

$$\begin{aligned} A_{(\phi)}(\sigma \vee \xi) &= \sum_{j=1}^{\infty}\sum_{i=0}^{m-1} c_i(\xi_j + j - \sigma)^i \lambda^{\xi_j + j - \sigma} \\ &= \sum_{i=0}^{m-1} c_i \sum_{j=1}^{\infty}(\xi_j + j - \sigma)^i \lambda^{\xi_j + j - \sigma} \\ &= \lambda^{-\sigma}\sum_{i=0}^{m-1} c_i \sum_{j=1}^{\infty}\sum_{k=0}^{i}\binom{i}{k}(-\sigma)^{i-k}(\xi_j + j)^k \lambda^{\xi_j + j} \\ &= \lambda^{-\sigma}\sum_{i=0}^{m-1} c_i \sum_{k=0}^{i}\binom{i}{k}(-\sigma)^{i-k}\pi_k(\xi) = \lambda^{-\sigma}\left(\mathbb{D}_{-\sigma}^{(m)}\pi(\xi) \mid c\right). \end{aligned}$$

□

According to Definition 3.2.2 we obtain the Ruelle-Mayer transfer operator $\tilde{\mathcal{M}}$ acting as a formal operator on $L^2(F, d\nu)\hat{\otimes}_\pi \mathcal{C}_b(\mathbb{C}^m)$ via

$$(\tilde{\mathcal{M}} f)(x,z) := \int_F \mathbb{A}_a(\sigma, x)\, \exp\Big(\big(\, z \mid \lambda^{-\sigma}(\mathbb{D}_{-\sigma}^{(m)})^\top c\big)\Big)\; f\big(\sigma, \big((\sigma+1)^k \lambda^{\sigma+1}\big)_{k=0,\dots,m-1} + \lambda \mathbb{B}^{(m)} z\big)\, d\nu(\sigma)$$

where $\mathbb{D}_x^{(m)} \in \mathrm{Mat}(m, m; \mathbb{R})$ as defined in Proposition 3.5.1. Motivated by the results for Ising type interactions we will study the spectral properties of the operator acting via this formula on the Hilbert space $L^2(F, d\nu)\hat{\otimes}\, \mathcal{F}(\mathbb{C}^m)$. We will show that this operator fulfills a dynamical trace formula provided some conditions on the measure ν on F which we explain first.

As an abbreviation we set $a_\sigma := \overline{\beta \lambda^{-\sigma}(\mathbb{D}_{-\sigma}^{(m)})^\top c}$ and $b_\sigma := \big((\sigma+1)^k \lambda^{\sigma+1}\big)_{k=0,\dots,m-1}$, which are continuous functions with respect to the parameter $\sigma \in \mathbb{R}$. We set

$$(\mathcal{M}_\sigma f)(z) := \exp(\langle z | a_\sigma \rangle)\; f(b_\sigma + \lambda \mathbb{B}^{(m)} z). \tag{82}$$

By Remark 2.6.4 the spectral radius $\rho_{\mathrm{spec}}(\mathbb{B}^{(m)})$ of $\mathbb{B}^{(m)}$ is equal to $\max\{|z| \in \mathrm{spec}(\mathbb{B}^{(m)})\} = 1$. We have $\|(\lambda \mathbb{B}^{(m)})^n\| < 1$ for sufficiently large powers n and $\mathbb{B}^{(m)} \in \mathcal{S}_1(\mathbb{C}^m)$ trivially, hence by Lemma B.3.10 the composition operator $\mathcal{M}_\sigma : \mathcal{F}(\mathbb{C}^{p+1}) \to \mathcal{F}(\mathbb{C}^m)$ belongs to the Schatten class $\mathcal{S}_n(\mathcal{F}(\mathbb{C}^m))$. We assume for a moment that $\|\lambda \mathbb{B}^{(m)}\| < 1$, hence $\mathcal{M}_\sigma$ is trace class (the general case goes analogous to the proof of Theorem 2.7.6). By Corollary B.4.4 the Hilbert-Schmidt norm (A.2.2) of $\mathcal{M}_\sigma : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ is equal to

$$\|\mathcal{M}_\sigma\|^2_{\mathcal{S}_2(\mathcal{F}(\mathbb{C}^m))} = \frac{\exp\Big(\pi\|\pi^{-1}a_\sigma\|^2 + \pi\|(1-\lambda^2\mathbb{B}^{(m)}(\mathbb{B}^{(m)})^\star)^{-1/2}(\pi^{-1}\mathbb{B}^{(m)}a_\sigma + b_\sigma)\|^2\Big)}{\det(1-\lambda^2\mathbb{B}^{(m)}(\mathbb{B}^{(m)})^\star)}. \tag{83}$$

By investigating the coefficients of a_σ and b_σ, one confirms that $\|b_\sigma\|$ is bounded as $\sigma \to \infty$ and of order $\lambda^{-\sigma}$ as $\sigma \to -\infty$. Similarly, $\|a_\sigma\|$ is bounded as $\sigma \to -\infty$ and of order λ^σ as $\sigma \to \infty$. In other word, the sum $\|a_\sigma\| + |b_\sigma\|$ is of order $\exp(|\sigma \log \lambda|)$ as $|\sigma| \to \infty$, and the Hilbert-Schmidt norm of $\mathcal{M}_\sigma : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ growths double-exponentially as $|\sigma| \to \infty$.

Corollary 3.5.2. *Let $a > 0$ be the length of the hard rods, $F \subset \mathbb{R}$ an interval containing zero equipped with a finite Borel measure ν, and $(\Omega_{\mathbb{A}_a}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided hard rod subshift (3.4.1). Suppose that $\int_F \|\mathcal{M}_\sigma\|^2_{\mathcal{S}_2(\mathcal{F}(\mathbb{C}^{p+1}))}\, d\nu(\sigma)$ given by (83) above is finite. Let ϕ be a two-body hard rod interaction (3.4.4) with distance function $\Delta : \mathbb{R} \to \mathbb{C},\ x \mapsto \lambda^{|x|}\sum_{i=0}^{m-1} c_i\, |x|^i$, where $0 < \lambda < 1$ and $c_i \in \mathbb{C}$ ($i = 0, \dots, m-1$). Let $A_{(\phi)}$ be the standard observable (81) and $\mathbb{D}_x^{(m)} \in \mathrm{Mat}(m, m; \mathbb{R})$ as defined in Proposition 3.5.1. Then there exists an index $n_0 \in \mathbb{N}$ depending on m and λ such that for all $n \geq n_0$ the Ruelle-Mayer transfer operator $\tilde{\mathcal{M}}_\beta : L^2(F, d\nu)\hat{\otimes}\, \mathcal{F}(\mathbb{C}^m) \to L^2(F, d\nu)\hat{\otimes}\, \mathcal{F}(\mathbb{C}^m)$ acting via*

$$(\tilde{\mathcal{M}}_\beta f)(x,z) := \int_F \mathbb{A}_a(\sigma, x)\, \exp\Big(\beta\big(\, z \,|\, \lambda^{-\sigma}(\mathbb{D}_{-\sigma}^{(m)})^\top c\big)\Big)\; f\big(\sigma, \big((\sigma+1)^k \lambda^{\sigma+1}\big)_{k=0,\dots,m-1} + \lambda \mathbb{D}_0^{(m)} z\big)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1-\lambda^n)^m\, \mathrm{trace}\, (\tilde{\mathcal{M}}_\beta)^n$.

Proof. Each of the operators $\mathcal{M}_\sigma : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ introduced in (82) satisfies the Atiyah-Bott trace formula (53). Then the assertion follows from Theorem 3.2.5 and the fact that $\det\big(1 - \lambda(\mathbb{D}_0^{(m)})^n\big) = \det(1 - (\lambda\mathbb{B}^{(m)})^n) = (1 - \lambda^n)^m$. $\square$

We will now deal with a special case, namely with exponentially decaying hard rod interactions. Let ϕ be a two-body hard rod interaction (3.4.4) with distance function $\Delta : \mathbb{R} \to \mathbb{R},\ x \mapsto \lambda^{|x|}$, where $0 < \lambda < 1$. In this situation our condition $\int_F \|\mathcal{M}_\sigma\|^2_{\mathcal{S}_2(\mathcal{F}(\mathbb{C}))}\, d\nu(\sigma) < \infty$ on the a priori measure ν can be well analysed. The matrix $\mathbb{D}_x^{(1)} = \mathbb{B}^{(1)} = 1 \in \mathrm{Mat}(1, 1; \mathbb{R})$ has operator norm equal to one and the auxiliary operators $\mathcal{M}_\sigma : \mathcal{F}(\mathbb{C}) \to \mathcal{F}(\mathbb{C})$ simplify to

$$(\mathcal{M}_\sigma f)(z) = \exp\big(\beta\lambda^{-\sigma} z\big)\ f(\lambda(\lambda^\sigma + z)).$$

By Corollary B.4.4 we can explicitly determine the Hilbert-Schmidt norm

$$\begin{aligned} \|\mathcal{M}_\sigma\|^2_{\mathcal{S}_2(\mathcal{F}(\mathbb{C}))} &= \frac{\exp\big(\pi|\pi^{-1}\beta\lambda^{-\sigma}|^2 + \pi|(1-\lambda^2)^{-1/2}(\pi^{-1}\beta\lambda^{-\sigma} + \lambda^{1+\sigma})|^2\big)}{1-\lambda^2} \\ &= \frac{\exp(\pi\left(|\beta/\pi|^2\lambda^{-2\sigma} + (1-\lambda^2)^{-1}|\beta\pi^{-1}\lambda^{-\sigma} + \lambda^{1+\sigma}|^2\right))}{1-\lambda^2}, \end{aligned}$$

which allows to determine the admissible sets $F \subset \mathbb{C}$ and Borel measures ν_F on F such that the Ruelle-Mayer transfer operator $\tilde{\mathcal{M}}_\beta : L^2(F, d\nu)\hat{\otimes}\,\mathcal{F}(\mathbb{C}) \to L^2(F, d\nu)\hat{\otimes}\,\mathcal{F}(\mathbb{C})$,

$$(\tilde{\mathcal{M}}_\beta f)(x, z) := \int_F \mathbb{A}_a(\sigma, x)\ \exp\big(\beta\lambda^{-\sigma} z\big)\ f(\sigma, \lambda(\lambda^\sigma + z))\, d\nu(\sigma)$$

belongs to the Hilbert-Schmidt class. Corollary 3.5.2 immediately implies the following:

Corollary 3.5.3. *Let $a > 0$ be the length of the hard rods, $F \subset \mathbb{R}$ an interval containing zero equipped with a finite Borel measure ν, and $(\Omega_{\mathbb{A}_a}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ a one-sided hard rod subshift (3.4.1). Let ϕ be a two-body hard rod interaction (3.4.4) with distance function $\Delta : \mathbb{R} \to \mathbb{R},\ x \mapsto \lambda^{|x|}$, where $0 < \lambda < 1$. Let $A_{(\phi)}$ be the standard observable (81). If $\int_F \|\mathcal{M}_\sigma\|^2_{\mathcal{S}_2(\mathcal{F}(\mathbb{C}))}\, d\nu(\sigma)$ is finite, then for all $n \in \mathbb{N}_{>1}$ the Ruelle-Mayer transfer operator $\tilde{\mathcal{M}}_\beta : L^2(F, d\nu)\hat{\otimes}\,\mathcal{F}(\mathbb{C}) \to L^2(F, d\nu)\hat{\otimes}\,\mathcal{F}(\mathbb{C})$,*

$$(\tilde{\mathcal{M}}_\beta f)(x, z) := \int_F \mathbb{A}_a(\sigma, x)\ \exp\big(\beta\lambda^{-\sigma} z\big)\ f(\sigma, \lambda(\lambda^\sigma + z))\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0}, \phi}_{\{1,\ldots,n\}}(\beta) = \det(1 - \lambda^n)\,\mathrm{trace}\ (\tilde{\mathcal{M}}_\beta)^n$. $\square$

4 The Dynamical Zeta Function

In this chapter we study the properties of the partition function by using a method from number theory. The dynamical zeta function ζ_R has been introduced by Ruelle in [Ru76], [Ru76a], [Ru94], [Ru02] and is defined as the generating function of the partition functions

$$\zeta_R(z,\beta) := \exp\left(\sum_{n=1}^{\infty} Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}(\beta)\,\frac{z^n}{n}\right).$$

From its definition and a standard estimate on the partition functions it is appearend that ζ_R is a holomorphic function in a neighbourhood of zero with finite radius of convergence. Methods from analytic number theory allow to relate analytic properties of zeta to average properties of the partition function and hence yield information about the dynamical system. For this reasoning one needs the existence of a meromorphic continuation of zeta beyond the first pole.

If the partition functions can be expressed via a dynamical trace formula of a transfer operator, then we will show that ζ_R has an Euler product and a meromorphic continuation to the entire complex plane. This will be done using a representation of Ruelle's zeta as a quotient of (regularised) Fredholm determinants. Hence the zeros and poles of ζ_R have a spectral interpretation, i. e., can be given in terms of the eigenvalues of the transfer operator. We refer to Appendix A.1 and A.2 for the definition and properties of regularised determinants.

In section 4.1 we define the dynamical zeta function and prove its meromorphic continuation in the easiest case. Section 4.2 investigates a special class of generating functions, which contains the dynamical zeta function in the case when a dynamical trace formula holds. To prove the meromorphic continuation in the general case we will also need certain limits of sequences of generating functions. This will be done in Section 4.3. These results will be applied in Section 4.4, where we show the following result: Suppose the spin systems satisfies a dynamical trace formula of the form

$$Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}} = \det(1-\Lambda^n)\,\mathrm{trace}\, G^n$$

for all $n \geq n_0$, where G is a transfer operator of class $\mathcal{S}_{n_0}(\mathcal{H})$ and $\Lambda \in \mathcal{S}_1(\mathcal{H}_0)$ with $\rho_{\mathrm{spec}}(\Lambda) < 1$. Then the dynamical zeta function has a meromorphic continuation to the entire plane and an Euler product.

4.1 Basic properties

In this section we define the dynamical zeta function and discuss some of its basic properties. Then we show that the dynamical zeta function can be represented in special cases via regularised Fredholm determinants which have an Euler product and thus a meromorphic continuation to the entire complex plane. This applies for the non-interacting case $\beta = 0$, finite range interactions, and superexponentially decreasing Ising type interactions due to the dynamical trace formulae proved in Corollary 2.8.3 and Proposition 2.9.3, respectively.

Definition 4.1.1. Given the sequence $\big(Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}\big)_{n\in\mathbb{N}}$ of partition functions (1.7.1) associated to a one-sided one-dimensional shift, the *dynamical zeta function* or *Ruelle zeta function* is defined as their generating function, i. e.

$$\zeta_R(z,\beta) := \exp\left(\sum_{n=1}^{\infty} Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}(\beta)\,\frac{z^n}{n}\right)$$

where $\beta \in \mathbb{C}$ is the complexification of the inverse temperature. For fixed β Ruelle's zeta defines a holomorphic function in a neighbourhood of $z = 0 \in \mathbb{C}$ (see Remark 4.1.2). □

A similar definition can be made by replacing the sequence of partition functions by the sequence of dynamical partition functions $\tilde{Z}^{b^{\mathbb{N}_0}}_n(A)$ (1.11.4) for an observable A. For the standard observable $\beta A_{(\phi)}$ (1.11.1) these two definitions coincide since $Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}(\beta) = \tilde{Z}^{b^{\mathbb{N}_0}}_n(\beta A_{(\phi)})$ as we have shown in Remark 1.11.5. We will drop the parameter β occasionally and write then $\zeta_R(z)$.

In the following Remark 4.1.2 we show that zeta has a finite non-zero radius of convergence, give then a physical interpretation of the first pole of zeta, and compute the zeta function in a trivial case.

Remark 4.1.2. (Properties of ζ_R)

(i) Let $\beta \in \mathbb{C}$. We use the representation of the partition function provided by Corollary 1.11.3

$$Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta) = \tilde{Z}^{b^{\mathbb{N}_0}}_n(\beta A_{(\phi)}) = \int_{F^n} \prod_{i=1}^{n} \mathbb{A}_{x_i,x_{i+1}} \exp\Big(\beta \sum_{k=0}^{n-1} A_{(\phi)}(\tau_k(\overline{x_1 \ldots x_n}))\Big)\, d\nu^n(x_1,\ldots,x_n),$$

from which it is apparent that $\big|Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)\big| \leq \big(\exp(\|\beta\, A_{(\phi)}\|_{\mathcal{C}_b(\Omega)})\, \nu(F)\big)^n$. Hence the radius of convergence $\rho(\beta)$ of zeta is non-zero, but finite. It is given by

$$\rho(\beta) := \lim_{n\to\infty} \big|Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)\big|^{-1/n}. \tag{84}$$

(ii) Suppose the sequence of partition functions consists of positive numbers. This happens for instance if the interaction and the parameter β are real. By a result of Pringsheim[30] a power series whose coefficients are positive has its first pole at the intersection of the positive real line and the boundary of the disk of convergence, i. e., precisely at $\rho(\beta)$ (84). This quantity is related to the *free energy* $f := -\frac{1}{\beta}\lim_{n\to\infty} \frac{1}{n} \log Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)$ of the system via $\rho(\beta) = \exp(\beta\, f)$. Recall from Theorem 2.1.4 the definition and characterisation of the topological pressure $P(A)$ of a real-valued observable A as

$$P(A) = \lim_{n\to\infty} \frac{1}{n} \log \widetilde{Z}^{b^{\mathbb{N}_0}}_n(A).$$

Hence using Definition 1.11.4 of the dynamical partition function $\widetilde{Z}^{b^{\mathbb{N}_0}}_n(A)$, the topological pressure of the standard observable $A_{(\phi)}$ satisfies $\rho(\beta) = \exp(-P(\beta A_{(\phi)})) = \exp(\beta f)$.

(iii) In Remark 1.7.2 and Corollary 3.2.7 we have seen that for the non-interacting case $\beta = 0$ we have $\widetilde{Z}^{b^{\mathbb{N}_0}}_n(0) = Z^{b^{\mathbb{N}_0},0}_{\{1,\ldots,n\}} = \nu^n(\rho_{\{1,\ldots,n\}}(\mathrm{Fix}(\tau^n : \Omega_{\mathbb{A}} \to \Omega_{\mathbb{A}})))$, which measures the number of closed orbits of with period length n with respect to the a priori measure ν. In particular, this quantity is independent of the specific interaction. The topological pressure of the zero observable, $h := P(0) = -\log\rho(0)$, is called the *entropy.*

(iv) In particular, if the spin system is a one-sided one-dim. full shift (1.2.6), then $Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(0) = \nu(F)^n$. The entropy h is thus $h = \log\nu(F)$. Using the definition of the dynamical zeta function (4.1.1) and the power series representation of $-\log(1-x)$, we obtain

$$\zeta_R(z,0) = \exp\Big(\sum_{n=1}^{\infty} \nu(F)^n \frac{z^n}{n}\Big) = \exp(-\log(1-\nu(F)z) = \frac{1}{1-\nu(F)z}.$$

Hence, in this very simple case, the zeta function is rational. We will consider the zeta function for the general non-interacting case in Remark 4.1.6. □

Unlike other kinds of zeta functions, as for instance Riemann's, Selberg's, or Artin's zeta, our dynamical zeta function is an exponential of a power series, hence itself a power series as the next remark shows. By considering $s \mapsto \zeta_R(e^{-s},\beta)$ one obtains a function which is holomorphic in the right half plane $\mathbf{Re}(s) > -\log\rho(\beta)$.

Remark 4.1.3. Lemma A.1.3 leads to an explicit power series expansion of zeta as $\zeta_R(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\, z^n$ with coefficients $a_0 = 1$ and for $n \geq 1$

$$a_n = \det \begin{pmatrix} Z_1 & n-1 & 0 & \cdots & 0 & 0 \\ -Z_2 & Z_1 & n-2 & \cdots & 0 & 0 \\ Z_3 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ (-1)^n Z_{n-1} & (-1)^{n-1} Z_{n-2} & (-1)^{n-2} Z_{n-3} & \cdots & Z_1 & 1 \\ (-1)^{n+1} Z_n & (-1)^n Z_{n-1} & (-1)^{n-1} Z_{n-2} & \cdots & -Z_2 & Z_1 \end{pmatrix},$$

where $Z_n := Z^{b^{\mathbb{N}_0},\phi}_{\{1,\ldots,n\}}(\beta)$ for abbreviation. □

[30] as cited in [May80a, p. 116], there reference to [La29].

A natural question is to ask whether ζ_R has a meromorphic continuation to a larger disk or even to the entire complex plane. The following remark gives a partial answer. Since we know the first pole of zeta implicitely, it can be separated. The remainder can again be written a generating function of almost the same type.

Remark 4.1.4. We combine the previous considerations of Remark 4.1.2 with an argument appearing in [PaPo90, p. 81] and obtain that for $\rho(\beta)$ as defined in (84) we have

$$\zeta_R(z,\beta) = \exp\Big(\sum_{n=1}^{\infty} \frac{(z/\rho(\beta))^n}{n}\Big) \exp\Big(\sum_{n=1}^{\infty} \Big(Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}(\beta) - \rho(\beta)^{-n}\Big) \frac{z^n}{n}\Big) = \frac{\eta(z,\beta)}{1 - z/\rho(\beta)},$$

where $\eta(\,\cdot\,,\beta)$ is holomorphic in a neighbourhood of $\rho(\beta)$. □

This splitting idea from Remark 4.1.4 now can be iterated, if the poles do not accumulate. For a special class of sequences $(Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}})_{n\in\mathbb{N}}$, namely those for which a dynamical trace formula holds, we will obtain a complete factorisation of the dynamical zeta function in terms of (regularised[31]) Fredholm determinants. The following lemma is the key idea towards proving the meromorphic continuation of the dynamical zeta functions for systems with exponentially decaying interactions. The case $n_0 = 1$ (i. e. G trace class) was observed by Moritz in [Mo89].

Lemma 4.1.5. *Suppose there exists a transfer operator $G \in \mathcal{S}_{n_0}(\mathcal{H})$ which satisfies the dynamical trace formula $Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}} = \operatorname{trace} G^n$ for all $n \geq n_0$. Then*

$$\zeta_R(z) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n} Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}\Big) \operatorname{det}_{n_0}(1 - zG)^{-1}$$

gives the meromorphic continuation of Ruelle's zeta to the entire $\mathbb{C}-$plane.

Proof. We write the dynamical zeta function as

$$\zeta_R(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n} Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}\Big) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n} Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}\Big)\Big(\exp\Big(-\sum_{n=n_0}^{\infty} \frac{z^n}{n} \operatorname{trace} G^n\Big)\Big)^{-1}.$$

Obviously the first factor is an entire function, so we have to analyse the second. For small $|z|$ it is given as the n_0-regularised determinant by Lemma A.1.2 which has a meromorphic continuation to the entire $\mathbb{C}$-plane. □

When combining Lemmas A.1.4 and 4.1.5 we obtain an Euler product expansion of ζ_R. Our first application of the previous lemma is the general non-interacting case, i. e., $\beta = 0$.

Remark 4.1.6. Let $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift (1.2.8) and ν a finite measure on F. Then by Corollary 3.2.7 the transfer operator

$$\mathcal{G}_{\mathbb{A}} : L^2(F, d\nu) \to L^2(F, d\nu),\ (\mathcal{G}_{\mathbb{A}} f)(x) = \int_F \mathbb{A}_{\sigma,x}\, f(\sigma)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $Z^{b^{\mathbb{N}_0},0}_{\{1,\dots,n\}}(0) = \nu^n(\rho_{\{1,\dots,n\}}(\operatorname{Fix}(\tau^n : \Omega_{\mathbb{A}} \to \Omega_{\mathbb{A}}))) = \operatorname{trace}\,(\mathcal{G}_{\mathbb{A}})^n$ for $n \geq 2$. Hence by Lemma 4.1.5

$$\zeta_R(z,0) = \exp\Big(z\, Z^{b^{\mathbb{N}_0},0}_{\{1\}}\Big) \exp\Big(\sum_{n=2}^{\infty} \frac{z^n}{n} \operatorname{trace} \mathcal{G}^n_{\mathbb{A}}\Big) = \frac{\exp\Big(z\, Z^{b^{\mathbb{N}_0},0}_{\{1\}}\Big)}{\operatorname{det}_2(1 - z\mathcal{G}_{\mathbb{A}})}.$$

□

As a consequence of Lemma 4.1.5 and Corollary 2.8.3 for finite range interactions and Proposition 2.9.3 for superexponentially decreasing distance functions we obtain the following result.

[31] In Appendix A.1 and A.2 we give an introduction to regularised determinants and their properties.

Corollary 4.1.7. *Let $F \subset \mathbb{C}$ be a bounded set and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with distance function d and potential $q \in \mathcal{C}_b(F)$.*

(i) *If d has finite range ρ_0, then the transfer operator*

$$\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^{\rho_0}) \to \mathcal{F}(\mathbb{C}^{\rho_0}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\bigl(\beta q(x) + \beta\, z_1\bigr)\, f(x\, w^d + \lambda \mathbb{S}_{\rho_0} z)\, d\nu(x)$$

defined in Corollary 2.8.3 satisfies $\zeta_R(z, \beta) = \det(1 - z\mathcal{M}_\beta)^{-1}$.

(ii) *If there exists $\gamma > 0$, $\delta > 1$ and $a : \mathbb{N} \to \mathbb{C}$ such that $\lim_{k\to\infty} a(k) \exp(-\epsilon_1 k^{\epsilon_2}) = 0$ for all $\epsilon_1, \epsilon_2 > 0$ and the distance function $d : \mathbb{N} \to \mathbb{C}$ is given as $d(k) := a(k) \exp(-\gamma\, k^\delta)$, then the transfer operator*

$$\mathcal{M}_\beta : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\bigl(\beta\, q(\sigma) + \beta\, \sigma\, z_1\bigr)\, f(\sigma\, \tilde{v}^d + \mathbb{S}z)\, d\nu(\sigma)$$

defined in Corollary 2.9.3 satisfies $\zeta_R(z, \beta) = \det(1 - z\mathcal{M}_\beta)^{-1}$. □

As a side remark: The provisional Ruelle-Mayer transfer operator

$$\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^{\rho_0}) \to \mathcal{F}(\mathbb{C}^{\rho_0}),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\bigl(\beta q(x) + \beta\, z_1\bigr)\, f(x\, v^d + \mathbb{S}_{\rho_0} z)\, d\nu(x)$$

for a finite-range interaction defined in (62) of Subs. 2.8 satisfies

$$\zeta_R(z, \beta) = \exp\Bigl(\sum_{n=1}^{\rho_0 - 1} \frac{z^n}{n}\, Z^{b^{\mathbb{N}_0}, \phi}_{\{1,\dots,n\}}(\beta)\Bigr)\, \det\nolimits_{\rho_0}(1 - z\mathcal{M}_\beta)^{-1}.$$

With the motivation of Lemma 4.1.5 we can now formulate the program of this chapter:

Remark 4.1.8. We suppose that there exists a transfer operator G such that for all $n \geq n_0$ we have the dynamical trace formula

$$Z^{b^{\mathbb{N}_0}, \phi}_{\{1,\dots,n\}} = \det(1 - \Lambda^n)\, \mathrm{trace}\, G^{\cdot}$$

In Chapter 2 we made a lot of effort to obtain such a situation. We will now benefit from this as follows: We proceed as in the proof of Lemma 4.1.5 and write

$$\zeta_R(z) = \exp\Bigl(\sum_{n=1}^{\infty} \frac{z^n}{n}\, Z^{b^{\mathbb{N}_0}, \phi}_{\{1,\dots,n\}}\Bigr) = \exp\Bigl(\sum_{n=1}^{n_0 - 1} \frac{z^n}{n}\, Z^{b^{\mathbb{N}_0}, \phi}_{\{1,\dots,n\}}\Bigr) \exp\Bigl(\sum_{n=n_0}^{\infty} \frac{z^n}{n}\, \det(1 - \Lambda^n)\, \mathrm{trace}\, G^n\Bigr).$$

The first factor on the right hand side is an entire function, so we are left with second one. In the following we will investigate under which conditions ζ_R has a meromorphic continuation. The answer needs some preparation which will be done in the following Section 4.2. Finally the result be proved in Corollary 4.4.2. We would like to point out that this method only depends on the fact that a dynamical trace formula holds. Hence this result can also be applied for the dynamical zeta function associated to a sequence of dynamical partition functions. □

4.2 Generating functions

In order to implement the program formulated in Remark 4.1.8 we will investigate generating functions of a special kind. In this section we will provide some first tools which will be further developed in Section 4.3. In Proposition 4.2.4 we will prove that the dynamical zeta function has an Euler product and a meromorphic continuation to the entire complex plane if the dynamical trace formula

$$Z^{b^{\mathbb{N}_0}, \phi}_{\{1,\dots,n\}} = \det(1 - \Lambda^n)\, \mathrm{trace}\, G^n$$

holds for all $n \geq n_0$, where $G \in \mathcal{S}_{n_0}(\mathcal{H})$ and Λ is a matrix. A typical example for this situation is the case of Ising type systems with polynomial-exponentially decaying distance function for which we showed this trace formula in Corollary 2.11.2.
We will now introduce this class of generating functions and discuss some basic properties.

Proposition 4.2.1. *Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with $r(a) := \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}} \neq 0$. For $u \in \mathbb{N}$, $G \in \mathcal{S}_u(\mathcal{H})$, $z \in \mathbb{C}$ let*

$$g_u(z, a, G) := \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, a_n \operatorname{trace} G^n \Big) \tag{85}$$

be the generating function associated to a. Then $g_u(\,\cdot\,, a, G)$ defines a holomorphic function on the ball $B(0; r(a)/\|G\|_{\mathcal{S}_u(\mathcal{H})})_{\mathbb{C}} = \{z \in \mathbb{C};\ |z| < r(a)/\|G\|_{\mathcal{S}_u(\mathcal{H})}\}$ with the properties: Let a, $b \in \mathbb{C}^{\mathbb{N}}$, then

(i) $g_u(z, c\,a, G) = g_u(z, a; G)^c$ *on* $B(0; r(a)/\|G\|_{\mathcal{S}_u(\mathcal{H})})$ *for all* $c \in \mathbb{C}$,

(ii) $g_u(z, a+b, G) = g_u(z, a, G)\, g_u(z, b, G)$ *on* $B(0; r(a)/\|G\|_{\mathcal{S}_u(\mathcal{H})}) \cap B(0; r(b)/\|G\|_{\mathcal{S}_u(\mathcal{H})})$, *and*

(iii) $g_u(cz, a, G) = g_u(z, b, G)$ *on* $B(0; r(a)/(c\|G\|_{\mathcal{S}_u(\mathcal{H})}))$, *where* $b_n := c^n\, a_n$, $c \in \mathbb{C}^\times$.

Proof. We recall that $\|A\|_{\mathcal{S}_p(\mathcal{H})} \leq \|A\|_{\mathcal{S}_q(\mathcal{H})}$ for all $A \in \mathcal{S}_q(\mathcal{H})$ and $p \geq q$. Then by standard estimates

$$|g_u(z, a, G)| \leq \exp\Big| \sum_{n=u}^{\infty} \frac{z^n}{n}\, a_n \operatorname{trace} G^n \Big| \leq \exp\Big(\sum_{n=u}^{\infty} \frac{|z|^n}{n}\, |a_n|\, \|G\|^n_{\mathcal{S}_n(\mathcal{H})} \Big) \leq \exp\Big(\sum_{n=1}^{\infty} \frac{|a_n|}{n}\, \|zG\|^n_{\mathcal{S}_u(\mathcal{H})} \Big),$$

hence $g_u(z, a, G)$ converges in a neighbourhood of zero with the claimed radius of convergence. Properties (i) - (iii) now follow from standard arguments:

$$g_u(z, c\,a, G) = \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, c\,a_n \operatorname{trace} G^n \Big) = \exp\Big(c \sum_{n=u}^{\infty} \frac{z^n}{n}\, a_n \operatorname{trace} G^n \Big) = \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, a_n \operatorname{trace} G^n \Big)^c,$$

hence $g_u(z, c\,a, G) = g_u(z, a, G)^c$. Let $a = (a_n)_{n\in\mathbb{N}}$, $b = (b_n)_{n\in\mathbb{N}}$, and $|z|$ be small, then

$$\begin{aligned} g_u(z, a+b, G) &= \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, (a_n + b_n) \operatorname{trace} G^n \Big) = \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, a_n \operatorname{trace} G^n + \sum_{n=u}^{\infty} \frac{z^n}{n}\, b_n \operatorname{trace} G^n \Big) \\ &= \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, a_n \operatorname{trace} G^n \Big) \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, b_n \operatorname{trace} G^n \Big) = g_u(z, a, G)\, g_u(z, b, G), \end{aligned}$$

and finally

$$g_u(z, b, G) = \exp\Big(\sum_{n=u}^{\infty} \frac{z^n}{n}\, c^n\, a_n \operatorname{trace} G^n \Big) = \exp\Big(c \sum_{n=u}^{\infty} \frac{(cz)^n}{n}\, a_n \operatorname{trace} G^n \Big) = g_u(cz, a, G).$$

□

The following proposition considers how the difference of two generating functions depends on their coefficients. This will be used for the approximation of generating functions in Section 4.3.

Proposition 4.2.2. *Let $q > 1$ and p be the dual exponent of q defined by $\frac{1}{p} + \frac{1}{q} = 1$. Let a, $b \in \ell^q\mathbb{N}$, $u \in \mathbb{N}$, and $G \in \mathcal{S}_u(\mathcal{H})$. Then for all $|z| < \min\big(r(a)/\|G\|_{\mathcal{S}_u(\mathcal{H})}, r(b)/\|G\|_{\mathcal{S}_u(\mathcal{H})}, 1\big)$ one has the following estimate for the difference of two generating functions (85)*

$$\begin{aligned} &|g_u(z, a, G) - g_u(z, b, G)| \\ &\leq\ \|a - b\|_{\ell^q\mathbb{N}}\, |\log(1 - \|zG\|^p_{\mathcal{S}_u(\mathcal{H})})|^{1/p} \exp\Big(\big|\log(1 - \|zG\|^p_{\mathcal{S}_u(\mathcal{H})})\big|^{1/p} \,(\|a\|_{\ell^q\mathbb{N}} + \|a - b\|_{\ell^q\mathbb{N}}) \Big). \end{aligned}$$

Proof. Using $e^x - 1 \leq x\, e^x$ for $x \geq 0$, one gets

$$|e^x - e^y| = |e^y|\, |e^{x-y} - 1| \leq |e^y|\, (e^{|x-y|} - 1) \leq |x - y|\, e^{|y| + |x-y|}.$$

With this preparation we conclude that

$$\begin{aligned} |g_u(z, a, G) - g_u(z, b, G)| &= \Big| \exp\Big(\sum_{k=m}^{\infty} \frac{a_k}{k}\, z^k \operatorname{trace} G^k \Big) - \exp\Big(\sum_{k=m}^{\infty} \frac{b_k}{k}\, z^k \operatorname{trace} G^k \Big) \Big| \\ &\leq \Big| \sum_{k=m}^{\infty} \frac{a_k - b_k}{k}\, z^k \operatorname{trace} G^k \Big| \exp\Big(\Big| \sum_{k=m}^{\infty} \frac{a_k}{k}\, z^k \operatorname{trace} G^k \Big| \Big) \exp\Big(\Big| \sum_{k=m}^{\infty} \frac{a_k - b_k}{k}\, z^k \operatorname{trace} G^k \Big| \Big) \\ &\leq \sum_{k=1}^{\infty} \Big(\frac{|a_k - b_k|}{k}\, \|zG\|^k_{\mathcal{S}_u(\mathcal{H})} \Big) \exp\Big(\sum_{k=1}^{\infty} \frac{|a_k|}{k}\, \|zG\|^k_{\mathcal{S}_u(\mathcal{H})} \Big) \exp\Big(\sum_{k=1}^{\infty} \frac{|a_k - b_k|}{k}\, \|zG\|^k_{\mathcal{S}_u(\mathcal{H})} \Big). \end{aligned} \tag{86}$$

Hence an estimate for the inner series is needed. For any small $c > 0$ and $(d_k)_{k\in\mathbb{N}} \in \ell^q\mathbb{N}$ one has by Hölder's inequality ($\frac{1}{p} + \frac{1}{q} = 1$, hence $p < \infty$)

$$\sum_{k=1}^{\infty} \frac{c^k}{k} |d_k| \leq \Big(\sum_{k=1}^{\infty} \big(\frac{c^k}{k}\big)^p \Big)^{1/p} \Big(\sum_{k=1}^{\infty} |d_k|^q \Big)^{1/q} \leq \Big(\sum_{k=1}^{\infty} \frac{c^{pk}}{k} \Big)^{1/p} \|d\|_{\ell^q\mathbb{N}} = |\log(1-c^p)|^{1/p} \|d\|_{\ell^q\mathbb{N}}.$$

This estimate together with (86) gives the assertion. □

In the applications we have in mind, the coefficients $(a_n)_{n\in\mathbb{N}}$ of the generating function $g(\,\cdot\,, a, G)$ (85) have a special form. In this section we assume them to be special values of a fixed polynomial. For those coefficients the generating function $g_u(\,\cdot\,, a, G)$ (85) can be represented as a quotient of u-regularised determinants.

Proposition 4.2.3. *Let $N \in \mathbb{N}$, $b_1, \ldots, b_N$, $c \in \mathbb{C}$ and $a^{(N)} = (a_n^{(N)})_{n\in\mathbb{N}}$ be defined via $a_n^{(N)} := \sum_{k=0}^{N} b_k\, c^{nk}$. Then for any $u \in \mathbb{N}$ and $G \in \mathcal{S}_u(\mathcal{H})$ the generating function $g_u(\,\cdot\,, a^{(N)}, G)$ (85) can be written as*

$$g_u(z, a^{(N)}, G) = \prod_{k=1}^{N} \det{}_u(1 - zc^k G)^{-b_k}.$$

Interpretation: Let $f(z) = \sum_{k=0}^{N} b_k\, z^k \in \mathbb{C}[z]$, then $a_n^{(N)} = f(c^n)$.

Proof. This follows for small $|z|$ from Proposition 4.2.1 and the calculation

$$g_u(z, a^{(N)}, G) = g_u\big(z, \big(\sum_{k=0}^{N} b_k c^{nk}\big)_{n\in\mathbb{N}}, G\big) = \prod_{k=0}^{N} g_u\big(z, \big(b_k c^{nk}\big)_{n\in\mathbb{N}}, G\big) = \prod_{k=0}^{N} g_u(c^k z, 1, G)^{b_k}.$$

Then Lemma 4.1.5 concludes the proof. □

We are now ready to compute the generating function $g_u(\,\cdot\,, a, G)$ whose coefficients $a = (a_n)_{\in\mathbb{N}}$ are given as $a_n = \det(1 - \Lambda^n)$, where Λ is a fixed matrix. This follows immediately from Proposition 4.2.3 for an appropriate polynomial function.

Proposition 4.2.4. *Let $\Lambda \in \mathrm{Mat}(M, M; \mathbb{C})$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_M \in \mathbb{C}$. Define $a : \mathbb{N} \to \mathbb{C}$, $n \mapsto a_n := \det(1 - \Lambda^n)$, then for any $u \in \mathbb{N}$ and $G \in \mathcal{S}_u(\mathcal{H})$*

$$g_u(z, a, G) = \prod_{\alpha\in\{0,1\}^M} \det{}_u \big(1 - z \prod_{\nu=1}^{M} \lambda_\nu^{\alpha_\nu}\, G\big)^{(-1)^{|\alpha|+1}}.$$

Proof. We expand the determinant in terms of the eigenvalues of Λ and apply Proposition 4.2.3 to

$$a_n = \det(1 - \Lambda^n) = \prod_{j=1}^{M} (1 - \lambda_j^n) = \sum_{\alpha\in\{0,1\}^M} (-1)^{|\alpha|} \big(\prod_{\nu=1}^{M} \lambda_\nu^{\alpha_\nu} \big)^n.$$

□

Proposition 4.2.4 performs the task of Remark 4.1.8 in the case where Λ is a matrix. We add a corollary for a specific form of coefficients which arise for polynomial-exponentially decaying Ising or Ising type interactions. In the first case the occurring M is linked to the degree of the polynomial, in the second case M is the rank of the interaction matrix.

Corollary 4.2.5. *Let $u \in \mathbb{N}$, $G \in \mathcal{S}_u(\mathcal{H})$, $\lambda \in \mathbb{C}$, $M \in \mathbb{N}$, and $a_n^{(M)} := (1 - \lambda^n)^M$ for all $n \in \mathbb{N}$. Then*

$$g_u(z, a^{(M)}, G) = \prod_{\alpha\in\{0,1\}^M} \det{}_u(1 - z\lambda^{|\alpha|}\, G)^{(-1)^{|\alpha|+1}} = \prod_{k=0}^{M} \det{}_u(1 - z\lambda^k G)^{(-1)^{k+1}\binom{M}{k}}.$$

In particular, $g_u(z, a^{(0)}, G) = \dfrac{1}{\det_u(1 - zG_{(0)})}$, $g_u(z, a^{(1)}, G) = \dfrac{\det_u(1 - z\lambda G)}{\det_u(1 - zG)}$.

Proof. This is a direct consequence of the binomial formula $a_n^{(M)} = (1-\lambda^n)^M = \sum_{k=0}^{M} \binom{M}{k}(-1)^k\lambda^{kn}$ and Proposition 4.2.3. □

As an immediate consequence of Corollary 4.2.5 and Example 2.11.2 we obtain a representation of the dynamical zeta function for polynomial-exponentially decaying Ising interactions.

Corollary 4.2.6. *Let $F \subset \mathbb{C}$ be a bounded set and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function of the form $d : \mathbb{N} \to \mathbb{C}$, $d(k) := \lambda^k \sum_{i=0}^{p} c_i\, k^i$ for some $0 < |\lambda| < 1$, $c_i \in \mathbb{C}$. Let $\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^{p+1}) \to \mathcal{F}(\mathbb{C}^{p+1})$,*

$$(\mathcal{M}_\beta f)(z) = \int_F \exp\big(\beta q(x) + \beta\, x\, \langle z|c\rangle\big)\, f(\lambda\, x\, \underline{1} + \lambda \mathbb{B}^{(p+1)} z)\, d\nu(x)$$

be the Ruelle-Mayer transfer operator from Example 2.11.2. Then

$$\zeta_R(z,\beta) = \prod_{k=0}^{p+1} \det(1 - z\lambda^k \mathcal{M}_\beta)^{(-1)^{k+1}\binom{p+1}{k}}$$

gives a meromorphic continuation to the entire complex plane. □

Propositions 4.2.3 and 4.2.4 and Corollary 4.2.5 show that for special coefficients $a = (a_n)_{n\in\mathbb{N}}$ the generating function $g_u(\,\cdot\,, a, G)$ (85) can be represented as a quotient of u-regularised determinants. This motivates the following definition.

