SOME RECENT RESULTS ON SOLITONS,SYMMETRIES AND CONSERVATION LAWS IN NONLINEAR DYNAMICS

Benno Fuchssteiner

University of Paderborn
D4790 Paderborn
GERMANY

ABSTRACT

Infinite dimensional abelian symmetry groups for nonlinear systems (like the KP equation or the anisotropic Heisenberg spin chain) are constructed. Aspects of linearization ,soliton solutions,and action angle variables are discussed.

1.INTRODUCTION

We are interested in nonlinear flows

$$u_t = K(u) \quad , \quad u \in M \qquad ,M \text{ some manifold.} \tag{1}$$

Here t stands for time, u(t) runs on the manifold,and K(u) is a suitable vectorfield.We assume that if u(0) is any initial condition then a one parameter group of diffeomorphisms is given by

$$u(0) \xrightarrow{R_K(t)} u(t) \tag{2}$$

having K as infinitesimal generator.
Most important are symmetry groups.They may be used for the construction of conservation laws , for the description of small invariant manifolds (soliton solutions), for the linearization of the system and for the diagonalization of the Hamiltonians (in the quantum mechanical case). The most accessible symmetry groups are those given by other flows $u_t = G(u)$ because then the corresponding R_K, and R_G commute if and only if $[K,G] = 0$ in the vectorfield Lie Algebra. So, the problem of finding the symmetry groups of (1) reduces to finding those Lie algebra elements G which commute with K. Related to symmetries are conserved quantities, where $I : M \to R$ is a conserved quantity if,for any solution u(t) of (1),the quantity I(u(t)) is time independent. Their infinitesimal description is reflected by their gradients :

$$< \text{grad } I , v > = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} I(u + \varepsilon v). \tag{3}$$

If I is conserved then $\{ u \in M \mid \text{grad } I(u) = 0 \}$ is invariant.In general soliton manifolds are of this form. For Hamiltonian systems there is a Lie algebra homomorphism from the Lie algebra of Poisson brackets onto the vectorfields such that conservation laws are mapped onto infinitesimal generators of symmetries (Noether's theorem)[4]. So, more or less, gradients of conservation laws reflect the same as generators of symmetries.

2. HEREDITARY STRUCTURES

The systematic search for symmetry generators or conservation laws is facilitated by the notion of hereditary symmetry. In order to understand what that means,we assume that (1) is linearizable, i.e. that there is a diffeomorphism Γ onto a linear space such that (1) has the form :

$$v_t = L\,v \quad , L \text{ linear.} \tag{4}$$

Then Γ' (variational derivative) is a Lie algebra isomorphism between the corresponding tangent bundles.The $L^n v$ are symmetry group generators for (4),hence they are generated out of a recursive application of the operator L. Now,transferring L back via Γ', we obtain an operator $\Phi(u)$ (depending on u) such that the $\Phi^n(u)K(u)$ are symmetry generators for (1). Therefore,Φ is said to be a recursion operator. Such a linearization may be extremely difficult to carry out,but it is worthwhile to study the algebraic properties which $\Phi(u)$ inherits from the linearizability (symmetry of Γ'' and linearity of L). From these algebraic properties we obtain for all vectorfields A and B

$$\Phi^2[A,B] + [\Phi A,\Phi B] = \Phi\{[\Phi A,B] + [A,\Phi B]\}. \tag{5}$$

Henceforth,Φ is called hereditary[2,3] if (5) is fulfilled. By simple computation :

THEOREM[3]:
If Φ is hereditary and if Φ commutes with K, ***in the sense that for all*** A

$$\Phi[K,A] = [K,\Phi A]$$

then the linear hull of $\{\Phi^n K \mid n \in N \text{ or } Z\}$ ***is an abelian Lie algebra. Furthermore Φ commutes with all the*** $\Phi^n K$.

A popular example[2] of such a hereditary operator (for the KdV) is given by $\Phi(u) = D^2 + 2u + 2DuD^{-1}$ where D^{-1} denotes integration from $-\infty$ to x. Translation invariance yields that Φ commutes with the generator u_x of x-translation. Since K_1 is the flow of the KdV we have found infinitely many symmetry generators. Furthermore, it turns out that $\Phi(u)$ also yields that the KdV has two compatible Hamiltonians[4]. And the spectral decomposition of u_x with respect to eigenvectors of $\Phi(u)$ characterizes the multisoliton solutions[2,3]. Also, it should be noted that the theorem above gives that Φ and $K_1'(u)$ constitute a Lax-pair for the KdV[2].

Although it looks as if the hereditary operators are completely resolving the task of finding the symmetries of soliton equations,certain difficulties still remain. For example, the recursion operator for the Kuperschmidt equation[4] is known, but checking whether or not it is hereditary amounts to check if a integro-differential operator with about 2600 terms is equal to zero. It is impossible to do that by inspection.We are producing computer programs (based on algebraic formula manipulation) performing the necessary computations. It is not easy to implement problems like this on the computer since, considered as a language problem, we have to do with a context sensitive problem. Another problem stems from the fact that hereditary operators are not always of polynomial type. Sometimes they seem to be given by complicated implicit functions.This is the case for the BO (Benjamin - Ono equation):

$$u_t = Hu_{xx} + 2uu_x \quad \text{,H Hilberttransform} \tag{6}$$

as well as for the KP (Kadomtsev-Petviashvili equation):

