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Formelsammlung und Repetitorium der Mathematik

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§ 61. Das schiefwinklige sphärische Dreieck.

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Determ. Sind β und γ gleichartig, so muss $\beta + \gamma > 90^0$ und $< 270^0$ sein; sind β und γ ungleichartig, so muss $\beta - \gamma$ oder $\gamma - \beta < 90^0$ sein (s. 53,13$_{c,\ d}$).

§ 61. Das schiefwinklige Dreieck.

A. Formeln.

I. $\begin{cases} \cos a = \cos b \cos c + \sin b \sin c \cos \alpha \\ \cos b = \cos c \cos a + \sin c \sin a \cos \beta \\ \cos c = \cos a \cos b + \sin a \sin b \cos \gamma \end{cases}$ a, b, c, α Cosinussatz.

II. $\begin{cases} \sin a : \sin b : \sin c = \sin \alpha : \sin \beta : \sin \gamma; \\ \sin a \sin \beta = \sin b \sin \alpha = h'' \\ \sin b \sin \gamma = \sin c \sin \beta = h \\ \sin c \sin \alpha = \sin a \sin \gamma = h' \end{cases}$ a, b, α, β; Sinussatz.

III. $\begin{cases} \sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha; \\ \sin a \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha; \\ \sin b \cos \gamma = \cos c \sin a - \sin c \cos a \cos \beta; \\ \sin b \cos \alpha = \cos a \sin c - \sin a \cos c \cos \beta; \\ \sin c \cos \alpha = \cos a \sin b - \sin a \cos b \cos \gamma; \\ \sin c \cos \beta = \cos b \sin a - \sin b \cos a \cos \gamma. \end{cases}$ a, b, c, α, β;

IV. $\begin{cases} \sin \frac{\alpha}{2} \sin \frac{b + c}{2} = \sin \frac{a}{2} \cos \frac{\beta - \gamma}{2}; \\ \sin \frac{\alpha}{2} \cos \frac{b + c}{2} = \cos \frac{a}{2} \cos \frac{\beta + \gamma}{2} \\ \cos \frac{\alpha}{2} \sin \frac{b - c}{2} = \sin \frac{a}{2} \sin \frac{\beta - \gamma}{2} \\ \cos \frac{\alpha}{2} \cos \frac{b - c}{2} = \cos \frac{a}{2} \sin \frac{\beta + \gamma}{2} \end{cases}$ a, b $\pm$ c, $\beta \pm \gamma$; Delambre'sche bezw. Gauss'sche Gleichungen

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V.

$$\operatorname{tg}\frac{b+c}{2}=\operatorname{tg}\frac{a}{2}\cdot\frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta+\gamma}{2}}\quad\Bigg|\quad\begin{matrix}a,\ b\pm c,\ \beta+\gamma,\ \beta-\gamma\\ a,\ \beta\pm\gamma,\ b+c,\ b-c\end{matrix}$$

$$\operatorname{tg}\frac{b-c}{2}=\operatorname{tg}\frac{a}{2}\cdot\frac{\sin\frac{\beta-\gamma}{2}}{\sin\frac{\beta+\gamma}{2}}$$

$$\operatorname{tg}\frac{\beta+\gamma}{2}=\operatorname{ctg}\frac{\alpha}{2}\cdot\frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}$$

Nepersche Gleichungen.

$$\operatorname{tg}\frac{\beta-\gamma}{2}=\operatorname{ctg}\frac{\alpha}{2}\cdot\frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}$$

VI. $a+b+c=2s.$

$$\sin\frac{\alpha}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}};\quad \alpha,\ a,\ b,\ c$$

$$\sin\frac{\beta}{2}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}};$$

$$\sin\frac{\gamma}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}};$$

$$\cos\frac{\alpha}{2}=\sqrt{\frac{\sin s.\sin(s-a)}{\sin b\sin c}};$$

$$\cos\frac{\beta}{2}=\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}};$$

$$\cos\frac{\gamma}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.$$

VII.
$$\begin{cases} S = \sqrt{\sin s \cdot \sin(s-a)\sin(s-b)\sin(s-c)} \\ \sin\alpha = \dfrac{2S}{\sin b \sin c} \qquad \alpha,\ a,\ b,\ c \\ \sin\beta = \dfrac{2S}{\sin c \sin a}; \quad \sin\gamma = \dfrac{2S}{\sin a \sin b}. \end{cases}$$

(S Eckensinus.)