Remark 4.2.7. Let $G \in \mathcal{S}_u(\mathcal{H})$. We say that the generating function $g_u(\,\cdot\,, a, G)$ (85) associated to a is of *rational type* p, q if there exists polynomials p, $q \in \mathbb{C}[z]$ with $p(0) = q(0) = 1$ such that

$$g_u(z, a, G) = \frac{\det_u p(zG)}{\det_u q(zG)}.$$

If $u = 1$, then the Fredholm determinant is multiplicative: $\det((1+A)(1+B)) = \det(1+A)\,\det(1+B)$ for all A, $B \in \mathcal{S}_1(\mathcal{H})$. Hence

$$\frac{\det p(zG)}{\det q(zG)} = \det \frac{p}{q}(zG).$$

This is wrong if $u > 1$.

(i) Let $a_n^{(M)} := (1-\lambda^n)^M$, then by Corollary 4.2.5 the generating function $g_u(\cdot, a^{(M)}, G)$ is of rational type p_M, q_M where

$$p_M(z) := \prod_{k=0}^{\lceil \frac{M}{2} \rceil - 1} (1 - z\lambda^{2k+1})^{\binom{M}{2k+1}}, \quad q_M(z) := \prod_{k=0}^{\lfloor \frac{M}{2} \rfloor} (1 - z\lambda^{2k})^{\binom{M}{2k}}, \tag{87}$$

since

$$\begin{aligned}\prod_{k=0}^{M} \det_u(1 - z\lambda^k G)^{(-1)^{k+1}\binom{M}{k}} &= \frac{\prod_{k=0;k\equiv 1\,(2)}^{M} \det_u(1 - z\lambda^k G)^{\binom{M}{k}}}{\prod_{k=0;k\equiv 0\,(2)}^{M} \det_u(1 - z\lambda^k G)^{\binom{M}{k}}} \\ &= \frac{\prod_{k=0}^{\lceil \frac{M}{2} \rceil - 1} \det_u(1 - z\lambda^{2k+1} G)^{\binom{M}{2k+1}}}{\prod_{k=0}^{\lfloor \frac{M}{2} \rfloor} \det_u(1 - z\lambda^{2k} G)^{\binom{M}{2k}}}.\end{aligned}$$

Examples: $M = 0$: $p_0(z) = 1$, $q_0(z) = 1 - z$; $M = 1$: $p_1(z) = 1 - \lambda z$, $q_1(z) = 1 - z$; $M = 2$: $p_2(z) = (1-\lambda z)^2$, $q_2(z) = (1-z)(1-\lambda^2 z)$.

(ii) Euler product: For every polynomial p and compact operator G with eigenvalues $(\mu_n)_{n\in\mathbb{N}}$, the operator $p(G)$ has eigenvalues $(p(\mu_n))_{n\in\mathbb{N}}$. Let $G \in \mathcal{S}_u(\mathcal{H})$ and $g_u(\,\cdot\,, a, G)$ be a generating function of rational type p, q. Then by Lemma A.1.4 we have

$$g_u(z, a, G) = \frac{\det_u p(zG)}{\det_u q(zG)} = \prod_{n=1}^{\infty} \frac{p(z\mu_n)\, \exp\big(\sum_{k=1}^{u-1} \frac{(-1)^k}{k}\, (p(z\mu_n) - 1)^k\big)}{q(z\mu_n)\, \exp\big(\sum_{k=1}^{u-1} \frac{(-1)^k}{k}\, (q(z\mu_n) - 1)^k\big)}.$$

Hence $g_u(\,\cdot\,,a,G)$ extends to a meromorphic function with poles of finite order. If one writes the polynomial q as $q(z)=\prod_{i=1}^{M}(1-q_i z)$, then the poles of g_u are contained (not necessarily all such points are poles because of possible cancellations) in the set $\{(\mu_n q_j)^{-1}|n\in\mathbb{N},\, j=1,\dots,q\}$.

(iii) In particular, if G is a trace class operator on $\mathcal{H}$, then for every $g_1(\,\cdot\,,a,G)$ of rational type p, q, we have

$$g_1(z,a,G)=\det\frac{p}{q}(zG)=\frac{\prod_{i=1}^{N}\det(1-\lambda_i zG)}{\prod_{j=1}^{M}\det(1-q_j zG)}=\prod_{n=1}^{\infty}\frac{p}{q}(z\mu_n).$$

(iv) Let

$$\tilde{g}(z,b)=\exp\Big(\sum_{n=u}^{\infty}\frac{b_n}{n}\,z^n\Big),\ f(z)=\sum_{n=u}^{\infty}\frac{b_n}{n}\,z^n.$$

Suppose $\tilde{g}(z,b)=\frac{p}{q}(z)$ for some polynomials p and q. Then for every $G\in\mathcal{S}_u(\mathcal{H})$ the operator $f(zG)=\sum_{n=u}^{\infty}\frac{b_n}{n}\,z^n\,G^n$ belongs to $\mathcal{S}_1(\mathcal{H})$ and the generating function $g(\,\cdot\,,b,G)$ satisfies

$$g(z,b,G)=\exp\Big(\sum_{n=u}^{\infty}\frac{b_n}{n}\,z^n\operatorname{trace}G^n\Big)=\exp\operatorname{trace}\Big(\sum_{n=u}^{\infty}\frac{b_n}{n}\,(zG)^n\Big)\overset{(102)}{=}\det\exp f(zG)=\det\frac{p}{q}(zG)$$

and hence is of rational type. □

We have seen a class of examples leading to generating functions of rational type 4.2.7. One may ask which sequences $a=(a_n)_{n\in\mathbb{N}}$ lead to generating functions $g(\,\cdot\,,b,G)$ of rational type. A desired result would be a result similar to [BoLa70, Lemmas 3, 4] which concerns the power series expansion of a rational function. However, we did not succeed in that direction.

4.3 Limits of zeta functions

In the last section we have studied generating functions $g_u(\,\cdot\,,a,G)$ (85) whose coefficients a are special values of a fixed polynomial. For applications this is a very restrictive requirement. Recall for instance Theorem 2.4.6 and its application in Corollary 2.10.5 to Ising interactions with distance function of class $\mathcal{D}_1^{(1),\Delta}$ (2.10.1). There we showed that the partition functions can be expressed via a dynamical trace formula of the form

$$Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta)=\det(1-\operatorname{diag}(\lambda)^n)\operatorname{trace}\,(\mathcal{M}_\beta)^n,$$

where $\mathcal{M}_\beta$ is the Ruelle-Mayer transfer operator defined in (66) and $\operatorname{diag}(\lambda):\ell^2\mathbb{N}\to\ell^2\mathbb{N}$ is a trace class operator. This motivates the investigation of the following situation: Let Λ be a fixed trace class operator on a Hilbert space $\mathcal{H}_0$ with spectral radius $\rho_{\text{spec}}(\Lambda)<1$ (2.6.4) and let the coefficients of the generating function be given as $a_n=\det(1-\Lambda^n)$. In Theorem 4.3.4 we will show that under these assumptions the generating function $g_u(\,\cdot\,,a,G)$ has a meromorphic continuation to the entire complex plane and can be represented as an Euler product.
Before proving the meromorphic continuation we make sure that the generating function $g_u(\,\cdot\,,a,G)$ (85) is at least holomorphic in a neighbourhood of zero. In view of Proposition 4.2.2 we need the following estimates on the coefficients $(a_n)_{n\in\mathbb{N}}$.

Lemma 4.3.1. *Let $\Lambda\in\mathcal{S}_p(\mathcal{H}_0)$ with $\rho_{\text{spec}}(\Lambda)<1$ and $G\in\mathcal{S}_u(\mathcal{H})$. Let $n_0\in\mathbb{N}$ be such that $G^n\in\mathcal{S}_1(\mathcal{H})$, $\Lambda^n\in\mathcal{S}_1(\mathcal{H}_0)$ and $\|\Lambda^n\|<1$ for all $n\geq n_0$. Set $a_n:=\det(1-\Lambda^n)$ for all $n\geq n_0$. Then $g_m(z,a,G)$ (85) converges for any $m\geq n_0$ at least for $|z|<\|G\|_{\mathcal{S}_m(\mathcal{H})}^{-1}$.*

Proof. By Proposition 4.2.1 we have to investigate the limit behaviour of $|a_n|^{-1/n}=|\det(1-\Lambda^n)|^{-1/n}$ as $n\to\infty$. We expand the determinants in terms of the eigenvalues and split the infinite product into two parts which will be considered separately. Observe that all eigenvalues of Λ have modulus strictly less than one by Remark 2.6.4. Choose $k_0\in\mathbb{N}$ such that $\sum_{k=k_0}^{\infty}|\lambda_k|^{n_0}<1$. Then using standard arguments from the subject of infinite products one obtains for $n\geq n_0$

$$\prod_{k=1}^{k_0-1}|1-\lambda_k^n|^{1/n}\leq\prod_{k=1}^{k_0-1}(1+|\lambda_k|^n)^{1/n}\leq\Big(\exp\sum_{k=1}^{k_0-1}|\lambda_k|^n\Big)^{1/n}\leq\exp\Big(\frac{1}{n}\sum_{k=1}^{k_0-1}|\lambda_k|^{n_0}\Big)\leq\exp\Big(\frac{k_0}{n}\Big),$$

which tends to one as $n \to \infty$. The tail product can be estimated by

$$\prod_{k=k_0}^{\infty} |1-\lambda_k^n|^{1/n} \le \prod_{k=k_0}^{\infty} (1+|\lambda_k|^n)^{1/n} \le \Big(\exp \sum_{k=k_0}^{\infty} |\lambda_k|^n\Big)^{1/n} \le \exp\Big(\frac{1}{n}\sum_{k=k_0}^{\infty} |\lambda_k|^{n_0}\Big),$$

which also tends to one as $n \to \infty$. Hence

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \limsup_{n\to\infty} \prod_{k=1}^{k_0-1} |1-\lambda_k^n|^{1/n} \limsup_{n\to\infty} \prod_{k=k_0}^{\infty} |1-\lambda_k^n|^{1/n} \le 1$$

and the generating function $g_m(\,\cdot\,, a, G)$ converges at least inside the stated disk. □

We want to investigate the generating function $g_u(\,\cdot\,, a, G)$ whose coefficients $a = (a_n)_{n\in\mathbb{N}}$ are given as the Fredholm determinants $a_n := \det(1-\Lambda^n)$ of a trace class operator $\Lambda \in \mathcal{S}_1(\mathcal{H}_0)$ with $\rho_{\text{spec}}(\Lambda) < 1$ and where $G \in \mathcal{S}_u(\mathcal{H})$. We will approximate the generating function $g_u(\,\cdot\,, a, G)$ via approximating its coefficients $a_n^{(N)} = \det(1-\Lambda_N^n)$, where Λ_N is an approximating finite rank operator. We will choose $\Lambda_N := \text{pr}_N \circ \Lambda \circ \text{pr}_N$ where $\text{pr}_N \in \text{End}(\mathcal{H}_0)$ is the orthogonal projection onto the space spanned by the first N generalised eigenvectors of Λ. Hence $a_n^{(N)} = \det(1-\Lambda_N^n) = \prod_{j=1}^{N}(1-\lambda_j^n)$. We now estimate the differences $a_n - a_n^{(N)}$ of the coefficients. In the forthcoming step, Lemma 4.3.3, we will consider the associated generating functions and estimate their differences.

Lemma 4.3.2. *Let $\Lambda \in \mathcal{S}_p(\mathcal{H}_0)$ with $\rho_{\text{spec}}(\Lambda) < 1$ and eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$. Let $n_0 \in \mathbb{N}$ be such that $\Lambda^n \in \mathcal{S}_1(\mathcal{H}_0)$ and $\|\Lambda^n\| < 1$ for all $n \ge n_0$. Let $\text{pr}_N \in \text{End}(\mathcal{H}_0)$ be the orthogonal projection onto the space spanned by the first N (generalised) eigenvectors and $\Lambda_N := \text{pr}_N \circ \Lambda \circ \text{pr}_N$. Let a, $a^{(N)} : \mathbb{N}_{\ge n_0} \to \mathbb{C}$ be defined via $a_n = \det(1-\Lambda^n)$ and $a_n^{(N)} = \det(1-\Lambda_N^n)$. Then*

(i) *a and $a^{(N)}$ belong to $\ell^\infty\mathbb{N}$ and $\|a - a^{(N)}\|_{\ell^\infty\mathbb{N}} \le \exp(2\|\Lambda^{n_0}\|_{\mathcal{S}_1(\mathcal{H}_0)}) \sum_{j=N+1}^{\infty} |\lambda_j|^{n_0} \longrightarrow_{N\to\infty} 0$, and*

(ii) *For N sufficiently large one has $(a^{(N)} - a) \in \ell^1\mathbb{N}$ and $\|a^{(N)} - a\|_{\ell^1\mathbb{N}} \longrightarrow_{N\to\infty} 0$.*

Proof. Let $(\lambda_i)_{i\in\mathbb{N}}$ be the eigenvalues of Λ. By assumption they all are in modulus smaller than one. Hence for $n \ge n_0$ and all $N \in \mathbb{N}$ we have the estimate

$$|a_n^{(N)}| = \prod_{j=1}^{N} |1-\lambda_j^n| \le \prod_{j=1}^{N}(1+|\lambda_j|^n) \le \prod_{j=1}^{N}(1+|\lambda_j|^{n_0}) \le \exp\Big(\sum_{j=1}^{N} |\lambda_j|^{n_0}\Big) \le \exp(\|\Lambda^{n_0}\|_{\mathcal{S}_1(\mathcal{H}_0)})$$

by arguments similar to the proof of Lemma 4.3.1. Thus $a^{(N)}$ and a belong to $\ell^\infty\mathbb{N}$. We expand the determinants in terms of the eigenvalues and obtain the following identity

$$a_n - a_n^{(N)} = \prod_{j=1}^{\infty}(1-\lambda_j^n) - \prod_{j=1}^{N}(1-\lambda_j^n) = \Big(\prod_{j=N+1}^{\infty}(1-\lambda_j^n) - 1\Big)\prod_{j=1}^{N}(1-\lambda_j^n).$$

Hence for all $n \ge n_0$, $N \in \mathbb{N}$ we can estimate the difference $|a_n - a_n^{(N)}|$ by

$$\begin{aligned} |a_n - a_n^{(N)}| &= \Big|\prod_{j=N+1}^{\infty}(1-\lambda_j^n) - 1\Big| \prod_{j=1}^{N}|1-\lambda_j^n| \\ &\le \Big(\exp\Big(\sum_{j=N+1}^{\infty}|\lambda_j|^n\Big) - 1\Big) \exp\Big(\sum_{j=1}^{N}|\lambda_j|^n\Big) \\ &\le \sum_{j=N+1}^{\infty}|\lambda_j|^n \, \exp\Big(\sum_{j=N+1}^{\infty}|\lambda_j|^n\Big) \exp\Big(\sum_{j=1}^{\infty}|\lambda_j|^n\Big) \\ &\le \sum_{j=N+1}^{\infty}|\lambda_j|^n \, \exp\Big(2\sum_{j=1}^{\infty}|\lambda_j|^n\Big), \end{aligned}$$

which converges to zero as $N \to \infty$, showing part (i). Since by assumption $\Lambda^{n_0} \in \mathcal{S}_1(\mathcal{H}_0)$ we can choose a (sufficiently large) N such that $\sum_{j=N+1}^{\infty} |\lambda_j|^{n_0} < 1$. Summing up the previous estimates we obtain - using the definition of the spectral radius (2.6.4) in the last step -

$$\begin{aligned}
\sum_{n=n_0}^{\infty} |a_n - a_n^{(N)}| &\leq \sum_{n=n_0}^{\infty} \sum_{j=N+1}^{\infty} |\lambda_j|^n \exp(2\|\Lambda^n\|_{\mathcal{S}_1(\mathcal{H}_0)}) \\
&\leq \exp(2\|\Lambda^{n_0}\|_{\mathcal{S}_1(\mathcal{H}_0)}) \sum_{j=N+1}^{\infty} \sum_{n=n_0}^{\infty} |\lambda_j|^n \\
&= \exp(2\|\Lambda^{n_0}\|_{\mathcal{S}_1(\mathcal{H}_0)}) \sum_{j=N+1}^{\infty} \frac{|\lambda_j|^{n_0}}{1-|\lambda_j|} \\
&\leq \frac{\exp(2\|\Lambda^{n_0}\|_{\mathcal{S}_1(\mathcal{H}_0)})}{1-\rho_{\text{spec}}(\Lambda)} \sum_{j=N+1}^{\infty} |\lambda_j|^{n_0},
\end{aligned}$$

which tends to zero as N tends to infinity. □

The previous result will now imply that in a neighbourhood of zero the generating function $g_m(\,\cdot\,, a, G)$ can be approximated by the sequence of generating functions $g_m(\,\cdot\,, a^{(M)}, G)$ under quite general conditions. In particular, $g_m(\,\cdot\,, a, G)$ is a holomorphic function near zero.

Lemma 4.3.3. *Let $\Lambda \in \mathcal{S}_p(\mathcal{H}_0)$ with $\rho_{\text{spec}}(\Lambda) < 1$ and $G \in \mathcal{S}_u(\mathcal{H})$. Let $n_0 \geq u$ be such that $\Lambda^n \in \mathcal{S}_1(\mathcal{H}_0)$ and $\|\Lambda^n\| < 1$ for all $n \geq n_0$. Let $\text{pr}_N \in \text{End}(\mathcal{H}_0)$ be the orthogonal projection onto the space spanned by the first N (generalised) eigenvectors and $\Lambda_N := \text{pr}_N \circ \Lambda \circ \text{pr}_N$. Let $a, a^{(N)} : \mathbb{N}_{\geq n_0} \to \mathbb{C}$ be defined via $a_n = \det(1 - \Lambda^n)$ and $a_n^{(N)} = \det(1 - \Lambda_N^n)$. Then the generating functions defined in (85) converge*

$$\lim_{N\to\infty} g_m(z, a^{(N)}, G) = g_m(z, a, G)$$

for any $m \geq n_0$ at least for $|z| < \|G\|_{\mathcal{S}_m(\mathcal{H})}^{-1}$.

Proof. By Proposition 4.2.2 for $\|zG\|_{\mathcal{S}_m(\mathcal{H})} \leq r < 1$ the difference of the generating functions can be estimated by

$$\begin{aligned}
&|g_m(z, a, G) - g_m(z, a^{(N)}, G)| \\
&\quad\leq \|a - a^{(N)}\|_{\ell^\infty\mathbb{N}} \,|\log(1 - \|zG\|_{\mathcal{S}_m(\mathcal{H})})|\, \exp\Big(|\log(1 - \|zG\|_{\mathcal{S}_m(\mathcal{H})})|\,(\|a\|_{\ell^\infty\mathbb{N}} + \|a - a^{(N)}\|_{\ell^\infty\mathbb{N}})\Big) \\
&\quad\leq \|a - a^{(N)}\|_{\ell^\infty\mathbb{N}} \,|\log(1 - r)|\, \exp\Big(|\log(1 - r)|\,(\|a\|_{\ell^\infty\mathbb{N}} + \|a - a^{(N)}\|_{\ell^\infty\mathbb{N}})\Big).
\end{aligned}$$

The latter tends to zero by Lemma 4.3.2. Hence $g_m(z, a^{(N)}, G)$ converges to $g_m(z, a, G)$ as $N \to \infty$. □

Now we are prepared to prove our main result of this section: The generating function is holomorphic in a neighbourhood of zero by Lemma 4.3.3 and can be approximated by a sequence of generating functions with polynomial coefficients. These have meromorphic continuations, and the sequence of the meromorphic continuations converges locally uniformly in the entire $\mathbb{C}$-plane.

Theorem 4.3.4. *Let $\Lambda \in \mathcal{S}_1(\mathcal{H}_0)$ be a trace class operator with $\rho_{\text{spec}}(\Lambda) < 1$ and $a_n := \det(1 - \Lambda^n)$. Let $(\lambda_i)_{i\in\mathbb{N}}$ be the eigenvalues of Λ. Then for any $G \in \mathcal{S}_u(\mathcal{H})$*

$$g_u(z, a, G) = \lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M} \det{}_u(1 - z\underline{\lambda}^\alpha\, G)^{(-1)^{|\alpha|+1}} = \frac{\lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M : |\alpha|\equiv 1\,(2)} \det_u(1 - z\underline{\lambda}^\alpha\, G)}{\lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M : |\alpha|\equiv 0\,(2)} \det_u(1 - z\underline{\lambda}^\alpha\, G)},$$

where we set $\underline{\lambda}^\alpha := \prod_{\nu=1}^M \lambda_\nu^{\alpha_\nu}$ for $\alpha \in \{0,1\}^M$. In particular, the generating function $g_u(\,\cdot\,, a, G)$ extends to a meromorphic function on the entire $\mathbb{C}$-plane.

Proof. By Lemma 4.3.3 the generating function $g_u(\,\cdot\,,a,G)$ (85) can be approximated by the sequence $\big(g_u(\,\cdot\,,a^{(N)},G)\big)_{N\in\mathbb{N}}$ in a neighbourhood of zero. For each $N\in\mathbb{N}$ the generating function $g_u(z,a^{(N)},G)$ has an Euler product. We now show that the limit $g_u(z,a,G)$ can be represented in the stated form. Hence one has to show that the infinite product converges. For any $q\in\mathbb{C}$ with $|q|\le 1$ one has the estimate

$$|\det_u(1-zqG)-1| \overset{(109)}{\le} \|zq\,G\|_{\mathcal{S}_u(\mathcal{H})}\exp(c_u\,(1+\|zqG\|_{\mathcal{S}_u(\mathcal{H})})^u) \le \|zqG\|_{\mathcal{S}_u(\mathcal{H})}\exp(c_u\,(1+\|zG\|_{\mathcal{S}_u(\mathcal{H})})^u).$$

Let $M\in\mathbb{N}$. Since $|\lambda_i|\le\rho_{\text{spec}}(\Lambda)<1$, we can apply the previous estimate for each $q_\alpha=\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}$ and then sum up over all $\alpha\in\{0,1\}^M$:

$$\begin{aligned}\sum_{\alpha\in\{0,1\}^M}\Big|\det_u\big(1-z\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}\,G\big)-1\Big| &\le \exp(c_u\,(1+\|zG\|_{\mathcal{S}_u(\mathcal{H})})^u)\,\|zG\|_{\mathcal{S}_u(\mathcal{H})}\sum_{\alpha\in\{0,1\}^M}\prod_{\nu=1}^M|\lambda_\nu|^{\alpha_\nu}\\ &= \exp(c_u\,(1+\|zG\|_{\mathcal{S}_u(\mathcal{H})})^u)\,\|zG\|_{\mathcal{S}_u(\mathcal{H})}\prod_{\nu=1}^M(1+|\lambda_\nu|). \qquad (88)\end{aligned}$$

The last expression is bounded as $M\to\infty$, since

$$\prod_{\nu=1}^M(1+|\lambda_\nu|)\le\prod_{\nu=1}^\infty(1+|\lambda_\nu|)\le\exp\big(\sum_{\nu=1}^\infty|\lambda_\nu|\big)\le\exp(\|\Lambda\|_{\mathcal{S}_1(\mathcal{H}_0)})<\infty$$

due to the assumption that Λ is trace class. The boundedness ensures the existence of the following infinite products

$$\begin{aligned} g_u(z,a,G) &= \lim_{M\to\infty} g_u(z,a^{(M)},G)\\ &= \lim_{M\to\infty}\prod_{\alpha\in\{0,1\}^M}\det_u\big(1-z\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}\,G\big)^{(-1)^{|\alpha|+1}}\\ &= \lim_{M\to\infty}\frac{\prod_{\alpha\in\{0,1\}^M:|\alpha|\equiv1\,(2)}\det_u(1-z\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}\,G)}{\prod_{\alpha\in\{0,1\}^M:|\alpha|\equiv0\,(2)}\det_u(1-z\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}\,G)}\\ &= \frac{\lim_{M\to\infty}\prod_{\alpha\in\{0,1\}^M:|\alpha|\equiv1\,(2)}\det_u(1-z\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}\,G)}{\lim_{M\to\infty}\prod_{\alpha\in\{0,1\}^M:|\alpha|\equiv0\,(2)}\det_u(1-z\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}\,G)}.\end{aligned}$$

Moreover, this limit exists locally uniformly in $z\in\mathbb{C}$. Hence $g_u(\,\cdot\,,a,G)$ is the locally uniform limit of meromorphic functions and thus itself meromorphic. □

We note that our proof essentially depends on the fact that Λ is trace class. We did not succeed to weaken that condition as we were able to do in the previous lemmas. For instance, look at the trivial example $G=\gamma\in\mathbb{C}$. Then the estimate (88) becomes $|\det(1-zq\,G)-1|=|(1-zq\gamma)-1|=|zq\gamma|$ and

$$\sum_{\alpha\in\{0,1\}^M}|\det(1-z\prod_{\nu=1}^M\lambda_\nu^{\alpha_\nu}\,G)-1|=|z\gamma|\sum_{\alpha\in\{0,1\}^M}\prod_{\nu=1}^M|\lambda_\nu|^{\alpha_\nu}=|z\gamma|\prod_{\nu=1}^M(1+|\lambda_\nu|)$$

is bounded in the limit $M\to\infty$ if and only if Λ is trace class. We think that our estimates are optimal also in the general case. - As an immediate consequence of Lemma A.1.4 and Theorem 4.3.4 we obtain the location of the poles and zeros of the generating function $g_u(\,\cdot\,,a,G)$:

Corollary 4.3.5. *Let $\Lambda\in\mathcal{S}_1(\mathcal{H}_0)$ be a trace class operator with $\rho_{\text{spec}}(\Lambda)<1$ with eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ and $a_n:=\det(1-\Lambda^n)$. Then for any $G\in\mathcal{S}_u(\mathcal{H})$ with eigenvalues $(\mu_i)_{i\in\mathbb{N}}$, the poles of the generating function $g_u(\,\cdot\,,a,G)$ (85) are contained in the set*

$$P_g:=\bigcup_{M\in\mathbb{N}}\{(\underline{\lambda}^\alpha\mu_j)^{-1}\mid\alpha\in\{0,1\}^M,\ |\alpha|\equiv0\,(2),\ j\in\mathbb{N}\}$$

where $\underline{\lambda}^\alpha := \prod_{\nu=1}^M \lambda_\nu^{\alpha_\nu}$ *for* $\alpha \in \{0,1\}^M$*; its zeros are contained in the set*

$$N_g := \bigcup_{M\in\mathbb{N}} \{(\underline{\lambda}^\alpha \mu_j)^{-1} \,|\, \alpha \in \{0,1\}^M,\ |\alpha| \equiv 1\,(2),\ j \in \mathbb{N}\}.$$

□

Note: Because of possible cancellations we cannot show that all elements of P_g, N_g are poles or zeros respectively. In a remark following Corollary 4.4.3 we will give an example.

4.4 Transfer operator method

With the preparation of the two previous sections we can now answer the question raised in Remark 4.1.8, namely the properties of the dynamical zeta function ζ_R as introduced in (4.1.1) in the presence of a dynamical trace formula. In Section 2.4 we have found two types of dynamical trace formulas. In the first case, Theorem 2.4.4, the partition function can be expressed via a family of transfer operators. This will result in a product representation of Ruelle's zeta, one factor for each transfer operator. In the second case, Theorem 2.4.6, one has a dynamical trace formula of the type $Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi} = \det(1-\Lambda^n)\,\mathrm{trace}\,G^n$. We show that in both cases the dynamical zeta function has a meromorphic continuation to the entire complex plane and a representation as an Euler product. The zeros and poles of zeta have a spectral interpretation. This result is one of the main applications of the transfer operator method.
The following result was observed by D. Mayer in the case $n_0 = 1$, $n_1 < \infty$. It is designed for spin systems for which one has a family of transfer operators as for instance in Theorem 2.4.4.

Corollary 4.4.1. *Suppose there exists a family of transfer operators* $G^{(\nu)} \in \mathcal{S}_{n_0}(\mathcal{H}^{(\nu)})$ *(*$\nu = 1,\dots,n_1$ *with possibly* $n_1 = \infty$*) such that for all* $n \geq n_0$ *we have the dynamical trace formula*

$$Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi} = \sum_{\nu=0}^{n_1} (-1)^\nu \,\mathrm{trace}\,(G^{(\nu)})^n$$

and $\sum_{\nu=0}^{n_1} \|G^{(\nu)}\|_{\mathcal{S}_{n_0}(\mathcal{H}^{(\nu)})}^{n_0} < \infty$*. Then*

$$\zeta_R(z) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n} Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}\Big) \prod_{\nu=0}^{\infty} \Big(\det{}_{n_0}(1 - zG^{(\nu)})\Big)^{(-1)^{\nu+1}}$$

gives the meromorphic continuation of Ruelle's zeta ζ_R *(4.1.1) to the entire* $\mathbb{C}$*-plane.*

Proof. First we assume $n_1 < \infty$. In this case the condition $\sum_{\nu=0}^{n_1} \|G^{(\nu)}\|_{\mathcal{S}_{n_0}(\mathcal{H}^{(\nu)})}^{n_0} < \infty$ is void. For small z, i. e., for $|z| < \min\{\|G^{(\nu)}\|_{\mathcal{S}_{n_0}(\mathcal{H}^{(\nu)})}^{-1} \,|\, \nu = 0,\dots,n_1\}$, one calculates similarly to Lemma 4.1.5

$$\begin{aligned}
\zeta_R(z) &= \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n} Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}\Big) \exp\Big(\sum_{n=n_0}^{\infty} \sum_{\nu=0}^{n_1} (-1)^\nu \frac{z^n}{n} \,\mathrm{trace}\,(G^{(\nu)})^n\Big) \\
&= \exp\Big(\sum_{n=n_0}^{n_0-1} \frac{z^n}{n} Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}\Big) \exp\Big(\sum_{\nu=0}^{n_1} (-1)^\nu \sum_{n=n_0}^{\infty} \frac{z^n}{n} \,\mathrm{trace}\,(G^{(\nu)})^n\Big) \\
&= \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n} Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}\Big) \prod_{\nu=0}^{n_1} \exp\Big(\sum_{n=n_0}^{\infty} \frac{z^n}{n} \,\mathrm{trace}\,(G^{(\nu)})^n\Big)^{(-1)^\nu} \\
&= \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n} Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}\Big) \prod_{\nu=0}^{n_1} \Big(\det{}_{n_0}(1 - zG^{(\nu)})\Big)^{(-1)^{\nu+1}}
\end{aligned}$$

By Lemma A.1.2 this is a finite product of meromorphic functions. We now turn the case $n_1 = \infty$. Let $n \in \mathbb{N}$. For each factor in the following product we apply the estimate (108) from A.2.6, hence

$$\prod_{\nu=0}^{n} |\det{}_{n_0}(1 - zG^{(\nu)})| \leq \prod_{\nu=0}^{n} \exp\big(c_{n_0} \|zG^{(\nu)}\|_{\mathcal{S}_{n_0}(\mathcal{H}^{(\nu)})}^{n_0}\big) = \exp\big(c_{n_0} |z|^{n_0} \sum_{\nu=0}^{n} \|G^{(\nu)}\|_{\mathcal{S}_{n_0}(\mathcal{H}^{(\nu)})}^{n_0}\big),$$

which is bounded by assumption as $n \to \infty$. Hence the infinite product

$$\prod_{\nu=0}^{\infty} \Big(\det_{n_0}(1 - zG^{(\nu)}) \Big)^{(-1)^{\nu+1}}$$

converges absolutely and locally uniformly. The sequence $\|G^{(\nu)}\|_{\mathcal{S}_{n_0}(\mathcal{H}^{(\nu)})}$ tends to zero as $\nu \to \infty$, hence the minimum $\min\{\|G^{(\nu)}\|^{-1}_{\mathcal{S}_m(\mathcal{H}^{(\nu)})} \,|\, \nu \in \mathbb{N}_0\} > 0$ exists, and thus zeta is holomorphic in a neighbourhood of zero. □

Using Lemma A.1.4 the previous corollary gives an Euler product expansion of ζ_R.
Our following corollary is an important result for systems for which Theorem 2.4.6 holds.

Corollary 4.4.2. *Suppose there is a transfer operator $G \in \mathcal{S}_{n_0}(\mathcal{H})$ which satisfies the dynamical trace formula*

$$Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}} = \det(1 - \Lambda^n)\,\mathrm{trace}\, G^n$$

for all $n \geq n_0$, where $\Lambda \in \mathcal{S}_1(\mathcal{H}_0)$ is a trace class operator with $\rho_{\mathrm{spec}}(\Lambda) < 1$. Set $a_n := \det(1 - \Lambda^n)$ and $a = (a_n)_{n\in\mathbb{N}}$. Then

$$\zeta_R(z) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n}\, Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}} \Big)\, g_{n_0}(z, a, G)$$

gives the meromorphic continuation of zeta (4.1.1) to the entire complex plane.

Proof. We write the dynamical zeta function as

$$\zeta_R(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n}\, Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}} \Big) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n}\, Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}} \Big)\, \exp\Big(\sum_{n=n_0}^{\infty} \frac{z^n}{n} \det(1 - \Lambda^n)\ \mathrm{trace}\ G^n \Big).$$

Obviously the first factor on the right hand side is an entire function. The second one has a meromorphic continuation by Theorem 4.3.4 via an Euler product. □

As a direct consequence of Corollaries 4.3.5 and 4.4.2 we can locate the poles and zeros of the dynamical zeta function. Recall, for $(\lambda_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and $\alpha \in \{0,1\}^M$ we set $\underline{\lambda}^\alpha := \prod_{\nu=1}^{M} \lambda_\nu^{\alpha_\nu}$.

Corollary 4.4.3. *Let $\Lambda \in \mathcal{S}_1(\mathcal{H}_0)$ be a trace class operator with $\rho_{\mathrm{spec}}(\Lambda) < 1$ and eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. Suppose there exists a transfer operator $G \in \mathcal{S}_{n_0}(\mathcal{H})$ with eigenvalues $(\mu_i)_{i\in\mathbb{N}}$ such that for all $n \geq n_0$ we have the dynamical trace formula $Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}} = \det(1 - \Lambda^n)\,\mathrm{trace}\, G^n$. Then the poles of Ruelle's zeta function (4.1.1) are contained in the set $P_\zeta := \bigcup_{M\in\mathbb{N}} \{(\underline{\lambda}^\alpha \mu_j)^{-1} \,|\, \alpha \in \{0,1\}^M,\ |\alpha| \equiv 0\,(2)\, j \in \mathbb{N}\}$, its zeros are contained in the set $N_\zeta := \bigcup_{M\in\mathbb{N}} \{(\underline{\lambda}^\alpha \mu_j)^{-1} \,|\, \alpha \in \{0,1\}^M,\ |\alpha| \equiv 1\,(2)\, j \in \mathbb{N}\}$.* □

As above, Cor. 4.3.5, we cannot prove that each point in the sets P_ζ, N_ζ is indeed a pole (a zero, respectively) because of possible cancellations. For example consider the situation of Remark 4.1.2: If $\beta = 0$, then we know that $\zeta_R(z, 0) = (1 - \nu(F)z)^{-1}$ and hence almost all poles and zeros of the dynamical zeta function cancel.
As an immediate consequence of Theorem 3.2.6, Corollary 2.10.5, Theorem 4.3.4, and Corollary 4.4.2 we obtain the following main result of this chapter. It contains the Corollaries 4.1.7 and 4.2.6 as special cases.

Corollary 4.4.4. *Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega_{\mathbb{A}}, F, \mathbb{N}, \mathbb{N}_0, \tau)$ be a one-sided one-dimensional matrix subshift (1.2.8). Let ϕ be a two-body Ising interaction (1.8.3) with distance function $d \in \mathcal{D}_1^{(1)}$ (2.7.1), say $d(k) = \langle \mathbb{B}^{k-1} v | w \rangle_{\ell^2\mathbb{N}}$, and potential $q \in \mathcal{C}_b(F)$. Denote by $(\lambda_i)_{i\in\mathbb{N}}$ the eigenvalues of $\mathbb{B}$. Let $\mathcal{M}_\beta : L^2(F,\nu) \hat{\otimes} \mathcal{F}(\ell^2\mathbb{N}) \to L^2(F,\nu) \hat{\otimes} \mathcal{F}(\ell^2\mathbb{N})$,*

$$(\mathcal{M}_\beta f)(x, z) = \int_F \mathbb{A}_{\sigma,x}\, \exp\Big(\beta q(\sigma) + \beta\sigma \langle z | w \rangle \Big)\, f(\sigma, \sigma\, v + \mathbb{B}\, z)\, d\nu(\sigma)$$

be the Ruelle-Mayer transfer operator defined in Theorem 3.2.6. Then there exists $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that the dynamical zeta function satisfies

$$\zeta_R(z, \beta) = \exp\Big(\sum_{n=1}^{n_0-1} \frac{z^n}{n}\, \widetilde{Z}^{b^{\mathbb{N}_0}}_n(\beta A_{(\phi)}) \Big)\ \lim_{M\to\infty} \prod_{\alpha\in\{0,1\}^M} \det_{n_0}(1 - z\underline{\lambda}^\alpha\, \mathcal{M}_\beta)^{(-1)^{|\alpha|+1}},$$

and the right hand side has an Euler product and a meromorphic continuation to $\mathbb{C}$. □

Analogously, by Theorem 3.2.6 the same result holds for Ising type interactions (say of rank M) when replacing the sequence $\underline{\lambda} := (\lambda_i)_{i\in\mathbb{N}}$ of eigenvalues of $\mathbb{B} : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ by the sequence $\underline{\lambda}_M$ of eigenvalues of the M-fold direct sum $\mathbb{B}_M : (\ell^2\mathbb{N})^M \to (\ell^2\mathbb{N})^M$ of $\mathbb{B}$. Obviously, $\mathbb{B}_M$ has the same eigenvalues as $\mathbb{B}$, but with the M-fold multiplicity. Note that if F is finite, then every interaction matrix has finite rank.

Corollaries 4.4.3 and 4.4.4 show that in order to understand the zeros and poles of Ruelle's zeta one has to investigate the spectrum of the Ruelle-Mayer transfer operator in detail. Whereas the spectrum of a generalised composition operator is well-understood, the spectrum of the Ruelle-Mayer operator is more difficult, since the operator is a sum (an integral) of composition operator which in general do not commute. For a certain class of interactions we will do a step of preparation in the next chapter.

5 The Extended Fock Oscillator semigroup and Kac-Gutzwiller transfer operators

In Chapters 2 and 3 we have shown the following result: Let $F \subset \mathbb{C}$ be a bounded set and $(\Omega = F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function given as $d(k) = (\Lambda^{k-1}v|w)$ for some $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$, $v, w \in \mathbb{C}^n$. Such a distance function belongs to the class $\mathcal{D}_1^{(1)}$ (2.7.1). There are two types of such distance functions (and the superposition of the two): Either d has finite range (Subs. 2.8) or it is polynomial-exponentially decaying. By the latter we mean a distance function $d : \mathbb{N} \to \mathbb{C}$ given as $d(k) := \lambda^k \sum_{i=0}^{n-1} c_i\, k^i$ for some fixed $n \in \mathbb{N}$, $c_i, \lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$. By Theorem 2.7.6 the Ruelle-Mayer transfer operator

$$\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\bigl(\beta q(\sigma) + \beta\sigma(z|w)\bigr)\, f(\sigma\, v + \Lambda\, z)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\ldots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1-\Lambda^n)\,\mathrm{trace}\,(\mathcal{M}_\beta)^n$ for all $n \in \mathbb{N}$ and Theorem 3.2.6 gives the analogue result for matrix subshifts.

There is a completely different type of transfer operators for Ising spin systems with exponentially decaying distance function, which is due to M. Gutzwiller [Gu82] building upon results of M. Kac [Ka66]. This integral operator satisfies the same dynamical trace formula as the Ruelle-Mayer operator for the same system. In [HiMay02] and [HiMay04] it was shown a correspondence which implies that the spectra of the both operators coincide, without proving that the operators are conjugate. The argument of J. Hilgert and D. Mayer uses the fact that there is a unitary isomorphism $B : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$, called the Bargmann transform. By conjugating the Ruelle-Mayer operator with the Bargmann transform they obtained an operator which is closely related to the original Kac-Gutzwiller operator. In this chapter we want to understand this correspondence better in the hope that this may lead to new classes of transfer operators for spin systems. Another application of this chapter arises from the fact that the spectral properties of the Ruelle-Mayer transfer operator can better be analysed via the corresponding Kac-Gutzwiller transfer operator and hence opens up the possibility to study zero statistics of the dynamical zeta function.

We use our results of the preceding chapters and define a *Kac-Gutzwiller transfer operator* to be a Bargmann conjugate of a Ruelle-Mayer transfer operator, i. e., $B^{-1} \circ \mathcal{M}_\beta \circ B$ acting on $L^2(\mathbb{R}^n)$. In this chapter we will compute the Kac-Gutzwiller transfer operator explicitly and study its properties. We consider both full and matrix subshifts with Ising type interaction, both with polynomial-exponential and finite-range distance function. In particular, we will compute the Kac-Gutzwiller transfer operator for the Potts model.

For this purpose we write the Ruelle-Mayer transfer operator as an integral over a family of generalised composition operators of the following type: For any $a, b \in \mathbb{C}^n$, $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$ we define a generalised composition operator $\mathcal{L}_{a,b,\Lambda} := m_{\exp_a} \circ C_\Lambda \circ \tau_{-b}$ acting via

$$(\mathcal{L}_{a,b,\Lambda} f)(z) = e^{(z|a)}\, f(\Lambda z + b)$$

on the Fock space $\mathcal{F}(\mathbb{C}^n)$ as a trace class operator. Thereby is $\tau_b : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$, $(\tau_b f)(z) := f(z-b)$ the translation operator, $m_{\exp_a} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$, $(m_{\exp_a} f)(z) := e^{(z|a)}\, f(z)$ the multiplication operator, and $C_\Lambda : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$, $(C_\Lambda f)(z) = f(\Lambda z)$ the composition operator. Our strategy is to show that $\mathcal{L}_{a,b,\Lambda}$ belongs to the so called extended Fock oscillator semigroup. Via the Bargmann transform the Fock oscillator semigroup is conjugate to the oscillator semigroup and explicit conjugation formulas are known in the literature.

