



Exceptional point topology in nonlinear and non-Hermitian systems

Dissertation

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by

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Kurzfassung

Viele physikalische Systeme tauschen Energie mit ihrer Umgebung aus und lassen sich daher nicht durch herkömmliche hermitesche Modelle beschreiben. Exzeptionelle Punkte (EPs) sind Singularitäten nicht-hermitescher Systeme, an denen Eigenwerte und Eigenvektoren verschmelzen. Ihre spektrale Topologie macht sie für Anwendungen in Modenumschaltung und Sensorik interessant. Während lineare EPs gut verstanden sind, sind die Entstehung und Topologie ihrer nichtlinearen Gegenstücke bislang nur unzureichend erforscht.

Diese Arbeit untersucht, wie Nichtlinearität die Topologie und Empfindlichkeit exzeptioneller Punkte verändert. Ein zentrales Ergebnis ist die universelle Entfaltung nichtlinearer exzeptioneller Punkte (NEPs) in der Nähe eines linearen EPs: Für eine breite Klasse nichtlinearer Zweimodensysteme wird die lokale Lösungsstruktur durch die D_4^- -Katastrophe bestimmt und führt zu einer konischen Anordnung von NEPs zweiter und dritter Ordnung. Dieses Ergebnis ist plattformübergreifend gültig, etwa in der Photonik sowie in der Festkörper- und Atomphysik. Aufbauend darauf wird die spektrale Topologie nichtlinearer exzeptioneller Punkte und Ringe im Hinblick auf Anwendungen zur Modenumschaltung analysiert. Im nichtlinearen Regime zeigt sich eine verstärkte und kontrollierbare Empfindlichkeit. Diese Ergebnisse schaffen ein allgemeines Verständnis für NEPs und zeigen, wie NEPs zur gezielten Modenumschaltung und Kontrolle von Sensorleistung genutzt werden können.

Abstract

Many physical systems exchange energy with their environment and therefore cannot be described by conventional Hermitian models. Exceptional points (EPs) are singularities of non-Hermitian systems at which eigenvalues and eigenvectors coalesce. Their spectral topology makes them attractive for mode switching and sensing applications. While linear EPs are well understood, the topology and physical consequences of their nonlinear counterparts remain largely unexplored.

This thesis investigates how nonlinearity modifies EP topology and responsivity in nonlinear systems. A central result is the universal unfolding of nonlinear exceptional points (NEPs) near a linear EP. It is shown that the local solution structure of a broad class of nonlinear two-mode systems is governed by the D_4^- (elliptic umbilic) catastrophe, giving rise to a cone-like arrangement of second- and third-order NEPs. This universal framework applies across diverse nonlinear platforms, including photonics, condensed-matter systems, and atomic physics. Building on this framework, the topology of NEP structures is analyzed in relation to mode-switching applications. Enhanced and adjustable responsivity is reported in the nonlinear regime. These results establish a unified framework for nonlinear exceptional-point physics and demonstrate how nonlinearity can be exploited to engineer the mode switching and sensing performance of non-Hermitian systems.

Scientific Publications

Main Publications

- [P1] **Jan Wingenbach**, Stefan Schumacher, Xuekai Ma. *Manipulating spectral topology and exceptional points by nonlinearity in non-Hermitian polariton systems*. [Physical Review Research](#) **6**, 013148, (2024).

X. Ma and I developed the concept and methodology of this work. I performed and analyzed all analytical and numerical calculations presented in the paper and wrote the initial draft for the manuscript.

- [P2] **Jan Wingenbach**, David Bauch, Xuekai Ma, Robert Schade, Christian Plessl, Stefan Schumacher. *PHOENIX – Paderborn highly optimized and energy efficient solver for two-dimensional nonlinear Schrödinger equations with integrated extensions*. [Computer Physics Communications](#) **315**, 109689, (2025).

I initiated the project and contributed to the conceptual design and coordination of the numerical implementation. I also contributed experience in the physical modeling framework and required program features. I contributed to the implementation of application examples and supported subsequent debugging and validation processes. I wrote the initial draft of the manuscript and prepared most of the figures. D. Bauch and R. Schade implemented and optimized the code. Benchmark studies were designed by R. Schade and performed in collaboration.

- [P3] **Jan Wingenbach**, Laura Ares, Xuekai Ma, Nai H. Kwong, Jan Sperling, Rolf Binder, Stefan Schumacher. *Exceptional rings in nonlinear non-Hermitian planar optical microcavities: implementation, signal enhancement, and topology*. [ACS Photonics](#) **13**, 3238-3251 (2026).

I initiated the project, performed numerical simulations, and carried out data analysis and visualization. I wrote the manuscript. L. Ares and N. H. Kwong contributed the analytical solutions. R. Binder contributed in particular to discussions on the topology of exceptional rings.

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- [P4] Nai H. Kwong, **Jan Wingenbach**, Laura Ares, Jan Sperling, Xuekai Ma, Stefan Schumacher, Rolf Binder. *Universal Topology of Exceptional Points in Nonlinear Non-Hermitian Systems*. [Nature Communications](#) **17**, 7051 (2026).

I initiated the project to investigate EP structures in nonlinear 2×2 systems in connection with [P3] and found initial solutions using numerical methods. L. Ares and J. Sperling derived parts of the analytical solutions, which I then analyzed in detail. The concept for this work emerged from joint discussions with N. H. Kwong and R. Binder based on my initial analysis. N. H. Kwong and R. Binder carried out the analytical derivations in close collaboration with S. Schumacher and myself. Specifically, I contributed the literature research and context on nonlinear exceptional points, performed the numerical simulations, and prepared the figures.

- [P5] **Jan Wingenbach**, Roman Lebs, Xuekai Ma, Ewan M. Wright, Nai. H. Kwong, Rolf Binder, Harald Giessen, Stefan Schumacher. *Skyrmions in scalar fields of non-Hermitian optical microcavities: spontaneous formation, nonlinear control, and optical forces*. [arXiv: 2606.02265](#) (2026).

S. Schumacher, R. Binder, and H. Giessen initiated the project. I subsequently developed the central concepts and obtained the main results. I performed the numerical simulations with support from R. Lebs. I wrote the manuscript and prepared the figures.

Other Publications

- [P6] Qiang Ai, **Jan Wingenbach**, Xinmiao Yang, Jing Wei, Zaharias Hatzopoulos, Pavlos G. Savvidis, Stefan Schumacher, Xuekai Ma, Tingge Gao. *Optically and remotely controlling localization of exciton-polariton condensates in a potential lattice*. [Physical Review Applied](#) **23**, 024029 (2025).

I performed the numerical simulations presented in this work in collaboration with X. Ma. Q. Ai performed the experiments and prepared the experimental data in collaboration with T. Gao. I wrote the section 'Theoretical investigation' for the Supplemental Material.

- [P7] Helene Wetter, **Jan Wingenbach**, Falk Rehberg, Wenlong Gao, Stefan Schumacher, Thomas Zentgraf. *Polarization- and wave-vector-selective optical metasurface with near-field coupling*. [ACS Photonics](#) **13**, 2128–2133 (2026).

T. Zentgraf conceived the project. H. Wetter prepared the sample and performed the experiments. I supervised F. Rehberg during the code implementation and generation of the numerical results presented in the paper. S. Schumacher and I developed the model extension used to localize the singularity in both the numerical simulations and the experiments.

Research Visits

- **Max Planck Institute for the Science of Light**, Erlangen, Germany
Hosted by: Dr. Flore Kunst
Duration: 02.02.–05.02.2026
Activity: Research collaboration on exceptional point topology; delivered invited seminar.
- **University of Arizona**, Tucson, USA
Hosted by: Prof. Rolf Binder
Duration: 30.09.–03.11.2025
Activity: Research collaboration on exceptional rings and unfolding topology of exceptional rings; delivered invited seminar.
- **University of Freiburg**, Freiburg, Germany
Hosted by: Prof. Andreas Buchleitner
Duration: 12.12.–15.12.2022
Activity: Research exchange on chaotic and non-Hermitian systems; delivered invited seminar.

Invited Talks

- **Exceptional point topology in nonlinear and non-Hermitian systems**, group of Flore Kunst, Max Planck Institute for the Science of Light, Erlangen, Germany, February 2026.
- **Nonlinear and non-Hermitian physics in polariton condensates**, Wyant College of Optical Sciences Seminar, University of Arizona, Tucson, USA, October 2024.
- **Non-Hermitian physics in polariton condensates**, group of Andreas Buchleitner, University of Freiburg, Freiburg, Germany, December 2022.

Conferences Contributions

- **Exceptional points and rings in exciton-polariton systems**, 13th International Conference on Spontaneous Coherence in Excitonic Systems, Las Palmas de Gran Canaria, Canary Islands, January 2026. (Contributed talk)
- **PHOENIX – Paderborn highly optimized solver for twodimensional nonlinear Schrödinger equations: applications in polariton systems**, 13th International Conference on Spontaneous Coherence in Excitonic Systems, Las Palmas de Gran Canaria, Canary Islands, January 2026. (Poster)
- **Exceptional points and rings in exciton-polariton systems**, 19th International Conference on Optics of Excitons in Confined Systems, Warsaw, Poland, July 2025. (Contributed talk)
- **PHOENIX – Paderborn highly optimized solver for twodimensional nonlinear Schrödinger equations: applications in polariton systems**, 19th International Conference on Optics of Excitons in Confined Systems, Warsaw, Poland, July 2025. (Poster, Best Poster Award)
- **Manipulating Spectral Topology and Exceptional Points by Nonlinearity**, 12th International Conference on Spontaneous Coherence in Excitonic Systems, Dublin, Ireland, June 2024. (Contributed talk)
- **Exceptional Points in Nonlinear Polariton Condensates: Tweaking Spectral Topology by Nonlinearity**, 18th International Conference on Optics of Excitons in Confined Systems, Lecce, Italy, June 2023. (Poster)
- **Dynamics of phase defects trapped in optically imprinted orbits in dissipative binary polariton condensates**, DPG Spring Meeting, Regensburg, Germany, September 2022. (Contributed talk)

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One of the fundamental axioms of quantum mechanics states that physical observables are described by Hermitian operators [1]. This guarantees real-valued eigenvalues corresponding to measurable quantities such as energy. Preserving probability, a Hermitian Hamiltonian generates unitary time evolution for closed systems. Furthermore, Hermiticity implies that the eigenstates form a complete orthonormal basis, providing a convenient mathematical framework for describing quantum systems.

However, many physically relevant systems are inherently open, exchanging particles, energy, or information with their environment. In such cases, an effective non-Hermitian description is required to fully capture the underlying physical processes. Early examples include the works of George Gamow [2] and Herman Feshbach [3], who employed non-Hermitian models to describe radioactive decay and resonance phenomena.

A major conceptual breakthrough was the realization by Carl Bender and Stefan Boettcher [4] that Hermiticity can be relaxed while still retaining entirely real spectra, provided the Hamiltonian obeys *parity–time* ($\mathcal{PT}$) symmetry. In this unbroken $\mathcal{PT}$ -symmetric phase, the eigenvalues remain real, and the system permits a consistent physical interpretation within the framework of a suitably defined scalar product.

When $\mathcal{PT}$ symmetry is broken, the spectrum undergoes a qualitative transition: the real-valued eigenvalues merge and become complex conjugates [5]. The phase transition is associated with a non-Hermitian square-root branch-point singularity, known as an *exceptional point* (EP), at which both eigenvalues and eigenvectors coalesce. In contrast to exceptional points, Hermitian degeneracies like *Diabolic points* (DPs) are associated with degenerate eigenvalues, while their eigenvectors stay orthogonal [6].

While exceptional points were first studied in a mathematical context by Kato [7], they were recognized as fundamental topological singularities in physical systems by Berry [8], based on an observation of Pancharatnam [9]. Moreover, their occurrence is not restricted to $\mathcal{PT}$ -phase transitions, but can be traced back to a wide variety of similarities [10]. The order η of an exceptional point is defined as the number η of coalescing eigenvalues and eigenvectors.

The nontrivial spectral topology of exceptional points gives rise to a range of unconventional physical phenomena and has motivated extensive studies across various platforms, including optical waveguides [11], microresonators [12–14], atomic systems [15], electric circuits [16], and polariton condensates [17–20].

Near an exceptional point, mode coalescence leads to counterintuitive effects arising from the interplay of gain and loss. Prominent examples include loss-induced suppression and revival of lasing [21], loss-induced transparency [22, 23], unidirectional invisibility [24, 25], and reflection coefficients exceeding unity [26]. Moreover, the non-trivial topology of bands surrounding exceptional points enables chiral mode evolution by encircling an exceptional point [27, 28]. These unique properties have made exceptional points a central concept in modern non-Hermitian physics [29].

Recently, operation near exceptional points has been explored as a mechanism for optimizing wireless power transfer in non-Hermitian devices [30, 31]. Here, wireless chargers operating in the $\mathcal{PT}$ -unbroken regime or in the vicinity of the corresponding exceptional point can strongly enhance transfer efficiency over large distances.

Another remarkable property of exceptional points is the enhanced response to perturbation. For an exceptional point of order η , the so-called *responsivity* (perturbation–response) to a small perturbation ϵ scales with $\epsilon^{1/\eta}$ near the exceptional point [32], exceeding the linearly scaling responsivity near conventional Hermitian degeneracies. This feature makes exceptional points promising candidates for novel sensor designs [33]. While the true sensing capabilities in the linear regime are limited due to noise and linewidth broadening [34], exceptional-point-based sensors have been broadly studied on a wide range of platforms, such as optical and electronic sensors [35]. Here, the exceptional point responsivity is used, among others, for pressure [36], magnetic field [37], temperature [38], phase [39], or nanoparticle sensing [40]. It has been argued that the noise constraints on sensitivity can be overcome by incorporating nonreciprocal coupling [41] or by operating near a lasing transition [42, 43].

These developments highlight the broad applicability of non-Hermitian physics beyond purely academic interest. Despite this progress, the vast majority of these results rely on an underlying linear description of the system. Understanding how nonlinearity affects exceptional points remains an intensively investigated question.

Nonlinearity affects a wide range of physical systems, including optical systems [44], Bose-Einstein condensates [45], electronic circuits [46], superfluids [47], and exciton-polariton

systems [48, 49]. Nonlinearity-induced amplitude-dependent responses lead to fascinating phenomena such as rogue waves [50], soliton and vortex formation [51, 52], harmonic generation [53], and multistability [54], among others.

In this context, it is important to highlight that nonlinear and non-Hermitian physics are deeply intertwined. For instance, gain in optical systems such as resonators and lasers is usually intensity-dependent, while Kerr-nonlinearity can couple modes of distinct frequencies. Moreover, since the principle of linear superposition no longer applies, the coalescence of modes at an exceptional point can lead to complex mode behavior [55].

Therefore, it is not surprising that considerable efforts have been devoted in recent years to understanding the interplay of nonlinear and non-Hermitian physics. Nonlinearity can provide additional control over non-Hermitian phase transitions [14, 56–58], including but not limited to $\mathcal{PT}$ -phase transitions. Particular attention has been given to the synergy between nonlinearity and exceptional points. For example, frequency combs [59] and exceptional point encircling in the nonlinear regime have been investigated [28, 60, 61]. With sensing applications in mind, higher-order exceptional points have been realized in low-dimensional systems [62–64]. Moreover, exceptional point shifting and rotation [65, 66], as well as stability properties [67], have been explored. Notably, exceptional-like points can emerge in nonlinear Hermitian systems, highlighting the strong connection between non-Hermiticity and nonlinearity [68]. Finally, wireless-charge transfer performance can be further enhanced in the nonlinear regime [69].

Despite this progress, a general and systematic understanding of exceptional points in nonlinear systems is still lacking. In particular, a unifying and system-independent framework that captures their structural and topological properties across a broad class of nonlinear systems remains an open challenge.

In this thesis, the influence of nonlinearities on exceptional points in non-Hermitian systems is systematically investigated. While previous studies in this direction usually focused on specific exceptional points in a certain nonlinear scenario, often incorporating nonlinearities through a perturbative approach, here the universal unfolding topology of exceptional points for a large class of nonlinear systems is developed using catastrophe theory. It is shown that the well-studied exceptional point in the linear regime is embedded in a higher-dimensional, cone-like structure of exceptional points in nonlinear parameter space. Thus, a second-order linear exceptional point acts as the organizing center of an elliptic-umbilic singularity set in nonlinear parameter space, where the linear exceptional point can be interpreted as the coalescence of four nonlinear eigenstates. Building on this result,

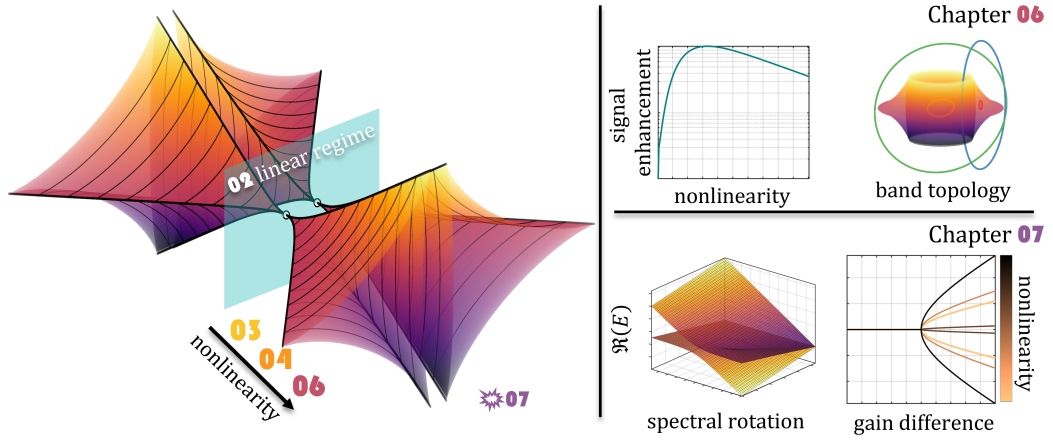


Figure 1.1: Chapter overview illustrating the transition from linear to nonlinear regime. Chapter 2: The central concepts of linear exceptional points are introduced. Chapter 3: Introducing catastrophe theory as a framework to study exceptional points in the nonlinear regime. Chapter 4: Universal unfolding topology of exceptional points in nonlinear systems is derived. Two exceptional points (white dots) of a linear system (blue rectangle) unfold into an elliptic umbilic topology of exceptional points of second and third order when entering the nonlinear regime. The exceptional-point related chapters are marked with respect to this topological structure as a guide for the reader. Chapter 6: Exceptional rings in nonlinear planar optical microcavities. Band topology and responsivity manipulation are investigated on the elliptic umbilic. Chapter 7: Single exceptional point localization in highly-complex nonlinear systems. The interplay of multiple nonlinearities enables responsivity and spectral manipulation. The left panel is adapted from Ref. [70]; the top-right panel is reproduced from Ref. [71]; the bottom-left panel is adapted from Ref. [66].

specific exceptional point configurations are analyzed in detail, and the analytical findings are supported by numerical simulations of the spatiotemporal dynamics in nonlinear and non-Hermitian systems.

The natural occurrence of exceptional rings in planar optical resonators exhibiting TE-TM splitting and a certain form of circular dichroism is demonstrated. Such gapless rings of exceptional points have been intensively studied for their topological properties, but their behavior in nonlinear systems remains unexplored. Going beyond the linear regime, the ring transforms into a higher-dimensional exceptional point structure. In addition, an isolated exceptional point in a highly complex system with multiple interacting nonlinearities is investigated under a perturbative approach, and a multi-step method for their localization is introduced.

With a view toward sensing applications, new mechanisms for controlling the responsiv-

ity near exceptional points are identified. In particular, it is shown that the square-root responsivity of an exceptional point can be scaled linearly by nonlinearity. Moreover, it is shown that responsivity scaling can go beyond square-root dependence and become dependent on the exceptional point location in nonlinear parameter space. These results open new avenues towards sensing at nonlinear exceptional points, with highly tunable responsivity. Moreover, the spectral topology of nonlinear exceptional points is analyzed in detail, including the calculation of topological invariants and the observation of spectral rotation.

The results presented in this work frequently refer to their implementation in hybrid light-matter exciton-polariton systems, which provide a natural platform combining non-Hermiticity (due to their photonic component) with strong nonlinear interactions (due to their excitonic component). Thus, they represent an outstanding platform to investigate the effects of nonlinearities at exceptional points. However, it must be emphasized that many of the presented results are general and transferable to other nonlinear and non-Hermitian systems.

In the following, the individual chapters of this thesis are briefly introduced. Figure 1.1 summarizes the key results on nonlinear exceptional points presented in this thesis and illustrates how the respective chapters are positioned relative to the exceptional-point unfolding topology developed in this work. This thesis is organized as follows: In Chapter 2, the relevant theoretical concepts of this work are introduced, including exceptional points and exceptional point structures such as rings, arcs, and nexus. The responsivity, spectral topology, and selected topological invariants of exceptional points and structures are discussed. As an important tool to characterize and localize singularities such as exceptional points in nonlinear systems, catastrophe theory is introduced, and simple examples of relevant catastrophes are provided in Chapter 3.

In Chapter 4, the universal unfolding topology of exceptional points is derived via catastrophe theory. In Chapter 5, exciton-polaritons are introduced as a physical platform for many investigations performed in this work. The concepts of polariton condensation, spin-orbit interaction in spinor polariton fields, and quantized vortices are introduced. Building on the framework and physical platform established in the preceding chapters, exceptional rings in planar optical microcavities are analyzed as a concrete photonic implementation in Chapter 6. Spectral topology and responsivity of nonlinear exceptional points are investigated based on the unfolding topology derived before. In Chapter 7, a nonlinear exceptional point in a nonresonantly driven polariton condensate is located in the perturbative nonlinear

regime. The interplay of nonlinear effects and their impact on the responsivity and spectral topology of the exceptional point are studied.

In Chapter 8, details about the numerical investigation of the spatiotemporal polariton dynamics are provided. In this chapter, PHOENIX, the Paderborn highly optimized and energy-efficient solver for two-dimensional nonlinear Schrödinger equations with integrated extensions, is introduced. Finally, in Chapter 9, PHOENIX is utilized to demonstrate the spontaneous formation and nonlinear control of Skyrmion patterns in scalar polariton condensates in a non-Hermitian platform. As this thesis is cumulative in nature, each results chapter is based on a corresponding publication, which is included at the end of the respective chapter.

Part I

Theoretical Framework

Theory of exceptional points in non-Hermitian systems

2

2.1 Introduction to exceptional points

This chapter introduces the fundamental mathematical concepts underlying exceptional points (EPs) and summarizes their key spectral and topological properties. We begin with a simple, effective two-mode system, with mode coupling β . The two modes x and y have energy $\pm\delta$ and experience balanced gain and loss $\pm\gamma$, as sketched in Figure 2.1(a). By suitable parameter transformation, this model can also describe the purely lossy and amplifying regime, and the overall energy scale can be shifted without loss of generality. The resulting 2×2 effective non-Hermitian Hamiltonian reads

$$\hat{H} = \begin{bmatrix} a & \beta \\ \beta & -a \end{bmatrix}, \quad (2.1)$$

with $a = \delta + i\gamma$. The corresponding right-eigenvalue problem is written as

Equation 2.2

$$\hat{H} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & \beta \\ \beta & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = E_{\pm} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x \\ y \end{bmatrix} = |R\rangle.$$

Solving the eigenvalue problem yields the right eigenvectors

$$|R_{\pm}\rangle \propto \begin{bmatrix} \beta \\ E_{\pm} - a \end{bmatrix}$$

and the eigenvalues are given by

$$E_{\pm} = \pm\sqrt{a^2 + \beta^2}.$$

The eigenvalues coalesce if $\beta^2 + a^2 = 0$ and hence at the points $(\beta, \delta) = (\pm\gamma, 0)$. This corresponds to degenerate energies and balanced mode gain/loss and coupling.

First, we consider the Hermitian limit $\gamma = 0$ and calculate the purely real eigenvalue spectrum, shown in Figure 2.1(b). The surface consists of two cones intersecting at $(\beta, \delta) = (0, 0)$, as marked by a green dot in Figure 2.1(b). The resulting degeneracy corresponds to a *diabolic point* (DP), characterized by degenerate eigenvalues but linearly independent eigenvectors [72]. At the DP, the Hamiltonian remains diagonalizable. Encircling a DP does not allow for switching from one sheet of the spectrum to another, as no branch-point singularity is enclosed, as illustrated by the black circle in Figure 2.1(b). DPs have been investigated in crystal optics [73] and magnetic molecules [74], among others. In Ref. [75], we have presented a metasurface hosting a DP that can act as a polarization switch.

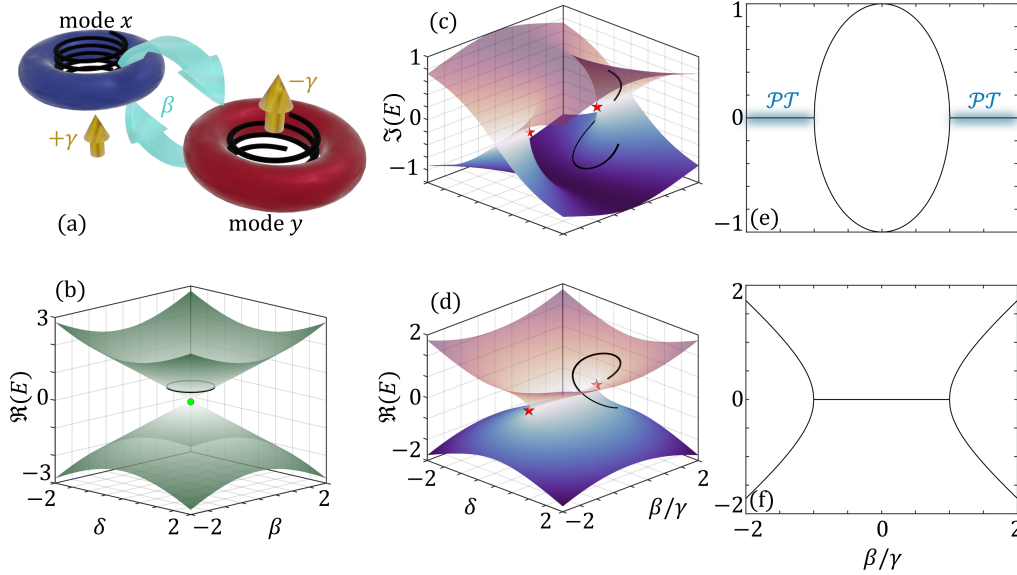


Figure 2.1: Degeneracies in 2×2 effective non-Hermitian Hamiltonian. (a) System sketch of two modes x and y with gain and loss γ , coupling β and detuning δ , represented by different springs within the modes. (b) Energy spectrum of a diabolic point (green dot). Encircling (black ring) preserves the state. (c-d) Imaginary and real parts of complex energy spectra holding exceptional points (red stars). Encircling (black line) leads to mode switching. (e-f) Central cuts of the energy spectra along $\delta = 0$. $\mathcal{PT}$ -unbroken regimes are highlighted in blue.

In contrast, for the non-Hermitian case (e.g., $\gamma = 1$), the eigenvalues become complex-valued. In Figure 2.1(c-f), the real and imaginary parts of the eigenvalue surfaces are depicted together with their central cuts. The condition for degeneracy is now fulfilled

at the exceptional points $(\beta, \delta) = (\pm\gamma, 0)$, where both real and imaginary parts of the eigenvalues coincide (highlighted by stars on the surfaces). In the regime $(|\beta| > |\gamma|, \delta = 0)$, the system is in the $\mathcal{PT}$ -unbroken regime. Mathematically, $\mathcal{PT}$ symmetry is expressed as

$$[\mathcal{PT}, \hat{H}] = 0. \quad (2.3)$$

Here, the parity operator $\mathcal{P}$ performs spatial inversion, while the time-reversal operator $\mathcal{T}$ implements complex conjugation and time reversal [76]. The Hamiltonian is said to be $\mathcal{PT}$ symmetric when it commutes with the combined operation. In the unbroken phase, the eigenstates are simultaneous eigenstates of both $\hat{H}$ and $\mathcal{PT}$, leading to entirely real eigenvalues, whereas in the broken phase this property is lost and the eigenvalues form complex-conjugate pairs [76]. The EPs mark the transition between these two phases. At an EP, the Hamiltonian becomes non-diagonalizable and can be transformed into its Jordan normal form,

$$\hat{H}_{\text{EP}} = \begin{bmatrix} E_{\text{EP}} & 1 \\ 0 & E_{\text{EP}} \end{bmatrix}, \quad (2.4)$$

with the EP-eigenvalue E_{EP} . This reflects the defectiveness of the Hamiltonian. Here, the algebraic multiplicity of the eigenvalue (degree of degeneracy in the characteristic polynomial) exceeds the geometric multiplicity (number of linearly independent eigenvectors). Consequently, only a single eigenvector exists at an EP, and the eigenbasis becomes incomplete. Another defining property of an EP is given by the complex square-root branching around it [7], visualized in Figure 2.1(e,f).

In the following, we discuss the implications of this topology. In Subsection 2.2, the concepts of perturbation-response and signal enhancement near EPs are introduced. In Subsection 2.3, the associated spectral topology is discussed.

2.2 Sensing capabilities of exceptional points

We now focus on the behavior of the system in the vicinity of an exceptional point. Since $\hat{H}$ is non-Hermitian, it is generally non-normal, i.e., $[\hat{H}, \hat{H}^\dagger] \neq 0$. As a consequence, one must distinguish between right $|R\rangle$ and left eigenvectors $|L\rangle$. The latter are given by the left eigenvalue problem $\hat{H}^\dagger |L\rangle = E^* |L\rangle$.

A useful measure of non-Hermitian effects is the Petermann factor

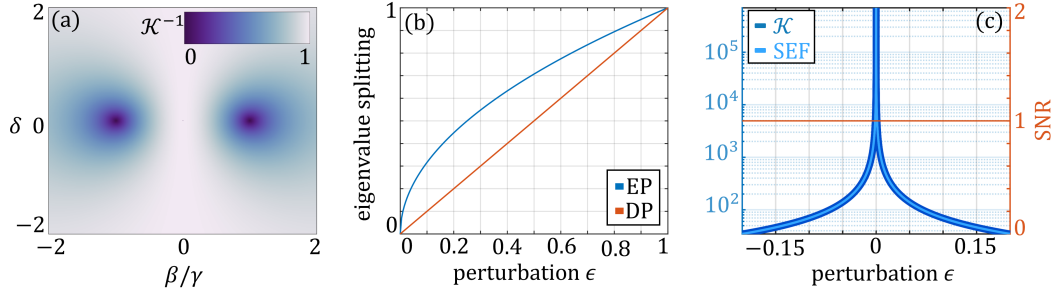


Figure 2.2: Sensitivity of exceptional points. (a) The Petermann factor $\mathcal{K}$ diverges at the exceptional points at $(\beta, \delta) = (\pm\gamma, 0)$. (b) Eigenvalue splitting under perturbation of exceptional points (EP) and diabolic points (DP). For weak perturbation, the EP square-root branching out-scales the linear scaling near the DP. (c) Petermann factor noise amplification $\mathcal{K}$, signal-enhancement factor SEF and signal-to-noise ratio SNR near the EP. In the EP vicinity, the noise amplification cancels the signal enhancement.

$$\mathcal{K} = \frac{\langle L|L \rangle \langle R|R \rangle}{|\langle L|R \rangle|^2}.$$

For normalized eigenvectors $\langle L|L \rangle = \langle R|R \rangle = 1$, this reduces to

$$\mathcal{K} = |\langle L|R \rangle|^{-2}. \quad (2.5)$$

At the exceptional point, the coalesced eigenvector becomes self-orthogonal, i.e. $\langle L|R \rangle \rightarrow 0$, resulting in a divergent Petermann factor at its location [77]. This quantity provides not only a measure for self-orthogonality of the eigenvectors but was originally introduced to describe quantum noise-induced linewidth broadening in lasers [78, 79]. The inverse of the Petermann factor for the solutions of Eq. (2.2) is depicted in Figure 2.2(a).

A striking consequence of the spectral topology around EPs becomes visible under perturbation. For a perturbation of strength ϵ , the eigenvalue splitting scales as $\sqrt{\epsilon}$, in contrast to the linear scaling near a diabolic point [40]. In the following, this enhanced response is referred to as the *responsivity*. In Figure 2.2(b), the eigenvalue splittings for both degeneracies are depicted against perturbation. Based on this, EPs are studied for novel sensor designs.

The signal-enhancement factor SEF can be defined as

$$\text{SEF} = \left| \frac{\partial(E - E_{\text{EP}})}{\partial\epsilon} \right|^2, \quad (2.6)$$

which diverges near the EP [62], as shown in light blue in Figure 2.2(c). However, the

non-orthogonality of the eigenvectors leads to a similar divergence of the Petermann factor, covering excess spontaneous emission and thus noise-amplification. In Figure 2.2(c) the Petermann factor scaling close to the EP is shown in dark-blue. In the idealized linear-response model considered here, the divergence of the signal-enhancement factor is compensated by the divergence of the Petermann factor, yielding an asymptotically constant signal-to-noise ratio

$$\text{SNR} = \sqrt{\frac{\text{SEF}}{\mathcal{K}}} \rightarrow 1, \quad (2.7)$$

where the Petermann noise amplification cancels signal enhancement [80]. While quantum shot noise is present in every system [41–43], significant effort has been devoted to improving EP-based sensors beyond this limitation [35].

2.3 Topological properties of exceptional points and exceptional structures

In this section, the framework of spectral topology is introduced for the example of isolated EPs as well as gapless rings of EPs. Relevant topological invariants are defined, and their impact on the non-Hermitian system is explained. In the end, we introduce the exceptional nexus as an example for a higher-order exceptional point that is directly linked to a well-studied singularity of the bifurcation theory.

A single exceptional point is a topological defect, thus a closed loop $\mathcal{L}$ enclosing an EP cannot be shrunk to a point without passing an EP [81–83]. A topological charge can be assigned to an EP through its spectral vorticity

$$v = -\frac{1}{2\pi} \oint_{\mathcal{L}} \nabla_k \arg(E_+ - E_-) \cdot d\mathbf{k} = \pm 0.5. \quad (2.8)$$

The integrand of Eq. (2.8) is plotted in Figure 2.3(a) with the phase defects marking the exceptional points. Similarly, using the Berry connection $A_{\pm}^{RR}(\mathbf{k}) = i\langle R_{\pm}(\mathbf{k}) | \nabla_k | R_{\pm}(\mathbf{k}) \rangle$, the Berry phase [84] can be determined as follows:

$$\gamma_{\text{Berry},\pm}^{RR} = \oint_{\mathcal{L}} A_{\pm}^{RR}(\mathbf{k}) \cdot d\mathbf{k} = \pm\pi. \quad (2.9)$$

In the present chapter, the Berry connection is calculated among right eigenvectors (RR). Alternative formulations based on left and right eigenvector combinations (LR) lead to

closely related topological invariants and are discussed in Ref. [71].

The intriguing spectral topology of the eigenvalues leads to characteristic behavior when an exceptional point is encircled in parameter space, as illustrated by the black curve in Figure 2.1(c,d). In the adiabatic (quasi-static) limit, one loop around the EP results in a permutation of the eigenstates under accumulation of a geometrical Berry phase of $\pm\pi$. Thus, repetitive encircling (“ $\cup$ ”) of the EP leads to the following state permutations:

$$\left\{ \begin{array}{c} |R_+\rangle \\ |R_-\rangle \end{array} \right\} \cup \left\{ \begin{array}{c} |R_-\rangle \\ -|R_+\rangle \end{array} \right\} \cup \left\{ \begin{array}{c} -|R_+\rangle \\ -|R_-\rangle \end{array} \right\} \cup \left\{ \begin{array}{c} -|R_-\rangle \\ |R_+\rangle \end{array} \right\} \cup \left\{ \begin{array}{c} |R_+\rangle \\ |R_-\rangle \end{array} \right\},$$

restoring the initial states after four turns around the EP [27]. In dynamically encircled systems, non-adiabatic effects become important and give rise to direction-dependent (chiral) mode switching [85].

Finally, these concepts extend naturally to higher-order exceptional points, where more than two eigenstates coalesce. The order determines both the number of encirclements required to return to the initial state and the scaling of the system’s responsivity [86, 87]. Specifically, an EP of order η exhibits eigenvalue splittings scaling as $\epsilon^{1/\eta}$.

While isolated EPs exhibit rich physics, they can be challenging to access experimentally, as multiple parameters must be tuned simultaneously. For an EP of order η , $2(\eta - 1)$ conditions must generally be satisfied [88]. Notably, the number of conditions can be reduced by symmetries [89]. Two prominent ways to simplify access to EPs are introduced in the following.

We extend our focus beyond isolated EPs towards *exceptional rings* (ERs), which are gapless rings of EPs that form in non-Hermitian systems with suitable symmetry constraints. A prominent platform to realize ERs is the momentum space of spin-orbit coupled systems [90–93]. As a minimal model, we introduce an isotropic momentum-dependent coupling $\beta = \mathbf{k}^2$ in Eq. (2.2). The precise functional form is not essential, provided it is isotropic and monotonic.

Solving the non-Hermitian eigenvalue problem (2.2) provides a gapless ring of EPs along $(\delta, k^2) = (0, |\gamma|)$ as depicted by a black ring in the complex energy spectra in Figure 2.3(c,d), with rotationally symmetrical central-cuts in $\mathbf{k}$ shown in Figure 2.3(e,f). ERs not only make EPs more accessible, but they also have been studied extensively for their non-trivial spectral topology. Calculating the Berry curvature

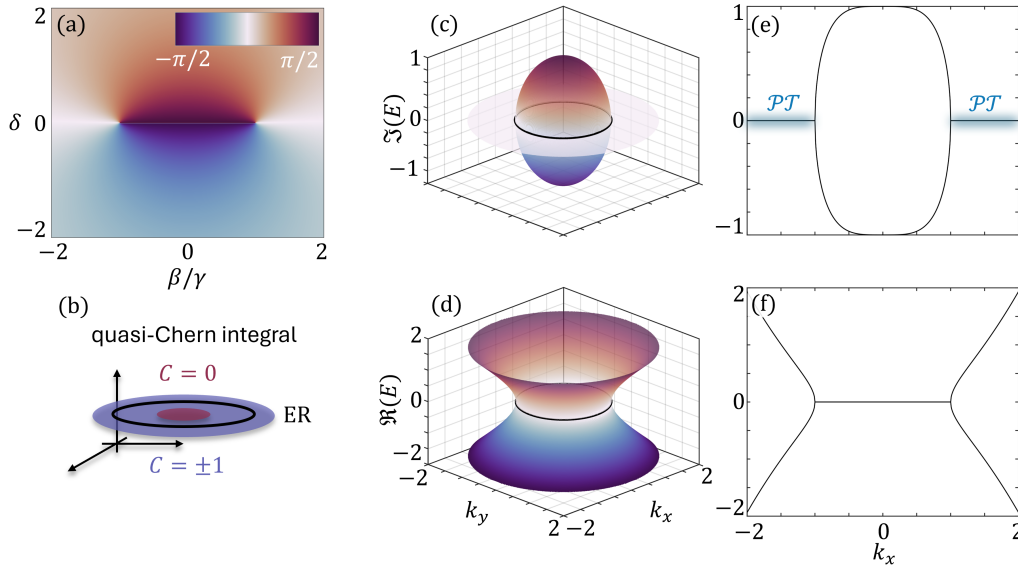


Figure 2.3: Spectral topology of exceptional points and rings in non-Hermitian systems. (a) Vorticity argument $\nabla_k \arg(E_+ - E_-)$ in the vicinity of an exceptional point pair positioned at the phase defects. (b) Sketched Chern number calculation on an exceptional ring (ER). Inside the ER, the quasi-Chern integral amounts to $C = 0$ and converges to $C = \pm 1$ when enclosing the ER. (c,d) Imaginary and real parts of complex-valued energy spectra with a gapless ring of exceptional points (exceptional ring) highlighted by a black ring. (e,f) Central cuts along k_x , visualizing the square-root branching around the opposing exceptional points. $\mathcal{PT}$ symmetric regimes are highlighted in blue. The results displayed in (c-d) are rotationally symmetric in $\mathbf{k}$.

$$\Omega_{\pm}(\mathbf{k}) = \frac{\partial A_{y,\pm}^{RR}(\mathbf{k})}{\partial k_x} - \frac{\partial A_{x,\pm}^{RR}(\mathbf{k})}{\partial k_y} \quad (2.10)$$

and integrating over a disk enclosing the ER, the quasi-Chern integral [94] provides the ring's quantized 1st Chern number

$$C_{\pm} = \frac{1}{2\pi} \int_S \Omega_{\pm}(\mathbf{k}) d^2 \mathbf{k} = \pm 1, \quad (2.11)$$

in agreement with Chern integrals over surfaces enclosing the ring [95]. Thus the ring carries a topological charge [95], as sketched in Figure 2.3(b). A full discussion on the behavior of the quasi-Chern integral is provided in the Supporting Information of Ref. [71]. The ER separates momentum space into distinct regions, leading to a Berry phase of 2π for loops encircling the entire ring. Encircling individual EPs on the ring yields a Berry phase of

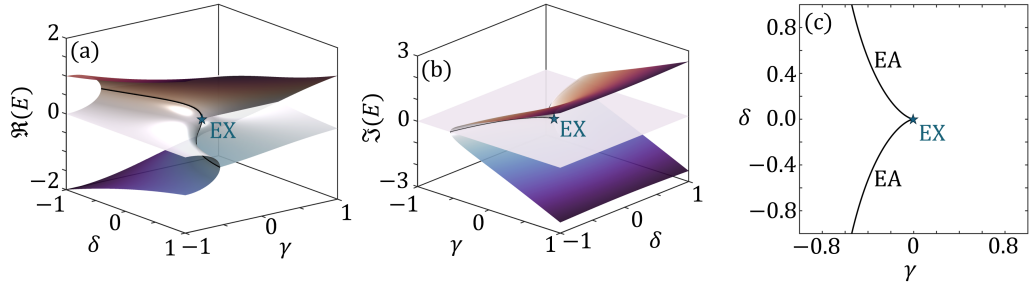


Figure 2.4: Exceptional arcs and nexus in three-level non-Hermitian systems. (a,b) Real and imaginary part of complex energy spectra with two exceptional arcs (EA) merging into an exceptional nexus (EX). (c) Mapping of the arcs and nexus onto the (γ, δ) plane to visualize the intersection pattern. In all panels, the EX and the EAs are marked by a blue star and black lines, respectively.

$\gamma_{\text{Berry}} = \pi$, analogous to isolated EPs. Encircling a single point on an ER in momentum space does not generally induce mode switching. However, introducing controlled anisotropy can break the ring into isolated EP pairs, thereby restoring mode-switching behavior [95]. Another important example of exceptional lines with intriguing spectral topology are so-called exceptional arcs, which have been observed in acoustic cavities and Bose-Einstein condensates [96, 97]. To model these structures, we start from the simple three-level non-Hermitian Hamiltonian

$$\hat{H} = \begin{pmatrix} \sqrt{2}i(1+\gamma) & -1 & 0 \\ -1 & -\delta & -1 \\ 0 & -1 & -\sqrt{2}i(1+\gamma) \end{pmatrix}, \quad (2.12)$$

with mode detuning δ , gain or loss γ , and fixed mode coupling. Solving the eigenvalue problem, the complex spectra exhibit lines of exceptional points [96]. These *exceptional arcs* are depicted as black lines in Figure 2.4(a-b). Here, these arcs are protected by $\mathcal{PT}$ symmetry, which becomes visible in the imaginary part of the spectra as shown in Figure 2.4(b). Along each arc, two modes of the three-level system coalesce. Thus, as the two arcs merge or intersect, all three modes coalesce into a so-called *exceptional nexus* forming a third-order exceptional point. The exceptional nexus represents a junction of exceptional arcs and constitutes a third-order spectral degeneracy. As such, it combines the topology of exceptional lines with the algebraic properties of higher-order exceptional points. The exceptional nexus is marked by a blue star in all panels of Figure 2.4. In Figure 2.4(c), the

arcs are mapped onto the (γ, δ) plane together with the nexus. Encircling the exceptional nexus on different planes of the parameter space results in distinct vorticities and state transfers, resulting in hybrid topological invariants [96].

Besides being a third-order exceptional point, the exceptional nexus is also associated with a cusp singularity [96]. This association provides a promising opportunity to study exceptional points by applying well-established methods of singularity analysis. To this end, an introduction to catastrophe theory is given in the following.

Interim Summary

Exceptional points are defective singularities in non-Hermitian systems at which both eigenvalues and eigenvectors coalesce. This coalescence leads to fascinating mode behavior close to the singularity and shows great promise for the design of novel devices actively using the dissipative nature of open systems. Moreover, their spectral topology and enhanced responsivity makes exceptional points potential candidates for mode switching and sensing applications. Symmetries can make exceptional points more accessible, leading to the formation of exceptional structures such as rings, arcs, or higher-order junctions like nexus points. These structures themselves carry nontrivial spectral topology and provide the grounds for the study of exceptional singularities in nonlinear and non-Hermitian systems.

Catastrophe theory provides a powerful framework for identifying and classifying singularities in systems with discontinuities, i.e., systems where small parameter changes lead to abrupt qualitative changes in system behavior [98]. In this chapter, catastrophe theory is introduced as a tool to locate and classify exceptional points in nonlinear systems. Exceptional points in nonlinear systems appear as singularities of nonlinear solutions. Since catastrophe theory classifies precisely such singularities and their unfoldings, it provides a natural framework for understanding nonlinear exceptional points. In the following, the method used to classify singularities within catastrophe theory is introduced, and two representative examples are discussed.

Consider a set of solutions $\Psi(x_i, u_i)$ of a set of equations $\mathcal{F}_j$. Here, a solution is described by the variable set x_i , called the *state variables*. The variables u_i influence the behavior or qualitative properties of the solutions and thus are called *control parameters* [99]. The equilibrium configurations of the system are assumed to be generated by a smooth scalar Lyapunov potential $V_L(x_i, u_i)$ whose stationary points define equilibrium configurations (critical points) of the system. Catastrophe theory studies the structure of these *critical points* under variation of the control parameters. Thus, critical points of the system are given by

Equation 3.1

$$\nabla_{x_i} V_L = 0 .$$

Degenerate critical points form the *singularities set* of the system for which

Equation 3.2

$$\det\{h\} = 0 ,$$

with h denoting the Hessian matrix of V_L . Catastrophe theory classifies these singularity sets around an *organizing point*.

A key result of catastrophe theory is that the categorization of such singularities depends

only on the number of control parameters, not on the dimensionality of the state space [100, 101]. The minimum number of control parameters required to unfold a singularity coincides with its co-dimension. The set of elementary catastrophes with co-dimension up to five was classified by Thom [99]. Through coordinate transformations and local Taylor expansions of V_L around the organizing point, many physical systems can be mapped onto one of the elementary catastrophes. Applications of catastrophe theory are found among others in mechanics [102], biology [103], economics [104], laser physics [105], and thermodynamics [106]. Moreover, optical phenomena such as rainbows and diffraction patterns can be identified as diffraction catastrophes [107–109].

To demonstrate this method, we consider the Lyapunov potential

$$V_L(x) = x^3 + ux, \quad (3.3)$$

The stationary condition yields $3x^2 + u = 0$ and degeneracy requires $6x = 0$. For $u < 0$, the potential V shows two stationary points (one minimum and one maximum), while no real stationary solutions exist for $u > 0$. At $(x, u) = (0, 0)$, the maximum and minimum coincide into an inflection point forming a fold catastrophe as depicted in Figure 3.1.

The second important catastrophe that requires introduction here is the cusp catastrophe. In contrast to the fold, the cusp will be introduced in a physical context to provide an example of how catastrophe theory can be used to investigate nonlinear systems. We consider the Duffing oscillator

$$\ddot{x} + k\dot{x} + x + \alpha x^3 = F \cos(\Omega t), \quad (3.4)$$

with damping term $k\dot{x}$ ($k > 0$) and nonlinearity α . The external driving force is defined by its amplitude F and frequency Ω . We define $\Omega = 1 + \omega$ and restrict ourselves to small k , α , and ω . Using a harmonic ansatz and standard approximations, one obtains a cubic equation for the steady-state intensity A^2 [100], with

$$A^2 \left(\frac{3}{4} \alpha A^2 - 2\omega \right)^2 = F^2 - k^2 A^2. \quad (3.5)$$

Next, catastrophe theory is applied to locate and characterize critical points in the system.

The intensity A^2 is treated as the state variable, while the detuning ω and nonlinearity α serve as control parameters, resulting in a co-dimension of 2. Thus, the critical points are determined by the simultaneous conditions that the first and second derivatives of the

response equation (3.5) vanish with respect to the state variable. Elimination of A^2 in the resulting equations yields that this system hosts a pair of cusp catastrophes at

$$(\alpha, \omega) = \pm \left(\frac{32k^3\sqrt{3}}{27F^2}, \frac{\sqrt{3}k}{2} \right). \quad (3.6)$$

At the cusp points [marked in red in Figure 3.1(b)] all three solutions of equation (3.5) coalesce [110]. To validate this result, we solve the cubic equation and plot the solutions for A as a surface in Figure 3.1(b) against the control parameters with selected branch cuts along ω depicted in Figure 3.1(c). In the linear regime ($\alpha = 0$), a conventional resonance behavior is observed, as shown by the blue line in Figure 3.1(c), while introducing nonlinearity leads to multistability [red and green lines in Figure 3.1(c)]. The cusp points predicted by catastrophe theory mark the boundary to the multistable regimes and correspond to the coalescence of two fold bifurcations in the steady-state amplitudes [100]. In Figure 3.1(b), the fold lines are marked by black lines and the cusp points by red dots. The saturated areas of the surface plot mark the regimes of multistability. Notably, the position of the cusp points in control space is fully defined by the linear parameters k and F .

This example illustrates how catastrophe theory allows to determine the structure and location of critical points directly from the organizing point, without explicitly solving the nonlinear equations. In later chapters, this framework will be used to identify and classify exceptional points in nonlinear systems.

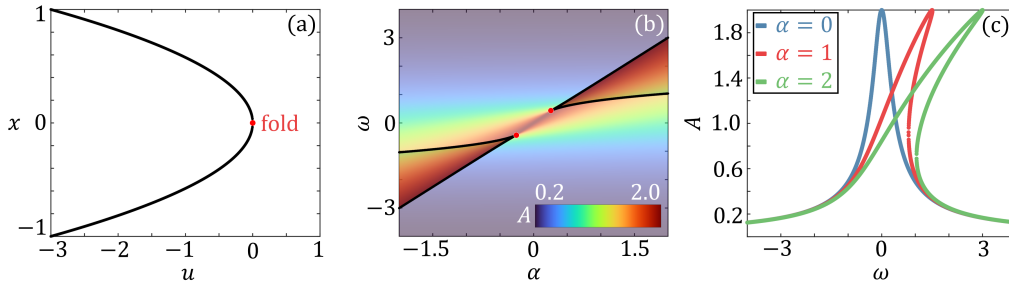


Figure 3.1: Introduction to fold and cusp catastrophe. (a) Fold catastrophe (red) for the textbook Lyapunov function $V_L(x) = x^3 + ux$. Stationary solutions are shown in black. (b) Surface: Cubic root steady state amplitudes A of a Duffing oscillator in the weak nonlinear regime for variable detuning. In the unsaturated (saturated) regions, one (three) steady-state solution is present. Black lines mark fold lines forming in the multistable regime. Red dots highlight cusp points located by catastrophe theory. (c) Selected branch cuts of (b) to demonstrate tilting of resonance peaks and emergence of multistability with folds under increasing nonlinearity α .

In Ref. [111], the connection between catastrophe theory and exceptional points was suggested. Since then, in linear and non-Hermitian systems, both the swallowtail and cusp catastrophes have been observed theoretically and experimentally [96, 97, 112], forming exceptional nexus and arcs as introduced in Chapter 2.3. Later, it was shown that in nonlinear non-Hermitian systems, exceptional nexus and arcs can be found even in 2×2 systems [61] and that so-called nonlinear exceptional points can be associated with fold points [68].

The concept of nonlinear exceptional points will be introduced in detail in Chapter 4, and we demonstrate that in the nonlinear regime the unfolding topology of a linear exceptional point forms one of the elementary catastrophes, i.e., the elliptic umbilic catastrophe [70]. Furthermore, in Chapter 6 it will be shown that in nonlinear and non-Hermitian 2×2 systems the concept of exceptional nexus points and arcs can extend to rings of nexus points and surfaces of folds by introducing spin-orbit-coupling induced symmetries [71].

