Qualitative analysis of some cross-diffusive evolution systems

Am Institut für Mathematik der Universität Paderborn unter der Betreuung durch Herrn Univ.-Prof. Dr. Michael Winkler erarbeitete und der Fakultät für Elektrotechnik, Informatik und Mathematik der Universität Paderborn im Februar 2017 vorgelegte Dissertation

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Zusammenfassung

In dieser Arbeit werden Anfangs-Randwertprobleme verschiedener kreuzdiffusiver Systeme von Evolutionsgleichungen aus dem Umfeld von Chemotaxis-Gleichungen hinsichtlich qualitativer Eigenschaften ihrer Lösungen untersucht. Insbesondere werden in einem parabolisch-elliptischen Chemotaxissystem mit logistischen Quelltermen kurzlebige Wachstumserscheinungen nachgewiesen, die Existenz globaler schwacher schließlich glatt werdender Lösungen eines parabolisch-parabolischen Chemotaxissystems mit logistischer Quelle und eines mit einem Navier-Stokes-Fluid gekoppelten solchen Systems wird gezeigt und ihr Langzeitverhalten wird untersucht; weiterhin wird die Beschränktheit der Lösungen eines Keller-Segel-artigen Systems mit logarithmischer Sensitivität bewiesen und die Existenz lokal beschränkter globaler Lösungen eines konsumptiven Chemotaxissystems mit logarithmischer Sensitivität und nichtlinearer Diffusion wird demonstriert.

Abstract

In this work, initial–boundary value problems of different cross-diffusive evolution systems from the context of chemotaxis equations are investigated with respect to qualitative properties of their solutions. In particular, transient growth phenomena are detected in a parabolic-elliptic chemotaxis system with logistic source terms, the existence of global weak solutions to a parabolic-parabolic chemotaxis system with logistic source and to such a system coupled with Navier-Stokes fluid which eventually become smooth is shown and their long term behaviour is described; furthermore, boundedness of solutions to a Keller-Segel type chemotaxis system with logarithmic sensitivity is proven and the existence of locally bounded solutions to a consumptive chemotaxis system with logarithmic sensitivity and nonlinear diffusion is demonstrated.

Danksagung

Eine Arbeit wie die vorliegende ist nicht nur das Ergebnis einer mehrjährigen Beschäftigung mit ihrem Thema, sondern darüber hinaus etwas, das auf vieles aufbaut, das vorher war. Ich möchte daher diese Gelegenheit nutzen, meiner Dankbarkeit Ausdruck zu verleihen.

Ich danke meiner Familie und meinen Freunden, ohne die ich nicht der wäre, der ich bin; vor allem meinen Eltern für alle Liebe und Unterstützung in den letzten knapp drei Jahrzehnten.

Was die Mathematik angeht, danke ich vor allem

denen, die mich lehrten,

beginnend mit der Schulzeit, in der ich u.a. sehr guten Mathematikunterricht genießen durfte (ich denke dabei besonders an B. Ritz, T. Neumark, K. Niemax, M. Winkler), betreffend aber vor allem auch viele Professoren, sowohl in Essen als auch in Paderborn als auch bei den Summer Schools in Jyväskylä, die mich an ihrem Wissen und ihren Sichtweisen auf die Mathematik teilhaben ließen. Besonders dankbar bin ich denen, die ihre Vorlesungen und Übungen auch für nur etwa ein oder zwei Teilnehmer abhielten. Erwähnen möchte ich hier außerdem Herrn Prof. Neff, der mich nicht nur während meiner Bachelor- und Masterarbeit betreut hat, sondern mich auch früh an Konferenzteilnahmen und das Schreiben mathematischer Artikel herangeführt hat.

denen, die von mir lernten,

für ihr Interesse und alle guten Fragen; dafür, mich motiviert und auch mir zu besserem Verständnis geholfen zu haben,

denen, die mit mir lernten,

vor allem Christian, Robert und Tobias, mit denen ich mir jeweils gerne das Büro geteilt und zahlreiche mathematische Gespräche über verschiedenste Themen geführt habe; and I thank the many colleagues with whom I also have had the pleasure to have fruitful discussions or to have collaborated on several topics (in some cases relatively unrelated to the present work); to name but very few in addition to those already mentioned: Xinru, Yulan, Yan, Masaaki, Giuseppe, Kentarou, Nikos, Evangelos, Sachiko, Dongmei, Bingran, Christian, Sebastian, Cristian, Tian, Marcel,...

I thank everyone working in the field who has made me feel welcome; my thanks in particular go to those who have invited me to a stay at their university (Shanghai), to workshops (Tokyo, Bedlewo, St. Petersburg) or conferences and to those who have made sure that I spent an enjoyable as well as productive time there.
I am grateful to D. Wrzosek and Y. Tao for agreeing to report on this thesis and to everyone serving on the thesis committee for their time and effort.

Ganz besonderer und herzlicher Dank gilt Michael Winkler, der nicht nur derjenige ist, der mich für partielle Differentialgleichungen und Analysis begeistert hat, sondern mich jederzeit motiviert, inspiriert und unterstützt hat, bei allen Fragen ein offenes Ohr und hilfreiche Denkanstöße hatte und mir wiederum interessante Fragen gestellt und mich bei ihrer Beantwortung begleitet und betreut hat, mir dabei aber auch die nötige Freiheit gelassen hat, selbst und in Ruhe daran zu arbeiten.
Zusätzlich zu einem großen Teil meiner mathematischen Kenntnisse und zahlreicher nützlicher Kommentare verdanke ich ihm viele der angedeuteten Kontakte und Reisemöglichkeiten und sicherlich nicht zuletzt, dass ich die letzten Jahre über gerne in Paderborn gearbeitet und mich dabei wohlgefühlt habe.

1 Introduction

An important tool for understanding the world around us – be it for either predicting the evolution of some quantity of interest or gaining insight into (the mechanisms of) some process – are evolution equations: partial differential equations linking the rate of change of some quantity with respect to time with a function of the quantity itself and its (spatial) derivatives.
The simplest and most classical example of such an equation is the heat equation $u_t = \Delta u$, which models the spread of heat or other diffusive processes.
More complicated instances may involve several interacting quantities, leading to systems of evolution equations that are intricately linked.
As soon as such systems are suggested as description, it becomes important to study them mathematically. Not only does the question arise whether there are solutions at all – which, of course, is an indispensable condition for the model to be used sensibly for anything –, but also the question about qualitative properties: Can anything be said about the solutions – even if they are not known explicitly?
Apart from mathematical understanding of the model, answers to this question can be used to predict the behaviour of the underlying process (if the description by that model is already known to be at least somewhat accurate) or to firstly evaluate whether the equations provide an appropriate description at all.
The systems to be considered in this work primarily originate from the area of mathematical biology, more precisely: the study of chemotaxis. Even some of the simplest lifeforms (whose density we will denote by u) are able to adjust their direction of movement to environmental conditions, i.e. to preferrably move toward higher concentrations of a chemical substance with concentration v – be it a nutrient or be it a signalling substance produced by themselves.
The evolution systems used to model this phenomenon approximately have the following form:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + f(u), \\ v_t = \Delta v + g(u,v), \end{cases} \tag{1.1}$$

and we will comment on the particular variants to be investigated (and in particular on the choices of f and g) in the corresponding chapters.
For now, let us focus on the term $-\nabla \cdot (u\nabla v) = -\nabla u \cdot \nabla v - u\Delta v$ in the first equation: In contrast to diffusion as modelled by the heat equation $u_t = \Delta u$, not only the second derivatives of the same component determine the rate of change, but also those of the second component. This effect is called cross-diffusion. (More precisely: weak cross-diffusion; in the case of strong cross-diffusion there would be an additional contribution of Δu in the second equation.)
Apart from making the analysis much more challenging, the probably most striking consequence the presence of such a cross-diffusive term can have is causing blow-up of solutions after finite time, which can be considered the desired behaviour when modelling the aggregation of *Dictyostelium discoideum*, as studied by Keller and Segel in the original model ((1.1) with $f(u) = 0$, $g(u, v) = u - v$), but may be less appropriate in other situations. There are several related models that arise quite naturally from biological motivation, e.g. from the desire to include population growth effects. In the simplest sensible form this leads to logistic source terms $f(u) = \kappa u - \mu u^2$, where the negative term possibly counteracts the aggregation effects of the cross-diffusion term

to some extent. It is not accidental that "to some extent" in the previous sentence is vague. The two mechanisms being of similar strength gives rise to a delicate, non-obvious interaction between them.

The main message of Chapter 2 is that even the behaviour of solutions to chemotaxis systems with logistic source terms which exist globally, are bounded, etc., is not entirely dominated by the logistic equation. We consider the (initial boundary value problem of the) parabolic-elliptic chemotaxis system

$$\begin{cases} u_t = \varepsilon\Delta u - \nabla\cdot(u\nabla v) + \kappa u - \mu u^2 \\ 0 = \Delta v - v + u, \end{cases}$$

and show that not only the carrying capacity $\frac{\kappa}{\mu}$, but every arbitrary bound can be surpassed if the diffusion is sufficiently weak. This is an extension of [117] to the radially symmetric higher-dimensional case.

Chapter 3 contains a first highlight: Without smallness conditions on the chemotactic strength, we prove existence of global weak solutions to a Keller-Segel type chemotaxis model with logistic source terms which (in the physically relevant case of space dimension 3) eventually become smooth, and consider their long-term behaviour.

Chapter 4 features a small result concerning the two-dimensional case of a chemotaxis system with singular sensitivity,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\left(\dfrac{u}{v}\nabla v\right) \\ v_t = \Delta v - v + u, \end{cases}$$

where it was known (and can easily be seen) that χ being smaller than 1 guarantees global existence and boundedness of solutions. We prove that 1 is not a critical number in this respect: There is some $\chi_0 > 1$ such that global existence and boundedness are also asserted whenever $\chi \in [1, \chi_0)$.

Of course, the effects of a singular sensitivity function become much more exciting when combined with a second equation which aims at forcing v to tend to zero. With such a model we will be dealing in Chapter 5: We derive a condition on the strength of nonlinear diffusion which is sufficient for solutions to

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot\left(\dfrac{u}{v}\nabla v\right) \\ v_t = \Delta v - uv \end{cases}$$

to (exist globally and) be locally (i.e. on every finite time-interval) bounded.

Systems with consumption of the chemoattractant in particular appear in the form of a chemotaxis model coupled with fluid, whose examination has spawned a multitude of mathematical works since ca. 2010, after experiments in 2006 suggested that in the study of chemotaxis the effects of surrounding fluid should not be neglected. Such a system – the three-dimensional case of a Navier-Stokes coupled chemotaxis system with logistic source terms – will be treated in Chapter 6. Again we prove existence of weak solutions that become eventually smooth, and show their convergence in the large-time limit.

For further details on these problems and the relevant literature, we refer the reader to the introductions of Chapters 2 to 6.

The material is arranged in such a way that the chapters can be read almost independently from each other. The amount of repetition this has caused is minuscule and seemed a small price to pay for better accessibility of the text.

1.1 Previous publications

Small changes notwithstanding, the chapters roughly coincide with the following publications. Quotations from or similarities with these articles will not be indicated separately:

Chapter 2:
[49]: J. Lankeit. Chemotaxis can prevent thresholds on population density. *Discrete Contin. Dyn. Syst. Ser. B*, 20(5):1499–1527, 2015.

Chapter 3:
[50]: J. Lankeit. Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source. *J. Differential Equations*, 258(4):1158–1191, 2015.

Chapter 4:
[52]: J. Lankeit. A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity. *Math. Methods Appl. Sci.*, 39(3):394–404, 2016.

Chapter 5:
[53]: J. Lankeit. Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion. *J. Differential Equations*, 262(7):4052–4084, 2017.

Chapter 6:
[51]: J. Lankeit. Long-term behaviour in a chemotaxis-fluid system with logistic source. *Math. Models Methods Appl. Sci.*, 26(11):2071–2109, 2016.

Contents

2 Chemotaxis can prevent thresholds on population density

2.1 Introduction

The Keller-Segel model

$$\begin{cases} u_t = \nabla \cdot (D_2(u,v)\nabla u) - \nabla \cdot (D_1(u,v)\nabla v) \\ v_t = D\Delta v - k(v)v + uf(v) \end{cases} \tag{KS}$$

of chemotaxis has been introduced by Keller and Segel in [43] to model the aggregation of bacteria (for instance, of the species *Dictyostelium discoideum*, with density denoted by u) in the presence of a signalling substance (the chemoattractant, with density v) they emit in case of food scarceness. Their movement is governed by random diffusion and chemotactically directed motion towards higher concentrations of the chemoattractant. The Keller-Segel model or variants thereof, as for example

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) \\ \tau v_t = \Delta v - v + u \end{cases}$$

where all coefficient functions appearing in (KS) have a simple form and diffusion of the signalling substance is assumed to occur fast (instantaneously if $\tau = 0$), have been widely used and incorporated in more complicated models in the mathematical depiction of biological phenomena, ranging from pattern formation in *E. coli* colonies [7] to angiogenesis in early stages of cancer [92] or HIV-infections [89]. For a survey of the extensive mathematical literature on the subject see the survey articles [32] or [34, 35].

Often the occurrence of the desired structure formation is identified with the blow-up of solutions to the model in finite time, i.e. the existence of some finite time $T > 0$ such that $\limsup_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty} = \infty$; and the model – both for $\tau = 0$ and $\tau > 0$ – is known to admit such solutions for every sufficiently large initial mass or in space dimensions larger than two, whereas in dimension 2 for small initial mass all solutions exist globally in time and are bounded [39, 30, 70, 73]. Moreover, blow-up of solutions has been shown to be a generic phenomenon of the equation in some sense even for the parabolic-parabolic version of the system, in dimensions $N \geq 3$ as well as for large enough initial mass in the two-dimensional case [69, 115].

Another point of view is that blow-up is "too much" and biologically inadequate, at least in some situations. Then, for example, terms preventing blow-up are added, e.g. some logistic growth term (cf. for example the tumor models in [6] or [90]), so that the model reads

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) + \kappa u - \mu u^2 \\ \tau v_t = \Delta v - v + u. \end{cases}$$

For this problem it is known [100, 78, 111] that classical solutions exist globally in time and are bounded if $N \leq 2$ or μ is large (where for $\tau = 0$, an explicit condition sufficient for this is $\mu > \frac{N-2}{N}$). For higher dimensions and small μ the existence or non-existence of exploding

solutions is unknown. As [112] seems to indicate, superlinear absorption does not necessarily imply global existence.
The important question is: To what extent does the logistic term render the chemotaxis-term innocuous? Does there still emerge some structure? Recently this question has been answered affirmatively by Winkler [117] in the one-dimensional case: If the death rate μ is small enough ($0 < \mu < 1$), then there is some criterion on (the L^p-norm with $p > \frac{1}{1-\mu}$ of) the initial data that ensures the existence of some time up to which any threshold of the population density will be surpassed - as long as the bacteria do not diffuse too fast. Of course, the biologically relevant situation is not that of only one space-dimension. With the present chapter we give an answer to the question whether this phenomenon is restricted to this case or if it also occurs in higher dimensions.
We shall confine ourselves to the prototypical radially symmetric setting and in the end obtain

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a ball. Let $\kappa \geq 0, \mu \in (0,1)$. Then for all $p > \frac{1}{1-\mu}$ there exists $C(p) > 0$ satisfying the following: Whenever $u_0 \in C^1(\overline{\Omega})$ is nonnegative, radially symmetric and satisfies $\partial_\nu u_0\big|_{\partial\Omega} = 0$ and such that*

$$\|u_0\|_{L^p(\Omega)} > C(p) \max\left\{\frac{1}{|\Omega|}\int_\Omega u_0, \frac{\kappa}{\mu}\right\},$$

there is $T > 0$ such that to each $M > 0$ there corresponds some $\varepsilon_0(M) > 0$ with the property that for any $\varepsilon \in (0, \varepsilon_0(M))$ one can find $t_\varepsilon \in (0,T)$ and $x_\varepsilon \in \Omega$ such that the solution (u,v) of

$$\begin{cases} u_t = \varepsilon\Delta u - \nabla\cdot(u\nabla v) + \kappa u - \mu u^2 \\ 0 = \Delta v - v + u \\ \partial_\nu v\big|_{\partial\Omega} = \partial_\nu u\big|_{\partial\Omega} = 0, \qquad u(0,\cdot) = u_0, \end{cases} \tag{2.1}$$

in $\Omega \times (0, T_{max})$, where $T_{max} \in (0,\infty]$ is its maximal time of existence, satisfies $u(x_\varepsilon, t_\varepsilon) > M$.

For this purpose we set out to find estimates finally leading to the crucial extensibility criterion (2.27) for solutions of the "$\varepsilon = 0$-limit" model

$$\begin{cases} u_t = -\nabla\cdot(u\nabla v) + \kappa u - \mu u^2 \\ 0 = \Delta v - v + u \\ \partial_\nu v\big|_{\partial\Omega} = \partial_\nu u\big|_{\partial\Omega} = 0, \qquad u(0,\cdot) = u_0 \end{cases} \tag{2.2}$$

of (2.1) in $\Omega\times(0,T)$ for some $T > 0$. The extensibility criterion is analogous to (1.6) of [117], that in turn is built upon estimates, some of which heavily rely on one-dimensionality of the problem. Cornerstone of our analysis therefore will be Section 2.3.5, where we craft the inequality which also allows for higher-dimensional and therefore more realistic scenarios.
We will introduce our concept of solutions of (2.2) in Definition 2.4.1 and show their uniqueness – if $u_0 \in W^{1,q}(\Omega)$ for some $q > N$ – in Theorem 2.4.3 and the existence of radially symmetric solutions that can be approximated by solutions of (2.1) in Theorem 2.4.7.
In contrast to the one-dimensional case, we are confronted with the challenge that we cannot, in general, rely on the existence of global classical bounded solutions to the approximate problem. Hence we prepare these results by finding a common existence time of such solutions – regardless of the value of ε (Theorem 2.3.14).
After collecting some additional boundedness property in Lemma 2.3.15, by a limiting procedure (Lemma 2.4.5) we can turn to solutions to (2.2).

If then μ is large enough, a global, in some cases even bounded solution is guaranteed to exist [Prop. 2.4.10]. However, if this is not the case, any radial solution to (2.2) with somehow (L^p-)large enough initial mass blows up in finite time (Theorem 2.4.13). In combination with the fact that solutions to (2.2) can be obtained as limits of solutions to (2.1), this yields the announced theorem about nonexistence of thresholds to population density: If $\mu < 1$ and $\|u_0\|_{L^p}$ (for $p > \frac{1}{1-\mu}$) is large enough, before some time T any threshold on the population density will be exceeded at least at one point by any population that diffuses slowly enough.
After the following short section which recalls a few basic properties of solutions to the second equation in (2.1) and equation (2.1) with $\varepsilon > 0$, in Section 2.3 we focus our attention on existence of solutions to (2.1) and estimates yielding a common existence time (Theorem 2.3.14) as well as preparing for compactness arguments (by the estimates of Lemma 2.3.15 and Corollary 2.3.12). Section 2.4 will be devoted to definition, uniqueness, existence, estimates and a blow-up result for solutions to (2.2), followed in Section 2.5 by, finally, the proof of the "no threshhold" theorem 2.1.1.
Throughout the chapter, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Often we will speak of radially symmetric (or, for short, radial) functions. In this case $\Omega = B_R(0)$ is to be understood to be a ball centered in the origin and we will interchange $u(x,t), x \in \Omega$, and $u(r,t), r = |x| \in [0,R)$.

2.2 Preliminaries: The elliptic equation

In the proofs we will mainly be concerned with u, therefore it would be desirable to estimate various terms involving the solution v of

$$-\Delta v + v = u \quad \text{in } \Omega, \qquad \partial_\nu v\big|_{\partial\Omega} = 0 \tag{2.3}$$

or its derivatives in terms of u. The following lemmata will be the tools to make this possible:

Lemma 2.2.1. *Let v solve (2.3) for a nonnegative right-hand side $u \in C^0(\overline{\Omega})$. Then also v is nonnegative.*

Proof. This is a consequence of the elliptic maximum principle. □

Lemma 2.2.2. *Let v solve (2.3) for some $u \in C^0(\overline{\Omega})$. Then for all $p \in [1,\infty]$,*

$$\|v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)}$$

Proof. As in [117, Lemma 2.1], for $p \in (1,\infty)$, testing the equation by $v(v^2+\eta)^{\frac{p}{2}-1}$ as $0 < \eta \to 0$ yields this estimate, which can then be extended to $p \in \{1,\infty\}$ by limiting procedures. □

Estimates involving the $L^1(\Omega)$-norm of u can be obtained as well:

Lemma 2.2.3. *For all $p > 0$, $\eta > 0$ there exists $C(\eta,p) > 0$ such that whenever $u \in C^0(\overline{\Omega})$ is nonnegative, the solution v of (2.3) satisfies*

$$\int_\Omega v^{p+1} \le \eta \int_\Omega u^{p+1} + C(\eta,p)\left(\int_\Omega u\right)^{p+1}.$$

Proof. (as in [117, Lemma 2.2]) Multiplication of (2.3) by v^p, integration over Ω and use of Young's inequality result in

$$p\int_\Omega v^{p-1}|\nabla v|^2 + \int_\Omega v^{p+1} = \int_\Omega uv^p \le \frac{1}{p+1}\int_\Omega u^{p+1} + \frac{p}{p+1}\int_\Omega v^{p+1},$$

and therefore

$$p\int_\Omega v^{p-1}|\nabla v|^2 \le \frac{1}{p+1}\int_\Omega u^{p+1}.$$

Because $\nabla(v^{\frac{p+1}{2}}) = \frac{p+1}{2}v^{\frac{p-1}{2}}\nabla v$ and $v^{p-1}|\nabla v|^2 = \frac{4}{(p+1)^2}|\nabla(v^{\frac{p+1}{2}})|^2$, we obtain

$$\frac{4p}{p+1}\int_\Omega |\nabla(v^{\frac{p+1}{2}})|^2 \le \int_\Omega u^{p+1}.$$

As now

$$W^{1,2}(\Omega) \overset{cpt}{\hookrightarrow} L^2(\Omega) \hookrightarrow L^{\frac{2}{p+1}}(\Omega),$$

by Ehrling's lemma there is $\widetilde{c}_1 = \widetilde{c}_1(\eta, p) > 0$ (and hence $c_1 > 0$) such that for all $\phi \in W^{1,2}(\Omega)$

$$\|\phi\|_{L^2}^2 \le \frac{4p}{p+1}\eta\,\|\phi\|_{W^{1,2}(\Omega)}^2 + \widetilde{c}_1\,\|\phi\|_{L^{\frac{2}{p+1}}(\Omega)}^2 \le \frac{4p}{p+1}\eta\,\|\nabla\phi\|_{L^2(\Omega)}^2 + c_1\,\|\phi\|_{L^{\frac{2}{p+1}}(\Omega)}^2.$$

Applying these two inequalities to $\phi = v^{\frac{p+1}{2}}$ and using Lemma 2.2.2 for $p = 1$, we arrive at

$$\int_\Omega v^{p+1} \le \frac{4p}{p+1}\eta\int_\Omega |\nabla(v^{\frac{p+1}{2}})|^2 + c_1\left(\int_\Omega v\right)^{p+1} \le \eta\int_\Omega u^{p+1} + c_1\left(\int_\Omega u\right)^{p+1}. \qquad \square$$

We also recall useful facts on maximal regularity for elliptic PDEs:

Lemma 2.2.4. *For $q \ge 1, \gamma > 0$, there is a constant $C > 0$ such that any (classical) solution v of (2.3) satisfies $\|v\|_{W^{2,q}(\Omega)} \le C\,\|u\|_{L^q(\Omega)}$ and $\|v\|_{C^{2+\gamma}(\Omega)} \le C\,\|u\|_{C^\gamma(\Omega)}$.*

Proof. [23, ch. 19] [28, Thm. 6.30] □

2.3 Parabolic-elliptic case

2.3.1 Existence

We prepare the following two lemmata with this estimate from [110, Lemma 1.3 iv)] about the (Neumann) heat semigroup:

Lemma 2.3.1. *Let $1 < q \le p < \infty$ or $1 < q < p = \infty$. Then there exists $C > 0$ such that for all $t > 0$ and for all $\varphi \in L^q(\Omega;\mathbb{R}^N)$*

$$\left\|e^{t\Delta}\nabla\cdot\varphi\right\|_{L^p(\Omega)} \le C(1 + t^{-\frac12-\frac N2(\frac1q-\frac1p)})\,\|\varphi\|_{L^q(\Omega)}.$$

Proof. [110, Lemma 1.3 iv)]. Although the lemma in that article is stated only for $p < \infty$, the proof actually already covers the case $p = \infty$, because $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$. □

One of the first steps in dealing with solutions of (2.1) is to show that they exist, at least locally. Let us briefly give the corresponding fixed point arguments.

Lemma 2.3.2. *Let $u_0 \in C^0(\overline{\Omega}), \varepsilon > 0, \kappa \ge 0, \mu > 0$. Then there is $T_{max} \in (0,\infty]$ such that (2.1) has a unique classical solution $(u, v) \in \left(C^{2,1}(\overline{\Omega}\times(0,T_{max})) \cap C^0(\overline{\Omega}\times[0,T_{max}))\right) \times C^{2,0}(\overline{\Omega}\times(0,T_{max}))$ and*

$$\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty \text{ or } T_{max} = \infty.$$

Proof. For $\widehat{u} \in C^0(\overline{\Omega})$, by $\widehat{v}_{\widehat{u}}$ we denote the solution $\widehat{v}$ of

$$0 = \Delta\widehat{v} + \widehat{u} - \widehat{v} \quad \text{in } \Omega, \qquad\qquad \partial_\nu \widehat{v}\big|_{\partial\Omega} = 0.$$

Assuming $R > 2\|u_0\|_{W^{1,q}(\Omega)}$ to be given, we fix constants C_1 as in Lemma 2.3.1, C_2 and the function $C\colon [0,1] \to \mathbb{R}$ such that

$$\|\nabla\widehat{v}_{\widehat{u}}\|_{L^q(\Omega)} \le C_2 \|\widehat{u}\|_{L^\infty(\Omega)}, \qquad C(t) = C_1 \int_0^t \big(1 + (\varepsilon(t-s))^{-\frac{1}{2}-\frac{N}{2q}}\big)\mathrm{d}s, t \in (0,1),$$

and note that C is monotone and continuous with $C(0) = 0$. We moreover choose $T \in (0,1)$ such that

$$(\kappa + 2\mu R)T + 2RC_2C(T) < \frac{1}{2}.$$

For $t \in [0,T]$ we then define

$$\begin{aligned}\Phi(u)(\cdot,t) :&= \mathrm{e}^{\varepsilon t\Delta}u_0 - \int_0^t \mathrm{e}^{\varepsilon(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla\widehat{v}_{u(\cdot,s)})\mathrm{d}s + \kappa\int_0^t \mathrm{e}^{\varepsilon(t-s)\Delta}u(\cdot,s)\mathrm{d}s \\ &\quad - \mu\int_0^t \mathrm{e}^{\varepsilon(t-s)\Delta}u^2(\cdot,s)\mathrm{d}s.\end{aligned}$$

Then $\Phi\colon C^0(\overline{\Omega}\times[0,T]) \to C^0(\overline{\Omega}\times[0,T])$ is well-defined and, in fact, even $\Phi(u) \in C^\infty(\Omega\times(0,T))$. In addition, Φ is a contraction in $M := \Big\{f \in C^0(\overline{\Omega}\times[0,T]); \|f\|_{L^\infty(\Omega\times(0,T))} \le R\Big\}$, as can be seen from the fact that

$$\begin{aligned}&\|\Phi(u) - \Phi(\widetilde{u})\|_{L^\infty(\Omega\times(0,T))} \\ &\le \sup_{0<t<T}\int_0^t \Big\|\mathrm{e}^{\varepsilon(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla\widehat{v}_{u(\cdot,s)} - \widetilde{u}(\cdot,s)\nabla\widehat{v}_{\widetilde{u}(\cdot,s)})\Big\|_{L^\infty(\Omega)}\mathrm{d}s \\ &\quad + \sup_{0<t<T}\int_0^t \Big\|\mathrm{e}^{\varepsilon(t-s)\Delta}(\kappa(u(\cdot,s) - \widetilde{u}(\cdot,s)) + \mu(u^2(\cdot,s) - \widetilde{u}^2(\cdot,s)))\Big\|_{L^\infty(\Omega)}\mathrm{d}s \\ &\le C(T)(\|u-\widetilde{u}\|_{L^\infty(\Omega\times(0,T))}C_2R + RC_2\|u-\widetilde{u}\|_{L^\infty(\Omega\times(0,T))}) + T(\kappa+2\mu R)\|u-\widetilde{u}\|_{L^\infty(\Omega\times(0,T))} \\ &\le \frac{1}{2}\|u-\widetilde{u}\|_{L^\infty(\Omega\times(0,T))}\end{aligned}$$

for $u, \widetilde{u} \in M$. Furthermore Φ maps M to M as well: If $u \in M$, then

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(\Omega\times(0,T))} &\le \|\Phi(u) - \Phi(0)\|_{L^\infty(\Omega\times(0,T))} + \|\Phi(0)\|_{L^\infty(\Omega\times(0,T))} \\ &\le \frac{1}{2}\|u\|_{L^\infty(\Omega\times(0,T))} + \|u_0\|_{L^\infty(\Omega)} \le R.\end{aligned}$$

With the aid of Banach's fixed point theorem, this procedure yields a unique solution of $\Phi(u) = u$, which can be seen to produce a solution (u,v) of (2.1) on $\Omega\times(0,T)$ if we let $v(\cdot,t) := \widehat{v}_{u(\cdot,t)}$ for $t \in (0,T)$. Successively employing the same reasoning on later time intervals (then with different u_0 and possibly larger R) the existence of a solution on a maximal time interval $(0,T_{max})$ is obtained where either $T_{max} = \infty$ or $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. □

2.3.2 L^p-bounds and global existence

Bounds on L^p-norms are of great utility, not only for the deduction of global existence. A standard testing procedure (see also [117]) yields

Lemma 2.3.3. *Let $\kappa \geq 0$, $\mu > 0$, $u_0 \in C^0(\overline{\Omega})$ be nonnegative. Let (u,v) solve* (2.1) *classically in $\Omega \times (0,T)$ for some $T > 0, \varepsilon > 0$. Then for $p \geq 1$ and on the whole time interval $(0,T)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p(p-1)\varepsilon \int_\Omega u^{p-2}|\nabla u|^2 \leq p\kappa \int_\Omega u^p - (1-p+\mu p) \int_\Omega u^{p+1}. \tag{2.4}$$

Proof. Multiplication of the first equation of (2.1) by u^{p-1} and integration by parts yield

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + (p-1)\varepsilon \int_\Omega u^{p-2}|\nabla u|^2 = (p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v + \kappa \int_\Omega u^p - \mu \int_\Omega u^{p+1}$$

on $(0,T)$. Another integration by parts in combination with the second equation of (2.1) and Lemma 2.2.1 show

$$(p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v = -\frac{p-1}{p}\int_\Omega u^p \Delta v = \frac{p-1}{p}\int_\Omega u^p(u-v) \leq \frac{p-1}{p}\int_\Omega u^{p+1}$$

on $(0,T)$, which gives formula (2.4). □

This estimate directly leads to the following bound on L^p-norms of u.

Corollary 2.3.4. *Let $\kappa \geq 0, \mu > 0, u_0 \in C^0(\overline{\Omega})$ nonnegative, suppose (u,v) is classical solution of* (2.1) *in $\Omega \times (0,T)$ for some $T > 0, \varepsilon > 0$. Let $p \in [1, \frac{1}{(1-\mu)_+})$.*
Then for all $t \in [0,T)$,

$$\int_\Omega u^p(\cdot,t) \leq \max\left\{\int_\Omega u_0^p, \left(\frac{p\kappa}{1-(1-\mu)p}\right)^p |\Omega|\right\}.$$

Proof. An application of Hölder's inequality gives $\int_\Omega u^{p+1} \geq |\Omega|^{-\frac{1}{p}}(\int_\Omega u^p)^{\frac{p+1}{p}}$ and transforms (2.4) into the differential inequality

$$y'(t) \leq p\kappa y(t) - (1-p+\mu p)|\Omega|^{-\frac{1}{p}}(y(t))^{1+\frac{1}{p}}), \quad t \in (0,T),$$

for $y = \int_\Omega u^p$. An ODE-comparison then yields the result. □

Corollary 2.3.5. *Let $\kappa \geq 0$, $q > N$, $u_0 \in C^0(\overline{\Omega})$. If $\mu > \frac{N-2}{N}$, the solutions of* (2.1) *are global.*

Proof. This arises from the bounds in Corollary 2.3.4 by arguments that can be found in Lemmata 2.3 and 2.4 of [100]. □

If even $\mu \geq 1$, bounds can be given in a more explicit form and independently of ε.

Lemma 2.3.6. *Let $\kappa \geq 0$, $\mu \geq 1$, $u_0 \in C^0(\overline{\Omega})$, $u_0 \geq 0$, $u_0 \not\equiv 0$, and let $(u_\varepsilon, v_\varepsilon)$ be a classical solution of* (2.1) *in $\Omega \times (0,\infty)$ for $\varepsilon > 0$. Then, for all $t > 0$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \begin{cases} \frac{\kappa}{\mu-1}(1+(\frac{\kappa}{(\mu-1)\|u_0\|_{L^\infty(\Omega)}}-1)\mathrm{e}^{-\kappa t})^{-1} & \kappa > 0, \mu > 1, \\ \frac{\|u_0\|_{L^\infty(\Omega)}}{1+(\mu-1)\|u_0\|_{L^\infty(\Omega)}t} & \kappa = 0, \mu > 1, \\ \|u_0\|_{L^\infty(\Omega)}\,\mathrm{e}^{\kappa t} & \kappa > 0, \mu = 1, \\ \|u_0\|_{L^\infty(\Omega)} & \kappa = 0, \mu = 1. \end{cases}$$

Proof. (Cf. [117, Lemma 4.6].) This can be obtained by comparison with the solution y of

$$y'(t) = \kappa y(t) - (\mu-1)y^2(t), \;\; t > 0, \qquad y(0) = \|u_0\|_{L^\infty(\Omega)}.$$ □

2.3.3 Radial solutions

In the following sections we will restrict ourselves to the prototypical radially symmetric situation. In this case, the first two equations in (2.1) can be rewritten in the form

$$\begin{cases} u_t = \varepsilon u_{rr} + \varepsilon \frac{N-1}{r} u_r - u_r v_r - u v_{rr} - \frac{N-1}{r} u v_r + \kappa u - \mu u^2 & (2.5a) \\ 0 = v_{rr} + \frac{N-1}{r} v_r - v + u, & (2.5b) \end{cases}$$

if – as announced earlier – we perform the common "abuse of notation", writing $u(r,t)$, $r \in [0,R)$, instead of $u(x,t)$, $|x| = r$. We begin by preparing an inequality for the derivative of v. Gained from the radial symmetry, it will be one of the most important tools for the calculations preparing the estimation of $\|\nabla u\|_{L^q(\Omega)}$ in terms of $\|u\|_{L^\infty(\Omega)}$.

Lemma 2.3.7. *Let (u,v) be a radially symmetric nonnegative classical solution of* (2.1) *in $\Omega \times (0,T)$ for some $T > 0$ and $\Omega = B_R$. Then for $r \in [0,R], t \in (0,T)$,*

$$v_r(r,t) \le \frac{1}{N} r \left\| u(\cdot,t) \right\|_{L^\infty(\Omega)}. \tag{2.6}$$

Proof. We fix $t > 0$. Equation (2.5b) can also be written in the form $\frac{1}{r^{N-1}}(r^{N-1}v_r)_r = v - u$ and implies

$$(r^{N-1}v_r)_r = r^{N-1}(v-u),$$

hence

$$\begin{aligned} r^{N-1}v_r(r,t) &= 0 + \int_0^r \rho^{N-1}(v(\rho,t) - u(\rho,t))\mathrm{d}\rho \le \left\| u(\cdot,t) \right\|_{L^\infty(\Omega)} \int_0^r \rho^{N-1}\mathrm{d}\rho \\ &= \frac{1}{N} r^N \left\| u(\cdot,t) \right\|_{L^\infty(\Omega)}, \end{aligned}$$

which leads to (2.6). □

2.3.4 Compatibility

We say that a function u_0 satisfies the compatibility criterion (or, for short, that u_0 is compatible) if $u_0 \in C^1(\overline{\Omega})$ and $\partial_\nu u_0\big|_{\partial\Omega} = 0$. If functions with this property are used as initial data in parabolic problems, the solutions they yield have bounded first (spatial) derivatives on a time interval containing 0 ([48]). This will be important in the derivation of the crucial estimate of $\int_\Omega |\nabla u|^q$ for solutions u of (2.1) (Corollary 2.3.12) in terms of $\|\nabla u_0\|_{L^q}$ instead of only $\|\nabla u(\cdot,\tau)\|_{L^q}$ for arbitrary small $\tau > 0$.
At first we show that any function $u_0 \in W^{1,q}(\Omega)$ can be approximated by compatible functions preserving all kind of "nice properties":

Lemma 2.3.8. *Let $q > N$, $u_0 \in W^{1,q}(\Omega)$ be radially symmetric and nonnegative, let $\varepsilon > 0$. There is $\widetilde{u}_0 \in C^1(\overline{\Omega})$ with $\partial_\nu \widetilde{u}_0\big|_{\partial\Omega} = 0$ such that $\|u_0 - \widetilde{u}_0\|_{W^{1,q}(\Omega)} < \varepsilon$ and also $\widetilde{u}_0$ is radial and nonnegative.*

Proof. Given $\varepsilon > 0$ we consider the standard mollifications $\eta^{\widetilde{\varepsilon}} * \widehat{u}_0$ (cf. [22, C.5]) of

$$\widehat{u}_0 := u_0 \mathbb{1}_{B_{R-\frac{\delta}{2}}} + u_0\left(R - \frac{\delta}{2}\right) \mathbb{1}_{\mathbb{R}^N \setminus B_{R-\frac{\delta}{2}}}$$

in Ω, where $\mathbb{1}$ denotes the characteristic function of a set, for an appropriate, small choice of δ and $\widetilde{\varepsilon}$. □

2.3.5 Preparations for the continuation estimate

In this section we are going to derive an inequality which shows that we can control the $\|\cdot\|_{W^{1,q}(\Omega)}$-norm of solutions to (2.1) by their $L^\infty(\Omega)$-norm. For the following computation we define, for $\eta > 0$,

$$\Phi_\eta(s) := (s^2+\eta)^{\frac{q}{2}}, \quad s \in \mathbb{R},$$

and compute

$$\Phi_\eta'(s) = qs(s^2+\eta)^{\frac{q}{2}-1},$$

which implies

$$s\Phi_\eta'(s) \le q\Phi(s) \qquad \text{as well as} \qquad s\Phi_\eta'(s) \ge 0 \tag{2.7}$$

for $s \in \mathbb{R}$ and

$$\Phi_\eta''(s) = q\left((q-1)s^2+\eta\right)(s^2+\eta)^{\frac{q}{2}-2} \ge 0, \qquad s \in \mathbb{R}. \tag{2.8}$$

In preparation for later calculations we also note that for $a, s \in \mathbb{R}$,

$$\Phi_\eta(s) - as\Phi_\eta'(s) = (1-aq)s^2(s^2+\eta)^{\frac{q}{2}-1} + \eta(s^2+\eta)^{\frac{q}{2}-1}. \tag{2.9}$$

Lemma 2.3.9. *Let $\kappa \ge 0, \mu > 0, q > N, T > 0$.*
For any radial classical solution u of (2.1) in $\Omega \times (0,T)$ with radial initial data $u_0 \in W^{1,q}(\Omega)$, and arbitrary $\tau \in (0,T)$, $t \in (\tau, T)$, we have (with $K^{\frac{1}{q}}$ as C from Lemma 2.2.4)

$$\int_\Omega \Phi_\eta(|\nabla u(\cdot,t)|) \le \int_\Omega \Phi_\eta(|\nabla u(\cdot,\tau)|) + \int_\tau^t \left(\left(5q + \frac{q-2}{q}\eta\right)\|u\|_{L^\infty(\Omega)} + \kappa q\right)\int_\Omega \Phi_\eta(|\nabla u|)$$
$$+ |\Omega| \int_\tau^t \left(K\|u\|_{L^\infty(\Omega)}^{1+q} + \frac{2\eta}{q}\|u\|_{L^\infty(\Omega\times(0,t))}\right)$$

Proof. We denote $\Omega_\delta = \Omega \backslash B_\delta(0)$, let $0 < \tau < t < T$ and use ω_N to denote the $(N-1)$-dimensional measure of the unit sphere $\partial B_1(0) \subset \mathbb{R}^N$.
We note that on $\Omega_\delta \times (\tau, t)$ all derivatives of u appearing in the following calculation are smooth and bounded, and we can hence change the order of integration and differentiation to start with

$$\int_{\Omega_\delta} \Phi_\eta(|\nabla u(\cdot,t)|) - \int_{\Omega_\delta} \Phi_\eta(|\nabla u(\cdot,\tau)|) = \omega_N \int_\tau^t \int_\delta^R r^{N-1}\Phi_\eta'(u_r)u_{rt}.$$

Here we use equation (2.5a) for u_t:

$$\begin{aligned}
&\int_{\Omega_\delta} \Phi_\eta(|\nabla u(\cdot,t)|) - \int_{\Omega_\delta} \Phi_\eta(|\nabla u(\cdot,\tau)|) \\
&\quad = \omega_N \int_\tau^t\int_\delta^R r^{N-1}\Phi_\eta'(u_r)\left(\varepsilon u_{rr} + \varepsilon\frac{N-1}{r}u_r - u_r v_r - uv_{rr} - \frac{N-1}{r}uv_r + \kappa u - \mu u^2\right)_r \\
&\quad = \varepsilon\omega_N \int_\tau^t \int_\delta^R r^{N-1}u_{rrr}\Phi_\eta'(u_r) + \varepsilon\omega_N \int_\tau^t \int_\delta^R r^{N-2}(N-1)\frac{\mathrm{d}}{\mathrm{d}r}(\Phi_\eta(u_r)) \\
&\quad - \varepsilon\omega_N \int_\tau^t \int_\delta^R r^{N-3}(N-1)u_r\Phi_\eta'(u_r) - \omega_N \int_\tau^t \int_\delta^R r^{N-1}v_r u_{rr}\Phi_\eta'(u_r) \\
&\quad - 2\omega_N \int_\tau^t \int_\delta^R r^{N-1}v_{rr}u_r\Phi_\eta'(u_r) - \omega_N \int_\tau^t \int_\delta^R r^{N-1}v_{rrr}u\Phi_\eta'(u_r)
\end{aligned}$$

$$
\begin{aligned}
&+\omega_N \int_\delta^R r^{N-3}(N-1)v_r u\Phi'_\eta(u_r) - \omega_N \int_\tau^t \int_\delta^R r^{N-2}(N-1)v_r u_r \Phi'_\eta(u_r)\\
&-\omega_N \int_\tau^t \int_\delta^R r^{N-2}(N-1)v_{rr}u\Phi'_\eta(u_r) + \kappa\omega_N \int_\tau^t \int_\delta^R r^{N-1}u_r\Phi'_\eta(u_r)\\
&-2\mu\omega_N \int_\tau^t \int_\delta^R r^{N-1}uu_r\Phi'_\eta(u_r) =: I_1 + I_2 + \ldots + I_{11}.
\end{aligned}
$$

Now we integrate by parts twice in the first term

$$
\begin{aligned}
I_1 =& -\varepsilon\omega_N \int_\tau^t \int_\delta^R r^{N-1}u_{rr}\Phi''_\eta(u_r)u_{rr} - \varepsilon\omega_N \int_\tau^t \int_\delta^R r^{N-2}u_{rr}\Phi'_\eta(u_r)(N-1)\\
&+\varepsilon\omega_N \int_\tau^t r^{N-1}u_{rr}\Phi'_\eta(u_r)\big|_\delta^R\\
\le& 0 + \varepsilon(N-1)(N-2)\omega_N \int_\tau^t \int_\delta^R r^{N-3}\Phi_\eta(u_r) - \varepsilon\omega_N \int_\tau^t r^{N-2}\Phi_\eta(u_r)(N-1)\big|_\delta^R\\
&+\varepsilon\omega_N \int_\tau^t r^{N-1}u_{rr}\Phi'_\eta(u_r)\big|_\delta^R,
\end{aligned}
$$

where we also used (2.8), and once in the second integral

$$
I_2 = +\varepsilon(N-1)\omega_N \int_\tau^t r^{N-2}\Phi_\eta(u_r)\big|_\delta^R - \varepsilon(N-1)(N-2)\omega_N \int_\tau^t \int_\delta^R r^{N-3}\Phi_\eta(u_r).
$$

Upon addition, some of these summands vanish and estimating $I_3 \le 0$ by (2.7), we obtain

$$
I_1 + I_2 + I_3 \le \varepsilon\omega_N \int_\tau^t r^{N-1}u_{rr}\Phi'_\eta(u_r)\big|_\delta^R = -\varepsilon\omega_N \int_\tau^t r^{N-1}u_{rr}\Phi'_\eta(u_r)\big|_{r=\delta},
$$

because $u_r(R,\widetilde{t}) = 0$ for all $\widetilde{t} \in (0,T)$.
Also the next term can be rewritten by integration by parts and using $v_r(R,\widetilde{t}) = 0$ for $\widetilde{t} \in (0,T)$.

$$
\begin{aligned}
I_4 =& \,\omega_N \int_\tau^t r^{N-1}v_r\Phi_\eta(u_r)\big|_{r=\delta} + (N-1)\omega_N \int_\tau^t \int_\delta^R r^{N-2}v_r\Phi_\eta(u_r)\\
&+\omega_N \int_\tau^t \int_\delta^R r^{N-1}v_{rr}\Phi_\eta(u_r).
\end{aligned}
$$

Inserting (2.5b) to express v_{rrr} in I_6 differently, (among others) we obtain terms to cancel I_7 and I_9:

$$
I_6 = -\omega_N \int_\tau^t \int_\delta^R r^{N-1}v_r u\Phi'_\eta(u_r) + \omega_N \int_\tau^t \int_\delta^R r^{N-1}uu_r\Phi'_\eta(u_r) - I_9 - I_7.
$$

Together with the trivial observation that $I_{11} \le 0$ by (2.7), these estimates and reformulations give

$$
\begin{aligned}
&\int_{\Omega_\delta} \Phi_\eta\left(|\nabla u(\cdot,t)|\right) - \int_{\Omega_\delta} \Phi_\eta(|\nabla u(\cdot,\tau)|)\\
\le& -\varepsilon\omega_N \int_\tau^t r^{N-1}u_{rr}\Phi'_\eta(u_r)\big|_{r=\delta} + \omega_N \int_\tau^t r^{N-1}v_r\Phi_\eta(u_r)\big|_{r=\delta}
\end{aligned}
$$

$$+ (N-1)\omega_N \int_\tau^t \int_\delta^R r^{N-2} v_r \Phi_\eta(u_r) + \omega_N \int_\tau^t \int_\delta^R r^{N-1} v_{rr} \Phi_\eta(u_r)$$
$$- 2\omega_N \int_\tau^t \int_\delta^R r^{N-1} v_{rr} u_r \Phi_\eta'(u_r) - \omega_N \int_\tau^t \int_\delta^R r^{N-1} v_r u \Phi_\eta'(u_r)$$
$$+ \omega_N \int_\tau^t \int_\delta^R r^{N-1} u u_r \Phi_\eta'(u_r) - \omega_N \int_\tau^t \int_\delta^R r^{N-2} (N-1) v_r u_r \Phi_\eta'(u_r)$$
$$+ \kappa\omega_N \int_\tau^t \int_\delta^R r^{N-1} u_r \Phi_\eta'(u_r).$$

Passing to the limit $\delta \searrow 0$ by boundedness of u_r, u_{rr}, v_r on $(0,R)\times(\tau,t)$ and the dominated convergence theorem we arrive at

$$\int_\Omega \Phi_\eta\left(|\nabla u(\cdot,t)|\right) - \int_\Omega \Phi_\eta(|\nabla u(\cdot,\tau)|)$$
$$= (N-1)\omega_N \int_\tau^t \int_0^R r^{N-2} v_r \left[\Phi_\eta(u_r) - u_r \Phi_\eta'(u_r)\right]$$
$$+ \omega_N \int_\tau^t \int_0^R r^{N-1} v_{rr} \left[\Phi_\eta(u_r) - 2u_r \Phi_\eta'(u_r)\right] - \omega_N \int_\tau^t \int_0^R r^{N-1} v_r u \Phi_\eta'(u_r)$$
$$+ \omega_N \int_\tau^t \int_0^R r^{N-1} u u_r \Phi_\eta'(u_r) + \kappa q \int_\tau^t \int_\Omega \Phi_\eta\left(|\nabla u|\right) \;=\; I_A + I_B + I_C + I_D + I_E$$

and with the help of (2.9), the first of these integrals can be rewritten as

$$I_A = (N-1)(1-q)\omega_N \int_\tau^t \int_0^R r^{N-2} v_r u_r^2 \left(u_r^2 + \eta\right)^{\frac{q}{2}-1}$$
$$+ \eta(N-1)\omega_N \int_\tau^t \int_0^R r^{N-2} v_r \left(u_r^2 + \eta\right)^{\frac{q}{2}-1}.$$

Treating the second term similarly and inserting (2.9) and (2.5b) gives

$$I_B = (1-2q)\omega_N \int_\tau^t \int_0^R r^{N-1} v_{rr} u_r^2 \left(u_r^2+\eta\right)^{\frac{q}{2}-1} + \eta\omega_N \int_\tau^t \int_0^R r^{N-1} v_{rr} \left(u_r^2+\eta\right)^{\frac{q}{2}-1}$$
$$= (2q-1)\omega_N \int_\tau^t \int_0^R r^{N-1} (u-v) u_r^2 \left(u_r^2+\eta\right)^{\frac{q}{2}-1}$$
$$+ (N-1)(2q-1)\omega_N \int_\tau^t \int_0^R r^{N-2} v_r u_r^2 \left(u_r^2+\eta\right)^{\frac{q}{2}-1}$$
$$+ \eta\omega_N \int_\tau^t \int_0^R r^{N-1} \left(v - u - \frac{N-1}{r} v_r\right) \left(u_r^2+\eta\right)^{\frac{q}{2}-1},$$

where also $(u-v)u_r^2 \le u\left(u_r^2+\eta\right)$. For the sum of these terms we are thereby led to

$$I_A + I_B \le (N-1)q\omega_N \int_\tau^t \int_0^R r^{N-2} v_r u_r^2 \left(u_r^2+\eta\right)^{\frac{q}{2}-1}$$
$$+ (2q-1)\omega_N \int_\tau^t \int_0^R r^{N-1} u \Phi_\eta(u_r) + \eta\omega_N \int_\tau^t \int_0^R r^{N-1} (v-u) \left(u_r^2+\eta\right)^{\frac{q}{2}-1},$$

where we can use Lemma 2.3.7 to infer

$$
\begin{aligned}
I_A + I_B &\le q\frac{N-1}{N}\omega_N \int_\tau^t \int_0^R r^{N-2} r \left\|u\right\|_{L^\infty(\Omega)} u_r^2 \left(u_r^2+\eta\right)^{\frac{q}{2}-1} \\
&\quad + (2q-1)\omega_N \int_\tau^t \int_0^R r^{N-1} u \Phi_\eta(u_r) + \eta\omega_N \int_\tau^t \int_0^R r^{N-1}(v-u)\left(u_r^2+\eta\right)^{\frac{q}{2}-1} \\
&\le (q\frac{N-1}{N} + 2q - 1)\omega_N \int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \int_0^R r^{N-1}\Phi_\eta(u_r) \\
&\quad + \eta\omega_N \int_\tau^t \int_0^R r^{N-1}(v-u)\left(u_r^2+\eta\right)^{\frac{q}{2}-1}.
\end{aligned}
$$

Furthermore adding the other terms and making use of (2.7) in I_D,

$$
\begin{aligned}
&I_A + \ldots + I_E \\
&\le \int_\tau^t \left(\left(\left(4-\frac{1}{N}\right)q-1\right)\left\|u\right\|_{L^\infty(\Omega)} + \kappa q\right) \int_\Omega \Phi_\eta\left(|\nabla u|\right) \qquad (2.10)\\
&\quad + q\int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \int_\Omega |\nabla v||\nabla u|(|\nabla u|^2+\eta)^{\frac{q}{2}-1} \\
&\quad + \eta\omega_N \int_\tau^t \int_0^R r^{N-1}(v-u)\left(u_r^2+\eta\right)^{\frac{q}{2}-1} \\
&\le \int_\tau^t \left(4q\left\|u\right\|_{L^\infty(\Omega)} + \kappa q\right) \int_\Omega \Phi_\eta\left(|\nabla u|\right) \\
&\quad + q\int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \int_\Omega |\nabla v|(|\nabla u|^2+\eta)^{\frac{q-1}{2}} + \eta\omega_N \int_\tau^t \int_0^R r^{N-1} v\left(u_r^2+\eta\right)^{\frac{q}{2}-1}.
\end{aligned}
$$

Here an application of Young's inequality gives

$$
\begin{aligned}
I_A + \ldots + I_E &\le \int_\tau^t (4q\left\|u\right\|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_\eta\left(|\nabla u|\right) + \int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \int_\Omega |\nabla v|^q \\
&\quad + q\frac{q-1}{q}\int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \int_\Omega \left(|\nabla u|^2+\eta\right)^{\frac{q}{2}} \qquad (2.11)\\
&\quad + \eta\omega_N \int_\tau^t \left\|v\right\|_{L^\infty(\Omega)} \int_0^R r^{N-1}\left(u_r^2+\eta\right)^{\frac{q}{2}-1}.
\end{aligned}
$$

Merging first and third term, with Lemma 2.2.4 (and K as provided by that lemma) and Lemma 2.2.2 we have

$$
\begin{aligned}
I_A + \ldots + I_E &\le \int_\tau^t (5q\left\|u\right\|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_\eta\left(|\nabla u|\right) + K\int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \left\|u\right\|_{L^q(\Omega)}^q \\
&\quad + \eta\omega_N \int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \int_0^R r^{N-1}\left(\frac{q-2}{q}\left(u_r^2+\eta\right)^{\frac{q-2}{2}\frac{q}{q-2}} + \frac{2}{q}\cdot 1^{\frac{q}{2}}\right) \\
&\le \int_\tau^t (5q\left\|u\right\|_{L^\infty(\Omega)} + \kappa q) \int_\Omega \Phi_\eta\left(|\nabla u|\right) + K\int_\tau^t |\Omega|\left\|u\right\|_{L^\infty(\Omega)}^{1+q} \\
&\quad + \eta\frac{q-2}{q}\int_\tau^t \left\|u\right\|_{L^\infty(\Omega)} \int_\Omega (|\nabla u|^2+\eta)^{\frac{q}{2}} + \eta\int_\tau^t \frac{2}{q}|\Omega|\left\|u\right\|_{L^\infty(\Omega\times(\tau,t))} \\
&\le \int_\tau^t \left(\left(5q + \frac{q-2}{q}\eta\right)\left\|u\right\|_{L^\infty(\Omega)} + \kappa q\right) \int_\Omega \Phi_\eta\left(|\nabla u|\right)
\end{aligned}
$$

$$+ |\Omega| \int_\tau^t \left(K \|u\|_{L^\infty(\Omega)}^{1+q} + \frac{2\eta}{q} \|u\|_{L^\infty(\Omega\times(0,t))} \right)$$

In total, these estimates show the claim. □

Remark 2.3.10. In the above proof (and all affected propositions), $5q$ could be replaced by $(5 - \frac{1}{N})q - 2$ by keeping the negative summands neglected in lines (2.10) and (2.11).