VIII.
$$\begin{cases} k = \sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}} \\ \operatorname{ctg}\dfrac{\alpha}{2} = \dfrac{\sin(s-a)}{k} \qquad \alpha,\ a,\ b,\ c \\ \operatorname{ctg}\dfrac{\beta}{2} = \dfrac{\sin(s-b)}{k}; \quad \operatorname{ctg}\dfrac{\gamma}{2} = \dfrac{\sin(s-c)}{k}. \end{cases}$$

IX.
$$\begin{cases} \varepsilon = \alpha + \beta + \gamma - 180^0 \quad \text{(sph. Exzess)} \\ \operatorname{tg}\dfrac{\varepsilon}{4} = \sqrt{\operatorname{tg}\dfrac{s}{2}\operatorname{tg}\dfrac{s-a}{2}\operatorname{tg}\dfrac{s-b}{2}\operatorname{tg}\dfrac{s-c}{2}} \\ \operatorname{tg}\left(\dfrac{\alpha}{2} - \dfrac{\varepsilon}{4}\right) = \sqrt{\dfrac{\operatorname{tg}\frac{s-b}{2}\operatorname{tg}\frac{s-c}{2}}{\operatorname{tg}\frac{s}{2}\operatorname{tg}\frac{s-a}{2}}}. \end{cases}$$

(L'Huilier'sche Gleichung.)

Polarformeln.

Ib)
$$\begin{cases} \cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a; \quad a,\ \alpha,\ \beta,\ \gamma. \\ \cos\beta = -\cos\gamma\cos\alpha + \sin\gamma\sin\alpha\cos b \\ \cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c. \end{cases}$$

III b)
$$\begin{cases} \sin\alpha\cos b = \cos\beta\sin\gamma + \sin\beta\cos\gamma\cos a; \quad a, b, \alpha, \beta, \gamma \\ \sin\alpha\cos c = \cos\gamma\sin\beta + \sin\gamma\cos\beta\cos a \\ \sin\beta\cos c = \cos\gamma\sin\alpha + \sin\gamma\cos\alpha\cos b \\ \sin\beta\cos a = \cos\alpha\sin\gamma + \sin\alpha\cos\gamma\cos b \\ \sin\gamma\cos a = \cos\alpha\sin\beta + \sin\alpha\cos\beta\cos c \\ \sin\gamma\cos b = \cos\beta\sin\alpha + \sin\beta\cos\alpha\cos c. \end{cases}$$

VII b)
$$\begin{cases} \alpha+\beta+\gamma=2\sigma; \\ \Sigma=\sqrt{-\cos\sigma\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)}; \\ \sin a=\dfrac{2\Sigma}{\sin\beta\sin\gamma}; \\ \sin b=\dfrac{2\Sigma}{\sin\gamma\sin\alpha};\ \sin c=\dfrac{2\Sigma}{\sin\alpha\sin\beta}. \end{cases}$$

VIII b)
$$\begin{cases} k'=\sqrt{\dfrac{\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)}{-\cos\sigma}} \\ \operatorname{tg}\dfrac{a}{2}=\dfrac{\cos(\sigma-\alpha)}{k'} \\ \operatorname{tg}\dfrac{b}{2}=\dfrac{\cos(\sigma-\beta)}{k'};\ \operatorname{tg}\dfrac{c}{2}=\dfrac{\cos(\sigma-\gamma)}{k'}. \end{cases}$$

