Arch. Math., Vol. 49, 189–195 (1987)

0003-889X/87/4903-0189 $ 2.90/0

Maximal semigroups and controllability in products of Lie groups

By

JOACHIM HILGERT

0. Introduction. In recent years authors from various fields of mathematics were lead to the study of subsemigroups of Lie groups (cf. [13], [2], [16, 17] [14, 15], [18] etc). In [11] a first attempt has been made to give a systematic approach to the study of subsemigroups of Lie groups.The basic idea is to associate with a subsemigroup S of a Lie group G a tangent object $L(S) = \{x \in L(G): x = \lim\limits_{n \to \infty} n x_n, \exp x_n \in S, n \in \mathbb{N}\}$ (where $L(G)$ is the Lie algebra of G and $\exp: L(G) \to G$ is the exponential map) and study the properties of $L(S)$ in order to get information about S. It turns out that $L(S)$ is a wedge, i.e. that it is a closed convex set which is also closed under addition and multiplication by positive scalars (cf. [11]). Moreover it satisfies

$$(1) \qquad e^{\operatorname{ad} x} L(S) = L(S) \quad \text{for all} \quad x \in L(s) \cap -L(S)$$

where $\operatorname{ad} x(y) = [x, y]$ with the bracket in $L(G)$. If S generates G as a group the tangent wedge can also be written as $L(S) = \{x \in L(G): \exp(\mathbb{R}^+ x) \subset \bar{S}\}$ where $\bar{S}$ is the closure of S in G.

An important concept in this context is that of a maximal open subsemigroup. It plays a role in the theoretical question for which wedges W in $L(G)$ satisfying (1) one can find semigroups S in G with $L(S) = W$ as well in deciding controllability questions on G. In this paper a control system will be simply a family F of left invariant vectorfields on G and the control system will be called controllable if for any point x in G we can find an integral curve for F connecting the identity with x.

In section one we will determine the maximal open subsemigroups of Lie groups which are the product of compact groups with a nilpotent normal subgroup. In section two we apply this result to give a complete characterization of controllable systems on such Lie groups. Finally, in an appendix we recall the proof of [4] Proposition 3.1 in order to make this paper as self-contained as possible.

1. Maximal open subsemigroups. A *maximal open* (*proper*) *subsemigroup* of a topological group G is an open subsemigroup S of G such that G is the only open subsemigroup of G strictly containing S. This concept has been used in [10] and we summarize some basic facts given there:

Proposition 1.1. *Let G be a connected topological group and S be a maximal open subsemigroup of G, then:*

i) *For any normal subgroup $N \lhd G$ we have $N \cap S = \emptyset$ if and only if $NS = SN = S$.*
ii) *For any closed normal subgroup $N \lhd G$ such that $N \cap S = \emptyset$ the subsemigroup SN/N of G/N is maximal open.*
iii) *There exists a unique largest normal subgroup N_S of G such that $N_S \cap S = \emptyset$. Moreover N_S is closed.* □

We start by considering a special case which will allow us to use inductive arguments later on. The first lemma is a slight variation of a result given in [2].

Lemma 1.2. *Let V be a finite dimensional vector space and C be a compact connected subgroup of* Aut V *that operates on V without non-trivial fixed points. If S is an open subsemigroup of $G = V \rtimes C$ which intersects $V \rtimes \{\mathbb{1}\}$ then $S = G$.*

Proof. Let $(v, \mathbb{1}) \in V \rtimes C \cap S$ then $\bar{v} = \int_C c \cdot v \, d\mu$, with μ Haarmeasure on C, is a fixed point of C in V, hence $\bar{v} = 0$. Now consider the orbit $M = \{c \cdot v;\ c \in C\}$ of v under the action of C. Since C is compact, M is also compact and therefore the convex hull conv M of M is closed. Hence $0 = \bar{v} \in \operatorname{conv} M$. Thus there exist

$$c_1, \ldots, c_k \in C \quad \text{and} \quad \lambda_1, \ldots, \lambda_k \in \,]0, 1] \quad \text{with} \quad 0 = \sum_{i=1}^{k} \lambda_i c_i \cdot v.$$