Lemma 2.3.11. *Under the assumptions of Lemma 2.3.9 the following holds:*

$$\int_\Omega |\nabla u(\cdot,t)|^q \le \left(\|\nabla u(\cdot,\tau)\|_{L^q(\Omega)}^q + \left(|\Omega| K \int_0^t \|u(\cdot,s)\|_{L^\infty(\Omega)}^{1+q} \, ds \right) \right) \cdot$$
$$\cdot \exp\left(5q \int_0^t \|u(\cdot,s)\|_{L^\infty(\Omega)} \, ds + \kappa q t \right).$$

Proof. Letting $\tau \in (0,T)$ and $t \in (\tau,T)$ and starting from Lemma 2.3.9, by Gronwall's inequality we can conclude

$$\int_\Omega \Phi_\eta(\nabla u(\cdot,t)) \le \left(\int_\Omega \Phi_\eta\left(\nabla u(\cdot,\tau)\right) + K \int_\tau^t |\Omega| \|u\|_{L^\infty(\Omega)}^{1+q} + \eta t \frac{2}{q} |\Omega| \|u\|_{L^\infty(\Omega\times(0,t))} \right) \cdot$$
$$\cdot \exp\left(\int_\tau^t \left(\left(5q + \frac{q-2}{q}\eta \right) \|u\|_{L^\infty(\Omega)} + \kappa q \right) \right).$$

By smoothness of u in $\overline{\Omega} \times (0,T)$, we have for all $s \in (0,T)$

$$\int_\Omega \Phi_\eta(\nabla u(\cdot,s)) \to \int_\Omega |\nabla u(\cdot,s)|^q \quad \text{as } \eta \searrow 0.$$

From this we gain

$$\int_\Omega |\nabla u(\cdot,t)|^q \le \left(\int_\Omega |\nabla u(\cdot,\tau)|^q + |\Omega| K \int_\tau^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp\left(\int_\tau^t (5q \|u\|_{L^\infty(\Omega)}) + \kappa q t \right),$$

which implies the assertion. □

Corollary 2.3.12. *In addition to the hypotheses of Lemma 2.3.9, let u_0 be compatible. Then*

$$\int_\Omega |\nabla u(\cdot,t)|^q \le \left(\int_\Omega |\nabla u_0|^q + |\Omega| K \int_0^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp\left(\int_0^t (5q \|u\|_{L^\infty(\Omega)}) + \kappa q t \right).$$

Proof. Since from the dominated convergence theorem we know that

$$\int_\Omega |\nabla u(\cdot,\tau)|^q \to \int_\Omega |\nabla u_0|^q$$

as $\tau \searrow 0$ due to the boundedness of ∇u for solutions of (2.1) with compatible initial data, this is a direct consequence of Lemma 2.3.11. □

2.3.6 Epsilon-independent time of existence

We begin this section with some Gronwall-type lemma which we will need during the next proof:

Lemma 2.3.13. *Let $f\colon [0,\infty) \to \mathbb{R}$ nondecreasing and locally Lipschitz continuous, let $y_0 \in \mathbb{R}$. Denote by y the solution of $y(0) = y_0$, $y'(t) = f(y(t))$ on some interval $(0,T)$ and assume that the continuous function $z\colon [0,T) \to \mathbb{R}$ satisfies*

$$z(t) \le z(0) + \int_0^t f(z(\tau))d\tau \quad \text{for all } t \in (0,T), \qquad z(0) < y_0.$$

Then $z(t) \le y(t)$ for all $t \in (0,T)$.

Proof. Let $T_0 := \inf\{t \in (0,T) : z(t) > y(t)\}$ and assume that $T_0 < T$ exists. Due to continuity, $z(T_0) = y(T_0)$, i.e.

$$\begin{aligned} z(0) + \int_0^{T_0} f(z(\tau))\mathrm{d}\tau &\ge z(T_0) = y(T_0) = y(0) + \int_0^{T_0} y'(\tau)\mathrm{d}\tau \\ &= y_0 + \int_0^{T_0} f(y(\tau))\mathrm{d}\tau \ge y_0 + \int_0^{T_0} f(z(\tau))\mathrm{d}\tau > z(0) + \int_0^{T_0} f(z(\tau))\mathrm{d}\tau, \end{aligned}$$

which is contradictory. □

The next theorem prepares the ground for the approximation procedure to be carried out in Theorem 2.4.7. It guarantees that solutions to (2.1) exist "long enough". Its proof is an adaption of that of [117, Lemma 4.5], where an assertion similar to our Theorem 2.4.7 is shown.

Theorem 2.3.14. *Let $\kappa \ge 0, \mu > 0, q > N$. Then for any $L > 0$ there are some numbers $T(L) > 0$ and $M(L) > 0$ such that for any radially symmetric nonnegative and compatible $u_0 \in W^{1,q}(\Omega)$ with $\|u_0\|_{W^{1,q}(\Omega)} \le L$, for any $\varepsilon > 0$ the classical solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) exists on $\Omega \times (0,T(L))$ and $\|u_\varepsilon\|_{L^\infty(\Omega\times(0,T(L))} \le M(L)$.*

Proof. For any $\varepsilon > 0$, the classical solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) exists on some interval $(0, T_{max,\varepsilon})$ and satisfies $\limsup_{t\nearrow T_{max,\varepsilon}} \|u_\varepsilon\|_{L^\infty(\Omega)} = \infty$, unless $T_{max,\varepsilon} = \infty$. It is therefore sufficient to show boundedness of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ on $(0,T(L))$ for some ε-independent $T(L) > 0$. For this, we fix constants c_1, c_2 such that for all $\phi \in W^{1,q}(\Omega)$

$$\|\phi\|_{L^\infty(\Omega)} \le c_1 \|\nabla\phi\|_{L^q(\Omega)} + c_1 \|\phi\|_{L^1(\Omega)} \qquad \text{and} \qquad \|\phi\|_{L^1(\Omega)} \le c_2 \|\phi\|_{W^{1,q}(\Omega)}, \tag{2.12}$$

where we use $q > N$, as well as K as in Lemma 2.3.9 and $c_3 = c_3(L)$ such that

$$\frac{c_3}{c_1} = \max\left\{c_2 L, \frac{\kappa|\Omega|}{\mu}\right\},$$

so that by Corollary 2.3.4 applied to $p = 1$ we obtain

$$\int_\Omega u_\varepsilon(\cdot,t) \le \frac{c_3}{c_1} \tag{2.13}$$

for any $t \in (0, T_{max,\varepsilon})$ and any solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) with u_0, κ, μ as specified. Furthermore, we let y_L denote the solution to

$$y_L'(t) = (6qc_1 + K|\Omega|(2c_1)^{1+q})y_L^{1+\frac{1}{q}} + (6qc_3 + \kappa q)y_L + |\Omega|(K(2c_3)^{1+q} + 1),$$

$$y_L(0) = (\sqrt{2}L)^q + 1$$

and let $T(L) > 0$ be a number such that $y_L(t) \le (\sqrt{2}L)^q + 2$ for all $t \in (0, T(L))$.
We then let $u_0 \in W^{1,q}(\Omega)$ be as specified in the lemma, which in particular entails that $\|u_0\|_{W^{1,q}(\Omega)} \le L$. For $\varepsilon > 0$ we denote by $(u_\varepsilon, v_\varepsilon)$ the solution of the corresponding equation (2.1). We apply Lemma 2.3.9 for conveniently small $\eta \in (0, \min\{q, \frac{1}{2}|\Omega|^{-\frac{2}{q}}, \frac{q}{2c_3}\})$ and arbitrary $t \in (0, T(L))$ and, due to compatibility of u_0, in the limit $\tau \searrow 0$ obtain

$$\begin{aligned}\int_\Omega \Phi_\eta(|\nabla u_\varepsilon(\cdot,t)|) &\le \int_\Omega \Phi_\eta(|\nabla u_0|) + \int_0^t \left(\left(6q\,\|u_\varepsilon\|_{L^\infty(\Omega)} + \kappa q\right)\int_\Omega \Phi_\eta(|\nabla u_\varepsilon|)\right)\\ &\quad + |\Omega| \int_0^t \left(K\,\|u_\varepsilon\|_{L^\infty(\Omega)}^{1+q} + \frac{2\eta}{q}\,\|u_\varepsilon\|_{L^\infty(\Omega)}\right)\\ &\le \int_\Omega \Phi_\eta(|\nabla u_0|) + \int_0^t \left((6q(c_1\,\|\nabla u_\varepsilon\|_{L^q(\Omega)} + c_3) + \kappa q)\int_\Omega \Phi_\eta(|\nabla u_\varepsilon|)\right)\\ &\quad + |\Omega| \int_0^t \left(K(c_1\,\|\nabla u_\varepsilon\|_{L^q(\Omega)} + c_3)^{1+q} + \frac{2\eta}{q}(c_1\,\|\nabla u_\varepsilon\|_{L^q(\Omega)} + c_3)\right).\end{aligned}$$

Here we abbreviate $\int_\Omega \Phi_\eta(|\nabla u_\varepsilon(\cdot,t)|) =: z(t)$ and estimate $\|\nabla u_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le z^{\frac{1}{q}}(t)$. Then

$$z(t) \le z(0) + \int_0^t \left((6qc_1 + K|\Omega|(2c_1)^{1+q})z^{1+\frac{1}{q}}(s) + (6qc_3 + \kappa q)z(s) + |\Omega|(K(2c_3)^{1+q} + \frac{2\eta}{q}c_3)\right)\mathrm{d}s.$$

Additionally

$$z(0) = \int_\Omega \Phi_\eta(\nabla u_0) \le 2^{\frac{q}{2}} \int_\Omega |\nabla u_0|^q + (2\eta)^{\frac{q}{2}}|\Omega| \le (\sqrt{2}L)^q + 1.$$

Lemma 2.3.13 therefore leads us to the conclusion that, for all $t \in (0, T(L))$ and independently of η, we have $\int \Phi_\eta(\nabla u_\varepsilon(\cdot,t)) \le y_L(t)$, which by Fatou's lemma implies

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^q \le (\sqrt{2}L)^q + 2 \qquad \text{for all } t \in (0, T(L)).$$

Along with (2.12) and (2.13), this shows that for all $\varepsilon > 0$

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_1((\sqrt{2}L)^q + 2)^{\frac{1}{q}} + c_3 =: M(L)$$

on $(0, T(L))$. □

2.3.7 Preparations for convergence: Boundedness of u_t in an appropriate space

In order to use the Aubin-Lions-type estimate of Lemma 2.4.4, we will need at least some regularity of the time derivative of bounded solutions.

Lemma 2.3.15. *Let $\varepsilon_0 > 0$, let $\mu > 0, T > 0, q > N, p \in (1,\infty), M > 0$. Let $u_0 \in W^{1,q}(\Omega)$ be radially symmetric and nonnegative. Then there is $C > 0$ such that the following holds for $\varepsilon \in (0, \varepsilon_0)$: If a solution u_ε of (2.1) in $\Omega \times (0,T)$ with compatible nonnegative radial initial data $u_{0,\varepsilon} \in W^{1,q}(\Omega)$, $\|u_{0,\varepsilon} - u_0\|_{W^{1,q}(\Omega)} < \varepsilon$, satisfies*

$$|u_\varepsilon(x,t)| < M$$

for all $(x,t) \in \Omega \times [0,T)$*, then*

$$\|u_{\varepsilon t}\|_{L^p((0,T);(W_0^{1,\frac{q}{q-1}}(\Omega))^*)} \le C.$$

Proof. We let $\varphi \in C_0^1(\overline{\Omega})$, multiply (2.1) by φ and integrate over Ω:

$$\begin{aligned}\left|\int_\Omega u_{\varepsilon t}\varphi\right| &= \left|\int_\Omega \varepsilon\Delta u_\varepsilon\varphi - \nabla\cdot(u_\varepsilon\nabla v_\varepsilon)\varphi + \kappa u_\varepsilon\varphi - \mu u_\varepsilon^2\varphi\right| \\ &\le \varepsilon\left|\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi\right| + \left|\int_\Omega u_\varepsilon\nabla v_\varepsilon\cdot\nabla\varphi\right| + \kappa\|u_\varepsilon\|_{L^\infty(\Omega)}\left|\int\varphi\right| + \mu\|u_\varepsilon\|_{L^\infty(\Omega)}^2\left|\int\varphi\right|.\end{aligned}$$

Invoking Corollary 2.3.12, Lemma 2.2.4 and Hölder's inequality, we infer the existence of constants with $\|\nabla u_\varepsilon\|_{L^q(\Omega)} \le \widetilde{M}$, $\|\nabla v_\varepsilon\|_{L^q(\Omega)} \le \widetilde{C}\|u_\varepsilon\|_\infty \le \widetilde{C}M$ and $\|\varphi\|_{L^1(\Omega)} \le \widehat{C}\|\varphi\|_{L^{\frac{q}{q-1}}}$ and conclude for $C := T^{\frac{1}{p}}\max\{\varepsilon_0\widetilde{M} + \widetilde{C}M^2, (\kappa M + \mu M^2)\widehat{C}\}$ that

$$\begin{aligned}\left|\int_\Omega u_{\varepsilon t}\varphi\right| &\le \varepsilon\|\nabla u_\varepsilon\|_{L^q(\Omega)}\|\nabla\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} + M\|\nabla v_\varepsilon\|_{L^q(\Omega)}\|\nabla\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} \\ &\quad + \kappa M\|\varphi\|_{L^1(\Omega)} + \mu M^2\|\varphi\|_{L^1(\Omega)} \\ &\le \left(\varepsilon\widetilde{M} + \widetilde{C}M^2\right)\|\nabla\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} + (\kappa M + \mu M^2)\widehat{C}\|\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} \\ &\le CT^{-\frac{1}{p}}\|\varphi\|_{W^{1,\frac{q}{q-1}}(\Omega)}.\end{aligned}$$

Here taking the supremum over φ with $\|\varphi\|_{W^{1,\frac{q}{q-1}}(\Omega)} = 1$ reveals

$$\|u_{\varepsilon t}\|_{(W^{1,\frac{q}{q-1}}(\Omega))^*} \le CT^{-\frac{1}{p}}$$

and hence

$$\left(\int_0^T \|u_{\varepsilon t}\|^p_{(W^{1,\frac{q}{q-1}}(\Omega))^*}\right)^{\frac{1}{p}} \le C.$$

□

2.4 Hyperbolic-elliptic case

2.4.1 What is a solution?

We want to call a function "solution" if it satisfies (2.2) in the following sense:

Definition 2.4.1. *Let* $T \in (0,\infty]$*. A strong* $W^{1,q}$*-solution of* (2.2) *in* $\Omega \times (0,T)$ *is a pair of functions* $u \in C^0(\overline{\Omega} \times [0,T)) \cap L^\infty_{loc}([0,T); W^{1,q}(\Omega))$ *and* $v \in C^{2,0}(\overline{\Omega} \times [0,T))$ *such that* u, v *are nonnegative,* v *classically solves* $0 = \Delta v - v + u$ *in* Ω*,* $\partial_\nu v\big|_{\partial\Omega} = 0$ *and*

$$-\int_0^T\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = \int_0^T\int_\Omega u\nabla v\cdot\nabla\varphi + \kappa\int_0^T\int_\Omega u\varphi - \mu\int_0^T\int_\Omega u^2\varphi \tag{2.14}$$

holds true for all $\varphi \in L^1((0,T); W^{1,1}(\Omega))$ *that have compact support in* $\overline{\Omega} \times [0,T)$ *and satisfy* $\varphi_t \in L^1(\Omega \times (0,T))$*. If additionally* $T = \infty$*, we call the solution global.*

Remark 2.4.2. Due to density arguments, it is of course possible to formulate Definition 2.4.1 for $\varphi \in C_0^\infty(\overline{\Omega} \times [0,T))$ and obtain the same solution concept.

2.4.2 Uniqueness

These solutions are unique as can be proven very similarly to the one-dimensional case.

Lemma 2.4.3. *Let $q > N$, $T \in (0, \infty]$ and $u_0 \in W^{1,q}(\Omega)$ with $u_0 \geq 0$. Then there is at most one $W^{1,q}$-solution of (2.2) in $\Omega \times (0, T)$.*

Proof. (Cf. [117, Lemma 4.2]). Letting $q > N$ and $u_0 \in W^{1,q}(\Omega)$ with $u_0 \geq 0$, we assume $(u, v), (\widetilde{u}, \widetilde{v})$ to be strong $W^{1,q}$-solutions of (2.2) and note that $(w, z) := (u - \widetilde{u}, v - \widetilde{v})$ satisfies

$$-\int_0^T \int_\Omega w\varphi_t = \int_0^T \int_\Omega (u\nabla v - \widetilde{u}\nabla\widetilde{v}) \cdot \nabla\varphi + \kappa \int_0^T \int_\Omega w\varphi - \mu \int_0^T \int_\Omega (u^2 - \widetilde{u}^2)\varphi \tag{2.15}$$

for all $\varphi \in L^1((0, T); W^{1,1}(\Omega))$ which have compact support in $\overline{\Omega} \times [0, T)$ and satisfy $\varphi_t \in L^1(\Omega \times (0, T))$ and

$$0 = \Delta z + w - z. \tag{2.16}$$

Letting $T_0 \in (0, T)$, by Definition 2.4.1 (and by Lemma 2.2.4), we can define constants such that

$$c_1 := \|v\|_{L^\infty(\Omega\times(0,T_0))}, \qquad c_2 := \|u\|_{L^\infty(\Omega\times(0,T_0))},$$
$$c_3 := \|\widetilde{u}\|_{L^\infty((0,T_0),W^{1,q}(\Omega))}, \qquad c_5 := \|\widetilde{u}\|_{L^\infty(\Omega\times(0,T_0))}.$$

According to Lemma 2.2.4, (2.16) implies $\|z\|_{W^{2,q}(\Omega)} \leq \widetilde{C}\, \|w\|_{L^q(\Omega)}$ for some $\widetilde{C} > 0$ and hence, as $W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $q > N$, there is $c_4 > 0$ such that

$$\|z\|_{W^{1,\infty}(\Omega)} \leq c_4 \|w\|_{L^q(\Omega)}.$$

With these constants, we set

$$C := q(c_1 + c_2 + \kappa + 2c_3c_4 + c_5). \tag{2.17}$$

In (2.15) we use some function φ we construct as follows: For $t_0 \in (0, T_0)$ we define $\xi_\delta \in W^{1,\infty}(\mathbb{R})$ by

$$\xi_\delta(t) := \begin{cases} 1, & t < t_0, \\ \frac{t_0 - t + \delta}{\delta}, & t \in [t_0, t_0 + \delta], \\ 0, & t > t_0 + \delta, \end{cases}$$

for $\delta \in (0, \frac{T_0 - t_0}{2})$, and let

$$\varphi(x, t) := \xi_\delta(t) \frac{1}{h} \int_t^{t+h} w(x, s)(w^2(x, s) + \eta)^{\frac{q}{2}-1} \mathrm{d}s.$$

Then for $\delta \in (0, \frac{T_0 - t_0}{2})$, $h \in (0, \frac{T_0 - t_0}{2})$, $1 > \eta > 0$, φ is a valid test function in (2.15), and inserting it yields

$$\begin{aligned}
&\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \int_\Omega w(x, t) \frac{1}{h} \int_t^{t+h} w(x, s)(w^2(x, s) + \eta)^{\frac{q}{2}-1} \mathrm{d}s\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \xi_\delta(t) w(x, t) \frac{w(x, t + h)(w^2(x, t + h) + \eta)^{\frac{q}{2}-1} - w(x, t)(w^2(x, t) + \eta)^{\frac{q}{2}-1}}{h} \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \xi_\delta(t) \left[-\nabla u \cdot \nabla v - u\Delta v + \nabla\widetilde{u} \cdot \nabla\widetilde{v} + \widetilde{u}\Delta\widetilde{v} + \kappa w - \mu w(u + \widetilde{u})\right]
\end{aligned}$$

$$\cdot \frac{1}{h}\int_t^{t+h} w(x,s)(w^2(x,s)+\eta)^{\frac{q}{2}-1}\mathrm{d}s\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_\Omega \xi_\delta(t)\left[-\nabla w\cdot\nabla v - w\Delta v - \nabla\widetilde{u}\cdot\nabla z - \widetilde{u}\Delta z + \kappa w - \mu w(u+\widetilde{u})\right]$$
$$\cdot \frac{1}{h}\int_t^{t+h} w(x,s)(w^2(x,s)+\eta)^{\frac{q}{2}-1}\mathrm{d}s\mathrm{d}x\mathrm{d}t.$$

Taking the limit $\delta \searrow 0$, which is possible for the first term, because the function $(x,t)\mapsto w(x,t)\frac{1}{h}\int_t^{t+h} w(x,s)(w^2(x,s)+\eta)^{\frac{q}{2}-1}\mathrm{d}s$ is continuous, and on the right hand side by Lebesgue's theorem, since $\nabla z, \Delta z, \widetilde{u}, \nabla v, \Delta v, w, u,$ are bounded and ∇w is uniformly bounded in $L^q(\Omega)$ up to time t_0+h (according to Definition 2.4.1), we obtain

$$\begin{aligned}
&\int_\Omega w(x,t_0)\frac{1}{h}\int_{t_0}^{t_0+h} w(x,s)(w^2(x,s)+\eta)^{\frac{q}{2}-1}\mathrm{d}s\mathrm{d}x\\
&-\int_0^{t_0}\int_\Omega w(x,t)\frac{w(x,t+h)(w^2(x,t+h)+\eta)^{\frac{q}{2}-1}-w(x,t)(w^2(x,t)+\eta)^{\frac{q}{2}-1}}{h}\mathrm{d}x\mathrm{d}t\\
&=\int_0^{t_0}\int_\Omega \left[-\nabla w\cdot\nabla v - w\Delta v - \nabla\widetilde{u}\cdot\nabla z - \widetilde{u}\Delta z + \kappa w - \mu w(u+\widetilde{u})\right]\cdot\\
&\qquad\cdot\frac{1}{h}\int_t^{t+h} w(x,s)(w^2(x,s)+\eta)^{\frac{q}{2}-1}\mathrm{d}s\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.18}$$

With the abbreviations $w=w(x,t)$, $w_h=w(x,t+h)$ we observe that

$$\begin{aligned}
&-\frac{1}{h}\int_0^{t_0}\int_\Omega ww_h(w_h^2+\eta)^{\frac{q}{2}-1}+\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1}\\
&\geq -\frac{1}{h}\int_0^{t_0}\int_\Omega (w^2+\eta)^{\frac{1}{2}}(w_h^2+\eta)^{\frac{1}{2}}(w_h^2+\eta)^{\frac{q}{2}-1}+\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1}\\
&\geq -\frac{1}{h}\frac{1}{q}\int_0^{t_0}\int_\Omega (w^2+\eta)^{\frac{q}{2}}-\frac{1}{h}\frac{q-1}{q}\int_0^{t_0}\int_\Omega (w_h^2+\eta)^{\frac{q}{2}}+\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1},
\end{aligned}$$

where we have used that $s\leq (s^2+\eta)^{\frac{1}{2}}$ and Young's inequality. Converting the time shift in the arguments to a change of integration limits, we obtain

$$\begin{aligned}
&-\frac{1}{h}\frac{1}{q}\int_0^{t_0}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1}-\frac{1}{h}\frac{1}{q}\eta\int_0^{t_0}\int_\Omega (w^2+\eta)^{\frac{q}{2}-1}-\frac{1}{h}\frac{q-1}{q}\int_h^{t_0+h}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1}\\
&\qquad-\frac{1}{h}\frac{q-1}{q}\eta\int_h^{t_0+h}\int_\Omega (w^2+\eta)^{\frac{q}{2}-1}+\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1}\\
&\qquad=\frac{q-1}{q}\Big(\frac{1}{h}\int_0^{h}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1}-\frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega w^2(w^2+\eta)^{\frac{q}{2}-1}\Big)\\
&\qquad\quad-\eta\left(\frac{1}{hq}\int_0^{t_0}\int_\Omega (w^2+\eta)^{\frac{q}{2}-1}+\frac{q-1}{qh}\int_h^{t_0+h}\int_\Omega (w^2+\eta)^{\frac{q}{2}-1}\right),
\end{aligned}$$

where in the limit $\eta\searrow 0$ the last line vanishes. Furthermore, by the continuity of w and because $|w(w^2+\eta)^{\frac{q}{2}-1}|$ can be bounded by the integrable function $(w^2+1)^{\frac{q}{2}}$,

$$\left\{\Omega\ni x\mapsto \frac{1}{h}\int_{t_0}^{t_0+h} w(x,s)(w^2(x,s)+\eta)^{\frac{q}{2}-1}\mathrm{d}s\right\}\overset{*}{\rightharpoonup}\left\{x\mapsto\frac{1}{h}\int_{t_0}^{t_0+h} w(x,s)|w(x,s)|^{q-2}\mathrm{d}s\right\}$$

in $L^\infty(\Omega)$ as well as

$$\left\{(x,t)\mapsto \frac{1}{h}\int_t^{t+h} w(x,s)(w^2(x,s)+\eta)^{\frac{q}{2}-1}\mathrm{d}s\right\} \overset{*}{\rightharpoonup} \left\{(x,t)\mapsto \frac{1}{h}\int_t^{t+h} w(x,s)|w(x,s)|^{q-2}\mathrm{d}s\right\}$$

in $L^\infty(\Omega\times(0,t_0))$ as $\eta\searrow 0$. Therefore we can conclude from (2.18) that

$$\begin{aligned}\int_\Omega w(x,t_0)\frac{1}{h}\int_{t_0}^{t_0+h} & w(x,s)|w(x,s)|^{q-2}\mathrm{d}s\mathrm{d}x + \frac{q-1}{qh}\int_0^h\int_\Omega |w(x,t)|^q \\ & - \frac{q-1}{qh}\int_{t_0}^{t_0+h}\int_\Omega |w(x,t)|^q\mathrm{d}x\mathrm{d}t \\ & \le \int_0^{t_0}\int_\Omega (-\nabla w\cdot\nabla v - w\Delta v - \nabla\widetilde{u}\cdot\nabla z - \widetilde{u}\Delta z + \kappa w - \mu w(u+\widetilde{u}))\cdot \\ & \qquad \cdot\frac{1}{h}\int_t^{t+h} w(\cdot,s)|w(\cdot,s)|^{q-2}\mathrm{d}s\end{aligned}$$

for $h\in(0,\frac{T_0-t_0}{2})$. As w is continuous on $\overline{\Omega}\times[0,T_0]$ and $w(\cdot,0)=0$ in Ω, $h\searrow 0$ yields

$$\begin{aligned}\frac{1}{q}\int_\Omega & |w(x,t_0)|^q \\ & \le \int_0^{t_0}\int_\Omega (-\nabla w\cdot\nabla v - w\Delta v - \nabla\widetilde{u}\cdot\nabla z - \widetilde{u}\Delta z + \kappa w - \mu w(u+\widetilde{u}))w(x,t)|w(x,t)|^{q-2}.\end{aligned}$$

Here we will estimate the integral on the right-hand side to obtain an expression that allows to conclude $w=0$ by means of Gronwall's lemma. We will consider the summands separately:

$$\begin{aligned}-\int_0^{t_0}\int_\Omega \nabla w\cdot\nabla v w|w|^{q-2} &= \frac{1}{q}\int_0^{t_0}\int_\Omega |w|^q\Delta v \\ &= \frac{1}{q}\int_0^{t_0}\int_\Omega |w|^q(v-u) \le \|v\|_{L^\infty(\Omega\times(0,T_0))}\int_0^{t_0}\int_\Omega |w|^q.\end{aligned}$$

Also for the next term, the second equation of (2.1) and nonnegativity of v are helpful:

$$-\int_0^{t_0}\int_\Omega w\Delta v w|w|^{q-2} = -\int_0^{t_0}\int_\Omega |w|^q(v-u) \le \|u\|_{L^\infty(\Omega\times(0,T_0))}\int_0^{t_0}\int_\Omega |w|^q.$$

For the last term we make use of the nonnegativity of both u and $\widetilde{u}$:

$$\int_0^{t_0}\int_\Omega (\kappa w - \mu w(u+\widetilde{u}))w|w|^{q-2} = \int_0^{t_0}\int_\Omega (\kappa - \mu(u+\widetilde{u}))|w|^q \le \kappa\int_0^{t_0}\int_\Omega |w|^q.$$

Boundedness of $\widetilde{u}$ in $W^{1,q}(\Omega)$ and Lemma 2.2.4 play the main role in the following estimate.

$$\begin{aligned}-\int_0^{t_0}\int_\Omega \nabla\widetilde{u}\cdot\nabla z w|w|^{q-2} &\le \int_0^{t_0} \|\nabla\widetilde{u}\|_{L^q(\Omega)}\|\nabla z\|_{L^\infty(\Omega)}\left\|w|w|^{q-2}\right\|_{L^{\frac{q}{q-1}}(\Omega)} \\ &\le \int_0^{t_0} c_3c_4\|w\|_{L^q(\Omega)}\|w\|_{L^q(\Omega)}^{q-1}.\end{aligned}$$

And finally, once more employing the second equation, we have

$$-\int_0^{t_0}\int_\Omega \widetilde{u}\Delta z w|w|^{q-2} = -\int_0^{t_0}\int_\Omega \widetilde{u} z w|w|^{q-2} + \int_0^{t_0}\int_\Omega \widetilde{u}|w|^q$$
$$\le (c_3c_4+c_5)\int_0^{t_0}\int_\Omega |w|^q.$$

Gathering all these estimates together, we see that with the constant C from (2.17) for all $t_0 \in (0,T_0)$

$$\int_\Omega |w(x,t_0)|^q \mathrm{d}x \le C\int_0^{t_0}\int_\Omega |w|^q,$$

hence by Gronwall's lemma $w = 0$ (and therefore also $z = 0$), which proves uniqueness of solutions. ☐

2.4.3 Local existence and approximation

We will prove existence of solutions to (2.2) by means of a compactness argument whose key lies in

Lemma 2.4.4. *Let X, Y, Z be Banach spaces such that $X \hookrightarrow Y \hookrightarrow Z$, where the embedding $X \hookrightarrow Y$ is compact. Then for any $T > 0, p \in (1,\infty]$, the space*

$$\{w \in L^\infty([0,T];X); w_t \in L^p((0,T);Z)\}$$

is compactly embedded into $C^0([0,T];Y)$.

Proof. The proof uses the Arzelá-Ascoli theorem and Ehrling's lemma and can be found in [117, Lemma 4.4], see also [86, Section 8]. ☐

We directly take this tool to its use and employ it with a slightly different choice of spaces than in the one-dimensional case to obtain a similar result.

Lemma 2.4.5. *(Cf. [117, Lemma 4.3]) Let $\kappa \ge 0, \mu > 0, q > N$, assume $u_0 \in W^{1,q}(\Omega)$ nonnegative and radially symmetric. Suppose that $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$, $(u_{0,\varepsilon_j})_j \subset W^{1,q}(\Omega)$, $T > 0$, $M > 0$ are such that $\varepsilon_j \searrow 0$ as $j \to \infty$, all $u_{0,\varepsilon}$ are compatible, nonnegative and radial with $\|u_{0,\varepsilon} - u_0\|_{W^{1,q}(\Omega)} < \varepsilon$ and such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, for the solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) with initial condition $u_{0,\varepsilon}$, we have*

$$u_\varepsilon(x,t) \le M \tag{2.19}$$

for all $(x,t) \in \Omega \times (0,T)$. Then there exists a strong $W^{1,q}$-solution (u,v) of (2.2) in $\Omega \times (0,T)$ such that

$$\begin{aligned} u_\varepsilon &\to u && \text{in } C^0(\overline{\Omega}\times[0,T]),\\ u_\varepsilon &\overset{*}{\rightharpoonup} u && \text{in } L^\infty((0,T);W^{1,q}(\Omega)),\\ v_\varepsilon &\to v && \text{in } C^{2,0}(\overline{\Omega}\times[0,T]). \end{aligned}$$

Proof. According to Corollary 2.3.12 and by (2.19),

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \text{ is bounded in } L^\infty((0,T);W^{1,q}(\Omega)). \tag{2.20}$$

Lemma 2.3.15 gives boundedness of the time derivatives: For some $p > 1$,

$$(u_{\varepsilon_j t})_{j\in\mathbb{N}} \text{ is bounded in } L^p((0,T);(W^{1,\frac{q}{q-1}}(\Omega))^*).$$

With the choice of $X = W^{1,q}(\Omega), Y = C^{\gamma}(\overline{\Omega}), Z = (W^{1,\frac{q}{q-1}}(\Omega))^*$, Lemma 2.4.4 allows to conclude relative compactness of $(u_{\varepsilon_j})_j$ in $C^0([0,T], C^{\gamma}(\Omega))$. Due to this and (2.20), given any subsequence of $(\varepsilon_j)_j$, we can pick a further subsequence thereof such that

$$u_{\varepsilon_{j_i}} \to u \qquad \text{in } C^0([0,T], C^{\gamma}(\Omega)), \tag{2.21}$$

$$u_{\varepsilon_{j_i}} \overset{*}{\rightharpoonup} u \qquad \text{in } L^{\infty}((0,T); W^{1,q}(\Omega)) \tag{2.22}$$

as $i \to \infty$ and also by the propagation of the Cauchy property from $(u_{\varepsilon_{j_i}})_{i\in\mathbb{N}}$ to $(v_{\varepsilon_{j_i}})_{i\in\mathbb{N}}$ via

$$\begin{aligned} \left\| v_{\varepsilon_{j_i}} - v_{\varepsilon_{j_k}} \right\|_{C^{2,0}(\overline{\Omega}\times[0,T])} \leq & \left\| v_{\varepsilon_{j_i}} - v_{\varepsilon_{j_k}} \right\|_{C^{2+\gamma,0}(\overline{\Omega}\times[0,T])} \\ \leq & C \left\| u_{\varepsilon_{j_i}} - u_{\varepsilon_{j_k}} \right\|_{C^{\gamma,0}(\overline{\Omega}\times[0,T])} = C \left\| u_{\varepsilon_{j_i}} - u_{\varepsilon_{j_k}} \right\|_{C^0([0,T],C^{\gamma}(\overline{\Omega}))} \end{aligned}$$

for all $i, k \in \mathbb{N}$, where C is the constant from Lemma 2.2.4,

$$v_{\varepsilon_{j_i}} \to v \qquad C^{2,0}(\overline{\Omega} \times [0,T]) \qquad \text{as } i \to \infty. \tag{2.23}$$

The limit (u,v) is a strong $W^{1,q}$-solution of (2.2), as can be seen by testing (2.1) by an arbitrary $\varphi \in C_0^{\infty}(\overline{\Omega} \times [0,T))$ and taking $\varepsilon = \varepsilon_{j_i} \to 0$ in each of the integrals separately, as possible by (2.21) to (2.23):

$$\begin{aligned} -\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,t) = -0 \cdot \int_0^T \int_\Omega \nabla u \cdot \nabla\varphi - \int_0^T \int_\Omega u \nabla v \cdot \nabla \varphi \\ + \kappa \int_0^T \int_\Omega u\varphi - \mu \int_0^T \int_\Omega u^2 \varphi. \end{aligned}$$

Hence the limit of all these subsequences $u_{\varepsilon_{j_i}}$ of subsequences is the same, namely the unique (Lemma 2.4.3) solution of (2.2) and therefore the whole sequence converges (in the spaces indicated in equations (2.21) to (2.23)) to the solution (u,v) of (2.2). □

In Lemma 2.4.5 we assumed uniform boundedness of the approximating solutions. Fortunately, on small time scales we are entitled to do so and can prove the following:

Lemma 2.4.6. *Let $\kappa \geq 0, \mu > 0, q > N$. Then for $D > 0$ there is some $T(D) > 0$ such that for any radial symmetric nonnegative $u_0 \in W^{1,q}(\Omega)$ fulfilling $\|u_0\|_{W^{1,q}(\Omega)} < D$ there is a unique $W^{1,q}(\Omega)$-solution (u,v) of (2.2) in $\Omega \times (0,T(D))$. Furthermore, if $u_{0,\varepsilon}$ are compatible functions satisfying $\|u_{0,\varepsilon} - u_0\|_{W^{1,q}(\Omega)} < \varepsilon$, this solution (u,v) can be approximated by solutions $(u_\varepsilon, v_\varepsilon)$ of (2.1) (with initial condition $u_{0,\varepsilon}$) in the following sense:*

$$u_\varepsilon \to u \qquad \text{in } C^0(\overline{\Omega} \times [0,T]), \tag{2.24}$$

$$u_\varepsilon \overset{*}{\rightharpoonup} u \qquad \text{in } L^{\infty}((0,T); W^{1,q}(\Omega)), \tag{2.25}$$

$$v_\varepsilon \to v \qquad \text{in } C^{2,0}(\Omega \times (0,T)). \tag{2.26}$$

Moreover, with K as in Lemma 2.3.9 this solution satisfies

$$\int_\Omega |\nabla u(\cdot,t)|^q \leq \left(\int_\Omega |\nabla u_0|^q + K|\Omega| \int_0^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp\left(\int_0^t (5q \|u\|_{L^\infty(\Omega)}) + \kappa q t \right)$$

for a.e. $t \in (0, T(D))$.

Proof. For $\varepsilon \in (0,1)$ we let $u_{0,\varepsilon}$ be compatible and $\|u_{0,\varepsilon} - u_0\|_{W^{1,q}(\Omega)} < \varepsilon$. We apply Theorem 2.3.14 with $L := D+1$ to obtain $T(L)$ such that the solutions u_ε to (2.1) with initial data u_0 exist on $(0, T(L))$ and are bounded by $M(L)$ on that interval. Here Lemma 2.4.5 applies to provide a strong $W^{1,q}$-solution with the claimed approximation properties. The inequality results from Corollary 2.3.12 as follows:
According to Corollary 2.3.12, for all $t \in (0, T(L+1))$,

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^q \le \left(\int_\Omega |\nabla u_{0,\varepsilon}|^q + K|\Omega| \int_0^t \|u_\varepsilon\|_{L^\infty(\Omega)}^{1+q} \right) \exp\left(\int_0^t (5q \|u_\varepsilon\|_{L^\infty(\Omega)}) + \kappa q t \right).$$

We let $t \ge 0$ be such that $u(\cdot,t) \in W^{1,q}(\Omega)$. Convergence of the right-hand side is obvious because of the uniform convergence $u_\varepsilon \to u$ and $u_{0,\varepsilon} \to u_0$ in $W^{1,q}(\Omega)$. This implies boundedness of $(\nabla u_{\varepsilon_j})_{j\in\mathbb{N}}$ in $L^q(\Omega)$, hence $L^q(\Omega)$-weak convergence along a subsequence and – due to the weak lower semicontinuity of the norm –

$$\begin{aligned}\int_\Omega |\nabla u(\cdot,t)|^q &\le \liminf_{k\to\infty} \int_\Omega |\nabla u_{\varepsilon_{j_k}}(\cdot,t)|^q \\ &\le \left(\int_\Omega |\nabla u_0|^q + |\Omega| K \int_0^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp\left(\int 5q \|u\|_{L^\infty(\Omega)} + \kappa q t \right).\end{aligned}$$

□

2.4.4 Continuation and existence on maximal time intervals

Solutions constructed up to now may only exist on very short time intervals. With the following theorem (which parallels [117, Thm. 1.2] in statement and proof) we ensure that they can be glued together to yield a solution on a maximal time interval – to all eternity or until blow-up.

