Formelsammlung und Repetitorium der Mathematik

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Sphärische Trigonomie.

Sphärische Trigonometrie.

§ 60. Das rechtwinklige sphärische Dreieck.

I. Formeln (a Hypotenuse).

1. $\cos a = \cos b \cos c$ (a, b, c).
2. $\cos a = \operatorname{ctg} \beta \operatorname{ctg} \gamma$ (a, β, γ).
3. $\cos \beta = \sin \gamma \cos b$ (β, γ, b).

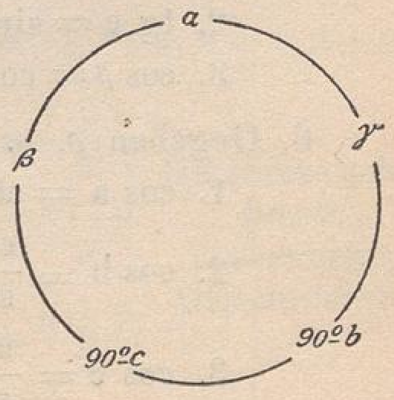
   $\cos \gamma = \sin \beta \cos c$.
4. $\sin \beta = \frac{\sin b}{\sin a}$ (β, b, a).

   $\sin \gamma = \frac{\sin c}{\sin a}$.
5. $\cos \beta = \frac{\operatorname{tg} c}{\operatorname{tg} a}$ (β, c, a).

   $\cos \gamma = \frac{\operatorname{tg} b}{\operatorname{tg} a}$.
6. $\operatorname{tg} \beta = \frac{\operatorname{tg} b}{\sin c}$ (β, b, c).

   $\operatorname{tg} \gamma = \frac{\operatorname{tg} c}{\sin b}$.

7. Nepers Regel. Der cos irgend eines der wie nebenstehend angeschriebenen Stücke ist gleich dem Produkt der sin der getrennten und gleich dem Produkt der ctg der anliegenden Stücke.



Hiedurch können die Formeln 1—6 mechanisch abgeleitet werden.

II. Berechnung des rechtwinkligen Dreiecks.

1. Gegeben a, b.

1. $\cos c = \frac{\cos a}{\cos b}$.

2. $\operatorname{tg} \beta = \frac{\operatorname{tg} b}{\sin c}$; $\operatorname{tg} \gamma = \frac{\operatorname{tg} c}{\sin b}$.

2. Gegeben b, c.

$\cos a = \cos b \cos c$.

$\operatorname{tg} \beta = \frac{\operatorname{tg} b}{\sin c}$; $\operatorname{tg} \gamma = \frac{\operatorname{tg} c}{\sin b}$.

3. Gegeben a, β.

1. $\operatorname{ctg} \gamma = \cos a \operatorname{tg} \beta$.

2. $\operatorname{tg} c = \operatorname{tg} a \cos \beta$; $\operatorname{tg} b = \operatorname{tg} a \cos \gamma$.

4. Gegeben b, β.

1. $\sin a = \frac{\sin b}{\sin \beta}$ (zwei Werte für a).

2. $\operatorname{ctg} \gamma = \cos a \operatorname{tg} \beta$.

3. $\operatorname{tg} c = \operatorname{tg} a \cos \beta$.

5. Gegeben b, γ.

1. $\operatorname{tg} a = \frac{\operatorname{tg} b}{\cos \gamma}$.

2. $\operatorname{tg} c = \sin b \operatorname{tg} \gamma$.

3. $\cos \beta = \cos b \sin \gamma$.

6. Gegeben β, γ.

1. $\cos a = \operatorname{ctg} \beta \operatorname{ctg} \gamma$.

2. $\cos b = \frac{\cos \beta}{\sin \gamma}$.

3. $\cos c = \frac{\cos \gamma}{\sin \beta}$.

