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Daniell lattices and adapted cones

By

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Adapted cones (Mokobozky-Sibony [8], see also [2], [3], and [7]) came up in potential theory, they were introduced to study integral-representation of linear functionals: Every monotone linear functional on an adapted cone P has a representing measure. The reason for that fact is, roughly speaking, that for every $f \in P$ there is a $g \in P$ which, compared to f, increases very rapidly at the points away from compactness. Hewitt's [5] (see also [9]) representation theorem for bounded functionals on $C(X)$ can be understood by the same intuitive argument although, from the technical point of view, if differs very much from the preceding result. In order to contribute to a unified treatment of integral representation we generalize the notion of "adaptedness" and extend the integral representation theorem. This theorem then covers both results which were mentioned above. At the same time this result simplifies the proof of Hewitt's theorem. For all proofs the ingredients are of an elementary nature (Hahn-Banach, Daniell-Stone).

Lattices. Let X be a nonempty set and $E(X)$ a truncated vector lattice (with respect to pointwise operations). Recall that truncated means that $1 \wedge f \in E(X)$ for all $f \in E(X)$. For a sequence $f_n \in E(X)$ we write $f_n \downarrow 0$ if f_n is decreasing (i.e. $f_{n+1} \leqq f_n$ for all $n \in N$) and converges pointwise to zero.

$E(X)$ is said to be a *Daniell lattice* if, for every positive linear functional $\mu\colon E(X) \to \mathbb{R}$, Daniell's condition is satisfied, i.e. for every sequence $f_n \downarrow 0$ we have $\inf\limits_{n \in \mathbb{N}} \mu(f_n) = 0$. From the Daniell-Stone theorem [1] we obtain:

Every positive linear functional on a Daniell lattice has a representing measure.

Recall that a σ-additive positive measure m on X (with respect to the σ-algebra generated by $E(X)$) is said to be a representing measure for μ if

$$\mu(f) = \int_X f\,dm \quad \text{for all} \quad f \in E(X).$$

A sequence $f_n \downarrow 0$ in $E(X)$ is said to be *Dini convergent* if there is some $0 \leqq \varphi \in E(X)$ such that $\varphi^{-1} f_n$ converges uniformly to zero. Here, as usual, we put

$$0 \cdot (+\infty) = 0 \quad \text{and} \quad \varphi^{-1}(x) = +\infty \quad \text{if} \quad \varphi(x) = 0.$$

If every sequence $f_n \downarrow 0$ in $E(X)$ is Dini convergent, then $E(X)$ is called a *Dini lattice*.

Observation. *Every Dini lattice is a Daniell lattice.*

Proof. Let μ be a positive linear functional on the Dini lattice $E(X)$ and consider $f_n \downarrow 0$. Take $0 \leqq \varphi \in E(X)$ such that $\varphi^{-1} f_n$ converges uniformly to zero. Then $\delta_n \downarrow 0$, where $\delta_n = \sup\{\varphi^{-1}(x) f_n(x) \mid x \in X\}$. Since μ is positive we obtain:

$$0 \leqq \mu(f_n) \leqq \mu(\delta_n \varphi) = \delta_n \mu(\varphi) \downarrow 0. \quad \square$$

We define a sequence $f_n \downarrow 0$ to be *almost Dini convergent* if there is $0 \leqq \Psi \in E(X)$ such that, for every $\varepsilon > 0$, the sequence

$$(f_n \vee (\varepsilon \Psi)) - \varepsilon \Psi$$

is Dini convergent. The lattice $E(X)$ is said to be *adapted* if every sequence $f_n \downarrow 0$ in $E(X)$ is almost Dini convergent.

Theorem 1. *Every adapted lattice is a Daniell lattice.*

Proof. Let μ be a positive linear functional on the adapted lattice $E(X)$. Consider $f_n \downarrow 0$ in $E(X)$. Let $0 \leqq \Psi \in E(X)$ such that for every $\varepsilon > 0$ the sequence $h_{n,\varepsilon} = (f_n \vee \varepsilon\Psi) - \varepsilon\Psi$ is Dini convergent. It suffices to show

$$\inf_{n \in \mathbb{N}} \mu(f_n) \leqq \varepsilon \mu(\Psi) \quad \text{for every} \quad \varepsilon > 0.$$