In Section 5.1 we compute the conjugate of the composition operator C_Λ under the Bargmann transform. It turns out that for Λ belonging to the unit disk $\{X \in \mathrm{Mat}(n,n;\mathbb{C}) \,|\, \|X\| < 1\}$ the operator C_Λ is an element of the Fock oscillator semigroup $\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ which is well-understood. In Section 5.2 we introduce the oscillator semigroup Ω_n. It consists of trace class integral operators with Gaussian kernels. The Fock oscillator semigroup and the oscillator semigroup are conjugate via the Bargmann transform, $\Omega_{n,\mathcal{F}(\mathbb{C}^n)} = \{B \circ T \circ B^{-1} \,|\, T \in \Omega_n\}$. The precise relation between the integral kernels of $T \in \Omega_n$ and $B \circ T \circ B^{-1} \in \Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ is given by the Cayley transform. This general result directly yields the Bargmann conjugate $B^{-1} \circ C_\Lambda \circ B$ of the composition operator C_Λ. Exemplarily, we compute the conjugate integral operators corresponding to $\Lambda = J_{(\lambda,n)}$ being a Jordan block with eigenvalue

$0 < |\lambda| < 1$ and, secondly, $\Lambda = sJ_{(0,n)}$ for some small scalar multiple of a Jordan block with eigenvalue zero. The first matrix corresponds to Ruelle-Mayer transfer operators for the one-dimensional Ising model with polynomial-exponentially decaying interaction. The second example corresponds to Ruelle-Mayer transfer operators for the one-dimensional Ising model with finite-range interaction.
In Section 5.3 we define the extended oscillator semigroup $E\Omega_n$ consisting of integral operators with (general) Gaussians as integral kernel. We introduce the extended Fock oscillator semigroup to be the image $E\Omega_{n,\mathcal{F}(\mathbb{C}^n)} := \{B \circ T \circ B^{-1} \,|\, T \in E\Omega_n\}$ under the conjugation with the Bargmann transform. During this section we will compute the conjugates of translations and of multiplication operators $B \circ \tau_r \circ B^{-1},\ B \circ m_{\exp_s} \circ B^{-1}$ both acting on $\mathcal{F}(\mathbb{C}^n)$ and $B^{-1} \circ m_{\exp_a} \circ B,\ B^{-1} \circ \tau_{-b} \circ B$ both acting on $L^2(\mathbb{R}^n)$, which will lead to an explicit description of the extended Fock oscillator semigroup. These results show that $\mathcal{L}_{a,b,\Lambda} \in E\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ and lead to an explicit formula (Prop. 5.3.5) for its Bargmann conjugate $B^{-1} \circ \mathcal{L}_{a,b,\Lambda} \circ B \in E\Omega_n$.
In Section 5.4 we will finally compute the Kac-Gutzwiller transfer operator for two-body Ising interaction with distance function $d \in \mathcal{D}_1^{(1)}$ given as $d(k) = (\Lambda^{k-1}v|w)$ for some $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$, v, $w \in \mathbb{C}^n$. We apply this to the two main cases of such distance functions, namely the polynomial-exponentially decaying interactions and the finite-range interactions. The action of the general linear group $\mathrm{Gl}(n;\mathbb{C})$ on $\mathrm{Mat}(n,n;\mathbb{C})$ by conjugation leads to the fact that for each distance function $d(k) = (\Lambda^{k-1}v|w)$ there is a family of generating triples parametrised by $\mathrm{Gl}(n;\mathbb{C})$. We show that for certain generating triples (a,b,Λ) the corresponding integral operator $B^{-1} \circ \mathcal{L}_{\pi a,b,\Lambda} \circ B$ has a special form. This will lead to Kac-Gutzwiller transfer operators whose integral kernels are of a simple form. This fact can be used to investigate the spectral properties of the Kac-Gutzwiller operator and hence of the Ruelle-Mayer transfer operator. This allows to determine the spectrum of an integral (or a weighted sum) of composition operators, for which we do not know another method. We end this chapter by giving examples of Kac-Gutzwiller transfer operators for the Potts model.

5.1 Composition operators on the Fock space

We briefly recall the definition of the *Bargmann-Fock space* as introduced in A.4.5. It consists of

$$\mathcal{F}(\mathbb{C}^n) := \mathcal{H}L^2(\mathbb{C}^n) := \Big\{ f \in \mathcal{O}(\mathbb{C}^n) \,|\, \|f\|^2_{\mathcal{F}(\mathbb{C}^n)} := \int_{\mathbb{C}^n} |f(z)|^2 \exp(-\pi\|z\|^2)\, dz < \infty \Big\}$$

where dz denotes Lebesgue measure on $\mathbb{C}^n$. The inner product is given by

$$\big\langle f \,\big|\, g \big\rangle_{\mathcal{F}(\mathbb{C}^n)} := \int_{\mathbb{C}^n} f(z)\, \overline{g(z)}\, \exp(-\pi\|z\|^2)\, dz.$$

The Bargmann-Fock space is a reproducing kernel Hilbert space with kernel $k(z,w) = \exp(\pi \langle z \,|\, w \rangle)$.
We will now introduce an operator semigroup acting on the Fock space.

Definition 5.1.1. (i) We equip $\mathrm{Mat}(n,n;\mathbb{C})$ with the operator norm. The symmetric unit ball $\Delta_n := B(0;1)_{\mathrm{Sym}(n;\mathbb{C})} = \{W \in \mathrm{Mat}(n,n;\mathbb{C}) \,|\, W = W^\top, 1 - W^\star W > 0\} \subset B(0;1)_{\mathrm{Mat}(n,n;\mathbb{C})}$ is called the *Siegel disk*. Here $1 - W^\star W > 0$ means that $\langle (1 - W^\star W)x|x\rangle > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$.

(ii) For any $D \in \Delta_{2n}$ the integral operator S_D acting on $\mathcal{F}(\mathbb{C}^n)$ is defined via its integral kernel[32]

$$\Gamma_D(z,\overline{w}) = \exp\Big(\frac{\pi}{2}\left(\begin{pmatrix} z \\ \overline{w}\end{pmatrix} \Big|\, D \begin{pmatrix} z \\ \overline{w}\end{pmatrix}\right)\Big).$$

(iii) The *Fock oscillator semigroup* is defined as the space $\Omega_{n,\mathcal{F}(\mathbb{C}^n)} := \{c\, S_D \,|\, D \in \Delta_{2n},\, c \in \mathbb{C}^\times\}$ of integral operators with Γ_D-kernels. □

We will now show that the composition operator $C_\Lambda : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (C_\Lambda f)(z) = f(\Lambda z)$ belongs to the Fock oscillator semigroup $\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ (5.1.1) if $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$. Therefor we need an embedding of the unit ball $\{\Lambda \in \mathrm{Mat}(n,n;\mathbb{C}) \,|\, \|\Lambda\| < 1\}$ into the Siegel disk Δ_{2n} (5.1.1).

[32]The Hermitian inner product on $\mathbb{C}^n$ is denoted by $\langle\cdot|\cdot\rangle$, the Euclidean inner product on $\mathbb{R}^n$ and its $\mathbb{C}$-bilinear extension are denoted by $(\cdot|\cdot)$, for $z \in \mathbb{C}^n$ we set $z^2 := (z|z)$ as an abbreviation.

Remark 5.1.2. The map

$$X : \mathrm{Mat}(n, n; \mathbb{C}) \to \mathrm{Sym}(2n; \mathbb{C}) := \{Y \in \mathrm{Mat}(2n, 2n; \mathbb{C}) \,|\, Y^\top = Y\},\ \Lambda \mapsto X_\Lambda := \begin{pmatrix} 0 & \Lambda^\top \\ \Lambda & 0 \end{pmatrix} \tag{89}$$

is injective and linear, it defines an embedding of the unit ball $\{\Lambda \in \mathrm{Mat}(n, n; \mathbb{C}) \,|\, \|\Lambda\| < 1\}$ into the Siegel disk Δ_{2n} (5.1.1)

$$X : B(0; 1)_{\mathrm{Mat}(n,n;\mathbb{C})} \to \mathrm{Sym}(2n; \mathbb{C}) \cap B(0; 1)_{\mathrm{Mat}(2n,2n;\mathbb{C})} \subset \Delta_{2n},\ \Lambda \mapsto X_\Lambda$$

with (using [Fo89, A. Lemma 4]) $\det X_\Lambda = \det(0 - \Lambda\Lambda^\top) = (\det \Lambda)^2$. □

Remark 5.1.3. Let $\Lambda \in B(0; 1)_{\mathrm{Mat}(n,n;\mathbb{C})}$. Using Remark A.6.1 the composition operator

$$C_\Lambda : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (C_\Lambda F)(z) := F(\Lambda z)$$

can be written as an integral operator with kernel

$$\exp(\pi\langle\Lambda z|w\rangle) = \exp(\pi(\Lambda z|\overline{w})) = \exp\Big(\frac{\pi}{2}\Big(\begin{pmatrix} z \\ \overline{w} \end{pmatrix}\Big|\begin{pmatrix} 0 & \Lambda^\top \\ \Lambda & 0 \end{pmatrix}\begin{pmatrix} z \\ \overline{w} \end{pmatrix}\Big)\Big).$$

This shows $C_\Lambda = S_{X_\Lambda}$ using the shortly introduced notations and hence this composition operator belongs to the Fock oscillator semigroup $\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ (5.1.1). □

5.2 The oscillator semigroup

In this section we introduce the oscillator semigroup Ω_n. It consists of trace class integral operators with Gaussian kernel. The Fock oscillator semigroup and the oscillator semigroup are conjugate via the Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$ as follows

$$\Omega_{n,\mathcal{F}(\mathbb{C}^n)} = \{B \circ T \circ B^{-1} \,|\, T \in \Omega_n\}.$$

Usually the Fock oscillator semigroup is defined to be the image under the conjugation with the Bargmann transform, but we preferred to define both semigroups separately. The precise relation between the integral kernels of $T_A \in \Omega_n$ and $B \circ T_A \circ B^{-1} \in \Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ is given by the Cayley transform which is a conformal map between the Siegel upper half plane $\mathfrak{S}$ and the Siegel disk Δ_{2n} (5.1.1). We end this section by computing the Bargmann conjugate $B^{-1} \circ C_\Lambda \circ B$ of the composition operator $(C_\Lambda f)(z) = f(\Lambda z)$ (5.1.3) which is our main motivation for introducing the oscillator semigroup. For two types of examples we compute the corresponding integral kernels in detail. The first example concerns $\Lambda = J_{(\lambda,n)}$ being a Jordan block with non-vanishing eigenvalue $0 < |\lambda| < 1$ which corresponds to Ruelle-Mayer transfer operators for the one-dimensional Ising model with polynomial-exponentially decaying interaction. The second type of examples concerns finite-range Ising interactions. By the generating triple found in Proposition 2.8.2 we have to study $\Lambda = sJ_{(0,n)}$ for some small parameter $0 < s < 1$.

Definition 5.2.1. (i) Let $\mathfrak{S}_n := \{\alpha \in \mathrm{Mat}(n, n; \mathbb{C}) \,|\, \alpha = \alpha^\top,\ \mathbf{Im}(\alpha) \text{ positive definite}\} \subset \mathrm{Gl}(n; \mathbb{C})$ be the *Siegel upper half plane*.

(ii) We introduce the *unnormalised Gaussian*[33]

$$g : \mathbb{C}^n \times \mathfrak{S}_n \to \mathbb{C},\ (z, A) \mapsto g(z, A) := g_A(z) := \exp(-\pi\,(z \,|\, Az)).$$

(iii) For $\mathcal{A} \in \mathfrak{S}_{2n}$ we define $T_\mathcal{A}$ to be the integral operator acting on $L^2(\mathbb{R}^n)$ with kernel

$$g_\mathcal{A}(x, y) = \exp\Big(\pi i\Big(\begin{pmatrix} x \\ y \end{pmatrix}\Big|\,\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix}\Big)\Big).$$

(iv) The *oscillator semigroup* is defined as the space $\Omega_n := \{cT_\mathcal{A} \,|\, \mathcal{A} \in \mathfrak{S}_{2n},\ c \in \mathbb{C}^\times\}$ of integral operators with Gaussian kernels. □

[33] In [Fo89] this function is denoted by γ_A.

Remark 5.2.2. For $\mathcal{A} = \mathcal{A}^\top \in \mathrm{Mat}(n,n;\mathbb{C})$ one has the equivalent characterisations

$$\mathcal{A} \in \mathfrak{S}_n \iff g_{\mathcal{A}} \in L^2(\mathbb{R}^n) \iff g_{\mathcal{A}} \in \mathcal{S}(\mathbb{R}^n),$$

where we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decaying smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. By the second characterisation Ω_n consists of Hilbert-Schmidt operators, whereas the last characterisation implies (see for instance [CoGr90, A.3.9]) that Ω_n consists of trace class operators. □

We will now show the relation between the oscillator semigroup Ω_n (Def. 5.2.1) and the Fock oscillator semigroup $\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ (Def. 5.1.1) which is given via the Bargmann transform. The Bargmann transform[34] $B_n : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$,

$$(B_n f)(z) = \int_{\mathbb{R}^n} \exp\Big(2\pi \left(z \,\middle|\, x\right) - \pi \left(x \,\middle|\, x\right) - \frac{\pi}{2} \left(z \,\middle|\, z\right)\Big)\, f(x)\, dx \tag{90}$$

is a unitary ismorphism, see for instance [Fo89, ch. 1.6].
The Siegel upper half plane $\mathfrak{S}_n$ (5.2.1) is the higher dimensional analogon of the upper half plane $\mathfrak{S}_1 = \{z \in \mathbb{C} \,|\, \mathbf{Im}(z) > 0\}$. As in the case $n = 1$ there is a conformal map between the upper half plane $\mathfrak{S}_n$ and the unit disk Δ_n (5.1.1), which is called the Cayley transform.[35]

Proposition 5.2.3. *([Fo89, 4.67]) The Cayley transform*

$$\mathbf{c}_n : \mathfrak{S}_n \to \Delta_n,\ \mathbf{c}_n(Z) := (1 + iZ)(1 - iZ)^{-1} = (1 - iZ)^{-1}(1 + iZ)$$

is conformal. Its inverse acts via $\mathbf{c}_n^{-1}(W) = i(1 - W)(1 + W)^{-1}$. □

Proposition 5.2.4. *([Fo89, 4.70]) For all $\mathcal{A} \in \mathfrak{S}_{2n}$ (5.2.1) one has*

$$B_n \circ T_{\mathcal{A}} \circ (B_n)^{-1} = 2^{n/2} \det(1 - i\mathcal{A})^{-1/2}\, S_{\mathbf{c}_{2n}(\mathcal{A})},$$

□

Because of this proposition the Fock oscillator semigroup $\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ (Def. 5.1.1) is often defined as the image under the conjugation

$$\Omega_{n,\mathcal{F}(\mathbb{C}^n)} = \{B_n \circ T \circ (B_n)^{-1} \,|\, T \in \Omega_n\}$$

of the oscillator semigroup Ω_n. In Section 5.1 we have seen that for $\Lambda \in B(0;1)_{\mathrm{Mat}(n,n;\mathbb{C})}$ the composition operator $C_\Lambda : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$, $(C_\Lambda f)(z) = f(\Lambda z)$ (5.1.3) belongs to the Fock oscillator semigroup $\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$. We will now use the previous Proposition 5.2.4 to compute its Bargmann conjugate acting on $L^2(\mathbb{R}^n)$.

Proposition 5.2.5. *Let $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$ and $C_\Lambda : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ $(C_\Lambda F)(z) = F(\Lambda z)$ be the associated composition operator (5.1.3). Then its Bargmann conjugate on $L^2(\mathbb{R}^n)$ is the operator*

$$B^{-1} \circ C_\Lambda \circ B = 2^{n/2} \det(1 - \Lambda\Lambda^\top)^{-1/2}\, T_{\mathbf{c}_{2n}^{-1}(X_\Lambda)},$$

which has the integral kernel

$$\begin{aligned} k_\Lambda(x,y) &= \frac{2^{n/2}}{\det(1 - \Lambda\Lambda^\top)^{1/2}}\, g_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}(x,y) \\ &= \frac{2^{n/2}}{\det(1 - \Lambda\Lambda^\top)^{1/2}} \exp\Big(-\pi\Big(\begin{pmatrix} x \\ y \end{pmatrix} \,\Big|\, \begin{pmatrix} (1+\Lambda^\top\Lambda)(1-\Lambda^\top\Lambda)^{-1} & -2\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} \\ -2\Lambda(1-\Lambda^\top\Lambda)^{-1} & (1+\Lambda\Lambda^\top)(1-\Lambda\Lambda^\top)^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\Big)\Big). \end{aligned}$$

Proof. Using Remark 5.1.2 one has $C_\Lambda = S_{X_\Lambda}$. Solve $\mathbf{c}_{2n}(\mathcal{A}) = X_\Lambda$, i. e., $\mathcal{A} = \mathbf{c}_{2n}^{-1}(X_\Lambda)$, and use Proposition 5.2.4 on the relation between the oscillator semigroup and the Fock oscillator semigroup. In order to avoid long computations we have to make a little excursus using the notation of [Fo89, ch. 4.5]. For $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n;\mathbb{R}) \subset \mathrm{Gl}(2n;\mathbb{R})$, $Z \in \{W \in \mathrm{Mat}(n,n;\mathbb{C}) \,|\, CW + D \in \mathrm{Gl}(n;\mathbb{C})\}$ set $m(\mathcal{A}, Z) := \det(CZ+D)^{-\frac{1}{2}}$. Then for $\mathcal{C}_n := \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} i & 1 \\ -i & 1 \end{smallmatrix}\right) \in \mathrm{Gl}(2n;\mathbb{R})$ being the so called Cayley element,

[34] We will omit the index n for the dimension most of the time.
[35] Folland writes $\alpha_n(\mathcal{C}_n)$, where we write $\mathbf{c}_n$.

one obtains $m(\mathcal{C}_n, Z) = 2^{n/4}\det(1-iZ)^{-\frac{1}{2}}$ The inverse of the Cayley element is $\mathcal{C}_n^{-1} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}-i & i\\ 1 & 1\end{smallmatrix}\right)$ and its multiplier acts via $m(\mathcal{C}_n^{-1}, Z) = 2^{n/4}\det(1+Z)^{-\frac{1}{2}}$. The determinant factor in Proposition 5.2.4 is thus $m(\mathcal{C}_{2n}, \mathcal{A}) = 2^{n/2}\det(1-i\mathcal{A})^{-1/2}$. The multiplier satisfies a cocycle identity which implies formula [Fo89, 4.63] which we use at $(\star)$

$$2^{n/4}\det(1-i\mathbf{c}_n^{-1}(Z))^{-1/2} = m(\mathcal{C}_n, \mathbf{c}_n^{-1}(Z)) \overset{(\star)}{=} m(\mathcal{C}_n^{-1}, Z)^{-1} = 2^{-n/4}\det(1+Z)^{1/2}.$$

Hence for $\mathcal{A}$ as chosen above one has

$$m(\mathcal{C}_{2n}, \mathcal{A}) = m(\mathcal{C}_{2n}, X_\Lambda)^{-1} = 2^{-n/2}\det(X_\Lambda+1)^{\frac{1}{2}} = 2^{-n/2}\det(1-\Lambda\Lambda^\top)^{\frac{1}{2}}. \tag{91}$$

Concerning the explicit formula for $\mathbf{c}_{2n}^{-1}(X_\Lambda)$:

$$\begin{aligned}
\mathbf{c}_{2n}^{-1}(X_\Lambda) &= i(1-X_\Lambda)(1+X_\Lambda)^{-1} = i\begin{pmatrix} 1 & -\Lambda^\top \\ -\Lambda & 1\end{pmatrix}\begin{pmatrix} 1 & \Lambda^\top \\ \Lambda & 1\end{pmatrix}^{-1} \\
&= i\begin{pmatrix} 1 & -\Lambda^\top \\ -\Lambda & 1\end{pmatrix}\begin{pmatrix} (1-\Lambda^\top\Lambda)^{-1} & -\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} \\ -\Lambda(1-\Lambda^\top\Lambda)^{-1} & (1-\Lambda\Lambda^\top)^{-1}\end{pmatrix} \\
&= i\begin{pmatrix} (1+\Lambda^\top\Lambda)(1-\Lambda^\top\Lambda)^{-1} & -2\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} \\ -2\Lambda(1-\Lambda^\top\Lambda)^{-1} & (1+\Lambda\Lambda^\top)(1-\Lambda\Lambda^\top)^{-1}\end{pmatrix} \qquad (92)
\end{aligned}$$

□

We end this section by studying two types of examples of composition operators C_Λ (5.1.3). The first example concerns $\Lambda = J_{(\lambda,n)}$ being a Jordan block (68) with eigenvalue $0 < |\lambda| < 1$. It corresponds to Ruelle-Mayer transfer operators for the one-dimensional Ising model with polynomial-exponentially decaying interaction (see 2.11). The second type of examples concerns $\Lambda = sJ_{(0,n)}$ for some small parameter $0 < s < 1$. It corresponds to Ruelle-Mayer transfer operators for the one-dimensional Ising model with finite-range interaction (see 2.8).
The following proposition is of preparatory nature.

Proposition 5.2.6. *Let $J_{(\lambda,n)}$ be a standard Jordan block of size n as defined in (68). Then*

$$J_{(\lambda,n)}J_{(\lambda,n)}^\top = \begin{pmatrix} 1+\lambda^2 & \lambda & & & \\ \lambda & 1+\lambda^2 & \ddots & & \\ & \ddots & \ddots & \lambda & \\ & & \lambda & 1+\lambda^2 & \lambda \\ & & & \lambda & \lambda^2 \end{pmatrix},$$

$$J_{(\lambda,n)}^\top J_{(\lambda,n)} = \begin{pmatrix} \lambda^2 & \lambda & & & \\ \lambda & 1+\lambda^2 & \ddots & & \\ & \ddots & \ddots & \lambda & \\ & & \lambda & 1+\lambda^2 & \lambda \\ & & & \lambda & 1+\lambda^2 \end{pmatrix}.$$

Proof. The Jordan block $J := J_{(\lambda,n)}$ has the entries $J_{i,j} = \lambda\,\delta_{i,j} + \delta_{i+1,j}$. Hence

$$\begin{aligned}
(JJ^\top)_{i,j} &= \sum_{k=1}^n J_{i,k}\,J_{j,k} \\
&= \sum_{k=1}^n (\lambda\,\delta_{i,k} + \delta_{i+1,k})(\lambda\,\delta_{j,k} + \delta_{j+1,k}) \\
&= \sum_{k=1}^n \left(\lambda^2\,\delta_{i,k}\,\delta_{j,k} + \lambda\,\delta_{j,k}\,\delta_{i+1,k} + \lambda\,\delta_{i,k}\,\delta_{j+1,k} + \delta_{j+1,k}\,\delta_{i+1,k}\right) \\
&= \lambda^2\,\delta_{j,i} + \lambda\,\delta_{i+1,j} + \lambda\,\delta_{j+1,i} + (1-\delta_{i,n})\,\delta_{i,j}
\end{aligned}$$

Let $\Phi \in \mathrm{Gl}(n;\mathbb{Z})$ be the flip matrix with ones along the main antidiagonal and zeros otherwise, i. e., $\Phi_{i,j} = \delta_{i+j,n+1}$. Then one confirms that $(\Phi\mathbb{B}\Phi)_{i,l} = \mathbb{B}_{n+1-i,n+1-j}$ for all $\mathbb{B} = (\mathbb{B}_{i,j}) \in \mathrm{Mat}(n,n;\mathbb{C})$. Hence $\Phi^2 = 1$, $\Phi J_{(\lambda,n)}\Phi = J_{(\lambda,n)}^\top$ and thus $J^\top J = \Phi J \Phi J = \Phi(J\Phi J\Phi)\Phi = \Phi(JJ^\top)\Phi$, which together with the first part gives the stated form of $J^\top J$. □

Let $d : \mathbb{N} \to \mathbb{C}$ be a polynomial-exponential decaying distance function, say $d(k) = \lambda^k\, p(k)$, where $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $p \in \mathbb{C}[X]$ is a polynomial of degree $n-1$. Then by Section 2.11 d has a representation $d(k) = \langle (J_{(\lambda,n)})^{k-1} v | w\rangle$ for some $v, w \in \mathbb{C}^n$, where $J_{(\lambda,n)}$ is the standard Jordan block (68) of dimension n with eigenvalue λ. The corresponding Ruelle-Mayer transfer operator is an integral over a family of generalised composition operators with composition part $C_{J_{(\lambda,n)}} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$. By Proposition 5.2.5 its conjugate on $L^2(\mathbb{R}^n)$ is the operator $B^{-1} \circ C_{J_{(\lambda,n)}} \circ B = 2^{n/2} \det(1 - J_{(\lambda,n)} J_{(\lambda,n)}^\top)^{-1/2}\, T_{\mathbf{c}_{2n}^{-1}(X_{J_{(\lambda,n)}})}$, which has the integral kernel

$$k_{J_{(\lambda,n)}}(x,y) = \frac{2^{n/2}}{\det(1 - J_{(\lambda,n)} J_{(\lambda,n)}^\top)^{1/2}}\, g_{\mathbf{c}_{2n}^{-1}(X_{J_{(\lambda,n)}})}(x,y).$$

The computations become much more complicated as the dimension n increases. Thus we exemplarily treat the cases $n = 1$ and $n = 2$. The following one-dimensional example has been studied in [Gu82] and [HiMay04].

Example 5.2.7. Let λ be a complex number with $0 < |\lambda| < 1$. Then by Proposition 5.2.5 the integral kernel of $H := B^{-1} \circ C_\Lambda \circ B : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given explicitly as

$$h(x,y) = \frac{2^{\frac{1}{2}}}{(1-\lambda^2)^{\frac{1}{2}}} \exp\Big(-\frac{\pi(1+\lambda^2)}{1-\lambda^2}\,(x^2+y^2) + \frac{4\pi\lambda}{1-\lambda^2}\, xy \Big).$$

For $c \in \mathbb{R}^+$ let $R_c\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(R_c f)(x) := \sqrt{c}\, f(cx)$ be the scaling operator. Then for the special choice $c = (4\pi)^{-1/2}$ we obtain from Lemma A.6.3 the integral kernel of the scaled operator $H_c := R_c \circ H \circ R_c^{-1}$ as $h_c(x,y) = c\, h(cx,cy)$. Similarly to [HiMay04] we write $\lambda = e^{-\gamma}$ and conclude that

$$\begin{aligned} h_{(4\pi)^{-1/2}}(x,y) &= \frac{(2/\pi)^{-1/2}}{(1-\lambda^2)^{\frac{1}{2}}} \exp\Big(-\frac{1}{4}\frac{1+\lambda^2}{1-\lambda^2}(x^2+y^2) + \frac{\lambda}{1-\lambda^2} xy \Big) \\ &= \frac{e^{\gamma/2}}{\sqrt{4\pi \sinh\gamma}} \exp\Big(\frac{1}{2\sinh\gamma}\big(-\frac{1}{2}(\cosh\gamma\,(x^2+y^2) + xy)\big)\Big) \\ &= \frac{e^{\gamma/2}}{\sqrt{4\pi \sinh\gamma}} \exp\Big(-\frac{1}{4}\big(\tanh\frac{\gamma}{2}\,(x^2+y^2) + \frac{(x-y)^2}{\sinh\gamma}\big)\Big), \end{aligned}$$

which both can be interpreted as variants of Mehler's s formula, cp. [Fo89, 1.87]. □

In Example 5.2.7 we have discussed the case $n = 1$. We will now compute the integral kernel $k_{J_{(\lambda,n)}}(x,y)$ for the next easiest case $n = 2$. This will take much more computational effort.

Example 5.2.8. Let $\lambda \in \mathbb{C}$ and $\Lambda := J_{(\lambda,n)}$ be a Jordan block (68). The inverse Cayley transform of X_Λ is given explicitly by (92)

$$\mathbf{c}_{2n}^{-1}(X_\Lambda) = i \begin{pmatrix} (1+\Lambda^\top\Lambda)(1-\Lambda^\top\Lambda)^{-1} & -2\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} \\ -2\Lambda(1-\Lambda^\top\Lambda)^{-1} & (1+\Lambda\Lambda^\top)(1-\Lambda\Lambda^\top)^{-1} \end{pmatrix}.$$

Thus we have to compute its (block) entries. Let $\Phi \in \mathrm{Gl}(n;\mathbb{Z})$ be the flip as introduced in the proof of Proposition 5.2.6. This proof also implies that $(1-\Lambda\Lambda^\top)^{-1} = \big(\Phi(1-\Lambda^\top\Lambda)\Phi\big)^{-1} = \Phi(1-\Lambda^\top\Lambda)^{-1}\Phi$, where Φ is the flip. Thus the lower right entry of $\mathbf{c}_{2n}^{-1}(X_\Lambda)$ is

$$i(1+\Lambda\Lambda^\top)(1-\Lambda\Lambda^\top)^{-1} = i\Phi(1+\Lambda^\top\Lambda)(1-\Lambda^\top\Lambda)^{-1}\Phi$$

a flip-conjugate of the upper left entry of $\mathbf{c}_{2n}^{-1}(X_\Lambda)$. The remaining off-diagonal entries are also flip-conjugate and their mutual transposes, since

$$\big(\Lambda^\top(1-\Lambda\Lambda^\top)^{-1}\big)^\top = (1-\Lambda^\top\Lambda)^{-1}\Lambda = \Lambda(1-\Lambda^\top\Lambda)^{-1} = \Phi\Lambda^\top(1-\Lambda\Lambda^\top)^{-1}\Phi.$$

Thus only two blocks are unknown. We will now study the smallest non-trivial example $\Lambda := J_{(\lambda,2)}$. By Proposition 5.2.6 we obtain $\Lambda^\top\Lambda$, hence $1-\Lambda^\top\Lambda = \begin{pmatrix} 1-\lambda^2 & -\lambda \\ -\lambda & -\lambda^2 \end{pmatrix}$, whose inverse is given as $(1-\Lambda^\top\Lambda)^{-1} = \frac{1}{\lambda^2(\lambda^2-2)}\begin{pmatrix} -\lambda^2 & \lambda \\ \lambda & 1-\lambda^2 \end{pmatrix}$. Thus the upper left entry of $\mathbf{c}_{2n}^{-1}(X_\Lambda)$ is

$$\begin{aligned} i(1+\Lambda^\top\Lambda)(1-\Lambda^\top\Lambda)^{-1} &= \frac{i}{\lambda^2(\lambda^2-2)}\begin{pmatrix} 1+\lambda^2 & \lambda \\ \lambda & 2+\lambda^2 \end{pmatrix}\begin{pmatrix} -\lambda^2 & \lambda \\ \lambda & 1-\lambda^2 \end{pmatrix} \\ &= \frac{i}{\lambda^2(\lambda^2-2)}\begin{pmatrix} -\lambda^4 & 2\lambda \\ 2\lambda & 2-\lambda^4 \end{pmatrix}. \end{aligned}$$

For the off-diagonal entries observe that

$$\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} = \frac{1}{\lambda^2(\lambda^2-2)}\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} 1-\lambda^2 & \lambda \\ \lambda & -\lambda^2 \end{pmatrix} = \frac{1}{\lambda^2(\lambda^2-2)}\begin{pmatrix} \lambda(1-\lambda^2) & \lambda^2 \\ 1 & \lambda(1-\lambda^2) \end{pmatrix}.$$

Thus finally

$$i\mathbf{c}_{2n}^{-1}(X_\Lambda) \overset{(92)}{=} \frac{1}{\lambda^2(\lambda^2-2)}\begin{pmatrix} \lambda^4 & -2\lambda & 2\lambda(1-\lambda^2) & 2 \\ -2\lambda & \lambda^4-2 & 2\lambda^2 & 2\lambda(1-\lambda^2) \\ 2\lambda(1-\lambda^2) & 2\lambda^2 & \lambda^4-2 & -2\lambda \\ 2 & 2\lambda(1-\lambda^2) & -2\lambda & \lambda^4 \end{pmatrix}$$

and the integral kernel of $B^{-1}\circ C_{J_{(\lambda,2)}}\circ B = 2\det(1-J_{(\lambda,2)}J_{(\lambda,2)}^\top)^{-1/2}\,T_{\mathbf{c}_2^{-1}(X_{J_{(\lambda,2)}})}$ is

$$\begin{aligned} &k_{J_{(\lambda,2)}}(x_1,x_2,y_1,y_2) = \frac{2}{\det(1-\Lambda\Lambda^\top)^{1/2}}\,g_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}(x_1,x_2,y_1,y_2) \\ &= \frac{2}{(\lambda^2\,(\lambda^2-2))^{1/2}}\,g_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}(x_1,x_2,y_1,y_2) \\ &= \frac{2}{\lambda\,(\lambda^2-2)^{1/2}}\exp\Big(\frac{-\pi}{\lambda^2(\lambda^2-2)}\Big(-x_1\lambda\left(\lambda^3x_1-2x_2-2y_1\lambda^2+2y_1+2y_2\lambda\right) \\ &\quad +x_2\left(2\lambda x_1+2x_2-x_2\lambda^4-2y_1+2y_2\lambda^3-2y_2\lambda\right)+y_1\left(2\lambda^3x_1-2\lambda x_1-2x_2+2y_1-y_1\lambda^4+2y_2\lambda\right) \\ &\quad -y_2\lambda\left(2\lambda x_1-2x_2\lambda^2+2x_2-2y_1+y_2\lambda^3\right)\Big)\Big). \end{aligned}$$

This long expression may be the reason that - unlike the case of a (pure) exponential distance function (5.2.7) - no (neither direct nor indirect) construction of this integral kernel was known before. □

Let $d:\mathbb{N}\to\mathbb{C}$ be a distance function with finite range n for some fixed $n\in\mathbb{N}$. For all $0<s<1$ the distance function d has a representation $d(k) = \langle (sJ_{(0,n)})^{k-1}v|w\rangle$ for some $v,\ w\in\mathbb{C}^n$, where $J_{(0,n)}$ is the standard Jordan block (68) of dimension n with eigenvalue zero. In the next example we will compute the integral kernel of $B^{-1}\circ C_{sJ_{(0,n)}}\circ B = 2^{n/2}\det(1-s^2J_{(0,n)}J_{(0,n)}^\top)^{-1/2}\,T_{\mathbf{c}_{2n}^{-1}(X_{sJ_{(0,n)}})}$.

Example 5.2.9. Let $0<s<1$ and $n\in\mathbb{N}$. The Bargmann conjugate $B^{-1}\circ C_{sJ_{(0,n)}}\circ B : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ of the composition operator $C_{sJ_{(0,n)}}$ has the integral kernel

$$(93)\quad k_{sJ_{(0,n)}}(x,y) = \frac{2^{n/2}}{(1-s^2)^{(n-1)/2}}\exp\Big(\frac{-\pi}{1-s^2}\big((1+s^2)(\|x\|^2+\|y\|^2)-2s^2(x_1^2+y_n^2)-4s\sum_{i=1}^{n-1}x_i\,y_{i+1}\big)\Big).$$

In fact: The integral kernel of $B^{-1}\circ C_{sJ_{(0,n)}}\circ B$ is $k_{sJ_{(0,n)}}(x,y) = 2^{n/2}\det(1-\Lambda\Lambda^\top)^{-1/2}\,g_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}(x,y)$. We set $\Lambda := s\,J_{(0,n)}\in\mathrm{Mat}(n,n;\mathbb{R})$ and show that

$$\begin{aligned} i\mathbf{c}_{2n}^{-1}(X_\Lambda) &\overset{(92)}{=} -\begin{pmatrix} (1+\Lambda^\top\Lambda)(1-\Lambda^\top\Lambda)^{-1} & -2\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} \\ -2\Lambda(1-\Lambda^\top\Lambda)^{-1} & (1+\Lambda\Lambda^\top)(1-\Lambda\Lambda^\top)^{-1} \end{pmatrix} \\ &= -\begin{pmatrix} \mathrm{diag}(1,\frac{1+s^2}{1-s^2},\ldots,\frac{1+s^2}{1-s^2}) & -\frac{2s}{1-s^2}\,J_{(0,n)} \\ -\frac{2s}{1-s^2}\,J_{(0,n)}^\top & \mathrm{diag}(\frac{1+s^2}{1-s^2},\ldots,\frac{1+s^2}{1-s^2},1) \end{pmatrix} \\ &= -\frac{1}{1-s^2}\begin{pmatrix} \mathrm{diag}(1-s^2,1+s^2,\ldots,1+s^2) & -2s\,J_{(0,n)} \\ -2s\,J_{(0,n)}^\top & \mathrm{diag}(1+s^2,\ldots,1+s^2,1-s^2) \end{pmatrix}. \end{aligned}$$

By Proposition 5.2.6 we have $J_{(0,n)}^\top J_{(0,n)} = \operatorname{diag}(0,1,\dots,1)$. One easily confirms that $(1 \pm \Lambda^\top\Lambda) = 1 \pm s^2\, J_{(0,n)}^\top J_{(0,n)} = \operatorname{diag}(1, 1\pm s^2, \dots, 1\pm s^2)$ which we use for the upper left entry of $\mathbf{c}_{2n}^{-1}(X_\Lambda)$. The lower right entry is the flip conjugate of the upper left entry, hence $1\pm\Lambda\Lambda^\top = \operatorname{diag}(1\pm s^2,\dots,1\pm s^2,1)$. Hence the determinant is equal to $\det(1-\Lambda\Lambda^\top) = \det\operatorname{diag}(1-s^2,\dots,1-s^2,1) = (1-s^2)^{n-1}$. For the off-diagonal blocks we note that $\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} = \frac{s}{1-s^2}\, J_{(0,n)}^\top \operatorname{diag}(1,\dots,1,1-s^2) = \frac{s}{1-s^2}\, J_{(0,n)}^\top$. By the symmetry of $\mathbf{c}_{2n}^{-1}(X_\Lambda)$ this implies that $\Lambda(1-\Lambda^\top\Lambda)^{-1} = (\Lambda^\top(1-\Lambda\Lambda^\top)^{-1})^\top = \frac{s}{1-s^2}\, J_{(0,n)}$. The sparse shape of the matrix $\mathbf{c}_{2n}^{-1}(X_\Lambda)$ reduces the number of terms in the quadratic form

$$
\begin{aligned}
& i\left(\begin{pmatrix} x \\ y \end{pmatrix}\middle|\, \mathbf{c}_{2n}^{-1}(X_\Lambda)\begin{pmatrix} x \\ y \end{pmatrix}\right) \\
= \; & \frac{-1}{1-s^2}\Big((x|\operatorname{diag}(1-s^2,1+s^2,..,1+s^2)x) - 4s(x|J_{(0,n)}y) + (y|\operatorname{diag}(1+s^2,,..,1+s^2,1-s^2)y)\Big) \\
= \; & \frac{-1}{1-s^2}\Big((1+s^2)(\|x\|^2+\|y\|^2) - 2s^2(x_1^2+y_n^2) - 4s\sum_{i=1}^{n-1} x_i\, y_{i+1}\Big).
\end{aligned}
$$

□

5.3 Extended oscillator semigroup

In the previous two Sections 5.1 and 5.2 we have investigated the composition operator $(C_\Lambda f)(z) = f(\Lambda z)$ (5.1.3) acting on the Fock space and its Bargmann conjugate. As explained in the introduction of this chapter our motivation for this is the study of the generalised composition operator

$$\mathcal{L}_{a,b,\Lambda} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (\mathcal{L}_{a,b,\Lambda} f)(z) = e^{(z|a)}\, f(\Lambda z + b)$$

(for some fixed $a,\ b \in \mathbb{C}^n$, $\Lambda \in \operatorname{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$) from which the Ruelle-Mayer transfer operator is built up. It turns out that $\mathcal{L}_{a,b,\Lambda}$ belongs to the so called extended Fock oscillator semigroup. We will first define the extended oscillator semigroup $E\Omega_n$ consisting of integral operators with (general) Gaussians as integral kernel. We introduce the extended Fock oscillator semigroup to be the image

$$E\Omega_{n,\mathcal{F}(\mathbb{C}^n)} := \{B \circ T \circ B^{-1} \,|\, T \in E\Omega_n\}$$

under the conjugation with the Bargmann transform. The task of this section is give an explicit description of this space of operators and to compute the Bargmann conjugate of $\mathcal{L}_{a,b,\Lambda}$. For this purpose we will compute the conjugates of translations and of multiplication operators $B\circ\tau_r\circ B^{-1}$, $B\circ m_{\exp_s}\circ B^{-1}$ both acting on $\mathcal{F}(\mathbb{C}^n)$ and obtain as a consequence an explicit formula for the operators $B^{-1}\circ m_{\exp_a}\circ B$, $B^{-1}\circ\tau_{-b}\circ B$ both acting on $L^2(\mathbb{R}^n)$.

Definition 5.3.1. (i) For $s \in \mathbb{C}^n$ (resp. $s \in \mathbb{R}^n$) let τ_s be the translation $(\tau_s f)(z) = f(z-s)$ on $\mathcal{F}(\mathbb{C}^n)$, respectively on $L^2(\mathbb{R}^n)$.

(ii) For any $s \in \mathbb{C}^n$ one defines the (unbounded) multiplication operator $m_{\exp_s}$ via $(m_{\exp_s} f)(z) = e^{(z|s)}\, f(z)$ which acts both on $L^2(\mathbb{R}^n)$ and on $\mathcal{F}(\mathbb{C}^n)$ via this formula.

(iii) For $p,\ q \in \mathbb{C}^n$, $\mathcal{A} \in \mathfrak{S}_{2n}$ we set $T_{\mathcal{A}}^{p,q} := m_{\exp_{2\pi i p}} \circ T_{\mathcal{A}} \circ m_{\exp_{2\pi i q}}$.

(iv) The *extended oscillator semigroup* is defined as the space of integral operators with (general) Gaussians as integral kernel, $E\Omega_n := \{c\, T_{\mathcal{A}}^{p,q} \,|\, \mathcal{A} \in \mathfrak{S}_{2n},\ c \in \mathbb{C}^\times,\ p,q \in \mathbb{C}^n\}$. Similarly to Remark 5.2.2 one shows that the extended oscillator semigroup consists of trace class operators.

(v) Via the Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$ from (90) one defines the *extended Fock oscillator semigroup* as $E\Omega_{n,\mathcal{F}(\mathbb{C}^n)} := \{B \circ T \circ B^{-1} \,|\, T \in E\Omega_n\}$. □

Occasionally, we will use the abbreviation $m_{\cosh_s} := \frac{1}{2}(m_{\exp_s} + m_{\exp_{-s}})$.
Our next aim is to give, similar to Proposition 5.2.4, an explicit description of the extended Fock oscillator semigroup $E\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ (5.3.1), i.e., one has to compute the image of $E\Omega_n$ under the conjugation by the Bargmann transform. This will be achieved in Corollary 5.3.4. We start with some commutation relations of compositions, translations, and multiplication operators, the latter ones defined in (5.3.1).