Interim Summary

For systems whose equations can be expressed in terms of the gradient of a suitably chosen Lyapunov potential, catastrophe theory provides a universal framework for describing how solution branches change under parameter variation. Singularities occur when stationary points of the effective potential become degenerate. Up to smooth coordinate transformations, and for co-dimension less than or equal to five, the complete list of degenerate critical points and their universal unfolding is given by the so-called elementary catastrophes. Because nonlinear exceptional points can be formulated as degeneracies of nonlinear solution branches, catastrophe theory provides the natural framework for identifying their universal topological structure of the unfolding.

Part II

A Study on Nonlinear Exceptional Points

Nonlinear and non-Hermitian: It's a catastrophe

4

Now that the fundamental concepts of exceptional points and catastrophe theory have been introduced, this thesis turns to the effects of nonlinearities on exceptional points. However, before their characteristic behavior is investigated in the nonlinear regime, the question of how exceptional points are organized in nonlinear systems must be addressed. To this end, the universal unfolding topology of exceptional points is derived in this chapter. In the following, we distinguish between conventional (linear) exceptional points introduced above and *nonlinear exceptional points* (NEPs), i.e., exceptional points in the solutions of nonlinear eigenvalue problems.

Starting from an isolated exceptional point of the linear eigenvalue problem, introduced in Chapter 2, the topological structure of the unfolding, hereafter referred to as the *unfolding topology*, describes the organization of NEPs that emerges when extending the parameter space by nonlinearity. In principle, this topological structure does not have to be unique. Different nonlinearities could, in principle, generate distinct local organizations of NEPs. The exceptional point could remain isolated, shift in parameter space, or unfold into a higher-dimensional NEP structure. Thus, each system would have to be analyzed by repeated numerical parameter sweeps or system modifications. However, we come to the conclusion that this is not the case for a broad class of nonlinear two-mode systems. Instead, for a large class of nonlinear 2×2 problems, the unfolding topology is universal and dictated by the local structure of the linear exceptional point itself, as discussed below [70].

Earlier works in this field focused on the idea of how exceptional points are shifted when nonlinearity is introduced [60, 113]. This idea is appealing, as saturable gain or Kerr-nonlinearity shifts the point in parameter space that fulfills mode degeneracy and balance in coupling and gain ratios. While this approach works within a perturbative framework and will be investigated further in Chapter 7, a complete description requires allowing changes to the singularity structure. Therefore, recent works started to detach NEPs from linear exceptional points, allowing them to exhibit different characteristics and organization in their respective regime [63, 68]. However, the relation between NEPs and linear exceptional points is still an active debate.

Past works aimed at solving specific nonlinear eigenvalue problems around an exceptional point with numerical iterative methods or analytical approaches. In Ref. [60], it has been demonstrated that numerical iterative methods can perform poorly around a NEP under Kerr-nonlinearity. Later, it was demonstrated that saturable gain allows for reliable convergence around the NEP [63, 64, 113].

In the presence of multistability, however, an n -dimensional system can possess more than n solutions, and new degeneracies can emerge. Moreover, while linear exceptional points exhibit coalescing eigenvectors, it was shown both theoretically and experimentally that NEPs can host a (partially) complete eigenbasis due to multistability and may exhibit a non-diverging Petermann factor [63, 64]. A mapping to a higher-dimensional linear auxiliary system can identify NEPs as exceptional points in this auxiliary system [63]. Analytically, this is done by construction of the Frobenius companion matrix [88]. Next to that, construction of the Sylvester matrix may provide the location of exceptional points even in nonlinear systems [68]. Exceptional-like points can emerge in nonlinear and Hermitian systems, where fold points can be mapped to exceptional points in linear non-Hermitian systems [68]. The NEPs investigated below naturally host coalescing eigenvectors and eigenvalues as they are formed by coalescing critical points (fold and cusp points), as discussed in Ref. [68].

In this chapter, we use catastrophe theory, introduced in Chapter 3, to derive the local unfolding topology of a second-order linear exceptional point for a broad class of nonlinear 2×2 eigenvalue problems. With a general ansatz for the eigenvectors, two eigenvector equations are derived from a general nonlinear eigenvalue problem. Next, the Lyapunov function is constructed so that the critical point equations match the eigenvector equations. Quantities defining the eigenstates are labeled as state variables, while physical parameters, including nonlinearity, are identified as control parameters. In the vicinity of the linear exceptional point, the Lyapunov function can be locally matched to the germ of the D_4^- singularity, i.e., an elliptic umbilic catastrophe. Given its construction, its singularity set provides the organization of NEPs in nonlinear parameter space.

In the full three-dimensional parameter space, the singularity set consists of fold surfaces that meet along three cusp ribs, forming a cone-like surface of NEPs. In the singularity set, fold and cusp points are identified with second- and third-order NEPs, respectively. Inside the cone, the Lyapunov potential has four critical points and two on the outside, corresponding to four solutions inside and two solutions outside the cone. For a fixed nonlinearity, the cone-like surface exhibits a deltoid cross-section, formed by three fold lines

that merge pairwise into three cusp points. The fold lines are characterized as exceptional arcs merging pairwise into an exceptional nexus (cusp points). While the germ represents a second-order exceptional point in the linear regime, it becomes effectively fourth-order in the extended nonlinear parameter space.

This unfolding topology is highly general and applies to any real-valued nonlinearity function that can be expressed by a Taylor expansion in the germ's vicinity. This includes, but is not limited to, Kerr-nonlinearities and saturable gain. Moreover, the unfolding topology holds true under asymmetric and complex mode coupling. While smooth changes of parameters may deform the precise geometry of the cone and its deltoid cross-section, the unfolding topology remains as long as the nonlinear eigenvalue problem belongs to the same catastrophe class. Thus, the elliptic umbilic determines the universal local organization of NEPs near the linear exceptional point.

Below, an overview of previous work on exceptional points in nonlinear systems is provided and placed within the context of the unfolding topology we have discovered. The key results of Refs. [60] and [113] will be briefly reproduced, and we will show that the elliptic umbilic singularity can be found in both nonlinear systems discussed in these works. To this end, let us investigate a generic nonlinear eigenvalue problem that hosts an exceptional point in the linear regime. In the two works mentioned, the authors used distinct nonlinearities, leading to similar results. The general nonlinear eigenvalue problem considered reads

$$\begin{bmatrix} \delta + \mathcal{L}_{\text{non}}^a - 2i\Gamma & \beta \\ \beta & -\delta + \mathcal{L}_{\text{non}}^b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = E \begin{bmatrix} a \\ b \end{bmatrix}, \quad (4.1)$$

with mode detuning δ and coupling β , as well as system loss $\Gamma = 1$. The nonlinearity $\mathcal{L}_{\text{non}}^x$ can be Kerr-type

$$\mathcal{L}_{\text{non}}^x = g|x|^2 \quad (4.2)$$

as done in Ref. [60] or saturable

$$\mathcal{L}_{\text{non}}^x = \frac{g_x}{1 + |x|^2} \quad (4.3)$$

as investigated in Ref. [113]. Previous studies solved this eigenvalue problem numerically using self-consistent-field methods. In the presence of saturable nonlinearity, an analytic

approach provides an eighth-order polynomial equation for the spectrum. The spectra for both nonlinearities look very similar and are exemplarily displayed for equation (4.3) in Figure 4.1(a-b) with fold points marked in red and the exceptional point by a star for $g_x = 1.5$.

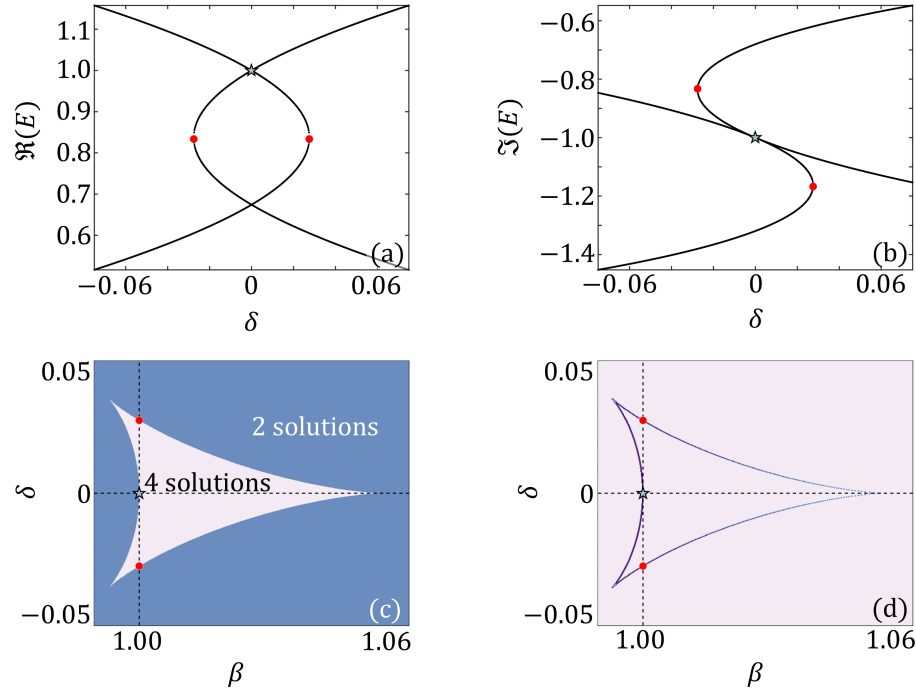


Figure 4.1: Revisiting previous works on nonlinear exceptional points to demonstrate the presence of an elliptic umbilic topology. (a-b) Real and imaginary parts of eigenvalues under saturable gain along $\beta = \Gamma$. The position of the preserved exceptional point is marked by a star; the two nonlinear exceptional points are marked in red. (c) Number of solutions found in the (δ, β) plane: four solutions are found inside the deltoid area (pink) and two outside of it (blue). (d) Coalescing critical points hit by numerical discretizations in the (δ, β) plane. The points marked in (a,b) are highlighted in (c,d), respectively. The results in (c,d) demonstrate the elliptic umbilic unfolding topology derived below. (a,b) were recalculated by the author following Ref. [113]

It must be emphasized that the critical points found in Refs. [60] and [113] are consistent with the fold structure of the elliptic umbilic topology found in the work introduced below, as shown in Figure 4.1(c-d). Figure 4.1(c) illustrates the regime of four solutions (pink) and two solutions (blue). In Figure 4.1(d), the critical points hit by discretizations are marked in blue; here, the number of solutions switches from two to four and vice versa. Thus, it is concluded that the elliptic umbilic topology is inherently present in both Refs. [60]

and [113]. Note that uneven nonlinearity $g_a \neq g_b$ shifts the elliptic umbilic towards finite mode detuning.

This observation demonstrates that the topology derived in Ref. [70] was already implicitly present in previously studied nonlinear and non-Hermitian models, even though it had not been recognized as such. The derived topology allows for the study of nonlinear exceptional points without relying solely on repetitive solving of the nonlinear eigenvalue problem by analytical or numerical means. Moreover, it shows the complexity that a single conventional exceptional point can lead to in the nonlinear regime. This point is illustrated by the poetic interlude at the end of the chapter.

Having established the universal unfolding topology of nonlinear exceptional points, the following chapters turn to its physical realization and broader consequences. The next chapter introduces polariton condensates as a natural platform for studying nonlinear exceptional points. This platform is then used in the subsequent chapter to apply the unfolding topology to another exceptional-point structure and to analyze its effects on exceptional-point responsivity and topology.

The results presented in this chapter were previously published and are reproduced below. Reprinted from N.-H. Kwong et al., Universal topology of exceptional points in nonlinear non-Hermitian systems, *Nature Communications*, **17**, 7051 (2026),

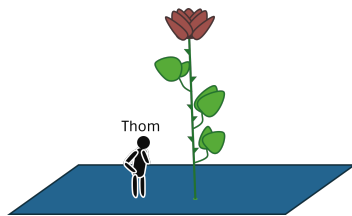
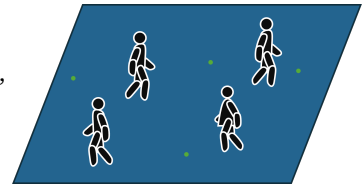
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A Catastrophe in Bloom

by Jan Wingenbach and Rolf Binder

*In a world as thin as a silken thread,
Where up and down are never said,
The flat folk lived in lines and squares,
With dotted things just everywhere.*

*A dot was naught—a common mark,
No bloom, no scent, no meaning, spark.*

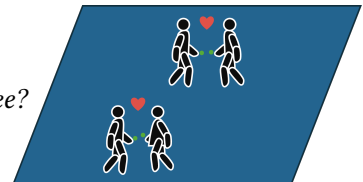


*Thom Zweistein, clever and bold,
Mapped folds from where the dot unrolled.*

*He saw in math and ink's design
A petal's curve, a thorny spine.*

*Now on birthdays, hand in hand,
They gift a rose across the land.*

*Two linear EPs in a 2D space
Seem simple, as if all's in place.
Hold fast, now wait and see,
For who knows what's in dimension three?*



*Cusp lines arise as EP threes align,
Fold surfaces form with curves refined.
Together they weave a pattern cryptic,
The elegant form of an umbilic elliptic.*

Story concept by Rolf Binder. Poem and illustration by Jan Wingenbach.

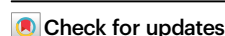
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Universal topology of exceptional points in nonlinear non-Hermitian systems

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Exceptional points are non-Hermitian degeneracies where eigenvalues and eigenvectors coalesce, giving rise to unusual physical effects across scientific disciplines. The concept of exceptional points has recently been extended to nonlinear physical systems. We theoretically demonstrate a universal topology in the nonlinear parameter space for a large class of physical systems that support second-order exceptional points in the linear regime. Knowledge of this topology (called elliptic umbilic singularity in bifurcation theory) deepens our understanding of second-order linear exceptional points, which here emerge as coalescence of four nonlinear eigenvectors. This helps guide future experimental discovery of nonlinear exceptional points and their classification, establish rigorous bounds of sensitivity enhancement of exceptional points in nonlinear systems, and helps envision and optimize technological applications of nonlinear exceptional points. Our theoretical approach is general and can be extended to nonlinear perturbations of third-order and higher-order exceptional points.

Exceptional points (EPs), which are spectral degeneracies of non-Hermitian systems at which both eigenvalues and eigenvectors coalesce (mathematical definitions and generalizations are in refs. 1–3), have been increasingly explored across a broad range of scientific areas, including anharmonic oscillators⁴, diffraction optics⁵, quantum chaos^{6,7}, conventional lasers^{8,9}, waveguides and photonic systems^{10–13}, atomic gases, condensates and high-Q cavity atomic lasers^{14–16}, polariton lasers, microlasers and polariton condensates^{17–25}, non-Hermitian Bose–Hubbard models²⁶, critical fluctuations and fluctuation spectra^{27,28}, and sensing applications^{29–34}, non-Hermitian parity-time (PT) symmetric quantum systems^{35–44}, and quantum state control and squeezing^{45,46}. EPs have also been studied in the context of topological insulators and other topological systems (for example refs. 47–56).

While in most of these physical realizations EPs emerge as eigenvalues λ of a linear non-Hermitian matrix M , in other words from the eigenvalue problem $M\mathbf{x} = \lambda\mathbf{x}$, more recent studies of EPs include nonlinear physical systems, such as coupled optical or electric resonators⁵⁷, models of population dynamics⁵⁸, as well as polariton

condensates in microresonators. In these systems, the complex interplay of nonlinear and non-Hermitian physics can lead to counter-intuitive phenomena, such as shifts^{59–62} and rotations⁶² of the EP in parameter space through blueshifts and saturable gain. Moreover, mode switching can be realized in bistable regions near the EP⁶³. In refs. 62,64, it was shown that nonlinearity can alter the eigenvalue splitting in the vicinity of an EP and highly sensitive EP-based sensors have been realized in nonlinear non-Hermitian systems^{65–67}. The underlying eigenvalue problem here may be nonlinear in the sense that the matrix M may depend on the eigenvalue λ (for example ref. 68), the eigenvector $\mathbf{x}$ (for example ref. 69), or both. In some instances, the nonlinear dependence on the eigenvector can be interchanged by that on the eigenvalue, for example ref. 34. Other kinds of nonlinear eigenvalue problems are discussed in ref. 70. In the following, we concentrate on the case where the nonlinear eigenvalue problem is formulated as a matrix equation where the matrix depends on the eigenvector. For example, the energies in an optical microcavity, which are described by the diagonal elements of M , can shift with the

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intensity of the light field described by $\mathbf{x}$, hence $M(\mathbf{x})\mathbf{x} = \lambda\mathbf{x}$. In the following, we refer to EPs that are solutions of a linear eigenvalue problem as ‘linear EPs,’ and those solving a nonlinear eigenvalue problem are called ‘nonlinear EPs.’

The wide range of physical systems in which EPs can occur raises the question whether there exist any guiding principles that help clarify the occurrence of EPs as the physical parameters (in particular the nonlinearity) entering the matrix M are varied.

In this paper, we demonstrate that important insight can be obtained on the distribution of EPs in parameter space, especially for nonlinear eigenvalue problems, with implications for photonic systems with Kerr-type nonlinearity, atomic^{71,72} and polariton condensates⁷³, superfluids⁷⁴ and coupled ring or micropillar optical⁷⁵ and electric resonators⁶⁶. Additional novel applications, such as robust wireless power transfer and enhanced frequency-comb generation are discussed in ref. 76. We show that a powerful way to do this is to consider the behavior of eigenvectors around EPs as a bifurcation problem and make use of established mathematical results in catastrophe theory.

We illustrate this rigorous procedure with an analysis of nonlinear parameter-dependent 2×2 matrices, schematically shown in Fig. 1, and show that a broad class of physically different nonlinear systems share the same EP neighborhood geometry that fixes one (and only one) topology, namely a cone-like structure (with deltoid cross section) that consists of three fold-bifurcation surfaces attached to each other at cusp-bifurcation ribs. As a corollary, we find the possible numbers of solutions (eigenvalues) in each region of parameter space. All these insights are based on one single equation (namely that for the so-called Lyapunov potential). Our findings can be used in future studies to establish rigorous bounds on the sensitivity enhancement of EPs in nonlinear systems. It also provides guardrails against possibly wrong conclusions, illustrated in Fig. 1b, that might be obtained from limited experimental observations or approximate theories.

To solve the highly sophisticated problem outlined above, we utilize the general theorems of a branch of bifurcation theory called catastrophe theory^{77–82}. Roughly speaking, the framework of catastrophe theory describes the shapes of singularity sets – or phase boundaries in nonequilibrium phase diagrams, to use a physicist’s language – related to singularities of potential functions V_L (Lyapunov potentials). The word catastrophe is used for the sudden qualitative change or bifurcation at a ‘phase boundary.’ Moreover, it lists all possible shapes (‘elementary catastrophes’) for cases in which the number of variables (often called state variables) and the number of control parameters (external parameters) are relatively small, see Supplementary Table 1 in the Supplementary Information.

Examples of systems that have been analyzed by catastrophe theory include dynamical systems obeying equations of motion of the form $\dot{\mathbf{x}} = -\text{grad}\{V_L(\mathbf{x})\}$, where the phase boundaries in the control parameter space follow from the assumption of degenerate stationary states (degenerate critical points), corresponding to $\text{grad}\{V_L(\mathbf{x})\} = 0$ and $\det\{h\} = 0$, where h denotes the Hessian matrix of V_L . Similarly, in optics, the solution of the Helmholtz equation, which describes the stationary solutions of a second-order time-differential equation, in contrast to the example of a first-order time-differential equation mentioned above, can be evaluated with the method of stationary phase and stationary points, as well as caustics occurring when two or more stationary points coalesce (p. 336 of ref. 83). This identifies rainbows and other optical phenomena as diffraction catastrophe, specifically fold and cusp bifurcations; see refs. 83–85 and Ch. 12 of ref. 82 for details. Higher bifurcations (catastrophes), such as the so-called elliptic umbilic, have also been studied in the context of optical diffraction^{85,86}. Many other areas of application of catastrophe theory can be found in refs. 7,11,77,80–82,87–90. This further extends to studies in biology, social sciences, and economics, with various degrees of acceptance because low-dimensional dynamical-systems models are not always appropriate for the simulation of complex systems in those sciences^{88,90}. In contrast, in many areas of physics low-

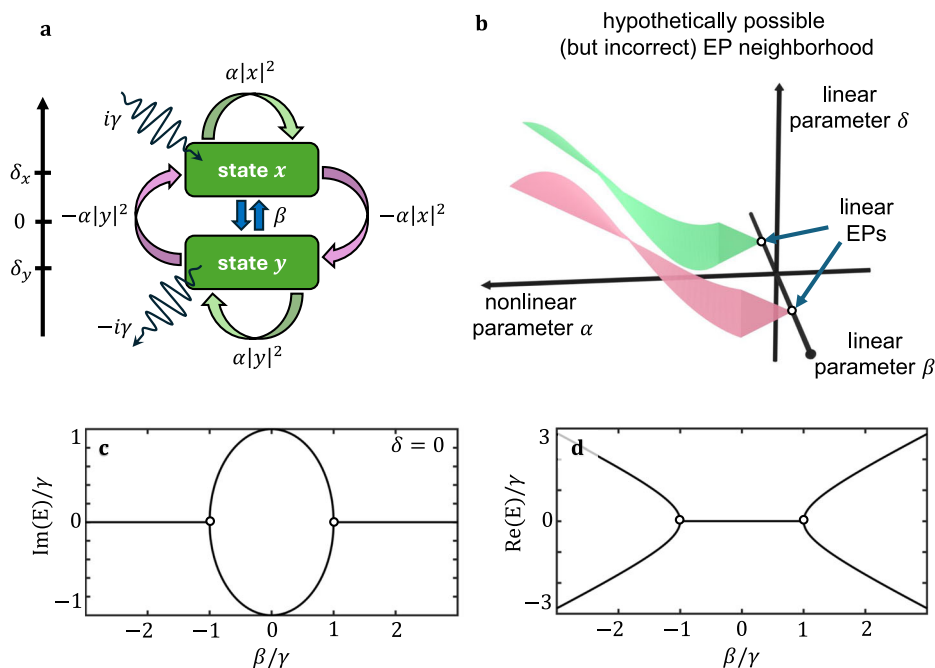


Fig. 1 | Non-Hermitian two-state system. **a** Sketch of a non-Hermitian and nonlinear two-state (or two-mode) system, with linear coupling β , energy splitting $2\delta = \delta_x - \delta_y$, and loss/gain difference 2γ , cf. Eq. (2). **b** Sketch of hypothetical location of exceptional points (EPs) as a function of physical parameters. Without a nonlinearity, $\alpha = 0$, isolated linear EPs are at $(\beta, \delta) = (\pm \gamma, 0)$; in the nonlinear regime,

$\alpha \neq 0$, EPs are implied on the colored surfaces. To illustrate the problem our study is solving, the surfaces we show here are one of infinitely many hypothetically possible (but generally wrong) EP neighborhoods. The correct geometric shape is shown in Fig. 2a. **c** and **d** show the eigenvalues in the linear case ($\alpha = 0$) with two isolated EPs at $(\beta, \delta) = (\pm \gamma, 0)$.

dimensional models and eigenvalue problems (for example two-level models) are well established.

In this study, we aim to understand the topological structure of the EP set around the linear EP in parameter space when nonlinearity in the matrix is turned on. Our strategy of utilizing catastrophe theory is as follows (more details are given in Methods and Supplementary Note 2 of the Supplementary Information). The layout of the catastrophe theory formulation consists of a state variable space and a control parameter space. The eigenvector elements of the nonlinear matrix are considered as values of state variables determined by the eigenvector equations, and the coordinates of the control parameter space are the parameters in the matrix. A Lyapunov potential function is constructed under the condition that its critical points (in state variable space) are equal to the eigenvector elements. The coalescing eigenvectors are accordingly given by the degenerate critical points of the Lyapunov potential. The set of degenerate critical points forms a hyper-surface in the combined state variable space and parameter space. The projection of this hyper-surface onto the parameter space yields the set of EPs in parameter space.

To obtain a topological characterization of the nonlinear EP set in parameter space around the linear EP, we Taylor expand the Lyapunov potential in both the state variables and the parameters about this point. This allows us to match our Lyapunov potential locally with one of the known catastrophes. See, for example, p. 137 of ref. 80 and p. 11 of ref. 81 for a list of the ‘elementary catastrophes’, which is also reproduced as Supplementary Table 1 in the Supplementary Information. We find that, for the broad class of nonlinear 2×2 matrices considered here, the matched catastrophe is the elliptic umbilic. Consequently, the EP sets (in parameter space) of these matrices share a common topological structure which is that of the set of degenerate critical points of this catastrophe.

Results

In the following, we present our approach for a large class of 2×2 matrices and analyze the EP structure in parameter space. We find that, for all matrices in this class, the linear EPs are the organizing points of an elliptic umbilic singularity. We stress that, in addition to the matrices discussed below, there may be other 2×2 matrices with elliptic umbilic nonlinear EP structures that we have not yet considered.

In the next subsection, *Example: quadratic nonlinearity*, we begin by illustrating our findings using a specific example, namely a symmetric matrix with Kerr-like nonlinearity, and in the following subsections we discuss extensions and generalizations of that example. All examples discussed below belong to the class of matrices that can be written in the following form (now using the symbol H as commonly used in the context of non-Hermitian Hamiltonians)

$$H = \begin{pmatrix} \delta + i\gamma + \alpha F(w) & b \\ c & -\delta - i\gamma - \alpha F(w) \end{pmatrix} \quad (1)$$

with $\delta, \gamma \in \mathbb{R}$, $b, c, \alpha \in \mathbb{C}$, and the nonlinearity $F(w)$ being a real-valued analytic function of the eigenvector parameter w in the vicinity of the linear EP. Specifically, the real-valued variable w is related to the (right) eigenvector of H , which, normalized to 1, is written as $[x \ y]^T = (1/A)[1 \ \tilde{w}]^T$, $\tilde{w} = we^{i\theta}$, $A = \sqrt{1 + w^2}$. The matrix in Eq. (1) is a useful starting point for our analysis. The fact that H is traceless, in other words that the average of the diagonal elements is zero, is not a restriction. Below, in the subsection *Extension beyond quadratic nonlinearities*, we will discuss in more detail the procedure of how to remove the average of the diagonal elements without changing the eigenvectors.

Example: quadratic nonlinearity

In order to give a concrete example, in this section we illustrate our approach for symmetric 2×2 matrices with a second-order linear EP

and a nonlinearity that is a quadratic function of the eigenvector variable w defined above (Kerr-like nonlinearity). Furthermore, we assume that at the EP, where $w = 1$, the lowest-order Taylor expansion of the nonlinearity function yields $F(1) = 0$ and $F'(1) \neq 0$. We write the nonlinear eigenvalue problem as

$$H \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & \beta \\ \beta & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = E \begin{bmatrix} x \\ y \end{bmatrix}, \quad (2)$$

where $a = \delta + i\gamma + \alpha(|x|^2 - |y|^2)$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are parameters, and all variables and parameters are assumed scaled and unitless. This parametric form is chosen for ease of analysis and the specific form of the nonlinearity includes a broad range of physical systems^{66,71–75}; the restrictions can be relaxed and the method is still applicable (see Sec. 2.2 and Supplementary Note 1 of the Supplementary Information for details). In the linear limit, the two states are separated by 2δ in frequency, and one state gains and the other dissipates at equal rates $|\gamma|$, Fig. 1c,d

Eliminating the eigenvalue E from Eq. (2) reduces the matrix equation to a nonlinear equation in the complex eigenvector element $\tilde{w}$, the real and imaginary parts of which are

$$2\alpha \left[\frac{1 - w^2}{1 + w^2} \right] + 2\delta + \beta \left[w - \frac{1}{w} \right] \cos \theta = 0, \quad (3)$$

$$2\gamma + \beta \left[w + \frac{1}{w} \right] \sin \theta = 0. \quad (4)$$

Formulating the eigenvalue problem this way enables the map to the function singularity problem to be constructed. In this context, we call (w, θ) the state variables and $(\alpha, \beta, \gamma, \delta)$ the control parameters. The first step in the map is to construct a Lyapunov function $V_L(w, \theta, \alpha, \beta, \gamma, \delta)$ such that its critical point equations in state variable space, $\partial V_L / \partial w = 0$, $\partial V_L / \partial \theta = 0$, are the same as the eigenvector equations (3) and (4). It can be directly verified that the Lyapunov function

$$V_L(w, \theta, \alpha, \beta, \gamma, \delta) = -2\alpha \ln[g/2] + 2\delta \ln w + \beta g \cos \theta - 2\gamma[\theta - \frac{3\pi}{2}], \quad (5)$$

with $g = w + w^{-1}$, satisfies this requirement.

To see the connection with catastrophe theory, we need to consider V_L in the vicinity of the linear EPs. Since V_L only has terms that are linear in the control parameters, we can reduce the dimensions of the parameter space by rescaling by one of the parameters, which we choose to be γ . The parameter space for the EP analysis, $(\alpha/\gamma, \beta/\gamma, \delta/\gamma)$, is thus three-dimensional. In the linear limit ($\alpha/\gamma = 0$), the matrix H has two separate EPs (bifurcation points) in the $(\beta/\gamma, \delta/\gamma)$ plane at $(\beta/\gamma, \delta/\gamma) = (\pm 1, 0)$. The coalescing eigenvector element (degenerate critical point) is $(w, \theta) = (1, 3\pi/2)$ for $\beta/\gamma = 1$ and $(w, \theta) = (1, \pi/2)$ for $\beta/\gamma = -1$. Since the critical point behavior around the two linear EPs is related by reflection symmetry about the plane $\beta = 0$, it suffices to analyze one, $\beta/\gamma = 1$, of the two. We choose, for convenience, the (w, θ) -independent terms in Eq. (5) for V_L to shift V_L to zero at the $\beta/\gamma = 1$ linear EP. We firstly consider the properties of the nonlinear EP set in the immediate neighborhood of this linear EP. We write $r = w - 1$, $\phi = \theta - 3\pi/2$, $\beta'/\gamma = \beta/\gamma - 1$ and denote by $V_{EU}(r, \phi, \alpha/\gamma, \beta'/\gamma, \delta/\gamma)$ the lowest-order terms in an expansion in $(r, \phi, \alpha/\gamma, \beta'/\gamma, \delta/\gamma)$ of V_L . Then, we find

$$V_{EU}(r, \phi, \alpha/\gamma, \beta'/\gamma, \delta/\gamma) = - \left[\frac{\phi^3}{3} - \phi r^2 + \frac{\alpha}{\gamma} r^2 - 2 \frac{\beta'}{\gamma} \phi - 2 \frac{\delta}{\gamma} r \right]. \quad (6)$$

The form of Eq. (6) matches that of a universal unfolding of the D_{-4} singularity, also called elliptic umbilic singularity, around the linear EP. Its degenerate critical points and bifurcations, which are well

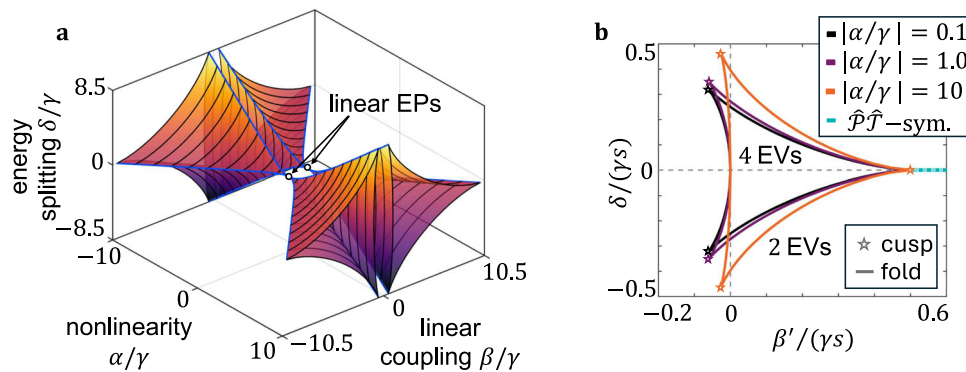


Fig. 2 | Elliptic umbilic singularity set. a Surface of exceptional point (EP) locations, shown as shaded areas, as a function of physical parameters (singularity set) explained in Fig. 1. In the weakly nonlinear regime, $\alpha \ll \gamma$, the shape approaches that of an elliptic umbilic bifurcation ('catastrophe'). The shape of each singularity set consists of two infinite three-cusped conical surfaces, the apices of which meet at the two linear EPs $(\alpha, \beta, \delta) = (0, \pm \gamma, 0)$. At large α , the three-cusped shape remains but is deformed guaranteeing that the two cone-like surfaces originating at $\beta = \pm 1$

do not overlap as shown in **b**. **b** Cross section of EP surface for fixed nonlinearity α ; axis scaled with scale factor $s = 2(\sqrt{1 + \alpha^2/\gamma^2} - 1)$. Here, $\beta' = \beta - \gamma$. Small $\alpha \ll \gamma$ (vicinity of linear EP, Eq. (6)): deltoid or three-cusped hypocycloid^{80,82,102} of the elliptic umbilic surface with exact scaling $s = \alpha^2/\gamma^2$. Large α (farther away from linear EP, Eq. (5)): exact scaling is lost, but three-cusped shape remains. The number of eigenvectors (EVs) inside and outside the cone is indicated. The half-infinite line where parity-time (PT) symmetry is unbroken is indicated by the blue line.

understood, are obtained as simultaneous solutions to three equations: $\partial V_{EU}/\partial r = 0$, $\partial V_{EU}/\partial \theta = 0$, and $\det[h] = 0$, where h is the Hessian matrix of V_{EU} . The corresponding set of three equations, with V_L taking the place of V_{EU} , give the EP set in the whole parameter space (the singularity set). Since there are five unknowns, $w, \theta, \alpha/\gamma, \beta'/\gamma, \delta/\gamma$, the solution set of the three equations forms a 2D surface in the 5D combined state variable and parameter space. The projection of this solution surface onto the parameter space is the singularity set. Algebraic detail of the solution set can be found in Supplementary Note 3 of the Supplementary Information.

The singularity set, plotted in Fig. 2a, consists of a pair of two conical surfaces with deltoid cross sections, the apices of which meet at the linear EPs at $(\alpha, \beta', \delta) = (0, 0, 0)$ and $(0, -2\gamma, 0)$. There are four and two critical points in state variable space at each control parameter point inside and outside the cones, respectively; also see Fig. 2b. The ribs of each cone are cusp points where three critical points inside the cone meet and emerge as one critical point outside the cone. Each point on the surface between two ribs is a fold point where two critical points inside the cone meet and annihilate each other, thus vanishing on the outside. This arrangement of the fold and cusp singularities is characteristic of the elliptic umbilic catastrophe (D_{-4}). We call the meeting point of the cones the 'organizing point' of the catastrophe. We show in Fig. 2b that the cone-like topological structure is not limited to the immediate neighborhood of the linear EP, but remains valid even at large nonlinearities (large α). More information on the EP set and the Lyapunov potential is given in Supplementary Note 4 and 5 of the Supplementary Information.

Interestingly, we see that the non-trivial topology of the set of EPs in the nonlinear parameters space (the parameter space that includes the nonlinearity α , as shown in Fig. 2a), is already fully determined by the EPs in the linear space (the subspace of parameters where the nonlinearity is zero).

In other words, there are infinitely many possibilities for the EP topology in the nonlinear case (cf. Fig. 1b), but already from the properties of the linear EP alone we find that the topology must be the cone-like structure. At small nonlinearities, this structure is of the form of an elliptic umbilic, a cone with a deltoid cross section (and the cone-like structure can persist at larger nonlinearities, as in our example, preserving the topology of the elliptic umbilic but not its precise geometry). This is a remarkable statement, since usually one would try to find the set of EPs in the nonlinear parameter space by searching or scanning the part of the parameter space where the nonlinearity is nonzero. To be more specific, if in Eq. (6) we set $\alpha = 0$,

$\beta' = 0$ and $\delta = 0$, we are left with the so-called 'germ' (sometimes called degenerate core) of the catastrophe. For catastrophes with codimension up to 5 (called elementary catastrophes), the germs can be found in textbooks, for example ref. 33,81 and are reproduced in Supplementary Table 1. Each germ gives an unambiguous assignment of the corresponding elementary catastrophe. Since, in our case, the Lyapunov potential (as a polynomial in the state variables given by the eigenvectors) that generates the eigenvector equations of the nonlinear eigenvalue problem exists, we can predict the topology of the EP singularity set in the full parameter space solely from the Lyapunov potential where all parameters (including the nonlinearity) have been set to zero. The precise geometric shape can change under smooth transformations of parameters, but its topological properties are determined by the universal unfolding (which is defined as the versal unfolding of minimal dimension, see, e.g., p. 621 of ref. 81) of the elliptic umbilic singularity.

Regarding PT symmetry^{35–44}, we note that the Hamiltonian in Eq. (2) is PT-symmetric, in other words $[H, PT] = 0$, with parity represented by the σ_1 Pauli matrix, if $\delta = 0$. In that case, PT symmetry is unbroken (the eigenenergies are real and the eigenvectors are common to H and PT) if $|\beta| > \sqrt{\alpha^2 + \gamma^2}$. This region is indicated as thick blue line in Fig. 2b. In Fig. 2a, it is the part of the $\delta = 0$ plane outside of the deltoid cone.

The power of using catastrophe theory in the context of EPs can be seen from the following considerations. First, the information on the geometric structure of the nonlinear EPs provides immediate knowledge of the number of eigenvalues as a function of the parameters used. The potential V_{EU} has four critical points inside the cone and two outside the cone; exemplary plots of V_{EU} are shown in Supplementary Fig. 3 of the Supplementary Information. This implies that we have four eigenvalues inside the cone and two outside, and the number changes at the folds and cusps, which represent the location of the EPs. This is illustrated in Fig. 3, which shows the eigenvalues of H along the straight lines in (δ, β') -parameter space as indicated in each row of subplots. At the apices of the cones that are shown in Fig. 2a, the four critical points coalesce, showing that, depending on how the limit is performed, the linear EP can be viewed as the limiting case of a fourth-order nonlinear EP. A stability analysis of the solutions is planned as future work. We also note that F might be a function of both w and θ , and the eigenvector equations may not be derivable from a Lyapunov potential. General bifurcation theory may be needed to study the conditions under which we still have an elliptic umbilic singularity.

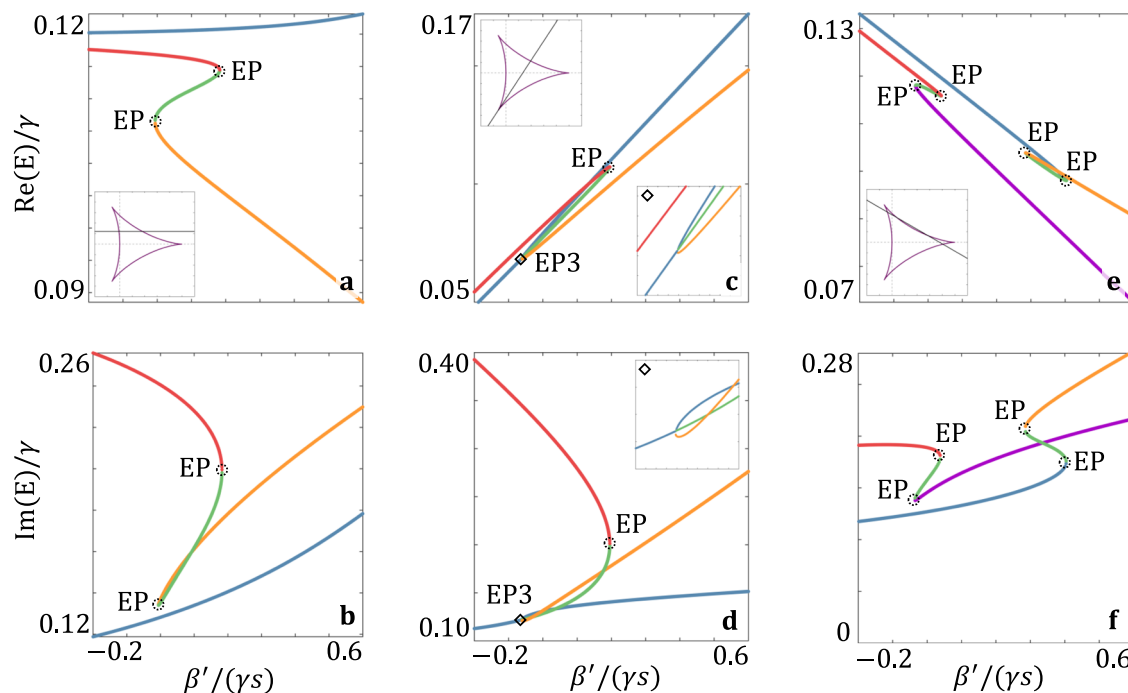


Fig. 3 | Selected complex eigenvalue traces. Complex eigenvalues (real and imaginary parts) of H in Eq. (2) at fixed, nonzero $\alpha/\gamma = 1$ across lines in the (β', δ) plane. The insets not marked by diamonds show the (β', δ) plane and the line along which the spectra shown in the main parts of the figures are evaluated. Regardless of the lines' directions, we have two and four eigenvalues outside and inside the cone, respectively, in agreement with the number of values of critical points of the

Lyapunov potential (see Supplementary Fig. 3 for examples of the potential landscape). At the smooth parts of the deltoid boundary (fold lines), two states are created/annihilated, and two states are unaffected by the boundary (a, b, e, f). Crossing the cusp, (c, d), three states coalesce at a third-order nonlinear exceptional point (EP), denoted as EP3, and one state is unaffected. The insets marked by diamonds show a zoomed-in plot of the eigenvalue traces close to the EP3.

Extension beyond quadratic nonlinearities

In the previous subsection we discussed the example of a Kerr-like nonlinearity. We note, however, that the power of catastrophe theory lies in the fact that the linear matrix already fixes the elliptic umbilic catastrophe, and a broad class of matrices extending our case given in Eq. (2) will have the same topological singularity structure. Let us first note that the matrix in Eq. (2) is equivalent, as far as the topology of the EP object in parameter space is concerned, to one that contains self coupling (α_c) and cross coupling (α_x) terms with quadratic nonlinearities:

$$H_1 = \begin{pmatrix} \delta + i\gamma + \alpha_c |x|^2 + \alpha_x |y|^2 & \beta \\ \beta & -\delta - i\gamma + \alpha_c |y|^2 + \alpha_x |x|^2 \end{pmatrix}. \quad (7)$$

Here, the parameter α used above corresponds to $\alpha = (\alpha_c - \alpha_x)/2$. The two matrices differ by a multiple of the unit matrix, and thus have the same eigenvectors while the eigenvalues differ only by a constant shift $\bar{\alpha} = (\alpha_c + \alpha_x)/2$. Here, we assume the couplings α_c and α_x to be real-valued, but below we will see that they also can be imaginary (which in some applications can be related to gain saturation) or complex.

We can generalize the matrix in Eq. (7) to allow for sufficiently smooth real-valued nonlinearity functions $f(u)$ (for which conditions will be given below) and with complex nonlinearity parameters α_c, α_x

$$H_1 = \begin{pmatrix} \delta + i\gamma + \alpha_c f(|x|) + \alpha_x f(|y|) & \beta \\ \beta & -\delta - i\gamma + \alpha_c f(|y|) + \alpha_x f(|x|) \end{pmatrix}. \quad (8)$$

Similar to the case discussed above, we can subtract a diagonal matrix that is a unit matrix multiplied by the (nonlinear) constant $g(\mathbf{x}) = \bar{\alpha}(f(|x|) + f(|y|))$, that is we define $H_1 = g(\mathbf{x})\hat{1} + H$, where $\hat{1}$ is the

unit matrix, and obtain

$$H = \begin{pmatrix} \delta + i\gamma + \alpha(f(|x|) - f(|y|)) & \beta \\ \beta & -\delta - i\gamma - \alpha(f(|x|) - f(|y|)) \end{pmatrix}. \quad (9)$$

Once the nonlinear eigenvalue problem of H has been solved and the eigenvalues λ_n and eigenvectors $\mathbf{x}_n$ are known, we evaluate the constant $g(\mathbf{x})$ with the eigenvectors $\mathbf{x}_n$ and obtain the eigenvalues of H_1 in Eq. (8) as $\lambda_n^{(1)} = \lambda_n + g(\mathbf{x}_n)$, while the eigenvectors for H_1 and H are the same. As in the previous section, we write the eigenvector elements as functions of w and θ , and note that the moduli of both components, $|x| = 1/\sqrt{1+w^2}$ and $|y| = w/\sqrt{1+w^2}$, only depend on the real variable w . Hence the nonlinear term can be written as

$$\alpha(f(|x|) - f(|y|)) = \alpha F(w) \quad (10)$$

with

$$F(w) = f\left(\frac{1}{\sqrt{1+w^2}}\right) - f\left(\frac{w}{\sqrt{1+w^2}}\right) \quad (11)$$

Similar to the example of the quadratic nonlinearity discussed in the previous section, the linear EP is at $w = 1$. As mentioned at the beginning of the Results section, the condition for the elliptic umbilic singularity is that at the linear EP (where $w = 1$ in the case of symmetric matrices), the function $F(w)$ must be approximated by a power expansion. Therefore, important examples of functions with elliptic umbilic singularity sets include all power functions $f(u) = u^n$ with nonzero exponent, $n \neq 0$, such as the quartic nonlinearity $f(u) = u^4$, and periodic functions, such as $f(u) = \cos(u^2)$.

Different Taylor expansions

So far, we have limited the analysis to symmetric matrices where, at the EP (that is for $w = 1$) the Taylor expansion of the nonlinearity function $F(w)$ yields $F(1) = 0$ and $F'(1) \neq 0$. We note, however, that the theory remains valid for any real-valued function $F(w)$ that can be approximated by a Taylor expansion in the vicinity of the EP. In this context, we consider separately four categories of functions that yield elliptic umbilic EP structures that include and extend the example discussed in Sec. 2. and 2.2. These are (1) $F(1) \neq 0$ and $F'(1) \neq 0$, (2) $F(1) = 0$ and $F'(1) \neq 0$, (3) $F(1) \neq 0$ and $F'(1) = 0$, and (4) $F(1) = 0$ and $F'(1) = 0$. Here category (2) is the one discussed above. We note that in the four categories the elliptic umbilic singularity sets emerge as functions of different parameters/variables. But common to these categories is the elliptic umbilic topology, in other words their singularity sets are topologically equivalent. Further details about the Lyapunov functions for the four categories are given in Supplementary Note 1, Sec. 1.B.1, of the Supplementary Information. Examples of matrices belonging to category (1) include the Hamiltonian in Eq. (1) of ref. 66, given for coupled electric resonators, evaluated without noise, and Eq. (5) of ref. 34, with applicability for example to ring or micropillar optical resonators.

Asymmetric matrices, complex nonlinear coupling constants

We have also considered the case of asymmetric matrices, with the off-diagonal parameters b and c in the Hamiltonian Eq. (1) being independent complex parameters. Our findings are summarized here, and the algebraic derivation is detailed in Supplementary Note 1, Secs. 1.B.3 and 1.B.4, in the Supplementary Information. As in the symmetric-matrix case, we construct a Lyapunov potential for the eigenvector elements of this asymmetric Hamiltonian in Supplementary Note 1, Sec. 1.B.3. A straightforward expansion around the linear EP shows that the germ (the limiting Lyapunov potential as a function of (r, ϕ) when the parameters are set at the EP values) formally appears as the sum of two elliptic umbilics. We derive in Supplementary Note 1, Sec. 1.B.4, a coordinate transformation that maps this structure into a canonical elliptic umbilic in the new coordinates. This shows that the (linear and nonlinear) EP sets also possess the topological structure of the elliptic.

We find that the elliptic umbilic topology is also retained when the nonlinear coupling constant α is imaginary or complex. We comment briefly on this case in Supplementary Note 1, Sec. 1.B.2, in the Supplementary Information.

Symmetry in nonlinear coefficient α

We also mention again that the EP structure shown in Fig. 2a is symmetric with respect to the Kerr-like nonlinearity α and the linear coupling β (in other words invariant under $\alpha \rightarrow -\alpha$, $\beta \rightarrow -\beta$). For real-valued self (α_c) and cross (α_x) couplings in Eq. (7), it is instructive to compare our nonlinear 2×2 eigenvalue problem with the nonlinear Schrödinger equation (NLS). If the NLS is derived from the Helmholtz equation in paraxial approximation, then the spatial propagation coordinate z plays the role of time and the diffraction (second-order derivative in the transverse direction) plays the role of the kinetic energy, see e.g. ref. 91. While the general solution structure of the NLS includes effects like soliton formation and is generally complicated (and the NLS is not integrable in more than one spatial dimensions), it has been shown in ref. 92 that coupled-mode analysis in the nonlinear regime can be done with the NLS under suitable conditions. If one would use two modes with different transverse extensions, effects analogous to self-focusing and self-defocusing could be described by a 2-mode model. In the language of our eigenvalue equation, the eigenvalue E would correspond to the wave vector in the propagation direction z , and the coupling coefficients α_c and α_x would depend on the spatial overlap between the two modes. More detailed studies of the discrete NLS (also called discrete self-trapping equation) are provided in ref. 93, but most studies of the discrete NLS, including 2-mode models that are equivalent to the (nonlinear) pendulum⁹⁴, do not include decay, which

is essential in our study of the EP structure. If, however, a self-focusing and self-defocusing system with decay can be described by our 2-mode model, Eq. (7), with the difference in the absorption coefficients of the two modes given by γ , then we predict that the EP structure is symmetric in $\alpha = \frac{1}{2}(\alpha_c - \alpha_x)$ as well as the linear coupling β and the energy splitting δ , as shown in Fig. 2a.

Larger matrix dimensions

For $N \times N$ matrices with $N > 2$, the dimension of the state variable space increases with N , the set of linear EPs is expected to have more structure than that of the $N = 2$ case. More complicated, higher-order singularities, e.g. parabolic umbilics or singularities outside of the elementary catastrophe set, may be involved, most of which have higher codimensions (minimum dimension of the parameter space to capture the full structure of the singularity set) than the elliptic umbilic. But even when a relatively complete analysis comparable to the 2×2 case cannot be done, using the tools of catastrophe theory and bifurcation theory may still provide fruitful insight into the topological/geometric structure of the singularity set. We also note that in higher dimensional systems that can, for example, systematically be realized and explored with coupled ring resonators^{10,75} (described by $N \times N$ matrices with $N > 2$), even in the linear case we can have EPs of various orders, for example a 3×3 matrix can have second and third-order linear EPs. The structure of the nonlinear EPs can then be expected to be more complex than the case of nonlinear EPs discussed above for the 2×2 matrix. We found that in the 2×2 case where a Lyapunov potential exists, the coalescence of eigenvectors agrees with the condition of singularities of the Lyapunov function, and - as is already known⁶⁵⁻⁶⁷ - the order of the nonlinear EPs can be higher than that of the linear EPs (in our case, the linear problem can only have second order EPs while the nonlinear problem has up to fourth-order nonlinear EPs).

Discussion

In summary, we have shown that the rich interplay of nonlinearity and non-Hermiticity at EPs results in a non-trivial topological structure of the EPs in the space of parameters including the nonlinearity parameter α . This structure (or singularity set) is called elliptic umbilic and shown in Fig. 2a. Our findings apply to a large class of nonlinear systems that include—but are not limited to—all power-law nonlinearity functions $f(u)$ in the effective non-Hermitian Hamiltonian Eq. (8) with self-coupling and cross-coupling, where the coupling coefficients can be real (in optics related to nonlinear refractive index coupling), imaginary (often related to nonlinear decay or gain), or complex. Asymmetric matrices with complex parameters in the off-diagonal elements also belong to this class. To the best of our knowledge, the class of matrices covered by our theory includes most, if not all, nonlinear EP systems that are presently discussed in the literature (Kerr-type nonlinearity, atomic and polariton condensates, superfluids, and coupled ring or micropillar optical and electric resonators). Further extensions of the class of matrices with the elliptic umbilic topological structure may be possible.

Specifically, we find that, by mapping EPs to degenerate critical points of a potential, catastrophe theory predicts that a second-order linear EP unfolds into the elliptic umbilic form. In practical terms, the EP perturbed by nonlinearity will spawn two touching cones of EPs with tri-cusped boundaries, independent of the underlying physical model. Interestingly, the non-trivial topological properties of the set of EPs in the space of parameters including the nonlinearity parameter α can be predicted solely from the Lyapunov potential at the linear EP (where $\alpha = 0$). This is a strong statement made possible by catastrophe theory, and in this sense, the topology can be regarded as 'universal'. From this topological viewpoint, the original linear EP acts as the organizing center of a higher-order singularity: a second-order EP becomes effectively fourth-order in the extended space of nonlinear

parameters. These conclusions are the same for all physical platforms, e.g., photonic, atomic, and condensed-matter platforms. Our approach can be extended to the case of third (or higher) order linear EPs (e.g., refs. 43,95–97) perturbed by nonlinearities. Since, in the linear case, we find the germ to be that of an elliptic umbilic catastrophe, we conjecture that all nonlinearities that are real or complex functions of the variable w (and possibly both w and θ) defining the eigenvectors, and that are analytic in a neighborhood around the linear EPs and fulfill the relaxed vanishing-curl condition (see discussion following Eq. (16) in Methods), will lead to locally elliptic umbilic singularity sets.

