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Dissertation

Circular flows on signed graphs

by Michael Schubert

Adviser: Prof. Dr. Eckhard Steffen

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Abstract

For planar graphs, integer nowhere-zero flows and face colorings are equivalent concepts. Considering embeddings in orientable surfaces every face coloring can be expressed as an integer nowhere-zero flow. However, not every integer nowhere-zero flow can be expressed as a face-coloring. Therefore, nowhere-zero flows generalize the concept of colorings.

In this thesis we study flows on signed graphs. Signed graphs (G, σ) generalize ordinary unsigned graphs G in such a way that each edge is either positive or negative. While positive edges must be directed in the ordinary way, negative edges must be either oriented introverted or extroverted. Nowhere-zero flows on signed graphs differ essentially from nowhere-zero flows on unsigned graphs because negative edges define a set of edges that act like sources and sinks.

We will motivate the study of flows on signed graphs over colorings and tensions. A graphs G has a nowhere-zero k-tension if and only if G has a k-coloring. For every graph G there exists one surface such that G can be embedded without crossing-edges and each circuit in this embedding is a contractible curve. Corresponding to one wisely chosen embedding, a face coloring of G always exist. We show that the corresponding tension leads in a natural way to an integer nowhere-zero flow on a certain dual signed graph.

Integer nowhere-zero flows on signed graphs establish a generalization for colorings. We also study a refinement, circular nowhere-zero r-flows where we allow edges to have real flow-values from the set $\{x \in \mathbb{R} | 1 \le |x| \le r - 1\}$. We study the relation between the circular flow number $F_c((G, \sigma))$ and the integer flow number $F((G, \sigma))$ that are defined as the infimum over all r such that (G, σ) admits a circular nowhere-zero r-flow or respectively an integer nowhere-zero r-flow. For flow-admissible graphs, Raspaud and Zhu proved that $F((G, \sigma)) \le 2\lceil F_c((G, \sigma))\rceil - 1$, and

they conjectured $F_c((G,\sigma)) > F((G,\sigma)) - 1$. We disprove this conjecture by showing that $sup\{F((G,\sigma)) - F_c((G,\sigma))\} \geq 2$. Furthermore, we show some sufficient conditions for $\lceil F_c((G,\sigma)) \rceil = F((G,\sigma))$.

The circular or integer flow spectrum ($\mathcal{S}(G)$ or $\overline{\mathcal{S}}(G)$, respectively) of a graph G is the set of all possible circular respectively integer flow numbers given due to arbitrary flow-admissible signatures. We study integer and circular flow spectra on regular graphs. The circular flow number $2 + \frac{1}{t}$ is the smallest possible flow number in a regular graph. We characterize $(2t + 1)$-regular graphs whose flow spectrum contains $2 + \frac{1}{t}$. Furthermore, we analyze some cases for the flow spectrum of a graph G if G has a 1-factor. By introducing the concept of r-minimal sets we prove for a graph $G \neq K_2^3$ the following statements are equivalent:

(1) G has a 1-factor. (2) $3 \in \mathcal{S}(G)$. (3) $3 \in \overline{\mathcal{S}}(G)$. (4) $4 \in \overline{\mathcal{S}}(G)$.

We find cubic graphs whose integer flow spectrum does not contain 5 or 6, and we construct an infinite family of bridgeless cubic graphs with integer flow spectrum $\{3, 4, 6\}$. We prove some sharp bounds for the cardinality of smallest 3-minimal and 4-minimal sets, respectively. If G is not 3-edge-colorable, then these bounds are formulated in terms of its resistance and oddness.

A *Kotzig graph* is a cubic graph that has three 1-factors such that the union of any two of them induces a hamiltonian circuit. We give a proof of Bouchet's conjecture for Kotzig-graphs.

Let $\mathcal{F}^c$ be the set of circular flow numbers that can be obtained by signed graphs and $\mathcal{F}_k^c$ be the set of circular flow numbers that can be obtained by k-regular signed graphs. We show that $\mathcal{F}_{2k+1}^c = (\mathcal{F}^c - [2; 2 + \frac{2}{2k-1}]) \cup \{2 + \frac{1}{k}\}$. As a corollary for unsigned graphs we entirely determine the set of flow numbers for regular graphs up to 5, which is best possible if Tutte's 5-flow conjecture is true.

Lastly, we relate the problem of finding certain circular and integer nowhere-zero flows to the problem of finding a set of orientations with special properties. In this regard, we characterize all nowhere-zero flows on signed graphs.

Zusammenfassung

Ganzzahlige nirgends-null Flüsse und Flächen-Färbungen sind equivalente Konzepte für planare Graphen. Hinsichtlich Einbettungen in orientierbaren Flächen kann jede Flächenfärbung als ganzzahliger nirgends-null Fluss dargestellt werden. Allerdings kann nicht jeder ganzzahlige nirgends-null Fluss als Flächenfärbung dargestellt werden. Nirgends-null Flüsse verallgemeinern also das Konzept von Färbungen.

In dieser Arbeit beschäftigen wir uns mit signierten Graphen. Signierte Graphen (G, σ) verallgemeinern gewöhnliche unsignierte Graphen G, sodass jede Kante entweder eine positive oder negative Kante ist. Positive Kanten werden wie im gewöhnlichen Fall orientiert. Negative Kanten sind entweder introvertiert oder extrovertiert. Nirgends-null Flüsse auf signierten Graphen unterscheiden sich wesentlich von Flüssen auf unsignierten Graphen, da negative Kanten eine Menge von Kanten bilden, die als Quellen und Senken interpretiert werden können.

Wir motivieren das Studium von Flüssen auf signierten Graphen über Färbungen und Tensionen. Ein Graph G hat eine nirgends-null k-Tension genau dann, wenn G eine k-Färbung hat. Für jeden Graphen G existiert eine Fläche, in die G ohne Kreuzungskanten eingebettet werden kann und jeder Kreis in dieser Einbettung bildet eine Kurve, die sich auf einen Punkt zusammenziehen lässt. Für eine mit Bedacht gewählte Einbettung existiert eine Flächenfärbung. Wir zeigen, dass die zugehörige Tension

in natürlicher Art und Weise zu einem ganzzahligen nirgends-null Fluss auf einem dualen signierten Graphen führt.

Ganzzahlige nirgends-null Flüsse auf signierten Graphen verallgemeinern den Färbungsbegriff. Wir untersuchen neben ganzzahligen nirgends-null Flüssen zudem eine Verfeinerung: zirkuläre nirgends-null r-Flüsse, bei denen wir reelle Flusswerte aus der Menge $\{x \in \mathbb{R} | 1 \leq |x| \leq r - 1\}$ erlauben. Wir betrachten die Beziehung zwischen der zirkulären Flusszahl $F_c((G, \sigma))$ und der ganzzahligen Flusszahl $F((G, \sigma))$. Für Graphen, die einen nirgends-null Fluss zulassen, bewiesen Raspaud and Zhu $F((G, \sigma)) \leq 2\lceil F_c((G, \sigma))\rceil - 1$ und stellten die Vermutung $F_c((G, \sigma)) > F((G, \sigma)) - 1$ auf. Wir widerlegen diese Vermutung, indem wir $sup\{F((G, \sigma)) - F_c((G, \sigma))\} \geq 2$ zeigen. Desweiteren stellen wir einige hinreichende Bedingungen für $\lceil F_c((G, \sigma))\rceil = F((G, \sigma))$ auf.

Das zirkuläre bzw. das ganzzahlige Flussspektrum $\mathcal{S}(G)$ bzw. $\overline{\mathcal{S}}(G)$ eines Graphen G ist die Menge aller möglichen zirkulären bzw. ganzzahligen Flusszahlen, die durch beliebige Signaturen, welche einen nirgends-null Fluss zulassen, gegeben sind. Wir untersuchen ganzzahlige und zirkuläre Flussspektren auf regulären Graphen. Der Wert $2 + \frac{1}{t}$ ist die kleinstmögliche Flusszahl eines regulären Graphen. Wir charakterisieren $(2t + 1)$-reguläre Graphen, deren Flussspektrum $2 + \frac{1}{t}$ enthält. Desweiteren untersuchen wir einige Fälle des Flussspektrums eines Graphen G, falls G einen 1-Faktor hat. Mithilfe des entwickelten Konzepts von r-minimalen Mengen beweisen wir, dass für einen Graphen $G \neq K_2^3$ folgende Aussagen equivalent sind:

(1) G hat einen 1-Faktor. (2) $3 \in \mathcal{S}(G)$. (3) $3 \in \overline{\mathcal{S}}(G)$. (4) $4 \in \overline{\mathcal{S}}(G)$.

Wir finden Graphen, deren ganzzahliges Flussspektrum nicht 5 oder 6 enthält und wir konstruieren eine unendliche Familie von brückenlosen kubischen Graphen mit ganzzahligem Flussspektrum $\{3, 4, 6\}$. Wir beweisen einige scharfe Schranken für die Kardinalität von kleinsten 3-

minimalen bzw. 4-minimalen Mengen. Falls G nicht 3-kantenfärbbar ist, sind diese Schranken über den Widerstand und die Ungradheit von G definiert.

Ein *Kotzig Graph* ist ein kubischer Graph, der drei 1-Faktoren besitzt, sodass die Vereinigung zweier beliebiger 1-Faktoren einen Hamiltonkreis induziert. Wir beweisen Bouchets Vermutung für Kotzig Graphen.

Sei $\mathcal{F}^c$ die Menge der zirkulären Flusszahlen, die durch signierte Graphen erhalten werden können und sei $\mathcal{F}^c_k$ die Menge der zirkulären Flusszahlen, die durch k-reguläre signierte Graphen erhalten werden können. Wir zeigen $\mathcal{F}^c_{2k+1} = (\mathcal{F}^c - [2; 2 + \frac{2}{2k-1}]) \cup \{2 + \frac{1}{k}\}$. Für unsignierte Graphen ist $\mathcal{F}^c_k$ damit vollständig bis 5 bestimmt. Dies ist bestmöglich, falls Tuttes 5-Fluss Vermutung wahr ist.

Abschließend überführen wir das Problem gewisse zirkuläre oder ganzzahlige nirgends-null Flüsse zu finden in das Problem eine Menge von Orientierungen mit speziellen Eigenschaften zu finden. In diesem Zusammenhang charakterisieren wir alle nirgends-null Flüsse.

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Chapter 1

Introduction

1.1 Preliminaries

One of the most famous issues in graph theory is the Four color theorem. Nearly one and a half century ago Francis Guthrie raised the conjecture that four colors are always sufficient to color a map in such a way, that adjacent regions receive different colors. Once the conjecture was stated, there appeared many proofs and counterexamples that emerge to be wrong until Appel and Haken finally proved the conjecture in 1976. The mathematical historical significance for that theorem established since this is one of the first major theorems which could just be prooven with the aid of a computer so far.

This thesis is about flows on signed graphs. On the one hand, flows can be interpreted as a generalization of map-colorings and on the other hand, signed graphs establish a more general concept for graphs. A more detailed explanation about the connection of colorings and flows is given in chapter 2. However, map-colorings are only defined for crossing-free embeddings of graphs. The fact that flows extend the intuitional concept

of map-colorings in the non-planar case made a great stir and caused flows to be a well-respected topic in graph theory.

In 1954, Tutte stated the conjecture that every bridgeless graph admits a nowhere-zero 5-flow. A nowhere-zero 5-flow of a graph is an assignment of a direction and a value from the set $\{1, 2, 3, 4\}$ for each edge, in such a way that for each vertex the sum over all incoming values equals the sum over all outgoing values. This conjecture became one of the most famous in flow theory and until now it is unsolved. However, there are several approaches for that conjecture as for example Seymour's 6-flow theorem, where the conjecture is prooven to be true by exchanging the set of numbers by $\{1, 2, 3, 4, 5\}$.

For signed graphs the concept of nowhere-zero flows is basically the same, except the fact that we allow graphs to have negative and positive edges. The orientation of a positive edge is always defined from one vertex towards another vertex, whereas the orientation of a negative edge can only be directed towards each incident vertex, or away from each incident vertex. Regarding that concept of flows on signed graphs, flows on unsigned graphs are a special case where all edges are positive.

Until now, no flow-admissible all-positive graph is known that does not have a nowhere-zero 5-flow. On the contrary, in the signed case infinitely many graphs exist that do not admit a signed nowhere-zero 5-flow.

In this context two famous conjectures remain unsolved, for all-positive graphs the above mentioned Tutte's 5-flow-conjecture and for signed graphs the conjecture that every flow-admissible graph admits a nowhere-zero 6-flow stated by Bouchet in 1983.

1.2   Basic Definitions

A *graph* G is an ordered pair $G = (V, E)$ where $V = V(G)$ is a set of vertices and $E = E(G)$ is a set of edges that are 2-element subsets of V. Two vertices $v, w \in V$ are *neighbors* and called *adjacent* if there exists an edge $e \in E$ with $e = (v, w)$ or short vw. We can construe and visualize graphs as vertices that are connected by edges. In this thesis we consider graphs to be finite.

For a vertex $v \in V(G)$ let $E(v)$ be the set of edges which are incident to v, and let $|E(v)|$ be the *degree* of v which is denoted by $d_G(v)$. A graph G is d-regular if all vertices of G have the degree d. The *maximum degree* in a graph is denoted by $\Delta(G)$ and the *minimum degree* in a graph is denoted by $\delta(G)$.

A *signed graph* (G, σ) consists of a graph G and a function σ from $E(G)$ into $\{-1, 1\}$. The function σ is called a *signature*. Let $e \in E(G)$ an edge. Either $\sigma(e) = 1$ in which case e is called a *positive edge*, or $\sigma(e) = -1$ in which case e is called a *negative edge*. For a graph G the set which consists of all negative edges is denoted by N_σ. It is called *the set of negative edges* while $E(G) - N_\sigma$ is called *the set of positive edges*. Every subset X of $E(G)$ defines a signature σ of G with $N_\sigma = X$. If all edges of (G, σ) are positive, i. e. when $N_\sigma = \emptyset$, we will call (G, σ) an *all-positive signed graph*. An unsigned graph is an all-positive signed graph. If we refer to the unsigned case we will denote an all-positive signed graph (G, σ) simply by G and if we refer to the signed case we will denote (G, σ) by $(G, 1)$.

Let $e \in E(G)$ be an edge which is incident to the vertices u and v. We divide e into two half-edges h_e^u and h_e^v, one incident to u and one incident to v. The set of the half-edges of G is denoted by $H(G)$. For each half-edge $h \in H(G)$, the corresponding edge in $E(G)$ is denoted by e_h. For a

vertex v, $H(v)$ denotes the set of half-edges incident to v. An *orientation* of (G, σ) is a function $\tau : H(G) \to \{\pm 1\}$ such that $\tau(h_e^u)\tau(h_e^v) = -\sigma(e)$ for each edge $e = uv$. The function τ can be interpreted as an assignment of a direction to each edge in the following way:

For a positive edge exactly one half-edge is incoming and the other one is outgoing. For a negative edge either both half-edges are incoming, in which case e is an *extroverted edge*, or both half-edges are outgoing, in which case e is an *introverted edge*. Taken together, a positive edge can be directed like ○—→○ or like ○←—○ and a negative edge can be directed like ○←—→○ (extroverted edge) or like ○—→←—○ (introverted edge). An oriented signed graph is also called a *bidirected graph*.

For a vertex v let $H^+(v)$ be the set of outgoing half-edges and $H^-(v)$ be the set of incoming half-edges which are incident to v.

Let $n \geq 1$ and $P = u_0u_1...u_n$ be a path. We say that P is *a v-w-path* if $v = u_0$ and $w = u_n$. Let (G, σ) be oriented. If a path P of G does not contain any negative edge and for every $i \in \{0, \ldots, n-1\}$ the edge u_iu_{i+1} is directed from u_i to u_{i+1}, then we say that P is *a directed v-w-path*. If P is not directed, then every edge of P that is directed from u_i to u_{i+1} will be called *forward-edge* and every edge of P that is directed from u_{i+1} to u_i will be called *backward-edge*.

Let r, r', x be real numbers with $0 \leq r' < r$. We write $x \equiv r' \pmod{r}$ if there is an integer t such that $x - r' = tr$

Let (G, σ) be a signed graph with orientation τ. The boundary of the orientation τ is a function $\delta\tau : V(G) \to \mathbb{R}$ with $\delta\tau = \sum_{h \in H(v)} \tau(h)$. The boundary of a function $f : E(G) \to \mathbb{R}$ is a function $\delta f : V(G) \to \mathbb{R}$ with $\delta f(v) = \sum_{h \in H(v)} \tau(h)f(e_h)$.

The function f is a (modular) r-flow on (G, σ), if $|f(e)| \in \{x : 1 \leq x \leq r-1\} \cup \{0\}$ for every $e \in E(G)$ and $\delta f(v) = 0$ ($\delta f(v) \equiv 0 \pmod{r}$) for every $v \in V(G)$. An r-flow f is also called an circular flow. If $f(e) \in \mathbb{Z}$

for all $e \in E(G)$ we also call f an integer r-flow. Every flow f has one underlying orientation τ_f. For defining one flow we also use the tupel (D, f) consisting of an orientation D and a flow f.

The set $\{e : f(e) \neq 0\}$ is the support of f, and f is a *nowhere-zero* (modular) r-flow on (G, σ) if $E(G)$ is the support of f.

A signed graph (G, σ) is *flow-admissible*, if there exists an orientation τ and a number $r \geq 2$ such that (G, σ) has a nowhere-zero r-flow. The *circular flow number* of a flow-admissible signed graph (G, σ) is

$$F_c((G, \sigma)) = \inf\{r : (G, \sigma) \text{ admits a nowhere-zero } r\text{-flow}\}.$$

If we restrict our studies on flows that are integer-valued functions, the corresponding *integer flow number* is denoted by $F(G, \sigma)$.

Let (D, ϕ) be a nowhere-zero k-flow on (G, σ). If we reverse the orientation of an edge e (or of the two half-edges, respectively) and replace $\phi(e)$ by $-\phi(e)$, then we obtain another nowhere-zero k-flow (D^*, ϕ^*) on (G, σ). Hence, if (G, σ) is flow-admissible, then it has always a nowhere-zero flow with all flow values positive. This also shows that if $F_c(G, \sigma) = r$, then there is a nowhere-zero r-flow for any prescribed orientation of (G, σ).

Let (G, σ) be a signed graph. For $i \in \{1, 2\}$ let ϕ_i be a flow on (G, σ) with underlying orientation τ_i. Note that for each edge $e = uv$ either $\tau_1(h_e^u) = \tau_2(h_e^u)$ and $\tau_1(h_e^v) = \tau_2(h_e^v)$ or $\tau_1(h_e^u) \neq \tau_2(h_e^u)$ and $\tau_1(h_e^v) \neq \tau_2(h_e^v)$. The sum $\phi_1 + \phi_2$ is the function ϕ on (G, σ) with orientation τ, where $\tau = \tau_1|_{\{h|\phi_1(e_h) \geq \phi_2(e_h)\}} \cup \tau_2|_{\{h|\phi_2(e_h) > \phi_1(e_h)\}}$, and $\phi(e) = \phi_1(e) + \phi_2(e)$ if e has the same direction in τ_1 and τ_2, and otherwise $\phi(e) = |\phi_1(e) - \phi_2(e)|$. Clearly, if $|\phi(e)| \geq 1$ for every edge with $\phi(e) \neq 0$, then ϕ is a flow.

Let (G, σ) be a signed graph. *A switching at v* defines a graph (G, σ^*) with $\sigma^*(e) = -\sigma(e)$ if e is incident to v, and $\sigma^*(e) = \sigma(e)$ otherwise. We say that signed graphs (G, σ) and (G, σ^*) are *switching equivalent* if they

can be obtained from each other by a sequence of switchings. We also say that σ and σ^* are *switching equivalent signatures* of G. If we consider a signed graph with an orientation D, then switching at v is a change of the orientations of the half-edges that are incident with v. If D^* is the resulting orientation, then we say that D and D^* are *switching equivalent orientations*.

A circuit in (G, σ) is *balanced*, if it contains an even number of negative edges; otherwise it is *unbalanced*. Note that a circuit of (G, σ) remains balanced (resp. unbalanced) after switching at any vertex of (G, σ). The signed graph (G, σ) is *an unbalanced graph*, if it contains an unbalanced circuit; otherwise (G, σ) is *a balanced graph*.

A signed graph (G, σ) is called a barbell if either (G, σ) consists of two circuits that intersect in exactly one vertex or (G, σ) consists of two disjoint circuits C_1, C_2 and a path P that connects C_1 with C_2 and $V(C_1) \cap V(C_2) \cap P$.

A circuit cover of a graph G is a set of circuits $C_1, C_2, ..., C_n$ in G such that the union $\bigcup_{i=1}^{n} E(C_i) = E(G)$. A signed circuit is a graph that is either a balanced circuit or a barbell. A graph which has a circuit that visits each vertex precisely once is a *hamilton graph*. The corresponding circuit is called a *hamiltonian circuit*.

An *eulerian* graph is a graph, that has one circuit that contains each edge of the graph precisely once. Clearly, if (G, σ) and (G, σ') are equivalent and H is an eulerian subgraph of G, then $|N_\sigma \cap E(H)|$ and $|N_{\sigma'} \cap E(H)|$ have the same parity. Hence, H is unbalanced in (G, σ) if and only if it is unbalanced in (G, σ').

1.3   Contribution of the thesis

Some results of this thesis have already been published:

- The results of section 3.1 (except 3.1.5 and 3.1.6) and of chapter 4 have been published in

  [32] M. Schubert, E. Steffen, Nowhere-zero flows on signed regular graphs, European Journal of Combinatorics, 48:34–47, 2015.

- The results of the parts 5.1.1 - 5.1.3 and section 5.2 have been published for unsigned graphs in

  [31] M. Schubert, E. Steffen, The Set of Circular Flow Numbers of Regular Graphs, Journal of Graph Theory, 76(4):297–308, 2014.

1.4 State of the art

Certainly, for studying flows we are mainly interested in flow-admissible graphs. In general, one can always construct a nowhere-zero flow by the summation of proper flows if the support of those covers the whole edge set. The question arises, what subgraphs are the smallest possible in such a way that we can define flows on these subgraphs in order to cover each edge at least once. It turns out that flow-admissibility is based on the collection of signed circuits. Let (G, σ) be a signed graph. (G, σ) is flow-admissible if and only if there exists a set of signed circuits in (G, σ) that covers $E(G)$. For an unsigned graph G it follows that G is flow-admissible if and only if G has a circuit cover what is equivalent of being bridgeless. If (G, σ) does not admit any nowhere-zero flow, we set $F_c((G, \sigma)) = \infty$. It is easy to see that two switching equivalent graphs (G, σ) and (G, σ') have the same (circular) flow numbers. Thus, switching is an equivalence relation. It is well known (see e.g. [30]) that (G, σ) is balanced if and only if it is switching equivalent to $(G, 1)$. Bouchet [3] showed that (G, σ) is not flow-admissible if and only if (G, σ) is switching equivalent to (G, σ') with $|N_{\sigma'}| = 1$ or G has a bridge b and a component of $G - b$ is balanced.

For an unsigned graph G, an orientation D, and a set $X \subseteq V(G)$ of G the set $D^+(X)$ denotes the set of edges oriented from $V(G) - X$ towards X and $D^-(X)$ denotes the set of edges oriented from X towards $V(G) - X$. From a theorem of [11] it can be shown that an unsigned graph G admits a circular nowhere-zero r-flow if and only if G has one orientation such that for all $X \subseteq V(G)$, $r - 1 \geq \frac{|D^+(X)|}{|D^-(X)|}$ and $r - 1 \geq \frac{|D^-(X)|}{|D^+(X)|}$ [6]. Thus, for flow-admissible unsigned graphs it can be seen that the infimum $F_c(G)$ is a minimum and it is obtained at a rational number. For signed graphs the authors of [30] generalize the above mentioned characterization for circular flows and deduce that for every flow-admissible graph (G, σ), $F_c((G, \sigma))$ is a minimum and it is obtained at a rational number.

In 1954, Tutte established the topic of flows. Flows on graphs constitute a very active field of research in graph theory and there are many difficult conjectures on circular flows on graphs, see [37] for a brief overview. The flow number of unsigned graphs has been studied intensively in the recent years. One of the most famous conjectures in this context is Tutte's 5-flow conjecture [41].

Conjecture 1.4.1. *[41] If G is flow-admissible, then G admits a nowhere-zero 5-flow.*

The smallest counterexample for that an unsigned flow-admissible graph does not admit a nowhere-zero k-flow for $k \geq 2$ is cubic [33]. Hence, it is sufficient to proof Conjecture 1.4.1 for cubic graphs.

There exist infinitely many graphs G for which the circular flow number $F_c(G) = 5$ (see [18]). The first graph that was known in this regard is the Petersen graph. A proof that for the Petersen graph P holds $F_c(P) = 5$ can be found in [35]. Thus, the bound in Tutte's conjecture is sharp. So far, the closest approach for Tutte's conjecture is Seymour's 6-Flow Theorem.

Theorem 1.4.2. *[33] If G is flow-admissible, then G admits a nowhere-zero 6-flow.*

In fact Tutte's 5-flow Conjecture as well as Seymours 6-flow Theorem were stated for integer flows. Until now, it is still an open issue if there exists an $\epsilon \in (0,1)$ such that every flow-admissible graph G admits a nowhere-zero $(5 + \epsilon)$-flow. Furthermore, Tutte's 5-flow conjecture and Seymours 6-flow Theorem were stated without the consideration of a graph to be signed, since that generalization was established later.

Naturally, the concept of nowhere-zero flows has been extended in several ways. In this thesis we study one generalization of these – nowhere-zero flows on signed graphs. Flows on signed graphs were introduced by Bouchet, who stated the following conjecture.

Conjecture 1.4.3. *[3] Let (G, σ) be a signed graph. If (G, σ) is flow-admissible, then (G, σ) admits a nowhere-zero 6-flow.*

The bound on Bouchet's conjecture is best possible, since the Petersen graph, equipped with a certain signature, has flow number 6. However, there exists no published proof, yet. To show that a flow number of a graph equals a specific value k we need to show that there exist one nowhere-zero k-flow on the graph and there exist no nowhere-zero k'-flow with $k' < k$. The latter case usually is quite hard to show since there are many possibilities. In Appendix A we will prove that the Petersen graph, equipped with a certain signature, has flow number 6.

