A Modified Trellis Coding Technique for Partial Response Channels

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Abstract— **The problem of trellis coding for multilevel baseband transmission over partial response channels with transfer polynomials of the form $(1 \pm D^N)$ is addressed. The novel method presented here accounts for the channel memory by using multidimensional signal sets and partitioning the signal set present at the noiseless channel output. It is shown that this coding technique can be viewed as a generalization of a well-known procedure for binary signaling, the concatenation of convolutional codes and inner block codes that are tuned to the channel polynomial. It results in high coding gains with moderate complexity if some bandwidth expansion is accepted.**

I. Introduction

TRELLIS coding techniques which increase the reliability of data transmission without increasing bandwidth requirements have been pioneered by Ungerboeck [1]. The basic idea is that by trellis coding onto an expanded modulation set (relative to that needed for uncoded transmission) and by designing the trellis codes to maximize the minimum free Euclidian distance between allowable code sequences, asymptotic (high signal-to-noise ratio) coding gains of 3–6 dB compared to an uncoded system can be achieved without bandwidth expansion. Spurred by the impressive performance on spectrally flat channels efforts have been made to apply trellis coding also to partial response (PR) channels, i.e., channels with a controlled amount of intersymbol interference [2]–[10].

Ungerboeck and Wolf [2] consider binary signaling. Through the use of a precoder which causes the channel to resemble a spectrally flat channel conventional convolutional codes can be applied. Karabed and Siegel [3] showed that the concatenation of convolutional codes and simple inner block codes that are tuned to the channel polynomial can yield large gains. Using the notion of matched spectral null codes they developed codes that achieve the same performance as codes of Ungerboeck and Wolf (in terms of rate, coding gain) with much smaller decoder complexity [4]. They proved that if the spectral null of the code spectrum matches the spectral null of the channel then the memory of the channel enhances the performance of the code.

Multilevel signaling on PR channels has also been considered [5]–[10]. Ketchum applies the precoding approach of Ungerboeck and Wolf to multilevel signaling [5]. His conclusion is that trellis coding on PR channels suffers a loss in performance relative to trellis coding on memoryless channels which typically approaches 3 dB. Further the extra memory introduced by the channel requires additional complexity for the maximum likelihood sequence detection. Ketchum uses a conventional precoder while Forney and Calderbank [6] attach a preprocessor to the channel which reduces to a conventional precoder when the power constraint is at the channel input.

Other approaches to trellis coding on partial response channels subdivide the channel and use ISI-free signaling on the subchannels. Cioffi and Ruiz [7] use frequency division. Kasturia *et al.* [8] divide the channel into a set of parallel independent channels using eigenvectors of CC^t where C specifies the pulse response of the channel over a finite input block. The achievable coding gains are larger than in the other approaches described. However this method requires additional complexity since a pre- and a post processor have to be attached to the channel.

Matched spectral null codes for a multilevel symbol alphabet have been studied by Eleftheriou and Cideciyan [9] and by the author [10]. Besides other sometimes desirable features (e.g., absence of a dc component) they offer increased Euclidian distance (coding gain) and code rates as close to the uncoded M-level system as desired. These codes seem to be, however, well suited only for applications where for some reason an extension of the channel input alphabet via coding is not desired or possible. The reason for this is that the coding gain is usually not large enough to compensate for the rate loss. No four-level code of rate 1 with a coding gain > 0 dB versus uncoded binary transmission could be found (neither in [9] nor in [10]). If an extension of the channel input alphabet is possible and if the line code properties of the matched spectral null codes are not required then trellis coding by set partitioning is therefore more attractive.

In this paper we apply the set partitioning idea to the signal set present at the noiseless channel output. After introducing the channel model in Section II we briefly review trellis coding incorporating a precoder. In Section III we present our new approach which avoids the use of a precoder. We use multidimensional trellis coding where the edge labels of the trellis are obtained by partitioning the set of noiseless channel output signals. It is shown that this technique may be regarded as a generalization of concatenating outer convolutional codes and inner block codes, which has been explored in the binary case by Karabed and Siegel [3], to multilevel signals. The best results are obtained for codes of

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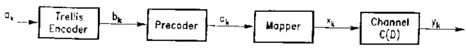

Fig. 1. Precoded partial response system with trellis coding.

rate $[(d-1)/d]\log_2(M/2)$ bits per M-ary symbol with some integer $d \geq 2$.