Beyond the conceptual advance, our results have practical implications. Knowing the canonical cone-and-cusp shape of nonlinear EPs provides a clear blueprint for engineering non-Hermitian devices. For example, one can predict how tuning parameters (gain saturation, coupling strengths, etc.) will move or create EPs in polariton systems and lasers⁹⁸, ring or micropillar optical resonators^{10,34,75}, or coupled electric resonators (including PT-symmetric circuits)⁶⁶. Crucially, because EP-based sensors and mode-control schemes depend on the topology of degeneracies, this universality will make it possible to establish rigorous bounds on the enhancement of sensitivity of EPs in nonlinear systems and enables the design of robust, high-sensitivity devices across different platforms. Our prediction of the singularity structure of EPs in the nonlinear regime should enable systematic studies and possible optimization of the renormalization of EPs due to noise and stochastic processes, e.g. refs. 34,57,66,97, as well as studies of Liouvillian EPs^{99,100}. From a broader perspective, identifying a common “fingerprint” for EPs highlights a deep unity in non-Hermitian physics. We expect these principles to guide future explorations – for instance in higher-dimensional multi-mode or many-body setups – and to inform the development of new spectroscopic and sensing techniques grounded in the universal topology of EPs.

Methods

The methodology of our theoretical approach is based on a branch of bifurcation theory called catastrophe theory. Here, we summarize the main aspects used in this method of analyzing nonlinear systems. More details are given in Supplementary Note 2 of the Supplementary Information.

We begin with a brief review of catastrophe theory. Consider a smooth function (Lyapunov potential) $V_L(x_1, x_2, \dots, x_n, a_1, a_2, \dots, a_m)$ of n state variables $x_i, i = 1, \dots, n$ and m control parameters $a_j, j = 1, \dots, m$. The critical points of V are points in state space where (solutions in state space to the equations)

$$\frac{\partial V_L}{\partial x_i} = 0, i = 1, \dots, n. \quad (12)$$

Catastrophe theory seeks to classify the qualitative behavior of the critical points when the parameters are varied. If we think of the parameter space as a ‘phase diagram’, qualitative changes of critical point behavior happen at parameter space points for which V has degenerate critical points (singularities). At these points (in parameter and state spaces), the Hessian matrix of second derivatives is singular,

$$\det \left[\frac{\partial^2 V_L}{\partial x_i \partial x_j} \right] = 0, \quad (13)$$

and multiple critical points coincide. We call the set of points in parameter space where V_L has degenerate critical points the singularity set. The theory classifies the structure of the singularity set locally around ‘organizing points’. Simple structures include folds, cusps, and umbilics. Complete tables for the elementary catastrophes with codimension (defined below) up to 5 were given in refs. 80,81. The table is reproduced as Supplementary Tab. 1 in the Supplementary

Information. For each catastrophe in the table, $x \equiv x_1, y \equiv x_2$ are state variables, and the other lower-case letters in the Perturbation column are parameters. For a given parameterized function V_L , an isolated point in the singularity set is fixed as the organizing point (in parameter space). At this parameter value, V_L as a function of the state variables has one or more degenerate critical points, one of which is chosen for consideration. The state coordinates are redefined with the origin shifted to this degenerate critical point, and the value of V_L is also shifted to zero at this point. To match a particular catastrophe, one may Taylor expand $V_L(x, y)$, with parameters set at the organizing point values, around the origin and compare the lowest-order terms with the standard forms in the Germ column in Supplementary Tab. 1. A smooth coordinate transformation may be needed to match the shown standard form. The singularity set, when restricted to the organizing point, may be structurally unstable: the geometric structure of the singularity set, and other properties, may change under small perturbations. The Perturbation column in Supplementary Tab. 1 shows some standard forms of perturbations that stabilize the singularity set: its geometric structure does not change under further small perturbations. The parameters equal zero at the organizing point. For each catastrophe, the number of parameters in the shown perturbation is the minimum number that can affect the stabilization. This minimum number is the codimension of the catastrophe. A more mathematically precise explanation of catastrophe theory terminology and concepts, such as germs and codimension, can be found in, for example, ref. 101.

Next, we discuss EPs of the (nonlinear) eigenvalue problem and the relation to catastrophe theory. Consider an eigenvalue problem with an $N \times N$ matrix which is nonlinear in the eigenvector. Suppose the eigenvector is defined by n real variables (a complex variable is considered as a pair of real variables), with $n \leq 2N$. Call these variables $x_i, i = 1, \dots, n$. Suppose also that the matrix depends on m parameters $a_j, j = 1, \dots, m$ and the eigenvector equation, after elimination of the eigenvalue, can be written as a set of n equations

$$f_k(x_1, \dots, x_n, a_1, \dots, a_m) = 0, k = 1, \dots, n \quad (14)$$

For each fixed parameter set $(a_j, j = 1, \dots, m)$, the solution $(x_i, i = 1, \dots, n)$ to Eq. (14) gives the eigenvector. The EPs are points in parameter space for which multiple eigenvectors, or equivalently, multiple solutions $(x_i, i = 1, \dots, n)$ to Eq. (14), coalesce.

If a potential function $V_L(x_1, x_2, \dots, x_n, a_1, a_2, \dots, a_m)$ such that

$$f_k(x_1, \dots, x_n, a_1, \dots, a_m) = \frac{\partial V_L}{\partial x_k}, k = 1, \dots, n \quad (15)$$

can be constructed, then the nonlinear eigenvalue problem can be ‘mapped’ onto the catastrophe theory formalism. The solutions $(x_i, i = 1, \dots, n)$ to Eq. (14) are the critical points of V_L , and the machinery of catastrophe theory can be used to analyze the structure of the set of EPs, which is just the singularity set (parameter space subset where V_L carries degenerate critical points). In our application, the linear EP is chosen as the organizing point (see previous subsection) and structurally stabilizing perturbations are identified.

A necessary condition for the potential V_L to exist is that the set $f_k, k = 1, \dots, n$ have ‘vanishing curl’:

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0, i, j = 1, \dots, n \quad (16)$$

But for our purpose here, this condition can be slightly relaxed. Even if the f_k in question does not satisfy Eq. (16), if a related set of functions, $f'_k(x_1, \dots, x_n, a_1, \dots, a_m), k = 1, \dots, n$, has the same solutions when used in Eq. (14) and is the gradient of a potential V_L , then the critical-point analysis of V_L is applicable to the EPs of f_k . This is the case for the

eigenvector equations considered in this paper. The left hand sides of the real and imaginary parts of the eigenvector equations are equal to $w\partial V_I/\partial w$ and $-\partial V_I/\partial\theta$ respectively.

Data availability

The data for the main figures of this paper, Fig. 2a and b, and 3, have been deposited in the GitHub database under accession code <https://doi.org/10.5281/zenodo.17589377>

Code availability

The Matlab code that generated the surface of the main figure (Fig. 2a), cross sections of the surface (Fig. 2b), and complex eigenvalue trajectories (Fig. 3) have been deposited in the GitHub database under accession code <https://doi.org/10.5281/zenodo.17589377>

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Author contributions

N.H.K., J.W., S.S. and R.B. conceived the project, N.H.K. developed the bifurcation theory, J.W. provided numerical solutions, L.A., J.S. and X.M. provided the solution algorithm for the nonlinear problem, R.B. and S.S. wrote the paper with input from all authors.

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Additional information

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Exciton-polariton condensates 5

This chapter introduces the physical platform used throughout this thesis to investigate exceptional points in nonlinear and non-Hermitian systems. Optical resonators present a prominent platform for this subject, with a special focus on ring resonators, which can exhibit nonlinearities such as gain saturation [60, 62, 63]. Here, however, we focus on exciton-polariton systems inside planar optical resonators, which intrinsically combine strong nonlinear interactions with non-Hermitian dynamics [66]. Exciton-polaritons are hybrid light-matter particles emerging in the strong-coupling regime in semiconductor microcavities [114, 115]. Semiconductor microcavities consist of two main components: (i) an optical resonator formed by reflective layers such as Bragg reflectors, typically distributed Bragg reflectors, and (ii) an excitonic active material embedded within the cavity, like semiconductor quantum wells, 2D materials, organic crystalline systems, or perovskite-based systems [116–118]. Figure 5.1(a) displays a sketch of a semiconductor microcavity. The system is inherently non-Hermitian due to the finite lifetime of cavity photon modes, arising from radiative and absorptive losses. Photons inside the cavity can be absorbed by the quantum wells, creating an exciton. An exciton is a bound electron-hole pair, i.e., an electron in the conduction band with angular momentum $J_z^e = \pm 1/2$ and a hole in the valence band with angular momentum $J_z^h = \pm 1/2; \pm 3/2$. Optical selection rules imply that only excitons with total angular momentum $J_z = \pm 1$ couple to circularly polarized cavity photons, while states with $J_z = \pm 2$ remain optically inactive and are therefore referred to as dark states [119, 120].

The excitons exhibit strong nonlinear interactions due to Coulomb interactions of their composite fermionic structure and phase-space filling effects [121]. Upon recombination, excitons emit photons that either escape the cavity or are reabsorbed, thereby sustaining light-matter coupling. The Q-factor of a cavity is given by the ratio of the stored energy to its power loss.

In the strong-coupling regime, the exciton-photon normal-mode splitting can be spectrally resolved, and the system eigenmodes describe a linear combination of excitons and photons [122, 123]. The resulting exciton-polaritons are referred to as polaritons in the further course of this work. The eigenvalue problem for the coupled exciton Ψ_{ex} and photon wavefunctions Ψ_{phot} in semiconductor microcavities reads [123]

$$\begin{bmatrix} \hbar\omega_k & g/2 \\ g/2 & \varepsilon \end{bmatrix} \begin{bmatrix} \Psi_{\text{phot}} \\ \Psi_{\text{ex}} \end{bmatrix} = E_k^{\text{LPB,UPB}} \begin{bmatrix} \Psi_{\text{phot}} \\ \Psi_{\text{ex}} \end{bmatrix}. \quad (5.1)$$

Here $\hbar\omega_k = \hbar\omega_0 + \hbar^2 \mathbf{k}^2 / 2m$ denotes the quadratic dispersion of cavity photon modes in the regime of small in-plane wave vectors $\mathbf{k}$, with the effective photon mass $m \approx 10^{-4} m_e$. Here m_e denotes the free electron mass. Compared to this, effective exciton masses are about $0.1m_e$ to $1.0m_e$, thus the exciton dispersion can be modeled as a constant ε (approximately flat). g describes the rate of conversion between excitons and photons and thus depends on their spatial overlap and the dipole matrix element for a photon to excite a single exciton.

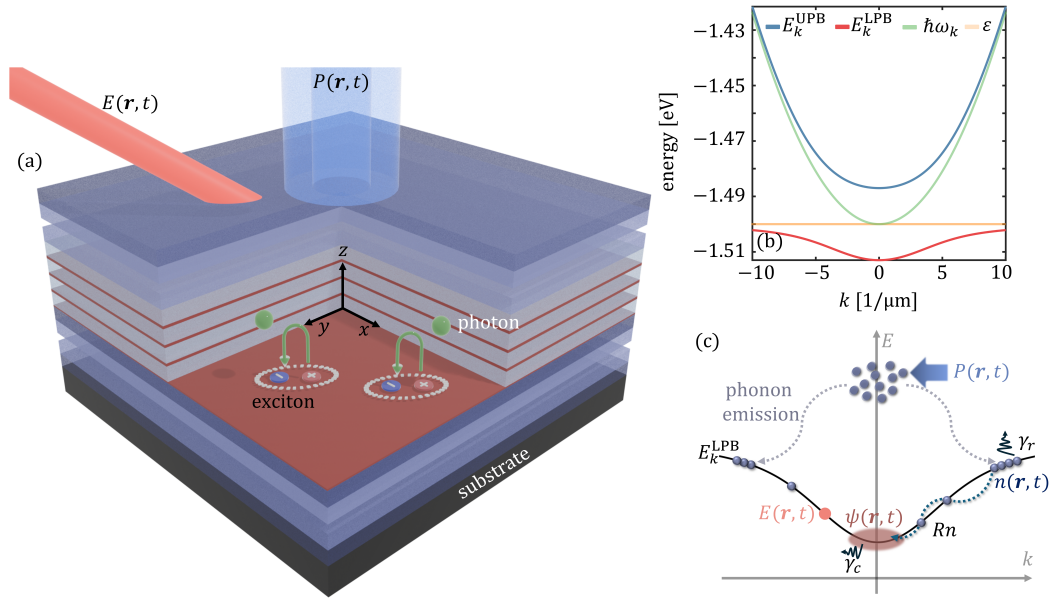


Figure 5.1: Overview on exciton-polariton condensates. (a) Semiconductor microcavity consisting of Bragg reflectors (blue) and quantum wells (red and gray). Resonant $E(\mathbf{r}, t)$ and nonresonant excitation $P(\mathbf{r}, t)$ can excite excitons. In the strong-coupling regime, exciton-polaritons form under the hybridization of excitons and photons. Inspired by [122]. (b) The exciton (orange) and photon dispersions (green) are displayed together with the hybridized upper (blue) and lower (red) polariton dispersion branches. (c) Polariton condensation under nonresonant excitation. While resonant excitation can form polaritons at defined phase and frequency, nonresonant excitation feeds an incoherent excitonic reservoir $n(\mathbf{r}, t)$ on the lower polariton branch (LPB). Stimulated in-scattering Rn leads to condensation $\psi(\mathbf{r}, t)$ in the minimum of the LPB. The system is inherently non-conservative: constant particle supply is required to compensate for the reservoir loss rate γ_r and condensate loss rate γ_c . Inspired by [124].

Normal-mode splitting can be resolved when the exciton-photon coupling exceeds the

relevant photon and exciton linewidths. In this so-called strong-coupling regime, the eigenmodes are no longer purely photonic or excitonic but hybridize into exciton-polaritons [125, 126]. Solving this eigenvalue problem leads to the upper (UPB) and lower branch (LPB) of the hybridized polariton dispersion

$$E_k^{\text{LPB,UPB}} = \frac{(\hbar\omega_k + \varepsilon) \mp \sqrt{(\hbar\omega_k - \varepsilon)^2 + g^2}}{2}. \quad (5.2)$$

In the following, we focus on the lower polariton branch. At resonance $\hbar\omega_0 = \varepsilon$, it exhibits nearly parabolic dispersion at low momenta with $m_{\text{pol}} = 2m$ [123]. At larger momentum, it asymptotically approaches the exciton dispersion, as shown in Figure 5.1(b).

Conventional resonant excitation with $E(\mathbf{r}, t)$ can drive polariton modes at defined phase and frequency, requiring fine tuning of the optical source [127]. However, polaritons can also condense under nonresonant excitation with $P(\mathbf{r}, t)$. This condensation process is explained following the sketch displayed in Figure 5.1(c). Nonresonant excitation creates high-energy excitations, which cool down by phonon emission until they occupy states on the lower polariton branch at finite momentum. Here, the particles form an incoherent reservoir $n(\mathbf{r}, t)$ with loss rate γ_r given by the inverse particle lifetime. Stimulated scattering between these particles leads to an in-scattering Rn into the minimum of the lower polariton branch, where the polaritons start forming a coherent polariton condensate exhibiting a loss rate γ_c [128, 129]. Due to the finite polariton lifetime (typically on the orders of 10^1 – 10^2 ps), polariton condensates are intrinsically dissipative systems and therefore operate out of equilibrium [130, 131]. Continuous pumping is required to balance losses, making them a natural playground for non-Hermitian physics.

In the following, the nonlinear character of polariton condensates is discussed. The excitonic component of polaritons gives rise to strong polariton-polariton interactions, driven by short-ranged electron-electron exchange interaction at low momenta [132, 133]. This results in an effective Kerr-like nonlinearity with interaction energies on the order of a few meV [134]. This enables fascinating phenomena including superfluidity [135], soliton formation [136], and multistability [137]. The latter arises from the interplay of nonlinear interactions and feedback; as a consequence, the control parameters of the system can exhibit regimes in which two or more states can be stable at the same time [138, 139]. In polariton systems, multistabilities have been used to design novel all-optical transistors and have been investigated for data storage and information processing applications [140].

The repulsive interaction with the reservoir can be used to optically trap and manipulate the condensate [141, 142].

In addition, the depletable reservoir forms a saturable gain for the condensate, which can give rise to modulational instabilities [143, 144]. By the feedback mechanism on the reservoir, the repulsive reservoir-condensate interaction can overcome the repulsive polariton-polariton interaction, rendering it effectively attractive. This focusing nonlinearity breaks the translational symmetry, leading to turbulent state formation. Here, phase turbulence forms vortices carrying quantized orbital-angular momentum [145]. These quantized vortices can also be generated in optically-induced potential traps and switched by optical means [118, 146, 147]. Moreover, vortices also feature strong interaction among each other and with the condensate [148, 149]. It has been demonstrated that an optically trapped vortex mode of a polariton condensate represents an exceptional point formed by two dipole modes [18]. Because of these properties, vortices in polariton condensates are discussed in the context of novel memory devices and information processing [52, 150]. In conclusion, optical control of the reservoir density allows for flexible manipulation of the condensate, making polariton systems a powerful platform for studying nonlinear dynamics [146].

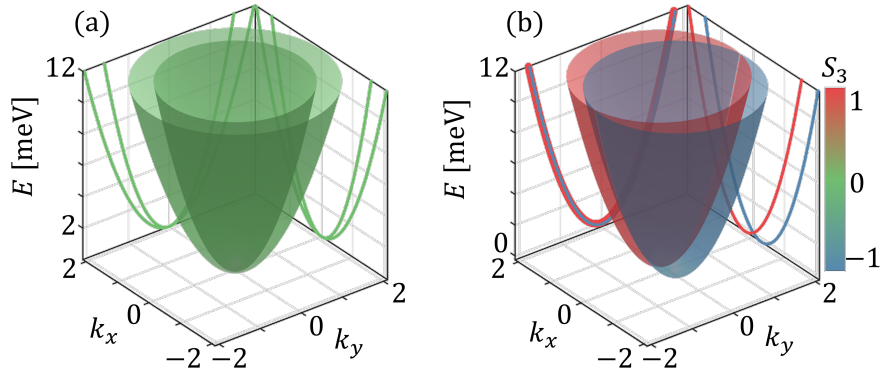


Figure 5.2: Spin-orbit interaction of polariton modes. Polariton spectra close to the minimum of the lower polariton dispersion branch are colored according to the respective eigenmode polarization. (a) TE-TM splitting acts as an effective magnetic field, splitting the polariton modes into TE and TM modes. (b) The Rashba-Dresselhaus spin-orbit interaction acts as opposing gauge field on the two circularly polarized polariton modes, leading to spin-dependent momentum splitting. The spectrum central cuts are mapped to the respective walls of the plot box.

Light-field polarization effects introduce an additional degree of freedom to polariton systems. In semiconductor microcavities, the polarization symmetry is often broken by

mechanical strain or polarization-dependent reflector penetration depth. As a consequence, the circularly polarized light field components experience different cavity lengths, expressed by distinct effective photon masses and dispersions. Thus, the transverse-electric (TE) and transverse-magnetic (TM) modes become non-degenerate. This so-called TE-TM splitting takes the same form as an effective in-plane magnetic field acting on the polariton pseudo-spin and leads to spin-orbit coupling [119, 151]. In GaAs microcavities, the strength of the TE-TM splitting is on the order of a few meV [152]. TE-TM splitting has been exploited to realize phenomena such as the optical spin Hall effect [153, 154] and phase defect manipulation [147, 149].

Recent advances have extended polariton physics to perovskite microcavities, which enable room-temperature condensation due to strong exciton binding energies [155]. These systems also provide additional control mechanisms. For instance, another form of spin-orbit coupling can be achieved by applying an electric field to perovskite microcavities filled with liquid-crystals. The crystals rotate under the impact of the electric field, modulating the anisotropic refractive index distribution within the cavity. This results in a shifting of the two lower polariton branches into opposite k directions. The resulting Rashba-Dresselhaus spin splitting splits polaritons of opposite spins in real and momentum space [156, 157].

In the presence of TE-TM splitting and Rashba-Dresselhaus spin-orbit coupling, the Hamiltonian reads as follows in the circular polarization basis

Equation 5.3

$$\hat{H}(\mathbf{k}) = \begin{bmatrix} \frac{\hbar^2 \mathbf{k}^2}{2m} - 2\alpha k_x & \beta \mathbf{k}^2 e^{2i\phi} \\ \beta \mathbf{k}^2 e^{-2i\phi} & \frac{\hbar^2 \mathbf{k}^2}{2m} + 2\alpha k_x \end{bmatrix}.$$

Here $\hbar^2 \mathbf{k}^2 / 2m$ describes the parabolic dispersion near the minimum of the lower polariton branch, with the effective polariton mass m and ϕ the polar angle. α and β denote the strengths of Rashba-Dresselhaus spin-orbit interaction and TE-TM splitting, respectively. Solving the corresponding eigenvalue problem and calculating the S_3 Stokes parameter provide the resulting dispersion branches and their polarization. In Figure 5.2, the polariton dispersion is plotted for (a) TE-TM splitting and (b) Rashba-Dresselhaus splitting. The central cuts along k_x and k_y are mapped to the walls of the plot box. The coloring of surfaces and line plots matches the eigenmode polarization.

To model the dynamics of the coherent polariton field $\psi_{\pm}(\mathbf{r}, t)$ in a planar optical microcavity, the well-established Gross-Pitaevskii equations are used. For the upcoming explanation of

these equations, the reader is encouraged to revisit Figure 5.1(c) to compare the mathematical expression with the respective illustration. The core of these equations is derived with a semiclassical approach, treating cavity photons with Maxwell's equations and excitons with Semiconductor Bloch equations in the Heisenberg picture [158, 159]. The terms introduced below are restricted to the contributions relevant to the dynamical studies presented in the subsequent chapters. The resulting equations read:

Equation 5.4

$$i\hbar\partial_t\psi_{\pm} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V + g_c|\psi_{\pm}|^2 + g_x|\psi_{\mp}|^2 + g_r n_{\pm} + \frac{i\hbar}{2}[Rn_{\pm} - \gamma_c] \right) \psi_{\pm} + J_{\pm}\psi_{\mp} + E_{\pm}.$$

The subscripts $\pm$ mark the two spin-components of the fields. Approximating a parabolic polariton dispersion of the lower polariton branch close to its minimum gives rise to the kinetic energy expression in the first term on the right-hand side, with the effective polariton mass m . $V = V(\mathbf{r})$ describes the external potential landscape fabricated onto the sample. The polariton-polariton interaction strength g_c gives rise to Kerr-like nonlinearity. g_x describes the interaction between the spin-components, which is usually weaker and has opposite sign compared to the interaction inside each component. Reservoir-polariton interaction $g_r n$ leads to the formation of an optically-induced potential. Non-Hermiticity enters via the system gain and loss, given by $Rn_{\pm}$ and γ_c , respectively. Spin-orbit interaction is introduced through TE-TM splitting. The respective operator reads $J_{\pm} = \Delta(\partial_x \mp i\partial_y)^2$, with TE-TM splitting strength Δ . Resonant driving of the polariton field is described by the resonant pump $E_{\pm}(\mathbf{r}, t)$.

The dynamics of the excitonic reservoir $n_{\pm}(\mathbf{r}, t)$ are phenomenologically incorporated [143, 160] with

Equation 5.5

$$\partial_t n_{\pm} = P_{\pm} - (\gamma_r + R|\psi_{\pm}|^2)n_{\pm}.$$

Here, $P_{\pm}(\mathbf{r}, t)$ describes the non-resonant pump that drives the reservoir. The reservoir loss is given by its loss rate γ_r (the inverse lifetime of reservoir particles) and the nonlinear gain saturation of the polariton field $R|\psi_{\pm}|^2$. In Chapter 8, the numerical implementation of these equations is discussed, and PHOENIX, the Paderborn highly optimized and energy efficient solver for two-dimensional nonlinear Schrödinger equations with integrated extensions, is introduced. Utilizing the computational speedup provided by PHOENIX, variations of these

equations will be used to study the dynamics of polariton systems in Chapter 6, 7, and 9.

Overall, exciton-polariton condensates provide a versatile platform that naturally combines non-Hermitian dynamics, strong nonlinearities, and spin-dependent effects. These features make exciton-polariton condensates ideally suited for the study of exceptional points in nonlinear optical systems. Exceptional points have been investigated in optically induced potential landscapes and the polariton excitation spectrum [17, 18, 20]. In Ref. [161], it is shown how limit cycle solutions merge into exceptional points, directly connecting bistability and exceptional points in polariton systems. Moreover, gain and blueshift control enabled tuning through an exceptional point to observe suppression and revival of the polariton condensate [19]. The described forms of spin-orbit coupling have been used to investigate exceptional points in reciprocal space [162–165].

Next to exceptional points, other non-Hermitian phenomena can be investigated in polariton systems. In Ref. [166], bound states in the continuum inside the polariton dispersion have been studied. The topological spin-Hall effect has been observed both theoretically and experimentally in spinor polariton systems. This opens up possibilities to investigate the non-Hermitian skin effect on micro-pillar chains under the introduction of circular dichroism [167, 168]. The non-Hermitian skin effect describes a phenomenon that is exclusive to non-Hermitian systems. Here, under open boundary conditions, eigenstates become localized non-Hermitian skin modes and thus drastically differ from extended Bloch waves in Hermitian systems [169, 170]. Notably, the first experimental measurement of the quantum metric in non-Hermitian systems was performed in a polariton system [171].

Interim Summary

Exciton-polaritons provide a natural platform to investigate exceptional points and exceptional structures in nonlinear optical systems. They are inherently non-Hermitian due to finite photon and polariton lifetimes and exhibit strong nonlinearities due to their excitonic component and reservoir feedback, leading to fascinating nonlinear phenomena such as the formation of quantized vortices and multistabilities. TE-TM splitting and Rashba-Dresselhaus coupling provide control knobs for spinor polariton modes.

Responsivity and topology of nonlinear exceptional rings

6

It has been shown that exceptional-point-based sensor performance can drastically increase while operating near the lasing threshold of a system [42, 43]. Operating near this threshold, however, poses new challenges, as systems become inherently nonlinear above it. In addition, the Petermann (noise amplification) factor becomes invalid in nonlinear systems where self-orthogonality is no longer determined solely by non-Hermiticity but also over-completeness of eigenbasis in multistable regimes [172]. Moreover, the interplay of exceptional-point encircling and multistability can potentially lead to even richer state transfer [63]. Now that we know where to find nonlinear exceptional points and how they are organized, we can apply this knowledge to the nonlinear and non-Hermitian system of our choice and systematically investigate the characteristics of nonlinear exceptional points.

Here, we show that exceptional rings form naturally in planar optical resonators that feature TE-TM splitting, which is usually intrinsically present, and a certain form of circular dichroism, i.e., distinct loss rates for circularly polarized light field components. Based on this, we provide a sophisticated analysis of exceptional rings in the nonlinear regime. In the linear regime, the system supports an exceptional ring in momentum space. Once nonlinearity is introduced, each point of this ring acquires the local unfolding structure derived in Chapter 4. Thus, the exceptional ring unfolds into a higher-dimensional exceptional structure whose cross-sections are governed by the elliptic umbilic. This produces concentric nonlinear exceptional rings associated with fold and cusp singularities.

We analyze the responsivity of exceptional points on the elliptic umbilic topology and demonstrate that in the nonlinear regime, this characteristic depends on their specific location on the cone-like structure. This result opens up new possibilities to design sensing devices based on nonlinear exceptional points. Moreover, we investigate the spectral topology of the quasi-torus of exceptional points that unfold as nonlinearity is introduced. We calculate topological invariants of the exceptional structure and show how different forms of anisotropy affect it.

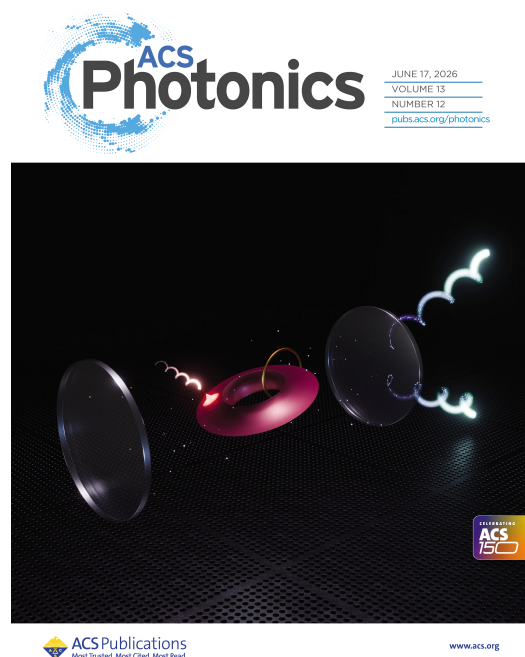


Figure 6.1: Artistic visualization of nonlinear exceptional ring topology in gyroscopic and spin-orbit coupled planar optical resonators. Each point of the ring turns into a cross-section of the elliptic umbilic topology. In the publication below, the resulting quasi-torus of exceptional points is investigated for its topological invariants and responsivity. Cover artwork associated with Ref. [71]. Reproduced from ACS Photonics, Vol. 13, Issue 12, 2026. Copyright 2026 American Chemical Society.

complex behavior such as multistability, hysteresis, or limit cycles [176]. In the literature, nonlinear exceptional points are found in both the self-consistent Hamiltonian and the instantaneous Hamiltonian. In the latter case, the nonlinear exceptional points may feature a (partially) complete eigenbasis [63].

In the following, we provide an example of how to locate nonlinear exceptional points of the self-consistent Hamiltonian within the instantaneous Hamiltonian. Due to the symmetry

It should be noted that we do not claim increased sensitivity in the form of an improved signal-to-noise ratio. The reason for this is the ongoing debate regarding the correct treatment of interactions between noise and nonlinearity [62, 173–175]. Recently, several articles have made very different claims on this topic. We have therefore decided that this topic warrants further research, independent of the analysis of nonlinear exceptional rings. Nevertheless, signal enhancement is an important component of sensitivity, and this debate emphasizes the timeliness of our work.

Given the unfolding topology of exceptional points in the nonlinear regime, a natural next step is to study the relation between the nonlinear eigenvectors derived with the unfolding topology and the steady states of a dynamical nonlinear system. Here we distinguish between (a) the self-consistent Hamiltonian of the eigenvalue problem providing eigenvectors and (b) the instantaneous Hamiltonian of the time-dependent Schrödinger equation providing steady states. While the relation of eigenvectors and steady states is trivial in the linear regime, resonance behavior can drastically change in nonlinear dynamics with

of our system, we validate the formation of the nonlinear exceptional rings corresponding, respectively, to the fold (cross) and cusp singularity (star) found in the unfolding topology along $\delta = 0$, as highlighted in Figure 6.2(a). This part of the discussion is taken from the Supporting Information of Ref. [71]. With respect to the unfolding topology, we expect the exceptional ring in our planar optical resonator to split into two concentric rings in the nonlinear regime. The parabolic dispersion background goes back to the parabolic dispersion of the cavity photon (or polariton) mode close to the dispersion minimum. Here, one exceptional ring is formed by the fold and one by the cusp on the umbilic along $\delta = 0$. The concentric exceptional rings are highlighted by a cross and star in the real and imaginary parts of the spectra displayed in Figure 6.2(b-c). The self-consistent solutions are colored according to their degree of circular polarization expressed by the S_3 Stokes parameter. The nonlinear eigenvalue problem provides two circularly polarized solutions inside the inner ring and two linearly polarized solutions outside the outer ring. Between the rings, the nonlinear eigenvalue problem yields four (two circularly and two linearly polarized) solutions. Inspired by the separability eigenvalue equations [177, 178], the complete analytical solutions and eigenvalues for this regime are derived and provided in the Supporting Information of Ref. [71].

To compare the self-consistent and instantaneous real-space descriptions, we first solve the nonlinear eigenvalue problem at fixed nonlinearity and momentum, obtaining the self-consistent eigenvectors ψ_n and eigenfrequencies ω_n . We then use these eigenvectors to define the amplitude and phase of a broad coherent pump field with wave vector $k_0 = k$, and scan the pump frequency ω_0 in the time-dependent two-dimensional nonlinear Schrödinger equation describing the cavity field. If a self-consistent solution is driven resonantly, and if the pump amplitude and phase are chosen proportional to the corresponding self-consistent eigenvector, the steady-state solution of the instantaneous Hamiltonian is expected to reproduce this self-consistent eigenvector.

Note that this method fixes the pump amplitude while keeping the nonlinearity variable. Therefore, the blueshift of the instantaneous solutions is expected to vary over the resonance peak, potentially leading to small mismatches with the self-consistent eigenvalues. However, while we believe that this method is experimentally feasible, fixing the nonlinearity instead would resolve these mismatches but requires optimization at each frequency and wave vector, resulting in a large optimization overhead in both experiment and numerics. All relevant parameter values are provided in the Supporting Information of Ref. [71].

In Figure 6.2(g), the surface plot shows the parameter sweep of wave vector k_0 and frequency

ω_0 for the resonant pump in the spatio-temporal dynamics. The real parts of the self-consistent eigenvalues are marked by the dots as a guide to the eye. The solution behavior along the highlighted vertical cuts is discussed in the following. In Figure 6.2(d-f), the resonance sweeps (d) inside the inner ring [solid line in (g)], (e) between the two rings [dashed line in (g)], and (f) outside the outer ring [dotted line in (g)] are displayed and colored according to their degree of circular polarization. In agreement with the self-consistent eigenvectors, two, four, and two resonance peaks are observed in panels (d) to (f), while their polarization behavior matches that of the self-consistent eigenvectors. In panel (e), linearly and circularly polarized resonances exhibit finite spectral overlap. This leads to a small polarization shift in the upper linearly polarized instantaneous state [turquoise peak in (e)]. The distinct peak intensities in panel (d) agree with the linewidth behavior of the imaginary part of the self-consistent eigenvalues, displayed in Figure 6.2(c). The distinct resonance peak heights in panel (f) are related to the decreasing momentum space density at high frequencies. This is due to the fact that time evolutions are not performed in 2×2 effective Hamiltonians, but from the full two-dimensional model.

All displayed results are rotationally symmetric in momentum space. These results visualize the agreement of the self-consistent and instantaneous eigenvectors and nonlinear exceptional points in the cavity field. Note that the surface in Figure 6.2(g) consists of about 3×10^4 time evolutions on a two-dimensional real-space grid. To make this feasible, we used PHOENIX, our CUDA-compatible and highly optimized solver for two-dimensional nonlinear and dissipative Schrödinger equations. A detailed overview of PHOENIX is provided in Chapter 8.

The resonance peaks in Figure 6.2 demonstrate that the steady states of the instantaneous Hamiltonian are in agreement with the solutions of the self-consistent Hamiltonian. If the system is driven slightly off-resonance, the number of solutions and their polarization still match the results of the self-consistent Hamiltonian while the nonlinearity remains strong. A natural next step is to investigate the system far off-resonance and to study the nonlinear exceptional point structures that can arise in the instantaneous Hamiltonian.

The following chapter provides a first step in this direction. Using a perturbative approach, we investigate exceptional points in the nonlinear regime under nonresonant excitation. Because the system is subject to multiple intertwined nonlinearities and off-resonant excitation, exceptional points can shift, and their spectral topology can rotate in parameter space. Moreover, responsivity can be scaled linearly close to the exceptional point.

The results presented in this chapter were previously published and are reproduced below.

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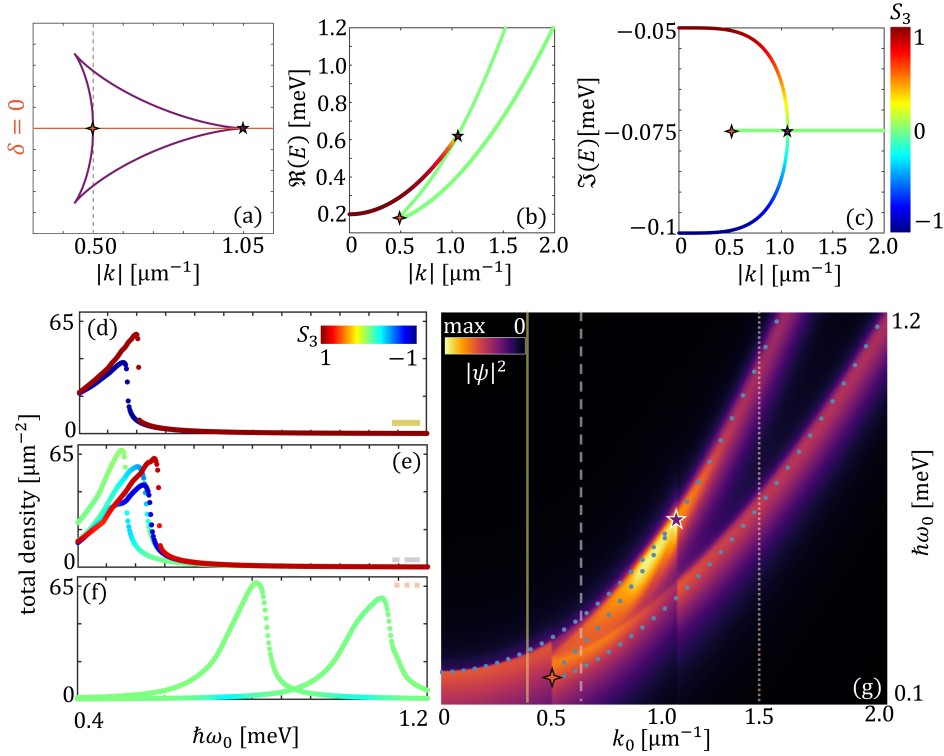


Figure 6.2: Concentric exceptional rings in the nonlinear system response. (a) Cross-section of the elliptic-umbilic unfolding with the fold and cusp points highlighted along $\delta = 0$. (b-c) Real and imaginary parts of self-consistent eigenvalues, colored according to the degree of circular polarization. The spectra show two concentric exceptional rings corresponding to the fold and cusp points highlighted in panel (a). (d-f) resonance peaks in the instantaneous solutions as a function of pump energy, scanned along the vertical lines highlighted in (g). The number and polarization of resonance peaks match the self-consistent solutions. In panel (e), finite spectral overlap modifies the polarization character of the upper linearly polarized peak. (g) Momentum-frequency scan of the resonance structure. The real parts of the self-consistent eigenvalues are included as dots as a guide to the eye. The two exceptional rings at $k_0 = 0.5 \mu\text{m}^{-1}$ and $k_0 \approx 1.05 \mu\text{m}^{-1}$ coincide with the fold and cusp points of the umbilic along $\delta = 0$. Adapted from Ref. [71].

Exceptional Rings in Nonlinear Non-Hermitian Planar Optical Microcavities: Implementation, Signal Enhancement, and Topology

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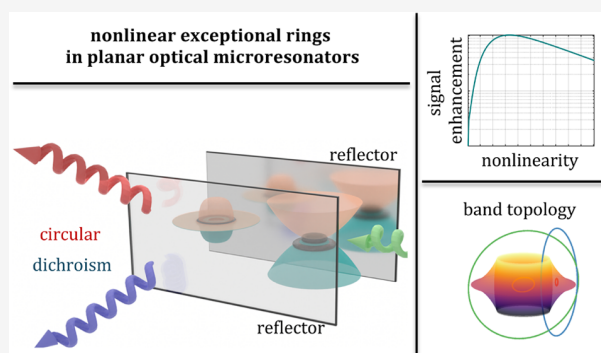
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Supporting Information

ABSTRACT: Non-Hermitian systems hosting exceptional points (EPs) exhibit signal enhancement and unconventional mode dynamics. Going beyond isolated EPs, here we report on the existence of exceptional rings (ERs) in planar optical resonators with a specific form of circular dichroism and TE–TM splitting. Such exceptional rings possess intriguing topologies as discussed earlier for condensed matter systems, but they remain virtually unexplored in the presence of nonlinearity, for which our photonic platform is ideal. We find that when Kerr-type nonlinearity (or saturable gain) is introduced, the linear ER splits into two concentric ERs, with the larger-radius ring being a ring of third-order EPs. Transitioning from linear to nonlinear regime, we present a rigorous analysis of (spectral and band) topologies and report an enhanced and adjustable perturbation response in the nonlinear regime. Whereas certain features are specific to our system, the results on non-Hermitian topologies and nonlinearity-enhanced perturbation responses are generic and equally relevant to a broad class of other nonlinear non-Hermitian systems, providing a universal framework for engineering ERs and EPs in nonlinear non-Hermitian systems.

KEYWORDS: non-Hermitian photonics, nonlinear physics, exceptional points, exceptional rings, topology, exciton–polaritons



Whereas certain features are specific to our system, the results on non-Hermitian topologies and nonlinearity-enhanced perturbation responses are generic and equally relevant to a broad class of other nonlinear non-Hermitian systems, providing a universal framework for engineering ERs and EPs in nonlinear non-Hermitian systems.

1. INTRODUCTION

Non-Hermitian systems have attracted significant attention in recent years. In contrast to Hermitian systems, they exhibit potentially complex eigenvalues and nonorthogonal eigenvectors, enabling the investigation of striking and often counterintuitive phenomena. Exceptional points are singularities at which two or more complex eigenvalues coalesce, and their respective eigenvectors become parallel.^{1–7} The number of modes n that coalesce at an exceptional point defines the order n of the exceptional point. Exceptional points have been investigated in various platforms, including microwave resonators,^{8–10} atomic systems,¹¹ plasmonic nanostructures,^{12,13} optical waveguides,¹⁴ microresonators,^{15–17} and nonreciprocal systems.¹⁸ So-called Liouvillian exceptional points can be investigated in systems obeying a Lindblad master equation.¹⁹ The coalescence of eigenvectors often leads to counterintuitive behavior close to the exceptional point such as loss-induced transparency,²⁰ loss-induced suppression and revival of lasing,^{21,22} unidirectional invisibility, and reflectivity exceeding unity.^{14,23} Unlike diabolic points in Hermitian systems, the eigenvalue splitting scales with the n th-root of the perturbation strength close to an exceptional point of order n , rendering them highly sensitive.^{24,25} Although the discussion about noise enhancement remains ongoing, this sensitivity makes exceptional points promising candidates for new sensor

designs,^{26–28} while some works demonstrate limitations to these prospects.^{29–36} Recent studies show that nonlinear systems provide excellent platforms to manipulate exceptional points and characteristic features. Prominent examples include coupled optical and electrical resonators and hybrid light-and-matter quasiparticles such as exciton–polaritons. Notable phenomena in these systems include mode-switching around exceptional points in bistable domains,^{37,38} abrupt global changes in the stability due to coinciding exceptional points,³⁹ phase transitions,^{40,41} robust wireless power transfer,⁴² as well as saturable gain and Kerr nonlinearity-induced shifting^{43,44} and rotation of exceptional points.⁴⁴ Remarkably, exceptional-like points can also be studied in nonlinear Hermitian systems.⁴⁵

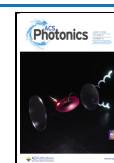
While the interplay of nonlinear and non-Hermitian physics is fascinating, in the nonlinear regime it is not always easy to tune the system to the vicinity of an isolated EP.⁴⁴ In contrast to distinct pairs of exceptional points, *exceptional rings* (ERs)

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form a continuous, gapless loop of coalescing eigenvectors and eigenvalues and offer a promising platform to drive a system to an exceptional point. It was shown that ERs may emerge from diabolic points in reciprocal space of non-Hermitian spin-orbit coupled systems.⁴⁶ To date, they have been studied in helical waveguide arrays,⁴⁷ high-spin ultracold atoms,⁴⁸ mechanical and thermal systems,^{49–51} honeycomb⁵² and superconducting Lieb lattices,⁵³ photonic and sonic crystals,^{54–60} as well as the excitation spectrum of light-matter systems.⁴⁰ In ref 61, a design for an ER-laser with pseudochiral optical response was presented in active plasmonic metasurfaces. By tweaking the gain and loss profiles, ERs can be tuned into different one-dimensional contours.⁶² In ref 63, topological phase transitions between ERs with opposite topological charges were investigated. Recently, the interplay of an ER and the nonlinear Hall effect was also investigated.⁶⁴

In the present paper, we show that the in-plane dispersion relations in spin-orbit coupled planar optical resonators with a specific form of circular dichroism naturally host ERs (Section II.A). For these we discuss in detail their topology and perturbation response in both linear and nonlinear regimes. In Section II.B, we introduce nonlinearity in the system and discuss the nature of the nonlinear ERs. We report that the introduction of a Kerr-type medium (or saturable gain) splits the ring into an inner ring of second- and an outer ring of third-order exceptional points. As derived in ref 65, finite mode detuning reveals more ERs of similar kind, whose radii form an elliptic umbilic singularity structure in nonlinear parameter space: for a given nonlinearity three fold lines of exceptional points (exceptional arcs) intersect at three exceptional points of third-order (exceptional nexus). In planar optical resonators with Kerr medium, this leads to the emergence of rings formed by points on exceptional arcs (exceptional arc rings) and rings formed by exceptional nexuses (exceptional nexus rings), respectively. In Section II.C, we discuss perturbation features of nonlinear ERs and show ultraenhanced and adjustable perturbation response. In Section II.D, we show that the linear ER can (partially) be recovered, even in anisotropic systems. With mode-switching applications in mind, we report that the ring forms a tilted quasi-toroidal structure in the nonlinear regime. We provide vorticity and Berry phase calculations both in the linear and nonlinear regime. While parts of our work focus on the specific system of planar optical resonators, and by that present a novel platform to study ERs, the ultraenhanced and adjustable perturbation response and complex spectral topological features introduced are very generic and apply to a wide range of nonlinear non-Hermitian Hamiltonians.⁶⁵

II. DISCUSSION

II.A. Exceptional Ring in Planar Resonators with Spin-Orbit Coupling

In spin-orbit coupled optical cavities and exciton-polariton systems, individual exceptional point pairs have been observed in k -space via linear dichroism, i.e., losses dependent on the linear polarization of the cavity mode.^{66,67} Pairs of exceptional points have also been observed for nonmonotonic momentum dependence on the mean polariton line width.^{68,69} In ref 40, a natural ER is observed in the excitation spectrum of an exciton-polariton system in the transition from weak to strong light-matter coupling (transition from exciton and photon to polariton modes). In the present paper, we introduce ERs

resulting from the interplay of coupled circular polarization components and circular dichroism and discuss in detail the impact of nonlinearities on these ERs.

We start with the linear problem, where we find that our microcavity shows the interesting feature of a natural ER. Only two basic ingredients are required for this observation: (i) TE-TM splitting and (ii) different losses for left- and right-circularly polarized field components ($\Gamma_+ \neq \Gamma_-$), a specific form of circular dichroism. We note that TE-TM splitting is typically intrinsically present (the specific functional form is not important here). Circular dichroism may be introduced, for example, through gyroscopic materials, assuming degeneracy of the resonances in the two circular components for the scalar case at zero TE-TM splitting.^{70,71} If nondegenerate, the modes would have to be tuned back into resonance for example, by applying an external magnetic field^{72,73} or external off-resonant pumping in active semiconductor materials.^{74,75} As a specific realization, in much of the following, we study exciton-polaritons in a semiconductor microcavity. The polariton amplitudes (or wave functions) of the lower polariton branch are denoted by ψ ; they depend on polariton wave vector $\mathbf{k}$. The two circular polarization states (right and left-circularly polarized) are indicated by the subscripts ψ_+ and ψ_- . The physics of interest comes from the (in general different) decay rates in the two polarization channels and the coupling of the two channels.

We model the linear system by an effective 2×2 Hamiltonian and solve the corresponding eigenvalue problem for each $\mathbf{k}$ on a discretized 2D k -space grid

$$\begin{bmatrix} \frac{\hbar^2 \mathbf{k}^2}{2m} + i\Gamma_+ & \Delta_{\text{LT}}(ik_x + k_y)^2 \\ \Delta_{\text{LT}}(ik_x - k_y)^2 & \frac{\hbar^2 \mathbf{k}^2}{2m} + i\Gamma_- \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \mu \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \quad (1)$$

Here, for sufficiently small $|\mathbf{k}|$, we assume parabolic in-plane dispersions for the left- and right-circularly polarized modes in the absence of spin-orbit interaction ($\Delta_{\text{LT}} = 0$) and set the dispersion minimum to zero. The dispersion curvature is given by the effective mass, $m = 10^{-4}m_e$, with m_e being the free electron mass. $\psi_{\pm}$ describes the field of the right- and left-circularly polarized modes with losses $\Gamma_{\pm}$, respectively. As a measure for the circular dichroism we define $\gamma = |(\Gamma_+ - \Gamma_-)|/2$. The mode coupling, TE-TM splitting, is introduced by the off-diagonal elements with the factor $(ik_x + k_y)^2$ signifying a spin-orbit coupling with an exchange of $2\hbar$ between the orbital motion and the polaritonic spin (circular polarization) states. The coupling strength is given by $\Delta_{\text{LT}} = 0.1 \text{ meV}\mu\text{m}^2$ and here assumed to be independent of the magnitude of the wavevector, $|\mathbf{k}|$. Such a dependence depends on the structural details of the system, and does not affect the occurrence of an ER as long as the total TE-TM splitting is a monotonically increasing function of $|\mathbf{k}|$ at least up to the $|\mathbf{k}|$ at which the ER is observed.^{76–79} In other words, the specific functional form of the dispersion is not important for the observation of the ER and even in the absence of k -space isotropy, a deformed ER structure is found. The value of TE-TM splitting strength chosen here is in good agreement with measured values in both GaAs and CsPbBr₃ perovskite microcavities.^{80,81} Note that while here we focus on the lossy case of negative $\Gamma_{\pm}$ in the following, the eigenvalue problem also shows an ER for overall gain.

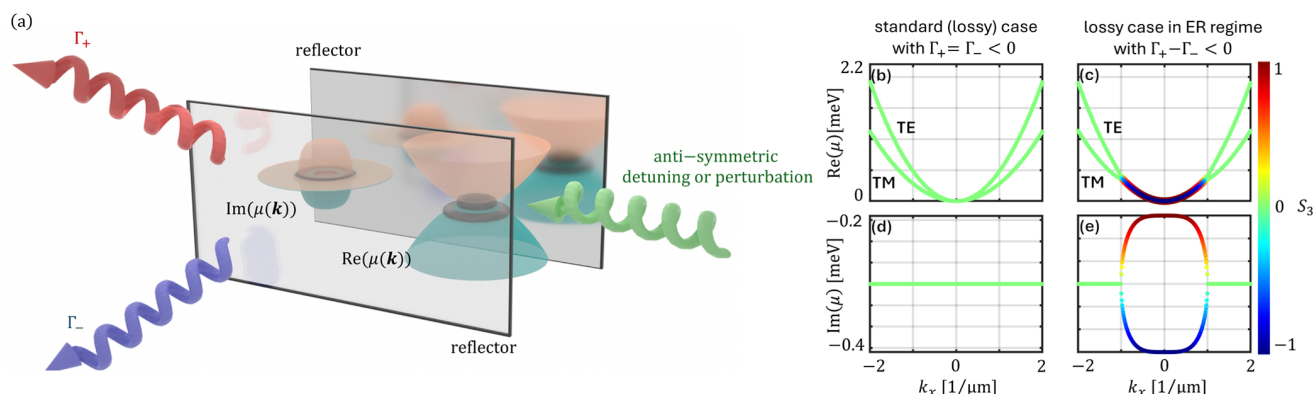


Figure 1. Concept of exceptional ring in planar microcavities with circular dichroism and spin-orbit coupling. (a) Sketch of a planar resonator with circular dichroism in the form $\Gamma_+ \neq \Gamma_-$ and finite TE-TM splitting. The complex-valued in-plane resonator dispersions, $\mu(\mathbf{k})$, host an ER centered at $k = 0$ in reciprocal space. Nonlinearity splits the ring into multiple rings (marked as black donuts). (b, d) Isotropic in-plane dispersions of the circularly polarized modes split into TE and TM modes by their coupling. (c, e) In the ER regime, when circular dichroism is introduced, $\Gamma_+ \neq \Gamma_-$, TE-TM splitting is negated inside an exceptional ring (ER) forming in k -space at $|k_{\text{ER}}| = \sqrt{|\gamma|/|\Delta_{\text{LT}}|} = 1/\mu\text{m}$; the dispersions become circularly/elliptically polarized as shown by their S_3 Stokes vector element. Dispersions are rotationally invariant in k -space, and y -axes in parts b and c are relative to the dispersion minimum.