For cubic graphs, Bouchet's conjecture 1.4.3 is as follows:

Conjecture 1.4.4. *Let (G, σ) be a signed cubic graph. If (G, σ) is flow-admissible, then (G, σ) admits a nowhere-zero 6-flow.*

It is well-known that Bouchet's conjecture is equivalent to its restriction on cubic graphs. Note, that for showing the equivalence for signed

graphs, a proof similar to that in [33] does not work. Since there seems
to be no proof published so far, we will give one proof in the following.

Theorem 1.4.5. *Conjecture 1.4.3 and Conjecture 1.4.4 are equivalent.*

Proof. If Conjecture 1.4.3 is true, then obviously Conjecture 1.4.4 is
true. Now, assume that Conjecture 1.4.4 is true. Let (G, σ) be a non-
cubic flow-admissible signed graph. We suppress all vertices of degree 2
and for each vertex $v \in V(G)$ with $d_G(v) \geq 4$ we subdivide each incident
edge e_i^v $(i \in \{1, \ldots, d_G(v)\})$ by adding a new vertex v_i. Now, delete v
and add the edges (e_i^v, e_{i+1}^v). The addition of the indices is taken modulo
$d_G(v)$. The edges joining the circuit are defined to be f_i^v, such that each
v_i has the edges e_i^v, e_{i+1}^v, f_i^v. We repeat this procedure for each vertex
$v \in V(G)$. We end up with a cubic graph G' and every vertex $v \in V(G)$
corresponds to a circuit C_v of G'. The edge set $E(G)$ induces a 1-factor
F_1 in G', and the circuits in G' which correspond to the vertices in G form
a 2-factor F_2. Let $\sigma' : E(G') \to \{\pm 1\}$ be the signature with $\sigma'(e) = \sigma(e)$
for each $e \in F_1$ and $\sigma'(e) = 1$ for each $e \in F_2$. We claim that (G', σ')
is flow-admissible. Construct a nowhere-zero flow ψ as follows: Let ϕ be
a nowhere-zero flow on an orientation (G, τ) of (G, σ). We may assume
that $\phi(f) \geq 1$ for all $f \in F_1$. As otherwise, we switch the direction of
f and set the flow value to its negative inverse. For every circuit C_v let
$\delta_G^+(v)$ be the set of edges such that the corresponding half-edge is directed
to the circuit. We define $\psi : E(G') \to \mathbb{R}$ with $\psi(e) = \phi(e)$ for $f \in F_1$.
For each circuit C_v in F_2 we direct all edges in one direction along the
circuit and set:

$$\psi(e_1) = \sum_{u \in \delta_G^+(v)} \phi(u) + 1$$

$$\psi(e_{i+1}) = \begin{cases} \psi(e_i) + f_i, \text{ if } f_i \text{ is directed towards } C_v \\ \\ \psi(e_i) - f_i, \text{ if } f_i \text{ is directed away from } C_v \end{cases}$$

Hence, (G', σ') is flow-admissible and due to Conjecture 1.4.4 (G', σ') admits a nowhere-zero 6-flow. Now we contract the circuits given by F_2 and insert the edges of degree 2 which were suppressed in G and by contraction we get $F((G, \sigma)) \leq 6$.

$\square$

The best published approximation to Bouchet's Conjecture 1.4.3 is proved by the following theorem of Zýka.

Theorem 1.4.6. *[52] Let (G, σ) be a signed graph. If (G, σ) is flow-admissible, then (G, σ) admits a nowhere-zero 30-flow.*

Moreover, DeVos proved in one manuscript that every flow-admissible graph (G, σ) admits a nowhere-zero 12-flow [5].

For 2-edge-connected signed graphs the following improvement was proved.

Theorem 1.4.7. *[4] Let (G, σ) be a 2-edge-connected signed graph. If (G, σ) is flow-admissible, then (G, σ) admits a nowhere-zero 11-flow.*

Bouchet's conjecture has been confirmed for particular classes of graphs [23, 21] and also for signed graphs with restricted edge-connectivity (for example [30]). By Theorem 1.4.2 it is also true for signed graphs with all edges positive, because it describes the special case for all-positive graphs. Furthermore, if for a graph (G, σ) there is an edge-cut $X \subseteq E(G)$ with $X = N_\sigma$, then (G, σ) is switching equivalent to $(G, 1)$ and is follows that their (circular) flow numbers coincide. This can be seen easily by switching all vertices of one side of the cut.

Another interesting fact about flows in unsigned graphs is, that whenever we consider one flow, we are able to scale the flow values in a specific way as shown by the following theorem.

Theorem 1.4.8. *[35] Let G be a graph and $F_c(G, \sigma) = \frac{p}{q} + 1$. Then there exists a nowhere-zero $\left(\frac{p}{q} + 1\right)$-flow ϕ on G, such that for each edge $e \in E(G)$ exists one $n_e \in \mathbb{Z}$ such that $\phi(e) = \frac{n_e}{q}$.*

For signed graphs this theorem does not hold in general.

Chapter 2

Motivation

There exist several studies that connect signed graphs with interesting applications from other branches besides mathematics. As an example, for politics we would like to refer to a structural analysis of the situation in the middle east in 1956 [9] or a graph-theoretical approach for the analysis of international relations [10]. In physics signed graphs are often used to study spin glasses (see [39] for example). Moreover, in chemistry signed graphs are used to study molecular structures (see [8] for example). This is just a short overview about topics where signed graphs are used. For more information in [49] a mathematical bibliography of signed and gain graphs and allied areas is given with more than 340 pages. Moreover, the author of [49] regularly updates a webpage which contains a list of hundreds of publications that fall in the area of signed graphs. For signed graphs pure mathematical concepts, as for example colorings, are also studied (see [47],[46],[48]). An interesting point is that there are even different concepts for colorings of signed graphs (see [19] and [16] for example). In this thesis we will focus on a pure mathematical analysis of flows of signed graphs. A survey article about flows on signed graphs is given in [15].

2.1   Flow and Coloring Duality

Throughout this chapter we consider flows to be integer flows. The motivation for studying flows emerges by the fact that flows and colorings are dual concepts. For unsigned graphs, this can be seen quite easily, while for signed graphs it turns out to be more complicated. However, the connection of colorings and flows, especially for signed graphs, establishes a great motivation. Therefore, we will give a brief overview about that topic.

Let G be a graph. An *embedding M* of G into a surface Σ is an injective function $\phi : V \to \Sigma$ together with a set of homomorphisms $\{\psi_e : e = (x,y) \in E\}$ such that:

1. $\psi_e : [0,1] \to \Sigma$

2. $\{\psi_e(0), \psi_e(1)\} = \{\phi(x), \phi(y)\}$

3. $\psi_e((0,1)) \cap (\{\phi(v) : v \in V\} \cup \{\psi_{e'} : e' \in E, e' \neq e, x \in [0,1]\}) = \emptyset$

A *crossing-free embedding* is an embedding of a graph such that for any two edges there is no intersection. With repect to one surface Σ we call a graph G crossing-free if there exists a crossing-free embedding of G into Σ. A graph G is planar if there exists one crossing-free embedding of G into a sphere.

The Jordan Curve Theorem allows us to define faces of a crossing-free embedding of a graph given by the connected components that are separated by edges which generate a Jordan curve.

Theorem 2.1.1 (Jordan Curve Theorem [14])**.** *Every simple closed curve y separates the 2-sphere into two connected components of which y is their common boundary.*

The first proof of Theorem 2.1.1 was submitted in [14], however, some cases were missing. For a detailed proof we refer to [44].

Defining faces in one embedding is only meaningful if the embedding is crossing-free. However, for every graph G exists a surface Σ_G such that G has a crossing-free embedding into Σ. For example, let M be an embedding of a graph G into S^2 with minimum number of crossing edges. For one intersection of two edges e, f we add one handle H to our surface, connecting the endvertices of e and exchange the Jordan curve that represents e by a Jordan curve that connects the end vertices of e over H. By applying this procedure we end up with one surface Σ^* that is orientable and one crossing-free embedding of G into Σ^*.

However, this procedure might lead to the problem that not every face is bounded by one circuit of the graph, because not every closed Jordan Curve in the resulting surface is contractible. For defining face-colorings we need to consider embeddings where each face is bounded by one circuit. Thus, we can use the above described procedure and to avoid crossing edges we insert cross caps instead of adding handles. Eventually, we reach to one non-orientable surface Σ^* and one crossing-free embedding of G into Σ^*. Furthermore, each Jordan Curve is contractible in Σ^*. In summary, to ensure that for every graph a surface exists where we can define a face coloring on a proper crossing free embedding, we cannot rely on orientable surfaces. Jaeger proposed the following conjecture.

Conjecture 2.1.2 (The strong embedding conjecture [13])**.** *Every 2-connected graph G has an embedding where each face boundary is a circuit in some orientable surface.*

Thus, whenever we consider a face-coloring of a graph, we may require the corresponding embedding to be crossing-free and each face bounded by one circuit. For an embedding of a graph G the set of faces is denoted by $F(G)$.

A *face-coloring* of an embedding M of a graph G is a mapping c that assigns a color to every face of G in M in such a way that any two

adjacent faces receive different colors. Whenever not stated explicitly, for convenience we will use the set of integer numbers $\{0, \ldots, k-1\}$ to describe a set of k colors. A graph regarding to one embedding has a *k-face-coloring*, if it has a face-coloring which uses at most k colors. In a similar manner we define edge-colorings and vertex-colorings. A *k-vertex-coloring* of a graph is an assignment $c : V(G) \to \{0, \ldots, k-1\}$ such that adjacent vertices receive different colors. Moreover, a *k-edge-coloring* of a graph is an assignment $c : E(G) \to \{0, \ldots, k-1\}$ such that adjacent edges receive different colors. The smallest number of colors needed to edge-color or vertex-color G is the *chromatic index* or the *chromatic number* of G, respectively.

Let G be a graph embedded in a surface Σ with faces $F = F_1, \ldots, F_n$. Regarding one corresponding embedding M of G, the dual graph G' is given by $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{(v_i, v_l) : F_i \text{ and } F_l \text{ are adjacent}\}$. By identifying each face of G as a vertex of G' and by connecting each pair of vertices if and only if the corresponding faces are adjacent we naturally get one embedding of G' in Σ as well. Clearly, G is k-face colorable if and only if G' is k-vertex colorable. For unsigned graphs Tutte proved the following.

Theorem 2.1.3. *[42] Let $k \in \mathbb{Z}$. A graph G embedded in S^2 has a k-face-coloring if and only if G admits a nowhere-zero k-flow.*

Since the definition of nowhere-zero flows is independent of embeddings, flows on unsigned graphs establish a generalization for the concept of face-colorings.

To fathom a similar relation for signed graphs we need the concept of tensions. Let G be a graph and for every circuit C let D_C be an orientation of C such that each vertex has one incoming and one outgoing edge. For an orientation D of G we partition the set of edges of each circuit C into two sets C_1 and C_2. C_1 contains all edges, that have

the same orientation in D and D_C, and C_2 contains all edges that are oriented differently. G has a *k-tension* if there exists an orientation and a function $\phi : E(G) \to \mathbb{R}$ such that for each circuit C holds $\sum_{e \in C_1} \phi(e) = \sum_{e \in C_2} \phi(e)$. An embedded graph G has a *local k-tension* if there exists an orientation and a function $\phi : E(G) \to \mathbb{R}$ such that for each circuit C, which is a contractible curve, holds $\sum_{e \in C_1} \phi(e) = \sum_{e \in C_2} \phi(e)$. A k-tension ϕ or a local k-tension ϕ where $1 \leq \phi(e) \leq k-1$ is a *nowhere-zero k-tension* or *nowhere-zero local k-tension*, respectively. One equivalence between colorings and tensions is given by the following theorem. Note, that the following statements are well known. For the sake of motivating flows on signed graph, the proofs are given.

Theorem 2.1.4. *Let G be a graph and $k \in \mathbb{Z}$. G has a nowhere-zero k-tension if and only if G has a k-coloring.*

   Proof. We may assume that G is connected. Let $f : E(G) \to \mathbb{Z}$ be a k-tension with an orientation D. We will define a k-coloring $c : E(G) \to \mathbb{Z}$ as follows: Choose a vertex $u \in V(G)$ and set $c(u) = 0$. To color the remaining vertices we successively apply the following rule: Let x, y be two vertices that are connected by one edge $e = (x, y)$ such that x is colored and y is not colored. We set $c(y) \equiv c(x) + f(e) \pmod{k}$ if e is directed towards y and $c(y) \equiv c(x) - f(e) \pmod{k}$ if e is directed towards x. Thus, for a vertex z the value of $c(x)$ is the sum of $f(e)$ of forward edges e minus the sum of $f(g)$ of backward edges g on a u-z-path. Next, we have to show that the coloring which we get, is well-defined. Let z be a vertex and u_1 and u_2 be two different neighbors of z. We assume that there is one u-u_1-path W_1 and one u-u_2-path W_2 in G, both paths do not contain z. Furthermore, we may assume that W_1 and W_2 only intersect in u and z and thus, we get a circuit C consisting of $E(W_1) \cup E(W_2)$. If W_1 and W_2 would intersect in another vertex, then the last $v \in V(W_1) \cup V(W_2) \setminus \{u, z\}$ on the u-u_1-path would replace

u. We may color z by a sequence of vertex colorings corresponding to W_1 or corresponding to W_2. We have to show, that, independent of the choice, $c(z)$ is uniquely defined. We define a direction D_C of C such that each vertex has one incoming and one outgoing edge and define $C_1 := \{e \in C | D_C(e) = D(e)\}$ and $C_2 := C \setminus C_1$. Note that we may change the direction of one edge regarding to the tension and exchange its value in f by its negative inverse and receive a valid tension again. Thus, we may assume that $C_1 = W_1 \cup \{u_1 z\}$ and $C_2 = W_2 \cup \{u_2 z\}$. Since f is a tension it follows

$$\sum_{e \in C_1} f(e) - \sum_{e \in C_2} f(e) = 0. \tag{2.1}$$

For the color of z regarding W_1 it is

$$c(z) \equiv c(u) + \sum_{e \in C_1} f(e) \pmod{k} \tag{2.2}$$

and with 2.1 it follows

$$c(z) \equiv c(u) + \sum_{e \in C_2} f(e) \pmod{k}. \tag{2.3}$$

Thus, independent of the choice, $c(z)$ is uniquely defined and therefore is c well-defined.

Now, let $c : V(G) \to \mathbb{Z}$ be a k-coloring. We will define a nowhere-zero k-tension $f : E(G) \to \mathbb{Z}$ as follows. Let $e = uv$ be an edge. If $c(v) > c(u)$, we direct e from u towards v and set $f(e) = c(v) - c(u)$ and if $c(u) > c(v)$, we direct e from v towards u and set $f(e) = c(u) - c(v)$. Since c is a proper k-coloring, f is a proper k-tension.

$$\square$$
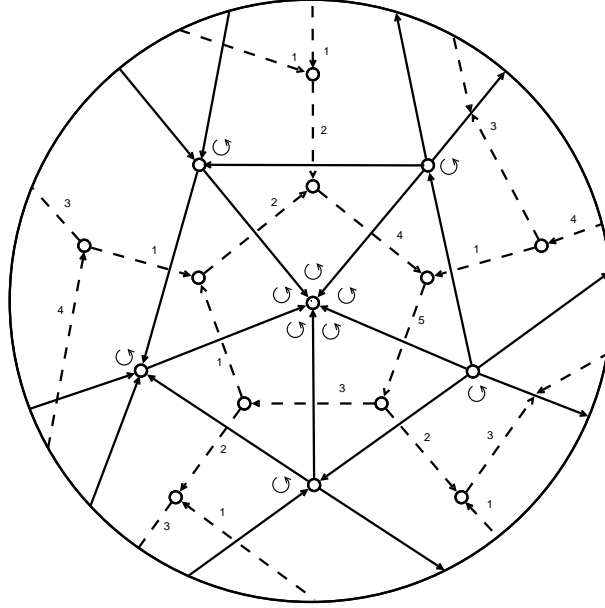
Figure 2.1: Embedding of the signed Petersen graph (P, σ) with $F(P, \sigma) = 6$ and its dual graph in the projective plane

In the following we observe the connection between colorings of embedded unsigned graphs and flows on the corresponding dual graphs. Therefore, we introduce a method for creating a bidirected dual signed graph from a directed unsigned graph embedded in one surface Σ. Let G be a directed unsigned graph. For each cylce C of G we predefine one orientation D_C such that each vertex of C has one outgoing and one incoming edge. Let G' be a dual graph of G in regard to Σ. For a vertex v' of G' with incident edges $e'_1, \ldots, e'_{d_G(v')}$ the corresponding edges $e_1, \ldots, e_{d_G(v')}$ in G are given by the circuit $C_{v'}$ which bounds the corresponding face. We define a direction of the incident half-edges of v' as follows: If e_l ($l \in \{1, \ldots, d_G(v')\}$) has the same orientation in G and D_C, then $h_v(e'_l)$ is incoming and if e_l has a different orientation in G and D_C, then $h_v(e'_l)$ is outgoing. Repeating this procedure for each vertex $u' \in G'$ leads to a bidirected dual graph with an induced signature. Let $\phi : E(G) \to \mathbb{R}$ be a mapping and let $\phi' : E(G') \to \mathbb{R}$ by $\phi'(e'_i) = \phi(e_i)$ for all $i \in \{1, \ldots, d_G(v)\}$. By this construction it is easy

to see that ϕ is a nowhere-zero k-flow if and only if ϕ' is a nowhere-zero local k-tension. Hence, a graph G has a nowhere-zero k-flow if and only if the corresponding dual Graph G' has a nowhere-zero local k-tension. In figure 2.1 one signed Petersen graph embedded in the projective plane and the corresponding dual graph is depicted with a nowhere-zero 6-flow and a nowhere-zero local 6-tension, respectively.

Chapter 3

Relations between the circular flow number and the integer flow number

The results of section 3.1, except 3.1.5 and 3.1.6, have already been published in [32]. In the previous chapter 2 we concluded that, regarding certain embeddings, colorings and integer flows are dual concepts. The definition of a coloring leads to integer valued flows in a natural way. When we allow flows to be real-valued, we get a refinement of the concept of flows and therefore a refinement for colorings as well. However, the focus of this thesis is on flows. In this chapter we study the relation between the integer flow number and its refinement, the circular flow number. Clearly, for a signed graph (G, σ) it holds $F_c((G, \sigma)) \leq F((G, \sigma))$. For an unsigned graph G, it is proven in [6] that $F(G) = \lceil F_c(G) \rceil$.

## 3.1   The difference between $F_c((G, \sigma))$ and $F((G, \sigma))$

Raspaud and Zhu [30] proved that $F((G, \sigma)) \leq 2\lceil F_c((G, \sigma)) \rceil - 1$, and
they conjectured the following.

Conjecture 3.1.1. *[30] For every graph (G, σ) holds*
$F_c((G, \sigma)) > F((G, \sigma)) - 1$.

We will show that this conjecture is not true. Let $\delta_F = \sup\{F((G, \sigma)) - F_c((G, \sigma)) : (G, \sigma) \text{ is flow-admissible}\}$. Let $t \geq 1$ be an integer and H_t be the graph which is obtained from $2t + 1$ triangles T_i, one vertex v and precisely one vertex of each triangle, say v_i, is adjacent to v. For $i \in \{1, \ldots, 2t + 1\}$ let $b_i = vv_i$. Clearly, each b_i is a bridge and H_t has no 1-factor. We define the signature σ^* by the set of negative edges N^σ which includes precisely each edge between the two bivalent vertices of each triangle.

Theorem 3.1.2. $F_c((H_t, \sigma^*)) = 3 + \frac{2}{t}$ *and* $F((H_t, \sigma^*)) = 5$, *for each integer* $t \geq 1$. *Furthermore, H_t has an integer nowhere-zero 5-flow ϕ such that $\phi(e) \in \{1, 2, 4\}$, and a $(3 + \frac{2}{t})$-flow with ϕ_c with $\phi_c(e) \in \{1, 1 + \frac{1}{t}, 2, 2 + \frac{2}{t}\}$ for all $e \in E(H_t)$*

Proof. We will construct an integer nowhere-zero 5-flow on (H_t, σ^*). Define an orientation τ on H_t as follows: Let $e_i \in N_{\sigma^*} \cap E(T_i)$ and let $e_1, \ldots, e_{t+1}$ be extroverted and $e_{t+2}, \ldots, e_{2t+1}$ be introverted. Orient the positive edges of the triangles such that the triangles are oriented like a "loop". For $i \in \{1, \ldots, t + 1\}$, v is the terminal end of b_i, and for $j \in \{t+1, \ldots, 2t+1\}$, v is the initial end of b_j. Let ϕ be the integer nowhere-zero 5-flow with $\phi(e) = 1$ if $e \in \bigcup_{i=1}^{2t} E(T_i)$, $\phi(e) = 2$, if $e \in \{b_1, \ldots, b_{2n}\} \cup E(T_{2t+1})$, and $\phi(e) = 4$, if $e = b_{2t+1}$. Hence, $F((H_t, \sigma^*)) \leq 5$.
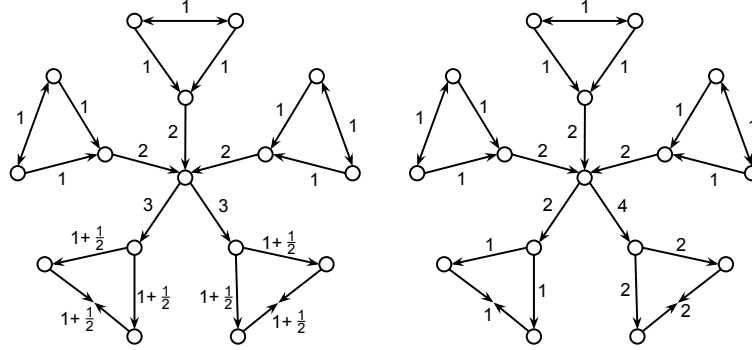
Figure 3.1: H_2

Let ψ be an integer nowhere-zero flow on (H_t, σ^*). Let $E^+(v)$ $(E^-(v))$ be the set of incoming (outgoing) edges at v. Assume that $|E^+(v)| \geq t + 1$. Since ψ is an integer flow it follows that $\psi(b_i)$ is even for every bridge. Hence, $\sum_{b \in E^+(v)} \psi(b) \geq 2t + 2$. Since $|E^-(v)| \leq t$ and ψ is an integer flow it follows that there is a bridge b with $\psi(b) \geq 4$. Hence, $F((H_t, \sigma^*)) = 5$.

We construct a nowhere-zero $(3 + \frac{2}{t})$-flow ϕ_c on (H_t, σ^*). Let τ be as above and $\phi_c(e) = \phi(e)$ if $e \in \bigcup_{i=1}^{t+1} E(T_i)$, $\phi_c(e) = 2$, if $e \in \{b_1, \ldots, b_{t+1}\}$, $\phi_c(e) = 2 + \frac{2}{t}$ if $e \in \{b_{t+2}, \ldots, b_{2t+1}\}$, and $\phi_c(e) = 1 + \frac{1}{t}$ if $e \in \bigcup_{i=t+2}^{2t+1} E(T_i)$. Hence, $F_c((H_t, \sigma^*)) \leq 3 + \frac{2}{t}$.

Let ψ_c be a nowhere-zero flow on (H_t, σ^*). Assume that $|E^+(v)| \geq t + 1$. Since $\psi_c(b_i) \geq 2$ for every bridge it follows that $\sum_{b \in E^+(v)} \psi_c(b) \geq 2t + 2$. Hence, there is a bridge b with $\psi(b) \geq 2 + \frac{2}{t}$. Therefore, $F((H_t, \sigma^*)) = 3 + \frac{2}{t}$.

□

For $t \geq 2$, the graphs $(H_t, \sigma*)$ are counterexamples to the conjecture of Raspaud and Zhu. One interesting fact about H_t is also, that H_t stays to be a counterexample under any arbitrary flow-admissible signature σ.

Lemma 3.1.3. *For $t \geq 2$, let (H_t, σ) be a flow admissible graph. Then,*
$F((H_t, \sigma)) = F((H_t, \sigma^*))$ and $F_c((H_t, \sigma)) = F_c((H_t, \sigma^*))$.

    Proof.   Let (H_t, σ) be flow-admissible.   We show that (H_t, σ) is
switching equivalent to (H_t, σ^*) where precisely the edge between the
two bivalent vertices of each triangle is negative. Switch, if necessary, at
vertices v_i to obtain a switching equivalent signature where all bridges
are positive. Clearly, each triangle is unbalanced. Hence, if three edges
of a triangle are negative, then switch at a bivalent vertex such that pre-
cisely one edge of that triangle is negative. Now, if necessary, switch at a
bivalent vertex to obtain (H_t, σ^*). Hence, $F((H_t, \sigma)) = F((H_t, \sigma^*))$ and
$F_c((H_t, \sigma)) = F_c((H_t, \sigma^*))$.

<div align="right">□</div>

We saw, that for the graph H_t exists only one flow number over all
signatures. In chapter 4 we will study possible flow numbers for fixed
graphs and variable signatures, and will introduce the term spectrum.
The observation of lemma 3.1.3 shows that, the circular flow spectrum
and the integer flow spectrum of H_t contains precisely one element which
is a seemingly rare property.

    Graph H_2 is shown in Figure 3.1. As a consequence of Theorem 3.1.2
we state:

Corollary 3.1.4. $\delta_F \geq 2$.

    The statement of Theorem 3.1.2 holds also for graphs obtained from
H_t by replacing the triangles by (negative) loops. Clearly, the argumen-
tation for the lower bounds of the flow numbers in the proof of Theorem
3.1.2 works also if replace the triangles of H_t by any other unbalanced
component and v by a balanced component. After this result was pub-
lished, Máčajová and Steffen [22] proved $\delta_F \geq 3$ by using a slightly

modified graph, where each negative loop is exchanged by $2t$ negative loops. Furthermore, when Bouchet's Conjecture is true, then $\delta_F = 3$. The constructed graphs have a special property.

Let $K_{1,t}$ be a graph consisting of one vertex, which is connected to t different vertices by a single edge.

