Formelsammlung und Repetitorium der Mathematik

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Ebene Trigonometrie.

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Ebene Trigonometrie.

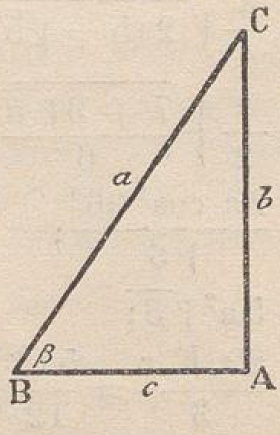
I. Goniometrie.

§ 56. Funktionen einfacher Winkel.

1. Erklärung der Funktionen.

a) Am rechtwinkligen Dreieck (a Hypotenuse, b und c Katheten.)

$$\begin{cases} \sin\beta = \frac{b}{a}; \quad \mathrm{tg}\,\beta = \frac{b}{c}; \quad \mathrm{cosec}\,\beta = \frac{a}{b}; \\ \cos\beta = \frac{c}{a}; \quad \mathrm{ctg}\,\beta = \frac{c}{b}; \quad \sec\beta = \frac{a}{c}. \end{cases}$$



b) Am Koordinatensystem (r Fahrstrahl, x und y Abscisse und Ordinate.)

$$\begin{cases} \sin\alpha = \frac{y}{r}; \quad \mathrm{tg}\,\alpha = \frac{y}{x}; \\ \cos\alpha = \frac{x}{r}; \quad \mathrm{ctg}\,\alpha = \frac{x}{y}. \end{cases}$$

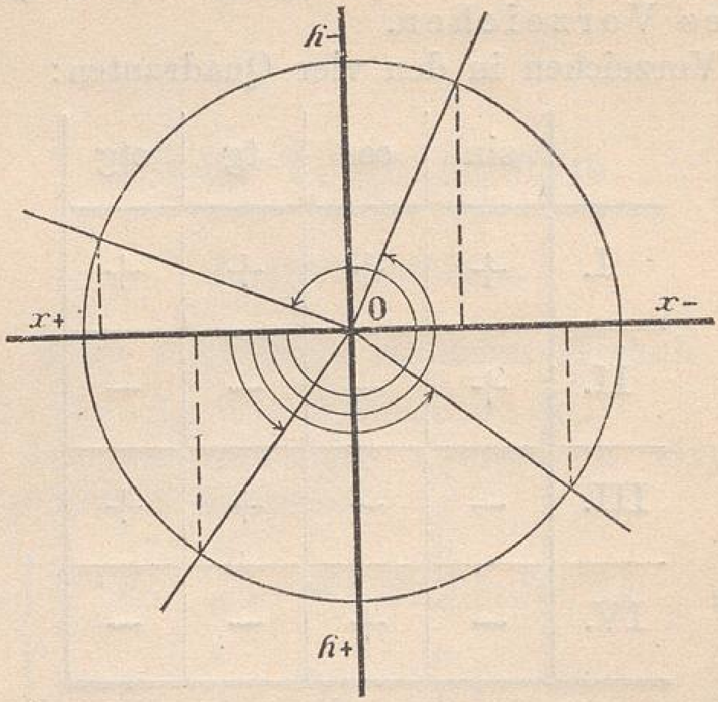
Der sin hat das Vorzeichen der Ordinate, der cos das der Abscisse, tg und ctg haben gleiches Vorzeichen.

c) Vorzeichen in den vier Quadranten:

	sin	cos	tg	ctg
I.	+	+	+	+
II.	+	—	—	—
III.	—	—	+	+
IV.	—	+	—	—

2.