In the linear regime, the eigenvalue problem (eq 1) can be solved straightforwardly. In Figure 1b,d, the real and imaginary parts of the solutions in the linear isotropic regime (and $\gamma = 0$) show well-established TE and TM frequency splitting and equal losses along k_x . The displayed results are rotationally invariant in k -space. The coloring of the lines indicates the $S_3 = (|\psi_+|^2 - |\psi_-|^2)/(|\psi_+|^2 + |\psi_-|^2)$ Stokes vector element for the respective eigenvector $|\psi\rangle = [\psi_+ \ \psi_-]^T$ as a measure for the degree of circular polarization. Left and right circular, as well as linear polarizations, are given by $S_3 = -1, 1$, and 0 , respectively.

In Figure 1c,e, the respective solutions are shown in the linear non-Hermitian regime with circular dichroism ($\gamma = 0.1$ meV) with $\{\Gamma_+, \Gamma_-\} = \{-0.2, -0.4\}$ meV. The corresponding lifetimes are in a realistic range as measured in GaAs and CsPbBr₃ perovskite microcavities.^{82,83} In this regime an ER with radius $k_{\text{ER}} = |k_{\text{ER}}| = \sqrt{|\gamma|/|\Delta_{\text{LT}}|} = 1/\mu\text{m}^{-1}$ is formed in the dispersion (the displayed results are rotationally symmetric in k -space). The mode splitting is canceled inside the ER where the two solutions become circularly polarized [Figure 1c]. The radius of the ER is determined by the ratio of TE-TM splitting Δ_{LT} and the degree of circular dichroism γ . Specific line widths only matter for the visual appearance of the ring; the dispersion and spectral properties are best visible when the lines close to the ER radius do not overlap too much.

We verify these results and our model Hamiltonian by simulating the complex spatiotemporal dynamics of the respective cavity fields in time-dependent, two-dimensional real-space calculations. In line with our simplified k -space model above, eq 1, the equations governing the in-plane field propagation and dispersions in the circular polarization basis read

$$i\hbar\partial_t\psi_{\pm} = \left(-\frac{\hbar^2\nabla^2}{2m} + i\Gamma_{\pm}\right)\psi_{\pm} + J_{\pm}\psi_{\mp} + R_{\pm} \quad (2)$$

In contrast to the eigenvalue problem in eq 1 this equation describes the field dynamics in two-dimensional real-space with $\psi_{\pm} = \psi_{\pm}(\mathbf{r}, t)$, which is driven by an external coherent field $R_{\pm} = R_{\pm}(\mathbf{r}, t)$ with $\mathbf{r} = (x, y)$. Consistent with the k -space representation in eq 1, the TE-TM splitting operator is defined as $J_{\pm} = \Delta_{\text{LT}}(\partial_x \mp i\partial_y)^2$. To extract the system response that

shows the dispersions, we pulse the system with a short resonant pulse

$$R_{\pm}(\mathbf{r}, t) = R_0 \exp\left(-\frac{\mathbf{r}^2}{p_w^2}\right) \exp(-i\omega t) \exp\left(-\frac{(t - t_0)^2}{\sigma^2}\right)$$

separately in each polarization channel in two independent time evolution calculations. The calculations are performed with our CUDA-compatible open-source solver for 2D nonlinear Schrödinger equations, “PHOENIX”,⁸⁴ and the code for the specific application shown here is included in the PHOENIX Github.⁸⁵ In Figure 2 we plot $-\text{Im}(\chi)$ for the

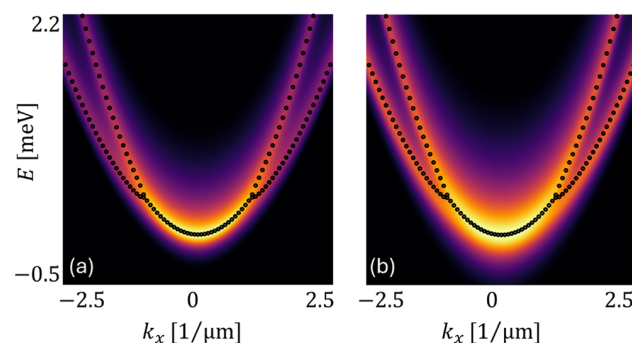


Figure 2. Exceptional ring in in-plane resonator dispersion in linear regime. Diagonal elements of response to left (a) and right (b) circularly polarized light, plotted along k_x . Comparing (a) and (b), the difference in line widths, as expected inside the exceptional ring, are clearly visible [cf. Figure 1e]. These full model calculations are in good agreement with the dispersions of the simplified 2×2 system in eq 1 (black circles). The displayed results are rotationally symmetric in k -space, and energies are measured relative to the dispersion minimum.

diagonal elements of the response tensor, $\chi_{++} = \mathcal{F}(\psi_+)/\mathcal{F}(R_+)$ and $\chi_{--} = \mathcal{F}(\psi_-)/\mathcal{F}(R_-)$, which dominate the system response for small TE-TM splitting. Here, $\mathcal{F}(X)$ is the Fourier transform of the respective quantity X into k - and ω -space.

The resulting dispersions in Figure 2 show perfect agreement with the results of the eigenvalue problem, eq 1.

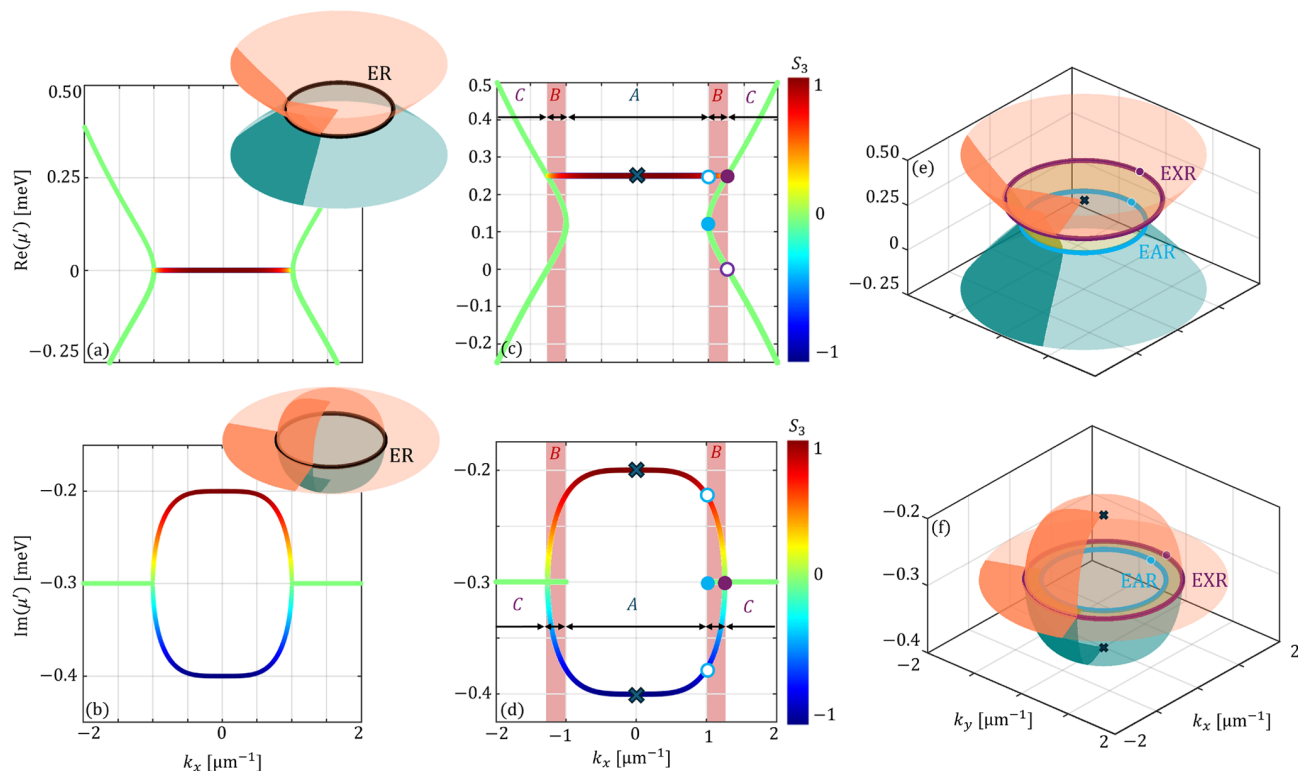


Figure 3. Exceptional rings in the complex-valued energy spectra in the nonlinear regime at zero detuning ($\Delta = 0$). Central cuts of the (a) real and (b) imaginary parts of energy spectra of the linear eigenvalue problem (neglecting the k^2 dispersion). The respective surfaces are depicted in the insets with the exceptional ring (ER) marked in black. Central cuts of the (c) real and (d) imaginary parts of the energy spectra of the nonlinear eigenvalue problem. The respective surfaces are depicted in (e and f). In contrast to the linear regime, the energy spectrum in the nonlinear regime hosts two exceptional rings, one with radius $|k|^2 = |\gamma|/|\Delta_{LT}| = 1 \mu\text{m}^{-1}$ (solid blue dot and labeled by EAR) and a second ring with radius $|k|^2 = \sqrt{\gamma^2 + \alpha^2}/|\Delta_{LT}|$ (solid violet dot and labeled by EXR). Between these two rings [marked in red in section B in panels (c and d)], four solutions are found. Inside the inner (outside the outer) ring, two circularly (linearly) polarized solutions are found [section A and C in panel (c,d)]. The circles mark points where a solution crosses the boundary between two segments without tracing through an exceptional ring and the cross marks $k = 0$. These markers are used in Figure 4(b) to analyze the bifurcation behavior near the two rings. The results displayed are rotationally symmetric in the k -space. Simulation parameters: $\{\alpha, g_c, g_x\} = \{0.125, 0.25, 0\} \text{ meV}\mu\text{m}^2$.

Both energy and line width behavior in the time-dependent calculations agree with the results of our reduced 2×2 model as displayed in Figure 1c,e. We note that, in principle, for suitably tailored $\gamma(k)$ or for non- k -isotropic systems, different shapes of the exceptional structure in 2D k -space can be realized, allowing for the formation of exceptional ellipses and more complex exceptional curves. In Supporting Information Note S1, for the specific realization in an exciton–polariton system, we explicitly show that off-resonant pumping may preserve the assumed k -independence of the polarization-dependent gain difference, γ , inferred in the present study. The results displayed in Figure 2 can be reproduced with a polarized reservoir as a source for circular dichroism, as discussed in Supporting Information Note S1.

II.B. Nonlinearity-Induced Third-Order Exceptional Ring

So far, the optical resonators considered here were assumed to be linear systems; however, their response can become significantly nonlinear if a nonlinear optical medium is inserted. The nonlinearity may result from a Kerr-type nonlinear material response or, in systems that form exciton–polaritons, stem from polariton–polariton interactions, which can typically lead to strong coherent optical nonlinearities. Besides symmetries,⁸⁶ nonlinearities allow for the realization of higher-order exceptional points in lower-

dimensional systems.^{87,88} This led to the investigation of an exceptional nexus, i.e., a higher-order exceptional point forming at the intersection of multiple lines of second-order exceptional points, so-called exceptional arcs.^{87,89}

Here, we take into account coherent Kerr-type nonlinearity and cross-mode coupling from TE-TM splitting. Note that in the specific case of exciton–polaritons the cross-nonlinearity between circularly polarized spinor components, ψ_+ and ψ_- , is known to be weaker and attractive compared to the repulsive self-interaction nonlinearity.⁷⁹ Also, in the nonlinear regime we make the (in most cases valid) assumption of unidirectional coherent light-field propagation with negligible scattering into other k -modes (typically also justified for polariton systems if exciting below the magic-angle;⁹⁰ numerically explicitly shown in Supporting Information Note S2). The extended nonlinear eigenvalue problem then reads

$$\begin{bmatrix} M_+ & \Delta_{LT}(ik_x + k_y)^2 \\ \Delta_{LT}(ik_x - k_y)^2 & M_- \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \mu \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \quad (3)$$

with $M_{\pm} = \frac{\hbar^2 k^2}{2m} + i\Gamma_{\pm} + g_c |\psi_{\pm}|^2 + g_x |\psi_{\mp}|^2 \pm \Delta$. Here, g_c and g_x define the Kerr-like nonlinearity and cross-mode interaction strength, respectively. We first focus on the degenerate case of

$\Delta = 0$ for which the fully analytic solutions of the nonlinear eigenvalue problem are of tractable length (see [Supporting Information Method S1](#)). Later, through $\Delta \neq 0$ we will include finite mode detuning of different origins; examples we discuss are through magnetic field-induced Zeeman splitting and electric field-tunable Rashba–Dresselhaus splitting.

To solve nonlinear eigenvalue problems, several approaches have been proposed, including the polar representation^{91,92} or Stokes parameters and momentum operators incorporation.^{93–96} However, these methods are often limited to specific Hamiltonian structures. More recently, iterative techniques based on self-consistent field approaches have been applied to non-Hermitian nonlinear eigenvalue problems.^{38,97} Despite their utility, iterative methods, such as the Newton–Raphson method, show convergence issues in bistable regimes close to the exceptional point. In *ref* 65, we provide the general solutions (with analytical expressions in the case of zero detuning) of the eigenvalue problem (eq 3); the solution approach used was inspired by the separability eigenvalue equations^{98,99} as discussed in more detail in the [Supporting Information Methods S1 and S2](#). The resulting solutions apply to any real normalization of $|\psi_+|^2 + |\psi_-|^2 = N^2$. The given solutions are also valid for the left eigenvalue problem under complex conjugation ([Supporting Information Method S1](#)). Our analytical solutions are validated using them as initial conditions for the Newton–Raphson method. Here, we scale the solutions to norm $N = 1 \mu\text{m}^{-1}$, and introduce $\alpha = (g_c - g_x)/2$ as a measure of nonlinearity.

In [Supporting Information Note S2](#), we verify our model assumptions by simulating the complex spatiotemporal dynamics of the respective cavity fields by solving the coherently driven spinor nonlinear Schrödinger equations (eq S27), including nonlinearity. These simulations allow for all four-wave mixing (FWM) or higher-order scattering events to occur. However, if initially we have only one wavevector (k_0) excited, i.e., the three waves on the rhs of eq S27 are plane waves at k_0 , then the condition of wavevector conservation restricts the wavevector on the lhs of eq S27 to k_0 . The results presented in [Figure S1](#) of the Supporting Information explicitly demonstrate that no new k -components with significant amplitudes are created under these conditions, justifying the assumptions of a two-mode system also in the nonlinear regime.

We begin the discussion of the nonlinear eigenvalue problem for the case of zero detuning, $\Delta = 0$, showing eigenvalues and further analysis in [Figures 3 and 4](#). As shown in [Supporting Information Method S1](#), we have two solutions in region A and C, and four solutions in region B. In the following figures, for clarity, we show the eigenvalues $\mu' = \mu - (\hbar^2 k^2)/2m$, where the parabolic dispersion is subtracted. For a better comparison with the results in [Section II.A](#), we set $\{\Gamma_+, \Gamma_-\} = \{-0.2, -0.4\}$ meV and $\Delta_{LT} = 0.1$ meV μm^2 for the remainder of this work. [Figure 3](#) shows the resulting real and imaginary parts of the complex-valued energy spectrum and central cuts for $\{\alpha, g_c, \text{ and } g_x\} = \{0.125, 0.25, 0\}$ meV μm^2 [see [Figure 3c–f](#)] in comparison to the linear results [see [Figure 3a,b](#)]. These parameter values are in good agreement with measured blueshifts in both GaAs and CsPbBr₃ perovskite microcavities.^{100,101} In the nonlinear regime, the complex-valued energy spectrum hosts two concentric rings marked by solid blue and violet dots in the central cuts in [Figure 3c,d](#), and labeled by EAR and EXR in [Figure 3e,f](#), separating three radial segments (A–C). The inner

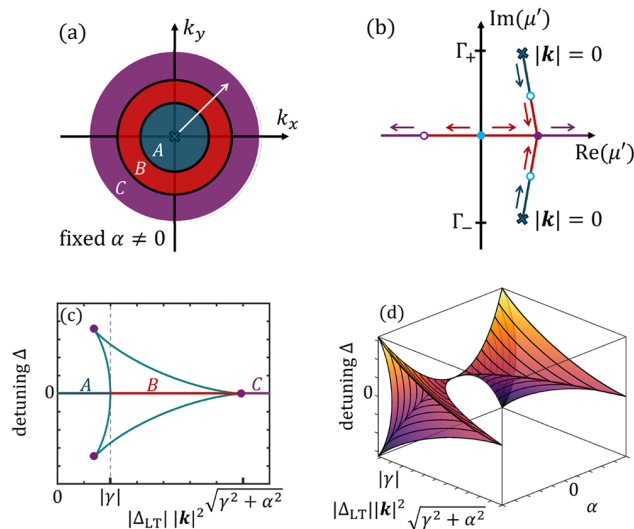


Figure 4. Elliptic umbilic structure of exceptional rings and exceptional points in the nonlinear regime. (a) Sketch of the three sections A–C [as displayed in [Figure 3c](#)] with two, four, and two solutions, respectively. (b) The bifurcation behavior on the edges of ring B characterizes the inner (outer) ring as a ring of fold (cusp) points. (c) Radii of fold rings (turquoise lines) and cusp rings (violet dots) for degenerate modes (segments marked as above) and for finite mode detuning Δ . The umbilic structure shows that fold rings are part of exceptional arcs, and cusp rings are located at exceptional nexuses. (d) Cross section in (c) for variable nonlinearity α . The depicted elliptic umbilic predicts exceptional ring radii and type in the full parameter space. The central organization point represents a point on the linear exceptional ring discussed in [Section II.A](#).

segment A hosts two circularly polarized solutions, the segment B between the rings, marked in red, features four solutions (two circularly and two linearly polarized), and the outer segment C shows two linearly polarized solutions. Note that nonlinearity shifts the non-linearly polarized real parts of the solutions in segments A and B by 2α , while the linearly polarized solutions in segment B and C are shifted by α . In [Figure 4a,b](#), these segments are sketched for an arbitrary nonlinearity $\alpha \neq 0$ to visualize the bifurcation behavior at the two rings. Analyzing this behavior, we conclude that in the nonlinear regime the linear ER splits into an inner ring of fold points (blue dot) along $|k|^2 = |\gamma|/|\Delta_{LT}|$ and an outer ring of cusp points (violet dot) along $\sqrt{(\alpha^2 + \gamma^2)/\Delta_{LT}^2}$, which grows with increasing Kerr-like nonlinearity g_c or cross-interaction strength g_x . The fold and cusp points can be identified with exceptional points of the second and third-order, respectively. Hence, the outer ring is a third-order exceptional ring. The circles mark points where a solution crosses the boundary between two segments without tracing through an ER (e.g., the two green colored solutions form an ER at the blue solid dot, at the border of segments A and B, while the blue and red colored solutions cross the border of segments A and B at the blue circle without forming an ER). For $\alpha < 0$ the cusp ring is red-shifted in relation to the fold ring [this scenario is displayed in [Figure 1a](#)]. While previous work realized individual higher-order exceptional points in two-dimensional nonlinear parameter space and thus reduced the parameter tuning requirements,^{87,88} the ring structure observed here reduces these parameter tuning requirements even further,

making higher-order exceptional point structures in low dimensions more accessible.

At this point, we need to discuss the nature of exceptional points of order $n > 2$ in nonlinear 2×2 systems. In previous work, it was exemplarily shown that mapping a nonlinear and non-Hermitian Hamiltonian to a $n \times n$ linear and $\mathcal{PT}$ -symmetric Hamiltonian features the same exceptional point of order n .^{87,88} Accordingly, the singularity in the nonlinear system is called a “nonlinear exceptional point”, containing at least one dynamically unstable solution. It is straightforward to show that the four solutions in Figure 3c,d, form an overcomplete basis. This is reflected in the Petermann factor $PF = |\langle \psi_L | \psi_R \rangle|^{-2}$, which serves as a measure for non-orthogonality of left $|\psi_L\rangle$ and right $|\psi_R\rangle$ eigenvectors.¹⁰² $PF = 1$ indicates orthogonal modes, whereas $PF > 1$ signals mode nonorthogonality, resulting in a divergent Petermann factor at linear exceptional points. At the inner ring, the PF diverges, highlighting the presence of an ER with coalescing solutions in the nonlinear regime. At low nonlinearity the nonlinear third-order ER exhibits diverging PF. With increasing nonlinearity, the Petermann factor at the nonlinear third-order ER decreases until it converges to 1. This means that the eigenvectors at the ER become increasingly orthogonal and do not coincide in the nonlinear regime. However, it has been shown that nonlinear exceptional points still feature an intriguing spectral topology and the n th-root perturbation response of their linear counterparts.²⁵ This suggests that the unstable mode still contributes to the order of the nonlinear exceptional point,^{87,88} which we will prove for our third-order ER in Section II.C. In that sense, we believe that the nonlinear ER appears to show sufficient similarities with its linear counterpart to justify its name and order classification.

To understand the nature of the two ERs in the nonlinear regime in a more general setting, we have to solve the nonlinear eigenvalue problem (eq 3) for finite detuning $\Delta \neq 0$, which, while not rendering it impossible, greatly complicates the fully analytic solution. However, in ref 65, we have derived the fourth-order polynomial that determines the solutions for this eigenvalue problem. Solving it numerically allows us to analyze the neighborhood of the two ERs. In ref 65, we have shown that for finite α a single linear exceptional point splits into three exceptional nexuses (third-order exceptional points) connected by three exceptional arcs (lines of second-order exceptional points) in the plane of mode coupling (here: TE-TM splitting strength Δ_{LT}) and mode detuning Δ . In 2D k -space, each exceptional point on this structure represents a radius of an ER. In Figure 4c the possible ER radii are sketched for finite α together with the segments along $\Delta = 0$ discussed above, turquoise lines depict exceptional arcs, and violet dots the exceptional nexuses. Thus, the inner and outer ER shown in Figure 3 are formed by the fold point of the exceptional arc and by the exceptional nexus along $\Delta = 0$, respectively.

Extending this picture for variable nonlinearity, the cross section in Figure 4c forms an elliptic umbilic in parameter space along the nonlinearity strength α [Figure 4d]. This shows that in contrast to the linear ER, the two rings in the nonlinear regime are not individual singularities in parameter space, but surrounded by a surface of other fold rings on the respective exceptional arcs and more cusp rings at the other exceptional nexuses on the umbilic even for finite detuning. Following this convention, we will refer to the rings formed by the fold rings on the umbilic as *exceptional arc rings* (EARs) and rings formed by cusp points of the umbilic as *exceptional*

nexus rings (EXRs). Note that these exceptional rings not only reduce the parameter number to tune the system toward an exceptional point but can even be found for finite detuning.

While in the linear regime the relation between eigenstates and stationary solutions is straightforward, their relation becomes nontrivial when moving to the nonlinear regime. However, in Supporting Information Method S3, we show analytically that the ER structures discussed above can also be found in the coherently driven nonlinear Schrödinger equations with resonant excitation. There, our solutions hold true up to a shift induced by the resonant driving force. Moreover, in Supporting Information Note S2, we provide one possible example to study the exceptional ring structure in the coherently driven nonlinear system. Specifically, we demonstrate that the resonance behavior in the spatiotemporal dynamics of the driven system exhibits the same features as the nonlinear eigenvalue problem discussed above. The analytic resonance spectrum with the two cocentric ERs in the case of zero detuning is included in Figure S1 in the Supporting Information as a guide to the eye. We emphasize that the systematic study of the nonlinear system dynamics with off-resonant excitation requires more work and, thus, needs to be addressed in future works. However, this outlook lays the foundation for more detailed investigations into nonlinear dynamics and nonlinear response behavior.

In previous work,⁴⁷ it was mentioned that the linear ER does not exist in the solutions of the eigenvectors for finite mode detuning. Here, we have shown that in the nonlinear regime, new ERs arise even for finite mode detuning. We use this knowledge to analyze the perturbation response of nonlinear ERs in Section II.C and their topological features under mode detuning in Section II.D. In Supporting Information Note S3, we discuss that the elliptic umbilic structure applies to a wide range of non-Hermitian Hamiltonians when introducing nonlinearity. Therefore, the topological features and perturbation response of nonlinear ERs on the umbilic we discuss below, are generic and can be applied to exceptional points in many other systems and constellations.

II.C. Enhanced Perturbation Response at Nonlinear Exceptional Rings and Points

Optical resonators are promising platforms to study the sensing capabilities of exceptional points.^{24,103,104} In the linear regime, the characteristic root-law eigenvalue response to perturbations is compensated by noise enhancement due to line width broadening, which results from the coalescence of the eigenvectors.^{29,105} This leads to limitations of the overall signal-to-noise ratio close to linear exceptional points. Recently, there has been growing interest in studying the effect of nonlinearity on the perturbation response of exceptional points. It was shown theoretically in ref 44 and later experimentally in ref 106, that nonlinearity can be used to tune the eigenvalue splitting close to an exceptional point.

In refs 88,107, it was proposed that nonlinear exceptional points can exhibit a complete basis of eigenstates. In laser systems, the Petermann factor can be investigated as a measure of the noise enhancement. Based on this, it was proposed in ref 87 that nonlinear exceptional points can exhibit a divergent signal-to-noise ratio. More recently, however, it was shown that the conventional Petermann factor does not cover the interplay of noise and nonlinearity and that nonlinear exceptional points exhibit a finite signal-to-noise ratio,¹⁰⁷ with a generalized form of the Petermann factor at exceptional points derived in ref

108. In the following, we discuss that nonlinear exceptional points possess adjustable and in some cases ultraenhanced perturbation response. We will not make claims about the resulting signal-to-noise ratio in our system, as further work would be needed in that direction. However, we note that our results demonstrate how the signal enhancement can not only be increased but also tuned in the nonlinear regime. Our analysis covers the group of both real- and imaginary-valued smooth nonlinearities that lead to the elliptic umbilic exceptional point structure described above; here, it is exemplarily evaluated for Kerr-type coherent nonlinearities. The elliptic umbilic structure significantly increases the complexity of an exceptional points neighborhood, permitting sensor design for exceptional points at different positions on the singularity structure, which changes the response to perturbations not limited to n th-root laws.

For now, we will focus on the EXR along $\Delta = 0$ (without detuning) and show that the nonlinearity leads to a drastically increased perturbation response at the EXR [marked by stars in Figure 5d]. We specifically analyze the eigenvalue splitting resulting from perturbations ϵ of the form

$$\hat{H}(\epsilon, \alpha) = \hat{H}_{\text{EP}}(\alpha) + \epsilon \hat{H}_1, \text{ with } \hat{H}_1 = \text{diag}(1, -1) \quad (4)$$

Here, $\hat{H}_{\text{EP}}(\alpha)$ denotes the Hamiltonian of the nonlinear eigenvalue problem (eq 3) at the investigated exceptional point. The perturbation ϵ is equivalent to a small additional detuning $\Delta \rightarrow \Delta + \epsilon$ in the nonlinear eigenvalue problem. We focus on perturbations of form $\hat{H}_1$ to investigate the straightforward interplay of perturbation and the elliptic umbilic singularity structure. We note that this interplay may become more complex for other forms of perturbation, as discussed in ref 25. In the latter case, the exceptional points on the umbilic surface can be perturbed into new dimensions in parameter space, making visualization difficult.

For now, we investigate the signal enhancement at the EXR. Aside from the obvious signal enhancement resulting from the increased order of the exceptional points on the EXR compared to the linear ER, we calculate the signal-enhancement factor defined by the change in eigenvalue with perturbation:⁸⁷

$$\text{SEF}(\epsilon, \alpha) = \left| \frac{\partial \mu(\epsilon, \alpha)}{\partial \epsilon} \right|^2 \quad (5)$$

with $\mu(\epsilon, \alpha)$ being the eigenvalue at the exceptional point described by $\hat{H}(\epsilon, \alpha)$. In Figure 5a the inverse signal-enhancement factor $(\text{SEF})^{-1}$ is displayed. Note that in the linear regime the noise enhancement covered by the nonorthogonality of eigenstates compensates the signal enhancement. In the nonlinear regime, the interplay between noise and nonlinearity does not allow conclusions to be drawn about noise amplification based solely on nonorthogonality.¹⁰⁷

In Figure 5b, the maximum of the signal-enhancement factor SEF_{max} is plotted against α . The maximum signal enhancement of $\text{SEF} = 1.9 \times 10^9$ is found for a nonlinearity strength $(g_c - g_x)/2 = 68 \mu\text{eV}\mu\text{m}^2$. Here, the signal enhancement scales with the perturbation as $\text{SEF} \propto |\epsilon|^{-1.29}$. In contrast to an isolated linear exceptional point or ER, the elliptic umbilic allows for different exceptional points to be perturbed, depending on the initial detuning Δ . Since the perturbation ϵ is parallel to Δ this means that depending on which exceptional point we choose to perturb, the response depends on its position on the umbilic. In planar nonlinear resonators,

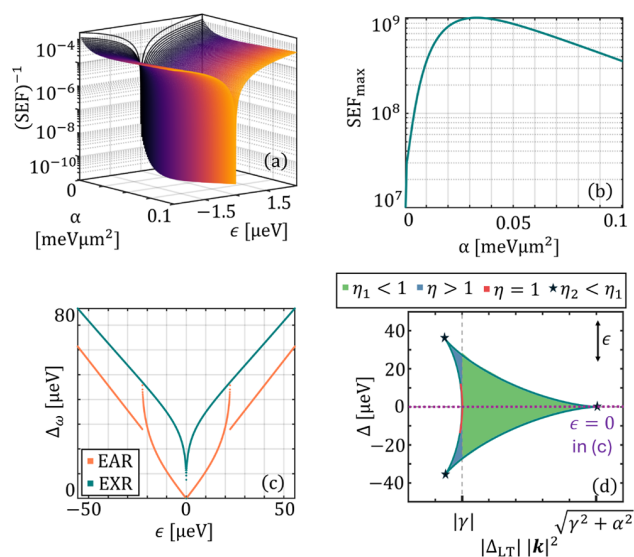


Figure 5. Enhanced perturbation response at nonlinear exceptional rings and points. In (a, b), a third-order exceptional point on the exceptional nexus ring [star in (d)] is traced for increasing α . (a) The inverse signal-enhancement factor $(\text{SEF})^{-1}$ decreases in the nonlinear regime. The coloring visualizes α in each curve (growing from black to yellow). (b) Maximum SEF_{max} at the exceptional point as a function of α . (c) Response Δ_ω of the exceptional points on the exceptional arc ring (EAR) and exceptional nexus ring (EXR) on the elliptic umbilic along $\Delta = 0$ [horizontal dashed line in (d) corresponds to $\epsilon = 0$ in (c)] for a fixed nonlinearity α . The response of the EAR significantly deviates from the square-root law of linear exceptional points. Here, small perturbations tune the system along the left exceptional arc of the umbilic [red line in (d)]. Larger perturbations tune it away from the arc due to its curvature [blue area in (d)], leading to a growing response. (d) Colored cut through elliptic umbilic at finite nonlinearity α . The coloring denotes the response function scaling factor η when an exceptional point on the umbilic singularity structure. Detuning allows for the realization of different rings on the umbilic, which feature distinct perturbation responses. Perturbation from the lower to the upper arc (or vice versa) leads to the characteristic square-root response η_1 . Nexus perturbation leads to a stronger response η_2 . The perturbation ϵ is defined in relation to the respective point on the umbilic. Simulation parameters: $\{g_c, g_x\} = \{2\alpha, 0\}$ $\mu\text{eV}\mu\text{m}^2$.

external electric and magnetic fields can be used to tune the mode detuning via Zeeman and Rashba–Dresselhaus splitting, for instance. The response is defined by the eigenvalue splitting $\Delta_\omega(\epsilon, \alpha)$ from the exceptional point eigenvalue $\mu(\epsilon = 0, \alpha)$ in dependence of the perturbation strength

$$\Delta_\omega(\epsilon, \alpha) = |\mu(\epsilon, \alpha) - \mu(0, \alpha)| \quad (6)$$

In Figure 5c, the response of the EXR (turquoise) is shown together with the response function of EAR (orange) for $\alpha = 0.1 \mu\text{eV}\mu\text{m}^2$ and $\Delta = 0$. Note that the response function of the EAR clearly deviates from the n th-root law of a linear exceptional point. This is due to the fact that the exceptional point is initially perturbed parallel to the exceptional arc of the umbilic [red line in Figure 5c], resulting in a linear response ($\Delta_\omega \propto |\epsilon|^\eta$ with $\eta = 1$) to perturbation. Only when the curved shape of the arc starts to deviate from the perturbation direction, the eigenvalues split faster ($\eta > 1$) [blue in Figure 5c]. If the perturbation pushes the system outside the elliptic umbilic, then the system jumps to another solution, which then has a linear response in relation to the perturbed exceptional

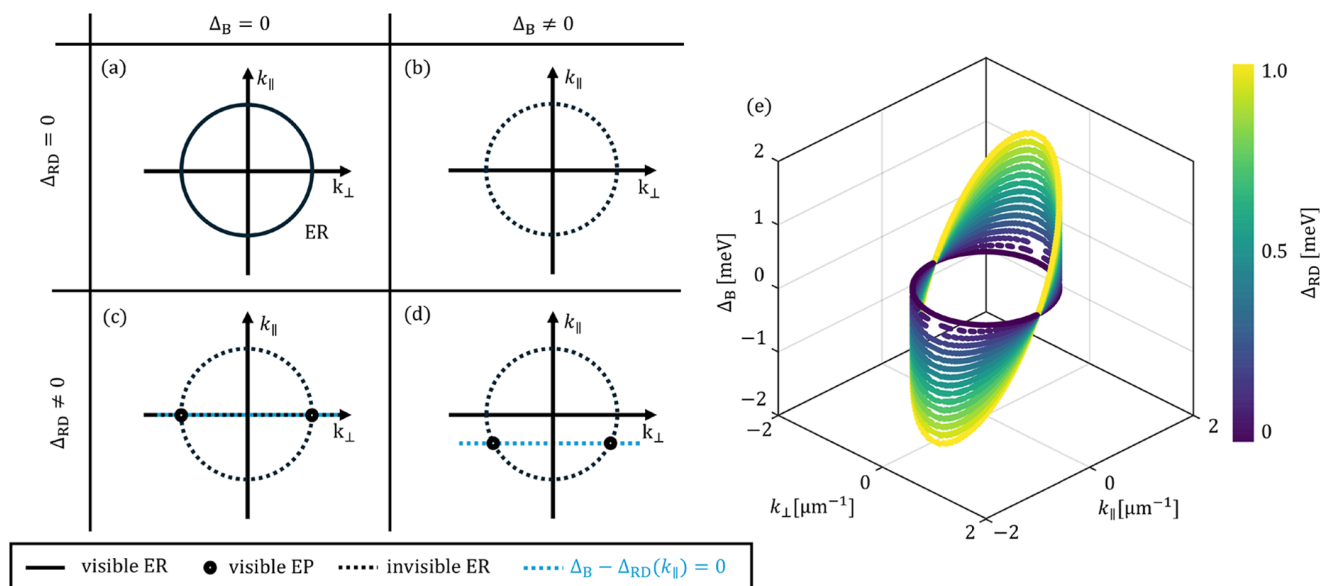


Figure 6. Topological neighborhood of linear exceptional rings in 4D parameter space. (a) For degenerate modes ($\Delta_B = \Delta_{RD} = 0$), the complete ER is visible in the original reduced parameter space (2D k -space). (b) In the case of finite Zeeman splitting ($\Delta_B \neq 0$), the ER vanishes from the reduced parameter space as the degeneracy is lifted for all k -values. (c) In the case of Rashba–Dresselhaus splitting ($\Delta_{RD} \neq 0$), the ER vanishes from the reduced parameter space, but a pair of exceptional points remains along $k_\perp$. (d) If Zeeman and Rashba–Dresselhaus splitting are combined ($\Delta_B \neq 0$, $\Delta_{RD} \neq 0$), the ER vanishes, but the pair of exceptional points remains along $k_\perp$ at $k_\parallel$ for which Zeeman and Rashba–Dresselhaus splitting compensate each other in the extended parameter space. (e) Exceptional-point surface consisting of tilted ERs (gaps in the surface result from limited discretization). The tilting in the $k_\parallel$ -direction is proportional to the Rashba–Dresselhaus splitting, Δ_{RD} . The exceptional points are located at those points in parameter space at which the effect of Zeeman and Rashba–Dresselhaus splitting compensates each other.

point [white area outside the umbilic in Figure 5c]. If finite detuning is used, the system can also be perturbed from one exceptional arc to another (top to bottom or vice versa). In this case, the exceptional point then shows a classic root-shaped response ($\eta_1 < 1$) [green in Figure 5c]. Perturbation of the EXR leads to stronger response $\eta_2 < \eta_1$. Importantly, these findings are valid for all exceptional points located along the ERs, owing to the rotational symmetry of the structure in the k -space. This underlines the versatility of exceptional points in nonlinear systems, opening the door to highly tunable sensor designs. These findings further illustrate that in the nonlinear regime the response of an exceptional point depends on its position in parameter space (relative to the other arcs on the umbilic) and the specific form of perturbation.

II.D. Topology of Exceptional Rings

In the literature, ERs are intensively studied for their nontrivial band topologies.¹⁰⁹ In the following, we discuss several related aspects of our system. We have evaluated the vorticity for bands $n = u$ (upper band) and $n = l$ (lower band) in the case without nonlinearity, defined in ref 110

$$\nu_{ul} = -\frac{1}{2\pi} \oint_{\mathcal{L}} \nabla_k \arg(\mu'_u(\mathbf{k}) - \mu'_l(\mathbf{k})) \cdot d\mathbf{k} \quad (7)$$

over a closed-loop $\mathcal{L}$ encircling one point of the ER once, without piercing the ER. In particular, we have used a loop in the $\Delta - k_y$ plane encircling the ER on the positive k_y axis, i.e., at $(k_y, \Delta) = (k_{ER}, 0)$. We call this configuration a Hopf–link configuration, because the ER and the loop $\mathcal{L}$ form a Hopf link. We obtain a vorticity of $\nu_{ul} = 0.5$, consistent with the square-root singularity of the exceptional point¹¹¹ and the results in refs 109,110.

We have numerically verified that the result is independent of the details of the loop,¹¹¹ for example, using circular loops

with various radii ($k_L < 2k_{ER}$) and elliptical loops. Without nonlinearity, the vorticity can be obtained for a loop in the $k_x - k_y$ plane with $\Delta = 0$ without a numerical evaluation. In this case, both eigenvalues are real (imaginary) for $k_L > k_{ER}$ ($k_L < k_{ER}$) and hence the argument of the eigenvalue difference is k -independent of $\mathcal{L}$, resulting in the vorticity being zero.

In Supporting Information Method S4 we provide more details regarding the topological properties of the ER in the $k_x - k_y$ plane, i.e., using $\Delta = 0$. We find that the closed-loop Berry phase and the integral over the z -component of the Berry curvature depend on the choice of eigenvector (left or right) used to calculate the Berry connection. Next, we calculate an integral over the Berry curvature that we call a quasi-Chern integral

$$C_n^{ab} = \frac{1}{2\pi} \int \Omega_n^{ab}(\mathbf{k}) d\mathbf{s} \quad (8)$$

If we use only right eigenvectors ($ab = RR$), we find that the integral over the z -component of the Berry curvature exists and, without nonlinearity, is equal to $C_n^{RR} = \pm 1$ if the integration area includes the ER. Note that for combinations of left and right eigenvectors (i.e., $ab = LR$ or RL), self-orthogonality leads to divergence of quasi-Chern integral at the ER. A detailed discussion of the scaling of the integral value is given in Supporting Information Method S4.

With the objective of analyzing the nonlinear solutions, the analytical results for the quasi-Chern integral are first validated numerically. For this purpose, we discretize the k -space grid and use the $U(1)$ link variable to calculate the Berry curvature for each lattice site $\mathbf{k}$, following ref 112

$$C_n^{ab} = \frac{1}{2\pi} \sum_{\mathbf{k}} \Omega_n^{ab}(\mathbf{k}) \delta k^2 \quad (9)$$

In the nonlinear regime, we compute the quasi-Chern integral on three types of round disks: (i) inside the inner ring, (ii) enclosing only the inner ring, and (iii) enclosing both the inner and outer rings. To calculate the quasi-Chern integral of the EAR and the EXR the integral is evaluated over the eigenvalue surfaces forming the respective ring. Here, both rings exhibit a quasi-Chern integral of $C_n^{RR} = \pm 1$. For weak nonlinearity, the proximity of the two rings leads to a significant overlap of their Berry curvature distributions. Consequently, integrating over the disk enclosing only the inner ring yields finite quasi-Chern integral values on the order of $\approx 10^{-1}$ for the outer ring, despite the integration disk lying entirely inside the ring.

As also discussed in ref 109, the ER only exists in the eigenvalues of the matrix in eq 1 if the two modes are degenerate in energy ($\Delta = 0$) over all k in the vicinity of the ER [Figure 6a]. Here, the eigenvalue problem (eq 1) exhibits passive $\mathcal{PT}$ -symmetry (the system is $\mathcal{PT}$ -symmetric up to a shift in eigenvalues), without detuning, resulting in a codimension of one for our exceptional points (which therefore form a gapless ring).¹¹³ Introducing detuning breaks the $\mathcal{PT}$ -symmetry and increases the exceptional points' codimension to two,^{86,114} resulting in isolated exceptional points as reported in ref 109. In microcavities, however, different physical effects (e.g., Zeeman and Rashba–Dresselhaus splitting) can induce different forms of, even k -dependent, detuning, which adds dimensions to the parameter space that can be accessed and explored.

To show that the ER is not destroyed with finite detuning but instead can be recovered in a higher-dimensional parameter space, we extend the linear eigenvalue problem (eq 1) by finite (k -dependent) detuning (with Zeeman and Rashba–Dresselhaus splittings as our specific examples of the physical causes of this detuning). To this end, $\Delta_{\pm} = \pm \Delta_B \pm \Delta_{RB}(k_{\parallel})$ is added to the first (+) and second (−) diagonal elements in eq 1. Here Δ_B describes the magnetic field-induced Zeeman splitting and $\Delta_{RB} = 2\alpha_{RD}k_{\parallel}$, the Rashba–Dresselhaus splitting^{115–117} with splitting strength α_{RD} . In specifically designed semiconductor microcavities α_{RD} is adjustable by an external electric field acting on an optically anisotropic liquid crystal inside the cavity.^{22,115,116,118} Here we denote the Rashba–Dresselhaus splitting direction in k -space as $k_{\parallel}$ and the direction perpendicular to the splitting as $k_{\perp}$. We emphasize that the exact mathematical form and physical origin of detuning is not important for our results; those transfer to any form of detuning breaking k -isotropy in a similar fashion.

By introducing the Zeeman shift Δ_B to the circularly polarized modes via a magnetic field (which may be enabled with an optically active matter component inside the resonator), the two modes become nondegenerate and the ER vanishes [Figure 6b]. As discussed, in certain structures and materials, an electric field-induced Rashba–Dresselhaus splitting Δ_{RD} along $k_{\parallel}$ may be used to lift the degeneracy of TE and TM modes in the $k_{\parallel}$ -direction.¹¹⁷ Thus, in the $k_x - k_y$ plane only the exceptional point pair along $k_{\perp} = 0$ remains visible [Figure 6c]. The combination of these two effects lets us see the two exceptional points on the ring positioned where $k_{\perp} \neq 0$ with $\Delta_{RD}(k_{\parallel}) - \Delta_B = 0$ [Figure 6d]. With this approach, we can validate that the ER structure stays intact in higher-dimensional parameter space (k_x, k_y, Δ_B , and α_{RD}) even for finite detuning. In Figure 6e the ER is shown in this 4D parameter space, and the dots mark the location of exceptional points and thus the points for which the Zeeman shift

compensates the Rashba–Dresselhaus splitting (note that the ERs form a continuous structure; for discretization reasons, this is not fully resolved in the plot). Here, the tilt of the individual representations of the ER is induced by the respective Rashba–Dresselhaus splitting marked in color. While we are at present unable to calculate the quasi-Chern integrals for the tilted rings, we note that we expect each colored ring to possess a first quasi-Chern integral of $C^{RR} = \pm 1$. Note that the radius of the ER projected onto the $k_x - k_y$ plane remains unchanged from these extensions. The detuning strengths chosen here and below are in agreement with possible Zeeman and Rashba–Dresselhaus splittings measured in microcavity systems with the perovskite semiconductor CsPbBr₃ as active medium.^{117,119} This observation shows that in higher-dimensional parameter space, the ER is part of an exceptional surface and the ring is resilient against detuning. Increasing both parameter space dimensionality and codimension of the exceptional points explains the occurrence of the exceptional surface, as discussed in ref 114. While detuning values might not be fully tunable in experiments, this result illustrates that the linear ER can be (partially) recovered even in k -anisotropic samples. Note that this analysis extends to any similar form of detuning or anisotropy in this and other physical systems. Moreover, the tilt of the ER permits new ways for tracing trajectories in higher-order dimensional space, such as Solomon's knots around the ER. This is promising for both mode-switching applications as well as the investigation of topological characteristics. In the following, we will extend this investigation into the nonlinear regime, to show that the elliptic umbilic leads to even richer (spectral and band) topologies and exceptional point structures in the nonlinear regime.

We use the solutions of the nonlinear eigenvalue problem (eq 3) with finite detuning to investigate topological features by applying Rashba–Dresselhaus and Zeeman splittings to the system. In contrast to the linear ER, the elliptic umbilic allows the formation of new ER constellations for finite detuning Δ . In Figure 7, the complex-valued energy spectrum and central cuts are displayed for a finite Zeeman splitting of $\Delta_B = 12.5 \mu\text{eV}$. The inset in Figure 7b shows the umbilic cut at $\alpha = 0.1 \text{ meV}\mu\text{m}^2$ in turquoise together with the chosen Zeeman splitting in orange. The intersections mark the visible ERs for a given setup. Here, two rings are formed by points on exceptional arcs. Notably, reversing the direction of the Zeeman-induced energy shift inverts the polarization in the complex-valued energy spectrum. We note that similar energy spectra can be achieved for stronger Zeeman splitting if the nonlinearity is increased simultaneously (to increase the size of the umbilic acid in the regime where its structure is valid). A relatively weak nonlinearity was selected to limit the blueshift, ensuring the resulting structures remain within an easily visualizable energy range. These findings highlight that while they both feature identical quasi-Chern integrals, the neighborhood topology of linear and nonlinear ERs is drastically different, as detuning is introduced.

Building on the understanding that the linear ER is tilted in parameter space for k -anisotropy (in this case, explored through Rashba–Dresselhaus and Zeeman splitting [see Figure 6e]), we now examine how this affects the arrangement of umbilic cross sections in the nonlinear regime. While the ER vanishes in the linear regime for finite Zeeman and vanishing Rashba–Dresselhaus splitting ($\Delta_B \neq 0, \Delta_{RD} = 0$) [see Figure 6b], in the nonlinear regime the elliptic umbilic profile forms a

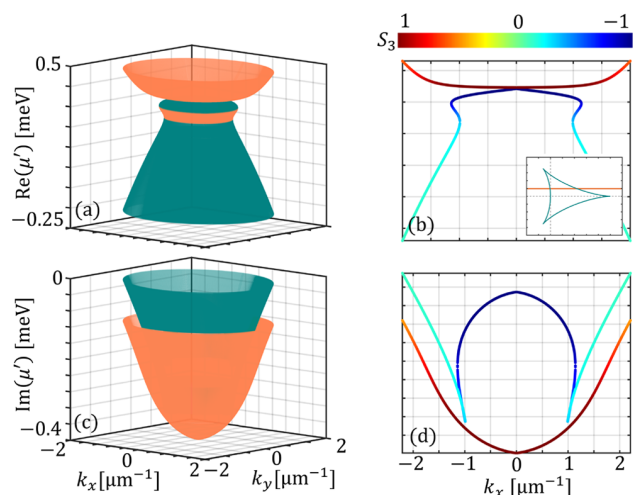


Figure 7. Complex-valued energy spectrum with exceptional rings for finite Zeeman splitting in nonlinear regime. (a, c) Real- and imaginary part of eigenvalues μ . (b, d) Cross sections of k -isotropic complex-valued energy spectrum in (a, c). The main panels show the real and imaginary parts of the energy spectrum and central cuts that host the two concentric arc rings. The turquoise line in the inset in (b) shows the cross section of the umbilic for finite $\alpha = 0.1 \text{ meV } \mu \text{ m}^2$. The orange line displays the Zeeman-splitting-induced detuning (increasing with increasing Δ_B). The displayed exceptional rings form at radii defined by the intersections with the two exceptional arcs. The coloring in parts b and d illustrates the mode polarization in the form of the S_3 Stokes vector element for positive Δ_B . Simulation parameters: $\{\alpha, g_\sigma, g_x\} = \{0.1, 0.2, 0\} \text{ meV } \mu \text{ m}^2$.

donut-like quasi-torus in parameter space, as displayed in Figure 8a. Here, the colored rings and the connecting black surfaces form EXRs and surfaces of exceptional arcs, respectively. To the best of our knowledge, this represents the first theoretical observation of an exceptional nexus ring, emerging from the intersection of two exceptional arc surfaces. This structure extends the concept of an exceptional nexus (previously point-like) into higher dimensions. The size of the quasi-torus scales with nonlinearity. These results demonstrate, that in contrast to linear ERs, finite detuning induces a change in topology in the nonlinear regime. In the case of finite Zeeman and Rashba–Dresselhaus splitting ($\Delta_B \neq 0$, $\Delta_{RD} \neq 0$), the quasi-torus is tilted in parameter space in $k_{||}$ -direction. In Figure 8b a tilted quasi-torus is displayed for Rashba–Dresselhaus splitting of $\alpha_{RD} = 0.1 \text{ meV}$. While the tilt of the linear ER allows for new loops such as Solomon’s knot, the topology of nonlinear ERs would allow for even more complex loops. Analogously to the linear ER, a stronger Rashba–Dresselhaus splitting leads to a stronger tilt of the quasi-torus. These findings demonstrate the robustness and versatility of exceptional points and ER singularity with the elliptic umbilic structure, even with variations of the simplest model. The highly tunable nature of energy spectra associated with exceptional points on the umbilic region presents exciting opportunities for mode-switching applications, where different trajectories on the quasi-torus can be strategically exploited and this represents an exciting avenue for future investigations.

Next, we extend our analysis of vorticity ν and Berry phase $\gamma_{\text{Berry}, n}^{ab}$ to the exceptional quasi-torus emerging from the ER when entering the nonlinear regime. To this end, we evaluate eqs 7 and S15 numerically using solutions of the nonlinear eigenvalue problem. In Figure 9a the exceptional quasi-torus is

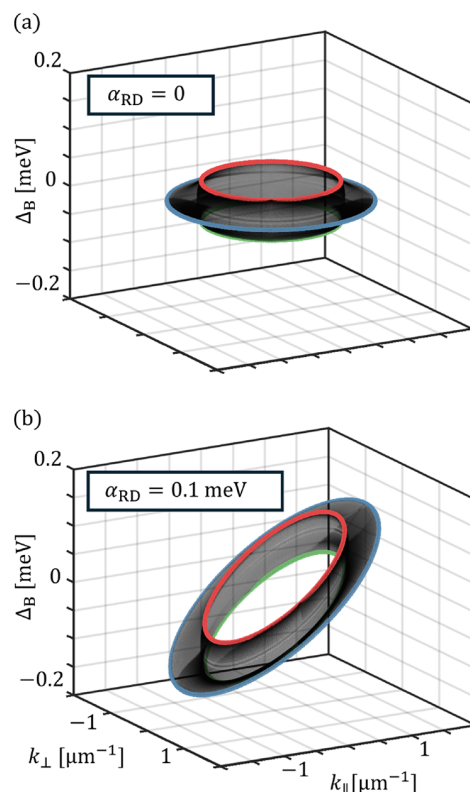


Figure 8. Topological structure of nonlinear exceptional rings in 4D parameter space. (a) With Zeeman splitting, a single linear exceptional ring at $\Delta_B = 0$, in the nonlinear regime, expands into a quasi-torus of exceptional nexuses (colored) connected by surfaces of exceptional arcs (black). (b) Rashba–Dresselhaus splitting in the $k_{||}$ -direction leads to a tilt of the quasi-torus along $k_{||}$ (shown for $\alpha_{RD} = 0.1 \text{ meV}$). Simulation parameters: $\{\alpha, g_\sigma, g_x\} = \{0.1, 0.2, 0\} \text{ meV } \mu \text{ m}^2$.