Proposition 5.3.2. *For $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ let $C_\Lambda : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$, $(C_\Lambda f)(z) = f(\Lambda z)$ be the associated composition operator (5.1.3). For $s \in \mathbb{C}^n$ (resp. $s \in \mathbb{R}^n$) let τ_s be the translation $(\tau_s f)(z) = f(z-s)$ and $(m_{\exp_s} f)(z) = e^{(s|z)} f(z)$ be the multiplication operator. Then*

(i) $\tau_r \circ C_\Lambda = C_\Lambda \circ \tau_{\Lambda r}$,

(ii) $C_\Lambda \circ m_{\exp_s} = m_{\exp_{\Lambda^\top s}} \circ C_\Lambda$,

(iii) $\tau_r \circ m_{\exp_s} = e^{-(r|s)}\, m_{\exp_s} \circ \tau_r$.

Proof. This follows from the following straight forward calculations:

$$(\tau_r \circ C_\Lambda f)(z) = f(\Lambda(z-r)) = f(\Lambda z - \Lambda r) = (C_\Lambda \circ \tau_{\Lambda r} f)(z),$$
$$(m_{\exp_{\Lambda^\top s}} \circ C_\Lambda f)(z) = e^{(\Lambda^\top s|z)} f(\Lambda z) = e^{(s|\Lambda z)} f(\Lambda z) = (C_\Lambda \circ m_{\exp_s} f)(z),$$
$$(\tau_r \circ m_{\exp_s} f)(z) = e^{(s|z-r)} f(z-r) = e^{-(r|s)} e^{(s|z)} f(z-r) = e^{-(r|s)}\,(m_{\exp_s} \circ \tau_r f)(z).$$

□

We will use the following lemma from [HiMay02]. It states that the Bargmann conjugates both of a translation $\tau_r : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $(\tau_r f)(z) = f(z-r)$ and a multiplication operator $m_{\exp_s} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $(m_{\exp_s} f)(z) = e^{(z|s)}\, f(z)$ are combinations of a translation and multiplication by an exponential.

Lemma 5.3.3. *([HiMay02, (5.5),(5.6)]) Let r, $s \in \mathbb{C}^n$. Then on $\mathcal{F}(\mathbb{C}^n)$*

(i) $B \circ \tau_r \circ B^{-1} = \exp\big(-\frac{\pi}{2}\,(r\,|\,r)\,\big)\, m_{\exp_{\pi r}} \circ \tau_r$,

(ii) $B \circ m_{\exp_s} \circ B^{-1} = \exp\big(\frac{1}{8\pi}\,(s\,|\,s)\,\big)\, m_{\exp_{s/2}} \circ \tau_{-s/(2\pi)}$. □

As a combination of Proposition 5.2.4 and Lemma 5.3.3 (ii) we obtain a full description of the extended Fock oscillator semigroup $E\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$ (5.3.1):

Corollary 5.3.4. *Let p, $q \in \mathbb{C}^n$, $\mathcal{A} \in \mathfrak{S}_{2n}$. Then the Bargmann conjugate of $T^{p,q}_{\mathcal{A}} \in E\Omega_n$ (5.3.1) on $\mathcal{F}(\mathbb{C}^n)$ is given as*

$$B \circ T^{p,q}_{\mathcal{A}} \circ B^{-1} = 2^{n/2}\,\det(1 - i\mathcal{A})^{-1/2} \exp\Big(-\frac{\pi}{2}[(p\,|\,p) + (q\,|\,q)]\Big)\, m_{\exp_{\pi i p}} \circ \tau_{-ip} \circ S_{\mathbf{c}_{2n}(\mathcal{A})} \circ m_{\exp_{\pi i q}} \circ \tau_{-iq}.$$

□

As an immediate consequence we obtain the Bargmann conjugate of the generalised composition operator

$$\mathcal{L}_{a,b,\Lambda} := m_{\exp_a} \circ C_\Lambda \circ \tau_{-b} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (\mathcal{L}_{a,b,\Lambda} F)(z) = e^{(z|a)}\, f(\Lambda z + b), \tag{94}$$

where a, $b \in \mathbb{C}^n$, and $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$ are fixed.

Proposition 5.3.5. *Let $a, b \in \mathbb{C}^n$, $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$. Then the Bargmann conjugate of the composition operator $\mathcal{L}_{a,b,\Lambda} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ from (94) is given as*

$$B^{-1} \circ \mathcal{L}_{\pi a,b,\Lambda} \circ B = \frac{2^{n/2}}{\det(1 - \Lambda\Lambda^\top)^{1/2}}\, \exp\Big(-\frac{\pi i}{2}\Big(\begin{pmatrix} a \\ b \end{pmatrix}\Big|\,\Psi_\Lambda \begin{pmatrix} a \\ b \end{pmatrix}\Big)\Big)\, T^{\Psi_\Lambda(a,b)}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)},$$

where $\Psi_\Lambda := -i(1 + X_\Lambda)^{-1}$.

Proof. Let $\mathcal{A} := \mathbf{c}_{2n}^{-1}(X_\Lambda)$ and $m(\mathcal{C}_{2n}, \mathcal{A}) = 2^{n/2}\,\det(1 - i\mathcal{A})^{-1/2}$ be as in Proposition 5.2.5. Then by Corollary 5.3.4 and Proposition 5.3.2

$$\begin{aligned}
B \circ T^{p,q}_{\mathcal{A}} \circ B^{-1} &= \exp\Big(-\frac{\pi}{2}[(p\,|\,p) + (q\,|\,q)]\Big)\ m(\mathcal{C}_{2n}, \mathcal{A})\, m_{\exp_{\pi i p}} \circ \tau_{-ip} \circ C_\Lambda \circ m_{\exp_{\pi i q}} \circ \tau_{-iq} \\
&= \exp\Big(-\frac{\pi}{2}[(p\,|\,p) + (q\,|\,q)]\Big)\, m(\mathcal{C}_{2n}, \mathcal{A})\, m_{\exp_{\pi i p}} \circ \tau_{-ip} \circ m_{\exp_{\pi i \Lambda^\top q}} \circ C_\Lambda \circ \tau_{-iq} \\
&= \exp\Big(-\frac{\pi}{2}[(p\,|\,p) + (q\,|\,q) + 2\,(p\,|\,\Lambda^\top q)]\Big)\, m(\mathcal{C}_{2n}, \mathcal{A})\, m_{\exp_{\pi i(p+\Lambda^\top q)}} \circ C_\Lambda \circ \tau_{-i(\Lambda p+q)} \\
&= \exp\Big(-\frac{\pi}{2}\Big(\begin{pmatrix} p \\ q \end{pmatrix}\Big|\begin{pmatrix} 1 & \Lambda^\top \\ \Lambda & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}\Big)\Big)\, m(\mathcal{C}_{2n}, \mathcal{A})\, m_{\exp_{\pi i(p+\Lambda^\top q)}} \circ C_\Lambda \circ \tau_{-i(\Lambda p+q)}.
\end{aligned}$$

Observe that
$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto i \begin{pmatrix} p+\Lambda^\top q \\ \Lambda p + q \end{pmatrix} = i \begin{pmatrix} 1 & \Lambda^\top \\ \Lambda & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = i(1+X_\Lambda) \begin{pmatrix} p \\ q \end{pmatrix}$$
defines a linear isomorphism $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$, since the estimate $\|\Lambda^\top \Lambda\| \le \|\Lambda\|^2 < 1$ makes its determinant $\det(i(1+X_\Lambda)) = i^{2n} \det(1-\Lambda\Lambda^\top)$ non-vanishing. Let $\Psi_\Lambda : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be its inverse

$$\Psi_\Lambda := (i(1+X_\Lambda))^{-1} = -i \begin{pmatrix} 1 & \Lambda^\top \\ \Lambda & 1 \end{pmatrix}^{-1} = -i \begin{pmatrix} (1-\Lambda^\top\Lambda)^{-1} & -\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} \\ -\Lambda(1-\Lambda^\top\Lambda)^{-1} & (1-\Lambda\Lambda^\top)^{-1} \end{pmatrix}.$$

Ψ_Λ is symmetric, hence $\Psi_\Lambda^\top(1+X_\Lambda)\Psi_\Lambda = -(1+X_\Lambda)^{-1} = -i\Psi_\Lambda$ and

$$\begin{aligned} B^{-1} \circ \mathcal{L}_{\pi a,b,\Lambda} \circ B &= m(\mathcal{C}_{2n}, \mathcal{A})^{-1} \exp\Big(\frac{\pi}{2} \Big(\begin{pmatrix} a \\ b \end{pmatrix} \Big| \Psi_\Lambda^\top \begin{pmatrix} 1 & \Lambda^\top \\ \Lambda & 1 \end{pmatrix} \Psi_\Lambda \begin{pmatrix} a \\ b \end{pmatrix} \Big) \Big) T_{\mathcal{A}}^{\Psi_\Lambda(a,b)} \\ &= m(\mathcal{C}_{2n}, \mathcal{A})^{-1} \exp\Big(-\frac{\pi i}{2} \Big(\begin{pmatrix} a \\ b \end{pmatrix} \Big| \Psi_\Lambda \begin{pmatrix} a \\ b \end{pmatrix} \Big) \Big) T_{\mathcal{A}}^{\Psi_\Lambda(a,b)}, \end{aligned}$$

which together with Proposition 5.2.5 and formula (91) gives the claim. □

In particular, Proposition 5.3.5 shows that $\mathcal{L}_{a,b,\Lambda}$ belongs to the extended Fock oscillator semigroup $E\Omega_{n,\mathcal{F}(\mathbb{C}^n)}$. - Lemma 5.3.3 has the following converse statement, which seems to be new in the literature: The (inverse) Bargmann conjugates both of a translation and a multiplication operator by an exponential function on $\mathcal{F}(\mathbb{C}^n)$ are combinations of a translation and multiplication by an exponential on $L^2(\mathbb{R}^n)$. The proof is based on our observation that the correspondence between the coefficients in Lemma 5.3.3 is linear and bijective.

Proposition 5.3.6. *Let a, $b \in \mathbb{C}^n$. Then on $L^2(\mathbb{R}^n)$*

(i) $B^{-1} \circ m_{\exp_a} \circ B = \exp\Big(-\frac{1}{4\pi} (a \,|\, a)\Big) m_{\exp_a} \circ \tau_{a/(2\pi)}$,

(ii) $B^{-1} \circ \tau_{-b} \circ B = \exp\Big(\frac{\pi}{4} (b \,|\, b)\Big) m_{\exp_{\pi b}} \circ \tau_{-b/2}$.

Proof. We multiply the equations of Lemma 5.3.3 and use Proposition 5.3.2 (i) on the commutation relations of the multiplication and the translation operator. This yields

$$\begin{aligned} B \circ m_{\exp_{\pi s}} \circ \tau_{-r} \circ B^{-1} &= \exp\Big(\frac{\pi}{8} (s \,|\, s) - \frac{\pi}{2} (r \,|\, r)\Big) m_{\exp_{\pi s/2}} \circ \tau_{-s/2} \circ m_{\exp_{-\pi r}} \circ \tau_{-r} \\ &= \exp\Big(\frac{\pi}{8} (s \,|\, s) - \frac{\pi}{2} (r \,|\, r) - \frac{\pi}{2}(r|s)\Big) m_{\exp_{\pi s/2-\pi r}} \circ \tau_{-s/2-r}. \end{aligned}$$

Set $V_{a,b} := m_{\exp_{\pi a}} \circ \tau_{-b}$ for all a, $b \in \mathbb{C}^n$, then the last equation reads as follows

$$B \circ V_{a,b} \circ B^{-1} = \exp\Big(-\frac{\pi}{4} \Big(\begin{pmatrix} a \\ b \end{pmatrix} \Big| \begin{pmatrix} -1/2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \Big) \Big) V_{E(a,b)}, \tag{95}$$

where $E(a,b) := \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ defines a linear isomorphism $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$ with inverse

$$E^{-1}(a,b) = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence we can invert (95) to get

$$\begin{aligned} B^{-1} \circ V_{a,b} \circ B &= \exp\Big(+\frac{\pi}{4} \Big(E^{-1}(a,b) \Big| \begin{pmatrix} -1/2 & 1 \\ 1 & 2 \end{pmatrix} E^{-1}(a,b) \Big) \Big) V_{E^{-1}(a,b)} \\ &= \exp\Big(-\frac{\pi}{4} \Big(\begin{pmatrix} a \\ b \end{pmatrix} \Big| \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \Big) \Big) V_{E^{-1}(a,b)}, \end{aligned} \tag{96}$$

since

$$E^{-\top} \begin{pmatrix} -1/2 & 1 \\ 1 & 2 \end{pmatrix} E^{-1} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} -1/2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

□

Proposition 5.3.5 gives an explicit description of the image

$$\{B^{-1}\circ \mathcal{L}_{a,b,\Lambda}\circ B \,|\, a,b\in\mathbb{C}^n,\ \Lambda\in \mathrm{Mat}(n,n;\mathbb{C}) \text{ with } \|\Lambda\|<1\}\subset E\Omega_n$$

of the set of composition operators $\mathcal{L}_{a,b,\Lambda}$ (94) under the conjugation. In particular, this image is small inside the extended oscillator semigroup, compare for instance the dimensions of the corresponding parameter spaces. This in turn lights our hope that also other types of operators from the extended (Fock) oscillator semigroup may serve as builing blocks for future transfer operators.
In the following we will discuss special cases of pairs $(a,b)\in\mathbb{C}^n\times\mathbb{C}^n$ (depending on a given matrix Λ) such that the integral operator $B^{-1}\circ\mathcal{L}_{a,b,\Lambda}\circ B$ has special properties. Namely, we will determine the pairs $(a,b)\in\mathbb{C}^n\times\mathbb{C}^n$ such that

$$\begin{pmatrix}p\\q\end{pmatrix} := \Psi_\Lambda\begin{pmatrix}a\\b\end{pmatrix} = -i(1+X_\Lambda)^{-1}\begin{pmatrix}a\\b\end{pmatrix}$$

satisfies that either 1.) $q=0$, 2.) $p=0$, or 3.) $p=q$. These pairs $(p,q)\in\mathbb{C}^n\times\mathbb{C}^n$ correspond to integral operators $T_{\mathcal{A}}^{p,q}$ (5.3.1) where either one of the multiplication operators vanishes 1.), 2.), or the multiplication operators are equal 3.). The corresponding integral kernels are of a simple form: 1.) $g_A(x,y)\exp(2\pi i\,(x\,|\,p))$, 2.) $g_A(x,y)\exp(2\pi i\,(y\,|\,p))$, or , 3.) $g_A(x,y)\exp(2\pi i\,(x+y\,|\,p))$. Since the correspondence $\mathbb{C}^2\to\mathbb{C}^2$, $\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right) := \Psi_\Lambda\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$ is bijective, there is a unique solution in each of the three cases.

Corollary 5.3.7. *Let $a,b\in\mathbb{C}^n$, $\Lambda\in\mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\|<1$ and $\mathcal{L}_{a,b,\Lambda}:\mathcal{F}(\mathbb{C}^n)\to\mathcal{F}(\mathbb{C}^n)$ be the corresponding composition operator (94). Then*

(i) $B^{-1}\circ\mathcal{L}_{\pi a,\Lambda a,\Lambda}\circ B = \dfrac{2^{n/2}\exp\big(-\frac{\pi}{2}\,(a\,|\,a)\big)}{\det(1-\Lambda\Lambda^\top)^{1/2}}\,T^{-ia,0}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}$,

(ii) $B^{-1}\circ\mathcal{L}_{\pi\Lambda^\top b,b,\Lambda}\circ B = \dfrac{2^{n/2}\exp\big(-\frac{\pi}{2}\,(b\,|\,b)\big)}{\det(1-\Lambda\Lambda^\top)^{1/2}}\,T^{0,-ib}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}$,

(iii) $B^{-1}\circ\mathcal{L}_{\pi(1+\Lambda^\top)(1+\Lambda)^{-1}b,b,\Lambda}\circ B = \dfrac{2^{n/2}\exp\big(-\frac{\pi}{2}\,(b\,|\,(1+\Lambda)^{-1}b)\big)}{\det(1-\Lambda\Lambda^\top)^{1/2}}\,T^{-i(1+\Lambda)^{-1}b,-i(1+\Lambda)^{-1}b}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}$.

Proof. By the Proposition 5.3.5 we have

$$B^{-1}\circ\mathcal{L}_{\pi a,b,\Lambda}\circ B = \frac{2^{n/2}}{\det(1-\Lambda\Lambda^\top)^{1/2}}\,\exp\Big(-\frac{\pi i}{2}\Big(\begin{pmatrix}a\\b\end{pmatrix}\Big|\,\Psi_\Lambda\begin{pmatrix}a\\b\end{pmatrix}\Big)\Big)\,T^{\Psi_\Lambda(a,b)}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)},$$

where $\Psi_\Lambda=-i(1+X_\Lambda)^{-1}$. Hence it remains to check that $\Psi_\Lambda(a,b)$ has the stated form for the special choices of (a,b) and to compute the inner product $\big(\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\big|\,\Psi_\Lambda\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\big)$. Concerning the first case we note that $\left(\begin{smallmatrix}a\\\Lambda a\end{smallmatrix}\right)=i(1+X_\Lambda)\left(\begin{smallmatrix}-ia\\0\end{smallmatrix}\right)$, hence $\Psi_\Lambda\left(\begin{smallmatrix}a\\\Lambda a\end{smallmatrix}\right)=\left(\begin{smallmatrix}-ia\\0\end{smallmatrix}\right)$ and $\big(\left(\begin{smallmatrix}a\\\Lambda a\end{smallmatrix}\right)\big|\,\Psi_\Lambda\left(\begin{smallmatrix}a\\\Lambda a\end{smallmatrix}\right)\big)=\big(\left(\begin{smallmatrix}a\\\Lambda a\end{smallmatrix}\right)\big|\left(\begin{smallmatrix}-ia\\0\end{smallmatrix}\right)\big)=-i\,(a\,|\,a)$. Similarly, $\left(\begin{smallmatrix}\Lambda^\top b\\b\end{smallmatrix}\right)=i(1+X_\Lambda)\left(\begin{smallmatrix}0\\-ib\end{smallmatrix}\right)$, hence $\Psi_\Lambda\left(\begin{smallmatrix}\Lambda^\top b\\b\end{smallmatrix}\right)=\left(\begin{smallmatrix}0\\-ib\end{smallmatrix}\right)$ and

$$\Big(\begin{pmatrix}\Lambda^\top b\\b\end{pmatrix}\Big|\,\Psi_\Lambda\begin{pmatrix}\Lambda^\top b\\b\end{pmatrix}\Big)=\Big(\begin{pmatrix}\Lambda^\top b\\b\end{pmatrix}\Big|\begin{pmatrix}0\\-ib\end{pmatrix}\Big)=-i\,(b\,|\,b)\,.$$

For the third case we observe that $\left(\begin{smallmatrix}(1+\Lambda^\top)(1+\Lambda)^{-1}b\\b\end{smallmatrix}\right)=i(1+X_\Lambda)\left(\begin{smallmatrix}-i(1+\Lambda)^{-1}b\\-i(1+\Lambda)^{-1}b\end{smallmatrix}\right)$, which implies that $\Psi_\Lambda\left(\begin{smallmatrix}(1+\Lambda^\top)(1+\Lambda)^{-1}b\\b\end{smallmatrix}\right)=\left(\begin{smallmatrix}-i(1+\Lambda)^{-1}b\\-i(1+\Lambda)^{-1}b\end{smallmatrix}\right)$ and hence

$$\begin{aligned}\Big(\begin{pmatrix}(1+\Lambda^\top)(1+\Lambda)^{-1}b\\b\end{pmatrix}\Big|\,\Psi_\Lambda\begin{pmatrix}-i(1+\Lambda)^{-1}b\\-i(1+\Lambda)^{-1}b\end{pmatrix}\Big) &= \Big(\begin{pmatrix}(1+\Lambda^\top)(1+\Lambda)^{-1}b\\b\end{pmatrix}\Big|\begin{pmatrix}-i(1+\Lambda)^{-1}b\\-i(1+\Lambda)^{-1}b\end{pmatrix}\Big)\\ &= -2i\,\big(b\,|\,(1+\Lambda)^{-1}b\big)\,.\end{aligned}$$

□

Corollary 5.3.7 will imply in the next section that certain generating triples (a,b,Λ) lead to Kac-Gutzwiller transfer operators of a simple form.

5.4 Kac-Gutzwiller transfer operators for the Ising model

Let $F \subset \mathbb{C}$ be a bounded set and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with potential $q \in \mathcal{C}_b(F)$ and distance function given as $d(k) = (\Lambda^{k-1}v|w)$ for some $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$, v, $w \in \mathbb{C}^n$. Then by Theorem 2.7.6 the Ruelle-Mayer transfer operator

$$\mathcal{M}_\beta : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (\mathcal{M}_\beta f)(z) = \int_F \exp\Big(\beta q(\sigma) + \beta\sigma(z|w)\Big)\, f(\sigma\, v + \Lambda\, z)\, d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}(\beta) = \det(1-\Lambda^n)\,\mathrm{trace}\,(\mathcal{M}_\beta)^n$ for all $n \in \mathbb{N}$. The conjugate integral operator $B^{-1}\circ\mathcal{M}_\beta\circ B$ on $L^2(\mathbb{R}^n)$ is called a *Kac-Gutzwiller transfer operator*. In this section we will use the results 5.3.5 and 5.3.7 from the previous section for the computation of the Kac-Gutzwiller transfer operator. We apply this to the two main cases of distance functions which have a finite-dimensional representation $d(k) = \langle \mathbb{B}^{k-1}v|w\rangle$, namely the polynomial-exponentially decaying interactions and the finite-range interactions. These results will be generalised in the next section to matrix subshifts with Ising type interactions. In particular, the finite state Potts model will be considered.
Using Propositions 3.3.4 and 5.3.5 we compute the corresponding Kac-Gutzwiller transfer operator

$$\begin{aligned} B^{-1}\circ\mathcal{M}_\beta\circ B &= \int_F e^{\beta q(\sigma)}\, B^{-1}\circ\mathcal{L}_{\beta\sigma w,\sigma v,\Lambda}\circ B\, d\nu(\sigma) \\ &= \frac{2^{n/2}}{\det(1-\Lambda\Lambda^\top)^{1/2}}\int_F e^{\beta q(\sigma)}\,\exp\Big(-\frac{\pi i\sigma^2}{2}\Big(\Big(\begin{smallmatrix}\frac{\beta}{\pi}w\\ v\end{smallmatrix}\Big)\Big|\,\Psi_\Lambda\Big(\begin{smallmatrix}\frac{\beta}{\pi}w\\ v\end{smallmatrix}\Big)\Big)\Big)\, T^{\sigma\,\Psi_\Lambda(\frac{\beta}{\pi}w,v)}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\, d\nu(\sigma), \end{aligned}$$

where $\Psi_\Lambda = -i(1+X_\Lambda)^{-1}$. We will now emploit the ambiguity of the generating triples. Namely, once one has one representation of a distance function, one obtains by conjugation a family of generating triples: Let v, $w \in \mathbb{C}^n$ and $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$. For all $S \in \mathrm{Gl}(n;\mathbb{C})$ one has

$$d(k) = \big(\Lambda^{k-1}v\,\big|\,w\big) = \big((S^{-1}\Lambda S)^{k-1}S^{-1}v\,\big|\,S^\top w\big) = \big(\Lambda_S^{k-1}v_S\,\big|\,w_S\big)$$

with $w_S := S^\top w$, $v_S := S^{-1}v$ and $\Lambda_S := S^{-1}\Lambda S$. For each $S \in \mathrm{Gl}(n;\mathbb{C})$ one has a Ruelle-Mayer transfer operator $\mathcal{M}_{\beta;S} = \int_F e^{\beta q(\sigma)}\,\mathcal{L}_{\beta\sigma w_S,\sigma v_S,\Lambda_S}\,d\nu(\sigma) : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ acting via

$$(\mathcal{M}_{\beta;S}f)(z) = \int_F \exp\Big(\beta q(\sigma) + \beta\sigma(z|w_S)\Big)\, f(\sigma\, v_S + \Lambda_S\, z)\, d\nu(\sigma).$$

Let $C_S : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$, $(C_S f)(z) = f(Sz)$ be the (unbounded) composition operator associated to $S \in \mathrm{Gl}(n;\mathbb{C})$, then $\mathcal{M}_{\beta;S} = C_S\circ\mathcal{M}_\beta\circ C_S^{-1}$ formally. We mention that between suitably weighted Fock spaces C_S becomes a bounded operator.[36]

Theorem 5.4.1. *In the above setting set* $\mathcal{G}_{\beta;S} := B^{-1}\circ\mathcal{M}_{\beta;S}\circ B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

(i) *If* $v \in \Lambda\mathbb{C}^n$, *then there exists* $S \in \mathrm{Gl}(n;\mathbb{C})$ *such that* $v = \frac{\beta}{\pi}\Lambda SS^\top w$ *and*

$$\mathcal{G}_{\beta;S} = 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\beta^2\sigma^2}{2\pi}\,\big(w_S\,\big|\,w_S\big)\Big)\, m_{\exp_{2\beta\sigma w_S}}\, d\nu(\sigma)\circ T_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}$$

with integral kernel $2^{n/2}\dfrac{g_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}(x,y)}{\det(1-\Lambda_S\Lambda_S^\top)^{1/2}}\displaystyle\int_F \exp\Big(\beta q(\sigma) - \frac{\beta^2\sigma^2}{2\pi}\,\big(w_S\,\big|\,w_S\big) + 2\beta\sigma\,\big(x\,\big|\,w_S\big)\Big)\, d\nu(\sigma)$.

(ii) *If* $w \in \Lambda^\top\mathbb{C}^n$, *then there exists* $S \in \mathrm{Gl}(n;\mathbb{C})$ *such that* $\frac{\beta}{\pi}w = \Lambda^\top(SS^\top)^{-1}v$ *and*

$$\mathcal{G}_{\beta;S} = 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\, T_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}\circ\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\,\big(v_S\,\big|\,v_S\big)\Big)\, m_{\exp_{2\pi\sigma v_S}}\, d\nu(\sigma)$$

with integral kernel $2^{n/2}\dfrac{g_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}(x,y)}{\det(1-\Lambda_S\Lambda_S^\top)^{1/2}}\displaystyle\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\,\big(v_S\,\big|\,v_S\big) + 2\pi\sigma\,\big(y\,\big|\,v_S\big)\Big)\, d\nu(\sigma)$.

[36] Set for instance $\mathcal{F}(\mathbb{C}^n, \mu_{SS^*}) := \big\{f \in \mathcal{O}(\mathbb{C}^n)\,\big|\,\|f\|^2_{\mathcal{F}(\mathbb{C}^n,\mu_{SS^*})} := \int |f(z)|^2\,|\det(S)|^2\,\exp(-\pi\langle z|SS^*z\rangle)\,dz < \infty\big\}$. Then $C_S : \mathcal{F}(\mathbb{C}^n, \mu_1) = \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n, \mu_{SS^*})$ is unitary because of the substitution rule.

(iii) *For all v, w there exists $S \in \mathrm{Gl}(n;\mathbb{C})$ such that $(1+\Lambda^\top)(SS^\top)^{-1}(1+\Lambda)^{-1}v = \frac{\beta}{\pi}w$. Set $\tilde{v} := S^{-1}(1+\Lambda)^{-1}v$. Then*

$$\mathcal{G}_{\beta;S} = 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\left(v_S \,\big|\, (1+\Lambda_S)^{-1}v_S\right)\Big)\, T^{-i\sigma\tilde{v},-i\sigma\tilde{v}}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} d\nu(\sigma)$$

with integral kernel

$$\frac{2^{n/2} g_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}(x,y)}{\det(1-\Lambda_S\Lambda_S^\top)^{1/2}}\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\left(v_S \,\big|\, (1+\Lambda_S)^{-1}v_S\right) + 2\pi\sigma\left(x+y \,\big|\, (1+\Lambda_S)^{-1}v_S\right)\Big)\, d\nu(\sigma).$$

Proof. For the proof we will assume that S solves the linear equation. The next lemma shows that one can always find such a matrix S. - If $v = \frac{\beta}{\pi}\Lambda SS^\top w$, then $v_S = S^{-1}v = \frac{\beta}{\pi}S^{-1}\Lambda SS^\top w = \frac{\beta}{\pi}\Lambda_S w_S$. Proposition 3.3.4 and Corollary 5.3.7 (i) applied for $a := \sigma\frac{\beta}{\pi}w_S$ yield

$$\begin{aligned}
\mathcal{G}_{\beta;S} &= B^{-1}\circ\mathcal{M}_\beta\circ B = \int_F e^{\beta q(\sigma)}\, B^{-1}\circ\mathcal{L}_{\beta\sigma w_S,\sigma v_S,\Lambda_S}\circ B\, d\nu(\sigma)\\
&= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\pi}{2}(a|a)\Big)\, T^{-ia,0}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} d\nu(\sigma)\\
&= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\beta^2\sigma^2}{2\pi}(w_S|w_S)\Big)\, T^{-i\frac{\beta\sigma}{\pi}w_S,0}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} d\nu(\sigma)\\
&= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\beta^2\sigma^2}{2\pi}\left(w_S \,\big|\, w_S\right)\Big)\, m_{\exp_{2\beta\sigma w_S}}\, d\nu(\sigma)\circ T_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}.
\end{aligned}$$

If $\frac{\beta}{\pi}w = \Lambda^\top S^{-\top}S^{-1}v$, then $\frac{\beta}{\pi}w_S = \frac{\beta}{\pi}S^\top w = S^\top\Lambda^\top S^{-\top}S^{-1}v = \Lambda_S^\top v_S$ and hence by Corollary 5.3.7 (ii) with $b := \sigma v_S$ one obtains

$$\begin{aligned}
\mathcal{G}_{\beta;S} &= \int_F e^{\beta q(\sigma)}\, B^{-1}\circ\mathcal{L}_{\beta\sigma w_S,\sigma v_S,\Lambda_S}\circ B\, d\nu(\sigma)\\
&= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\pi}{2}(b|b)\Big)\, T^{0,-ib}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} d\nu(\sigma)\\
&= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\left(v_S \,\big|\, v_S\right)\Big)\, T^{0,-i\sigma v_S}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} d\nu(\sigma)\\
&= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\, T_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}\circ\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\left(v_S \,\big|\, v_S\right)\Big)\, m_{\exp_{2\pi\sigma v_S}} d\nu(\sigma).
\end{aligned}$$

If $\frac{\beta}{\pi}w_S = (1+\Lambda_S^\top)(1+\Lambda_S)^{-1}v_S$, then $(1+\Lambda_S)^{-1}v_S = \big(S(1+\Lambda)S^{-1}\big)^{-1}S^{-1}v = S^{-1}(1+\Lambda)^{-1}v = \tilde{v}$ and hence $(1+\Lambda_S^\top)(1+\Lambda_S)^{-1}v_S = S^\top(1+\Lambda^\top)(SS^\top)^{-1}(1+\Lambda)^{-1}v = \frac{\beta}{\pi}w_S$ and Corollary 5.3.7 (iii) applied for $b := \sigma v_S$ yields

$$\begin{aligned}
\mathcal{G}_{\beta;S} &= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\pi}{2}(b|(1+\Lambda_S)^{-1}b)\Big)\, T^{-i(1+\Lambda_S)^{-1}b,-i(1+\Lambda_S)^{-1}b}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} d\nu(\sigma)\\
&= 2^{n/2}\det(1-\Lambda_S\Lambda_S^\top)^{-1/2}\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\left(v_S \,\big|\, (1+\Lambda_S)^{-1}v_S\right)\Big)\, T^{-i\sigma\tilde{v},-i\sigma\tilde{v}}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} d\nu(\sigma).
\end{aligned}$$

□

In order to complete the proof of Theorem 5.4.1 we have to investigate if the occurring linear equations for the matrix S can be solved. We answer this question with the help of the following lemma.

Lemma 5.4.2. *Let $a, b \in \mathbb{C}^n$. Then there exists a symmetric matrix $\Sigma \in \mathrm{Gl}(n;\mathbb{C})$ such that $\Sigma a = b$.*

Proof. Let V be the (at most) two-dimensional space spanned by a and b. We define Σ on $V^\perp$ to be the identity. By change of basis in $V \cong \mathbb{C}^2$ (the case $\dim V = 1$ is trivial) we can assume that $a = \binom{a_1}{a_2}$, $b = \binom{b_1}{b_2}$. We make the Ansatz $\Sigma = \begin{pmatrix} x & y \\ y & z\end{pmatrix}$ and obtain a system of linear equations $\begin{pmatrix} x & y \\ y & z\end{pmatrix}\binom{a_1}{a_2} = \binom{b_1}{b_2}$ in the unknowns x, y and z. This can be rewritten as

$$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & a_1 & a_2\end{pmatrix}\begin{pmatrix} x \\ y \\ z\end{pmatrix} = \begin{pmatrix} b_1 \\ b_2\end{pmatrix}$$

from which it is appearent that it is solvable. Except for $z = \frac{y^2}{x}$ the matrix Σ is invertible. □

Since every symmetric invertible matrix Σ can be written as $\Sigma = SS^\top$, we obtain as an immediate consequence of the previous lemma that the equations for S in parts (i) and (ii) of Theorem 5.4.1 can be fulfilled provided v belongs to the image of Λ (i) and w belongs to the image of $\Lambda^\top$ (ii), respectively. Since $\|\Lambda\| = \|\Lambda^\top\| < 1$, both $(1+\Lambda^\top)$ and $(1+\Lambda)$ are bijective. Hence the equation for S in part (iii) can always be solved. This concludes the proof of Theorem 5.4.1.
In particular, if one considers a two-state Ising model with spin values in $F = \{\pm 1\}$ equipped with the uniform distribution on F, then the integrals in Theorem 5.4.1 simplify and one obtains the following expressions.

Corollary 5.4.3. *Let $F = \{\pm 1\}$ be equipped with the uniform distribution and $(F^{\mathbb{N}}, \mathbb{N}_0, \tau)$ a one-sided one-dimensional full shift (1.2.6). Let ϕ be a two-body Ising interaction (1.8.3) with vanishing potential and distance function given as $d(k) = (\Lambda^{k-1}v|w)$ for some $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$, v, $w \in \mathbb{C}^n$. For all $S \in \mathrm{Gl}(n;\mathbb{C})$ set $w_S := S^\top w$, $v_S := S^{-1}v$ and $\Lambda_S = S^{-1}\Lambda S$. Let*

$$\mathcal{M}_{\beta;S} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n),\ (\mathcal{M}_{\beta;S}f)(z) = \int_F \exp\big(\beta q(\sigma) + \beta\sigma(z|w_S)\big)\, f(\sigma\, v_S + \Lambda_S\, z)\, d\nu(\sigma)$$

be the corresponding Ruelle-Mayer transfer operator and $\mathcal{G}_{\beta;S} := B^{-1} \circ \mathcal{M}_{\beta;S} \circ B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the Kac-Gutzwiller transfer operator.

(i) *If $v \in \Lambda\mathbb{C}^n$, then there exists $S \in \mathrm{Gl}(n;\mathbb{C})$ such that $v = \dfrac{\beta}{\pi}\Lambda SS^\top w$ and*

$$\mathcal{G}_{\beta;S} = 2^{n/2} \det(1 - \Lambda_S\Lambda_S^\top)^{-1/2} \exp\Big(-\frac{\beta^2}{2\pi}\,(w_S\,|\,w_S)\Big)\, m_{\cosh_{2\beta w_S}} \circ T_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}$$

with integral kernel $2^{n/2} \det(1 - \Lambda_S\Lambda_S^\top)^{-1/2} \exp\big(-\frac{\beta^2}{2\pi}(w_S|w_S)\big) \cosh\big(2\beta(x|w_S)\big)\, g_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}(x,y)$.

(ii) *If $w \in \Lambda^\top\mathbb{C}^n$, then there exists $S \in \mathrm{Gl}(n;\mathbb{C})$ such that $\dfrac{\beta}{\pi}w = \Lambda^\top(SS^\top)^{-1}v$ and*

$$\mathcal{G}_{\beta;S} = 2^{n/2} \det(1 - \Lambda_S\Lambda_S^\top)^{-1/2} \exp\big(-\frac{\pi}{2}\,(v_S\,|\,v_S)\big)\, T_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} \circ m_{\cosh_{2\pi v_S}}$$

with integral kernel $2^{n/2} \det(1 - \Lambda_S\Lambda_S^\top)^{-1/2} \exp\big(-\frac{\pi}{2}\,(v_S\,|\,v_S)\big)\, g_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}(x,y) \cosh\big(2\pi\,(y\,|\,v_S)\big)$.

(iii) *For all v, w there exist $S \in \mathrm{Gl}(n;\mathbb{C})$ such that $(1+\Lambda^\top)(SS^\top)^{-1}(1+\Lambda)^{-1}v = \frac{\beta}{\pi}w$. Then*

$$\mathcal{G}_{\beta;S} = \frac{\exp\big(-\frac{\pi}{2}\,(v_S\,|\,(1+\Lambda_S)^{-1}v_S)\big)}{2^{1-n/2}\det(1-\Lambda_S\Lambda_S^\top)^{1/2}}\,\big(T^{i(1+\Lambda_S)^{-1}v_S,\,i(1+\Lambda_S)^{-1}v_S}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})} + T^{-i(1+\Lambda_S)^{-1}v_S,\,-i(1+\Lambda_S)^{-1}v_S}_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}\big)$$

with integral kernel

$$\frac{\exp\Big(-\frac{\pi}{2}\,(v_S\,|\,(1+\Lambda_S)^{-1}v_S)\Big)}{2^{-n/2}\det(1-\Lambda_S\Lambda_S^\top)^{1/2}}\, g_{\mathbf{c}_{2n}^{-1}(X_{\Lambda_S})}(x,y)\, \cosh\Big(2\pi\,(x+y\,|\,(1+\Lambda_S)^{-1}v_S)\Big).$$

□

As shown in Remark 2.12.3 there are only two types of distance functions which have an irreducible representation $d(k) = \big(\mathbb{B}^{k-1}v\,\big|\,w\big)$ on a finite dimensional space, namely the polynomial-exponentially decaying and the finite-range distance functions. First we consider polynomial-exponentially decaying distance functions: Let $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$, $c_i \in \mathbb{C}$ and $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto \lambda^k \sum_{i=0}^{n-1} c_i\, k^i$, then by Remark 2.11.1 d can be represented with the help of the matrix $\mathbb{B}^{(n)} \in \mathrm{Gl}(n;\mathbb{C})$ (2.11.1) as $d(k) = \lambda\big((\lambda\mathbb{B}^{(n)})^{k-1}\underline{1} \mid c\big)$, where $\underline{1} : \{0,\dots,n-1\} \to \mathbb{C}$ is the constant function one and $c = (c_0,\dots,c_{n-1}) \in \mathbb{C}^n$. Since $\mathbb{B}^{(n)} \in \mathrm{Gl}(n;\mathbb{C})$ is invertible, we can find generating triples by conjugation such that each of the three cases of Theorem 5.4.1 can be applied in order to obtain a Kac-Gutzwiller transfer operator with a simple integral kernel.
We specialise to two-state Ising interactions with exponential decaying distance function $d(k) = J\lambda^k$, which is the model firstly studied by M. Kac and later (for periodic boundary condition) by M. Gutzwiller, D. Mayer, and J. Hilgert in [Gu82] and [HiMay04]. In the context of the dynamical zeta function the question arised whether the Ruelle-Mayer transfer operator has real spectrum. We will now give a direct answer to that question.

Example 5.4.4. (Two-state Ising model with pure exponential interaction) Let $F = \{\pm 1\}$ and ν be the uniform distribution on F. For every $s \in \mathbb{C}^\times$ we have a representation $d(k) = J\lambda^k = \lambda^{k-1}\frac{\lambda}{s}Js$ and a corresponding Ruelle-Mayer transfer operator $(\mathcal{M}_{\beta;s}f)(z) = e^{\beta Jsz} f\left(\frac{\lambda}{s} + \lambda z\right) + e^{-\beta Jsz} f\left(\frac{-\lambda}{s} + \lambda z\right)$ for the two-state Ising spin model.

(i) Let $s_1 = \sqrt{\frac{\pi}{\beta J}}$, then $\mathcal{G}_{\beta;s_1} := B^{-1} \circ \mathcal{M}_{\beta;s_1} \circ B$ has the integral kernel

$$\frac{2^{1/2}}{(1-\lambda^2)^{1/2}} \exp\Big(-\frac{\beta^2 s_1^2 J^2}{2\pi}\Big)\, g_{\mathcal{A}}(x,y) \cosh(2\beta s_1 J x) = \frac{2^{1/2} e^{-J\beta/2}}{(1-\lambda^2)^{1/2}}\, g_{\mathcal{A}}(x,y)\, \cosh(2\sqrt{\pi\beta J}\, x).$$

(ii) Let $s_2 = \sqrt{\frac{\pi\lambda^2}{\beta J}}$, then $\mathcal{G}_{\beta;s_2} := B^{-1} \circ \mathcal{M}_{\beta;s_2} \circ B$ has the integral kernel

$$\frac{2^{1/2}}{(1-\lambda^2)^{1/2}} \exp\Big(-\frac{\pi\lambda}{2s_2^2}\Big)\, g_{\mathcal{A}}(x,y)\, \cosh\Big(\frac{2\pi\lambda y}{s_2}\Big) = \frac{2^{1/2} e^{-J\beta/2}}{(1-\lambda^2)^{1/2}}\, g_{\mathcal{A}}(x,y)\, \cosh(2\sqrt{\pi\beta J}\, y).$$

(iii) Let $s_0 = \sqrt{\frac{\pi\lambda}{\beta J}}$, then $\mathcal{G}_{\beta;s_0} := B^{-1} \circ \mathcal{M}_{\beta;s_0} \circ B$ has the integral kernel

$$\begin{aligned} g_{\beta;s_0}(x,y) &= \frac{2^{1/2}}{(1-\lambda^2)^{1/2}} \exp\Big(-\frac{\pi\lambda^2}{2s_0^2(1+\lambda)}\Big)\, g_{\mathcal{A}}(x,y)\, \cosh!\Big(\frac{2\pi\lambda}{s_0(1+\lambda)}(x+y)\Big) \\ &= \frac{2^{1/2}}{(1-\lambda^2)^{1/2}} \exp\Big(-\frac{\beta J\lambda}{2(1+\lambda)}\Big)\, g_{\mathcal{A}}(x,y)\, \cosh(2\sqrt{\pi\beta J\lambda}\,(x+y)), \end{aligned}$$

which is symmetric. Hence for $\beta J \in \mathbb{R}^+$, $\lambda \in]-1,1[$ the corresponding Kac-Gutzwiller transfer operator $\mathcal{G}_{\beta;s_0}$ is symmetric and hence has real spectrum.