	$-\alpha$	$R-\alpha$	$R+\alpha$	$2R-\alpha$	$2R+\alpha$
sin	$-\sin\alpha$	$\cos\alpha$	$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$
cos	$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$
tg	$-\text{tg}\,\alpha$	$\text{ctg}\,\alpha$	$-\text{ctg}\,\alpha$	$-\text{tg}\,\alpha$	$+\text{tg}\,\alpha$
ctg	$-\text{ctg}\,\alpha$	$\text{tg}\,\alpha$	$-\text{tg}\,\alpha$	$-\text{ctg}\,\alpha$	$+\text{ctg}\,\alpha$
	$3R-\alpha$	$3R+\alpha$	$4nR\pm\alpha$		
sin	$-\cos\alpha$	$-\cos\alpha$	$\sin(\pm\alpha)$		
cos	$-\sin\alpha$	$+\sin\alpha$	$\cos(\pm\alpha)$		
tg	$+\text{ctg}\,\alpha$	$-\text{ctg}\,\alpha$	$\text{tg}(\pm\alpha)$		
ctg	$+\text{tg}\,\alpha$	$-\text{tg}\,\alpha$	$\text{ctg}(\pm\alpha)$		



3. Grenzwerte und besondere Werte:

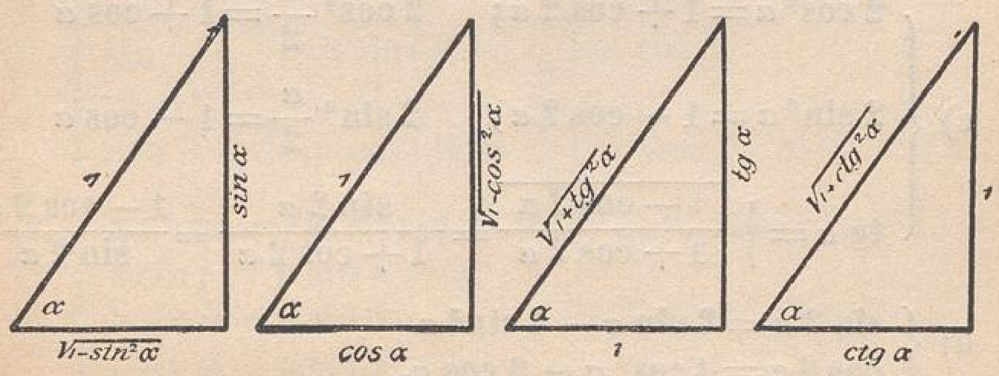
	0° 360°	90°	180°	270°	45°	30°	60°
sin	0	1	0	— 1	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
cos	1	0	— 1	0	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$
tg	0	∞	0	— ∞	1	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$
ctg	∞	0	— ∞	0	1	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$

4. Zusammenhang der Funktionen:

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\left\{\begin{aligned} \operatorname{tg}\alpha &= \frac{\sin\alpha}{\cos\alpha} = \frac{1}{\operatorname{ctg}\alpha} \\ \operatorname{ctg}\alpha &= \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\operatorname{tg}\alpha} \\ \operatorname{tg}\alpha \cdot \operatorname{ctg}\alpha &= 1 \\ 1 + \operatorname{tg}^2\alpha &= \frac{1}{\cos^2\alpha} \\ 1 + \operatorname{ctg}^2\alpha &= \frac{1}{\sin^2\alpha}. \end{aligned}\right.$$

	$\sin\alpha$	$\cos\alpha$	$\operatorname{tg}\alpha$	$\operatorname{ctg}\alpha$
$\sin\alpha =$		$\sqrt{1-\cos^2\alpha}$	$\frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}}$	$\frac{1}{\sqrt{1+\operatorname{ctg}^2\alpha}}$
$\cos\alpha =$	$\sqrt{1-\sin^2\alpha}$		$\frac{1}{\sqrt{1+\operatorname{tg}^2\alpha}}$	$\frac{\operatorname{ctg}\alpha}{\sqrt{1+\operatorname{ctg}^2\alpha}}$
$\operatorname{tg}\alpha =$	$\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$	$\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$		$\frac{1}{\operatorname{ctg}\alpha}$
$\operatorname{ctg}\alpha =$	$\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$	$\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}$	$\frac{1}{\operatorname{tg}\alpha}$	



§ 57. Funktionen zusammengesetzter Winkel.

