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ON THE "FUNDAMENTAL PRINCIPLE" OF L. EHRENPREIS

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1. Introduction and statement of the "Fundamental Principle"

It is very well known that every solution u to a homogeneous linear ordinary differential equation with constant coefficients

$$P\left(\frac{d}{dx}\right)u = 0,$$

P a non-trivial polynomial, is an exponential-polynomial of the form

$$u(x) = \sum_{P(\lambda)=0,\lambda\in C} q_\lambda(x)e^{\lambda x}, \quad x \in \boldsymbol{R},$$

where the polynomials q_λ have degree strictly less than the multiplicity of the root λ. Around 1960 L. Ehrenpreis [3] stated a "Fundamental Principle" which enabled him to give a genuine extension of this result to homogeneous linear partial differential equations with constant coefficients. In this paper we shall give a detailed proof of the "Fundamental Principle" for a single equation. The proof will be self-contained except for some standard facts from functional analysis and for solvability results on the Cauchy–Riemann equations.

Let $n \in \boldsymbol{N}$ and let P be a polynomial in n variables. We have

$$P(\partial/\partial x)\exp\langle z, x\rangle = P(z)\exp\langle z, x\rangle \quad \text{for } x \in \boldsymbol{R}^n,\ z \in \boldsymbol{C}^n.$$

Hence the equation

$$P(\partial/\partial x)u = 0 \tag{1.1}$$

holds if $u = \exp\langle z, \cdot\rangle$ for some $z \in \boldsymbol{C}^n$ with $P(z) = 0$. Roughly speaking the "Fundamental Principle" states that any solution u of (1.1) is a superposition of such special solutions. Since the zero-variety of P

$$V = \{z \in \boldsymbol{C}^n;\ P(z) = 0\}$$

is non-discrete if $n > 1$ we may not only form finite sums but also integrals over exponential (-polynomial) solutions to obtain new solutions of equation (1.1). But since V is also unbounded if $n > 1$ one has to be careful that these integrals exist.

Let us fix some more notation. Choose a non-characteristic vector $N \in \boldsymbol{C}^n$ for P. This means that $P_m(N) \neq 0$ where P_m denotes the principal part of P; m is the degree of P. Let ∂_N denote the following differential operator acting on functions f which are holomorphic in some domain $U \subset \boldsymbol{C}^n$

$$(\partial_N f)(z) = \frac{d}{d\lambda} f(z+\lambda N)|_{\lambda=0}, \quad z \in U.$$

Let $P = P_1^{l_1} \cdot \ldots \cdot P_r^{l_r}$ be the (essentially unique) factorization of P into distinct irreducible factors P_i, $l_i \in \boldsymbol{N}$. For $i = 1, \ldots, r$ consider the subvariety

$$V_i = \{z \in \boldsymbol{C}^n;\ P_i(z) = 0,\ (\partial_N P_i)(z) \neq 0,\ P_j(z) \neq 0 \text{ if } j \neq i\}$$

of the zero-variety of the polynomial P_i. The V_i's are contained in V and $(n-1)$-dimensional complex submanifolds of $\boldsymbol{C}^n$.

For a compact set $K \subset \boldsymbol{R}^n$ let H_K denote its supporting function, i.e.

$$H_K(\xi) = \max\{\langle x, \xi\rangle;\ x \in K\}, \quad \xi \in \boldsymbol{R}^n.$$

We are now able to state the "Fundamental Principle" or

INTEGRAL REPRESENTATION THEOREM. *Let $\Omega \subset \boldsymbol{R}^n$ be open and convex. Let $u \in C^\infty(\Omega)$. Then u is a solution to the equation*

$$P(\partial/\partial x)u = 0 \quad \text{in } \Omega$$

if and only if u has a representation

$$u(x) = \sum_{i=1}^{r} \sum_{j=0}^{l_i-1} \langle x, N\rangle^j \int_{V_i} e^{\langle x,z\rangle} d\mu_{ij}(z), \quad x \in \Omega, \tag{1.2}$$

with Radon measures $d\mu_{ij}$ which are supported by V_i and which satisfy for some positive continuous function ω on C_n with

$$(1+|z|)^M \exp H_K(\operatorname{Re} z) = o(\omega(z)) \quad as \quad |z| \to \infty,$$

for every $M > 0$ and every compact set $K \Subset \Omega$, the condition

$$\int_{V_i} \omega(z)\,|d\mu_{ij}(z)| < +\infty; \quad i = 1, \ldots, r;\ j = 0, 1, \ldots, l_i-1. \tag{1.3}$$

From the Integral Representation Theorem we can immediately derive the following corollary on decomposition of solutions which is of independent interest.

COROLLARY. *Let $\Omega \subset \mathbf{R}^n$ be open and convex, and let Q_1, Q_2 be relatively prime polynomials. Then $u \in C^\infty(\Omega)$ is a solution to $(Q_1Q_2)(\partial/\partial x)u = 0$ in Ω if and only if there exist $u_1, u_2 \in C^\infty(\Omega)$ such that $Q_1(\partial/\partial x)u_1 = 0$, $Q_2(\partial/\partial x)u_2 = 0$ and $u = u_1+u_2$.*

Let us illustrate this corollary by the following well-known examples:

(i) $\Delta = 4\partial\bar{\partial}$. Therefore, every harmonic function of $\mathbf{R}^2$ is the sum of a holomorphic and an antiholomorphic function.

(ii) $\partial^2/\partial t^2 - \partial^2/\partial x^2 = (\partial/\partial t - \partial/\partial x)(\partial/\partial t + \partial/\partial x)$. Therefore, every solution $u \in C^\infty(\mathbf{R}^2)$ to the one-dimensional wave equation can be written as $u(t, x) = f(t+x)+g(t-x)$ with $f, g \in C^\infty(\mathbf{R}^1)$.

(iii) Let $c_1, c_2 \in \mathbf{R}$, $c_1^2 \neq c_2^2$. Let $(\partial^2/\partial t^2 - c_1^2\Delta)(\partial^2/\partial t^2 - c_2^2\Delta)u = 0$, $u \in C^\infty(\mathbf{R}^{n+1})$. Then $u = u_1+u_2$ where $(\partial^2/\partial t^2 - c_i^2\Delta)u_i = 0$, $i = 1, 2$.