Now it suffices to show that there is an $r \in \mathbb{R}^+$ such that $r \lambda_i c_i \cdot v \in S$ for all $i \in \{1, \ldots, k\}$. In fact, then we have $\sum_{i=1}^{k} r \lambda_i \cdot v$ so that $(0, \mathbb{1}) = \prod_{i=1}^{k} (r \lambda_i c_i \cdot v, \mathbb{1}) \in S$ since any neighborhood of the identity generates G as a semigroup. Note that the complement of an open subsemigroup of $(\mathbb{R}^+, +)$ is always bounded, hence for any $(x, \mathbb{1}) \in S$ there is an $r_x \in \mathbb{R}^+$ such that $(r x, \mathbb{1}) \in S$ for all $r > r_x$. Thus it only remains to show that for any $c \in C$ there is an $m_c \in \mathbb{N}$ with $(m_c c \cdot v, \mathbb{1}) \in S$, since then we can choose

$$r = (\max \{r_{(m_{c_i} c_i \cdot v)}\}) (\min \{\lambda_i\})^{-1} \prod_{i=1}^{k} m_{c_i}.$$

But the existence of m_c is shown in [2] Lemma 4 so the proof of Lemma 1.2 is complete. □

We are now ready to describe the tangent wedge of maximal open subsemigroups in semidirect products of compact groups and vector groups.

Proposition 1.3. *Let V be a finite dimensional vector space and C a compact connected group of automorphisms of V. If S is a maximal open subsemigroup of $G = V \rtimes C$ then the tangent wedge $\mathbf{L}(S)$ of S is a halfspace in $\mathbf{L}(G)$ bounded by an ideal.*

Proof. Let G be a counterexample of minimal dimension. Then for any C-invariant subspace I of V we have $(I \rtimes \{\mathbb{1}\}) \cap S \neq \emptyset$. In fact $N = I \rtimes \{\mathbb{1}\}$ is a closed normal subgroup of G hence $N \cap S = \emptyset$ implies $SN = NS = S$ by Proposition 1.1. Moreover the subsemigroup S/N is maximal open in G/N. Since G was a counterexample of minimal dimension $\mathbf{L}(S/N)$ is a halfspace bounded by an ideal in $\mathbf{L}(G/N) = \mathbf{L}(G)/\mathbf{L}(N)$. But [4] Proposition 3.1 applied to $T = \bar{S}$ implies that $\mathbf{L}(S) = \pi^{-1}(\mathbf{L}(S/N))$ where $\pi: \mathbf{L}(G) \to \mathbf{L}(G/N)$ is the canonical projection. Thus $\mathbf{L}(S)$ is a halfspace bounded by an ideal in $\mathbf{L}(G)$ contradicting our hypotheses. Now consider $G_1 = [\mathbf{L}(G), V] \rtimes C$ and note that [4] Lemma 2.1 shows that C operates without nonzero fixed points on $[\mathbf{L}(G), V]$. Moreover $S_1 = G_1 \cap S$ is an open proper semigroup that intersects $[\mathbf{L}(G), V] \rtimes \{\mathbb{1}\}$ by the above, so that Lemma 1.2 applies and we see that $(0, \mathbb{1}) \in S_1 \subset S$ whence $S = G$. This final contradiction to our assumptions proves the proposition. $\square$

More generally we obtain

Proposition 1.4. *Let G be a finite dimensional Lie group, C a connected compact subgroup of G and A an abelian analytic normal subgroup of G such that $G = C\bar{A}$. If S is a maximal open subsemigroup of G then $\mathbf{L}(S)$ is a halfspace bounded by an ideal.*

Proof. Let $\bar{A} = TV$ where T is the maximal torus in $\bar{A}$ and V is a vector group. Note that T is characteristic in $\bar{A}$ so it is normal in G. Moreover $S \cap T = \emptyset$ since otherwise $S \supseteq T$ which would imply $S = G$. Thus $ST = S$ and S/T is a maximal open subsemigroup of G/T by Proposition 1.1. But $G/T = (C\bar{A})/T \cong ((CT)/T)(\bar{A}/T)$ with $CT/T \cong C/_{C \cap T}$ and $\bar{A}/T \cong V$. Therefore $(CT/T) \cap (\bar{A}/T) = \{T\}$ and we may apply Proposition 1.3 to G/T and S/T. Thus $\mathbf{L}(S/T)$ is a halfspace bounded by an ideal and as before [4], Proposition 3.1 applied to $T = \overline{S/T}$ shows that $\mathbf{L}(S)$ is a halfspace bounded by an ideal. $\square$