Definition 3.1.5. *A star-cut is an induced subgraph S isormorphic to $K_{1,t}$ of G such that every vertex and every edge of H is a cut.*

Up to today, all examples with the property that $\lceil F_c((G, \sigma)) \rceil < F((G, \sigma))$ contain a star-cut. It becomes natural to ask whether for each 2-edge-connected signed graph (G, σ) the numbers $\lceil F_c((G, \sigma)) \rceil$ and $F((G, \sigma))$ are the same. We deny this question by giving a counterexample.

By K_4 we denote the complete graph on four vertices.

Proposition 3.1.6. *Let (G, σ) be the signed graph obtained from the all-positive $(K_4, 1)$ by deleting an edge $v_1 v_2$ and adding two negative loops, l_1 at v_1 and l_2 at v_2. It holds $F_c((G, \sigma)) = 3$ and $F((G, \sigma)) = 4$.*

Proof. Claim that (G, σ) admits a circular nowhere-zero 3-flow. The graph $(K_4, 1)$ admits a nowhere-zero 4-flow f with precisely one edge $v_1 v_2$ of flow value 3. Hence, define ϕ of G from f assigning $1 + \frac{1}{2}$ to each loop. We claim that (G, σ) does not admit an integer nowhere-zero 3-flow. Suppose that ϕ is one positive integer nowhere-zero 3-flow on (G, σ). Since (G, σ) has only two negative edges, both loops are oriented in opposite directions and get the same flow value x with $x \in \{1, 2\}$. An integer flow f of $(K_4, 1)$ can be obtained from ϕ on G by assigning $2x + \phi(v_1 v_2)$ to the edge $v_1 v_2$. If $x = 1$, then f itself is a nowhere-zero 3-flow of $(K_4, 1)$, which forms a contradiction. If $x = 2$, then $f(v_1 v_2) = 4$ and $f : E(K_4) \to \{\pm 1, \pm 2, 4\}$. Hence, f is a modulo 3-flow of $(K_4, 1)$

(here, $f(v_1v_2) \equiv 1 \pmod{3}$), thus, by Lemma 3.2 of [45] $(K_4, 1)$ has an integer nowhere-zero 3-flow, which forms a contraction.

$\square$

3.2   Some sufficient conditions for

$$\lceil F_c((G, \sigma)) \rceil = F((G, \sigma))$$

In 3.1.2 it was shown that the supremum of $F((G, \sigma)) - F_c((G, \sigma))$ is at least 2. In the following we give some sufficient conditions for $\lceil F_c((G, \sigma)) \rceil = F((G, \sigma))$.

A signed graph (G, σ) has a circular-flow-$\frac{1}{\mu k}$-property if for each nowhere-zero $\frac{n}{k}$-flow ϕ on (G, σ) exists a nowhere-zero $\frac{n}{k}$-flow ϕ' on (G, σ) such that $\mu k \phi'(e) \in \mathbb{Z}$ for each edge e. The family of signed graphs that have a circular-flow-$\frac{1}{\mu k}$-property is denoted by $\mathcal{G}_\mu$. With this notion, we can restate Theorem 1.4.8 as follows: All unsigned flow-admissible graphs are members of $\mathcal{G}_1$. Now, we will show the corresponding result for general signed graphs that all signed graphs are members of $\mathcal{G}_2$. This is best possible since in general flow-values can not be scaled by mutiples of $\frac{1}{q}$, as for example figure 3.1 shows. It is easy to see that every circular nowhere-zero 4-flow on (H_2, σ) must contain an edge with the flow value $1 + \frac{1}{2}$.

For the next theorem we need the concept of pseudoflows:

Let (G, σ) be a signed graph. A function $f : E(G) \rightarrow \mathbb{R}$ with orientation τ_f which fulfills $0 = \sum_{h \in H(v)} \tau(h) f(e_h)$ for each vertex $v \in V(G)$ is a pseudo-r-flow if $0 \leq |f(e)| \leq r - 1$ for each edge $e \in E(G)$. Obviously, a pseudo-r-flow is a nowhere-zero r-flow if $|f(e)| \geq 1$ for each $e \in E(G)$.

For two nowhere-zero flows f and g with orientations τ_f and τ_g, the difference $f - g$ is defined for all edges $e \in E(G)$ as

- an orientation τ_h where $\tau_h(e) = \tau_f(e)$ if $f(e) \geq g(e)$ or $\tau_h(e) = \tau_g(e)$ if $f(e) < g(e)$ and

- a function $h : E(G) \to \mathbb{R}$ with $h(e) = f(e) - g(e)$ if $f(e) \geq g(e)$ or $h(e) = g(e) - f(e)$ if $f(e) < g(e)$.

For a signed graph (G, σ) with two pseudo flows f and g, it is easy to see that $f - g$ is also a pseudo flow.

Theorem 3.2.1. *Let (G, σ) be a graph and $F_c((G, \sigma)) = \frac{p}{q} + 1$. Then there exists a nowhere-zero $(\frac{p}{q} + 1)$-flow ϕ on (G, σ), such that $\forall e \in E(G)$ $\exists n_e \in \mathbb{Z} : \phi(e) = \frac{n_e}{2q}$.*

Proof. Given a nowhere-zero $(\frac{p}{q} + 1)$-flow ψ let $F_\psi = \{e \in E(G) : 2q\psi(e) \notin \mathbb{Z}\}$. Choose a nowhere-zero $(\frac{p}{q} + 1)$-flow ϕ of (G, σ) for which F_ϕ has minimum cardinality. We may assume that $\phi(e) \geq 1$ for each $e \in E(G)$. If $F_\phi = \emptyset$ then the flow value on each edge is a multiple of $\frac{1}{2q}$ and we are done. Thus, assume F_ϕ is not empty. Furthermore, each vertex of G is incident to either zero or at least two edges of F_ϕ.

Claim 1: There is no subset of F_ϕ which induces a signed circuit. Suppose to the contrary that there is a subset $F' \subseteq F_\phi$ that induces a signed circuit C. First, we consider the case that C is a balanced circuit or consists of two unbalanced circuits that intersect in exactly one vertex. Since C admits a nowhere-zero 2-flow let D_B be an orientation corresponding to a nowhere-zero 2-flow of C in G. Let $\epsilon = min_{e \in F_\phi}\{\frac{p}{q} - \phi(e), \phi(e) - 1\}$. Let ϕ_p be a pseudoflow on G with orientation $D_{\phi_p} = D_B \cup D_\phi|_{E(G) - E(C)}$, $\phi_p(e) = \epsilon$ for $e \in C$, and $\phi_p(e) = 0$

for $e \in E(G) \backslash C$. Then, the mappings $\phi + \phi_p$ and $\phi - \phi_p$ are nowhere-zero
$(\frac{p}{q} + 1)$-flows and $F_{\phi + \phi_p}$ or $F_{\phi - \phi_p}$ is a proper subset of F_ϕ contradicting the
choice of ϕ. It remains to show that C is not a barbell. Suppose C con-
sists of two unbalanced circuits C_1 and C_2 and a path P connecting them.
Define a parameter α_e with $\alpha_e = \frac{1}{2}$ for $e \in P$ and $\alpha_e = 1$ for $e \notin P$. Since
C admits a nowhere-zero 3-flow let D_B be an orientation corresponding
to a nowhere-zero 3-flow. Let $\epsilon = min_{e \in F_\phi} \{\alpha_e(\frac{p}{q} - \phi(e)), \alpha_e(\phi(e) - 1)\}$.
Let ϕ_p be a pseudoflow on G with orientation $D_{\phi_p} = D_B \cup D_\phi|_{E(G)-E(C)}$,
$\phi_p(e) = \epsilon$ for $e \in C_1 \cup C_2$, $\phi_p(e) = 2\epsilon$ for $e \in P$, and $\phi_p(e) = 0$ for
$e \in E(G) \backslash C$. Again, $\phi + \phi_p$ and $\phi - \phi_p$ are nowhere-zero $(\frac{p}{q} + 1)$-flows
and $F_{\phi + \phi_p}$ or $F_{\phi - \phi_p}$ is a proper subset of F_ϕ contradicting the choice of ϕ.

Claim 2: For each $v \in V$, $|\delta(v) \cap F_\phi| \in \{0, 2\}$. For sure, $|\delta(v) \cap F_\phi| \neq 1$.
Assume to the contrary that for $v_1 \in V(G)$ holds $|\delta(v_1) \cap F_\phi| \geq 3$.

If v_1 belongs to a path in $G[F_\phi]$, that connects two circuits, then
either one circuit is balanced or v_1 belongs to a barbell. That contra-
dicts Claim 1. Thus, v_1 belongs to one unbalanced circuit C and any
path from v_1 must end in C. Let $P = v_1, w_1, \ldots, w_l, v_k$ ($k \in [2, n]$,
$l \in \mathbb{Z}$) (the w_i's and v_j's are pairwise disjoint) be such a path and let
$C = v_1, \ldots, v_n$. However, either the circuit $C_1 = v_1, \ldots, v_k, w_l, \ldots, w_1, v_1$
or $C_2 = v_k, \ldots, v_n, v_1, w_1, \ldots, w_l, v_k$ is balanced contradicting Claim 1.

To complete the proof it remains to show that there is no unbal-
anced circuit in F_ϕ. Assume $C = v_1, \ldots, v_n$ ($n \in \mathbb{Z}$) is an unbalanced
circuit in F_ϕ with edges $e_l = (v_l, v_{l+1})$, $l \mod n$. We may change the
orientation of C such that each vertex but v_1 is incident to exactly
one incoming and one outgoing half-edge. For each edge for which we
change the direction we also exchange its flow-value by its negative in-
verse and ϕ remain a proper nowhere-zero flow. Since a vertex in C

has exactly two incident edges from F_ϕ we get $2q(\phi(e_n) + \phi(e_1)) \in \mathbb{Z}$ and $2q(\phi(e_k) - \phi(e_{k-1})) \in \mathbb{Z}$ for $k \in \{2, \ldots, n\}$. By induction we get $2q(\phi(e_n) - \phi(e_1)) \in \mathbb{Z}$ and hence, $4q\phi(e_n) \in \mathbb{Z}$ and it follows that each $\phi(e_k)$ is an odd multiple of $\frac{1}{4q}$ for $k \in \{1, \ldots, n\}$. Since C has an odd number of negative edges the minimum cut X that separates C from G carries an excess $\sum_{e \in X \cap C^+} \phi(e) - \sum_{e \in X \cap C^-} \phi(e) = \frac{2(2m+1)}{4q} = \frac{2m+1}{2q}$ $(m \in \mathbb{Z})$. Let $F' = \{e \in E(G) : \phi(e) = \frac{2k+1}{2q}, k \in \mathbb{Z}\}$.

We get $|F' \cap X| \equiv 1 \mod 2$ and for each $v \in V(G) - V(G[F_\phi])$, $|\delta(v) \cap F'| \equiv 0 \mod 2$. Therefore, not every path with edges of F' that begins with a vertex in C ends in C as well. There exists a barbell B consisting of a path $P \subseteq F'$ that connects C with another unbalanced circuit $C' \subseteq F_\phi$. Since B admits a nowhere-zero 3-flow let D_B be an orientation corresponding to a nowhere-zero 3-flow. Let ϕ_p be a pseudoflow on G with orientation $D_{\phi_p} = D_B \cup D_\phi|_{E(G)-E(B)}$, $\phi_p(e) = \frac{1}{4q}$ for $e \in C \cup C'$, $\phi_p(e) = \frac{1}{2q}$ for $e \in P$, and $\phi_p(e) = 0$ for $e \in E(G) \setminus B$. Again, $\phi + \phi_p$ and $\phi - \phi_p$ are nowhere-zero $(\frac{p}{q} + 1)$-flows and $F_{\phi+\phi_p}$ or $F_{\phi-\phi_p}$ is a proper subset of F_ϕ contradicting the choice of ϕ which completes the proof.

$\square$

Lemma 3.2.2. *Let* $(G, \sigma) \in \mathcal{G}_1$. *Then* $\lceil F_c((G, \sigma)) \rceil = F((G, \sigma))$.

Proof. Let $(G, \sigma) \in \mathcal{G}_1$ with a circular nowhere-zero $\frac{p}{q}$-flow f and let $h = \lceil \frac{p}{q} \rceil$. Since f can also be considered as a circular nowhere-zero h-flow, (G, σ) admits a circular nowhere-zero h-flow f' with rational flow values from the set $\{1, 1 + \frac{1}{1}, 1 + \frac{2}{1}, \cdots, h - 1\}$. Obviously, f' is an integer valued nowhere-zero h-flow.

$\square$

Lemma 3.2.3. *If (G, σ) has no long barbell, then $(G, \sigma) \in \mathcal{G}_1$.*

Proof. Given a nowhere-zero $(\frac{p}{q} + 1)$-flow ψ let $w(e) \equiv \phi(e) \pmod{1}$ and $F_\psi = \{e \in E(G) : w(e) \neq 0\}$. Choose a nowhere-zero $(\frac{p}{q} + 1)$-flow ϕ of (G, σ) for which F_ϕ has minimum cardinality. We may assume that $\phi(e) \geq 1$ for each $e \in E(G)$. If $F_\phi = \emptyset$ then we are done. Thus, we suppose to the contrary, that $F_\phi \neq \emptyset$.

Claim 1: F_ϕ contains neither a balanced circuit nor consists of two unbalanced circuits that intersect in exactly one vertex.

Suppose that there is a subset $F' \subseteq F_\phi$ that induces a signed circuit C that is either a balanced circuit or consists of two unbalanced circuits that intersect in exactly one vertex. Since C admits a nowhere-zero 2-flow let D_B be an orientation corresponding to a nowhere-zero 2-flow of C in G. Let $\epsilon = min_{e \in F_\phi}\{\frac{p}{q} - \phi(e), \phi(e) - 1\}$. Let ϕ_p be a pseudoflow on G with orientation $D_{\phi_p} = D_B \cup D_\phi|_{E(G) - E(C)}$, $\phi_p(e) = \epsilon$ for $e \in C$, and $\phi_p(e) = 0$ for $e \in E(G) \setminus C$. Then, the mappings $\phi + \phi_p$ and $\phi - \phi_p$ are nowhere-zero $(\frac{p}{q} + 1)$-flows and $F_{\phi + \phi_p}$ or $F_{\phi - \phi_p}$ is a proper subset of F_ϕ contradicting the choice of ϕ.

Thus, F contains an unbalanced circuit C_1. By switching, we may assume that in C_1 there is precisely one negative edge e_1. We may assume that e_1 is extroverted. Since ϕ is balanced at every vertex, the total in-flow and out-flow of all negative edges is zero.

$$\sum_{e \in F_\phi \cap N_\sigma} 2\phi(e) = 0.$$

That is,

$$\sum_{e \in F_\phi \cap N_\sigma} \phi(e) = 0.$$

Thus, there must be another negative edge e_2 in F_ϕ. The edges e_1, e_2 are contained in different unbalanced circuits C_1 and C_2 of F_ϕ. Joining C_1 and C_2 by a path of G, we get a long barbell, forming a contradiction.

$\square$

With lemma 3.2.2 and 3.2.3 it simply follows the next Corollary.

Corollary 3.2.4. *If (G,σ) has no long barbell, then*

$$\lceil F_c((G,\sigma)) \rceil = F((G,\sigma)).$$

Chapter 4

Flows on signed regular graphs

4.1 Preface

The results of this chapter have been published in [32]. Let G be a graph and $X \subseteq E(G)$. Let $\Sigma_X(G)$ be the set of signatures σ of G, for which (G, σ) is flow-admissible and $N_\sigma \subseteq X$. We define $\mathcal{S}_X(G) = \{r :$ there is a signature $\sigma \in \Sigma_X(G)$ such that $F_c((G, \sigma)) = r\}$ to be the X-*flow spectrum* of G. The $E(G)$-flow spectrum is the *flow spectrum* of G and it is denoted by $\mathcal{S}(G)$. If we consider integer-valued flows, then $\overline{\mathcal{S}}_X(G)$ denotes the *integer X-flow spectrum* of G.

Section 4.2 characterizes $(2t+1)$-regular graphs whose flow spectrum contains $2 + \frac{1}{t}$. Furthermore, if a $(2t + 1)$-regular graph has a 1-factor, then its integer flow spectrum contains 3. However, for every $t \geq 2$, there is a signed $(2t + 1)$-regular graph (H, σ) with integer flow number 3 and H does not have a 1-factor.

One of the earliest results on flows on graphs is Tutte's characterization of bipartite cubic graphs [40]. His observation that a cubic graph

is bipartite if and only if it admits a nowhere-zero 3-flow motivated the following statement.

Theorem 4.1.1. *[35] Let $t \geq 1$ be an integer. A $(2t+1)$-regular graph G is bipartite if and only if $F_c((G,1)) = 2 + \frac{1}{t}$. Furthermore, if G is not bipartite, then $F_c((G,1)) \geq 2 + \frac{2}{2t-1}$.*

The the situation does not change in the more general case of flow numbers on signed $(2t+1)$-regular graphs. It is proven that if r is an element of the flow spectrum of a $(2t+1)$-regular graph, then $r = 2 + \frac{1}{t}$ or $r \geq 2 + \frac{2}{2t-1}$. In order to generalize the structural part of Theorem 4.1.1 we will need the following definition: Let $r \geq 2$ be a real number and G be a graph. A set $X \subseteq E(G)$ is *r-minimal* if

1) there is a signature σ of G such that $F_c((G,\sigma)) = r$ and $N_\sigma = X$, and

2) $F_c((G,\sigma')) \neq r$ for every signature σ' of G with $N_{\sigma'} \subset X$.

In Section 4.3 we show that a set $X \subseteq E(G)$ is a minimal set such that $G - X$ is bipartite if and only if X is $(2 + \frac{1}{t})$-minimal.

Since Bouchet's conjecture is equivalent to its restriction on cubic graphs we study flows on signed cubic graphs in Section 4.4. Let K_2^3 be the unique cubic graph on two vertices which are connected by three edges. We study the relation between 3- and 4-minimal sets and deduce that if G has a 1-factor and $G \neq K_2^3$, then $\{3,4\}$ is a subset of its flow spectrum and of its integer flow spectrum. Furthermore, if $G \neq K_2^3$, then the following four statements are equivalent: (1) G has a 1-factor. (2) $3 \in \mathcal{S}(G)$ (3) $3 \in \overline{\mathcal{S}}(G)$. (4) $4 \in \overline{\mathcal{S}}(G)$. There are cubic graphs whose integer flow spectrum does not contain 5 or 6, and we construct an infinite family of bridgeless cubic graphs with integer flow spectrum $\{3,4,6\}$.

We prove some sharp bounds for the cardinality of smallest 3-minimal and 4-minimal sets, respectively. If G is not 3-edge-colorable, then these bounds are formulated in terms of its resistance and oddness.

A *Kotzig graph* is a cubic graph that has three 1-factors such that the union of any two of them induces a hamiltonian circuit. The chapter concludes with a proof of Bouchet's conjecture for Kotzig-graphs.

4.2 Smallest possible flow numbers of signed $(2t+1)$-regular graphs

This section characterizes $(2t + 1)$-regular graphs whose flow spectrum contains $2 + \frac{1}{t}$.

Let (G, σ) be a signed graph that admits a (modular) r-flow ϕ. Let e be an edge of the support of ϕ. If we reverse the orientation of e and replace $\phi(e)$ by $-\phi(e)$ $(r - \phi(e))$ then we obtain another (modular) r-flow ϕ' with the same support as ϕ. Hence, we can assume that a graph which has a (modular) r-flow ϕ has a (modular) r-flow ϕ' with the same support and $\phi'(e) \geq 0$ for all $e \in E(G)$.

Lemma 4.2.1. *Let $t \geq 1$ be an integer and (G, σ) be a signed $(2t + 1)$-regular graph. If (G, σ) admits a modular nowhere-zero $(2 + \frac{1}{t})$-flow ϕ, then $|\phi(e)| \in \{1, 1 + \frac{1}{t}\}$ for every $e \in E(G)$.*

Proof. Let ϕ be a modular nowhere-zero $(2 + \frac{1}{t})$-flow on (G, σ). Suppose to the contrary that there is an edge e' with $|\phi(e')| \notin \{1, 1 + \frac{1}{t}\}$. By the remark above there is modular nowhere-zero $(2 + \frac{1}{t})$-flow ψ with $\psi(e) > 0$ for every $e \in E(G)$. Since $-\phi(e') \notin \{1, 1 + \frac{1}{t}\}$, and $2 + \frac{1}{t} - \phi(e') \notin \{1, 1 + \frac{1}{t}\}$ it follows that $\psi(e') \notin \{1, 1 + \frac{1}{t}\}$.

Let $e' \in E(v)$. We can assume that there is an orientation of the half-edges of (G, σ) such that $|H^+(v)| = 2t + 1$ and a modular nowhere-

zero $(2 + \frac{1}{t})$-flow ψ' such that $\psi'(e') \notin \{1, 1 + \frac{1}{t}\}$ and $\psi'(e) > 0$ for every $e \in E(G)$. It follows that there is a positive integer k such that $\sum_{h \in H^+(v)} \psi'(e_h) = k(2 + \frac{1}{t})$. We have $\sum_{h \in H^+(v)} \psi'(e_h) > 2t + 1$ and hence, $k > t$. On the other side $\sum_{h \in H^+(v)} \psi'(e_h) < 2t + 3 + \frac{1}{t}$ and hence, $k < t + 1$, a contradiction. Therefore, $|\phi(e)| \in \{1, 1 + \frac{1}{t}\}$ for every $e \in E(G)$.

□

The next theorem is a generalization of Lemma 3.2 of [45].

Theorem 4.2.2. *Let $t \geq 1$ be an integer. A signed $(2t + 1)$-regular graph (G, σ) admits a nowhere-zero $(2 + \frac{1}{t})$-flow if and only if (G, σ) admits a modular $(2 + \frac{1}{t})$-flow and G has a t-factor.*

Proof. Let ϕ be a nowhere-zero $(2 + \frac{1}{t})$-flow on (G, σ) with $\phi(e) > 0$ for every $e \in E(G)$. Since every $(2 + \frac{1}{t})$-flow is a modular $(2 + \frac{1}{t})$-flow it follows with Lemma 4.2.1 that $\phi(e) \in \{1, 1 + \frac{1}{t}\}$ for every $e \in E(G)$. The set of edges with flow value $1 + \frac{1}{t}$ induces a t-factor of G, and ϕ is a modular nowhere-zero $(2 + \frac{1}{t})$-flow.

If (G, σ) admits a modular nowhere-zero $(2 + \frac{1}{t})$-flow, then it follows with Lemma 4.2.1 there is one, say ϕ, such that $\phi(e) = 1$ for every $e \in E(G)$. Since G is $(2t + 1)$-regular, it follows that every vertex of G is incident to either incoming or outgoing edges, only. Let F be a t-factor of G. If we reverse the orientation of the edges of F, then the function ϕ' with $\phi'(e) = 1$ if $e \in E(G) - E(F)$ and $\phi'(e) = 1 + \frac{1}{t}$ if $e \in E(F)$ is a nowhere-zero $(2 + \frac{1}{t})$-flow on (G, σ).   □

We will need the following result of Petersen.

Theorem 4.2.3. *[29] Let k be a positive integer and G a k-regular graph. If k is even, then G has a 2-factor.*

Theorem 4.2.4. *Let $t \geq 1$ be an integer.*

1) A $(2t+1)$-regular graph G has a t-factor if and only if $2 + \frac{1}{t} \in \mathcal{S}(G)$.

*2) If G is $(2t+1)$-regular and has a 1-factor, then $3 \in \overline{\mathcal{S}}(G)$.*

Furthermore, for each $t > 1$ there is a $(2t+1)$-regular graph G_t which has no 1-factor and $3 \in \overline{\mathcal{S}}(G_t)$.

Proof. 1) Let G have a t-factor. Let σ be the signature of G with $N_\sigma = E(G)$. The function ϕ with $\phi(e) = 1$ for every edge e is a modular nowhere-zero $(2 + \frac{1}{t})$-flow on G. It follows with Theorem 4.2.2 that $2 + \frac{1}{t} \in \mathcal{S}(G)$. If $2 + \frac{1}{t} \in \mathcal{S}(G)$, then it follows with Theorem 4.2.2 that G has a t-factor.

2) If $t = 1$, then ϕ' is an integer 3-flow and it follows from 1) that $3 \in \overline{\mathcal{S}}(G)$. Let $t \geq 2$ and F_1 be a 1-factor of G. By Theorem 4.2.3, $G - F_1$ has a 2-factor F_2. Hence, $F_1 \cup F_2$ induces a spanning cubic subgraph H of G which has a 1-factor. By 1), H has a signature σ such that (H, σ) has an integer nowhere-zero 3-flow. Furthermore, $H' = G - E(H)$ is $(2t - 2)$-regular and hence, $(H', 1)$ has a nowhere-zero 2-flow. Thus, (G, σ) has an integer nowhere-zero 3-flow. Since a 3-flow is the smallest possible integer flow on a $(2t+1)$-regular graph it follows that $3 \in \overline{\mathcal{S}}(G)$.

It remains to construct the graph G_t for $t > 1$. Let T be a triangle where exactly two vertices are joined by two parallel edges, all other vertices are connected by a single edge. Take four copies $T_1, \ldots, T_4$ of T and connect each bivalent vertex of T_1, T_2, T_3 with the bivalent vertex y of T_4 by an edge. Let H be this graph. Let σ be the signature of H where N_σ is the set of the parallel edges of $T_1, \ldots, T_4$. Graph (H, σ) with nowhere-zero 3-flow is shown in Figure 4.1. Since H has a vertex of degree 3, it follows that $F((H, \sigma)) = 3$.

Let $K'_{n,n}$ be the complete bipartite graph on $2n$ vertices where one edge uv is replaced by a path uxv. For $t > 1$ and for each trivalent vertex z of H take $t - 1$ copies of $K'_{2t+1,2t+1}$ and identify z and the bivalent
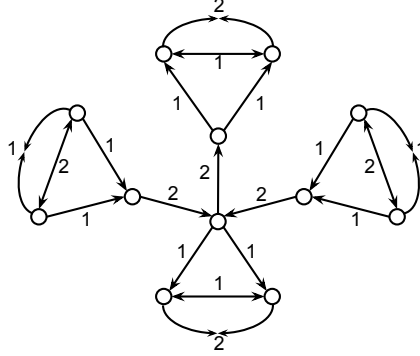
Figure 4.1: A nowhere-zero 3-flow on (H, σ)

vertices of the copies of $K'_{2t+1,2t+1}$. Do the same with $t-2$ copies of $K'_{2t+1,2t+1}$ and y. The resulting graph G_t is $(2t+1)$-regular. We have $N_\sigma \subseteq E(G_t)$, and since $F((K'_{2t+1,2t+1}, 1)) = 3$ it follows that $3 \in \overline{\mathcal{S}}(G_t)$.