The Integral Representation Theorem and the Corollary are in general not true if $\Omega \subset \mathbf{R}^n$ is not convex.

An Integral Representation Theorem is still true if, more generally, we consider an arbitrary homogeneous system of linear differential equations with constant coefficients. This fact constitutes the full "Fundamental Principle". It was proved by L. Ehrenpreis [4] and, independently, by V. I. Palamodov [8]. The proof of the general "Fundamental Principle" is considerably more difficult than the proof of the special case which we give here. In particular, the varieties, the polynomials and the differential operators (called "Noetherian operators" by Palamodov [8]) which are to play the role for a general system which for a single equation is played by the varieties V_i, the polynomials $\langle \cdot, N\rangle^j$ and the differential operators ∂_N^j cannot so easily be given explicitly. Furthermore, it may happen that the polynomials generalizing $\langle \cdot, N\rangle^j$ have to have coefficients depending nontrivially on z; polynomial dependence on z suffices, however. Another ingredient in the proof of the general "Fundamental Principle" which does not appear in this paper is an "Oka Theorem with bounds" (essentially given in Hörmander [5], Proposition 7.6.5, for example). These new features appear when one wants to give the analogs to the Division Lemma (Lemma 2.2) and the Extension Theorem (Theorem 2.3) in Section 2. Other proofs of the general "Fundamental Principle" have been given by O. Liess [7] and, relying on results of Hörmander ([5]; Chapter 7, Section 6) which already cover a large part of the proof of the "Fundamental Principle", by J.-E. Björk [2].

I like to thank Professor Bierstedt who called my attention to his joint work with Meise and Summers on projective descriptions of inductive limits of locally convex function spaces. This helped me to improve the functional analytic part in the proof given below.

2. Deduction of the "Fundamental Principle" from an Extension Theorem

Roughly speaking we have to exhibit the surjectivity on $\ker P(\partial/\partial x)$ of a linear map which maps Radon measures carried by the sets V_i into the space $C^\infty(\Omega)$. This mapping will be the dual $\mathscr{N}'$ to the mapping which we shall define below.

Fix an increasing sequence of compact, convex sets $K_j \Subset \Omega$, $j \in \boldsymbol{N}$, which exhausts Ω. Denote the supporting function H_{K_j} of K_j by H_j for short. For $W \subset \boldsymbol{C}^n$ let $\mathscr{C}(W)$ be the linear space of all continuous complex valued functions f on W which satisfy for some $j \in \boldsymbol{N}$

$$q_j(f) = \sup_{z\in W} |f(z)|(1+|z|)^{-j}\exp\big(-H_j(\operatorname{Re} z)\big) < +\infty. \tag{2.1}$$

Equip $\mathscr{C}(W)$ with the natural locally convex inductive limit topology. As a locally convex space $\mathscr{C}(W)$ only depends on W and on Ω, not on the particular choice of the sequence $(K_j)_{j\in\boldsymbol{N}}$. The linear map

$$\mathscr{N}\nu = (\partial_N^j \tilde{\nu}|_{V_i})_{i=1,j=1}^{r,l_i-1}, \qquad \nu \in \mathscr{E}'(\Omega),$$

is continuous from $\mathscr{E}'(\Omega)$ into $\prod_{i=1}^{r} \mathscr{C}(V_i)^{l_i}$ because of the Paley–Wiener theorem.

Here $\tilde{\nu}(z) = \langle \nu, e^{\langle z,\cdot\rangle}\rangle = \hat{\nu}(iz)$, $z \in \boldsymbol{C}^n$, $\nu \in \mathscr{E}'(\Omega)$, where $\hat{}$ denotes the Fourier transform.

Let us call those positive continuous functions ω on $\boldsymbol{C}^n$ which satisfy for every $j \in \boldsymbol{N}$,

$$(1+|z|)^j \exp H_j(\operatorname{Re} z) = o\big(\omega(z)\big) \quad \text{as} \quad |z| \to \infty,$$

weight functions. Then we can give the following description of the space $\mathscr{C}(W)$ which is needed for the proof of the Integral Representation Theorem. This description is intimately connected with Ehrenpreis' PLAU-space approach to the "Fundamental Principle" (see Ehrenpreis [4] and Berenstein–Dostal [1]), however, we shall not go into this here.

PROPOSITION 2.1. *The natural locally convex inductive limit topology on $\mathscr{C}(W)$ coincides with the seminorm topology given by the seminorms*

$$p_\omega(f) = \sup_{z\in W} |f(z)|/\omega(z), \quad f \in \mathscr{C}(W),$$

where ω runs through all weight functions. $\mathscr{C}(W)$ coincides with the space of all continuous functions f on W satisfying $|f(z)|/\omega(z) = o(1)$ as $|z| \to \infty$ for every weight function ω. Every bounded set in $\mathscr{C}(W)$ is already bounded with respect to some seminorm q_k, and $\mathscr{C}(W)$ is a (DF)-space.

Proof. The inductive limit topology on $\mathscr{C}(W)$ is clearly finer than the topology defined by all seminorms p_ω. Therefore, let $U \subset \mathscr{C}(W)$ be a closed neighbourhood of zero with respect to the inductive limit topology. We have to find a weight function ω such that $\{f \in \mathscr{C}(W);\ |f| < \omega\} \subset U$. With a suitable sequence of positive numbers $\varepsilon_j \searrow 0$ we have

$$\operatorname{ch} \bigcup_{j=1}^{\infty} \{f \in \mathscr{C}(W);\ q_j(f) \leqslant \varepsilon_j\} \subset U. \tag{2.2}$$

Here ch stands for convex hull. Choose a continuous partition of unity $(\varphi_j)_{j=1}^{\infty}$ on C^n such that the supports of the functions φ_j are bounded and

$$(1+|z|)^{-1} \leqslant 2^{-j}\varepsilon_j \quad \text{if} \quad \varphi_j(z) \neq 0,\ j \geqslant 2. \tag{2.3}$$