Theorem 2.4.7. *Let $\kappa \ge 0, \mu > 0$, for some $q > N$ suppose $u_0 \in W^{1,q}(\Omega)$ is nonnegative and radially symmetric. Then there exist $T_{max} \in (0,\infty]$ and a unique pair (u,v) of functions*

$$\begin{aligned}&u \in C(\overline{\Omega} \times [0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,q}(\Omega)),\\ &v \in C^{2,0}(\overline{\Omega} \times [0,T_{max}))\end{aligned}$$

that form a strong $W^{1,q}$-solution of (2.2) and which are such that

$$\textit{either} \quad T_{max} = \infty \quad \textit{or} \quad \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.27}$$

Proof. We apply Lemma 2.4.6 to $D := \|u_0\|_{W^{1,q}(\Omega)}$ to gain $T > 0$ and a strong $W^{1,q}$-solution (u,v) of (2.2) in $\Omega \times (0,T)$ fulfilling

$$\int_\Omega |\nabla u|^q \le \left(\int_\Omega |\nabla u_0|^q + K|\Omega| \int_0^t \|u\|_{L^\infty(\Omega)}^{1+q} \right) \exp(5q \int_0^t \|u\|_{L^\infty(\Omega)} + \kappa q t) \tag{2.28}$$

for almost every $t \in (0,T)$. Accordingly, the set

$$\begin{aligned}S := \{\widetilde{T} > 0 \mid \exists \text{ strong } W^{1,q}\text{-solution of (2.2) in } \Omega \times (0,\widetilde{T})\\ \text{with initial condition } u_0 \text{ and satisfying (2.28) for a.e. } t \in (0,\widetilde{T})\}\end{aligned}$$

is not empty and $T_{max} := \sup S \le \infty$ is well-defined. According to Lemma 2.4.3, the strong $W^{1,q}$-solution on $\Omega \times (0, T_{max})$, which obviously exists, is unique. We only have to verify the extensibility criterion (2.27). If $T_{max} < \infty$ and $\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$, then there would exist $M > 0$ such that for all $(x,t) \in \Omega \times (0, T_{max})$

$$u(x,t) \le M.$$

We could let $N \subset (0, T_{max})$ be a set of measure zero, as provided by the definition of S, such that (2.28) holds for all $t \in (0, T_{max}) \setminus N$. Together with $u \le M$ this would imply

$$\|u(\cdot, t_0)\|_{W^{1,q}(\Omega)} \le D_1$$

for some positive D_1 and for each $t_0 \in (0, T_{max}) \setminus N$. Lemma 2.4.6 would yield the existence of a strong $W^{1,q}$-solution of

$$\begin{cases} \widehat{u}_t = -\nabla\cdot(\widehat{u}\nabla\widehat{v}) + \kappa\widehat{u} - \mu\widehat{u}^2 \\ 0 = \Delta\widehat{v} - \widehat{v} + \widehat{u} \\ 0 = \partial_\nu \widehat{u}\big|_{\partial\Omega} = \partial_\nu\widehat{v}\big|_{\partial\Omega}, \\ \widehat{u}(x,0) = u(x,t_0) \end{cases}$$

on $\Omega \times (0, T(D_1))$, which would satisfy

$$\int_\Omega |\nabla\widehat{u}(\cdot,t)|^q \le \left(\int_\Omega |\nabla u(\cdot,t_0)|^q + K|\Omega|\int_0^t \|\widehat{u}\|_{L^\infty(\Omega)}^{1+q}\right)\exp\left(5q\int_0^t \|\widehat{u}\|_{L^\infty(\Omega)} + \kappa q t\right) \tag{2.29}$$

for almost every $t \in (0, T(D_1))$.
Upon the choice of $t_0 \in (0, T_{max}) \setminus N$ with $t_0 > T_{max} - \frac{T(D_1)}{2}$,

$$(\widetilde{u},\widetilde{v})(\cdot,t) = \begin{cases} (u,v)(\cdot,t) & t \in (0,t_0) \\ (\widehat{u},\widehat{v})(\cdot, t-t_0) & t \in [t_0, t_0 + T(D_1)) \end{cases}$$

would define a strong $W^{1,q}$-solution of (2.2) in $\Omega \times (0, t_0 + T(D_1))$ which clearly would satisfy (2.28) for a.e. $t < t_0$. For $t > t_0$ on the other hand, a combination of (2.29) and (2.28) would give

$$\begin{aligned} \int_\Omega |\nabla\widetilde{u}(\cdot,t)|^q \le& \Bigg(\left(\int_\Omega |\nabla u_0|^q + K|\Omega|\int_0^{t_0} \|\widetilde{u}\|_{L^\infty(\Omega)}^{1+q}\right)\exp\left(5q\int_0^{t_0} \|\widetilde{u}\|_{L^\infty(\Omega)} + \kappa q t_0\right) \\ &+ K|\Omega|\int_{t_0}^t \|\widetilde{u}\|_{L^\infty(\Omega)}^{1+q}\Bigg)\cdot\exp\left(5q\int_{t_0}^t \|\widetilde{u}\|_{L^\infty(\Omega)} + \kappa q(t-t_0)\right) \\ \le& \left(\int_\Omega |\nabla u_0|^q + K|\Omega|\int_0^t \|\widetilde{u}\|_{L^\infty(\Omega)}^{1+q}\right)\exp\left(5q\int_0^t \|\widetilde{u}\|_{L^\infty(\Omega)} + \kappa q t\right). \end{aligned}$$

This would finally lead to a contradiction to the definition of T_{max} as supremum, because then obviously $(\widetilde{u},\widetilde{v})$ would satisfy (2.28) for a.e. $t \in (0, t_0 + T(D_1))$. □

2.4.5 An estimate for strong solutions: Boundedness in L^1

Having confirmed existence and uniqueness of solutions, we set out to explore some more of their properties. And as in [117, Lemma 4.1], one of the first facts that can be observed (and proven like in the 1-dimensional case) is their boundedness in $L^1(\Omega)$.

Lemma 2.4.8. *Let $\kappa \geq 0, \mu > 0$, assume that for $T > 0, q > N$, u is a strong $W^{1,q}$-solution of (2.2) in $\Omega \times (0,T)$ with $u_0 \in W^{1,q}(\Omega), u_0$ nonnegative. Then for all $t \in (0,T)$*

$$\int_\Omega u(x,t)\,dx \leq \max\left\{\int_\Omega u_0, \frac{\kappa|\Omega|}{\mu}\right\}.$$

Proof. We define $y(t) = \int_\Omega u(x,t)\mathrm{d}x$. Then y is continuous on $[0,T)$, as $u \in C^0(\overline{\Omega} \times [0,T))$ and $u \in L^\infty_{loc}([0,T); W^{1,q}(\Omega)) \hookrightarrow L^\infty_{loc}([0,T); L^1(\Omega)) \hookrightarrow L^1_{loc}([0,T) \times \overline{\Omega})$, and it is sufficient to show that $y \in C^1((0,T))$ and for all $t \in (0,T)$

$$y'(t) \leq \kappa y(t) - \frac{\mu}{|\Omega|} y^2(t).$$

To see this, we let $t_0 \in (0,T), t_1 \in (t_0,T)$ and let $\xi_\delta \in W^{1,\infty}(\mathbb{R})$ be given by

$$\xi_\delta(t) = \begin{cases} 0 & t < t_0 - \delta \vee t > t_1 + \delta \\ \frac{t-t_0+\delta}{\delta} & t \in [t_0 - \delta, t_0] \\ 1 & t \in (t_0, t_1) \\ \frac{t_1 - t + \delta}{\delta} & t \in [t_1, t_1 + \delta] \end{cases}$$

for $\delta \in (0, \delta_0)$ with $\delta_0 = \min\{t_0, T - t_1\}$. Then $\varphi(x,t) := \xi_\delta(t)$, $(x,t) \in \Omega \times (0,T)$ defines an admissible test function in (2.14) and we have

$$-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) = \int_0^T \int_\Omega u \nabla v \cdot \nabla \varphi + \kappa \int_0^T \int_\Omega u\varphi - \mu \int_0^T \int_\Omega u^2 \varphi$$

so that, by $\xi_\delta(0) = 0$ and $\nabla \varphi = 0$,

$$-\frac{1}{\delta}\int_{t_0-\delta}^{t_0} \int_\Omega u + \frac{1}{\delta}\int_{t_1}^{t_1+\delta} \int_\Omega u = \kappa \int_0^T \int_\Omega \xi_\delta(t) u(x,t)\mathrm{d}x\mathrm{d}t - \mu \int_0^T \int_\Omega u^2 \xi_\delta(t)\mathrm{d}x\mathrm{d}t.$$

Since u is continuous, the left hand side converges to $y(t_1) - y(t_0)$, whereas the right hand side makes application of the dominated convergence theorem possible due to the boundedness of u on $[0, T - \delta_0]$ as $\delta \searrow 0$ and we arrive at

$$y(t_1) - y(t_0) = \kappa \int_{t_0}^{t_1} \int_\Omega u - \mu \int_{t_0}^{t_1} \int_\Omega u^2 \mathrm{d}x\mathrm{d}t \leq \kappa \int_{t_0}^{t_1} u - \mu \int_{t_0}^{t_1} \frac{1}{|\Omega|}\left(\int_\Omega u\right)^2.$$

Upon division by $t_1 - t_0$ and taking limits $t_1 \to t_0$, we infer that indeed $y \in C^1((0,T))$ with

$$y'(t) \leq \kappa y(t) - \frac{\mu}{|\Omega|} y^2(t) \qquad \text{for all } t \in (0,T).$$ □

2.4.6 Global existence for large μ

Bounds on the $L^\infty(\Omega)$-norm are the only thing we need to guarantee existence of solutions for longer times. They arise as a corollary to Lemma 2.3.6, which directly implies the following.

Corollary 2.4.9. *Let $\kappa \geq 0, \mu \geq 1, q > N$. For each nonnegative, radial $u_0 \in W^{1,q}(\Omega)$, (2.2) has a unique global strong $W^{1,q}$-solution (u,v). Furthermore, if $u_0 \not\equiv 0$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \begin{cases} \frac{\kappa}{\mu-1}\left(1 + \left(\frac{\kappa}{(\mu-1)\|u_0\|_{L^\infty(\Omega)}} - 1\right)\mathrm{e}^{-\kappa t}\right)^{-1}, & \kappa > 0, \mu > 1, \\ \frac{\|u_0\|_{L^\infty(\Omega)}}{1 + (\mu-1)\|u_0\|_{L^\infty(\Omega)} t}, & \kappa = 0, \mu > 1, \\ \|u_0\|_{L^\infty(\Omega)} \mathrm{e}^{\kappa t}, & \kappa > 0, \mu = 1, \\ \|u_0\|_{L^\infty(\Omega)}, & \kappa = 0, \mu = 1. \end{cases}$$

Proof. Local existence up to a maximal time $T_{max} \leq \infty$ is given by Theorem 2.4.7. For each $T \in (0, T_{max})$, there are solutions of (2.1) converging to (u, v) in $C^0(\overline{\Omega} \times (0,T))$, hence u inherits the bounds from Lemma 2.3.6. By (2.27), (u, v) must thus be global. □

Without further labour, we can state what we have obtained so far:

Proposition 2.4.10. *Let $\kappa \geq 0, \mu \geq 1, q > N$. Then for each nonnegative radial $u_0 \in W^{1,q}(\Omega)$, (2.2) has a unique global strong $W^{1,q}$-solution. Furthermore, if $\mu > 1$ or $\kappa = 0$, u, v are bounded in $\Omega \times (0, \infty)$.*

2.4.7 Blow-up for small μ

The contrasting – and more interesting – case is that of small values of μ. Here we will show blow-up. We borrow the following technical tool from [117]:

Lemma 2.4.11. *Let $a > 0$, $b \geq 0$, $d > 0$, and $\kappa > 1$ be such that*

$$a > \left(\frac{2b}{d}\right)^{\frac{1}{\kappa}}.$$

Then if for some $T > 0$ the function $y \in C^0([0,T))$ is nonnegative and satisfies

$$y(t) \geq a - bt + d\int_0^t y^{\kappa}(s)ds$$

for all $t \in (0,T)$, we necessarily have

$$T \leq \frac{2}{(\kappa-1)a^{\kappa-1}d}.$$

Proof. [117, Lemma 4.9]. □

To be of any use to us, this estimate must be accompanied by lower bounds for (some norm of) u. We prepare those by the following lemma

Lemma 2.4.12. *Let $\kappa \geq 0$ and $\mu > 0$. For all $p > 1$ and $\eta > 0$ there is $B(p, \eta) > 0$ such that for all $q > 1$ all $W^{1,q}$-solutions (u, v) of (2.2) with nonnegative u_0 in $\Omega \times (0,T)$ satisfy*

$$\int_\Omega u^p(\cdot,t) \geq \int_\Omega u_0^p + ((1-\mu)p - 1 - \eta)\int_0^t\int_\Omega u^{p+1} - B(p,\eta)\int_0^t \left(\int_\Omega u\right)^{p+1}$$

for all $t \in (0,T)$.

Proof. The same testing procedure as in [117, Lemma 4.8] leads to success. We repeat it (with the necessary adaptions) for the sake of completeness, because Lemma 2.4.12 is a main building block of the blow-up result.
Let $T_0 \in (0,T)$, $t_0 \in (0, T_0)$, $\delta \in (0, T - t_0)$, ξ_δ be as in the proof of Lemma 2.4.3:

$$\xi_\delta(t) := \begin{cases} 1 & t < t_0, \\ \frac{t_0 - t + \delta}{\delta} & t \in [t_0, t_0 + \delta], \\ 0 & t > t_0 + \delta. \end{cases}$$

For each $\zeta > 0$, the function $(u+\zeta)^{p-1}$ belongs to $L^\infty_{loc}([0,T); W^{1,q}(\Omega))$ and for $\delta \in (0, T_0 - t_0)$, $h \in (0,1)$,$\zeta > 0$,

$$\varphi(x,t) := \xi_\delta(t)\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-1}\mathrm{d}s, \qquad (x,t) \in \Omega \times (0,T),$$

is a test function for (2.14), if we set $u(\cdot,t) = u_0$ for $t < 0$. This yields

$$\begin{aligned}
\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega u(x,t)\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-1}ds\mathrm{d}x\mathrm{d}t - \int_\Omega u_0(x)(u_0(x)+\zeta)^{p-1}\mathrm{d}x \\
- \int_0^T\int_\Omega \xi_\delta(t)u(x,t)\frac{(u(x,t)+\zeta)^{p-1}-(u(x,t-h)+\zeta)^{p-1}}{h}\mathrm{d}x\mathrm{d}t \\
= (p-1)\int_0^T\int_\Omega \xi_\delta(t)u(x,t)\nabla v(x,t)\cdot\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-2}\nabla u(x,s)ds\mathrm{d}x\mathrm{d}t \\
+ \kappa\int_0^T\int_\Omega \xi_\delta(t)u(x,t)\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-1}ds\mathrm{d}x\mathrm{d}t \\
- \mu\int_0^T\int_\Omega \xi_\delta(t)u^2(x,t)\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-1}ds\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Letting δ tend to 0, we use continuity of u and Lebesgue's theorem to obtain

$$\begin{aligned}
\int_\Omega u(x,t_0)\frac{1}{h}\int_{t_0-h}^{t_0}(u(x,s)+\zeta)^{p-1}\mathrm{d}s\mathrm{d}x \\
- \int_0^{t_0}\int_\Omega u(x,t)\frac{(u(x,t)+\zeta)^{p-1}-(u(x,t-h)+\zeta)^{p-1}}{h}\mathrm{d}x\mathrm{d}t - \int_\Omega u_0(u_0+\zeta)^{p-1} \\
= (p-1)\int_0^{t_0}\int_\Omega u(x,t)\nabla v(x,t)\cdot\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-2}\nabla u(x,s)\mathrm{d}s\mathrm{d}x\mathrm{d}t \\
+ \kappa\int_0^{t_0}\int_\Omega u(x,t)\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-1}\mathrm{d}s\mathrm{d}x\mathrm{d}t \\
- \mu\int_0^{t_0}\int_\Omega u^2(x,t)\frac{1}{h}\int_{t-h}^{t}(u(x,s)+\zeta)^{p-1}\mathrm{d}s\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.30}$$

We consider the second integral on the left hand side:

$$\begin{aligned}
-\int_0^{t_0}\int_\Omega u(x,t)\frac{(u(x,t)+\zeta)^{p-1}-(u(x,t-h)+\zeta)^{p-1}}{h}\mathrm{d}x\mathrm{d}t \\
= -\frac{1}{h}\int_0^{t_0}\int_\Omega (u(x,t)+\zeta)^p\mathrm{d}x\mathrm{d}t + \frac{1}{h}\int_0^{t_0}\int_\Omega (u(x,t)+\zeta)(u(x,t-h)+\zeta)^{p-1}\mathrm{d}x\mathrm{d}t \\
+ \frac{\zeta}{h}\int_0^{t_0}\int_\Omega (u(x,t)+\zeta)^{p-1}\mathrm{d}x\mathrm{d}t - \frac{\zeta}{h}\int_0^{t_0}\int_\Omega (u(x,t-h)+\zeta)^{p-1}\mathrm{d}x\mathrm{d}t = I_1 + I_2,
\end{aligned}$$

where, upon an application of Young's inequality, we obtain

$$\begin{aligned}
I_1 \le -\frac{1}{h}\int_0^{t_0}\int_\Omega (u(x,t)+\zeta)^p\mathrm{d}x\mathrm{d}t + \frac{1}{ph}\int_0^{t_0}\int_\Omega (u(x,t)+\zeta)^p\mathrm{d}x\mathrm{d}t \\
+ \frac{p-1}{p}\frac{1}{h}\int_0^{t_0}\int_\Omega (u(x,t-h)+\zeta)^p\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$= -\frac{p-1}{ph}\int_0^{t_0}\int_\Omega (u(x,t)+\zeta)^p \mathrm{d}x\mathrm{d}t + \frac{p-1}{ph}\int_0^{t_0}\int_\Omega (u(x,t-h)+\zeta)^p \mathrm{d}x\mathrm{d}t$$
$$= \frac{p-1}{ph}\left[h\int_\Omega (u_0(x)+\zeta)^p \mathrm{d}x - \int_{t_0-h}^{t_0}\int_\Omega (u(x,t)+\zeta)^p \mathrm{d}x\mathrm{d}t\right]$$

and by similar cancellations as in the last step,

$$I_2 = -\zeta\int_\Omega (u_0(x)+\zeta)^{p-1}\mathrm{d}x + \frac{\zeta}{h}\int_{t_0-h}^{t_0}\int_\Omega (u(x,t)+\zeta)^{p-1}\mathrm{d}x\mathrm{d}t.$$

Hence, again by continuity of u,

$$\limsup_{h\to 0}\left(-\int_0^{t_0}\int_\Omega u(x,t)\frac{(u(x,t)+\zeta)^{p-1}-(u(x,t-h)+\zeta)^{p-1}}{h}\mathrm{d}x\mathrm{d}t\right)$$
$$\le \frac{p-1}{p}\int_\Omega (u_0+\zeta)^p - \frac{p-1}{p}\int_\Omega (u(\cdot,t_0)+\zeta)^p - \zeta\int_\Omega (u_0+\zeta)^{p-1}$$
$$+\zeta\int_\Omega (u(\cdot,t_0)+\zeta)^{p-1}.$$

For any $\psi \in L^{q'}(\Omega\times(0,T_0);\mathbb{R}^N)$, where $\frac{1}{q'}+\frac{1}{q}=1$, by $u\in L^\infty((-1,T);W^{1,q}(\Omega))$,

$$\int_\Omega\int_0^t \frac{1}{h}\int_{t-h}^t (u(x,s)+\zeta)^{p-2}\nabla u(x,s)ds\cdot\psi(x,t)\mathrm{d}x\mathrm{d}t$$
$$\to \int_\Omega\int_0^t (u(x,t)+\zeta)^{p-2}\nabla u(x,t)\cdot\psi(x,t)\mathrm{d}x\mathrm{d}t.$$

Therefore, setting $\psi = u\nabla v$ and similarly dealing with the first integral, (2.30) becomes

$$\int_\Omega u(\cdot,t_0)(u(\cdot,t_0)+\zeta)^{p-1} + \frac{p-1}{p}\int_\Omega (u_0+\zeta)^p - \frac{p-1}{p}\int_\Omega (u(\cdot,t_0)+\zeta)^p$$
$$-\zeta\int_\Omega (u_0+\zeta)^{p-1} + \zeta\int_\Omega (u(\cdot,t_0)+\zeta)^{p-1} - \int_\Omega u_0(u_0+\zeta)^{p-1}$$
$$\ge (p-1)\int_0^{t_0}\int_\Omega u\nabla v\cdot(u+\zeta)^{p-2}\nabla u + \kappa\int_0^{t_0}\int_\Omega u(u+\zeta)^{p-1} - \mu\int_0^{t_0}\int_\Omega u^2(u+\zeta)^{p-1}$$
$$\ge (p-1)\int_0^{t_0}\int_\Omega u\nabla v\cdot(u+\zeta)^{p-2}\nabla u - \mu\int_0^{t_0}\int_\Omega u^2(u+\zeta)^{p-1}$$

as $h\to 0$. Because $p>1$, $u(u+\zeta)^{p-2}\to u^{p-1}$ uniformly as $\zeta\to 0$. In this limit we therefore obtain

$$\int_\Omega u^p(\cdot,t_0) + \frac{p-1}{p}\int_\Omega u_0^p - \frac{p-1}{p}\int_\Omega u^p(\cdot,t_0) - \int_\Omega u_0^p$$
$$\ge (p-1)\int_0^{t_0}\int_\Omega u^{p-1}\nabla v\cdot\nabla u - \mu\int_0^{t_0}\int_\Omega u^{p+1}.$$

This is equivalent to the following inequality, where we can use the elliptic equation of (2.2) to express Δv differently.

$$\frac{1}{p}\int_\Omega u^p(\cdot,t_0) - \frac{1}{p}\int_\Omega u_0^p \ge -\frac{p-1}{p}\int_0^{t_0}\int_\Omega u^p\Delta v - \mu\int_0^{t_0}\int_\Omega u^{p+1}$$

$$= \frac{p-1}{p}\int_0^{t_0}\int_\Omega u^{p+1} - \frac{p-1}{p}\int_0^{t_0}\int_\Omega u^p v - \mu\int_0^{t_0}\int_\Omega u^{p+1}.$$

Here Young's inequality and Lemma 2.2.3 provide constants C_1 and $\widetilde{c}$ respectively, such that

$$\begin{aligned}\frac{1}{p}\int_\Omega u^p(\cdot,t_0) - \frac{1}{p}\int_\Omega u_0^p & \\ &\geq \left(\frac{p-1}{p}-\mu\right)\int_0^{t_0}\int_\Omega u^{p+1} - \frac{\eta}{2p}\int_0^{t_0}\int_\Omega u^{p+1} - C_1\int_0^{t_0}\int_\Omega v^{p+1} \\ &\geq \left(\frac{p-1}{p}-\mu-\frac{\eta}{2p}\right)\int_0^{t_0}\int_\Omega u^{p+1} - \frac{\eta}{2p}\int_0^{t_0}\int_\Omega u^{p+1} - \widetilde{c}\int_0^{t_0}\left(\int_\Omega u\right)^{p+1} \\ &= \frac{1}{p}\left((1-\mu)p-1-\eta\right)\int_0^{t_0}\int_\Omega u^{p+1} - \widetilde{c}\int_0^{t_0}\left(\int_\Omega u\right)^{p+1}. \qquad \square\end{aligned}$$

These lemmata can be utilized to decide which alternative of Theorem 2.4.7 occurs for $\mu < 1$. It is the same as in case of dimension one (see [117, Thm. 1.4]) and can be proven almost identically:

Theorem 2.4.13. *Let $\kappa \geq 0, \mu \in (0,1)$. For all $p > \frac{1}{1-\mu}$ there is $C(p) > 0$ with the following property: Whenever $q > N$ and $u_0 \in W^{1,q}(\Omega)$ is nonnegative, radial and*

$$\|u_0\|_{L^p(\Omega)} > C(p)\max\left\{\frac{1}{|\Omega|}\int u_0, \frac{\kappa}{\mu}\right\},$$

the strong $W^{1,q}$-solution of (2.2) blows up in finite time, i.e. in Theorem 2.4.7, we have $T_{max} < \infty$ and $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

Proof. We let $\eta = \frac{(1-\mu)p-1}{2} > 0$, $B(p,\eta)$ as in Lemma 2.4.12,

$$C(p) := \left(\frac{4B(p,\eta)}{(1-\mu)p-1}\right)^{\frac{1}{p+1}}|\Omega|^{1+\frac{1}{p(p+1)}}.$$

Supposing that $\|u_0\|_{L^p(\Omega)} > C(p)\max\{\frac{1}{|\Omega|}\int_\Omega u_0, \frac{\kappa}{\mu}\}$ and the corresponding $W^{1,q}$-solution of (2.2) (from Theorem 2.4.7) was global in time, i.e. $T_{max} = \infty$, we let $y(t) := \int_\Omega u^p(x,t)\mathrm{d}x$ for $t \geq 0$. This would define a continuous function on $[0,\infty)$.
According to Lemma 2.4.12, $\kappa \geq 0, \mu > 0, p > \frac{1}{1-\mu} > 1$ and the choice of $B(p,\eta)$ would make y satisfy

$$\begin{aligned}y(t) &\geq \int_\Omega u_0^p + ((1-\mu)p-1-\eta)\int_0^t\int_\Omega u^{p+1}\mathrm{d}s - B(p,\eta)\int_0^t\left(\int_\Omega u\right)^{p+1} \\ &\geq y(0) + \frac{(1-\mu)p-1}{2}|\Omega|^{-\frac{1}{p}}\int_0^t [y(s)]^{\frac{p+1}{p}} - B(p,\eta)\int_0^t\left(\int_\Omega u\right)^{p+1}\end{aligned}$$

for all $t \geq 0$. By Lemma 2.4.8, for all $t \geq 0$ we would obtain

$$\int_0^t\left(\int_\Omega u\right)^{p+1} \leq \int_0^t (|\Omega|\widehat{m})^{p+1} \leq |\Omega|^{p+1}\widehat{m}^{p+1}t,$$

where $\widehat{m} = \max\{\frac{1}{|\Omega|}\int_\Omega u_0, \frac{\kappa}{\mu}\}$. Therefore

$$y(t) \geq y(0) + \frac{(1-\mu)p-1}{2}|\Omega|^{-\frac{1}{p}}\int_0^t y^{\frac{p+1}{p}}(s)ds - B(p,\eta)|\Omega|^{p+1}\widehat{m}^{p+1}t$$

for all $t \geq 0$. An application of Lemma 2.4.11 with $a = y(0)$, $b = B(p,\eta)|\Omega|^{p+1}\widehat{m}^{p+1}$, $d = \frac{(1-\mu)p-1}{2}|\Omega|^{-\frac{1}{p}}$, and $\kappa = \frac{p+1}{p}$ now would allow to conclude from

$$a\left(\frac{2b}{d}\right)^{-\frac{1}{\kappa}} = \|u_0\|_{L^p(\Omega)}^p \left(\frac{4B(p,\eta)|\Omega|^{p+1}\widehat{m}^{p+1}|\Omega|^{\frac{1}{p}}}{(1-\mu)p-1}\right)^{-\frac{p}{p+1}} > 1$$

that – contradicting our assumption and proving the theorem – T_{max} must be finite. □

2.5 No thresholds on population density. Proof of Theorem 2.1.1

Let us now, finally, prove the main result, corresponding to [117, Thm. 1.1] and expanding this to higher dimensional space.

Proof of Theorem 2.1.1. (See [117, Thm. 1.1]). Let $u_0 \in W^{1,q}(\Omega)$ be as in the statement of the theorem and let $T > 0$ denote the maximal existence time of the corresponding solution of (2.2). We then know by Theorem 2.4.13 that $T < \infty$ and

$$\limsup_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.31}$$

In the following we only consider solutions of (2.1) that exist at least until time T. All other solutions blow up earlier according to Lemma 2.3.2 and therefore trivially satisfy the theorem. Let us assume that Theorem 2.1.1 were not true. Then there would be $M > 0$ and a sequence $\varepsilon_j \to 0$ such that

$$u_{\varepsilon_j}(x,t) \leq M$$

for all $(x,t) \in \Omega \times (0,T)$ and $j \in \mathbb{N}$. Therefore, we would obtain convergence by Lemma 2.4.5:

$$u_{\varepsilon_j} \to \widetilde{u} \quad \text{in } C^0(\overline{\Omega} \times [0,T])$$

and

$$v_{\varepsilon_j} \to \widetilde{v} \quad \text{in } C^{2,0}(\overline{\Omega} \times [0,T]),$$

as $j \to \infty$, where $(\widetilde{u},\widetilde{v})$ is a strong solution of (2.2). Because such solutions are unique, $(\widetilde{u},\widetilde{v}) = (u,v)$ and in particular $u = \widetilde{u} \leq M$ in $\Omega \times (0,T)$, contradicting (2.31). □

3 Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source

3.1 Introduction

Starting from the pioneering work of Keller and Segel [43], an extensive mathematical literature has grown on the Keller-Segel model and its variants, mathematical models describing chemotaxis, that is the tendency of (micro-)organisms to adapt the direction of their (otherwise random) movement toward increasing concentrations of a signalling substance (see also Section 2.1).

If biological phenomena where chemotaxis plays a role are modelled on not only small time scales, often growth of the population, whose density we will denote by u, must be taken into account. A prototypical choice to accomplish this is the addition of logistic growth terms $+\kappa u - \mu u^2$ in the evolution equation for u. Here $+\kappa u$, with $\kappa \in \mathbb{R}$ being the difference between birth rate and death rate of the population, is used to describe population growth, and the term $-\mu u^2$ models additional overcrowding effects. Negative values of κ can be used to include effects like spontaneous degradation into the model (e.g. in the case of a starving population) that – in contrast to the effects modelled by the quadratic term – take place also in regions with small population density. Unfortunately, it is unclear whether global classical solutions to the chemotaxis-system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + \kappa u - \mu u^2 \\ v_t = \Delta v - v + u \\ \partial_\nu u\big|_{\partial\Omega} = \partial_\nu v\big|_{\partial\Omega} = 0 \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, \end{cases} \tag{3.1}$$

where u_0, v_0 are given functions, exist in the smooth, bounded domain $\Omega \subset \mathbb{R}^N$ if $N \geq 3$ and $\mu > 0$ is small.

The parabolic-elliptic simplification (where v_t is replaced by 0) of (3.1) has been considered in [100], where – besides some study of asymptotic behaviour – it is shown that weak solutions exist for arbitrary $\mu > 0$ and that they are smooth and globally classical if $\mu > \frac{N-2}{N}$. In [109] the existence of (very) weak solutions is proven under more general conditions. Under additional assupmtions, also the existence of a bounded absorbing set in $L^\infty(\Omega)$ is shown.

Turning to the parabolic-parabolic system, important findings are given in [111], which assert existence and uniqueness of global, smooth, bounded solutions to (3.1) under the condition that μ be large enough.

Additional results on existence of global solutions or even of an exponential attractor have been given in the two-dimensional case (see e.g. [78, 79]). In this case, global solutions exist for arbitrary $\mu > 0$.

But not only the restriction to dimension 2, also the inclusion of some kind of saturation effect in the chemotactic sensitivity [8], sublinear dependence of the chemotactic sensitivity on u [13] or

even changing the second equation into one that models the consumption of the chemoattractant (as done in [93, 95] for $\kappa = \mu = 0$) can make it possible to derive the global existence of solutions. The same can be accomplished by replacement of the secretion term $+u$ in the second equation of (3.1) by $+\frac{u}{(1+u)^{1-\beta}}$ with some $0 < \beta < \frac{9}{10}$, which enables the authors of [76] to show the existence of attractors in the corresponding dynamical system.
On the other hand, the model

$$\begin{cases} u_t = \varepsilon\Delta u - \nabla\cdot(u\nabla v) + \kappa u - \mu u^2 \\ 0 = \Delta v - v + u \end{cases} \tag{3.2}$$

has recently been shown to exhibit the following property (see Chapter 2): If $\mu \in (0,1)$ and the (radially symmetric) initial datum u_0 is large in a certain $L^p(\Omega)-$space, there exists some finite time such that up to this time any given threshold will be surpassed by solutions to (3.2) for sufficiently small $\varepsilon > 0$. Although this demeanour may be interesting from an emergence-of-patterns point-of-view and although solutions become very large, it still is not the same as blow-up and, in fact, also occurs in case of bounded solutions, even in space-dimension 1 [117].
In [112] it is shown that in another related model,

$$u_t = \Delta u - \nabla\cdot(u\nabla v) + \kappa u - \mu u^\beta, \qquad v_t = \Delta v - m(t) + u, \qquad m(t) = \frac{1}{|\Omega|}\int_\Omega u,$$

blow-up may occur for space-dimension $N \geq 5$ and exponents $1 < \beta < \frac{3}{2} + \frac{1}{2N-2}$.
Consequently, the supposition that any superlinear growth restriction already signifies the existence of a global, bounded solution does not stand unchallenged; and the question whether the above-mentioned results on the presence of global smooth solutions in similar situations find their analogue in the case of (3.1), the most prototypical chemotaxis system including logistic growth, is not clear at all.
In the present chapter, we therefore investigate the existence of solutions to (3.1). More precisely, we will construct weak solutions in the sense of Definition 3.5.1 below. We shall show that, in dimension 3 and under a smallness condition on κ, they become smooth after some time, which also excludes finite-time blow-up from then on. Note that this, however, does not provide any information on a small timescale.
To the aim sketched above we will then consider the approximate system

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla\cdot(u_\varepsilon\nabla v_\varepsilon) + \kappa u_\varepsilon - \mu u_\varepsilon^2 - \varepsilon u_\varepsilon^\theta \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon \\ \partial_\nu u_\varepsilon\big|_{\partial\Omega} = \partial_\nu v_\varepsilon\big|_{\partial\Omega} = 0 \\ u_\varepsilon(\cdot,0) = u_{0,\varepsilon}, \quad v(\cdot,0) = v_{0,\varepsilon}, \end{cases} \tag{3.3}$$

for $\theta > N+2$ with nonnegative initial values $u_{0,\varepsilon} \in C^0(\overline{\Omega})$ and $v_{0,\varepsilon} \in W^{1,N+1}(\Omega)$, where global classical solutions are quickly seen to exist, and derive estimates finally allowing for compactness arguments, which will provide the existence of a weak solution to (3.1) in Proposition 3.6.1 and Lemma 3.6.2.
We will employ the estimates from Section 3.4 to conclude that a solution must become small in an appropriate sense after some time. This, in turn, will be the starting point for an ODE comparison argument for the quantity $\int_\Omega u_\varepsilon^2(\cdot,t) + \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4$, whose thereby-obtained boundedness in conjunction with estimates on the Neumann heat semigroup results in eventual boundedness and hence in eventual smoothness of (u,v). We finally arrive at the following result:

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a smooth, bounded, convex domain and $u_0 \in L^2(\Omega)$, $v_0 \in W^{1,2}(\Omega)$ be nonnegative. Let $\kappa \in \mathbb{R}$, and $\mu > 0$.*
Then there is a nonnegative weak solution (u, v) (in the sense of Definition 3.5.1 below) to (3.1) *with initial data u_0, v_0.*
It can be approximated in the sense of a.e.-convergence by solutions of (3.3) *(and moreover in the sense detailed in Proposition 3.6.1).*
Furthermore, if $N = 3$, for any $\mu > 0$ there exists $\kappa_0 > 0$ such that if $\kappa < \kappa_0$, there is $T > 0$ such that u and v are a classical solution of (3.1) *for $t > T$.*
Moreover, in this case, there are $C > 0$ and $\gamma > 0$ such that for any $t > T$

$$\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \leq C.$$

Remark 3.1.2. Because we have adopted a weak concept of solution, it is conceivable that solutions to (3.1) are not unique. Investigation of this issue is beyond the scope of the present work and we state the following theorems only for solutions as provided by Theorem 3.1.1.

Besides the aforementioned results about attractors, little is known about asymptotic behaviour of solutions to models like (3.1). Recently, in [116] convergence to the positive homogeneous equilibrium was found for values of μ being sufficiently large as compared to the chemotactic sensitivity.
The richness of dynamics and pattern formation exhibited by chemotaxis models with growth [80, 47] however indicates that any speculation about asymptotical behaviour, especially about convergence to homogeneous states, should be backed by rigorous examinations.
In the situation of (3.1), we can summarize the long-term behaviour as follows: If $\kappa \leq 0$, solutions will converge to the trivial steady state - and any formation of interesting patterns has to take place on intermediary timescales.

Theorem 3.1.3. *Let $\mu > 0$, and $\kappa \leq 0$. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded convex domain and let (u, v) be the solution to* (3.1) *provided by Theorem 3.1.1. Then*

$$(u(\cdot,t), v(\cdot,t)) \to (0,0) \qquad \text{as } t \to \infty$$

in the sense of uniform convergence.

Remark 3.1.4. The same convergence result can be given for any classical solution of (3.1) for $\mu > 0$ and $\kappa \leq 0$ in $\Omega \subset \mathbb{R}^3$ as above. In this case, only minor adaptions of the proofs become necessary.

If κ is positive and sufficiently small, we can assert the existence of an absorbing set in the following sense:

Theorem 3.1.5. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded convex domain. Then for any $\mu > 0$ there is $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0)$, there is $\gamma > 0$ and a bounded set $B_{\mu,\kappa} \subset (C^{2+\gamma}(\overline{\Omega}))^2$ such that for all $(u_0, v_0) \in L^2(\Omega) \times W^{1,2}(\Omega)$, the corresponding solution (u, v) as constructed in Theorem 3.1.1 admits the existence of $T > 0$ such that*

$$(u(\cdot,t), v(\cdot,t)) \in B_{\mu,\kappa} \qquad \text{for all } t > T.$$

Moreover, for each fixed $\mu > 0$,

$$\operatorname{diam}_{L^\infty(\Omega)\times W^{1,\infty}(\Omega)}(B_{\mu,\kappa}) \to 0 \qquad \text{as } \kappa \searrow 0.$$

Further steps in this direction may hopefully lead to an even more detailed insight, much in the spirit of [75, 4], into the long-time behaviour of solutions to (3.1) in dimension 3 for small, positive μ.

Remark 3.1.6. In the calculations below, we will assume that $\mu > 0$ is a fixed number.
Throughout the chapter, we fix $\Omega \subset \mathbb{R}^N$ to be a convex bounded domain with smooth boundary and $u_0 \in L^2(\Omega)$, $v_0 \in W^{1,2}(\Omega)$ nonnegative.
Also, we let θ denote a number satisfying $\theta > N + 2$.

3.2 Existence of approximate solutions

The system (3.3) has a unique, global, classical solution. At a first glance, the source term $f(s) = \kappa s - \mu s^2 - \varepsilon s^\theta$ seems to satisfy the condition $f(s) \leq a - \mu_0 s^2$ from Theorem 0.1 of [111], which would provide a global solution, but as μ_0 depends on a, this theorem is not applicable in the present case. Even tracing the dependece of μ_0 on a does not improve the situation.
We therefore use Lemma 1.1 of the same article, which asserts the local existence of a unique classical solution $(u_\varepsilon, v_\varepsilon)$ to (3.3) for initial data $u_{0,\varepsilon} \in C^0(\overline{\Omega})$, $v_{0,\varepsilon} \in W^{1,N+1}(\Omega)$. More specifically, it implies that this solution exists on a time interval $[0, T_{max})$, $T_{max} \in (0, \infty]$, and satisfies

$$\limsup_{t \nearrow T_{max}} \Big(\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \Big) = \infty$$

if $T_{max} < \infty$. Hence, in order to show the global existence of this solution, it is sufficient to derive boundedness of $u_\varepsilon, v_\varepsilon$ and ∇v_ε.
Our means of pursuing this aim will be

Proposition 3.2.1. *Let $q > N + 2$. Let (u, v) be a nonnegative classical solution of*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + f(u), \\ v_t = \Delta v - v + u \end{cases}$$

in $\Omega \times [0, T)$, $T > 0$, with homogeneous Neumann boundary conditions, for initial data $v_0 \in W^{1,\infty}(\Omega)$, $u_0 \in L^\infty(\Omega)$ and some function f satisfying $f(s) \leq C_0$ for all $s > 0$ with some $C_0 > 0$. Furthermore, assume that there exists $C > 0$ such that u satisfies

$$\left(\int_0^T \int_\Omega u^q \right)^{\frac{1}{q}} \leq C.$$

Then u, v and ∇v are bounded in $\Omega \times [0, T)$.

Proof. We denote by C_1, C_2, C_3 the constants provided by Lemma 1.3 of [110] such that

$$\left\| \nabla e^{\tau \Delta} \phi \right\|_{L^\infty(\Omega)} \leq C_1 \left\| \nabla \phi \right\|_{L^\infty(\Omega)} \tag{3.4}$$

for all $\phi \in W^{1,\infty}(\Omega)$ and

$$\left\| \nabla e^{\tau \Delta} \phi \right\|_{L^\infty(\Omega)} \leq C_2 (1 + \tau^{-\frac{1}{2} - \frac{n}{2q}}) \left\| \phi \right\|_{L^q(\Omega)} \tag{3.5}$$

for all $\phi \in L^q(\Omega)$ as well as

$$\left\| e^{\tau \Delta} \nabla \cdot \text{♀} \right\|_{L^\infty(\Omega)} \leq C_3 (1 + \tau^{-\frac{1}{2} - \frac{n}{2q}}) \left\| \text{♀} \right\|_{L^q(\Omega)} \tag{3.6}$$

for $\varphi \in L^q(\Omega;\mathbb{R}^N)$. Here, $e^{\tau\Delta}\nabla\cdot$ signifies the extension of the corresponding operator on $C_0^\infty(\Omega;\mathbb{R}^N)$ to a continuous operator from $L^q(\Omega;\mathbb{R}^N)$ to $L^\infty(\Omega)$, see [110, Lemma 1.3]. Since $(-\frac{1}{2}-\frac{N}{2q})\cdot\frac{q}{q-1} = -\frac{1}{2}\frac{q+N}{q}\frac{q}{q-1} = -\frac{1}{2}(1+\frac{N+1}{q-1}) > -\frac{1}{2}(1+\frac{N+1}{(N+2)-1}) = -1$,

$$C_4 = \left(\int_0^T (1+(T-s)^{-\frac{1}{2}-\frac{N}{2q}})^{\frac{q}{q-1}}\mathrm{d}s\right)^{\frac{q-1}{q}} \tag{3.7}$$

is finite. We let $t\in[0,T]$. Employing (3.4) and (3.5) in the variations-of-constants formula for v, we obtain

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leq& \left\|\nabla e^{t(\Delta-1)}v_0\right\|_{L^\infty(\Omega)} + \int_0^t \left\|\nabla e^{(t-s)(\Delta-1)}u(\cdot,s)\right\|_{L^\infty(\Omega)}\mathrm{d}s\\ \leq& C_1\|\nabla v_0\|_{L^\infty(\Omega)} + C_2\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}})\|u(\cdot,s)\|_{L^q(\Omega)}\mathrm{d}s\\ \leq& C_1\|\nabla v_0\|_{L^\infty(\Omega)} + C_2\left(\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}})^{\frac{q}{q-1}}\mathrm{d}s\right)^{\frac{q-1}{q}}\left(\int_0^t \|u(\cdot,s)\|_{L^q(\Omega)}^q\mathrm{d}s\right)^{\frac{1}{q}}\\ \leq& C_1\|\nabla v_0\|_{L^\infty(\Omega)} + C_2C_4C =: C_5. \end{aligned}\tag{3.8}$$

We represent also u in terms of the semigroup, use the order-preserving property of the heat semigroup and estimate with the help of (3.6) to see that

$$\begin{aligned}0 \leq u(\cdot,t) =& e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\mathrm{d}s + \int_0^t e^{(t-s)\Delta}f(u(\cdot,s))\mathrm{d}s\\ \leq& \|u_0\|_{L^\infty(\Omega)} + C_3\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}})\|u(\cdot,s)\|_{L^q(\Omega)}\|\nabla v(\cdot,s)\|_{L^\infty(\Omega)}\mathrm{d}s + TC_0.\end{aligned}$$

Another application of Hölder's inequality, now in time, in combination with (3.8) and (3.7) gives

$$\begin{aligned}0 \leq u(\cdot,t) \leq& \|u_0\|_{L^\infty(\Omega)} + C_3C_5\left(\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}})^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\left(\int_0^t \|u(\cdot,s)\|_{L^q(\Omega)}^q\right)^{\frac{1}{q}} + TC_0\\ \leq& \|u_0\|_{L^\infty(\Omega)} + C_3C_5C_4C + TC_0 =: C_6.\end{aligned}$$

Boundedness of v on $\Omega\times[0,T]$ then is an easy consequence:

$$0 \leq v(\cdot,t) \leq \left\|e^{t(\Delta-1)}v_0\right\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}u(\cdot,s)\right\|_{L^\infty(\Omega)}\mathrm{d}s \leq \|v_0\|_{L^\infty(\Omega)} + \int_0^t C_6\mathrm{d}s$$

for all $t\in[0,T]$. □

For given nonnegative $u_0\in L^2(\Omega)$, $v_0\in W^{1,2}(\Omega)$ and $\varepsilon>0$, we choose nonnegative functions $u_{0,\varepsilon}\in C^0(\overline{\Omega})$, $v_{0,\varepsilon}\in W^{1,N+1}(\Omega)$ such that

$$\|u_0-u_{0,\varepsilon}\|_{L^2(\Omega)} \leq \min\{\varepsilon,1\}, \qquad \|v_0-v_{0,\varepsilon}\|_{W^{1,2}(\Omega)} \leq \min\{\varepsilon,1\}. \tag{3.9}$$

From now on, by $(u_\varepsilon,v_\varepsilon)$ we denote the unique classical solution on $[0,T_{max})$ to (3.3) with initial data $u_{0,\varepsilon}$ and $v_{0,\varepsilon}$. Proposition 3.2.1 in conjunction with the next two lemmata and Lemma 1.1 of [111] will show that, indeed, $T_{max}=\infty$.
Note that, by (3.9), in the following lemmata estimates in terms of $u_{0,\varepsilon}$ or $v_{0,\varepsilon}$ can be made ε-independent by retreating to the corresponding integral of u_0 or v_0 plus 1.

3.3 Estimates

In this section we present estimates for different quantities involving u_ε and v_ε respectively, which can be obtained more or less directly from (3.3) together with ODE comparison arguments. In the following, we let

$$\kappa_+ := \max\{\kappa, 0\}.$$

Lemma 3.3.1. *For any $\varepsilon > 0$, the function u_ε satisfies*

$$\int_\Omega u_\varepsilon(\cdot,t) \le \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}$$

for $t > 0$. Furthermore,

$$\limsup_{t\to\infty} \int_\Omega u_\varepsilon(\cdot,t) \le \frac{\kappa_+|\Omega|}{\mu},$$

uniformly in $\varepsilon > 0$.

Proof. By Hölder's inequality, $\left(\int_\Omega u_\varepsilon\right)^2 \le \left(\int_\Omega u_\varepsilon^2\right)|\Omega|$. Hence, integration of the first equation of (3.3) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon = \int_\Omega u_{\varepsilon t} &\le 0 - 0 + \kappa_+\int_\Omega u_\varepsilon - \mu\int_\Omega u_\varepsilon^2 - \varepsilon\int_\Omega u_\varepsilon^\theta \\ &\le \kappa_+\int_\Omega u_\varepsilon - \frac{\mu}{|\Omega|}\left(\int_\Omega u_\varepsilon\right)^2 \qquad \text{on } (0,\infty).\end{aligned} \tag{3.10}$$

The claim can be seen by solving the logistic ODE. ☐

Lemma 3.3.2. *Let $\kappa > 0$, let $T > 0$. Then there exists $C > 0$ such that for all $\varepsilon > 0$*

$$\int_0^T\int_\Omega u_\varepsilon^2 + \frac{\varepsilon}{\mu}\int_0^T\int_\Omega u_\varepsilon^\theta \le \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}T + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} \le C.$$

Proof. The estimate

$$\begin{aligned}\int_0^T\int_\Omega u_\varepsilon^2 + \frac{\varepsilon}{\mu}\int_0^T\int_\Omega u_\varepsilon^\theta &\le \frac{\kappa_+}{\mu}\int_0^T\int_\Omega u_\varepsilon + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} - \frac{1}{\mu}\int_\Omega u_\varepsilon(\cdot,T) \\ &\le \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}T + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}\end{aligned}$$

results from (3.10) after time-integration. ☐

Also for the second component of the solution some basic estimates are available:

Lemma 3.3.3. *Let $\kappa \in \mathbb{R}$ and $\varepsilon > 0$. The inequality*

$$\int_\Omega v_\varepsilon(\cdot,t) \le \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}, \int_\Omega v_{0,\varepsilon}\right\}$$

holds as well as

$$\int_\Omega v_\varepsilon^2(\cdot,t) + \int_0^t\int_\Omega v_\varepsilon^2 \le \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}t + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} + \int_\Omega v_{0,\varepsilon}^2$$

for all $t > 0$.

Proof. Integrating the second equation of (3.3) gives, by Lemma 3.3.1,

$$\begin{aligned}\frac{d}{dt}\int_\Omega v_\varepsilon(\cdot,t) = \int_\Omega v_{\varepsilon t}(\cdot,t) &= \int_\Omega \Delta v_\varepsilon(\cdot,t) - \int_\Omega v_\varepsilon(\cdot,t) + \int_\Omega u_\varepsilon(\cdot,t)\\ &\le -\int_\Omega v_\varepsilon(\cdot,t) + \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}\end{aligned}$$

for $t > 0$, an ODI for $\int_\Omega v_\varepsilon$, whose solution directly shows

$$\int_\Omega v_\varepsilon(\cdot,t) \le \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} + \mathrm{e}^{-t}\int_\Omega v_{0,\varepsilon} \qquad \text{for all } t>0 \tag{3.11}$$

and hence the first part of the assertion.
As to the second part, we derive an ODI for $\frac{1}{2}\int_\Omega v_\varepsilon^2$ in quite the same way: For $t > 0$, by Young's inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2(\cdot,t) = \int_\Omega v_\varepsilon(\cdot,t)v_{\varepsilon t}(\cdot,t) &= \int_\Omega v_\varepsilon(\cdot,t)\Delta v_\varepsilon(\cdot,t) - \int_\Omega v_\varepsilon^2(\cdot,t) + \int_\Omega u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t)\\ &\le -\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 - \int_\Omega v_\varepsilon^2(\cdot,t) + \frac{1}{2}\int_\Omega u_\varepsilon^2(\cdot,t) + \frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,t)\\ &\le -\frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,t) + \frac{1}{2}\int_\Omega u_\varepsilon^2(\cdot,t).\end{aligned}$$

Integrating this with respect to the time variable, so that we can use the bound from Lemma 3.3.2 on u_ε^2, we obtain

$$\begin{aligned}\int_\Omega v_\varepsilon^2(\cdot,t) - \int_\Omega v_{0,\varepsilon}^2 &\le -\int_0^t\int_\Omega v_\varepsilon^2 + \int_0^t\int_\Omega u_\varepsilon^2\\ &\le -\int_0^t\int_\Omega v_\varepsilon^2 + \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}t + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}\end{aligned}$$

for any $t > 0$ and the claim follows. □

The next lemma gives estimates on the derivatives of v.