Since $G_t - y$ has more than one odd component it follows that G_t does not have a 1-factor.

$\square$

The second part of Theorem 4.1.1 can be generalized to signed graphs.

Theorem 4.2.5. *[32] Let $t \geq 1$ be an integer and (G, σ) be a signed $(2t+1)$-regular graph. If $F_c((G, \sigma)) = r$, then $r = 2 + \frac{1}{t}$ or $r \geq 2 + \frac{2}{2t-1}$.*

Theorems 4.2.4 and 4.2.5 imply the following corollary.

Corollary 4.2.6. *Let $t \geq 1$ be an integer and (G, σ) be a flow-admissible signed $(2t+1)$-regular graph. If G does not have a t-factor, then $F_c((G, \sigma)) \geq 2 + \frac{2}{2t-1}$.*

4.3 r-minimal sets

This section studies the structural implications of the existence of a nowhere-zero $(2 + \frac{1}{t})$-flow on a signed $(2t+1)$-regular graph. Hence, it extends the first part of Theorem 4.1.1 to signed graphs.

Proposition 4.3.1. *Let $r \geq 2$ and G be graph.*

1) The empty set is r-minimal if and only if $F_c((G, 1)) = r$.

2) $F_c((G, 1)) \in \mathcal{S}_X(G)$ for every r-minimal set X.

Let G be a $(2t + 1)$-regular graph and $X \subseteq E(G)$. Let H and H' be two copies of $G - X$. For $v \in V(H)$ and $e \in E(H)$, let v' (e') be the corresponding vertex (edge) in H'. For each edge $uv \in X$ add edges uu' and vv' (between H and H') to obtain a new $(2t + 1)$-regular graph G_X^2. Let E_X^2 be the set of the added edges. Let σ be a signature on G, and $\sigma|_{G-X}$ be the restriction of σ on $G - X$. Let σ_X^2 be the signature on G_X^2 which is equal to $\sigma|_{G-X}$ on H and H' and all edges of E_X^2 are positive. Note that $|N_{\sigma_X^2}| = 2|N_\sigma - X|$. In particular, if $N_\sigma \subseteq X$, then σ_X^2 is the empty signature.

Lemma 4.3.2. *Let $t \geq 1$ be an integer and (G, σ) be a signed $(2t + 1)$-regular graph. Let $r \geq 2$ and $X \subseteq E(G)$. Every nowhere-zero r-flow on (G, σ) induces a nowhere-zero r-flow on (G_X^2, σ_X^2).*

Proof. If τ is an orientation of the half-edges of (G, σ), then $\overline{\tau}$ denotes the orientation of the half-edges of (G, σ) which is obtained from τ by reversing the orientation of every half-edge. Now, if ϕ is a flow on (G, σ) with orientation τ, then ϕ is also a flow on (G, σ) with orientation $\overline{\tau}$.

If ϕ is a nowhere-zero r-flow with orientation τ, then define a nowhere-zero r-flow on (G_X^2, σ_X^2) as follows. Let ψ be the restriction of ϕ on (H, σ) with orientation τ and ψ' be the restriction of ψ on (H', σ') with orientation $\tau' = \overline{\tau}$. Extend these orientations to an orientation τ_X^2 on (G_X^2, σ_X^2) as follows. The orientation of the half-edges of H or H' is unchanged, and for an edge $e \in X$ with $e = uv$ orient the edges uu' and vv' of E_X^2 as follows: $\tau_X^2(h_{uu'}^u) = \tau(h_e^u)$, $\tau_X^2(h_{uu'}^{u'}) = \tau'(h_{e'}^{u'})$, $\tau_X^2(h_{vv'}^v) = \tau(h_e^v)$, and $\tau_X^2(h_{vv'}^{v'}) = \tau'(h_{e'}^{v'})$. To obtain a nowhere-zero r-flow ϕ_X^2 on

(G_X^2, σ_X^2) let ψ and ψ' be unchanged and for $uu', vv' \in E_X^2$ which are obtained from edge $e \in X$ with $e = uv$ let $\phi_X^2(uu') = \phi_X^2(vv') = \phi(e)$.

$\square$

Theorem 4.3.3. *Let $t \geq 1$ be an integer and G be a $(2t+1)$-regular graph. A set $X \subseteq E(G)$ is $(2+\frac{1}{t})$-minimal if and only if G has a t-factor and X is a minimal set such that $G - X$ is bipartite.*

Proof. Let X be $(2+\frac{1}{t})$-minimal. By definition, there is a signature σ of G, such that $F((G,\sigma)) = 2 + \frac{1}{t}$, $N_\sigma = X$ and $F((G,\sigma')) \neq 2 + \frac{1}{t}$ for every signature σ' of G with $N_{\sigma'} \subset X$. By Theorem 4.2.4, G has a t-factor.

Let ϕ be a nowhere-zero $(2 + \frac{1}{t})$-flow on (G,σ). By Lemma 4.3.2, ϕ induces a nowhere-zero $(2 + \frac{1}{t})$-flow on $(G_X^2, 1)$. By Theorem 4.1.1, G_X^2 is bipartite and therefore $G - X$ as well.

Now suppose to the contrary that there is a proper subset X^* of X such that $G - X^*$ is bipartite.

(*) It easily follows that $G_{X^*}^2$ is bipartite and therefore, $(G_{X^*}^2, 1)$ has a nowhere-zero $(2 + \frac{1}{t})$-flow by Theorem 4.1.1. This $(2 + \frac{1}{t})$-flow can be modified to a modular nowhere-zero $(2 + \frac{1}{t})$-flow ϕ such that $\phi(e) = 1$ for every edge $e \in E_{X^*}^2$. Hence, if we reconstruct G from $G_{X^*}^2$ by keeping the orientation of the half-edges appropriately, we obtain a signature σ^* of G with $N_{\sigma^*} \subseteq X^*$ and a modular nowhere-zero $(2 + \frac{1}{t})$-flow ϕ^* on G. By Theorem 4.2.2, (G, σ^*) has a nowhere-zero $(2 + \frac{1}{t})$-flow. But N_{σ^*} is a proper subset of X, contradicting the fact that X is $(2 + \frac{1}{t})$-minimal.

Let X be a minimal set such that $G - X$ is bipartite. Clearly, if $X \neq \emptyset$, then $|X| \geq 2$. As above (*), we deduce that G has a signature σ with $N_\sigma \subseteq X$ and a nowhere-zero $(2+\frac{1}{t})$-flow ϕ on (G,σ). Suppose to the contrary that there is an edge $e \in X - N_\sigma$. By Lemma 4.3.2, ϕ induces a nowhere-zero $(2+\frac{1}{t})$-flow on $(G_{X-e}^2, 1)$. Hence, $G - (X - e)$ is bipartite,

contradicting the minimality of X. Therefore, X is a $(2 + \frac{1}{t})$-minimal set.

$\square$

An exhaustive survey on sufficient conditions for the existence of factors in regular graphs is given in [1].

4.4 The flow spectrum of cubic graphs

This section studies the flow spectrum and r-minimal sets of cubic graphs. We will construct some flows on cubic graphs.

Theorem 4.4.1. *Let G be a cubic graph. If G is bipartite, then it has signature σ with $|N_\sigma| = 2$ and $F_c((G, \sigma)) = F((G, \sigma)) = 4$.*

Proof. By Theorem 4.1.1, $(G, 1)$ has a nowhere-zero 3-flow ϕ. We can assume that $\phi(e) \geq 1$ for each $e \in E(G)$. The edges with flow value 1 induce a 2-factor F of G. Let C be a circuit of F. Any two adjacent edges of C are both oriented towards the vertex they share or both away from it. Hence, ϕ can be modified to a nowhere-zero 3-flow ϕ', where the circuits of F are directed circuits and the flow values on the edges are 1 and -1, alternately.

Claim 4.4.2. *There is a circuit C of F and a path P of three consecutive edges e_1, e_2, e_3 of C such that $G - \{e_1, e_3\}$ is connected. Furthermore, G contains a circuit D such that $e_1 \in E(D)$ and $e_3 \notin E(D)$.*

Proof. Every circuit of F has a length of at least 4. Let C be a circuit of F, and $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$ and $e_4 = v_4v_5$ ($v_1 = v_5$ is not excluded) four consecutive edges in C. Furthermore, v_2 has a neighbor $x \notin \{v_1, v_3, v_4\}$. If $\{e_1, e_3\}$ is a 2-edge-cut in G, then choose P' with edges e_2, e_3, e_4. Suppose to the contrary that $\{e_2, e_4\}$ is an edge-cut. It

follows that e_2 is simple and $v_2 x$ is a bridge, contradicting the fact that G is bridgeless. Hence, there is a path as claimed. Furthermore, $G - e_3$ is bridgeless and hence, $G - e_3$ contains a circuit D with $e_1 \in E(D)$.

$\square$

Let P be the path in C with three consecutive edges e_1, e_2, e_3 ($e_i = v_i v_{i+1}$) such that $G - \{e_1, e_3\}$ is connected. We can assume that e_2 is directed from v_2 to v_3 and that $\phi'(e_2) = 1$. Let τ be the orientation of $H(G)$ which is obtained from the underlying orientation for ϕ' by reversing the orientation of $h_{e_1}^{v_1}$ and $h_{e_3}^{v_4}$. Hence, we obtain a signature σ of G with $N_\sigma = \{e_1, e_3\}$, where e_1 is extroverted and e_3 introverted. Consider ϕ' on (G, σ), then $\delta\phi'(v_1) = -2$, $\delta\phi'(v_4) = 2$, and $\delta\phi'(v) = 0$ if $v \in V(G) - \{v_1, v_4\}$. Let $E(\overline{P}) = E(C) - E(P)$. The function $\psi : E(G) \to \{1, 2, 3\}$ with $\psi(e) = \phi'(e)$ if $e \in E(G) - E(\overline{P})$ and $\psi(e) = \phi'(e) + 2$ if $e \in E(\overline{P})$ is a nowhere-zero 4-flow on (G, σ). Since ψ is an integer flow it follows that $F((G, \sigma)) \leq 4$.

By Claim 4.4.2 and Theorem 4.2.5 it follows that $F_c((G, \sigma)) \geq 4$ and hence, $F_c((G, \sigma)) = F((G, \sigma)) = 4$.

$\square$

We will use the strict form of Petersen's Theorem on 1-factors in cubic graphs.

Theorem 4.4.3 ([29])**.** *Let G be a bridgeless cubic graph. For every $e \in E(G)$ there is a 1-factor of G that contains e.*

The minimum number of odd circuits of a 2-factor of a cubic graph G is the *oddness* of G and it is denoted by $\omega(G)$. A 2-factor that has precisely $\omega(G)$ odd circuits is a minimum 2-factor of G.

Theorem 4.4.4. *Every non-bipartite cubic graph G with a 1-factor has a signature σ such that $|N_\sigma| = \omega(G)$ and $F_c((G, \sigma)) = F((G, \sigma)) = 4$. Furthermore, if G is bridgeless, then for every 3-minimal set X_3 there is a 4-minimal set X_4 with $X_4 \subset X_3$ and $|X_4| \leq \omega(G) < |X_3|$.*

Proof. If G is 3-edge-colorable, then the empty set is 4-minimal and the statement follows.

Thus, we assume that G is not 3-edge-colorable in the following. Let $\omega(G) = 2n$, F_2 be a 2-factor with odd circuits $C_1, \ldots, C_{2n}$, and F_1 be the complementary 1-factor. If G has a bridge, then there is one, say b, such that one component of $G - b$ is bridgeless. Such components will be called end-components.

We first show that there is a signature σ with $|N_\sigma| = \omega(G)$ and $F_c((G, \sigma)) = F((G, \sigma)) = 4$. For $i \in \{1, \ldots, 2n\}$ choose $f_i \in E(C_i)$ with the following restrictions if G has bridges or if an odd circuit of F_2 contains a multi-edge.

(1) One of the odd circuits of F_2, say C_k, has two vertices which are connected by two edges in G. Choose f_k to be one of these two edges, and for $i \neq k$ choose $f_i \in E(C_i)$ arbitrarily.

(2) All edges of the odd circuits of F_2 are simple in G and G has an end-component K such that the bivalent vertex x is contained in an odd circuit C_k of F_2. Let x_1 and x_2 be the two neighbors of x in K. Then choose f_k to be an edge of C_k which is incident to x_1 and different from xx_1. For $i \neq k$ choose $f_i \in E(C_i)$ arbitrarily.

In all other cases choose $f_i \in E(C_i)$ arbitrarily.

Subdivide f_i by a vertex u_i and add edges $e_k = u_{2k-1}u_{2k}$, for $k \in \{1, \ldots, n\}$. The resulting graph G' is cubic. The set $F_1' = F_1 \cup \{e_k : k = 1, \ldots, n\}$ is a 1-factor and $F_2' = E(G') \setminus F_1'$ is an even 2-factor of G'. The odd circuits C_i of F_2 are transformed into even circuits C_i' of F_2'. Let f_i' and f_i'' be the two edges of C_i' which are incident to u_i. Let c

be a proper 3-edge-coloring of G' that colors the edges of F_1' with color
1, and the edges of the circuits of F_2' with colors 2 and 3. Assume that
$f_i' \in c^{-1}(2)$. Let ϕ_1 be a nowhere-zero 2-flow on $G'[c^{-1}(1) \cup c^{-1}(2)]$. Let
ϕ_2 be a nowhere-zero 2-flow on $G[c^{-1}(2) \cup c^{-1}(3)]$, with the additional
property that the orientation of f_i' is different in ϕ_1 and ϕ_2. Flow ϕ_2
exists, since for every $i \in \{1, \ldots, 2n\}$ there is precisely one edge f_i' in C_i'.

Now, $2\phi_1 + \phi_2$ is a nowhere-zero 4-flow ψ on G' with the additional
property that for each $i \in \{1, \ldots, 2n\}$ the vertex u_i is either the terminal
vertex of f_i' and of f_i'' or it is the initial vertex of both of these edges.
Furthermore, $\psi(f_i') = \psi(f_i'') = 1$.

For each $i \in \{1, \ldots, n\}$, remove edge e_i, suppress vertices u_{2i-1}, u_{2i}
and consider f_i' and f_i'' as two half-edges of f_i to construct a signature
σ of G with $N_\sigma = \{f_1, \ldots, f_{2n}\}$. Furthermore, ψ induces an integer
nowhere-zero 4-flow on (G, σ). Hence, $F((G, \sigma)) \leq 4$.

It remains to show that $F_c((G, \sigma)) \geq 4$. Suppose to the contrary that
$F_c((G, \sigma)) < 4$. Then, $F_c((G_{N_\sigma}^2)) < 4$ and hence, $G - N_\sigma$ is bipartite by
Theorem 4.1.1.

(1') If there is $k \in \{1, \ldots, 2n\}$ such that C_k has two vertices which are
connected by two edges in G, then it follows from the construction of ψ,
that $G - N_\sigma$ contains an odd circuit, a contradiction. Hence, $F_c((G, \sigma)) =
F((G, \sigma)) = 4$.

We may assume that all edges of the odd circuits of F_2 are simple in
G.

(2') If G has an end-component, then there is one, say K, which is
respected in the construction of ψ. Let $F_2[E(K)]$ be the 2-factor of K
which is a subgraph of F_2. It follows with Theorem 4.4.3 (suppress the
bivalent vertex), that there is a 2-factor F' of K that does not contain f_k.
Since K has odd order and F contains at least as many odd circuits as
$F_2[E(K)]$ it follows that F' contains an odd circuit that does not contain

any edge of N_σ. Thus, $G - N_\sigma$ contains an odd circuit, a contradiction. Hence, $F_c((G,\sigma)) = F((G,\sigma)) = 4$.

It remains to consider the case when G is bridgeless. Let X_3 be a 3-minimal set. Since X_3 contains an edge of every odd circuit of F_2 it follows that $|X_3| \geq \omega(G)$. For $i \in \{1, \ldots, 2n\}$ let $f_i \in X_3 \cap E(C_i)$. Let σ' be the signature on G with $N_{\sigma'} = \{f_1, \ldots, f_{2n}\}$ and construct an integer nowhere-zero 4-flow on (G, σ') as above. Hence, X_3 contains a 4-minimal set X_4 with $|X_4| \leq \omega(G)$.

We will show that $|X_3| > \omega(G)$. Suppose to the contrary that $|X_3| = \omega(G)$. Then $\{f_i\} = E(C_i) \cap X_3$ for each $i \in \{1, \ldots, 2n\}$. By Theorem 4.4.3 there is a 1-factor of G that contains f_1. The complementary 2-factor F' has at least $\omega(G)$ odd circuits. Thus, there is an odd circuit of F' that does not contain an edge of X_3 which implies that $G - X_3$ is not bipartite, a contradiction. Thus, $|X_3| > \omega(G)$, $X_4 \subset X_3$, and $F_c((G,\sigma')) = F((G,\sigma')) = 4$.

$\square$

Every signature of K_2^3 is either equivalent to a signature with no negative edges or to a signature with precisely one negative edge. Hence, $\mathcal{S}(K_2^3) = \overline{\mathcal{S}}(K_2^3) = \{3\}$. The following statement follows with Theorems 4.2.4, 4.4.1, and 4.4.4.

Theorem 4.4.5. *Let G be a cubic graph which has a 1-factor. If $G \neq K_2^3$, then $\{3, 4\} \subseteq \mathcal{S}(G) \cap \overline{\mathcal{S}}(G)$.*

Theorem 4.4.6. *Let G be a cubic graph. If $G \neq K_2^3$, then the following statements are equivalent.*

1) G has a 1-factor.

2) $3 \in \mathcal{S}(G)$.

3) $3 \in \overline{\mathcal{S}}(G)$.

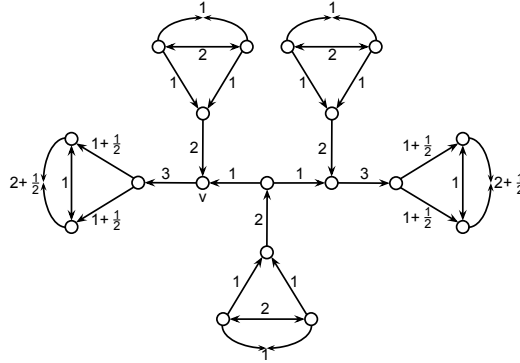*4) $4 \in \overline{\mathcal{S}}(G)$.*

Figure 4.2: A graph H which has no 1-factor and $4 \in \mathcal{S}(H)$

Proof. Statement 1) implies 2) by Theorem 4.4.5. By Lemma 4.2.1 it follows that every circular nowhere-zero 3-flow on G is an integer nowhere-zero 3-flow. Hence, statement 2) implies statement 3), which implies statement 1 by Theorem 4.2.4.

We show the equivalence of statements 1) and 4). If G has a 1-factor, then $4 \in \overline{\mathcal{S}}(G)$ by Theorems 4.4.1 and 4.4.4. If $4 \in \overline{\mathcal{S}}(G)$, then G has an integer nowhere-zero 4-flow ϕ with $\phi(e) > 0$ for each edge e. It is easy to see that $F = \{e : \phi(e) = 2\}$ is a 1-factor of G.

$\square$

Theorem 4.4.5 says that if G has a 1-factor, then $4 \in \mathcal{S}(G)$. However, the other direction is not true.

Proposition 4.4.7. *There is a cubic graph H which has no 1-factor and $4 \in \mathcal{S}(H)$.*

Proof. The graph H in Figure 4.2 has a nowhere-zero 4-flow. It has no 1-factor since $H - v$ has more than one odd component. By Theorem 4.4.6, it has no nowhere-zero 3-flow. Hence, $4 \in \mathcal{S}(G)$ by Theorem 4.2.5.

$\square$

Corollary 4.4.8. *Let G be a cubic graph that does not have a 1-factor. If $k \in \overline{\mathcal{S}}(G)$, then $k \geq 5$.*

Smallest r-minimal sets

Let $r \geq 2$. An r-minimal set X is a *smallest r-minimal set* of G if $|X| \leq |X'|$ for every r-minimal set X' of G.

Proposition 4.4.9. *Let $t \geq 1$ be an integer, G a $(2t+1)$-regular graph and $r \geq 2$. If $X \subseteq E(G)$ is a smallest r-minimal set, then $\Delta(G[X]) \leq t$.*

Proof. Suppose to the contrary that $\Delta(G[X]) > t$. Then there is $v \in V(G)$ such that $d_{G[X]}(v) > t$. If we switch at v, then we obtain an equivalent graph (G, σ) with $|N_\sigma| < |X|$ and $F_c((G, \sigma)) = r$. But N_σ contains an r-minimal set X', contradicting the fact that X is a smallest r-minimal set. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We will prove some bounds for the cardinality of smallest r-minimal sets. The independence number of G is denoted by $\alpha(G)$.

Proposition 4.4.10. *Let $t \geq 1$ be an integer and G be a $(2t+1)$-regular graph. If $X \subseteq E(G)$ is a smallest $(2 + \frac{1}{t})$-minimal set, then $|X| \leq \min\{(\frac{1}{2}|V(G)| - \alpha(G))(2t+1), \frac{t}{2}|V(G)|\}$.*

Proof. Since G has a $(2 + \frac{1}{t})$-minimal set, it follows by Theorem 4.2.4 that G has a t-factor. Let (G, σ) be the graph with $N_\sigma = E(G)$ and $V \subseteq V(G)$ be an independent set with $|V| = \alpha(G)$. The function $\phi : E(G) \to \{1\}$ is a modular nowhere-zero $(2 + \frac{1}{t})$-flow on (G, σ). Switch at every vertex of V to obtain a switching equivalent graph (G, σ') with a modular nowhere-zero $(2 + \frac{1}{t})$-flow ϕ' and $|N_{\sigma'}| \leq |E(G)| - \alpha(G)(2t+1) =$

$(\frac{1}{2}|V(G)| - \alpha(G))(2t+1)$. It follows with Theorem 4.2.2 that (G, σ') has a nowhere-zero $(2 + \frac{1}{t})$-flow. Therefore, $|X| \leq (\frac{1}{2}|V(G)| - \alpha(G))(2t+1)$. Proposition 4.4.9 implies that $|X| \leq \frac{t}{2}|V(G)|$.

$\square$

Let G be a bridgeless cubic graph. The *resistance* $r(G)$ is the cardinality of a minimum color class, where the minimum is taken over all proper 4-edge-colorings of G. It is easy to see that $r(G) \leq \omega(G)$ and if $r(G) \neq 0$, then $r(G) \geq 2$ (see [34]). A bridgeless cubic graph which is not 3-edge-colorable is called a *snark*.

Theorem 4.4.11. *Let G be a cubic graph which has a 1-factor and $G \neq K_2^3$. For each $i \in \{3, 4\}$ there is a smallest i-minimal set X_i in G, and*

1) if G is bipartite, then $|X_3| = 0$ and $|X_4| = 2$.

2) if G is 3-edge-colorable and not bipartite, then $2 \leq |X_3| \leq 3(\frac{1}{2}|V(G)| - \alpha(G))$ and $|X_4| = 0$.

3.1) if G is not 3-edge-colorable, then $r(G) \leq |X_4| \leq \omega(G) \leq |X_3| \leq \min\{3(\frac{1}{2}|V(G)| - \alpha(G)), \frac{1}{2}|V(G)|\}$.

3.2) if G is a snark, then $r(G) \leq |X_4| \leq \omega(G) < |X_3| \leq \min\{3(\frac{1}{2}|V(G)| - \alpha(G)), \frac{1}{2}|V(G)|\}$.

Proof. It follows from Theorem 4.4.5 that there is a smallest i-minimal set X_i in G for each $i \in \{3, 4\}$. By Proposition 4.4.10, $|X_3| \leq 3(\frac{1}{2}|V(G)| - \alpha(G))$.

1) If G is bipartite, then $|X_3| = 0$ and it follows with Theorem 4.4.1 that G has a signature σ with $|N_\sigma| = 2$ and $F((G, \sigma)) = 4$. Since every signature of a flow-admissible graph has at least two edges it follows that $|X_4| = 2$.

2) If G is not bipartite and 3-edge-colorable, then $|X_4| = 0$, and as above we get that $|X_3| \geq 2$.

3) Let G be not 3-edge-colorable. Suppose to the contrary that there is a smallest 4-minimal set X such that $|X| < r(G)$. By Lemma 4.3.2, $(G_X^2, 1)$ has a nowhere-zero 4-flow and hence, it is 3-edge-colorable. By Proposition 4.4.9, X is an independent set. Hence, a 3-edge-coloring of G_X^2 induces a proper 4-edge-coloring of G which has a minimal color class c with $|c| \leq |N_\sigma| < r(G)$, contradiction. Thus, $r(G) \leq |X_4|$. The other inequalities follow by Theorem 4.4.4.

$\square$

The bounds of Theorem 4.4.11 are sharp for the Petersen graph. In [36] it is shown that for every positive integer k there is a cubic graph G such that $\omega(G) - r(G) \geq k$ and that there is cyclically 5-edge-connected cubic graph H and $r(H) \geq k$. If G is not 3-edge-colorable and $r(G) > 2$, then there is a signature σ such that (G, σ) is flow-admissible and $2 \leq |N_\sigma| < r(G)$. Theorem 4.4.11 implies $F((G, \sigma)) > 4$. Hence, we obtain the following corollary which is similar to Corollary 4.4.8.

Corollary 4.4.12. *Let G be a cubic graph. If $\omega(G) > 2$, then for every k with $2 \leq k < r(G)$ there is a signature σ such that $|N_\sigma| = k$ such that $F_c((G, \sigma)) > 4$ and $F((G, \sigma)) \geq 5$.*

The integer flow spectrum of a class of cubic graphs

It is quite difficult to determine the flow spectrum of a graph. Indeed even for the integer flow spectrum it is difficult. So far the integer flow spectrum has been determined only for eulerian graphs in [23], and for complete and complete bipartite graphs in [20]. For instance, it is known that $\overline{\mathcal{S}}(G) = \{3, 4, 5, 6\}$, if G is the Petersen graph.

For $n \geq 1$, let G_n be the cubic graph which is obtained from a circuit of length $2n$, where every second edge is replaced by two parallel edges.