Choose $0 < \delta_j \leqslant 1$, $j \in N$, such that

$$\delta_j(1+|z|)^j \exp H_j(\operatorname{Re} z) \leqslant 2^{-k}\varepsilon_k(1+|z|)^k \exp H_k(\operatorname{Re} z) \tag{2.4}$$

if $\varphi_k(z) \neq 0$, $1 \leqslant k \leqslant j$. This implies immediately that

$$\omega(z) := \sup_{j \in N} \delta_j(1+|z|)^j \exp H_j(\operatorname{Re} z) < +\infty, \qquad z \in C^n, \tag{2.5}$$

defines a weight function. Let $f \in \mathscr{C}(W)$ satisfy $|f(z)| \leqslant \omega(z)$, $z \in W$. Using $\delta_j \leqslant 1$ and $H_j \leqslant H_k$, $j < k$, we then obtain with the definitions (2.1) and (2.5)

$$\begin{aligned} q_k(\varphi_k f) &\leqslant \sup_{\varphi_k(z)\neq 0} \omega(z)\big((1+|z|)^k \exp H_k(\operatorname{Re} z)\big)^{-1} \\ &\leqslant \sup_{\varphi_k(z)\neq 0} \max\Big(\sup_{1\leqslant k\leqslant j} \delta_j(1+|z|)^{j-k} \exp\big(H_j(\operatorname{Re} z) - H_k(\operatorname{Re} z)\big),\ (1+|z|)^{-1}\Big). \end{aligned}$$

Hence it follows from (2.3) and (2.4) that

$$q_k(\varphi_k f) \leqslant 2^{-k}\varepsilon_k$$

for $k = 2, 3, \ldots$ and also for $k = 1$ as is obvious from (2.4). Therefore, by (2.2) we have

$$f = \sum_{1}^{\infty} 2^{-k}(2^k \varphi_k f) \in U, \tag{2.6}$$

provided the series converges in the inductive limit topology. As for this note that $q_j(f) < +\infty$ for some j. From this it easily follows that the partial sums in (2.6) converge to f with respect to the seminorm q_{j+1}.

Now, let B be a set of continuous complex valued functions on W which is bounded with respect to all seminorms p_ω. We have to show that for some $k \in N$

$$B \subset \{f \in \mathscr{C}(W);\ q_k(f) \leqslant k\}. \tag{2.7}$$

Suppose the contrary is true. This means that we can find a sequence $(f_k)_{k=1}^{\infty}$ of continuous functions on W which is bounded with respect to every p_ω and an unbounded sequence $(z_k)_{k=1}^{\infty}$, $z_k \in W$, such that

$$|f_k(z_k)| > k(1+|z_k|)^k \exp H_k(\operatorname{Re} z_k), \quad k = 1, 2, \ldots \tag{2.8}$$

Choose a sequence $(\delta_j)_{j=1}^{\infty}$ which satisfies

$$0 < \delta_j(1+|z_k|)^j \exp H_j(\operatorname{Re} z_k) \leqslant (1+|z_k|)^k \exp H_k(\operatorname{Re} z_k), \quad 1 \leqslant k \leqslant j, \tag{2.9}$$

and is such that

$$\omega(z) = \sup_{j=1,2,\ldots} \delta_j(1+|z|)^j \exp H_j(\operatorname{Re} z)$$

is a weight function. Since $H_j \leqslant H_k$, $j < k$, and $\delta_j \leqslant 1$ by (2.9) it follows from (2.9) that we have

$$\omega(z_k) \leqslant (1+|z_k|)^k \exp H_k(\operatorname{Re} z_k), \quad k = 1, 2, \ldots$$

But this contradicts (2.8) since $|f_k(z_k)| = O(\omega(z_k))$ as $k \to \infty$ and therefore (2.7) holds with some $k \in \mathbf{N}$. We have thus in particular shown that $\mathscr{C}(W)$ has a fundamental sequence of bounded sets. Therefore, being a countable inductive limit of Banach spaces and thus in particular barrelled, $\mathscr{C}(W)$ is a (DF)-space (see e.g. Köthe [6], p. 400). This completes the proof of Proposition 2.1.

We shall show that

$$\ker P(\partial/\partial x) = \operatorname{im} \mathscr{N}'. \tag{2.10}$$

Applying this together with Proposition 2.1 and the Riesz Representation Theorem we arrive at a proof of the "Fundamental Principle": $u \in C^\infty(\Omega)$ satisfies $P(\partial/\partial x)u = 0$ if and only if there exist Radon measures $d\mu_{ij}$ supported by V_i such that

$$\langle u, \nu\rangle = \sum_{i=1}^{r}\sum_{j=0}^{l_i-1} \int_{V_i} (\partial_N^j \tilde{\nu})(z)\, d\mu_{ij}(z), \quad \nu \in \mathscr{E}'(\Omega), \tag{2.11}$$

and (1.3) holds for some weight function ω. Inserting Dirac measures for ν in (2.11) we obtain (1.2).

Let us first show that $\ker P(\partial/\partial x)$ equals the weak closure of $\operatorname{im} \mathscr{N}'$. This assertion follows by duality from

LEMMA 2.2 (Division Lemma). *Let $\nu \in \mathscr{E}'(\Omega)$. Then $\mathscr{N}\nu = 0$ if and only if there exists $\mu \in \mathscr{E}'(\Omega)$ such that $\nu = P(-\partial/\partial x)\mu$.*