II. Channel Model and Trellis Coding with Precoder

The term partial response channel is used to denote channels that exhibit a certain amount of controlled intersymbol interference (ISI). This is practically accomplished by an equalizer at the receiver front end which equalizes the channel impulse response to a predetermined shape which gives rise to a known amount of ISI. Such a configuration is described by the polynomial $C(D)$, the D-transform of the discrete-time impulse response of the combination of channel and equalizer.

In this paper we will confine ourselves to the "dicode" $(1-D)$ partial response channel, i.e.,

$$C(D) = (1-D)/\sqrt{2}. \tag{1}$$

The same codes that will be developed for the $(1-D)$ channel can be used on the $(1+D)$ channel if the coded sequence is multiplied by $(-1)^k$ (k: time index). The codes can also be used on channels of the form $(1 \pm D^N)$ since it is well known that these channels can be regarded as N interleaved $(1 \pm D)$ channels with independent coding/decoding on each of the interleaved channels. We assume baseband transmission with an M-ary signal alphabet (M being a power of 2). The channel input sequence $\{x_k\}$ assumes values from the set $x_k \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$. The $(1-D)$ channel produces at its output the noiseless sequence $\{y_k\}$ and the noisy sequence $\{r_k\}$ where

$$\sqrt{2}y_k = x_k - x_{k-1} \tag{2}$$

$$r_k = y_k + n_k. \tag{3}$$

The n_k denote independent Gaussian noise samples with zero mean and variance σ^2.

Fig. 1 shows the block diagram of a precoded partial response trellis coding system. $\{a_k\}$ is the binary input sequence where each element is a vector of m bits, i.e., $a_k = (a_k^m, a_k^{m-1}, \cdots, a_k^1)$. $\{b_k\}$ is the trellis encoded sequence where $b_k = (b_k^m, b_k^{m-1}, \cdots, b_k^0)$. The binary encoder has therefore the rate $m/(m+1)$. Instead of the binary vector representation b_k is interchangeably considered to be an integer in the range $[0, 2^{m+1}-1]$.

For the $(1-D)$ channel under consideration the corresponding precoder performs the operation

$$c_k = (b_k + c_{k-1}) \bmod M. \tag{4}$$

Thus, c_k is the integer in the range $[0, 2^{m+1}-1]$ that is congruent modulo M to $b_k + c_{k-1}$. The task of the precoder is to avoid quasicatastrophic error events and to make the precoded channel resemble a spectrally flat channel such that conventional coding techniques for memoryless channels can be applied [2]. Subsequently c_k is mapped into a M-ary symbol x_k where $M = 2^{m+1}$. The information rate is therefore $m = \log_2(M/2)$ bits/symbol.

Using polynomial notation, let $b(D)$ and $b'(D) = b(D) \oplus e(D)$ be two similar code sequences where $\oplus$ denotes modulo-2 addition. And let $e(D)$ describe an error event of length $L+1$, i.e.,

$$e(D) = e_k D^k + \cdots + e_{k+L} D^{k+L}; \quad e_k, e_{k+L} \neq 0$$
$$e_{k-1} = e_{k+L+1} = 0; \quad L \geq 0. \tag{5}$$

In order to lowerbound the Euclidian distance (ED) between the channel symbol sequences $x(D)$ and $x'(D)$ obtained from $b(D)$ and $b'(D)$ [absence of precoder] Ungerboeck defined the Euclidian weight [1]

$$w(e_i) = \min(d(x(b_i), x(b_i \oplus e_i))) \tag{6}$$

where the minimization goes over all possible b_i and $d(\cdots)$ is the Euclidian distance between the channel symbols specified. Ketchum [5] showed that in the presence of a precoder the squared ED associated with any error event at the channel output is lowerbounded by

$$d^2(y, y') = \sum_i |y_i - y'_i|^2 \geq \frac{1}{2} \sum_{i=k}^{k-L} w^2(e_i). \tag{7}$$

In particular, (7) holds also for the minimum distance error event. For a spectrally flat channel a 1 would occur instead of 1/2 on the right-hand side of (7), [1]. From this observation Ketchum concluded that coding on PR channels delivers typically 3 dB less coding gain than on spectrally flat channels.