Determ. Sind β und γ gleichartig, so muss $\beta + \gamma > 90^0$ und $< 270^0$ sein; sind β und γ ungleichartig, so muss $\beta - \gamma$ oder $\gamma - \beta < 90^0$ sein (s. $53{,}13_{c,\,d}$).

§ 61. Das schiefwinklige Dreieck.

A. Formeln.

I. $$\begin{cases} \cos a = \cos b \cos c + \sin b \sin c \cos \alpha \\ \cos b = \cos c \cos a + \sin c \sin a \cos \beta \\ \cos c = \cos a \cos b + \sin a \sin b \cos \gamma \end{cases}$$ a, b, c, α Cosinussatz.

II. $$\begin{cases} \sin a : \sin b : \sin c = \sin \alpha : \sin \beta : \sin \gamma; \\ \sin a \sin \beta = \sin b \sin \alpha = h'' \\ \sin b \sin \gamma = \sin c \sin \beta = h \\ \sin c \sin \alpha = \sin a \sin \gamma = h' \end{cases}$$ a, b, α, β; Sinussatz.

III. $$\begin{cases} \sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha; \\ \sin a \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha; \\ \sin b \cos \gamma = \cos c \sin a - \sin c \cos a \cos \beta; \\ \sin b \cos \alpha = \cos a \sin c - \sin a \cos c \cos \beta; \\ \sin c \cos \alpha = \cos a \sin b - \sin a \cos b \cos \gamma; \\ \sin c \cos \beta = \cos b \sin a - \sin b \cos a \cos \gamma. \end{cases}$$ a, b, c, α, β;

IV. $$\begin{cases} \sin \frac{\alpha}{2} \sin \frac{b+c}{2} = \sin \frac{a}{2} \cos \frac{\beta-\gamma}{2}; \\ \sin \frac{\alpha}{2} \cos \frac{b+c}{2} = \cos \frac{a}{2} \cos \frac{\beta+\gamma}{2} \\ \cos \frac{\alpha}{2} \sin \frac{b-c}{2} = \sin \frac{a}{2} \sin \frac{\beta-\gamma}{2} \\ \cos \frac{\alpha}{2} \cos \frac{b-c}{2} = \cos \frac{a}{2} \sin \frac{\beta+\gamma}{2} \end{cases}$$ a, $b \pm c$, $\beta \pm \gamma$; Delambre'sche bezw. Gauss'sche Gleichungen

V.

$$\operatorname{tg}\frac{b+c}{2}=\operatorname{tg}\frac{a}{2}\cdot\frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta+\gamma}{2}}\quad\begin{array}{|l}a,\ b\pm c,\ \beta+\gamma,\ \beta-\gamma\\ a,\ \beta\pm\gamma,\ b+c,\ b-c\end{array}$$

$$\operatorname{tg}\frac{b-c}{2}=\operatorname{tg}\frac{a}{2}\cdot\frac{\sin\frac{\beta-\gamma}{2}}{\sin\frac{\beta+\gamma}{2}}$$

$$\operatorname{tg}\frac{\beta+\gamma}{2}=\operatorname{ctg}\frac{\alpha}{2}\cdot\frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}$$

Nepersche Gleichungen.

$$\operatorname{tg}\frac{\beta-\gamma}{2}=\operatorname{ctg}\frac{\alpha}{2}\cdot\frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}$$

VI. $a+b+c=2\,s.$

$$\sin\frac{\alpha}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}};\quad \alpha,\ a,\ b,\ c$$

$$\sin\frac{\beta}{2}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}};$$

$$\sin\frac{\gamma}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}};$$

$$\cos\frac{\alpha}{2}=\sqrt{\frac{\sin s\,.\,\sin(s-a)}{\sin b\sin c}};$$

$$\cos\frac{\beta}{2}=\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}};$$

$$\cos\frac{\gamma}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.$$

VII. $$\begin{cases} S = \sqrt{\sin s \cdot \sin(s-a)\sin(s-b)\sin(s-c)} \\ \sin\alpha = \dfrac{2S}{\sin b \sin c} \qquad \alpha,\ a,\ b,\ c \\ \sin\beta = \dfrac{2S}{\sin c \sin a}; \quad \sin\gamma = \dfrac{2S}{\sin a \sin b}. \end{cases}$$

(S Eckensinus.)