Lemma 3.3.4. *Let $\kappa \in \mathbb{R}$ and $\varepsilon > 0$. The solutions of* (3.3) *satisfy, for all $t > 0$,*

$$\left[\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \frac{1}{\mu}\int_\Omega u_\varepsilon(\cdot,t)\right] \le \max\left\{\int |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+ + 1}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}\right\}$$

and

$$\int_0^t\int_\Omega |\Delta v_\varepsilon(\cdot,t)|^2 \le \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}t + \int_\Omega |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}.$$

Proof. Integration by parts and Young's inequality result in

$$\begin{aligned}\frac{d}{dt}\left[\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{\mu}\int_\Omega u_\varepsilon\right] &= -2\int_\Omega \Delta v_\varepsilon v_{\varepsilon t} + \frac{1}{\mu}\int_\Omega u_{\varepsilon t}\\ &\le -2\int_\Omega |\Delta v_\varepsilon|^2 + 2\int_\Omega \Delta v_\varepsilon v_\varepsilon - 2\int_\Omega \Delta v_\varepsilon u_\varepsilon + \frac{\kappa_+}{\mu}\int_\Omega u_\varepsilon - \int_\Omega u_\varepsilon^2 - \frac{\varepsilon}{\mu}\int_\Omega u_\varepsilon^\theta\\ &\le -2\int_\Omega |\Delta v_\varepsilon|^2 - 2\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega u_\varepsilon^2 - \int_\Omega u_\varepsilon^2 + \frac{\kappa_+}{\mu}\int_\Omega u_\varepsilon\end{aligned}$$

$$\leq -\int_\Omega |\Delta v_\varepsilon|^2 - \int_\Omega |\nabla v_\varepsilon|^2 - \frac{1}{\mu}\int_\Omega u_\varepsilon + \frac{\kappa_+ + 1}{\mu}\int_\Omega u_\varepsilon \tag{3.12}$$

on $(0,\infty)$. From this, we can conclude by Lemma 3.3.1 that

$$\frac{d}{dt}\left[\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{\mu}\int_\Omega u_\varepsilon\right] \leq -\left[\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{\mu}\int_\Omega u_\varepsilon\right] + \frac{\kappa_+ + 1}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}$$

on $(0,\infty)$ and hence the claim follows by comparison with the solution of $y' = -y + const$. Re-sorting the terms in (3.12) moreover gives

$$\int_\Omega |\Delta v_\varepsilon(\cdot,t)|^2 \leq -\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \frac{\kappa_+}{\mu}\int_\Omega u_\varepsilon(\cdot,t) - \frac{d}{dt}\left[\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \frac{1}{\mu}\int_\Omega u_\varepsilon(\cdot,t)\right]$$

for $t > 0$, and therefore

$$\begin{aligned}
&\int_0^t \int_\Omega |\Delta v_\varepsilon|^2 \\
&\leq \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} t - \left[\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \frac{1}{\mu}\int_\Omega u_\varepsilon(\cdot,t)\right] + \int_\Omega |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} \\
&\leq \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} t + \int_\Omega |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} \quad \text{for all } t > 0.
\end{aligned}$$

□

The bounds that have been derived so far can be combined to yield

Lemma 3.3.5. *Let $\kappa \in \mathbb{R}$. Then for any positive time $T > 0$, there exists a positive constant $C = C(T, \mu, \kappa_+, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{W^{1,2}(\Omega)})$ such that for all $\varepsilon > 0$*

$$\frac{1}{2}\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \mu\int_0^T \int_\Omega u_\varepsilon^2 \ln(1+u_\varepsilon) + \varepsilon\int_0^T \int_\Omega u_\varepsilon^\theta \ln(1+u_\varepsilon) \leq C.$$

In particular: The families $\{u_\varepsilon^2\}_{\varepsilon\in(0,1)}$ and $\{\varepsilon u_\varepsilon^\theta\}_{\varepsilon\in(0,1)}$ are equi-integrable over $\Omega \times (0,T)$.

Proof. Testing the first equation of (3.3) with $\ln(1+u_\varepsilon)$ and integrating by parts gives

$$\begin{aligned}
\int_\Omega u_{\varepsilon t}\ln(1+u_\varepsilon) \leq &-\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_\Omega \frac{u_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_\Omega u_\varepsilon \ln(1+u_\varepsilon) \\
&- \mu\int_\Omega u_\varepsilon^2 \ln(1+u_\varepsilon) - \varepsilon\int_\Omega u_\varepsilon^\theta \ln(1+u_\varepsilon) \qquad \text{on } (0,\infty),
\end{aligned}$$

which, using $((1+u_\varepsilon)\ln(1+u_\varepsilon) - u_\varepsilon)_t = u_{\varepsilon t}\ln(1+u_\varepsilon)$, can be turned into

$$\begin{aligned}
&\mu\int_\Omega u_\varepsilon^2 \ln(1+u_\varepsilon) + \varepsilon\int_\Omega u_\varepsilon^\theta \ln(1+u_\varepsilon) \\
&\leq -\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_\Omega \frac{u_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_\Omega u_\varepsilon \ln(1+u_\varepsilon) - \int_\Omega [(1+u_\varepsilon)\ln(1+u_\varepsilon) - u_\varepsilon]_t
\end{aligned}$$

on $(0,\infty)$. If we fix $T > 0$, integration in time hence shows that

$$I := \mu\int_0^T \int_\Omega u_\varepsilon^2 \ln(1+u_\varepsilon) + \varepsilon\int_0^T \int_\Omega u_\varepsilon^\theta \ln(1+u_\varepsilon)$$

$$
\begin{aligned}
&\le -\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_0^T\int_\Omega \frac{u_\varepsilon \nabla v_\varepsilon\cdot\nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_0^T\int_\Omega u_\varepsilon \ln(1+u_\varepsilon)\\
&\quad -\int_\Omega ((1+u_\varepsilon(\cdot,T))\ln(1+u_\varepsilon(\cdot,T)) - u_\varepsilon(\cdot,T)) + \int_\Omega ((1+u_{0,\varepsilon})\ln(1+u_{0,\varepsilon}) - u_{0,\varepsilon})\\
&\le -\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_0^T\int_\Omega \frac{u_\varepsilon \nabla v_\varepsilon\cdot\nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_0^T\int_\Omega u_\varepsilon \ln(1+u_\varepsilon)\\
&\quad +\int_\Omega u_\varepsilon(\cdot,T) + \int_\Omega (1+u_{0,\varepsilon})\ln(1+u_{0,\varepsilon}). \qquad (3.13)
\end{aligned}
$$

We integrate the second term by parts:

$$
\begin{aligned}
\int_0^T\int_\Omega \frac{u_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon}{1+u_\varepsilon} &= -\int_0^T\int_\Omega \frac{u_\varepsilon^2}{1+u_\varepsilon}\Delta v_\varepsilon - \int_0^T\int_\Omega u_\varepsilon \nabla v_\varepsilon\cdot\nabla\left(\frac{u_\varepsilon}{1+u_\varepsilon}\right)\\
&= -\int_0^T\int_\Omega \frac{u_\varepsilon^2}{1+u_\varepsilon}\Delta v_\varepsilon - \int_0^T\int_\Omega u_\varepsilon \nabla v_\varepsilon\cdot\nabla u_\varepsilon \frac{1+u_\varepsilon-u_\varepsilon}{(1+u_\varepsilon)^2}.
\end{aligned}
$$

Inserting this into (3.13) then results in

$$
\begin{aligned}
I \le &-\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} - \int_0^T\int_\Omega \frac{u_\varepsilon^2}{1+u_\varepsilon}\Delta v_\varepsilon - \int_0^T\int_\Omega \frac{u_\varepsilon\nabla v_\varepsilon}{(1+u_\varepsilon)^{\frac{3}{2}}}\cdot\frac{\nabla u_\varepsilon}{(1+u_\varepsilon)^{\frac{1}{2}}}\\
&+\kappa_+\int_0^T\int_\Omega u_\varepsilon\ln(1+u_\varepsilon) + \int_\Omega u_\varepsilon(\cdot,T) + \int_\Omega (1+u_{0,\varepsilon})\ln(1+u_{0,\varepsilon}),
\end{aligned}
$$

where application of the trivial inequality $\frac{u_\varepsilon}{(1+u_\varepsilon)^{\frac{3}{2}}} \le 1$ gives rise to

$$
\begin{aligned}
I \le &-\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} - \int_0^T\int_\Omega \frac{u_\varepsilon}{1+u_\varepsilon}u_\varepsilon\Delta v_\varepsilon + \int_0^T\int_\Omega |\nabla v_\varepsilon|\frac{|\nabla u_\varepsilon|}{\sqrt{1+u_\varepsilon}}\\
&+\kappa_+\int_0^T\int_\Omega u_\varepsilon\ln(1+u_\varepsilon) + \int_\Omega u_\varepsilon(\cdot,T) + \int_\Omega (1+u_{0,\varepsilon})\ln(1+u_{0,\varepsilon}).
\end{aligned}
$$

Estimating $\frac{u_\varepsilon}{1+u_\varepsilon} \le 1$, $\ln(1+u_\varepsilon) \le u_\varepsilon$ and employing Young's inequality shows

$$
\begin{aligned}
I \le &-\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \frac{1}{2}\int_0^T\int_\Omega u_\varepsilon^2 + \frac{1}{2}\int_0^T\int_\Omega |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_0^T\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon}\\
&+\kappa_+\int_0^T\int_\Omega u_\varepsilon^2 + \int_\Omega u_\varepsilon(\cdot,T) + \int_\Omega (1+u_{0,\varepsilon})\ln(1+u_{0,\varepsilon})\\
= &-\frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \left(\kappa_+ + \frac{1}{2}\right)\int_0^T\int_\Omega u_\varepsilon^2 + \int_\Omega u_\varepsilon(\cdot,T) + \frac{1}{2}\int_0^T\int_\Omega |\Delta v_\varepsilon|^2\\
&+\frac{1}{2}\int_0^T\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega (1+u_{0,\varepsilon})\ln(1+u_{0,\varepsilon}).
\end{aligned}
$$

And if we compile the bounds provided by Lemmata 3.3.2, 3.3.4 and 3.3.1, we arrive at

$$
\begin{aligned}
&I + \frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon}\\
&\le \left(\kappa_+ + \frac{1}{2}\right)\frac{1}{\mu}\left(\kappa_+ \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}T + \int_\Omega u_{0,\varepsilon}\right) + \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}
\end{aligned}
$$

$$+\frac{1}{2}\left(\frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon},\frac{\kappa_+|\Omega|}{\mu}\right\}T+\int_\Omega|\nabla v_{0,\varepsilon}|^2+\frac{1}{\mu}\int_\Omega u_{0,\varepsilon}\right)$$
$$+\frac{1}{2}T\max\left\{\int|\nabla v_{0,\varepsilon}|^2+\frac{1}{\mu}\int_\Omega u_{0,\varepsilon},\frac{\kappa_++1}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon},\frac{\kappa_+|\Omega|}{\mu}\right\}\right\}$$
$$+\int_\Omega(1+u_{0,\varepsilon})\ln(1+u_{0,\varepsilon})=:C. \qquad \square$$

From the bound on $\int_0^T\int_\Omega\frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon}$ we can extract information on the behaviour of the spatial gradient of u.

Lemma 3.3.6. *Let $\kappa\in\mathbb{R}$. For all $T>0$ there is $C>0$ such that for all $\varepsilon>0$*

$$\|u_\varepsilon\|_{L^{\frac{4}{3}}((0,T),W^{1,\frac{4}{3}}(\Omega))}\le C.$$

Proof. We denote by C_1 the constant provided by Lemma 3.3.2 and by C_2 that of Lemma 3.3.5. Then, by Hölder's and Young's inequalities,

$$\begin{aligned}\|u_\varepsilon\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((0,T),W^{1,\frac{4}{3}}(\Omega))}&=\int_0^T\|u_\varepsilon\|^{\frac{4}{3}}_{W^{1,\frac{4}{3}}(\Omega)}=\int_0^T\left(\int_\Omega u_\varepsilon^{\frac{4}{3}}+\int_\Omega|\nabla u_\varepsilon|^{\frac{4}{3}}\right)\\&\le\left(\int_0^T\int_\Omega u_\varepsilon^2\right)^{\frac{2}{3}}(|\Omega|T)^{\frac{1}{3}}+\int_0^T\int_\Omega\frac{|\nabla u_\varepsilon|^{\frac{4}{3}}}{(1+u_\varepsilon)^{\frac{2}{3}}}(1+u_\varepsilon)^{\frac{2}{3}}\\&\le C_1^{\frac{2}{3}}(|\Omega|T)^{\frac{1}{3}}+\frac{2}{3}\int_0^T\int_\Omega\frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon}+\frac{1}{3}\int_0^T\int_\Omega(1+u_\varepsilon)^2\\&\le C_1^{\frac{2}{3}}(|\Omega|T)^{\frac{1}{3}}+\frac{4}{3}C_2+\frac{2}{3}T|\Omega|+\frac{2}{3}C_1=:C.\end{aligned} \qquad \square$$

In order to gain convergence results from an Aubin-Lions-type lemma, we need some information on the time derivative. The following lemma provides this kind of information.

Lemma 3.3.7. *Let $\kappa\in\mathbb{R}$ and $T>0$. Then there is $C>0$ such that for all $\varepsilon>0$*

$$\|u_{\varepsilon t}\|_{L^1((0,T);(W_0^{3,N+1}(\Omega))^*)}\le C.$$

Proof. Definition of the norm and integration by parts in (3.3) lead us to

$$\begin{aligned}&\int_0^T\sup_\varphi\left|\int_\Omega u_{\varepsilon t}\varphi\right|\\&\quad\le\int_0^T\sup_\varphi\left(\left|\int_\Omega u_\varepsilon\Delta\varphi\right|+\left|\int_\Omega u_\varepsilon\nabla v_\varepsilon\cdot\nabla\varphi\right|+\left|\kappa\int_\Omega u_\varepsilon\varphi\right|+\mu\left|\int_\Omega u_\varepsilon^2\varphi\right|+\varepsilon\left|\int_\Omega u_\varepsilon^\theta\varphi\right|\right),\end{aligned}$$

where the suprema are taken over all functions $\varphi\in C_0^\infty(\Omega)$ satisfying $\|\varphi\|_{W^{3,N+1}(\Omega)}\le 1$, and where we can use Young's inequality and the fact that, due to the embedding $W_0^{3,N+1}(\Omega)\hookrightarrow W^{2,\infty}(\Omega)$, with some $c_1>0$ we have $\|\varphi\|_{W^{2,\infty}(\Omega)}\le c_1$ for all such φ, to see

$$\begin{aligned}&\|u_{\varepsilon t}\|_{L^1((0,T);(W_0^{3,N+1}(\Omega))^*)}\\&\quad\le c_1\int_0^T\left(\int_\Omega u_\varepsilon+\frac{1}{2}\int_\Omega u_\varepsilon^2+\frac{1}{2}\int_\Omega|\nabla v_\varepsilon|^2+|\kappa|\int_\Omega u_\varepsilon+\mu\int_\Omega u_\varepsilon^2+\varepsilon\int_\Omega u_\varepsilon^\theta\right)\end{aligned}$$

and infer boundedness of this norm, independent of ε, from Lemmata 3.3.1, 3.3.2 and 3.3.4. $\square$

The space in which the spatial gradient is known to be bounded can be improved if a bound on u is assumed.

Lemma 3.3.8. *Let $\kappa \in \mathbb{R}$ and let $[T_1, T_2]$, $T_2 > T_1$, be an interval such that*

$$\sup_{\varepsilon>0} \sup_{t\in[T_1,T_2]} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

Then there is $C > 0$ such that for all $\varepsilon > 0$

$$\|\nabla u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))} \leq C.$$

Proof. By Lemma 3.3.5, given $T > T_2 > T_1$, we can find $\widetilde{C} > 0$ such that

$$\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} \leq \widetilde{C}$$

for all $\varepsilon > 0$, ergo, setting $C = (1+M)\widetilde{C}$,

$$\int_{T_1}^{T_2} \int_\Omega |\nabla u_\varepsilon|^2 \leq \int_{T_1}^{T_2} \int_\Omega \frac{1+M}{1+u_\varepsilon} |\nabla u_\varepsilon|^2 \leq (1+M)\widetilde{C} = C. \qquad \square$$

Under similar conditions, also the time derivative is bounded in a better space.

Lemma 3.3.9. *Let $\kappa \in \mathbb{R}$ and let $[T_1, T_2]$, $T_2 > T_1$, be an interval such that*

$$\sup_{\varepsilon>0} \sup_{t\in[T_1,T_2]} \Big(\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\Big) < \infty.$$

Then there is $C > 0$ such that for all $\varepsilon > 0$

$$\|u_{\varepsilon t}\|_{L^2((T_1,T_2);(W^{1,2}(\Omega))^*)} \leq C.$$

Proof. Let φ be an element of $L^2((T_1, T_2); W^{1,2}(\Omega)$ with norm 1.
Let $\widetilde{C}$ be the bound on $\|\nabla u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))}$ provided by Lemma 3.3.8. Then

$$\begin{aligned}
\left|\int_{T_1}^{T_2} \int_\Omega u_{\varepsilon t}\varphi\right| \leq{}& \left|\int_{T_1}^{T_2} \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\right| + \left|\int_{T_1}^{T_2} \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi\right| + \left|\int_{T_1}^{T_2} \int_\Omega (\kappa u_\varepsilon - \mu u_\varepsilon^2 - \varepsilon u_\varepsilon^\theta)\varphi\right| \\
\leq{}& \|\nabla u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))} \|\nabla\varphi\|_{L^2((T_1,T_2);L^2(\Omega))} \\
&+ (\sup_{t\in[T_1,T_2]} \|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}) \|u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))} \|\nabla\varphi\|_{L^2((T_1,T_2);L^2(\Omega))} \\
&+ \sqrt{(T_2-T_1)|\Omega|} \left\|\kappa u_\varepsilon + \mu u_\varepsilon^2 + u_\varepsilon^\theta\right\|_{L^\infty(\Omega\times(T_1,T_2))} \|\varphi\|_{L^2((T_1,T_2);L^2(\Omega))} \\
\leq{}& \widetilde{C} + M\sqrt{(T_2-T_1)|\Omega|}M + \sqrt{(T_2-T_1)|\Omega|}(|\kappa|M + \mu M^2 + M^\theta) =: C
\end{aligned}$$

and hence boundedness of $u_{\varepsilon t}$ in $(L^2((T_1, T_2); W^{1,2}(\Omega)))^*$ follows. $\square$

3.4 Preservation of smallness

In the last two lemmata, we have seen that boundedness can provide bounds also for derivatives. It will as well be important in establishing regularization effects. Therefore, in this section we

will derive this boundedness and to this aim proceed as follows: At first we will prepare some estimates on $y_\varepsilon(t) := \int_\Omega u_\varepsilon^2(\cdot,t) + \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4$. These will establish that y_ε satisfies a differential inequality with a polynomial right hand side; we will show that this polynomial has a positive root and y_ε eventually undermatches its value. Finally, we will use the bounds just gained to improve them to L^∞-bounds for the solutions under consideration.
At first we state the following easy consequence of Poincaré's inequality.

Lemma 3.4.1. *If we denote $\bar\psi = \frac{1}{|\Omega|}\int_\Omega \psi$, then*

$$\int_\Omega \psi^2 \le C_P \int_\Omega |\nabla\psi|^2 + |\Omega|\bar\psi^2,$$

for all $\psi \in W^{1,2}(\Omega)$, where C_P is the Poincaré constant of Ω, which is defined to be the smallest number for which the inequality $\int_\Omega (\psi - \bar\psi)^2 \le C_P \int_\Omega |\nabla\psi|^2$ holds true for all $\psi \in W^{1,2}(\Omega)$.

Proof. As announced, this is a direct consequence of Poincaré's inequality:

$$C_P \int_\Omega |\nabla\psi|^2 \ge \int_\Omega (\psi-\bar\psi)^2 = \int_\Omega \psi^2 - 2\int_\Omega \psi\bar\psi + \int_\Omega \bar\psi^2 = \int_\Omega \psi^2 - 2\bar\psi|\Omega|\bar\psi + |\Omega|\bar\psi^2 = \int_\Omega \psi^2 - |\Omega|\bar\psi^2.$$

□

Another elementary but useful identity is the following:

Lemma 3.4.2. *Let $\psi \in C^3(\Omega)$. Then*

$$\Delta|\nabla\psi|^2 = 2\nabla\psi\cdot\nabla\Delta\psi + 2|D^2\psi|^2.$$

Proof. For a simple proof we consider the components of these expressions

$$\begin{aligned}\Delta|\nabla\psi|^2 &= \sum_{i,j=1}^N \partial_i^2(\partial_j\psi)^2 = \sum_{i,j=1}^N \partial_i(2\partial_j\psi\partial_i\partial_j\psi)\\ &= 2\sum_{i,j=1}^N (\partial_i\partial_j\psi\partial_i\partial_j\psi + \partial_j\psi\partial_i^2\partial_j\psi) = 2|D^2\psi|^2 + 2\nabla\psi\nabla\Delta\psi\end{aligned}$$

□

In the proof of Lemma 3.4.4 we will also make use of the well-known Gagliardo-Nirenberg inequality, which we recall in order to fix notation:

Lemma 3.4.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $p,q,r,s \ge 1$, $j,k \in \mathbb{N}_0$ and $a \in [\frac{j}{k},1)$ satisfying $\frac{1}{p} = \frac{j}{k} + (\frac{1}{r} - \frac{k}{N})a + \frac{1-a}{q}$. Then there are positive constants C_1 and C_2 such that for all functions $\psi \in L^q(\Omega)\cap L^s(\Omega)$ with $\nabla\psi \in L^r(\Omega)$,*

$$\left\|D^j\psi\right\|_{L^p(\Omega)} \le C_1 \left\|D^k\psi\right\|_{L^r(\Omega)}^a \left\|\psi\right\|_{L^q(\Omega)}^{1-a} + C_2\left\|\psi\right\|_{L^s(\Omega)}.$$

Proof. See [77, p.126]. □

We are aiming for an estimate for $\int_\Omega |\nabla v_\varepsilon|^4$. During the calculations we therefore will have to get rid of integrals of $|\nabla v_\varepsilon|^6$. The Gagliardo-Nirenberg inequality enables us to replace them by more convenient terms.

Lemma 3.4.4. *Let $N=3$. For any $\alpha > 0$ there is $C(\alpha) > 0$ such that, for any $\kappa \in \mathbb{R}$, $\varepsilon > 0$,*

$$\int_\Omega |\nabla v_\varepsilon|^6 \le \alpha\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 + C(\alpha)\left[\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}}\right] \quad on\ (0,\infty).$$

Proof. For given $j = 0$, $k = 1$, Ω, $p = 3$, $r = q = 2$, $s = 2$, the Gagliardo-Nirenberg inequality (Lemma 3.4.3) provides constants C_1 and C_2 such that for $\phi \in L^2(\Omega)$ and with $a = \frac{1}{2}$ the inequality

$$\|\phi\|^3_{L^3(\Omega)} \le 8C_1^3 \|\nabla\phi\|^{\frac{3}{2}}_{L^2(\Omega)} \|\phi\|^{\frac{3}{2}}_{L^2(\Omega)} + 8C_2^3 \|\phi\|^3_{L^2(\Omega)},$$

holds true (where we at the same time have used $(x+y)^3 < 8(x^3+y^3)$). Applied to $\phi = |\nabla v_\varepsilon(\cdot,t)|^2$ for any $t \in (0,\infty)$, this means

$$\int_\Omega |\nabla v_\varepsilon|^6 \le 8C_1^3 \left(\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2\right)^{\frac{3}{4}} \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{4}} + 8C_2^3 \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}} \quad \text{on } (0,\infty).$$

With $p = \frac{4}{3}$, $q = 4$, corresponding to $\alpha > 0$ Young's inequality provides $\widetilde{C}(\alpha) > 0$ such that

$$\int_\Omega |\nabla v_\varepsilon|^6 \le \alpha \int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 + \widetilde{C}(\alpha) \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{3} + 8C_2^3 \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}} \quad \text{on } (0,\infty)$$

and the claim results with the choice of $C(\alpha) = \max\{8C_2^3, \widetilde{C}(\alpha)\}$. □

With the help of Lemma 3.4.4, we separate u_ε, ∇u_ε and ∇v_ε in one of the terms arising from differentiation of $\int_\Omega u_\varepsilon^2$.

Lemma 3.4.5. *Let $N = 3$. Corresponding to $\mu > 0$ there exists $C > 0$ such that for any $\kappa \in \mathbb{R}$ and $\varepsilon > 0$ the estimate*

$$\int_\Omega u_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon \le \frac{1}{4}\int_\Omega |\nabla u_\varepsilon|^2 + \mu \int_\Omega u_\varepsilon^3 + \frac{1}{2}\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 + C\left(\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}}\right)$$

holds on $(0,\infty)$.

Proof. Double application of Young's inequality yields a constant $\widetilde{C} > 0$ such that on $(0,\infty)$

$$\int_\Omega u_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon \le \frac{1}{4}\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^2 \le \frac{1}{4}\int_\Omega |\nabla u_\varepsilon|^2 + \mu \int_\Omega u_\varepsilon^3 + \widetilde{C}\int_\Omega |\nabla v_\varepsilon|^6.$$

Using Lemma 3.4.4 with $\alpha = \frac{1}{2}$ to estimate $\int_\Omega |\nabla v_\varepsilon|^6$ this produces the assertion, with the choice $C = \widetilde{C}C(\frac{1}{2})$. □

The term $\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2$, known to us from Lemma 3.4.4, arises from the following estimate with the "correct" sign.

Lemma 3.4.6. *Let $\kappa \in \mathbb{R}$, let $q \ge 1$. Then on $(0,\infty)$, for any $\varepsilon > 0$,*

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q} \le -q(q-1)\int_\Omega |\nabla v_\varepsilon|^{2q-4}|\nabla|\nabla v_\varepsilon|^2|^2 - 2q\int_\Omega |\nabla v_\varepsilon|^{2q} + 2q\int_\Omega |\nabla v_\varepsilon|^{2q-1}|\nabla u_\varepsilon|.$$

Proof. Evaluating the derivative and inserting the second equation of (3.3) gives

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q} = 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon \cdot \nabla\Delta v_\varepsilon - 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon \cdot \nabla v_\varepsilon + 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon \cdot \nabla u_\varepsilon$$

on $(0,\infty)$. Here, Lemma 3.4.2 and integration by parts eventuate

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q} = q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\Delta|\nabla v_\varepsilon|^2 - 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}|D^2 v_\varepsilon|^2 - 2q\int_\Omega |\nabla v_\varepsilon|^{2q}$$

$$\begin{aligned}&+2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon\cdot\nabla u_\varepsilon\\ \le&-q(q-1)\int_\Omega |\nabla v_\varepsilon|^{2q-4}|\nabla|\nabla v_\varepsilon|^2|^2-2q\int_\Omega|\nabla v_\varepsilon|^{2q}+2q\int_\Omega|\nabla v_\varepsilon|^{2q-1}|\nabla u_\varepsilon|.\end{aligned}$$

In this step we used convexity of Ω to estimate the boundary integral

$$\int_{\partial\Omega}|\nabla v_\varepsilon(\cdot,t)|^{2q-2}\nabla(|\nabla v_\varepsilon(\cdot,t)|^2)\cdot\nu\le 0\qquad\text{for all } t>0,$$

due to the fact that in convex domains $\partial_\nu|\nabla v_\varepsilon|^2\big|_{\partial\Omega}\le 0$ follows from $\partial_\nu v\big|_{\partial\Omega}=0$, confer [94, Lemma 3.2]. □

The other summand arising in the calculation of $y_\varepsilon'(t)$ can be estimated as follows:

Lemma 3.4.7. *For any $\kappa\in\mathbb{R}$ and $\varepsilon>0$,*

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2\le -2\int_\Omega|\nabla u_\varepsilon|^2+2\int_\Omega u_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon+2\kappa\int_\Omega u_\varepsilon^2-2\mu\int_\Omega u_\varepsilon^3\quad\text{on } (0,\infty).$$

Proof. This results from integration by parts and estimation of the negative last term in the inequality

$$2\int_\Omega u_\varepsilon u_{\varepsilon t}\le 2\int_\Omega u_\varepsilon\Delta u_\varepsilon-2\int_\Omega u_\varepsilon\nabla\cdot(u_\varepsilon\nabla v_\varepsilon)+2\kappa\int_\Omega u_\varepsilon^2-2\mu\int_\Omega u_\varepsilon^3-2\varepsilon\int_\Omega u_\varepsilon^\theta,$$

which by (3.3) is valid on $(0,\infty)$. □

We put the estimates that we have found so far to their use and state

Proposition 3.4.8. *Let $N=3$ and $\mu>0$. There is a constant $A>0$ such that for all $\varepsilon>0$, for all $\nu>0$, $\eta\in(0,4]$ and $\widehat{\kappa}>0$ the following holds: If $\kappa\in\mathbb{R}$ satisfies $\kappa<\widehat{\kappa}$ and $2\kappa+\eta\le\frac{1}{C_P}$, where C_P is the Poincaré constant associated with Ω, then the quantity*

$$y_\varepsilon(t):=\int_\Omega u_\varepsilon^2(\cdot,t)+\int_\Omega|\nabla v_\varepsilon(\cdot,t)|^4 \tag{3.14}$$

satisfies the differential inequality

$$y_\varepsilon'(t)\le \nu-\eta y_\varepsilon(t)+A\left(1+\frac{1}{4\nu}\right)y_\varepsilon^3(t)+\frac{4\widehat{\kappa}^2|\Omega|}{C_P\mu^2}=:p(y_\varepsilon(t))$$

for all $t>T_0$ with some $T_0=T_0(\mu,\kappa,\widehat{\kappa})>0$ depending on $\mu,\kappa,\widehat{\kappa}$ only.

Proof. With the aid of Lemma 3.3.1, fix $T_0>0$ such that

$$\int_\Omega u_\varepsilon(\tau)<\frac{2\widehat{\kappa}|\Omega|}{\mu}\qquad\text{for all } \tau>T_0,\ \varepsilon>0. \tag{3.15}$$

By Lemma 3.4.7 and Lemma 3.4.6 with $q=2$, we have

$$\begin{aligned}y_\varepsilon'&=\frac{d}{dt}\left(\int_\Omega u_\varepsilon^2+\int_\Omega|\nabla v_\varepsilon|^4\right)\\ &\le -2\int_\Omega|\nabla u_\varepsilon|^2+2\int_\Omega u_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon+2\kappa\int_\Omega u_\varepsilon^2-2\mu\int_\Omega u_\varepsilon^3\end{aligned}$$

$$-2\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 - 4\int_\Omega |\nabla v_\varepsilon|^4 + 4\int_\Omega |\nabla v_\varepsilon|^3|\nabla u_\varepsilon| \quad \text{on } (0,\infty).$$

By application of Lemma 3.4.5 to the second and Young's inequality and Lemma 3.4.4 to the last term, this becomes

$$\begin{aligned}
y_\varepsilon' \leq & -2\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + 2\mu\int_\Omega u_\varepsilon^3 + \int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 \\
& + 2C\left(\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}}\right) \\
& + 2\kappa\int_\Omega u_\varepsilon^2 - 2\mu\int_\Omega u_\varepsilon^3 - 2\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 - 4\int_\Omega |\nabla v_\varepsilon|^4 + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \\
& + 8\left(\frac{1}{8}\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 + C(\tfrac{1}{8})\left(\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}}\right)\right) \\
\leq & 2\kappa\int_\Omega u_\varepsilon^2 + A\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + A^{\frac{1}{2}}\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}} - 4\int_\Omega |\nabla v_\varepsilon|^4 - \int_\Omega |\nabla u_\varepsilon|^2,
\end{aligned}$$

on $(0,\infty)$, where we denoted $A^{\frac{1}{2}} = \max\{2C + 8C(\frac{1}{8}), 1\} \leq A$, C being the constant from Lemma 3.4.5 and $C(\frac{1}{8})$ taken from Lemma 3.4.4.
Another application of Young's inequality with $\nu > 0$ – so as to remove the unsolicited exponent $\frac{3}{2}$ – and sorting other terms, in order that the term $-\eta y$ appears, leave us with

$$y_\varepsilon' \leq (2\kappa+\eta)\int_\Omega u_\varepsilon^2 + A\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \nu + \frac{A}{4\nu}\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 - \eta\int_\Omega u_\varepsilon^2 - 4\int_\Omega |\nabla v_\varepsilon|^4 - \int_\Omega |\nabla u_\varepsilon|^2$$

on $(0,\infty)$, where we apply Lemma 3.4.1 to the last summand and use that by (3.15) we have $\overline{u}_\varepsilon(t) = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon(\cdot,t) \leq \frac{2\widehat{\kappa}}{\mu}$ for $t > T_0$ to arrive at

$$\begin{aligned}
y_\varepsilon'(t) \leq & \left((2\kappa+\eta) - \frac{1}{C_P}\right)\int_\Omega u_\varepsilon^2(\cdot,t) + A\left(1 + \frac{1}{4\nu}\right)\left(\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4\right)^3 \\
& + \nu - \eta\left(\int_\Omega u_\varepsilon^2(\cdot,t) + \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4\right) + \frac{|\Omega|}{C_P}\overline{u}_\varepsilon^2(t) \\
\leq & \nu - \eta y_\varepsilon(t) + A\left(1 + \frac{1}{4\nu}\right)(y_\varepsilon(t))^3 + \frac{4|\Omega|\widehat{\kappa}^2}{C_P\mu^2} \quad \text{for } t \in (T_0,\infty),
\end{aligned}$$

as long as $(2\kappa+\eta)C_P \leq 1$ and $\eta \in (0,4]$. □

The function y_ε satisfies a differential inequality with polynomial right hand side. This information is not very useful in obtaining boundedness if not accompanied by the statement that comparison with a stationary solution to the differential equation might be possible, i.e. that there is a root of the polynomial. Such is provided by the following lemma.

Lemma 3.4.9. *For any $\mu > 0$ there exists $\nu_0 > 0$ such that for all $\nu \in (0,\nu_0]$ there are $\widetilde{\kappa} > 0$, $\eta \in (0,4]$ such that the polynomial*

$$p(x) = \nu - \eta x + A\left(1 + \frac{1}{4\nu}\right)x^3 + \frac{4\widehat{\kappa}^2|\Omega|}{C_P\mu^2}$$

defined in Proposition 3.4.8 has a positive root for $\widehat{\kappa} = \widetilde{\kappa}$.

Furthermore, for each $\widehat{\kappa} \in [0, \widetilde{\kappa}]$ it has a largest positive root $\delta_\nu(\widehat{\kappa})$ as well, satisfying

$$\delta_\nu(\widetilde{\kappa}) \le \delta_\nu(\widehat{\kappa}) \le \sqrt{\frac{4}{A(1+\frac{1}{4\nu})}}.$$

Proof. Because $p(x)$ is increasing in $\widehat{\kappa}$, the estimate $\delta_\nu(\widetilde{\kappa}) \le \delta_\nu(\widehat{\kappa})$ for $\widehat{\kappa} \in [0, \widetilde{\kappa}]$ is obvious. We choose $\nu_0 > 0$ such that

$$\nu_0^2 + \frac{\nu_0}{4} < \min\left\{\frac{4}{27AC_P^3}, \frac{256}{27A}\right\}$$

and let $\nu \in (0, \nu_0]$. Then the inequality

$$\left(\nu + \frac{4|\Omega|}{\mu^2 C_P}\widehat{\kappa}^2\right)^2 < \frac{4\min\left\{\left(\frac{1}{C_P} - 2\widehat{\kappa}\right)^3, 4^3\right\}}{27A(1+\frac{1}{4\nu})} \tag{3.16}$$

is satisfied with $\widehat{\kappa} = 0$. Let $\widetilde{\kappa} \in (0, \frac{1}{2C_P})$ be such that (3.16) is still satisfied for $\widehat{\kappa} = \widetilde{\kappa}$. This is possible due to continuity of the expressions in $\widehat{\kappa}$. Additionally, let $\eta = \min\{4, \frac{1}{C_P} - 2\widetilde{\kappa}\}$. Consequently, the inequality

$$\left(\nu + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2\right)^2 < \frac{4\eta^3}{27A(1+\frac{1}{4\nu})}, \quad \text{that is} \quad \nu - \frac{2}{3}\eta\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2 < 0 \tag{3.17}$$

holds. We observe that p attains a local minimum at

$$x_m = \sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} > 0,$$

where

$$\begin{aligned} p(x_m) &= \nu - \eta\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + A(1+\frac{1}{4\nu})\frac{\eta}{3A(1+\frac{1}{4\nu})}\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2 \\ &= \nu - \frac{2}{3}\eta\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2 \end{aligned}$$

is negative by (3.17) and therefore p has a root in (x_m, ∞). For any $\widehat{\kappa} \in [0, \widetilde{\kappa}]$ this root is smaller than $\sqrt{\frac{4}{A(1+\frac{4}{\nu})}}$, because for $x > \sqrt{\frac{4}{A(1+\frac{4}{\nu})}} > \sqrt{\frac{\eta}{A(1+\frac{4}{\nu})}}$ we have

$$p(x) > A\left(1 + \frac{1}{4\nu}\right)x^3 - \eta x \ge 0. \qquad \square$$

We use this root for a comparison argument:

Proposition 3.4.10. *Let $N = 3$ and $\mu > 0$, let ν_0, η, $\widetilde{\kappa}$ and $\delta_\nu(\widetilde{\kappa})$ for some $\nu \in (0, \nu_0]$ be as in Lemma 3.4.9. Then for any $0 \le \widehat{\kappa} \le \widetilde{\kappa}$, $\delta_\nu(\widehat{\kappa}) > \delta_\nu(\widetilde{\kappa}) > 0$ is such that for every $\kappa \le \widehat{\kappa}$ every $\varepsilon > 0$ has the following property: If y_ε from* (3.14) *satisfies*

$$y_\varepsilon(T) \le \delta_\nu(\kappa)$$

for some $T > T_0$ (with $T_0 = T_0(\mu, \kappa, \widehat{\kappa})$ from Proposition 3.4.8), then $y_\varepsilon(t) \le \delta_\nu(\kappa)$ for all $t > T$.

Proof. We choose as $\delta = \delta_\nu(\widehat{\kappa})$ the largest root of p from Lemma 3.4.9 and observe that according to Proposition 3.4.8 and the assumption on T

$$y_\varepsilon'(t) \le p(y(t)) \qquad \text{for all} \quad t > T \qquad \text{and} \qquad y_\varepsilon(T) \le \delta.$$

The comparison principle for ordinary differential equations therefore shows by means of comparison with $\overline{y} \equiv \delta$ that $y_\varepsilon(t) \le \delta$ for all $t > T$ as well. $\square$

3.4.1 Eventual boundedness of y_ε

Proposition 3.4.10 asserts that y_ε stays small, should it ever fall below a certain value. We still have to ensure that the condition actually occurs.

Proposition 3.4.11. *Let $N = 3$. Let $\nu \in (0, \nu_0]$ with ν_0 as in Lemma 3.4.9. Then there exists $\kappa_0 \in (0, \frac{1}{8})$ such that for any $\kappa < \widehat{\kappa} \in (0, \kappa_0]$ there is $t_0 > 0$ such that for all $\tau > t_0$, for all $\varepsilon > 0$*

$$\int_\Omega \left(u_\varepsilon^2(\cdot,\tau) + |\nabla v_\varepsilon(\cdot,\tau)|^4\right) < \delta_\nu(\widehat{\kappa})$$

where $\delta_\nu(\widehat{\kappa}) > 0$ is the positive root of p given by Lemma 3.4.9.
Furthermore, $\widehat{\kappa}$ satisfies

$$\widehat{\kappa} \leq \sqrt{\frac{\delta_\nu(\widehat{\kappa})\mu^2}{(4 + 8C_\Omega)|\Omega|}}, \tag{3.18}$$

where C_Ω is a constant depending on the domain Ω only.

Proof. Due to the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$, there is $C_\Omega > 0$ such that

$$\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}} \leq C_\Omega \int_\Omega (w^2 + |\Delta w|^2) \qquad \text{for all} w \in W^{2,2}(\Omega). \tag{3.19}$$

Let ν be as given in the statement of the proposition, let $\widetilde{\kappa} > 0$ be as provided by Lemma 3.4.9 and let $\delta = \min\{\delta_\nu(\widetilde{\kappa}), 1\}$. Choose

$$0 < \kappa_0 < \min\left\{\widetilde{\kappa}, \sqrt{\frac{\delta\mu^2}{(4 + 8C_\Omega)|\Omega|}}, \frac{1}{8}\right\} \tag{3.20}$$

and let $\widehat{\kappa} \in (0, \kappa_0]$ and $\kappa < \widehat{\kappa}$. (This already ensures (3.18) as well as the applicability of Proposition 3.4.10.)
We then let $T_0 = T_0(\mu, \kappa, \widehat{\kappa})$ be as provided by Proposition 3.4.8, let $t > T_0$ and note that as a result of (3.15) this entails

$$\int_\Omega u_\varepsilon(\cdot, t) < \frac{2\widehat{\kappa}|\Omega|}{\mu}. \tag{3.21}$$

Furthermore we denote

$$C_0 = \max\left\{1 + \int |\nabla v_0|^2 + \frac{1}{\mu}\int_\Omega u_0 + \frac{1}{\mu}, \frac{\widehat{\kappa}+1}{\mu} \max\left\{1 + \int_\Omega u_0, \frac{\widehat{\kappa}|\Omega|}{\mu}\right\}\right\} \tag{3.22}$$

and choose $T > 0$ so large that

$$\begin{aligned}\frac{1}{T}\Bigg(&\frac{2\widehat{\kappa}|\Omega|}{\mu^2} + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2} + C_\Omega\frac{\widehat{\kappa}}{\mu}\max\left\{1 + \int_\Omega u_0, \frac{\widehat{\kappa}|\Omega|}{\mu}\right\} t \\ &+ C_\Omega\frac{1}{\mu}\int_\Omega u_0 + \frac{C_\Omega}{\mu} + C_\Omega\int_\Omega v_0^2 + C_\Omega + 2C_\Omega C_0 + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2}\Bigg) < \frac{\delta}{2}.\end{aligned} \tag{3.23}$$

Combining (3.19) with Lemmata 3.3.2, 3.3.3 and 3.3.4 gives

$$\int_t^{t+T}\left(\int_\Omega u_\varepsilon^2 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{1}{2}}\right)$$

$$\begin{aligned}
&\le \int_t^{t+T}\int_\Omega u_\varepsilon^2 + C_\Omega \int_t^{t+T}\int_\Omega v_\varepsilon^2 + C_\Omega \int_t^{t+T}\int_\Omega |\Delta v_\varepsilon|^2 \\
&\le \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_\varepsilon(\cdot,t), \frac{\kappa_+|\Omega|}{\mu}\right\} T + \frac{1}{\mu}\int_\Omega u_\varepsilon(\cdot,t) \\
&\quad + C_\Omega \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_\varepsilon(\cdot,t), \frac{\kappa_+|\Omega|}{\mu}\right\} T + C_\Omega \frac{1}{\mu}\int_\Omega u_\varepsilon(\cdot,t) + C_\Omega \int_\Omega v_\varepsilon^2(\cdot,t) \\
&\quad + C_\Omega \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_\varepsilon(\cdot,t), \frac{\kappa_+|\Omega|}{\mu}\right\} T + C_\Omega \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \frac{C_\Omega}{\mu}\int_\Omega u_\varepsilon(\cdot,t).
\end{aligned}$$

Due to (3.21), upon another application of Lemmata 3.3.3 and 3.3.4 and taking into account that $\kappa_+ \le \widehat{\kappa}$, this reduces to

$$\begin{aligned}
&\int_t^{t+T}\left(\int_\Omega u_\varepsilon^2 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{1}{2}}\right) \\
&\quad \le \frac{\widehat{\kappa}}{\mu}\frac{2\widehat{\kappa}|\Omega|}{\mu}T + \frac{2\widehat{\kappa}|\Omega|}{\mu^2} \\
&\qquad + C_\Omega \frac{\widehat{\kappa}}{\mu}\frac{2\widehat{\kappa}|\Omega|}{\mu}T + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2} + C_\Omega\frac{\widehat{\kappa}}{\mu}\max\left\{1+\int_\Omega u_0, \frac{\widehat{\kappa}|\Omega|}{\mu}\right\} t \\
&\qquad + C_\Omega \frac{1}{\mu}\left(1+\int_\Omega u_0\right) + C_\Omega + C_\Omega\int_\Omega v_0^2 + C_\Omega\frac{\widehat{\kappa}}{\mu}\frac{2\widehat{\kappa}|\Omega|}{\mu}T + 2C_\Omega C_0 + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2},
\end{aligned}$$

where C_0 is as defined in (3.22). Therefore,

$$\begin{aligned}
&\frac{1}{T}\int_t^{t+T}\left(\int_\Omega u_\varepsilon^2 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{1}{2}}\right) \\
&\quad \le \frac{(2+4C_\Omega)|\Omega|}{\mu^2}\widehat{\kappa}^2 + \frac{1}{T}\Bigg(\frac{2\widehat{\kappa}|\Omega|}{\mu^2} + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2} + C_\Omega\frac{\widehat{\kappa}}{\mu}\max\left\{1+\int_\Omega u_0, \frac{\widehat{\kappa}|\Omega|}{\mu}\right\} t \\
&\qquad\qquad + C_\Omega\frac{1}{\mu}\int_\Omega u_0 + \frac{C_\Omega}{\mu} + C_\Omega\int_\Omega v_0^2 + C_\Omega + 2C_\Omega C_0 + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2}\Bigg).
\end{aligned}$$

Our choices of κ_0 and T in (3.20) and (3.23), respectively, now entail

$$\frac{1}{T}\int_t^{t+T}\left(\int_\Omega u_\varepsilon^2 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{1}{2}}\right) \le \delta.$$

Accordingly, for at least one $t_0 \in (t, t+T)$

$$\int_\Omega u_\varepsilon^2(\cdot,t_0) + \left(\int_\Omega |\nabla v_\varepsilon(\cdot,t_0)|^4\right)^{\frac{1}{2}} \le \delta$$

holds as well. Because $\delta \le 1$, for the same t_0 we may infer

$$\int_\Omega \left(u_\varepsilon^2(\cdot,t_0) + |\nabla v_\varepsilon(\cdot,t_0)|^4\right) \le \delta \le \delta(\widetilde{\kappa}).$$

Due to $\delta_\nu(\widetilde{\kappa}) \le \delta_\nu(\kappa_0) \le \delta_\nu(\widehat{\kappa})$ for $0 < \widehat{\kappa} < \kappa_0$, the claimed inequality for larger times τ is a direct consequence of Proposition 3.4.10. $\square$

3.4.2 Eventual boundedness of $(u_\varepsilon, v_\varepsilon)$ in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$

The next step is to refine these bounds on y_ε to bounds on u, v and ∇v. L^p-L^q-estimates for the heat semigroup will be the cornerstone of this procedure.