Finally we obtain

Theorem 1.5. *Let G be a connected finite dimensional Lie group, C a compact subgroup of G and N a nilpotent analytic normal subgroup of G such that $G = C\bar{N}$. If S is a maximal open subsemigroup of G then $\mathbf{L}(S)$ is a halfspace bounded by an ideal in $\mathbf{L}(G)$.*

Proof. Note first that we may assume that C is connected. In fact if C_m is a maximal compact subgroup of G containing C then C_m is connected and $G = C_m\bar{N}$. Thus we can replace C by C_m. Moreover we may assume that N is closed.

Now consider the commutatorgroup N' of N. If $S \cap N' \neq \emptyset$ then $S \cap N'$ is contained in some maximal open subsemigroup S_N of N (cf. [10]). But then by [10] we have that $\mathbf{L}(S_N)$ is a halfspace bounded by an ideal which must then contain $\mathbf{L}(N') = [\mathbf{L}(N), \mathbf{L}(N)]$. Hence $\exp \mathbf{L}(N') \subset \bar{S}$ so that $N' \subset \bar{S}$. Therefore $SN' = N'S = S$ since $\bar{S}S, S\bar{S} \subset S$ and the identity cannot be in S. Thus Proposition 1.1. shows that $S \cap N' = \emptyset$. Since S is open we have also $H \cap S = \emptyset$ where H is the closure of N' in G. Again we apply Proposition 1.1. to see that $SH/H = S/H$ is a maximal open subsemigroup of G/H. Note that $G/H \cong (CH/H)(\bar{N}/H)$ where CH/H is compact and $\bar{N}/H$ is abelian by [12] Theorem 2.1 in Chap. XVI. Hence Proposition 1.4 shows that $\mathbf{L}(S/H)$ is a halfspace bounded by an ideal and consequently $\mathbf{L}(S)$ is a halfspace bounded by an ideal in $\mathbf{L}(G)$. $\square$

It now only remains to translate the information we have on the tangent wedges of maximal open semigroups into information on the semigroups themselves:

Remark 1.6. Let G be a connected Lie group then for a non empty open subsemigroup S of G the following statements are equivalent:

(1) $L(S)$ is a halfspace bounded by an ideal.
(2) $S = \varphi^{-1}(\mathbb{R}^+\backslash\{0\})$ for some continuous homomorphism $\varphi: G \to \mathbb{R}$.

These properties imply that S is maximal open.

Proof. (1) $\Rightarrow$ (2) Let I be the ideal that bounds $L(S)$ and N be the subgroup generated by $\exp I$ where $\exp: L(G) \to G$ is the exponential function. Then $N \cap S = \emptyset$ hence $\bar{N} \cap S = \emptyset$ so that $\dim N \leqq \dim \bar{N} < \dim S$ and thus N is closed. Moreover Proposition 1.1 implies $SN = NS = S$ or, in other words, $S = \pi^{-1}(SN/N)$ where $\pi: G \to G/N$ is the quotient map. Therefore $G/N \cong \mathbb{R}$ since there are no proper open subsemigroups in the torus. Moreover [4] Proposition 3.1 shows that $L(S) = (L(\pi))^{-1} L(S/N)$. Thus $L(S/N)$ is a halfline and the claim is proved.

(2) $\Rightarrow$ (1) Conversely since $S \neq \emptyset$ the map φ is a quotient map and hence [4] Proposition 3.1 implies that $L(S)$ is a halfspace bounded by $L(\ker \varphi)$. Note finally that any open subsemigroup T, which contains $S = \varphi^{-1}(\mathbb{R}^+\backslash\{0\})$ strictly, must satisfy $\varphi(T) = \mathbb{R}$ since any open subsemigroup of $\mathbb{R}$ containing positive and negative elements must be all of $\mathbb{R}$. Since G is connected and $G/\ker \varphi \cong \mathbb{R}$ is simply connected we conclude that $\ker \varphi$ is connected and contained in $\bar{S}$ hence in $\bar{T}$. Thus $\bar{T} = \varphi^{-1}(\varphi(\bar{T})) = G$ and T is dense open in G. But then T^{-1} is open dense and also $T \cap T^{-1}$ is open dense, hence $T \cap T^{-1} = G$ since $T \cap T^{-1}$ is a group. Thus $T = G$. □