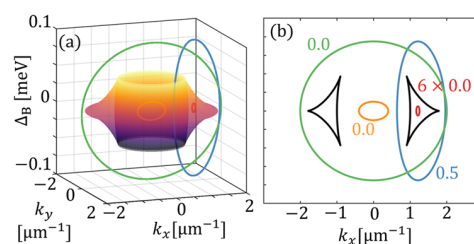


Figure 9. Vorticity of nonlinear exceptional quasi-torus. (a) Full and (b) central cut of the exceptional quasi-torus arising from an exceptional ring in the nonlinear regime. Closed loops, as highlighted, are used to calculate the vorticity of the torus. The quasi-torus carries a vorticity of $\nu_{ul} = 0.5$, while the bands inside the torus carry no vorticity. Simulation parameters $\{\alpha, g_\sigma, g_x\} = \{0.15, 0.3, 0\} \text{ meV } \mu \text{ m}^2$ were used.

shown together with multiple closed loops in the $\Delta - k_x$ plane used to calculate the vorticity. Note that the blue ring represents a Hopf-link configuration for the linear ER, while the red ring would not form a Hopf link with the linear ER, since it is fully enclosed by the umbilic and thus would run outside the linear ER. If the Hopf link is large enough to enclose all exceptional points on the umbilic, the vorticity amounts to $\nu_{ul} = 0.5$ for the nonlinear exceptional quasi-torus. Any loop that does not encircle the torus (orange and green in Figure 9) amounts to vorticities of $\nu_{ul} = 0$. Lastly, all unique

permutations of bands inside the quasi-torus (6 in total) amount to vorticities of $\nu_{ul} = 0$ as indicated in the figure. We note that depending on the radius chosen numerical inaccuracies of about 1% may occur.

These results show that certain topological invariants (here, the vorticity) can be preserved even if nonlinearity is introduced into the system. In the present case, the second-order ER in the linear case is a point-like singularity (e.g., in the $k_x - \Delta_B$ plane) and obstruction to Stokes theorem and therefore yields the expected vorticity of 0.5. In contrast, in the nonlinear regime, the cross section of the torus (again, e.g., in the $k_x - \Delta_B$ plane) is an extended object with singularities on the surface of the triangular cross section. Nevertheless, we find from our numerical studies that this extended object in the nonlinear regime has the same vorticity as the isolated point-like singularity in the linear regime.

Numerical evaluation of the Berry phase $\gamma_{\text{Berry}, n}^{ab}$ in the $k_x - k_y$ plane for EAR and EXR in the nonlinear regime leads to $\gamma_{\text{Berry}, \text{EAR}}^{RR} = 0$ and $\gamma_{\text{Berry}, \text{EXR}}^{RR} \in \{0, 2\pi\}$. Here the indices “EAR” and “EXR” indicate that we are using bands that are forming the corresponding exceptional ring. Since the EXR features reduced self-orthogonality, we can calculate the Berry phase for mixed left and right eigenvectors. Then the Berry phase amounts to $\gamma_{\text{Berry}, \text{EXR}}^{RL} \in \{0, 2\pi\}$.

III. CONCLUSION

We have demonstrated that exceptional rings (ERs) naturally emerge inside the in-plane dispersion of planar optical resonators with TE-TM splitting and circular dichroism (leading to different losses in the opposite circularly polarized resonator modes). These ERs can even be recovered in k -space-anisotropic systems, where adjusting mode detuning can be used to recover them in higher-dimensional parameter space. Introducing a Kerr-type medium (or saturable gain) splits the linear ER into multiple ERs forming an elliptic umbilic in nonlinear parameter space. Here, third-order exceptional nexus rings and second-order exceptional arc rings form concentrically in k -space. We performed a rigorous analysis of the nonlinear ERs, showing diverse perturbation-response enhancement at different points on the umbilic singularity structure and intriguing topological features. We show that a linear ER in higher-dimensional parameter space splits into a complex quasi-toroidal structure of ERs, making nonlinear ERs promising candidates for mode-switching applications. Since a wide range of nonlinear non-Hermitian Hamiltonians feature exceptional points or contours that are the organization point of an elliptic umbilic,⁶⁵ our results on topology and perturbation response can be applied to a large group of physical systems hosting exceptional points that are subject to real (or imaginary) valued smooth nonlinearities. Examples are Kerr-type nonlinearity or saturable gain but also higher-order nonlinearities and nonlinearities of different functional forms. Our work lays the foundation for a range of further research questions: the experimental realization of ERs in planar optical resonators and exciton–polariton systems, the comprehensive investigation of the sensitivity properties (such as signal-to-noise ratio) of exceptional points on the elliptic umbilic, taking into account noise–nonlinearity interactions, and the analysis of topological invariants in other nonlinear and non-Hermitian systems. Moreover, we also lay the foundation for a detailed investigation of the relation of the nonlinear eigenvalue problem and the system dynamics for different possible excitation setups.

■ ASSOCIATED CONTENT

Supporting Information

The Supporting Information is available free of charge at <https://pubs.acs.org/doi/10.1021/acsp Photonics.5c02049>.

Methods: analytical solutions and solution approach for the nonlinear eigenvalue problem (3) without detuning; separability eigenvalue equations as a means to solve the nonlinear eigenvalue problem (3); stationary solutions of the resonantly cw-driven nonlinear system; ER topology at zero detuning. Notes: example of non-resonantly pumped system where the pump induces circular dichroism; ERs in the cw-driven nonlinear spatiotemporal dynamics; and generalization of reported (spectral and band) topological features to a broad class of nonlinear and non-Hermitian Hamiltonians (PDF) Additional details on spectral bands (ZIP)

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Notes

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Exceptional points in nonresonant nonlinear systems

7

In the previous chapter, it was demonstrated that, for nonlinear and non-Hermitian systems, the results obtained from the self-consistent Hamiltonian remain valid in the vicinity of the resonance peaks of the resonantly driven instantaneous Hamiltonian. This allowed us to study nonlinear exceptional points for their topological features and responsivity under resonant excitation. However, a key feature of polariton systems stems from their condensation under nonresonant excitation. Thus, in the following, we show how this excitation mechanism leads to multiple nonlinear effects that are deeply intertwined with the formation of exceptional points. Note that in this case, the results of the self-consistent and instantaneous Hamiltonian may differ. Thus, here we do not draw any conclusions about the unfolding topology, but take a perturbative approach to study exceptional points directly in the instantaneous Hamiltonian.

Nonresonant pumping generates an additional degree of freedom. The reservoir dynamics regulate the saturable gain, the optically-induced potential, and the polariton mode densities. Thus, the instantaneous mode dynamics are no longer fully covered by a self-consistent description. Here, we employ a toy model to investigate the behavior in the vicinity of the exceptional point within the perturbative nonlinear regime. Figure 7.1(a) illustrates how nonlinear effects relate to the important quantities of exceptional point formation in the context of an effective two-level system of two coupled polariton modes $\psi_j(\mathbf{r}, t)$. Each mode features a fixed loss rate γ_c defined by the polariton lifetime. Saturable gain and an optically induced potential $g_r n(\mathbf{r}, t)$ are provided by the excitonic reservoir $n(\mathbf{r}, t)$, while Kerr-type nonlinearity relates mode energies to polariton densities $g_c |\psi_j(\mathbf{r}, t)|^2$. The polariton-polariton interaction strength g_c affects mode profiles and thus indirectly modifies the effective mode coupling μ . Moreover, the saturable gain leads to density differences, which in turn lead to changes in mode coupling and energy due to Kerr-type nonlinearity. Thus, all quantities that require control to tune the system towards the exceptional point are connected. Hence, we developed a toy model to relocate exceptional points in the nonlinear regime of the system. The key idea of the toy model is presented here, while detailed explanations are provided in the work included below.

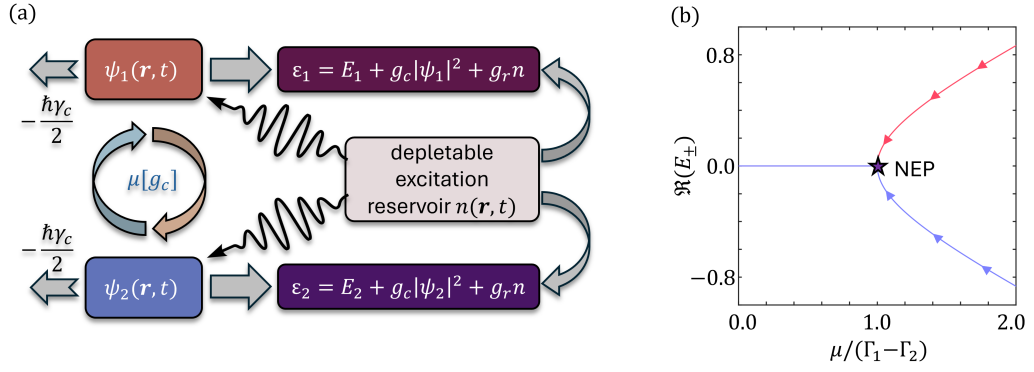


Figure 7.1: Nonlinear and non-Hermitian interactions in a nonresonantly driven system near an exceptional point. (a) Two polariton modes $\psi_j(\mathbf{r}, t)$ are fed by a depletable reservoir $n(\mathbf{r}, t)$, which provides saturable gain to counteract the loss rate γ_c . Polariton-polariton interactions induce Kerr-type nonlinearity, which affects the mode coupling μ . Together with polariton-reservoir interaction strengths, the Kerr-type nonlinearity affects the mode energy $\varepsilon_j = E_j + g_c |\psi_j|^2 + g_r n$. (b) Schematic eigenvalue structure in the vicinity of a nonlinear exceptional point. Assuming perturbative preservation of the square-root topology, a nonlinear two-level toy model is used to relocate the exceptional point (star) and analyze the influence of nonlinear interactions on the eigenvalue splitting and spectral topology.

We begin by designing a potential landscape that can feature an exceptional point of second order in the linear regime between modes ψ_1 and ψ_2 . Now the challenge is to relocate the exceptional point after introducing nonlinearity. To address this issue, we developed a toy model by starting from a 2×2 non-Hermitian Hamiltonian to describe a linear system in the vicinity of the exceptional point. Then we replace the scalar complex-valued energies on the main diagonal with the nonlinear instantaneous Hamiltonian of the Gross-Pitaevskii equation for stationary excitation. This leaves us with two coupled time-dependent nonlinear and non-Hermitian Schrödinger equations that describe the dynamics of the two polariton modes. Here, the coupling on the off-diagonal provides a control knob to tune the system towards the exceptional point. We assume that, within the perturbative nonlinear regime, the vicinity of the exceptional point can be locally described by the square-root topology known from the linear eigenvalue problem.

Figure 7.1(b) shows the eigenvalue real part close to the exceptional point (marked by a star) with preserved square-root topology. The arrowheads on the two branches visualize how we tune the system towards the exceptional point by tuning the control parameter μ . To demonstrate this toy model, we investigate the tri- and dipole modes of a rectangular

external potential. At the exceptional point, the two modes coalesce into a hybridized state carrying a vortex-antivortex pair. These vortex pairs are usually hard to stabilize, since they can annihilate each other by recombination [149].

Once the exceptional point is localized, we validate the results of our toy model with the full dissipative Gross-Pitaevskii model, which covers the full mode dynamics of the system. Note that the full nonlinear dynamics of the system can also cover complex nonlinear phenomena such as attractors and limit cycles. Thus, our toy model provides a powerful tool to filter the relevant dynamics of the two modes and guide them to the exceptional point. Moreover, by tuning the pump parameters, we encircle the exceptional point to demonstrate the characteristic mode switching and to validate that the found singularity describes an exceptional point.

After validating the nonlinear exceptional point with the full dissipative Gross-Pitaevskii model, we analyze how the nonlinear interaction strengths modify its spectral topology. In particular, the exceptional point rotates in parameter space as the Kerr-type nonlinearity connects the mode detuning to the density difference resulting from distinct mode gain. This opens new methods of encircling exceptional points, designing the potential or excitation to realize finite detuning, and the gain difference can directly control the energy difference, potentially reducing the number of control parameters to achieve mode switching. Moreover, the blueshift difference does not reflect an independent external system parameter. Instead, it results from a gain imbalance leading to distinct polariton densities. Hence, the observed spectral rotation of the exceptional point is not a mere visualization feature but represents a renormalization of system parameters.

Furthermore, we show that the characteristic eigenvalue splitting can be scaled in magnitude by nonlinear interaction strengths. Here, nonlinearity manipulates mode frequencies, leading to a linear scaling of the eigenvalue splitting while conserving the characteristic square-root dependence.

Two phrases must be clarified to align the following work with the format of this thesis. First, the phrase *Riemann surface* relates to the complex-valued energy spectra of the non-Hermitian system. Second, the expression *sensitivity* relates to the responsivity of the exceptional point to external perturbation. We do not claim any enhancement of the signal-to-noise ratio in this work.

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Manipulating spectral topology and exceptional points by nonlinearity in non-Hermitian polariton systems

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Exceptional points (EPs), with their intriguing spectral topology, have attracted considerable attention in a broad range of physical systems, with potential sensing applications driving much of the present research in this field. Here, we investigate spectral topology and EPs in systems with significant nonlinearity, exemplified by a nonequilibrium exciton-polariton condensate. With the possibility to control loss and gain and nonlinearity by optical means, this system allows for a comprehensive analysis of the interplay of nonlinearities (Kerr type and saturable gain) and non-Hermiticity. Not only do we find that EPs can be intentionally shifted in parameter space by the saturable gain, but we also observe intriguing rotations and intersections of Riemann surfaces and find nonlinearity-enhanced sensing capabilities. With this, our results illustrate the potential of tailoring spectral topology and related phenomena in non-Hermitian systems by nonlinearity.

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I. INTRODUCTION

Exceptional points (EPs) are singularities in parameter space at which two or more eigenvalues and their corresponding eigenvectors coalesce [1–4]. Such singularities occur exclusively in non-Hermitian systems which are subject to gain and loss and exhibit nonorthogonal eigenvectors and complex eigenvalues [5,6]. Compared to conventional singularities in Hermitian systems, known as Diabolic points, EPs show intriguing properties due to the spectral topology of their Riemann surfaces [7,8]. EPs have been widely investigated in a variety of physical systems such as microwave resonators [9–11], atomic systems [12], plasmonic nanostructures [13,14], optical waveguides [15], microresonators [16–18], and nonreciprocal systems [19]. Near an EP, a range of counterintuitive phenomena has been reported in optical systems due to the coalescence of the eigenvectors, including loss- and optomechanically induced transparency [20,21], unidirectional invisibility, and reflectivity exceeding unity [15,22]. Moreover, sensing enhancement can be realized close to an EP of order n where the frequency splitting scales as the n th root of the perturbation [23–26]. This makes EPs promising candidates for a new generation of sensors, with the potential to outperform their Hermitian counterparts [27–30].

Recently, there has been growing interest in the interplay of nonlinearity and non-Hermiticity. For instance, the influence of nonlinear effects on the $\mathcal{PT}$ symmetry in lasing systems

was studied [31–34]. EP-based sensors with saturable gain [35], saturable-gain-induced energy shift [36,37], and the encircling of EPs in bistable domains [38] were investigated. To systematically study the interplay of nonlinear and non-Hermitian physics, systems with variable nonlinearity and controllable gain and loss are required. An example of such systems is exciton polaritons, in which rich nonlinear physics have been reported, such as nonlinear emission [39], stimulation of photoluminescence [40], angle-resonant amplification [41,42], and strong stimulated scattering under resonant excitation [43].

Exciton polaritons are hybrid light-matter quasiparticles that form due to strong light-matter coupling, pairing finite lifetimes on a picosecond scale, and thus non-Hermiticity, with strong nonlinearity from polariton-polariton interactions. For nonresonant optical excitation, spontaneous macroscopic coherence can form, known as polariton condensation [44–46]. In that case, the significant interaction of the condensate and the exciton reservoir induces a repulsive potential energy landscape, enabling optical trapping [47–51] as well as the optical control of the polariton condensate [52–56]. Moreover, the nonlinearity of polariton condensates permits multistabilities [57–60] and hence offers new design avenues for all-optical transistors [61] and allows for the creation and control of vortices [56,62,63]. The nonlinearity inside the polariton condensate can be tuned with static electric fields [64–66]. With their nonequilibrium nature and the possibility of both structural and optical control, polariton systems offer a natural playground for the study of non-Hermitian physics. For instance, non-Hermitian topological corner modes [67], topological interface states [68], $\mathcal{PT}$ symmetry [69], and EPs [70–74] have been investigated. Moreover, the first measurement of the quantum metric in a non-Hermitian system was

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conducted on exciton-polariton eigenstates [75]. However, the interplay of nonlinearity and non-Hermiticity has not been fully explored or exploited.

In the present work, we investigate the manipulation of EPs and surrounding Riemann surfaces through the nonlinearities in the system. We find that the saturable gain shifts the EP in parameter space, leading to a variation in the mode coupling requirement for the observation of the EP. The repulsive polariton-polariton nonlinearity (akin to a Kerr-type nonlinearity) can induce not only an energy blueshift but also a simultaneous rotation of the Riemann surface and movement of the EP. We show that this also applies to higher-order EPs where the Riemann surfaces can show complex intersection patterns. With potential applications in mind, we further demonstrate that sensing sensitivity near the EP can be significantly enhanced by nonlinearity.

II. THEORETICAL MODEL

To study the dynamics of polariton condensates under non-resonant excitation, we use the dissipative Gross-Pitaevskii (GP) model [63,76]. Within this framework, the dynamics of the coherent polariton field $\psi = \psi(\mathbf{r}, t)$ and the density of the exciton reservoir $n = n(\mathbf{r}, t)$ are

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi &= \left(H_0(\mathbf{r}) + g_c |\psi|^2 + g_r n + \frac{i\hbar}{2} [Rn - \gamma_c] \right) \psi, \\ \frac{\partial n}{\partial t} &= P(\mathbf{r}) - (\gamma_r + R|\psi|^2)n. \end{aligned} \quad (1)$$

The linear operator $H_0(\mathbf{r}) = -\frac{\hbar^2}{2m_{\text{eff}}} \nabla^2 + V(\mathbf{r})$ describes the kinetic energy and the external potential $V(\mathbf{r})$ which confines the condensate spatially. $m_{\text{eff}} = 10^{-4}m_e$ is the effective polariton mass, with m_e being the free electron mass. g_c is the strength of the polariton-polariton interaction (akin to a Kerr-type nonlinearity), while g_r is the strength of the polariton-reservoir interaction. $\gamma_c = 0.3 \text{ ps}^{-1}$ is the loss constant of the condensate in the quasimode approximation [77], counteracted by the stimulated in-scattering with $R = 0.01 \text{ ps}^{-1} \mu\text{m}^2$. $\gamma_r = 0.45 \text{ ps}^{-1}$ is the loss constant of the reservoir, supplemented by the incoherent pump $P(\mathbf{r})$. For stationary excitation and solution we obtain $n(\mathbf{r}) = \frac{P(\mathbf{r})}{\gamma_r + R|\psi(\mathbf{r})|^2}$; the term $R|\psi(\mathbf{r})|^2$ holds the saturable gain.

A system containing an EP of two coalescing eigenvectors is approximated in its vicinity by a two-level effective Hamiltonian,

$$H_{\text{EP2}} = \begin{pmatrix} E_\alpha + i\Gamma_\alpha & \mu/2 \\ \mu/2 & E_\beta + i\Gamma_\beta \end{pmatrix}. \quad (2)$$

Here, E_α and E_β are the energies of modes α and β . The gain and loss are included in Γ_α and Γ_β , and μ characterizes the coupling strength of the two modes. The eigenvalues of the Hamiltonian (2) read $E_\pm = [E_\alpha + E_\beta + i(\Gamma_\alpha + \Gamma_\beta)]/2 \pm \sqrt{|\mu|^2 + [E_\alpha - E_\beta + i(\Gamma_\alpha - \Gamma_\beta)]^2}/2$. The second term describes the energy splitting on one side of the EP, which shows the characteristic square-root dependence. Plotting the eigenvalues depending on the energy difference $\Delta E_L = E_\alpha - E_\beta$ and the loss and gain difference $\Delta\Gamma = \Gamma_\alpha - \Gamma_\beta$ visualizes the Riemann surface of the system, as shown in Fig. 1(a) for the real part. The EP is localized at $\Delta E_L = 0$ and $\mu = \Delta\Gamma$.

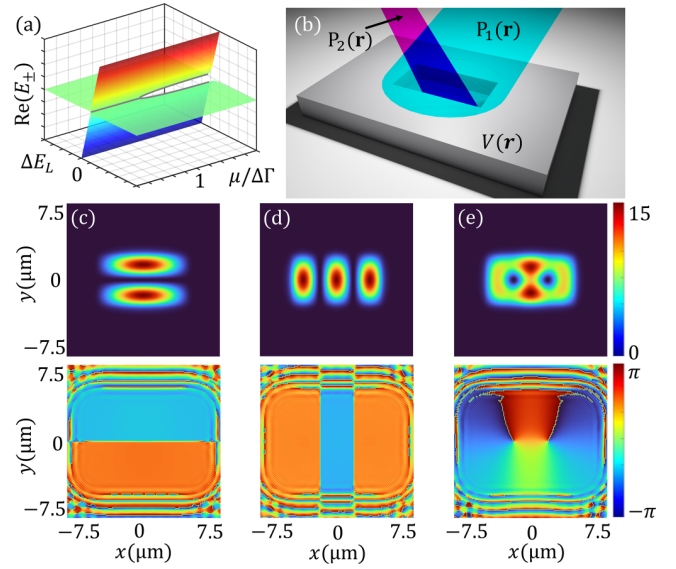


FIG. 1. System and excitation setup. (a) Sketch of the Riemann surface (real part only) of a two-level non-Hermitian system containing an EP for varying level separation $\Delta E_L = E_\alpha - E_\beta$ and varying relative level coupling $\mu/\Delta\Gamma$. (b) Scheme for realization with polariton condensates, including the rectangular external potential $V(\mathbf{r})$ and excitation with two nonresonant pump beams, $P_1(\mathbf{r})$ and $P_2(\mathbf{r})$. Polariton density (top row; in μm^{-2}) and phase (bottom row) of the (c) third and (d) fourth modes of the potential $V(\mathbf{r})$ that are chosen here as an example for the realization of an EP. (e) Vortex-antivortex mode at the EP. Further details are given in the text.

To study the dynamics of two coupled modes $\psi_{\alpha,\beta}$ of a polariton condensate in the vicinity of an EP, we extend the effective non-Hermitian Hamiltonian (2) by including the nonlinearity of the GP model. Consequently, the two-level non-Hermitian Hamiltonian and the effective nonlinear Schrödinger equation (NSE) read

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix} = \begin{pmatrix} H(\psi_\alpha) & \mu/2 \\ \mu/2 & H(\psi_\beta) \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix}, \quad (3)$$

where $H(\psi) = H_0(\mathbf{r}) + g_c |\psi|^2 + g_r n + \frac{i\hbar}{2} [Rn - \gamma_c]$. It should be noted that this model can also be applied to other nonlinear systems by substituting the Hamiltonian. In the further course of this work, the eigenenergies of the system are denoted by $E_{\alpha,\beta}$, and their difference is denoted by ΔE_L . The blueshift-induced correction to this difference is defined as ΔE_{NL} , such that the energy difference in the nonlinear regime is $\Delta\mathcal{E} = \Delta E_L + \Delta E_{\text{NL}} = \mathcal{E}_\alpha - \mathcal{E}_\beta$. In this work, we investigate the third, $\alpha = 3$, and fourth, $\beta = 4$, modes [as shown in Figs. 1(c) and 1(d)] of a rectangular external potential with a width of $5 \mu\text{m}$, length of $8.9 \mu\text{m}$, and depth of 2 meV , as sketched in Fig. 1(b). These two modes are chosen because their energies are close to each other and their degeneracies can be realized in experiments. Provided these conditions are fulfilled, the model can be applied to arbitrary modes [70]. We use a nonresonant pump $P(\mathbf{r})$ which consists of a broad flat top pump $P_1(\mathbf{r})$ [light blue cone in Fig. 1(b)] with a diameter of $12 \mu\text{m}$ and intensity of $I_1 = 16 \text{ ps}^{-1} \mu\text{m}^{-2}$. An additional elliptical pump $P_2(\mathbf{r})$ [light pink cone in Fig. 1(b)] with a height of

2 μm and width of 5 μm is used to tune the mode energy difference $\Delta\mathcal{E}$. Mathematical expressions for the pump profiles are given in Appendix A 2. The total energies $\mathcal{E}_{\alpha,\beta}$ of the two modes are obtained by solving the GP equation, Eq. (1), on a discrete spatial grid using the Dormand-Prince method for the time integration and time-domain Fourier transformation for the respective steady states [Figs. 1(c) and 1(d)]. Their scalar gain and loss rates can be approximated by $\Gamma_{\alpha,\beta} \approx N^{-1} \frac{\hbar}{2} \int [Rn_{\alpha,\beta}(\mathbf{r}) - \gamma_c] |\psi_{\alpha,\beta}(\mathbf{r})|^2 d^2\mathbf{r}$, with $N^{-1} = \int |\psi_{\alpha}(\mathbf{r})|^2 d^2\mathbf{r}$ [71]. The complex eigenvalues $E_{\pm}$ of the corresponding effective two-level system, Eq. (2), can then be determined for a given $\Delta\mathcal{E} = \mathcal{E}_{\alpha} - \mathcal{E}_{\beta}$ and as a function of the mode coupling strength μ with the nonlinear energies as input for $E_{\alpha,\beta}$. For the coupled two-level NSE, Eq. (3), with $\mu = \Delta\Gamma$ close to the degeneracy $\Delta\mathcal{E} = 0$, the two modes coalesce into the superposition mode in which a vortex-antivortex pair forms [see Fig. 1(e)]. We note that in our simulations, the superposition mode is resilient to a random disorder potential (explicitly tested for random external potentials with a correlation length of 1 μm and average depth of about 0.1 meV). By encircling the EP, supported by P_2 , the two vortices swap their topological charge, which is a typical property of a topological Riemann surface [71]. Details of the encircling are given in Appendix A 2 in Fig. 6.

III. SENSING ENHANCEMENT

We first focus on the influence of nonlinearities on the complex eigenvalues along the degeneracy of real-valued mode energies $\Delta\mathcal{E} = 0$. The eigenvalues $E_{\pm}$ of the two-dimensional system are determined as a function of μ by evaluating the eigenvalues according to the method introduced above. Exposed to a saturable gain, which in our case is expressed by $R|\psi(\mathbf{r})|^2$, the gain difference of the modes in a nonlinear system depends on their intensities. The coupling of the modes in our multistable system is also influenced by the strength of the polariton-polariton interaction. Thus, the necessary coupling strength μ to reach the EP in such a nonlinear system is changed by the intensity of the modes or, in this case, the two-dimensional polariton density. Hence, we observe a shift of the EP along the $\Delta\Gamma$ or μ axis as a function of the nonlinear interaction strength (see Fig. 5). A similar effect was recently observed in Ref. [37] in a lasing system. For stationary solutions with $|\psi_{\alpha,\beta}|^2 = \max[|\psi_{\alpha,\beta}(\mathbf{r})|^2]$ and $n_{\alpha,\beta} = \max[n_{\alpha,\beta}(\mathbf{r})]$, the eigenvalues of Eq. (2) with respect to the nonlinear terms of Eq. (1) read

$$E_{\pm,\text{NL}} \approx \frac{[E_{\alpha} + E_{\beta} + g_c|\psi_{\text{tot}}|^2 + g_r n_{\text{tot}} + i(\Gamma_{\alpha} + \Gamma_{\beta})]}{2} \pm \frac{\sqrt{|\mu|^2 + (\Delta E_L + \Delta E_{\text{NL}} + i\Delta\Gamma)^2}}{2}. \quad (4)$$

Here, the polariton- and reservoir-polariton interactions induce an energy blueshift on the Riemann surface proportional to the total polariton $|\psi_{\text{tot}}|^2 = |\psi_{\alpha}|^2 + |\psi_{\beta}|^2$ and reservoir density $n_{\text{tot}} = n_{\alpha} + n_{\beta}$. $\Delta E_{\text{NL}} = g_c \Delta|\psi|^2 + g_r \Delta n$ is characterized by the density difference $\Delta|\psi|^2 = |\psi_{\alpha}|^2 - |\psi_{\beta}|^2$ and $\Delta n = n_{\alpha} - n_{\beta}$. We note that the expression in Eq. (4) represents a scalar approximation of the problem described in

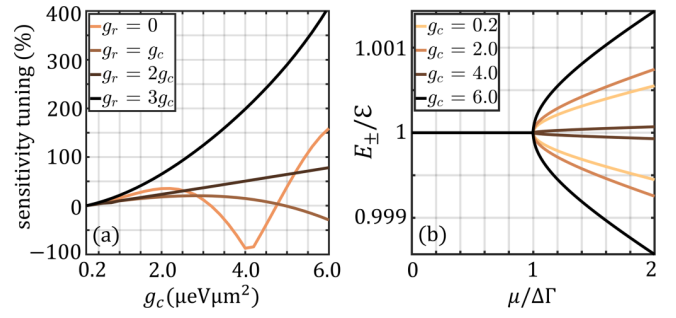


FIG. 2. Nonlinearity-induced enhancement of sensing at the EP. (a) Deviation of the eigenvalue splitting (real part) from the linear case at $\mu/\Delta\Gamma = 2$, close to the EP as a function of the polariton-polariton interaction g_c and for different polariton-reservoir interactions g_r . The values of the splitting deviation and the corresponding increased or decreased EP sensitivity are given as a percentage of the energy splitting in the linear case. (b) Eigenvalues $E_{\pm}$ along the mode degeneracy line $\Delta\mathcal{E} = 0$ for different interaction strengths g_c and $g_r = 0$, illustrating the change in the energy splitting. Eigenvalues are renormalized to the total mode energy $\mathcal{E}$ at the EP.

Eq. (3). In this framework, the density of the modes is not subject to perturbation because the model describes the system in the vicinity of the EP, which limits the variation of the mode gain (see also details in Appendix A 1) and hence the density. From this discussion it follows that while the two modes coalesce at the EP with $\Delta|\psi|^2 = 0$ and $\Delta n = 0$, tuning the system away from the EP also lifts the coalescence of the two modes, resulting in nonvanishing contributions of $\Delta|\psi|^2$ and Δn . Depending on the exact shape of the two modes, this can lead to an increase ($\Delta E_{\text{NL}} > 0$) or decrease ($\Delta E_{\text{NL}} < 0$) of the eigenvalue splitting. The sensitivity change κ for a given ratio $\mu/\Delta\Gamma$ can be derived as discussed in Appendix A 1.

After Eq. (3) is solved, the eigenvalues of Eq. (2) are calculated. Then their splitting κ is determined for $\mu/\Delta\Gamma = 2$, and the percentage deviation from the splitting at the reference value is calculated. The results are plotted in Fig. 2(a) as a function of g_c for different g_r . For $g_r = 0$, the sensitivity of the EP can increase by up to 150% and decrease to about 10%. For comparison, the resulting EPs are shown renormalized by their respective mode energies at $\Delta\mathcal{E} = 0$ and their gain differences [see Fig. 2(b)]. For $g_r \neq 0$, the oscillation of the sensitivity decreases. For values of $g_r \geq 2g_c$, the sensitivity increases monotonically as the reservoir interaction dominates. It is worth pointing out that g_c depends on the detuning of the exciton and photon states and the choice of material [78] and can be controlled during sample preparation or in certain systems by applying a static electric field [64–66].

We note that the results shown here can also be validated with the dissipative Gross-Pitaevskii model in Eq. (1) once the EP is localized in the parameter space. There, the time evolution initialized with the third or fourth mode yields the superposition mode at the EP. In order not to duplicate results already shown in Fig. 1(e), the results of the GP model calculation are not shown explicitly. In essence, we find that the approximations introduced above for the derivation and evaluation of a simplified model description and the reduction to the third and fourth modes in the framework of the effective

Hamiltonian in Eq. (2) can also be applied to our nonlinear system in the vicinity of said EP, validating the model in Eq. (3). The added value of our model, which focuses on the relevant parameter domain and relevant modes, lies in its simplicity and low dimension. This renders the localization of the EP in the parameter space of the nonlinear system easier and the analysis more systematic and less specific to the particular physical system investigated here.

IV. RIEMANN SURFACE ROTATION

We extend our investigation from the bifurcation of the eigenvalues at the EP along the trace of degeneracy to the entire Riemann surface in the nonlinear regime. Here, we determine the eigenvalues by using Eq. (4) depending on ΔE_L and $\Delta\Gamma$. Spatially varying quantities are replaced by scalar approximations. Thus, the flat top pump P is approximated by its peak intensity I_1 , and the mode densities are approximated by their average values $|\bar{\psi}|^2 \approx 3 \mu\text{m}^{-2}$ inside the external potential. E_α is fixed to the eigenenergy of the dipole mode, as shown in Fig. 1(c), while E_β is varied close to this value. The resulting energy difference is linked to the ratio of the length and width of the rectangular potential [70]. The mode coupling is extracted from the calculations performed in the previous section. To construct the Riemann surface, the gain and loss difference is varied independently. To this end, we assume that the total density $|\psi_{\text{tot}}|^2$ is resilient to small variations of $\Delta\Gamma$. The remaining parameters are taken from the GP model calculations. The nonlinear correction to the energy difference $\Delta E_{\text{NL}} = g_c \Delta|\psi|^2$ is determined by deriving an expression for the polariton density contrast resulting from a gain and loss difference $\Delta\Gamma$, which reads (details are given in Appendix A 3)

$$\Delta|\psi|^2 = \frac{2\Delta\Gamma}{\left[\frac{\hbar PR}{\gamma_r} - \hbar\gamma_c\right] + \sum_{m=2}^{\infty} \frac{\hbar PR^m}{\gamma_r^m} (-|\psi_{\text{tot}}|^2)^{m-1}}. \quad (5)$$

The terms within the sum in the denominator follow from the Taylor series of the average gain and loss difference $\bar{\Gamma} = \frac{\hbar}{2} \left[\frac{RP}{\gamma_r + R|\bar{\psi}|^2} - \gamma_c \right] |\bar{\psi}|^2$, including the saturable gain. For the following study, we consider terms up to $m = 6$. Note that the density difference $\Delta|\psi|^2$, which mainly depends on $\Delta\Gamma$, has to be in a reasonable range for convergence. The influence of the Kerr-type nonlinearity $|\psi|^2\psi$ and the saturable gain $R|\psi|^2$ on the Riemann surface is illustrated in Fig. 3. Remarkably, the Kerr-type nonlinearity causes the trace of $\Delta\mathcal{E} = 0$ not to be parallel to the $\Delta\Gamma$ axis, which leads to rotation of the Riemann surface. In the presence of the saturable gain the rotation angle becomes density dependent and can thus be increased even further [see Figs. 3(a)–3(c)]; the nonlinearity-induced blueshift leads to a tilting of the $\Delta\mathcal{E} = 0$ trace and movement of the EP. In the particular case of the polariton condensate, the polariton-reservoir interaction g_r can induce an additional blueshift and thus a rotation of the Riemann surface, as shown in Figs. 3(d)–3(f). The relation between the total reservoir density and the reservoir density difference derived from Eq. (5) is discussed in detail in Appendix A 3. It shows that the Riemann surface can even be rotated into the opposite direction by virtue of the polariton-reservoir interaction. In polariton condensates the pump intensity can control

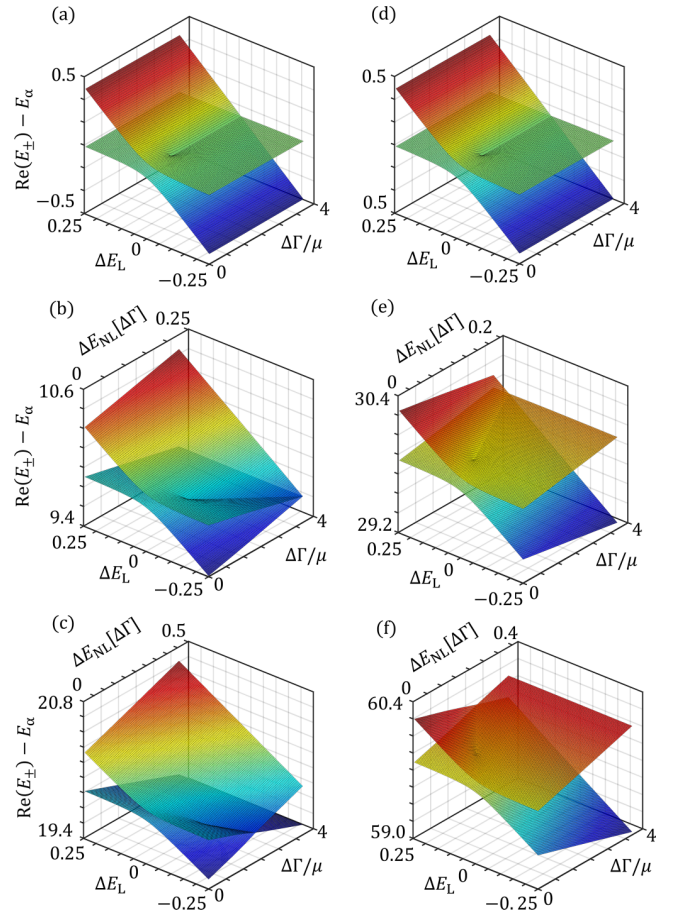


FIG. 3. Nonlinearity-induced rotation of Riemann surfaces. Riemann surfaces (real parts) resulting from the eigenvalue calculations, Eq. (4), for (a)–(c) $g_r = 0$ and (d)–(f) $g_r = 2g_c$ for different polariton-polariton interactions: (a) and (d) $g_c = 0 \mu\text{eV} \mu\text{m}^2$, (b) and (e) $g_c = 3 \mu\text{eV} \mu\text{m}^2$ and (c) and (f) $g_c = 6 \mu\text{eV} \mu\text{m}^2$. The eigenvalues are depicted against the energy difference ΔE_L and relative gain and loss difference $\Delta\Gamma/\mu$. For $g_c \neq 0$ the nonlinear correction ΔE_{NL} to the energy difference is plotted with $\Delta\Gamma$. Energies are given in μeV .

both nonlinear effects by tuning the density of the polariton condensate, which can provide an efficient method to actively tweak the spectral topology and the corresponding EP since in this case no other system parameters need to be varied.

From the rotation of the Riemann surface, it is worth asking whether more complicated higher-order EPs and multiple layer Riemann surfaces can also be manipulated. Here, we consider a parametrized and general three-level non-Hermitian Hamiltonian (with no direct relation to the specific modes in Fig. 1), as motivated by a linear system in Ref. [79], which reads

$$H_{\text{EP3}} = \begin{pmatrix} \mathcal{E}_\alpha + i\Gamma_\alpha & \mu_{\alpha\beta}/2 & 0 \\ \mu_{\alpha\beta}/2 & \mathcal{E}_\beta + i\Gamma_\beta & \mu_{\beta\gamma}/2 \\ 0 & \mu_{\beta\gamma}/2 & \mathcal{E}_\gamma + i\Gamma_\gamma \end{pmatrix}. \quad (6)$$

$\mathcal{E} = E + g_c|\psi|^2$ denotes the mode energy with nonlinearity. The mode energies E_α and E_β and their coupling $\mu_{\alpha\beta}$ refer to the values used above, while E_γ is set to 1.75 meV and $\mu_{\beta\gamma} = 0.75\mu_{\alpha\beta}$, providing a clear picture of the resulting

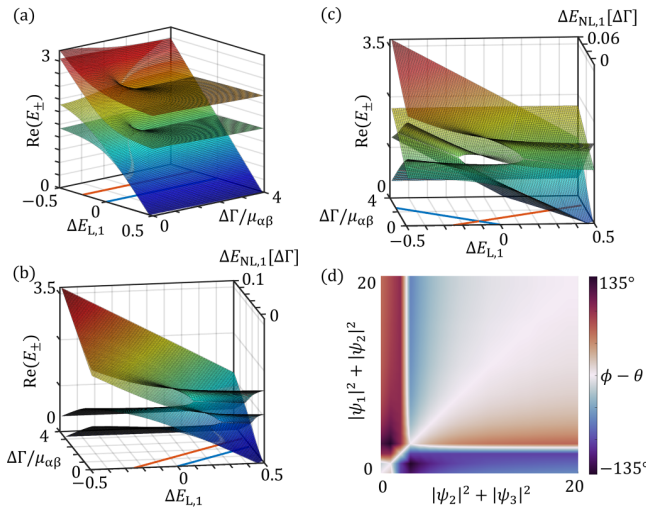


FIG. 4. Rotation of the Riemann surfaces with higher-order EPs. Riemann surfaces (real parts) containing two EPs (a) in the linear regime and (b) and (c) under the impact of the polariton-polariton interaction $g_c = 6 \mu\text{eV} \mu\text{m}^2$ and the saturable gain. In (b) the total densities of the modes forming the EPs are identical, whereas in (c) the total densities inside each EP are distinct. The eigenvalues are depicted against the eigenenergy difference E_L and the gain and loss difference $\Delta\Gamma$. For $g_c \neq 0$ the nonlinear correction to the energy difference is plotted with $\Delta\Gamma$. The lines at the bottom of the surface plot illustrate the orientation of the $\Delta\mathcal{E} = 0$ trace to visualize the rotation of the Riemann surfaces. (d) Deflection angle of the two projected lines as a function of the total densities of the adjacent modes for a nonlinearity of $g_c = 6 \mu\text{eV} \mu\text{m}^2$. Energies are given in meV.

Riemann surfaces. In Figs. 4(a)–4(c), the Riemann surfaces are shown for different nonlinear contributions. Here, the traces $\Delta\mathcal{E}_{1,2} = 0$ are projected onto the bottom of the surface plots to indicate the rotation of the Riemann surfaces. Remarkably, the Riemann surfaces can be rotated as a whole with the two projected lines being parallel [Figs. 4(a) and 4(b)]. Due to the saturable gain, however, the two Riemann surfaces can also be moved towards opposite directions with the two lines crossing each other [Fig. 4(c)], leading to complicated intersection behavior of the Riemann surfaces. Figure 4(d) shows the displacement angle difference of the two projected lines in Figs. 4(a)–4(c) as a function of the total densities of the two adjacent modes. It can be seen that if the saturable gain of the system is negligible or is identical for both EPs, the two lines are not deflected against each other, whereas they are deflected otherwise. It can also be inferred that if the energy blueshift is significant for one or two modes, it may be possible to observe coalescence of the two EPs, consequently leading to a phase transition in the spectral topology.

V. CONCLUSION

We have investigated the influence of nonlinearity on non-Hermitian spectral topology in microcavity polariton condensates. The nonlinearities lead to a significant change in the eigenvalue splitting near the EP as a function of density difference, leading to a clear nonlinearity-induced sensing

capability enhancement. The Kerr-type nonlinearity of the condensate (from polariton-polariton interactions) and the saturable gain (from polariton-reservoir interactions) lead to a rotation of the Riemann surfaces and shift the EP in parameter space. This can give rise to a complex intersection pattern of Riemann surfaces for higher-order EPs. In nonlinear mode control this offers interesting insights for encircling higher-order EPs and for phase transitions at EPs. These results are generic enough to be applied to other non-Hermitian systems with similar nonlinearities such as in nonlinear optics and atomic systems.

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APPENDIX

This Appendix is structured as follows. In Appendix A1, the EP shift along the $\Delta\Gamma$ or μ axis is presented, and the expression for the sensitivity change κ in the vicinity of the EP is derived and introduced. In Appendix A2, the encircling of the EP and the resulting vortex-antivortex switching process are discussed. In Appendix A3, the derivation of the relevant equations for Riemann surface rotation is given.

1. EP shift and sensitivity tuning

With saturable gain, the gain difference of the modes in a nonlinear system depends on their intensity. Since the coupling of the modes in the multistable system is also affected by the strength of the polariton-polariton interaction g_c , the necessary coupling strength μ to achieve an EP at $\mu = \Delta\Gamma$ in such a nonlinear system depends on the intensity of the modes or, in this case, on the two-dimensional polariton density. Hence, we observe a shift of the EP along the $\Delta\Gamma$ or μ axis, as shown in Fig. 1(a), as a function of the nonlinear interaction strength g_c , as shown in Fig. 5. The EP position in parameter space is shown on the x axis, and the polariton-polariton interaction-strength parameter g_c is plotted on the y axis. It can be seen that the gain and loss difference of the two modes not only increases or decreases but can also switch sign for increasing (Kerr-type) nonlinearity.

In the vicinity of an EP in a system subject to a Kerr-type nonlinearity the sensitivity change κ for a given ratio $\mu/\Delta\Gamma$ can be derived from Eq. (4) for $\Delta E_L \approx 0$ and reads

$$\kappa = \pm \frac{\sqrt{|\frac{\mu}{\Delta\Gamma}|^2 + (g_c \Delta|\psi|^2 + g_r \Delta n + i \Delta\Gamma)^2}}{2\mathcal{E}} \mp \frac{\sqrt{|\frac{\mu}{\Delta\Gamma'}|^2 - \Delta\Gamma'^2}}{2\mathcal{E}'}. \quad (\text{A1})$$

Here, the $\pm$ and $\mp$ signs describe the eigenvalue splitting in both directions. The first term describes the eigenvalue splitting under the influence of the nonlinear interactions g_c and

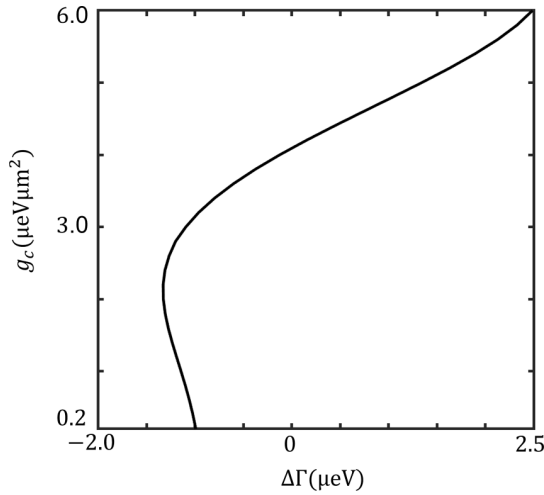


FIG. 5. Nonlinearity-induced shift of the required gain difference $\Delta\Gamma$ to access the EP. Inverted plot to illustrate the EP-position (x -axis) shifting in parameter space under variation of the nonlinearity parameter g_c .

g_r , while the second term describes the eigenvalue splitting in the linear case. The resulting EPs are superimposed by renormalizing the two scenarios with the respective mode energies $\mathcal{E}$ and $\mathcal{E}'$ at $\Delta\mathcal{E} = 0$ and their gain differences $\Delta\Gamma$ and $\Delta\Gamma'$. Since multistability of modes is required in the system under investigation, the eigenvalue bifurcation for a weak nonlinearity of $g_c = 0.2 \mu\text{eV} \mu\text{m}^2$ is chosen as a reference value for the sensitivity tuning in Fig. 2(a).

2. EP encircling

The encircling of an EP and the resulting state transitions, including the accumulation of geometric Berry phases, were originally discussed in Ref. [80]. Here, for completeness, we show that in the system studied in the present work, the encircling process can lead to a change in the topological charge of the vortex-antivortex pair upon reentry into the EP. To this end, the width of the rectangular potential is slightly reduced to $8.7 \mu\text{m}$ to lift the degeneracy of the two modes. This shifts the tripole mode [Fig. 1(d)] above the dipole mode [Fig. 1(c)] of the system. In addition to the excitation pump $P_1(x, y)$, a second elliptical control pump $P_2(\mathbf{r})$ is introduced. The total pump profile is then given as

$$P(x, y) = I_1 \exp\left(-\frac{x^2 + y^2}{\sigma^2}\right) + I_2 \exp\left(-\left|\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right|\right). \quad (\text{A2})$$

Here, $I_1 \in [14; 16] \text{ps}^{-1} \mu\text{m}^{-2}$ and $\sigma = 12 \mu\text{m}$ describe the intensity and width of the excitation pump. The shape of the elliptic pump is defined by its width $\sigma_x = 5 \mu\text{m}$ in the x direction and $\sigma_y = 2 \mu\text{m}$ in the y direction. $I_2 \in [0; 2] \text{ps}^{-1} \mu\text{m}^{-2}$ describes the intensity of the elliptic pump. Since the saturable gain affects the gain difference of the two modes, the excitation pump is used in the following to control these differences. In addition, the elliptic pump induces a spatially modulated blueshift which, due to its shape, mainly affects the dipole mode of the system. Thus, both the energy and gain difference of the modes can be controlled, which allows for an encircling of the EP that is induced all optically.

We note that in this case the encircling trajectories are not parallel to the ΔE and $\Delta\Gamma$ axes since both pumps cause an individual blueshift and gain shift to the two modes. Figure 6 shows the first and second encircling processes. It is emphasized that due to the nonlinearity of the system, the states exemplified at the corners of the encircling process are stationary before converging to the EP mode. During the first encircling, the dipole mode gathers a $-\pi$ Berry phase, while the tripole mode and hence the EP mode remain unchanged. Only after a second encircling does the tripole mode also collect a geometric Berry phase, leading to the switching of the topological charge of the vortex-antivortex pair.