As in Example 5.2.7 we prefer to write $\lambda = e^{-\gamma}$ for some complex number γ which will lead to a Mehler type kernel. In combination with scaling by $c = (4\pi)^{-1/2}$ one obtains the kernels the scaled integral operators $R_c \circ \mathcal{G}_{\beta;s} \circ R_c^{-1}$ corresponding to (i) - (iii) as

(i')

$$\frac{e^{\gamma/2}}{\sqrt{4\pi \sinh\gamma}} \exp\Big(-\frac{1}{4}(\tanh\frac{\gamma}{2}\,(x^2+y^2) + \frac{(x-y)^2}{\sinh\gamma})\Big) \cosh(\sqrt{\beta J}\, x),$$

which is one of the asymmetric Kac-Gutzwiller type operators occurring in [HiMay04].

(ii')

$$\frac{e^{\gamma/2}}{\sqrt{4\pi \sinh\gamma}} \exp\Big(-\frac{1}{4}(\tanh\frac{\gamma}{2}\,(x^2+y^2) + \frac{(x-y)^2}{\sinh\gamma})\Big) \cosh(\sqrt{\beta J}\, y),$$

(iii')

$$\frac{e^{\gamma/2}}{\sqrt{4\pi \sinh\gamma}} \exp\Big(-\frac{1}{4}(\tanh\frac{\gamma}{2}\,(x^2+y^2) + \frac{(x-y)^2}{\sinh\gamma})\Big)\, \cosh\Big(\frac{\sqrt{\beta J\lambda}}{1+\lambda}\,(x+y)\Big),$$

i. e. one has a *natural* symmetric Kac-Gutzwiller operator which was not known before. □

Remark 5.4.5. It is quite easy to see, cf. [HiMay04], that the operators $\mathcal{G}_{\beta;s_i} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ from the preceding Remark 5.4.4 have the same spectrum as the original Kac-Gutzwiller operator given in [Gu82], which is the symmetric integral operator $\mathcal{G} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given via its integral kernel

$$\frac{e^{\gamma/2}}{\sqrt{4\pi \sinh\gamma}} \exp\Big(-\frac{1}{4}(\tanh\frac{\gamma}{2}\,(x^2+y^2) + \frac{(x-y)^2}{\sinh\gamma})\Big) \sqrt{\cosh(\sqrt{\beta J x})}\sqrt{\cosh(\sqrt{\beta J y})}.$$

□

We will now use our machinery developed in this chapter to compute the Kac-Gutzwiller transfer operator for finite range Ising interactions.

Example 5.4.6. Let $d : \mathbb{N} \to \mathbb{C}$ be a finite range distance function, say $d(k) = 0$ for all $k > n$. Let $J_{(0,n)} = \mathbb{S}_n$ be the standard n-step nilpotent matrix from (61) and $0 < \lambda < \|J_{(0,n)}\|^{-1}$. Define $w^d \in \mathbb{C}^n$ with entries $w^d(k) = \lambda^{1-k}\, d(k)$. Then by Proposition 2.8.2 $d(k) = \langle (\lambda \mathbb{S}_n)^{k-1} w^d | e_1\rangle$ for all $k \in \mathbb{N}$. The integral kernel of the corresponding Kac-Gutzwiller transfer operator $\mathcal{G}_\beta = B^{-1} \circ \mathcal{M}_\beta \circ B$ is given as

$$\frac{2^{n/2}}{(1-\lambda^2)^{n/2}} \exp\Big(\frac{-\pi}{1-\lambda^2}\big((1+\lambda^2)(\|x\|^2 + \|y\|^2) - 2\lambda^2(x_1^2 + y_n^2) - 4\lambda \sum_{i=1}^{n-1} x_i\, y_{i+1}\big)\Big)$$
$$\int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\big(\frac{\beta^2}{\pi^2} + c_{\lambda,d}\big)\Big) \exp\Big(\frac{2\pi\sigma}{1-\lambda^2}\left(\begin{pmatrix} x \\ y\end{pmatrix} \,\middle|\, \begin{pmatrix} \frac{\beta(1-\lambda^2)}{\pi}\, e_1 - \lambda\, J_{(0,n)} w^d \\ w^d - \lambda^2 w^d(n) e_n\end{pmatrix}\right)\Big)\, d\nu(\sigma),$$

where $c_{\lambda,d} := (1-\lambda^2)^{-1}\big(\big(w^d \,\big|\, w^d\big) - \lambda^2(w^d(n))^2\big)$. In fact: By Proposition 5.3.5 the Kac-Gutzwiller operator is given as

$$\begin{aligned} B^{-1} \circ \mathcal{M}_\beta \circ B &= \int_F e^{\beta q(\sigma)}\, B^{-1} \circ \mathcal{L}_{\beta\sigma e_1, \sigma w^d, \lambda J_{(0,n)}} \circ B\, d\nu(\sigma) \\ &= \frac{2^{n/2}}{(1-\lambda^2)^{n/2}} \int_F e^{\beta q(\sigma)} \exp\Big(-\frac{\pi i \sigma^2}{2}\left(\begin{pmatrix} \frac{\beta}{\pi} e_1 \\ w^d\end{pmatrix} \,\middle|\, \Psi_{\lambda J_{(0,n)}} \begin{pmatrix} \frac{\beta}{\pi} e_1 \\ w^d\end{pmatrix}\right)\Big)\, T^{\sigma\, \Psi_{\lambda J_{(0,n)}}(\frac{\beta}{\pi} e_1, w^d)}_{\mathbf{c}_{2n}^{-1}(X_{\lambda J_{(0,n)}})}\, d\nu(\sigma), \end{aligned}$$

where $\Psi_{\lambda J_{(0,n)}} = -i(1 + X_{\lambda J_{(0,n)}})^{-1}$.
Applying some ideas of the proofs of Proposition 5.2.5 and of Example 5.2.9 we obtain

$$\begin{aligned} (1 + X_\Lambda)^{-1} &= \begin{pmatrix} 1 & \Lambda^\top \\ \Lambda & 1\end{pmatrix}^{-1} = \begin{pmatrix} (1-\Lambda^\top\Lambda)^{-1} & -\Lambda^\top(1-\Lambda\Lambda^\top)^{-1} \\ -\Lambda(1-\Lambda^\top\Lambda)^{-1} & (1-\Lambda\Lambda^\top)^{-1}\end{pmatrix} \\ &= \begin{pmatrix} \mathrm{diag}(1, 1-\lambda^2, \dots, 1-\lambda^2)^{-1} & -\frac{\lambda}{1-\lambda^2} J_{(0,n)} \\ -\frac{\lambda}{1-\lambda^2} J_{(0,n)}^\top & \mathrm{diag}(1-\lambda^2, \dots, 1-\lambda^2, 1)^{-1}\end{pmatrix} \\ &= \frac{1}{1-\lambda^2}\begin{pmatrix} \mathrm{diag}(1-\lambda^2, 1, \dots, 1) & -\lambda\, J_{(0,n)} \\ -\lambda\, J_{(0,n)}^\top & \mathrm{diag}(1, \dots, 1, 1-\lambda^2)\end{pmatrix}. \end{aligned}$$

From this expression we deduce that

$$\begin{pmatrix} \mathrm{diag}(1-\lambda^2, 1, \dots, 1) & -\lambda\, J_{(0,n)} \\ -\lambda\, J_{(0,n)}^\top & \mathrm{diag}(1, \dots, 1, 1-\lambda^2)\end{pmatrix}\begin{pmatrix} \frac{\beta}{\pi} e_1 \\ w^d\end{pmatrix} = \begin{pmatrix} \frac{\beta}{\pi}(1-\lambda^2)\, e_1 - \lambda\, J_{(0,n)} w^d \\ 0 + \mathrm{diag}(1, \dots, 1, 1-\lambda^2) w^d\end{pmatrix},$$

hence

$$\begin{aligned} \left(\begin{pmatrix} \frac{\beta}{\pi} e_1 \\ w^d\end{pmatrix} \,\middle|\, \Psi_\Lambda \begin{pmatrix} \frac{\beta}{\pi} e_1 \\ w^d\end{pmatrix}\right) &= \frac{-i}{1-\lambda^2}\left(\begin{pmatrix} \frac{\beta}{\pi} e_1 \\ w^d\end{pmatrix} \,\middle|\, \begin{pmatrix} \frac{\beta}{\pi}(1-\lambda^2)\, e_1 - \lambda\, J_{(0,n)} w^d \\ \mathrm{diag}(1, \dots, 1, 1-\lambda^2) w^d\end{pmatrix}\right) \\ &= \frac{-i}{1-\lambda^2}\Big((1-\lambda^2)\frac{\beta^2}{\pi^2} + \big(w^d \,\big|\, \mathrm{diag}(1, \dots, 1, 1-\lambda^2) w^d\big) - \frac{2\beta\lambda}{\pi}\,\big(w^d \,\big|\, J_{(0,n)}^\top e_1\big)\Big) \\ &= -i\Big(\frac{\beta^2}{\pi^2} + \frac{\big(w^d \,\big|\, w^d\big)}{1-\lambda^2} - \frac{\lambda^2}{1-\lambda^2}(w^d(n))^2 + 0\Big) = -i\frac{\beta^2}{\pi^2} - i c_{\lambda,d} \end{aligned}$$

and the integral kernel of the Kac-Gutzwiller transfer operator $\mathcal{G}_\beta = B^{-1} \circ \mathcal{M}_\beta \circ B$ is

$$\begin{aligned} &\frac{2^{n/2}}{(1-\lambda^2)^{n/2}} \exp\Big(\frac{-\pi}{1-\lambda^2}\big((1+\lambda^2)\|x\|^2 - 2\lambda^2(x_1^2 + y_n^2) + (1+\lambda^2)\|y\|^2 - 4\lambda \sum_{i=1}^{n-1} x_i\, y_{i+1}\big)\Big) \\ &\qquad \int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\big(\frac{\beta^2}{\pi^2} + c_{\lambda,d}\big)\Big) \exp\Big(2\pi i \sigma \left(\begin{pmatrix} x \\ y\end{pmatrix} \,\middle|\, \Psi_\Lambda \begin{pmatrix} \frac{\beta}{\pi} e_1 \\ w^d\end{pmatrix}\right)\Big)\, d\nu(\sigma) \\ &= \frac{2^{n/2}}{(1-\lambda^2)^{n/2}} \exp\Big(\frac{-\pi}{1-\lambda^2}\big((1+\lambda^2)\|x\|^2 - 2\lambda^2(x_1^2 + y_n^2) + (1+\lambda^2)\|y\|^2 - 4\lambda \sum_{i=1}^{n-1} x_i\, y_{i+1}\big)\Big) \\ &\qquad \int_F \exp\Big(\beta q(\sigma) - \frac{\pi\sigma^2}{2}\big(\frac{\beta^2}{\pi^2} + c_{\lambda,d}\big)\Big) \exp\Big(\frac{2\pi\sigma}{1-\lambda^2}\left(\begin{pmatrix} x \\ y\end{pmatrix} \,\middle|\, \begin{pmatrix} \frac{\beta(1-\lambda^2)}{\pi}\, e_1 - \lambda\, J_{(0,n)} w^d \\ \mathrm{diag}(1, \dots, 1, 1-\lambda^2) w^d\end{pmatrix}\right)\Big)\, d\nu(\sigma). \end{aligned}$$

In order to apply Theorem 5.4.1 (i) and (ii) we have to investigate the range of $J_{(0,n)}$ and of its transpose, part (iii) is always applicable. Clearly, $J_{(0,n)}\mathbb{C}^n = \mathbb{C}^{n-1}\times\{0\}$ and $(J_{(0,n)})^\top\mathbb{C}^n = \{0\}\times\mathbb{C}^{n-1}$, but $w^d(n) \neq 0$ and $e_1 \notin \{0\}\times\mathbb{C}^{n-1}$. Hence we cannot apply the first two cases of Thm. 5.4.1 without any adaptations. However, we observe that $d(k) = \langle(\lambda J_{(0,n)})^{k-1}w^d|e_1\rangle_{\mathbb{C}^n} = \langle(\lambda J_{(0,n+1)})^{k-1}\tilde{w}^d|e_1\rangle_{\mathbb{C}^{n+1}}$ gives a representation suitable for case (i), where $\tilde{w}^d \in \mathbb{C}^{n+1}$ is the vector with entries $\tilde{w}^d(k) = w^d(k) = \lambda^{1-k}\, d(k)$ for $k \le n$ and zero otherwise. - Using Example 5.2.9 the corresponding Kac-Gutzwiller transfer operator can be written out explicitely, but the formulas become quite long. □

5.5 Kac-Gutzwiller transfer operators for Ising type interactions

In this section we generalise the results of the previous section to matrix subshifts with Ising type interactions. Recall Theorem 3.2.6 which provides us with a Ruelle-Mayer transfer operator for the one-sided one-dimensional matrix subshift (1.2.8) with two-body Ising type interaction (1.8.3) if the distance function d belongs to $\mathcal{D}_1^{(p)}$ (2.7.1) for some $p < \infty$. In this section we suppose that d is given as $d(k) = \left(\Lambda^{k-1}v \,\middle|\, w\right)$ where $\Lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$ and that the interaction matrix $r \in \mathcal{C}_b(F\times F)$ has a decomposition $r(x,y) = \sum_{j=1}^M s_j(x)\, t_j(y)$ with $s_j,\ t_j \in \mathcal{C}_b(F)$. By Theorem 3.2.6 the Ruelle-Mayer transfer operator $\tilde{\mathcal{M}}_\beta : L^2(F,\nu)\hat{\otimes}\mathcal{F}((\mathbb{C}^n)^M) \to L^2(F,\nu)\hat{\otimes}\mathcal{F}((\mathbb{C}^n)^M)$,

$$(\tilde{\mathcal{M}}_\beta f)(x;z_1,..,z_M) = \int_F \mathbb{A}_{\sigma,x}\exp\Big(\beta q(\sigma)+\beta\sum_{j=1}^M s_j(\sigma)(z_j|w)\Big)\, f(\sigma;t_1(\sigma)v+\Lambda z_1,..,t_M(\sigma)v+\Lambda z_M)\,d\nu(\sigma)$$

satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z_{\{1,\dots,n\}}^{b^{\mathbb{N}_0},\phi}(\beta) = \det(1-\Lambda)^M \operatorname{trace}(\tilde{\mathcal{M}}_\beta)^n$. In this section we compute the corresponding Kac-Gutzwiller transfer operator by which we mean the partial Bargmann[37] conjugate of the Ruelle-Mayer operator $\tilde{\mathcal{M}}_\beta$ with respect to the z-variable, i. e., the operator $\tilde{\mathcal{G}}_\beta := (\mathrm{id}\otimes B_{nM})^{-1}\circ\tilde{\mathcal{M}}_\beta\circ(\mathrm{id}\otimes B_{nM}) : L^2(F,\nu)\hat{\otimes}L^2((\mathbb{R}^n)^M) \to L^2(F,\nu)\hat{\otimes}L^2((\mathbb{R}^n)^M)$.
First we will consider the transfer operators for the full shift. The Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}((\mathbb{C}^n)^M) \to \mathcal{F}((\mathbb{C}^n)^M)$ is given by Theorem 2.13.8 as

$$(\mathcal{M}_\beta f)(z_1,\dots,z_M) = \int_F \exp\Big(\beta q(\sigma) + \beta\sum_{l=1}^M s_l(\sigma)\,\langle z_l|w\rangle\Big)\, f\Big(\,(t_1(\sigma)v,\dots,t_M(\sigma)v) + \underline{\mathbb{B}}z\Big)\,d\nu(\sigma).$$

We recall Definition (94) of the generalised composition operator $\mathcal{L}_{a,b,\Lambda} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$, $(\mathcal{L}f)(z) = e^{(z|a)}\,f(b+\Lambda z)$ where $a,\ b \in \mathbb{C}^n$ and $\lambda \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\Lambda\| < 1$. We rewrite $\mathcal{M}_\beta$ as

$$\mathcal{M}_\beta = \int_F e^{\beta q(\sigma)} \bigotimes_{j=1}^M \mathcal{L}_{\beta s_j(\sigma)w,t_j(\sigma)v,\mathbb{B}}\,d\nu(\sigma).$$

This expression allows us to apply Propositions 3.3.4 and 5.3.5 and to compute the corresponding Kac-Gutzwiller transfer operator $\mathcal{G}_\beta = (B_{nM})^{-1}\circ\mathcal{M}_\beta\circ B_{nM} \in \mathrm{End}(L^2((\mathbb{R}^n)^M))$ for the full shift

$$\begin{aligned}\mathcal{G}_\beta &= \int_F e^{\beta q(\sigma)}\bigotimes_{j=1}^M \left((B_n)^{-1}\circ\ \mathcal{L}_{\beta s_j(\sigma)w,t_j(\sigma)v,\mathbb{B}}\circ B_n\right)d\nu(\sigma)\\
&= 2^{nM/2}\det(1-\Lambda\Lambda^\top)^{-M/2}\ \times \qquad\qquad (97)\\
&\quad \int_F \exp\Big(\beta q(\sigma) - \frac{\pi i}{2}\sum_{j=1}^M \Big(\begin{pmatrix}\frac{\beta}{\pi}s_j(\sigma)w\\ t_j(\sigma)v\end{pmatrix}\,\Big|\,\Psi_\Lambda\begin{pmatrix}\frac{\beta}{\pi}s_j(\sigma)w\\ t_j(\sigma)v\end{pmatrix}\Big)\Big)\bigotimes_{j=1}^M T^{\Psi_\Lambda(\frac{\beta}{\pi}s_j(\sigma)w,t_j(\sigma)v)}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\,d\nu(\sigma),\end{aligned}$$

where $\Psi_\Lambda = -i(1+X_\Lambda)^{-1}$. We now consider the transfer operators for the matrix subshift. The Ruelle-Mayer transfer operator is $\mathrm{pr}_x\tilde{\mathcal{M}}_\beta = \int_F \mathbb{A}_{\sigma,x}\,e^{\beta q(\sigma)}\left(\bigotimes_{j=1}^M \mathcal{L}_{\beta s_j(\sigma)w,t_j(\sigma)v,\mathbb{B}}\right)\circ\mathrm{pr}_\sigma\,d\nu(\sigma)$. By Proposition 3.3.4 we have the following characterisation of the corresponding Kac-Gutzwiller transfer

[37] In order to clarify the arguments we will use a lower index to indicate the dimension of the base space on which the Bargmann transform acts, e. g. $B_n : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$.

operator $\tilde{\mathcal{G}}_\beta : L^2(F,\nu)\hat{\otimes}L^2((\mathbb{R}^n)^M) \to L^2(F,\nu)\hat{\otimes}L^2((\mathbb{R}^n)^M)$ via

$$\begin{aligned} \mathrm{pr}_x\tilde{\mathcal{G}}_\beta &= \mathrm{pr}_x\big((B_{nM})^{-1}\circ\tilde{\mathcal{M}}_\beta\circ B_{nM}\big) \\ (98)\quad &= 2^{nM/2}\det(1-\Lambda\Lambda^\top)^{-M/2}\ \times \end{aligned}$$

$$\int_F \mathbb{A}_{\sigma,x}\exp\Big(\beta q(\sigma) - \frac{\pi i}{2}\sum_{j=1}^M \Big(\Big(\begin{smallmatrix}\frac{\beta}{\pi}s_j(\sigma)w\\ t_j(\sigma)v\end{smallmatrix}\Big)\,\Big|\,\Psi_\Lambda\Big(\begin{smallmatrix}\frac{\beta}{\pi}s_j(\sigma)w\\ t_j(\sigma)v\end{smallmatrix}\Big)\Big)\Big)\Big(\bigotimes_{j=1}^M T^{\Psi_\Lambda(\frac{\beta}{\pi}s_j(\sigma)w,t_j(\sigma)v)}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\Big)\circ\mathrm{pr}_\sigma\,d\nu(\sigma).$$

Since the Kac-Gutzwiller transfer operator is by definition conjugate to the Ruelle-Mayer transfer operator, it satisfies the dynamical trace formula $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}(\beta) = \det(1-\Lambda)^M\,\mathrm{trace}\,(\tilde{\mathcal{G}}_\beta)^n$ for the matrix subshift, $\widetilde{Z}_n^{b^{\mathbb{N}_0}}(\beta A_{(\phi)}) = Z^{b^{\mathbb{N}_0},\phi}_{\{1,\dots,n\}}(\beta) = \det(1-\Lambda)^M\,\mathrm{trace}\,(\mathcal{G}_\beta)^n$ for the full shift, respectively. The integral kernel of $\mathcal{G}_\beta$ is a Schwartz function, hence the trace of $(\mathcal{G}_\beta)^n$ can be evaluated by integrating the integral kernel over the diagonal, see Remark 3.3.3 or [CoGr90, A.3.9]. We ask whether one can find a direct way to show that the occurring integrals express the partition function. This could be used to construct new types of transfer operators.
We recall a couple of techniques introduced and used throughout this dissertation and compute the transfer operators for the M-state Potts model.

(i) The decomposition of an operator as an integral over a family of basic operators, Theorem A.7.6,

(ii) The superposition principle: The integral over the tensor product of the basic operators for the treatment of Ising type interactions, Proposition 2.3.9 and Subsection2.13,

(iii) The tensor product with the transition matrix for the treatment of matrix subshifts, Lemma 3.3.1,

(iv) The choice of a suitable generating triple leading to a transfer operator with nicer properties, Theorem 5.4.1, and

(v) The scaling of an integral operator, Lemma A.6.3 applied in Example 5.2.7.

Example 5.5.1. (Potts model) Let $F = \{1,\dots,M\}$ be finite and ν the a priori measure ν identified with its distribution vector. Let ϕ be a two-body Potts interaction with potential $q \in \mathcal{C}_b(F)$ and distance function $d(k) = \big(\mathbb{B}^{k-1}v\,\big|\,w\big)$ where $\mathbb{B} \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|\mathbb{B}\| < 1$.

(i) Example 2.13.10 yields the Ruelle-Mayer transfer operator $\mathcal{M}_\beta : \mathcal{F}((\mathbb{C}^n)^M) \to \mathcal{F}((\mathbb{C}^n)^M)$,

$$(\mathcal{M}_\beta f)(z_1,\dots,z_M) = \sum_{j=1}^M \nu_j\,\exp\Big(\beta q_j + \beta\,\big(z_j\,\big|\,w\big)\Big)\,f\Big((\delta_{j,m}w + \Lambda z_m)_{m=1,\dots,M}\Big)$$

for the full shift Potts model. Introducing the notation $\bigotimes^j A := A\otimes\ldots\otimes A$ (j-times) for the j-fold tensor product of an operator A, we can write $\mathcal{M}_\beta = \sum_{j=1}^M \nu_j\,e^{\beta q_j}\bigotimes^{j-1}C_\Lambda\otimes\mathcal{L}_{\beta w,v,\Lambda}\otimes\bigotimes^{M-j}C_\Lambda$. By the above formula (97) we get the Bargmann conjugate of $\mathcal{M}_\beta$ as

$$\begin{aligned}(B_{nM})^{-1}\circ\ \mathcal{M}_\beta\circ B_{nM} \quad &= \quad \frac{2^{nM/2}}{\det(1-\Lambda\Lambda^\top)^{M/2}}\exp\Big(-\frac{\pi i}{2}\Big(\Big(\begin{smallmatrix}\frac{\beta}{\pi}w\\ v\end{smallmatrix}\Big)\,\Big|\,\Psi_\Lambda\Big(\begin{smallmatrix}\frac{\beta}{\pi}w\\ v\end{smallmatrix}\Big)\Big)\Big)\ \times \\ &\quad \sum_{j=1}^M \nu_j\,e^{\beta q_j}\bigotimes^{j-1}T_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\otimes T^{\Psi_\Lambda(\frac{\beta}{\pi}w,v)}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\otimes\bigotimes^{M-j}T_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}.\end{aligned}$$

(ii) We now generalise (i) to matrix subshifts. Since the alphabet F is finite, the Hilbert space on which the Ruelle-Mayer transfer operator acts is $\mathcal{F}((\mathbb{C}^n)^M)^{|F|} \cong L^2(F,\nu)\hat{\otimes}\mathcal{F}((\mathbb{C}^n)^M)$ as pointed out in Remark 3.2.8. By Remark 3.2.8 the components of the Ruelle-Mayer transfer operator $\tilde{\mathcal{M}}_\beta : \mathcal{F}((\mathbb{C}^n)^M)^M \to \mathcal{F}((\mathbb{C}^n)^M)^M$ for the matrix subshift Potts spin model are

$$(\tilde{\mathcal{M}}_\beta(f_1,\dots,f_M)(z_1,\dots,z_M))_l = \sum_{k=1}^M \mathbb{A}_{k,l}\,\nu_k\,\exp\Big(\beta q_k + \beta\,(z_k|w)\Big)\,f_k\Big((\delta_{k,m}v + \Lambda z_m)_{m=1,\dots,M}\Big)$$

for $l = 1,\ldots,M$. By formula (98) we get the corresponding Kac-Gutzwiller transfer operator $\tilde{\mathcal{G}}_\beta = (\mathrm{id}\otimes B_{nM})^{-1}\circ\tilde{\mathcal{M}}_\beta\circ(\mathrm{id}\otimes B_{nM}) : L^2((\mathbb{R}^n)^M)^M \to L^2((\mathbb{R}^n)^M)^M$, in components written as

$$\mathrm{pr}_l\circ\tilde{\mathcal{G}}_\beta = \frac{2^{nM/2}}{\det(1-\Lambda\Lambda^\top)^{M/2}}\exp\Big(-\frac{\pi i}{2}\Big(\Big(\begin{smallmatrix}\frac{\beta}{\pi}w\\ v\end{smallmatrix}\Big)\Big|\Psi_\Lambda\Big(\begin{smallmatrix}\frac{\beta}{\pi}w\\ v\end{smallmatrix}\Big)\Big)\Big)\times$$
$$\sum_{k=1}^M \mathbb{A}_{k,l}\,\nu_k\,e^{\beta q_k}\Big(\bigotimes^{k-1} T_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\otimes T^{\Psi_\Lambda(\frac{\beta}{\pi}w,v)}_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\otimes\bigotimes^{M-k} T_{\mathbf{c}_{2n}^{-1}(X_\Lambda)}\Big)\circ\mathrm{pr}_k.$$

(iii) Now we specialise (i) to Potts interactions with distance function $d(k) = J\lambda^k$ where $0 < |\lambda| < 1$. In order to reduce the notational effort we restrict to the one-sided one-dimensional full shift. As in Example 5.4.4 we consider the scaled Kac-Gutwiller operator $R_c\circ\mathcal{G}_{\beta;s}\circ R_c^{-1} = R_c\circ B^{-1}\circ\mathcal{M}_{\beta;s}\circ B\circ R_c^{-1}$ where $\mathcal{M}_{\beta;s} : \mathcal{F}(\mathbb{C}^M)\to\mathcal{F}(\mathbb{C}^M)$ is the Ruelle-Mayer transfer operator

$$(\mathcal{M}_{\beta;s}f)(z) = \sum_{k=1}^M \nu_k\,\exp(\beta q_k + \beta J s\langle z|e_k\rangle)\,f\Big(\frac{\lambda}{s}\,e_k + \lambda z\Big).$$

Let γ be a complex number with $\lambda = e^{-\gamma}$ and $c := (4\pi)^{-1/2}$.

(a) Let $s_1 = \sqrt{\frac{\pi}{\beta J}}$, then $R_c\circ\mathcal{G}_{\beta;s_1}\circ R_c^{-1}$ has the integral kernel

$$\frac{e^{M\gamma/2}}{(4\pi\sinh\gamma)^{M/2}}\exp\Big(-\frac{1}{4}\sum_{k=1}^M(\tanh\frac{\gamma}{2}\,(x_k^2+y_k^2)+\frac{(x_k-y_k)^2}{\sinh\gamma})\Big)\sum_{j=1}^M\nu_j\,e^{\beta q_j}\,\exp(\sqrt{\beta J}\,x_j).$$

(b) Let $s_2 = \sqrt{\frac{\pi\lambda^2}{\beta J}}$, then $R_c\circ\mathcal{G}_{\beta;s_2}\circ R_c^{-1}$ has the integral kernel

$$\frac{e^{M\gamma/2}}{(4\pi\sinh\gamma)^{M/2}}\exp\Big(-\frac{1}{4}\sum_{k=1}^M(\tanh\frac{\gamma}{2}\,(x_k^2+y_k^2)+\frac{(x_k-y_k)^2}{\sinh\gamma})\Big)\sum_{j=1}^M\nu_i\,e^{\beta q_j}\,\exp(\sqrt{\beta J}\,y_j).$$

(c) Let $s_0 = \sqrt{\frac{\pi\lambda}{\beta J}}$, then $R_c\circ\mathcal{G}_{\beta;s_0}\circ R_c^{-1}$ has the integral kernel

$$\frac{e^{M\gamma/2}}{(4\pi\sinh\gamma)^{M/2}}\exp\Big(-\frac{1}{4}\sum_{k=1}^M(\tanh\frac{\gamma}{2}\,(x_k^2+y_k^2)+\frac{(x_k-y_k)^2}{\sinh\gamma})\Big)\sum_{j=1}^M\nu_i\,e^{\beta q_j}\,\exp\Big(\frac{\sqrt{\beta J\lambda}}{1+\lambda}\,(x_j+y_j)\Big).$$

□

A Miscellaneous topics from functional analysis

In this chapter we collect several ingredients from functional analysis we used in this dissertation. The first three sections deal with classes of operators which admit a spectral trace and a determinant. First we recall the axiomatic approach of embedded subalgebras with the approximation property. In Section A.2 we recall the normal form of a compact operator on a Hilbert space and define in this way the singular numbers and the Schatten classes, in particular the trace class and the Hilbert-Schmidt class. We state the Lidskii trace theorem as an example of a spectral trace, i. e., the trace is given as the (absolutely convergent) sum over all eigenvalues. In Section A.3 we comment on the situation in Banach spaces and briefly recall the concept of nuclear operators and Grothendieck's 2/3-trace theorem.

In Section A.4 we introduce reproducing kernel Hilbert spaces and discuss some important properties. We give a couple of examples and then focus on the classification of Fock spaces. The use of the Fock space the context of Ruelle-Mayer transfer operators has been proposed in [HiMay02] and [HiMay04]. In section B.3 we will essentially emploit the properties of a reproducing kernel Hilbert space for the determination of the trace norm of certain composition operators acting on the Fock space.

The short fifth section provides some identities on Gaussian integrals which we need at several points in this dissertation.

Section A.6 collects some tools dealing with integral operators. Integral operators arise in this dissertation, since every bounded linear operator on a reproducing kernel Hilbert space can be written as an integral operator and, secondly, by the study of the Kac-Gutzwiller transfer operator and the extended oscillator semigroup.

In the last section we give a proof for a folklore theorem which states that an operator defined as an integral over an integrable family of trace class operators is trace class and its trace can be computed by integrating the family of traces, i. e.,

$$\operatorname{trace} \int_Y \mathcal{L}_y \, dy = \int_Y \operatorname{trace} \mathcal{L}_y \, dy.$$

A.1 (Regularised) determinants

We briefly recall the definition and properties of regularised determinants of higher order. Our representation is based on [GoGoKr00, Ch. XI]. We will use the theory of regularised determinants in cases where the dynamical trace formula (only) holds for sufficiently large powers of the transfer operator, a phenomenon which appears for instance for all transfer operators for matrix subshifts.

Let $\mathcal{B}$ be a Banach space and $\mathcal{B}'$ its dual. An operator $F : \mathcal{B} \to \mathcal{B}$ of the form $F = \sum_{i=1}^n \phi_k' \otimes f_k$ with $f_k \in \mathcal{B}$, $\phi_k' \in \mathcal{B}'$ is called a *finite rank operator*. Denote by $\lambda_j(F)$ its eigenvalues (counted with multiplicity). Let $\operatorname{trace} F = \sum_{i=1}^n \langle \phi_k, f_k \rangle_{\mathcal{B}',\mathcal{B}} = \sum_j \lambda_j$ be its trace[38].

For any finite rank operator F acting on a Banach space $\mathcal{B}$ and any $u \in \mathbb{N}$ one defines the *u-regularised determinant*

$$\det_u(1-F) := \det(1-F) \, \exp\Big(- \sum_{k=1}^{u-1} \frac{1}{k} \operatorname{trace} F^k \Big). \tag{99}$$

A natural question concerns the (continuous) extension of the trace and the u-regularised determinant to wider algebras of linear operators. For this continuity we need other norms than the operator norm for which neither trace nor determinant are continuous on the finite rank operators acting in infinite dimensional spaces. The right setting is the following:

Definition A.1.1. Let $\mathcal{B}$ be a Banach space. We denote by $\operatorname{End}(\mathcal{B})$ the algebra of bounded linear operators on $\mathcal{B}$ and by $\operatorname{End}_f(\mathcal{B})$ the subalgebra of finite rank operators. We say that a subalgebra $\mathcal{E} \subset \operatorname{End}(\mathcal{B})$ is an *embedded subalgebra*, if $\mathcal{E}$ carries a norm $\| \cdot \|_{\mathcal{E}}$ such that

$$\|A\|_{\operatorname{End}(\mathcal{B})} \leq c_{\mathcal{E}} \, \|A\|_{\mathcal{E}}, \quad \|AB\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}} \, \|B\|_{\mathcal{E}}$$

for all A, $B \in \mathcal{E}$. If, in addition, the set $\mathcal{E} \cap \operatorname{End}_f(\mathcal{B})$ is dense in $\mathcal{E}$ with respect to the norm $\| \cdot \|_{\mathcal{E}}$, we say that $\mathcal{E}$ has the *approximation property*. □

[38] The second equality is [GoGoKr00, Thm. I.3.1].

[GoGoKr00, Thm. 1.1, Ch. XI] gives a characterisation when the u-regularised determinant $\det_u(1+\cdot)$ from (99) admits a continuous extension to an embedded subalgebra $\mathcal{E} \subset \mathrm{End}(\mathcal{B})$. In this case also the functions $\mathcal{E} \cap \mathrm{End}_f(\mathcal{B}) \to \mathbb{C},\ F \mapsto \mathrm{trace}\ F^k$ admit continuous extensions to $\mathcal{E}$ for all $k \geq u$.

Lemma A.1.2. *Let $\mathcal{E} \subset \mathrm{End}(\mathcal{B})$ be an embedded subalgebra with the approximation property. Suppose the function $\det_u(1+\cdot)$ from (99) admits a continuous extension to $\mathcal{E}$. Then the function $z \mapsto \det_u(1+zA)$ is entire for every fixed $A \in \mathcal{E}$ and has the following representations:*

$$\det_u(1+zA) = 1 + \sum_{k=u}^{\infty} \frac{c_k(A)}{k!}\, z^k,$$

where the coefficients $c_k(A)$ are defined by

$$c_n(A) := \det \begin{pmatrix} b_1 & n-1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & n-2 & \dots & 0 & 0 \\ b_3 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_2 & b_1 \end{pmatrix}$$

and

$$b_k := \begin{cases} \mathrm{trace}\ A^k, & \text{if } k \geq u, \\ 0, & \text{otherwise.} \end{cases}$$

For $|z|$ sufficiently small one has

$$\det_u(1-zA) = \exp\Big(-\sum_{k=u}^{\infty} \frac{z^k}{k}\, \mathrm{trace}\ A^k\Big). \tag{100}$$

Proof. [GoGoKr00, Theorem XI.2.1]. □

The proof of Lemma A.1.2 uses the following analytic lemma.

Lemma A.1.3. *Let $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$, $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} b_n}{n} z^n$ be analytic functions in a neighbourhood of zero with $g(z) = \exp f(z)$. Then $a_0 = 1$ and for $n \geq 1$*

$$a_n = \sum_{k=1}^{n} (-1)^{k+1}\, b_k\, a_{n-k}\, \frac{(n-1)!}{(n-k)!} = \det \begin{pmatrix} b_1 & n-1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & n-2 & \dots & 0 & 0 \\ b_3 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_2 & b_1 \end{pmatrix}.$$

Proof. [GoGoKr00, Lemma I.7.1 and its proof]. □

One has the following generalisation of the theorem of Lidskii (Prop. A.2.4), which is contained as the special case $u = 1$. In particular the zeros of $z \mapsto \det_u(1+zA)$ are in one-to-one correspondence with the eigenvalues of A.

Lemma A.1.4. *Let $\mathcal{E} \subset \mathrm{End}(\mathcal{B})$ be an embedded subalgebra with the approximation property. Suppose the function $\det_u(1+\cdot)$ admits a continuous extension to $\mathcal{E}$. Then $\det_u(1+\cdot)$ has the Euler product*

$$\det_u(1+A) = \prod_j \Big((1+\lambda_j)\, \exp\big(\sum_{k=1}^{u-1} \frac{(-1)^k}{k}\, \lambda_j^k\big)\Big),$$

where λ_j are the eigenvalues of $A \in \mathcal{E}$. □

This representation of the regularised determinant is true on the level of finite rank operators and hence by continuation on an embedded subalgebra with the approximation property.
In the following two sections we will introduce two families of examples of embedded subalgebra with the approximation property and their (regularised) determinants. We consider the Schatten classes $\mathcal{S}_p(\mathcal{H}) \subset \mathrm{End}(\mathcal{H})$, which we use for throughout this dissertation. We also comment on the space of nuclear operators, which has been used by D. Mayer in his basal work on this subject.

A.2 Schatten classes

As we pointed out in the introduction chapter we prefer to use Hilbert space techniques in this dissertation. Of particular interest are the Schatten classes $\mathcal{S}_p(\mathcal{H}) \subset \mathrm{End}(\mathcal{H})$ which are embedded subalgebras with the approximation property. For the definition of the Schatten classes we recall (see for instance [We00, p. 245]) the normal form of a compact operator on a Hilbert space.

Proposition A.2.1. *(Schmidt expansion) Let $K : \mathcal{H} \to \mathcal{H}$ be a compact operator on a Hilbert space $\mathcal{H}$. Let $(s_n(K))_{n\in\mathbb{N}}$ be the sequence of singular numbers of K, i. e., the eigenvalues of $|K| = \sqrt{K^\star K}$, $(e_n)_{n\in\mathbb{N}}$ the orthonormal system consisting of eigenvectors of $K^\star K$, and $(f_n)_{n\in\mathbb{N}}$ the orthonormal system consisting of eigenvectors of $KK^\star$. Then the expansion*

$$K = \sum_n s_n(K) \left\langle \cdot \,\middle|\, e_n \right\rangle f_n \tag{101}$$

converges to K in operator norm. □

The singular numbers can be characterised by an approximation problem, too, cf. Lemma A.7.4 and [GoGoKr00, IV.2, IV.3]. If K happens to be an integral operator, the Schmidt expansion leads to an expansion of the integral kernel which is often called Mercer expansion.
The sequence of singular numbers of a compact operator tends to zero. If one moreover requires a certain summability, this leads to definition of the so called Schatten classes, named after R. Schatten.

Definition A.2.2. For $1 \le p < \infty$ the Schatten ideal $\mathcal{S}_p(\mathcal{H})$ is defined as the space of all operators K such that

$$\|K\|_{\mathcal{S}_p(\mathcal{H})} := \|(s_n(K))_{n\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} < \infty.$$

The elements of $\mathcal{S}_1(\mathcal{H})$ are called *trace class operators*, the elements of $\mathcal{S}_2(\mathcal{H})$ are called *Hilbert-Schmidt operators*. For $K \in \mathcal{S}_i(\mathcal{H})$ $(i = 1, 2)$ the Schmidt expansion converges in $\mathcal{S}_i(\mathcal{H})$ to K. Given a trace class operator in Schmidt expansion, $K = \sum_n s_n(K) \left\langle \cdot \,\middle|\, e_n \right\rangle f_n$, one defines its trace via

$$\operatorname{trace} K = \sum_n s_n(K) \left\langle f_n \,\middle|\, e_n \right\rangle.$$

□

Remark A.2.3. Let K be a compact operator on a Hilbert space $\mathcal{H}$. Computing the adjoint of K via (101) gives $K^\star = \sum_n s_n(K) \left\langle \cdot \,\middle|\, f_n \right\rangle e_n$, hence the operators $K^\star K$ and $KK^\star$ have the same non-zero spectrum (counted with multiplicities). □

Let $\mathcal{H}$ be a Hilbert space. As shown in a side remark following [GoGoKr00, Theorem XI.2.1] the Schatten classes $\mathcal{S}_p(\mathcal{H}) \subset \mathrm{End}(\mathcal{H})$ for $1 \le p < \infty$ are embedded subalgebras and have the approximation property. Hence we can apply Lemma A.1.2 to the operator algebra $\mathcal{S}_p(\mathcal{H})$. In particular for $p = 1$ one obtains

Proposition A.2.4. *(Lidskii trace theorem) Let $T \in \mathcal{S}_1(\mathcal{H})$ be a trace class operator and $(\lambda_k(T))_{k\in\mathbb{N}}$ the sequence of eigenvalues of T counted with multiplicities, then*

$$\operatorname{trace} T = \sum_k \lambda_k(T) = \sum_k \left\langle Te_k \,\middle|\, e_k \right\rangle$$

for any orthonormal basis $(e_k) \subset \mathcal{H}$. □

Lidskii's trace theorem roughly states that the trace of a trace class operator behaves like the trace of a square-matrix. In particular the trace is independent of the choice of the representation or of the chosen orthonormal basis. A direct proof is given for instance in [GoGoKr00, IV.6]. We say that a trace class operator admits a *spectral trace*, i. e., the trace is given as the sum over the eigenvalues.
One has the following estimates for the trace and the determinant function.

Remark A.2.5. Recall (see for instance [GoGoKr00]) that for matrices and hence for trace class operators $A,\, B \in \mathcal{S}_1(\mathcal{H})$ the following relations hold:

$$\begin{aligned}
&(102) & \exp \operatorname{trace} A &= \det \exp A \\
&(103) & \operatorname{trace}\ \log(1+A) &= \log\det(1+A) \\
&(104) & |\det(1+A)-\det(1+B)| &\le \|A-B\|_{\mathcal{S}_1(\mathcal{H})} \exp\big(1+\|A\|_{\mathcal{S}_1(\mathcal{H})}+\|B\|_{\mathcal{S}_1(\mathcal{H})}\big), \\
&(105) & |\det(1+A)| &\le \exp(\|A\|_{\mathcal{S}_1(\mathcal{H})}), \\
&(106) & |\det(1+A)-1| &\le \exp(\|A\|_{\mathcal{S}_1(\mathcal{H})})-1 \le \|A\|_{\mathcal{S}_1(\mathcal{H})}\ \exp(\|A\|_{\mathcal{S}_1(\mathcal{H})}).
\end{aligned}$$

Let $(\lambda_j)_{j\in\mathbb{N}}$ be the sequence of eigenvalues of A and $(s_j)_{j\in\mathbb{N}}$ be the sequence of singular numbers of A. Observe that $e^x - 1 \le xe^x$ for $x \ge 0$, then the last estimate (106) comes from

$$|\det(1+A)-1| = |\prod_j (1+\lambda_j) - 1| \le \exp(\sum_j |\lambda_j|) - 1 \le \exp(\sum_j |s_j|) - 1 = \exp(\|A\|_{\mathcal{S}_1(\mathcal{H})}) - 1.$$

□

We end this section with the regularised determinant of order $1 \le p < \infty$. Notice that the case $\lfloor p \rfloor = 1$ is the theory of trace class operators and their Fredholm determinants.