Theorem 4.4.13. *If $n = 1$, then $\overline{\mathcal{S}}(G_n) = \{3\}$. If $n = 2$, then $\overline{\mathcal{S}}(G_n) = \{3, 4\}$, and if $n \geq 3$, then $\overline{\mathcal{S}}(G_n) = \{3, 4, 6\}$.*

Proof. If $n = 1$, then $G_1 = K_2^3$. Hence, $\overline{\mathcal{S}}(G_1) = \{3\}$.

Let $n \geq 2$. Theorem 4.4.5 implies that $\{3, 4\} \subseteq \overline{\mathcal{S}}(G_n)$. Let $V(G_n) = \{v_0, \ldots, v_{2n-1}\}$. For $i \in \{0, \ldots, n-1\}$ the vertices v_{2i}, v_{2i+1} are connected by two parallel edges and the vertices v_{2i+1}, v_{2i+2} are connected by a simple edge (indices are added modulo $2n$). Every signature of G_n is equivalent to a signature σ where for each $i \in \{0, \ldots, n-1\}$ at most one edge between v_{2i}, v_{2i+1} is negative and all other edges are positive. We call σ a normal signature of G_n. We say that σ is odd or even, depending on whether $|N_\sigma|$ is odd or even. Hence, we have only to consider the two cases, whether σ is even or odd.

If σ is even, then G_n is the union of two balanced eulerian graphs and hence, $F((G, \sigma)) \leq 4$ by Lemma 4.4.14. Hence, $\overline{\mathcal{S}}(G_2) = \{3, 4\}$.

It remains to consider the case when $n \geq 3$ and σ is odd. Then $|N_\sigma| \geq 3$. Let $e_1, e_2 \in N_\sigma$. There are hamiltonian circuits C_1, C_2 such that $E(C_i) \cap N_\sigma = N_\sigma - \{e_i\}$. Both circuits are balanced and hence, there are nowhere-zero 2-flows ϕ_i on C_i. Let e_1' be the positive edges which is parallel to e_1. Then $\psi = 2\phi_1 + \phi_2$ is a 4-flow on G_n, and $\psi(e) \neq 0$ if $e \in N_\sigma$, $\psi(e) \in \{0, 1, 3\}$ if $e \in E(G_n) - (N_\sigma \cup \{e_1'\})$, and $\psi(e_1') = 2$. Let τ_ψ be the underlying orientation of $H(G)$ for ψ. There is a hamiltonian circuit C that consists only of positive edges, that contains e_1' and all edges e with $\psi(e) = 0$. Let τ be an orientation of $H(C)$ such that C is a directed circuit and τ and τ_ψ coincide on e_1'. Let ψ' be a nowhere-zero 2-flow on C with orientation τ. Then, $\psi + 2\psi'$ is a nowhere-zero 6-flow on G_n.

Suppose to the contrary that $F((G_n, \sigma)) < 6$. Let $k < 6$ and ψ_n be nowhere-zero k-flow on G_n. Without loss of generality we assume in the following that all flow values are positive.

We first show that if $n > 3$, then $F((G_n, \sigma)) < 6$ implies that there is a $m < n$ such that $F((G_m, \sigma_m)) < 6$, where σ_m is normal and odd.

Let $n > 3$. It is easy to see that if one edge of the two edges between v_{2i} and v_{2i+1} is negative, then $\psi_n(v_{2i-1}v_{2i}) \neq \psi_n(v_{2i+1}v_{2i+2})$. If there are $i, j \in \{0, \ldots, n-1\}$ with $i < j$ and $\psi_n(v_{2i+1}v_{2i+2}) = \psi_n(v_{2j+1}v_{2j+2})$, then remove these two edges and add edges $v_{2i+2}v_{2j+1}$ and $v_{2i+1}v_{2j+2}$ to obtain two graphs G_{n_1} and G_{n_2} with nowhere-zero k-flows ψ_{n_1} and ψ_{n_2}, respectively. Depending on the orientation of the half-edges of $v_{2i+1}v_{2i+2}$ and of $v_{2j+1}v_{2j+2}$ the new edges might be negative. For one of these two graphs, say G_{n_1}, ψ_{n_1} is equivalent to a k-flow ψ'_{n_1} on an odd normal signature σ'_{n_1} of G_{n_1}, since for otherwise G_n and an even normal signature which is equivalent to σ_n could be reconstructed from (G_{n_1}, σ_{n_1}) and (G_{n_2}, σ_{n_2}). Hence, $|N_{\sigma_{n_1}}| \geq 3$. Thus, we can assume that G_n has an odd normal signature σ_n with $|N_\sigma| = n$. In particular, n is odd.

Since $k < 6$, it follows that if G_n is not reducible to a smaller graph G_m, then $n = 3$. Consider (G_3, σ_3). Since σ_3 is normal it follows that the difference between the flow values of any two simple edges is at least 2. Since $\psi_3(e) \geq 1$ for every edge $e \in G_3$, it follows that there is a simple edge with $\psi_3(e) \geq 5$, contradicting our assumption that $k < 6$.

Therefore, $F((G_n, \sigma)) = 6$ and $\overline{\mathcal{S}}(G_n) = \{3, 4, 6\}$ if $n > 2$.

$\square$

Every graph G_n is bipartite. Hence, the empty set is a smallest 3-minimal set for all $n \geq 1$. If $n \geq 2$, then a smallest 4-minimal set contains precisely two edges, and if $n \geq 3$, then a smallest 6-minimal set consists of three edges.

Bouchet's conjecture

To prove Bouchet's conjecture it is sufficient to prove it for cubic graphs (see Theorem 1.4.5). Hence, we are in a similar situation as for Tutte's 5-flow conjecture. However, as Section 3.1 shows Bouchet's conjecture has to be proven for integer flows explicitly.

Lemma 4.4.14. *Let (G, σ) be a signed graph. If (G, σ) is the union of two balanced eulerian graphs, then $F((G, \sigma)) \leq 4$.*

    Proof. Let H_1 and H_2 be eulerian graphs such that $E(H_1) \cup E(H_2) = E(G)$. For $i \in \{1, 2\}$, let σ_i be the restriction of σ to H_i. Since H_i is balanced it follows that there is a nowhere-zero 2-flow ϕ_i on (H_i, σ_i). Hence, $\phi_1 + 2\phi_2$ is a nowhere-zero 4-flow on (G, σ).

                                                                         □

Theorem 4.4.15. *Let (G, σ) be a flow-admissible signed cubic graph. If G is a Kotzig-graph, then $F((G, \sigma)) \leq 6$.*

    Proof. Since G is a Kotzig-graph, G has three 1-factors M_1, M_2, M_3 such that the union of any two of them induces a hamiltonian circuit of G. It follows that there are two, say M_1 and M_2, such that $|N_\sigma \cap M_1|$ and $|N_\sigma \cap M_2|$ have the same parity. Hence, $|N_\sigma \cap M_1| + |N_\sigma \cap M_2|$ is even, and $(G[M_1 \cup M_2], \sigma_{1,2})$ with $\sigma_{1,2} = \sigma|_{M_1 \cup M_2}$ is balanced. Clearly, (G, σ) is equivalent to (G, σ') with $N_{\sigma'} \cap (M_1 \cup M_2) = \emptyset$.

    If $|M_3 \cap N_{\sigma'}|$ is even, then $(G[M_1 \cup M_3], \sigma_{1,3})$ with signature $\sigma_{1,3} = \sigma'|_{M_1 \cup M_3}$ is balanced. Hence, $F(G, \sigma) \leq 4$ by Lemma 4.4.14.

    If $|M_3 \cap N_{\sigma'}|$ is odd, then $|N_{\sigma'}| \geq 3$. Let $e = xy$ be an extroverted edge of (G, σ'), and let (G, σ^*) be the graph which is obtained from (G, σ') by changing the direction of h_e^x. Then e is a positive edge which is directed from x to y in (G, σ^*). It follows as above, that (G, σ^*) has a nowhere-zero 4-flow ϕ. Without loss of generality we can assume that all flow

values are positive, all edges of M_3 have flow value 1, and $M_1 \cup M_2$ is a directed circuit C. For two vertices a, b, let $P(a, b)$ denote the directed path from a to b in C.

If we consider ϕ on (G, σ'), then $\delta\phi(x) = 2$, and $\delta\phi(v) = 0$ for all $v \in V(G) \setminus \{x\}$. Since all flow values are positive, it follows that there is an introverted edge $f = uw$. Let x, u, w be the sequent order of these three vertices in C. We define a nowhere-zero 6-flow ϕ^* on (G, σ') as follows: $\phi^*(e) = \phi(e)$, if $e \in E(G) - (E(P(x, w)) \cup \{f\})$, $\phi^*(e) = \phi(e) + 2$, if $e \in E(P(x, u))$, $\phi^*(e) = \phi(e) + 1$, if $e \in E(P(u, w))$, and $\phi^*(f) = 2$.

$\square$

Greenwell and Kronk [7] proved that every uniquely 3-edge-colorable cubic graph has precisely three hamiltonian circuits, which are induced by the three color classes. Hence, we obtain the following corollary.

Corollary 4.4.16. *Let (G, σ) be a flow-admissible signed cubic graph. If G is uniquely 3-edge-colorable, then $F((G, \sigma)) \leq 6$.*

Further results and references on Kotzig-graphs and uniquely 3-edge-colorable graphs can be found in [51].

Remark: For $t \geq 2$ it can be proven by a slight modification of the proof of Theorem 4.4.15 that $F((G, \sigma)) \leq 4$ for every flow-admissible $(2t + 1)$-regular Kotzig-graph (G, σ). However, this is also a simple consequence of a result of Raspaud and Zhu [30] that every flow-admissible 4-edge-connected graph has a nowhere-zero 4-flow.

Chapter 5

The set of circular flow numbers of regular graphs

5.1 Circular flow numbers of d-regular graphs

The parts 5.1.1 - 5.1.3 and section 5.2 have been published for unsigned graphs in [31]. In [31], the graphs are considered to be unsigned; this section is the generalization of that study for signed graphs.

In this chapter we consider graphs to be loopless. The constructions that we will use require multiedges, so-called t-edges. For $t \geq 1$, a t-edge between two vertices u and v is a set of t (parallel) edges between u and v; that is, an edge is a 1-edge.

If we assign a direction to each edge of (G, σ), then we obtain a directed graph $D((G, \sigma))$. For $k \geq 1$, a k-edge e in a directed graph is denoted by $(u, v)_k$, if all k edges of e are directed from u to v. The shorthand (u, v) is used for a 1-edge $(u, v)_1$. We say that u is the *initial vertex* and v is the *end vertex* of e.

A *network* is an ordered pair $((G, \sigma), U)$ consisting of a signed graph (G, σ) and a subset $U \subseteq V(G)$ whose elements are called *terminals*. A *nowhere-zero r-flow* ϕ on a network $((G, \sigma), U)$ is an assignment of a direction and a real flow value $\phi(e)$ to each edge $e \in E(G)$ such that $1 \leq \phi(e) \leq r - 1$ and the *excess flow* $\delta\phi(v)$ is equal to 0 for every $v \in V(G) - U$, where $\delta\phi(v) = \sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h)$. Considering flows on terminals we will use the more intuitive term excessflow instead of boundary. Clearly, if ϕ is a nowhere-zero r-flow on a network $((G, \sigma), U)$, then $\sum_{v \in U} \delta\phi(v) = 0$.

We say that an edge e has flow value x in $((G, \sigma), U)$ if $\phi(e) = x$. Moreover, we say that a k-edge $(u, v)_k$ has flow value x if each of its k edges has flow value x. In this chapter terminals will always have degree 1. The set of terminals U is partitioned into two sets U^+ and U^-, where U^+ denotes the set of terminals which are end vertices and U^- denotes the set of terminals which are initial vertices of an edge. Note that a signed graph (G, σ) is a network $((G, \sigma), \emptyset)$.

For a set A, let $A_{\mathbb{Q}} = A \cap \mathbb{Q}$ and

$$\mathcal{F}^c = \{r : \text{there is a signed graph } (G, \sigma) \text{ with } F_c(G, \sigma) = r\}.$$

If we restrict our studies on all-positive graphs and include Seymour's 6-flow Theorem 1.4.2, then $\mathcal{F}^c \subseteq [2; 6]$. In general, using the 12-flow theorem of DeVos [5] we get for signed graphs $\mathcal{F}^c \subseteq [2; 12]$.

Theorem 5.1.1. *[28] For every $r \in [2; 5]_{\mathbb{Q}}$, there is an all-positive graph $(G, 1)$ with $F_c(G, 1) = r$.*

If Bouchet's 6-flow conjecture is true, then $\mathcal{F}^c \subseteq [2; 6]_{\mathbb{Q}}$.

Many flow conjectures are equivalent to their restrictions to regular graphs. For instance, Tutte's 5-flow conjecture and Bouchet's 6-flow conjecture are equivalent to its restrictions to cubic graphs (see Theorem

1.4.5), and the 3-flow conjecture [43] is equivalent to its restriction to 5-regular graphs. However, the set of flow values of graphs with a specific odd regularity is not a closed interval, as Theorem 4.2.5 shows.

In this chapter we determine the set of flow numbers $\mathcal{F}_d^c$ of d-regular graphs for odd d. Clearly, 1-regular graphs do not have any nowhere-zero flows.

The initial motivation of the study in this chapter, which has its origin in [31], is to estimate the set of flow numbers for regular all-positive graphs. Nonetheless, in case for graphs of odd regularities these results can be obtained for signed graphs as well.

If d is even, then $F_c(G, 1) = 2$ for every d-regular graph G.

Therefore, we focus on the case when d is odd and $d \geq 3$. Let $k \geq 1$ and $\mathcal{F}_{2k+1}^c = \{r : \text{there is a signed } (2k + 1)\text{-regular graph } (G, \sigma) \text{ with } F_c(G, \sigma) = r\}$. In [17] it is proven that for every $r \in [4; 5]_{\mathbb{Q}}$ there is a cubic graph $(G, 1)$ with $F_c(G, 1) = r$. Indeed there is the additional constraint, that these graphs are cyclically 4-edge connected and that they have girth at least 5. We prove a similar result for $(2k + 1)$-regular signed graphs, and we show that in case that Tutte's 5-flow conjecture is false, then gaps for circular flow numbers in the interval $[5; 6]$ are due for all graphs and not just for regular graphs.

Theorem 5.1.2. *For all* $k \geq 1$ *:* $\mathcal{F}_{2k+1}^c = (\mathcal{F}^c - [2; 2 + \frac{2}{2k-1})) \cup \{2 + \frac{1}{k}\}$.

Theorems 5.1.1 and 5.1.2 imply the following result.

Corollary 5.1.3. *For every integer* $k \geq 1$ *and every rational number* $r \in \{2 + \frac{1}{k}\} \cup [2 + \frac{2}{2k-1}; 5]$, *there exists a* $(2k + 1)$-*regular graph* (G, σ) *with* $F_c(G, \sigma) = r$.

In [35], Theorem 2.6, it is proven that if there is a graph $(G, 1)$ with $F_c(G, 1) = r$, then there is a simple graph G' with $F_c(G', 1) = r$. The

construction of [35] that originates from [27] preserves d-regularity. However, that proof uses a complicated method that is just applicable for all-positive graphs. Here, we will give an easier proof for the general case.

Let (G, σ) be a signed graph and v be a vertex with $\delta(v) = n$. Let $v_1, \ldots, v_n$ be the neighbors of v that are not necessarily distinct. Further, let $(H, 1)$ be the complete all-positive bipartite graph $K_{n,n-1}$ and $u_1, \ldots, u_n$ be the vertices of degree $n - 1$. The graph (G^M, σ^M) obtained from $(G - v, \sigma)$ and $(H, 1)$ by adding the edges $v_i u_i$ with signature $\sigma(v_i u_i) = \sigma(v_i v)$ for $i \in \{1, \ldots, n\}$ is called a Meredith-extension of G.

Theorem 5.1.4. *If* (G^M, σ^M) *is a Meredith-extension of a signed graph* (G, σ) *then* $F_c((G, \sigma)) = F_c((G^M, \sigma^M))$.

Proof. Let v be a vertex with $\delta(v) = n$, $v_1, \ldots, v_n$ be the neighbors of v, $(H, 1)$ be the complete all-positive bipartite graph $K_{n,n-1}$, $u_1, \ldots, u_n$ be the vertices of degree $n - 1$ and $x_1, \ldots, x_{n-1}$ be the vertices of degree n. (G^M, σ^M) is obtained from $(G - v, \sigma)$ and $(H, 1)$ by adding the edges $v_i u_i$ with signature $\sigma(v_i u_i) = \sigma(v_i v)$ for $i \in \{1, \ldots, n\}$.

Suppose ϕ is a nowhere-zero r-flow on (G^M, σ^M). Since $(H, 1)$ is an all-positive graph, we get a nowhere-zero r-flow ϕ' on (G, σ) by defining $\phi'(v_i v) = \phi(v_i u_i)$ and $\phi'(e) = \phi(e)$ if $e \neq (v_i u_i)$ and corresponding orientation $\tau_{\phi'}(v_i v) = \tau_\phi(v_i u_i)$ and $\tau_\phi(e) = \tau_{\phi'}(e)$ if $e \neq (v_i u_i)$ for $i \in \{1, \ldots, n\}$. Hence, $F_c((G, \sigma)) \leq F_c((G^M, \sigma^M))$.

To show the other direction we will construct a nowhere-zero r-flow on $F_c((G^M, \sigma^M))$ from a given nowhere-zero r-flow ϕ on (G, σ). Define a nowhere-zero r-flow on $F_c((G^M, \sigma^M))$ as follows: For $i \in \{1, \ldots, n\}$ we define the corresponding orientation $\tau_\phi(v_i u_i) = \tau_{\phi'}(v_i v)$, $\tau_{\phi'}(u_i, x_l) = \tau_\phi(v_{i+l}, v)$ (the summation of the indices is taken modulo n), and $\tau_{\phi'}(e) = \tau_\phi(e)$ for $e \notin \{v_i u_i : i \in \{1, \ldots, n\}\} \cup E(H)$. For $i \in \{1, \ldots, n\}$ let $\phi'(v_i u_i) = \phi(v_i v)$, $\phi'(u_i, x_l) = \phi(v_{i+l}, v)$ (the summation of the indices is

taken modulo n), and $\phi'(e) = \phi(e)$ for $e \notin \{v_i u_i : i \in \{1, \ldots, n\}\} \cup E(H)$. We show that ϕ' is a proper flow. Kirchoff's law is obviously fulfilled at all vertices $v \in V(G^M) - V(H)$. For every u_i we get

$$
\sum_{h \in H(u_i)} \tau'(e_h)\phi'(e_h) = \tau(u_i x_i)\phi(u_i x_i) + \sum_{k \in \{1, \ldots, n\} - i} \tau(u_i x_k)\phi(u_i x_k)
$$

$$
= \sum_{h \in H(v)} \tau(e_h)\phi(e_h) = 0
$$

(5.1)

Similarly we get $\sum_{h \in H(x_i)} \tau'(e_h)\phi'(e_h) = \sum_{h \in H(v)} \tau(e_h)\phi(e_h) = 0$. Hence, ϕ' is a proper nowhere-zero r-flow on (G^M, σ^M) and $F_c((G, \sigma)) \geq F_c((G^M, \sigma^M))$.

$\square$

In this regard it is interesting to note that a more general concept of the Meredith-extension as given in [38] and [2] will not keep the flow number in general.

Let (G, σ) be a signed graph, v be a vertex with $\delta(v) = n$ and $\Delta = \Delta(G)$ be the maximum degree of G. Let $v_1, \ldots, v_n$ be the neighbors of v that are not necessarily distinct. Further, let $(H, 1)$ be the complete all-positive bipartite graph $K_{\Delta, \Delta-1}$ and $u_1, \ldots, u_n$ be the vertices of degree $n - 1$. The graph $(G^{M'}, \sigma^{M'})$ optained from $(G - v, \sigma)$ and $(H, 1)$ by adding the edges $v_i u_i$ with signature $\sigma(v_i u_i) = \sigma(v_i v)$ for $i \in \{1, \ldots, n\}$ is called a general Δ-Meredith-extension of G.

Let $(J, 1)$ be the all-positive graph given by $K_{3,3}$ with one edge subdivided by a vertex v. Since $F_c(K_{3,3}, 1) = 3$ [35], we get $F_c(J, 1) = 3$. Now, let $(H, 1)$ be the graph obtained by applying a Δ-Meredith-extension on v in $(J, 1)$. We will show that $F_c(H, 1) \geq 4$. After contraction of $(K_{3,3}, 1)$ in $(J, 1)$ and suppression of two vertices of degree two we get the complete

graph on four vertices $(K_4, 1)$. Since we consider all-positive graphs, it follows $F_c(H, 1) \geq F_c(K_4, 1) = 4$.

Hence, the Meredith-extension keeps the flow-value for every graph, whereas there exist graphs where a Δ-Meredith-extension does not keep the flow value.

Thus, Theorem 5.1.1, Theorem 5.1.2 and Corollary 5.1.3 are true for the restriction on simple graphs as well.

5.2   Proof of Theorem 5.1.2

Let $k \geq 1$ be an integer and (G, σ) be a signed $(2k + 1)$-regular graph. If $F_c(G, \sigma) = r$, then $r = 2 + \frac{1}{t}$ or $r \geq 2 + \frac{2}{2t-1}$. Hence, we need to consider the case that $r \geq 2 + \frac{2}{2k-1}$. Let (G, σ) be a graph with $F_c(G, \sigma) = r$. We have to show that there is a $(2k + 1)$-regular graph (H, τ) with $F_c(H, \tau) = r$.

Lemma 5.2.1. *For every $r \in \mathcal{F}^c$, there is a graph (H, τ) with $F_c(H, \tau) = r$ and a nowhere-zero r-flow ϕ such that the following holds: If $v \in V(H)$ and $d_H(v) \notin \{2, 3\}$, then $\phi(e) < 2$ for every edge e which is incident to v.*

Proof. If $r < 3$, then there is nothing to prove. Let $r \in \mathcal{F}^c \cap [3; 6]$, (G, σ) be a graph with $F_c(G, \sigma) = r$, and ϕ be a nowhere-zero r-flow on (G, σ). There is an edge e with $\phi(e) \geq 2$. Let $t_{(G,\sigma)}(v) = \sum_{e \in E(v)} (\lfloor \phi(e) \rfloor - 1)$, $V^-(G) = \{v : v \in V(G) \text{ and } d_G(v) \notin \{2, 3\}\}$, and $t((G, \sigma)) = \sum_{v \in V^-(G)} t_{(G,\sigma)}(v)$.

Among all graphs with circular flow number r let H be a graph with $t((H, \tau))$ minimum. We claim that $t((H, \tau)) = 0$ and hence (H, τ) has the desired property.

Suppose to the contrary that $t((H, \tau)) > 0$. Then there is $v \in V^-(H)$ with $t_{(H,\tau)}(v) > 0$. Let $e \in E(v)$ and $\phi(e) \geq 2$.

We assume that v is the end vertex of e. Subdivide e by a vertex w and add an positive edge wv to obtain a new graph (H', τ'). Then wv is a 2-edge, say $(w, v)_2 = \{e_1, e_2\}$ and both edges, e_1 and e_2, are directed from w to v. Extend ϕ to a nowhere-zero r-flow ϕ' on (H', τ') with $\phi'(e_1) = 1$ and $\phi'(e_2) = \phi(e) - 1$. Since (H, τ) can be obtained from (H', τ') by contraction of an all-positive subgraph, it follows that $F_c(H', \tau') = r$. Furthermore, $t_{(H', \tau')}(v) = t_{(H, \tau)}(v) - 1$ and hence, $t((H', \tau')) < t((H, \tau))$, which contradicts our choice of (H, τ).

$\square$

Let $((G, \sigma), U)$ be a network with a nowhere-zero r-flow ϕ. Let $v \in V(G) - U$, and $E(v) = \{e_1, \ldots, e_n\}$. For each $i \in \{1, \ldots, n\}$, subdivide e_i by a new vertex v_i and delete v, to obtain a network $((G', \sigma'), U')$. Let e'_i be the edge incident to v_i in $((G', \sigma'), U')$. We extend the direction of $((G, \sigma), U)$ to $((G', \sigma'), U')$ by letting the direction of all edges which are not in $\{e'_1, \ldots, e'_n\}$ unchanged and giving e'_i the same direction as e_i, that is, if v is the initial (end) vertex of e_i, then v_i is the initial (end) vertex of e'_i. Furthermore, ϕ is extended to a nowhere-zero r-flow ϕ' on $((G', \sigma'), U')$ as follows: Let $\phi'(e) = \phi(e)$ if $e \notin \{e'_1, \ldots, e'_n\}$ and $\phi'(e'_i) = \phi(e_i)$, for $i \in \{1, \ldots, n\}$. If v_i is an end vertex, then $\delta\phi'(v_i) = \phi'(e'_i)$ and otherwise $\delta\phi'(v_i) = -\phi'(e'_i)$. Therefore $\sum_{i=1}^{n} \delta\phi'(v_i) = \sum_{e \in E^+(v)} \phi(e) - \sum_{e \in E^-(v)} \phi(e) = 0$. We say that the network $((G', \sigma'), U')$ is obtained from $((G, \sigma), U)$ by *splitting up v into terminals*.

Construction 5.2.2. *For $k \geq 1$, let $(T_k, 1)$ be the triangle where any two vertices are connected by a k-edge, let (G, τ) be a signed graph, and let v be a trivalent vertex of G. Let $V(T_k) = \{w_1, w_2, w_3\}$. Split up v into terminals v_1, v_2, v_3 and identify v_i and w_i for $i \in \{1, 2, 3\}$ to obtain the graph (G_v^k, σ).*

Note that if $k \geq 2$, then (G_v^k, σ) has one trivalent vertex less than (G, τ). Furthermore, all vertices of T_k have degree $2k + 1$ in G_v^k.

Lemma 5.2.3. *Let $k \geq 2$, (G, τ) be a graph with $F_c(G, \tau) = r$, and let $v \in V(G)$. If $d_G(v) = 3$, then $F_c(G, \tau) = F_c(G_v^k, \sigma)$.*

Proof. Since G has a vertex v of degree 3, it follows that $r \geq 3$. We first consider the case when $k = 2$. Let $E(v) = \{e_1, e_2, e_3\}$, without loss of generality we assume that v is the end vertex of e_1 and e_2 and the initial vertex of e_3.