VIII. $$\begin{cases} k = \sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}} \\ \operatorname{ctg}\dfrac{\alpha}{2} = \dfrac{\sin(s-a)}{k} \qquad \alpha,\ a,\ b,\ c \\ \operatorname{ctg}\dfrac{\beta}{2} = \dfrac{\sin(s-b)}{k}; \quad \operatorname{ctg}\dfrac{\gamma}{2} = \dfrac{\sin(s-c)}{k}. \end{cases}$$

IX. $$\begin{cases} \varepsilon = \alpha + \beta + \gamma - 180^0 \quad \text{(sph. Exzess)} \\ \operatorname{tg}\dfrac{\varepsilon}{4} = \sqrt{\operatorname{tg}\dfrac{s}{2}\operatorname{tg}\dfrac{s-a}{2}\operatorname{tg}\dfrac{s-b}{2}\operatorname{tg}\dfrac{s-c}{2}} \\ \operatorname{tg}\left(\dfrac{\alpha}{2} - \dfrac{\varepsilon}{4}\right) = \sqrt{\dfrac{\operatorname{tg}\frac{s-b}{2}\operatorname{tg}\frac{s-c}{2}}{\operatorname{tg}\frac{s}{2}\operatorname{tg}\frac{s-a}{2}}}. \end{cases}$$

(L'Huilier'sche Gleichung.)

Polarformeln.

Ib) $$\begin{cases} \cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a; \quad a,\ \alpha,\ \beta,\ \gamma. \\ \cos\beta = -\cos\gamma\cos\alpha + \sin\gamma\sin\alpha\cos b \\ \cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c. \end{cases}$$

III b) $$\begin{cases} \sin\alpha\cos b = \cos\beta\sin\gamma + \sin\beta\cos\gamma\cos a; \quad a, b, \alpha, \beta, \gamma \\ \sin\alpha\cos c = \cos\gamma\sin\beta + \sin\gamma\cos\beta\cos a \\ \sin\beta\cos c = \cos\gamma\sin\alpha + \sin\gamma\cos\alpha\cos b \\ \sin\beta\cos a = \cos\alpha\sin\gamma + \sin\alpha\cos\gamma\cos b \\ \sin\gamma\cos a = \cos\alpha\sin\beta + \sin\alpha\cos\beta\cos c \\ \sin\gamma\cos b = \cos\beta\sin\alpha + \sin\beta\cos\alpha\cos c. \end{cases}$$

VII b)
$$\left\{\begin{aligned} &\alpha+\beta+\gamma=2\sigma;\\ &\Sigma=\sqrt{-\cos\sigma\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)};\\ &\sin a=\frac{2\Sigma}{\sin\beta\sin\gamma};\\ &\sin b=\frac{2\Sigma}{\sin\gamma\sin\alpha};\ \sin c=\frac{2\Sigma}{\sin\alpha\sin\beta}.\end{aligned}\right.$$

VIII b)
$$\left\{\begin{aligned} &k'=\sqrt{\frac{\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)}{-\cos\sigma}}\\ &\operatorname{tg}\frac{a}{2}=\frac{\cos(\sigma-\alpha)}{k'}\\ &\operatorname{tg}\frac{b}{2}=\frac{\cos(\sigma-\beta)}{k'};\ \operatorname{tg}\frac{c}{2}=\frac{\cos(\sigma-\gamma)}{k'}.\end{aligned}\right.$$