1. a)
$$\begin{cases} \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \operatorname{tg}(\alpha \pm \beta) = \dfrac{\operatorname{tg}\alpha \pm \operatorname{tg}\beta}{1 \mp \operatorname{tg}\alpha\operatorname{tg}\beta} \\ \operatorname{ctg}(\alpha \pm \beta) = \dfrac{\operatorname{ctg}\alpha\operatorname{ctg}\beta \mp 1}{\operatorname{ctg}\beta \pm \operatorname{ctg}\alpha} \end{cases}$$

b)
$$\begin{cases} \sin 2\alpha = 2\sin\alpha\cos\alpha; & \sin\alpha = 2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2} \\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha; & \cos\alpha = \cos^2\dfrac{\alpha}{2} - \sin^2\dfrac{\alpha}{2} \\ \operatorname{tg} 2\alpha = \dfrac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}; & \operatorname{tg}\alpha = \dfrac{2\operatorname{tg}\dfrac{\alpha}{2}}{1-\operatorname{tg}^2\dfrac{\alpha}{2}} \\ \operatorname{ctg} 2\alpha = \dfrac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}; & \operatorname{ctg}\alpha = \dfrac{\operatorname{ctg}^2\dfrac{\alpha}{2} - 1}{2\operatorname{ctg}\dfrac{\alpha}{2}} \end{cases}$$

c)
$$\begin{cases} 2\cos^2\alpha = 1 + \cos 2\alpha; & 2\cos^2\dfrac{\alpha}{2} = 1 + \cos\alpha \\ 2\sin^2\alpha = 1 - \cos 2\alpha; & 2\sin^2\dfrac{\alpha}{2} = 1 - \cos\alpha \\ \operatorname{tg}\alpha = \sqrt{\dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}} = \dfrac{\sin 2\alpha}{1+\cos 2\alpha} = \dfrac{1-\cos 2\alpha}{\sin 2\alpha} \end{cases}$$

d)
$$\begin{cases} \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \\ \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha. \end{cases}$$

2. Umformung von Summen und Differenzen.

a)
$$\begin{cases} \sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\ \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \end{cases}$$

b)
$$\begin{cases} \cos\alpha + \sin\alpha = \sqrt{2}\,.\sin(45^0 + \alpha) \\ \cos\alpha - \sin\alpha = \sqrt{2}\,.\cos(45^0 + \alpha) \\ \frac{1+\operatorname{tg}\alpha}{1-\operatorname{tg}\alpha} = \operatorname{tg}(45 + \alpha) \\ \frac{\operatorname{ctg}\alpha + 1}{\operatorname{ctg}\alpha - 1} = \operatorname{ctg}(45^0 - \alpha) \end{cases}$$

c)
$$\begin{cases} \operatorname{ctg}\alpha + \operatorname{tg}\alpha = \frac{2}{\sin 2\alpha} \\ \operatorname{ctg}\alpha - \operatorname{tg}\alpha = 2\operatorname{ctg} 2\alpha. \end{cases}$$

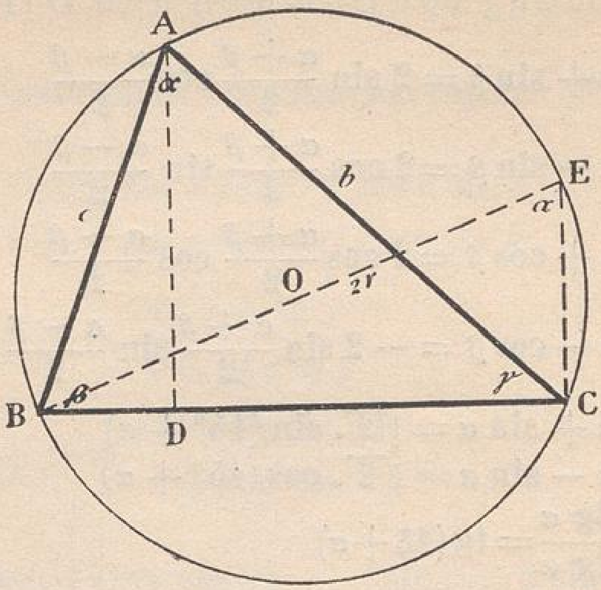
II. Das Dreieck etc.

§ 58. Formeln über das schiefwinklige Dreieck.