Let $q(e_i)$ be the number of trailing zeros in the binary representation of e_i. Ungerboeck showed that from the mapping by set partitioning it follows that $w(e_i) \geq \Delta_{q(e_i)}$; $\Delta_{q(e_i)}$ results from the set partitioning procedure [1]. It is the minimum intraset distance after $q(e_i)$ partitioning levels, see later examples. Therefore the squared free ED can be lowerbounded as follows:

$$d_{\text{free}}^2 \geq \min \frac{1}{2} \sum_{i=k}^{k+L} \Delta_{q(e_i)}^2. \tag{8}$$

This bounding technique will be used in the subsequent sections.

III. Trellis Coding without Precoder

A. A Motivating Example

Here the aim is to adapt the code to the channel polynomial rather than to eliminate the channel memory by the use of a precoder. In the following we assume therefore absence of a precoder. A means of adapting the code to the channel is the use of a concatenated coding scheme, which has been studied for the binary case, i.e., $x_k \in \{1, -1\}$, by Siegel and Karabed [3]. The codes comprise convolutional outer codes

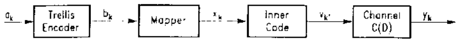

Fig. 2. Partial response trellis coding with inner block code.

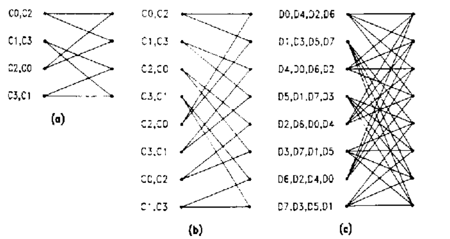

Fig. 3. Sample trellis codes used in Section III (from [1]). (a) and (b): Codes for one-dimensional signal constellation. (a) and (c): Codes for two-dimensional signal constellation.

TABLE I
PERFORMANCE OF 4-(ENCODER)-STATE TRELLIS CODE WITH INNER CODES ON $(1-D)$ CHANNEL

	IC1	IC2
Rate (bits/symbols)	1/2	2/3
d^2_{free}	92	28
Gain versus uncod. 2-AM	(92/5)/4	(28/5)/4
Gain in dB	6.6	1.5

and simple nonlinear block inner codes. The simplest inner codes proposed are

$$IC1 : x_1 \rightarrow x_1\bar{x}_1 = \nu_1\nu_2 \tag{9}$$

$$IC2 : x_1x_2 \rightarrow x_1x_2\bar{x}_2 = \nu_1\nu_2\nu_3. \tag{10}$$

where $\bar{x} = -x$. IC1 is known as biphase code [3]. The motivation for these simple codes is the observation that the squared ED at the channel output can be written as

$$\begin{aligned} d^2(y, y\prime) &= \sum_i (y_i - y_i')^2 \\ &= \frac{1}{2}\sum_i \left((\nu_i - \nu_{i-1}) - (\nu_i' - \nu_{i-1}')\right)^2 \\ &= d^2(\nu, \nu') - \sum_i (\nu_i - \nu_i')(\nu_{i-1} - \nu_{i-1}'). \end{aligned} \tag{11}$$

The second term on the right-hand side of the last equality has the desired sign if successive symbols ν_{i-1}, ν_i have opposite sign and that is just what the inner codes try to achieve. It is interesting to note that the spectral density of the block codes is minimum where the channel transfer function is minimum, i.e., at dc.

Next, the performance of these inner codes is studied in the multilevel case. Fig. 2 shows the block diagram of the system. Using Ungerboeck's 4-state 4-AM code [1], see Fig. 3(a) as the outer code gives the results summarized in Table I.

Here gain means the asymptotic (high signal-to-noise ratio) coding gain which is determined by the minimum free Euclidian distance between allowed noisefree sequences at the channel output. The average power of the coded systems is $5/T$ as opposed to $1/T$ in the uncoded case. Note that due to the inner code the information rate of the coded system is smaller than in the uncoded case. Further note that the optimal decoder requires 8 states because of the channel memory.

The baseline system against which the coded systems are compared consists of a binary signal source, a precoder, the $(1 - D)$ channel, and a two-state Viterbi decoder. The precoder is used to avoid quasicatastrophic error events, and the Viterbi decoder is the maximum likelihood sequence detector for this configuration. The squared minimum free ED of this system is 4.