IX b)
$$\left\{\begin{aligned} &d=360^0-(a+b+c)\ \text{(sph. Defekt)}\\ &\operatorname{tg}\frac{d^0}{4}=\\ &\sqrt{-\operatorname{tg}\left(45^0+\frac{\sigma}{2}\right)\operatorname{tg}\left(45^0-\frac{\sigma-\alpha}{2}\right)\operatorname{tg}\left(45^0-\frac{\sigma-\beta}{2}\right)\operatorname{tg}\left(45^0-\frac{\sigma-\gamma}{2}\right)}\\ &\operatorname{tg}\left(\frac{a}{2}-\frac{d}{4}\right)=\sqrt{\frac{\operatorname{tg}\frac{\sigma-\beta}{2}\operatorname{tg}\frac{\sigma-\gamma}{2}}{\operatorname{tg}\frac{\sigma}{2}\operatorname{tg}\frac{\sigma-\alpha}{2}}}\end{aligned}\right.$$

X. Sphärischer Umkreishalbmesser R.

$$\left\{\begin{aligned} &\frac{\sin a}{\sin\alpha}=\frac{\sin b}{\sin\beta}=\frac{\sin c}{\sin\gamma}=2\operatorname{tg}R\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\\ &\operatorname{ctg}R=\sqrt{\frac{\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)}{-\cos\sigma}}=k'\end{aligned}\right.$$

XI. Sphärischer Inkreishalbmesser ϱ.

$$\operatorname{tg}\varrho=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}=k$$

XII. Inhalt des sphär. Dreiecks s. § 53_{15}.

B. Berechnungen.

1. Gegeben a, b, c.

1. $a+b+c=2s,\ s-a=\ldots,\ s-b=\ldots,\ s-c=\ldots$

2. $$k=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

3. $$\operatorname{ctg}\frac{\alpha}{2}=\frac{\sin(s-a)}{k};\quad \operatorname{ctg}\frac{\beta}{2}=\frac{\sin(s-b)}{k};$$
$$\operatorname{ctg}\frac{\gamma}{2}=\frac{\sin(s-c)}{k}.$$

Proben: 1. $(s-a)+(s-b)+(s-c)=s$

2. $$\frac{1}{\sin s}\cdot\operatorname{ctg}\frac{\alpha}{2}\cdot\operatorname{ctg}\frac{\beta}{2}\cdot\operatorname{ctg}\frac{\gamma}{2}=\frac{1}{k}$$

2. Gegeben α, β, γ.

1. $2\sigma=\alpha+\beta+\gamma;\ \sigma-\alpha=\ldots,\ \sigma-\beta=\ldots,\ \sigma-\gamma=\ldots$

2. $k'=$ s. VIII b,

3. $$\operatorname{tg}\frac{a}{2}=\frac{\cos(\sigma-\alpha)}{k'},$$ u. s. w. s. VIII b.

Proben: 1. $(\sigma-\alpha)+(\sigma-\beta)+(\sigma-\gamma)=\sigma$

2. $$-\frac{1}{\cos\sigma}\cdot\operatorname{tg}\frac{a}{2}\cdot\operatorname{tg}\frac{b}{2}\cdot\operatorname{tg}\frac{c}{2}=\frac{1}{k'}.$$

3. Gegeben b, c, α.

1. $$\operatorname{tg}\frac{\beta+\gamma}{2}=\frac{\cos\frac{\alpha}{2}\cos\frac{b-c}{2}}{\sin\frac{\alpha}{2}\cos\frac{b+c}{2}}=\frac{Z}{N}\quad\text{(s. V.)}$$

2. $$\operatorname{tg}\frac{\beta-\gamma}{2}=\frac{\cos\frac{\alpha}{2}\sin\frac{b-c}{2}}{\sin\frac{\alpha}{2}\sin\frac{b+c}{2}}=\frac{Z'}{N'}\quad\text{"}$$