1.
$$\begin{cases} \alpha + \beta + \gamma = 2\,\mathrm{R};\ \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \mathrm{R} \\ \sin(\beta+\gamma) = \sin(2\,\mathrm{R} - \alpha) = \sin\alpha \\ \cos(\beta+\gamma) = \cos(2\,\mathrm{R} - \alpha) = -\cos\alpha \\ \sin\frac{\beta+\gamma}{2} = \sin\left(\mathrm{R} - \frac{\alpha}{2}\right) = \cos\frac{\alpha}{2} \\ \cos\frac{\beta+\gamma}{2} = \cos\left(\mathrm{R} - \frac{\alpha}{2}\right) = \sin\frac{\alpha}{2}. \end{cases}$$

2. $a : b : c = \sin\alpha : \sin\beta : \sin\gamma.$ (Sinussatz.)

$$\begin{cases} a\sin\beta = b\sin\alpha = h'' \\ b\sin\gamma = c\sin\beta = h \\ c\sin\alpha = a\sin\gamma = h'. \end{cases}$$ (Höhenformel.)



$$\begin{cases} \dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma} = 2r \\ a = 2r\sin\alpha \\ b = 2r\sin\beta \\ c = 2r\sin\gamma. \end{cases}$$ (Sehnenformel.)

3. $$\begin{cases} a = b\cos\gamma + c\cos\beta \\ b = c\cos\alpha + a\cos\gamma \\ c = a\cos\beta + b\cos\alpha. \end{cases}$$ (Projektionssatz.)

4. $$\begin{cases} \dfrac{b+c}{b-c} = \dfrac{\operatorname{tg}\dfrac{\beta+\gamma}{2}}{\operatorname{tg}\dfrac{\beta-\gamma}{2}} \\ \dfrac{c+a}{c-a} = \dfrac{\operatorname{tg}\dfrac{\gamma+\alpha}{2}}{\operatorname{tg}\dfrac{\gamma-\alpha}{2}} \\ \dfrac{a+b}{a-b} = \dfrac{\operatorname{tg}\dfrac{\alpha+\beta}{2}}{\operatorname{tg}\dfrac{\alpha-\beta}{2}} \end{cases}$$ (Neper'sche Gleichgn.)

5. $$\begin{cases} (b+c)\sin\frac{\alpha}{2} = a\cos\frac{\beta-\gamma}{2} \\ (c+a)\sin\frac{\beta}{2} = b\cos\frac{\gamma-\alpha}{2} \\ (a+b)\sin\frac{\gamma}{2} = c\cos\frac{\alpha-\beta}{2} \\ (b-c)\cos\frac{\alpha}{2} = a\sin\frac{\beta-\gamma}{2} \\ (c-a)\cos\frac{\beta}{2} = b\sin\frac{\gamma-\alpha}{2} \\ (a-b)\cos\frac{\gamma}{2} = c\sin\frac{\alpha-\beta}{2} \end{cases}$$ (Mollweide'sche Gleichgn.)

6. Pythagoräischer Lehrsatz für das schiefwinklige Dreieck.

1. $$\begin{cases} \mathbf{a^2 = b^2 + c^2 - 2\,bc \cdot \cos}\,\alpha \\ \mathbf{b^2 = c^2 + a^2 - 2\,ca \cdot \cos}\,\beta \\ \mathbf{c^2 = a^2 + b^2 - 2\,ab \cdot \cos}\,\gamma. \end{cases}$$

Folgerungen:

2. $$a^2 = \begin{cases} (b+c)^2 - 4\,b\,c\cos^2\frac{\alpha}{2} \\ (b-c)^2 + 4\,b\,c\sin^2\frac{\alpha}{2}. \end{cases}$$

$$a+b+c = 2s \qquad a-b+c = 2(s-b)$$
$$-a+b+c = 2(s-a) \qquad a+b-c = 2(s-c).$$

3. $$\begin{cases} \sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}; & \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}; \\ \sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}; & \cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ca}}; \\ \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}; & \cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}. \end{cases}$$

4. $$\begin{cases} \sin\alpha = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \\ \sin\beta = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} \\ \sin\gamma = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}. \end{cases}$$

5. $$\begin{cases} \operatorname{tg}\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\varrho}{s-a} \\ \operatorname{tg}\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\varrho}{s-b} \\ \operatorname{tg}\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\varrho}{s-c}. \end{cases}$$

7. Inhalt, In- und Ankreishalbmesser.

1. $\mathbf{2\,J = a\,b\,\sin\gamma} = b\,c\sin\alpha = c\,a\sin\beta$.