Proposition 3.4.12. *Let $N = 3$. Then there exists a function $K\colon [0,\infty) \to [0,\infty)$ satisfying $\lim_{\delta\to 0} K(\delta) = 0$ with the following properties:*
Assume, $\nu \in (0, \nu_0)$, $\kappa < \kappa_0$ with ν_0, κ_0 as in Lemma 3.4.9 and Proposition 3.4.11 respectively. Choose $\widehat{\kappa} \in (\kappa_+, \kappa_0]$ and let $\delta_\nu(\widehat{\kappa})$ be as given by Proposition 3.4.11. Then there are $T_* > 0$ and $C > 0$ such that for all $t > T_*$ and for all $\varepsilon > 0$

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < K(\delta_\nu(\widehat{\kappa})).$$

Furthermore, corresponding to any $\|u_0\|_{L^2(\Omega)}$, $\|v_0\|_{L^2(\Omega)}$, there is $C > 0$ such that for all $t > T_*$, $\varepsilon > 0$

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C\mathrm{e}^{-(t-T_*)} + K(\delta_\nu(\widehat{\kappa})).$$

Proof. Let $\nu_0, \nu, \kappa \le \kappa_+ < \widehat{\kappa} < \kappa_0$ and $\delta := \delta_\nu(\widehat{\kappa})$ be as indicated in the statement of the proposition. Let $T_* - 2$ be the number from Proposition 3.4.11, let $t_0 \ge T_* - 2$ and let us first show boundedness of u_ε on $[t_0+1, t_0+2]$. To this aim, we define

$$M := \sup_{\tau\in(t_0,t_0+2]} \left\|(\tau - t_0)^{\frac{3}{4}} u_\varepsilon(\cdot,\tau)\right\|_{L^\infty(\Omega)}$$

and let $p \in (3,4)$.
From the choice of T_* and δ and Proposition 3.4.11, we know that $\int_\Omega u_\varepsilon^2(\cdot,t) + \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4 \le \delta$ for $t > T_*$. Together with Hölder's inequality this implies

$$\begin{aligned}
\|u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \le & \|u_\varepsilon(\cdot,s)\|_{L^{\frac{4p}{4-p}}(\Omega)} \|\nabla v_\varepsilon(\cdot,s)\|_{L^4(\Omega)} \\
\le & \delta^{\frac{1}{4}} \left(\int_\Omega u_\varepsilon^{\frac{4p}{4-p}}(\cdot,s)\right)^{\frac{4-p}{4p}} \\
\le & \delta^{\frac{1}{4}} \left(\int_\Omega u_\varepsilon^2(\cdot,s)\cdot u_\varepsilon^{\frac{4p}{4-p}-2}(\cdot,s)\right)^{\frac{4-p}{4p}} \\
\le & \delta^{\frac{1}{4}} \left(\int_\Omega u_\varepsilon^2(\cdot,s)\right)^{\frac{4-p}{4p}} \left(\sup_\Omega u_\varepsilon^{\frac{4p}{4-p}-2}(\cdot,s)\right)^{\frac{4-p}{4p}} \\
\le & \delta^{\frac{1}{4}+\frac{4-p}{4p}} \sup_\Omega \left(u_\varepsilon^{1-\frac{4-p}{2p}}(\cdot,s)\right) \\
\le & \delta^{\frac{1}{p}} (s-t_0)^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}} \sup_\Omega \left((s-t_0)^{\frac{3}{4}} u_\varepsilon(\cdot,s)\right)^{1-\frac{4-p}{2p}}
\end{aligned} \tag{3.24}$$

for $s \in (t_0, t_0+2]$. Triangle inequality and L^p-L^q-estimates [110, Lemma 1.3] give a constant $C_1 > 0$ such that

$$\left\|\mathrm{e}^{\tau\Delta} u_\varepsilon(\cdot,t_0)\right\|_{L^\infty(\Omega)} \le C_1(1+\tau^{-\frac{3}{4}}) \|u_\varepsilon(\cdot,t_0) - \overline{u}_\varepsilon(t_0)\|_{L^2(\Omega)} + \|\overline{u}_\varepsilon(t_0)\|_{L^\infty(\Omega)},$$

where $\|\overline{u}_\varepsilon(t_0)\|_{L^\infty(\Omega)} = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon(\cdot,t_0) \le |\Omega|^{-\frac{1}{2}} \|u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)} \le |\Omega|^{-\frac{1}{2}}\sqrt{\delta}$ and its consequence $\|\overline{u}_\varepsilon(t_0)\|_{L^2(\Omega)} \le \sqrt{\delta}$ lead to

$$\left\|\mathrm{e}^{\tau\Delta} u_\varepsilon(\cdot,t_0)\right\|_{L^\infty(\Omega)} \le C_1(1+\tau^{-\frac{3}{4}})2\sqrt{\delta} + |\Omega|^{-\frac{1}{2}}\sqrt{\delta}. \tag{3.25}$$

Again, by semigroup representation and the fact that the heat semigroup is order-preserving,

$$\begin{aligned}0 \le \tau^{\frac{3}{4}} u_\varepsilon(\cdot, t_0+\tau) \le& \tau^{\frac{3}{4}} \mathrm{e}^{\tau\Delta} u_\varepsilon(\cdot, t_0) - \tau^{\frac{3}{4}} \int_0^\tau \mathrm{e}^{(\tau-s)\Delta} \nabla\cdot (u_\varepsilon(\cdot, t_0+s)\nabla v_\varepsilon(\cdot, t_0+s))\mathrm{d}s\\ &+ \tau^{\frac{3}{4}} \int_0^\tau \mathrm{e}^{(\tau-s)\Delta} \left(\kappa_+ u_\varepsilon(\cdot, t_0+s) - \mu u_\varepsilon^2(\cdot, t_0+s)\right)\mathrm{d}s\\ \le& \tau^{\frac{3}{4}} \left\|\mathrm{e}^{(\tau-s)\Delta} u_\varepsilon(\cdot, t_0)\right\|_{L^\infty(\Omega)} + \tau^{\frac{3}{4}} \int_0^\tau \left\|\mathrm{e}^{(\tau-s)\Delta}\nabla\cdot (u_\varepsilon(\cdot, t_0+s)\nabla v_\varepsilon(\cdot, t_0+s))\right\|_{L^\infty(\Omega)}\mathrm{d}s\\ &+ \tau^{\frac{3}{4}} \int_0^\tau \kappa_+ M s^{-\frac{3}{4}}\mathrm{d}s.\end{aligned}$$

Together with L^p-L^q-estimates, (3.24) and (3.25) this entails for $\tau \in [0,2]$ and some $C_2 > 0$ from [110, Lemma 1.3]

$$\begin{aligned}\left\|\tau^{\frac{3}{4}} u_\varepsilon(\cdot, t_0+\tau)\right\|_{L^\infty(\Omega)} \le& \tau^{\frac{3}{4}} C_1(1+\tau^{-\frac{3}{4}})2\sqrt{\delta} + \tau^{\frac{3}{4}}|\Omega|^{-\frac{1}{2}}\sqrt{\delta} + 8\kappa_+ M\\ &+ 2\int_0^\tau (1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}}) \left\|u_\varepsilon(\cdot, t_0+s)\nabla v_\varepsilon(\cdot, t_0+s)\right\|_{L^p(\Omega)}\mathrm{d}s\\ \le& 2C_1(1+\tau^{\frac{3}{4}})\sqrt{\delta} + \tau^{\frac{3}{4}}|\Omega|^{-\frac{1}{2}}\sqrt{\delta} + 8\kappa_0 M\\ &+ C_2\int_0^\tau (1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}})\delta^{\frac{1}{p}} s^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}} \left\|s^{\frac{3}{4}} u_\varepsilon(\cdot, t_0+s)\right\|_{L^\infty(\Omega)}^{1-\frac{4-p}{2p}}\mathrm{d}s\\ \le& 2C_1(1+\tau^{\frac{3}{4}})\sqrt{\delta} + \tau^{\frac{3}{4}}|\Omega|^{-\frac{1}{2}}\sqrt{\delta} + 8\kappa_0 M\\ &+ C_2\int_0^\tau (1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}})\delta^{\frac{1}{p}} s^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}} M^{1-\frac{4-p}{2p}}\mathrm{d}s,\end{aligned}$$

As $\int_0^2 (1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}})s^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}}\mathrm{d}s$ is finite and $\frac{1}{1-8\kappa_0} > 0$, taking the supremum over $\tau \in [0,2]$, we infer

$$M \le C_3\sqrt{\delta} + C_4\delta^{\frac{1}{p}} M^{1-\frac{4-p}{2p}}$$

with obvious choices of the constants $C_3, C_4 > 0$. Therefore

$$M \le D(\delta) := \sup\{s \in [0,\infty) : s - C_4\delta^{\frac{1}{p}} s^{1-\frac{4-p}{2p}} \le C_3\sqrt{\delta}\} < \infty.$$

Note that $D(\delta)$ tends to 0 as δ becomes small. For $t \in [t_0, t_0+2]$

$$(t-t_0)^{\frac{3}{4}} \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} < D(\delta),$$

meaning that for $t \in [t_0+1, t_0+2]$

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} < D(t-t_0)^{-\frac{3}{4}} \le D(\delta).$$

$D(\delta)$ is independent of the choice of $t_0 > T_* - 2$, therefore we can conclude

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le D(\delta) \tag{3.26}$$

for any $t > T_* - 1$.

Boundedness of $\{\nabla v(\cdot, \tau)\}_{\tau > T_*}$ in $L^\infty(\Omega; \mathbb{R}^N)$ can be achieved from the following estimates: We let $t_0 = T_* - 1$ and $t = \tau - t_0$. Then [110, Lemma 1.3] provides $C_5 > 0$ such that

$$\|\nabla v_\varepsilon(\cdot, \tau)\|_{L^\infty(\Omega)} \le \left\|\nabla \mathrm{e}^{t(\Delta-1)} v_\varepsilon(\cdot, t_0)\right\|_{L^\infty(\Omega)} + \int_0^t \left\|\nabla \mathrm{e}^{(t-s)(\Delta-1)} u_\varepsilon(\cdot, t_0+s)\right\|_{L^\infty(\Omega)}\mathrm{d}s$$

$$\leq C_5 t^{-\frac{7}{8}} \|\nabla v_\varepsilon(\cdot,t_0)\|_{L^4(\Omega)} + C_5 \int_0^t (1+(t-s)^{-\frac{1}{2}})\mathrm{e}^{-(t-s)} \|u_\varepsilon(\cdot,t_0+s)\|_{L^\infty(\Omega)} \mathrm{d}s$$
$$\leq C_5 \delta^{\frac{1}{4}} t^{-\frac{7}{8}} + D(\delta) C_5 \int_0^\infty (1+\sigma^{-\frac{1}{2}})\mathrm{e}^{-\sigma} \mathrm{d}\sigma \tag{3.27}$$

is bounded uniformly in $\tau \in [T_*,\infty)$. By similar reasoning together with Lemma 3.3.3, we obtain bounds on $\|v_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)}$. In preparation for these estimates, let $t_* > t_0$ and let us note that by (3.11) and Lemma 3.3.3

$$\begin{aligned}
\frac{1}{|\Omega|}\int_\Omega v_\varepsilon(\cdot,t_*) &\leq \frac{1}{|\Omega|}\left(\int_\Omega v_\varepsilon(\cdot,t_0)\right)\mathrm{e}^{-(t_*-t_0)} + \frac{\kappa_+}{\mu} + \frac{1}{|\Omega|}\int_\Omega u_\varepsilon(\cdot,t_0)\\
&\leq \left(\frac{\kappa_+}{\mu} + C_6\right)\mathrm{e}^{-(t_*-t_0)} + \frac{\kappa_+}{\mu} + \frac{1}{|\Omega|^{\frac{1}{2}}}\|u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)}\\
&\leq C_6 \mathrm{e}^{-(t_*-t_0)} + 2\frac{\widehat{\kappa}}{\mu} + \frac{\delta^{\frac{1}{2}}}{|\Omega|^{\frac{1}{2}}}\\
&\leq C_6 \mathrm{e}^{-(t_*-t_0)} + C_7\delta^{\frac{1}{2}},
\end{aligned}$$

where C_6 depends on $\|u_0\|_{L^1(\Omega)}$ and $\|v_0\|_{L^1(\Omega)}$ (and $|\Omega|$) only, and where we have applied (3.18) in the last step, so that $C_7 = \left(1 + \frac{1}{\sqrt{(4+8C_\Omega)}}\right)\frac{1}{\sqrt{|\Omega|}}$ with C_Ω as in (3.18).

Lemma 1.3 of [110] yields $C_8 > 0$, which, in conjunction with Poincaré's inequality and (3.26), gives

$$\begin{aligned}
\|v_\varepsilon(\cdot,t_*+t)\|_{L^\infty(\Omega)} &\leq \left\|\mathrm{e}^{t(\Delta-1)}v_\varepsilon(\cdot,t_*)\right\|_{L^\infty(\Omega)} + \int_0^t \left\|\mathrm{e}^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,t_*+s)\right\|_{L^\infty(\Omega)}\mathrm{d}s\\
&\leq \left\|\mathrm{e}^{t(\Delta-1)}(v_\varepsilon(\cdot,t_*) - \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(\cdot,t_*))\right\|_{L^\infty(\Omega)} + \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(\cdot,t_*) + tD(\delta)\\
&\leq C_8(1+t^{-\frac{3}{4}})\left\|v_\varepsilon(\cdot,t_*) - \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(\cdot,t_*)\right\|_{L^2(\Omega)} + \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(\cdot,t_*) + 2D(\delta)\\
&\leq C_8(1+t^{-\frac{3}{4}})\sqrt{C_P}\,\|\nabla v_\varepsilon(\cdot,t_*)\|_{L^2(\Omega)} + \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(\cdot,t_*) + 2D(\delta)\\
&\leq C_8(1+t^{-\frac{3}{4}})\sqrt{C_P}|\Omega|^{\frac{1}{4}}\delta^{\frac{1}{4}} + C_6\mathrm{e}^{-(t_*-t_0)}\mathrm{e}^{2-t} + C_7\delta^{\frac{1}{2}} + 2D(\delta)
\end{aligned}$$

for any $t \in (0,2]$ and therefore

$$\|v_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \leq 2C_8\sqrt{C_P}|\Omega|^{\frac{1}{4}}\delta^{\frac{1}{4}} + C_7\delta^{\frac{1}{2}} + 2D(\delta) + C_6\mathrm{e}\mathrm{e}^{-(\tau-(t_0+1))} \tag{3.28}$$

for any $\tau > t_0 + 1 = T_*$. Collecting terms from (3.26), (3.27) and (3.28), we obtain a suitable definition of C and of $K(\delta)$ – and because $\delta^{\frac{1}{4}}$, $\delta^{\frac{1}{2}}$ and $D(\delta)$ tend to 0 as $\delta \searrow 0$, indeed, $\lim_{\delta\searrow 0} K(\delta) = 0$. □

3.5 Definition of solutions

Definition 3.5.1. *A pair of functions* $(u,v) \in L^2_{loc}([0,\infty); L^2(\Omega)) \times L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ *is called* ***weak solution*** *of* (3.1) *for initial data* $(u_0,v_0) \in L^2(\Omega)\times W^{1,2}(\Omega)$ *if for all test functions* $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ *with* $\partial_\nu\varphi\big|_{\partial\Omega} = 0$ *the following holds:*

$$-\int_0^\infty\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(0) = \int_0^\infty\int_\Omega u\Delta\varphi - \int_0^\infty\int_\Omega u\nabla v\cdot\nabla\varphi + \kappa\int_0^\infty\int_\Omega u\varphi - \mu\int_0^\infty\int_\Omega u^2\varphi \tag{3.29}$$

and, for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$,

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega v\varphi + \int_0^\infty \int_\Omega u\varphi. \tag{3.30}$$

3.6 Convergence to a solution

Purpose of the estimates from section 3.3 was to make the extraction of convergent sequences of approximate solutions $(u_\varepsilon, v_\varepsilon)$ possible. The following proposition lists, in which sense we have obtained convergence.

Proposition 3.6.1. *There exist* $u \in L^2_{loc}([0,\infty); L^2(\Omega))$ *and* $v \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ *and a sequence* $\varepsilon_j \searrow 0$ *such that for any* $T > 0$

$$u_{\varepsilon_j} \to u \quad \text{a.e. in } \Omega \times [0,T], \tag{3.31}$$

$$u_{\varepsilon_j} \to u \quad \text{in } L^2(\Omega \times (0,T)), \tag{3.32}$$

$$\varepsilon_j u_{\varepsilon_j}^\theta \rightharpoonup 0 \quad \text{in } L^1(\Omega \times (0,T)), \tag{3.33}$$

$$v_{\varepsilon_j} \rightharpoonup v \quad \text{in } L^2((0,T); W^{1,2}(\Omega)), \tag{3.34}$$

$$v_{\varepsilon_j} \to v \quad \text{in } L^2(\Omega \times (0,T)), \tag{3.35}$$

$$v_{\varepsilon_j} \to v \quad \text{a.e. in } \Omega \times [0,T], \tag{3.36}$$

$$\Delta v_{\varepsilon_j} \rightharpoonup \Delta v \quad \text{in } L^2(\Omega \times (0,T)), \tag{3.37}$$

$$v_{\varepsilon_j t} \rightharpoonup v_t \quad \text{in } L^2(\Omega \times (0,T)), \tag{3.38}$$

$$u_{\varepsilon_j} \nabla v_{\varepsilon_j} \rightharpoonup u\nabla v \quad \text{in } L^1(\Omega \times (0,T); \mathbb{R}^N). \tag{3.39}$$

Proof. Lemmata 3.3.6 and 3.3.7 show boundedness of $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ in $L^{\frac{4}{3}}((0,T); W^{1,\frac{4}{3}}(\Omega))$ and of the derivatives $\{u_{\varepsilon t};\ \varepsilon \in (0,1)\}$ in $L^1((0,T); (W_0^{3,N+1}(\Omega))^*)$ so that by a variant of the Aubin-Lions-Lemma [15, Prop. 6], $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ is relatively compact in $L^{\frac{4}{3}}(\Omega \times (0,T))$; in particular, there is a sequence $\varepsilon_j \searrow 0$ (of which we will, without relabeling, choose further subsequences in the following) such that $u_{\varepsilon_j} \to u$ almost everywhere in $\Omega \times (0,T)$ for some $u \in L^{\frac{4}{3}}(\Omega \times (0,T))$. Boundedness of $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ in $L^2(\Omega \times (0,T))$ due to Lemma 3.3.2 yields a subsequence along which $u_{\varepsilon_j} \rightharpoonup u$ in $L^2(\Omega \times (0,T))$.
By Lemma 3.3.5, $\{u_\varepsilon^2;\ \varepsilon \in (0,1)\}$ is equi-integrable and thus, according to [21, Thm. IV.8.9], weakly sequentially precompact in $L^1(\Omega \times (0,T))$. Along a subsequence, $u_{\varepsilon_j}^2 \rightharpoonup u^2$ in $L^1(\Omega \times (0,T))$ and hence

$$\left\|u_{\varepsilon_j}\right\|^2_{L^2(\Omega\times(0,T))} = \int_{\Omega\times(0,T)} u_{\varepsilon_j}^2 \cdot 1 \to \int_{\Omega\times(0,T)} u^2 \cdot 1 = \|u\|^2_{L^2(\Omega\times(0,T))}.$$

The combination of $u_{\varepsilon_j} \rightharpoonup u$ in $L^2(\Omega \times (0,T))$ and $\left\|u_{\varepsilon_j}\right\|_{L^2(\Omega\times(0,T))} \to \|u\|_{L^2(\Omega\times(0,T))}$ shows that actually (3.32) holds.
Similarly, we see that $\{\varepsilon u_\varepsilon^\theta;\ \varepsilon \in (0,1)\}$ is equi-integrable (Lemma 3.3.5) and hence is weakly convergent along a subsequence. Pointwise a.e. convergence of u_ε^θ to u^θ identifies the weak limit of $\varepsilon_j u_{\varepsilon_j}^\theta$ as 0, which is (3.33).
According to Lemmata 3.3.3 and 3.3.4, $\{v_\varepsilon;\ \varepsilon \in (0,1)\}$ is bounded in $L^\infty((0,T); W^{1,2}(\Omega))$, which is continuously embedded into $L^2((0,T); W^{1,2}(\Omega))$, and a subsequence with (3.34) can be found. Furthermore, $\{v_{\varepsilon t};\ \varepsilon \in (0,1)\} = \{\Delta v_\varepsilon - v_\varepsilon + u_\varepsilon;\ \varepsilon \in (0,1)\}$ is bounded in $L^2((0,T); L^2(\Omega))$ due to Lemmata 3.3.4, 3.3.3, 3.3.2, and the Aubin-Lions lemma yields (3.35) as well as, along another subsequence, (3.36). At the same time, we can conclude (3.37) and (3.38).

The statement (3.39), finally, results from a combination of (3.32) and (3.34). □

From now on, by (u, v) we will denote the limit provided by Proposition 3.6.1. Of course, it would be desirable for (u, v) to be a solution to the original problem. That is the case.

Lemma 3.6.2. *The pair (u, v) is a solution to* (3.1) *in the sense of Definition 3.5.1.*

Proof. We take φ as specified in Definition 3.5.1 and test the equations of (3.3) against it. The convergence results of Proposition 3.6.1 then produce (3.29) and (3.30). □

Remark 3.6.3. None of the arguments used for Proposition 3.6.1 and Lemma 3.6.2 depend on dimension N nor on the specific values of $\mu > 0$, $\kappa \in \mathbb{R}$.

3.7 Eventual smoothness. Proof of Theorem 3.1.1

In the most important scenario of spatial dimension 3, we can show that these solutions are not only solutions in some weak sense, but possess the property of eventual smoothness: From some time on, they are classical solutions. Our preparations from Section 3.4 that have provided boundedness of $(u_\varepsilon, v_\varepsilon)$ constitute the first step. The next proposition transfers these properties to (u, v).

Proposition 3.7.1. *Let $N = 3$ and assume $\kappa < \kappa_0$ with κ_0 from Proposition 3.4.11. With T_* denoting the number from Proposition 3.4.12,*

$$u \in L^2_{loc}([T_*, \infty); W^{1,2}(\Omega)), \qquad u_t \in L^2_{loc}([T_*, \infty), (W^{1,2}(\Omega))^*).$$

Furthermore $u, v \in L^\infty(\Omega \times [T_*, \infty))$, $\nabla v \in L^\infty(\Omega \times [T_*, \infty); \mathbb{R}^N)$.

Proof. On the interval $[T_*, \infty)$, from Proposition 3.4.12 we obtain boundedness of $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$, $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{|\nabla v_\varepsilon|\}_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega \times [T_*, \infty))$ and hence can choose a subsequence $\varepsilon_j \searrow 0$ of the sequence from Proposition 3.6.1 such that $u_\varepsilon, v_\varepsilon, \nabla v_\varepsilon$ are weak-*-convergent in this space. For $T > 0$, boundedness of $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{u_{\varepsilon t}\}_{\varepsilon\in(0,1)}$ in $L^2_{loc}([T_*, T_* + T]; W^{1,2}(\Omega))$ and $L^2_{loc}([T_*, T_* + T], (W^{1,2}(\Omega))^*)$, respectively, are guaranteed by Lemma 3.3.8 and 3.3.9 and the choice of a weakly convergent subsequence yields the assertion. □

Corollary 3.7.2. *Under the conditions of Proposition 3.7.1, $u \in C^0([T_*, \infty); L^2(\Omega))$.*

Proof. For any $T > 0$, $u \in L^2([T_*, T_* + T]; W^{1,2}(\Omega))$ and $u_t \in L^2([T_*, T_* + T], (W^{1,2}(\Omega))^*)$. By Proposition 23.23 of [125], u is L^2-continuous on $[T_*, T_* + T]$. □

Actually, u and v are even Hölder continuous.

Lemma 3.7.3. *Let $N = 3$. Assume, $\kappa < \kappa_0$ with κ_0 from Proposition 3.4.11 and let T_* be as in Proposition 3.4.12. There is $\gamma > 0$ such that $u, v \in C^{\gamma,\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times [T_* + 1, \infty))$. Moreover, there is $C > 0$ such that for every $T > T_* + 1$,*

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[T,T+1])} + \|v\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[T,T+1])} \le C.$$

Proof. We let T_* be as in Proposition 3.4.12 and let $t \ge T_*$ be given such that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega\times(T_*,\infty))}$ and $\|v(\cdot, t)\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega\times(T_*,\infty))}$, which is the case for almost every such t. Definition 3.5.1, Corollary 3.7.2 and Proposition 3.7.1 enable us to interpret u as a local weak solution in the sense of [83] of the equation

$$\widetilde{u}_t - \nabla \cdot (\nabla \widetilde{u} - \widetilde{u} \nabla v) = \kappa u - \mu u^2, \tag{3.40}$$

for $\widetilde{u}$ on $[T_*, \infty)$.
Using boundedness of $\kappa u - \mu u^2$ and ∇v, an application of Theorem 1.3 of [83] ensures that $u \in C^{\gamma', \frac{\gamma'}{2}}(\overline{\Omega} \times [T_* + \frac{1}{2}, \infty))$ for some $\gamma' > 0$. Theorem 1.3 of [83] additionally asserts that the norm $\|u\|_{C^{\gamma, \frac{\gamma}{2}}(\overline{\Omega}\times[t+\frac{1}{2}, t+2])}$ can be estimated by a constant C_u which depends on the $L^\infty(\Omega)$-norm of $u(\cdot, t)$ and some "data" of the problem, a term condensing structural information on the equation (such as exponents) and certain L^r-norms of coefficients and the right-hand-side in (3.40).
Important to note is that, due to Proposition 3.7.1, $u, v \in L^\infty(\Omega \times [T_*, \infty))$ and $\nabla v \in L^\infty(\Omega \times [T_*, \infty); \mathbb{R}^N)$ and therefore the restrictions of these functions to $\overline{\Omega} \times [t, t+2]$ are bounded in $L^\infty(\overline{\Omega} \times [t, t+2])$ and $L^\infty(\overline{\Omega} \times [t, t+2]; \mathbb{R}^N)$, respectively, independently of $t > T_*$. Hence C_u can be chosen independently of t.
Similar to Corollary 3.7.2, from (3.35), (3.38) and (3.34), we infer $v \in L^2_{loc}((0, \infty); W^{1,2}(\Omega)) \cap C^0((0, \infty); L^2(\Omega))$ and boundedness of u, v on $[T_* + \frac{1}{2}, \infty)$ imply, again by Theorem 1.3 of [83] applied to the solution v of

$$\widetilde{v}_t - \nabla \cdot (\nabla \widetilde{v}) = u - v \tag{3.41}$$

for $\widetilde{v}$, that $v \in C^{\gamma'', \frac{\gamma''}{2}}(\overline{\Omega} \times [t+1, t+2])$ for some $\gamma'' > 0$ – and that

$$\|v\|_{C^{\gamma'', \frac{\gamma''}{2}}(\overline{\Omega}\times[t+1,t+2])} \leq C_v,$$

with some constant C_v which can be chosen independently of t.
Letting $\gamma = \min\{\gamma', \gamma''\}$, deriving a suitable constant C from the values of C_u and C_v and taking the arbitrariness of t into account, the claim follows. □

Thanks to the regularity of u and v that we have gained so far, we can interpret u and v as generalized solutions in the sense of [48] of the homogeneous Neumann boundary value problem with initial value $u(T_* + 1)$, $v(T_* + 1)$ to (3.40) or (3.41). As the coefficients are bounded, these problems are known to be uniquely solvable [48, Thm. III.5.1]. Therefore we can use existence theorems for smoother solutions to establish higher regularity of u and v.
Theorem IV.5.3 of [48] asserts the existence of $C^{2+\gamma, 1+\frac{\gamma}{2}}$ solutions, albeit under stronger smoothness assumptions on the initial datum than we can guarantee so far. In order to nevertheless apply this theorem, let us, for $t_0 > 0$, $T > t > 0$, introduce a smooth monotone function $\xi_{t_0,t,T}\colon [t_0, t_0 + T] \to \mathbb{R}$ satisfying $\xi(t_0) = 0$ and $\xi \equiv 1$ on $[t_0 + t, t_0 + T]$ as well as $\|\xi_{t_0,t,T}\|_{C^1([t_0,t_0+T])} \leq 1 + \frac{2}{t}$.

Proposition 3.7.4. *Let $N = 3$ and assume that $\kappa < \kappa_0$ with κ_0 from Proposition 3.4.11. Then there are $T^* > 0$ and $\gamma > 0$ such that $u, v \in C^{2+\gamma, 1+\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times [T^*, \infty))$. Moreover, there exists $C > 0$ such that for all $t > T^*$*

$$\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \leq C.$$

Proof. Let T_* be as in Proposition 3.4.12 and $T_0 > T_* + 1$. We let $\xi = \xi_{T_0, \frac{1}{2}, 2}$ as defined above and observe that $(\xi v)(T_0) = 0$, $\partial_\nu(\xi v)\big|_{\partial\Omega} = 0$ and $\widetilde{v} := \xi v$ satisfies

$$\widetilde{v}_t - \Delta \widetilde{v} = \xi_t v + \xi u - \xi v \qquad \text{on } (T_0, T_0 + 2), \tag{3.42}$$

a parabolic PDE with smooth coefficients and Hölder continuous right-hand side (due to Lemma 3.7.3). Theorem IV.5.3 of [48] in conjunction with the above-mentioned uniqueness property makes ξv an element of $C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T_0, T_0+2])$ and therefore $v \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T_0+\frac{1}{2}, T_0+2])$,

where, according to the aforementioned theorem, its norm can be estimated by the $C^{\gamma,\frac{\gamma}{2}}$-norm of the right-hand-side in (3.42) and therefore independently of $T_0 > T_* + 1$, cf. Lemma 3.7.3.
For an analogous procedure concerning u, we let $\xi = \xi_{T_0+\frac{1}{2},\frac{1}{2},\frac{3}{2}}$ and consider $\widetilde{u} = \xi u$, satisfying $\widetilde{u}(T_0 + \frac{1}{2}) = 0$, $\partial_\nu \widetilde{u}\big|_{\partial\Omega} = 0$ and solving

$$\widetilde{u}_t - \Delta\widetilde{u} - \nabla\widetilde{u} \cdot \nabla v - \widetilde{u}\Delta v = \xi_t u + \xi(\kappa u - \mu u^2),$$

where the coefficients are Hölder continuous as well as the right hand side and, by the same argument as before, [48, Thm. IV.5.3] asserts $u \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega} \times [T_0 + 1, T_0 + 2])$ with a T_0-independent estimate on the norm. The claim follows upon the choice $T^* = T_0 + 1$ and due to the independence of the Hölder norm of T_0. □

After these preparations, the proof of our main result consists in nothing more than collating the right statements:

Proof of Theorem 3.1.1. Existence of a solution is given by Proposition 3.6.1 in combination with Lemma 3.6.2, eventual smoothness and bounds on the Hölder norms by Proposition 3.7.4. □

3.8 Asymptotic behaviour

Now that existence and smoothness of (u, v) have been ensured, let us concentrate on the long time behaviour of solutions.

3.8.1 The case $\kappa \leq 0$. Proof of Theorem 3.1.3

Proof of Theorem 3.1.3. Let $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}_{j\in\mathbb{N}}$ be a sequence of solutions to (3.3) approaching (u, v) in the sense of Proposition 3.6.1. Let $\vartheta > 0$.
From Proposition 3.4.12 we can infer $\delta_0 > 0$ such that $K(\delta)$ from Proposition 3.4.12 satisfies $K(\delta) < \frac{\vartheta}{3}$ for any $\delta \in [0, \delta_0)$.
Now we apply Lemma 3.4.9 with $\nu \in (0, \nu_0]$ so small that $\sqrt{\frac{4}{A(1+\frac{1}{4\nu})}} < \delta_0$ and choose $\widetilde{\kappa} > 0$ and $\eta \in (0, 4]$ as provided therupon. In particular, this implies $\delta_\nu(\widehat{\kappa}) \leq \delta_0$ for any $\widehat{\kappa} \in (0, \widetilde{\kappa})$.
We let $\widehat{\kappa} \in (0, \widetilde{\kappa})$ and let $T_0 = T_0(\mu, 0, \widehat{\kappa})$ be as in Proposition 3.4.8. As $\kappa \leq 0 < \widehat{\kappa}$, Proposition 3.4.12 implies that there is $T > 0$ such that, independently of $j \in \mathbb{N}$,

$$\left\|u_{\varepsilon_j}(\cdot, t)\right\|_{L^\infty(\Omega)} + \left\|v_{\varepsilon_j}(\cdot, t)\right\|_{W^{1,\infty}(\Omega)} \leq 2K(\delta_\nu(\widehat{\kappa})) + C\mathrm{e}^{-(t-T)} \qquad \text{for all } t > T, \tag{3.43}$$

where C is a constant depending on the norm of the initial data (u_0, v_0).
Choose $T_\vartheta > T$ in such a way that $C\mathrm{e}^{-(T_\vartheta - T)} < \frac{\vartheta}{3}$ and that u, v are continuous on $[T_\vartheta, \infty)$ by Theorem 3.1.1. Our choice of δ_0 thus shows that, independently of $j \in \mathbb{N}$,

$$\left\|u_{\varepsilon_j}(\cdot, t)\right\|_{L^\infty(\Omega)} + \left\|v_{\varepsilon_j}(\cdot, t)\right\|_{W^{1,\infty}(\Omega)} \leq 2\frac{\vartheta}{3} + \frac{\vartheta}{3} = \vartheta \qquad \text{for all } t > T_\vartheta.$$

Almost everywhere convergence of $(u_{\varepsilon_j}, v_{\varepsilon_j}) \to (u, v)$ (as stated by Proposition 3.6.1 in (3.31), (3.36)) and continuity of u and v hence imply that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq \vartheta \qquad \text{for all } t > T_\vartheta. \tag{3.44}$$

In conclusion,

$$(u(\cdot, t), v(\cdot, t)) \to 0 \qquad \text{as } t \to \infty$$

in the sense of uniform convergence on Ω. □

3.8.2 Asymptotics for positive κ. Proof of Theorem 3.1.5

Proof of Theorem 3.1.5. Under the condition of κ being sufficiently small, Theorem 3.1.1 shows that the solutions constructed above enter some bounded set $B_{\mu,\kappa} \subset (C^{2+\gamma}(\overline{\Omega}))^2$, where $\gamma > 0$ is chosen as in Proposition 3.7.4.
As to the statement about the diameter of $B_{\mu,\kappa}$ in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ as $\kappa \to 0$, we can proceed almost exactly as in the proof of Theorem 3.1.3: Let $\vartheta > 0$. From Proposition 3.4.12 we can infer $\delta_0 > 0$ such that $K(\delta)$ from Proposition 3.4.12 satisfies $K(\delta) < \frac{\vartheta}{3}$ for any $\delta \in [0, \delta_0)$. The application of Lemma 3.4.9 with $\nu \in (0, \nu_0]$ satisfying $\sqrt{\frac{4}{A(1+\frac{1}{4\nu})}} < \delta_0$ provides $\eta \in (0, 4]$ and $\widetilde{\kappa} > 0$. Let $\widehat{\kappa} \in (0, \widetilde{\kappa})$. We will prove that $\operatorname{diam} B_{\mu,\kappa} \leq 2\vartheta$ if $\kappa < \widehat{\kappa}$.
For this, we assume that $\kappa < \widehat{\kappa}$ and let $T_0 = T_0(\mu, \kappa, \widehat{\kappa})$ be as in Proposition 3.4.8. As $\kappa < \widehat{\kappa}$, Proposition 3.4.12 implies that there is $T > 0$ such that, independent of $j \in \mathbb{N}$,

$$\left\|u_{\varepsilon_j}(\cdot, t)\right\|_{L^\infty(\Omega)} + \left\|v_{\varepsilon_j}(\cdot, t)\right\|_{W^{1,\infty}(\Omega)} \leq 2K(\delta_\nu(\widehat{\kappa})) + C\mathrm{e}^{-(t-T)} \qquad \text{for all } t > T, \tag{3.45}$$

where C is a constant depending on the norm of the initial data (u_0, v_0).
We choose $T_\vartheta > T$ in such a way that $C\mathrm{e}^{-(T_\vartheta - T)} < \frac{\vartheta}{3}$ and that u, v are continuously differentiable on $[T_\vartheta, \infty)$ by Theorem 3.1.1.
Our choice of δ_0 thus shows that, independently of $j \in \mathbb{N}$,

$$\left\|u_{\varepsilon_j}(\cdot, t)\right\|_{L^\infty(\Omega)} + \left\|v_{\varepsilon_j}(\cdot, t)\right\|_{W^{1,\infty}(\Omega)} \leq 2\frac{\vartheta}{3} + \frac{\vartheta}{3} = \vartheta \qquad \text{for all } t > T_\vartheta.$$

We make use of the almost everywhere convergence of $(u_{\varepsilon_j}, v_{\varepsilon_j}) \to (u, v)$ (as stated by Proposition 3.6.1 in (3.31), (3.36)) and the fact that ∇v_{ε_j} is essentially bounded by some constant $\widetilde{C}$ on $\Omega \times [T_\vartheta, \infty)$ uniformly in j, which allows us to extract a L^∞-weak*-convergent subsequence leading to $\|\nabla v\|_{L^\infty(\Omega \times (T_\vartheta, \infty))} \leq \widetilde{C}$.
Together with the continuity of u, v and ∇v these convergence results hence imply that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq \vartheta \qquad \text{for all } t > T_\vartheta.$$

In terms of $B_{\mu,\kappa}$ this means

$$B_{\mu,\kappa} \subset B_\vartheta^{L^\infty(\Omega) \times W^{1,\infty}(\Omega)}(0)$$

and hence $\operatorname{diam}(B_{\mu,\kappa}) \leq 2\vartheta$ for sufficiently small $\kappa > 0$. $\square$

4 A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity

4.1 Introduction

Numerous phenomena in connection with spontaneous aggregation can be described by PDE models incorporating a cross-diffusion mechanism. A prototypical example, which lies at the core of models used for a variety of purposes and to so different aims as the description of pattern formation of bacteria or slime mold in biology [43] or the prediction of burglary in criminology [64], is the following variant of the Keller-Segel system of chemotaxis:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\, S(v)\, \nabla v) \\ v_t = \Delta v - v + u \\ \partial_\nu u\big|_{\partial\Omega} = \partial_\nu v\big|_{\partial\Omega} = 0 \\ u(\cdot,0) = u_0,\, v(\cdot,0) = v_0 \end{cases} \tag{4.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary, with given nonnegative initial data u_0, v_0. We shall be concerned with the case of the singular sensitivity function S given by

$$S(v) = \frac{\chi}{v} \tag{4.2}$$

for a constant $\chi > 0$, which is in compliance with the Weber-Fechner law of stimulus perception (see [44]).
One of the first questions of mathematical interest with respect to this model is that of existence of a global classical solution, as opposed to blow-up of solutions in finite time. For the vast mathematical literature on chemotaxis, a large part of which is concerned with this question, see one of the survey articles [34, 35, 32, 9] and references therein.
According to the standard reasoning in the realm of chemotaxis equations (as e.g. formulated in [9]), in order to obtain global existence of classical solutions, for the two-dimensional case considered here, it is sufficient to derive t-independent bounds on the quantities $\int_\Omega u(\cdot,t)\ln u(\cdot,t)$ and $\int |\nabla v(\cdot,t)|^2$.
To achieve this in the particular context of (4.1), it has proven useful to consider the expression

$$\int_\Omega u \ln u - a \int_\Omega u \ln v, \tag{4.3}$$

as it has been done by Nagai, Senba, Yoshida [74] or Biler [10]. In these works, global existence of solutions has been derived for $\chi \leq 1$.

In the present chapter we shall answer the question whether $\chi = 1$ is a critical value in this regard in the negative. This question had been left open in [113], where the above-mentioned results have been generalized to higher dimension N, then obtaining existence in the case $\chi < \sqrt{2/N}$.
Let us mention some more results concerning equation (4.1): That the classical solutions for $\chi < \sqrt{2/N}$ are global-in-time bounded has been shown in [24]. In [113] also weak solutions have been shown to exist for (4.1), as long as $\chi < \sqrt{\frac{N+2}{3N-4}}$. In the radially symmetric setting, moreover, certain global weak "power-λ-solutions" exist ([91]).
In [3] the authors prove global existence and investigate the asymptotic behaviour of solutions to a system incorporating logistic growth terms in addition to general sensitivities with singularity at $v = 0$. Parabolic-elliptic chemotaxis models related to (4.1) are investigated, e.g. in [25], where the presence of such growth terms is used to ensure global existence and boundedness of classical solutions. In [26] global existence and boundedness of classical solutions to the parabolic-elliptic counterpart of (4.1) are obtained for even more singular sensitivities of the form $0 < S(v) \le \frac{\chi}{v^k}$, $k \ge 1$, under a smallness condition on χ, which for $k = 1$ and $N = 2$ amounts to $\chi < 1$.
Also concerning classical solutions of the fully parabolic system (4.1), to the best of our knowledge, the assertions for $\chi \le 1$ are the best known so far.
Since the new possible values for χ are but slightly larger than 1, rather than these values it is the method that can be considered the new contribution of the present chapter: Key to our approach toward the expansion of the interval of values for χ known to yield global solutions, namely, shall be the employment of an additional summand

$$b \int |\nabla \sqrt{v}|^2$$

in (4.3). Functionals containing this term have successfully been used in the context of coupled chemotaxis-fluid systems (see [114]) or of chemotaxis models with consumption of the chemoattractant [95] (e.g. obtained from the aforementioned system upon neglection of the fluid).
In the end we will arrive at the following

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain with smooth boundary. Then there exists $\chi_0 > 1$ such that for any $\chi \in (0, \chi_0)$ and any initial data satisfying $0 \le u_0 \in C^0(\overline{\Omega})$, $u_0 \not\equiv 0$, and $v_0 \in W^{1,q}(\Omega)$ for some $q > 2$ and $v_0 > 0$ in $\overline{\Omega}$, the system (4.1) has a global classical solution which is bounded.*

The plan of the chapter is as follows: In the next section we will discuss local existence of and an extensibility criterion for solutions to (4.1). Section 4.3 provides identities and estimates that will facilitate the usage of the additional term at the center of the proof of Theorem 4.1.1, to which Section 4.4 will be devoted.

4.2 How to ensure global existence

A general existence theorem for chemotaxis models is the following, taken from [9]:

Theorem 4.2.1. *Let $N \ge 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and let $q > N$. For some $\omega \in (0,1)$ let $S \in C^{1+\omega}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$, $f \in C^{1-}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$ and $g \in C^{1-}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$, and assume that $f(x,t,0,r) \ge 0$ for all $(x,t,r) \in \overline{\Omega} \times [0,\infty)^2$ and that $g(x,t,s,0) \ge 0$ for any $(x,t,s) \in \overline{\Omega} \times [0,\infty)^2$. Then for all nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ there exist $T_{max} \in (0,\infty]$ and a uniquely determined pair of nonnegative functions*

$$u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \tag{4.4}$$

$$v \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})) \cap L^{\infty}_{loc}([0, T_{max}); W^{1,q}(\Omega)),$$

such that (u, v) *solves*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (uS(x,t,u,v)\nabla v) + f(x,t,u,v), \\ v_t = \Delta v - v + g(x,t,u,v), \\ 0 = \partial_\nu u\big|_{\partial\Omega} = \partial_\nu v\big|_{\partial\Omega}, \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 \end{cases} \tag{4.5}$$

classically in $\Omega \times (0, T_{max})$ *and such that*

$$\text{if } T_{max} < \infty, \text{ then } \limsup_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \Big) = \infty. \tag{4.6}$$

Proof. A Banach-type fixed point argument provides existence of mild solutions on a short time interval whose length T depends on $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{W^{1,q}(\Omega)}$. Standard bootstrapping arguments ensure the regularity properties listed above. It follows from the dependence of T on the norms of u_0 and v_0 that the solution can be extended to $T_{max} \in (0, \infty]$ satisfying (4.6), see [9, Lemma 3.1]. □

This theorem is not directly applicable to (4.1), because it does not cover the case of singular functions S. We will remove this obstruction via use of the following lemma, which is a generalization of Lemma 2.2 of [24].

Lemma 4.2.2. *Let the conditions of Theorem 4.2.1 be satisfied, let* Ω *be convex, and let* $\zeta > 0$. *Then there is* $\eta = \eta(u_0, v_0, \zeta) > 0$ *such that if* v_0 *and the solution* (u, v) *to* (4.5) *satisfy*

$$\inf v_0 > 0 \qquad \text{and} \qquad \inf_{s \in [0, T_{max})} \int_\Omega g(x, s, u(x,s), v(x,s))dx \geq \zeta,$$

the second component of the solution also fulfils

$$v(x,t) \geq \eta \qquad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{max}).$$

Proof. Let us fix $\tau = \tau(u_0, v_0) > 0$ such that

$$\inf_\Omega v(\cdot, t) \geq \frac{1}{2} \inf_\Omega v_0 \qquad \text{for all } t \in [0, \tau].$$

Employing the pointwise estimate

$$(\mathrm{e}^{t\Delta} w)(x) \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \mathrm{e}^{-\frac{d^2}{4t}} \int_\Omega w \qquad \text{for nonnegative } w \in C^0(\overline{\Omega})$$

for the Neumann heat semigroup $\mathrm{e}^{t\Delta}$ with $d = \operatorname{diam}\Omega$, as provided in [24, Lemma 2.2] for convex domains following [33, Lemma 3.1], we can then conclude that

$$\begin{aligned} v(\cdot, t) &= \mathrm{e}^{t(\Delta-1)} v_0 + \int_0^t \mathrm{e}^{(t-s)(\Delta-1)} g(\cdot, s, u(\cdot,s), v(\cdot,s)) \mathrm{d}s \\ &\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} \mathrm{e}^{-\frac{d^2}{4(t-s)} - (t-s)} \int_\Omega g(\cdot, s, u(\cdot,s), v(\cdot,s)) \mathrm{d}s \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^t \frac{1}{(4\pi r)^{\frac{N}{2}}} \mathrm{e}^{-(r+\frac{d^2}{4r})} \mathrm{d}r \inf_{s\in[0,t]} \int_\Omega g(x,s,u(x,s),v(x,s)) \mathrm{d}x \\
&\geq \zeta \int_0^\tau \frac{1}{(4\pi r)^{\frac{N}{2}}} \mathrm{e}^{-(r+\frac{d^2}{4r})} \mathrm{d}r \qquad \text{in } \Omega
\end{aligned}$$

for any $t \in [\tau, T_{max})$. With $\eta = \min\left\{\frac{\inf_\Omega v_0}{2}, \zeta \int_0^{\tau(u_0,v_0)} \frac{1}{(4\pi r)^{\frac{N}{2}}} \mathrm{e}^{-(r+\frac{d^2}{4r})} \mathrm{d}r\right\}$ this proves the claim. □

With this lemma we can weaken the assumptions on the sensitivity S so as to allow for a singularity at $v = 0$.

Theorem 4.2.3. *i) Let $S \in C^{1+\omega}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R} \times (0,\infty))$ for some $\omega \in (0,1)$ and apart from the condition on S let the assumptions of Theorem 4.2.1 be satisfied.*
Additionally, assume that f is nonnegative and $g(x,t,s,r) \geq cs$ for some $c > 0$ and any $(x,t,s,r) \in \overline{\Omega} \times [0,\infty) \times \mathbb{R}^2$ and that $\inf_\Omega v_0 > 0$ and $\int_\Omega u_0 =: m > 0$. Then there is $T_{max} > 0$ such that (4.1) has a unique solution (u,v) as in (4.4) and such that (4.6) holds.
ii) Furthermore, if there are $K_1, K_2 > 0$ such that $f(x,t,s,r) \leq K_1$ and $g(x,t,s,r) \leq K_2(1+s)$ for all $(x,t,s,r) \in \Omega \times (0,\infty)^3$, and for every $\eta > 0$, $|S|$ is bounded on $\Omega \times (0,\infty)^2 \times (\eta,\infty)$, and if $N = 2$ and there is $M > 0$ such that

$$\int_\Omega u(\cdot,t) \ln u(\cdot,t) \leq M, \qquad \text{and} \qquad \int_\Omega |\nabla v(\cdot,t)|^2 \leq M \qquad \text{for all } t \in [0,T_{max}) \tag{4.7}$$

then (u,v) is global and bounded.