IX b)
$$\begin{cases} d=360^0-(a+b+c)\ \text{(sph. Defekt)} \\ \operatorname{tg}\dfrac{d^0}{4}= \\ \sqrt{-\operatorname{tg}\left(45^0+\dfrac{\sigma}{2}\right)\operatorname{tg}\left(45^0-\dfrac{\sigma-\alpha}{2}\right)\operatorname{tg}\left(45^0-\dfrac{\sigma-\beta}{2}\right)\operatorname{tg}\left(45^0-\dfrac{\sigma-\gamma}{2}\right)} \\ \operatorname{tg}\left(\dfrac{a}{2}-\dfrac{d}{4}\right)=\sqrt{\dfrac{\operatorname{tg}\dfrac{\sigma-\beta}{2}\operatorname{tg}\dfrac{\sigma-\gamma}{2}}{\operatorname{tg}\dfrac{\sigma}{2}\operatorname{tg}\dfrac{\sigma-\alpha}{2}}} \end{cases}$$

X. Sphärischer Umkreishalbmesser R.

$$\begin{cases} \dfrac{\sin a}{\sin\alpha}=\dfrac{\sin b}{\sin\beta}=\dfrac{\sin c}{\sin\gamma}=2\operatorname{tg}R\cos\dfrac{a}{2}\cos\dfrac{b}{2}\cos\dfrac{c}{2} \\ \operatorname{ctg}R=\sqrt{\dfrac{\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)}{-\cos\sigma}}=k' \end{cases}$$

XI. Sphärischer Inkreishalbmesser ϱ.

$$\operatorname{tg}\varrho=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}=k$$

XII. Inhalt des sphär. Dreiecks s. § 53_{15}.

B. Berechnungen.

1. Gegeben a, b, c.

1. $a+b+c=2s,\ s-a=\ldots,\ s-b=\ldots,\ s-c=\ldots$

2. $$k=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

3. $$\operatorname{ctg}\frac{\alpha}{2}=\frac{\sin(s-a)}{k};\quad \operatorname{ctg}\frac{\beta}{2}=\frac{\sin(s-b)}{k};$$
$$\operatorname{ctg}\frac{\gamma}{2}=\frac{\sin(s-c)}{k}.$$

Proben: 1. $(s-a)+(s-b)+(s-c)=s$

2. $$\frac{1}{\sin s}\cdot\operatorname{ctg}\frac{\alpha}{2}\cdot\operatorname{ctg}\frac{\beta}{2}\cdot\operatorname{ctg}\frac{\gamma}{2}=\frac{1}{k}$$

2. Gegeben α, β, γ.

1. $2\sigma=\alpha+\beta+\gamma;\ \sigma-\alpha=\ldots,\ \sigma-\beta=\ldots,\ \sigma-\gamma=\ldots$

2. $k'=$ s. VIII b,

3. $$\operatorname{tg}\frac{a}{2}=\frac{\cos(\sigma-\alpha)}{k'},$$ u. s. w. s. VIII b.

Proben: 1. $(\sigma-\alpha)+(\sigma-\beta)+(\sigma-\gamma)=\sigma$

2. $$-\frac{1}{\cos\sigma}\cdot\operatorname{tg}\frac{a}{2}\cdot\operatorname{tg}\frac{b}{2}\cdot\operatorname{tg}\frac{c}{2}=\frac{1}{k'}.$$

3. Gegeben b, c, α.

1. $$\operatorname{tg}\frac{\beta+\gamma}{2}=\frac{\cos\frac{\alpha}{2}\cos\frac{b-c}{2}}{\sin\frac{\alpha}{2}\cos\frac{b+c}{2}}=\frac{Z}{N}$$ (s. V.)