Consider (G_v^2, σ), and let v_1, v_2, v_3 be the vertices of T_2. We assume that v_1 and v_2 are end vertices of e_1' and e_2', respectively, and that v_3 is the initial vertex of e_3'. Furthermore, denote the two edges between v_i and v_j $(i \neq j)$ by $e_{i,j}$ and $f_{i,j}$. Let us orient the new edges as follows (note that all of those are positive): Let $e_{1,3} = (v_1, v_3)$, $f_{1,3} = (v_1, v_3)$, $e_{1,2} = (v_2, v_1)$, $f_{1,2} = (v_2, v_1)$, $e_{2,3} = (v_2, v_3)$ and $f_{2,3} = (v_3, v_2)$. Extend ϕ to a nowhere-zero r-flow ϕ_v on (G_v^2, σ) with $\phi_v(e_{1,3}) = \phi(e_1)$, $\phi_v(f_{1,3}) = \phi_v(f_{2,3}) = 2$, $\phi_v(e_{2,3}) = \phi(e_2)$ and $\phi_v(e_{1,2}) = \phi_v(f_{1,2}) = 1$. Clearly, ϕ_v is a nowhere-zero r-flow on (G_v^2, σ).

If $k > 2$, then first construct (G_v^2, σ) as above and then add $k - 2$ directed (all-positive) triangles with a nowhere-zero 2-flow to the triangle T_2 of (G_v^2, σ) to obtain a graph (G_v^k, σ'). Clearly, $F_c(G_v^k, \sigma') = r$, for all $k \geq 2$. The replacement of v by T_k is shown in Figure 5.1.

$\square$

We first consider the interval $[4; 6]_{\mathbb{Q}}$. Let $((G, \sigma), U)$ be a network, $k \geq 1$ be an integer, and $V'(G) = \{v : v \in V(G) - U \text{ and } d_G(v) \notin \{2, 2k + 1\}\}$.

Lemma 5.2.4. *For all $k \geq 1$: $\mathcal{F}^c \cap [4; 6] \subseteq \mathcal{F}_{2k+1}^c$*
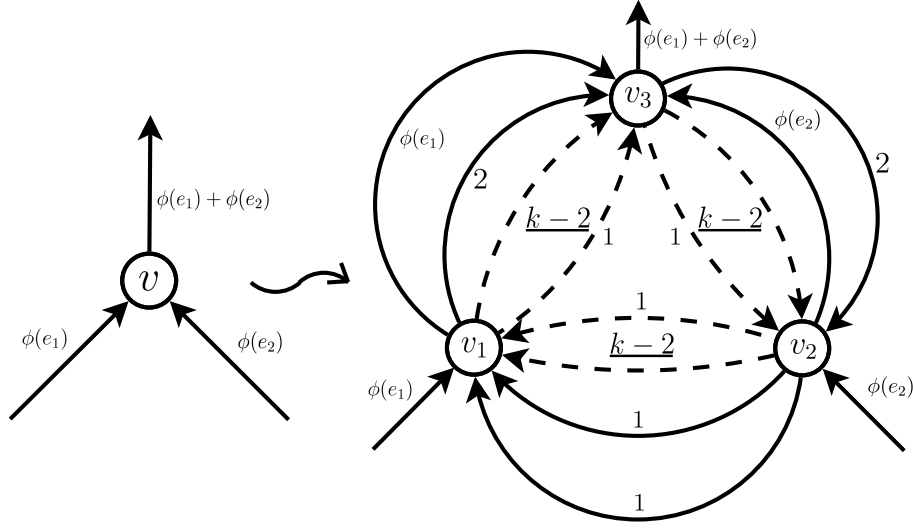
Figure 5.1: Replacement of v by T_k; the underlined term between two dashed lines shows the multiplicity of edges, non-underlined terms represent the flow values on the edges

Proof. We first prove the statement for $k = 1$. Let $r \in \mathcal{F}^c \cap [4; 12]$, (G, σ) be a signed graph with $F_c(G, \sigma) = r$, and let ϕ be a nowhere-zero r-flow on (G, σ). We assume that (G, σ) has no bivalent vertices and that it satisfies the conditions of Lemma 5.2.1. If $V'(G) = \emptyset$, then we are done.

If there is a vertex $v \in V'(G)$, then let $((G_1, \sigma_1), U_1)$ be the network which is obtained from (G, σ) by splitting up v into terminals $v_1, \ldots, v_l$. For $i \in \{1, \ldots l\}$ let $E(v_i) = \{e'_i\}$, and let ϕ_1 be the corresponding nowhere-zero r-flow on (G_1, U_1). Note, that $|V'(G_1)| = |V'(G)| - 1$.

Step 1: Consider terminal v_i. We assume that v_i is the end vertex of e'_i. The argumentation in the other case is similar. If $\delta\phi_1(v_i) > \frac{3}{2}$, then add six vertices $x_1, x_2, u_1, u_2, u_3, u_4$ and the positive edges $(v_i, x_1), (x_1, x_2), (u_1, v_i), (x_1, u_2), (x_2, u_3), (x_2, u_4)$. Extend ϕ_1 to a nowhere-zero r-flow ϕ_2 on the new network $((G_2, \sigma_2), U_2)$, where $\phi_2((x_1, u_2)) = \phi_2((x_2, u_3)) = \phi_2((x_2, u_4)) = 1$, $\phi_2((v_i, x_1)) = 3$, $\phi_2((x_1, x_2)) = 2$ and $\phi_2((u_1, v_i)) = 3 - \delta\phi_1(v_i)$. Then $U_2 = (U_1 - \{v_i\}) \cup$
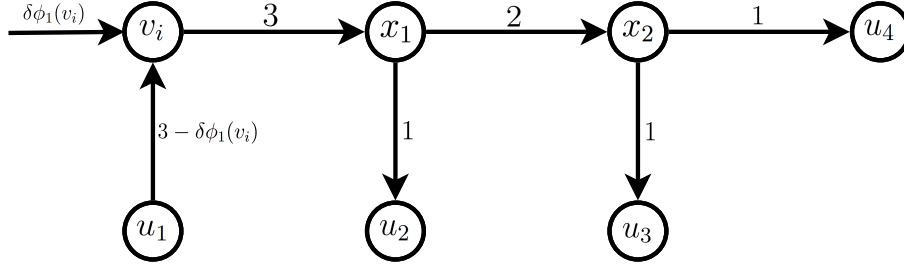
Figure 5.2: Construction if $\delta\phi_1(v_i) > \frac{3}{2}$

$\{u_1, \ldots, u_4\}$, $\delta\phi_2(u_1) \leq \frac{3}{2}$ and $\delta\phi_2(u_j) = 1$ for $j \in \{2, 3, 4\}$. By Lemma 5.2.1 we have $1 < \phi_2((u_1, v_i)) < \frac{3}{2}$. The construction is depicted in Figure 5.2. Repeat Step 1 until we obtain a network $((G_3, \sigma_3), U_3)$ with a nowhere-zero flow ϕ_3 where the absolute value of the excess flow of all terminals is at most $\frac{3}{2}$. Note that all vertices of $((G_3, \sigma_3), U_3)$ have degree 1 or 3, and that the number of terminals with excess flow different from 1 is unchanged.

Step 2: If there are two terminals w_1, w_2 of (G_3, U_3) with an excess flow greater than 1, then identify w_1 and w_2 to a single vertex y_1 and add vertices y_2, y_3, y_4 and positive edges (y_1, y_2), (y_2, y_3), (y_2, y_4). Extend the nowhere-zero r-flow on (G_3, U_3) to a nowhere-zero r-flow ϕ_4 on the new network $((G_4, \sigma_4), U_4)$, where $\phi_4((y_1, y_2)) = \delta\phi_3(w_1) + \delta\phi_3(w_2)$, $\phi_4((y_2, y_3)) = 1$ and $\phi_4((y_2, y_4)) = \delta\phi_3(w_1) + \delta\phi_3(w_2) - 1$. The construction is depicted in Figure 5.3.

If $\delta\phi_4(y_4) = 2$, then add two further vertices y_4', y_4'' and positive edges (y_4, y_4'), (y_4, y_4'') and define the flow value on these two edges to be 1. Then we have $U_4 = (U_3 - \{w_1, w_2\}) \cup \{y_3, y_4', y_4''\}$ and all terminals have excess flow 1. Hence, the number of terminals with an excess flow different from 1 is reduced by 2.

If $\frac{3}{2} < \delta\phi_4(y_4) < 2$, then we have $U_4 = (U_3 - \{w_1, w_2\}) \cup \{y_3, y_4\}$, and the number of terminals with excess flow different from 1 is reduced by 1. Apply Step 1 again to obtain a new network with the same number of
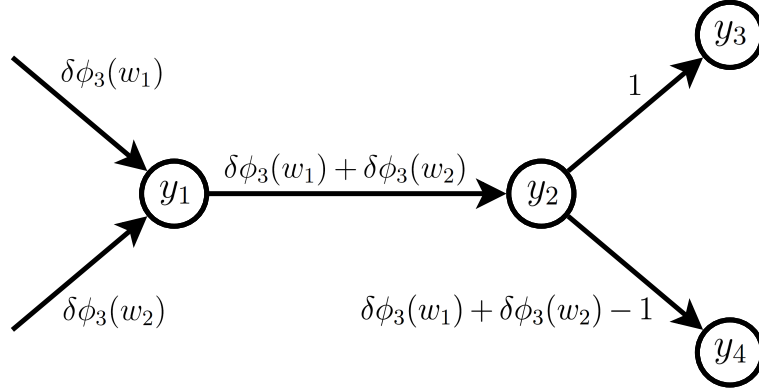
Figure 5.3: Construction for two terminals with an excess flow greater than 1

terminals with excess flow different from 1 and where all terminals have an excess flow of at most $\frac{3}{2}$.

Since the number of terminals with excess flow different from 1 is unchanged in Step 1 and reduced in Step 2 we eventually obtain (after a finite number of applications of these two steps) a network $((G_5, \sigma_5), U_5)$ which has a nowhere-zero r-flow ϕ_5 and the following property: Let $U_5^+ = \{z_1, \ldots, z_n\}$, and $U_5^- = \{z_1', \ldots, z_m'\}$. At most one terminal of U_5^+, say z_1, has an excess flow greater than 1 and all the other terminals have an excess flow of 1. By our construction we have that $\delta\phi_5(z_1) < 2$. Since ϕ_5 is a nowhere-zero r-flow on $((G_5, \sigma_5), U_5)$ which is successively constructed from the nowhere-zero r-flow ϕ on (G, σ), the same holds true for U_5^-. Hence, $n = m$ and $\delta\phi_5(z_1) = \delta\phi_5(z_1')$. Identify terminals z_i and z_i', to obtain a cubic graph H and a nowhere-zero r-flow on H. Since (G, σ) can be obtained from (H, τ) by contracting an all-positive subgraph of H it follows that $F_c(H, \tau) = r$.

For $k \geq 2$ apply Construction 5.2.2 to every vertex of (H, τ) to obtain a $(2k + 1)$-regular graph, which has circular flow number r by Lemma 5.2.3.

$\square$

Since an all-positive bipartite cubic graph has circular flow number 3 and by Lemma 5.2.4 there is a cubic graph with circular flow number r for every $r \in \mathcal{F}^c \cap [4; 6]$, it follows that Theorem 5.1.2 is proven for $k = 1$.

We now consider the interval $[2 + \frac{2}{2k-1}; 4)_\mathbb{Q}$ for $k \geq 2$. Clearly, Construction 5.2.2 and Lemma 5.2.3 allow us to eliminate trivalent vertices. The following lemma describes a "target substructure" in a graph (G, σ), which allows to replace a specific vertex v with $d_G(v) \neq 2k+1$ by a graph (H_v, τ_v), such that the new graph (H, τ) has the same circular flow number as (G, σ) and $|V'(H)| = |V'(G)| - 1$.

Lemma 5.2.5. *Let $n \geq 1$ and $k \geq 2$ be integers, $r \in [2 + \frac{2}{2k-1}; 4)$ and (G, σ) be a graph with $F_c((G, \sigma)) = r$. If there is a nowhere-zero r-flow ϕ of (G, σ) and a vertex $v \in V'(G)$ with $|E^+(v)| = n(2k-1)$, $|E^-(v)| = 2kn$ and $\phi(e) = 1 + \frac{1}{2k-1}$ if $e \in E^+(v)$ and $\phi(e) = 1$ if $e \in E^-(v)$, then there is a graph (H, ρ) with $F_c((H, \rho)) = F_c((G, \sigma))$ and $|V'(H)| = |V'(G)| - 1$.*

Proof. Let v be a vertex of a graph G with $d_G(v) = n(4k - 1)$, and $E(v) = \{e_1, \ldots, e_{n(4k-1)}\}$. Assume that the directions and the flow values on the edges of $E(v)$ are as stated above.

Let $((G_1, \sigma_1), U_1)$ be the network which is obtained from (G, σ) by splitting up v into terminals $v_1, \ldots, v_{n(4k-1)}$. Let $E(v_i) = \{e'_i\}$ and ϕ_1 be the corresponding nowhere-zero r-flow on $((G_1, \sigma_1), U_1)$. Note that $|U_1^+| = n(2k - 1)$ and $|U_1^-| = 2kn$.

We transform $((G_1, \sigma_1), U_1)$ into a network $((G_2, \sigma_2), U_2)$ with $|U_2^+| = (n-1)(2k-1)$ and $|U_2^-| = 2k(n-1)$. Let X^+ be a set of $2k - 1$ elements of U_1^+ and X^- be a set of $2k$ elements of U_1^-.

Step 1: For $u_1, u_2 \in X^+$, we add six vertices $v_1, v_2, v_3, v_4, u_3, u_4$. We add the positive edges $(v_2, v_1), (v_4, v_3), (v_1, v_4)_{k+1}, (v_4, v_1)_{k-2}, (v_2, v_3)_{k-1},$
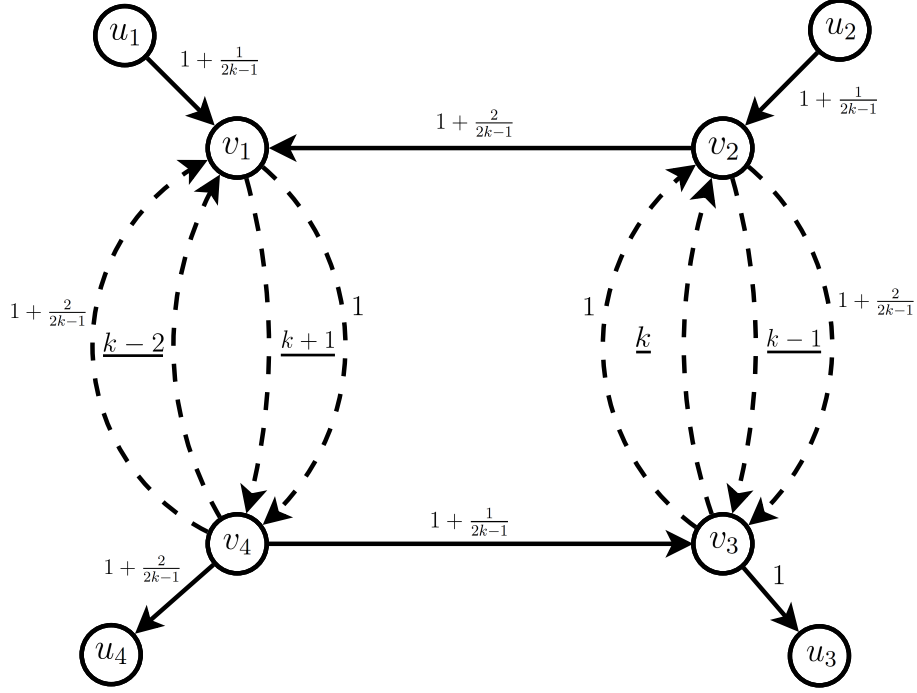
Figure 5.4: Step 1 of the construction for the target substructure

$(v_3, v_2)_k$, (u_1, v_1), (u_2, v_2), (v_3, u_3) and (v_4, u_4). The construction is depicted in Figure 5.4. We extend ϕ_1 to a nowhere-zero r-flow ϕ_2 on $((G_2, \sigma_2), U_2)$ as follows:

Let $\phi_2((v_3, u_3)) = \phi_2((v_3, v_2)_k) = \phi_2((v_1, v_4)_{k+1}) = 1$, $\phi_2((u_1, v_1)) = \phi_2((u_2, v_2)) = \phi_2((v_4, v_3)) = 1 + \frac{1}{2k-1}$, and $\phi_2((v_2, v_1)) = \phi_2((v_4, v_1)_{k-2}) = \phi_2((v_2, v_3)_{k-1}) = \phi_2((v_4, u_4)) = 1 + \frac{2}{2k-1}$. Note that we have changed two terminals of X^+ with excess flow $1 + \frac{1}{2k-1}$ into two terminals, where one has excess flow 1 and the other has excess flow $1 + \frac{2}{2k-1}$. The successive application of Step 1 on $k - 1$ pairs of terminals of X^+ transfer $2k - 2$ terminals with excess flow $1 + \frac{1}{2k-1}$ into $k - 1$ terminals with excess flow 1 and $k - 1$ terminals with excess flow $1 + \frac{2}{2k-1}$. Let X_1^+ be the set of terminals with excess flow 1 and X_2^+ be the set of terminals with excess flow $1 + \frac{2}{2k-1}$. We continue with Step 2.
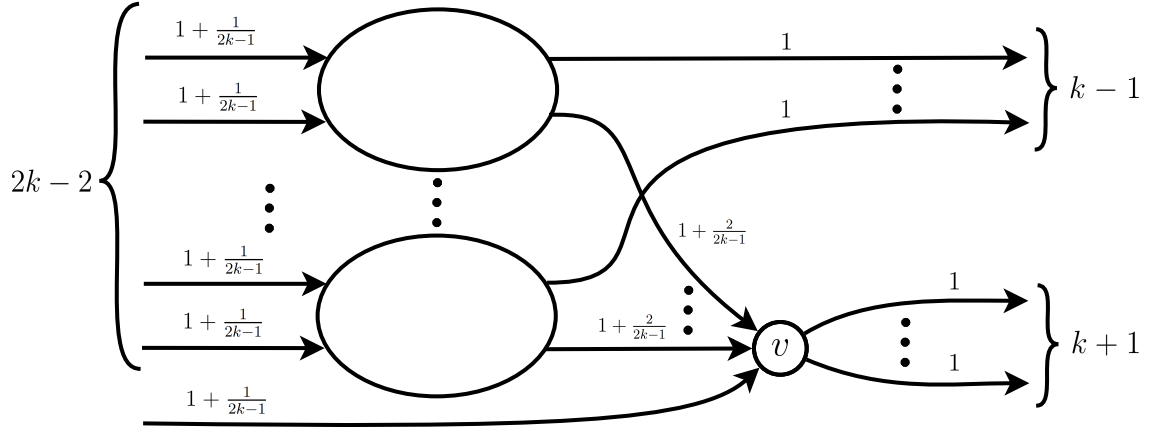
Figure 5.5: Step 2 of the construction for the target substructure

Step 2: First, identify each of the $k-1$ elements of X_1^+ with precisely one terminal of X^-, and identify the $k+1$ remaining elements of X^- to a single vertex v^-. Then identify the elements of X_2^+ and the remaining terminal of X^+ with excess flow $1 + \frac{2}{2k-1}$ to a single vertex v^+. Finally identify v^+ and v^- to a single vertex v to obtain the desired network $((G_2, \sigma_2), U_2)$ (see Figure 5.5).

Repeat Step 1 and Step 2 until we get a graph H with $F_c((H, \rho)) = r$, and $|V'(H)| = |V'(G)| - 1$.

$\square$

Lemma 5.2.6. *For all $k \geq 2$: $\mathcal{F}^c \cap [2 + \frac{2}{2k-1}; 4) \subseteq \mathcal{F}^c_{2k+1}$*

Proof. Let $k \geq 2$, $r \in \mathcal{F}^c \cap [2 + \frac{2}{2k-1}; 4)$ and (G, σ) be a graph with $F_c((G, \sigma)) = r$ and let ϕ be a nowhere-zero r-flow on (G, σ). Due to Lemmas 5.2.1, 5.2.3, and Construction 5.2.2 we can assume that if $e = uv$ is an edge and $\phi(e) > 2$, then $d_G(v) = d_G(u) = 2k + 1$. If $V'(G) = \emptyset$, then we are done. If $V'(G) \neq \emptyset$, then we proceed as follows:

Step 1: Let $v \in V'(G)$, and $((G_1, \sigma_1), U_1)$ be the network which is obtained from (G, σ) by splitting up v into terminals $v_1, \ldots, v_n$. Let $E(v_i) =$
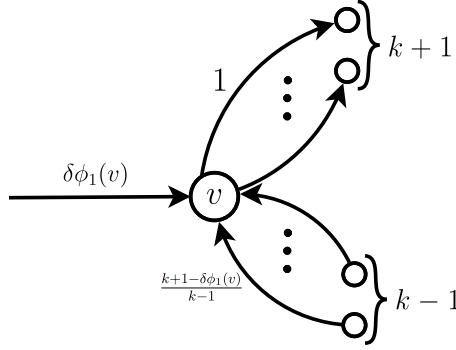
Figure 5.6: Construction for v if $\delta\phi_1(v) \geq 1 + \frac{2}{2k-1}$

$\{e_i'\}$ and ϕ_1 be the corresponding nowhere-zero r-flow on (G_1, U_1). Note that $|V'(G_1)| = |V'(G)| - 1$.

Step 2: We are now going to transfer $((G_1, \sigma_1), U_1)$ stepwise into a network $((G_4, \sigma_4), U_4)$ with $|V'(G_1)| = |V'(G_4)|$ and a nowhere-zero r-flow ϕ_4 with $|\delta\phi_4(u)| \leq 1 + \frac{1}{2k-1}$ for every terminal u. For the constructions we only consider positive terminals, for negative terminals the constructions work analogously.

Step 2.1: If there is a terminal v in $((G_1, \sigma_1), U_1)$ with $\delta\phi_1(v) \geq 1 + \frac{2}{2k-1}$, then add $2k$ new vertices and connect each new vertex with v by a single positive edge. We extend the nowhere-zero r-flow as follows: Exactly $k + 1$ of these edges have v as initial vertex and they have flow value 1 and exactly $k - 1$ edges have v as end vertex and they have flow value $\frac{k+1-\delta\phi_1(v)}{k-1}$. Since $1 + \frac{1}{2k-1} < \delta\phi_1(v) < 2$, it follows that $1 < \frac{k+1-\delta\phi_1(v)}{k-1} < 1 + \frac{1}{2k-1} \leq r - 1$. The construction is shown in Figure 5.6. We apply this construction to each terminal u of $((G_1, \sigma_1), U_1)$ with $\delta\phi_1(u) \geq 1 + \frac{2}{2k-1}$ to obtain a network $(G_2, \sigma_2), U_2)$ with a nowhere-zero r-flow ϕ_2 such that the excess flow is smaller than $1 + \frac{2}{2k-1}$ for every terminal. Note that $|V'(G_2)| = |V'(G)|$. If $\delta\phi_2(v) \leq 1 + \frac{1}{2k-1}$, for all $v \in U_2$, then set $((G_4, \sigma_4), U_4) = ((G_2, \sigma_2), U_2)$.
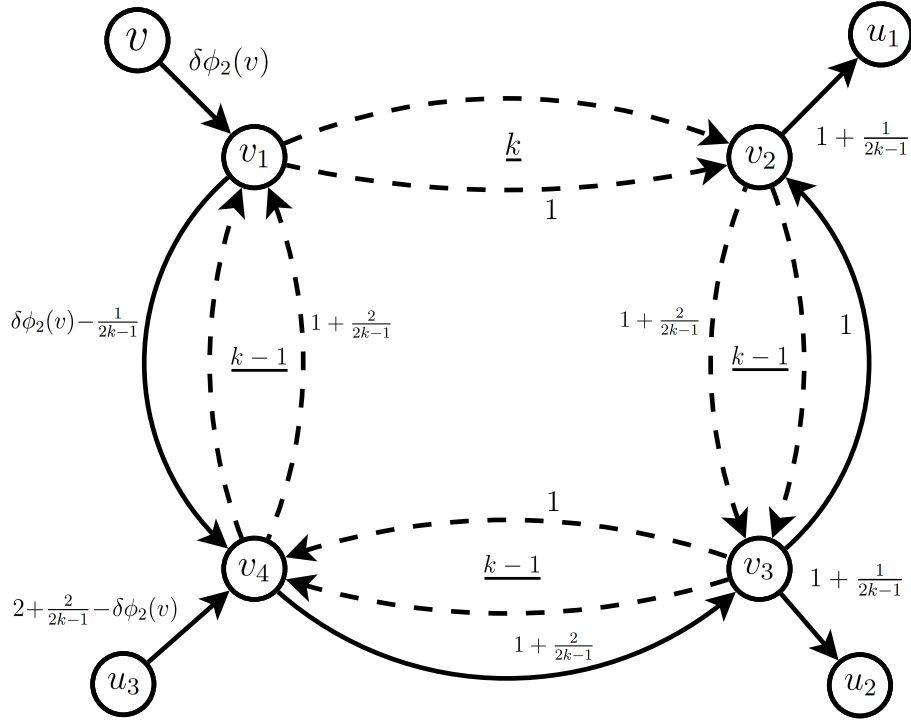
Figure 5.7: Construction for v if $1 + \frac{1}{2k-1} < \delta\phi_2(v) < 1 + \frac{2}{2k-1}$

Step 2.2: If there is a terminal $v \in U_2$ with $1 + \frac{1}{2k-1} < \delta\phi_2(v) < 1 + \frac{2}{2k-1}$, then construct a new network $((G_3, \sigma_3), U_3)$ by adding seven new vertices $v_1, v_2, v_3, v_4, u_1, u_2, u_3$. We add the positive edges $(v_1, v_2)_k$, $(v_2, v_3)_{k-1}$, $(v_3, v_4)_{k-1}$, $(v_4, v_1)_{k-1}$, (v, v_1), (v_2, u_1), (v_3, v_2), (v_3, u_2), (v_4, v_3), (u_3, v_4) and (v_1, v_4). The construction is shown in Figure 5.7. Extend ϕ_2 to a nowhere-zero r-flow ϕ_3 on $((G_3, \sigma_3), U_3)$ as follows: Let $\phi_3((v, v_1)) = \delta\phi_2(v)$, $\phi_3((u_3, v_4)) = 2 + \frac{2}{2k-1} - \delta\phi_2(v)$, $\phi_3((v_1, v_4)) = \delta\phi_2(v) - \frac{1}{2k-1}$, $\phi_3((v_2, v_3)_{k-1}) = \phi_3((v_4, v_1)_{k-1}) = \phi_3((v_4, v_3)) = 1 + \frac{2}{2k-1}$, $\phi_3((v_2, u_1)) = \phi_3((v_3, u_2)) = 1 + \frac{1}{2k-1}$, and $\phi_3((v_3, v_2)) = \phi_3((v_1, v_2)_k) = \phi_3((v_3, v_4)_{k-1}) = 1$. Terminal v is replaced by three new terminals u_1, u_2, u_3 and all other vertices of $((G_2, \sigma_2), U_2)$ are unchanged. The absolute value of the excess flow at two terminals is $1 + \frac{1}{2k-1}$ and the other one is $2 + \frac{2}{2k-1} - \delta\phi_2(v)$, which is at least 1 and and smaller than $1 + \frac{1}{2k-1}$. Repeat this construction to obtain a
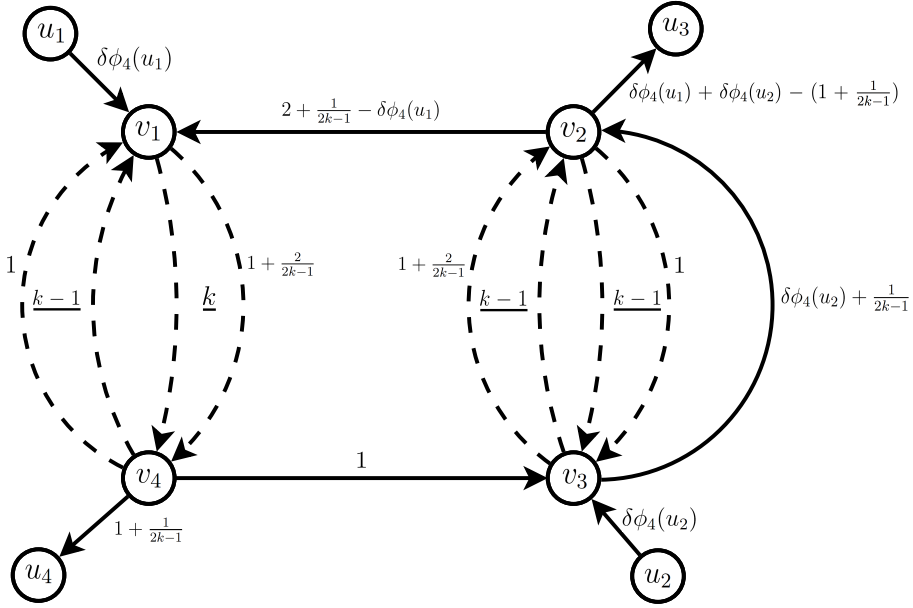
Figure 5.8: Construction if $2 + \frac{1}{2k-1} < \delta\phi_4(u_1) + \delta\phi_4(u_2) < 2 + \frac{2}{2k-1}$

network $((G_4, \sigma_4), U_4)$, where the excess flow of every terminal is at most $1 + \frac{1}{2k-1}$.