Proof. Let $\nu = P(-\partial/\partial x)\mu$ with $\mu \in \mathscr{E}'(\Omega)$. Then $\tilde{\nu}(z) = \hat{\nu}(iz) = P(z)\tilde{\mu}(z)$, $z \in \mathbf{C}^n$. From the definition of $\mathscr{N}$ it is clear that $\mathscr{N}\nu = 0$. Now, assume that $\mathscr{N}\nu = 0$. Consider $\tilde{\nu}/P$ which is a holomorphic function outside V. If $z \in V$ does not belong to set

$$V^* = \{z \in V;\ P_i(z) = P_j(z) = 0 \text{ for some } i \neq j \text{ or } P_i(z) = \partial_N P_i(z) = 0 \text{ for some } i\}$$

then by Riemann's theorem on removable singularities the function

$$\frac{1}{2\pi i}\int_{|\lambda|=\delta}\frac{\tilde{\nu}(\cdot+\lambda N)}{P(\cdot+\lambda N)}\frac{d\lambda}{\lambda}, \qquad 0<\delta<\inf_{z'\in V^*}|z-z'|,$$

gives a holomorphic extension of $\tilde{\nu}/P$ to a neighbourhood of z. By Theorem A.3 in the Appendix we may regard $\tilde{\nu}/P$ as an entire function since the possible singularities of $\tilde{\nu}/P$ are contained in the "essentially 2-codimensional" subvariety V^* of C^n. Using the well-known Ehrenpreis–Malgrange inequality (see Lemma A.1, Appendix) we obtain with some constant $c>0$ independent of $z\in C^n$

$$|\tilde{\nu}(z)/P(z)|\leqslant c\sup_{|z'|\leqslant 1}|\tilde{\nu}(z+z')|.$$

Therefore, by the Paley–Wiener theorem, there is a distribution $\mu\in\mathscr{E}'(\Omega)$ such that $\nu=P(-\partial/\partial x)\mu$. Thus Lemma 2.2 is proved.

To complete the proof of the "Fundamental Principle" we have to show that im $\mathscr{N}'$ is closed in $C^\infty(\Omega)$. We shall deduce this fact from the following result which is the major step in the proof of the "Fundamental Principle".

THEOREM 2.3 (Extension Theorem). *Let φ be a plurisubharmonic function on C^n which satisfies for some positive constant $C>0$*

$$|\varphi(z_1)-\varphi(z_2)|<C \quad \text{if} \quad |z_1-z_2|<1,\ z_1,z_2\in C^n. \tag{2.12}$$

Let f be an entire function such that

$$\sup_{z\in V_i}|(\partial_N^j f)(z)|e^{-\varphi(z)}\leqslant 1 \quad \text{for all } i,j. \tag{2.13}$$

Then there exists an entire function g such that

$$\begin{aligned}&\partial_N^j(f-g)_{|V_i}=0 \quad \text{for all } i,j,\\ &\sup_{z\in C^n}|g(z)|e^{-\varphi(z)}(2+|z|)^{-M}\leqslant 1.\end{aligned} \tag{2.14}$$

Here $M>0$ is independent of f.

For the definition and some properties of plurisubharmonic functions, see e.g. Hörmander [5]. Note that

$$\varphi(z)=M\log(1+|z|^2)+H(\mathrm{Re}\,z), \qquad z\in C^n,$$

is plurisubharmonic if $M>0$ and if H is the supporting function of a compact, convex set. Hence, using the Paley–Wiener theorem, Proposition 2.1 and Theorem 2.3, we can apply the following lemma from functional analysis with $E=\mathscr{E}'(\Omega)$, $F=\prod_{i=1}^{r}\mathscr{C}(V_i)^{l_i}$ and $T=\mathscr{N}$ to conclude that the range of $\mathscr{N}'$ is (weakly) closed in $C^\infty(\Omega)$. Thus (2.10) holds.

LEMMA 2.4. *Let E and F be (DF)-spaces. Assume that every closed bounded subset of E is compact. Let $T\colon E \to F$ be a bounded linear map such that for every bounded $B_F \subset F$ we can find a bounded $B_E \subset E$ with $B_F \subset T(B_E)$. Then* $\operatorname{im} T'$ *is weakly closed in E'.*

Proof. The strong duals E'_β and F'_β are Fréchet spaces (see e.g. Köthe [6], p. 400). Let R denote the closure of $\operatorname{im} T'$ in E'_β. Note that, since all closed bounded subsets of E are compact, the weak-$*$-closure and the strong closure of an absolutely convex subset in E'_β coincide. In particular, R equals the polar $(\ker T)^0$. Let V be a neighbourhood of zero in F'_β. There is an absolutely convex bounded subset $B_F \subset F$ such that the polar B_F^0 is contained in V. By assumption there is a bounded set $B_E \subset E$ with $T^{-1}(B_F) \subset B_E + \ker T$. Using the Bipolar theorem we have therefore in the weak-$*$-topology of E'

$$\overline{T'(V)} \supset \overline{T'(B_F^0)} = \big(T^{-1}(B_F)\big)^0 \supset B_E^0 \cap (\ker T)^0. \tag{2.15}$$

Since (2.15) also holds with closures taken with respect to the strong topology we may conclude by the Banach–Schauder theorem (e.g. Köthe [6], pp. 169–170) that T' considered as a mapping from F'_β into R is open and onto. This proves Lemma 2.4.

3. Construction of local extensions

The proof of Theorem 2.3 will be based on its following local version which is essentially what Ehrenpreis calls the "semilocal quotient structure theorem" for principal ideals (see [4], Theorem 3.1).

LEMMA 3.1. *Let f be an entire function on C^n and let $\zeta \in C^n$. Then there exists a function g which is defined and holomorphic in $U_\zeta = \{z \in C^n;\ |z-\zeta| < (1+|\zeta|)^{-2m}\}$ and which satisfies*

$$\begin{gathered} \partial_N^j (f-g)_{|V_i \cap U_\zeta} = 0, \quad \text{for all } i, j, \\ \sup_{z \in U_\zeta} |g(z)| \leqslant (2+|\zeta|)^M \max_{i,j} \sup_{z \in V_i, |z-\zeta| \leqslant M} |\partial_N^j f(z)|. \end{gathered} \tag{3.1}$$

Here i, j vary according to $i = 1, \ldots, r$ and $j = 0, 1, \ldots, l_i - 1$, and $M > 0$ is a constant independent of f and ζ.