It might be surprising that the gain of the trellis code with IC2 is so much smaller than with IC1. The reason for this is that there is no fixed alignment between the state transitions of the trellis code and the block boundaries of the inner code IC2. Consider the minimum distance error event between the two sequences $\{x\}$ and $\{x\prime\}$:

$$\{x\} = \{\cdots, x_{k-1}, x_k, -1, -3, 1, x_{k+1}, \cdots\}$$
$$\{x'\} = \{\cdots, x_{k-1}, x_k, 3, -1, -3, x_{k+4}, \cdots\}$$

There are two possible alignments of the inner code IC2 with the trellis coded sequence: either the negative of x_k is inserted or the negative of x_{k+1}. The first choice delivers a squared ED between $\{x\}$ and $\{x'\}$ of 28 whereas the second alignment yields a squared ED of 116! In the case of IC1 such an ambiguity is not present: the combination of trellis code and inner code can be well described by an overall trellis diagram whose edges are labeled with two-dimensional signals (two consecutive symbols per branch).

Fig. 4 shows the trellis diagram obtained by combining the four-state 4-AM trellis code of Fig. 3(a) with the biphase inner code. Four branch labels are used: $(-3, 3)$, $(-1, 1)$, $(1, -1)$ and $(3, -3)$. Transitions originating from the same state or joining in the same state are labeled either with a signal of the subset $\{(-3, 3), (1, -1)\}$ or $\{(3, -3), (-1, 1)\}$ which corresponds to the set of differences between consecutive symbols $\sqrt{2}y_k \in B0 = \{-6, 2\}$ or $\sqrt{2}y_k \in B1 = \{6, -2\}$. This shows that the same labeling of the trellis can be obtained by successive two-way partitioning of the set of possible differences between successive symbols from the four-level symbol set such that the intraset distance increases with every partitioning step, see Fig. 5

$$\begin{aligned} \Delta_0 &= 2 \\ \Delta_1(1) &= 2\Delta_0 \\ \Delta_2(1) &= 2\Delta_1(1). \end{aligned} \tag{12}$$

The argument "1" denotes the dimensionality of the signal set to be partitioned. Note that the dimensionality d of the channel input signals is 2, whereas the dimensionality at the channel output is considered to be 1 (there is only one difference between the two symbols associated with a trellis edge). The minimum distance Δ_0 of the total signal set is always normalized to be 2.

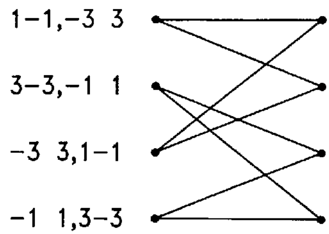

Fig. 4. Representation of concatenation of four-state four-level code and inner block code IC1 as two-dimensional trellis code.

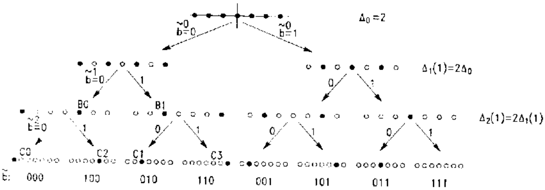

Fig. 5. Set partitioning of one-dimensional channel output signal set when channel input signal consists of two successive four-level symbols ($d = 2$).

The performance of the codes can be lowerbounded by using (8). Let $\tilde{b}_k$ denote the binary representation of the channel output signal y_k. $\tilde{b}_k$ results from the set partitioning exemplified in Fig. 5. Using polynomial notation, let $\tilde{b}(D)$ and $\tilde{b}'(D) = \tilde{b}(D) \oplus \tilde{e}(D)$ be two similar code sequences (at the channel output). Let $q(\tilde{e}_i)$ be the number of trailing zeros in the binary representation of the error polynomial $\tilde{e}(D)$. Using the code trellis of Fig. 3(a) with the above set partitioning (Fig. 5) the minimum squared free ED is lowerbounded by

$$\begin{aligned} d_{\text{free}}^2 &\geq \frac{1}{2}\left[\Delta_2^2(1) + \Delta_1^2(1) + \Delta_2^2(1)\right] \\ &= \frac{1}{2}\left[(4\Delta_0)^2 + (2\Delta_0)^2 + (4\Delta_0)^2\right] = 72. \end{aligned} \tag{13}$$

This gives a lower bound for the asymptotic coding gain over uncoded binary signaling of

$$\left(\frac{d_{\text{free}}^2}{P_{\text{ave}}}\right)_{\text{cod}} \left(\frac{P_{\text{ave}}}{d_{\text{free}}^2}\right)_{\text{uncod}} \geq \frac{72}{5/T}\,\frac{1/T}{4} = 3.6 \tag{14}$$

which corresponds to 5.6 dB (the actual asymptotic coding gain is 6.6 dB, see Table I). The decoder requires 8 states, the rate is 1/2 bit/symbol. Using the code trellis of Fig. 3(b) gives 7.3 dB asymptotic coding gain (16 state decoder required). Note that the coded system does not exhibit quasicatastrophic error events since the sets of differences are disjoint.