Step 3: Now we are going to construct a new network where all but at most two (one positive and one negative) terminals have excess flow in $\{\pm 1, \pm(1 + \frac{1}{2k-1})\}$. If one of the aforementioned networks has this property, then we are done. Again, for the constructions we only consider positive terminals, for negative terminals it works analogously.

Suppose that $((G_4, \sigma_4), U_4)$ does not have this property. Then there are two (positive) terminals u_1 and u_2 with $\delta\phi_4(u_1), \delta\phi_4(u_2) \notin \{1, 1 + \frac{1}{2k-1}\}$.

Step 3.1: If $2 + \frac{1}{2k-1} < \delta\phi_4(u_1) + \delta\phi_4(u_2) < 2 + \frac{2}{2k-1}$, then add six new vertices $v_1, v_2, v_3, v_4, u_3, u_4$ to obtain a new network $((G_5, \sigma_5), U_5)$. We add the positive edges (v_2, v_1), (v_4, v_3), (v_3, v_2), $(v_2, v_3)_{k-1}$, $(v_3, v_2)_{k-1}$, $(v_4, v_1)_{k-1}$, $(v_1, v_4)_k$, (u_1, v_1), (v_2, u_3), (u_2, v_3) and (v_4, u_4). The construction is depicted in Figure 5.8.

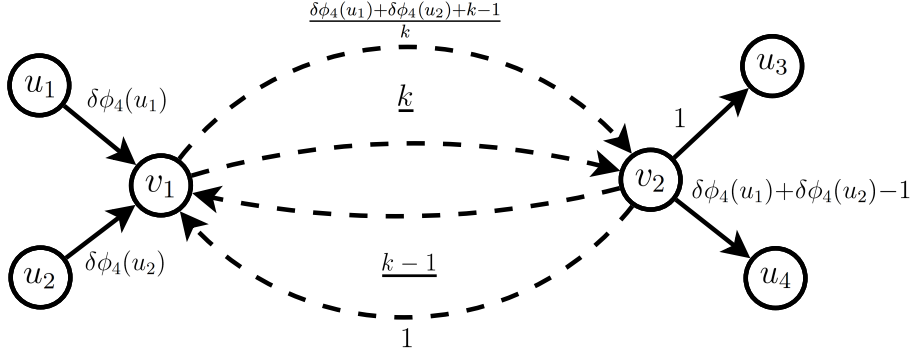Extend ϕ_4 to a nowhere-zero r-flow ϕ_5 on $((G_5, \sigma_5), U_5)$ as follows:

Figure 5.9: Construction if $2 < \delta\phi_4(u_1) + \delta\phi_4(u_2) \leq 2 + \frac{1}{2k-1}$

Let $\phi_5((u_1, v_1)) = \delta\phi_4(u_1)$, $\phi_5((v_2, u_3)) = \delta\phi_4(u_1) + \delta\phi_4(u_2) - (1 + \frac{1}{2k-1})$, $\phi_5((u_2, v_3)) = \delta\phi_4(u_2)$, $\phi_5((v_2, v_1)) = 2 + \frac{1}{2k-1} - \delta\phi_4(u_1)$, $\phi_5((v_3, v_2)) = \delta\phi_4(u_2) + \frac{1}{2k-1}$, $\phi_5((v_4, u_4)) = 1 + \frac{1}{2k-1}$, $\phi_5((v_1, v_4)_k) = \phi_5((v_3, v_2)_{k-1}) = 1 + \frac{2}{2k-1}$, and $\phi_5((v_4, v_3)) = \phi_5((v_4, v_1)_{k-1}) = \phi_5((v_2, v_3)_{k-1}) = 1$.

Step 3.2: If $2 < \delta\phi_4(u_1) + \delta\phi_4(u_2) \leq 2 + \frac{1}{2k-1}$, then add four vertices v_1, v_2, u_3, u_4 and the following edges to obtain a new network $((G_6, \sigma_6), U_6)$: We add the positive edges (u_1, v_1), (u_2, v_1), (v_2, u_3), (v_2, u_4), $(v_2, v_1)_{k-1}$ and $(v_1, v_2)_k$. The construction is shown in Figure 5.9.

Extend ϕ_4 to a nowhere-zero r-flow ϕ_6 on $((G_6, \sigma_6), U_6)$ as follows: Let $\phi_6((u_1, v_1)) = \delta\phi_4(u_1)$, $\phi_6((u_2, v_1)) = \delta\phi_4(u_2)$, $\phi_6((v_2, u_4)) = \delta\phi_4(u_1) + \delta\phi_4(u_2) - 1$, $\phi_6((v_1, v_2)_k) = \frac{\delta\phi_4(u_1) + \delta\phi_4(u_2) + k - 1}{k}$, and $\phi_6((v_2, v_1)_{k-1}) = \phi_6((v_2, u_3)) = 1$. Then u_3 and u_4 are terminals in (G_6, U_6) and their excess flow is at most $1 + \frac{1}{2k-1}$.

Repeat these two constructions until we get a network (G_7, U_7) where all terminals but at most two (one positive and one negative) terminals have excess flow in $\{\pm 1, \pm(1 + \frac{1}{2k-1})\}$.

Now we are going to reduce the number of terminals so that we eventually get a signed graph (H^*, σ^*) with $F_c((H^*, \sigma^*)) = r$ and $|V'(H^*) = |V'(G)| - 1$.

Let $((G_7, \sigma_7), U_7)$ be the network where the only possible excess flow values are ± 1 or $\pm(1 + \frac{1}{2k-1})$ except for at most two terminals (one with positive and one with negative excess flow) x and y with $|\delta\phi_7(x)|, |\delta\phi_7(y)| < 1 + \frac{1}{2k-1}$. Let $^a U_7^+$ be the set of terminals in U_7^+ which have excess flow 1, and $^b U_7^+$ be the set of terminals in U_7^+ which have excess flow $1 + \frac{1}{2k-1}$; $^a U_7^-$ and $^b U_7^-$ are defined analogously.

If only one of x and y exists, say x, then, since ϕ_7 is a nowhere-zero r-flow on $((G_7, \sigma_7), U_7)$, x is a multiple of $\frac{1}{2k-1}$, contradicting the fact that $\delta\phi_7(x) \in (1; 1 + \frac{1}{2k-1})$.

If x and y exist then it follows that

$$|^b U_7^+|(1 + \frac{1}{2k-1}) + |^a U_7^+| + |\delta\phi_7(x)| = |^b U_7^-|(1 + \frac{1}{2k-1}) + |^a U_7^-| + |\delta\phi_7(y)|$$

and therefore

$$|\delta\phi_7(y)| - |\delta\phi_7(x)| = (|^b U_7^+| - |^b U_7^-|)(1 + \frac{1}{2k-1}) + |^a U_7^+| - |^a U_7^-|.$$

Hence, $|\delta\phi_7(y)| - |\delta\phi_7(x)|$ is a multiple of $\frac{1}{2k-1}$. Since $|\delta\phi_7(x)|, |\delta\phi_7(y)| \in (1; 1 + \frac{1}{2k-1})$ it follows that $|\delta\phi_7(y)| - |\delta\phi_7(x)| \in (-\frac{1}{2k-1}; \frac{1}{2k-1})$, and therefore $|\delta\phi_7(y)| = |\delta\phi_7(x)|$.

Without loss of generality we assume that $|^a U_7^+| \leq |^a U_7^-|$. Since $((G_7, \sigma_7), U_7)$ has a nowhere-zero r-flow, it follows that $|^b U_7^+| \geq |^b U_7^-|$.

Step 4: Identify x and y. Then identify every terminal of $^a U_7^+$ with precisely one terminal of $^a U_7^-$, and every terminal of $^b U_7^-$ with precisely one terminal of $^b U_7^+$ to obtain a network $((G_8, \sigma_8), U_8)$ where all positive terminals have excess flow $1 + \frac{1}{2k-1}$ and all negative terminals have excess

flow -1. Then identify these remaining terminals in one vertex v and suppress the bivalent vertices to obtain a signed graph (G_9, σ_9) with $F_c((G_9, \sigma_9)) = r$. If $|V'(G_9)| = |V'(G)| - 1$, then $(G_9, \sigma_9) = (H^*, \sigma^*)$ and we are done. Otherwise, $|V'(G_9)| = |V'(G)|$ and there is an integer $n \geq 1$ such that $d_G(v) = n(4k - 1)$, $|E^+(v)| = n(2k - 1)$, $|E^-(v)| = 2kn$, and if $e \in E^+(v)$, then e has flow value $1 + \frac{1}{2k-1}$, and if $e \in E^-(v)$, then e has flow value 1. By Lemma 5.2.5 there is a graph (H^*, σ^*) with $F_c((H^*, \sigma^*)) = F_c((G_9, \sigma_9)) = r$ and $|V'(H^*)| = |V(G_9)| - 1 = |V(G)| - 1$.

Repeat Step 1 to Step 4 until we get a graph (H, τ) with $V'(H) = \emptyset$; i.e. H is $(2k + 1)$-regular and $F_c((H, \tau)) = r$.

$\square$

Theorem 5.1.2 follows from Lemmas 5.2.4 and 5.2.6.

Chapter 6

Flows and orientations

6.1 Flows and signed circuits

Let G be a graph and $e_1, e_2 \in E(G)$ be two edges incident with a vertex v_1. We say, under an orientation D of G, e_1, e_2 are consistent at v_1 if either

- $h_{e_1}^{v_1}$ is directed towards v_1 and $h_{e_2}^{v_1}$ is directed away from v_1, or

- $h_{e_1}^{v_1}$ is directed away from v_1 and $h_{e_2}^{v_1}$ is directed towards v_1.

We also say that v_1 is consistently balanced. A trail or a balanced cycle S is consistent, if S is consistently balanced in each vertex. An odd cycle C is consistent if C is consistently balanced at each vertex but one.

We say that a nowhere-zero flow ϕ on a graph (G, σ) is A-decomposable, if there exist a set A of subgraphs of G endowed with certain nowhere-zero flows, such that

$$\phi = \sum_{S \in A} n_S \phi_S$$

where ϕ_S is the corresponding nowhere-zero flow on S and n_S is an positive integer.

For a graph (G, σ) let $\mathscr{F}_{(G,\sigma)}$ be the set of signed circuits in (G, σ). For a signed graph (G, σ) with a positive integer nowhere-zero flow ϕ let $F^i_{(G,\sigma)} \in \mathscr{F}_{(G,\sigma)}$ ($i \in \mathbb{Q}$) be the set of signed circuits that are endowed with pseudo flows with the following properties:

- if $S \in F^i_{(G,\sigma)}$ is a barbell, then all edges in the unbalanced circuits receive the value i and all edges on the path (possibly zero edges) that connects these circuits receive the value $2i$.

- if $S \in F^i_{(G,\sigma)}$ is a balanced circuit, then all edges receive value 1

Máčajová and Škoviera proved the following theorem:

Theorem 6.1.1. *[25] Let ϕ be a positive integer nowhere-zero flow on a signed graph (G, σ). Then ϕ is $F^{\frac{1}{2}}_{(G,\sigma)}$-decomposable such that all graphs in $F^{\frac{1}{2}}_{(G,\sigma)}$ have the same orientation as in (G, σ).*

The issue that orientations are kept seems to be a strong condition. However, here, we decompose a nowhere-zero flow into non-proper nowhere-zero flows. The question at hand is, whether the use of non-proper nowhere-zero flows with fractional values is necessary. As an example for the necessity the authors of [25] gave the following example.

Let (K_2^*, σ) be the graph consisting of two vertices connected by two parallel positive edges, where a negative loop is attached to each vertex. It is easy to see that (K_2^*, σ) admits a nowhere-zero 2-flow, but each decomposition of this flow into flows of signed circuits contains two distinct barbells that share both negative loops.

Thus, in this setting we do not get rid of non-proper flows with fractional flow values. It seems natural to express a nowhere-zero flow of a graph as the sum of nowhere-zero flows of signed circuits. For this

purpose, in the further study of decompositions of a signed graph (G, σ) we do not want to fix the orientation of the signed circuits in $F^i_{(G,\sigma)}$ corresponding to (G, σ). It turns out that this will give us the flexibility to express a nowhere-zero flow by the sum of proper nowhere-zero flows of signed circuits.

Theorem 6.1.2. *Let (G, σ) be a flow-admissible signed graph and ϕ be a positive integer nowhere-zero flow of (G, σ). Then ϕ is $F^1_{(G,\sigma)}$-decomposable.*

Proof. Let ϕ be a positive integer nowhere-zero flow on (G, σ).

We will prove the statement by induction on $\sum_{e \in E(G)} \phi(e)$. Therefore, in each step we are going to find a directed signed circuit in (G, σ) and choose one corresponding $C \in \mathscr{F}_{(G,\sigma)}$ together with a nowhere-zero flow ϕ_C such that $\tau_{\phi_C}(e) = \tau_\phi(e)$ for each $e \in E(G)$. Next, we consider the integer nowhere-zero flow $\phi' = \phi - \phi_C$ on the directed graph (G', σ') with $E(G') = \{e \in E(G) | \phi(e) - \phi_C(e) \neq 0\}$ and $\sigma' = \sigma|_{E(G')}$. If C is a balanced circuit or a short barbell, then

$$\sum_{e \in E(G')} \phi'(e) = \sum_{e \in E(G)} \phi(e) - |E(C)|.$$

Assume C is a barbell consisting of two disjoint unbalanced circuits C_1 and C_2 and a path T connecting C_1 and C_2. It follows

$$\sum_{e \in E(G')} \phi'(e) = \sum_{e \in E(G)} \phi(e) - |E(C_1) \cup E(C_2)| - |\{e \in E(T) : \phi(e) \neq 1\}|.$$

Note, that in (G', σ') edges in T with flow value 1 reversed their direction.

It remains to show that in each step we can find a signed circuit $C \in \mathscr{F}_{(G,\sigma)}$ as a subgraph. Suppose after a series of steps we get one graph (G', σ') with a pseudoflow ϕ' and there are no signed circuits $C \in \mathscr{F}_{(G,\sigma)}$

as subgraphs in (G', σ'). Let $v \in V(G')$. Note, since ϕ' is a positive nowhere-zero flow, for each vertex v exist two edges that are consistent in v. Beginning with a vertex v and an outgoing half-edge we successively add consistent vertices and construct a maximum consistent trail P_1. The trail P_1 must contain at least one vertex, say u_1 twice. The corresponding circuit C_1 must be unbalanced.

Next, beginning with v and an incoming half-edge we successively add consistent vertices and construct a maximum consistent trail P_2. Similarly P_2 must contain at least one vertex, say w_1 twice, forming an unbalanced circuit C_2. If C_1 and C_2 are pairwise disjoint or intersect in precisely one vertex, then (G', σ') contains a barbell, which contradicts our assumption. Thus, there are two circuits $C_u = u_1, \ldots, u_p, \ldots, u_n, u_1$ and $C_w = w_1, \ldots, w_p, \ldots, w_m, w_1$ $(p, n, m \in \mathbb{Z})$ and $u_i = w_i$ for $i \in \{1, \ldots, p\}$. The circuit C_t formed by the the trails $P_1 = u_p, u_{p+1}, \ldots, u_n$ and $P_2 = w_p, w_{p+1}, \ldots, w_m$ must also be unbalanced. Therefore, precisely one trail is balanced. Suppose P_1 is balanced and P_2 is unbalanced. However, since C_1 and C_2 are unbalanced, the trail $u_1, \ldots, u_p$ cannot be balanced, nor unbalanced, a contradiction.

<div align="right">□</div>

6.2   Characterization of flows

As seen in Theorem 6.1.2 orientations play an important role of the way we are able to decompose a flow. Just by allowing a higher flexibility we are able to prove a decomposition into proper nowhere-zero flows. In this section, for a signed graph (G, σ) we characterize a nowhere-zero flow ϕ by finding a set Q of orientations of (G, σ). In this regard for the orientations in Q there are no constraints. An orientation $D \in Q$ even

does not have to admit a positive nowhere-zero flow. We will relate the problem of finding a nowhere-zero k-flow to the problem of finding a set of orientations with the property that a linear combination with given coefficients over the boundary of these so called reorientations equals zero. Let (G, σ) be an directed signed graph with orientation D. A *reorientation* of (G, σ) is an assignment $f_D : E(G) \to \{-1, +1\}$. We interpret f_D as a new orientation of (G, σ) itself. If $f_D(e) = 1$ for an edge e, then each half-edge of e is oriented as in D and if $f(e) = -1$, then each half-edge of e is oriented in the opposite direction. We may view each orientation as an reorientation of one fixed orientation. Here, we use the expression ∂f to denote the corresponding boundary of an orientation f.

The following theorem characterizes unsigned graphs that admit a nowhere-zero 5-flow by orientations.

Theorem 6.2.1. *[12] Let G be a directed unsigned graph. G admits a nowhere-zero 5-flow if and only if G has three reorientations f_1, f_2, f_3, such that $\partial f_1 + 2\partial f_2 + 5\partial f_3 = 0$*

In the following we will give a similar characterization which is more general for unsigned graphs that admit a nowhere-zero $(\frac{p}{q} + 1)$-flow.

Theorem 6.2.2. *A directed unsigned graph G admits a nowhere-zero $(\frac{p}{q} + 1)$-flow if and only if G has reorientations $f_1, \ldots, f_{p-q+1}$, such that $\sum_{i=1}^{p-q+1} \alpha_i \partial f_i = 0$ with $\alpha_1 = \frac{q+p}{2q}$ and $\alpha_l = \frac{1}{2q}$ for $l \in \{2, \ldots, p-q+1\}$*

Proof. Let ϕ be a nowhere-zero $(\frac{p}{q} + 1)$-flow on G which is chosen according to Theorem 1.4.8, such that for each edge e, $\phi(e)$ is a multiple of $\frac{1}{q}$ and let D be the underlying orientation. We define the required reorientations in the following way:

For $e \in E(G)$ and $i \in \{1, \ldots, p-q+1\}$ let $f_i(e) = 1$ if $1 + \frac{i-1}{q} \leq \phi(e)$ or $-1 - \frac{i-1}{q} < \phi(e) \leq -1$ and $f_i(e) = -1$ otherwise. Next, we consider the vector consisting of all possible flow values of ϕ. We are going to express

each flow value of an edge e by one unique set of reorientations regarding e. We will determine the coefficients α_i of the equation $\sum_{i=1}^{p-q+1} \alpha_i f_i(e) = \phi(e)$. For reasons of symmetry we can restrict the linear system to only positive flow values.

$$
\begin{pmatrix}
1 & -1 & \cdots & -1 \\
1 & 1 & -1 & \vdots \\
\vdots & \ddots & 1 & -1 \\
1 & \cdots & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{p-q+1}
\end{pmatrix}
= \frac{1}{q}
\begin{pmatrix}
q \\
q+1 \\
\vdots \\
p
\end{pmatrix}. \tag{6.1}
$$

The unique solution is given by $\alpha_1 = \frac{q+p}{2q}$ and $\alpha_l = \frac{1}{2q}$ for $l \in \{2, \ldots, p-q+1\}$. Hence,

$$
\begin{aligned}
\sum_{i=1}^{p-q+1} \alpha_i \partial f_i(v) &= \sum_{i=1}^{p-q+1} \alpha_i \left(\sum_{h \in H(v)} f_i(h_e) \right) \\
&= \sum_{i=1}^{p-q+1} \sum_{h \in H^+(v)} \alpha_i f_i(h_e) - \sum_{i=1}^{p-q+1} \sum_{h \in H^-(v)} \alpha_i f_i(h_e) \\
&= \sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0
\end{aligned} \tag{6.2}
$$

To prove the other direction, let $\alpha_1 = \frac{p+q}{2q}$ and $\alpha_l = \frac{1}{2q}$ for $l \in \{2, \ldots, p-q+1\}$ and $f_1, \ldots, f_{p-q+1}$ be reorientations of (G, σ) such that $\sum_{i=1}^{p-q+1} \alpha_i \partial f_i = 0$. Let $\phi : E(G) \to \mathbb{R}$ be an assignment with $\phi(e) = \sum_{i=1}^{p-q+1} \alpha_i f_i(e)$. We need to show, that ϕ is a nowhere-zero flow. Due to the choice of ϕ we get the same linear system as before. We get $\sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0$ for every $v \in V(G)$ and from equation 6.1 follows $\phi^{-1}(E) \subseteq \{\pm\frac{q}{q}, \pm\frac{q+1}{q}, \ldots, \pm\frac{p}{q}\}$. Thus, ϕ is a proper nowhere-zero $(\frac{p}{q}+1)$-flow. □

In a similar way we can find a characterization for general signed graphs. For flows on signed graphs we need more possible flow values. That will result in an increasing number of reorientations.

Theorem 6.2.3. *A directed signed graph* (G, σ) *admits a nowhere-zero* $(\frac{p}{q} + 1)$*-flow if and only if* (G, σ) *has reorientations* $f_1, \ldots, f_{2p-2q+1}$*, such that* $\sum_{i=1}^{2p-2q+1} \alpha_i \partial f_i = 0$ *with* $\alpha_1 = \frac{p+q}{2q}$ *and* $\alpha_l = \frac{1}{4q}$ *for* $l \in \{2, \ldots, 2p - 2q + 1\}$*.*

Proof. Let ϕ be a nowhere-zero $(\frac{p}{q} + 1)$-flow of (G, σ) which is chosen according to Theorem 3.2.1, such that for each edge e, $\phi(e)$ is a multiple of $\frac{1}{2q}$ and let D be the underlying orientation. We define the required reorientations in the following way:

For $e \in E(G)$ and $i \in \{1, \ldots, 2p - 2q + 1\}$ let $f_i(e) = 1$ if $1 + \frac{i-1}{2q} \leq \phi(e)$ or $-1 - \frac{i-1}{2q} < \phi(e) \leq -1$ and $f_i(e) = -1$ otherwise. Next, we consider the vector consisting of all possible flow values of ϕ. We are going to express each flow value of an edge e by one unique set of reorientations regarding e.