Proof. Let us introduce coordinates in C^n such that $N = (0, \ldots, 0, 1)$. For $z \in C^n$ we shall denote by $z' \in C^{n-1}$ the first $n-1$ coordinates and by z_n the last coordinate. With polynomials a_k of degree at most k we have

$$P(z', z_n) = \sum_{j=0}^{m} a_{m-j}(z') z_n^j, \quad z \in C^n.$$

Let $\zeta = (\zeta', \zeta_n) \in C^n$. Introduce the set

$$W_\zeta = \{z' \in C^{n-1};\ |z'-\zeta'| < 2(1+|\zeta|)^{-2m}\}.$$

By the mean value theorem we have with a constant $c > 0$ depending only on P

$$|a_k(z') - a_k(\zeta')| \leqslant c|z'-\zeta'|(1+|\zeta'|)^m \leqslant 2c(1+|\zeta|)^{-m},$$

for $z' \in W_\zeta$, $k = 0, 1, \ldots, m$. Since a_0 is constant we therefore obtain for $s \in C$ and $z' \in W_\zeta$

$$\text{(3.2)} \qquad |P(z', s) - P(\zeta', s)| \leqslant 2c(1+|\zeta|)^{-m} m(1+|s|)^{m-1} \leqslant 2mc(1+|s-\zeta_n|)^{m-1}.$$

Now choose a positive constant b independent of ζ such that

$$\text{(3.3)} \qquad 2mc(2+2mb)^{m-1} < |P_m(N)|b^m.$$

Let K_ζ be the union of all connected components of the set

$$\{s \in C;\ |s-s_0| < b \text{ for some } s_0 \in C \text{ with } P(\zeta', s_0) = 0\}$$

which have a non-empty intersection with the ball $\{s \in C;\ |s-\zeta_n| < 1\}$. It is geometrically obvious that

$$\text{(3.4)} \qquad |s-\zeta_n| \leqslant 1+2mb \quad \text{if} \quad s \in \overline{K}_\zeta.$$

Since $a_0(\zeta') = P_m(N) \neq 0$ we obtain after factorizing the polynomial $P(\zeta', s)$ in the variable s into linear factors

$$\text{(3.5)} \qquad |P(\zeta', s)| \geqslant |P_m(N)|b^m, \quad s \in \partial K_\zeta.$$

Using (3.2), (3.4) and (3.5) we arrive in view of the choice of b (3.3) at

$$\text{(3.6)} \qquad |P(z', s) - P(\zeta', s)| < |P(\zeta', s)|, \quad z' \in W_\zeta,\ s \in \partial K_\zeta.$$

Let D be the set of all $z' \in C^{n-1}$ for which there exists a common zero to $P_i(z', \cdot)$, $\dfrac{d}{dz_n} P_i(z', \cdot)$ for some i or to $P_i(z', \cdot)$, $P_j(z', \cdot)$ for some $i \neq j$. D is the zero variety of discriminants and resultants and is not equal to all C^{n-1}. From (3.6) and Rouché's theorem it therefore follows that the polynomial $P(z', \cdot)$ has exactly $q = \sum_{i=1}^{r} \deg P_i$ distinct roots $s_i(z')$, $z' \in C^{n-1} \setminus D$, with multiplicity m_i, $i = 1, \ldots, q$, which are ordered such that with some integer p, $1 \leqslant p \leqslant q$, we have for $z' \in W_\zeta \setminus D$

$$\text{(3.7)} \qquad \begin{aligned} &s_i(z') \in K_\zeta \quad \text{if} \quad 1 \leqslant i \leqslant p, \\ &s_i(z') \notin \overline{K}_\zeta \quad \text{if} \quad p < i \leqslant q. \end{aligned}$$

Furthermore, note that $|P(z', s) - P_m(N)s^m| < |P_m(N)s^m|$, $|s| = c(1+|z'|)$, with $c > 0$ depending only on P, and therefore we have by Rouché's theorem

$$|s_i(z')| < c(1+|z'|), \quad z' \in C^{n-1}\setminus D,\ i = 1, \dots, q. \tag{3.8}$$

After these preparations we can now give the construction of the desired function g for the given entire function f. g will be given as a polynomial in z_n

$$g(z) = \sum_{k=0}^{m-1} g_k(z') z_n^k \tag{3.9}$$

where the functions g_k have to be determined such that (3.1) holds. This leads in view of (3.7) to the following interpolation problem

$$\frac{d^j}{dz_n^j} g(z', s_i(z')) = \sum_{k=j}^{m-1} g_k(z') \frac{k!}{(k-j)!} s_i(z')^{k-j} = \begin{cases} \dfrac{d^j}{dz_n^j} f(z', s_i(z')), & 1 \leqslant i \leqslant p, \\ 0, & p < i \leqslant q, \end{cases}$$

$0 \leqslant j < m_i$, $z' \in W_\zeta \setminus D$. The determinant of this $m \times m$-system of linear equations for the g_k equals $\pm \Delta(s_1(z'), \dots, s_q(z'))$, where

$$\Delta(\sigma_1, \dots) = \begin{vmatrix} 1 & \sigma_1 & \sigma_1^2 & \dots & & & \sigma_1^{m-1} \\ 0 & 1 & & & & & \\ \vdots & & & & & & \\ 0 & & 0 & 1 & \dots & \dfrac{(m-1)!}{(m-m_1)!} & \sigma_1^{m-m_1} \\ 1 & \sigma_2 & & & & & \\ \vdots & & & & & & \end{vmatrix}.$$

$(\sigma_1, \dots, \sigma_q) \in C^q$, is a generalized Vandermonde determinant which vanishes if and only if $\sigma_i = \sigma_j$ for some $i \neq j$. By Cramer's rule we thus have

$$g_k(z') = \delta_k(z')/\delta(z'), \tag{3.10}$$

$z' \in W_\zeta \setminus D$; $k = 0, 1, \dots, m-1$; where

$$\delta(z') = \left(\Delta(s_1(z'), \dots, s_q(z'))\right)^2,$$
$$\delta_k(z') = \Delta_k(z', s_1(z'), \dots, s_q(z')) \cdot \Delta(s_1(z'), \dots, s_1(z')).$$

$\Delta_k(z', \sigma_1, \dots, \sigma_q)$ is the determinant of the $m \times m$-matrix which equals the matrix defining $\Delta(\sigma_1, \dots)$ with the $(k+1)$-st column replaced by

$$^t\left(f(z', \sigma_1), \dots, \frac{d^{m_p-1}}{d\sigma_p^{m_p-1}} f(z', \sigma_q), 0, \dots, 0\right).$$

Since they are symmetric in the zeros $s_1(z'), \dots, s_p(z')$ and in the zeros $s_{p+1}(z'), \dots, s_q(z')$ the functions δ and δ_k are well-defined and holomorphic in $W_\zeta \setminus D$. Furthermore δ and δ_k stay bounded near D and can therefore by Riemann's theorem on removable singularities be continued to holomorphic functions in W_ζ. Because its definition does not depend on ζ the function δ is entire. It is in fact a polynomial since it grows only polynomially by (3.8).