B. Trellis Codes Based on Channel Output Set Partitioning

The example of the last section showed that the use of good inner block codes can be viewed as labeling the trellis with multidimensional signal points where the edge labels are obtained by partitioning the set of noiseless channel output signals. Then the edge labels of the trellis diagram which consist of two or more consecutive symbols have the task of the inner codes: adapt the "outer" trellis codes to the PR polynomial.

The code construction can be generalized to higher dimensions and arbitrary M. Coset codes [11] for a PR $(1 - D)$ channel can be found by partition the $(d - 1)$-dimensional regular signal point constellation present at the channel output when a d-dimensional regular lattice is present at the channel input. The region R which contains the finite set of channel output signal points whose corresponding channel input signal points are used in the transmitter, has to be chosen such that the average transmitted power is minimum. This design criterion may be traded off against the decoder complexity, see later examples. Generally set partitioning of the channel output signal set is harder than set partitioning of the channel input signal set. For the examples considered here R was found by hand or by computer search. A systematic way that is applicable to arbitrary dimensions could not be found.

It is well known that optimum convolutional codes are found by using the sequence of intraset distances obtained from the set partitioning [see, e.g., (12)] in a code search program [1]. The same codes that have been previously found for $(d - 1)$-dimensional constellations on memoryless channels are therefore also suited here when the channel output signal set has dimension $(d - 1)$.

Fig. 6 shows the set partitioning for a four-level signal in the case of $d = 3$. The edges of the trellis diagram at the channel input are labeled with three-dimensional signals (3 successive symbols). This corresponds to a two-dimensional channel output signal because there are two differences between the three consecutive input symbols associated with one edge.

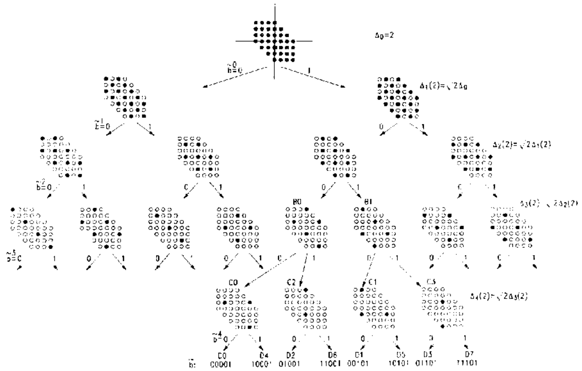

Fig. 6. Set partitioning of two-dimensional channel output signal set when channel input signal consists of three successive four-level symbols ($d = 3$).

TABLE II
SIGNAL SET FOR RATE 2/3 CODE

	$D0$	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$	$D7$
2-dim. ch. output sign.	$-4, 2$	$-4, -2$	$0, 6$	$0, 2$	$4, 2$	$1, -2$	$0, -2$	$0, -6$
3-dim. ch. input sign.	$1, -3, -1$	$3, -1, -3$	$-3, -3, 3$	$-1, -1, 1$	$-3, 1, 3$	$-1, 3, 1$	$1, 1, -1$	$3, 3, -3$

The difference between the second and the first channel input symbols is considered to be the first dimension and the difference between the third and the second symbol is considered to be the second dimension of the channel output signal. The $M^d = 4^3$ possible input signals transform into 37 possible channel output signals. Fig. 6 shows the successive two-way set partitioning applied to these 37 channel output signals. At the third level of set partitioning there still exist two subsets with enough (i.e., $2^{d-1} = 4$) signal points to support a rate $(d-1)/d = 2/3$ bit/symbol code. From Fig. 6 we choose the signals $D0 \cdots D7$ as shown in Table II.