2. $$\operatorname{tg}\frac{\beta-\gamma}{2}=\frac{\cos\frac{\alpha}{2}\sin\frac{b-c}{2}}{\sin\frac{\alpha}{2}\sin\frac{b+c}{2}}=\frac{Z'}{N'}$$ "

3. $$\begin{cases}\beta=\frac{\beta+\gamma}{2}+\frac{\beta-\gamma}{2}\\ \gamma=\frac{\beta+\gamma}{2}-\frac{\beta-\gamma}{2}\end{cases}$$

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$$4. \begin{cases} \cos\frac{a}{2} = \frac{Z}{\sin\frac{\beta+\gamma}{2}} \overset{\text{oder}}{=} \frac{N}{\cos\frac{\beta+\gamma}{2}}, & \text{(s. IV.), oder} \\ \sin\frac{a}{2} = \frac{Z'}{\sin\frac{\beta-\gamma}{2}} = \frac{N'}{\cos\frac{\beta-\gamma}{2}} \end{cases}$$

Ist nur a verlangt, dann dies aus I.

4. Gegeben β, γ, a.

$$1.\ \operatorname{tg}\frac{b+c}{2} = \frac{\sin\frac{a}{2}\cos\frac{\beta-\gamma}{2}}{\cos\frac{a}{2}\cos\frac{\beta+\gamma}{2}} = \frac{Z}{N}\ \text{(s. V).}$$

$$2.\ \operatorname{tg}\frac{b-c}{2} = \frac{\sin\frac{a}{2}\sin\frac{\beta-\gamma}{2}}{\cos\frac{a}{2}\sin\frac{\beta+\gamma}{2}} = \frac{Z'}{N'}.$$

$$3. \begin{cases} b = \frac{b+c}{2} + \frac{b-c}{2} \\ c = \frac{b+c}{2} - \frac{b-c}{2}. \end{cases}$$

$$4.\ \sin\frac{\alpha}{2} = \frac{Z}{\sin\frac{b+c}{2}} \text{ oder} = \frac{N}{\cos\frac{b+c}{2}} \text{ s. IV, oder}$$

$$\cos\frac{\alpha}{2} = \frac{Z'}{\sin\frac{b-c}{2}} = \frac{N'}{\cos\frac{b-c}{2}}.$$

Ist nur α verlangt, dann dieses aus Ib.

5. Gegeben a, b, α.

$$1.\ \sin\beta = \frac{\sin b \sin\alpha}{\sin a}.$$

$$2.\ \operatorname{tg}\frac{c}{2} = \operatorname{tg}\frac{a+b}{2} \cdot \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \quad \text{oder}$$

$$= \operatorname{tg} \frac{a - b}{2} \cdot \frac{\sin \frac{\alpha + \beta}{2}}{\sin \frac{\alpha - \beta}{2}} \text{ (s. V).}$$

$$3. \operatorname{tg} \frac{\gamma}{2} = \operatorname{ctg} \frac{\alpha + \beta}{2} \cdot \frac{\cos \frac{a - b}{2}}{\cos \frac{a + b}{2}}$$

$$= \operatorname{ctg} \frac{\alpha - \beta}{2} \cdot \frac{\sin \frac{a - b}{2}}{\sin \frac{a + b}{2}}.$$

Determination. Für β ergeben sich aus 1. im allgemeinen zwei Werte. Bei der Bestimmung ist zu berücksichtigen, dass

$$\text{wenn } a \gtreqless b, \text{ dann } \alpha \gtreqless \beta$$

$$\text{und } a + b \gtrless 180^0, \text{ dann } \alpha + \beta \gtrless 180^0$$

(s. § 53, 13 b und e).

6. Gegeben α, β, a.

$$1. \sin b = \frac{\sin a \sin \beta}{\sin \alpha}.$$

$$2. \operatorname{tg} \frac{\gamma}{2} = \text{s. } 5{,}3.$$

$$3. \operatorname{tg} \frac{c}{2} = \text{s. } 5{,}2.$$

Determination s. ebenfalls vorige Aufgabe.

Anmerkung. Die Aufgaben 2, 4, 6 sind die Polarfälle zu den Aufgaben 1, 3. 5; ihre Lösung kann daher durch Uebergang auf das Polardreieck auf die Lösung der Aufgaben 1, 3, 5 zurückgeführt werden.

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