We now want to show that the quotients (3.10) can also be continued holomorphically across D. Again, by Riemann's theorem this follows once we have shown that the functions

$$(3.11) \qquad (z', \sigma_1, \ldots, \sigma_q) \to \Delta_k(z', \sigma_1, \ldots, \sigma_q)/\Delta(\sigma_1, \ldots, \sigma_q),$$

$k = 0, 1, \ldots, m-1$, can be extended to holomorphic functions in $W_\zeta \times \times (K_\zeta)^p \times (C \setminus \overline{K}_\zeta)^{q-p}$. This is a problem only for those points $(z', \sigma_1, \ldots \ldots, \sigma_q)$ where $\sigma_i = \sigma_j$ for some $i < j$ such that $i, j \leqslant p$ or $p < i, j$. Let such a point be given where in addition all $\sigma_1, \overset{i}{\ldots}, \sigma_q$ are distinct. Recall the rule for differentiating the determinant of a matrix depending on a parameter λ

$$\frac{d^l}{d\lambda^l}\begin{vmatrix} A_1(\lambda) \\ \vdots \\ A_m(\lambda) \end{vmatrix} = \sum_{|a|=l} \begin{vmatrix} A_1^{(a_1)}(\lambda) \\ \vdots \\ A_m^{(a_m)}(\lambda) \end{vmatrix}, \qquad l = 0, 1, \ldots$$

Here $A_1(\lambda), \ldots, A_m(\lambda)$ are the rows of the matrix and the a's are multi-indices, $a \in (N \cup \{0\})^m$. Therefore, the lth derivative at $\lambda = 0$ of the following functions of λ

$$(3.12) \qquad \Delta(\sigma_1, \ldots, \sigma_i + \lambda, \sigma_{i+1}, \ldots, \sigma_j, \ldots),$$

$$(3.13) \qquad \Delta_k(z', \sigma_1, \ldots, \sigma_i + \lambda, \sigma_{i+1}, \ldots, \sigma_j, \ldots),$$

$k = 0, 1, \ldots, m-1$, is a sum over $a \in (N \cup \{0\})^{m_i}$, $|a| = l$, of terms which vanish when the following $m_i + m_j$ nonnegative integers

$$(3.14) \qquad a_0, 1 + a_1, \ldots, m_i - 1 + a_{m_i - 1}, 0, 1, 2, \ldots, m_j - 1$$

are not all distinct; for the function (3.12) the terms vanish only in this case because they are generalized Vandermonde determinants. The sum of all integers in the list (3.14) is $l + \big((m_i - 1)m_i + (m_j - 1)m_j\big)/2$. Since the sum of $m_i + m_j$ distinct nonnegative integers is at least $(m_i + m_j - 1)(m_i + m_j)/2$ it follows that the lth derivative at $\lambda = 0$ of the functions (3.12) and (3.13) vanishes if $0 \leqslant l < m_i m_j$. If now $l = m_i m_j$ then all integers in the list (3.14) are different for $a \in (N \cup \{0\})^{m_i}$, $|a| = l$, if and only if $a_k = m_j$ for $k = 0, 1, \ldots, m_i - 1$. Hence

$$\frac{d^l}{d\lambda^l} \Delta(\sigma_1, \ldots, \sigma_i + \lambda, \sigma_{i+1}, \ldots, \sigma_j, \ldots)|_{\lambda=0} \neq 0, \quad l = m_i m_j.$$

Thus it follows from Riemann's theorem that the function (3.11) can be regarded as a holomorphic function outside the union of the 2-codimensional submanifolds $\sigma_i = \sigma_j$, $i \neq j$, of $W_\zeta \times (K_\zeta)^p \times (C \setminus \overline{K}_\zeta)^{q-p}$. The remaining singularities of (3.11) can be removed with a theorem of Hartogs (Lemma A.2 in the Appendix). Thus we have shown that g_k has a holomorphic extension to W_ζ for $k = 0, 1, \ldots, m-1$.

So we may apply the Ehrenpreis–Malgrange inequality (Lemma A.1 in the Appendix) to the quotient (3.10) and obtain using (3.8)

$$\sup_{|z'-\zeta'|<(1+|\zeta|)^{-2m}} |g_k(z')| \leqslant (2+|z|)^M \max_{1\leqslant i\leqslant p, 0\leqslant j<m_i} \sup_{|z'-\zeta'|\leqslant M} \left| \frac{d^j}{dz_n^j} f(z', s_i(z')) \right|,$$

$k = 0, 1, \ldots, m-1$. Here $M > 0$ depends only on P. In view of (3.4) and (3.7) Lemma 3.1 follows with g given by (3.9) and some larger constant M.

4. Proof of the Extension Theorem

The local extensions obtained in Lemma 3.1 will be pasted together to give the desired global extension of the entire function f. This pasting procedure follows the pattern given by Čech cohomology theory for coherent analytic sheaves. Here, however, we have to be careful with the bounds on the various cochains which occur in the procedure.