2. Controllability of systems in $G = C\bar{N}$. It has been pointed out in the introduction that a control system on a Lie group G can be viewed as a family of left invariant vectorfields on G, hence as a subset F of $L(G)$. It is well known (cf. [13]) that the system described by F is controllable if and only if the system given by the wedge $W = \overline{\mathrm{conv}}(\mathbb{R}^+ F)$ is controllable. Therefore the question of controllability reduces to the question whether the semigroup S_w generated by $\exp W$ in G is all of G or not (cf. also [2]). Certainly S_w is contained in the analytic group generated by $W - W$ so that a necessary condition for controllability is that the smallest Lie algebra L_w containing W is all of L. Conversely if $L_w = L$ it has been noted by various authors (cf. [11]) that the interior $\mathring{S}_w$ of S_w is dense in S_w, hence $\mathring{S}_w$ is contained in a maximal open subsemigroup S of G unless $S_w = G$. We find

Remark 2.1. Let G be a connected Lie group such that for any maximal open subsemigroup S we have $L(S)$ is a halfspace bounded by an ideal in $L(G)$. If W is a wedge in $L(G)$ which generates $L(G)$ as a Lie algebra, then the following statements are equivalent:

(1) The semigroup S_w generated by $\exp W$ in G is not equal to G (hence not even dense).
(2) There exists a continuous non trivial homomorphism $\varphi: G \to \mathbb{R}$ such that $L(\varphi)(W) \in \mathbb{R}^+$.

Proof. (1) $\Rightarrow$ (2) If $S_w \neq G$ then by the above $\mathring{S}_w$ is contained in a maximal open subsemigroup S of G. By hypothesis $L(S)$ is a halfspace bounded by an ideal so that Remark 1.6 shows the existence of $\varphi: G \to \mathbb{R}$ such that $S_w \subset S = \varphi^{-1}(\mathbb{R}^+ \setminus \{0\})$ and hence $W \subset L(S_w) \subset L(S) = (L(\varphi))^{-1}(\mathbb{R}^+)$ by [4] Proposition 3.1.

(2) $\Rightarrow$ (1) Conversely let $S := \varphi^{-1}(\mathbb{R}^+ \setminus \{0\})$ then again $L(S) = (L(\varphi))^{-1}(\mathbb{R}^+)$ and therefore $W \subset L(S)$. Thus $\exp W \subset \exp L(S) \subset \bar{S}$ which is a subsemigroup of G, strictly contained in G. □

Recall that $L(\varphi): L(G) \to \mathbb{R}$ is a Lie algebra morphism and since $\mathbb{R}$ is abelian we know that $L(G)' = [L(G), L(G)]$ is contained in the kernel of $L(\varphi)$. Hence the relative interior $\operatorname{int}_{w-w} W$ in the vector space $W - W$ cannot intersect $L(G)'$ unless $L(\varphi)(W)$ contains positive and negative values or is completely contained in $L(G)'$. Moreover if C is a maximal compact subgroup of G then $C \subset \ker \varphi$ since $\varphi(C)$ is a compact subgroup of $\mathbb{R}$. Thus in the situation of Remark 2.1 we have $L(C) + L(G)' \subset \ker L(\varphi)$. Even more is true:

Lemma 2.2. *Let G be a connected Lie group, C a maximal compact subgroup of G and W a wedge in $L(G)$ which generates $L(G)$ as a Lie algebra. Then the following statements are equivalent*

(1) $\operatorname{int}_{w-w} W \cap (L(C) + L(G)') = \emptyset$.
(2) *There exists a continuous non trivial homomorphism $\varphi: G \to \mathbb{R}$ such that $L(\varphi)(W) \subset \mathbb{R}^+$.*

Proof. (2) $\Rightarrow$ (1) We know that $L(C) + L(G)' \subset \ker L(\varphi)$. Moreover W is not contained in $\ker L(\varphi)$ since it generates $L(G)$.

If now $\operatorname{int}_{w-w} W \cap \ker L(\varphi) = \emptyset$, then $L(\varphi)(W)$ contains positive and negative values contradicting our hypothesis. Thus $\operatorname{int}_w W \cap (L(C) + L(G)') = \emptyset$.