3. Mathematical description of the Riemann surface rotation

To describe the rotation of the Riemann surface in the nonlinear regime as shown in Fig. 3, the expression for $\Delta|\psi|^2$ depending on $\Delta\Gamma$ can be derived by performing the Taylor expansion on the average mode gain and loss ratio $\bar{\Gamma} = \frac{\hbar}{2} \left[\frac{RP}{\gamma_r + R|\bar{\psi}|^2} - \gamma_c \right] |\bar{\psi}|^2$ at $|\bar{\psi}|^2 = 0$ given the approximation of constant average polariton density $|\bar{\psi}|^2$ and pump intensity $P = I_1$. The series expansion converges for $|\bar{\psi}|^2 < \gamma_r/R$. We note that this does not limit the strength of nonlinearity that can be studied within this approximation, as g_c can be tuned freely to model strong nonlinear effects. The Taylor expansion of the gain and loss ratio reads

$$\bar{\Gamma} = \frac{\hbar}{2} \left(\left[\frac{RP}{\gamma_r} - \gamma_c \right] |\bar{\psi}|^2 - \frac{PR^2}{\gamma_r^2} |\bar{\psi}|^4 + \frac{PR^3}{\gamma_r^3} |\bar{\psi}|^6 \mp \dots \right). \quad (\text{A3})$$

From this the mode density difference of the two modes, α and β , as a function of their gain and loss difference can be derived as

$$\begin{aligned} \Delta\Gamma &= \frac{\hbar}{2} \left(\left[\frac{RP}{\gamma_r} - \gamma_c \right] (|\bar{\psi}_\alpha|^2 - |\bar{\psi}_\beta|^2) - \frac{PR^2}{\gamma_r^2} (|\bar{\psi}_\alpha|^4 - |\bar{\psi}_\beta|^4) + \frac{PR^3}{\gamma_r^3} (|\bar{\psi}_\alpha|^6 - |\bar{\psi}_\beta|^6) \mp \dots \right) \\ \Leftrightarrow \Delta\Gamma &= \frac{\hbar}{2} \left(\left[\frac{RP}{\gamma_r} - \gamma_c \right] \Delta|\psi|^2 - \frac{PR^2}{\gamma_r^2} (|\bar{\psi}_\alpha|^2 + |\bar{\psi}_\beta|^2) \Delta|\psi|^2 + \frac{PR^3}{\gamma_r^3} (|\bar{\psi}_\alpha|^4 + |\bar{\psi}_\alpha|^2 |\bar{\psi}_\beta|^2 + |\bar{\psi}_\beta|^4) \Delta|\psi|^2 \mp \dots \right) \\ \Leftrightarrow \Delta|\psi|^2 [\Delta\Gamma] &= \frac{2\Delta\Gamma}{\left[\frac{\hbar RP}{\gamma_r} - \hbar\gamma_c \right] - \frac{\hbar PR^2}{\gamma_r^2} (|\bar{\psi}_\alpha|^2 + |\bar{\psi}_\beta|^2) + \frac{\hbar PR^3}{\gamma_r^3} (|\bar{\psi}_\alpha|^4 + |\bar{\psi}_\alpha|^2 |\bar{\psi}_\beta|^2 + |\bar{\psi}_\beta|^4) \mp \dots} \end{aligned}$$

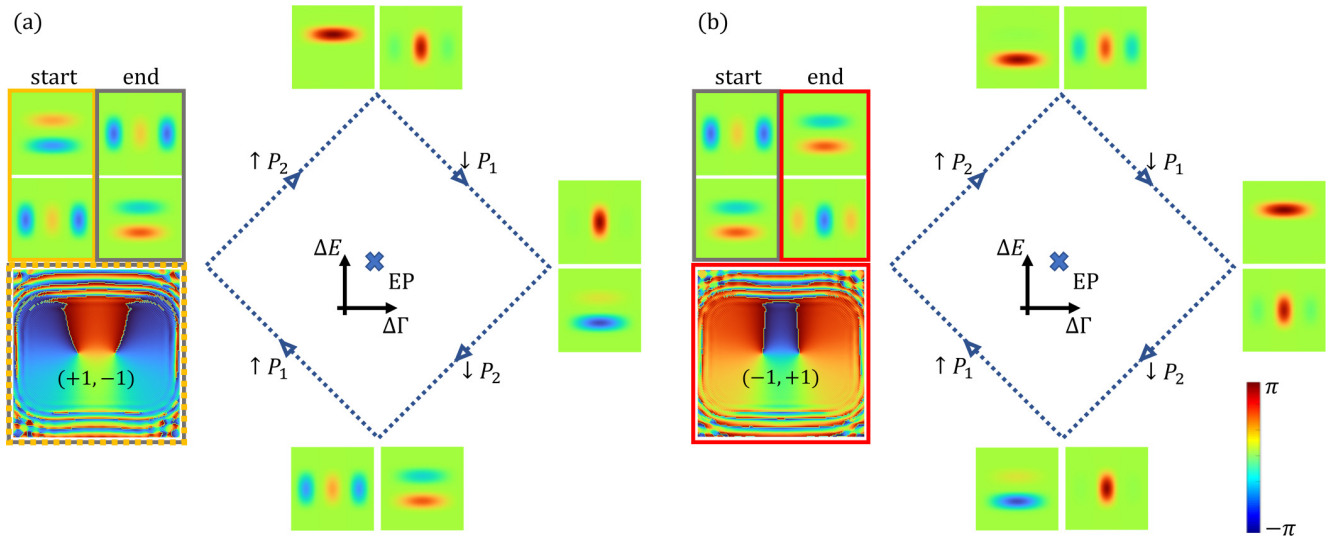


FIG. 6. All-optical vortex-antivortex switching under EP encircling. Illustration of the (a) first and (b) second EP encirclings. The contours of the stationary states are displayed at the corners of the sketched trajectory. The initial and final states are highlighted, and the phase information of the vortex-antivortex pair at the EP is displayed at the bottom. After the first encircling the topological charge of the vortex pair remains unchanged (highlighted by gray and yellow boxes). After the second encircling the topological charge of the vortex pair is switched (highlighted by a red box).

$$\Rightarrow \Delta|\psi|^2[\Delta\Gamma] \approx \frac{2\Delta\Gamma}{\left[\frac{\hbar RP}{\gamma_r} - \hbar\gamma_c\right] - \frac{\hbar PR^2}{\gamma_r^2}(|\overline{\psi}_\alpha|^2 + |\overline{\psi}_\beta|^2) + \frac{\hbar PR^3}{\gamma_r^3}(|\overline{\psi}_\alpha|^2 + |\overline{\psi}_\beta|^2)^2 \mp \dots}$$

$$\Rightarrow \Delta|\psi|^2[\Delta\Gamma] \approx \frac{2\Delta\Gamma}{\left[\frac{\hbar PR}{\gamma_r} - \hbar\gamma_c\right] + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{\hbar PR^m}{\gamma_r^m} (|\psi_{\text{tot}}|^2)^{m-1}}.$$

In the second step, the terms of the form $(|\overline{\psi}_\alpha|^2)^m - (|\overline{\psi}_\beta|^2)^m$ are factorized with respect to the density difference $\Delta|\psi|^2$, according to

$$x^m - y^m = (x - y) \sum_{l=0}^{m-1} x^l y^{m-1-l}. \quad (\text{A4})$$

In the last step, the factors of the m th-order terms are approximated by the total density $\psi_{\text{tot}} = |\psi_\alpha|^2 + |\psi_\beta|^2$ to the $(m-1)$ th power. By this approximation, the denominator is always larger than its exact value for the parameter range bounded by the Taylor expansion. Accordingly, this approximation describes a lower bound for the mode density difference under the independent variation of $\Delta\Gamma$. We emphasize that this approximation is exact for the first two terms in the denominator and the strengths of the higher-order terms decrease by one order of magnitude for each order m . In this sense this approximation does not significantly affect our results. We note that the denominator vanishes when the saturable gain compensates the linear gain. For larger densities the direction of the rotation is inverted. Keeping in mind the approximations discussed above, we collect our results away from this divergence.

To determine the rotation and blueshift of the Riemann surface [see Eq. (4)] induced by the excitation reservoir n in a polariton system, we derive the expression for $\Delta n[\Delta|\psi|^2]$ and $n_{\text{tot}}[|\psi_{\text{tot}}|^2]$ from the Taylor expansion of $\overline{\Gamma}$ in Eq. (A3) and the relation $\overline{\Gamma} = \frac{\hbar}{2}[Rn - \gamma_c]|\overline{\psi}|^2$. The blueshift $g_r n_{\text{tot}}$ of the Riemann surface is then derived as

$$g_r n_{\text{tot}} = g_r \left(\left[\frac{2P}{\gamma_r} - \frac{2\gamma_c}{R} \right] + \sum_{m=1}^{\infty} (-1)^m \frac{PR^m}{\gamma_r^{m+1}} (|\psi_{\text{tot}}|^2)^m \right). \quad (\text{A5})$$

The rotation induced by the excitation reservoir is given by

$$\Delta n = \sum_{m=1}^{\infty} (-1)^m \frac{PR^m}{\gamma_r^{m+1}} (|\psi_{\text{tot}}|^2)^{m-1} \Delta|\psi|^2. \quad (\text{A6})$$

From the given expression and the results shown in Figs. 3(e) and 3(f) it can be seen that the rotation induced by the reservoir not only can increase the rotation angle of the Riemann surface but can also switch the rotation direction, depending on the density and reservoir differences.

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Part III

From Solver Development to Skyrmion Patterns

In the preceding part of this thesis, nonlinear exceptional points were studied in polariton systems. Several of the presented results required solving the extended Gross-Pitaevskii equations (5.4) and (5.5). Conventionally, these equations are evolved on a two-dimensional discretized real-space grid using Runge-Kutta or split-step Fourier methods [179]. Higher-order adaptive Runge-Kutta methods are useful in scenarios that demand time-step variation, for instance, sweeping of the polariton mass or the TE-TM splitting strength [149]. For many applications, however, the classical fourth-order Runge-Kutta method provides an efficient compromise between accuracy, stability, and computational cost. Nevertheless, large spatial domains, fine grid spacing, or small time steps can lead to substantial computation times.

In this chapter, PHOENIX, the *Paderborn highly optimized and energy efficient solver for two-dimensional nonlinear Schrödinger equations with integrated extensions*, is introduced. With a focus on GPU acceleration, solvers for numerous variants of the Schrödinger equation have been published in the past [180–187]. However, the impact of single-precision calculations has only been investigated to a limited extent, and the speedup-induced energy savings have rarely been quantified systematically. PHOENIX addresses these aspects by providing a solver that is optimized for both CPUs and GPUs by exploiting their hardware-specific architectures. Due to its memory-efficient implementation, PHOENIX achieves speedups of three orders of magnitude and energy savings of up to 99.8% on high-end consumer hardware, compared to conventional MATLAB implementations [179].

PHOENIX covers a broad class of pre-implemented extensions, including the terms required for equations (5.4) and (5.5), and provides several solver schemes beyond the standard Runge-Kutta methods. PHOENIX also permits the investigation of very large numerical domains that were unfeasible to treat with conventional implementations. Here, a key feature is the memory-efficient subgrid decomposition, which partitions the spatial domain into smaller regions that are iterated sequentially and thereby improves cache efficiency. PHOENIX can be easily incorporated into preexisting code and comes with a flexible framework to construct matrix structures, such as spatial and temporal pump envelopes, pulses, potentials, and initial conditions. In addition, PHOENIX features integrated post-processing routines that allow selected output quantities to be monitored and stored during

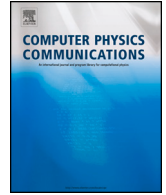
the time evolution. Benchmark results for representative hardware ranging from consumer GPUs to high-performance computing systems are provided in the manuscript.

Since the first release of PHOENIX, several features have been further optimized. Now, output during the time evolution can be performed asynchronously, and the results can be saved in compact format [188]. These improvements are especially beneficial for output-heavy calculations. Among others, the output of these calculations can be used for Fourier analysis that allows for determining mode frequencies, or to visualize the field evolution in video files.

The PHOENIX repository [188] provides a complete list of applications that have been published to date, together with ready-to-use code files. In Chapter 6, PHOENIX was used to calculate the linear response of the linear and non-Hermitian system and to sweep the resonance behavior of the nonlinear and non-Hermitian system shown in Figure 6.2(d-g). To this end, PHOENIX was incorporated in a MATLAB environment and executed on the system GPU. Parallel `parfor` loops enabled the concurrent execution of multiple PHOENIX simulations, limited by the number of available CPU cores and GPU resources. PHOENIX was also used in Chapter 7 to validate the toy model results with full model calculations.

In the following chapter, a recent PHOENIX application is presented to demonstrate a selection of key features. With fine spatial discretizations, large grid sizes, and precise frequency-resolved analysis, we demonstrate the spontaneous formation of Skyrmion patterns in the scalar fields of non-Hermitian optical microcavities.

The results presented in this chapter were previously published and are reproduced below. Reprinted from J. Wingenbach et al., PHOENIX – Paderborn highly optimized and energy efficient solver for two-dimensional nonlinear Schrödinger equations with integrated extensions, *Computer Physics Communications* **315**, 109689 (2025), <https://doi.org/10.1016/j.cpc.2025.109689>, Copyright © 2025 The Author(s). Published by Elsevier B.V. under the terms of the Creative Commons Attribution 4.0 International licence (CC BY 4.0). To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.



Computer Programs in Physics

PHOENIX – Paderborn highly optimized and energy efficient solver for two-dimensional nonlinear Schrödinger equations with integrated extensions

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ABSTRACT

In this work, we introduce PHOENIX, a highly optimized explicit open-source solver for two-dimensional nonlinear Schrödinger equations with extensions. The nonlinear Schrödinger equation and its extensions (Gross-Pitaevskii equation) are widely studied to model and analyze complex phenomena in fields such as optics, condensed matter physics, fluid dynamics, and plasma physics. It serves as a powerful tool for understanding nonlinear wave dynamics, soliton formation, and the interplay between nonlinearity, dispersion, and diffraction. By extending the nonlinear Schrödinger equation, various physical effects such as non-Hermiticity, spin-orbit interaction, and quantum optical aspects can be incorporated. PHOENIX is designed to accommodate a wide range of applications by a straightforward extensibility without the need for user knowledge of computing architectures or performance optimization. The high performance and power efficiency of PHOENIX are demonstrated on a wide range of entry-class to high-end consumer and high-performance computing GPUs and CPUs. Compared to a more conventional MATLAB implementation, a speedup of up to three orders of magnitude and energy savings of up to 99.8% are achieved. The performance is compared to a performance model showing that PHOENIX performs close to the relevant performance bounds in many situations. The possibilities of PHOENIX are demonstrated with a range of practical examples from the realm of nonlinear (quantum) photonics in planar microresonators with active media including exciton-polariton condensates. Examples range from solutions on very large grids, the use of local optimization algorithms, to Monte Carlo ensemble evolutions with quantum noise enabling the tomography of the system's quantum state.

Program summary

Program Title: PHOENIX

CPC Library link to program files: <https://doi.org/10.17632/kthy8wj3n5.1>

Developer's repository link: <https://github.com/Schumacher-Group-UPB/PHOENIX/>

Licensing provision: MIT

Programming language: C++, CUDA

Nature of problem: Time evolution of two-dimensional nonlinear systems such as Bose-Einstein condensates, nonlinear optical systems or hybrid light-matter systems (e.g., exciton-polariton condensates).

Solution method: Solving the extended two-dimensional nonlinear Schrödinger equation (Gross-Pitaevskii equation) on uniformly discretized grids in real-space with a wide range of Runge-Kutta schemes. The use of CPU and GPU and the precision used (fp32/fp64) can be set when compiling the code. Sub-grid decomposition is possible for optimal cache efficiency. The solver provides a framework that allows users unfamiliar with the details of GPU and CPU parallelization to extend the set of equations with additional terms.

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1. Introduction

The nonlinear Schrödinger equation describes the evolution of complex scalar fields and has applications in many areas in biology [1], engineering [2], mathematics [3], and physics [4]. In physics, it is commonly used to model the behavior of wave packets in nonlinear media, Bose-Einstein condensates [5], superfluids [6], and to investigate phenomena like solitons [7], and rogue waves [8]. Furthermore, loss terms can be included to model dissipative systems in the field of nonlinear optics [9] or hybrid light-matter systems such as exciton-polariton condensates [10,11]. In the latter case, the general dissipative nonlinear Schrödinger equation (dNLSE) reads

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\nabla_{\perp}^2 + V + g|\psi|^2 - \frac{i\hbar\gamma}{2}\right)\psi. \quad (1)$$

Here $\psi = \psi(\mathbf{r}, t)$ is the complex valued scalar field in 2D in the $x - y$ plane with $|\mathbf{r}| = \sqrt{x^2 + y^2}$ and $\nabla_{\perp}^2 = \partial_x^2 + \partial_y^2$. m denotes the effective particle mass. The field may be subjected to an external potential $V = V(\mathbf{r})$, and g determines the strength of a Kerr-type nonlinearity. The loss γ specifies the decay of the field. Note that Eq. (1) is generally used for the study of the time evolution of the field ψ in specific 2D systems. However, it could also be switched to 3D spatial systems, such as in nonlinear optics, to study the propagation of the field ψ along the perpendicular z direction by simply replacing the left-hand term by $i\hbar\partial_z\psi$. Since solving the nonlinear Schrödinger equation in multidimensional systems is very time-consuming, especially when additional terms are taken into account, significant attention has been paid to optimizing this process. Sophisticated analytical approaches, such as the construction of integrable models using Lax pairs [12], permit to determine exact solutions of a range of nonlinear integrable models including multi-component nonlinear Schrödinger equations [13,14]. For instance, soliton solutions have been explored by using Darboux transformations [15]. In the recent past, solving differential equations such as variants of the Schrödinger equation on graphics processing units (GPUs) has become increasingly popular.

GPUs are devices composed of a co-processor and tightly coupled high-speed memory that have originally been designed for image processing. While the central processing unit (CPU) is optimized for data caching and flow control, GPUs are designed for highly parallel processes executed on a large number of cores. To achieve that, a large number of transistors on the GPU is devoted to data processing [16]. CUDA is a parallel computing platform introduced by NVIDIA Corporation and allows GPUs to execute programs written in commonly used programming languages including C, C++, and packages like MATLAB. CUDA allows users to implement highly parallelized solvers to be used on the GPU, solving numerical problems such as large systems of coupled differential equations [17]. Thanks to modern GPU architecture, matrix operations can be performed in a fraction of the time that would be required on a CPU. In addition, the higher memory bandwidth of GPUs enables the rapid processing of large amounts of data, as it often occurs in multidimensional systems. In recent years, the use of GPUs to solve variants of the time-dependent Schrödinger equation has attracted more and more attention. This includes solving the 1D time-dependent Schrödinger equation [18], the 2D nonlinear Schrödinger equation with a spin degree of freedom [19], the 2D and 3D Gross-Pitaevskii equations on single and multiple GPUs [20–23], and rotating Gross-Pitaevskii equation [24].

In these works, speedups of one [25,20] to two [25,26] orders of magnitude were reported. However, the works in this area restrict their equations to a few additional terms, which limits the flexibility of the code. To our knowledge, the impact of single-precision calculations on the speedup factor for the NLSE was only investigated in Ref. [23] to some extent. The energy efficiency of a GPU-assisted solution is rarely investigated. In the present work both CPU and GPU code are optimized to a high level to take advantage of the hardware specifics of the dif-

ferent architectures and to allow for a fair and transparent comparison. This opens up choices for users based on available hardware.

In the present work, we introduce the open-source code “Paderborn highly optimized and energy efficient solver for two-dimensional nonlinear Schrödinger equations with integrated extensions” (PHOENIX, [27]). This tool allows GPU-assisted solution of an extended dNLSE and is implemented in C++. We note that PHOENIX can also be executed via MATLAB or Python, which allows the user to process the results by familiar methods, example codes are provided [27]. It is therefore feasible to embed PHOENIX into pre-existing code where it replaces the execution of the conventional Runge-Kutta method. PHOENIX is tailored for fast and efficient single-node CPU and single-GPU simulations because for the system sizes of interest here distributed memory is not required. However, the future extension of PHOENIX to a multi-node/multi-GPU solver is straightforward by using a space decomposition approach. We provide a wide range of benchmarks by comparing the iteration time using the entire CPU and a single GPU. This gives a measure of resource efficiency, as well as of the speedup factor for realistic simulations. We benchmark the performance of PHOENIX based on Eq. (1) on a variety of CPUs and GPUs from entry-level consumer to high-performance computing hardware. To illustrate the potential benefits of switching to PHOENIX for users of programming tools like MATLAB, we also compare PHOENIX to a straightforward MATLAB implementation of the fourth-order Runge Kutta method [details about the implementation are given in Sec. 3.1]. The resulting time and energy savings are shown in Fig. 1 for a large range of sizes for the two-dimensional spatial grid. We run the PHOENIX code both in double and single floating-point precision. In Fig. 1 we note a speedup of up to three orders of magnitude and substantial energy savings of up to 99.8%. Moreover, the comparison with the memory and cache bandwidth limits show that PHOENIX executed on the RTX 4090 is cache bandwidth bound, highlighting the cache-efficient implementation of PHOENIX. It should be noted that PHOENIX on the GPU still shows a significant speedup of one order of magnitude compared to the featured Matlab code when executed on the GPU. In addition to the rigorous performance benchmarks in the first part of the present paper, in the second part we provide explicit applications of various extensions to the implemented dNLSE as detailed below in Eqs. (16) and (17) to demonstrate the advantages and performance of PHOENIX in practical examples.

The remainder of the present work is structured as follows: in Sec. 2 the fourth-order Runge-Kutta integration method is introduced to solve the differential Eq. (1) on a discretized real-space grid. In Sec. 3.1 the more conventional numerical implementation in MATLAB is described. In Sec. 3.2 the numerical implementation of PHOENIX is covered. Sec. 4 contains detailed performance and efficiency results on a variety of hardware components. Sec. 5 gives an overview of possible applications of the extended equations covered by PHOENIX; it also illustrates the wide range of possibilities and exceptional performance of our code with overall only moderate hardware requirements. The range of applications discussed in the present work are from the area of (quantum) photonics in planar microresonators with active media including exciton-polariton condensates. For the more use-case oriented reader, in a first read we would recommend to focus on the introductory parts of the present paper and Sec. 5. Detailed understanding of the more technical sections in between are not required for efficient use or even extensions to our code. All codes for the described applications and detailed instructions on how to use PHOENIX, including calls of the core numerical routine and pre- and post-processing of data with easy to use and customize MATLAB and Python scripts, respectively, can also be found on our Github repository [27].

2. Time evolution schemes

The time evolution of the wavefunction $\psi(\mathbf{r}, t)$ is governed by Eq. (1) and following discretization on a uniform real-space grid is solved by explicit integration schemes like the Runge-Kutta integration method.

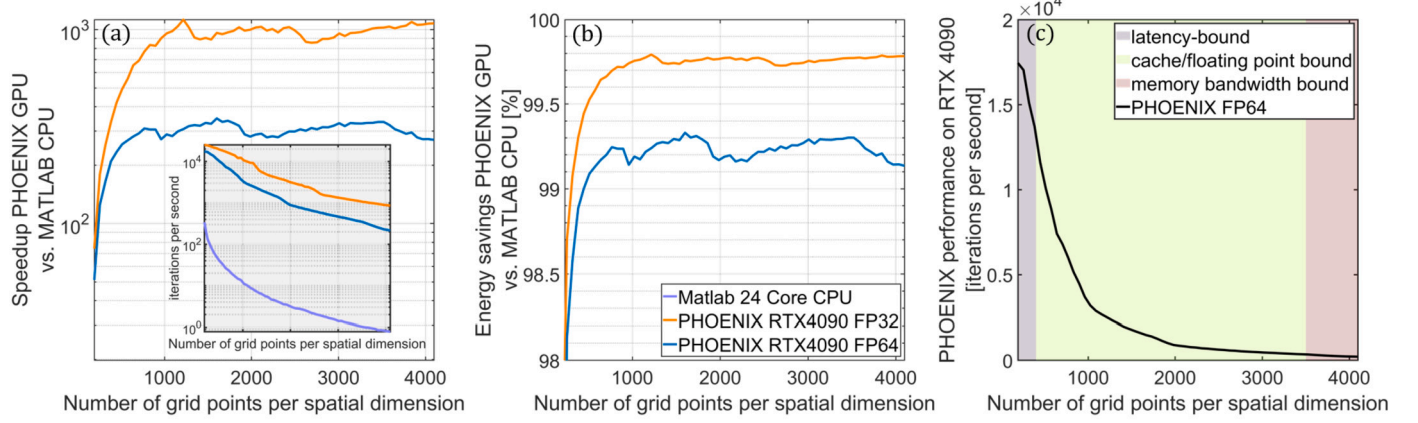


Fig. 1. Performance comparison on high-end consumer hardware. Results illustrate (a) speedup and iterations per second and (b) energy savings of PHOENIX compared to a conventional Runge-Kutta CPU implementation in MATLAB for various grid sizes of a square grid, using an AMD EPYC 7443P 24core CPU and an NVIDIA RTX 4090 GPU. Simulations employ MATLAB R2024a and PHOENIX compiled with GCC and CUDA. Grid sizes ranged from 192×192 to 4096×4096 , demonstrating PHOENIX's efficiency in evaluating a wide range of problems, achieving microsecond iteration times even for large grids. Performance optimization involved dividing grids into subgrids of up to 500×500 cells to minimize overhead. Both single- and double-precision calculations were performed. Energy savings are determined by the speedup and the peak power consumption for large grids manually measured using HWINFO. (c) PHOENIX performance on RTX 4090 in double precision float and computational bounds for achievable performance according to the Roofline model introduced in Sec. 3.2.8. On the RTX 4090, PHOENIX is floating point (cache bandwidth) bound in fp64 (fp32, not shown) for intermediate grid sizes.

We outline the process with the example of the fourth-order Runge-Kutta scheme (RK4): Let $f(t) = -\frac{i}{\hbar} \partial_t \psi(\mathbf{r}, t)$ represent the time derivative of the wavefunction for a single grid cell, and $\psi_i = \psi(\mathbf{r}, t)$ denote the current wavefunction value at that grid cell. The RK4 method advances the system by the time step Δt by solving the following equations:

$$K_1 = f(\psi_i) \quad (2)$$

$$K_2 = f(\psi_i + 0.5\Delta t K_1) \quad (3)$$

$$K_3 = f(\psi_i + 0.5\Delta t K_2) \quad (4)$$

$$K_4 = f(\psi_i + \Delta t K_3) \quad (5)$$

$$\psi_{i+\Delta t} = \psi_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)\Delta t. \quad (6)$$

While the standard RK4 algorithm is computationally efficient, higher-order methods like the RK-Fehlberg or Dormand-Prince can be used for systems requiring adaptive time-stepping.

We discretize the two-dimensional system into an $N_x \times N_y$ grid, where each cell corresponds to a point in the spatial domain. The time evolution of the system is computed iteratively by solving the extended dissipative nonlinear Schrödinger equation in each cell.

3. Numerical implementation

3.1. MATLAB implementation

As a comparison case of how this type of simulation is conventionally handled numerically, we start by describing a typical implementation in MATLAB. A well-established method to solve partial differential equations like Eq. (1) is the discretization of the spatial domain and approximation of the Laplacian operator via the central difference scheme. This transforms the problem into a set of coupled ordinary differential equations that can be solved explicitly in the time domain by the Runge-Kutta integration scheme. In the MATLAB implementation, we construct the Laplacian operator as a large tridiagonal sparse matrix holding nearest neighbor couplings between grid points. The integration of the differential equation then requires a series of matrix-vector multiplications. To increase computational efficiency, MATLAB's GPU computational toolbox can be used to move variables to the GPU memory.

3.2. PHOENIX

3.2.1. Design principles and usage

The PHOENIX solver presented in this work is implemented in C++ and relies on OpenMP for CPU-based parallelization and CUDA for GPU-based acceleration. Both CPU and GPU implementations share the same core code, with architecture-specific optimizations accounting for differences in memory hierarchy and execution models. PHOENIX can be easily connected to pre- and post-processing with other tools, e.g., the generation of pulse shapes or potential inputs with Python or plotting programs. Examples are included with the code [27].

A specific simulation is configured in PHOENIX by different means: The usage of CPU or GPU and the used precision (fp32/fp64) are set when compiling PHOENIX. Grid sizes, boundary conditions, prefactors of terms in the Hamiltonian, integrator settings, etc. are set via command line parameters. Input states, pulse shapes, and potentials can either be read from input files or generated in PHOENIX at the start of the simulation.

A primary design goal of the PHOENIX solver is to provide a framework where additional time evolution approaches or terms in the dNLSE Hamiltonian can be implemented by users unfamiliar with details of GPU and CPU parallelization approaches, while still using the hardware to its full extent. Examples in Sec. 3.2.2 demonstrate the simplicity of extensions to our implementation.

In PHOENIX, the time evolution of the dNLSE is decomposed into elementary kernels so that each kernel can process all cells in parallel either with OpenMP for CPU-based parallelization or with CUDA for GPU-based acceleration. The structure of the kernel steps in Eq. (2)–(6) is schematically shown in Fig. 2 for the example of the RK4 time step.

3.2.2. Code extensions

PHOENIX is designed for straightforward code extensions. Code Listing 1 presents the implementation of the RK4 time-stepping algorithm in PHOENIX, corresponding to Eq. (2)–(6). Due to its expressive and compact formulation, other integrators can be implemented without requiring background knowledge of GPU/CPU parallelization or prior experience with C++ programming.

In addition to alternative integrators, simulations may necessitate the inclusion of different terms in the Hamiltonian defined by Eq. (1). Such terms can be seamlessly added to the kernel, as illustrated in Code Listing 2.

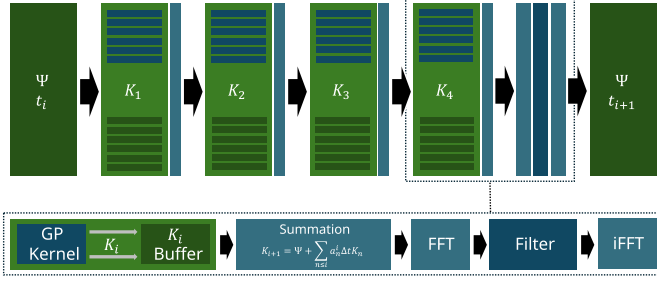


Fig. 2. Schematic of the RK4 steps. Dark green objects mark real or complex buffers while blue objects mark the execution of a kernel. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The extensibility of time-stepping algorithms and Hamiltonian terms in PHOENIX is as straightforward as in conventional MATLAB code. In the remainder of this paper, we focus on the well-established RK4, as it offers a good balance between stability and computation speed. However, it is worth noting that other integrators can offer larger stability regions and support adaptive time-stepping (see e.g. Ref. [28,29]). Hence, we have implemented a wide range of alternative time-stepping algorithms for the user to choose from, including but not limited to Dormand-Prince and Fehlberg methods, covering both fixed and adaptive time steps (for full details see [27]). Furthermore, the standardized implementation [see Listing 1] provides a perfect platform for users to add their own integration procedures if required.

```

1 // Calculate K1 from input "wavefunction"
2 CALCULATE_K( 1, wavefunction );
3 // Calculate Psi + dt/2*K1
4 // Saved into "buffer_wavefunction"
5 INTERMEDIATE_SUM_K( 1, 0.5f );
6 // Calculate K2 from input "buffer_wavefunction"
7 CALCULATE_K( 2, buffer_wavefunction );
8 // Calculate Psi + dt/2*K2
9 // Saved into "buffer_wavefunction"
10 INTERMEDIATE_SUM_K( 2, 0.5f );
11 // Calculate K3 from input "buffer_wavefunction"
12 CALCULATE_K( 3, buffer_wavefunction );
13 // Calculate Psi + dt*K3
14 // Saved into "buffer_wavefunction"
15 INTERMEDIATE_SUM_K( 3, 1.0f );
16 // Calculate K4 from input "buffer_wavefunction"
17 CALCULATE_K( 4, buffer_wavefunction );
18 // Sum all K's into new Psi
19 FINAL_SUM_K( 4, 1.0f/6.0f, 1.0f/3.0f,
20               1.0f/3.0f, 1.0f/6.0f );

```

Listing 1: Sample implementation of the RK4 time step in PHOENIX. This highlights how easily modifications can be made to the existing code.

```

1 const Type::real in_psi_norm = CUDA::abs2( in_wf ); //|Psi|^2
2 const Type::complex in_wf_mi = CUDA::mi_conjugate( in_wf );
3 // -i*in_wf
3 wf_plus += one_over_h_bar_s * g_c * in_psi_norm * in_wf_mi;
4 wf_plus -= Type::real( 0.5 ) * gamma_c * in_wf;

```

Listing 2: Implementation of the terms $g|\Psi|^2$ and $-i\hbar\gamma/2$ in PHOENIX.

3.2.3. Smoothing

Finally, a Fast Fourier Transform (FFT) can be applied to ψ to filter the resulting wavefunction in k -space, followed by an inverse FFT to return the data to real space. We use the library cuFFT [17] in GPU and FFTW [30] in CPU calculations. For systems with spin degrees of freedom, equivalent steps are applied for the two spinor components $\psi_{\pm}$. This filtering may be necessary to suppress numerical artifacts forming due to various reasons, including certain shapes of external pumping sources and boundary effects in k -space.

3.2.4. Subgrid decomposition

The computations required for the time steps, i.e., the evaluation of the derivative $f(t) = -\frac{i}{\hbar}\partial_t\psi(\mathbf{r},t)$ and the scaling/additions in the five steps of the RK4 schema in Eq. (2)-(6) have a low computational intensity like most other numerical stencil problems. A low computational intensity means that for the required floating-point operations a comparatively large amount of data needs to be transferred from the caches or memory. Thus, typically the bottleneck of the calculation is not the floating-point throughput but the useable bandwidth of caches or the bandwidth of the memory interface. The bandwidth of the memory interface of GPUs or CPUs is often up to an order of magnitude lower than the usable cache bandwidths. Hence, optimizing a code to use the caches more efficiently can deliver a drastic speed-up in situations with low computational intensity like the time evolution of the dNLSE in this work.

For this purpose, we have implemented a subgrid decomposition where the primary $N_x \times N_y$ grid is decomposed into smaller subgrids of size $N_{x,sg} \times N_{y,sg}$. Each subgrid includes a *halo* of grid cells corresponding to neighboring subgrids so that the calculations for the time step, i.e., Eq. (2)-(6), is independent for each subgrid. Thus, during a time step on a subgrid instead of the full grid of $N_x \times N_y$ points only the points of a subgrid, i.e., $N_{x,sg} \times N_{y,sg}$ need to be processed in the individual steps making the caches much more efficient due to the reduced data size. For a more readable notation, we assume that the grid dimensions are multiples of the subgrid dimensions for each spatial dimension, i.e., $N_x/N_{x,sg} = n_{x,sg} \in \mathbb{N}$ and $N_y/N_{y,sg} = n_{y,sg} \in \mathbb{N}$. The solver can also handle situations where this is not the case.

It should be noted that the halo region around the subgrids introduces a slight overhead in terms of the number of computed cells in a time step because instead of $N_{x,sg} \times N_{y,sg}$ cells per subgrid now $(N_{x,sg} + 2N_{halo}) \times (N_{y,sg} + 2N_{halo})$ cells per subgrid need to be computed.

After a time step has been applied to each subgrid individually, a halo exchange step is introduced in which boundary values are synchronized by copying updated boundary values between neighboring subgrids. This technique also enables efficient parallel execution on CPUs using OpenMP, with each subgrid being assigned to a separate CPU core such that the number of subgrids n_{sg} is a multiple of the number of CPU cores.

3.2.5. Lower precision floating-point formats

While the use of double-precision floating point numbers (fp64) was prevalent in computational simulations in the past, large improvements in throughput and computational efficiency can be achieved in situations where lower precision calculations are possible or the lower precision can be compensated by changes in the algorithms [31–35]. A straightforward reduction from double precision to single precision (fp32) in the simulation of the dNLSE with RK schemas is accurate except (for our examples) in cases where sharp spectral resonances appear and need to be matched to a high precision [see Sec. 5.1]. Thus, the solver presented in this paper can use either single or double precision for the calculation of the time evolution steps, Eqs. (2)-(6). The step from double precision to single precision already gives a number of advantages: On the one hand, all data objects are reduced in size by a factor of two so that transfer times from caches or memory are reduced. On the other hand, consumer GPUs typically only have a low double-precision performance because single-precision calculations are sufficient for graphical applications. Thus, using single-precision instead of double-precision formats ensures that the floating point throughput doesn't bottleneck the solver on consumer-grade GPUs.

3.2.6. Data volume

The data required for example in the GPU memory consists of buffers of complex single- or double-precision floating-point numbers with $n_{x,sg} \times n_{y,sg} \times (N_{x,sg} + 2N_{halo}) \times (N_{y,sg} + 2N_{halo})$ elements, where $n_{x,sg}$ and $n_{y,sg}$ are the numbers of subgrid dimensions in x and y dimension, respectively. For scalar systems with only the wavefunction ψ six buffers suffice. However, more complex systems, such as those with a

reservoir (see example in Section 5.2), require an additional six buffers, and systems including spin-orbit interaction (see example in Section 5.3) require 24 buffers in total.

Thus, the required CPU main memory or GPU memory for a scalar simulation without a reservoir using RK4 ($N_{\text{halo}} = 4$) in single precision can be determined as

$$M = 48n_{x,\text{sg}} \times n_{y,\text{sg}} \times (N_{x,\text{sg}} + 2N_{\text{halo}}) \times (N_{y,\text{sg}} + 2N_{\text{halo}}) \text{ byte} \quad (7)$$

Thus except for extremely large simulations with grid sizes beyond $\approx 10^4$, the memory requirement doesn't limit the choice of, for example, consumer GPUs.

3.2.7. Vectorization and parallelization

GPU parallelization consists of individual kernels that loop over subgrids of cells. The kernel launch latencies are mitigated by using CUDA graphs. In the GPU implementation, no special consideration of the storage format of complex numbers is required. So whether the complex numbers are stored as interleaved real and imaginary parts or separate arrays for the real and imaginary parts doesn't need additional attention. Thus in practice, the interleaved format is preferred due to the better readability of the program code. In contrast, for CPUs the interleaved format leads to challenges with vectorization if complex conjugation or multiplications are used. In such situations, the available compilers can often not sufficiently vectorize the code which can lead to unvectorized floating-point throughput becoming a bottleneck. For PHOENIX this problem is circumvented with temporary buffers for example for $-i\psi_i$ which by itself is computed in an efficiently vectorized way for example with AVX2 compiler intrinsics.

3.2.8. Performance modeling

The time step in Eq. (2)-(6) requires three types of kernels on the grid. If multiple subgrids are used, also the halo exchange kernel is required. We limit the performance modeling to the scalar case without a reservoir and no smoothing.

The derivative kernel, i.e., $f(t) = -\frac{1}{\hbar}\nabla_i\psi(\vec{r},t)$ used in (2)-(3) requires the evaluation of the right side of Eq. (1) at the grid points $\mathbf{r}_{i,j} = (i\Delta x, j\Delta y)$ for $\psi(i,j) = \psi(\mathbf{r}_{i,j},t)$ and most importantly a five-point stencil

$$\begin{aligned} \nabla^2\psi(i,j) &= \frac{1}{\Delta x^2}[\psi(i+1,j) + \psi(i-1,j) - 2\psi(i,j)] \\ &+ \frac{1}{\Delta y^2}[\psi(i,j+1) + \psi(i,j-1) - 2\psi(i,j)] \end{aligned} \quad (8)$$

for the spatial derivative. This stencil will effectively require three reads of the wavefunction ψ because $\psi(i+1,j)$, $\psi(i-1,j)$, and $\psi(i,j)$ are contained in one cache line or adjacent cache lines whereas $\psi(i,j+1)$ and $\psi(i,j-1)$ require additional reads of cache lines.

The two-term summation kernels used in Eq. (3)-(5) with the structure

$$z = x + \alpha y, \quad (9)$$

are identical to the well-known STREAM triad [36,37] case. The five-term summation kernel in Eq. (6) with the structure

$$z = x + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \alpha_4 y_4, \quad (10)$$

can be seen as a generalization of the STREAM triad case.

While the summation kernels could be integrated into the call of the derivative kernel, we maintain a separation to allow flexibility and straightforward extendability, such as implementing higher-order Runge-Kutta methods.

Table 1 lists the required effective read/write counts of buffers in the kernels and the computational intensity, i.e., the ratio of performed floating-point operations to data transfers.

Table 1

Effective expected numbers of reads n_r and writes n_w as well as the computational intensity for the kernels for the scalar case without a reservoir (cf. examples below in Section 5) in single precision.

Kernel	Number of Reads n_r	Number of Writes n_w	Comp. Intensity B (Flops/byte)
derivative kernel	3	1	$\approx \frac{1}{2}$
two-term sum	2	1	$\frac{1}{6}$
five-term sum	5	1	$\frac{1}{4}$
RK4 iteration	23	8	$\approx \frac{1}{3}$

Table 2

Hardware specifications of the used CPUs and GPUs. For CPUs, the peak cache bandwidths $b_{\text{max}}^{\text{cache}}$ (L3), memory bandwidths $b_{\text{max}}^{\text{mem}}$, and the peak floating-point throughputs $P_{\text{max}}^{\text{fp32/64}}$ in single (SP) and double (DP) precision have been measured LIKWID (5.3, [38], triad_avx_fma, triad_mem_avx_fma, peakflops_avx_fma, peakflops_sp_avx_fma). For NVIDIA GPUs, the maximal useable cache/memory bandwidths have been measured with gpu-benches ([39], gpu-stream/gpu-l2-cache), and non-tensor floating-point throughputs are according to the device specifications [40,41]. The AMD 7763 CPU consists of 8 core complex dies (CCDs) with 8 CPU cores each.

Hardware	Cache Size (MiB)	$b_{\text{max}}^{\text{cache}}$ (GB/s)	$b_{\text{max}}^{\text{mem}}$ (GB/s)	$P_{\text{max}}^{\text{fp32}}$ (TFlop/s)	$P_{\text{max}}^{\text{fp64}}$ (TFlop/s)
NVIDIA RTX 4060 TI 16 GB	32	1600	275	22	0.3
NVIDIA RTX 4090 24 GB	72	5300	950	82.6	1.3
NVIDIA H100 SXM5 80 GB	50	8000	3200	66.9	33.5
AMD 5800X3D	96	580	31	1.1	0.5
AMD 7763	32	520	38	0.9	0.4
CCD, 8 cores					
AMD 7763 full, 64 cores	256	2600	155	2.5	5.1

The computational intensity B of the derivative kernel depends on the terms on the right side of Eq. (1). We have assumed the case of $g \neq 0$ and $\gamma \neq 0$.

To put the computational intensities into perspective it is useful to consider the properties of typical hardware listed in Table 2 that are characterized by the peak floating-point throughput $P_{\text{max}}^{\text{fp32/64}}$ in single or double precision and the cache/memory transfer bandwidths $b_{\text{max}}^{\text{cache/mem}}$. If the computational intensity B of the code is larger than the computational intensity $B_{\text{H}}^{\text{fp32/64}} = P_{\text{max}}^{\text{fp32/64}}/b_{\text{max}}^{\text{cache/mem}}$ of the hardware then the code is limited by the floating-point throughput, otherwise it is limited by transfer bandwidth. For all hardware listed in Table 2 the solver is estimated to be transfer bound in single precision.

Using the Roofline model the maximal performance of the program can be estimated by considering the maximal useable floating-point throughput P_{max} and maximal useable memory bandwidth $b_{\text{max}}^{\text{mem}}$ or cache transfer bandwidth $b_{\text{max}}^{\text{cache}}$ in relation to the computational intensity B of the code. The assumptions of the Roofline model that either the floating-point throughput or the transfers are the bottlenecks lead to the performance estimate

$$P^{\text{mem/cache}} = \min(P_{\text{max}}, B \cdot b_{\text{max}}^{\text{cache/mem}}) \quad (11)$$

for the floating-point performance assuming that either memory or cache are the relevant transfer stages for the buffers. This estimate excludes the halo synchronization step which would require a more elaborate treatment.

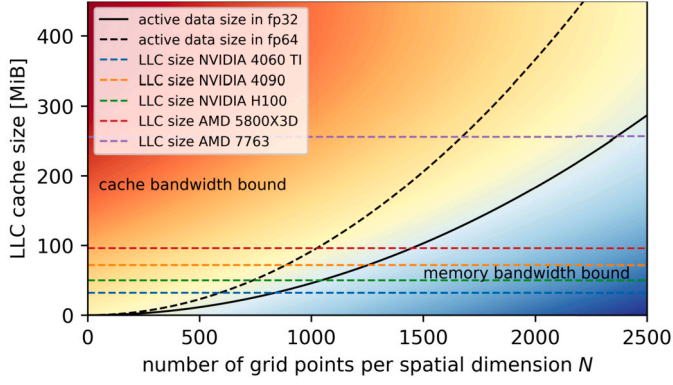


Fig. 3. The last-level cache (LLC) sizes of the devices in Table 2 is shown in comparison to the active data size, Eq. (7), for single precision (fp32) and double precision (fp64) calculations of $N \times N$ grids. The cache-bandwidth-bound regions (red) and memory-bandwidth-bound regions (blue) are shown as shaded areas.

Which transfer stage, either one of the cache levels or main memory, is relevant depends on the size of data being processed in parallel. Fig. 3 compares the active data size given by Eq. (7) to the last-level cache (LLC) sizes of the devices in Table 2. If the data size is smaller than the cache size, the cache bandwidth determines the maximal achievable performance in Eq. (11) otherwise the memory bandwidth is the relevant bandwidth. The subgrid decomposition described in Sec. 3.2.4 increases the cache locality within each time step.

The Roofline model predicts for hardware with $b_{\max}^{\text{mem}} = 1 \text{ TB/s}$ memory bandwidth like high-end consumer GPUs that the solver performs about 330 GFlops/s and, thus, would not be floating-point-throughput bound in single precision on a typical consumer GPU but transfer bandwidth bound. For a more immediate use and comparison, the execution time of a time step can be estimated for the scalar case without a reservoir to be at least

$$t_{\text{iter,roofline}} = \frac{2 \cdot (n_r + n_w) N_x N_y \cdot d}{b_{\max}} \quad (12)$$

$$= \begin{cases} \frac{248 \text{ byte} \cdot N_x N_y}{b_{\max}}, & \text{single prec.} \\ \frac{496 \text{ byte} \cdot N_x N_y}{b_{\max}}, & \text{double prec.} \end{cases} \quad (13)$$

according to the Roofline model in the transfer-bound case, where n_r and n_w are the numbers of reads and writes given in Table 1 and d is the data size of the floating-point number format. For other cases, for example with a reservoir, the additional buffers described in Sec. 3.2.6 have to be considered.

4. Performance and efficiency results

In this section, detailed performance results are presented to evaluate the PHOENIX implementation in relation to the performance estimates of the performance model in Sec. 3.2.8.

All benchmark runs in this chapter have been performed with GCC 13.2.0 and CUDA 12.6. Detailed documentation of the environments can be found in the benchmark documentation included in the PHOENIX release [27].

The main performance indicator is the number U of updated points per second (UP/s). It is related to the runtime t_{iter} of timestep for a given grid size $N_x \times N_y$ as

$$t_{\text{iter}} = N_x N_y / U. \quad (14)$$

The scalar case without a reservoir and smoothing is considered here for quadratic grids with the grid size N , i.e., $N_x = N_y = N$ with $N_x \cdot N_y = N^2$ total grid points. Hence, the total data size per grid point in all

required buffers is 48 byte in single precision and 96 byte in double precision.

The grid point update rates U measured on the devices listed in Table 2 are shown in Fig. 4 for different choices of subgrids and floating-point precisions. In addition, the grid point update rates estimated with the roofline model are shown for comparison. The performance of PHOENIX on CPUs and GPUs shows three characteristic regions:

Small grid sizes The simulation is latency-bound due to parallelization overhead from kernel launches on GPUs and OpenMP-parallelization loops on CPUs.

Intermediate grid sizes The highest performance is achieved because the simulation can efficiently utilize the cache.

Large grid sizes The memory bandwidth of the device limits the performance.

4.1. Comparison to roofline bounds

The performance bounds due to the cache bandwidths and memory bandwidths are shown in Fig. 4 in comparison to the measured performance results. GPU-based simulations with PHOENIX can reach between 51% (NVIDIA H100, Hopper architecture) and 94% (NVIDIA 4060 TI, Ada Lovelace architecture) of the cache bandwidth bound for intermediate grid sizes and saturate the memory bandwidth bound for large grid sizes in single precision. In double precision, the situation is similar for the NVIDIA H100 with 54% of the cache bandwidth bound but different for the NVIDIA 4060 TI and NVIDIA 4090 where the floating-point throughput bound limits the performance to 40% of the cache bandwidth bound resp. 70% of the floating-point bound. In all other cases, the comparison from single precision (fp32) simulations to double precision simulations (fp64) shows a difference of a factor of two for investigated GPUs and CPUs due to the doubling of the data size.

CPU-based simulations with PHOENIX reach 35-62% of the cache-transfer bound in cache-bandwidth bound situations and saturate the memory bandwidth for large grid sizes.

4.2. Cache-size dependence

The extent of regions with high performances at intermediate grid sizes, e.g., $500 \leq 1500$ for the NVIDIA 4060 TI (fp32, no subgrids) directly depends on the size of the last-level cache of the devices. This effect is especially pronounced for the consumer-grade GPUs and the CPUs because of the high ratio between cache bandwidths and memory bandwidths in these devices. A comparison between the case of using just one CCD of the AMD 7763, i.e., 8 CPU cores to the AMD 5800X3D (8 CPU cores) shown in the last two rows of Fig. 4 demonstrates the cache size-effect because the number and type of used CPU cores (AMD Zen3) as well as the effective memory bandwidth of the cores is very similar. The region of high-performing intermediate grid sizes is extended from $N \approx 700$ for the AMD 7763-CCD (32 MB L3 cache) to $N \approx 1200$ for the AMD 5800X3D (96 MB L3 cache).

4.3. Subgrid dependence

Decomposing the grid into more subgrids leads to similar effects on GPUs and CPUs by reducing the performance of small grid sizes but increasing the performance of large grid sizes. For small grid sizes, a higher number of subgrids increases the parallelization overhead. The computational overhead added by the subgrid halos increases this effect because the ratio of halo points to subgrid points is unfavorable for small sizes of subgrids.

For large grid sizes, the higher cache locality from the subgrid decomposition increases the performance so that the memory bandwidth bound limits only for larger grid sizes. This effect is especially noticeable for the NVIDIA 4060 TI and NVIDIA 4090 where in single precision

the sub-grid decomposition can lead to a roughly threefold increase in performance.

In contrast to GPUs, the CPUs show an overall weaker dependence on the number of subgrids. One reason is that at least one subgrid per CPU core is required for the simulation in the first place and the number of subgrids is required to be a multiple of the number of CPU cores for a balanced computational load.

4.4. Kernel performance

In this section, the kernels are considered individually to identify the reason for the difference between the roofline model estimates in Sec. 3.2.8 and the measured performance in Fig. 4. Fig. 5 compares the update rates of RK4-time steps that had all kernels except one deactivated to the corresponding estimates from the corresponding cache bandwidth bounds and memory bandwidth bounds in single precision.

For large grid sizes, most of the individual kernel performances approach the estimates from the memory bandwidth bounds. One exception is the derivative kernel on the H100 which shows a higher than expected performance. For intermediate grid sizes where the discrepancy between the cache-bandwidth bound and measured performance was observed in Fig. 4, on the NVIDIA H100 the two- and five-term-sum kernels show a large performance difference compared to the cache bandwidth bound. This is in stark contrast to the NVIDIA 4060 TI and NVIDIA 4090 GPUs where this behavior is not observed. The detailed investigation and optimization of the cache efficiency of PHOENIX is a topic for future research. For the CPU execution, all three kernels perform similarly when compared to the corresponding cache bandwidth bound.

4.5. Energy usage

In addition to time-to-solution also the energy-to-solution becomes more and more relevant in high-performance computing applications and computational research. A useful metric is the energy usage per grid point update E_{point} . It determines the energy E_{sim} used for a full simulation on a $N_x \times N_y$ grid with N_{iter} time steps as

$$E_{\text{sim}} = E_{\text{point}} N_x N_y N_{\text{iter}}. \quad (15)$$

Fig. 6 shows the measured energy usages per grid point update for the different devices in single and double precision simulations using RK4. For CPU runs, the power consumption is measured as the CPU-socket power consumption with RAPL counters (Running Average Power Limit). For GPU runs only the power consumption of the GPU itself according to the NVIDIA system management interface is used because the performance impact of the CPU in GPU-based simulations with PHOENIX is negligible.

For small grid sizes, i.e., latency-bound situations, the power consumption during the simulation is dominated by the idle power consumption of the devices. Consequently, the NVIDIA 4060 TI can outperform other GPUs in this region due to its low idle power consumption. GPUs show only a weak dependency on the grid size for large grid sizes compared to CPUs. For the large grid sizes shown in Fig. 6, i.e., up to $N_x = N_y = 4096$ the NVIDIA 4090 outperforms the NVIDIA H100 in terms of energy efficiency because the subgrid decomposition can efficiently use the large available L2 cache (72 MiB) to avoid the memory bandwidth bound. In the limit of very large grid sizes, the energy efficiency can be estimated from the ratio of the maximal power consumption to the memory bandwidth of the devices.

The lowest energy per grid point update of 24–30 nJ at $N \approx 600$ for single-precision simulations is similar among the tested GPUs. The NVIDIA 4090 shows the highest single-precision energy efficiency of 24 nJ per grid point update. In double precision, the NVIDIA H100 (46 nJ) is more energy efficient than the NVIDIA 4060 TI (79 nJ) and the NVIDIA 4090 (58 nJ).

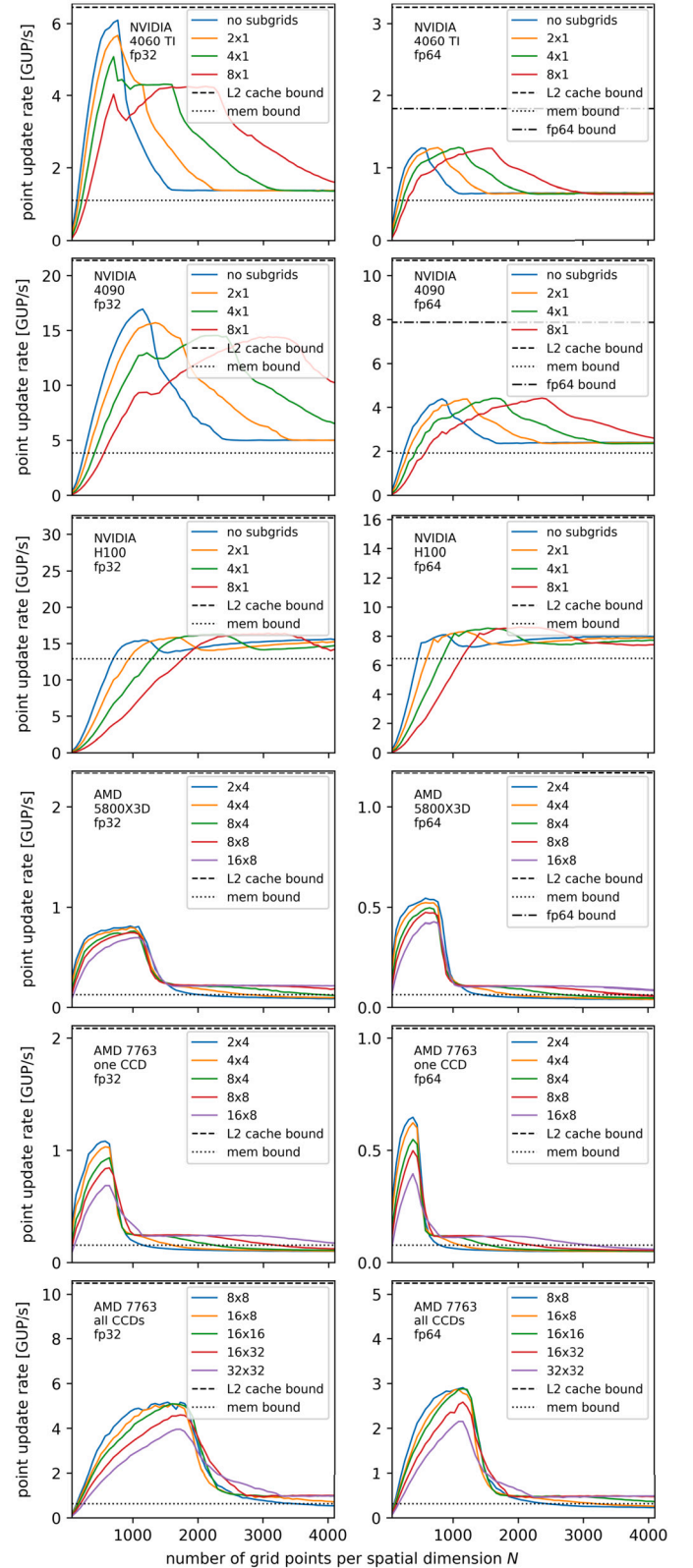


Fig. 4. The grid point update rates measured on the devices listed in Table 2 are shown for simulations in single precision (fp32, left column) and double precision (fp64, right column) for different subgrid settings of the $N \times N$ grid. The corresponding performance estimate according to the roofline model for the cache bandwidth bound (dashed lines) and memory bandwidth bound (dotted lines) cases are shown for comparison.