2. $$J = \begin{cases} 2r^2\sin\alpha\sin\beta\sin\gamma \\ \frac{abc}{4r}. \end{cases}$$

3. $$J = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \varrho\,.\,s = \varrho_1(s-a) = \varrho_2(s-b) = \varrho_3(s-c).$$

4. $\varrho\cdot\varrho_1\cdot\varrho_2\cdot\varrho_3 = J^2$.

5. $$\begin{cases} \varrho = (s-a)\operatorname{tg}\frac{\alpha}{2} = (s-b)\operatorname{tg}\frac{\beta}{2} = (s-c)\operatorname{tg}\frac{\gamma}{2} \\ \varrho_1 = s\operatorname{tg}\frac{\alpha}{2};\quad \varrho_2 = s\operatorname{tg}\frac{\beta}{2};\quad \varrho_3 = s\operatorname{tg}\frac{\gamma}{2}. \end{cases}$$

6. $$\begin{cases} \varrho = 4r\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \\ s = 4r\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}. \end{cases}$$

§ 59. Berechnungen.

I. Das rechtwinklige Dreieck,

a Hypotenuse.

1. Gegeben a, β.

$$b = a \sin\beta; \quad c = a \cos\beta.$$

2. Gegeben b, β.

$$a = \frac{b}{\sin\beta}, \quad c = b \operatorname{ctg}\beta.$$

3. Gegeben a, b.

$$\sin\beta = \frac{b}{a}, \; c = a\cos\beta = b \operatorname{ctg}\beta; \; c = \sqrt{a^2 - b^2}.$$

4. Gegeben b, c.

$$\operatorname{tg}\beta = \frac{b}{c}; \; a = \frac{b}{\sin\beta} = \frac{c}{\cos\beta}; \; a = \sqrt{b^2 + c^2}$$

$$2\,J = b\,c = a\,b\cos\beta = a\,b\sin\gamma = b^2 \operatorname{tg}\gamma.$$

II. Das gleichschenklige Dreieck.

1. Gegeben b, β.

$$a = 2\,b\cos\beta; \quad h = b\sin\beta.$$

2. Gegeben a, α.

$$b = \frac{a}{2\cos\beta}; \; h = \frac{a}{2}\operatorname{tg}\beta.$$

3. Gegeben a und b.

$$\cos\beta = \frac{a}{2b}; \quad h = b\sin\beta = \frac{a}{2}\operatorname{tg}\beta = \sqrt{b^2 - \frac{a^2}{4}}$$

$$2\,J = b^2 \sin\alpha = \frac{a^2}{4}\operatorname{tg}\beta.$$

III. Das regelmässige Vieleck.

1. Gegeben a.

$$r = \frac{a}{2} : \sin\frac{180^0}{n}$$

$$\varrho = \frac{a}{2} \operatorname{ctg} \frac{180^0}{n}$$

$$J = \frac{n a^2}{4} \operatorname{ctg} \frac{180^0}{n}.$$

2. Gegeben r.

$$a = 2 r \sin \frac{180^0}{n}$$

$$\varrho = r \cos \frac{180^0}{n}$$

$$J = \frac{n r^2}{2} \sin \frac{360^0}{n}.$$

3. Gegeben ϱ.

$$r = \varrho : \cos \frac{180^0}{n}$$

$$a = 2 \varrho \operatorname{tg} \frac{180^0}{n}$$

$$J = n \varrho^2 \operatorname{tg} \frac{180^0}{n}.$$

IV. Segment.

$$\text{Sektor} = \frac{r^2 \pi \alpha^0}{360^0} = \frac{r^2}{2} \operatorname{arc} \alpha.$$

$$\Delta = \frac{r^2}{2} \sin \alpha$$

$$\text{Segment} = \frac{r^2}{2} \left(\frac{\pi \alpha^0}{180^0} - \sin \alpha \right) = \frac{r^2}{2} (\operatorname{arc} \alpha - \sin \alpha).$$