(1) $\Rightarrow$ (2) To show the converse note first that the analytic subgroup A of G with $L(A) = L(C) + L(G)'$ is normal and contains C since G, hence C, is connected. Therefore A contains all compact subgroups of G and hence A is closed (cf. [12] Theorem 2.4 Chap. XVI and Proposition 2.3 Chap. XVI). Moreover the quotient group G/A is a vectorgroup. In fact, since G/A is abelian connected it is isomorphic to $T \times V$ where T is a torus and V is a vectorgroup. If $\pi: G \to T \times V$ is the quotient map with kernel A and B is the identity component of $\pi^{-1}(V)$ then B is a closed connected normal subgroup of G and $\pi(B) = V$ since π was a quotient map. Thus G/B is compact and [12] Theorem 2.3 Chap. III implies that $CB = G$ so that $G = CB \subset AB \subset B \subset G$. But this just means that $T = \{0\}$. Thus we may identify G/A with $L(G/A) = L(G)/L(A)$.

But now by condition (1) the geometric version of the Hahn-Banach Theorem implies that W is contained in a halfspace H with $L(A) \subset H$ so that we can find a linear functional $\bar{\varphi}: L(G)/L(A) \to \mathbb{R}$ with $H = \ker \bar{\varphi}$ and $\bar{\varphi}(W + L(A)/L(A)) \subset \mathbb{R}^+$. But then $\varphi = \bar{\varphi} \circ \pi: G \to \mathbb{R}$ is the desired homomorphism if we identify G/A and $L(G)/L(A)$. □

We can now summarize our results to

Theorem 2.3. *Let G be a connected Lie group such that for any maximal open semigroup S we have that $\mathbf{L}(S)$ is a halfspace bounded by an ideal in $L(G)$. Moreover let C be any maximal compact subgroup of G. If W is a wedge in $L(G)$ which generates $L(G)$ as a Lie algebra, then the following statements are equivalent:*

(1) $\operatorname{Int}_{W-W}(W) \cap (L(C) + [L(G), L(G)]) \neq \emptyset$.
(2) $\exp W$ *generates G as a semigroup.*
(3) *The system described by W is controllable.*

Appendix.

Proposition 3.1. ([4]) *Let G and H be Lie groups and $q\colon G \to H$ a quotient map. If S is a subsemigroup of G generating G as a group, then $L(q)(\mathbf{L}(S)) \subset \mathbf{L}(q(S))$ where $L(q)\colon L(G) \to L(H)$ is the morphism associated with q. The converse need not be true. If T is a subsemigroup of H generating H as a group and containing the identity then $\mathbf{L}(q^{-1}(T)) = (L(q))^{-1}\mathbf{L}(T)$.*

Proof. Note first that we may assume that S is closed since $q(\bar{S}) \subset q(S)^-$ so that $L(q)(\mathbf{L}(\bar{S})) \subset \mathbf{L}(q(\bar{S}))$ implies $L(q)\mathbf{L}(S) = L(q)(\mathbf{L}(\bar{S})) \subset \mathbf{L}(q(\bar{S})) \subset \mathbf{L}(S)^- = \mathbf{L}(q(S))$. If now $\exp_H\colon L(H) \to H$ and $\exp_G\colon L(G) \to G$ are the respective exponential functions then $\exp_G \mathbb{R}^+ x \subset S$ implies $\exp_H \mathbb{R}^+ L(q)(x) = q(\exp_G \mathbb{R}^+ x) \subset q(S) \subset q(S)^-$, hence by $\mathbf{L}(S) = \{x \in L(G) \colon \exp \mathbb{R}^+ x \subset \bar{S}\}$ we obtain $x \in \mathbf{L}(q(S))$. To see that the converse is not true consider an icecream cone W in $\mathbb{R}^3$ and factor a discrete subgroup of a line whose intersection with W is a halfline in the boundary of W; then W is a semigroup with $\mathbf{L}(W) = W$, whereas the quotient semigroup has a halfspace as tangent wedge.