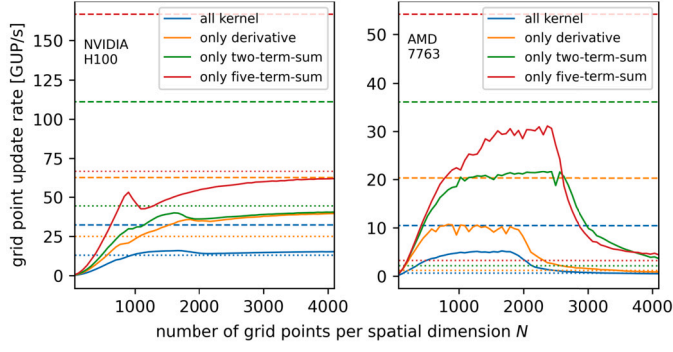


Fig. 5. The grid point update rates for single-precision RK4-time steps where only the individual kernels given in Table 1 were active for the NVIDIA H100 GPU (2x1 subgrids) and the AMD 7763 CPU (8x8 subgrids). Dashed lines show the corresponding cache-bandwidth-bound performance estimates and dotted lines show the corresponding memory-bandwidth-bound performance estimates.

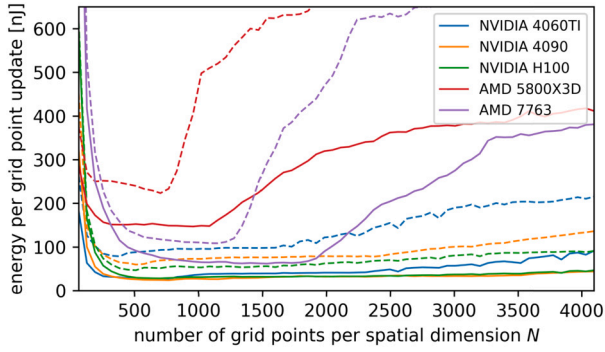


Fig. 6. Energy usage per grid point update E_{point} for the devices listed in Table 2 for different grid sizes for single-precision (solid lines) and corresponding double-precision calculations (dashed lines). For each grid size, the best energy efficiency of different subgrid settings is shown.

For CPU-based simulations, the behavior of the energy per grid point update has three regions similar to the grid point update rates: For small grid sizes the simulation is latency-bound and the power consumption mostly stems from the idle power consumption. Upon increasing the grid size the idle power gets less dominant compared to the dynamical power and a plateau of power consumption is entered where the simulation is bound by cache transfers instead of latency. In this intermediate grid size, e.g., $400 \leq N \leq 1800$ for the AMD 7763 CPU the CPUs can achieve their optimal power efficiency with PHOENIX of about 60 nJ per grid point update for single-precision simulations. Beyond the cache limit, the power consumption increases rapidly due to the memory bandwidth bound.

4.6. Conclusion

The performance model and results detailed in this section lead to the conclusion that entry-to-midrange consumer GPUs can be recommended for simulations with PHOENIX. Device architectures with large L2 caches are especially advantageous for simulations. However, data-center GPUs like the NVIDIA H100 are faster and more efficient for double-precision simulations. It should also be noted that due to their integration into HPC environments, their overall efficiency including cooling and maintenance expenses can be significantly better than for a large setup consisting of air-cooled consumer GPUs.

5. Applications

In the following, several applications of PHOENIX are presented. These examples are included with the code [27] and can be run using

the corresponding MATLAB and Python scripts. The selection of examples is designed to illustrate the capabilities of PHOENIX and to cover various variations of the equations implemented as well as a number of different variants, types, and possibilities of post-processing that can be performed. Applications include extensions of our previous research [42–46] covering the fields of topology, non-Hermitian physics, nonlinear physics, and quantum-state tomography based on Monte Carlo simulations with quantum noise. This relatively broad spectrum of examples is intended to help the easy and widespread use of PHOENIX and its adaptation to other interesting scenarios. These examples also showcase the exceptional performance and adaptability of the PHOENIX code and its potential on high-end consumer hardware for a wide range of practical applications.

First, the full set of equations covered by PHOENIX is introduced. For widespread applicability, we include a number of extensions going beyond Eq. (1) that was used for benchmarking purposes above. This enables the application of PHOENIX to a wider class of effectively two-dimensional systems with coherent driving or incoherent source or excitation as well as to other nonlinear systems or condensate systems described by a variation or subset of Eq. (16) and (17). Including the extensions to the dNLSE in Eq. (1) currently implemented, the set of equations covered by PHOENIX reads:

$$d\psi_{\pm} = -\frac{i}{\hbar} \left(-\frac{\hbar^2 \nabla^2}{2m} + V + g|\psi_{\pm}|^2 + g_{\times}|\psi_{\mp}|^2 + g_r n_{\pm} + \frac{i\hbar}{2} [Rn_{\pm} - \gamma] \right) \psi_{\pm} dt - \frac{i}{\hbar} (J_{\pm} \psi_{\mp} + E_{\pm}) dt + dW, \quad (16)$$

$$\frac{dn_{\pm}}{dt} = P_{\pm} - (\gamma_r + R|\psi_{\pm}|^2) n_{\pm}. \quad (17)$$

Here $\psi_{\pm} = \psi_{\pm}(\mathbf{r}, t)$ is a complex-valued spinor field, with $\pm$ denoting the two spin or polarization components and $n_{\pm} = n_{\pm}(\mathbf{r}, t)$ are spin-dependent densities of incoherent excitation reservoirs. Parameters g and $g_{\times}$ scale the self- and cross-interaction of the two spin-components of $\psi_{\pm}$, respectively, while g_r scales the interaction with excitation reservoirs $n_{\pm}$ (here included in equal spin channels, which can be generalized). The gain to $\psi_{\pm}$ from the reservoir scales with R while driving by resonant coherent excitation is included through $E_{\pm} = E_{\pm}(\mathbf{r}, t)$. If not independent from each other, the two spin components are coupled for example through $J_{\pm} = \Delta_{\text{LT}}(\partial_x \mp i\partial_y)^2$, here included to introduce a typical spin-orbit interaction from transverse-longitudinal splitting of the cavity field with magnitude Δ_{LT} . $P_{\pm} = P_{\pm}(\mathbf{r})$ is an external pump that replenishes the reservoir, while γ_r introduces loss from the reservoir. The complex Wiener noise is denoted by dW . When PHOENIX is executed, the boundary conditions in the x - and y -directions can be set independently periodic or zero. This allows to apply the truncated Wigner approximation and to investigate classical and quantum fluctuations [47–50,44].

We note that Eqs. (16) and (17) can describe a variety of two-dimensional physical systems governed by variants of the NLSE. Besides the polariton system discussed below this also includes systems such as conventional Bose-Einstein condensates (BECs), superfluids, and quantum gases. The introduction of additional terms going beyond Eq. (16) and (17) is user-friendly and detailed instructions are available in the PHOENIX documentation. In the following, multiple examples for potential applications that also require different types of pre- and post-processing are illustrated and the code to run these examples, including easily accessible Python and Matlab scripts, respectively, that call the pre-compiled main numerical PHOENIX routines, is provided together with the PHOENIX release [27].

As a prominent application area of the set of Eq. (16) and (17), in the following we will focus on the description of exciton-polaritons in planar microresonators. Exciton-polaritons are hybrid light-matter quasi-particles resulting from the strong coupling of exciton and cavity photons in semiconductor microcavities [51]. For both resonant and

nonresonant optical excitation of the cavity, macroscopic coherence of polaritons can form known as polariton condensation [52–54]. In case of nonresonant excitation, the interaction of the condensate and the excitation reservoir can be used to induce a repulsive potential energy landscape for the condensate polaritons to move in, allowing for optical trapping and control [55–59]. In materials with larger exciton binding energies polariton condensation even occurs at relatively high temperatures up to room temperature [60,61]. In contrast to conventional BECs, these condensates are inherently non-Hermitian as a constant supply of particles is required to compensate for the finite excitonic and polaritonic lifetimes. The nonlinear nature of polariton condensates leads to a variety of intriguing phenomena such as the formation of solitons and vortices [11]. Furthermore, multistabilities in polariton condensates allow for novel designs of all-optical transistors [62–65]. The strong nonlinearity in polariton condensates enables new designs for neuromorphic networks [66]. Due to the two active exciton spin states coupled to the two circular polarizations of the cavity light field, polaritons possess a spin degree of freedom, which permits the emergence of new vortex states, such as half- and spin-vortices [67,68,43]. The splitting of circular polarization modes into TE and TM modes that typically occurs in planar photonic resonators leads to spin-orbit coupling within the condensate [69]. This permits the observation of interesting phenomena such as the optical spin Hall effect [70–73] and oblique half-dark-solitons [74,75]. Polaritons in the different spin components also interact with each other, inducing a nonlinearity that can be attractive or focusing in nature [76]. Under resonant excitation, squeezed states were realized [77] and quantum correlations of the light source were mapped to the matter system in polariton condensates [78]. In Refs. [44,79,80] the quantum coherence of a non-resonantly excited polariton condensate was quantified, illustrating polariton condensates as a promising hybrid light and matter platform also for quantum information processing.

5.1. Example 1: Coherent excitation of polariton topological corner states in large AAH-SSH potential lattices

Over the past years, numerous studies have focused on the formation and utility of topological states in both electronic condensed matter and optical systems, for the latter including photonic crystals, coupled waveguides, meta-surfaces, and exciton-polaritons. In two dimensions topologically protected states may manifest as edge or corner states, respectively. A well-known example of a structure in which these states appear is the Su-Schrieffer-Heeger (SSH) model in 1D and 2D [81,82]. The topologically protected states are highly robust [83] which makes them promising candidates for example for unidirectional signal propagation [84], optical communication, stable single-mode lasing, and quantum information processing [81]. Another structure recently investigated are coupled AAH-SSH potential lattices in which both edge and corner states may be observed [46]. In two dimensions, structures investigated typically consist of a domain in space with a large number of potential wells [85], rendering full numerical investigations of large structures time-consuming.

In Fig. 7 we show results for an example in which a significantly larger structure than in Ref. [46] is investigated, enabled by the efficiency of PHOENIX. We demonstrate that the corner states discussed in Ref. [46] are indeed also stable in much larger systems of the same AAH-SSH structure. The physical parameters used to solve Eq. (16) are given in Ref. [86]. Parameters set to zero reduce the set of equations, Eq. (16) and (17), to the subset of equations required to describe the physical setup in the present example. Given a very large grid size of $N \times N = 9000 \times 9000$ grid points, the resulting system state vector propagating in time covers 0.81×10^8 distinct elements. Due to the large GPU memory-band width, however, calculations on systems of this size are now very much feasible. For the version of Eq. (16) used in this example we note that the maximum grid size compatible with the total memory

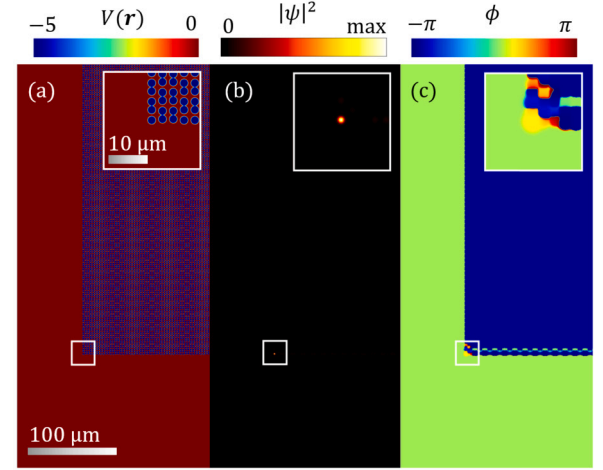


Fig. 7. (a) Extended double-wave lattice with 9000×9000 grid points and 272×272 potential wells. (b-c) Density and phase of topological corner state localized in the corners of the real space grid. For illustrative purposes, a segment around the bottom left corner of the potential lattice is displayed. The insets show a zoom onto one corner of the potential lattice and the corresponding corner state.

of 24GB of one single RTX 4090 GPU is about 2.25×10^8 (corresponding to a 15000×15000 square grid).

The polariton condensate is excited coherently to drive the system near the frequency of a pre-determined corner state. This frequency is determined by eigenvalue calculations for the linearized system operator, which is numerically quite costly and does not deliver the correct resonance frequency in the nonlinear regime. Utilizing the PHOENIX speedup in the time-domain calculations, optimizing the driving frequency is feasible using local optimizers like *fminbnd*. The optimized coherent pump underlying Fig. 7 features a frequency of $\omega = -5.6883$ THz and an amplitude of $E_0 = 5.6856$ μeV . Analytically, the potential well distribution in the 2D double-wave structure is given by

$$V(x, y) = \sum_n V_0 \exp \left(-\frac{(x - x_n)^2 + (y \pm y_n)^2}{w_v^2} \right)^{10} \quad (18)$$

as depicted in Fig. 7(a). Each of the 272 potential wells in x - and y -direction feature a depth of $V_0 = -5$ meV and a width of $w_v = 1$ μm .

After 5 ns the system converges into the corner state displayed in Fig. 7(b) and (c). Note that for illustrative purposes only a segment around the bottom left corner of the potential lattice is shown. Using AMD EPYC 7443P CPU and execution on the NVIDIA RTX 4090 GPU, time evolution of the corner state for a given frequency takes about 7.5 hours in real time. Here, the subgrid decomposition is used to minimize the overhead. A total number of 18×18 subgrids are used by setting the flag `-subgrids 18 18` when executing PHOENIX.

5.2. Example 2: Algorithm-assisted localization of exceptional points in reservoir-coupled polariton condensates

In recent years, there has been a growing interest in investigating non-Hermitian phenomena in dissipative systems. In these open systems, loss and gain lead to complex eigenvalues and potentially nonorthogonal eigenvectors of the operator governing the time evolution of a given system [87,88], permitting the observation of striking phenomena like the non-Hermitian skin effect or exceptional points [89,90]. The latter are singularities at which two or more eigenvalues are degenerate and their respective eigenvectors become parallel [91,92]. Exceptional points have been investigated in a variety of physical systems like microwave resonators [93], atomic systems [94], plasmonic nanostructures [95], optical waveguides [96], microresonators [97],

and non-reciprocal systems [98]. Near the exceptional point, the coalescence of the eigenvectors leads to counterintuitive system behavior, such as loss-induced transparency [99], unidirectional invisibility and reflectivity exceeding unity [100], and lasing self-termination [101]. The exceptional point order η is determined by the number of eigenvectors and eigenvalues coalescing at the exceptional point [102]. Exceptional points are promising candidates for a new generation of sensors, as the frequency splitting near exceptional points scales with the η^{th} -root of the perturbation and thus outperforms conventional Hermitian sensors [103,104].

Due to their non-equilibrium nature, polariton condensates offer a natural playground to study non-Hermitian phenomena such as exceptional points. By brining two modes into resonance with each other and fulfilling the condition that the mode coupling corresponds to the modes' gain difference, exceptional points have been realized in polariton condensates in the past [105–107,45,108]. Moreover, the nonlinearity of polariton condensates allows for the investigation of the interplay of nonlinear and non-Hermitian physics [109]. However, localizing η^{th} order exceptional points in parameter space can be challenging, as $2\eta - 2$ parameters require precise adjustment [104]. In nonlinear systems, this becomes even more challenging as the nonlinear characteristics such as the saturable gain and Kerr-nonlinearities couple mode gain, loss, and energy [109]. In the following, we show an example where the short computation times of PHOENIX permit the application of numerical optimization algorithms to localize exceptional points in parameter space consuming only a relatively small amount of time.

In Ref. [45], we realized an optical switch for the polariton condensation in a double-well potential featuring a second-order exceptional point. The double-well potential is given by

$$V(x, y) = V_1 f(x_1, w_p) + V_2 f(x_2, w_p) \quad (19)$$

with the envelope function $f(x_{\text{shift}}, w) = \exp(-[(x - x_{\text{shift}})^2 + y^2]/w^2)^2$. Here, $w_p = 1.5 \mu\text{m}$ describes the radius of the potential wells. Furthermore, $V_1 = -2.2$ and $V_2 = -2$ meV define the depths of the left and right potential well, respectively. Their distance is described by $d = x_2 - x_1 = 4 \mu\text{m}$, which directly influences the mode coupling of the condensates confined by the individual potential wells. The system is driven by non-resonant excitation with one Gaussian-shaped pump in each potential well:

$$P(x, y) = P_1 f(x_1, w_p) + P_2 f(x_2, w_p). \quad (20)$$

$P_1 = 12 \text{ ps}^{-1} \mu\text{m}^{-2}$ defines the pump intensity in the left, while P_2 defines the intensity in the right well. The pump width is given by $w_p = 1 \mu\text{m}$ and their separation is set to $d = x_2 - x_1 = 4 \mu\text{m}$. The choice of parameters used to solve Eq. (16) and (17) are provided in Ref. [110]. Parameters set to zero reduce the set of equations, Eq. (16) and (17), to the subset of equations required to describe the physical setup in the present example. The spatial distribution of pump and potential are shown in Fig. 8(a-b). The two lowest-energy eigenmodes are taken as the initial condition for the system evolution. These are the binding and anti-binding modes of the potential wells, which are focused in the left or right well respectively [see insets in Fig. 8(c)]. The pump intensity P_2 is then gradually increased, which not only shifts the gain and loss difference between the two modes, but also the energy difference due to a nonlinear blueshift. In Fig. 8(c) the tracing of the exceptional point in dependence of the pump intensity P_2 is illustrated. Here, increasing the pump intensity initially leads to a counterintuitively decreased integrated density in the system and the energy spectrum shows only the bonding mode. At the exceptional point, the density falls below the condensation threshold leading to a rapid increase in the mode linewidth and thus no observable energy spectrum in the numerics. Further increasing the pump intensity leads to recondensation and appearance of the antibonding mode in the energy spectrum above its threshold of $P_{\text{thr}} = 10 \text{ ps}^{-1} \mu\text{m}^{-2}$. The mode energy bifurcation and condensate switching-off is in good agreement with the experimen-

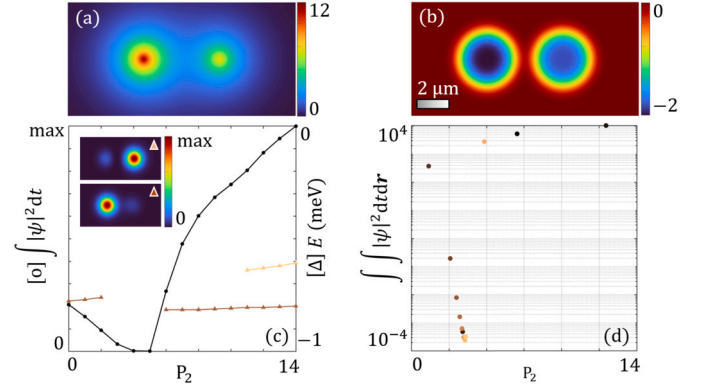


Fig. 8. (a) Spatial pump in $\text{ps}^{-1} \mu\text{m}^{-2}$ and (b) double-well potential distribution in meV. (c) The integrated density of the polariton modes in μm^{-2} (inserted plots) in black and the mode energies in scales of brown, indicating the condensation switch-off and the energy bifurcation at the exceptional point. Note that the second mode passes the condensation threshold at $P_2 > 10 \text{ ps}^{-1} \mu\text{m}^{-2}$. (d) Algorithm-assisted localization of the exceptional point using local optimizer “fminbnd” on an intensity interval $P_2 = [0; 20] \text{ ps}^{-1} \mu\text{m}^{-2}$. The increasing iteration number is illustrated by the marker color (from dark to light).

tal data shown in Ref. [45]. Due to the PHOENIX speedup, it is now feasible to localize the exceptional point automatically with optimizer algorithms. In Fig. 8(d) the exceptional point is localized with the MATLAB built-in one-dimensional minimizer for a pump intensity interval $P_2 = [0; 20] \text{ ps}^{-1} \mu\text{m}^{-2}$. Here, the minimizer finds the local minimum in the integrated density to localize the EP. The iteration number is illustrated by the lightening of the marker color (dark to light, marks an increase in the iteration number). The very fast evaluation of each step in the optimization process promises feasibility also for other tasks requiring optimization or complex inverse design [111] in a complex multi-dimensional parameter space and could be adapted to wavefunction tuning by adjusting the system parameters to minimize the discrepancy between the resulting density and the desired density profile. Using an AMD EPYC 7443P CPU and NVIDIA RTX 4090 GPU the EP localization takes about 1 hour in real time.

5.3. Example 3: Localized vortices in spinor polariton condensates with spin-orbit coupling

In this chapter, we investigate the multistability of localized vortices in spinor polariton condensates, as done in Ref. [43]. Quantized vortices are a well-studied phenomenon in conventional Bose-Einstein condensates and driven-dissipative polariton condensates [112–115]. Quantized vortices are characterized by a funnel-shaped deepening and a phase singularity in the so-called vortex core surrounded by a phase winding of $2\pi m$. Here, the integer number m is referred to as the topological charge of the vortex. In the case of polariton condensates the vortex results in a quantized orbital angular momentum (OAM) which can be extracted from the cavity-emitted light by interferometry methods [116]. Thanks to that, quantized vortices in polariton condensates are promising candidates for novel information processing and data storage applications, in which their topological charge act as bit values [117] with the potential also for qubit analogue realizations [118,119]. Hence, recent research on polariton condensates aims at reliable trapping and manipulating of quantized vortices. For example, quantized vortices can be trapped, moved, and switched between different topological charges by designing optically-induced potentials arising from the repulsive interaction with the excitonic reservoir [120–122,50].

Polariton condensates exhibit a spin degree of freedom, which results from the two optically active exciton spin states which are coupled to the two circular polarization states of the light field. This gives rise to novel vortex states such as half-quantum vortices, i.e. a vor-

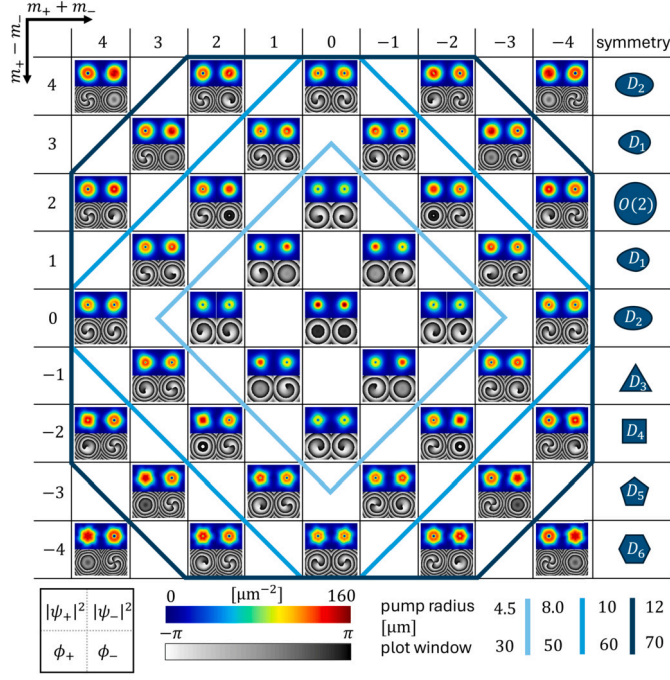


Fig. 9. Overview of the first seven groups of approximate real-space symmetry for different stable vortex states. The states are arranged according to the topological charge difference $m_+ - m_-$ (vertical) and the sum $m_+ + m_-$ (horizontal). The resulting symmetries are exemplified on the right-hand side. For each state, the densities and phase distributions of the circular polarizations are shown as in the scheme at the bottom left. The respective color bars are located at the bottom, together with the pump radii and the sizes of the plot windows used for the respective regions marked in blue.

tex in one spin component and a non-vortex state in the other ($m_{\pm} \neq 0 \wedge m_{\mp} = 0$) [123,124], as well as full- and spin-vortices, featuring parallel ($m_{\pm} = m_{\mp}$) or antiparallel ($m_{\pm} = -m_{\mp}$) circulating vortices in the two spin components [125,50]. Here $m_{\pm}$ denotes the topological charge of the respective spin component of the condensate. Vortices in different spin components are coupled via so-called TE-TM (or longitudinal-transverse) splitting, which describes the energy splitting of perpendicularly polarized cavity photon modes and induces a spin-orbit interaction of polaritons.

Investigating spinor polariton condensates demands solution of the dNLSE shown in Eq. (16) and (17) for both spin components $\pm$ for the polariton field and the reservoir which doubles the dimension of the numerical problem to be solved; in addition smaller time steps may be required to suitably resolve the system dynamics including the spin-orbit interaction operator. Therefore, a resource-efficient implementation such as PHOENIX is needed to allow for a time-efficient and systematic scanning of different vortex states while varying the topological charge configuration. By adding *-tetm* to the runstring, PHOENIX solves the set of coupled equations for the two spinor components $\psi_{\pm}$ of the polariton condensate and their respective reservoirs (for simplicity here assuming spin preservation in the relaxation from the reservoir).

In the numerical simulations, we use a x -linearly polarized continuous-wave pump with a ring-shaped profile

$$P_{\pm}(\mathbf{r}) = P_0 \frac{r^2}{w_p^2} \exp\left(-\frac{r^2}{w_p^2}\right), \quad (21)$$

with a radius w_p and a pump intensity $P_0 = 100 \mu\text{m}^{-2}\text{ps}^{-1}$. The choice of parameters used to solve Eq. (16) and (17) are provided in Ref. [126]. Parameters set to zero reduce the set of equations, Eq. (16) and (17), to the subset of equations required to describe the physical setup in the present example. The dynamically stable vortex states are depicted in Fig. 9 together with their topological charge difference $m_+ - m_-$ and sum

$m_+ + m_-$, as well as for different pump widths w_p . The spin-orbit interaction breaks the symmetry of the vortex states leading to a variety of real-space symmetries in the polariton density. These results were reproduced from Ref. [127]. We emphasize that in this application the high computational speed of PHOENIX enables the systematic investigation and classification of a large number of vortex shapes in the spinor system in a remarkably short time. Here, the temporal evolution over 10 ns is evaluated for each subplot to ensure convergence on a medium-sized grid of 500×500 , resulting in a runtime of 1 minute in real time on an AMD EPYC 7443P CPU and NVIDIA RTX 4090 GPU for each individual run, with 41 runs in total for the displayed results.

5.4. Example 4: Phase-space sampling and quantum-state tomography of polariton condensates with Monte Carlo simulations

In this chapter we show how PHOENIX can be used to analyze statistical properties by taking into account classical and quantum fluctuations. We reproduce the central results in Ref. [44] and determine the quantum coherence as a measure for the amount of Fock-state superpositions within the polariton condensate using the truncated Wigner approximation (TWA).

Polariton condensates have been extensively studied as examples of spontaneous buildup of macroscopic coherence under nonresonant excitation. The degree of spatial or temporal coherence is usually characterized by normalized correlation functions. Most prominent are the first- and second-order correlation functions $g^{(1)}$ and $g^{(2)}$ which capture phase and intensity correlations. However, the $g^{(1)}$ function does not contain information of a state being classic or quantum, and $g^{(2)}$ only relates to the diagonal elements of the density operator in Fock space. In contrast, the quantum coherence gives a measure for the off-diagonal contributions of the density operator, i.e. the extent of quantum superpositions of particle-number states [128–131].

Coherence properties can be quantified with Monte Carlo methods. By adding *-dW v* to the runstring, PHOENIX solves the stochastic differential Eq. (16) and (17) for a renormalized density $|\bar{\psi}|^2 = |\psi|^2 - (\Delta V)^{-1}$, with $\Delta V = \frac{L^2}{N^2}$ denoting the unit-cell volume of the two-dimensional grid. The correlations of the complex Wiener noise contribution dW follow

$$\langle dW(\mathbf{r})dW(\mathbf{r}') \rangle = 0$$

$$\text{and } \langle dW(\mathbf{r})dW^*(\mathbf{r}') \rangle = (Rn + \gamma_c) \frac{\delta_{\mathbf{r},\mathbf{r}'}}{2\Delta V}. \quad (22)$$

The parameter v in the runstring scales the Wiener noise intensity. For further details, discussion and for other applications of this theoretical approach, please refer to Refs. [132–134,44,135].

Here, we excite a system with a continuous-wave pump with super-Gaussian profile

$$P(\mathbf{r}) = P_0 \exp\left(-\frac{r^4}{w_p^4}\right), \quad (23)$$

with a width $w_p = 65 \mu\text{m}$. The choice of parameters used to solve Eq. (16) and (17) are provided in Ref. [136]. Parameters set to zero reduce the set of equations, Eq. (16) and (17), to the subset of equations required to describe the physical setup in the present example. To calculate the average polariton condensate excitation number, the stochastic differential equations are solved 150 times and the wavefunction is sampled every 5 ps for 1 ns. To ensure convergence the sampling starts after 1 ns. Using the sampled data, the expectation values $\langle \hat{n}_k \rangle$, the mean $\langle \hat{n}_c \rangle$ and variance $\langle (\Delta \hat{n}_c)^2 \rangle$ of the averaged polariton excitation number, the second-order correlation function $g^{(2)}(\tau = 0)$, and the amount of quantum coherence are determined for different pump intensities P_0 . A large number of repetitions of the calculations is necessary to ensure precise results of the stochastic evaluations. Here we profit substantially of the high computation speed of PHOENIX.

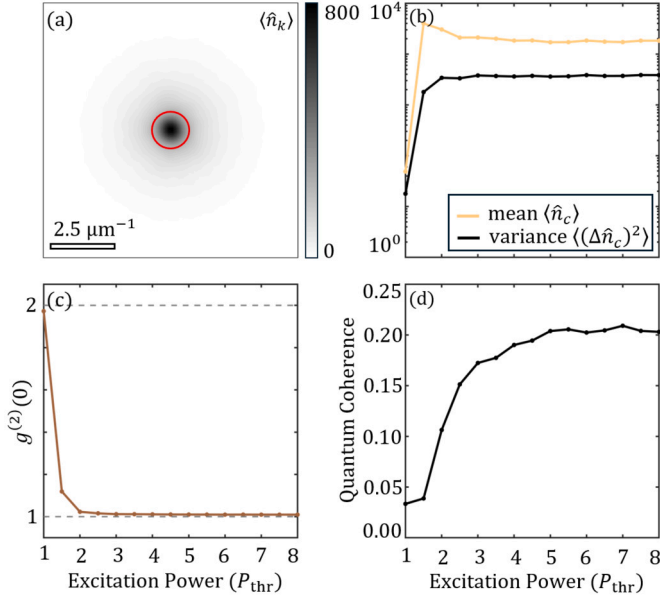


Fig. 10. (a) k -space expectation value above the condensation threshold $P_0 = P_{thr}$. The red square indicates the selected signal for the mode occupation. (b) Mean and variance of averaged polariton excitation number and (c) second-order correlation function $g^{(2)}(0)$ as a function of the excitation power in multiples of the pump threshold. (d) Resulting quantum coherence ζ in dependence of the excitation power.

We average the polariton number over a ring covering N_p discrete modes in the vicinity of $k \approx 0$ as displayed by the red ring in Fig. 10(a). The resulting mean polariton excitation number saturates for excitation powers $P_0 \geq 4P_{thr}$; see Fig. 10(b). Fig. 10(c) illustrates the transition of a thermal ($g^{(2)} = 2$) to a coherent ($g^{(2)} = 1$) state under increasing excitation power. The system's quantum coherence saturates at around $\zeta = 0.2$ as shown in Fig. 10(d). The displayed results are in good agreement with our results in Ref. [44].

We emphasize that PHOENIX enables the evaluation of stochastic processes even for large two-dimensional systems and high numbers of repetitions. Using AMD EPYC 7443P CPU and NVIDIA RTX 4090 GPU one sampling for the stochastic calculation is performed in about 95 seconds in real time calculated on the GPU. In total 2250 samples are computed for the displayed results, with a total compute time of approximately 60 hours. The target quantities shown in Fig. 10 are then evaluated with the post-processing program included in Ref [27]. Other possible applications would include the actual quantum state tomography and density matrix reconstruction in polariton number space as demonstrated in Refs. [79,80].

6. Conclusion

In this paper, we have presented the open-source code PHOENIX as a solver for two-dimensional Schrödinger equations with extensions. This solver can be executed and is highly performant on both CPUs and GPUs. For parallelization and efficient cache usage, PHOENIX supports subgrid decomposition.

We have performed a wide range of benchmarks showing that PHOENIX enables very time- and energy-efficient simulations in single and double precision on entry-class to high-end consumer GPUs. Practically, a workstation configuration with an inexpensive CPU and a few mid-level consumer GPUs can be recommended in case no access to an existing HPC cluster with GPUs is available. For double-precision simulations and on medium to large grids, the lower double-precision floating-point throughput of consumer-grade GPUs limits the performance and efficiency. In this case, datacenter-grade GPUs used in HPC clusters show to be the more time- and energy-efficient option. We note

that even in this scenario consumer GPUs exceed the CPU performance by a significant margin.

The CPU version of PHOENIX also delivers good performance on intermediate grid sizes and, thus, can be a suitable option if GPUs are not available. It should be emphasized here that PHOENIX is easy to extend by additional terms. In direct comparison to a more conventional Runge-Kutta implementation in MATLAB, PHOENIX shows a speedup of up to three orders of magnitude and energy savings of up to 99.8%.

PHOENIX is currently tailored for single-CPU and single-GPU based simulations for grid sizes up to the order of 10^4 in one of the two spatial dimensions (total grid size on the order of 10^8). Thus, the natural limit for the grid is the main memory in CPU-based simulations and the GPU memory in GPU-based simulations. For very large grid sizes exceeding the available device or node memory, PHOENIX can be straightforwardly generalized to multi-node or multi-GPU usage by distributing the subgrid decomposition. Although PHOENIX currently uses CUDA for the GPU implementation, care was taken to use only “portable” CUDA functionality which makes automated porting to GPU architectures from other vendors straightforward.

Furthermore, we provide a series of examples for describing exciton-polariton condensates including codes that illustrate the use of the different terms already available in PHOENIX. Here we show that the high efficiency of PHOENIX makes computing on large grids, local optimization algorithms, as well as large-scale parameter sweeps and Monte Carlo simulations with quantum noise for large statistical ensembles feasible. With this PHOENIX renders large system sizes and certain types of (such as statistical or iterative) evaluations possible that were previously virtually inaccessible to the typical user. We would further like to point out that in addition to the highly optimized functions described, PHOENIX also masters conventional forms of the imaginary-time algorithm or the split-step Fourier method. For further information on more specific capabilities, please refer to the most recent version of the PHOENIX repository [27].

As a natural next step, further equations such as the coupled set of equations of motion to separately handle two-dimensional light field and induced polarization to describe in particular coherent excitations beyond the lower-polariton branch's parabolic dispersion approximation, could be implemented. The presented set of equations can be further extended for example by additional terms such as the Rashba-Dresselhaus spin-orbit coupling or to the study of multi-component Hamiltonian systems [14] for example for the exploration of complex solitary solutions [15]. Moreover, adaptive Runge-Kutta methods or the imaginary-time algorithm could be benchmarked in detail within the framework of PHOENIX. Since very large grid sizes can be handled by PHOENIX, medium sized three-dimensional real-space grids (on the order of 500 grid points per spatial dimension) could also be investigated with PHOENIX with minor adaptation of the code.

CRediT authorship contribution statement

Jan Wingenbach: Writing – review & editing, Writing – original draft, Visualization, Validation, Investigation. **David Bauch:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Investigation. **Xuekai Ma:** Writing – review & editing, Resources. **Robert Schade:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Investigation, Formal analysis, Data curation. **Christian Plessl:** Writing – review & editing, Supervision, Resources, Conceptualization. **Stefan Schumacher:** Writing – review & editing, Supervision, Resources, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

The data repository is linked in the manuscript.

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Skyrmions in non-Hermitian polariton condensates

9

After studying nonlinear exceptional points in nonlinear and non-Hermitian systems in the previous part, the final chapter of this thesis turns to another manifestation of non-Hermitian polariton dynamics: the spontaneous formation of Skyrmion textures and possibilities to control them by nonlinear means. Topological textures in continuous fields are intensively studied because of their potential applications in information processing. Skyrmions are a prominent example of such textures, which are characterized by a swirling field configuration with a nontrivial topological charge (the so-called *Skyrmion number*). Skyrmions have been intensively studied in a wide range of two-dimensional vector fields, ranging from magnetism [189] and optics [190–192] to coherent Bose-Einstein condensates [193] and exciton-polariton condensates [194].

Recently, it was theoretically proposed that Skyrmion structures can be realized in interference patterns of scalar fields by mapping complex wavefunctions to a three-component displacement field [195]. This initiated rapid developments in hydrodynamic surface waves [196, 197], plasmonic nanostructures [198–200], surface phonon polariton systems [201, 202]. In these systems, however, scalar-field Skyrmions have primarily been generated through externally engineered linear interference patterns, where the required phase structure is imposed by the excitation geometry. Moreover, scalar-field Skyrmions have not yet been explored in full nonlinear condensate dynamics.

In the publication below, a theoretical framework is established for the formation of Skyrmion textures in scalar complex-valued optical fields. Here, the evaluation of the gradient-based displacement field and determination of the mode frequency require fine spatial and temporal domains. Moreover, large numerical domains are required to resolve the Skyrmion moiré lattice structure discussed below. These challenges are perfectly addressed by PHOENIX [179].

It is demonstrated below that Skyrmion textures can arise spontaneously in nonresonantly pumped exciton-polariton condensates. In contrast to previous realizations of Skyrmion textures based on linear interference, the topology here emerges from the non-Hermitian

condensate dynamics itself. Thus, no active phase-structuring or resonant excitation is required. A non-Hermitian Schrödinger equation is derived to show that for a simple Gaussian-shaped nonresonant pump, the non-Hermiticity-induced phase gradient of the resulting condensate imprints the Skyrmionic texture on the resulting condensate [203]. Building on this mechanism, spatial modulation of the pump profile enables the formation of complex Skyrmion moiré lattices.

Starting from resonant excitation schemes, it is shown how optically-induced potential and dispersion renormalization can be used to actively manipulate Skyrmion numbers and pattern formation with nonlinearity. This study demonstrates that nonlinear and non-Hermitian planar microcavities pose a promising platform to study Skyrmion pattern formation and manipulation by optical means. Furthermore, this work links the established displacement field to optical forces, potentially opening up Skyrmion-centered approaches to trapping and controlling atoms or quasi-particles in an adjacent layer.

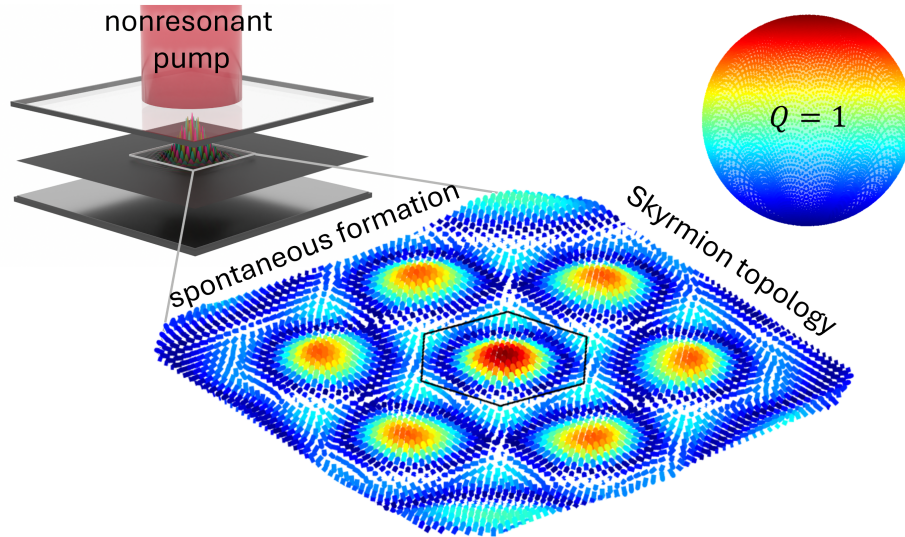


Figure 9.1: Spontaneously formed Skyrmion textures under nonresonant excitation. (left) Nonresonantly pumped polariton condensate in a planar microcavity. (center) Real-valued displacement vector field of the scalar and complex-valued wavefunction. (right) Mapping of the displacement field inside the Skyrmion cell (hexagon in the central plot) to the unit sphere, illustrating Skyrmion topology with Skyrmion number $Q = 1$. In the central and right plot, the color encodes the $\mathcal{Z}$ -component of the respective displacement field. This figure was submitted as the Table of Contents graphic for the corresponding manuscript, which is currently under review.

The results presented in this chapter are based on the preprint reproduced below. Reprinted from J. Wingenbach et al., Skyrmions in scalar fields of non-Hermitian optical microcavities: spontaneous formation, nonlinear control, and optical forces, arXiv:2606.02265v1 (2026), <https://doi.org/10.48550/arXiv.2606.02265>.

Skyrmions in scalar fields of non-Hermitian optical microcavities: spontaneous formation, nonlinear control, and optical forces

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Topological textures of light offer powerful routes for structuring optical fields, controlling wave transport, and manipulating matter. Skyrmions, long studied as topological solitons in vector fields, have recently been extended to scalar wave systems, including acoustics, hydrodynamics, and plasmonics. However, their realization in two-dimensional scalar wave propagation with nonlinearities and in quantum fluids remains uncharted. Here, we establish such a Skyrmion framework for scalar fields in optical microcavities. With focus on exciton-polaritons, we show that nonresonant excitation without imposed phase can spontaneously generate isolated Skyrmions and self-organized Skyrmion lattices in a polariton condensate. We trace this mechanism to gain- and loss-induced phase curvature together with outward polariton flow. We further demonstrate that polariton nonlinearities provide all-optical control of these topological textures, enabling switching of the Skyrmion number and reconfiguration of Skyrmion moiré lattices through resonant and nonresonant excitation schemes. These results establish nonlinear non-Hermitian resonators as a versatile platform for the spontaneous generation and active control of scalar topological light fields.

Introduction – Topological textures in continuous fields play a central role across condensed matter, photonics, and fluid dynamics. Among them, Skyrmions, topological excitations and textures characterized by a swirling, particle-like field configuration, have been extensively studied in two-dimensional vector fields, ranging from magnetism [1] and optics [2–6] to Bose-Einstein condensates (BECs) [7], phonons [8], and spinor exciton-polariton condensates [9]. In this context, exciton-polaritons in two-dimensional planar microcavities as realized in traditional semiconductor platforms, transition metal dichalcogenides (TMDs), perovskites, and organic materials, provide an exceptional optically accessible platform to investigate topological excitations and their dynamics, including targeted nonlinear control [10–15].

Beyond vector fields, recent work has shown that scalar fields can also host Skyrmion textures when embedded into an auxiliary vector representation constructed from amplitude and phase gradients [16]. Such Skyrmions have been observed in plasmonic nanostructures [17–19], hydrodynamic surface waves [20, 21], and surface phonon polariton systems [22–24] where carefully engineered multi-beam interference imposes the required phase structure. These studies raise a fundamental question: can nontrivial Skyrmion topology also emerge spontaneously in scalar complex-valued fields, and can those topologies be controlled by nonlinear interactions rather than external geometrical constraints?

In the present work we answer this question. We develop a general theoretical framework for Skyrmion formation in scalar driven-dissipative microcavity fields, and quantum fluids, such as polariton or Bose-Einstein condensates. As a main outcome of the present study, we show that in a nonresonantly driven exciton-polariton condensate spontaneous formation of Skyrmion topologies can indeed be observed, in a well controlled and tailorable manner. This elevates Skyrmions in scalar fields from externally engineered interference patterns to spontaneously forming and optically reconfigurable topological fields in an active nonlinear photonic platform. This observation opens the path for future explorations into the controlled realization and functionalization of a Skyrmion quantum-fluidic matter, enabling a novel type of photonic Skyrmionics with potential applications in information processing.

While the nonlinear interactions in our system are not required for the existence and formation of such Skyrmion topologies they provide us with a powerful control handle [12, 25]. This we use to demonstrate their efficient nonlinear optical control, including resonant and nonresonant optical excitation schemes. We envision that some of the nonlinear control concepts we introduce may in the future also be transferrable to other platforms such as plasmonic nanostructures combined with an active nonlinear medium. One further avenue to explore may also derive from the optical force fields generated by the Skyrmion textures. These could potentially be used

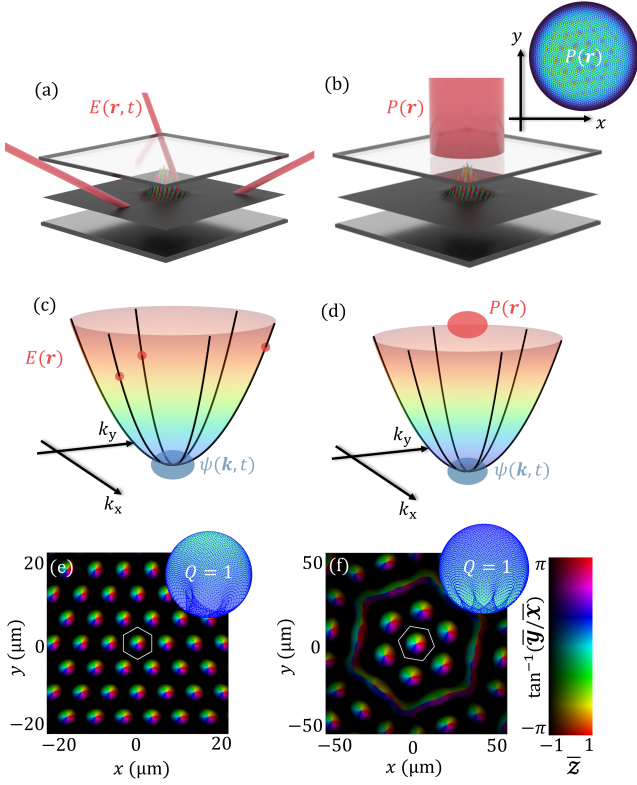


FIG. 1. **Excitation schemes and Skyrmion formation.** Resonant (left column) and nonresonant (right column) excitation of planar microcavity. (Left) N resonant pump beams (a) arranged on a ring in momentum space (c) excite propagating polariton waves that interfere in real space leading to formation of a Skyrmion lattice with corresponding displacement field $\mathcal{R}$ (e); example of hexagonal Skyrmion lattice for three input beams ($N = 3$) is shown. (Right) Spatially broad and structured nonresonant pump profile $P(\mathbf{r})$ (b) induces condensate formation in $\psi(\mathbf{k}, t)$ near the bottom of polariton dispersion (d). (f) Skyrmion bag in coherent condensed polariton system, spontaneously formed for reservoir with moiré lattice structure as indicated in (b) (for details see text). (e,f) Brightness encodes out-of-plane component $\bar{z}$, color encodes in-plane angle $\tan^{-1}(\bar{y}/\bar{x})$ of displacement field $\bar{\mathcal{R}}$. Mapping the normalized field $\bar{\mathcal{R}}$ inside the highlighted areas to the unit sphere provides a Skyrmion with Skyrmion number of $Q = 1$.

to trap and control atoms or quasi-particles that reside in an adjacent layer.

Theoretical Foundations – We model the coherent spatio-temporal dynamics of the polariton condensate based on the extended Gross–Pitaevskii equation coupled to an incoherent excitation reservoir. The dynamics of the condensate order parameter or wave function $\psi = \psi(\mathbf{r}, t)$ and the incoherent reservoir density $n = n(\mathbf{r}, t)$ are governed by [26]

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + g_c|\psi|^2 + g_r n + \frac{i\hbar}{2}[R_c n - \gamma_c]\right)\psi + E,$$

$$\partial_t n = P - (\gamma_r + R_c|\psi|^2)n, \quad (1)$$

with $\mathbf{r} = (x, y)$. Here, $m = 10^{-4}m_e$ denotes the effective particle mass of the lower polariton branch. The interaction strength between condensed polaritons is $g_c = 0.6 \mu\text{eV}\mu\text{m}^2$, while $g_r = 2g_c$ characterizes the condensate-reservoir interaction. Coherent (resonant) driving of the condensate enters via the complex-valued field $E = E(\mathbf{r}, t)$, while incoherent (nonresonant) pumping is described by the real-valued field $P = P(\mathbf{r}, t)$, which drives the reservoir. Stimulated in-scattering with $R_c = 0.01 \text{ ps}^{-1}\mu\text{m}^2$ feeds the condensate from the reservoir which in turn depletes the reservoir density. Dissipation enters through the condensate decay $\gamma_c = 0.3 \text{ ps}^{-1}$, and reservoir decay $\gamma_r = 1.5\gamma_c$. Here, these parameters are chosen to be in a realistic range for typical GaAs microcavities [27, 28]. Moreover, the results presented below remain qualitatively the same under variations of the system parameters. We also tested that our results are robust against random disorder potentials at least up to 0.2 meV with correlation lengths of about 1 μm .

To calculate the Skyrmion structure for stationary states ψ_s (for persistent coherent or incoherent driving) with frequency ω in the rotating frame, we adapt and apply the mathematical approach given in Ref. [16]. To this end we define the complex-valued displacement field $R = (\partial_x, \partial_y, 1)^T \psi_s$. This construction embeds the complex scalar field into a three-component vector field whose orientation encodes local amplitude and phase gradients. From this, the real-valued displacement field $\bar{\mathcal{R}} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}})^T = \Re(e^{i\omega t} R)$ is obtained, where $e^{i\omega t}$ eliminates the temporal evolution of ψ_s . This elimination permits calculation of a stationary topological charge Q , which is a crucial requirement for example for the Skyrmion switching investigated below. The topological charge of the Skyrmion (“Skyrmion number”) is calculated over one Skyrmion cell by

$$Q = \frac{1}{4\pi} \iint \bar{\mathcal{R}}(\partial_x \bar{\mathcal{R}} \times \partial_y \bar{\mathcal{R}}) dx dy, \quad (2)$$

with $\bar{\mathcal{R}} = \mathcal{R}/|\mathcal{R}|$. The bounds of the integration domain, i.e. the Skyrmion cell, are defined by critical points in the displacement field in agreement with previous Skyrmion-related works [2, 4, 18–21, 29, 30]. Mapping the directions of $\bar{\mathcal{R}}$ to the unit sphere visualizes the Skyrmion vector topology. In similar fashion, the topology of meron (half-Skyrmion) structures can be visualized via the spin density $S = (S_x, S_y, S_z)^T \propto \Im(R^* \times R)$. While outside the scope of this work, we use this to also validate the existence of meron lattices for three-wave interference in SM Note S3.