V. Das schiefwinklige Dreieck.

1. Gegeben a, β, γ.

$$\alpha = 180^0 - (\beta + \gamma); \quad b = \frac{a \sin \beta}{\sin \alpha}; \quad c = \frac{a \sin \gamma}{\sin \alpha}.$$

2. Gegeben, b, c, α.

$$\frac{\beta+\gamma}{2}=R-\frac{\alpha}{2}$$

$$\operatorname{tg}\frac{\beta-\gamma}{2}=\frac{b-c}{b+c}\operatorname{tg}\frac{\beta+\gamma}{2}$$

$$\begin{cases}\beta=\dfrac{\beta+\gamma}{2}+\dfrac{\beta-\gamma}{2}\\ \gamma=\dfrac{\beta+\gamma}{2}-\dfrac{\beta-\gamma}{2}\end{cases}$$

$$a=\frac{b\sin\alpha}{\sin\beta}$$

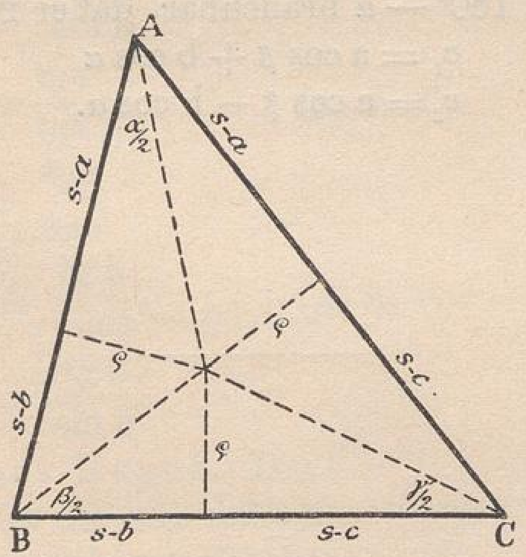
$$\left(=\sqrt{b^2+c^2-2bc.\cos\alpha}\right)$$

oder:

$$\operatorname{tg}\beta=\frac{b\sin\alpha}{c-b\cos\alpha}$$

$$a=\frac{b\sin\alpha}{\sin\beta}=\frac{c-b\cos\alpha}{\cos\beta}.$$

3. Gegeben a, b, c.



1. $J=\sqrt{s(s-a)(s-b)(s-c)}.$

2. $\varrho=\dfrac{J}{s}.$ $(2s=a+b+c).$

3. $\operatorname{tg}\dfrac{\alpha}{2}=\dfrac{\varrho}{s-a}$; $\operatorname{tg}\dfrac{\beta}{2}=\dfrac{\varrho}{s-b}$; $\operatorname{tg}\dfrac{\gamma}{2}=\dfrac{\varrho}{s-c}.$

Proben: 1. $(s-a)+(s-b)+(s-c)=s.$

$$2.\ s \cdot \operatorname{tg} \frac{\alpha}{2} \cdot \operatorname{tg} \frac{\beta}{2} \cdot \operatorname{tg} \frac{\gamma}{2} = \varrho.$$

$$3.\ \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^0.$$

4. Gegeben a, b, β.

a) **b > a.**

$$1.\ \sin \alpha = \frac{a \sin \beta}{b};\quad \alpha < 90^0$$

$$2.\ \gamma = 180^0 - (\alpha + \beta).$$

$$3.\ c = \frac{b \sin \gamma}{\sin \beta} = \frac{a \sin \gamma}{\sin \alpha}.$$

b) **b < a**; $\sin \alpha = \frac{a \sin \beta}{b}$;

hiebei α und $180^0 - \alpha$ brauchbar, daher 2 Werte für c:

$$c_1 = a \cos \beta + b \cos \alpha$$

$$c_2 = a \cos \beta - b \cos \alpha.$$