We will determine the coefficients α_i of the equation $\sum_{i=1}^{2p-2q+1} \alpha_i f_i(e) = \phi(e)$. For reasons of symmetry we can restrict the linear system to only positive flow values.

$$
\begin{pmatrix}
1 & -1 & \ldots & -1 \\
1 & 1 & -1 & \vdots \\
\vdots & \ddots & 1 & -1 \\
1 & \ldots & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{2p-2q+1}
\end{pmatrix}
= \frac{1}{2q}
\begin{pmatrix}
2q \\
2q + 1 \\
\vdots \\
2p
\end{pmatrix}. \tag{6.3}
$$

The unique solution is given by $\alpha_1 = \frac{1}{2q}(p + q)$ and $\alpha_l = \frac{1}{4q}$ for $l \in \{2, \ldots, 2(p - q) + 1\}$. Hence,

$$\sum_{i=1}^{2p-2q+1} \alpha_i \partial f_i(v) = \sum_{i=1}^{2p-2q+1} \alpha_i \left(\sum_{h \in H(v)} f_i(h_e) \right)$$

$$= \sum_{i=1}^{2p-2q+1} \sum_{h \in H^+(v)} \alpha_i f_i(h_e) - \sum_{i=1}^{2p-2q+1} \sum_{h \in H^-(v)} \alpha_i f_i(h_e)$$

$$= \sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0$$

$$(6.4)$$

To prove the other direction, let $\alpha_1 = \frac{p+q}{2q}$ and $\alpha_l = \frac{1}{4q}$ for $l \in \{2, \ldots, 2p - 2q + 1\}$ and $f_1, \ldots, f_{2p-2q+1}$ be reorientations of (G, σ) such that $\sum_{i=1}^{2p-2q+1} \alpha_i \partial f_i = 0$. Let $\phi : E(G) \to \mathbb{R}$ be an assignment with $\phi(e) = \sum_{i=1}^{2p-2q+1} \alpha_i f_i(e)$. We need to show, that ϕ is a nowhere-zero flow. Due to the choice of ϕ we get the same linear system as before. We get $\sum_{h \in H^+(v)} \phi(e_h) - \sum_{h \in H^-(v)} \phi(e_h) = 0$ for every $v \in V(G)$ and from equation 6.3 it follows $\phi^{-1}(E) \subseteq \{\pm\frac{2q}{2q}, \pm\frac{2q+1}{2q}, \ldots, \pm\frac{2p}{2q}\}$. Thus, ϕ is a proper nowhere-zero $(\frac{p}{q} + 1)$-flow.

$\square$

Chapter 7

Conclusion

Nowhere-zero flows on signed graphs generalize the coloring concept in a natural way. Recall that $\delta_F = \sup\{F((G, \sigma)) - F_c((G, \sigma)) : (G, \sigma)$ is flow-admissible$\}$. In Chapter 3 we showed $\delta_F \geq 2$. The strenghtening in [22] cannot be improved anymore in the case if Bouchet's Conjecture 1.4.3 is true. We also showed that not every counterexample to Conjecture 3.1.1 which states $F_c((G, \sigma)) > F((G, \sigma)) - 1$ has a star-cut. However, up to now the only graphs that are known to be counterexamples to Conjecture 3.1.1 contain a vertex cut consisting of precisely one vertex (we may interpret a vertex with a negative loop as such a vertex cut). In this context it is interesting to study whether these graphs are the only counterexamples or not.

In Chapter 4 we studied the flow spectrum of graphs and established the concept of r-minimal sets. If a graph H has an r-minimal set X of cardinality 2, then $\mathcal{S}_X(H) = \{r, F_c((H, 1))\}$. Theorem 4.4.11 implies that every 3-minimal set of a bridgeless non-3-edge-colorable cubic graph contains at least three edges. The Petersen graph P has a 3-minimal set X with $|X| = 3$. Hence, $\mathcal{S}_X(P) = \{3, 4, 5\}$. On the other hand, P has a 6-minimal set X' with $|X'| = 3$. It follows that $3 \notin \mathcal{S}_{X'}(P)$; indeed
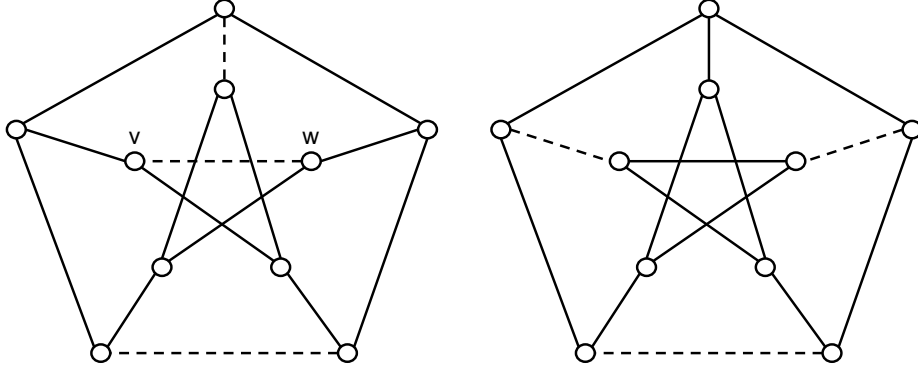
Figure 7.1: The dashed lines represent set X on the left and X' on the right on the Petersen graph

$\overline{\mathcal{S}}_{X'}(P) = \{4,5,6\}$. The sets X and X' are indicated in Figure 7.1. Two switches at v and w yield a signature σ of P such that (P,σ) and (P,X) are equivalent. But $X' \subset N_\sigma$ and therefore, $\mathcal{S}_X(P) \neq \mathcal{S}_{N_\sigma}(P)$.

Some further problems are:

- Let $r \geq 2$, (G,σ) be a flow-admissible signed graph and X a (non-empty) r-minimal set. Determine the (integer) X-flow spectrum of G.

- Let $r \geq 2$ and G be a graph. Let (G,σ) and (G,σ') be flow-admissible. Is it true that if N_σ and $N_{\sigma'}$ are (both) smallest r-minimal sets, then $\mathcal{S}_{N_\sigma}(G) = \mathcal{S}_{N_{\sigma'}}(G)$ (or $\overline{\mathcal{S}}_{N_\sigma}(G) = \overline{\mathcal{S}}_{N_{\sigma'}}(G)$, if r is an integer)?

- Let G be a snark and X_3 a 3-minimal set. Is it true that there exists a 4-minimal set X_4 and a 5-minimal set X_5 such that $X_5 \subset X_4 \subset X_3$?

According to Theorem 4.4.11 for snarks every 3-minimal set contains a proper 4-minimal set. If Tutte's 5-flow conjecture is true, then the last problem has an affirmative answer.

In Chapter 5 we determined the set of circular flow numbers for $(2k+1)$-regular signed graphs of odd regularity. An eulerian signed graph (G, σ) with an even number of negative edges has $F_c((G, \sigma)) = 2$. Let (H, τ) be the signed graph consisting of two vertices that are connected by $2k+1$ positive parallel edges and $2k+1$ negative parallel edges. H is eulerian and it is easy to see that $F_c(H, \tau) = 2 + \frac{1}{k}$. It is an interesting question whether for each $r \in [2, 6]$ there exist one eulerian graph (G, σ) with $F_c((G, \sigma)) = r$. Until now, only the integer flow numbers of eulerian graphs have been characterized [24], [26].

In Chapter 6 we showed that a positive flow ϕ on a signed graph (G, σ) is $F^1_{(G,\sigma)}$-decomposable. In our concept the orientation of the edges in the decomposition are free. Therefore, we are able to decompose (G, σ) into proper nowhere-zero flows. The authors of [25] proved that if a signed graph G admits an integer nowhere-zero k-flow, then it has a signed circuit cover of total length at least $2(k-1)|E(G)|$.

Suppose $k \geq 3$. It would be interesting to analyse if we can lower the total length of a signed circuit cover with the new concept.

Appendix A

Details

A.1   A graph (G, σ) with $F((G, \sigma)) = 6$

In the following we will show that the Petersen graph equipped with a special signature has integer flow number 6. The proof via case differentiation is quite technical. Considering, that we prove that a certain highly symmetric graph with ten vertices and 15 edges has a specific flow number the afford for proving turns out to be huge. Moreover, in the proof we use the fact that the Petersen graph is highly symmetric in order to shorten the proof. This gives a feeling of how hard it is in general to determine the flow number for a given graph. Up to switching isomorphism, there are six ways to put signs on the edges of the Petersen graph [50], thus, the spectrum of the Petersen graph P consists at most of five elements, since at least one switching equivalence class contains no flow-admissible graph. Indeed, it is well known, that $\overline{\mathcal{S}}(P) = \{3, 4, 5, 6\}$, and in the following we determine one element of the spectrum.

In the following the summation of the indices is taken modulo 5. Let (P, σ^*) be the Petersen graph consisting of two circuits $C_1 = u_1, u_2, u_3, u_4, u_5$ and $C_2 = v_1, v_2, v_3, v_4, v_5$ that are connected
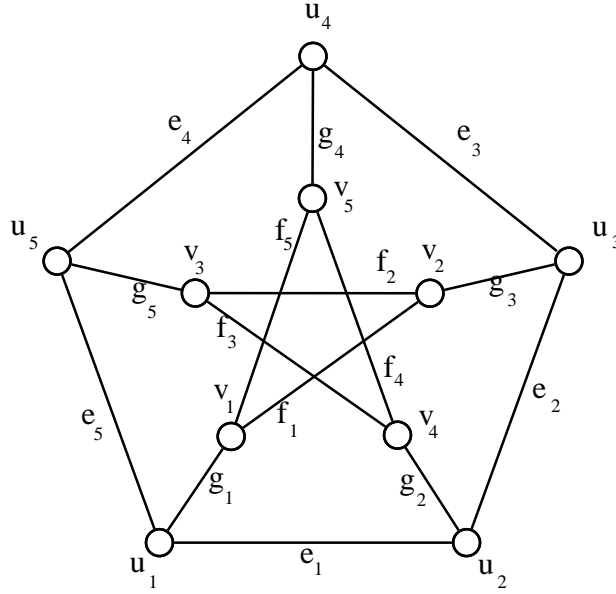
Figure A.1: (P, σ^*) with labels

by the edges $g_1 = u_1 v_1, g_2 = u_2 v_4, g_3 = u_3 v_2, g_4 = u_4 v_5$ and $u_5 v_3$. The summation of the indices is taken modulo 5. Further, let $e_i = u_i u_{i+1}$ and $f_i = v_i v_{i+1}$ for $i \in \{1, ..., 5\}$. The signature is given by $N_{\sigma^*} = \{e_1, e_2, e_3, e_4, e_5\}$. The signed Petersen graph (P, σ^*) is depicted in figure A.1.

Lemma A.1.1. *The signed Petersengraph (P, σ^*) has flow number $F((P, \sigma^*)) = 6$.*

  Proof. Suppose (P, σ^*) has a nowhere zero k-flow ϕ and $k \geq 6$. Let N_{ex} and N_{in} be the sets of extroverted and introverted edges of ϕ, respectively. Without loss of generality we may assume that either $|N_{ex}| = 4$ and $|N_{in}| = 1$ or $|N_{ex}| = 3$ and $|N_{in}| = 2$. First, we suppose $|N_{ex}| = 4$ and $|N_{in}| = 1$. Due to symmetry we may assume $N_{ex} = \{e_1, e_2, e_3, e_4\}$ and $N_{in} = \{e_5\}$. Since $4 \leq \sum_{e \in N_{ex}} \phi(e) = \sum_{e \in N_{in}} \phi(e) \leq 4$, it follows $\phi(e_i) = 1$ for $i = 1..4$ and $\phi(e_5) = 4$. It follows that g_1 and g_5 are directed towards C_1 and g_2, g_3, g_4 are directed towards C_2 with $\phi(g_1) = \phi(g_5) = 3$ and $\phi(g_2) = \phi(g_3) = \phi(g_4) = 2$. Due to symmetry we may assume that f_4

is directed towards v_4 and hence f_3 is directed towards v_3. If $\phi(f_4) = 1$, it follows $\phi(f_3) = 3$ and $\phi(f_2) = 0$. If $\phi(f_4) = 2$ we get $\phi(f_5) = 0$. Finally, if $\phi(f_4) \geq 3$, it follows $\phi(f_3) \geq 5$. In all cases we obtain a contradiction to the fact, that ϕ is a proper nowhere zero k-flow and $k \geq 6$. Hence, $|N_{ex}| = 3$ and $|N_{in}| = 2$. Let (a, b, c, d, e) be a 5-tuple for ϕ in such a way that a, b and c represent the flow values for the extroverted edges and d and e represent the flow values of introverted edges. Taken into account that $a + b + c = \sum_{e \in N_{ex}} \phi(e) = \sum_{e \in N_{in}} \phi(e) = d + e \leq 8$ and $\phi(e) \leq 4$ there are without regarding permutations 19 possible 5 tuples: $(1, 1, 1, 1, 2)$, $(1, 1, 2, 1, 3)$, $(1, 1, 2, 2, 2)$, $(1, 1, 3, 1, 4)$, $(1, 1, 3, 2, 3)$, $(1, 1, 4, 2, 4)$, $(1, 1, 4, 3, 3)$, $(1, 2, 2, 1, 4)$, $(1, 2, 2, 2, 3)$, $(1, 2, 3, 2, 4)$, $(1, 2, 3, 3, 3)$, $(1, 2, 4, 3, 4)$, $(1, 3, 3, 3, 4)$, $(1, 3, 4, 4, 4)$, $(2, 2, 2, 2, 4)$, $(2, 2, 2, 3, 3)$, $(2, 2, 3, 3, 4)$, $(2, 2, 4, 4, 4)$, $(2, 3, 3, 4, 4)$.   Next, we will diminsh that list of possible flow values for negative edges. If one flow value occurs more than 3 time in the 5-tuple, then there must be one introverted edge incident to one extroverted edge with the same flow value, but that would force another incident edge to have no proper flow value and forms a contradiction. Thus, we get rid of $(1, 1, 1, 1, 2)$ and $(2, 2, 2, 2, 4)$. If $d + e \geq 5$, the introverted edges must not be adjacent, since the value on the third incident edge would be ≥ 5. If additionally the value d or e occurs at at least three times in the 5-tuple, there will always be one incident edge forced to get flow value zero, which generate a contradiction. Thus, we get rid of $(1, 1, 3, 1, 4)$, $(1, 2, 2, 2, 3)$, $(1, 2, 3, 3, 3)$, $(1, 3, 4, 4, 4)$, $(1, 3, 3, 3, 4)$, $(2, 2, 4, 4, 4)$. We can also exclude the tuples $(2, 3, 3, 4, 4), (1, 1, 4, 2, 4)$ because at least two extroverted edges are adjacent, but the sum of flow values for any two extroverted edges is ≥ 5.

Thus, there are seven tuples left, where exists a possible distribution for the flow values in C_1, $(1, 1, 2, 1, 3)$, $(1, 1, 2, 2, 2)$, $(1, 1, 3, 2, 3)$,

$(1, 1, 4, 3, 3)$, $(1, 2, 2, 1, 4)$, $(1, 2, 3, 2, 4)$, $(2, 2, 2, 3, 3)$. We will show that for every tuple up to symmetry just exists one possible distribution in the circuit. We consider the following cases:

(i) Suppose $(a, b, c, d, e) = (1, 1, 2, 1, 3)$. The introverted edge with with value 1, say e_1 must not be adjacent to any introverted edge with value 1, hence e_3 and e_4 are extroverted with $\phi(e_3) = \phi(e_4) = 1$. Due to symmetry we may assume e_2 is introverted, e_5 is extroverted, $\phi(e_2) = 3$, and $\phi(e_5) = 2$.

(ii) Suppose $(a, b, c, d, e) = (1, 1, 2, 2, 2)$. The introverted edges with value 2 must be adjacent, otherwise, no extroverted edge could receive value 2. Hence, e_1 and e_2 are introverted, e_3, e_4 and e_5 are extroverted, and $\phi(e_1) = \phi(e_2) = \phi(e_4) = 2$ and $\phi(e_3) = \phi(e_5) = 1$.

In the following we may consider $(1, 2, 2, 1, 4)$ represented by $(2, 2, 1, 4, 1)$. For all remaining tuples $(a, b, c, d, e) \in \{(1, 1, 3, 2, 3)$, $(1, 1, 4, 3, 3)$, $(2, 2, 1, 4, 1)$, $(1, 2, 3, 2, 4)$, $(2, 2, 2, 3, 3)\}$ we have $d + e \geq 5$. Thus, the introverted edges cannot be incident. Let e_1 and e_3 be the introverted edges with $\phi(e_1) = d$ and $\phi(e_3) = e$ and e_2, e_4 and e_5 are extroverted.

(iii) Suppose $(a, b, c, d, e) = (1, 1, 3, 2, 3)$ or $(a, b, c, d, e) = (2, 2, 1, 4, 1))$. Since $c = e$ we get $\phi(e_5) = c$ and because of $a = b$, $\phi(e_2) = \phi(e_4) = a$.

(iv) Suppose $(a, b, c, d, e) = (1, 1, 4, 3, 3)$. It follows $\phi(e_2) = 4$ and $\phi(e_4) = \phi(e_5) = 1$ since the edge with value 4 must not be adjacent to any other extroverted edge.

(v) and (vi) Suppose $(a, b, c, d, e) = (1, 2, 3, 2, 4)$. Since the extroverted edges with values 2 and 3 must not be adjacent and the extroverted
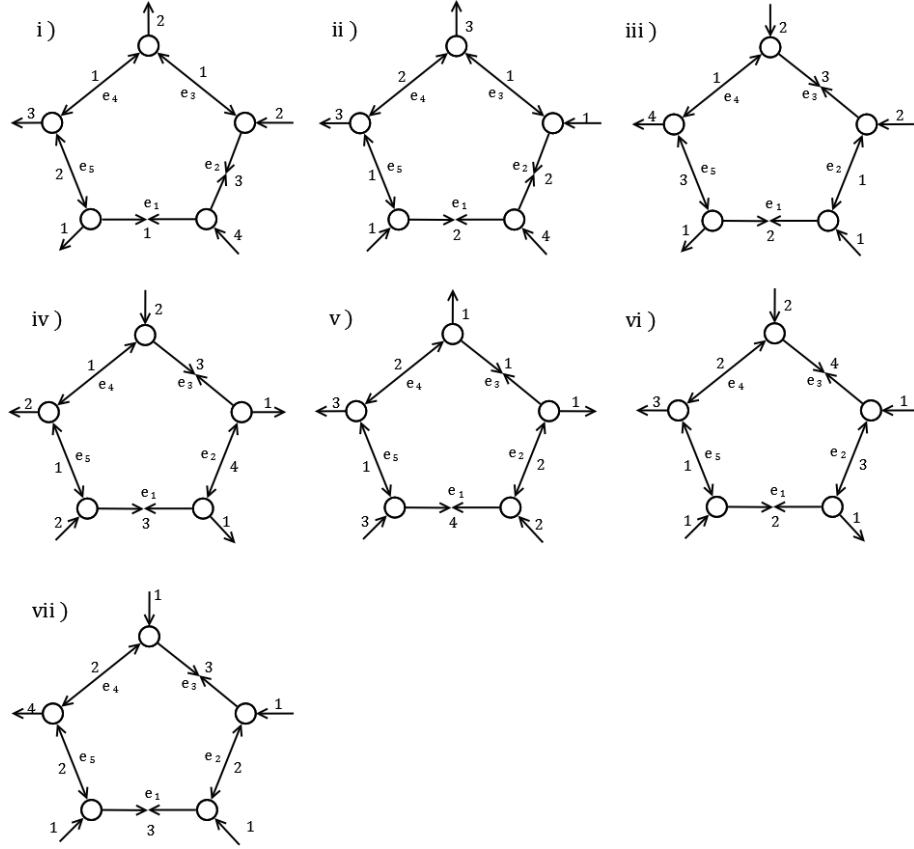
Figure A.2: Cases for a 6-flow on (P, σ^*)

and introverted edge with value 2 must not be adjacent we get $\phi(e_4) = 2, \phi(e_2) = 3$ and $\phi(e_5) = 1$.

(vii) Suppose $(a, b, c, d, e) = (2, 2, 2, 3, 3)$. We get $\phi(e_2) = \phi(e_4) = \phi(e_5) = 2$.

In figure A.2 the possibilities for C_1 with corresponding edges g_i $(i = 1..5)$ are depicted.

Note, for further consideration, that whenever there is one incoming and one outgoing edge adjacent with same the flow value, we may exclude that case, since the flow cannot be extended properly.

(i) $E^+(C_1) = \{e_1, e_4, e_5\}$ and $E^-(C_1) = \{e_2, e_3\}$ and $\phi(g_1) = 1, \phi(g_2) = 4, \phi(g_3) = \phi(g_4) = 2, \phi(g_5) = 3$: The edges f_3 and f_4 must be directed towards v_4. It follows $\phi(f_3) \notin \{3, 4\}$ and f_2 is directed towards v_2. If $\phi(f_3) = 1$, then $\phi(f_2) = 2$, and if $\phi(f_3) = 2$, then $\phi(f_4) = 2$, which forms a contradiction.

(ii) $E^+(C_1) = \{e_4, e_5\}$ and $E^-(C_1) = \{e_1, e_2, e_3\}$ and $\phi(g_1) = \phi(g_3) = 1, \phi(g_2) = 4, \phi(g_4) = \phi(g_5) = 3$: The edges f_3 and f_4 must be directed towards v_4. It follows $\phi(f_3) \notin \{3, 4\}$ and f_2 is directed towards v_2. If $\phi(f_3) = 1$, then $\phi(f_4) = 3$, and if $\phi(f_3) = 2$, then $\phi(f_2) = 1$, which forms a contradiction.

(iii) $E^+(C_1) = \{e_1, e_5\}$ and $E^-(C_1) = \{e_2, e_3, e_4\}$ and $\phi(g_1) = \phi(g_2) = 1, \phi(g_3) = \phi(g_4) = 4, \phi(g_5) = 4$: The edges f_2 and f_3 must be directed away from v_3. It follows $\phi(f_3) \notin \{1, 4\}$ and f_4 is directed towards v_5. If $\phi(f_3) = 2$, then $\phi(f_2) = 2$, and if $\phi(f_3) = 3$, then $\phi(f_4) = 2$, which forms a contradiction.

(iv) $E^+(C_1) = \{e_2, e_3, e_5\}$ and $E^-(C_1) = \{e_1, e_4\}$ and $\phi(g_1) = \phi(g_4) = \phi(g_5) = 2, \phi(g_2) = \phi(g_3) = 1$: Due to symmetry we can assume that f_5 is directed towards v_1. It follows that $\phi(f_5) = 1$, and hence f_1 is directed towards v_1 with $\phi(f_1) = 1$, which forms a contradiction.

(v) $E^+(C_1) = \{e_3, e_4, e_5\}$ and $E^-(C_1) = \{e_1, e_2\}$ and $\phi(g_1) = \phi(g_5) = 1, \phi(g_2) = 2, \phi(g_3) = \phi(g_4) = 3$: The edge f_2 must be directed towards v_2 and $\phi(f_2) \notin \{3, 4\}$. If $\phi(f_2) = 1$ then f_3 must be directed towards v_4 with $\phi(f_3) = 2$, a contradiction. If $\phi(f_2) = 2$ then f_1 must be directed towards v_1 with $\phi(f_1) = 3$, that forms a contradiction.

(vi) $E^+(C_1) = \{e_1, e_3, e_4\}$ and $E^-(C_1) = \{e_2, e_5\}$ and $\phi(g_1) = \phi(g_2) = \phi(g_3) = 1, \phi(g_4) = 2, \phi(g_5) = 3$: The edge f_3 must be directed

towards v_4 and $\phi(f_3) \notin \{3, 4\}$. If $\phi(f_3) = 1$ then f_4 must be directed towards v_5 with $\phi(f_4) = 2$, a contradiction. If $\phi(f_3) = 2$ then f_2 must be directed towards v_2 with $\phi(f_2) = 1$, a contradiction.

(vii) $E^+(C_1) = \{e_1, e_2, e_3, e_4\}$ and $E^-(C_1) = \{e_5\}$ and $\phi(g_1) = \phi(g_2) = \phi(g_3) = \phi(g_4) = 1, \phi(g_5) = 4$: The edge f_3 must be directed towards v_4, hence f_4 must be directed towards v_5, f_1 must be directed towards v_1 and $\phi(f_3) \notin \{1, 4\}$. If $\phi(f_3) = 2$ then $\phi(f_4) = 1$, a contradiction. If $\phi(f_3) = 3$ then f_2 must be directed towards v_2 with $\phi(f_2) = 1$, a contradiction.

To show that $F((P, \sigma^*)) \leq 6$ it is sufficient to find a nowhere-zero 6-flow as depicted in figure A.3.
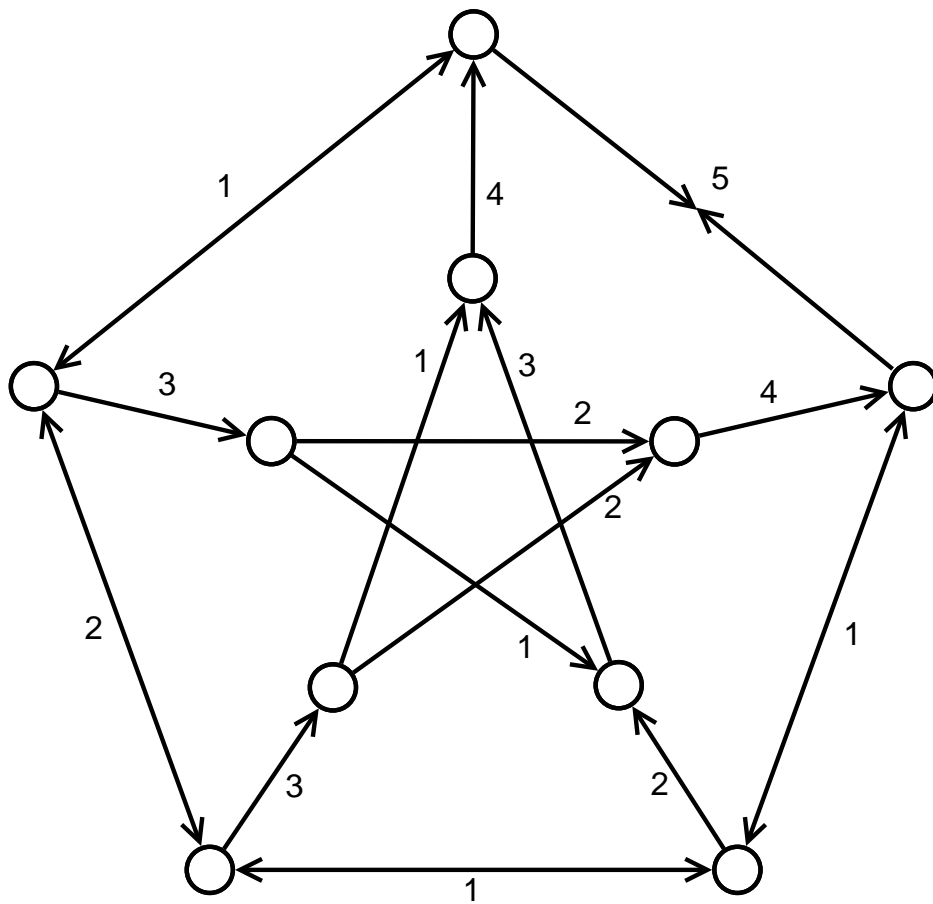
$\square$

Figure A.3: A nowhere-zero 6 flow on (P, σ^*)

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