Proof. i) We let $\eta := \eta(u_0, v_0, cm)$ be as in Lemma 4.2.2 and let $\xi \colon \mathbb{R} \to [0,1]$ be a smooth, monotone decreasing function with $\xi(r) = 1$ for $r \leq \frac{\eta}{2}$ and $\xi(r) = 0$ for $r \geq \eta$. We define

$$S_\eta(x,t,s,r) := \begin{cases} S(x,t,s,\frac{\eta}{2}), & (x,t,s,r) \in \overline{\Omega} \times [0,\infty) \times \mathbb{R} \times (-\infty, \frac{\eta}{2}), \\ \xi(r) S(x,t,s,\frac{\eta}{2}) + (1-\xi(r)) S(x,t,s,r), & (x,t,s,r) \in \overline{\Omega} \times [0,\infty) \times \mathbb{R} \times [\frac{\eta}{2}, \infty). \end{cases}$$

Then $S_\eta \in C^{1+\omega}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$ and S and S_η agree for $r \geq \eta$. Let us denote by $(4.5)_\eta$ problem (4.5) with S replaced by S_η. Then we can apply Theorem 4.2.1 to $(4.5)_\eta$ and obtain a solution (u,v) with the required properties (4.4) and (4.6). Nonnegativity of f and integration of the first equation of $(4.5)_\eta$ entail that $\int_\Omega u(\cdot,t) \geq m$ for all $t \in [0,T_{max})$ and accordingly $\int_\Omega g(x,t,u(x,t),v(x,t)) \mathrm{d}x \geq cm > 0$ for all $t \in [0,T_{max})$. Therefore, by Lemma 4.2.2, $v \geq \eta$ and hence (u,v) solves (4.5) as well.
In order to carry over the uniqueness statement from Theorem 4.2.1, we ensure that any solution of (4.5) also solves $(4.5)_\eta$ in $\Omega \times [0,T_{max})$: Let v be a solution of (4.5). Let $\varepsilon \in (0,\frac{\eta}{2})$ and define $t_0 = \inf\{t : \inf_\Omega v(\cdot,t) < \varepsilon\} \in (0,\infty]$. Then (u,v) solves $(4.5)_\eta$ in $\Omega \times (0,t_0)$. Assume $t_0 < \infty$. Then by Lemma 4.2.2 and continuity of v, $v(x,t_0) \geq \eta > \varepsilon = \inf_\Omega v(\cdot,t)$ for all $x \in \Omega$, a contradiction.
ii) Since (u,v) is a solution of $(4.5)_\eta$, we can apply [9, Lemma 3.3], which turns (4.7) into a uniform-in-time bound on $\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)}$, thus asserting global existence by means of (4.6) and boundedness. □

Remark 4.2.4. Throughout the remaining part of the chapter, we will assume that $\Omega \subset \mathbb{R}^2$ is a bounded, smooth, and convex domain, that $0 \leq u_0 \in C^0(\overline{\Omega})$, $q > 2$ and $v_0 \in W^{1,q}(\Omega)$,

$\inf_\Omega v_0 > 0$ and that $\int_\Omega u_0 =: m > 0$.
Then, in particular, Theorem 4.2.3 is applicable to (4.1). We will denote the unique solution of (4.1) by (u, v) and by T_{max} its maximal time of existence as in (4.6). It satisfies

$$\int_\Omega u(\cdot, t) = m \qquad \text{for all } t \in [0, T_{max}). \tag{4.8}$$

Lemma 4.2.5. *Let* $\tau = \min\{1, \frac{T_{max}}{2}\}$*, and assume there exists* $C > 0$ *such that*

$$\int_t^{t+\tau} \int_\Omega \frac{|\nabla u|^2}{u} \le C \quad \text{for any } t \in (0, T_{max} - \tau)$$

and that

$$\int_\Omega u(\cdot, t) \ln u(\cdot, t) \le C \quad \text{for any } t \in (0, T_{max})$$

Then $T_{max} = \infty$ *and* (u, v) *is bounded.*

Proof. Let $c_1, c_2 > 0$ be the constants yielded by the Gagliardo-Nirenberg inequality (Lemma 3.4.3, with parameter names as introduced there) for $j = 0$, $k = 1$, $q = 2$, $r = 2$, $s = 2$, $a = \frac{1}{2}$. Then with m from (4.8),

$$\begin{aligned}\int_t^{t+\tau} \int_\Omega u^2 &= \int_t^{t+\tau} \left\|\sqrt{u}\right\|_{L^4(\Omega)}^4 \le \int_t^{t+\tau} \left(c_1 \left\|\nabla\sqrt{u}\right\|_{L^2(\Omega)}^a \left\|\sqrt{u}\right\|_{L^2(\Omega)}^{1-a} + c_2 \left\|\sqrt{u}\right\|_{L^2(\Omega)}\right)^4 \\ &\le \int_t^{t+\tau} \left(c_1 \left(\int_\Omega \frac{|\nabla u|^2}{4u}\right)^{\frac{1}{4}} m^{\frac{1}{4}} + c_2 m^{\frac{1}{2}}\right)^4 \le \frac{c_1^4 m C}{4} + c_2^4 m^2\end{aligned}$$

holds for any $t \in (0, T_{max} - \tau)$.
Multiplying the second equation of (4.1) by $-\Delta v$ and integrating, from Young's inequality we obtain

$$\int_\Omega |\nabla v(\cdot, t)|^2 \le \int_\Omega |\nabla v_0|^2 - \int_0^t \int_\Omega |\Delta v|^2 - \int_0^t \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_0^t \int_\Omega u^2 + \frac{1}{2} \int_0^t \int_\Omega |\Delta v|^2$$

on $(0, T_{max})$, that is, $y(t) := \int_\Omega |\nabla v(\cdot, t)|^2$ satisfies the differential inequality $y' + y \le f$ on $[0, T_{max})$, where $f = \frac{1}{2} \int_\Omega u^2$ satisfies $\int_t^{t+\tau} |f(s)| \mathrm{d}s \le C$ for all $t \in (0, T_{max} - \tau)$ and with some constant $C > 0$. Letting z be a solution to $z' + z = f$, $z(0) = z_0 = \int_\Omega |\nabla v_0|^2$, we observe that the variation-of-constants formula entails

$$\begin{aligned}z(t) - \mathrm{e}^{-t} z_0 &= \int_0^t \mathrm{e}^{-s} f(t - s) \mathrm{d}s \le \sum_{k=0}^{\lfloor t/\tau \rfloor - 1} \int_{k\tau}^{(k+1)\tau} \mathrm{e}^{-s} |f(t - s)| \mathrm{d}s + \int_{\tau\lfloor t/\tau \rfloor}^t \mathrm{e}^{-s} |f(t - s)| \mathrm{d}s \\ &\le \sum_{k=0}^{\lfloor t/\tau \rfloor - 1} \mathrm{e}^{-k\tau} C + C \le C\left(1 + \frac{1}{1 - \mathrm{e}^{-\tau}}\right) \qquad \text{for } t \in (0, T_{max}),\end{aligned}$$

so that an ODE comparison yields boundedness of $y = \int_\Omega |\nabla v(\cdot, t)|^2$.
Together with the second assumption, the bound on $\int_\Omega u \ln u$, this is sufficient to conclude global existence and boundedness of solutions by Theorem 4.2.3 ii). □

4.3 Some useful general estimates and identities

Lemma 4.3.1. *Let Ω be convex and let $\psi \in C^2(\overline{\Omega})$ satisfy $\partial_\nu \psi\big|_{\partial\Omega} = 0$. Then for all $x \in \partial\Omega$ we have $\partial_\nu |\nabla\psi(x)|^2 \le 0$.*

Proof. This is Lemme 2.I.1 of [61], see also [94, Lemma 3.2]. □

Lemma 4.3.2. *For all positive $\psi \in C^2(\overline{\Omega})$ satisfying $\partial_\nu \psi\big|_{\partial\Omega} = 0$*

$$\int_\Omega \psi |D^2 \ln\psi|^2 = \int_\Omega \frac{1}{\psi}|D^2\psi|^2 + \int_\Omega \frac{1}{\psi^2}|\nabla\psi|^2\Delta\psi - \int_\Omega \frac{|\nabla\psi|^4}{\psi^3}.$$

Proof. This proof is also contained in the proof of [114, Lemma 3.2]. The equality rests on the pointwise identity

$$\begin{aligned}
\psi|D^2\ln\psi|^2 &= \psi\left|D\left(\frac{1}{\psi}\nabla\psi\right)\right|^2 = \psi\left|\nabla\left(\frac{1}{\psi}\right)(\nabla\psi)^T + \frac{1}{\psi}D^2\psi\right|^2\\
&= \psi\left|-\frac{1}{\psi^2}\nabla\psi(\nabla\psi)^T + \frac{1}{\psi}D^2\psi\right|^2\\
&= \psi\frac{|\nabla\psi|^4}{\psi^4} + \psi\frac{1}{\psi^2}|D^2\psi|^2 - \psi\frac{2}{\psi^3}(D^2\psi\nabla\psi)\cdot\nabla\psi\\
&= \frac{|\nabla\psi|^4}{\psi^3} + \frac{1}{\psi}|D^2\psi|^2 - \frac{1}{\psi^2}\nabla|\nabla\psi|^2\cdot\nabla\psi
\end{aligned}$$

and integration by parts in the last term giving

$$-\int_\Omega \frac{1}{\psi^2}\nabla|\nabla\psi|^2\cdot\nabla\psi = \int_\Omega \frac{1}{\psi^2}|\nabla\psi|^2\Delta\psi - 2\int_\Omega \frac{|\nabla\psi|^4}{\psi^3}.$$ □

Lemma 4.3.3. *Let $\psi \in C^2(\overline{\Omega})$ be positive and satisfy $\partial_\nu \psi\big|_{\partial\Omega} = 0$. Then*

$$-\int_\Omega \frac{1}{\psi}|\Delta\psi|^2 = -\int_\Omega \frac{1}{\psi}|D^2\psi|^2 - \frac{3}{2}\int_\Omega \frac{|\nabla\psi|^2\Delta\psi}{\psi^2} + \int_\Omega \frac{|\nabla\psi|^4}{\psi^3} + \frac{1}{2}\int_{\partial\Omega}\frac{1}{\psi}\partial_\nu|\nabla\psi|^2.$$

Proof. This results from [114, Lemma 3.1] upon the choice of $h(\psi) = \frac{1}{\psi}$. The proof can be found in [18, Lemma 2.3]. □

Lemma 4.3.4. *(i) For all positive $\psi \in C^2(\overline{\Omega})$ satisfying $\partial_\nu \psi\big|_{\partial\Omega} = 0$,*

$$-\int_\Omega \frac{1}{\psi}|\Delta\psi|^2 = -\int_\Omega \psi|D^2\ln\psi|^2 - \frac{1}{2}\int_\Omega \frac{1}{\psi^2}|\nabla\psi|^2\Delta\psi + \frac{1}{2}\int_{\partial\Omega}\frac{1}{\psi}\partial_\nu|\nabla\psi|^2.$$

(ii) If furthermore Ω is convex, then

$$-\int_\Omega \frac{1}{\psi}|\Delta\psi|^2 \le -\int_\Omega \psi|D^2\ln\psi|^2 - \frac{1}{2}\int_\Omega \frac{1}{\psi^2}|\nabla\psi|^2\Delta\psi.$$

Proof. This is a direct consequence of the previous three lemmata. □

Lemma 4.3.5. *There is $c_0 > 0$ such that for all positive $\psi \in C^2(\overline{\Omega})$ fulfilling $\partial_\nu \psi\big|_{\partial\Omega} = 0$ the following estimate holds:*

$$\int_\Omega \psi|D^2\ln\psi|^2 \ge c_0\int_\Omega \frac{|\nabla\psi|^4}{\psi^3}.$$

Proof. An even more general version of this lemma and its proof can be found in [114, Lemma 3.3]. □

Remark 4.3.6. As can be seen from the referenced lemma, the constant in the above statement can be chosen to be $\frac{1}{(2+\sqrt{2})^2}$.

4.4 The energy functional. Proof of Theorem 4.1.1

In this section let us investigate the energy functional defined by

$$\mathcal{F}_{a,b}(\widetilde{u},\widetilde{v}) = \int_\Omega \widetilde{u}\ln\widetilde{u} - a\int_\Omega \widetilde{u}\ln\widetilde{v} + b\int_\Omega |\nabla\sqrt{\widetilde{v}}|^2, \qquad 0<\widetilde{u}\in C^0(\overline{\Omega}), \quad 0<\widetilde{v}\in C^1(\overline{\Omega}) \tag{4.9}$$

for nonnegative parameters a, b.
If we want to gain useful information from this functional, the upper bounds on its derivative that we will derive should be accompanied by bounds for $\mathcal{F}_{a,b}$ from below. In order to ensure those, let us first provide the following estimate for solutions of (4.1).

Lemma 4.4.1. *For any $p>0$ there is $C_p>0$ such that*

$$\int_\Omega v^p(\cdot,t) \le C_p \qquad \text{for any } t\in[0,T_{max}).$$

Proof. Since $t\mapsto \|u(\cdot,t)\|_{L^1(\Omega)}$ is constant by (4.8), for $p\ge 1$ this is a consequence of Duhamel's formula for the solution of the second equation of (4.1) and estimates for the Neumann heat semigroup, which can e.g. be found in [110, Lemma 1.3]: They provide $C>0$ such that for all $t\in(0,T_{max})$,

$$\begin{aligned}\|v(\cdot,t)\|_{L^p(\Omega)} &\le \left\|e^{t(\Delta-1)}v_0\right\|_{L^p(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}(u(\cdot,s)-m|\Omega|)\right\|_{L^p(\Omega)} + \left\|e^{-(t-s)}m|\Omega|\right\|_{L^p(\Omega)} ds\\ &\le \|v_0\|_{L^p(\Omega)} + \int_0^t \left(C(1+(t-s)^{-\frac{N}{2}(1-\frac{1}{p})})e^{-(t-s)}\|u(\cdot,s)-m|\Omega|\|_{L^1(\Omega)} + e^{-(t-s)}m|\Omega|^{1+\frac{1}{p}}\right) ds.\end{aligned}$$

The case $p\in(0,1)$ then follows from $v^p\le 1+v$. □

The following lemma gives bounds from below as well as means to turn boundedness of $\mathcal{F}_{a,b}(u,v)$ into boundedness of $\int_\Omega u\ln u$.

Lemma 4.4.2. *Let $a,b\ge 0$. There is $\gamma\in\mathbb{R}$ such that*

$$\mathcal{F}_{a,b}(u,v) \ge \frac{1}{2}\int_\Omega u\ln u - \gamma \qquad \text{on } (0,T_{max}).$$

Proof. With m as in (4.8), we have

$$\mathcal{F}_{a,b}(u,v) \ge \frac{1}{2}\int_\Omega u\ln u + \int_\Omega u\ln\frac{u^{\frac{1}{2}}}{v^a} = \frac{1}{2}\int_\Omega u\ln u + m\int_\Omega\left(-\ln\frac{v^a}{u^{\frac{1}{2}}}\right)\frac{u}{m},$$

similar as in the proof of [10, Thm. 3]. Hence, following an idea from the proof of [74, Lemma 3.3] in applying Jensen's inequality with the probability measure $\frac{u}{m}d\lambda$ and the convex function $-\ln$, we obtain

$$\mathcal{F}_{a,b}(u,v) \ge \frac{1}{2}\int_\Omega u\ln u - m\ln\int_\Omega \frac{v^a u^{\frac{1}{2}}}{m}$$

$$\geq \frac{1}{2}\int_\Omega u\ln u - m\ln\left(\frac{1}{m}\left(\int_\Omega v^{2a}\int_\Omega u\right)^{\frac{1}{2}}\right)$$
$$\geq \frac{1}{2}\int_\Omega u\ln u + \frac{m}{2}\ln m - \frac{m}{2}\ln C_{2a}$$

after applying Hölder's inequality and with C_{2a} as in Lemma 4.4.1. □

Lemma 4.4.3. *Let $a, b \geq 0$. Then*
i) $\mathcal{F}_{a,b}(u,v)$ is bounded below.
ii) If $\sup_{t\in[0,T_{max})}\mathcal{F}_{a,b}(u(\cdot,t),v(\cdot,t)) < \infty$ then $\sup_{t\in[0,T_{max})}\int_\Omega u(\cdot,t)\ln u(\cdot,t) < \infty$.

Proof. Both statements are immediate consequences of Lemma 4.4.2. □

Lemma 4.4.1 as well enables us to control the first two summands of $\mathcal{F}_{a,b}(u,v)$ from above by $\int_\Omega \frac{u^2}{v}$.

Lemma 4.4.4. *Let $a > 0$. Then for any $\delta > 0$ there is $c_\delta > 0$ such that*

$$\int_\Omega u\ln u - a\int_\Omega u\ln v \leq \delta\int_\Omega \frac{u^2}{v} + c_\delta \qquad \text{on } [0,T_{max}).$$

Proof. Given $a > 0$ let $\varepsilon \in (0,1)$ be so small that $\frac{1+\varepsilon-2a\varepsilon}{1-\varepsilon} > 0$. There is $C_\varepsilon > 0$ such that for any $x > 0$ we have $\ln x \leq C_\varepsilon x^\varepsilon$. Therefore for any $\delta > 0$ Young's inequality and Lemma 4.4.1 provide $C_\delta > 0$ and $c_\delta > 0$ satisfying

$$\int_\Omega u\ln u - a\int_\Omega u\ln v = \int_\Omega u\ln\frac{u}{v^a} \leq C_\varepsilon\int_\Omega \frac{u^{1+\varepsilon}}{v^{a\varepsilon}} \leq \delta\int_\Omega (u^{1+\varepsilon}v^{-\frac{1+\varepsilon}{2}})^{\frac{2}{1+\varepsilon}} + C_\delta\int_\Omega (v^{\frac{1+\varepsilon-2a\varepsilon}{2}})^{\frac{2}{1-\varepsilon}}$$
$$\leq \delta\int_\Omega \frac{u^2}{v} + c_\delta \qquad \text{on} [0,T_{max}).$$ □

With these preparations, we turn to the time derivative of $\mathcal{F}_{a,b}(u,v)$, beginning with the already investigated first part:

Lemma 4.4.5. *For any $a \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,0}(u,v) = -\int_\Omega \frac{|\nabla u|^2}{u} + (\chi+2a)\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - a(\chi+1)\int_\Omega \frac{u|\nabla v|^2}{v^2} + a\int_\Omega u - a\int_\Omega \frac{u^2}{v}$$

holds on $(0,T_{max})$.

Proof. Using the first equation of (4.1) in $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u\ln u - a\int_\Omega u\ln v\right)$ and integrating by parts we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u\ln u - a\int_\Omega u\ln v\right) &= \int_\Omega u_t\ln u + \int_\Omega u_t - a\int_\Omega u_t\ln v - a\int_\Omega \frac{u}{v}v_t \\ &= -\int_\Omega \frac{\nabla u}{u}\cdot\left(\nabla u - \chi\frac{u}{v}\nabla v\right) + a\int_\Omega \frac{\nabla v}{v}\cdot\left(\nabla u - \chi\frac{u}{v}\nabla v\right) \\ &\quad - a\int_\Omega \frac{u}{v}(\Delta v - v + u) \\ &= -\int_\Omega \frac{|\nabla u|^2}{u} + \chi\int_\Omega \frac{\nabla u\cdot\nabla v}{v} + a\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - a\chi\int_\Omega \frac{u|\nabla v|^2}{v^2} \\ &\quad + a\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - a\int_\Omega \frac{u|\nabla v|^2}{v^2} + a\int_\Omega u - a\int_\Omega \frac{u^2}{v}.\end{aligned}$$ □

Since we do not know the sign of $\int_\Omega \frac{\nabla u \cdot \nabla v}{v}$ and, in this situation, cannot control $\int_\Omega \frac{u|\nabla v|^2}{v^2}$, we are left with Young's inequality, hoping that the resulting coefficient $\frac{(\chi+2a)^2}{4} - a(\chi+1)$ of $\int_\Omega \frac{u|\nabla v|^2}{v^2}$ turns out to be negative. This can be achieved if $\chi < 1$.
However, it becomes possible to cope with larger parameters if $\int_\Omega \frac{u|\nabla v|^2}{v^2}$ can be controlled, e.g. by having control over $\int_\Omega \frac{|\nabla v|^4}{v^3}$ and $\int_\Omega \frac{u^2}{v}$. The second term already being in place, fortunately, the first is one of the terms arising from the following:

Lemma 4.4.6. *The estimate*

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) \leq -2c_0\int_\Omega \frac{|\nabla v|^4}{v^3} - \int_\Omega \frac{|\nabla v|^2}{v} + 2\int_\Omega \frac{\nabla u \cdot \nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2} \tag{4.10}$$

holds on $(0, T_{max})$*, where* c_0 *is the constant provided by Lemma 4.3.5.*

Proof. From the second equation of (4.1), we obtain

$$\begin{aligned}4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla v|^2}{v} = \int_\Omega \frac{2\nabla v\cdot\nabla v_t}{v} - \int_\Omega \frac{|\nabla v|^2 v_t}{v^2}\\ &= \int_\Omega \frac{2\nabla v\cdot\nabla\Delta v}{v} - \int_\Omega \frac{2|\nabla v|^2}{v} + \int_\Omega \frac{2\nabla v\cdot\nabla u}{v} - \int_\Omega \frac{|\nabla v|^2\Delta v}{v^2}\\ &\quad + \int_\Omega \frac{|\nabla v|^2 v}{v^2} - \int_\Omega \frac{|\nabla v|^2 u}{v^2} \quad \text{on } (0, T_{max}).\end{aligned}$$

Integration by parts in the first integral and merging the second and second to last summand lead us to

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) = -2\int_\Omega \frac{|\Delta v|^2}{v} + \int_\Omega \frac{|\nabla v|^2\Delta v}{v^2} - \int_\Omega \frac{|\nabla v|^2}{v} + 2\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2} \tag{4.11}$$

on $(0, T_{max})$. By Lemma 4.3.4 and due to the convexity of Ω we can transform the first summand according to

$$-2\int_\Omega \frac{|\Delta v|^2}{v} \leq -2\int_\Omega v|D^2\ln v|^2 - \int_\Omega \frac{1}{v^2}|\nabla v|^2\Delta v,$$

making the second term in the right hand side of (4.11) vanish:

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) \leq -2\int_\Omega v|D^2\ln v|^2 - \int_\Omega \frac{|\nabla v|^2}{v} + 2\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2}. \tag{4.12}$$

We are left with a term we can estimate with the help of Lemma 4.3.5:

$$-2\int_\Omega v|D^2\ln v|^2 \leq -2c_0\int_\Omega \frac{|\nabla v|^4}{v^3},$$

thereby gaining the term which will make the crucial difference in the estimates to come and arriving at

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) \leq -2c_0\int_\Omega \frac{|\nabla v|^4}{v^3} - \int_\Omega \frac{|\nabla v|^2}{v} + 2\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2}. \qquad \square$$

If we combine the previous two lemmata, we are led to:

Lemma 4.4.7. *Let $a, b \geq 0$, $\delta \in (0,1)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v) \leq \left(\frac{1}{4a(1-\delta)}\left(\frac{(\chi+2a+\frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4}\right)_+^2 - \frac{bc_0}{2}\right)\int_\Omega \frac{|\nabla v|^4}{v^3}$$
$$- \delta\int_\Omega \frac{|\nabla u|^2}{u} - a\delta\int_\Omega \frac{u^2}{v} + a\int_\Omega u - \frac{b}{4}\int_\Omega \frac{|\nabla v|^2}{v} \qquad \text{on } (0, T_{max}). \tag{4.13}$$

Proof. An estimate for $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u(\cdot,t), v(\cdot,t))$ is given by the sum of the terms from Lemma 4.4.5 and Lemma 4.4.6:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v) \leq -\int_\Omega \frac{|\nabla u|^2}{u} + \left(\chi + 2a + \frac{b}{2}\right)\int_\Omega \frac{\nabla u \cdot \nabla v}{v} - \left(a\chi + a + \frac{b}{4}\right)\int_\Omega \frac{u|\nabla v|^2}{v^2}$$
$$+ a\int_\Omega u - a\int_\Omega \frac{u^2}{v} - \frac{b}{2}c_0\int_\Omega \frac{|\nabla v|^4}{v^3} - \frac{b}{4}\int_\Omega \frac{|\nabla v|^2}{v}.$$

In order to finally still have some control over $\int \frac{|\nabla u|^2}{u}$, as required for Lemma 4.2.5, we retain a small portion of this term when applying Young's inequality:

$$-\int_\Omega \frac{|\nabla u|^2}{u} + \left(\chi + 2a + \frac{b}{2}\right)\int_\Omega \frac{\nabla u \cdot \nabla v}{v} \leq \big(-1 + (1-\delta)\big)\int_\Omega \frac{|\nabla u|^2}{u} + \frac{(\chi+2a+\frac{b}{2})^2}{4(1-\delta)}\int_\Omega \frac{u|\nabla v|^2}{v^2},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v) \leq -\delta\int_\Omega \frac{|\nabla u|^2}{u} + \left(\frac{(\chi+2a+\frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4}\right)\int_\Omega \frac{u|\nabla v|^2}{v^2}$$
$$+ a\int_\Omega u - a\int_\Omega \frac{u^2}{v} - \frac{b}{2}c_0\int_\Omega \frac{|\nabla v|^4}{v^3} - \frac{b}{4}\int_\Omega \frac{|\nabla v|^2}{v} \qquad \text{on } (0, T_{max}).$$

By virtue of the presence of $-\int_\Omega \frac{|\nabla v|^4}{v^3}$, which originates from the additional summand of the energy functional and the preparations of Section 4.3, we can continue estimating $\int_\Omega \frac{u|\nabla v|^2}{v^2}$ by $\int_\Omega \frac{u^2}{v}$ and $\int_\Omega \frac{|\nabla v|^4}{v^3}$ and still hope for negative coefficients in front of the integrals, in contrast to the situation of Lemma 4.4.5. In doing so we keep some part of $\int_\Omega \frac{u^2}{v}$ for the sake of a later application of Lemma 4.4.4 and arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) \leq -\delta\int_\Omega \frac{|\nabla u|^2}{u} + a\int_\Omega u - a\int_\Omega \frac{u^2}{v} - \frac{b}{2}c_0\int_\Omega \frac{|\nabla v|^4}{v^3} - \frac{b}{4}\int_\Omega \frac{|\nabla v|^2}{v}$$
$$+ a(1-\delta)\int_\Omega \frac{u^2}{v} + \frac{1}{4a(1-\delta)}\left(\frac{(\chi+2a+\frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4}\right)_+^2\int_\Omega \frac{|\nabla v|^4}{v^3},$$

which amounts to (4.13). □

Lemma 4.4.8. *Let $a > 0$, $b \geq 0$, $\chi > 0$ be such that*

$$\varphi(a,b;\chi) := \left(\frac{1}{4a}\left(\frac{(\chi+2a+\frac{b}{2})^2}{4} - a\chi - a - \frac{b}{4}\right)_+^2 - \frac{bc_0}{2}\right) < 0. \tag{4.14}$$

Then there are $\kappa > 0$, $\delta > 0$ and $c > 0$ such that for any $t \in (0, T_{max})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) + \kappa\mathcal{F}_{a,b}(u,v)(t) + \delta\int_\Omega \frac{|\nabla u(\cdot,t)|^2}{u(\cdot,t)} \leq c.$$

Proof. By continuity of

$$\delta \mapsto \varphi_\delta(a,b;\chi) := \left(\frac{1}{4a(1-\delta)} \left(\frac{(\chi + 2a + \frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4} \right)_+^2 - \frac{bc_0}{2} \right)$$

in $\delta = 0$, for fixed a, b, χ, negativity of $\varphi(a,b;\chi)$ entails the existence of $\delta > 0$ so that $\varphi_\delta(a,b;\chi)$ is negative as well. Therefore, by Lemma 4.4.7,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v) + \delta \int_\Omega \frac{|\nabla u|^2}{u} + a\delta \int_\Omega \frac{u^2}{v} + b \int_\Omega |\nabla \sqrt{v}|^2 \le a \int_\Omega u \qquad \text{on } (0, T_{max}).$$

Since $\int_\Omega u$ is constant in time by (4.8), Lemma 4.4.4 implies the assertion. □

Lemma 4.4.9. *If*

$$\chi_0 \in \{\chi > 0; \textit{there are } a > 0 \textit{ and } b \ge 0 \textit{ such that } \varphi(a,b;\chi) < 0\} =: X,$$

then

$$(0, \chi_0) \subset X.$$

Proof. Since, for any fixed $a > 0$, $b \ge 0$,

$$\chi \mapsto \varphi(a,b;\chi) = \frac{1}{64a}\left(\left(\chi^2 + 4a^2 + \frac{b^2}{4} + b\chi + 2ab - 4a - b \right)_+^2 - 32abc_0 \right)$$

is monotone, for any $a > 0$, $b \ge 0$

$$\varphi(a,b;\chi_0) < 0 \text{ implies } \varphi(a,b;\chi) < 0 \text{ for any } 0 < \chi < \chi_0.$$ □

Lemma 4.4.10. *There is $\chi_0 > 1$ such that $\varphi(a,b;\chi_0) < 0$ for some $a > 0, b > 0$.*

Proof. Since $\varphi(\frac{1}{2}, 0, 1) = 0$ and

$$\frac{d}{db}\varphi\left(\frac{1}{2}, b, 1\right)\Bigg|_{b=0} = \frac{d}{db}\left(\frac{1}{32}\left(\frac{b^2}{4} + b\right)^2 - \frac{1}{2}c_0 b \right)\Bigg|_{b=0} = -\frac{c_0}{2} < 0,$$

there is $b > 0$ such that $\varphi(\frac{1}{2}, b, 1) < 0$ and by continuity of φ with respect to χ, the assertion follows. □

Proof of Theorem 4.1.1. By Lemma 4.4.10, there are $a, b > 0$, $\chi_0 > 1$ such that $\varphi(a,b,\chi_0) < 0$ and hence, by Lemma 4.4.9, also $\varphi(a,b,\chi) < 0$ for $\chi \in (0,\chi_0)$. An application of Lemma 4.4.8 thus reveals that for all $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) + \kappa \mathcal{F}_{a,b}(u,v)(t) + \delta \int_\Omega \frac{|\nabla u|^2}{u} \le c \tag{4.15}$$

for some $\kappa, \delta, c > 0$. Together with the boundedness of $\mathcal{F}_{a,b}(u,v)$ from below by Lemma 4.4.3 i) this ensures that $\mathcal{F}_{a,b}(u,v)$ is bounded so that an integration of (4.15) also shows the boundedness of $\int_t^{t+1}\int_\Omega \frac{|\nabla u|^2}{u}$.

Since $\mathcal{F}_{a,b}(u,v)$ is bounded, by Lemma 4.4.3 ii) the same holds true for $\int_\Omega u \ln u$ and so the conditions of Lemma 4.2.5 are met and Theorem 4.1.1 follows. □

Remark 4.4.11. Assuming $c_0 = \frac{1}{(2+\sqrt{2})^2}$, as permitted by Remark 4.3.6,

$$-1.1 \cdot 10^{-5} \approx \varphi(0.49, 0.001; 1.015) < 0,$$

i.e. $\chi_0 > 1.015$.

5 Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion

5.1 Introduction

Even simple, small organisms can exhibit comparatively complex and macroscopically apparent collective behaviour. Bacteria of the species *E. coli*, for example, when set in a capillary tube featuring a gradient of nutrient concentration form bands that are visible to the naked eye and migrate with constant speed. Following experimental works of Adler (see e.g. [1, 2]), in 1971 Keller and Segel ([44]) introduced a phenomenological model to capture this kind of behaviour, a prototypical version of which is given by

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\frac{u}{v}\nabla v) & \text{in } \Omega \times (0,\infty) \\ v_t = \Delta v - uv & \text{in } \Omega \times (0,\infty) \end{cases} \tag{5.1}$$

with $D(u) \equiv 1$. Herein, u represents the density of bacteria and v is used to denote the concentration of the nutrient. In the model in [44], the diffusion coefficient $D(u)$ is supposed to be constant, thus leading to the typical effect of linear diffusion which causes any population to spread with infinite speed of propagation. In order to avoid this (biologically clearly unrealistic) behaviour, it might be desirable to allow for diffusion of porous medium type (i.e. $D(u) = u^{m-1}$), cf. also [9, p. 1665].
Nevertheless, starting with [44], the model with linear diffusion has successfully been employed to find travelling wave solutions (see e.g. the overview in [105] and references cited therein) and also their stability has been investigated ([55],[71]).
In spite of the rich literature concerned with travelling wave solutions (for such solutions to related systems see also [65],[66], [56], or [36], [67]), little is known about existence of solutions for more general initial data (see below).
The difficulty lies in the hazardous combination of the consumptive effect of the second equation on the nutrient concentration with the singular chemotactic sensitivity in the first: While the second equation compels v to shrink, it is the cross-diffusive contribution of the chemotaxis term that seeks to enlarge the solutions to (5.1). And it is this very term that is furnished with a large coefficient whenever v becomes small.
For a moment leaving aside the logarithmic shape of the sensitivity in $\nabla \cdot (\frac{u}{v}\nabla v) = \nabla \cdot (u\nabla \ln v)$, we are led to the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{5.2}$$

which also appears as part of chemotaxis fluid systems intensively studied during the past six years. (The interested reader can consult the introduction to Chapter 6.) Even in (5.2), global

existence of classical solutions is not yet known, apart from 2-dimensional settings ([114]) or under smallness conditions on v_0 ([93]).
Although the mathematical difficulty in treating the system vastly increases when a logarithmic sensitivity is included, this form is important. Not only is it needed for the emergence of travelling waves ([44, 42, 85]), there are also models giving a detailled mechanistic basis ([123]) and experimental evidence asserting this form ([41]).
In those Keller-Segel models (cf. [34, 32, 9]) where v does not stand for a nutrient to be consumed but a signalling substance produced by the bacteria themselves, i.e. the evolution is governed by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

the singularity in the sensitivity function is mitigated by v tending to stay away from 0 thanks to the production term in the second equation. (For this system, global solutions are known to exist if χ is sufficiently small, where the precise condition depends on the dimension as well as on whether classical (Chapter 4, [113, 10]) or weak solutions ([91, 113]) are considered and on radial symmetry of initial data ([10, 72]); but for large χ also blow-up may occur in the corresponding parabolic-elliptic system ([72]).) The proof of boundedness of solutions for $\chi < \sqrt{\frac{2}{N}}$ in [24] even relies on the second equation actually ensuring a positive pointwise lower bound for v.
In (5.1), we cannot hope for such a convenient bound and thus have to deal with the influence of the actual singularity in the sensitivity function.
Nevertheless, for $D \equiv 1$, in the domains $\mathbb{R}^2$ and $\mathbb{R}^3$ a global existence result was achieved for initial data that are $H^1 \times H^1$-close to $(\overline{u}, 0)$ for some $\overline{u} > 0$ ([106]). The proof rests on energy estimates for a hyperbolic system into which (5.1) can be converted by means of the Hopf-Cole type transformation $q := \frac{\nabla v}{v}$ that had been introduced in [54] for the treatment of an angiogenesis model.
More recently it has become possible to treat general initial data (the only restrictions being positivity and regularity assumptions) for the system in bounded planar domains ([121]), where it was shown that global generalized solutions to (5.1) with $D \equiv 1$ exist whose second component v moreover converges to 0 with respect to the norm in any $L^p(\Omega)$ for $p \in [1, \infty)$ and to the weak-$*$–topology of $L^\infty(\Omega)$. If, moreover, the initial mass of bacteria is small, the solution becomes eventually smooth ([122]) and converges to the homogeneous steady state. In [122] also an explicit smallness condition on u_0 in $L \ln L(\Omega)$ and $\nabla \ln v_0$ in $L^2(\Omega)$ has been found that ensures the global existence of classical solutions.
Solutions emanating from large data, however, have not been proven to be bounded and might blow up and cease to exist as classical solutions after a finite time, continuing only as generalized solutions in the sense of [121]. In higher-dimensional domains, even the existence of such solutions is unknown. Only in a radially symmetric setting "renormalized solutions" have been constructed ([120]).
In the present chapter, we aim to find solutions to (5.1) that are locally bounded and hence do not blow up in finite time. For this, we will rely on stronger growth of D, i.e. on the nonlinear diffusion we want to include. More precisely, we assume that with some $m \geq 1$, which will be subject to further conditions, and $\delta > 0$

$$D \in \mathcal{C}_{\delta,m} := \left\{ d \in C^1([0,\infty)); d(s) \geq \delta s^{m-1} \text{ for all } s \in [0,\infty) \right\}.$$

In a first step we will additionally require strict positivity of D, i.e.

$$D \in \mathcal{C}^+_{\delta,m} := \left\{ d \in C^1([0,\infty)); d(s) \geq \delta s^{m-1} \text{ for all } s \in [0,\infty) \text{ and } d(0) > 0 \right\}$$

and prove global existence of classical solutions to (5.1):

Theorem 5.1.1. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Then for every $\delta > 0$ and $m \geq 1$ satisfying*

$$m > 1 + \frac{N}{4}, \tag{5.3}$$

every $D \in \mathcal{C}^+_{\delta,m}$ and every pair (u_0, v_0) of initial data fulfilling

$$u_0 \in C^\gamma(\overline{\Omega}) \text{ for some } \gamma \in (0,1), \qquad v_0 \in W^{1,\infty}(\Omega), \qquad u_0 \geq 0, \quad v_0 > 0 \quad \text{in } \overline{\Omega} \tag{5.4}$$

the initial boundary value problem

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot \left(\frac{u}{v}\nabla v\right) & \text{in } \Omega \times (0, T_{max}) & (5.5a) \\ v_t = \Delta v - uv & \text{in } \Omega \times (0, T_{max}) & (5.5b) \\ \partial_\nu u = 0 & \text{in } \partial\Omega \times (0, T_{max}) & (5.5c) \\ \partial_\nu v = 0 & \text{in } \partial\Omega \times (0, T_{max}) & (5.5d) \\ u(\cdot, 0) = u_0 & \text{in } \Omega & (5.5e) \\ v(\cdot, 0) = v_0 & \text{in } \Omega & (5.5f) \end{cases}$$

has a classical solution $(u, v) \in (C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})))^2$ which is global (i.e. $T_{max} = \infty$).

Afterwards dropping the strict positivity asumption on D, we will use an approximation procedure and finally prove the existence of global weak solutions that are locally bounded:

Theorem 5.1.2. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Then for every $\delta > 0$ and $m > 1 + \frac{N}{4}$, every initial data*

$$u_0 \in L^{\max\{1,m-1\}}(\Omega), \qquad v_0 \in W^{1,\infty}(\Omega), \qquad u_0 \geq 0, \qquad v_0 > 0 \tag{5.6}$$

and every $D \in \mathcal{C}_{\delta,m}$, (5.5) *has a global locally bounded weak solution (u, v) (in the sense made precise in Definition 5.4.1), which in particular satisfies*

$$\|u\|_{L^\infty(\Omega\times(0,T))} < \infty \qquad \text{for every } T \in (0, \infty).$$

We will devote Section 5.2 to the proof of local existence of solutions and an extensibility criterion. In the proof of boundedness that follows, we will sometimes use the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\nabla w) \\ w_t = \Delta w - |\nabla w|^2 + u \end{cases} \tag{5.7}$$

obtained from the transformation $w = -\ln(\frac{v}{\|v_0\|_{L^\infty(\Omega)}})$, which has also been used in [121, 122]. We note that while the first equation seems more accessible in (5.7) due to the lack of any singularity, it is (5.5), where the second equation is more amenable to the derivation of estimates on ∇v.

The first stepping stone for the proof will be a spatio-temporal L^2-bound for ∇w (Lemma 5.3.2), already giving some boundedness information for $\int_0^t \int_\Omega |\nabla u^{m-1}|$ and $\int_\Omega u^{m-1}(\cdot, t)$ for $t > 0$, which we can use to obtain bounds on $\int_0^t \int_\Omega |\nabla u^{m-1}|$ (Lemma 5.3.3) and thereby on $\int_0^t \|u\|_p^r$ for certain p, r and any $t > 0$ (Lemma 5.3.5). One consequence of such bounds is a spatio-temporal L^q-bound on ∇v (see Lemma 5.3.7), derived with the help of maximal Sobolev regularity properties

of the heat equation (cf. Lemma 5.3.6). Another is the (local-in-time) boundedness of w (Lemma 5.3.9). This is important, as it will enable us to transfer bounds from ∇v to ∇w (Lemma 5.3.10). Bounds on $\int_0^t \int_\Omega |\nabla w|^q$ now in turn will translate into control over $\int_\Omega u^p$ for some p (Lemma 5.3.12). If p is sufficiently large, this entails $L^\infty(\Omega \times (0,T))$–boundedness of $|\nabla v|$ and $|\nabla w|$ and thus finally of u (Lemma 5.3.11 and Lemma 5.2.1 v)). Thereby, the solution is not only locally bounded, but moreover exists globally, according to the extensibility criterion (5.15).
In Section 5.4 we rely on bounds already derived in the previous section to construct locally bounded weak solutions to (5.1) with functions D causing possibly degenerate diffusion.
Notation. Throughout the chapter we fix $N \in \mathbb{N}$, $N \geq 2$, and $\Omega \subset \mathbb{R}^N$ as a bounded, smooth domain. When dealing with the solution to a differential equation, we will use T_{max} to denote its maximal time of existence; in the case of (5.5) such T_{max} is provided by Lemma 5.2.4. By $\hookrightarrow$ and $\overset{cpt}{\hookrightarrow}$ we refer to continuous and compact embeddings of Banach spaces, respectively. We will sometimes write $D(u)$ for the concatenation $D \circ u$ of functions. The number $\lambda_1 > 0$ will always be the first positive eigenvalue of the Neumann Laplacian in Ω.

5.2 Local existence

We begin the proof by ensuring local existence of classical solutions in the non-degenerate case. As a first step let us, for easier reference, collect some basic results on existence of and estimates for solutions of certain parabolic PDEs.

Lemma 5.2.1. *i) For any $T > 0$, $q > N$ and $r > N$ and every $M > 0$ there are $C_i > 0$ and $\gamma > 0$ such that for all nonnegative functions $v_0 \in W^{1,q}(\Omega)$ and $u \in L^\infty((0,T); L^r(\Omega))$ satisfying $\|v_0\|_{W^{1,q}(\Omega)} \leq M$ and $\|u\|_{L^\infty((0,T);L^r(\Omega))} \leq M$ for the solution $v \in V_2 = \{v \in L^\infty((0,T); L^2(\Omega)); \nabla v \in L^2(\Omega \times (0,T); \mathbb{R}^N)\}$ of*

$$v_t = \Delta v - uv, \quad \partial_\nu v\big|_{\partial\Omega} = 0, \quad v(\cdot,0) = v_0 \tag{5.8}$$

one has $\|v\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])} < C_i$. If, moreover, $u \in C^{\beta,\frac{\beta}{2}}(\overline{\Omega} \times (0,T])$ for some $\beta \in (0,\gamma)$, then $v \in C^0(\overline{\Omega} \times [0,T]) \cap C^{2+\beta,1+\frac{\beta}{2}}(\overline{\Omega} \times (0,T])$. If $u \in L^\infty(\Omega \times (0,T))$, then $v \in C^1(\overline{\Omega} \times (0,T])$.

ii) For any $r \in (N,\infty]$ there is $C_{ii} = C_{ii}(r) > 0$ such that for any $T > 0$ and any $q \in [2,\infty]$ for all nonnegative functions $v_0 \in W^{1,q}(\Omega)$ and $u \in C^{\beta,\frac{\beta}{2}}(\overline{\Omega} \times (0,T))$ for some $\beta \in (0,1)$, the solution $v \in C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times [0,T])$ (for some $\gamma \in (0,\beta)$) of (5.8) satisfies

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \leq C_{ii} \|\nabla v_0\|_{L^q(\Omega)} + C_{ii} \|v_0\|_{L^\infty(\Omega)} \|u\|_{L^\infty((0,T);L^r(\Omega))}.$$

iii) For every $T > 0$, $\delta_0 > 0$, $M > 0$ and $K > 0$ there is $C_{iii} > 0$ such that for every $u_0 \in L^\infty(\Omega)$ satisfying $0 \leq u_0 \leq M$ and every $g \in C^0(\overline{\Omega} \times (0,T); \mathbb{R}^N)$ fulfilling $g \cdot \nu = 0$ on $\partial\Omega$ and $\|g\|_{L^\infty(\Omega)} \leq K$, and for all $A \in L^\infty(\Omega \times (0,T))$ with $A > \delta_0$ in $\Omega \times (0,T)$, the unique weak solution of

$$u_t = \nabla \cdot (A\nabla u - g) \text{ in } \Omega \times (0,T), \quad \partial_\nu u\big|_{\partial\Omega} = 0 \text{ in } (0,T), \quad u(\cdot,0) = u_0 \text{ in } \Omega, \tag{5.9}$$

satisfies

$$\|u\|_{L^\infty(\Omega\times(0,T))} \leq C_{iii} \quad \text{and} \quad \|\nabla u\|_{L^2(\Omega\times(0,T))} \leq C_{iii}. \tag{5.10}$$

iv) For any $T > 0$, for any $D_0 > \delta_0 > 0$, $M > 0$, $K > 0$ and $\beta \in (0,1)$ there are $C_{iv} > 0$ and $\gamma \in (0,1)$ such that for every $A \in L^\infty(\Omega\times(0,T))$ fulfilling $\delta_0 < A < D_0$ a.e. in $\Omega\times(0,T)$, and for *all* $g \in \left(C^0(\overline{\Omega} \times (0,T); \mathbb{R}^N)\right)$ *with* $g \cdot \nu = 0$ *on* $\partial\Omega$ *and* $\|g\|_{L^\infty(\Omega\times(0,T))} \le M$ *and all* $u_0 \in C^\beta(\overline{\Omega})$ *with* $\|u_0\|_{C^\beta(\overline{\Omega})} \le M$*, any solution* u *of* (5.9) *that obeys the estimate* $\|u\|_{L^\infty(\Omega\times(0,T))} \le K$ *satisfies*

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])} \le C_{iv}. \tag{5.11}$$

Moreover, if $g \in C^{\beta',\frac{\beta'}{2}}(\overline{\Omega} \times (0,T])$ *for some* $\beta' > 0$*, then* $u \in C^{2,1}(\overline{\Omega} \times (0,T])$.

v) For every $m \ge 1$, $\delta > 0$, $K > 0$, $p_0 \ge 1$, $q_1 > N+2$ *and* $T \in (0,\infty]$ *there is* $C_v > 0$ *such that for every* $D \in C^1(\overline{\Omega} \times [0,T) \times [0,\infty))$ *which obeys* $D \ge 0$, $D(x,t,s) \ge \delta s^{m-1}$ *for all* $(x,t,s) \in \Omega \times (0,T) \times (0,\infty)$ *and every* $f \in C^0((0,T); C^0(\overline{\Omega}) \cap C^1(\Omega))$, $f \cdot \nu \le 0$ *on* $\partial\Omega \times (0,T)$ *satisfying* $\|f\|_{L^\infty((0,T);L^{q_1}(\Omega))} \le K$*, for every nonnegative function* $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *that satisfies* $\|u\|_{L^\infty((0,T);L^{p_0}(\Omega))} \le K$ *and* $u_t \le \nabla\cdot(D(x,t,u)\nabla u) + \nabla \cdot f(x,t)$ *in* $\Omega \times (0,T)$ *and* $\partial_\nu u\big|_{\partial\Omega} \le 0$ *on* $(0,T)$*, we have* $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_v$ *for every* $t \in (0,T)$.

Proof. **i)** According to [48, III.5.1], (5.8) has a unique weak solution $v \in V_2$ in $\Omega \times (0,T)$. The first part of the statement thus immediately results from [83, Thm. 1.3 and Remarks 1.3, 1.4], whereas the second is a consequence of a uniqueness statement ([48, III.5.1]) combined with the existence assertion for classical solutions in [48, IV.5.3] (applied to $\xi_\varepsilon(t)v(x,t)$ for some cutoff function $\xi_\varepsilon \in C_0^\infty([0,\infty))$, $\xi_\varepsilon\big|_{(0,\frac{\varepsilon}{2})} \equiv 0$, $\xi_\varepsilon\big|_{(\varepsilon,\infty)} \equiv 1$ for arbitrary $\varepsilon > 0$). The third part – actually, even Hölder-continuity of ∇v – is provided by [58, Thm. 1.1].
ii) Existence of a solution ensured as in the proof of i), we may rely on [81, Cor. 4.3.3] to represent v as mild solution via the variation of constants formula, and invoking [110, Lemma 1.3 iii)] and [110, Lemma 1.3 ii)], we gain $c_1 > 0$ and $c_2 > 0$, respectively, such that with $\rho := \max\{q, r\}$ and by Hölder's inequality

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^q(\Omega)} &\le c_1 \|\nabla v_0\|_{L^q(\Omega)} + \int_0^t c_2 |\Omega|^{\frac{1}{q}-\frac{1}{\rho}} \left(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{r}-\frac{1}{\rho})}\right) \|uv(\cdot,s)\|_{L^r(\Omega)} \mathrm{e}^{-\lambda_1(t-s)} \mathrm{d}s \\ &\le c_1 \|\nabla v_0\|_{L^q(\Omega)} + c_3 \|v_0\|_{L^\infty(\Omega)} \|u\|_{L^\infty((0,T);L^r(\Omega))},\end{aligned}$$

where we have used that $\sigma^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{r}-\frac{1}{\rho})} \le 1 + \sigma^{-\frac{1}{2}-\frac{N}{2r}}$ for all $\sigma > 0$ and all $\rho \in [1,\infty]$ and set $c_3 := \int_0^\infty c_2 \left(2 + \sigma^{-\frac{1}{2}-\frac{N}{2r}}\right) \mathrm{e}^{-\lambda_1\sigma} \mathrm{d}\sigma$, which is finite because of $r > N$, and where we have taken into account that by comparison arguments $0 \le v(\cdot,t) \le \|v_0\|_{L^\infty(\Omega)}$ in Ω for all $t \in (0,T)$.
iii) This is a combination of [59, Thm. 6.38] (estimate for ∇u), [59, Thm. 6.39] (existence and uniqueness) and [59, Thm. 6.40] (uniform boundedness).
iv) The same theorems as in the proof of i) apply.
v) This is (part of) Lemma A.1 in [94]. □

Lemma 5.2.2. *For every positive function* $D \in C^0([0,\infty))$*, for every* $L > 0$ *and* $\gamma \in (0,1)$ *there is* $T > 0$ *such that for every* $u_0 \in C^\gamma(\overline{\Omega})$ *satisfying* $\|u_0\|_{C^\gamma(\overline{\Omega})} \le L$ *and every* $v_0 \in W^{1,\infty}(\Omega)$ *which satisfies* $v_0 > \frac{1}{L}$ *in* $\overline{\Omega}$ *and fulfils* $\|v_0\|_{W^{1,\infty}(\Omega)} \le L$ *there is a pair of functions* $(u,v) \in C^0(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T])$ *solving* (5.5) *in* $\Omega \times (0,T)$.