Since set partitioning has been performed on the two-dimensional channel output signal set a trellis code suited for two-dimensional signal constellations has to be used, e.g., the codes of Fig. 3(a) and (c). Note that the dth symbols of the d-dimensional signal points that are assigned to transitions joining into the same encoder state can assume two different values (e.g., $D0$, $D2$, $D4$, $D6$: -1 or 3). Therefore the number of decoder states is twice (not four times) the number of encoder states. With the code trellis of Fig. 3(c) the lower bound on the minimum free ED turns out to be

$$d_{\text{free}}^2 \geq \frac{1}{2}\left[\Delta_3^2(2) + \Delta_2^2(2) + \Delta_3^2(2)\right]$$
$$= \frac{1}{2}\left[\left(2\sqrt{2}\Delta_0\right)^2 + (2\Delta_0)^2 + \left(2\sqrt{2}\Delta_0\right)^2\right] = 40, \tag{15}$$

which corresponds to 3 dB coding gain over uncoded 2-AM. The actual asymptotic coding gain is 3.4 dB. The rate of the code is 2/3 bit/symbol and the decoder has 16 states. The same performance can be achieved with the 4-state trellis structure of Fig. 3(a) if two parallel transitions per branch are used. This results in 8 decoder states. Note that the gain achieved with this set partitioning is considerably larger than the gain obtained with the same trellis encoder structure and the same rate with the inner code IC2 (see Section III-A).

As a comparison consider trellis coding with precoding as investigated by Ketchum [5]. In this case set partitioning is done in the ordinary way, i.e., with the four-level channel input symbols. With a d-dimensional signal space (i.e., d symbols per trellis branch) $(d-1)$ two-way partitionings can be performed until the subset contains 2^{d-1} elements, i.e., as many as required for rate $(d-1)/d$. For example, for a two-dimensional signal space this results in an minimum intraset distance of $\sqrt{8}\Delta_0$. Using the same 4-state trellis encoder as before (Fig. 3a) results in a lower bound for the squared free ED of this rate 1/2 code of

$$d_{\text{free}}^2 \geq \frac{1}{2}\left[\Delta_3^2(2) + \Delta_2^2(2) + \Delta_3^2(2)\right] = 40 \tag{16}$$

or 3 dB asymptotic coding gain as opposed to the 5.6 dB of (14). As a further example consider $d = 3$. For a rate 2/3 code $2^{d-1} = 4$ partitioning steps of the three-dimensional channel input signal set have to be performed. With the 8-state trellis encoder of Fig. 3(c) [16-state decoder required] one obtains the lower bound

$$d_{\text{free}}^2 \geq \frac{1}{2}\left[\Delta_4^2(3) + \Delta_3^2(3) + \Delta_4^2(3)\right] = 40. \tag{17}$$

TABLE III
SIGNAL SET FOR RATE 2/3 CODE WITH LARGER GAIN THAN WITH SET PART. OF TABLE II

	$D0$	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$	$D7$
2-dim. ch. output sign.	$-4, -2$	$-4, 0$	$-2, 4$	$-2, 6$	$4, 0$	$4, 2$	$2, -6$	$2, -4$
3-dim. ch. input sign.	$3, -1, -3$	$3, -1, -1$	$-1, -3, 1$	$-1, -3, 3$	$-3, 1, 1$	$-3, 1, 3$	$1, 3, -3$	$1, 3, -1$

($\Delta_i(3)$ are obtained by partitioning a three-dimensional signal set.) The number of states cannot be decreased since parallel transitions would not have enough distance.

Now let's turn to rate 1 codes. Fig. 5 shows that with $d = 2$ and channel output signal set partitioning no rate 1 code is possible since two disjoint subsets of $2^d = 4$ signal points are required. However the channel output signal set for $d = 2$ only contains 7 symbols. With $d = 3$ a rate 1 code is possible since there is more than one subset of at least $2^d = 8$ signal points. Using the code of Fig. 3(a) with four parallel transitions per branch (16-state decoder required) the lower bound of the squared ED turns out to be 20. Employing set partitioning of the channel input signals the same lower bound can only be obtained with a 32-state decoder.

These examples show that set partitioning of the $(d-1)$-dimensional channel output signal set can provide codes with larger gain or less complexity than set partitioning of the corresponding channel input signal set. Similar relations hold for codes with a larger number of states. High coding gains with fairly low complexity are achievable when some bandwidth expansion is accepted, i.e., when the rate of the coded system is $[(d-1)/d]\log_2(M/2)$ as opposed to $\log_2(M/2)$ in the absence of coding.