Remark A.2.6. Let $\mathcal{H}$ be a Hilbert space and $1 \le p < 1$. Let $\lceil p \rceil := \min\{n \in \mathbb{N} \,|\, n \ge p\}$ denote the minimal integer greater or equal to p, then the regularised determinant $\det_{\lceil p \rceil}(1+\cdot)$ admits a continuous extension to $\mathcal{S}_p(\mathcal{H})$. For all $A \in \mathcal{S}_p(\mathcal{H})$ the operator $R_{\lceil p \rceil}(A) := (1+A)\ \exp\big(\sum_{j=1}^{\lceil p \rceil -1} \frac{(-1)^j}{j} A^j\big) - 1$ is trace class and

$$\det_{\lceil p \rceil}(1+A) = \det(1 + R_{\lceil p \rceil}(A)).$$

The following estimates ([GoGoKr00]) hold for all $A,\, A_1, \ldots, A_{\lceil p \rceil},\, B \in \mathcal{S}_p(\mathcal{H})$

$$\begin{aligned}
&(107) & |\operatorname{trace}\ (A_1 \ldots A_{\lceil p \rceil})| &\le \prod_{j=1}^{\lceil p \rceil} \|A_j\|_{\mathcal{S}_{\lceil p \rceil}(\mathcal{H})}, \\
&(108) & |\det_{\lceil p \rceil}(1+A)| &\le \exp\big(c_{\lceil p \rceil} \|A\|^{\lceil p \rceil}_{\mathcal{S}_{\lceil p \rceil}(\mathcal{H})}\big), \\
&(109) & |\det_{\lceil p \rceil}(1+A) - \det_{\lceil p \rceil}(1+B)| &\le \|A-B\|_{\mathcal{S}_{\lceil p \rceil}(\mathcal{H})} \exp\Big(c_{\lceil p \rceil}(1+\|A\|_{\mathcal{S}_{\lceil p \rceil}(\mathcal{H})} + \|B\|_{\mathcal{S}_{\lceil p \rceil}(\mathcal{H})})^{\lceil p \rceil}\Big).
\end{aligned}$$

A more refined analysis yields

$$(110) \qquad |\det_u(1+A)| \le \exp\big(c_u \|A^u\|_{\mathcal{S}_1(\mathcal{H})}\big) \le \exp\big(c_u \|A\|^u_{\mathcal{S}_u(\mathcal{H})}\big),$$

□

A.3 Nuclear operators

In Banach spaces it is a more difficult task to find the appropriate spaces of operators for which one can find a meaningfull determinant or trace. We briefly recall some aspects of nuclear operators. An excellent reference on this topic is the book [GoGoKr00] or [May80a, Appendix A].

Definition A.3.1. Let $\mathcal{B}$ be a Banach space. Suppose that the bounded linear operator $T \in \operatorname{End}(\mathcal{B})$ admits a representation $T = \sum_{k=1}^{\infty} \phi'_k \otimes f_k$, where $f_k \in \mathcal{B}$, $\phi'_k \in \mathcal{B}'$ and

$$(111) \qquad \sum_{k=1}^{\infty} \|f_k\|^r \|\phi_k\|^r < \infty$$

for some $0 < r \le 1$. Then T is called a (r-summable) *nuclear operator*. We define

$$\|T\|_{L^{(r)}(\mathcal{B})} := \inf \sum_{k=1}^{\infty} \|f_k\|^r \|\phi_k\|^r,$$

where the infimum is taken over all representations $T = \sum_{k=1}^{\infty} f_k \otimes \phi_k$. The infimum over the r satisfying (111) is called the *order* of T. □

The algebra of 1-summable nuclear operators endowed with the norm $\|\cdot\|_{L^{(1)}(\mathcal{B})}$ is an embedded subalgebra of $\mathrm{End}(\mathcal{B})$ with the approximation property, see [GoGoKr00, Thm.V.1.1]. The setting

$$\operatorname{trace} T := \sum_{k=1}^{\infty} \langle \phi_k', f_k \rangle_{\mathcal{B}',\mathcal{B}}$$

defines a continuous extension of the trace functional and the value trace T does not depend on the chosen representation.
Whereas in a Hilbert space the $\ell^1\mathbb{N}$-convergence of the sequence of singular numbers guarantees a spectral trace, we need in the Banach space setting a stronger decay. The result is known as Grothendieck's 2/3-trace theorem which we state next.

Theorem A.3.2. *(Grothendieck's 2/3-trace theorem) Let $\mathcal{B}$ be a Banach space and $T \in \mathrm{End}(\mathcal{B})$ be a 2/3-summable nuclear operator (A.3.1). Then $\sum_{k=1}^{\infty} |\lambda_k(T)| < \infty$ and for all 2/3-summable representations $T = \sum_{k=1}^{\infty} f_k \otimes \phi_k$ one has*

$$\operatorname{trace} T = \sum_{k=1}^{\infty} \langle \phi_k', f_k \rangle_{\mathcal{B}',\mathcal{B}} = \sum_{k=1}^{\infty} \lambda_k(T),$$

where $(\lambda_k(T))_{k\in\mathbb{N}}$ is the sequence of eigenvalues of T counted with multiplicities.

Proof. Firstly appeared in [Gro55, II.1. No. 4, Cor. 4]. A nicely written proof is given in [GoGoKr00, V.Thm.3.1]. □

[GoGoKr00, Thm. V 4.2] states the following generalisation of Grothendieck's 2/3-trace theorem to the analogue of Schatten class $\mathcal{S}_p$-operators.

Theorem A.3.3. *Let $T \in \mathrm{End}(\mathcal{B})$ be a r-summable nuclear operator for some $0 < r \leq 1$ (A.3.1). Then the sequence $(\lambda_k(T))_{k\in\mathbb{N}}$ of eigenvalues of T counted with multiplicities belongs to $\ell^p\mathbb{N}$ for $\frac{1}{p} = \frac{1}{r} - \frac{1}{2}$.* □

In Appendix B we will encounter composition operators which are nuclear of order zero. Their eigenvalues have a rapide decay, as the following corollary states.

Corollary A.3.4. *Let $\mathcal{B}$ be a Banach space and $T \in \mathrm{End}(\mathcal{B})$ a nuclear operator of order zero. Then the sequence $(\lambda_k(T))_{k\in\mathbb{N}}$ of eigenvalues of T satisfies $\sum_{k=1}^{\infty} |\lambda_k(T)|^p < \infty$ for all $p > 0$.*

A.4 Reproducing kernel Hilbert spaces

We change the topic and give a short introduction to reproducing kernel Hilbert spaces, which are Hilbert spaces of functions such that the point evaluation is given by an inner product. The subject was originally and simultaneously developed by N. Aronszajn and S. Bergman in 1950. We discuss some elementary properties and give some examples in A.4.5 and A.4.6, which are for instance the Hardy space, the (weighted) Bergman spaces, and Fock spaces. We will mainly focus on the family of Fock spaces, which turn out to be useful in connection with Ruelle-Mayer transfer operators. Combining results known in the literature we obtain the classification A.4.8 of all Fock spaces over separable Hilbert spaces. We start with the basic definitions and discuss in the following Remark A.4.2 their relations.

Definition A.4.1. Let $\mathcal{H} \subset \mathbb{C}^E$ be a Hilbert space consisting of complex valued functions on a set E.

(i) The space $\mathcal{H}$ is called a *functional Hilbert space*, if for each $x \in E$ the evaluation functional

$$\mathrm{ev}_x : \mathcal{H} \to \mathbb{C},\ f \mapsto f(x)$$

is continuous.

(ii) A function $k : E \times E \to \mathbb{C}$ is called a *reproducing kernel* of $\mathcal{H}$, if for all $y \in E$ the function $k_y := k(\,\cdot\,, y) : E \to \mathbb{C}$ belongs to $\mathcal{H}$ and if for all $f \in \mathcal{H}$, $y \in E$: $\langle f \mid k_y \rangle_{\mathcal{H}} = f(y)$.

(iii) A function $p : E \times E \to \mathbb{C}$ is *of positive type* on E, if for all $n \in \mathbb{N}$, $a_j \in \mathbb{C}$, $x_j \in E$ $(j = 1, \dots, n)$

$$\sum_{k,l=1}^{n} \overline{a_k}\, a_l\, p(x_k, x_l) \geq 0.$$

□

We will use the standard abbreviation *rkhs* for reproducing kernel Hilbert space occasionally.

Remark A.4.2. By the Riesz representation theorem the evaluation functional $\mathrm{ev}_x \in \mathcal{H}'$ (A.4.1) is given via $\mathrm{ev}_x = \langle \cdot \,|\, k_x \rangle_{\mathcal{H}}$ for some $k_x \in \mathcal{H}$, hence $\mathcal{H}$ is a functional Hilbert space iff it has a reproducing kernel. A reproducing kernel Hilbert space is uniquely determined by its kernel.
A reproducing kernel is a function of positive type on E and each function of positive type induces uniquely a functional Hilbert space [Ar50, I.2.4]. □

The kernel of a reproducing kernel Hilbert space has the following important properties.

Proposition A.4.3. *Let $k : E \times E \to \mathbb{C}$ be a reproducing kernel. Then for all $x, y \in E$*

(i) $k(x, y) = \overline{k(y, x)}$,

(ii) $\|k_y\|^2 = k(y, y) \geq 0$,

(iii) $|k(x, y)|^2 \leq k(x, x)\, k(y, y)$.

Proof. Note that $k(x, y) = k_y(x) = \langle k_y \,|\, k_x \rangle$. Hence the properties of the inner product $\langle \cdot \,|\, \cdot \rangle_{\mathcal{H}}$ yield

(i) $k(x, y) = \langle k_y \,|\, k_x \rangle = \overline{\langle k_x \,|\, k_y \rangle} = \overline{k(y, x)}$,

(ii) $k(y, y) = \langle k_y \,|\, k_y \rangle = \|k_y\|^2 \geq 0$, and

(iii) $|k(x, y)|^2 = |\langle k_y \,|\, k_y \rangle|^2 \leq \|k_x\|^2\, \|k_y\|^2$. □

On a reproducing kernel Hilbert space one has the following standard estimate:

Corollary A.4.4. *Let $\mathcal{H} \subset \mathcal{C}(E)$ be an rkhs with reproducing kernel $k : E \times E \to \mathbb{C}$. Then for all $f \in \mathcal{H}$, $x \in E$ one has*

$$|f(x)| \leq \|f\|\, \sqrt{k(x, x)}.$$

Proof. By the reproducing kernel property (A.4.1), Cauchy-Schwarz inequality, and the previous Proposition A.4.3 we get

$$|f(x)| = |\langle f \,|\, k_x \rangle| \leq \|f\|\, \|k_x\| = \|f\|\, \sqrt{k(x, x)}.$$

□

Many of the examples of reproducing kernel Hilbert spaces are spaces of analytic functions. We introduce two types, the Fock space, and the Bergmann space. Another example is the Hardy space $H^2(\mathbb{D})$ on the disk, other examples of rkhs are given in [Ma88].

Example A.4.5. (i) The *Bargmann-Fock space* $\mathcal{F}(\mathbb{C}^m)$ is defined as the space of entire functions $F : \mathbb{C}^m \to \mathbb{C}$ with

$$\|F\|^2_{\mathcal{F}(\mathbb{C}^m)} := \int_{\mathbb{C}^m} |F(z)|^2 \, \exp(-\pi \|z\|^2)\, dz < \infty.$$

The Fock space is a Hilbert space with respect to the (weighted) L^2 inner product

$$\langle F | G \rangle_{\mathcal{F}(\mathbb{C}^m)} := \int_{\mathbb{C}^m} F(z)\, \overline{G(z)}\, \exp(-\pi \|z\|^2)\, dz,$$

where dz denotes Lebesgue measure on $\mathbb{C}^m$. The Fock space is an rkhs with reproducing kernel $k(z, w) = \exp\big(\pi (z|\overline{w})\big) = \exp\big(\pi \langle z|w \rangle\big)$. The Fock space $\mathcal{F}(\mathbb{C}^m)$ has the standard orthonormal basis (ONB) consisting of the monomials $\zeta_\alpha(z) = \sqrt{\frac{\pi^\alpha}{\alpha!}}\, z^\alpha$, where we use the standard multiindex notations: For $z \in \mathbb{C}^m$, $\alpha \in \mathbb{N}_0^m$ we define the factorial $\alpha! := \prod_{i=1}^m \alpha_i!$, the length $|\alpha| := \sum_{i=1}^m \alpha_i$, and the power $z^\alpha := \prod_{i=1}^m z_i^{\alpha_i}$.

(ii) Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and $A^2(\mathbb{D}) = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$ be the *Bergman space* consisting of holomorphic functions on $\mathbb{D}$ with are square-integrable with respect to Lebesgue measure on $\mathbb{D}$. The function $k(z,w) = \dfrac{1}{\pi}\dfrac{1}{(1-z\overline{w})^2}$ is the reproducing kernel of $A^2(\mathbb{D})$. □

The examples from the previous Example A.4.5 are prototypes of certain types of rkhs and chave the following generalisations:

Remark A.4.6. (i) Fock space: For each base Hilbert space $\mathcal{H}_0$ the function $k : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{C}$, $(z,w) \mapsto \exp(\pi\langle z|w\rangle_{\mathcal{H}_0})$ is a function of positive type, hence is a reproducing kernel of some rkhs. One defines the Fock space to be the unique reproducing kernel Hilbert space $\mathcal{F}(\mathcal{H}_0) := \mathcal{H}_k \subset \mathbb{C}^{\mathcal{H}_0}$ with reproducing kernel $k(z,w) = \exp(\pi\langle z|w\rangle_{\mathcal{H}_0})$. A priori it is not clear that $\mathcal{H}_k$ has a realisation as a space of holomorphic functions as we have for the special choice $\mathcal{H}_0 = \mathbb{C}^m$ (A.4.5). This we will show in Theorem A.4.8. In [Fo89, p. 48] the connection between our Fock space and the physicists' (bosonic) Fock space is explained.

(ii) Bergman space: Let $U \subset \mathbb{C}^m$ be a (finite-dimensional) domain and $w : U \to [0,\infty[$ be a *regular weight*, by which we mean a measurable function which is locally bounded from below by some positive constant $w \geq c > 0$. Let du denote the Lebesgue measure on U. Then the *weighted Bergman space* $A^2_w(U) := \mathcal{O}(U) \cap L^2(U, w\,du)$ is an rkhs, see [Ne00, Appendix.III.12]. Note that depending on the weight and on the domain $A^2_w(U)$ might be the zero space. We give some examples of Bergman spaces:

(a) If U is bounded and the weight is constant, one obtains the unweighted Bergman space $A^2(U) := \mathcal{O}(U) \cap L^2(U, du)$.

(b) If U has infinite volume, then $A^2(U) = \{0\}$.

(c) Another example is the Bargman-Fock space from Example A.4.5 (i). □

Since the point evaluations are continuous, we know that a (non-trivial) Bergman space possesses a reproducing kernel, but in general it is not known explicitly. The following lemma(see for instance [He78, Ch. VIII. 3.3.]) gives an abstract way how to find the reproducing kernel for a class of reproducing kernel Hilbert spaces.

Lemma A.4.7. *Let $D \subset \mathbb{C}^m$ be a domain, $\mathcal{H} \subset \mathcal{O}(D)$ be an rkhs with reproducing kernel $k : D \times D \to \mathbb{C}$. Let $(h_k)_k$ be an orthonormal basis in $\mathcal{H}$. Then*

$$k(z,w) = \sum_k h_k(z)\,\overline{h_k(w)},$$

where the convergence is absolute and uniform on compact subsets. □

In example A.4.5 (i) we have seen the analytic realisation of the Fock space for *all* finite-dimensional Hilbert spaces: Choose a unitary isomorphism from a given m-dimensional Hilbert space to $\mathbb{C}^m$ and apply the Bargmann-Fock realisation A.4.5 (i). We will now give the analogue for all infinite-dimensional ones. Since we always assume that a Hilbert space is separable, the reference Hilbert space is in this case $\ell^2\mathbb{N}$.

Theorem A.4.8. *(Ri)*[39] *Let $\iota_m : \mathbb{C}^m \to \ell^2\mathbb{N}$, $(z_1,\ldots,z_m) \mapsto (z_1,\ldots,z_m,0,\ldots)$. A function f belongs to the Fock space $\mathcal{F}(\ell^2\mathbb{N})$, defined as the rkhs with reproducing kernel $k(z,w) = \exp(\pi\langle z|w\rangle)$, if and only if the following three conditions hold:*

(i) *$f : \ell^2\mathbb{N} \to \mathbb{C}$ is continuous,*

(ii) *For all $m \in \mathbb{N}$ $f \circ \iota_m : \mathbb{C}^m \to \mathbb{C}$, $(z_1,\ldots,z_m) \mapsto f(z_1,\ldots,z_m,0,\ldots)$ is analytic, and*

(iii) $\displaystyle\sup_{m\in\mathbb{N}} \int_{\mathbb{C}^m} |f \circ \iota_m(z_1,\ldots,z_m)|^2 \exp(-\pi\|(z_1,\ldots,z_m)\|^2)\,dz_1\ldots dz_m < \infty.$

In this case, $\displaystyle\|f\|^2_{\mathcal{F}(\ell^2\mathbb{N})} = \lim_{m\in\mathbb{N}} \int_{\mathbb{C}^m} |f \circ \iota_m(z_1,\ldots,z_m)|^2 \exp(-\pi\|(z_1,\ldots,z_m)\|^2)\,dz_1\ldots dz_m.$

[39] The equivalence seems to be not have been noticed before.

Proof. Since the reproducing kernel $k : \ell^2\mathbb{N} \times \ell^2\mathbb{N} \to \mathbb{C},\ (z, w) \mapsto \exp(\pi\langle z|w\rangle_{\ell^2\mathbb{N}})$ is continuous, holomorphic in the first, anti-holomorphic in the second variable, we have by [Ne00, Proposition A.3.10] that $\mathcal{F}(\ell^2\mathbb{N}) \subset \mathcal{O}(\ell^2\mathbb{N})$, hence (i) and (ii). By [Ma88, Corollary II,3.10] one gets

$$\|f\|^2_{\mathcal{F}(\ell^2\mathbb{N})} = \lim_{m\in\mathbb{N}} \int_{\mathbb{C}^m} |f \circ \iota_m(z_1, \ldots, z_m)|^2 \exp(-\pi\|(z_1, \ldots, z_m)\|^2)\, dz_1 \ldots dz_m.$$

Concerning the converse: By [Ma88, Corollary II,3.11] we get $f \in \mathcal{F}(\ell^2\mathbb{N})$ and

$$\|f\|^2_{\mathcal{F}(\ell^2\mathbb{N})} = \sup_{m\in\mathbb{N}} \int_{\mathbb{C}^m} |f \circ \iota_m(z_1, \ldots, z_m)|^2 \exp(-\pi\|(z_1, \ldots, z_m)\|^2)\, dz_1 \ldots dz_m.$$

It remains to show that the sequence

$$c_m(f) := \int_{\mathbb{C}^m} |f \circ \iota_m(z_1, \ldots, z_m)|^2 \exp(-\pi\|(z_1, \ldots, z_m)\|^2)\, dz_1 \ldots dz_m$$

indexed by $m \in \mathbb{N}$ is monotonically increasing: For all $f \in \mathcal{F}(\ell^2\mathbb{N})$, $m \in \mathbb{N}$, and $\underline{z_m} \in \mathbb{C}^m$ let $f^{(\underline{z_m})}$ be the function $f^{(\underline{z_m})} : \mathbb{C} \to \mathbb{C},\ z \mapsto f(\underline{z_m}, z, 0, \ldots)$. Obviously, $f^{(\underline{z_m})} \in \mathcal{O}(\mathbb{C})$.

$$c_{m+1}(f) = \int_{\mathbb{C}^{m+1}} |f(z_{m+1})|^2\, e^{-\pi\|z_{m+1}\|^2}\, dz_{m+1} = \int_{\mathbb{C}^m} \int_{\mathbb{C}} |f(\underline{z_m}, u)|^2\, e^{-\pi\|u\|^2}\, du\, e^{-\pi\|\underline{z_m}\|^2}\, d\underline{z_m} < \infty,$$

hence by Fubini's theorem

$$\int_{\mathbb{C}} |f(\underline{z_m}, z, 0, \ldots)|^2\, e^{-\pi\|z\|^2}\, dz < \infty$$

for almost every $\underline{z_m}$. In other words: For almost all $\underline{z_m} \in \mathbb{C}^m$ the function $f^{(\underline{z_m})}$ belongs to $\mathcal{F}(\mathbb{C})$. Hence for such $\underline{z_m}$ we use the standard rkhs estimate A.4.4 to obtain

$$|f(\underline{z_m}, 0)|^2 = |f^{(\underline{z_m})}(0)|^2 \leq \|f^{(\underline{z_m})}\|^2_{\mathcal{F}(\mathbb{C})}\, e^0 = \int_{\mathbb{C}} |f(\underline{z_m}, u)|^2\, e^{-\pi\|u\|^2}\, du.$$

Integrating this estimate concludes the proof

$$c_m(f) = \int_{\mathbb{C}^m} |f(\underline{z_m}, 0)|^2\, e^{-\pi\|\underline{z_m}\|^2}\, d\underline{z_m} \leq \int_{\mathbb{C}^m} \int_{\mathbb{C}} |f(\underline{z_m}, u)|^2\, e^{-\pi\|u\|^2}\, du\, e^{-\pi\|\underline{z_m}\|^2}\, d\underline{z_m} = c_{m+1}(f).$$

□

The Fock space $\mathcal{F}(\ell^2\mathbb{N})$ is a Hilbert space with inner product explicitly given by

$$\langle f \,|\, g \rangle_{\mathcal{F}(\ell^2\mathbb{N})} = \lim_{m\to\infty} \langle C_{\iota_m} f \,|\, C_{\iota_m} g \rangle_{\mathcal{F}(\mathbb{C}^m)} = \lim_{m\to\infty} \int_{\mathbb{C}^m} f(\underline{z_m}, 0)\, \overline{g(\underline{z_m}, 0)}\, \exp(-\pi\|\underline{z_m}\|^2)\, d\underline{z_m},$$

where $C_{\iota_m} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\mathbb{C}^m),\ f \mapsto f \circ \iota_m$ and $\iota_m : \mathbb{C}^m \to \ell^2\mathbb{N},\ (z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_m, 0, \ldots)$. The characterisation of the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ by the previous Theorem A.4.8 will now be applied to prove an auxiliary result which is invoked by the integral representation of the leading eigenfunction of the Ruelle transfer operator in Corollary 2.6.11. Another main application of Theorem A.4.8 is the trace and trace norm formula B.4.3 for a certain class of composition operators acting on $\mathcal{F}(\ell^2\mathbb{N})$.

Proposition A.4.9. *Let Y be a topological space with a finite measure μ, $c \in \mathcal{C}_b(Y)$ and $z : Y \to \ell^2\mathbb{N}$ a bounded function. Then*

$$\Theta_z : \mathcal{C}_b(Y) \to \mathcal{F}(\ell^2\mathbb{N}),\ (T_z c)(w) := \int_Y c(y)\, e^{2\pi\langle w, z(y)\rangle}\, d\mu(y)$$

is a bounded operator.

Proof. We have to check that $f : \ell^2\mathbb{N} \to \mathbb{C},\ f(w) := \int_Y c(y)\, e^{2\pi\langle w|z(y)\rangle}\, d\mu(y)$ belongs to $\mathcal{F}(\ell^2\mathbb{N})$, then the linearity of Θ_z is obvious. Using Theorem A.4.8 we investigate whether

(i) $f : \ell^2\mathbb{N} \to \mathbb{C}$ is continuous,

(ii) For all $m \in \mathbb{N}$ $f \circ \iota_m : \mathbb{C}^m \to \mathbb{C},\ (z_1, \ldots, z_m) \mapsto f(z_1, \ldots, z_m, 0, \ldots)$ is analytic, and

(iii) $\sup_{m\in\mathbb{N}} \int_{\mathbb{C}^m} \Big| f \circ \iota_m(z_1, \ldots, z_m)\Big|^2 \exp(-\pi\|(z_1, \ldots, z_m)\|^2)\, dz_1 \ldots dz_m < \infty.$

Since the integral converges locally uniformly in $w \in Y$ and the estimate

$$|f(w)| \leq \mu(Y)\, \|c\|_{\mathcal{C}_b(Y)} \exp(2\pi\|w\| \sup_{y\in Y} \|z(y)\|),$$

the first two conditions are satisfied. Let $f_m := f \circ \iota_m$ for all m and $y_m := \rho_{\{1,\ldots,m\}}(z(y)) : \{1, \ldots, m\} \to \mathbb{C}$ be the ρ-restriction of $z(y)$ to the index set $\{1, \ldots, m\}$ as defined in Remark 1.1.3, then by Hölder's inequality (for $\frac{1}{p} + \frac{1}{q} = 1$)

$$\begin{aligned}
\int_{\mathbb{C}^m} |f_m(w)|^p e^{-\pi\|w\|^2} dw &= \int_{\mathbb{C}^m} \Big| \int_Y c(y) \exp(2\pi\langle w|y_m\rangle)\, d\mu(y)\Big|^p e^{-\pi\|w\|^2} dw \\
&\leq \|c\|^p_{\mathcal{C}_b(Y)} \int_{\mathbb{C}^m} \mu(Y)^{p/q} \int_Y |\exp(2\pi\langle w|y_m\rangle)|^p\, d\mu(y)\, e^{-\pi\|w\|^2} dw \\
&= \|c\|^p_{\mathcal{C}_b(Y)}\, \mu(K)^{p/q} \int_Y \int_{\mathbb{C}^m} |\exp(2\pi\langle w|y_m\rangle)|^p e^{-\pi\|w\|^2} dw\, d\mu(y) \\
&\overset{A.5.1}{=} \|c\|^p_{\mathcal{C}_b(Y)}\, \mu(Y)^{p/q} \int_Y e^{p^2\pi\|y_m\|^2} d\mu(y) \\
&\leq \|c\|^p_{\mathcal{C}_b(Y)}\, \mu(Y)^p \sup_{y\in Y} e^{p^2\pi\|y_m\|^2} \\
&\leq \|c\|^p_{\mathcal{C}_b(Y)}\, \mu(Y)^p \exp(p^2\pi \sup_{y\in Y} \|z(y)\|^2).
\end{aligned}$$

In particular, for $p = 2$ one has $\|\Theta_z c\|_{\mathcal{F}(\ell^2\mathbb{N})} \leq \|c\|_{\mathcal{C}_b(Y)}\, \mu(Y) \exp(2\pi \sup_{y\in Y} \|z(y)\|^2)$. □

A.5 Gaussians

In this section we collect some elementary properties of Gaussian integrals. For instance we give a proof of an identity used in the previous proposition. Let $(\cdot|\cdot)$ be the usual scalar product on $\mathbb{R}^n$, respectively its $\mathbb{C}$-bilinear extension to $\mathbb{C}^n$. We denote by dw both Lebesgue measure on $\mathbb{C}^n$ and on $\mathbb{R}^n$.

Proposition A.5.1. *Let $\alpha \in -i\mathfrak{S}_n$ (5.2.1), i. e., $\alpha \in \mathrm{Sym}(n;\mathbb{C})$ with positive definite real part, $\mathbb{B} \in \mathrm{Mat}(n,n;\mathbb{C})$ with $\|B\| < 1$, $0 \leq p < \infty$, and z_0, $w_0 \in \mathbb{C}^n$. Then*

(i) $\int_{\mathbb{R}^n} e^{-\pi(x|\alpha x) - 2\pi i(z_0|x)}\, dx = (\det\alpha)^{-\frac{1}{2}}\, e^{-\pi(z_0|\alpha^{-1}z_0)}.$

(ii) $\int_{\mathbb{C}^n} \left|e^{2\pi\langle w|z_0\rangle}\right|^p e^{-\pi\|w\|^2} dw = \int_{\mathbb{C}^n} \left|e^{2\pi(w|z_0)}\right|^p e^{-\pi\|w\|^2} dw = e^{\pi p^2\|z_0\|^2}.$

(iii) $\int_{\mathbb{C}^n} \left|e^{\pi\langle w|z_0\rangle}\right|^p e^{\pi\|w_0 + \mathbb{B}w\|^2 - \pi\|w\|^2} dw = \dfrac{\exp\Big(\pi\|w_0\|^2 + \pi\|(1-\mathbb{B}^\star\mathbb{B})^{-1/2}(\frac{p}{2}z_0 + \mathbb{B}^\star w_0)\|^2\Big)}{\det(1-\mathbb{B}^\star\mathbb{B})}.$

Proof. The first assertion is proved for instance in [Fo89, Appendix A]. - Note that $k(z,w) = e^{\pi\langle z|w\rangle}$ is the reproducing kernel of the Fock space $\mathcal{F}(\mathbb{C}^n)$, hence by Proposition A.4.3

$$\int_{\mathbb{C}^n} \left|e^{\pi\frac{2}{p}\langle w|z_0\rangle}\right|^p e^{-\pi\|w\|^2} dw = \int_{\mathbb{C}^n} e^{\pi\langle w|z_0\rangle} e^{\pi\langle z_0|w\rangle}\, e^{-\pi\|w\|^2} dw = e^{\pi\langle z_0|z_0\rangle}.$$

We transform the third assertion in such a way that we can apply the second one:

$$\begin{aligned}
\int_{\mathbb{C}^n} \left|e^{\pi\langle w|z_0\rangle}\right|^p e^{\pi\|w_0+\mathbb{B}w\|^2-\pi\|w\|^2}dw &= e^{\pi\|w_0\|^2}\int_{\mathbb{C}^n} \exp\Big(\pi\mathbf{Re}(\langle w|pz_0+2\mathbb{B}^\star w_0\rangle)+\pi\|\mathbb{B}w\|^2-\pi\|w\|^2\Big)dw \\
&= e^{\pi\|w_0\|^2}\int_{\mathbb{C}^n} \big|\exp\Big(\pi\langle w|\frac{p}{2}z_0+\mathbb{B}^\star w_0\rangle\Big)\big|^2 \exp(-\pi\langle w|(1-\mathbb{B}^\star\mathbb{B})w\rangle)dw \\
&\overset{u=\sqrt{1-\mathbb{B}^\star\mathbb{B}}w}{=} e^{\pi\|w_0\|^2}\int_{\mathbb{C}^n} \big|\exp\Big(\pi\langle(1-\mathbb{B}^\star\mathbb{B})^{-1/2}u|\frac{p}{2}z_0+\mathbb{B}^\star w_0\rangle\Big)\big|^2 \exp(-\pi\|u\|^2)\det(1-\mathbb{B}^\star\mathbb{B})^{-1}du \\
&= \det(1-\mathbb{B}^\star\mathbb{B})^{-1}e^{\pi\|w_0\|^2}\int_{\mathbb{C}^n} \big|\exp\Big(\pi\langle u|(1-\mathbb{B}^\star\mathbb{B})^{-1/2}(\frac{p}{2}z_0+\mathbb{B}^\star w_0)\rangle\Big)\big|^2\exp(-\pi\|u\|^2)du \\
&\overset{A.5.1(\mathrm{ii})}{=} \det(1-\mathbb{B}^\star\mathbb{B})^{-1}e^{\pi\|w_0\|^2}\exp\Big(\pi\|(1-\mathbb{B}^\star\mathbb{B})^{-1/2}(\frac{p}{2}z_0+\mathbb{B}^\star w_0)\|^2\Big)
\end{aligned}$$

□

A.6 Integral Operators

In Section A.4 we have introduced reproducing kernel Hilbert spaces. We will now use the additional structure provided by the reproducing kernel to investigate operators acting on an rkhs. The main idea is that every bounded linear operator on an rkhs can be written as an integral operator which can be nicely analysed via the properties of its kernel. Then we study scaling, i.e. the unitary isomorphism induced by change of coordinates. - The following remark is based on [Fo89, p. 42 f.] where only the Fock space is concerned. The arguments of the proof do not rely on the specific rkhs.

Remark A.6.1. Let T be a bounded linear operator on a reproducing kernel Hilbert space of holomorphic functions. Let $k(x,y)=k_y(x)$ be its reproducing kernel. Then T acts via

$$(Tf)(x)=\langle Tf|k_x\rangle=\langle f|T^*k_x\rangle.$$

If $\mathcal{H}\subset L^2(M,dm)$ is a Hilbert space with reproducing kernel, then T is an integral operator with kernel

$$k_T(x,y)=\overline{(T^*k_x)(y)}=\overline{\langle T^*k_x|k_y\rangle}=\langle k_y|T^*k_x\rangle=(Tk_y)(x).$$

If $\mathcal{H}\subset L^2(M,dm)\cap\mathcal{O}(M)$ is an rkhs consisting of holomorphic functions, then T is uniquely determined by the values $k_T(x,x)$ of its integral kernel along the diagonal. Since

$$k_{T^\star}(x,y)=(T^\star k_y)(x)=\langle T^\star k_y|k_x\rangle=\langle k_y|Tk_x\rangle=\overline{k_T(y,x)},$$

the operator T is selfadjoint if and only if the kernel $k_T(x,x)$ is real-valued on the diagonal. □

Given a composition operator T (B.1.1), then it is in general quite difficult to determine TT^*, T^*T, and its positive part $|T|:=\sqrt{T^*T}$. In particular the trace of $|T|$ is of interest, since it coincides with the trace norm (A.2.2) of T. In a reproducing kernel Hilbert space (A.4.1), however, the computations can be carried out quite easily just by using the reproducing kernel property. The following formulas will be used in Proposition B.3.10 and finally in Theorem B.4.3 for a special class of composition operators acting on the Fock space.

Proposition A.6.2. *Let $\mathcal{H}\subset L^2(M,dm)$ be an rkhs with reproducing kernel $k:M\times M\to\mathbb{C}$ (A.4.1). Let $\psi:M\to M$ and $\phi:M\to\mathbb{C}$ be fixed functions such that the composition operator $T\in\mathrm{End}(\mathcal{H})$ defined via $(Tf)(z):=\phi(z)\,f(\psi(z))$ is bounded. Let $T^\star\in\mathrm{End}(\mathcal{H})$ be the Hilbert space adjoint of T. Then the integral kernels of T, T^*, $TT^\star$, and $T^\star T$ are*

(i) $k_T(v,w)=\phi(v)\,k(\psi(v),w)$,

(ii) $k_{T^\star}(v,w)=\overline{\phi(w)}\,k(v,\psi(w))$,

(iii) $k_{TT^\star}(v,w)=\phi(v)\,\overline{\phi(w)}\,k(\psi(v),\psi(w))$, *and*

(iv) $k_{T^\star T}(v,w)=\displaystyle\int_M|\phi(u)|^2\,k(v,\psi(u))\,k(\psi(u),w)\,dm(u)$.

Proof. By Remark A.6.1 the operator T has the integral kernel $k_T(v,w) = (Tk_w)(v) = \phi(v)\,k(\psi(v),w)$. Hence the adjoint $T^\star : \mathcal{H} \to \mathcal{H}$ of T has the integral kernel

$$k_{T^\star}(v,w) = \overline{k_T(w,v)} = \overline{\phi(w)}\,k(v,\psi(w)) = \overline{\phi(w)}\,k_{\psi(w)}(v).$$

Using the properties of a reproducing kernel, see Proposition A.4.3, we compute the integral kernel of $TT^\star : \mathcal{H} \to \mathcal{H}$ as

$$\begin{aligned}
k_{TT^\star}(v,w) &= \int_M k_T(v,u)\,k_{T^\star}(u,w)\,dm(u) \\
&= \int_M \phi(v)\,k(\psi(v),u)\,\overline{\phi(w)}\,k(u,\psi(w))\,dm(u) \\
&= \phi(v)\,\overline{\phi(w)}\int_M \overline{k_{\psi(v)}(u)}\,k_{\psi(w)}(u)\,dm(u) \\
&= \phi(v)\,\overline{\phi(w)}\,\langle k_{\psi(w)}|k_{\psi(v)}\rangle_{\mathcal{H}} \\
&= \phi(v)\,\overline{\phi(w)}\,k(\psi(v),\psi(w)).
\end{aligned}$$

The operator $T^\star T : \mathcal{H} \to \mathcal{H}$ has the integral kernel

$$\begin{aligned}
k_{T^\star T}(v,w) &= \int_M k_{T^\star}(v,u)\,k_T(u,w)\,dm(u) \\
&= \int_M \overline{\phi(u)}\,k(v,\psi(u))\phi(u)\,k(\psi(u),w)\,dm(u) \\
&= \int_M |\phi(u)|^2\,k(v,\psi(u))\,k(\psi(u),w)\,dm(u),
\end{aligned}$$

which in general cannot be simplified further. □

In Chapter 5 we use the notion of scaling. By this we mean the unitary transform which is induced by a linear change of coordinates.

Lemma A.6.3. *Let $c \in \mathrm{Gl}(n;\mathbb{R})$. Then*

(i) $R_c \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $(R_c f)(x) := \sqrt{|\det c|}\,f(cx)$ *defines a unitary isomorphism.*

(ii) *Let* $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $(Kf)(\xi) = \int_{\mathbb{R}^n} K(\xi,\eta)\,f(\eta)\,d\eta$ *be an integral operator, then the induced operator* $K_c := R_c \circ K \circ R_c^{-1}$ *on* $L^2(\mathbb{R}^n, dx)$ *is given by the kernel*

$$K_c(x,y) = |\det(c)|\,K(cx,cy).$$

Proof. The first part is a reformulation of the change of variables. The second part follows from the calculation

$$\begin{aligned}
\left((R_c \circ K \circ R_c^{-1}(f)\right)(x) &= \sqrt{|\det c|}\,\left(K(R_c^{-1}f)\right)(cx) \\
&= \sqrt{|\det c|}\int_{\mathbb{R}^n} K(cx,\eta)(R_c^{-1}f)(\eta)\,d\eta \\
&= \sqrt{|\det c|}\int_{\mathbb{R}^n} K(cx,\eta)\frac{1}{\sqrt{|\det c|}}f(c^{-1}\eta)\,d\eta \\
&\overset{\eta=cy}{=} \int_{\mathbb{R}^n} K(cx,cy)f(y)\,|\det c|\,dy.
\end{aligned}$$

□

A.7 A trace formula

In this section we will give a proof the following folklore result which we use as an essential tool for our proof of the dynamical trace formula (Theorems 2.7.6 and 2.13.8): Let $(\mathcal{L}_y)_{y\in Y}$ be a measurable family of trace class operators on a separable Hilbert space $\mathcal{H}$ with the property that $\int_Y \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}\, dy$ is finite, i.e. that $\|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}$ is integrable over Y. Let $\mathcal{L}$ be the linear operator on $\mathcal{H}$ defined as

$$\mathcal{L}f = \int_Y \mathcal{L}_y f\, dy.$$

Then Theorem A.7.6 states that $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is a trace class operator with

$$\operatorname{trace} \mathcal{L} = \int_Y \operatorname{trace} \mathcal{L}_y\, dy.$$

This theorem is useful to a great extent: It both provides a criterion if a given operator is trace class and a recipe to compute its trace. In Section 3.3 we formulate and prove a similar looking theorem for operators acting on tensor products (by using quite different techniques).
If Y is a finite set, then the assertion is nothing but the linearity of the trace. In general, the problem is more subtle, since there are many different notions of measurability. The stated theorem is a direct consequence of Bochner integration theory, if we require measurability of the function $Y \to \mathbb{R},\ y \mapsto \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}$. Fortunately, many notions of measurability coincide, see [DeFl93, B11]. We will give a direct proof. Our strategy to show the announced theorem is to

(i) prove a similar trace formula for operators of finite rank,

(ii) show the traceability of $\mathcal{L}$, and

(iii) compute the trace by a limit of traces of operators of finite rank.

One of the tools will be Lebegue's theorem on the dominated convergence, which can be stated as follows.

Theorem A.7.1. *(Dominated convergence theorem) Let be $(Y, \mathcal{Y}, dy)$ be a measure space. Let $f_n : Y \to \mathbb{C}$ be a sequence of integrable functions which is dominated by an integrable function g, i. e., $|f_n| \leq g$ almost everywhere. Suppose that $f_n \to f$ almost everywhere, then $f_n \to f$ in norm and $\int f_n \to \int f$.* □

We will need a vector-valued version of Lebegue's theorem. Let be $(Y, \mathcal{Y}, dy)$ be a measure space and $\mathcal{H}$ be a separable Hilbert space. A function $f : Y \to \mathcal{H}$ is called *integrable* if the scalar-valued function $y \mapsto \langle f(y) \,|\, h \rangle$ is integrable for all $h \in \mathcal{H}$. Then we define $\int_Y f(y)\, dy \in \mathcal{H}$ to be the unique vector which satisfies

$$\left\langle \textstyle\int_Y f(y)\, dy \,\middle|\, h \right\rangle = \int_Y \langle f(y) \,|\, h \rangle\, dy \tag{112}$$

for all $h \in \mathcal{H}$.

Theorem A.7.2. *(Dominated convergence theorem) Let be $(Y, \mathcal{Y}, dy)$ be a measure space and $\mathcal{H}$ be a separable Hilbert space. Let $f_n : Y \to \mathcal{H}$ a sequence of integrable functions which is dominated by an integrable function $g : y \to \mathbb{C}$, i. e., $\|f_n\| \leq g$ almost everywhere. Suppose that $f_n \to f$ almost everywhere, then $f_n \to f$ in norm and $\int f_n \to \int f$.*

Proof. Using Theorem A.7.1 one easily shows that

$$\lim_{n\to\infty} \int_Y \langle f_n(y) \,|\, h \rangle\, dy = \int_Y \langle f(y) \,|\, h \rangle\, dy$$

for all $h \in \mathcal{H}$. The Riesz representation theorem implies that $\lim_{n\to\infty} \int f_n = \int f$. □

A family $(\mathcal{L}_y)_{y\in Y}$ of bounded operators on a separable Hilbert space $\mathcal{H}$ is called *measurable* if for all $f, g \in \mathcal{H}$ the function $Y \to \mathbb{C},\ y \mapsto \langle \mathcal{L}_y f \,|\, g \rangle$ is measurable.