Proof. We let $R := L + 1 \ge \|u_0\|_{L^\infty(\Omega)} + 1$. For the choice of $r := \infty$ we obtain $c_1 := C_{ii} > 0$ with properties as described in Lemma 5.2.1 ii), and thereupon invoking Lemma 5.2.1 iii) for parameters $T = 1$, $\delta_0 = \inf_{s>0} D(s)$, $M = R$ and $K = c_1 R(1+R)L^2\mathrm{e}^R$, we are given $c_2 := C_{iii} > 0$ as in (5.10). An application of Lemma 5.2.1 iv) for $T = 1$, $\delta_0 = \inf_{s>0} D(s)$, $D_0 = \sup_{0<s<R} D(s)$, $M = \max\{c_1 R(1+R)L^2\mathrm{e}^R, L\}$ and $K = c_2$ provides us with $c_3 := C_{iv} > 0$

and $\gamma \in (0,1)$ as in (5.11). With these, we choose $T \in (0,1)$ such that $\|u_0\|_{L^\infty(\Omega)} + c_3 T^{\frac{\gamma}{2}} < R$ and introduce

$$S := \left\{ \widehat{u} \in C^0(\overline{\Omega} \times [0,T]); 0 \le \widehat{u} \le R, u(\cdot,0) = u_0, \|\widehat{u}\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])} \le c_3 \right\} \subset C^0(\overline{\Omega} \times [0,T]). \tag{5.12}$$

For any $\widehat{u} \in S$ we define $\widehat{u}(\cdot,t) := u(\cdot,T)$ for $t \in (T,1]$ and note that the solution v of

$$v_t = \Delta v - \widehat{u} v \quad \text{in } \Omega \times (0,1), \quad \partial_\nu v\big|_{\partial\Omega} = 0, \quad v(\cdot,0) = v_0 \text{ in } \Omega, \tag{5.13}$$

satisfies

$$\|v_0\|_{L^\infty(\Omega)} \ge v(\cdot,t) \ge (\inf v_0)\, \mathrm{e}^{-R} \ge \frac{1}{L} \mathrm{e}^{-R} \tag{5.14}$$

in Ω for all $t \in [0,1]$ and, by definition of c_1, $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le c_1(1+R)\|v_0\|_{W^{1,\infty}(\Omega)} \le c_1(1+R)L$.
We let u be the solution of

$$u_t = \nabla \cdot \left(D(\widehat{u}) \nabla u - \frac{\widehat{u}}{v} \nabla v \right) \text{ in } \Omega \times (0,1), \quad \partial_\nu u\big|_{\partial\Omega} = 0, \quad u(\cdot,0) = u_0 \text{ in } \Omega.$$

Then by definition of c_2 and c_3 (with $g = \frac{\widehat{u}}{v}\nabla v$ and $A = D \circ \widehat{u}$ in (5.9)), $\|u\|_{L^\infty(\Omega\times(0,1))} \le c_2$ and $\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,1])} \le c_3$. Hence if we define $\Phi(\widehat{u}) := u\big|_{\Omega\times(0,T)}$, we have $\|\Phi(\widehat{u})(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + c_3 t^{\frac{\gamma}{2}} \le R$ for every $t \in (0,T)$ and every $\widehat{u} \in S$, and thus Φ is a function mapping S into itself, where S is a closed convex set in $C^0(\overline{\Omega} \times [0,T])$. Moreover, $\Phi\colon S \to S$ is continuous: We let $\overline{u} \in S$ and $\widehat{u}^k \in S$ for all $k \in \mathbb{N}$ such that $\widehat{u}^k \to \overline{u}$ in $C^0(\overline{\Omega} \times [0,T])$. Then, with respect to $\|\cdot\|_{L^\infty(\Omega\times(0,T))}$ and with respect to the weak-*-topology of $L^\infty((0,T);W^{1,\infty}(\Omega))$, the solutions v^k of (5.13) with $\widehat{u}$ replaced by $\widehat{u}^k$ converge to $\bar{v}$ solving (5.13) with $\overline{u}$ instead of $\widehat{u}$: Assuming on the contrary that there were a sequence $(k_l)_{l\in\mathbb{N}}$ such that for each subsequence $(k_{l_m})_{m\in\mathbb{N}}$ thereof the sequence $(v^{k_{l_m}})_{m\in\mathbb{N}}$ did not converge in the indicated topologies, from the uniform bounds on $\left\|v^{k_l}\right\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])}$ and on $\left\|\nabla v^{k_l}\right\|_{L^\infty((0,T);L^\infty(\Omega))}$ asserted by Lemma 5.2.1 i) and Lemma 5.2.1 ii), respectively, we could conclude the existence of some subsequence $(v^{k_{l_n}})_{n\in\mathbb{N}}$ being uniformly convergent in $\Omega \times [0,T]$ and weakly-*-convergent in $L^\infty((0,T);W^{1,\infty}(\Omega))$. By passing to the limit in the weak formulation in the equations of the form (5.13) satisfied by v^{k_l}, the limit can easily be seen to coincide with the unique weak solution $\bar{v}$ of (5.13) with $\overline{u}$ replacing $\widehat{u}$, contradicting the choice of $(v^{k_l})_{l\in\mathbb{N}}$. We observe that hence and by (5.14), $\frac{u^k}{v^k}\nabla v^k \overset{*}{\rightharpoonup} \frac{\overline{u}}{\bar{v}}\nabla\bar{v}$ in $L^\infty((0,T);L^\infty(\Omega;\mathbb{R}^N))$. Similarly taking into account bounds on $\left\|\Phi(\widehat{u}^k)\right\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])}$ and $\left\|\nabla\Phi(\widehat{u}^k)\right\|_{L^2(\Omega\times(0,T))}$ as obtained from Lemma 5.2.1 iv) and 5.2.1 iii) and again employing the weak formulation of the equations defining $\Phi(\widehat{u}^k)$ and uniqueness of the solution $\Phi(\overline{u}) = u$ of $u_t = \nabla \cdot (D(\overline{u})\nabla u - \frac{\overline{u}}{\bar{v}}\nabla\bar{v})$, $\partial_\nu u\big|_{\partial\Omega} = 0$, $u(\cdot,0) = u_0$, we finally see that $\Phi(\widehat{u}^k) \to \Phi(\overline{u})$ in $C^0(\overline{\Omega} \times [0,T])$.
We note that $S \subset C^0(\overline{\Omega} \times [0,T])$ is a closed bounded convex set and $\Phi(S)$ is relatively compact in $C^0(\overline{\Omega} \times [0,T])$, owing to the uniform Hölder bound c_3 and Arzelà-Ascoli's theorem, so that we can apply Schauder's fixed point theorem to find $u \in S$ such that $\Phi(u) = u$. Due to the regularity assertions in Lemma 5.2.1 iv) even $u \in C^{2,1}(\overline{\Omega} \times (0,T])$; also the corresponding solution v of the second equation belongs to this space by 5.2.1 i). □

Lemma 5.2.3. *Let $T > 0$. If on $\Omega \times (0,T)$ there is a solution (u,v) to (5.5) such that*

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty,$$

then there is $\widetilde{T} > T$ such that there is a solution to (5.5) *in $\Omega \times (0, \widetilde{T})$ which on $\Omega \times (0, T)$ coincides with (u, v).*

Proof. Successive application of comparison arguments in (5.5b) and of Lemma 5.2.1 parts ii), iv) and i) show the existence of $\gamma > 0$ and $M > 0$ such that

$$\inf_{\Omega\times(0,T)} v > \frac{1}{M}, \quad \|v\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \le M, \quad \|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])} \le M, \quad \|v\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[0,T])} \le M.$$

Due to the uniform continuity of u and v,

$$\widetilde{u}_0(x) := \lim_{t\nearrow T} u(x,t), \qquad \widetilde{v}_0(x) := \lim_{t\nearrow T} v(x,t), \qquad x \in \overline{\Omega},$$

are well-defined and satisfy $M \ge \widetilde{v}_0 \ge \frac{1}{M}$ in $\overline{\Omega}$ as well as $\|\widetilde{u}_0\|_{C^\gamma(\overline{\Omega})} \le M$.
Picking a sequence $(t_k)_{k\in\mathbb{N}} \nearrow T$ and referring to $\|v\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \le M$, we may conclude the existence of a subsequence $(t_{k_l})_{l\in\mathbb{N}}$ such that $\nabla v(\cdot, t_{k_l}) \overset{*}{\rightharpoonup} \nabla\widetilde{v}_0$ in $L^\infty(\Omega;\mathbb{R}^N)$ as $l \to \infty$, and thus infer $\|\widetilde{v}_0\|_{W^{1,\infty}(\Omega)} \le M$. According to Lemma 5.2.2, we can find $\tau > 0$ and $(\widetilde{u}, \widetilde{v}) \in \left(C^0(\overline{\Omega}\times[0,\tau]) \cap C^{2,1}(\overline{\Omega}\times(0,\tau])\right)^2$ solving

$$\widetilde{u}_t = \nabla\cdot\left(D(\widetilde{u})\nabla\widetilde{u} - \frac{\widetilde{u}}{\widetilde{v}}\nabla\widetilde{v}\right), \qquad \widetilde{v}_t = \Delta\widetilde{v} - \widetilde{u}\widetilde{v} \qquad \text{in } \Omega\times(0,\tau),$$
$$\partial_\nu\widetilde{u}\big|_{\partial\Omega} = \partial_\nu\widetilde{v}\big|_{\partial\Omega} = 0, \;\; \widetilde{u}(\cdot,0) = \widetilde{u}_0, \;\; \widetilde{v}(\cdot,0) = \widetilde{v}_0.$$

Letting

$$(\overline{u},\overline{v})(\cdot,t) := \begin{cases} (u,v)(\cdot,t), & t < T, \\ (\widetilde{u},\widetilde{v})(\cdot,t-T), & t \in [T, T+\tau), \end{cases}$$

we obtain a weak solution of (5.5) in $\Omega\times(0,T+\tau)$, which by Lemma 5.2.1 parts iv) and i) is classical. $\square$

Lemma 5.2.4. *Let $\gamma \in (0,1)$, $m \ge 1$ and $\delta > 0$. For every $D \in \mathcal{C}_{\delta,m}$, $u_0 \in C^\gamma(\overline{\Omega})$, $v_0 \in W^{1,\infty}(\Omega)$, $u_0 \ge 0$, $v_0 > 0$ in $\overline{\Omega}$, there is $T_{max} > 0$ and $(u,v) \in (C^0(\overline{\Omega}\times[0,T_{max})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max})))^2$ such that (u,v) solves* (5.5) *and*

$$T_{max} = \infty \quad \text{or} \quad \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{5.15}$$

Moreover, $u \ge 0$ and $0 < v \le \|v_0\|_{L^\infty(\Omega)}$ throughout $\Omega\times(0,T_{max})$.

Proof. We let $u_0 \in C^\gamma(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$, define

$$\mathcal{S} = \left\{(t,u,v);\, t\in(0,\infty), u,v \in C^0(\overline{\Omega}\times[0,T_{max})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max}))\;\; (u,v) \text{ solves (5.5)}\right\}$$

and introduce the order relation $\preceq$ given by

$$(t_1,u_1,v_1) \preceq (t_2,u_2,v_2) :\Longleftrightarrow\; t_1 \le t_2, u_2\big|_{(0,t_1)} = u_1,\; v_2\big|_{(0,t_1)} = v_1.$$

Every totally ordered set $M_I = \{(t_i,u_i,v_i);\; i \in I\}$ with arbitrary index set I has an upper bound $(\sup_{i\in I} t_i, u, v)$, where $u(\tau) = u_i(\tau)$ if $\tau \in (0,t_i)$ and v is defined analogously. (This yields well-defined functions, since $u_{i_1}(\tau) = u_{i_2}(\tau)$ if $\tau \in (0,t_{i_1}) \cap (0,t_{i_2})$, because M_I is totally ordered.) Moreover, $\mathcal{S}$ is not empty, according to Lemma 5.2.2. By Zorn's lemma there is some maximal element $(T_{max},u,v) \in \mathcal{S}$. Assume that $\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$. Then Lemma 5.2.3 immediately yields $\widetilde{T} > T$ such that $(\widetilde{T},\widetilde{u},\widetilde{v}) \succeq (T_{max},u,v)$, contradicting the maximality of (T_{max},u,v). $\square$

5.3 The nondegenerate case

This section is devoted to the derivation of estimates for the solutions, so as to finally obtain their global existence by means of the extensibility criterion (5.15).
For some manipulations in (5.5a) it would be more convenient to deal with a nonsingular chemotaxis term of the form $\nabla\cdot(u\nabla w)$ instead of $\nabla\cdot(\frac{u}{v}\nabla v)$. For this purpose, we employ the following transformation and, given a solution $(u,v)\in(C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})))^2$ of (5.5) for initial data (u_0,v_0) as in (5.4), let

$$w := -\ln\left(\frac{v}{\|v_0\|_{L^\infty(\Omega)}}\right) \qquad \text{in } \Omega\times[0,T_{max}). \tag{5.16}$$

Then $w\geq 0$ in $\Omega\times(0,T_{max})$ and $(u,w)\in(C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})))^2$ solves

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u + u\nabla w) & \text{in } \Omega\times(0,T_{max}) & (5.17\text{a})\\ w_t = \Delta w - |\nabla w|^2 + u & \text{in } \Omega\times(0,T_{max}) & (5.17\text{b})\\ \partial_\nu u = 0 = \partial_\nu w & \text{in } \partial\Omega\times(0,T_{max}) & (5.17\text{c})\\ u(\cdot,0)=u_0, \quad w(\cdot,0)=w_0 := -\ln\dfrac{v_0}{\|v_0\|_{L^\infty(\Omega)}} & \text{in } \Omega, & (5.17\text{d}) \end{cases}$$

where (u_0,w_0) satisfy

$$u_0\geq 0, \quad w_0\geq 0, \quad w_0\in W^{1,\infty}(\Omega), \quad u_0\in C^\gamma(\overline{\Omega}) \text{ for some } \gamma\in(0,1). \tag{5.18}$$

Evidently, given $\|v_0\|_{L^\infty(\Omega)}$, every solution (u,w) to (5.17) yields a solution to (5.5) via $v := \|v_0\|_{L^\infty(\Omega)}\,\mathrm{e}^{-w}$.

5.3.1 First estimates

We will proceed in several steps, the first being the following simple observation that the bacterial mass is conserved throughout evolution:

Lemma 5.3.1. *Let $T>0$, $\delta>0$, $m\in\mathbb{R}$, $D\in\mathcal{C}_{\delta,m}$ and let $u\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ solve* (5.5c), (5.5a) *with some $v\in C^{2,1}(\overline{\Omega}\times(0,T))$ or* (5.17a), (5.17c) *with some $w\in C^{2,1}(\overline{\Omega}\times(0,T))$. Then, with $u_0:=u(\cdot,0)$,*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \textit{for every } t\in(0,T).$$

Proof. This is an immediate consequence of (5.5a) or (5.17a), obtained upon integration over Ω. ☐

In (5.17), a spatio-temporal L^2-bound for ∇w can be inferred rather directly:

Lemma 5.3.2. *Let $m\in\mathbb{R}$, $\delta>0$, $T>0$ and $(u,w)\in(C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T)))^2$ be a solution to* (5.17) *for any $D\in\mathcal{C}_{\delta,m}$. Then, with $u_0:=u(\cdot,0)$, $w_0:=w(\cdot,0)$,*

$$\int_0^t\int_\Omega|\nabla w|^2 \leq \int_\Omega w_0 + t\int_\Omega u_0 \qquad \textit{for every } t\in(0,T).$$

Proof. In order to see this, it is sufficient to integrate the second equation of (5.17) and take into account Lemma 5.3.1. □

This bound can be transformed into a first information on derivatives of u:

Lemma 5.3.3. *Let $m > 1$, $\delta > 0$. For any $K > 0$ there is $C > 0$ such that for any $D \in \mathcal{C}_{\delta,m}$ any solution (u,w) of (5.17) emanating from initial data (u_0,w_0) as in (5.18) with $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, $\|w_0\|_{L^1(\Omega)} \le K$ obeys the estimates*

$$\int_0^t \int_\Omega u^{2m-4}|\nabla u|^2 \le C(1+t) \qquad \text{for all } t \in (0,T_{max})$$

and

$$\int_0^t \int_\Omega D(u)u^{m-3}|\nabla u|^2 \le C(1+t) \qquad \text{for all } t \in (0,T_{max})$$

as well as

$$\int_\Omega u^{m-1}(\cdot,t) \le C(1+t) \qquad \text{for all } t \in (0,T_{max}). \tag{5.19}$$

Proof. Due to (5.17a), on $(0,T_{max})$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{m-1} &= (m-1)\int_\Omega u^{m-2}\nabla\cdot(D(u)\nabla u + u\nabla w)\\ &= (m-1)(2-m)\int_\Omega D(u)u^{m-3}|\nabla u|^2 + (m-1)(2-m)\int_\Omega u^{m-2}\nabla u\cdot\nabla w.\end{aligned} \tag{5.20}$$

We note that by Young's inequality

$$\left|\int_\Omega u^{m-2}\nabla u\cdot\nabla w\right| \le \frac{\delta}{3}\int_\Omega u^{2m-4}|\nabla u|^2 + \frac{3}{4\delta}\int_\Omega |\nabla w|^2 \quad \text{ on } (0,T_{max}). \tag{5.21}$$

The sign of $(m-1)(2-m)$ in (5.20) depends on the size of m and we therefore distinguish the following cases:
If $m \in (1,2)$, (5.20) together with (5.21) and Lemma 5.3.2 yields

$$\begin{aligned}&\frac{1}{3}\int_0^t\int_\Omega D(u)u^{m-3}|\nabla u|^2 + \frac{\delta}{3}\int_0^t\int_\Omega u^{2m-4}|\nabla u|^2\\ &\le \frac{1}{(m-1)(2-m)}\int_\Omega u^{m-1}(\cdot,t) - \frac{1}{(m-1)(2-m)}\int_\Omega u_0 + \frac{3}{4\delta}\int_0^t\int_\Omega |\nabla w|^2\\ &\le \frac{1}{(m-1)(2-m)}|\Omega|^{\frac{m-2}{m-1}}\left(\int_\Omega u_0\right)^{\frac{1}{m-1}} + \frac{3}{4\delta}\left(\int_\Omega w_0 + t\int_\Omega u_0\right) \quad \text{for any } t \in (0,T_{max}).\end{aligned} \tag{5.22}$$

If $m > 2$, (5.20) and (5.21) can be combined to give

$$\begin{aligned}\frac{1}{(m-1)(m-2)}\int_\Omega u^{m-1}(\cdot,t) - \frac{1}{(m-1)(m-2)}\int_\Omega u_0^{m-1} + \frac{1}{3}\int_0^t\int_\Omega D(u)u^{m-3}|\nabla u|^2\\ + \frac{\delta}{3}\int_0^t\int_\Omega u^{2m-4}|\nabla u|^2 \le \frac{3}{4\delta}\int_0^t\int_\Omega |\nabla w|^2 \quad \text{for any } t \in (0,T_{max}),\end{aligned}$$

which allows for a similarly obvious definition of C as (5.22). This inequality also entails (5.19) for $m > 2$, the only case that does not immediately result from Lemma 5.3.1.
If $m = 2$, apparently the consideration of $\frac{d}{dt}\int_\Omega u^{m-1} = \frac{d}{dt}\int_\Omega u = 0$ does not help in achieving an estimate for $\int_0^t\int_\Omega u^{2m-4}|\nabla u|^2 = \int_0^t\int_\Omega |\nabla u|^2$. From the analogously obtained inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u + \frac{1}{3}\int_\Omega \frac{D(u)}{u}|\nabla u|^2 + \frac{2\delta}{3}\int_\Omega |\nabla u|^2 \le \frac{\delta}{3}\int_\Omega |\nabla u|^2 + \frac{3}{4\delta}\int_\Omega |\nabla w|^2 \text{ on } (0,T_{max}),$$

however, we can derive the same form of estimates as in the other cases. □

For convenience let us recall those special cases of the Gagliardo-Nirenberg inequality we are going to use in the following:

Lemma 5.3.4. *i) Let $0 < q \le p \le \frac{2N}{N-2}$ (or $0 < q \le p < \infty$ if $N = 2$) and let $s > 0$ and $\gamma > 0$. Then there is $c > 0$ such that*

$$\|\psi\|^\gamma_{L^p(\Omega)} \le c\,\|\nabla\psi\|^{a\gamma}_{L^2(\Omega)}\,\|\psi\|^{(1-a)\gamma}_{L^q(\Omega)} + c\,\|\psi\|^\gamma_{L^s(\Omega)} \qquad \text{for all } \psi \in W^{1,2}(\Omega)\cap L^q(\Omega)\cap L^s(\Omega),$$

where

$$a = \frac{\frac{1}{q}-\frac{1}{p}}{\frac{1}{q}+\frac{1}{N}-\frac{1}{2}}.$$

ii) Let $p, q \in (1,\infty)$ be such that $p(q-N) = q(2p-N)a$ for some $a \in [\frac{1}{2}, 1)$. Then there is $c > 0$ such that

$$\|\nabla\psi\|^q_{L^q(\Omega)} \le c\,\|\Delta\psi\|^{qa}_{L^p(\Omega)}\,\|\psi\|^{q(1-a)}_{L^\infty(\Omega)} + c\,\|\psi\|^q_{L^\infty(\Omega)}$$

for all $\psi \in W^{2,p}(\Omega)\cap W^{1,q}(\Omega)\cap L^\infty(\Omega)$ with $\partial_\nu\psi = 0$ on $\partial\Omega$.

Proof. The Gagliardo-Nirenberg inequality (cf. Lemma 3.4.3) can be found in [77, p. 125], [23, Thm. 10.1] or in [57, Lemma 2.3] (where also the case of $p, q < 1$ in i) is covered); replacing $D^2\psi$ by $\Delta\psi$ in the standard formulation of ii) is possible by, e.g., [23, Thm. 19.1]. □

Aided by the Gagliardo-Nirenberg inequality, in the next step, as consequence of the estimates from Lemma 5.3.3 we shall acquire the bound (5.23), which will be featured as condition in Lemmata 5.3.7 and 5.3.8, and can be seen as an important ingredient of the proof of Theorem 5.1.1.

Lemma 5.3.5. *Let $K > 0$, $T \in (0,\infty)$, $\delta > 0$ and $m > 1$. If either*
i) $m \le 2$, $r > 1$ and $p \ge 1$ satisfy $p \le \frac{2N}{N-2}(m-1)$ and $r(1-\frac{1}{p}) \le 2m-3+\frac{2}{N}$ or
ii) $m \ge 2$, $r > 1$ and $p \in [m-1, \frac{2N}{N-2}(m-1)]$ are such that $(\frac{1}{m-1}-\frac{1}{p})r \le 1+\frac{2}{N}$,
then there is $C > 0$ such that whenever $(u,w) \in (C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T)))^2$ solves (5.17) for some $D \in \mathcal{C}_{\delta,m}$ and for initial data (u_0,w_0) with (5.18) and $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, $\|w_0\|_{L^1(\Omega)} \le K$, we have

$$\int_0^t \|u\|^r_{L^p(\Omega)} < C(1+t^{r+1}) \quad \text{for all } t \in (0,T). \tag{5.23}$$

Proof. **i)** Due to $m > 1$, the inequality $p \le \frac{2N}{N-2}(m-1)$ is equivalent to $\frac{p}{m-1} \le \frac{2N}{N-2}$, and $p \ge 1$ ensures $\frac{p}{m-1} \ge \frac{1}{m-1}$. Thus, the Gagliardo-Nirenberg inequality (Lemma 5.3.4 i)) yields $c_1 > 0$ such that with

$$a := \frac{m-1-\frac{m-1}{p}}{m-1+\frac{1}{N}-\frac{1}{2}},$$

and hence $\frac{r}{m-1}a \le 2$, for all $t \in (0, T_{max})$ we obtain

$$\int_0^t \|u\|_{L^p(\Omega)}^r = \int_0^t \left\|u^{m-1}\right\|_{L^{\frac{p}{m-1}}(\Omega)}^{\frac{r}{m-1}}$$
$$\le c_1 \int_0^t \left\|\nabla u^{m-1}\right\|_{L^2(\Omega)}^{\frac{r}{m-1}a} \left\|u^{m-1}\right\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{r}{m-1}(1-a)} + c_1 \int_0^t \left\|u^{m-1}\right\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{r}{m-1}}$$
$$\le c_1 \left(\int_\Omega u_0\right)^{\frac{r}{m-1}(1-a)} \int_0^t \left(1 + \left\|\nabla u^{m-1}\right\|_{L^2(\Omega)}^2\right) + c_1 \int_0^t \left(\int_\Omega u_0\right)^{\frac{r}{m-1}},$$

where we have used Lemma 5.3.1, and can conclude the proof with applications of Lemma 5.3.3 and Young's inequality.
ii) From Lemma 5.3.3 we obtain $c_2 > 0$ such that

$$\int_\Omega u^{m-1}(\cdot,t) \le c_2(1+t) \quad \text{and} \quad \int_0^t \int_\Omega |\nabla u^{m-1}|^2 \le c_2(1+t) \qquad \text{for all } t \in (0, T_{max}).$$

The fact that $p \in [m-1, \frac{2N}{N-2}(m-1)]$ entails both $\frac{p}{m-1} \ge 1$ and $\frac{p}{m-1} \le \frac{2N}{N-2}$. Therefore, with

$$a := \frac{1 - \frac{m-1}{p}}{1 + \frac{1}{N} - \frac{1}{2}},$$

Lemma 5.3.4 i) produces $c_3 > 0$ such that for all $t \in (0, T_{max})$

$$\int_0^t \|u\|_{L^p(\Omega)}^r = \int_0^t \left\|u^{m-1}\right\|_{L^{\frac{p}{m-1}}(\Omega)}^{\frac{r}{m-1}}$$
$$\le c_3 \int_0^t \left\|\nabla u^{m-1}\right\|_{L^2(\Omega)}^{\frac{r}{m-1}a} \left\|u^{m-1}\right\|_{L^1(\Omega)}^{\frac{r(1-a)}{m-1}} + c_3 \int_0^t \left\|u^{m-1}\right\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{r}{m-1}}$$
$$\le c_2 \left(c_3(1+t)\right)^{\frac{r(1-a)}{m-1}} \int_0^t \left(\left\|\nabla u^{m-1}\right\|_{L^2(\Omega)}^2 + 1\right) + c_3 \|u_0\|_{L^1(\Omega)}^r t \le c_4 + c_5 t^{r+1},$$

where we have used that $\frac{ra}{m-1} \le 2$ and, aided by Lemma 5.3.1 and the trivial inequality $r\frac{1-a}{m-1} \le r$, chosen suitable positive constants c_4 and c_5. □

5.3.2 Estimates for ∇v

As preparation for exploiting (5.23) in the second equation of (5.5), we recall

Lemma 5.3.6. *Let $p, q \in (1, \infty)$. Then for every $T > 0$ there exists $C > 0$ such that for every $z \in L^q((0,T); L^p(\Omega))$ the unique solution of*

$$v_t = \Delta v - z \quad \text{in } \Omega \times (0,T), \quad \partial_\nu v\big|_{\partial\Omega} = 0, \quad v(\cdot, 0) = 0$$

satisfies

$$\int_0^T \|\Delta v\|_{L^p(\Omega)}^q \le C \int_0^T \|z\|_{L^p(\Omega)}^q.$$

Proof. We obtain this lemma as straightforward consequence of well-known maximal Sobolev regularity assertions, cf. [27],[31]. □

Lemma 5.3.6 empowers us to develop (5.23) into useful knowledge about the gradient of v:

Lemma 5.3.7. *Let $p \geq \frac{N}{2}$, $r \geq p$ and*

$$\begin{cases} q \in (1, N + (2 - \frac{N}{p})r], & \text{if } p \geq N, \\ q \in (1, N + (2 - \frac{N}{p})r] \cap (1, \frac{Np}{N-p}), & \text{if } \frac{N}{2} < p < N. \end{cases}$$

Then for every $K > 0$ and $T > 0$ there is $C > 0$ such that for every $v_0 \in W^{1,\infty}(\Omega)$ with $\|v_0\|_{W^{1,\infty}(\Omega)} \leq K$, and every $0 \leq u \in L^r((0,T); L^p(\Omega))$ for which

$$\int_0^T \|u\|_{L^p(\Omega)}^r < K \tag{5.24}$$

is satisfied, the solution v of (5.5b) *fulfils*

$$\int_0^T \int_\Omega |\nabla v|^q < C. \tag{5.25}$$

Proof. In order to prepare the application of Lemma 5.3.6, we decompose $v(\cdot,t) = \widetilde{v}(\cdot,t) + \mathrm{e}^{t\Delta}v_0$ in $\Omega \times (0,T)$, where $\widetilde{v}$ solves

$$\widetilde{v}_t = \Delta\widetilde{v} - uv \quad \text{in } \Omega \times (0,T), \quad \partial_\nu \widetilde{v}\big|_{\partial\Omega} = 0, \quad \widetilde{v}(\cdot,0) = 0.$$

By nonnegativity of v_0 and uv, we clearly have $0 \leq \widetilde{v} \leq v \leq K$ in $\Omega \times (0,T)$.
We let $q \leq N + (2 - \frac{N}{p})r$ and without loss of generality assume $q \geq 2p$ (which is possible since $2p = N + 2p - N \leq N + (2p - N)\frac{r}{p} = N + (2 - \frac{N}{p})r$ and also $2p < \frac{Np}{N-p}$ if $p \in (\frac{N}{2}, N)$). We note that $q \leq N + (2 - \frac{N}{p})r$ implies that $r \geq \frac{q-N}{2-\frac{N}{p}} = \frac{(q-N)p}{2p-N}$ and hence with

$$a := \frac{p(q-N)}{q(2p-N)}$$

we have $aq \leq r$. Moreover, $q \geq 2p$ ensures that $pq - Np \geq pq - \frac{Nq}{2} = \frac{1}{2}q(2p-N)$ and thus $a \geq \frac{1}{2}$, and, furthermore, $(p-N)q > -Np$, which is obvious for $p > N$ and holds by assumption on q if $p < N$, entails $2pq - Nq > pq - Np$ and hence $a < 1$. Accordingly, from [110, Lemma 1.3 iii)] and the Gagliardo-Nirenberg inequality (Lemma 5.3.4 ii)) we obtain $c_1 > 0$, $c_2 > 0$, respectively, such that we have

$$\begin{aligned} \int_0^T \int_\Omega |\nabla v|^q &\leq 2^q \int_0^T \int_\Omega |\nabla \mathrm{e}^{t\Delta} v_0|^q + 2^q \int_0^T \int_\Omega |\nabla\widetilde{v}|^q \\ &\leq c_1 T \|\nabla v_0\|_{L^q(\Omega)}^q + 2^q \int_0^T \|\nabla\widetilde{v}\|_{L^q(\Omega)}^q \\ &\leq c_1 T |\Omega|^{\frac{1}{q}} \|\nabla v_0\|_{L^\infty(\Omega)} + c_2 \int_0^T \|\Delta\widetilde{v}\|_{L^p(\Omega)}^{aq} \|\widetilde{v}\|_{L^\infty(\Omega)}^{(1-a)q} + c_2 \int_0^T \|\widetilde{v}\|_{L^\infty(\Omega)}^q . \end{aligned}$$

Since $aq < r$, due to Young's inequality and boundedness of $\widetilde{v}$ this estimate can be turned into

$$\int_0^T \int_\Omega |\nabla v|^q \leq c_3 + c_4 \int_0^T \|\Delta\widetilde{v}\|_{L^p(\Omega)}^r ,$$

for some $c_3 > 0$, $c_4 > 0$, where we may invoke the maximal Sobolev result of Lemma 5.3.6 for $z = uv$ and hence $\int_0^T \|z\|_{L^p(\Omega)}^r \leq K^r \int_0^T \|u\|_{L^p(\Omega)}^r$ to conclude (5.25) from (5.24). □

5.3.3 Bounds on w

Another consequence of (5.23) is (local-in-time) boundedness of w:

Lemma 5.3.8. *Assume that $r \in (1,\infty)$, $p \in [1,\infty)$ are such that $\frac{Nr}{2p(r-1)} < 1$. Then for every $K > 0$ there is $C > 0$ such that whenever, for some $T > 0$, $w \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ solves* (5.17b), (5.17c), (5.17d) *for some w_0 as in* (5.18) *and some $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ such that $\|w_0\|_{L^\infty(\Omega)} \le K$, $\frac{1}{|\Omega|}\int_\Omega u(\cdot,t) \le K$ for $t \in (0,T)$ and moreover*

$$\int_0^T \|u\|_{L^p(\Omega)}^r < K,$$

then

$$w(x,t) \le C(1+t) \qquad \textit{for all } (x,t) \in \Omega \times (0,T).$$

Proof. By nonpositivity of $-|\nabla w|^2$, we have that according to the variation-of-constants formula, for any $t \in (0,T)$,

$$\begin{aligned} 0 \le w(\cdot,t) &= e^{t\Delta}w_0 - \int_0^t e^{(t-s)\Delta}|\nabla w(\cdot,s)|^2 \mathrm{d}s + \int_0^t e^{(t-s)\Delta}(u(\cdot,s) - \overline{u}(s))\mathrm{d}s + \int_0^t \overline{u}(s)\mathrm{d}s \\ &\le \|w_0\|_{L^\infty(\Omega)} + \int_0^t \left\| \mathrm{e}^{(t-s)\Delta}\left(u(\cdot,s) - \overline{u}(s)\right)\right\|_{L^\infty(\Omega)} \mathrm{d}s + Kt \qquad \text{in } \Omega, \end{aligned} \tag{5.26}$$

where $\overline{u}(t) := \frac{1}{|\Omega|}\int_\Omega u(\cdot,t) \le K$ for $t \in (0,T)$ by assumption. For assessing the remaining integral in (5.26) we invoke [110, Lemma 1.3 i)] to obtain $c_1 > 0$ such that

$$\begin{aligned} &\int_0^t \left\| \mathrm{e}^{(t-s)\Delta}\left(u(\cdot,s) - \overline{u}(s)\right)\right\|_{L^\infty(\Omega)} \mathrm{d}s \\ &\le c_1 \int_0^t \left(1 + (t-s)^{-\frac{N}{2p}}\right) \mathrm{e}^{-\lambda_1(t-s)} \left\|u(\cdot,s) - \overline{u}(s)\right\|_{L^p(\Omega)} \mathrm{d}s \qquad (5.27) \\ &\le c_1 \int_0^t \left(1 + (t-s)^{-\frac{N}{2p}}\right) \mathrm{e}^{-\lambda_1(t-s)} \left\|u(\cdot,s)\right\|_{L^p(\Omega)} \mathrm{d}s + c_1 K|\Omega|^{\frac{1}{p}} \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right) \mathrm{e}^{-\lambda_1\sigma}\mathrm{d}\sigma \\ &\le c_1 \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right)^{\frac{r}{r-1}} \mathrm{e}^{-\lambda_1\sigma\frac{r}{r-1}}\mathrm{d}\sigma + c_1 \int_0^t \left\|u(\cdot,s)\right\|_{L^p(\Omega)}^r + c_1 K|\Omega|^{\frac{1}{p}} \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right) \mathrm{e}^{-\lambda_1\sigma}\mathrm{d}\sigma \end{aligned}$$

for all $t \in (0,T)$. Collecting the constants in (5.26) and (5.27), we see that for all $(x,t) \in \Omega \times (0,T)$

$$w(x,t) \le C(1+t),$$

where

$$C := K + c_1 K + c_1 K|\Omega|^{\frac{1}{p}} \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right) \mathrm{e}^{-\lambda_1\sigma}\mathrm{d}\sigma + k_1 \int_0^\infty \left(1 + \sigma^{-\frac{N}{2p}}\right)^{\frac{r}{r-1}} \mathrm{e}^{-\lambda_1\sigma\frac{r}{r-1}}\mathrm{d}\sigma,$$

which is finite due to $\frac{Nr}{2p(r-1)} < 1$ (and its consequence $\frac{N}{2p} < 1$). □

If we can find parameters that allow for an application of Lemma 5.3.5 and Lemma 5.3.8 at the same time, we can conclude boundedness of w. This is the goal we pursue in the following lemma:

Lemma 5.3.9. *Let*

$$m > 1 + \frac{N}{4} \tag{5.28}$$

and $\delta > 0$. Then for all $T \in (0,\infty)$ there is $C > 0$ such that for every $D \in \mathcal{C}_{\delta,m}$ and every (u_0, w_0) as in (5.18) *with $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, $\|w_0\|_{L^\infty(\Omega)} \le K$, any solution $(u,w) \in (C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T)))^2$ of* (5.17) *satisfies*

$$w(x,t) \le C \qquad \textit{for all } x \in \Omega \textit{ and all } t \in (0,T).$$

Proof. Let us first consider the case $m \in (2 - \frac{1}{N}, 2]$ (that is of interest only if $N < 4$, because m is supposed to satisfy $m > 1 + \frac{N}{4}$) and observe that by (5.28), we have

$$m > \begin{Bmatrix} \frac{3}{2} & \text{if } N = 2, \\ \frac{7}{4} & \text{if } N = 3 \end{Bmatrix} = \frac{5}{4} - \frac{1}{2N} + \frac{N}{8} + \sqrt{\left(\frac{5}{4} - \frac{1}{2N} + \frac{N}{8}\right)^2 - \frac{5}{4} + \frac{1}{N} - \frac{3}{8}N}.$$

Therefore, we see that

$$4m^2 - 10m + \frac{4}{N}m - Nm + 5 - \frac{4}{N} + \frac{3}{2}N > 0$$

and hence

$$\begin{aligned}(N-2)\left(m - \frac{3}{2}\right) &= Nm - \frac{3}{2}N - 2m + 3 \\ &< 4m^2 - 8m + \frac{4}{N}m - 4m + 8 - \frac{4}{N} = 2(m-1)\left(2m - 4 + \frac{2}{N}\right),\end{aligned}$$

so that

$$\frac{N(m - \frac{3}{2})}{2m - 4 + \frac{2}{N}} < \frac{2N}{N-2}(m-1).$$

Since moreover $\frac{2N}{N-2}(m-1) > 1$, it is possible to choose some number $p \ge 1$ which satisfies $p \in \left(\frac{N(m-\frac{3}{2})}{2m-4+\frac{2}{N}}, \frac{2N}{N-2}(m-1)\right)$. With this choice of p we let

$$r := \frac{2m - 3 + \frac{2}{N}}{1 - \frac{1}{p}}$$

and note that $2m - 3 + \frac{2}{N} > 4 - \frac{2}{N} - 3 + \frac{2}{N} = 1 > 1 - \frac{1}{p}$ entails $r > 1$. Hence Lemma 5.3.5 i) is applicable. Moreover,

$$\frac{2p}{N}\left(1 - \frac{1}{r}\right) = \frac{2p}{N}\left(1 - \frac{1 - \frac{1}{p}}{2m - 3 + \frac{2}{N}}\right) = \frac{2}{N} \cdot \frac{p(2m - 3 + \frac{2}{N}) - p + 1}{2m - 3 + \frac{2}{N}} > \frac{2}{N} \cdot \frac{N(m - \frac{3}{2}) + 1}{2(m - \frac{3}{2} + \frac{1}{N})} = 1$$

and we can additionally invoke Lemma 5.3.8 so as to obtain the desired boundedness of w on $\Omega \times (0,T)$.
If $m \ge 2$ (and $m > 1 + \frac{N}{4}$), we note that

$$\frac{N^2(m-1)}{2N(m-1) + 4(m-1) - 2N} < \frac{N^2(m-1)}{2N\frac{N}{4} + 4\frac{N}{4} - 2N} = \frac{N^2(m-1)}{\frac{N^2}{2} - N} = \frac{2N}{N-2}(m-1).$$

Since $m \geq 2$,

$$\frac{1+\frac{2}{N}}{m-1} \leq 1+\frac{2}{N} < 1+\frac{4}{N}+\frac{4}{N^2},$$

and hence

$$\frac{1}{m-1} - 1 - \frac{2}{N} < \frac{2}{N} + \frac{4}{N^2} - \frac{2}{N(m-1)} = \frac{2N(m-1)+4(m-1)-2N}{N^2(m-1)}.$$

Therefore we can pick $p \in \left(\frac{N^2(m-1)}{2N(m-1)+4(m-1)-2N}, \frac{2N}{N-2}(m-1)\right)$ such that $\frac{1}{p} > \frac{1}{m-1} - 1 - \frac{2}{N}$ and $p > m-1$, and we let $r := \frac{1+\frac{2}{N}}{\frac{1}{m-1}-\frac{1}{p}}$. Then $r > 1$ and, apparently, $(\frac{1}{m-1} - \frac{1}{p})r \leq 1 + \frac{2}{N}$, warranting applicability of Lemma 5.3.5. Moreover, $p > \frac{N^2(m-1)}{2N(m-1)+4(m-1)-2N}$ entails $\frac{1}{p} < \frac{2}{N} + \left(\frac{2}{N}\right)^2 - \frac{2}{N(m-1)}$ and thus $\frac{N}{2p}(1+\frac{2}{N}) = \frac{N}{2p} + \frac{1}{p} < 1 + \frac{2}{N} - \frac{1}{m-1} + \frac{1}{p}$ and hence, finally,

$$\frac{N}{2p} < 1 - \frac{\frac{1}{m-1} - \frac{1}{p}}{1+\frac{2}{N}} = 1 - \frac{1}{r},$$

which permits us to employ Lemma 5.3.8 and conclude. □

Lemma 5.3.10. *For every $K > 0$ and every $q \in (0, \infty]$ there is $C > 0$ such that for all $T > 0$ and all $v \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$*

$$\|v_0\|_{L^\infty(\Omega)} \geq \frac{1}{K}, \quad w \leq K \quad \text{in } \Omega \times (0,T), \quad \text{and} \quad \|\nabla v\|_{L^q(\Omega\times(0,T))} \leq K$$

implies

$$\|\nabla w\|_{L^q(\Omega\times(0,T))} \leq C$$

Proof. Since $w \leq K$, we have $v = \|v_0\|_{L^\infty(\Omega)} \mathrm{e}^{-w} \geq \|v_0\|_{L^\infty(\Omega)} \mathrm{e}^{-K}$, and immediately obtain $\frac{1}{v} \leq \|v_0\|^{-1}_{L^\infty(\Omega)} \mathrm{e}^K \leq K\mathrm{e}^K$ in $\Omega \times (0,T)$. Thus

$$\|\nabla w\|_{L^q(\Omega\times(0,t))} \leq \left\|\frac{1}{v}\nabla v\right\|_{L^q(\Omega\times(0,t))} \leq K\mathrm{e}^K \|\nabla v\|_{L^q(\Omega\times(0,t))} \leq K^2\mathrm{e}^K =: C.$$ □

5.3.4 L^p-bounds on u

Lemma 5.3.11. *Let $\delta > 0$, $m \geq 1$, $q > 2$ and $p > 1$. Then for every $K > 0$ and $T > 0$ there is $C > 0$ such that the following holds: If $q \geq N$ and*

$$m \leq 2, \qquad p \geq m - \frac{2}{q}, \qquad p \leq (q-1)(m-1) + \frac{q-2}{N}, \tag{5.29}$$

$$\text{or} \qquad m \geq 2, \qquad p \geq 2\left(1-\frac{1}{q}\right)(m-1), \qquad p \leq (m-1)\left(\frac{q}{2} + \frac{(q-2)(N+2)}{2N}\right), \tag{5.30}$$

then for every function $w \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ with

$$\int_0^t \int_\Omega |\nabla w|^q \leq K \text{ for all } t \in (0,T),$$

any solution $u \in C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T))$ of (5.17a), (5.17c), (5.17d) with some $D \in \mathcal{C}_{\delta,m}$ and $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \leq K$ fulfils

$$\int_\Omega u^p(\cdot,t) \leq C \qquad \text{for all } t \in (0,T).$$

Proof. Either of (5.29) and (5.30) implies $p \geq m-1$. Moreover,

$$\frac{N-2}{2N} \leq \frac{q-2}{2q} \cdot \frac{p+m-1}{p-m+1}. \tag{5.31}$$

Let us first consider the case $m \leq 2$. Then $p \geq m - \frac{2}{q}$ implies $p-m+1 \geq \frac{q-2}{q}$ and hence

$$\frac{2}{m+p-1} \leq \frac{2q}{q-2} \cdot \frac{p-m+1}{p+m-1}. \tag{5.32}$$

We now let

$$a := \frac{\frac{m+p-1}{2} - \frac{q-2}{2q} \cdot \frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2} + \frac{1}{N} - \frac{1}{2}}$$

and observe that

$$a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} \leq 2, \tag{5.33}$$

because $p \leq (q-1)(m-1) + \frac{q-2}{N}$ implies that $\frac{q}{q-2}p - p = (\frac{q}{q-2}-1)p = \frac{2}{q-2}p \leq \frac{2(q-1)}{q-2}(m-1) + \frac{2}{N} = (1+\frac{q}{q-2})(m-1) + \frac{2}{N} = m-1+\frac{q}{q-2}(m-1)+\frac{2}{N}$, that is, $\frac{q}{q-2}(p-m+1) \leq m+p-1+\frac{2}{N}$ and hence $\frac{q}{q-2}(p-m+1) - 1 \leq (m+p-1) + \frac{2}{N} - 1$, which leads to

$$a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} = \frac{\frac{m+p-1}{2} - \frac{q-2}{2q}\frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2} + \frac{1}{N} - \frac{1}{2}} \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} = \frac{\frac{p-m+1}{2} \cdot \frac{2q}{q-2} - 1}{\frac{1}{2}((m+p-1) + \frac{2}{N} - 1)} \leq 2.$$

From Lemma 5.3.1 we obtain $c_1 > 0$ such that

$$\left\| u^{\frac{m+p-1}{2}}(\cdot,t) \right\|_{L^{\frac{2}{m+p-1}}(\Omega)} = c_1 \qquad \text{for all } t \in (0,T).$$

Due to (5.31) and (5.32) we can apply the Gagliardo-Nirenberg inequality in the form of Lemma 5.3.4 i) to obtain $c_2 > 0$ such that

$$\begin{aligned}
\int_\Omega u^{(p+1-m)\frac{q}{q-2}} &= \int_\Omega u^{\frac{m+p-1}{2}(\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1})} \\
&= \left\| u^{\frac{m+p-1}{2}} \right\|^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}_{L^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}(\Omega)} \\
&\leq c_2 \left\| \nabla u^{\frac{m+p-1}{2}} \right\|^{a\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}_{L^2(\Omega)} \left\| u^{\frac{m+p-1}{2}} \right\|^{(1-a)\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} + c_2 \left\| u^{\frac{m+p-1}{2}} \right\|^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\
&= c_2 c_1^{(1-a)\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}} \left\| \nabla u^{\frac{m+p-1}{2}} \right\|^{a\cdot\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}_{L^2(\Omega)} + c_2 c_1^{\frac{2q}{q-2}\cdot\frac{p-m+1}{m+p-1}}
\end{aligned} \tag{5.34}$$

on $(0,T)$.

In obtaining such an estimate for $m \geq 2$ we could use the same argument. It is, however, possible to obtain better conditions by relying on Lemma 5.3.3 instead of Lemma 5.3.1. Apart from that, the reasoning is analogous: We have $p \geq 2(1-\frac{1}{q})(m-1)$, which implies $qp \geq (q-2+q)(m-1)$, thus $q(p-m+1) \geq (m-1)(q-2)$ and hence

$$\frac{2(m-1)}{m+p-1} \leq \frac{2q}{q-2} \cdot \frac{p-m+1}{p+m-1} \tag{5.35}$$

and let

$$b := \frac{\frac{m+p-1}{2(m-1)} - \frac{q-2}{2q} \cdot \frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2(m-1)} + \frac{1}{N} - \frac{1}{2}},$$

noting that

$$b \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} \le 2, \tag{5.36}$$

because $p \le (m-1)(\frac{q}{2} + \frac{(q-2)(N+2)}{2N})$ implies that $(\frac{q}{q-2} - 1)(\frac{p}{m-1}) = \frac{2}{q-2}\frac{p}{m-1} \le \frac{N+2}{N} + \frac{q}{q-2}$ and hence $\frac{q(p-m+1)}{(m-1)(q-2)} = \frac{q}{q-2}(\frac{p}{m-1} - 1) \le \frac{p}{m-1} + \frac{N+2}{N} = \frac{m+p-1}{m-1} + \frac{2}{N}$, which shows that $\frac{p-m+1}{2(m-1)} \cdot \frac{2q}{q-2} - 1 \le \frac{m+p-1}{m-1} + \frac{2}{N} - 1$ and therefore also

$$\begin{aligned} b \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} &= \frac{\frac{m+p-1}{2(m-1)} - \frac{q-2}{2q} \cdot \frac{p+m-1}{p-m+1}}{\frac{m+p-1}{2(m-1)} + \frac{1}{N} - \frac{1}{2}} \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1} \\ &= \frac{\frac{q}{q-2} \cdot \frac{p-m+1}{2(m-1)} - 1}{\frac{m+p-1}{2(m-1)} + \frac{1}{N} - \frac{1}{2}} \le 2. \end{aligned}$$

Lemma 5.3.3 yields $c_3 > 0$ such that

$$\left\| u^{\frac{m+p-1}{2}}(\cdot,t) \right\|_{L^{\frac{2(m-1)}{m+p-1}}(\Omega)} \le c_3 \qquad \text{for all } t \in (0,T)$$

and hence (5.31) and (5.35) enable us to invoke the Gagliardo-Nirenberg inequality and obtain $c_4 > 0$ such that on $(0,T)$

$$\begin{aligned} \int_\Omega u^{(p+1-m)\frac{q}{q-2}} &= \int_\Omega u^{\frac{m+p-1}{2}(\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1})} \\ &= \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}}(\Omega)}^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \\ &\le c_4 \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2(m-1)}{m+p-1}}(\Omega)}^{(1-a) \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} + c_4 \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \\ &\le c_4 c_3^{(1-a) \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{a \cdot \frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}} + c_4 c_3^{\frac{2q}{q-2} \cdot \frac{p-m+1}{m+p-1}}. \end{aligned} \tag{5.37}$$

From either (5.34) and (5.33) or (5.37) and (5.36) (and possibly Young's inequality) we hence find that with some $c_5 > 0$ we have

$$\int_\Omega u^{(p+1-m)\frac{q}{q-2}} \le c_5 \left\| u^{\frac{m+p-3}{2}} \nabla u \right\|_{L^2(\Omega)}^2 + c_5 \qquad \text{on } (0,T). \tag{5.38}$$

In

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p-1)\delta \int_\Omega u^{p+m-3}|\nabla u|^2 \le (p-1)\left| \int_\Omega u^{p-1} \nabla u \cdot \nabla w \right| \qquad \text{on } (0,T)$$

we can apply Young's inequality to see that, on $(0,T)$,

$$(p-1)\left| \int_\Omega u^{p-1} \nabla u \cdot \nabla w \right| \le \frac{(p-1)\delta}{4} \int_\Omega u^{p+m-3}|\nabla u|^2 + \frac{p-1}{\delta} \int_\Omega u^{p-m+1}|\nabla w|^2.$$

A further application of Young's inequality allows us to separate u and $|\nabla w|$ in the last integral according to

$$\frac{p-1}{\delta}\int_\Omega u^{p-m+1}|\nabla w|^2 \le \frac{c_5(p-1)}{\delta^3}\int_\Omega |\nabla w|^q + \frac{(p-1)\delta}{4c_5}\int_\Omega u^{(p+1-m)\frac{q}{q-2}}, \quad \text{on } (0,T).$$

Therefore, due to (5.38), in total,

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{(p-1)\delta}{2}\int_\Omega u^{p+m-3}|\nabla u|^2 \le \frac{(p-1)\delta}{4} + \frac{c_5(p-1)}{\delta^3}\int_\Omega |\nabla w|^q \qquad \text{on } (0,T).$$

Integration with respect to time produces the lemma. □

We are particularly interested in applying the previous lemma for some $p > N$, because for such p, a bound on $\int_\Omega u^p$ on some interval $[0,T]$ already ensures uniform boundedness of ∇v (and hence ∇w) on $\overline{\Omega} \times [0,T]$.