It is essential to emphasize that the Skyrmions considered here are not spin-texture Skyrmions associated with internal polarization degrees of freedom, as they are widely investigated in many systems including spinor-polariton condensates [9, 31–34]. Instead, they arise from a mapping of a complex-valued scalar field into a three-component vector field constructed from spatial derivatives. This defines a continuous mapping $\mathbb{R}^2 \rightarrow S^2$,

allowing a topological charge analogous in mathematical structure to that of conventional Skyrmions in vector fields, when evaluated over an appropriately chosen Skyrmion cell or integration domain. While the vector field in the present work is derived from amplitude and phase gradients rather than intrinsic spin, the resulting topology is genuine in the sense that it is invariant under smooth deformations of the scalar field that preserve boundary conditions. This construction therefore represents a scalar-field realization of local Skyrmion topology [16]. As in other optical and interference-based Skyrmion textures, the integer-valued charge is meaningful when the integration domain is chosen to match the physically defined cell of the texture; arbitrary deformations of this domain generally lead to non-integer values but are not to be interpreted as invalidation of the local topological structure.

Moreover, the complex-valued displacement field R can be related to the phase- and intensity-gradient forces of the polariton field. In SM Note S1 the relation between these forces and the in-plane displacement field $\mathcal{R}_2 = (\mathcal{X}, \mathcal{Y})$ is derived. The phase-gradient force is proportional to

$$F_\phi \propto I(x, y) \nabla \phi(x, y) = \Im \{ \psi_s^*(x, y) R_2(x, y) \} \quad (3)$$

and the intensity-gradient force is proportional to

$$F_I \propto \nabla I(x, y) = 2\Re \{ \psi_s^*(x, y) R_2(x, y) \}. \quad (4)$$

Accurate post-processing and calculation of the required gradient fields relies on the accurate evaluation of spatial gradients along both x and y , requiring sufficiently finely discretized grids. At the same time, the simulation domain must be sufficiently large to accommodate the entire pumping region while preventing boundary artifacts. For persistent driving (resonant coherent or off-resonant incoherent, depending on the scenario investigated as detailed below), here we evolve the system in time until stationary behavior is observed, on large two-dimensional grids with on the order of $\sim 10^7$ to $\sim 10^8$ grid points. To render these computationally intense calculations feasible, we use PHOENIX, our high-performance solver for two-dimensional nonlinear Schrödinger equations [35].

Skyrmions from Linear Interference – First we reproduce established Skyrmion and meron patterns arising from three-wave interference and more complex moiré-type interference configurations as previously reported in hydrodynamic [20, 21] and plasmonic wave systems [18, 19]. In planar optical microcavities resonant driving can be used to generate the desired propagating waves. The source inducing resonant excitation, $E(\mathbf{r}, t)$, then consists of N Gaussian-profiled (in real space) excitation beams arranged on a ring in real- and reciprocal-space as sketched in Fig. 1(a); mathematical description given in SM Note S2. Three-wave interference is obtained with three spots separated by 120° angles [20]. A moiré lattice as in Ref. [20] is realized with a total of $N = 18$ excitation beams forming three hexagons with relative

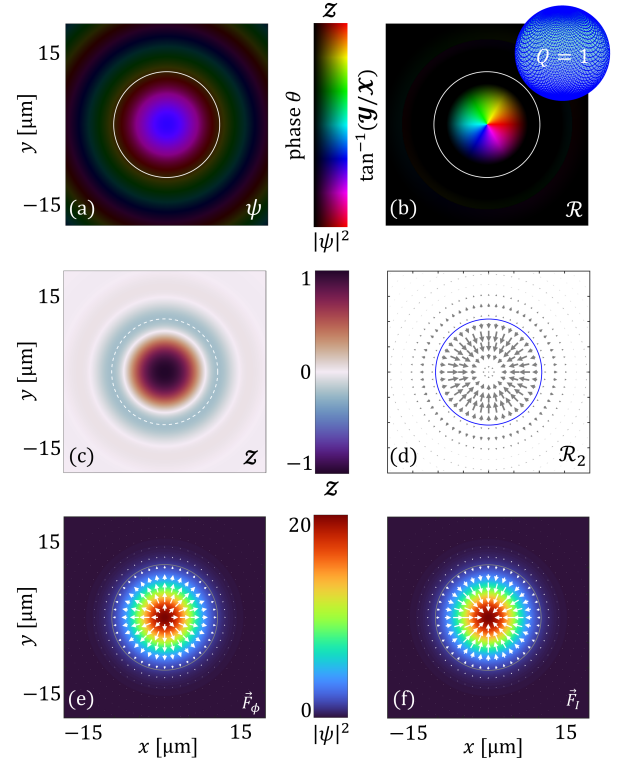


FIG. 2. Spontaneous Skyrmion formation for localized nonresonant pump. (a) An individual continuous Gaussian-shaped (in real space) nonresonant pump beam $P(\mathbf{r})$ induces a stationary condensate; density (brightness), phase (color). Finite lifetime and reservoir-induced effective external potential induce radial outflow and phase gradients. (b) Corresponding displacement field $\mathcal{R} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})^T$ for stationary condensate wavefunction, normalized to peak density. (c) Out-of-plane component $\mathcal{Z}$ showing sign reversal. (d) In-plane component $\mathcal{R}_2 = (\mathcal{X}, \mathcal{Y})$ revealing angular winding. Mapping $\mathcal{R}$ to the unit sphere yields Skyrmion charge $Q = 1$ (radius marked in each panel). (e) Phase-gradient force $\vec{F}_\phi$ and (f) density-gradient force $\vec{F}_I$ (vector fields) and density distribution (color) in the Skyrmion vicinity. $\vec{F}_I$ acts inward towards the Skyrmion center and $\vec{F}_\phi$ away from the center. Note that a small brightness offset is used in (a) to visualize the radial phase gradient around the condensate induced by non-Hermiticity.

angle of $\phi_{12} = \phi_{23} = 9.4^\circ$ (as indicated in Fig. 4(a) below).

In Fig. 1(e) the interference induced real-valued displacement field in the center of the spatial domain is shown. Here, brightness encodes the $\mathcal{Z}$ component and coloring encodes $\text{atan}^{-1}(\vec{\mathcal{Y}}/\vec{\mathcal{X}})$. The bright features that contain all the colors represent Skyrmion signatures with Skyrmion number $Q = 1$ (the hexagonal unit cell used for Skyrmion number evaluation is highlighted in white). In agreement with Refs. [16, 18, 20, 21] the unit-cell boundary is defined by the points in radial direction (away from the respective Skyrmion center) where the displacement

field changes sign. For illustration, in Fig. 1(f) the displacement field for a moiré lattice excitation is illustrated (albeit here obtained for a different, off-resonant excitation setup as discussed in detail in the following section). For a detailed analysis of the corresponding wavefunction, in-plane displacement field and meron structure are shown in SM Note S4. The Skyrmion structures observed here are equivalent in topology and displacement field to those reported for plasmonic or water wave systems [18, 20, 21].

Spontaneous Formation of Skyrmions – As demonstrated in the previous section, Skyrmion patterns in wave systems are typically engineered through carefully designed beam or wave interference, with input wave geometries that impose specific phase relations between multiple propagating coherent waves or beams.

As an essential and hitherto unexpected result of the present study, here we demonstrate that such external phase control is not necessarily required when turning to non-Hermitian exciton–polariton condensates, where condensate formation from noise and phase synchronization over large spatial domains may occur spontaneously. For that system, only using nonresonant excitation that does not carry phase information, we demonstrate that Skyrmions can emerge spontaneously in the driven–dissipative dynamics of the system. As a side note we would like to mention that this nonresonant excitation method is in principle also compatible with spatially structured electrical (rather than optical) pumping of the condensate.

Let us first consider a single Gaussian (in real space) nonresonant pump, with no imposed phase information. The pump feeds the incoherent reservoir $n(\mathbf{r}, t)$, which (if above condensation threshold) in turn populates the condensate via stimulated scattering [cf. Fig. 2(a)] triggered by initial random noise. Once reaching a stationary state, condensed particles generated inside the localized pumping region propagate radially outward before decaying. This outward flux leads to a non-vanishing probability current $\mathbf{j} = (\hbar/m) \text{Im}(\psi^* \nabla \psi)$ and therefore a radial phase gradient $\nabla \phi \propto \mathbf{j}/|\psi|^2$. The resulting steady-state solution for the coherent condensate exhibits oscillatory density modulations (in radial direction) and continuously varying phase around the pump center. The associated displacement field $\mathcal{R}$ in Fig. 2(b) develops sign changes in its out-of-plane component $\mathcal{Z}$ in Fig. 2(c) and a Skyrmion-cell border in its in-plane component $\mathcal{R}_2 = (\mathcal{X}, \mathcal{Y})$ in Fig. 2(d). As a result, the normalized vector field shows the characteristic vector topology to span the full unit sphere and form a Skyrmion with $Q = 1$. The Skyrmion-cell boundary is defined in agreement with optical Skyrmions observed in the photonic spin structure [2, 4, 29, 30]. In agreement with Ref. [30], increasing the integration radius beyond the inner Skyrmion cell in evaluating Eq. (2), the outcome alternates between 0 and 1, as plotted in Fig. S6 of the SM Note 7. A comprehensive approach to topological invariants for optical Skyrmions has recently been provided

in Ref. [30].

In addition to the crucial observation of spontaneously formed Skyrmion textures in the displacement field, the resulting optical forces (3) and (4) can be evaluated. These are displayed as vector fields in front of the density distribution in Fig. 2(e,f). While the phase-gradient force acts outward from the Skyrmion center, the density-gradient force acts inward towards the Skyrmion center. This observation may open up the possibility to trap atoms or quasi-particles (that may reside in an adjacent layer) with or inside the Skyrmion texture. Combining the possible quantum nature of the trapped particles with the topological character of the Skyrmion, with sufficiently strong coupling, this could motivate future studies into novel quasi-particles with intriguing fractional anyonic statistics [36–40]. It is worth noting that the phase-gradient force (albeit pointing away from the Skyrmion center for an isolated Skyrmion) may be used to trap particles in between Skyrmions in extended lattice structures (illustrated for a bag of Skyrmions in a moiré lattice in SM Note S5). We further note that the absolute and relative strength of phase-gradient and density-gradient forces will depend on the specific physical realization and material platform used. In this context also other nonlinear topological modes, such as vortex states with their winding phase-gradient force fields, may be worth exploring in the future.

The system dynamics described by the driven–dissipative Gross–Pitaevskii equations is intrinsically nonlinear due to condensate–reservoir coupling and intra-condensate interaction. However, it is important to note that the emergence of the Skyrmion topology does not rely on these nonlinearities. We have verified that nontrivial Skyrmion charge persists in the limit $g_c = g_r = 0$, as well as in the low-density regime where gain saturation is negligible ($R_c |\psi|^2 \ll \gamma_r$). Thus, while the dynamical equations remain nonlinear in general, the mechanism underlying spontaneous Skyrmion formation is fundamentally rooted in phase gradients generated by localized gain and outwards propagation with finite lifetime. In SM Note 6 we demonstrate that a non-Hermitian Schrödinger equation provides the analytical solution close to the condensation threshold that relates the Skyrmion properties of the solution to the phase curvature induced by the non-Hermiticity. This is validated in the full numerical calculations as the Skyrmion size is anti-proportional to m and γ_c , which directly control the radial propagation distance of the polaritons. On the other hand nonlinear interactions, which are a particular strength of our microcavity polariton platform, provide powerful knobs to control the Skyrmions formed as we will demonstrate further below.

Another important aspect to note is that for resonant excitation with a Gaussian-shaped beam (in real space) centered around $\mathbf{k} = 0$ Skyrmion textures are not found. In the resonantly driven case, the external field fixes the condensate phase, suppressing self-organization of the re-

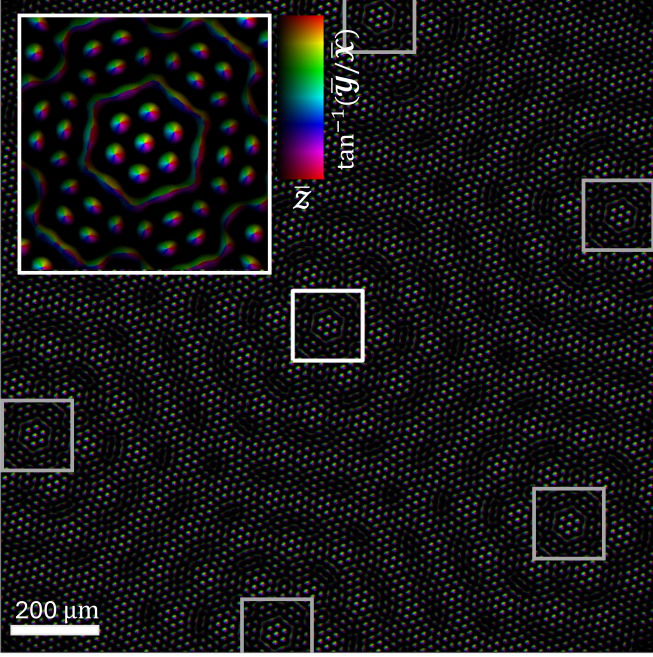


FIG. 3. **Spontaneously formed Skyrmion bags in polariton condensate in a moiré-shaped reservoir.** On the large two-dimensional spatial grid with $\sim 10^8$ grid points multiple Skyrmion bags are resolved as indicated by grey squares. The Skyrmion structure is shown via the displacement field $\mathcal{R} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})^T$ ($\mathcal{Z}$ encoded in brightness and $\text{atan}^{-1}(\mathcal{Y}/\mathcal{X})$ in color). The inset shows the central Skyrmion bag. Details of structured excitation reservoir in text.

quired phase gradients. The resulting steady state that is reached for cw coherent excitation in normal incidence inherits the Gaussian envelope of the pump, and the out-of-plane displacement component remains strictly positive, in turn yielding a trivial topology.

Based on the observed spontaneous formation of a Skyrmion for a single Gaussian-shaped nonresonant pump we now explore more complex spatial profiles of the nonresonant pump feeding the condensate. The nonresonant pump profile density is generated by superposition of cos-shaped waves with planar wave fronts with different spatial orientation (details given in SM Note S3). To prevent boundary effects in our finite computational domain, a very broad super-Gaussian envelope is applied to the pump profile. One possible boundary effect is that due to polariton flow at the envelope's edges, the outermost Skyrmions can be distorted, which is why we focus on the observations in the center of the spatial domain. The three-wave-interference results with nonresonant excitation are displayed in detail in SM Note S4. We note that also with nonresonant pumping they very closely resemble the results for resonant coherent excitation as displayed in Fig. 1(e).

For the nonresonant excitation we now turn to a more complex spatial profile of the pump for which phase-synchronization of the condensate – in a way such that

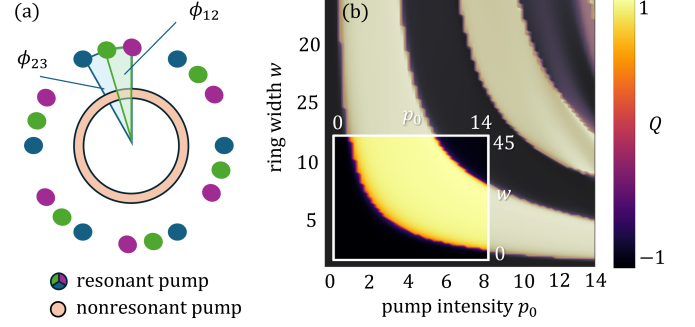


FIG. 4. **Nonresonant optical control of Skyrmion number.** (a) Optical excitation and switching setup: Skyrmion bags driven with resonant beams (dots) forming twisted hexagons (marked by color). A nonresonant ring pump (orange) creates a phase-shift to switch the Skyrmion numbers inside the ring. (b) For a twisting-angle $\phi_{12} = \phi_{23} = 19.4^\circ$ a single-Skyrmion bag is switched by varying ring-pump intensity p_0 and width w .

spatially extended Skyrmionic textures would form – is not trivially expected. Here it is important to emphasize that while the nonresonant pump may define an amplitude landscape, the condensate phase, phase gradients, and resulting Skyrmion topology emerge spontaneously in the driven-dissipative dynamics of the condensate from initial noise. To demonstrate this kind of self-organization on large scales and show the periodic nature of the system we impose the moiré grid structure of Fig. 1(f) on a very large spatial grid. The resulting spontaneously formed Skyrmion structure is displayed in Fig. 3 on a $1500 \times 1500 \mu\text{m}^2$ grid, showing the periodicity of the moiré structure with repetitions of the central Skyrmion bag also far away from the origin.

For completeness we note that similar Skyrmionic structures in the condensate can also form in non-optically induced, external potential lattice structure excited with a spatially homogeneous nonresonant pump (not shown). In that case careful parameter optimization is required. We further note that in the nonresonant excitation scenarios Skyrmion formation is more resilient against pumping imperfections. For resonant excitation, wave vector deviations larger than 1% can lead to destruction of the Skyrmion patterns. As no external phase control is required for nonresonant pumping, Skyrmion patterns are preserved for nonresonant pump amplitude deviations of up to $\sim 30\%$ and deviations in the cos argument for lattice generation of more than 5%.

Nonlinear Control of Skyrmions – While Skyrmions emerge already in the linear wave dynamics of the system, the intrinsic nonlinearities of the polariton condensate provide a powerful control knob over their topology and spatial structure. We now demonstrate two approaches illustrating the possibilities. The first lets us switch the topological Skyrmion charge in a controlled manner (Fig. 4). The second leads to global reorganization of the Skyrmion patterns in the nonlinear regime

(Fig. 5). We begin with resonant excitation inducing a moiré Skyrmion lattice (also see SM Note S2). For relative angles between hexagons of $\phi_{12} = \phi_{23} = 19.4^\circ$, a central bag containing a single Skyrmion forms in the center as shown in Fig. 5(a) below.

To actively control the topology of this central Skyrmion we now introduce an additional nonresonant ring-shaped pump surrounding the central region as sketched in Fig. 4(a). This ring does not directly interfere with the resonantly injected waves. Instead, it modifies their propagation before they meet or overlap at the center due to a reservoir-induced effective external potential. Depending on strength p_0 and width w of the reservoir ring, this induces a controllable phase delay for the propagating waves.

Figure 4(b) shows the resulting Skyrmion number in the (p_0, w) -plane for varying accumulated phase shift. In the bright area the Skyrmion number for $g_c = 0.6 \mu\text{eV}\mu\text{m}^2$ (consistent with the parameters used in the present paper) is shown. We display the extension for larger nonlinearity parameter $g_c = 6 \mu\text{eV}\mu\text{m}^2$ in the grayed-out area. We note that for $g_c = 0.6 \mu\text{eV}\mu\text{m}^2$ in the parameter regime belonging to the grayed-out area condensation inside the ring is observed, rendering the system dynamics much more complicated. In the (p_0, w) -plane, as the nonlinear phase delay increases, the topological charge of the central Skyrmion switches abruptly. The transitions occur when the accumulated phase shift

$$\varphi \approx \frac{g_r}{\hbar v_g} \int_{\text{ring}} n(\mathbf{r}) d\mathbf{r} \quad (5)$$

changes the interference condition in the center. Here, $v_g = \hbar k/m$ is the group velocity of the propagating polariton wave on the parabolic dispersion. These results demonstrate that optically induced changes (by nonresonant pumping) to the propagation outside the interference region enables controlled switching of the topological information contained in the central Skyrmion (without modifying the resonant pumping geometry itself).

We further note that for sufficiently strong ring-pump, the induced potential forms an optical trap, leading to multistability inside the ring (not shown). This regime where the nonlinear dynamics becomes more complex is beyond the scope of the present work but highlights the broader potential for nonlinear topological engineering in our microcavity polariton system.

Beyond discrete topological switching, Kerr-type nonlinearities directly reshape the Skyrmion lattice itself. For resonant excitation only, increasing the pump intensity enhances the effective intra-condensate interaction, interaction-induced blueshifts, and thereby re-normalizes the local wave vectors participating in the interference. The moiré patterns rely sensitively on precise wave vector control; even moderate nonlinear shifts can significantly alter the resulting Skyrmion topology. Because the interference condition depends on relative wave vectors k and interactions re-normalize the dispersion, effective shifts in k alter the moiré periodicity.

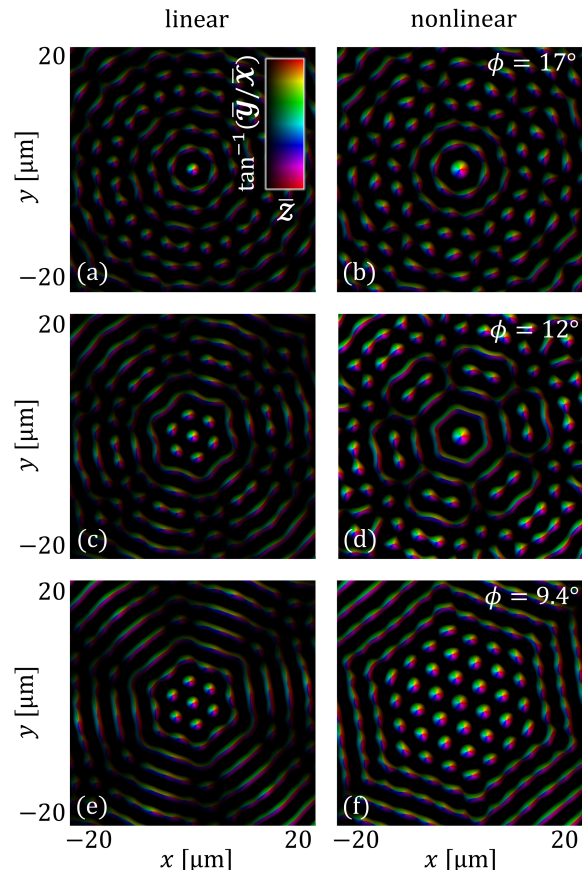


FIG. 5. **Skyrmion manipulation enabled by Kerr non-linearity.** Left: resonantly excited Skyrmion bags in moiré lattices induced by interference of 18 resonant beams (cf. sketch in Fig. 4(a)) with twist angles of (a) $\phi = 17^\circ$, (c) $\phi = 12^\circ$, and (e) $\phi = 9.4^\circ$ in the linear regime. Right: same as left but in the nonlinear regime with elevated beam intensities. Nonlinearity induces increased size of individual Skyrmions as visible in (b) and significant alterations to bag structures in (d) and (f).

Figure 5 compares several configurations with different relative twist angles in the linear and in the nonlinear regimes (coherent excitation). For $\phi = 17^\circ$, the single-Skyrmion bag expands substantially as the nonlinearity increases [cf. Figs. 5(a) and (b)], consistent with a reduction of the effective in-plane wave vector magnitude. More strikingly, configurations that appear nearly identical in the linear regime [cf. Figs. 5(c) and (e)] evolve into qualitatively distinct topological structures at elevated pump intensity [cf. Figs. 5(d) and (f)].

Conclusions – We have presented a theoretical framework for Skyrmion analysis, formation, and nonlinear control in scalar two-dimensional wave systems, with a focus on exciton-polariton condensates in planar semiconductor microcavities. Unlike previous Skyrmion realizations based on imposed interference geometries, here the topology can emerge spontaneously from non-Hermitian

condensate dynamics and can be reconfigured through optical nonlinearities, without the need to externally control the specific phase structure and without the need for resonant coherent optical excitation. In particular, we demonstrate controlled switching of topological charge and collective reconfiguration of Skyrmion moiré lattices by introducing additional spatially structured nonresonant optical pumping or by utilizing Kerr-type nonlinearities at elevated pumping intensities. We also show how the displacement fields in which these Skyrmionic topologies are observed can be linked to the physical quantities of intensity-gradient and phase-gradient optical forces.

Our findings combine linear interference-based Skyrmion generation with nonlinear topological engineering, establishing driven-dissipative polariton systems as a versatile and promising platform for studying collective Skyrmion states and their dynamical evolution. Future work may also explore the dynamical stability of spontaneously formed Skyrmions, the role of quantum fluctuations [41] and condensate lifetime, and the interplay of nonlinear non-Hermitian physics [42, 43] and Skyrmion topology. Our study also reveals direct implications for possible experimental realizations in planar photonic resonators, but also in plasmonic systems, and other two-dimensional (quantum) fluids, where the nonlinear control of topological excitations could enable new approaches to information processing and programmable photonics. While going beyond the scope of the present paper we briefly note that we

also find that saturable gain may lead to inversion of the Skyrmion structures observed at elevated densities, and multistability may provide additional intriguing control mechanisms for the Skyrmion topologies. Additionally, phase synchronization in specifically designed spontaneously formed Skyrmion lattices may permit controlled switching of Skyrmion phases and complex self-organization by inter-Skyrmion coupling.

Projecting further ahead, the introduced relation between force fields and Skyrmions may open up new possibilities to trap and manipulate quantum particles inside the Skyrmion force field with mutual coupling, which may motivate future studies into (lattices of) anyons with mixed statistics as topologically protected entities. Along the same lines also the optical force fields generated by other topological nonlinear excitations, such as vortices [15, 44, 45] with their winding phase-gradient forces, should be explored.

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Supporting Information:

Skyrmions in scalar fields of non-Hermitian optical microcavities: spontaneous formation, nonlinear control, and optical forces

S1. LIGHT FORCES AND DISPLACEMENT FIELD RELATIONS

The displacement field of the stationary-mode polariton field ψ_s is defined as

$$R = \begin{pmatrix} \partial_x \psi_s \\ \partial_y \psi_s \\ \psi_s \end{pmatrix} \quad (\text{S1})$$

and the polariton field is defined by

$$\psi_s(x, y) = A(x, y) e^{i\phi(x, y)}, \quad (\text{S2})$$

with an amplitude and phase distribution $A(x, y)$ and $\phi(x, y)$, as well as the polariton density $I(x, y) = |\psi_s(x, y)|^2 = A^2(x, y)$. Insertion into the displacement field provides

$$R = e^{i\phi} \begin{pmatrix} \partial_x A + iA\partial_x \phi \\ \partial_y A + iA\partial_y \phi \\ A \end{pmatrix} \quad (\text{S3})$$

and respectively

$$\psi_s^* R = A \begin{pmatrix} \partial_x A + iA\partial_x \phi \\ \partial_y A + iA\partial_y \phi \\ A \end{pmatrix}. \quad (\text{S4})$$

The in-plane displacement field R_2 , i.e. the (x, y) components, then reads

$$\psi_s^* R_2 = A \begin{pmatrix} \partial_x A + iA\partial_x \phi \\ \partial_y A + iA\partial_y \phi \end{pmatrix}. \quad (\text{S5})$$

Hence, we can write down the relation between the displacement field and the light forces:

The phase-gradient force is proportional to

$$F_\phi \propto I(x, y) \nabla \phi(x, y) = \Im \{ \psi_s^*(x, y) R_2(x, y) \} \quad (\text{S6})$$

and the intensity-gradient force is proportional to

$$F_I \propto \nabla I(x, y) = 2\Re \{ \psi_s^*(x, y) R_2(x, y) \}. \quad (\text{S7})$$

S2. RESONANT PUMP PROFILE DEFINITION

In this section we provide the mathematical description of the resonant pump profile used to investigate both three-wave interference and moiré lattices. The resonant pump profile is described by

$$E(\mathbf{r}, t) = E_0 e^{i\omega f(t)} \sum_{j=1}^N \exp \left[-\frac{|\mathbf{r} - \mathbf{r}_j|^2}{2\sigma^2} + i\mathbf{k}_j \cdot \mathbf{r} \right], \quad (\text{S8})$$

where

$$\mathbf{r} = (x, y), \quad \mathbf{r}_j = (x_0(j), y_0(j)), \quad \mathbf{k}_j = (k_x(j), k_y(j)).$$

E_0 , σ , and $\omega = \frac{\hbar|\mathbf{k}|^2}{2m}$ are the amplitude, width, and frequency of each resonant pump spot. $f(t)$ holds the temporal envelope of the pump. $\mathbf{r}_j$ and $\mathbf{k}_j$ hold the positions of each pump spot in direct and reciprocal space.

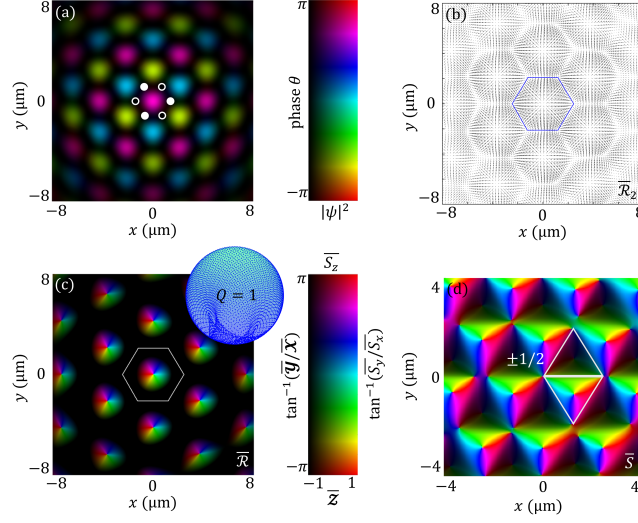


FIG. S1. **Skyrmion pattern under three-wave interference (resonant).** (a) Density (brightness) and phase (color) of the interference pattern. Vortex and anti-vortex pairs are highlighted by circles and dots. (b) In-plane displacement field, with Skyrmion cell marked in blue. (c) Full displacement field, with Skyrmion cell marked in white and unit-sphere mapping shown in the inset together with the Skyrmion number Q . (d) Spin density illustrating meron structure.

S3. NON-RESONANT PUMP PROFILE DEFINITION

In this section we provide the mathematical description of the nonresonant pump profile used to investigate spontaneous Skyrmion formation in both three-wave interference and moiré lattices in polariton systems. The nonresonant pump profile is described by

$$P(x, y) = \sum_{m=0}^M \sum_{n=0}^N 0.5[1 + \cos(k(x\cos(\alpha_{n,m})) + y\sin(\alpha_{n,m}))], \quad (\text{S9})$$

with $\alpha_{n,m} = \theta_n + m\phi$ and

$$\theta_n = \frac{2\pi n}{3} \quad M = 0, \quad N = 5.$$

for the three-wave interference setting and

$$\theta_n = \frac{n\pi}{3}, \quad M = 2, \quad N = 5, \quad \phi = 9.4^\circ.$$

for the moiré lattice setting.

S4. DETAILS ABOUT SKYRMION PATTERNS IN THREE-WAVE INTERFERENCE AND MOIRÉ LATTICES

Here we show additional details about the arising Skyrmion patterns in the two excitation scenarios discussed in the main text. Around the system center both resonant and nonresonant excitations lead to quantitatively identical results. Note, that as stated in the main part, nonresonant excitation allows for a more homogeneous density distribution, while the resonantly-driven interference patterns fade faster to the outside. In Fig. S1 and S2 the results for three-wave interference and moiré lattice formation under resonant excitation are shown. In Fig. S3 and S4 the results for three-wave interference and moiré lattice formation under nonresonant excitation are shown. The panels show the wavefunction information, the in-plane displacement field, the full displacement field and (in case of the three-wave interference) the spin density illustrating potential meron structures. These results are in very good agreement with the features measured in water and plasmonic systems as discussed in the main part of our work.

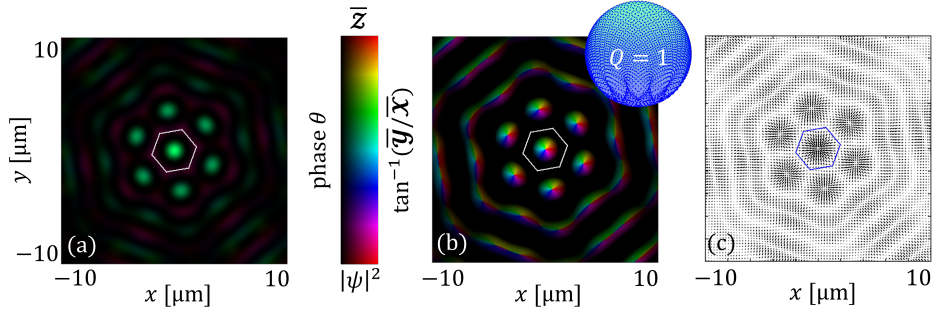


FIG. S2. **Skymion bag in moiré lattice (resonant).** (a) Density (brightness) and phase (color) of the interference pattern. (b) Full displacement field, with Skymion cell marked in white and unit-sphere mapping shown in the inset together with the Skymion number Q . (c) In-plane displacement field, with Skymion cell marked in blue.

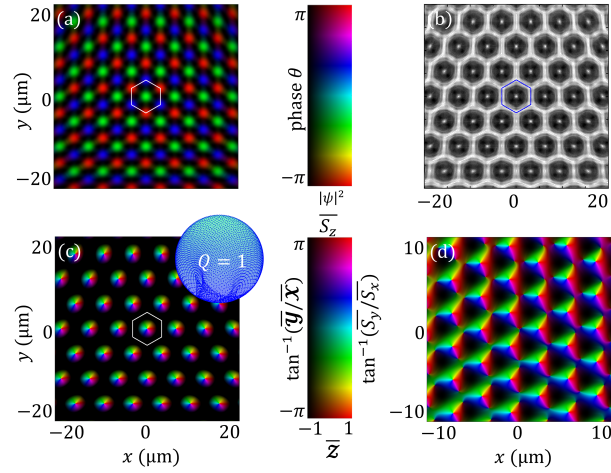


FIG. S3. **Skymion pattern under three-wave interference (nonresonant).** (a) Density (brightness) and phase (color) of the interference pattern. (b) In-plane displacement field, with Skymion cell marked in blue. (c) Full displacement field, with Skymion cell marked in white and unit-sphere mapping shown in the inset together with the Skymion number Q . (d) Spin density illustrating meron structure.

S5. OPTICAL FORCES IN SKYRMION MOIRÉ LATTICE

In this section the results for the optical forces for non-isolated Skyrmions is introduced. In Fig. S5 the optical forces on the Skymion moiré lattice are displayed. Notably the density-gradient force F_I forms traps in the Skymion center, while the phase-gradient force F_ϕ permits trapping between Skyrmions.

S6. SIMPLIFIED THEORETICAL MODEL FOR SINGLE SKYRMION CASE

Here we present a simplified theoretical description of an isolated Skymion under nonresonant pumping, with the aim of enlightening the fundamental physical mechanisms underlying the solution. The starting equations are those in eq. (1) of the main paper. For an incoherent pump P we may set $E = 0$. Furthermore, we focus on threshold conditions, where $|\psi|^2 \rightarrow 0$ can be assumed while retaining a finite field amplitude ψ . For a cylindrically symmetric pump $P(r)$ we further seek an eigenstate for which

$$\psi(r, t) = e^{-i\mu t/\hbar} U(r),$$

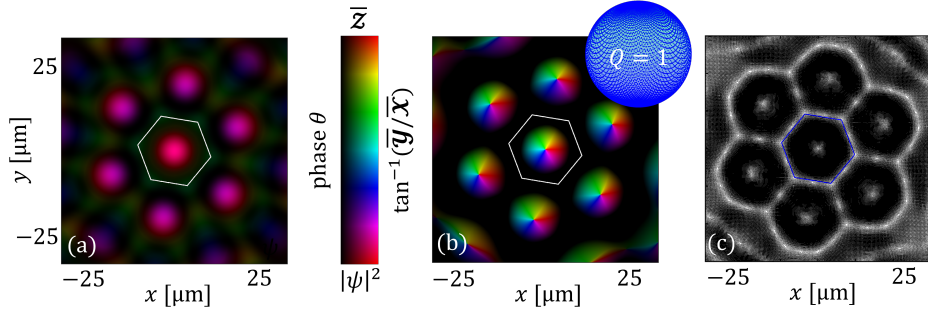


FIG. S4. **Skymion bag in moiré lattice (nonresonant).** (a) Density (brightness) and phase (color) of the interference pattern. (b) Full displacement field, with Skymion cell marked in white and unit-sphere mapping shown in the inset together with the Skymion number Q . (c) In-plane displacement field, with Skymion cell marked in blue.

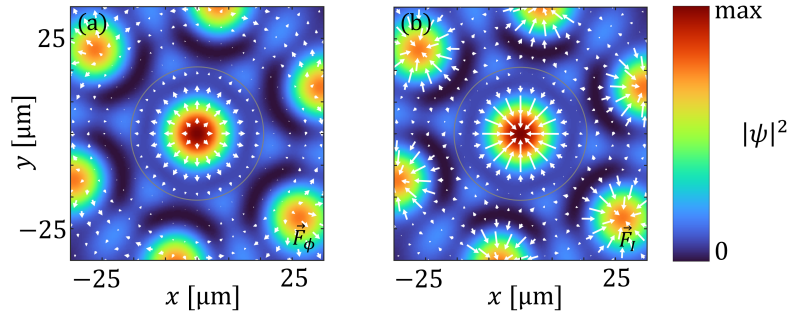


FIG. S5. **Optical forces in Skymion moiré lattice.** (a) Phase-gradient force and (b) Density-gradient force as vector fields in front of the density distribution (color) forming the Skymion moiré lattice. For visualization the inner bag is displayed.

with chemical potential μ . Under steady-state conditions, and neglecting nonlinear effects, the reservoir density is obtained as

$$n(r) = \frac{P(r)}{\gamma_r}.$$

The equation for the mode $U(r)$ then becomes

$$\mu U = \left(-\frac{\hbar^2}{2m} \nabla^2 + \frac{g_r}{\gamma_r} P(r) + \frac{i\hbar}{2} \left[\frac{R_c}{\gamma_r} P(r) - \gamma_c \right] \right) U. \quad (\text{S10})$$

This non-Hermitian Schrödinger equation (NHSE) can be used to determine the threshold mode $U(r)$. The underlying motivation is that the threshold solution may already reveal key features of the system that persist above threshold, where nonlinear interactions become significant. Skymionic structures are known to exist even in passive externally driven cavities. The objective here is therefore to examine the corresponding incoherently pumped system at threshold, before nonlinear effects dominate the dynamics. In this capacity we note that the chemical potential μ introduced above is complex to allow for loss and gain in the system. At threshold, the imaginary part of the complex chemical potential, $\mu = \mu' + i\mu''$, must vanish, ensuring an exact balance between gain and loss. Otherwise, the mode would either grow exponentially or decay in time.

To derive an approximate analytical solution of the NHSE, we consider a Gaussian pump profile

$$P(r) = P_0 e^{-r^2/r_p^2} \approx P_0 \left(1 - \frac{r^2}{r_p^2} + \dots \right),$$

where P_0 denotes the peak pump intensity and r_p is the characteristic pump radius. A Taylor expansion around $r = 0$ is employed to describe the mode behavior near the center of the pump spot. Using the Taylor expansion eq. (S10) can be approximated as

$$\mu U \approx \left(-\frac{\hbar^2}{2m} \nabla^2 + \frac{g_r P_0}{\gamma_r} \left(1 - \frac{r^2}{r_p^2} \right) + \frac{i\hbar}{2} \left[\frac{R_c P_0}{\gamma_r} \left(1 - \frac{r^2}{r_p^2} \right) - \gamma_c \right] \right) U. \quad (\text{S11})$$

A key observation is that the resulting equation can be mapped onto a quantum harmonic oscillator with a complex frequency, defined by

$$\frac{1}{2}m\omega_c^2 = -\frac{g_r P_0}{\gamma_r r_p^2} - \frac{i\hbar}{2} \frac{R_c P_0}{\gamma_r r_p^2}. \quad (\text{S12})$$

Then along with the definition

$$\mu^{(0)} = \frac{g_r P_0}{\gamma_r} + \frac{i\hbar}{2} \left[\frac{R_c P_0}{\gamma_r} - \gamma_c \right], \quad (\text{S13})$$

the NHSE can be written as

$$\mu U = \mu^{(0)} U - \underbrace{\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) U + \frac{1}{2} m \omega_c^2 r^2 U}_{\text{complex quantum harmonic oscillator with unnormalized Gaussian mode solution}}. \quad (\text{S14})$$

The underbraced term in this NHSE is the complex quantum harmonic oscillator with unnormalized Gaussian mode solution

$$U(r) = e^{-r^2/2r_c^2}, \quad r_c = \sqrt{\frac{\hbar}{m\omega_c}}, \quad (\text{S15})$$

and complex eigenvalue $\hbar\omega_c$. In this context,

$$\frac{1}{r_c^2} = \frac{m\omega_c}{\hbar} = \frac{m\omega'_c}{\hbar} + i \frac{m\omega''_c}{\hbar},$$

must be chosen with the root such that $\omega'_c > 0$ so that the Gaussian mode

$$\begin{aligned} U(r) &= e^{-r^2/2r_c^2} \\ &= e^{-\frac{m\omega'_c}{2\hbar} r^2 - i \frac{m\omega''_c}{2\hbar} r^2}, \end{aligned} \quad (\text{S16})$$

corresponds to a spatially confined mode. With this in mind the complex chemical potential for the model is given by

$$\mu = \mu^{(0)} + \hbar\omega_c,$$

and the threshold condition $\mu'' = 0$ can be satisfied for a chosen set of parameters by suitable choice of γ_c .

The central feature of the problem is its non-Hermitian character and the resulting modification of the mode structure. For the usual real quantum harmonic oscillator the modes can be chosen to have uniform phases, but here this is not possible even for the Gaussian mode, this is akin to the modes of so called gain guided semiconductor lasers where the oscillating mode has a phase curvature. The phase curvature associated with $\omega''_c \neq 0$ plays a crucial role in enabling the Skyrmionic structure discussed below.

Next, we analyze the components of the real-valued displacement field and relate them to the behavior shown in Fig. 2 of the main manuscript for the Gaussian solution derived above. First, the density profile in Fig. 2(a) will have the radial variation $\propto e^{-\frac{m\omega'_c}{\hbar} r^2}$. The components of the real valued displacement field (for $t = 0$) are:

- the x-component

$$\begin{aligned} \mathcal{X} &\propto \Re \left(\frac{\partial U}{\partial x} \right) \propto x e^{-\frac{m\omega'_c}{2\hbar} r^2} \left[\cos \left(\frac{m\omega''_c}{2\hbar} r^2 \right) + \frac{\omega''_c}{\omega'_c} \sin \left(\frac{m\omega''_c}{2\hbar} r^2 \right) \right] \\ &\propto x e^{-\frac{m\omega'_c}{2\hbar} r^2} \cos \left(\frac{m\omega''_c}{2\hbar} r^2 - \alpha \right), \end{aligned} \quad (\text{S17})$$

where $\alpha = \tan^{-1} \frac{\omega''_c}{\omega'_c}$,

- the y-component

$$\begin{aligned} \mathcal{Y} &\propto \Re \left(\frac{\partial U}{\partial y} \right) \propto y e^{-\frac{m\omega'_c}{2\hbar} r^2} \left[\cos \left(\frac{m\omega''_c}{2\hbar} r^2 \right) + \frac{\omega''_c}{\omega'_c} \sin \left(\frac{m\omega''_c}{2\hbar} r^2 \right) \right] \\ &\propto y e^{-\frac{m\omega'_c}{2\hbar} r^2} \cos \left(\frac{m\omega''_c}{2\hbar} r^2 - \alpha \right), \end{aligned} \quad (\text{S18})$$

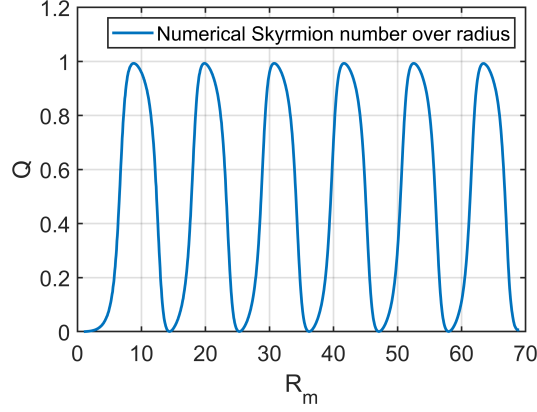


FIG. S6. Calculated Skyrmion number Q as a function of the Skyrmion cell radius R_m . Due to the normalization of the displacement field, the resulting topological charge oscillates between $Q = 0$ and $Q = 1$ depending on the selected integration radius.

- and the z-component

$$\mathcal{Z} \propto \Re(U) \propto e^{-\frac{m\omega'_c}{2\hbar}r^2} \cos\left(\frac{m\omega''_c}{2\hbar}r^2\right). \quad (\text{S19})$$

In each component, the trigonometric terms account for the radial sign reversals observed in the field distribution, as illustrated for $\mathcal{Z}$ in Fig. 2(c) of the main text. Similarly, the prefactors appearing in $\mathcal{X}$ and $\mathcal{Y}$ have the form of first-order Hermite-Gaussian modes along the x and y directions. This explains why $\mathcal{R}_2$ in Fig. 2(d) vanishes at the origin, while the cosine modulation accounts for the additional nodal ring structure. The Gaussian approximation should not be interpreted quantitatively for radii significantly larger than the pump radius r_p . This limitation arises from the parabolic approximation of the gain profile, which introduces nonphysical absorption at sufficiently large distances from the pump center.

The purpose of the simplified model is to demonstrate that the Skyrmionic character of the solutions is intimately connected to the distinctive modal properties of the NHSE. In particular, the analysis links the emergence of Skyrmionic features to the phase curvature of the non-Hermitian eigenmodes, a quantity that is, in principle, experimentally accessible. The model is not intended to provide quantitatively precise predictions, rather, its value lies in the physical insight it offers and in its potential to reveal how the modal properties scale with the relevant system parameters. Although the approach could be extended through a variational Gaussian treatment, such a refinement is unlikely to yield substantially deeper physical insight.

S7. SKYRMION NUMBER AS A FUNCTION OF SKYRMION CELL RADIUS

Here we discuss the scaling of the Skyrmion number with the radius of the circular integration domain in case of the single spontaneous Skyrmion formed in Fig. 2 of the main manuscript. To visualize the dependence of the Skyrmion number on the chosen integration domain size, Figure S6 shows the cell radius as a function of the resulting Skyrmion number. The Skyrmion number was calculated using the normalized displacement field. The computed value oscillates between $Q = 0$ and $Q = 1$ depending on the selected radius.

Importantly, the radius used for the Skyrmion number calculation is not chosen arbitrarily, nor is it adjusted iteratively until $Q = 1$ is obtained. Instead, the radius is determined from the first minimum of the displacement field component $\mathcal{Z}$. In practice, the extrema of the displacement field are used to define the boundaries of the Skyrmion cell, ensuring that the smallest physically meaningful cell is selected for the calculation.

Part IV

Final Remarks

Non-Hermitian physics has attracted increasing attention. In particular, exceptional points have been intensively studied for potential mode switching and sensing applications. While exceptional points in linear systems are well understood, their nonlinear counterparts require a broader framework: nonlinear eigenvalue problems can possess more solutions than the underlying linear mode dimension, and the relation between eigenstates, steady states, and dynamical response becomes nontrivial. This thesis investigated how nonlinearity reshapes the singularity structure of non-Hermitian systems. A central result of this thesis is that, for a broad class of nonlinear two-mode systems, the local organization of nonlinear exceptional points near a second-order linear exceptional point is governed by the D_4^- elliptic umbilic catastrophe [70], forming a cone-like structure with a deltoid cross-section. The three ribs formed by cusp points and the connecting surfaces of fold points are associated with third- and second-order nonlinear exceptional points, respectively. This result provides direct access to the complex and versatile structure of exceptional points in the nonlinear regime. Moreover, the derived topology and the solutions of the nonlinear eigenvalue problem apply to a large group of nonlinear systems such as optical resonators, electric circuits, Bose-Einstein and polariton condensates.

Beyond isolated exceptional points, higher-dimensional exceptional structures can render these singularities more accessible. A prominent example is exceptional rings, i.e., gapless rings of exceptional points. These are intensively investigated for their nontrivial topology. It is demonstrated how exceptional rings emerge naturally in planar optical resonators that feature TE-TM splitting and a certain form of circular dichroism [71].

Building on the unfolding topology derived before, a sophisticated analysis of responsivity and topology is performed on the quasi-torus of exceptional points that emerges from a single exceptional ring in the nonlinear regime according to the elliptic umbilic topology. We find that the deltoid cross-section of the elliptic umbilic carries finite vorticity, and the nonlinear exceptional rings found on the quasi-torus carry a finite first Chern number. With sensing applications in mind, it is found that the responsivity of nonlinear exceptional points depends on their relative position to other nonlinear exceptional points on the torus. While in the linear regime, the relations between eigenstates of the eigenvalue problem and steady states of the driven system are trivial, this is not the case in the nonlinear regime. To

confirm that the self-consistent nonlinear eigenvalue problem captures the driven system near resonance, the nonlinear eigenvectors and eigenvalues forming exceptional rings on the quasi-torus are validated by systematic resonance sweeps of a nonlinear polariton system [71].

The interplay of exceptional points and nonlinearity is investigated in a nonresonantly pumped polariton condensate. Here, the complex reservoir dynamics do not allow a self-consistent description of the polariton modes in a low-dimensional framework. Instead, saturable gain, spatial mode overlap, Kerr-type nonlinearity, and optically-induced potential landscapes are deeply intertwined and affect all crucial quantities of the exceptional point formation.

To analyze this regime, a toy model is employed to describe an exceptional point in the perturbative nonlinear regime. To validate the presence of an exceptional point, it is encircled, demonstrating the mode switching and accumulation of the geometric Berry phase characteristic for exceptional points. After re-localizing the exceptional point in the nonlinear regime with the toy model, the resulting mode structure is validated within the full nonlinear dynamics described by the Gross-Pitaevskii equations. It is demonstrated how nonlinear interactions scale the characteristic eigenvalue splitting close to an exceptional point, implying possible sensing applications. Moreover, the interplay of nonlinear effects leads to a spectral rotation of the exceptional point in parameter space, by tuning the polariton densities [66]. This could potentially permit all-optical mode switching by rotating the exceptional point in parameter space, rather than by externally encircling it.

This thesis also presents PHOENIX, an open-access solver for extended two-dimensional nonlinear Schrödinger equations. PHOENIX combines CPU and GPU optimization, selectable numerical precision, memory-efficient implementation, and integrated tools for post-processing. Benchmarks demonstrated speedups of up to three orders of magnitude and substantial energy savings compared with straightforward MATLAB implementations. Beyond its role in the simulations presented here, PHOENIX provides a reusable numerical framework for studying nonlinear and non-Hermitian dynamics [179].

As an application of PHOENIX, this thesis demonstrates the emergence of Skyrmions in the displacement field of scalar polariton condensates. The results show that Skyrmions in scalar fields can arise not only from engineered interference, as previously demonstrated in other platforms, but also arise spontaneously in a nonresonantly pumped polariton condensate due to the condensate's non-Hermitian dynamics. This establishes nonlinear

and non-Hermitian microcavities as a platform for studying optically controllable Skyrmion textures and their relation to optical force fields [203].

In conclusion, the results presented here and the associated publications provide a universal framework for studying nonlinear exceptional points across complementary physical regimes. Starting from a linear exceptional point, the complete solution structure and unfolding topology in the nonlinear regime are presented for coherent driving. For non-resonantly driving by a depletable reservoir, a toy model is presented to study nonlinear exceptional points in the perturbative nonlinear regime. These approaches reveal how non-linearity modifies exceptional-point topology, responsivity, mode structure, and dynamical accessibility.

Several directions follow naturally from these results. First, to deepen the understanding of the effects of the elliptic umbilic topology, complex knot trajectories should be evaluated in its vicinity to gain insight into its other topological invariants. This naturally leads to the development of a complete mode switching protocol combining the effects of chiral mode switching around exceptional points and mode hysteresis on the umbilic edges.

Second, building on the position-dependent responsivity and nonlinearity-dependent signal enhancement of nonlinear exceptional points that have been reported above, a complete description of noise-nonlinearity interactions needs to be developed to draw conclusions on the topology-dependent signal-to-noise ratio of sensor designs based on nonlinear exceptional points. Time-dependent nonlinear and non-Hermitian systems also remain an important open direction. In the strongly nonlinear regime, resonance peaks may tilt and enter hysteretic regimes, increasing the number of solutions and potentially giving rise to new topological structures. Stability analysis will therefore be essential for understanding the resulting topology and mode-switching dynamics.

Third, the unfolding topology of higher-order exceptional points should be investigated. It would also be natural to extend the nonlinear framework developed here to other non-Hermitian phenomena, such as bound states in the continuum or the non-Hermitian skin effect.

Finally, spontaneous Skyrmion formation in dissipative polariton condensates opens up new possibilities to study topological textures in nonlinear systems. Here, multistabilities may allow for optically-induced switching between Skyrmion states, and non-Hermiticity can be utilized to study Skyrmion-based exceptional points. The relation of the Skyrmions

reported above, and the optical force fields, could allow optical trapping of particles, potentially allowing the formation of new quasi-particles with mixed statistics as topologically protected entities.

To address the dynamics-related questions posed above, PHOENIX can be extended further. To make it more accessible to new users, a graphical user interface can be developed. More extensions and variations of prominent equations can be added, and non-CUDA GPU support can be implemented. More explicit implementation of common post-processing procedures, such as FFT-based algorithms, can provide additional tools for new PHOENIX users.

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