Proposition A.7.3. *Let $(\mathcal{L}_y)_{y\in Y}$ be a measurable family of bounded operators on a separable Hilbert space $\mathcal{H}$ with the property that $\int_Y \|\mathcal{L}_y\|\, dy < \infty$.*

(i) *Then there is a unique bounded operator on* $\mathcal{H}$ *denoted by* $\mathcal{L} = \int_Y \mathcal{L}_y \, dy$ *which satisfies*

$$\langle \mathcal{L}f \,|\, g \rangle := \int_Y \langle \mathcal{L}_y f \,|\, g \rangle \, dy$$

for all $f,\, g \in \mathcal{H}$.

(ii) *Let* $(h_n)_{n\in\mathbb{N}}$ *be a Hilbert basis of* $\mathcal{H}$. *For* $n \in \mathbb{N}$ *define* $P_n = \sum_{i=1}^{n} \langle \cdot \,|\, h_i \rangle \, h_i : \mathcal{H} \to \mathcal{H}$ *to be the orthogonal projection onto the* n*-dimensional subspace* $\operatorname{span}\{h_1, \ldots, h_n\} \subset \mathcal{H}$. *Then*

$$\mathcal{L}^{(n)} := \int_Y P_n \mathcal{L}_y P_n \, dy = P_n \mathcal{L} P_n$$

and

$$\operatorname{trace} \mathcal{L}^{(n)} = \int_Y \operatorname{trace} \mathcal{L}_y^{(n)} \, dy$$

Proof. The first part will be a consequence of the Lax-Milgram lemma. For this we have to confirm that

$$B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \; B(f, g) := \int_Y \langle \mathcal{L}_y f \,|\, g \rangle \, dy$$

defines a continuous sesquilinear form. This is obvious. Let $f,\, g \in \mathcal{H}$, then

$$\langle P_n \mathcal{L} P_n f \,|\, g \rangle = \langle \mathcal{L} P_n f \,|\, P_n^* g \rangle = \int_Y \langle \mathcal{L}_y P_n f \,|\, P_n^* g \rangle \, dy = \int_Y \left\langle \mathcal{L}_y^{(n)} f \,\middle|\, g \right\rangle dy$$

hence $P_n \mathcal{L} P_n = \int_Y \mathcal{L}_y^{(n)} \, dy$, which is a finite rank operator. We easily compute the trace by using the orthonormal basis $(h_n)_{n\in\mathbb{N}}$ observing that all sums are indeed finite sums:

$$\begin{aligned} \operatorname{trace} \mathcal{L}^{(n)} &= \sum_{m=1}^{\infty} \langle \mathcal{L}^{(n)} h_m \,|\, h_m \rangle \\ &= \sum_{m=1}^{n} \int_Y \langle \mathcal{L}_y h_m \,|\, h_m \rangle \, dy \\ &= \int_Y \sum_{m=1}^{n} \langle \mathcal{L}_y h_m \,|\, h_m \rangle \, dy = \int_Y \operatorname{trace} \mathcal{L}_y^{(n)} \, dy. \end{aligned}$$

□

The last assertion of Proposition A.7.3 states that the desired trace formula holds for the finite rank approximations $\mathcal{L}^{(n)}$ of $\mathcal{L}$. We now will show that $\mathcal{L} = \int_Y \mathcal{L}_y \, dy$ is a trace class operator under additional assumptions on the coefficients $(\mathcal{L}_y)_{y\in Y}$. This will be done using the following result of K. Fan.

Lemma A.7.4. *Let* A *be a compact operator on a Hilbert space* $\mathcal{H}$. *The sequence* $(s_i(A))_{i\in\mathbb{N}}$ *of singular numbers of* A *can be characterised as follows: For any* $n = 1, 2, \ldots (\leq \dim \mathcal{H})$ *one has*

$$\max \left| \sum_{i=1}^{n} \langle U A \phi_i \,|\, \phi_i \rangle \right| = \sum_{i=1}^{n} s_i(A),$$

where the maximum is taken over all unitary operators U *and orthonormal systems* $\phi_1, \ldots, \phi_n$.

Proof. See for instance [GoGoKr00, Thm. IV 3.5]. □

Proposition A.7.5. *Let* $(\mathcal{L}_y)_{y\in Y}$ *be a family of trace class operators on a separable Hilbert space* $\mathcal{H}$ *such that* $\int_Y \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})} \, dy < \infty$. *Then the linear operator* $\mathcal{L} : \mathcal{H} \to \mathcal{H}$, $\mathcal{L}f = \int_Y \mathcal{L}_y f \, dy$ *is a trace class operator with*

$$\|\mathcal{L}\|_{\mathcal{S}_1(\mathcal{H})} \leq \int_Y \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})} \, dy.$$

Proof. For all $f,\, g \in \mathcal{H}$ and an arbitrary unitary operator U on $\mathcal{H}$ one has

$$\big\langle U\mathcal{L}f \,\big|\, g\big\rangle = \big\langle \mathcal{L}f \,\big|\, U^\star g\big\rangle = \int_Y \big\langle \mathcal{L}_y f \,\big|\, U^\star g\big\rangle\, dy = \int_Y \big\langle U\mathcal{L}_y f \,\big|\, g\big\rangle\, dy.$$

Let $(v_j)_j$ be a Hilbert basis in $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ be unitary. Then standard estimates yield

$$\begin{aligned}
\Big|\sum_{j=1}^n \big\langle U\mathcal{L}v_j \,\big|\, v_j\big\rangle\Big| &= \Big|\sum_{j=1}^n \int_Y \big\langle U\mathcal{L}_y v_j \,\big|\, v_j\big\rangle\, dy\Big| \\
&\le \int_Y \sum_{j=1}^n \big|\big\langle U\mathcal{L}_y v_j \,\big|\, v_j\big\rangle\big|\, dy \\
&\le \int_Y \sum_{j=1}^n s_j(\mathcal{L}_y)\, dy \\
&\le \int_Y \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}\, dy < \infty.
\end{aligned}$$

Hence we have shown that for all $n \in \mathbb{N}$

$$\sum_{j=1}^n s_j(\mathcal{L}) = \max\Big|\sum_{j=1}^n \big\langle U\mathcal{L}v_j \,\big|\, v_j\big\rangle\Big| \le \int_Y \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}\, dy < \infty$$

and thus $\|\mathcal{L}\|_{\mathcal{S}_1(\mathcal{H})} = \sum_{j=1}^\infty s_j(\mathcal{L}) \le \int_Y \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}\, dy < \infty$ and $\mathcal{L}$ is a trace class operator. □

Let $\mathcal{L}f = \int_Y \mathcal{L}_y f\, dy$ be an operator satisfying the hypotheses of Proposition A.7.5. Let $P_n : \mathcal{H} \to \mathcal{H}$ be the sequence of orthogonal projections from Proposition A.7.3 (ii). Obviously, P_n tends to the identity on $\mathcal{H}$ as n tends to infinity, hence

$$\mathcal{L} = \lim_{n\to\infty} P_n \mathcal{L} P_n = \lim_{n\to\infty} \mathcal{L}^{(n)}$$

pointwisely. Since $\mathcal{L}$ is trace class, [GoGoKr00, Thm. IV 5.5] shows that the pointwise convergence is indeed in trace norm which gives a recipe to compute the trace of Λ, namely

$$\operatorname{trace} \mathcal{L} = \lim_{n\to\infty} \operatorname{trace} \mathcal{L}^{(n)} = \lim_{n\to\infty} \operatorname{trace} P_n \mathcal{L} P_n.$$

Similarly one proceeds for $\mathcal{L}_y$. By Proposition A.7.3 (ii) the trace formula holds for the finite rank approximations $P_n \mathcal{L} P$. Hence the following theorem will be a consequence of the dominated convergence theorem A.7.1.

Theorem A.7.6. *Let $(\mathcal{L}_y)_{y\in Y}$ be a family of trace class operators on a separable Hilbert space $\mathcal{H}$ such that $\int_Y \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}\, dy < \infty$. Then the linear operator $\mathcal{L} : \mathcal{H} \to \mathcal{H},\ \mathcal{L}f = \int_Y \mathcal{L}_y f\, dy$ is a trace class operator with*

$$\operatorname{trace} \mathcal{L} = \int_Y \operatorname{trace} \mathcal{L}_y\, dy.$$

Proof. By the previous considerations one has

$$\operatorname{trace} \mathcal{L} = \lim_{n\to\infty} \operatorname{trace} \mathcal{L}^{(n)} \overset{A.7.3}{=} \lim_{n\to\infty} \int_Y \operatorname{trace} \mathcal{L}_y^{(n)}\, dy$$

and

$$\int_Y \operatorname{trace} \mathcal{L}_y\, dy = \int_Y \operatorname{trace} \lim_{n\to\infty} \mathcal{L}_y^{(n)}\, dy = \int_Y \lim_{n\to\infty} \operatorname{trace} \mathcal{L}_y^{(n)}\, dy.$$

We will use the dominated convergence theorem A.7.1 in order to show that both expressions are equal.

The operators $\mathcal{L}_y^{(n)}$ converge to $\mathcal{L}_y$ in trace class norm, hence the integrands converge pointwisely. The standard estimate $|\operatorname{trace} \mathcal{L}_y^{(n)}| \le \|\mathcal{L}_y\|_{\mathcal{S}_1(\mathcal{H})}$ gives by assumption an integrable majorant, thus the assertion follows from A.7.1. □

We use Theorem A.7.6 to give a proof of Lemma 2.4.1 which we used for the proof of the abstract dynamical trace formulas formulated in Theorems 2.4.4 and 2.4.6.

Corollary A.7.7. *Let ν be a Borel measure on F and $(T_x)_{x\in F}$ a measurable family of trace class operators on a Hilbert space $\mathcal{H}$ with $\int_F \|T_x\|_{\mathcal{S}_1(\mathcal{H})}\,d\nu(x) < \infty$. Then $T:\mathcal{H}\to\mathcal{H},\ Tg := \int_F T_x g\,d\nu(x)$ is a trace class operator with*

$$T^n f = \int_{F^n} T_{x_n}\circ\ldots\circ T_{x_1} f\,d\nu(x_1)\ldots d\nu(x_n)$$

and

$$\text{trace}\, T^n = \int_{F^n} \text{trace}\,(T_{x_n}\circ\ldots\circ T_{x_1})\,d\nu(x_1)\ldots d\nu(x_n).$$

Proof. It remains to prove the formula for T^n. The basis of the induction ($n=1$) is trivial. The induction step is a consequence of the vector-valued integration, see Prop. A.7.3 (i),

$$T^{n+1} f = \int_F T_x(T^n f)\,d\nu(x) = \int_F\int_{F^n} T_x\circ T_{x_n}\circ\ldots\circ T_{x_1} f\,d\nu(x_1)\ldots d\nu(x_n)\,d\nu(x)$$

By Theorem A.7.6 the second assertion is an immediate consequence. □

B Composition operators

In many branches of mathematics one encounters (generalised) composition operators. These are linear operators defined on function spaces which act by composing with a fixed self-map ψ of the base space and multiplying the result by another complex-valued fixed function ϕ, i. e.,

$$(Tf)(z) = \phi(z)\,(f \circ \psi)(z).$$

Our interest is founded in the Ruelle transfer operator (2.1.3) and the Ruelle-Mayer transfer operator (2.3.7, 2.6.15) which are integrals over families of composition operators.
Composition operators are widely studied in the literature. One is interested in understanding the spectrum of the operator depending on the geometric properties of the self-map ψ and the function space on which the operator acts. In this chapter we will concentrate on trace class composition operators and the question on which function spaces the Atiyah-Bott fixed point formula

$$\text{trace } T = \frac{\phi(z^*)}{\det(1 - \psi'(z^*))}$$

holds. In Section B.1 we will deal with algebraic properties of composition operators and prove a trace formula for a class of degenerate composition operators. Section B.2 recalls the classical formulation of the Atiyah-Bott fixed point formula.
As we have explained in the introduction chapter, our strategy of proving the dynamical trace formula (Theorem 2.13.8) requires a Hilbert space which is invariant under the composition operator and has the property that the operator is trace class on it and satisfies the Atiyah-Bott formula. It turns out that such a Hilbert space possesses a reproducing kernel. In the following we will first investigate analogues of the Atiyah-Bott fixed point formula on function spaces over finite-dimensional domains, and then discuss the infinite-dimensional case in Section B.4. In particular we prove that the special class of composition operators acting on the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ via

$$(Tf)(z) = e^{\langle z|a\rangle}\, f(\mathbb{B}z + b)$$

for some fixed a, $b \in \ell^2\mathbb{N}$ and $\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}\| < 1$, is trace class and satisfies the Atiyah-Bott fixed point formula which is a key ingredient of our proof of dynamical trace formula Theorem 2.7.6.

B.1 Definition and elementary properties

In this section we recall the definition of a (generalised) composition operator. We show that composing two generalised composition operators gives again a composition operator, we compute the n-th (mixed) iterate of a composition operator, and prove a trace formula for a class of degenerate composition operators.

Definition B.1.1. Let E be a set and V a space of complex valued functions on E. A *(generalised) (or weighted) composition operator* is an operator $T : V \to V$ of the form

$$(Tf)(z) = \phi(z)\,(f \circ \psi)(z),$$

where $\phi : E \to \mathbb{C}$, $\psi : E \to E$ are fixed functions. If the multiplication part is trivial, i. e., $\phi \equiv 1$, then T is called a *(classical) composition operator.* □

Given a family of generalised composition operators acting on the same space one can form their product. It turns out that it is again a composition operator.

Lemma B.1.2. *Let E, F be non-empty sets. Let $\phi_x : E \to \mathbb{C}$, $\psi_x : E \to E$ for each $x \in F$. Let $T_x : \mathbb{C}^E \to \mathbb{C}^E$ be defined via $(T_x f)(z) = \phi_x(z)\,(f \circ \psi_x)(z)$. Then*

$$(T_{x_n} \circ \ldots \circ T_{x_1} f)(z) = \prod_{k=1}^{n} (\phi_{x_k} \circ \psi_{x_{k+1}} \circ \ldots \circ \psi_{x_n})(z)\,(f \circ \psi_{x_1} \circ \ldots \circ \psi_{x_n})(z).$$

Proof. Induction: $n = 1$ ✓; $n \to n+1$:

$$\begin{aligned} T_x(T_{x_n} \circ \ldots \circ T_{x_1} f)(z) &= \phi_x(z)\,(T_{x_n} \circ \ldots \circ T_{x_1} f)(\psi_x z) \\ &= \phi_x(z)\,\phi_{x_n}(\psi_x z)\,(\phi_{x_{n-1}} \circ \psi_{x_n})(\psi_x z)\,\ldots(\phi_{x_1} \circ \psi_{x_2} \circ \ldots \circ \psi_{x_n})(\psi_x z)\,(f \circ \psi_{x_1} \circ \ldots \circ \psi_{x_n})(\psi_x z). \end{aligned}$$

□

We call an expression of the form $T_{x_n} \circ \ldots \circ T_{x_1}$ which appeared in the previous lemma an n-th mixed iterate. We frequently use Lemma B.1.2 for Ruelle and Ruelle-Mayer transfer operators where $\phi_x = \exp(A_x)$. Then

Corollary B.1.3. *Let E, F be non-empty sets. Let $A_x : E \to \mathbb{C}$, $\psi_x : E \to E$ for each $x \in F$. Let $T_x : \mathbb{C}^E \to \mathbb{C}^E$ be defined via $(T_x f)(z) = \exp(A_x(z))\,(f \circ \psi_x)(z)$. Then*

$$(T_{x_n} \circ \ldots \circ T_{x_1} f)(z) = \exp\Big(\sum_{k=1}^{n} (A_{x_k} \circ \psi_{x_{k+1}} \circ \ldots \circ \psi_{x_n})(z)\Big)\,(f \circ \psi_{x_1} \circ \ldots \circ \psi_{x_n})(z).$$

□

A rather degenerate case of a composition operator is the following, where the composition part is constant. Such a (generalised) composition operator is nuclear and satisfies a simple version of the Atiyah-Bott fixed point formula. In this dissertation one encounters these degenerate composition operators in Sections 2.5 and 2.8: The high powers of the Ruelle-Mayer transfer operator for finite range interactions are integrals over families of degenerate composition operators.

Corollary B.1.4. *Let E be a topological space and $V \subset \mathcal{C}(E)$ a Banach space of continuous complex valued functions on E. Let $\phi \in V$ and $z_0 \in E$ be fixed, and $T : V \to V$ be the composition operator $(Tf)(z) = \phi(z)\, f(z_0)$. Then T is nuclear of order zero (A.3.1) with* trace $T = \phi(z_0)$.

Proof. The range of T is the span of the vector ϕ, hence T is a finite rank operator and thus nuclear (of any order). From $T\phi = \phi(z_0)\,\phi$ we can read off the only eigenvalue $\phi(z_0)$ and thus the trace. The nuclear norm of T is (at most) the norm of ϕ. □

Combining the previous Lemma B.1.2 and Corollary B.1.4 we obtain the following simple trace formula.

Corollary B.1.5. *Let E be a topological space and $V \subset \mathcal{C}(E)$ a Banach space of continuous complex valued functions on E. Let F be a non-empty set. Let $\phi_x : E \to \mathbb{C}$, $\psi_x : E \to E$ for each $x \in F$. Let $T_x : V \to V$ defined via $(T_x f)(z) = \phi_x(z)\,(f \circ \psi_x)(z)$. Suppose there exists $n \in \mathbb{N}$ such that for all choices $x_1, \ldots, x_n \in F$ the map $\psi_{x_1} \circ \ldots \circ \psi_{x_n} : E \to E$ is constant. Then $T_{x_n} \circ \ldots \circ T_{x_1}$ is a nuclear operator with*

$$\operatorname{trace} T_{x_n} \circ \ldots \circ T_{x_1} = \Phi_{x_1,\ldots,x_n}(z_{x_1,\ldots,x_n}),$$

where

$$\Phi_{x_1,\ldots,x_n} : E \to \mathbb{C},\ \ z \mapsto \prod_{k=1}^{n} (\phi_{x_k} \circ \psi_{x_{k+1}} \circ \ldots \circ \psi_{x_n})(z)$$

with $\|\Phi_{x_1,\ldots,x_n}\| \leq (\sup_{x\in F} \|\phi_x\|)^n$ and where $z_{x_1,\ldots,x_n}$ is the (constant) value of $\psi_{x_1} \circ \ldots \circ \psi_{x_n}$. □

B.2 The Atiyah-Bott type fixed point formula

The following sections are devoted to various versions of the following Atiyah-Bott type trace formula: Given a contraction $\psi : U \to U$ (we will specify the contraction later), a fixed function $\phi : U \to \mathbb{C}$, let

$$(Tf)(z) := \phi(z)\,(f \circ \psi)(z)$$

be the generalised composition operator (B.1.1) associated with ψ and ϕ. We will investigate under which circumstances (smoothness of the maps ϕ and ψ, contraction property of ψ, suitable function space) the operator T is a trace class operator (nuclear operator (A.3.1), respectively) such that the following trace formula holds

$$\operatorname{trace} T = \frac{\phi(z^*)}{\det(1 - \psi'(z^*))}, \tag{113}$$

where $z^* \in U$ is the unique fixed point of ψ. Note that even the trivial examples of Corollary B.1.4 and B.1.5 fit into this scheme. More precisely the above formula is the holomorphic version of the fixed point formula of Atiyah and Bott. We will state it as Theorem B.2.4. We use the following notion of holomorphicity.

Definition B.2.1. Let $U \subset X$ be an open domain in some (possibly infinite dimensional) Banach space X. A function $f : U \to \mathbb{C}$ is called *holomorphic* if $f \in \mathcal{C}(U)$ and for each finite-dimensional affine subspace $A \subset X$ the function $f|_{A \cap U}$ is holomorphic. The space of all holomorphic functions with compact open topology is denoted by $\mathcal{O}(U)$. □

We are interested in working with composition operators on Banach or, even better, Hilbert spaces. Hence we recall the definitions of the following spaces consisting of holomorphic functions.

Remark B.2.2. Let $U \subset X$ be a bounded open domain in a complex Banach space X.

(i) $\mathcal{O}^b(U) := \mathcal{C}^b(U) \cap \mathcal{O}(U)$ is a Banach space[40] (w. r. t. the supremum norm).

In fact: Let $f_n \in \mathcal{O}^b(U)$ a sequence converging uniformly to some f. Since $\mathcal{C}^b(U)$ is complete, the limit function f belongs to $\mathcal{C}^b(U)$. For any finite-dimensional affine subspace $A \subset X$ the restrictions $f_n|_{A\cap U}$ converge uniformly to $f|_{A\cap U}$, hence $f|_{A\cap U}$ is holomorphic, hence $f \in \mathcal{O}(U)$.

(ii) $A^\infty(U) := \mathcal{C}(\overline{U}) \cap \mathcal{O}(U)$ is (in general) not a Banach space.

(iii) $A^\infty(U)$ is a Banach space if X is finite dimensional, since $\overline{U}$ is compact.

(iv) Let $X \supset U_1 \supset U_2 \supset \overline{U_3}$ be bounded open domains, then the inclusion maps $A^\infty(U_1) \hookrightarrow \mathcal{O}^b(U_2)$, $\mathcal{O}^b(U_2) \hookrightarrow A^\infty(U_3)$ are obviously continuous. □

An important step in proving Theorem B.2.4 is the following theorem (see [EH70] or [May80b, Thm. 1]) which states that a strictly contractive map has a unique fixed point.

Theorem B.2.3. *(Earle-Hamilton fixed point theorem) Let $U \subset X$ be an open bounded domain in the complex Banach space X. Let $\psi : U \to U$ be a holomorphic mapping which is strictly contractive, i. e.* $\operatorname{dist}(\psi(U), X \setminus U) \geq \epsilon > 0$. *Then ψ has a unique fixed point $z^* \in U$ and the eigenvalues of the derivative $\psi'(z^*)$ are all strictly smaller than one in absolute value.* □

The following theorem is the classical formulation of the Atiyah-Bott fixed point formula. It is often attributed to D. Ruelle [Ru76], although it is quite similar to results in [AtBo67]. It has been generalised by D. Mayer [May80a, Appendix B], [May80b] whose results we will mention in Section B.4.

Theorem B.2.4. *Let $U \subset \mathbb{C}^k$ be an open bounded complex domain. Let $\phi \in A^\infty(U)$ and $\psi : U \to U$ be holomorphic and strictly contractive, i. e., $\psi(\overline{U}) \subset U$. Then ψ has a unique fixed point $z^* \in U$ and the generalised composition operator (B.1.1)*

$$T : A^\infty(U) \to A^\infty(U),\ (Tf)(z) := \phi(z)\,(f \circ \psi)(z)$$

is nuclear of order zero (A.3.1) with trace given by the Atiyah-Bott fixed point formula

$$\operatorname{trace} T = \frac{\phi(z^*)}{\det(1 - \psi'(z^*))}. \tag{114}$$

[40] This space was denoted by $A^\infty(U)$ in [May80a] and [May80b].

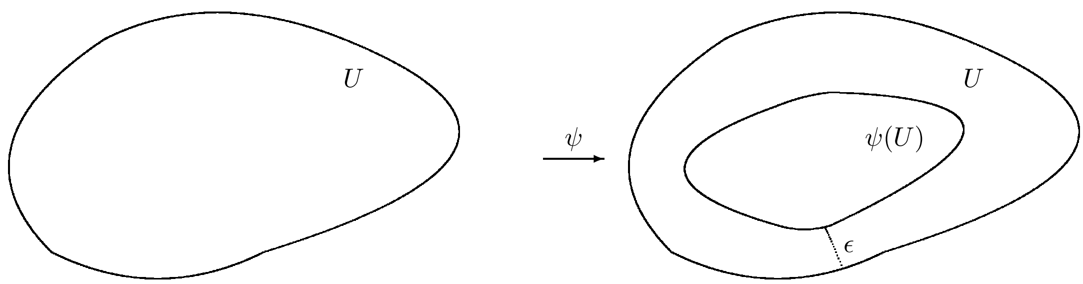

Figure 11: A strictly contractive map

Proof. (Idea): The easiest proof uses the nuclearity of the space $\mathcal{O}(U)$ and the boundedness of all maps in the following diagram (ι denotes the embedding)

$$A^\infty(U) \overset{\iota}{\hookrightarrow} \mathcal{O}(U) \xrightarrow{f\mapsto f\circ\psi} A^\infty(U) \xrightarrow{f\mapsto \phi\cdot f} A^\infty(U).$$

Hence A is nuclear. Using an explicit nuclear representation one shows that $\mathcal{L}$ is nuclear of order zero, hence we can apply Grothendieck's 2/3-trace theorem A.3.2: trace $\mathcal{L} = \sum_i \lambda_i(\mathcal{L})$, where $\lambda_i(\mathcal{L})$ are the eigenvalues of $\mathcal{L}$. Differentiation of an eigenfunction at the fixed point z^* of ψ determines the possible eigenvalues of $\mathcal{L}$: $\mathrm{spec}(\mathcal{L}) \subseteq \Lambda := \{0\} \cup \{\phi(z^*)\, \mu_{i_1} \cdot \ldots \cdot \mu_{i_k} \,|\, k \in \mathbb{N},\, \mu_j \in \mathrm{spec}(\psi'(z^*)\}$. To show $\mathrm{spec}(\mathcal{L}) \supseteq \Lambda$ it suffices to find for each $\lambda \in \Lambda$ a function g which is not in the image of $\mathcal{L} - \lambda$. In the case of $A^\infty(U)$ and $\lambda = \phi(z^*)\, \mu_{i_1} \cdot \ldots \cdot \mu_{i_k}$, one chooses $g \in A^\infty(U)$ such that

$$D^r g(z^*) = 0 \text{ for } r = 0, \ldots, k-1 \text{ and } D^k g(z^*)(e_{i_1}, \ldots, e_{i_k}) \neq 0,$$

where e_i is the eigenvector of $\psi'(z^*)$ corresponding to the eigenvalue μ_i. By the Earle-Hamilton fixed point theorem B.2.3 the eigenvalues of $\psi'(z^*)$ are all strictly smaller than one in absolute value, which makes a geometric series convergent and thus the trace formula. □

Remark B.2.5. Note that the space $\mathcal{O}(U)$ with its usual compact open topology is nuclear if and only if the dimension of U is finite, [Sch75]. Hence the proof of Theorem B.2.4 does not work in infinite dimensions. - The above proof fails even in the finite-dimensional case, when one tries to replace $A^\infty(U)$ by $\mathcal{O}^b(U)$. □

B.3 The trace formula on the Fock space

In this section we prove a trace formula for a certain class of composition operators where the composing part is a global contraction. Provided that the coefficients ϕ and ψ are entire functions we show that the eigenfunctions of the composition operator extend to entire functions and satisfy certain growth conditions. This permits to show that the Atiyah-Bott fixed point formula also holds on a much smaller space of functions. A typical application are composition operators of the type

$$T : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m),\; (Tf)(z) = e^{\pi\langle z|a\rangle} f(\mathbb{B}z + b)$$

for some fixed $a,\, b \in \mathbb{C}^m,\ \mathbb{B} \in \mathrm{Mat}(m, m; \mathbb{C})$. These operators appeared in Theorem 2.7.6 and will be investigated in the second part of this section. Using reproducing kernel techniques (see A.4) we will determine a formula for the trace norm for these composition operators. We start with a little proposition on the invariant sets of a contraction. It states that sufficiently large balls are invariant and each point is attracted by a neighbourhood of the fixed point in a finite number of steps because of a Banach fixed point argument.

Proposition B.3.1. *Let $0 < q < 1$ and $\psi : X \to X$ be a function on a normed spaces $(X, \|\cdot\|)$ with*

$$\|\psi(z) - \psi(w)\| \leq q\, \|z - w\|$$

for all $z,\, w \in X$. Then ψ is called a (global) contraction. Set

$$r_0 := \frac{\|\psi(0)\|}{1-q}. \tag{115}$$

Then for all $r \geq r_0$ the closed ball $K_r := \{z \in X;\ \|z\| \leq r\}$ is ψ-invariant, moreover $\psi(K_r) \subset K_{qr+\|\psi(0)\|} \subset K_r$. For all $r > r_0$ and all $z \in X$ there is an index $n_0 \in \mathbb{N}$ such that

$$(\forall m \geq n_0) \quad \psi^{(m)}(z) \in K_r,$$

where $\psi^{(m)} := \psi \circ \ldots \circ \psi$ (m-times) is the m-th iterate of ψ.

Proof. Let $r > r_0$ and $|z| \leq r$. Then

$$\|\psi(z)\| \leq \|\psi(z) - \psi(0)\| + \|\psi(0)\| \leq q\, \|z\| + \|\psi(0)\| < r$$

shows that $\psi(z) \in K_r$ for all $z \in K_r$. Let $r > r_0$. Since $0 \in K_{r_0}$, hence also $\psi^{(m)}(0) \in K_{r_0}$ by the first assertion. For any $z \in X$, $m \geq n_0 \geq \frac{\ln(\frac{r-r_0}{\|z\|})}{\ln q}$ we have

$$\|\psi^{(m)}(z)\| \leq \|\psi^{(m)}(z) - \psi^{(m)}(0)\| + \|\psi^{(m)}(0)\| \leq q^m\, \|z\| + r_0 \leq r.$$

□

We will now study composition operators $(Tf)(z) = \phi(z)\,(f \circ \psi)(z)$ whose composition part ψ is a globally contracting map. We will apply these results for the Ruelle-Mayer transfer operator which is an integral over a family of composition operators, each of them with an affine globally contracting map ψ.

Proposition B.3.2. *Let $(X, \|\cdot\|)$ be a normed space and $K_r := \{z \in X;\ \|z\| \le r\}$. Let $\psi : X \to X$ be a contraction in the sense of Proposition B.3.1, and $\phi : X \to \mathbb{C}$ a continuous function. Let $r > r_0$ with r_0 as in (115), and T be the generalised composition operator*

$$T : \mathcal{C}(K_r) \to \mathcal{C}(K_r),\ \ (Tf)(z) = \phi(z)\,(f \circ \psi)(z).$$

(i) *Let $g \in \mathcal{C}(K_r)$, then Tg belongs to $\mathcal{C}(K_{\delta r})$ with $\delta > 1$.*

(ii) *Every eigenfunction of T for a non-zero eigenvalue belongs to $\mathcal{C}(X)$.*

Proof. If $z \in K_{\delta r}$, then

$$|\psi(z)| \le q\, r\delta + \|\psi(0)\| \overset{!}{<} r,$$

if $\delta < \frac{r - \|\psi(0)\|}{rq}$. Since $\frac{r-\|\psi(0)\|}{rq} > 1$, one can choose $\delta > 1$. Let $f \in \mathcal{C}(K_r)$ be an eigenfunction of T for a non-zero eigenvalue ρ. Hence by iterating relation (i) n-times we get $f = \rho^{-n} T^n f \in \mathcal{C}(K_{\delta^n r})$ for some $\delta > 1$. Hence $f \in \mathcal{C}(X)$. □

Given a composition operator which satisfies a trace formula on a certain function space one is often interested in finding a smaller space which contains the eigenfunctions to non-zero eigenvalues. We will assume that the composing part is a contraction in the sense of Proposition B.3.1. By the previous Proposition B.3.2 the eigenfunctions extend to the whole base space. If the coefficients ϕ and ψ are smooth, then also the eigenfunctions will be smooth as well. Moreover, they satisfy certain growth estimates. This growth estimate can be used to form a weighted L^2-space on which the operator acts.

Lemma B.3.3. *Let $\psi : \mathbb{C}^m \to \mathbb{C}^m$ and $\phi : \mathbb{C}^m \to \mathbb{C}$ be entire functions, and ψ a contraction in the sense of Proposition B.3.1. Let T be the composition operator acting via*

$$(Tf)(z) = \phi(z)\,(f \circ \psi)(z).$$

Let $r > r_0$ with r_0 as in (115) and f an eigenfunction of $T : A^\infty(B(0;r)) \to A^\infty(B(0;r))$ for a non-zero eigenvalue ρ. Then f is entire and there exist $c_1, c_2 > 0$ such that for all $z \in \mathbb{C}^m$

$$|f(z)| \le \|z\|^{-c_1 \ln \rho} \sup_{|w| \le r} |f(w)| \max_{t \in [0,2\pi]} |\phi(e^{it} z))|^{c_2 \ln \|z\|}.$$

Moreover, if $A^2(U) := \mathcal{O}(U) \cap L^2(U, dz)$ denotes the Bergmann space, then

$$\text{trace}_{\,A^\infty(U)} T = \text{trace}_{\,A^2(U)} T$$

for all ψ-invariant bounded domains $U \subset \mathbb{C}^m$.

Proof. The operator T leaves the Banach space $A^\infty(B(0;r))$ invariant. Let f be an eigenfunction of $T : A^\infty(B(0;r)) \to A^\infty(B(0;r))$ for a non-zero eigenvalue ρ. For $n \in \mathbb{N}$ we have $f = \rho^{-n}\, T^n f$ which by (B.1.2) is given as

$$f(z) = \rho^{-n} \prod_{k=0}^{n-1} (\phi \circ \psi^{(k)})(z)\,(f \circ \psi^{(n)})(z),$$

where $\psi^{(k)}$ is the k-th iterate of ψ. As in Remark B.3.2 (ii) one shows that f is entire, thus belongs to $A^2(U)$ for all bounded domains $U \subset \mathbb{C}^m$. Hence every eigenvalue of $T|_{A^\infty(U)}$ belongs to the spectrum of $T|_{A^2(U)}$, thus by Lidskii's Trace Theorem the traces coincide. Let $r > r_0 = \frac{\|\psi(0)\|}{1-q}$ and $z \in \mathbb{C}^d$ with $\|z\| > r_0$. Choose $n(z) := \left\lceil \frac{\ln(\frac{r-r_0}{\|z\|})}{\ln q} \right\rceil$. One can find constants $c_1, c_2 > 0$ such that

$c_1 \ln\|z\| \le n(z) \le c_2 \ln\|z\|$ for all $\|z\| > r_0$. Remark B.3.1 implies that $\|\psi^{(n(z))}(z)\| \le r$, and hence

$$\begin{aligned}
|f(z)| &= |\rho|^{-n(z)} \Big| \prod_{k=0}^{n(z)-1} (\phi \circ \psi^{(k)})(z) \Big| \, \big|(f \circ \psi^{(n(z))}(z)\big| \\
&\le |\rho|^{-n(z)} \sup_{|w| \le r} |f(w)| \sup_{\|w\| \le \|z\|} \Big| \prod_{k=0}^{n(z)-1} (\phi \circ \psi^{(k)})(w) \Big| \\
&\le |\rho|^{-n(z)} \sup_{|w| \le r} |f(w)| \sup_{\|w\| \le \|z\|} |\phi(w)|^{n(z)} \\
&\le |\rho|^{-c_1 \ln\|z\|} \sup_{|w| \le r} |f(w)| \sup_{\|w\| \le \|z\|} |\phi(w)|^{c_2 \ln\|z\|}.
\end{aligned}$$

By the maximum principle of complex variables we know that the supremum $\sup_{\|w\| \le \|z\|} |\phi(w)|$ is attained for some w with $\|w\| = \|z\|$. □

We now apply Lemma B.3.3 to a certain class of composition operators acting on the Fock space $\mathcal{F}(\mathbb{C}^m)$ in finitely many variables as defined in Example A.4.5 (i).

Theorem B.3.4. *Let $b \in \mathbb{C}^m$, $\mathbb{A} \in \mathrm{Gl}(m;\mathbb{C})$ with $\|\mathbb{A}\| < 1$, and $\phi : \mathbb{C}^m \to \mathbb{C}$ an entire function which can be estimated by $|\phi(z)| \le c\exp(a\,\|z\|)$ for some constants a, $c > 0$. Let T be the composition operator given by*

$$(Tf)(z) = \phi(z)\, f(\mathbb{A}z + b).$$

Then $T : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ is a trace class operator with

$$\mathrm{trace}_{\,\mathcal{F}(\mathbb{C}^m)} T = \mathrm{trace}_{\,A^\infty(B(0;r))} T = \frac{\phi((1-\mathbb{A})^{-1}b)}{\det(1-\mathbb{A})}$$

for all $B(0;r) := \big\{ z \in \mathbb{C}^m \;\big|\; \|z\| < r \big\}$ with $r > \frac{\|b\|}{1-\|\mathbb{A}\|}$.

Proof. The affine map $\psi(z) = \mathbb{A}z + b$ is a contraction with $q = \|\mathbb{A}\| < 1$ and $r_0 = \frac{\|b\|}{1-\|\mathbb{A}\|}$. We claim that the Fock space $\mathcal{F}(\mathbb{C}^m)$ is a T-invariant Hilbert subspace of $A^\infty(B(0;r))$ for any $r > r_0 = \frac{\|b\|}{1-\|\mathbb{A}\|}$. In fact, for $f \in \mathcal{F}(\mathbb{C}^m)$ the standard estimate (A.4.4) yields

$$\begin{aligned}
\|Tf\|^2 &= \int_{\mathbb{C}^m} \Big| \phi(z)\, f(\mathbb{A}z+b) \Big|^2 e^{-\pi\|z\|^2}\, dz \\
&\le c^2 \int_{\mathbb{C}^m} e^{2a\|z\|}\, |f(\mathbb{A}z+b)|^2\, e^{-\pi\|z\|^2}\, dz \\
&\le c^2\, \|f\|^2 \int_{\mathbb{C}^m} e^{2a\|z\|}\, e^{\pi\|\mathbb{A}z+b\|^2}\, e^{-\pi\|z\|^2}\, dz \\
&\le \|f\|^2 \Big(C + \int_{\mathbb{C}^m \setminus B(0;r)} e^{2a\|z\|}\, e^{-\pi(1-\|\mathbb{A}\|^2)\|z\|^2}\, dz \Big) < \infty.
\end{aligned}$$

Thus $T|_{\mathcal{F}(\mathbb{C}^m)}$ is a nuclear map on a Hilbert space, and hence of trace class. Let $f \in A^\infty(B(0;r))$ be an eigenfunction of T corresponding to a non-zero eigenvalue ρ. By Lemma B.3.3 the eigenfunction f satisfies the estimate

$$|f(z)|^2 \exp(-\pi\|z\|^2) \le \|z\|^{-c_1 \ln\rho}\ \exp\big((a\,\|z\| + \ln c) c_2 \ln\|z\|\big)\, \exp(-\pi\|z\|^2).$$

This upper bound is Lebesgue-integrable on $\mathbb{C}^m$, and thus f belongs to $\mathcal{F}(\mathbb{C}^m)$. This shows that every non-zero eigenvalue of $T|_{A^\infty(B(0;r))}$ is an eigenvalue of $T|_{\mathcal{F}(\mathbb{C}^m)}$, hence the traces coincide and by Theorem B.2.4 they have the stated value. □

The assumptions made in Theorem B.3.4 on the multiplication part $\phi : \mathbb{C}^m \to \mathbb{C}$ imply that ϕ belongs to the Fock space. We disbelieve that the assertion is true for all $\phi \in \mathcal{F}(\mathbb{C}^m)$. A typical application of Theorem B.3.4 is the case where the multiplication part $\phi : \mathbb{C}^m \to \mathbb{C}$ is given as $\phi(z) = p(z)\, \exp(\langle z|a\rangle)$ for some polynomial $p \in \mathbb{C}[z_1, \dots, z_m]$ and $a \in \mathbb{C}^m$.
We add a remark concerning the proof of the previous theorem.

Remark B.3.5. Let $b \in \mathbb{C}^m$, $\mathbb{B} \in \mathrm{Gl}(m;\mathbb{C})$ with $\|\mathbb{B}\| < 1$, and $\phi : \mathbb{C}^m \to \mathbb{C}$ an entire function which can be estimated by $|\phi(z)| \le c\exp(a\,\|z\|)$ for all z. Let $T : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$, $(Tf)(z) = \phi(z)\, f(\mathbb{B}z + b)$. By Theorem B.3.4 the composition operator T is trace class and satisfies the Atiyah-Bott trace formula. Of course one would like to have a direct proof for the trace formula. There are two other ways to compute the trace:

(i) Choose an orthonormal basis $(e_\alpha)_\alpha$ in $\mathcal{F}(\mathbb{C}^m)$ and determine

$$\operatorname{trace} T = \sum_\alpha \langle Te_\alpha | e_\alpha\rangle.$$

(ii) T is a bounded operator on $\mathcal{F}(\mathbb{C}^m)$ and hence, by the reproducing kernel property A.6.1, can be written as an integral operator with integral kernel $k_T(z,w) = \phi(z)\, \exp(\pi\langle \mathbb{B}z + b|w\rangle)$. If ϕ can be estimated by $|\phi(z)|c\, \exp(a\|z\|)$, then the kernel is rapidely decaying. Hence by [CoGr90, A.3.9] the trace is given as the integral over the diagonal

$$\operatorname{trace} T = \int_{\mathbb{C}^m} k_T(z,z)\, \exp(-\pi\|z|^2)\, dz = \int_{\mathbb{C}^m} \phi(z)\, \exp(\pi\langle \mathbb{B}z + b|z\rangle)\, \exp(-\pi\|z|^2)\, dz.$$

□

In principle both ways described in Remark B.3.5 can be used to obtain a direct proof of Theorem B.3.4, but we can carry out these ideas only for a special class of linear maps $\mathbb{B} : \mathbb{C}^m \to \mathbb{C}^m$, namely if $\mathbb{B}$ is normal (semisimple), see Propositions B.3.6 and B.3.9.

Proposition B.3.6. *Let $\psi : \mathbb{C}^m \to \mathbb{C}^m$, $\psi(z) = \mathbb{B}z + b$ such that the linear part $\mathbb{B}$ is normal and $\|\mathbb{B}\| < 1$, and $\phi \in \mathcal{F}(\mathbb{C}^m)$ such that $p\,\phi \in \mathcal{F}(\mathbb{C}^m)$ for all polynomials $p \in \mathbb{C}[z]$. Then one has the fixed point formula*

$$\operatorname{trace} T = \int_{\mathbb{C}^m} \phi(z)\, e^{\pi\langle \mathbb{B}z+b|z\rangle}\, e^{-\pi\|z\|^2}\, dz = \frac{\phi(z^*)}{\det(1-\psi'(z^*))} = \frac{\phi((1-\mathbb{B})^{-1}b)}{\det(1-\mathbb{B})}, \tag{116}$$

where $z^* = (1-\mathbb{B})^{-1}b$ is the unique fixed point of ψ and $T : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ is the composition operator acting via $(Tf)(z) := \phi(z)\,(f\circ\psi)(z)$.