Let us first give suitable coverings of C^n and an associated partition of unity. Fix a sequence $(\zeta_k)_{k\in N}$, $\zeta_k \in C^n$, such that for the balls

$$U_k^a = \{z \in C^n;\ |z-\zeta_k| < a(1+|\zeta_k|)^{-2m}\}, \quad k \in N,\ a > 0,$$

we have

$$\bigcup_{k\in N} U_k^{1/4} = C^n$$

and such that the number of distinct balls $U_k^{1/2}$ which have non-empty intersection is bounded by a fixed number m_0. We leave it to the reader to show that such a sequence $(\zeta_k)_{k\in N}$ can be found. To construct a partition of unity we choose $\psi \in C_0^\infty(C^n)$, $0 \leqslant \psi \leqslant 1$, with $\psi(z) = 1$ if $|z| \leqslant 1/4$ and $\operatorname{supp}\psi \subset \{z \in C^n;\ |z| < 1/2\}$. We then have

$$1 \leqslant \Psi(z) = \sum_{k\in N} \psi\big((\cdot - \zeta_k)(1+|\zeta_k|)^{2m}\big) \leqslant m_0.$$

For the partition of unity

$$\Phi_k = \psi\big((\cdot - \zeta_k)(1+|\zeta_k|)^{2m}\big)/\Psi, \quad k \in N,$$

it follows that with some constant $M_0 > 0$

$$(4.1) \qquad |\bar\partial \Phi_k(z)| \leqslant (2+|z|)^{M_0}, \quad z \in C^n,\ k \in N.$$

Here $\bar{\partial}$ denotes the Cauchy–Riemann operator acting from (p, q)-forms into $(p, q+1)$-forms (see Hörmander [5]).

Now, let φ plurisubharmonic and f entire be given such that (2.12) and (2.13) hold. From (2.12) we can immediately derive

$$(4.2)\qquad |\varphi(z_1)-\varphi(z_2)| < l\cdot C \quad \text{if} \quad |z_1-z_2| < l; \quad z_1, z_2 \in \boldsymbol{C}^n;\ l \in \boldsymbol{N}.$$

From Lemma 3.1 we obtain functions g_k, $k \in \boldsymbol{N}$, holomorphic in U_k^1 such that

$$(4.3)\qquad \partial_N^j(f-g_k)_{|V_i\cap U_k^1} = 0; \quad i = 1, \ldots, r;\ j = 0, 1, \ldots, l_i-1;$$

and which by (2.13) and (4.2) can be estimated

$$(4.4)\qquad |g_k(z)| \leqslant (2+|z|^2)^M e^{\varphi(z)}, \quad z \in U_k^1,\ k \in \boldsymbol{N}.$$

Here and in the following M denotes a positive constant which only depends on P; however, M may well be different in different formulas.

We want to find functions h_k holomorphic in $U_k^{1/2}$ which satisfy "good bounds" and are such that $P(h_k-h_l) = g_k-g_l$ holds in $U_k^{1/2}\cap U_l^{1/2}$ for all k, l. Then the entire function g which equals g_k-Ph_k in $U_k^{1/2}$ will be the desired extension.

Arguing as in the proof of Lemma 2.2 we obtain from (4.3) that the quotient $(g_k-g_l)/P$ can be extended to a holomorphic function in $U_k^1\cap U_l^1$. By Lemma A.1 (see Appendix) and (4.2), (4.4) we have for all $k, l \in \boldsymbol{N}$

$$(4.5)\qquad \left|\big(g_k(z)-g_l(z)\big)/P(z)\right| \leqslant (2+|z|^2)^M e^{\varphi(z)}, \quad z \in U_k^{1/2}\cap U_l^{1/2}.$$

The function

$$(4.6)\qquad h'_k = \sum_{j\in \boldsymbol{N}} \Phi_j(g_k-g_j)/P, \quad k \in \boldsymbol{N},$$

is well-defined in $U_k^{1/2}$ and gives the decomposition

$$(4.7)\qquad h'_k-h'_l = (g_k-g_l)/P \quad \text{in } U_k^{1/2}\cap U_l^{1/2}.$$

Furthermore, using (4.5) and the fact that at most m_0 terms in (4.6) are non-zero for every $z \in U_k^{1/2}$, we have

$$(4.8)\qquad |h'_k(z)| \leqslant (2+|z|^2)^M e^{\varphi(z)}, \quad z \in U_k^{1/2},\ k \in \boldsymbol{N}.$$

Since the right-hand side in (4.7) is holomorphic the $(0,1)$-form H with $H = \bar{\partial}h'_k$ in $U_k^{1/2}$, $k \in \boldsymbol{N}$, is well-defined in $\boldsymbol{C}^n$. Again, since the number of non-zero terms in the sums

$$(4.9)\qquad (\bar{\partial}h'_k)(z) = \sum_{j\in \boldsymbol{N}} \big((g_k-g_j)/P\big)(z)(\bar{\partial}\Phi_j)(z), \quad z \in U_k^{1/2},\ k \in \boldsymbol{N},$$

is bounded by m_0, it follows from (4.1) and (4.5) that

$$|H(z)| \leqslant (2+|z|^2)^M e^{\varphi(z)}, \quad z \in \mathbf{C}^n. \tag{4.10}$$

With the constant M in (4.10) now define the plurisubharmonic function

$$\sigma = (2M+n+1)\log(2+|\cdot|^2)+2\varphi.$$

Applying $\bar\partial$ to (4.9) we see that H satisfies the compatibility conditions $\bar\partial H = 0$. By Theorem 4.4.2 in Hörmander [5] we can therefore find a function $v \in L^2_{\text{loc}}(\mathbf{C}^n)$ which is a solution to the equation

$$\bar\partial v = H \tag{4.11}$$

and satisfies

$$\int_{\mathbf{C}^n} |v|^2 e^{-\sigma}(1+|z|^2)^{-2} d\lambda \leqslant \int_{\mathbf{C}^n} |H|^2 e^{-\sigma} d\lambda. \tag{4.12}$$

Here $d\lambda$ denotes the Lebesgue measure in $\mathbf{C}^n$.