To see the last statement note first that $q^{-1}(T)$ generates G as a group since T generates H and $\ker q \subset q^{-1}(T)$ so that $\mathbf{L}(q^{-1}(T))$ makes sense. Moreover $q(q^{-1}(T)) = T$ so that the inclusion $\mathbf{L}(q^{-1}(T)) \subset L(q)^{-1}\mathbf{L}(T)$ follows from the first part. Conversely if $x \in L(q)^{-1}(\mathbf{L}(T))$ then $\exp_H \mathbb{R}^+ L(q)x \subset \bar{T}$ so that $\exp_G \mathbb{R}^+ x \subset q^{-1}(\bar{T})$. But since H is metrizable [1] (Cap. IX, § 2, Prop. 1.8) implies that $q^{-1}(\bar{T}) \subset (q^{-1}(T))^-$, since any Cauchy sequence in $\bar{T}$ can be lifted to a Cauchy sequence in $q^{-1}(\bar{T})$. In fact, for any $s \in q^{-1}(\bar{T})$ we find a sequence h_n in T converging to $q(s)$ and hence a sequence $s_n \in q^{-1}(h_n) \subseteq q^{-1}(T)$ converging to s, i.e. $s \in (q^{-1}(T))^-$. Thus $\exp_G \mathbb{R}^+ x \subseteq (q^{-1}(T))^-$ and hence, $x \in \mathbf{L}(q^{-1}(T))$. □

References

[1] N. BOURBAKI, Topologie génerale. Paris 1958.
[2] B. BONNARD, V. JURDJEVIC, I. KUPKA and G. SALLET, Transitivity of families of invariant vectorfields on the semidirect product of Lie groups. Trans. Amer. Math Soc. **271**, 525–535 (1982).
[3] J. GAUTHIER, I. KUPKA and G. SALLET, Controllability of right invariant systems on real simple Lie groups. Preprint Institut Fourier 1984.
[4] J. HILGERT, Infinitesimally generated semigroups in motion groups. Preprint THD 1986.
[5] J. HILGERT and K. H. HOFMANN, Lie semialgebras are real phenomena. Math. Ann. **270**, 97–103 (1985).
[6] J. HILGERT and K. H. HOFMANN, Semigroups in Lie groups, semialgebras in Lie algebras. Trans. Amer. Math. Soc. **288**, 481–503 (1985).

[7] J. Hilgert and K. H. Hofmann, Old and new on SL(2). Manuscripta Math. **54**, 17–52 (1985).
[8] J. Hilgert and K. H. Hofmann, On the automorphism group of cones and wedges. Geom. Dedicata **21**, 205–217 (1986).
[9] J. Hilgert and K. H. Hofmann, On Sophus Lie's Fundamental Theorems. J. Funct. Anal. **67**, 293–319 (1986).
[10] J. Hilgert, K. H. Hofmann and J. Lawson, Controllability of systems on a nilpotent Lie group. Beiträge Algebra Geom. **20**, 185–190 (1985).
[11] K. H. Hofmann and J. Lawson, Foundations of Lie semigroups. LNM **998**, 128-201, Berlin-Heidelberg-New York 1983.
[12] G. Hochschild, The structure of Lie groups. San Francisco 1965.
[13] V. Jurdjevic and H. Sussmann, Control systems on Lie groups. J. Diff. Equations **12**, 313–329 (1972).
[14] G. Ol'shanskií, Convex cones in symmetric Lie algebras, Lie semigroups, and invariant causal (order) structures on pseudo-Riemannian symmetric spaces. Dokl. Sov. Math. **26**, 97–101 (1982).
[15] G. Ol'shanskií, Invariant cones in symmetric Lie algebras, Lie semigroups, and the holomorphic discrete series. Functional Anal. Appl. **15**, 275–285 (1981).
[16] S. Paneitz, Invariant convex cones and causality in semisimple Lie algebras and groups. J. Funct. Anal. **43**, 313–359 (1981).
[17] S. Paneitz, Classification of invariant convex cones in simple Lie algebras. Ark. Mat. **21**, 217–228 (1984).
[18] L. Rothkrantz, Transformatiehalfgroepen van niet-compacte hermitesche symmetrische Ruimten. Dissertation Univ. Amsterdam 1984.

Eingegangen am 4. 3. 1986

Anschrift des Autors:
Joachim Hilgert
Fachbereich Mathematik
Technische Hochschule Darmstadt
Schloßgartenstr. 7
D-6100 Darmstadt