C. Low Complexity Codes with Higher Gain

Codes with larger gain can be found by exploiting the fact that for the rates considered here not every signal point within a region is actually used. First consider a four-level symbol alphabet and rate $(d-1)/d$ codes. In the previous rate 1/2 code only 4 out of the total of 7 channel output symbols are used, only 8 out of 37 are used for the rate 2/3 code. Using only a fraction of the total number of signal points allows us to increase the intraset distance by employing a different set partitioning and still having disjoint subsets.

Consider again the rate 1/2 code of the last chapter. Instead of using the subsets $B0 = \{-6, 2\}$ and $B1 = \{6, -2\}$ use the subsets $\{-6, 4\}$ and $\{6, -4\}$. The intraset distance of these subsets is the largest possible such that there exist two subsets of that distance. Since the two subsets are disjoint there are no quasicatastrophic error events. Note that $\Delta_1(1)$ is reduced to Δ_0 and $\Delta_2(1)$ is increased compared to the previous set partitioning. That is, the ED contribution of the beginning (transitions originate from the same state) and the end (transitions joining in the same state) of the error event have been increased. From inspection of (13) it may be concluded that this results in an overall increase in minimum squared free ED when the same encoder is used. The four-level channel input signals are assigned to these channel output signals again such that the average transmitter power is minimum

$$
\begin{aligned}
-6 &\rightarrow 3, -3\\
4 &\rightarrow -3, 1\\
6 &\rightarrow -3, 3\\
-4 &\rightarrow 3, -1.
\end{aligned}
\tag{18}
$$

A computer search for the minimum squared free ED resulted in $d_{\text{free}}^2 = 144$. The asymptotic coding gain is

$$
\left(\frac{d_{\text{free}}^2}{P_{\text{ave}}}\right)_{\text{cod}} \left(\frac{P_{\text{ave}}}{d_{\text{free}}^2}\right)_{\text{uncod}} = \frac{144}{7/T}\frac{1/T}{4} \rightarrow 7.1 \text{ dB} \tag{19}
$$

(compared to 6.6 dB with the biphase labeling). Note that since $\Delta_1(1)$ has been reduced it takes longer time to accumulate a certain amount of Euclidian distance. Therefore, the survivor path memory of the decoder has to be larger than in the previous case. Further note that with increasing encoder complexity the asymptotic coding gain does not increase as fast as with the set partitioning of the last chapter because $\Delta_1(1)$ has been reduced to Δ_0. Hence this modified set partitioning is only advantageous for low complexity codes.

Using the same approach of maximizing the intraset distance of the subsets of signal points that leave the same state (i.e., the subsets denoted $B0$ and $B1$) set partitionings can be obtained for higher dimensions. As a further constraint in the selection of signal points we require that the dth symbol of the d-dimensional signal point joining into the same encoder state shall only assume two different values (instead of possibly four). Then the number of decoder trellis states is again twice (instead of four times) the number of states of the encoder. Note that this constraint sometimes conflicts with the goal of achieving the largest possible intraset distance and assigning channel input signals to these channel output signals such that the average transmitter power is minimum. The channel output and corresponding channel input signal set for a rate 2/3 code is given in Table III.

The four signal points belonging to the subset $B0 = D0 \cup D2 \cup D4 \cup D6$ (or $B1$, respectively) have the largest intraset distance possible if they are constrained to end in only two different (channel input) symbols. With the code trellis of Fig. 3(a) and with two parallel transitions per branch the minimum squared free ED turns out to be 60 (this is the minimum distance associated with parallel transitions and at the same time the minimum distance of longer error events). The corresponding asymptotic coding gain is 4.8 dB (compared to 3.4 dB of the rate 2/3 code with the set part. of Table II. The decoder requires 8 states.