Fix $\varepsilon > 0$ and take $0 \leqq \varphi \in E(X)$ such that $\varphi^{-1} h_{n,\varepsilon}$ is uniformly convergent. We define $T = f_1 \vee \varphi \vee \Psi$ and consider the subspace

$$\tilde{E} = \{g \in E(X) \mid |g| \leqq \lambda T \text{ for some } \lambda \in \mathbb{R}_+\}.$$

The functions φ, Ψ and all the f_n are in this subspace. Let $\tilde{\mu}$ be the restriction of μ to $\tilde{E}$ and consider the sublinear functionals on $\tilde{E}$ given by:

$$p_n(g) = \sup\{T(x)^{-1} g(x) \mid x \in X \text{ with } f_n(x) \leqq \varepsilon \Psi(x)\},$$
$$q_n(g) = \sup\{T(x)^{-1} g(x) \mid x \in X \text{ with } f_n(x) \geqq \varepsilon \Psi(x)\}.$$

Then $\tilde{\mu} \leqq \varrho \max(p_n, q_n)$, where $\varrho = \tilde{\mu}(T) = \mu(T)$. By König's Maximumsatz [6] (see also [4]) we can decompose $\tilde{\mu} = \nu_n + \eta_n$ into linear ν_n, η_n with $\nu_n \leqq \varrho p_n$ and $\eta_n \leqq \varrho q_n$. In particular ν_n and η_n are positive. Put $\delta_n = \sup(\varphi^{-1} h_{n,\varepsilon})$ then $\delta_n \downarrow 0$. And from

$$p_n(f_n - \varepsilon\Psi) \leqq 0 \quad \text{and} \quad q_n(f_n - \delta_n \varphi - \varepsilon \Psi) \leqq 0$$

we obtain:

$$\nu_n(f_n) \leqq \varepsilon \nu_n(\Psi),$$
$$\eta_n(f_n) \leqq \delta_n \eta_n(\varphi) + \varepsilon \eta_n(\Psi) \leqq \varepsilon \eta_n(\Psi) + \delta_n \mu(\varphi).$$

Hence we obtain the desired inequality

$$\inf_{n \in \mathbb{N}} \mu(f_n) \leqq \varepsilon \mu(\Psi) + \inf_{n \in \mathbb{N}} \delta_n \mu(\varphi) = \varepsilon \mu(\Psi). \quad \square$$

Cones. A convex subcone $F(X)$ of a Daniell lattice or an adapted lattice $E(X)$ is said to be a *Daniell cone* or an *adapted cone*, respectively, if for every $g \in E(X)$ there is some $f \in F(X)$ with $|g| \leqq f$. As a consequence of theorem 1 we have that every adapted cone is a Daniell cone.

Recall that a functional $\mu: F(X) \to \mathbb{R}$ is said to be monotone if $f \leqq g$ implies $\mu(f) \leqq \mu(g)$.

Theorem 2. *Every monotone linear functional on a Daniell cone has a representing measure.*

Proof. Consider $F(X)$ and $E(X)$ as above and let μ be a monotone linear functional on $F(X)$. From the definition above follows that

$$p(g) = \inf\{\mu(f) \mid g \leqq f \in F(X)\}$$

defines a sublinear functional on $E(X)$. Furthermore $\mu = p|_{F(X)}$. Hence [4], μ can be extended to a linear $\tilde{\mu} \leqq p$ on $E(X)$. $\tilde{\mu}$ must be monotone since $g \leqq 0$ implies $\tilde{\mu}(g) \leqq p(g) \leqq \mu(0) = 0$. $\square$

Examples.

Example 1. Let $E(X)$ be an algebra and a lattice with $1 \in E(X)$ and assume that $E(X)$ has the following properties:

i) If $g \in E(X)$ with $g(x) > 0$ $\forall x \in X$ then there is some $\varphi \in E(X)$ with $\varphi \cdot g \geqq 1$.

ii) If $\tau_n \in E(X)$ with $0 \leqq \tau_n \leqq 1$ for all $n \in \mathbb{N}$, then there are $\lambda_n > 0$ with

$$\sum_{n \in \mathbb{N}} \lambda_n < \infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} \lambda_n \tau_n \in E(X).$$