Proof. We compute the trace via Remark B.3.5 (ii). By change of variables we can assume that $\mathbb{B} = \operatorname{diag}(\lambda_i)$ with respect to the standard basis. We use the standard basis $(\zeta_\alpha)_{\alpha\in\mathbb{N}_0^m}$ of the Fock space given in Example A.4.5. Observe that

$$(\zeta_\alpha\circ\psi)(z) = \sqrt{\frac{\pi^\alpha}{\alpha!}}\,(\mathbb{B}z+b)^\alpha = \sqrt{\frac{\pi^\alpha}{\alpha!}}\prod_{i=1}^m(\lambda_i z_i + b_i)^{\alpha_i} = \sqrt{\frac{\pi^\alpha}{\alpha!}}\prod_{i=1}^m\sum_{k=0}^{\alpha_i}\binom{\alpha_i}{k}(\lambda_i z_i)^k b_i^{\alpha_i-k}.$$

Let $\phi(z) = \sum_{\alpha\in\mathbb{N}_0^m}\phi_\alpha\, z^\alpha$ be the Taylor series expansion of ϕ. Hence by formal calculation we obtain

$$\begin{aligned}
\langle Te_\alpha|e_\alpha\rangle &= \int_{\mathbb{C}^m}\phi(z)\,(\zeta_\alpha\circ\psi)(z)\,\overline{\zeta_\alpha(z)}\,e^{-\pi\|z\|^2}\,dz \\
&= \sum_{\beta\in\mathbb{N}_0^m}\phi_\beta\frac{\pi^\alpha}{\alpha!}\int_{\mathbb{C}^m}\prod_{i=1}^m\sum_{k=0}^{\alpha_i}\binom{\alpha_i}{k}(\lambda_i z_i)^k\, b_i^{\alpha_i-k}\, z_i^{\beta_i}\,\overline{z_i}^{\alpha_i}\,e^{-\pi\|z\|^2}\,dz \\
&= \sum_{\beta\in\mathbb{N}_0^m}\phi_\beta\prod_{i=1}^m\sum_{k=0}^{\alpha_i}\binom{\alpha_i}{k}\lambda_i^k\, b_i^{\alpha_i-k}\frac{\pi^{\alpha_i}}{\alpha_i!}\int_{\mathbb{C}} z_i^{k+\beta_i}\,\overline{z_i}^{\alpha_i}\,e^{-\pi|z_i|^2}\,dz_i \\
&= \sum_{\beta\in\mathbb{N}_0^m}\phi_\beta\prod_{i=1}^m\sum_{k=0}^{\alpha_i}\binom{\alpha_i}{k}\lambda_i^k\, b_i^{\alpha_i-k}\,\delta_{k+\beta_i,\alpha_i}.
\end{aligned}$$

The latter step follows from the orthonormality of the basis ζ_α. The formal calculation holds, since for every $f \in \mathcal{F}(\mathbb{C}^m)$ its Taylor series converges to f in $\mathcal{F}(\mathbb{C}^m)$-norm and we assumed that $\zeta_\alpha\,\phi \in \mathcal{F}(\mathbb{C}^m)$.

The series in the following are absolutely convergent, since $|\lambda_i| < 1$, hence we can interchange the summation order to get

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}_0^m} \langle Te_\alpha | e_\alpha\rangle &= \sum_{\alpha\in\mathbb{N}_0^m}\sum_{\beta\in\mathbb{N}_0^m} \phi_\beta \prod_{i=1}^m \sum_{k=0}^{\alpha_i} \binom{\alpha_i}{k} \lambda_i^k\, b_i^{\alpha_i-k}\, \delta_{k+\beta_i,\alpha_i} \\
&= \sum_{\alpha\in\mathbb{N}_0^m}\sum_{\beta\in\mathbb{N}_0^m} \phi_\beta \sum_{k\in\mathbb{N}_0^m} \prod_{i=1}^m \binom{\alpha_i}{k_i} \lambda_i^{k_i}\, b_i^{\alpha_i-k_i}\, \delta_{k_i+\beta_i,\alpha_i} \\
&= \sum_{\beta\in\mathbb{N}_0^m} \phi_\beta \sum_{k\in\mathbb{N}_0^m} \prod_{i=1}^m \binom{k_i+\beta_i}{k_i} \lambda_i^{k_i}\, b_i^{\beta_i} \\
&= \sum_{\beta\in\mathbb{N}_0^m} \phi_\beta \prod_{i=1}^m \sum_{k\in\mathbb{N}_0} \binom{k+\beta_i}{k} \lambda_i^{k}\, b_i^{\beta_i} \\
&= \sum_{\beta\in\mathbb{N}_0^m} \phi_\beta \prod_{i=1}^m (1-\lambda_i)^{-\beta_i-1}\, b_i^{\beta_i},
\end{aligned}$$

where we finally used the series representation

$$(1-q)^{-m-1} = \sum_{n=0}^{\infty} \binom{m}{n}\, q^n$$

for $|q| < 1$ and the convention $\binom{m}{n} = 0$, whenever $n > m$. Using the explicit form of the fixed point $z^* = (1-\mathbb{B})^{-1}b$, which is the vector with entries $(1-\lambda_i)^{-1}b_i$, one can write down the right hand side of equation (116) as

$$\begin{aligned}
\frac{\phi(z^*)}{\det(1-\mathbb{B})} &= \det(1-\mathbb{B})^{-1} \sum_{\beta\in\mathbb{N}_0^m} \phi_\beta\, ((1-\mathbb{B})^{-1}b)^\beta \\
&= \prod_{i=1}^m (1-\lambda_i)^{-1} \sum_{\beta\in\mathbb{N}_0^m} \phi_\beta \prod_{i=1}^m ((1-\lambda_i)^{-1}b_i)^{\beta_i} \\
&= \sum_{\beta\in\mathbb{N}_0^m} \phi_\beta \prod_{i=1}^m (1-\lambda_i)^{-\beta_i-1} b_i^{\beta_i}.
\end{aligned}$$

□

Remark B.3.7. Let $\psi : \mathbb{C}^m \to \mathbb{C}^m$, $\psi(z) = \mathbb{B}z + b$ such that the linear part $\mathbb{B} \in \mathrm{Mat}(m,m;\mathbb{C})$ satisfies $\|\mathbb{B}\| < 1$, and $\phi \in \mathcal{F}(\mathbb{C}^m)$. Then our direct approach of Proposition B.3.6 fails to prove the desired fixed point formula (116) caused by the dramatically increasing complexity. We suggest the following idea: Find an interpretation of (116) via theory of complex variables, or Gauss-Green-Stokes theorem to prove the conjectural formula

$$\int_{\mathbb{C}^m} \phi(z)\, e^{\pi\langle \mathbb{B}z+b|z\rangle}\, e^{-\pi\|z\|^2}\, dz = \frac{\phi((1-\mathbb{B})^{-1}b)}{\det(1-\mathbb{B})}$$

without nasty computation. □

We will now specialise to a certain class of composition operators, namely those acting via

$$\mathcal{K}_{a,b,\Lambda} : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m),\ (\mathcal{K}_{a,b,\Lambda} f)(z) = e^{\pi\langle z|a\rangle}\, f(\Lambda z + b) \tag{117}$$

for some fixed $a, b \in \mathbb{C}^m$, $\Lambda \in \mathrm{Mat}(m,m;\mathbb{C})$ with $\|\Lambda\| < 1$. They arise as (parts of) Ruelle-Mayer transfer operators which we intensively studied in Section 2.7. There we needed precise information about the spectrum of $\mathcal{K}_{a,b,\Lambda}$ and in particular a formula for its trace norm which we will derive in the sequel. In the notation of Chapter 5 (94) we have $\mathcal{K}_{a,b,\Lambda} = \mathcal{L}_{\pi\overline{a},b,\Lambda}$. In this chapter the usage of the hermitian inner product $\langle \cdot \,|\, \cdot \rangle$ and hence of $\mathcal{K}_{a,b,\Lambda}$ seems to be preferable. First we will investigate the composition law for such operators. Then we use the reproducing kernel techniques from Proposition A.6.2 and compute the Hilbert space adjoint $(\mathcal{K}_{a,b,\Lambda})^\star$ of $\mathcal{K}_{a,b,\Lambda}$. This allows to determine the selfadjoint and the positive operators belonging to that class.

Proposition B.3.8. *For $a, b \in \mathbb{C}^m$, $\Lambda \in \mathrm{Mat}(m, m; \mathbb{C})$ with $\|\Lambda\| < 1$, let $\mathcal{K}_{a,b,\Lambda} : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ (117) be the corresponding composition operator. Then $(\mathcal{K}_{a,b,\Lambda})^\star = \mathcal{K}_{b,a,\Lambda^\star}$ and*

$$\mathcal{K}_{a_1,b_1,\Lambda_1}\mathcal{K}_{a_2,b_2,\Lambda_2} = e^{\pi\langle b_1|a_2\rangle}\,\mathcal{K}_{a_1+\Lambda_1^\star a_2,\Lambda_2 b_1+b_2,\Lambda_2\Lambda_1}$$

for all a_i, $b_i \in \mathbb{C}^m$, $\Lambda_i \in \mathrm{Mat}(m, m; \mathbb{C})$ with $\|\Lambda_i\| < 1$ ($i = 1, 2$). $\mathcal{K}_{a,b,\Lambda}$ is selfadjoint if and only if Λ is selfadjoint and $a = b$. If Λ is positive and $a = b$, then $\mathcal{K}_{a,b,\Lambda}$ belongs to the cone of positive composition operators.

Proof. For all $f \in \mathcal{F}(\mathbb{C}^m)$ one has

$$\begin{aligned}(\mathcal{K}_{a_1,b_1,\Lambda_1}\mathcal{K}_{a_2,b_2,\Lambda_2} f)(z) &= e^{\pi\langle z|a_1\rangle}\,(\mathcal{K}_{a_2,b_2,\Lambda_2} f)(b_1 + \Lambda_1 z)\\ &= e^{\pi\langle z|a_1\rangle}\,e^{\pi\langle b_1+\Lambda_1 z|a_2\rangle}\,f(\Lambda_2(\Lambda_1 z + b_1) + b_2)\\ &= e^{\pi\langle b_1|a_2\rangle}\,e^{\pi\langle z|a_1+\Lambda_1^\star a_2\rangle}\,f(\Lambda_2\Lambda_1 z + \Lambda_2 b_1 + b_2)\end{aligned}$$

Using the reproducing kernel property A.6.1, the operator $\mathcal{K} := \mathcal{K}_{a,b,\Lambda}$ is uniquely determined by its integral kernel

$$k_{\mathcal{K}}(z, w) = e^{\pi\langle z|a\rangle}\,e^{\pi\langle\Lambda z+b|w\rangle} = \exp\Big(\pi(\langle z|a\rangle + \langle b|w\rangle + \langle\Lambda z|w\rangle)\Big),$$

from which one easily gets the integral kernel

$$k_{\mathcal{K}^\star}(z, w) = \overline{k_{\mathcal{K}}(w, z)} = \exp\Big(\pi\big(\langle z|b\rangle + \langle a|w\rangle + \langle\Lambda^\star z|w\rangle\big)\Big) = \exp\big(\pi\langle z|b\rangle\big)\,\exp\big(\pi\langle a + \Lambda^\star z|w\rangle\big)$$

of its adjoint $\mathcal{K}^\star$. In particular, (using the properties of the reproducing kernel $k(x, y) = \exp(\pi\langle x|y\rangle)$ of the Fock space) the operator $\mathcal{K}^\star$ acts via

$$(\mathcal{K}^\star f)(z) = \int_{\mathbb{C}^m} e^{\pi\langle z|b\rangle}\,e^{\pi\langle a+\Lambda^\star z|w\rangle}\,f(w)\,e^{-\pi\|w\|^2}\,dw = e^{\pi\langle z|b\rangle}\,f(a + \Lambda^\star z),$$

i. e. $(\mathcal{K}_{a,b,\Lambda})^\star = \mathcal{K}_{b,a,\Lambda^\star}$. Hence $\mathcal{K}$ is selfadjoint iff Λ is selfadjoint and $a = b$. A compact selfadjoint operator is positive iff all its eigenvalues are positive. By the proof of Theorem B.2.4 we know that the spectrum of $\mathcal{K}_{b,b,\Lambda}$ is contained in $\{0\} \cup \big\{\phi(z^*)\,\mu_{i_1}\cdot\ldots\cdot\mu_{i_k}\,|\,k \in \mathbb{N}_0,\ \mu_j \in \mathrm{spec}(\psi'(z^*))\big\}$ where $\phi(z) := e^{\pi\langle z|b\rangle}$, $\psi(z) := \Lambda z + b$, $z^* = (1 - \Lambda)^{-1}b$. Thus

$$\mathrm{spec}(\mathcal{K}_{b,b,\Lambda}) \subseteq \{0\} \cup \big\{e^{\pi\langle(1-\Lambda)^{-1}b|a\rangle}\,\mu_{i_1}\cdot\ldots\cdot\mu_{i_k}\,|\,k \in \mathbb{N}_0,\ \mu_j \in \mathrm{spec}(\Lambda)\big\}.$$

If Λ is positive, then $(1-\Lambda)^{-1}$ is positive and thus all eigenvalues of $\mathcal{K}_{b,b,\Lambda}$ are necessarily positive. □

Proposition B.3.9. *Let $\Lambda \in \mathrm{Mat}(m, m; \mathbb{C})$ with $\|\Lambda\| < 1$ be positive, $\Lambda = \Lambda^\star > 0$, and $\beta \in \mathbb{C}^m$, then the corresponding composition operator $\mathcal{K}_{\beta,\beta,\Lambda} : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ (117) is positive and trace class with*

$$\mathrm{trace}\ \mathcal{K}_{\beta,\beta,\Lambda} = \|\mathcal{K}_{\beta,\beta,\Lambda}\|_{\mathcal{S}_1(\mathcal{F}(\mathbb{C}^m))} = \frac{\exp\Big(\pi\|(1 - \Lambda)^{-1/2}\beta\|^2\Big)}{\det(1 - \Lambda)}.$$

Proof. The operator $\mathcal{K}_{\beta,\beta,\Lambda}$ is positive by Proposition B.3.8. Hence the trace norm of $\mathcal{K}_{\beta,\beta,\Lambda}$ is equal to the trace, which is given by the Atiyah-Bott fixed point formula. We give an alternative proof which uses the fact that $\Lambda > 0$, hence we can apply Proposition A.5.1:

$$\begin{aligned}\mathrm{trace}\ \mathcal{K}_{\beta,\beta,\Lambda} &= \int_{\mathbb{C}^m} e^{\pi\langle z|\beta\rangle}\,e^{\pi\langle\Lambda z+\beta|z\rangle}e^{-\pi\|z\|^2}\,dz\\ &= \int_{\mathbb{C}^m} \big|e^{\pi\langle z|\beta\rangle}\big|^2\,e^{\pi\|\sqrt{\Lambda}z\|^2}\,e^{-\pi\|z\|^2}\,dz\,dz\\ &\overset{A.5.1}{=} \det(1 - \Lambda)^{-1}\exp\Big(\pi\|(1 - \Lambda)^{-1/2}\beta\|^2\Big)\\ &= \det(1 - \Lambda)^{-1}\exp\Big(\pi\langle(1 - \Lambda)^{-1}\beta|\beta\rangle\Big)\end{aligned}$$

□

We now use the previous lemmas to to determine a formula for the trace norm of the composition operator $\mathcal{K}_{a,b,\Lambda}$ introduced in (117). Given such a composition operator $\mathcal{K} = \mathcal{K}_{a,b,\Lambda}$ we will determine its positive part $|\mathcal{K}| = \sqrt{\mathcal{K}^\star\mathcal{K}}$ which happens to be composition operator of the same type as $\mathcal{K}$. Hence we can compute the trace of $|\mathcal{K}|$ via the Atiyah-Bott fixed point formula.

Lemma B.3.10. *Let* $\mathbb{B} \in \mathrm{Mat}(m, m; \mathbb{C})$ *with* $\|\mathbb{B}\| < 1$, a, $b \in \mathbb{C}^m$. *Set*

$$\Lambda_1 = |\mathbb{B}| = \sqrt{\mathbb{B}^\star\mathbb{B}},\ \beta_1 = (1 + |\mathbb{B}|)^{-1}(\mathbb{B}^\star b + a),\ \gamma_1 = \exp\Big(\frac{\pi}{2}(\|b\|^2 - \|\beta_1\|^2)\Big)$$

and

$$\Lambda_2 = \sqrt{\mathbb{B}\mathbb{B}^\star},\ \beta_2 = (1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b),\ \gamma_2 = \exp\Big(\frac{\pi}{2}(\|a\|^2 - \|\beta_2\|^2)\Big).$$

Let $\mathcal{K} := \mathcal{K}_{a,b,\mathbb{B}}$ *and* $K_i := \gamma_i\, \mathcal{K}_{\beta_i,\beta_i,\Lambda_i} : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ $(i = 1, 2)$ *be the corresponding composition operators (117). Then* $K_1 = \sqrt{\mathcal{K}\mathcal{K}^\star}$, $K_2 = |\mathcal{K}| = \sqrt{\mathcal{K}^\star\mathcal{K}}$, *and*

$$\|\mathcal{K}\|_{\mathcal{S}_1(\mathcal{F}(\mathbb{C}^m))} = \frac{\exp\Big(\frac{\pi}{2}\|a\|^2 + \frac{\pi}{2}\|(1 - \mathbb{B}\mathbb{B}^\star)^{-1/2}(\mathbb{B}a + b)\|^2\Big)}{\det(1 - |\mathbb{B}|)} = \frac{\gamma_i}{\det(1 - \Lambda_i)}\ \exp\Big(\pi\|(1 - \Lambda_i)^{-1/2}\beta_i\|^2\Big)$$

for $i = 1, 2$.

Proof. The composition operators $K_i : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ $(i = 1,\ 2)$ are positive by Proposition B.3.9. We will compare the squares $(K_i)^2$ with $\mathcal{K}\mathcal{K}^\star$ and $\mathcal{K}^\star\mathcal{K}$ given by Proposition B.3.8. For all $f \in \mathcal{F}(\mathbb{C}^m)$ one has

$$\begin{aligned}
(K_i^2 f)(z) &= \gamma_i^2\, e^{\pi\|\beta_i\|^2}\, e^{\pi\langle z|(1+\Lambda_i^\star)\beta_i\rangle}\, f(\Lambda_i^2 z + (1 + \Lambda_i)\beta_i),\\
(\mathcal{K}\mathcal{K}^\star f)(z) &= e^{\pi\|b\|^2}\, e^{\pi\langle z|\mathbb{B}^\star b+a\rangle}\, f(\mathbb{B}^\star\mathbb{B}z + \mathbb{B}^\star b + a),\\
(\mathcal{K}^\star\mathcal{K} f)(z) &= e^{\pi\|a\|^2} e^{\pi\langle z|\mathbb{B}a+b\rangle}\, f(\mathbb{B}\mathbb{B}^\star z + \mathbb{B}a + b).
\end{aligned}$$

For Λ_1, β_1, γ_1 chosen as above we get $(K_1)^2 = \mathcal{K}\mathcal{K}^\star$, hence $K_1 = \sqrt{\mathcal{K}\mathcal{K}^\star}$, which concludes the first part of the proof. Similarly, for Λ_2, β_2, γ_2 chosen as above we get $(K_2)^2 = \mathcal{K}^\star\mathcal{K}$, i. e., $K_2 = |\mathcal{K}| = \sqrt{\mathcal{K}^\star\mathcal{K}}$. By Remark A.2.3 the trace norm of $\mathcal{K}$ is equal to the trace of K_i, which is given by Proposition B.3.9. For a better understanding of $\gamma_2\ \exp\Big(\pi\|(1 - \Lambda_2)^{-1/2}\beta_2\|^2\Big)$, we compute

$$\begin{aligned}
&2\|(1 - \Lambda_2)^{-1/2}\beta_2\|^2 - \|\beta_2\|^2\\
&= 2\|(1 - \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1/2}(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b)\|^2 - \|(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b)\|^2\\
&= \langle(2(1 - \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1} - 1)(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b)|(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b)\rangle\\
&= \langle(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})(1 - \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b)|(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b)\rangle\\
&= \langle(1 - \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b)|\mathbb{B}a + b\rangle\\
&= \langle(1 - \mathbb{B}\mathbb{B}^\star)^{-1}(\mathbb{B}a + b)|\mathbb{B}a + b\rangle\\
&= \|(1 - \mathbb{B}\mathbb{B}^\star)^{-1/2}(\mathbb{B}a + b)\|^2.
\end{aligned}$$

Hence $\gamma_2\ \exp\Big(\pi\|(1 - \Lambda_2)^{-1/2}\beta_2\|^2\Big) = \exp\Big(\frac{\pi}{2}\|a\|^2 + \frac{\pi}{2}\|(1 - \mathbb{B}\mathbb{B}^\star)^{-1/2}(\mathbb{B}a + b)\|^2\Big)$. □

With some approximation arguments the result of Proposition B.3.10 can be extended to the infinite dimensional setting, which we will present in Theorem B.4.3.

B.4 Spectral properties of composition operators: infinite-dim. case

In this section we will deal with composition operators acting on function spaces over infinite-dimensional domains. In 1980 D. Mayer published his results [May80a] and [May80b] on composition operators acting on $\mathcal{O}^b(U)$ where $U \subset \mathcal{B}$ is a bounded open domain in a complex Banach space $\mathcal{B}$. We will specialise to a special class of composition operators acting on the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ in infinitely many variables via

$$\mathcal{K}_{a,b,\mathbb{B}} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{K}_{a,b,\mathbb{B}} f)(z) = e^{\pi\langle z|a\rangle}\, f(\mathbb{B}z + b)$$

for some fixed a, $b \in \ell^2\mathbb{N}$ and $\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}\| < 1$. In Theorem B.4.3 we prove the analogue of Lemma B.3.10 in the case of $\mathcal{F}(\ell^2\mathbb{N})$ and thus determine the trace and the trace norm of such operators $\mathcal{K}_{a,b,\mathbb{B}}$.

Our strategy for proving the Atiyah-Bott type fixed point formula (Theorem B.4.3) is to show that a given composition operator $\mathcal{K}$ on $\mathcal{F}(\ell^2\mathbb{N})$ can be approximated by a sequence of composition operators $\mathcal{K}_m$ which act on $\mathcal{F}(\mathbb{C}^m)$. The following lemma provides the tools how to embed $\mathcal{F}(\mathbb{C}^m)$ into $\mathcal{F}(\ell^2\mathbb{N})$ and how to project down from $\mathcal{F}(\ell^2\mathbb{N})$ to $\mathcal{F}(\mathbb{C}^m)$. The proof uses the characterisation of the Fock space $\mathcal{F}(\ell^2\mathbb{N})$ given in Theorem A.4.8.

Lemma B.4.1. (i) *Let $\iota_n : \mathbb{C}^n \to \ell^2\mathbb{N},\ (z_1,\dots,z_n) \mapsto (z_1,\dots,z_n,0,\dots)$ be the embedding of $\mathbb{C}^n$ into $\ell^2\mathbb{N}$. Then the orthogonal projection $C_{\iota_n} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\mathbb{C}^n) \subset \mathcal{F}(\ell^2\mathbb{N}),\ f \mapsto f\circ\iota_n$ is linear and continuous.*

(ii) *Let $\mathrm{pr}_n : \ell^2\mathbb{N} \to \mathbb{C}^n,\ z = (z_k)_k \mapsto (z_1,\dots,z_n)$ be the projection onto the first n components. Then the embedding $C_{\mathrm{pr}_n} : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\ell^2\mathbb{N}),\ f \mapsto f\circ\mathrm{pr}_n$ is linear, continuous, and the adjoint of C_{pr_n}.*

(iii) *The sequence C_{ι_n} converges pointwise to the identity id on $\mathcal{F}(\ell^2\mathbb{N})$ as $n \to \infty$.*

Proof. $C_{\iota_n} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\mathbb{C}^n) \subset \mathcal{F}(\ell^2\mathbb{N}),\ f \mapsto f\circ\iota_n$ is linear and continuous, since

$$\|C_{\iota_n} f\|_{\mathcal{F}(\mathbb{C}^n)} = \|f\circ\iota_n\|_{\mathcal{F}(\mathbb{C}^n)} \le \sup_m \|f\circ\iota_m\|_{\mathcal{F}(\mathbb{C}^m)} = \|f\|_{\mathcal{F}(\ell^2\mathbb{N})}$$

by Theorem A.4.8. For all $f \in \mathcal{F}(\mathbb{C}^n)$ one has

$$(C_{\iota_m} C_{\mathrm{pr}_n} f)(z_m) = (f\circ\mathrm{pr}_n)(z_m,0) = \begin{cases} f(z_n), & \text{if } n \le m,\\ f(z_m,0), & \text{otherwise.}\end{cases}$$

Hence for all $g \in \mathcal{F}(\mathbb{C}^n)$

$$\|C_{\mathrm{pr}_n} g\|_{\mathcal{F}(\ell^2\mathbb{N})} = \lim_m \|g\circ\mathrm{pr}_n\circ\iota_m\|_{\mathcal{F}(\mathbb{C}^m)} = \|g\|_{\mathcal{F}(\mathbb{C}^n}$$

and thus $\|C_{\mathrm{pr}_n}\| = 1$. For all $f \in \mathcal{F}(\mathbb{C}^n)$ and $g \in \mathcal{F}(\ell^2\mathbb{N})$ one has

$$\begin{aligned}
\big\langle C_{\mathrm{pr}_n} f \,\big|\, g\big\rangle_{\mathcal{F}(\ell^2\mathbb{N})} &= \lim_m \big\langle C_{\iota_m} C_{\mathrm{pr}_n} f \,\big|\, C_{\iota_m} g\big\rangle_{\mathcal{F}(\mathbb{C}^m)}\\
&= \lim_{m\ge n} \int_{\mathbb{C}^m} f(z_n)\, \overline{g(z_m,0)}\, e^{\pi\|z_m\|^2}\, dz_m\\
&= \lim_{m\ge n} \int_{\mathbb{C}^n} f(z_n) \int_{\mathbb{C}^{m-n}} \overline{g(z_n,z',0)}\, e^{\pi\|z'\|^2}\, dz'\, e^{\pi\|z_n\|^2}\, dz_n\\
&= \lim_{m\ge n} \int_{\mathbb{C}^n} f(z_n)\, \overline{\int_{\mathbb{C}^{m-n}} g(z_n,z',0)\, e^{\pi\langle 0|z'\rangle}\, e^{\pi\|z'\|^2}\, dz'}\, e^{\pi\|z_n\|^2}\, dz_n\\
&= \lim_{m\ge n} \int_{\mathbb{C}^n} f(z_n)\, \overline{g(z_n,0,0)}\, e^{\pi\|z_n\|^2}\, dz_n\\
&= \int_{\mathbb{C}^n} f(z_n)\, \overline{g(z_n,0)}\, e^{\pi\|z_n\|^2}\, dz_n\\
&= \int_{\mathbb{C}^n} f(z_n)\, \overline{(g\circ\iota_n)(z_n)}\, e^{\pi\|z_n\|^2}\, dz_n\\
&= \big\langle f \,\big|\, g\circ\iota_n\big\rangle_{\mathcal{F}(\mathbb{C}^n)} = \big\langle f \,\big|\, C_{\iota_n} g\big\rangle_{\mathcal{F}(\mathbb{C}^n)},
\end{aligned}$$

which shows that C_{pr_n} and C_{ι_n} are adjoints of each other and $\|C_{\iota_n}\| = 1$. Hence C_{ι_n} is an orthogonal projection. Let $f \in \mathcal{F}(\ell^2\mathbb{N})$. Given $\epsilon > 0$, we can find a polynomial g such that $\|f-g\|_{\mathcal{F}(\ell^2\mathbb{N})} < \epsilon$. Then

$$\|C_{\iota_n} f - f\|_{\mathcal{F}(\ell^2\mathbb{N})} \le \|C_{\iota_n}(f-g)\|_{\mathcal{F}(\ell^2\mathbb{N})} + \|C_{\iota_n} g - g\|_{\mathcal{F}(\ell^2\mathbb{N})} + \|f-g\|_{\mathcal{F}(\ell^2\mathbb{N})} \le 2\epsilon + \|C_{\iota_n} g - g\|_{\mathcal{F}(\ell^2\mathbb{N})}.$$

We have to show that $\|C_{\iota_n} g - g\|_{\mathcal{F}(\ell^2\mathbb{N})} \to 0$ as $n \to \infty$ for all polynomials g. Observe that for $m \ge n$ one has

$$(\iota_n\circ\mathrm{pr}_n\circ\iota_m)(z_m) = (z_n,0) = \iota_n(z_n) \in \ell^2\mathbb{N}.$$

Hence, for all m greater than the number d_g of variables of g, one has $g \circ \iota_m = g$. Thus

$$\begin{aligned} \|C_{\iota_n} g - g\|_{\mathcal{F}(\ell^2\mathbb{N})} &= \lim_m \|C_{\iota_m} C_{\mathrm{pr}_n} C_{\iota_n} g - C_{\iota_m} g\|_{\mathcal{F}(\mathbb{C}^m)} \\ &= \lim_{m \geq \max(d_g, n)} \|g \circ \iota_n \circ \mathrm{pr}_n \circ \iota_m - g \circ \iota_m\|_{\mathcal{F}(\mathbb{C}^m)} \\ &= \|g \circ \iota_n - g\|_{\mathcal{F}(\mathbb{C}^m)} \end{aligned}$$

tends to zero as n goes to infinity. □

Using the operators C_{ι_m} and C_{pr_m} we can "sandwich" a given operator $\mathcal{K}$ on $\mathcal{F}(\ell^2\mathbb{N})$ to obtain an operator $\mathcal{K}_m := C_{\iota_m} \circ \mathcal{K} \circ C_{\mathrm{pr}_m}$ on $\mathcal{F}(\mathbb{C}^m)$. The following proposition gives an explict formula for $\mathcal{K}_m$.

Proposition B.4.2. *Let $a, b \in \ell^2\mathbb{N}$ and $\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}\| < 1$. Let*

$$\mathcal{K} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}), \ (\mathcal{K}f)(z) = e^{\pi\langle z|a\rangle} f(\mathbb{B}z + b)$$

be the corresponding composition operator. For all $m \in \mathbb{N}$ let $\mathbb{B}_m := \mathrm{pr}_m \mathbb{B} \iota_m \in \mathrm{Mat}(m, m; \mathbb{C})$, $a_m := \mathrm{pr}_m a$, and $b_m := \mathrm{pr}_m b \in \mathbb{C}^m$. Then $\mathcal{K}_m := C_{\iota_m} \circ \mathcal{K} \circ C_{\mathrm{pr}_m} = C_{\iota_m} \circ \mathcal{K} \circ C^*_{\iota_m}$ acts via

$$(\mathcal{K}_m f)(z_m) = e^{\pi\langle z_m|a_m\rangle} f(\mathbb{B}_m z_m + b_m),$$

i. e. $\mathcal{K}_m = \mathcal{K}_{a_m, b_m, \mathbb{B}_m}$. If $\mathcal{K}$ is positive, then $\mathcal{K}_m$ is positive.

Proof. Let $f \in \mathcal{F}(\mathbb{C}^m)$, then

$$(\mathcal{K}_m f)(z_m) = (\phi \circ \iota_m)(z_m)\, (f \circ \mathrm{pr}_m \circ \psi \circ \iota_m)(z_m) = e^{\pi\langle \iota_m z_m|a\rangle} (f \circ \mathrm{pr}_m)(\mathbb{B}\iota_m z_m + b)$$

together with $(\iota_m)^* = \mathrm{pr}_m$ shows the stated formula. Since $C^*_{\iota_m} = C_{\mathrm{pr}_m}$ by Lemma B.4.1, with $\mathcal{K}$ is also $\mathcal{K}_m$ positive. □

We are now prepared to prove the main theorem of this appendix which is one of the key ingredients in our construction of trace class Ruelle-Mayer transfer operators satisfying a dynamical trace formula (Theorem 2.7.6). It yields the Atiyah-Bott fixed point formula for a special type of composition operators and a formula for their trace norm, which in general is difficult to determine.

Theorem B.4.3. *Let $a, b \in \ell^2\mathbb{N}$ and $\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}\| < 1$. Let*

$$\mathcal{K} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}), \ (\mathcal{K}f)(z) = e^{\pi\langle z|a\rangle} f(\mathbb{B}z + b)$$

be the corresponding composition operator. Set

$$\Lambda = \sqrt{\mathbb{B}\mathbb{B}^\star}, \ \beta = (1 + \sqrt{\mathbb{B}\mathbb{B}^\star})^{-1}(\mathbb{B}a + b), \ \gamma = \exp\Big(\frac{\pi}{2}(\|a\|^2 - \|\beta\|^2)\Big)$$

and $K : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N})$, $(Kf)(z) := \gamma\, e^{\pi\langle z|\beta\rangle} f(\Lambda z + \beta)$. Then $K = |\mathcal{K}| = \sqrt{\mathcal{K}^\star\mathcal{K}}$,

$$\|\mathcal{K}\|_{\mathcal{S}_1(\mathcal{F}(\ell^2\mathbb{N}))} = \frac{\exp\Big(\frac{\pi}{2}\|a\|^2 + \frac{\pi}{2}\|(1 - \mathbb{B}\mathbb{B}^\star)^{-1/2}(\mathbb{B}a + b)\|^2\Big)}{\det(1 - |\mathbb{B}|)} = \frac{\gamma}{\det(1 - \Lambda)} \exp\Big(\pi\|(1 - \Lambda)^{-1/2}\beta\|^2\Big),$$

and $\mathcal{K}$ is trace class with

$$\mathrm{trace}\, \mathcal{K} = \frac{\exp\big(\pi\langle(1 - \mathbb{B})^{-1}b|a\rangle\big)}{\det(1 - \mathbb{B})}.$$

Proof. As in Lemma B.3.10 one gets $K = |\mathcal{K}| = \sqrt{\mathcal{K}^\star\mathcal{K}}$. It remains to show that the trace norm of $\mathcal{K}$, i. e., the trace of K is finite. By change of variables we can assume that the positive operator $\Lambda : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ is indeed diagonal with respect to the standard basis. Let $\zeta_\alpha(z) = \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}}\, z^\alpha$ be the orthonormal basis ($\alpha \in \bigcup_{m\in\mathbb{N}} \mathbb{N}_0^m$) of the Fock space $\mathcal{F}(\ell^2\mathbb{N})$. For all $m \in \mathbb{N}$ the set $\{\zeta_\alpha \,|\, \alpha \in \mathbb{N}_0^m\}$ is an orthonormal basis of $\mathcal{F}(\mathbb{C}^m)$. Hence, using the orthogonality of C_{ι_m} (Lemma B.4.1), we get

$$\big\langle K\zeta_\alpha \,\big|\, \zeta_\alpha\big\rangle_{\mathcal{F}(\ell^2\mathbb{N})} = \big\langle C_{\iota_m} K\zeta_\alpha \,\big|\, C_{\iota_m}\zeta_\alpha\big\rangle_{\mathcal{F}(\mathbb{C}^m)} = \big\langle C_{\iota_m} K C_{\mathrm{pr}_m}\zeta_\alpha \,\big|\, C_{\iota_m}\zeta_\alpha\big\rangle_{\mathcal{F}(\mathbb{C}^m)} = \big\langle K_m\zeta_\alpha \,\big|\, \zeta_\alpha\big\rangle_{\mathcal{F}(\mathbb{C}^m)},$$

where $K_m : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ is given by Proposition B.4.2 as $(K_m f)(z_m) = \gamma\, e^{\pi\langle z_m|\beta_m\rangle} f(\Lambda_m z_m + \beta_m)$ with $\Lambda_m := \mathrm{pr}_m \Lambda \iota_m \in \mathrm{Mat}(m, m; \mathbb{C})$ and $\beta_m := \mathrm{pr}_m \beta \in \mathbb{C}^m$. Hence

$$\sum_{\alpha\in\mathbb{N}_0^m} \langle K\zeta_\alpha \,|\, \zeta_\alpha\rangle = \sum_{\alpha\in\mathbb{N}_0^m} \langle K_m\zeta_\alpha \,|\, \zeta_\alpha\rangle_{\mathcal{F}(\mathbb{C}^m)} = \text{trace } K_m \overset{(*)}{=} \gamma\, \frac{\exp\Big(\pi\|(1-\Lambda_m)^{-1/2}\beta_m\|^2\Big)}{\det(1-\Lambda_m)},$$

where we have applied the Atiyah-Bott formula from Proposition B.3.9 at $(*)$. As m goes to infinity we obtain the trace of K:

$$\begin{aligned} \text{trace } K &= \lim_{m\to\infty} \sum_{\alpha\in\mathbb{N}_0^m} \langle K\zeta_\alpha \,|\, \zeta_\alpha\rangle \\ &= \gamma \lim_{m\to\infty} \frac{\exp\big(\pi\|(1-\Lambda_m)^{-1/2}\beta_m\|^2\big)}{\det(1-\Lambda_m)} \\ &\overset{(\dagger)}{=} \gamma\, \frac{\exp\big(\pi\|(1-\Lambda)^{-1/2}\beta\|^2\big)}{\det(1-\Lambda)} \\ &= \frac{\exp\Big(\frac{\pi}{2}\|a\|^2 + \frac{\pi}{2}\|(1-\mathbb{B}\mathbb{B}^\star)^{-1/2}(\mathbb{B}a+b)\|^2\Big)}{\det(1-|\mathbb{B}|)} < \infty. \end{aligned}$$

For $(\dagger)$ we used Lemma B.4.1 together with [GoGoKr00, Thm. IV 5.5] showing that the pointwise convergence $\Lambda_m \to \Lambda$ is in trace norm such that the limit exists. Thus $\mathcal{K}$ and K are trace class. By Lemma B.4.1 the sequence of trace class operators $\mathcal{K}_m := C_{\iota_m} \circ \mathcal{K} \circ C_{\mathrm{pr}_m}$ converges to $\mathcal{K}$. By [GoGoKr00, Thm. IV 5.5] this convergence is in trace norm, hence

$$\begin{aligned} \text{trace } \mathcal{K} &= \lim_{m\to\infty} \text{trace } \mathcal{K}_m \\ &\overset{(*)}{=} \lim_{m\to\infty} \frac{\exp\big(\pi\,\langle (1-\mathbb{B}_m)^{-1} b_m \,|\, a_m\rangle\big)}{\det(1-\mathbb{B}_m)} \\ &= \frac{\exp\big(\pi\,\langle (1-\mathbb{B})^{-1} b \,|\, a\rangle\big)}{\det(1-\mathbb{B})}, \end{aligned}$$

where $(*)$ is the Atiyah-Bott fixed point theorem for each m combined with Proposition B.4.2. □

As a corollary we obtain an exact formula for the Hilbert-Schmidt norm of a generalised composition operator of the form $\mathcal{K}_{a,b,\Lambda}$. We will use this result in our proof of the dynamical trace for the matrix subshift (Theorem 3.2.6).

Corollary B.4.4. *Let a, $b \in \ell^2\mathbb{N}$ and $\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}\| < 1$. Let*

$$\mathcal{K} : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{K}f)(z) = e^{\pi\langle z|a\rangle} f(\mathbb{B}z + b)$$

be the corresponding composition operator. Then

$$\|\mathcal{K}\|^2_{\mathcal{S}_2(\mathcal{F}(\ell^2\mathbb{N}))} = \frac{\exp\Big(\pi\|a\|^2 + \pi\|(1-\mathbb{B}\mathbb{B}^*)^{-1/2}(\mathbb{B}a+b)\|^2\Big)}{\det(1-\mathbb{B}\mathbb{B}^*)}.$$

Proof. We will use that $\|\mathcal{K}\|^2_{\mathcal{S}_2(\mathcal{F}(\ell^2\mathbb{N}))} = \text{trace } \mathcal{K}\mathcal{K}^*$ together with Theorem B.4.3. By the arguments as in the proof of Lemma B.3.8 one has

$$(\mathcal{K}^\star\mathcal{K}f)(z) = e^{\pi\|a\|^2} e^{\pi\langle z|\mathbb{B}a+b\rangle} f(\mathbb{B}\mathbb{B}^\star z + \mathbb{B}a + b).$$

Hence the assertion follows from the Atiyah-Bott fixed point formula from Theorem B.4.3. □

We remark that for the above Corollary B.4.4 the preassumption $|\mathbb{B}| \in \mathcal{S}_2(\ell^2\mathbb{N})$ would be sufficient, since in this case $|\mathbb{B}|^2 = \mathbb{B}^\star\mathbb{B} \in \mathcal{S}_1(\ell^2\mathbb{N})$.
As an immediate consequence of the preceding Theorem B.4.3, the canonical isomorphism $\mathcal{F}((\ell^2\mathbb{N})^n) \cong \mathcal{F}(\ell^2\mathbb{N})^{\hat{\otimes} n}$, and the properties of the tensor product of operators we obtain the trace and the trace norm of certain tensor products of composition operators. These are used as building blocks of Ruelle-Mayer transfer operators for Ising type interactions. The stated trace norm formula is the reason that we cannot deal with interaction matrices which are not of Ising type as we point out in Remark 2.13.9.

Corollary B.4.5. *Let* $a_i,\, b_i \in \ell^2\mathbb{N}$ *and* $\mathbb{B}_i \in \mathcal{S}_1(\ell^2\mathbb{N})$ *with* $\|\mathbb{B}_i\| < 1$ *for* $i = 1, \ldots, n$. *Let*

$$\mathcal{K}_i : \mathcal{F}(\ell^2\mathbb{N}) \to \mathcal{F}(\ell^2\mathbb{N}),\ (\mathcal{K}_i f)(z) = e^{\pi\langle z|a_i\rangle} f(\mathbb{B}_i z + b_i)$$

be the corresponding composition operators. Let $K_n := \mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n : \mathcal{F}((\ell^2)\mathbb{N}^n) \to \mathcal{F}((\ell^2\mathbb{N})^n)$,

$$(K_n f)(z_1, \ldots, z_n) = \exp\bigl(\pi \sum_{i=1}^{n} \langle z_i|a_i\rangle\bigr)\, f(\mathbb{B}_1 z_1 + b_1, \ldots, \mathbb{B}_n z_n + b_n)$$

be the tensor product of the $\mathcal{K}_i$. *Then*

$$\|K_n\|_{\mathcal{S}_1(\mathcal{F}((\ell^2\mathbb{N})^n))} = \frac{\exp\Bigl(\frac{\pi}{2}\sum_{i=1}^{n}(\|a_i\|^2 + \|(1-\mathbb{B}_i\mathbb{B}_i^\star)^{-1/2}(\mathbb{B}_i a_i + b_i)\|^2)\Bigr)}{\prod_{i=1}^{n}\det(1-|\mathbb{B}_i|)}$$

and K_n *is trace class with*

$$\operatorname{trace} K_n = \frac{\exp\bigl(\pi\sum_{i=1}^{n}\langle(1-\mathbb{B}_i)^{-1}b_i|a_i\rangle\bigr)}{\prod_{i=1}^{n}\det(1-\mathbb{B}_i)}.$$

□

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