TABLE IV
SIGNAL SET FOR RATE 3/4 CODE WITH MODIFIED SET PARTITIONING

	$D0$	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$	$D7$
3-dim. ch.	4, −4, −2	4, −4, 0	−2, −2, −2	−2, −2, 0	2, 2, 0	2, 2, 2	−4, 4, 0	−4, 4, 2
output sign.	−2, 4, −6	−2, 4, −4	4, 2, −6	4, 2, −4	−4, −2, 4	−4, −2, 6	2, −4, 4	2, −4, 6
4-dim. ch.	−1, 3, −1, −3	−1, 3, −1, −1	3, 1, −1, −3	3, 1, −1, −1	−3, −1, 1, 1	−3, −1, 1, 3	1, −3, 1, 1	1, −3, 1, 3
input sign.	1, −1, 3, −3	1, −1, 3, −1	−3, 1, 3, −1	−3, 1, 3, −1	3, −1, −3, 1	3, −1, −3, 3	−1, 1, −3, 1	−1, 1, −3, 3

For a rate $3/4$ code and a four-dimensional signal set Table IV shows the chosen signal points. Using two parallel branches with the code trellis of Fig. 3(c) this code achieves a minimum squared free ED of 52 which is again the min. distance associated with parallel branches and at the same time that of longer error events. This gives rise to an asymptotic coding gain of 4.6 dB. The decoder requires 16 states.

For higher dimensions the search for the set partitioning that achieves the largest coding gain becomes very computationally complex and has not been pursued.

It is not obvious how to apply the above modified set partitioning to obtain rate 1 codes. With the regular set part. described in the last chapter the signal sets used for the rate 1 code are simply one depth less in the set partitioning tree. The total number of signals used in the $(d-1)/d$ rate code, $B0 \cup B1$, form one subset of signals used in the rate 1 code, say the subset that is assigned to the branches leaving the even numbered encoder states (where it is assumed that the states are numbered). This method of extension is not feasible with the modified set partitioning since the intraset distance of the set consisting of the union of the subset for the even states $B0 = D0 \cup D2 \cup D4 \cup D6$ and the odd states $B1 = D1 \cup D3 \cup D5 \cup D7$ has the very poor intraset distance of Δ_0.

Fig. 7 shows a way how to obtain a rate 1 code starting from the previously presented rate $2/3$ code (see Table III) while preserving the distance structure of the modified set partitioning. The number of signaling levels is extended. Now it is easy to find signal points in this signal space such that the minimum intraset distance of the subsets for the even numbered states and the odd numbered states remains unchanged compared to the rate $2/3$ code of Table III. Only four additional signal points per subset are required to support a rate 1 code. The four signal points actually chosen, see Fig. 7, are those for which the corresponding channel input signal set has minimum power and simultaneously ends in only two different symbols per subset $\{D0 \cup D2 \cup D4 \cup D6\}$ and $\{D1 \cup D3 \cup D5 \cup D7\}$, respectively. Therefore, the number of decoder states is only double the number of encoder states, i.e., 8.

The chosen signal points have an average power of $35/37$ and the minimum squared ED, when the code trellis of Fig. 3(a) is used with four-parallel transitions, is still 60. Thus, the coding gain of this rate 1 code is 1.1 dB. Ketchums best code with 32 decoder states has 0.8 dB gain [5]. Note that the channel input symbols are no longer confined to the four-level signal set.

The rate $3/4$ code previously described can also be extended to a rate 1 code by applying the same design procedure. A rate

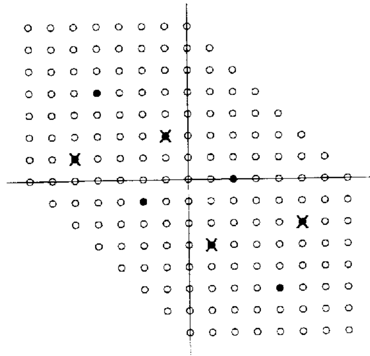

Fig. 7. Channel output signal set assigned to the even numbered encoder states. Signal points with the same marker are assigned to parallel transitions.

1 code with 16 decoder states and asymptotic coding gain of 1.5 dB results.

IV. CONCLUSION

This paper described a novel approach to trellis coding on partial response channels: optimum convolutional codes for a $(d-1)$-dimensional signal constellation are adapted to the PR channel by using multidimensional signal sets and partitioning the $(d-1)$-dimensional constellation present at the noiseless channel output. This partitioning, however, turns out to be in general a harder problem than partitioning the channel input signal set. It was shown that this coding technique can be viewed as a generalization of the concatenation of an outer convolutional code and an inner block code, which is one method proposed for binary coding on PR channels. High coding gains are achieved when some rate loss is accepted. If no bandwidth expansion is acceptable the proposed set partitioning renders codes with slightly better performance than applying the classical set partitioning of the signal set at the channel input and adapting the code to the channel by means of a precoder.

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