Observation. *$E(X)$ is a Daniell lattice.*

Proof. Let $f_n \downarrow 0$ and $\varepsilon > 0$. Consider the sets

$$Y_n = \left\{x \mid f_n(x) \leqq \frac{\varepsilon}{2}\right\}, \qquad Z_n = \{x \mid f_n(x) \geqq \varepsilon\}.$$

Define $\tau_n = \dfrac{2}{\varepsilon}\left(\varepsilon - \left(f_n \vee \dfrac{\varepsilon}{2}\right) \wedge \varepsilon\right)$. Then $\tau_n \in E(X)$ with $0 \leqq \tau_n \leqq 1$ and $\tau_{n|Y_n} = 1$ and $\tau_{n|Z_n} = 0$. Condition i) yields some $h \in E(X)$ with $h(f_1 + 1) \geqq 1$. Condition ii) yields $\lambda_n > 0$ with $\sum \lambda_n < \infty$ such that

$$g = \sum \lambda_n h \tau_n \in E(X).$$

Observe that $g(x) > 0$ $\forall x \in X$ since every $x \in X$ is in some Y_n. Again, by i), there is a $\varphi \in E(X)$ with $\varphi \cdot g \geqq 1$. Put $\Psi = 1$ then we obtain:

$$\begin{aligned}\varphi^{-1}((f_n \vee \varepsilon\Psi) - \varepsilon\Psi) \leqq g((f_n \vee \varepsilon) - \varepsilon) &= \sum_{k=n+1}^{\infty} \lambda_k \tau_k h((f_n \vee \varepsilon) - \varepsilon) \\ &\leqq \sum_{k=n+1}^{\infty} \lambda_k .\end{aligned}$$

Hence, f_n is almost Dini convergent. Since the sequence was arbitrarily chosen, $E(X)$ is adapted and a Daniell lattice by theorem 1. $\square$

Example 2 (Hewitt [5], see also [9]). Let $C(X)$ be the space of continuous real-valued functions on a topological space X. Then every positive linear functional $\mu\colon C(X) \to \mathbb{R}$ can be represented by a measure with respect to the σ-algebra generated by $C(X)$.

Proof. Trivially $C(X)$ fulfills the conditions considered in example 1. Hence $C(X)$ is a Daniell lattice. $\square$

Example 3. Let $E(X)$ be a truncated vector lattice. For $\emptyset \neq Y \subset X$ we denote by $E(Y)$ the space of restrictions of $f \in E(X)$ to Y. Assume that $E(X)$ has the following property: For every $0 \leqq f \in E(X)$ we can find some $0 \leqq \Psi \in E(X)$ such that for every $\varepsilon > 0$ there is a nonempty $Y \subset X$ with $f \leqq \varepsilon \Psi$ outside Y and such that $E(Y)$ is a Dini lattice. Then $E(X)$ is clearly adapted.

Example 4 (Mokobozky-Sibony [8], see also [3, p. 283]). Let X be a locally compact space and $P(X)$ a cone of continuous non-negative functions on X such that:

i) For every $x \in X$ there is some $f \in P(X)$ with $f(x) > 0$.
ii) For every $f \in P(X)$ there is some $p \in P(X)$ such that for every $\varepsilon > 0$ there is a compact set $K \subset X$ with $f \leqq \varepsilon p$ outside K.

Denote by $E(X)$ the following space

$$E(X) = \{f \in C(X) \mid \text{there is a } p \in P(X) \text{ with } |f| \leqq p\}.$$

Then $E(X)$ is a Daniell lattice. Hence $P(X)$ is a Daniell cone.

Proof. Obviously $E(X)$ is a truncated vector lattice. Take a sequence $f_n \downarrow 0$ in $E(X)$. From condition ii) and the construction of $E(X)$ it is clear that we can find a $p \in P(X)$ such that for every $\varepsilon > 0$ there is a compact $K_\varepsilon \subset X$ with $f_1 \leqq \varepsilon p$ outside K_ε. We claim that the sequence $h_n = (f_n \vee (\varepsilon p)) - \varepsilon p$ is Dini convergent. To see this, take some $\varphi_\varepsilon \in P(X)$ with $\varphi_\varepsilon(k) > 0$ for all $k \in K_\varepsilon$ (since K_ε is compact such a φ exists by i)). Now, by Dini's lemma, $\varphi^{-1} h_n$ converges uniformly. $\square$

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