3. $$\begin{cases}\beta=\dfrac{\beta+\gamma}{2}+\dfrac{\beta-\gamma}{2}\\[2mm]\gamma=\dfrac{\beta+\gamma}{2}-\dfrac{\beta-\gamma}{2}\end{cases}$$

4. $$\begin{cases} \cos\frac{a}{2} = \dfrac{Z}{\sin\frac{\beta+\gamma}{2}} \overset{\text{oder}}{=} \dfrac{N}{\cos\frac{\beta+\gamma}{2}}, \\ \sin\frac{a}{2} = \dfrac{Z'}{\sin\frac{\beta-\gamma}{2}} = \dfrac{N'}{\cos\frac{\beta-\gamma}{2}} \end{cases}$$ (s. IV.), oder

Ist nur a verlangt, dann dies aus I.

4. Gegeben β, γ, a.

1. $$\operatorname{tg}\frac{b+c}{2} = \frac{\sin\frac{a}{2}\cos\frac{\beta-\gamma}{2}}{\cos\frac{a}{2}\cos\frac{\beta+\gamma}{2}} = \frac{Z}{N} \text{ (s. V).}$$

2. $$\operatorname{tg}\frac{b-c}{2} = \frac{\sin\frac{a}{2}\sin\frac{\beta-\gamma}{2}}{\cos\frac{a}{2}\sin\frac{\beta+\gamma}{2}} = \frac{Z'}{N'}.$$

3. $$\begin{cases} b = \dfrac{b+c}{2} + \dfrac{b-c}{2} \\ c = \dfrac{b+c}{2} - \dfrac{b-c}{2}. \end{cases}$$

4. $$\sin\frac{\alpha}{2} = \frac{Z}{\sin\frac{b+c}{2}} \text{ oder} = \frac{N}{\cos\frac{b+c}{2}}$$ s. IV, oder

$$\cos\frac{\alpha}{2} = \frac{Z'}{\sin\frac{b-c}{2}} = \frac{N'}{\cos\frac{b-c}{2}}.$$

Ist nur α verlangt, dann dieses aus Ib.

5. Gegeben a, b, α.

1. $$\sin\beta = \frac{\sin b \sin\alpha}{\sin a}.$$

2. $$\operatorname{tg}\frac{c}{2} = \operatorname{tg}\frac{a+b}{2} \cdot \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$ oder

$$= \operatorname{tg} \frac{a - b}{2} \cdot \frac{\sin \frac{\alpha + \beta}{2}}{\sin \frac{\alpha - \beta}{2}} \text{ (s. V).}$$

$$3. \operatorname{tg} \frac{\gamma}{2} = \operatorname{ctg} \frac{\alpha + \beta}{2} \cdot \frac{\cos \frac{a - b}{2}}{\cos \frac{a + b}{2}}$$

$$= \operatorname{ctg} \frac{\alpha - \beta}{2} \cdot \frac{\sin \frac{a - b}{2}}{\sin \frac{a + b}{2}}.$$

Determination. Für β ergeben sich aus 1. im allgemeinen zwei Werte. Bei der Bestimmung ist zu berücksichtigen, dass

$$\text{wenn } a \gtreqless b, \text{ dann } \alpha \gtreqless \beta$$

$$\text{und } a + b \gtrless 180^0, \text{ dann } \alpha + \beta \gtrless 180^0$$

(s. § 53, 13 b und e).

6. Gegeben α, β, a.

1. $\sin b = \frac{\sin a \sin \beta}{\sin \alpha}$.

2. $\operatorname{tg} \frac{\gamma}{2} =$ s. 5,3.

3. $\operatorname{tg} \frac{c}{2} =$ s. 5,2.

Determination s. ebenfalls vorige Aufgabe.

Anmerkung. Die Aufgaben 2, 4, 6 sind die Polarfälle zu den Aufgaben 1, 3. 5; ihre Lösung kann daher durch Uebergang auf das Polardreieck auf die Lösung der Aufgaben 1, 3, 5 zurückgeführt werden.