The right-hand side in (4.12) is by (4.10) and the definition of σ smaller than the finite constant

$$\int_{\mathbf{C}^n} (2+|z|^2)^{-n-1} d\lambda(z).$$

Hence the function $g \in L^2_{\text{loc}}(\mathbf{C}^n)$ given by

$$g = g_k - P(h_k' - v) \quad \text{in } U_k^{1/2}, \quad k \in \mathbf{N}, \tag{4.13}$$

which is well-defined by (4.7), satisfies in view of (4.4), (4.8) and (4.12) with another constant $M > 0$ still depending only on P (and on n, of course)

$$\int_{\mathbf{C}^n} |g|^2 (2+|z|^2)^{-2M} e^{-2\varphi(z)} d\lambda(z) \leqslant 1. \tag{4.14}$$

Since v is a solution to the equation (4.11) we have $\bar\partial g = 0$ which implies by the regularity theory for the Cauchy–Riemann equations that g is an entire function. From the definition (4.13) of g and from (4.3) we obtain (2.14). Because g is holomorphic the well-known inequality

$$|g(z)| \leqslant c_n \Big(\int_{|z'-z|<1} |g(z')|^2 d\lambda(z') \Big)^{1/2}, \quad z \in \mathbf{C}^n,$$

holds. With (4.14) this implies

$$|g(z)| \leqslant c_n \cdot \sup_{|z'-z|<1} (2+|z'|^2)^M e^{\varphi(z')}, \quad z \in \mathbf{C}^n.$$

An application of the assumption (2.12) completes the proof of Theorem 2.3.

Appendix

For the convienience of the reader we shall state and prove some classical results on division by polynomials and on removability of singularities which are frequently used throughout the paper.

LEMMA A.1 (Ehrenpreis–Malgrange). *Let P be a polynomial of degree m. Then there exists a constant $C > 0$, such that for every $r > 0$, $z \in C^n$, and every function f which is defined for all $z' \in C^n$ with $|z'-z| < r$ and is such that f/P is holomorphic there, we have*

$$|f(z)/P(z)| \leqslant Cr^{-m} \sup_{|z'-z|<r} |f(z')|.$$

Proof. Fix a non-characteristic vector N for P, i.e. $P_m(N) \neq 0$. Let $r > 0$ and $z \in C^n$ be given. Consider $P(z+\lambda N)$ as a polynomial in $\lambda \in C$. If we denote its roots by $\lambda_1(z), \ldots, \lambda_m(z)$ we have the factorization

$$P(z+\lambda N) = P_m(N) \prod_{j=1}^{m} \big(\lambda - \lambda_j(z)\big), \qquad \lambda \in C.$$

We can find $0 < \varrho < r/|N|$ such that

$$|\lambda - \lambda_j(z)| \geqslant r\big(2(m+1)|N|\big)^{-1} \quad \text{if } |\lambda| = \varrho.$$

From the maximum principle we obtain

$$|f(z)/P(z)| \leqslant \max_{|\lambda|=\varrho} |f(z+\lambda N)/P(z+\lambda N)| \leqslant Cr^{-m} \sup_{|z'-z|<r} |f(z')|$$

with $C = \big(2(m+1)|N|\big)^m |P_m(N)|^{-1}$. The proof is complete.

Now, let us give the results which show that a holomorphic function can be extended holomorphically across "small" exceptional sets.

LEMMA A.2. *Let $U \subset C^n$ be open. Let $V_1, \ldots, V_k \subset U$ be complex submanifolds of codimension 2 satisfying $\bar{V}_i \subset \bigcup_{j>i} V_j$ for all i. Set $V = V_1 \cup \ldots \cup V_k$. Then every holomorphic function in $U \setminus V$ has a unique holomorphic extension to U.*

Proof. Let us first assume that V is a complex submanifold of codimension 2. Let f be holomorphic in $U \setminus V$. Let $\zeta \in V$. We introduce coordinates such that ζ becomes the origin and $V = \{z = (z_1, \ldots, z_n) \in C^n;\ z_1 = z_2 = 0\}$. Choose $\delta > 0$ such that $z \in U$ if $|z| \leqslant 2\delta$. Hence

$$\frac{1}{2\pi i} \int_{|\lambda|=\delta} \frac{f(\lambda, z')}{\lambda - z_1}\, d\lambda$$

is a well-defined holomorphic function of $z = (z_1, z') \in C^n$, $|z| < \delta$. By Cauchy's integral formula this function coincides with f in the open set $\{z \in C^n;\ |z| < \delta,\ z_2 \neq 0\}$, and thus gives an extension of f to a neighbourhood of ζ.

By assumption all $W_i = \bigcup_{j>i} V_j$ are closed. Therefore, using the construction above, we can successively extend f holomorphically across $V_1 \setminus W_1, V_2 \setminus W_2, \ldots, V_k$. Thus f extends to a holomorphic function on U. This extension is clearly unique. The proof of the lemma is complete.

Lemma A.2 is essentially a special case of

THEOREM A.3. *Let $U \subset C^n$ be open and let $V \subset C^n$ be a finite union of varieties $\{z \in C^n;\ P_1(z) = P_2(z) = 0\}$ where P_1 and P_2 are relatively prime polynomials. Then every holomorphic function in $U \setminus V$ can be continued to a unique holomorphic function in U.*

Proof. We shall prove Theorem A.3 by applying Lemma A.2 to some "stratification" of V. Let us introduce coordinates $z = (z_1, \ldots, z_n) \in C^n$ such that P_1 and P_2 are normalized in the z_1-direction and such that the resultant $R(z_2, \ldots, z_n)$ for the polynomials P_1, P_2 in the variable z_1 is normalized the z_2-direction. Recall from algebra that R is not identically zero if P_1 and P_2 are relatively prime, and, furthermore, that

$$\{z \in C^n;\ P_1(z) = P_2(z) = 0\} \subset \{z \in C^n;\ P_1(z) = 0,\ R(z_2, \ldots, z_n) = 0\}.$$

Therefore, we may well replace P_2 by R. So let us now assume that P_1 and P_2 are normalized in the z_1- and z_2-direction, respectively, and that P_2 does not depend on z_1. From the first property it follows that V is contained in the union over $k, l \in N$ of the sets

$$V_{kl} = \{z \in C^n;\ (\partial^{k-1}P_1/\partial z_1^{k-1})(z) = (\partial^{l-1}P_2/\partial z_2^{l-1})(z) = 0 \text{ and } (\partial^k P_1/\partial z_1^k)(z) \neq 0,\ (\partial^l P_2/\partial z_2^l)(z) \neq 0\}.$$

Only finitely many V_{kl} are non-empty. If V_{kl} is not empty then it is a complex submanifold of codimension 2. This follows – after we have made use of the fact that P_2 is independent of z_1 – from the Implicit Function Theorem. Hence Theorem A.3 follows from Lemma A.2.

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