$$u_{tx} = (6uu_x - u_{xxx})_x - 3u_{yy}. \tag{7}$$

3. MASTERSYMMETRIES

A way out of these difficulties is the introduction of mastersymmetries. In order to explain this we change the point of view by speaking of conservation laws instead of symmetries. We fix Poisson brackets and consider a system where the dynamic for scalar functions G in the field variable u(t) is given by

$$G(u(t))_t = -\{H,G\} \quad \text{,H some Hamiltonian.} \tag{8}$$

A function in u and t is said to be a time-dependent conservation law if

$$(d/dt)F(u(t),t) = 0 \tag{9.1}$$

which is equivalent to

$$(\partial/\partial t)F = \{H,F\}. \tag{9.2}$$

Formally, such quantities are easily found. Take any function $M_0(u)$ and apply the exponential of the adjoint map $\bar{H} = \{H, . \}$ of H, i.e. take $F = \exp(t\bar{H})M_0$ then this is a time-dependent conservation law. Convergence difficulties are avoided if one considers those F where the sum is finite

$$F = \sum_{n=0}^{N} M_n t^n. \tag{10}$$

In this case we call M_0 a mastersymmetry (of order N)[5]. Take for example a mastersymmetry $F = M_0 + tM_1$ of order one. Then, by virtue of (9) $H_1 = \{H,M_0\}$ must be conserved and the Jacobi identity yields that $H_2 = \{H_1,M_0\}$ is again conserved.Now, for example, if the conservation laws for (8) are in involution,we can proceed further producing a set $H_{n+1} = \{H_n,M_0\}$ of conservation laws. Fortunately, even if we do not know in advance whether the conservation laws are in involution we can check this by structural properties of M_0.So, one mastersymmetry of first order yields a set of conserved quantities (mastersymmetries of order 0). And this can be continued: One mastersymmetry of second order yields (eventually infinitely) many mastersymmetries of first order, etc.

But, there seem to be serious difficulties. Consider, for example, a system with compact orbits, then a mastersymmetry of first order obviously cannot exist, since its absolute term remains bounded whereas the first order term grows linearily in time . In order to understand this we consider the classical harmonic oscillator $x_t = y$,$y_t = -x$ which certainly has compact orbits.Easily we find that $F_1 = \sin(2t)(x^2-y^2)(x^2+y^2)^{-1} + \cos(2t)(2xy)(x^2+y^2)^{-1}$ is time independent.But then, we also can find a conservation law of first order, namely , $F_2 = 1/2\arcsin(F_1) = t - \arcsin(y/x)$ which seems to be in contradiction to the reasoning above. This apparent contradiction easily resolves by the observation that F_1 is not globally defined. But, certainly we can use F_1 for the construction of further conserved quantities since its gradient is globally defined, and gradients are all what we need in computing the Poisson

brackets. So, a way out of these difficulties is to work with external derivations on the Lie algebra of Poisson brackets. Furthermore, if one looks at the absolute term of F_2 one discovers that this is the angle-variable of the system. That this is not globally defined just reflects the fact that the motion is periodic. This carries over to the general case, where we take a suitable series of first-order mastersymmetries $F_1, F_2, \ldots$ and the corresponding conserved quantities $E_i = \{H, F_i\}$. Then we parametrize our manifold by the new coordinates given by the E_i and the $T_i = M_i E_i^{-1}$. Now, in the coordinates E_i, T_i the flow goes with constant velocity on lines being parallel to the T_i - axes, i.e. the flow is represented in action-angle variables.
But how to find all these quantities? Simply, find one mastersymmetry of second order and take commutators with H (as described before). In order to give a meaningful example we consider the BO (eq. (6)), where Hamiltonian and Poisson brackets are given by

$$H = \int_{-\infty}^{\infty} (\frac{1}{2} u H u_x - \frac{1}{3} u^3) dx, \quad \{A, B\} = \int_{-\infty}^{\infty} (\text{grad } A)(\text{grad } B)_x dx.$$

Then a mastersymmetry of second order is[5]:

$$M_0 = \frac{1}{2} \int_{-\infty}^{\infty} x^2 u \, dx.$$

From this we obtain the action-angle variables as described above.
Other examples where this concept has been applied successfully are the KP[5], the anisotropic Heisenberg spin chain[7])

$$H = \sum_n (J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z)$$

the XY-spin chain with external magnetic field[1] and the Landau- Lifshitz equation[6].

4. REFERENCES

[1] Barouch, E. and Fuchssteiner, B.: Mastersymmetries and similarity Equations of the XYh-model, preprint, Paderborn (1985)
[2] Fuchssteiner, B.; Application of Hereditary Symmetries to Nonlinear Evolution equations, Nonlinear Analysis TMA 3, 849 - 862 (1979)
[3] Fuchssteiner, B.: The Lie algebra structure of Nonlinear Evolution Equations admitting Infinite Dimensional Abelian Symmetry Groups, Progr. Theor. Phys. 65, 861 - 876 (1981)
[4] Fuchssteiner, B.: The Lie algebra structure of Degenerate Hamiltonian and Bi-Hamiltonian systems, Progr. Theor. Phys. 68, 1082 - 1104(1982)
[5] Fuchssteiner, B.: Mastersymmetries, Higher-order Time-dependent symmetries and conserved Densities of Nonlinear Evolution Equations, Progr. Theor. Phys. 70, 1508 - 1522 (1983)
[6] Fuchssteiner,B.:On the Hierarchy of the Landau-Lifshitz equation, Physica 13D,387-394(1984)
[7] Fuchssteiner, B.: Mastersymmetries for completely integrable systems in statistical mechanics.(In Lecture Notes in Physics 216, Ed. L. Garrido), 305 - 315 (1985)