Lemma 5.3.12. *Let $\delta > 0$. Assume that either*
i) $2 - \frac{1}{N} < m \le 2$, $N \ge 2$, $q > N$ and $q > 1 + \frac{N^2+1}{Nm-N+1}$, or
ii) $m \ge 2$, $N \ge 2$, $q > N$ and $q > \frac{2N^2+2m^2+2m-4}{(m-1)(N+m+2)}$.
Then there is $p > N$ and for every $K > 0$ and $T \in (0,\infty)$ there is $C > 0$ such that whenever $u \in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ solves (5.17a), (5.17c), (5.17d), *with some $D \in \mathcal{C}_{\delta,m}$, some u_0 as in* (5.18) *and such that $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, and some $w \in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ satisfying*

$$\int_0^T\int_\Omega |\nabla w|^q \le K,$$

then

$$\int_\Omega u^p(\cdot,t) \le C \qquad \text{for all } t \in (0,T).$$

Proof. **i)** For $\widetilde{q} = 2$ we have $m - \frac{2}{\widetilde{q}} = m-1 = (\widetilde{q}-1)(m-1) + \frac{\widetilde{q}-2}{N}$ and because $m > 2 - \frac{1}{N}$, for every $\widetilde{q} \ge 2$ we have

$$\frac{d}{d\widetilde{q}}\left(m - \frac{2}{\widetilde{q}}\right) = \frac{2}{\widetilde{q}^2} \le 1 = 2 - \frac{1}{N} - m + m - 1 + \frac{1}{N} < m - 1 + \frac{1}{N} = \frac{d}{d\widetilde{q}}\left((\widetilde{q}-1)(m-1) + \frac{\widetilde{q}-2}{N}\right).$$

Therefore $m - \frac{2}{q} < (q-1)(m-1) + \frac{q-2}{N}$. Furthermore $q > 1 + \frac{N^2+1}{mN-N+1} = \frac{m-1+\frac{2}{N}+N}{m-1+\frac{1}{N}}$ implies that

$$(q-1)(m-1) + \frac{q-2}{N} = q\left(m-1+\frac{1}{N}\right) + 1 - m - \frac{2}{N} > N.$$

Hence it is possible to find $p > N$ such that $p > m - \frac{2}{q}$ and $p < (q-1)(m-1) + \frac{q-2}{N}$ and an application of Lemma 5.3.11 proves the statement.
ii) Since $x + \frac{1}{x} \ge 2$ for all $x > 0$, and since $q \ge 2$, we have

$$2 - \frac{2}{q} \le \frac{q}{2} + \frac{(q-2)(m+2)}{2N}$$

and hence $2(1-\frac{1}{q})(m-1) \le (m-1)(\frac{q}{2} + \frac{(q-2)(m+2)}{2N})$. The fact that $q > \frac{2N^2+2m^2+2m-4}{(m-1)(N+m+2)} = \frac{1}{(m-1)(N+m+2)}(2N^2 + (2m+4)(m-1)) = (\frac{2N^2}{m-1} + 2m + 4)\frac{1}{N+m+2}$ shows that $q(N+m+2) >$

$2m+4+\frac{2N^2}{m-1}$ and hence $N < \frac{m-1}{2N}(q(N+m+2)-2m-4) = \frac{m-1}{2N}(Nq+(q-2)(m+2)) = (m-1)(\frac{q}{2}+\frac{(q-2)(m+2)}{2N})$. Therefore we can choose $p > N$ such that

$$p < (m-1)\left(\frac{q}{2}+\frac{(q-2)(m+2)}{2N}\right) \qquad \text{and} \qquad p > 2\left(1-\frac{1}{q}\right)(m-1)$$

and apply Lemma 5.3.11 for this choice of p to obtain the assertion. □

The previous lemma requires a bound on some $\int_0^T\int_\Omega |\nabla w|^q$. Fortunately, this is exactly what we have prepared in Lemma 5.3.5, Lemma 5.3.7, Lemma 5.3.9, and Lemma 5.3.10.

Lemma 5.3.13. *Let $m > 1+\frac{N}{4}$ and $\delta > 0$. Then there is $p > N$ and for every $K > 0$ and $T > 0$ there is $C > 0$ such that every solution $(u,w) \in (C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T)))^2$ of (5.17) with initial data (u_0,w_0) as in (5.18) and with $\|u_0\|_{L^1(\Omega)} \le K$, $\|w_0\|_{W^{1,\infty}(\Omega)} \le K$ and any $D \in \mathcal{C}^+_{\delta,m}$ satisfies*

$$\int u^p(\cdot,t) \le C \qquad \textit{for every } t \in (0,T).$$

Proof. By the choice of m, from Lemma 5.3.9 we know that we can find $C > 0$ such that for any u_0, w_0 and D as above, any solution $(u,w) \in (C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T)))^2$ of (5.17) satisfies $0 \le w \le C$ in $\Omega\times(0,T)$. Lemma 5.3.10 therefore warrants that the desired conclusion results from a combination of Lemma 5.3.5 and Lemma 5.3.7 with Lemma 5.3.12 – provided that there are parameters p,q,r that simultaneously satisfy all conditions posed by these lemmata. This is what we ensure in the remainder of the proof:

Case $N=2$, $m \in (\frac{3}{2},2]$: We let $r = 4(m-1)$, $p=2$, and $q = 4m-2$. Then $r(1-\frac{1}{p}) = 4(m-1)(1-\frac{1}{2}) = 2(m-1) \le 2m-2 = 2m-3+\frac{2}{2}$, which enables us to invoke Lemma 5.3.5 i). Moreover, $m > \frac{3}{2}$ implies $4m-4 > 2$ and thus $r > p$, and we have $q = 4m-2 = 2+4m-4 \le 2+(2-\frac{2}{2})4(m-1) = N+(2-\frac{N}{p})r$. Therefore, Lemma 5.3.7 becomes applicable. Thanks to $q = 4m-2 \ge 4\cdot\frac{3}{2}-2 = 4 > 2 = N$ and thanks to $m \ge \frac{3}{2}$, hence $q = 4m-2 > 4\cdot 32-2 > \frac{7}{2} = 1+\frac{5}{2\cdot\frac{3}{2}-1} > 1+\frac{5}{2m-1} = 1+\frac{N^2+1}{Nm-N+1}$ holds true, facilitating the use of Lemma 5.3.12 i).

Case $N=3$, $m \in (\frac{7}{4},2]$: Here we let $r = p = 2m-\frac{4}{3}$. Then $p \ge 1$, $r \ge 1$, $p = 2m-\frac{4}{3} < 6m-6 = \frac{2N}{N-2}(m-1)$ and $r(1-\frac{1}{p}) = r-1 = 2m-\frac{7}{3} = 2m-3+\frac{2}{N}$, so that Lemma 5.3.5 i) can be used. Since $m > \frac{7}{4} = \frac{21}{12} > \frac{19}{12}$, we have that $12m^2-19m-\frac{8}{3} = 12(m-\frac{19}{12})m-\frac{8}{3} \ge 12(\frac{7}{4}-\frac{19}{12})\frac{7}{4}-\frac{8}{3} = \frac{7}{2}-\frac{8}{3} = \frac{21-16}{6} > 0$ and thus $3m+8 < 12m^2-16m+\frac{16}{3} = (4m-\frac{8}{3})(3m-2)$, i.e. $2p > \frac{3m+8}{3m-2}$. Furthermore, $p = 2m-\frac{4}{3} \le 4-\frac{4}{3} = \frac{8}{3} < 3$ and $p = 2m-\frac{4}{3} \ge \frac{7}{2}-\frac{4}{3} = \frac{21-8}{6} = \frac{13}{6} > \frac{3}{2}$, so that consequently, also $\frac{3p}{3-p} > 2p$ holds. We choose $q \in (\frac{3m+8}{3m-2}, 2p)$, thereby ensuring the applicability of Lemma 5.3.7.
Since finally $q > \frac{3m+8}{3m-2} = \frac{m+\frac{8}{3}}{m-\frac{2}{3}} = 1+\frac{\frac{10}{3}}{m-\frac{2}{3}} = 1+\frac{3+\frac{1}{3}}{m-1+\frac{1}{3}}$ and $q > \frac{3m+8}{3m-2} = 1+\frac{10}{3m-2} \ge 1+\frac{10}{6-2} = \frac{7}{2} > 3 \ge 2$ we may also draw on Lemma 5.3.12 i).

Case $N \ge 2$, $m \ge 2$, $m \ge 1+\frac{N}{4}$: Let $r := p := 2\frac{N+1}{N}(m-1)$. Then obviously $p = r > 1$. Moreover, $p \le \frac{2N}{N-2}(m-1)$ (because $\frac{2N}{N-2} > \frac{2+2N}{N}$ is equivalent to $2N^2 > 2N^2+2N-4N-4$ and hence to $0 > -2N-4$) and

$$\left(\frac{1}{m-1}-\frac{1}{p}\right)r = \frac{p}{m-1}-1 = 2\frac{N+1}{N}-1 = \frac{N+2}{N} \le 1+\frac{2}{N},$$

so that the conditions of Lemma 5.3.5 ii) are satisfied. We furthermore let $q := 2p = \frac{4(N+1)}{N}(m-1)$ and note that $p > \frac{N}{2}$, since $2\frac{N+1}{N}(m-1) > 2 \cdot \frac{N+1}{N}\frac{N}{4} = \frac{N+1}{2} > \frac{N}{2}$, and that $q \le 2p = 2r + N - N\frac{r}{p}$, that moreover either $p \ge N$ or $p < N$ and $q = 2p < \frac{Np}{N-p}$, because $p > \frac{N}{2}$, and therefore Lemma 5.3.7 is applicable. In order to see that these choices also make the use of Lemma 5.3.12 ii) viable, we first investigate the polynomial

$$P_N(m) := (2N+2)m^3 + (2N^2+N)m^2 + (-4N^2-11N-6)m - N^3 + 2N^2 + 8N + 4. \quad (5.39)$$

It is extremal whenever $P_N'(m) = (6N+6)m^2 + (4N^2+2N)m + (-4N^2-11N-6) = 0$, which is the case for exactly two real numbers that lie in $(-\infty, 2)$, because for $m \ge 2$ we have $P_N'(m) \ge (24N+24) + (8N^2+4N) + (-4N^2-11N-6) > 0$. We claim that $P_N(m) > 0$ for any $m > \max\{2, 1+\frac{N}{4}\}$ and for this compute $P_N(\max\{2, 1+\frac{N}{4}\})$:

$$\begin{aligned} P_N(2) &= 16N + 16 + 8N^2 + 4N - 8N^2 - 22N - 12 - N^3 + 2N^2 + 8N + 4 \\ &= -N^3 + 2N^2 + 6N + 8 \\ &= \begin{cases} -8+8+12+8 > 0, & N = 2, \\ -27+18+18+8 > 0, & N = 3, \\ -64+32+24+8 = 0, & N = 4, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &P_N\left(1+\frac{N}{4}\right) = \\ &\frac{1}{4^3}\Big((2N+2)(N+4)^3 + 4(2N^2+N)(N+4)^2 + 16(-4N^2-11N-6)(N+4) \\ &\qquad - 64N^3 + 128N^2 + 512N + 256\Big) \\ &= \frac{2}{4^3}N^2(5N+3)(N-4), \end{aligned}$$

which is nonnegative for $N \ge 4$. Since P_N is nonnegative in $\max\{2, 1+\frac{N}{4}\}$ and strictly increasing on $(2,\infty)$, we conclude that $P_N(m) > 0$ for any $m > \max\{2, 1+\frac{N}{4}\}$. Positivity of $P_N(m)$ is equivalent to

$$2(N+1)(m-1)^2(N+m+2) > N^3 + m^2N + mN - 2N$$

and hence

$$q = \frac{4(N+1)}{N}(m-1) > \frac{2N^2 + 2m^2 + 2m - 4}{(m-1)(N+m+2)}.$$

Furthermore by the fact that $p > \frac{N}{2}$, we also have $q > N$, and can invoke Lemma 5.3.12 ii). □

Remark 5.3.14. The condition $m > 1 + \frac{N}{4}$ in Lemma 5.3.13 is first and foremost employed to guarantee boundedness of w, that is, boundedness of v from below by a positive constant. Therefore it seems reasonable to ask whether it would be possible to soften the assumption on m if we already knew that w be bounded. It turns out that the condition $m > 2 - \frac{1}{N}$ of Lemma 5.3.12 is as strict as $m > 1 + \frac{N}{4} = \frac{3}{2}$ if $N = 2$, whereas for $N = 3$ we see that for any value of p choosing $r = \frac{2m-3+\frac{2}{3}}{1-\frac{1}{p}}$ is optimal (cf. Lemma 5.3.5 i)). Then $r \ge p$ (required by Lemma 5.3.7) entails $p \le 2m - \frac{4}{3}$. Other conditions on p are either obviously satisfied with this choice of $p = 2m - \frac{4}{3}$ (namely $p < 6m - 6$) or are essentially largeness conditions on p. Thus the

choice of p, r in the second case in the proof of Lemma 5.3.13 was optimal and we can follow the calculations there, which leaves us with two more necessary conditions: $p > \frac{3}{2}$ leading to $m > \frac{17}{12}$, and positivity of $12m^2 - 19m - \frac{8}{3}$, requiring $m > \frac{19}{24} + \frac{1}{8}\sqrt{\frac{163}{3}}$, which therefore remains as condition on m if one already supposes boundedness of w. For $N = 4$, $\frac{7}{4} \le m \le 2$, similarly choosing $p = r = 2m - \frac{3}{2}$ admits application of Lemma 5.3.5 i), whereas for invoking Lemma 5.3.7 and Lemma 5.3.12 we need some q between $1 + \frac{N^2+1}{Nm-N+1} = \frac{4m+14}{4m-3}$ and $2p = 4m-3$, which exists if $4m - 3 > \frac{4m+14}{4m-3}$, i.e. $m > \frac{1}{8}(7 + \sqrt{69})$. The conditions $r \ge p \ge \frac{N}{2}$ of Lemma 5.3.7 and $r(1-\frac{1}{p}) \le 2m - 3 + \frac{2}{N}$ (Lemma 5.3.5i)) imply $\frac{N}{2} - 1 \le 2m - 3 + \frac{2}{N}$ and hence $m \ge \frac{N}{4} - \frac{1}{N} + 1$ so that for $N \ge 5$ any choice of $m \le 2$ is impossible and for these dimensions we may restrict our attention to Lemma 5.3.7 and the second parts of Lemmata 5.3.5 and 5.3.12. The assumptions of Lemma 5.3.5 ii) combined with the condition $r \ge p$ of 5.3.7 imply that $p \le 2\frac{N+1}{N}(m-1)$. Therefore it is necessary that $2\frac{N+1}{N}(m-1) > \frac{N}{2}$, i.e. $m > \frac{(N+2)^2}{4(N+1)}$ – apart from this condition we are led to follow the case "$N \ge 2$, $m \ge 2$" of the proof of Lemma 5.3.13.
In conclusion: If boundedness of w were known a priori, the present proof would be applicable if

$$m > \begin{cases} \frac{3}{2}, & N = 2, \\ \frac{19}{24} + \frac{1}{8}\sqrt{\frac{163}{3}}, & N = 3, \\ \frac{1}{8}(7+\sqrt{69}), & N = 4, \\ \max\left\{\frac{(N+2)^2}{4(N+1)}, \text{largest root of } P_N \text{ from (5.39)}\right\}, & N \ge 5. \end{cases} \qquad \square$$

5.3.5 Global solutions. Proof of Theorem 5.1.1

Having completed the necessary preparations, we can now turn to the proof of existence of a global solution. In order to lay the groundwork for compactness arguments in Section 5.4, at the same time we derive a batch of estimates for the solutions.

Lemma 5.3.15. *Let* $\delta > 0$, $m > 1 + \frac{N}{4}$.
i) For any (u_0, v_0) *as in* (5.4) *and any* $D \in \mathcal{C}^+_{\delta,m}$ *there is a global classical solution* $(u, v) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^2$ *to* (5.5).
ii) Moreover, for every $T > 0$, $K > 0$ *there is* $C_T > 0$ *such that for every* $D \in \mathcal{C}_{\delta,m}$ *and* (u_0, v_0) *as in* (5.4) *with* $\|u_0\|_{L^{\max\{1,m-1\}}(\Omega)} \le K$, $\frac{1}{K} \le \|v_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{W^{1,\infty}(\Omega)} \le K$, *every solution* $(u, v) \in (C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T)))^2$ *to* (5.5) *satisfies*

$$\|u\|_{L^\infty(\Omega\times(0,T))} \le C_T \tag{5.40}$$

$$\|v\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \le C_T \tag{5.41}$$

$$\left\|\frac{1}{v}\nabla v\right\|_{L^\infty(\Omega\times(0,T))} \le C_T \tag{5.42}$$

$$\|D(u)\nabla u\|_{L^2(\Omega\times(0,T))} \le C_T \tag{5.43}$$

$$\left\|\nabla u^{m-1}\right\|_{L^2(\Omega\times(0,T))} \le C_T, \tag{5.44}$$

$$\int_0^T \int_\Omega D(u)u^{m-3}|\nabla u|^2 \le C_T, \tag{5.45}$$

$$\|v_t\|_{L^2((0,T);(W_0^{1,1}(\Omega))^*)} \le C_T, \tag{5.46}$$

$$\|u_t\|_{L^1((0,T);(W_0^{1,N+1}(\Omega))^*)} \le C_T\left(1 + \sup_{s\in[0,C_T]} D(s)\right). \tag{5.47}$$

Proof. According to Lemma 5.2.4, corresponding to (u_0, v_0) and D as in the hypothesis of the present lemma, there is a local solution $(u, v) \in (C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})))^2$. We now let $T \in (0, T_{max}] \cap (0, \infty)$ and $K > 0$. By $\mathcal{ID}_K$ let us abbreviate the set of initial data

$$\mathcal{ID}_K := \left\{ (u_0, v_0) \in C^\gamma(\overline{\Omega}) \times W^{1,\infty}(\Omega) \text{ for some } \gamma \in (0,1); \|u_0\|_{L^\infty(\Omega)} \le K, \|v_0\|_{W^{1,\infty}(\Omega)} \le K \right\}.$$

Lemma 5.3.13 provides us with $p > N$ and $c_1 > 0$ such that for every $D \in \mathcal{C}_{\delta,m}$ and every $(u_0, v_0) \in \mathcal{ID}_K$, every classical solution $(u, v) \in (C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)))^2$ of (5.5) satisfies

$$\|u\|_{L^\infty((0,T);L^p(\Omega))} \le c_1$$

and hence

$$\|\nabla v\|_{L^\infty(\Omega \times (0,T))} \le c_2 \quad \text{and} \quad \|w\|_{L^\infty(\Omega \times (0,T))} \le c_3$$

as well as

$$\|\nabla w\|_{L^\infty(\Omega \times (0,T))} \le c_4$$

with some c_2, c_3 and c_4 obtained from Lemma 5.2.1 ii), Lemma 5.3.9 and Lemma 5.3.10, respectively, (all of which, accordingly, do not depend on the precise value of $D \in \mathcal{C}_{\delta,m}$ or $(u_0, v_0) \in \mathcal{ID}_K$) and with w being defined as in (5.16). This asserts (5.41) and (5.42). An application of Lemma 5.3.11 for sufficiently large values of q and p then ascertains the existence of $c_5 > 0$ such that for all $D \in \mathcal{C}_{\delta,m}$ and all $(u_0, v_0) \in \mathcal{ID}_K$ any classical solution (u, v) of (5.5) satisfies

$$\|u \nabla w\|_{L^\infty((0,T);L^{N+3}(\Omega))} \le c_5,$$

again with w as in (5.16). Additionally taking into account Lemma 5.3.1, we can apply Lemma 5.2.1 v) with $f := u\nabla w$ so as to obtain $c_6 > 0$ such that for all $D \in \mathcal{C}_{\delta,m}$ and all $(u_0, v_0) \in \mathcal{ID}_K$ every classical solution (u, v) of (5.5) satisfies

$$\|u\|_{L^\infty(\Omega \times (0,T))} \le c_6,$$

which shows (5.40) and – in light of the extensibility criterion in (5.15) – also proves i). Given $D \in \mathcal{C}_{\delta,m}$ we let $\overline{D}(s) := \int_0^s D(\sigma) \mathrm{d}\sigma$ and $\overline{\overline{D}}(s) := \int_0^s \overline{D}(\sigma) \mathrm{d}\sigma$ for $s > 0$. Then for every $D \in \mathcal{C}_{\delta,m}$ and $(u_0, v_0) \in \mathcal{ID}_K$, any classical solution (u, v) of (5.5) obeys $u_t = \Delta \overline{D}(u) + \nabla \cdot (u \nabla w)$ with w as in (5.16), and testing this equation by $\overline{D}(u)$ we obtain

$$\int_0^T \int_\Omega (\overline{\overline{D}}(u))_t = -\int_0^T \int_\Omega |\nabla \overline{D}(u)|^2 - \int_0^T \int_\Omega u \nabla w \cdot \nabla \overline{D}(u),$$

which, by Young's inequality, turns into

$$\int_\Omega \overline{\overline{D}}(u(\cdot, T)) + \frac{1}{2} \int_0^T \int_\Omega |D(u) \nabla u|^2 \le \int_\Omega \overline{\overline{D}}(u_0) + \frac{1}{2} \int_0^T \int_\Omega u^2 |\nabla w|^2 \le |\Omega| \overline{\overline{D}}(K) + \frac{1}{2} |\Omega| T c_6^2 c_4^2,$$

due to nonnegativity of D proving (5.43). The existence of $c_7 > 0$, $c_8 > 0$ such that for any $D \in \mathcal{C}_{\delta,m}$ and any $(u_0, v_0) \in \mathcal{ID}_K$ every solution of (5.5) satisfies

$$\left\| \nabla u^{m-1} \right\|_{L^2(\Omega \times (0,T))} \le c_7, \qquad \int_0^T \int_\Omega D(u) u^{m-3} |\nabla u|^2 \le c_8$$

immediately results from Lemma 5.3.3, so that (5.44) and (5.45) have been shown. For every $\varphi \in C_0^\infty(\Omega)$ we have that any solution (u, v) of (5.5) for any $D \in \mathcal{C}_{\delta,m}$, $(u_0, v_0) \in \mathcal{ID}_K$ satisfies

$$\left| \int_\Omega v_t \varphi \right| = \left| -\int_\Omega \nabla \varphi \cdot \nabla v - \int_\Omega u v \varphi \right| \le c_2 \|\nabla \varphi\|_{L^1(\Omega)} + K c_6 \|\varphi\|_{L^1(\Omega)}$$

and we can conclude (5.46). We let $c_9 > 0$ be such that $\|\phi\|_{L^\infty(\Omega)} \le c_9$ for every $\phi \in W_0^{1,N+1}(\Omega)$ with $\|\phi\|_{W_0^{1,N+1}(\Omega)} \le 1$ and $c_{10} > 0$, $c_{11} > 0$ such that $\|\phi\|_{L^2(\Omega\times(0,T))} \le c_{10}$, $\|\phi\|_{L^1(\Omega\times(0,T))} \le c_{11}$ for every $\phi \in L^\infty((0,T); L^{N+1}(\Omega))$ with $\|\phi\|_{L^\infty((0,T);W_0^{1,N+1}(\Omega))} \le 1$. We denote $X := L^1((0,T);(W_0^{1,N+1}(\Omega))^*)$ and thus have $X^* = L^\infty((0,T); W_0^{1,N+1}(\Omega))$. Taking $\varphi \in X^*$ with $\|\varphi\|_{X^*} \le 1$, for any solution (u,v) of (5.5) for $D \in \mathcal{C}_{\delta,m}$ and $(u_0,v_0) \in \mathcal{ID}_K$ we have

$$\begin{aligned}
\frac{1}{m-1}\left|\int_0^T\int_\Omega (u^{m-1})_t\varphi\right| &= \left|\int_0^T\int_\Omega u^{m-2}u_t\varphi\right| \\
&\le \left|\int_0^T\int_\Omega u^{m-2}\varphi\nabla\cdot(D(u)\nabla u)\right| + \left|\int_0^T\int_\Omega u^{m-2}\varphi\nabla\cdot\Big(\frac{u}{v}\nabla v\Big)\right| \\
&\le |m-2|\left|\int_0^T\int_\Omega u^{m-3}\varphi D(u)|\nabla u|^2\right| + \left|\int_0^T\int_\Omega u^{m-2}D(u)\nabla u\cdot\nabla\varphi\right| \\
&\quad + |m-2|\left|\int_0^T\int_\Omega \frac{u^{m-2}\varphi}{v}\nabla v\cdot\nabla u\right| + \left|\int_0^T\int_\Omega \frac{u^{m-1}}{v}\nabla v\cdot\nabla\varphi\right| =: I_1+I_2+I_3+I_4,
\end{aligned}$$

where we can estimate $I_1 \le |m-2|c_8c_9$,

$$I_2 \le \frac12\int_0^T\int_\Omega u^{m-3}D(u)|\nabla u|^2 + \frac12\int_0^T\int_\Omega u^{m-1}D(u)|\nabla\varphi|^2 \le \frac{c_8}{2} + \frac12 c_6^{m-1}c_{10}^2\sup_{s\in[0,c_6]}D(s),$$

moreover

$$\begin{aligned}
I_3 &\le c_9c_4|m-2|\left\|u^{m-2}\nabla u\right\|_{L^1(\Omega\times(0,T))} \le c_9c_4|m-2|\sqrt{|\Omega|T}\left\|u^{m-2}\nabla u\right\|_{L^2(\Omega\times(0,T))} \\
&= \frac{c_9c_4|m-2|\sqrt{|\Omega|T}}{m-1}\left\|\nabla u^{m-1}\right\|_{L^2(\Omega\times(0,T))} \le \frac{c_9c_4c_7|m-2|\sqrt{|\Omega|T}}{m-1}
\end{aligned}$$

and $I_4 \le c_6^{m-1}c_4c_{11}$, so that finally

$$\left\|(u^{m-1})_t\right\|_{L^1((0,T);(W_0^{1,N+1}(\Omega))^*)} \le c_{12} + c_{13}\sup_{s\in[0,c_6]}D(s),$$

where $c_{12} := c_8c_9|m-2| + \frac{c_8}{2} + \frac{c_9c_4c_7|m-2|\sqrt{|\Omega|T}}{m-1} + c_6^{m-1}c_4c_{11}$ and $c_{13} := \frac12 c_6^{m-1}c_{10}^2$, holds for any solution (u,v) of (5.5) for any $(u_0,v_0)\in\mathcal{ID}_K$ and any $D\in\mathcal{C}_{\delta,m}$. ☐

Proof of Theorem 5.1.1. Lemma 5.3.15 i) together with (5.40) contains Theorem 5.1.1. ☐

5.4 Weak solutions in the degenerate case. Proof of Theorem 5.1.2

If the diffusion becomes degenerate at points where $u = 0$, we can no longer hope for classical solutions. Therefore we introduce the following definition of weak solutions that are – in line with our goal of finding solutions that do not blow up in finite time – locally bounded.

Definition 5.4.1. *Let $\delta > 0$, $m \ge 1$ and $D \in \mathcal{C}_{\delta,m}$ and define $\overline{D}(s) := \int_0^s D(\sigma)\mathrm{d}\sigma$ for $s\in[0,\infty)$. Moreover, let (u_0,v_0) be as in* (5.4)*. By a* locally bounded global weak solution *to* (5.5) *we mean a pair of functions $(u,v)\colon \Omega\times[0,\infty)\to\mathbb{R}^2$ such that*

$$u \in L^\infty_{loc}([0,\infty); L^\infty(\Omega))$$

$$\overline{D}(u) \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$$
$$v \in L^\infty_{loc}([0,\infty); W^{1,\infty}(\Omega))$$

and for every $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ *we have*

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla\overline{D}(u)\cdot\nabla\varphi + \int_0^\infty \int_\Omega \frac{u}{v}\nabla v\cdot\nabla\varphi \tag{5.48}$$

and

$$-\int_0^\infty \int_\Omega v\varphi - \int_\Omega v_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v\cdot\nabla\varphi - \int_0^\infty \int_\Omega uv\varphi. \tag{5.49}$$

Having prepared a lot of bounds on solutions to (5.5) for $D \in \mathcal{C}^+_{\delta,m}$ that are uniform in $D \in \mathcal{C}^+_{\delta,m}$ (Lemma 5.3.15), we approximate $D \in \mathcal{C}_{\delta,m}$ and find a limit of the corresponding solutions.

Proof of Theorem 5.1.2. Let $D \in \mathcal{C}_{\delta,m}$. For any $\varepsilon > 0$ we define $D_\varepsilon(s) := D(s+\varepsilon)$, $s \in [0,\infty)$, and note that, for any $\varepsilon > 0$, $D_\varepsilon \in \mathcal{C}^+_{\delta,m}$. We choose $(u_{0,\varepsilon}, v_{0,\varepsilon}) \in (C^1(\overline{\Omega}))^2$ such that $u_{0,\varepsilon} \to u_0$ and $v_{0,\varepsilon} \to v_0$ in $L^1(\Omega)$ as $\varepsilon \searrow 0$ and that there is $K > 0$ such that for all $\varepsilon \in (0,1)$ we have $\|u_{0,\varepsilon}\|_{L^{\max\{1,m-1\}}(\Omega)} + \|v_{0,\varepsilon}\|_{W^{1,\infty}(\Omega)} \le K$ and $\|v_{0,\varepsilon}\|_{L^\infty(\Omega)} > \frac{1}{K}$, and let $(u_\varepsilon, v_\varepsilon) \in (C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2$ denote a solution to

$$\begin{cases} u_{\varepsilon t} = \nabla\cdot(D_\varepsilon\nabla u_\varepsilon) - \nabla\cdot\left(\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon\right) & \text{in } \Omega\times(0,\infty), \\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon & \text{in } \Omega\times(0,\infty), \\ u_\varepsilon(\cdot,0) = u_{0,\varepsilon}, \quad v_\varepsilon(\cdot,0) = v_{0,\varepsilon} & \text{in } \Omega, \\ \partial_\nu u_\varepsilon\big|_{\partial\Omega} = 0 = \partial_\nu v_\varepsilon\big|_{\partial\Omega} & \text{in } (0,\infty), \end{cases} \tag{5.50}$$

which exists due to 5.3.15 i).
Let us define $\overline{D}_\varepsilon(s) := \int_0^s D_\varepsilon(\sigma)\mathrm{d}\sigma$, $s \in [0,\infty)$. We claim that for every $n \in \mathbb{N}$ there is a sequence $(\varepsilon_{n,k})_{k\in\mathbb{N}}$ such that $\varepsilon_{n,k} \to 0$ as $k \to \infty$ for any $n \in \mathbb{N}$, that for $n > 1$ the sequence $(\varepsilon_{n,k})_{k\in\mathbb{N}}$ is a subsequence of $(\varepsilon_{n-1,k})_{k\in\mathbb{N}}$ and that for any $n \in \mathbb{N}$

$$\begin{cases} u_{\varepsilon_{n,k}} & \text{converges a.e. in } \Omega\times(0,n) \text{ and in } L^1(\Omega\times(0,n)) \\ \overline{D}_{\varepsilon_{n,k}}(u_{\varepsilon_{n,k}}) & \text{converges weakly in } L^2((0,n); W_0^{1,2}(\Omega)) \\ v_{\varepsilon_{n,k}} & \text{converges uniformly in } \Omega\times(0,n) \\ \nabla v_{\varepsilon_{n,k}} & \text{converges weakly}^* \text{ in } L^\infty(\Omega\times(0,n);\mathbb{R}^N) \\ \frac{1}{v_{\varepsilon_{n,k}}}\nabla v_{\varepsilon_{n,k}} & \text{converges weakly}^* \text{ in } L^\infty(\Omega\times(0,n);\mathbb{R}^N) \end{cases} \tag{5.51}$$

as $k \to \infty$. For $n = 0$ we choose an arbitrary monotone sequence $(\varepsilon_{0,k})_{k\in\mathbb{N}} \subset (0,1)$ which converges to 0. Let $n \in \mathbb{N}$ and let us assume that some sequence $(\varepsilon_{n-1,k})_{k\in\mathbb{N}}$ with properties as in (5.51) is given. Then by Lemma 5.3.15 ii), more precisely, by (5.40), there is $c_1(n) > 0$ such that

$$\left\|u_{\varepsilon_{n-1,k}}\right\|_{L^\infty(\Omega\times(0,n))} \le c_1(n) \qquad \text{for all } k \in \mathbb{N}. \tag{5.52}$$

We abbreviate

$$d_n := \sup_{\varepsilon\in(0,1)} \sup_{0\le s\le c_1(n)} D_\varepsilon(s) \le \sup_{0\le s\le c_1(n)+1} D(s).$$

Then

$$\overline{D}_{\varepsilon_{n-1,k}}(u_{\varepsilon_{n-1,k}}(x,t)) \le \int_0^{c_1(n)} d_n = c_1(n)d_n \quad \text{ for all } (x,t) \in \Omega\times(0,n)$$

and combining this with (5.43), we find $c_2(n) > 0$ such that for all $k \in \mathbb{N}$

$$\left\|\overline{D}_{\varepsilon_{n-1,k}}(u_{\varepsilon_{n-1,k}})\right\|_{L^2((0,n);W^{1,2}(\Omega))} \le c_2(n).$$

Hence there is a subsequence $(\varepsilon^{(1)}_{n,k})_{k\in\mathbb{N}}$ of $(\varepsilon_{n-1,k})_{k\in\mathbb{N}}$ such that $(\overline{D}_{\varepsilon^{(1)}_{n,k}}(u_{\varepsilon^{(1)}_{n,k}}))_{k\in\mathbb{N}}$ is weakly convergent in $L^2((0,n);W^{1,2}(\Omega))$. Moreover, (5.44), (5.40) and (5.47) show that there is $c_3(n)$ such that for all $k \in \mathbb{N}$

$$\left\|u^{m-1}_{\varepsilon^{(1)}_{n,k}}\right\|_{L^2((0,n);W^{1,2}(\Omega))} \le c_3(n), \qquad \left\|\left(u^{m-1}_{\varepsilon^{(1)}_{n,k}}\right)_t\right\|_{L^1((0,n);(W_0^{1,N+1})^*)} \le c_3(n).$$

Since $W^{1,2}(\Omega) \overset{cpt}{\hookrightarrow} L^2(\Omega) \hookrightarrow (W_0^{1,N+1}(\Omega))^*$, we can invoke a version of the Aubin-Lions lemma ([86, Cor. 8.4]) to find a subsequence $(\varepsilon^{(2)}_{n,k})_{k\in\mathbb{N}}$ of $(\varepsilon^{(1)}_{n,k})_{k\in\mathbb{N}}$ such that $(u^{m-1}_{\varepsilon^{(2)}_{n,k}})_{k\in\mathbb{N}}$ is convergent in $L^2((0,n);L^2(\Omega))$, and a further subsequence $(\varepsilon^{(3)}_{n,k})_{k\in\mathbb{N}}$ of $(\varepsilon^{(2)}_{n,k})_{k\in\mathbb{N}}$ such that $(u^{m-1}_{\varepsilon^{(3)}_{n,k}})_{k\in\mathbb{N}}$ and thus, by continuity of $[0,\infty) \ni x \mapsto x^{\frac{1}{m-1}}$, also $(u_{\varepsilon^{(3)}_{n,k}})_{k\in\mathbb{N}}$ converge a.e. in $\Omega \times (0,n)$ as well as with respect to the norm of $L^1(\Omega \times (0,n))$ due to Lebesgue's dominated convergence theorem and the fact that the constant $c_1(n)$ is integrable over $\Omega \times (0,n)$.
Moreover, (5.41) and (5.46) ensure the existence of $c_4(n) > 0$ such that

$$\left\|v_{\varepsilon^{(3)}_{n,k}}\right\|_{L^\infty((0,n);W^{1,\infty}(\Omega))} \le c_4(n), \qquad \left\|\left(v_{\varepsilon^{(3)}_{n,k}}\right)_t\right\|_{L^2((0,n);(W_0^{1,1}(\Omega))^*)} \le c_4(n) \qquad \text{for all } k \in \mathbb{N}$$

and again due to $W^{1,\infty}(\Omega) \overset{cpt}{\hookrightarrow} C^0(\overline{\Omega}) \hookrightarrow (W_0^{1,1}(\Omega))^*$ and [86, Cor. 8.4] we find a subsequence $(\varepsilon^{(4)}_{n,k})_{k\in\mathbb{N}}$ of $(\varepsilon^{(3)}_{n,k})_{k\in\mathbb{N}}$ such that $(v_{\varepsilon^{(4)}_{n,k}})_{k\in\mathbb{N}}$ converges uniformly in $\Omega \times (0,n)$. Additionally, (5.41) produces another subsequence $(\varepsilon^{(5)}_{n,k})_{k\in\mathbb{N}}$ of $(\varepsilon^{(4)}_{n,k})_{k\in\mathbb{N}}$ such that $(\nabla v_{\varepsilon^{(5)}_{n,k}})_{k\in\mathbb{N}}$ converges weakly* in $L^\infty(\Omega \times (0,n))$. Finally, owing to the bound in (5.42), we can extract a further subsequence $(\varepsilon_{n,k})_{k\in\mathbb{N}}$ of $(\varepsilon^{(5)}_{n,k})_{k\in\mathbb{N}}$ such that also $\left(\frac{1}{v_{\varepsilon_{n,k}}}\nabla v_{\varepsilon_{n,k}}\right)_{k\in\mathbb{N}}$ is weakly* convergent in $L^\infty(\Omega \times (0,n))$.
We then use the diagonal sequence $(\widetilde{\varepsilon}_k)_{k\in\mathbb{N}} := (\varepsilon_{k,k})_{k\in\mathbb{N}}$ to find functions $u, v, z \colon \Omega \times [0,\infty) \to \mathbb{R}$ and $\zeta, \xi \colon \Omega \times [0,\infty) \to \mathbb{R}^N$ such that

$$u_{\widetilde{\varepsilon}_k} \to u \qquad \text{in } L^1_{loc}([0,\infty);L^1(\Omega)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{5.53}$$

$$v_{\widetilde{\varepsilon}_k} \to v \qquad \text{in } L^\infty_{loc}([0,\infty);C^0(\overline{\Omega})), \tag{5.54}$$

$$\overline{D}_{\widetilde{\varepsilon}_k}(u_{\widetilde{\varepsilon}_k}) \rightharpoonup z \qquad \text{in } L^2_{loc}([0,\infty);W^{1,2}(\Omega)), \tag{5.55}$$

$$\nabla v_{\widetilde{\varepsilon}_k} \overset{*}{\rightharpoonup} \zeta \qquad \text{in } L^\infty_{loc}([0,\infty);L^\infty(\Omega;\mathbb{R}^N)) \text{ and} \tag{5.56}$$

$$\frac{1}{v_{\widetilde{\varepsilon}_k}}\nabla v_{\widetilde{\varepsilon}_k} \overset{*}{\rightharpoonup} \xi \qquad \text{in } L^\infty_{loc}([0,\infty);L^\infty(\Omega;\mathbb{R}^N)) \tag{5.57}$$

as $k \to \infty$. Since $u_{\widetilde{\varepsilon}_k} + \widetilde{\varepsilon}_k \to u$ a.e. and $\overline{D}$ is continuous, also $\overline{D}_{\widetilde{\varepsilon}_k}(u_{\widetilde{\varepsilon}_k}) = \overline{D}(u_{\widetilde{\varepsilon}_k} + \widetilde{\varepsilon}_k) - \overline{D}(\widetilde{\varepsilon}_k) \to \overline{D}(u) - \overline{D}(0) = \overline{D}(u)$ a.e., and hence $z = \overline{D}(u)$. Also, (5.54) and (5.56) imply $\zeta = \nabla v$ and the combination of (5.54) and (5.57) shows that $\xi = \nabla \ln v$.
We let $\varphi \in C_0^\infty(\Omega \times [0,\infty))$. Then (5.50) entails that

$$-\int_0^\infty \int_\Omega u_{\widetilde{\varepsilon}_k}\varphi_t - \int_\Omega u_{0,\widetilde{\varepsilon}_k}\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla\overline{D}_{\widetilde{\varepsilon}_k}(u_{\widetilde{\varepsilon}_k}) \cdot \nabla\varphi + \int_0^\infty \int_\Omega \frac{u_{\widetilde{\varepsilon}_k}}{v_{\widetilde{\varepsilon}_k}}\nabla v_{\widetilde{\varepsilon}_k} \cdot \nabla\varphi$$

and
$$-\int_0^\infty \int_\Omega v_{\widetilde{\varepsilon}_k}\varphi - \int_\Omega v_{0,\widetilde{\varepsilon}_k}\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v_{\widetilde{\varepsilon}_k} \cdot \nabla\varphi - \int_0^\infty \int_\Omega u_{\widetilde{\varepsilon}_k} v_{\widetilde{\varepsilon}_k}\varphi,$$
so that passing to the limit as $k \to \infty$ in each of these integrals shows that (u, v) satisfies (5.48) and (5.49). That $u \in L^\infty_{loc}([0,\infty); L^\infty(\Omega))$ is also entailed by (5.53) and (5.52). Hence (u, v) is a locally bounded global weak solution to (5.5) in the sense of Definition 5.4.1. □

6 Long-term behaviour in a chemotaxis-fluid system with logistic source

6.1 Introduction

Bacteria and sand are different. Although both are heavier than water and will tend to sink if dispersed in it, bacteria may possess the ability to swim – and to direct their movement toward more favorable environmental conditions, i.e. for example toward higher concentration of oxygen, thus instigating the emergence of bioconvective patterns (see [82, Sec. 4.2]). Such behaviour can, e.g., be observed if colonies of *Bacillus subtilis* are suspended in a drop of water (see e.g. [45]), and models describing this phenomenon, that is, the model proposed by Tuval et al. in [101] and variants thereof, have received much attention from the mathematical community over the past few years.

Before we recall some of the progress made in the analysis of such models, let us briefly motivate the form of the system we want to investigate in the present chapter. In order to describe the interaction between bacteria, their fluid environment and oxygen (or another nutrient) contained therein, we introduce scalar-valued functions u and v standing for the concentration of bacteria and oxygen, respectively, and a vector-valued function U representing the velocity field of the surrounding water. The fluid motion is supposed to be governed by the incompressible Navier-Stokes equations

$$U_t + (U \cdot \nabla)U = \Delta U + \nabla P + u\nabla\Phi + f, \qquad \nabla \cdot U = 0$$

where we have allowed for an external force f (which nevertheless might best be thought of as being zero in the most prototypical case) and, more importantly, where buoyancy effects are included, which arise from density differences between fluid with and without bacteria, as mandated by the presence of a given gravitational potential Φ. P symbolizes the pressure of the fluid, another unknown quantity.

Oxygen is assumed to diffuse in the manner of linear diffusion, as described by the heat equation. It is moreover transported in the direction of the fluid flow and, finally, consumed with a rate proportional to the amount of bacteria present. Combining these effects, the resulting equation is the following:

$$v_t = \underbrace{\Delta v}_{\text{diffusion}} \underbrace{-uv}_{\text{consumption}} \underbrace{-U \cdot \nabla v}_{\text{transport}}.$$

The evolution of the bacterial concentration is also influenced by diffusion and transport along the velocity field of the fluid. The cells moreover steer their motion in the direction of the concentration gradient of oxygen, by means of chemotaxis. This gives rise to a contribution $-\chi\nabla \cdot (u\nabla v)$ to the time derivative of u, thus introducing cross-diffusive effects into the model, which lie at the core of the mathematical difficulties accompanying the analysis of chemotaxis systems like the famous Keller-Segel model ([34, 9]). Therein $\chi > 0$ is a parameter regulating the strength of the chemotactic attraction. In addition we want to allow for population growth to take place in the simplest conceivable manner, namely according to a logistic law, where we denote by κ the effective growth rate of the population and by μ a parameter controlling death by overcrowding. Evidently, each subdomain cannot sustain infinitely many bacteria, but has a finite carrying capacity only. Therefore, death effects taking place at high population densities cannot be neglected. Indeed, the subsequent analysis will strongly rely on positivity of μ, whereas κ, small whenever typical time-scales of bacterial reproduction are substantially deceeded such as in application contexts like bioconvection, may attain any nonnegative value. Accounting for these effects leads to the equation

$$u_t = \underbrace{\Delta u}_{\text{diffusion}} \underbrace{-\chi\nabla\cdot(u\nabla v)}_{\text{chemotaxis}} \underbrace{-U\cdot\nabla u}_{\text{transport}} \underbrace{+\kappa u - \mu u^2}_{\text{logistic growth}}.$$

With time starting at 0, spatially the whole scenario is to take place in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, which we want to think of as drop of water resting on a surface. Thus it is quite natural to assume that no fluid motion takes place on the surface of the drop, that is,

$$U = 0 \qquad \text{on } \partial\Omega,$$

and that no bacteria cross the boundary between the drop and its surroundings,

$$\partial_\nu u = 0 \qquad \text{on } \partial\Omega.$$

We will also assume that

$$\partial_\nu v = 0 \qquad \text{on } \partial\Omega,$$

that is, that no exchange of oxygen takes place between the fluid environment and its exterior. This assumption is less natural, at least for the part of the boundary that separates water and air, but so far has been employed in almost all papers dealing with chemotaxis fluid interaction from a mathematical viewpoint (exceptions being early existence results for weak solutions in 2-dimensional bounded domains [63], numerical experiments like in [16] and, most notably, a recent work by Braukhoff [12], where it was shown that in 2- or 3-dimensional convex bounded domains classical or weak solutions, respectively, exist for a chemotaxis-Navier-Stokes model with logistic source if the boundary condition for v is $\partial_\nu v = 1 - v$).

Thus, in total, the system to be considered here is

$$\begin{cases} u_t + U\cdot\nabla u = \Delta u - \chi\nabla\cdot(u\nabla v) + \kappa u - \mu u^2 & \text{in } \Omega\times(0,\infty)\\ v_t + U\cdot\nabla v = \Delta v - uv & \text{in } \Omega\times(0,\infty)\\ U_t + (U\cdot\nabla)U = \Delta U + \nabla P + u\nabla\Phi + f, \qquad \nabla\cdot U = 0 & \text{in } \Omega\times(0,\infty)\\ U = 0, \quad \partial_\nu u = \partial_\nu v = 0 & \text{in } \partial\Omega\times(0,\infty)\\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, \quad U(\cdot,0) = U_0 & \text{in } \Omega \end{cases} \tag{6.1}$$

for some initial data

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), \quad v_0 \in W^{1,q}(\Omega), \quad U_0 \in \mathcal{D}(A^\alpha)\\ u_0 > 0, \qquad v_0 > 0 \qquad \text{in } \overline{\Omega} \end{cases} \tag{6.2}$$

with $q > 3$, $\alpha \in (\frac{3}{4}, 1)$, where A denotes the realization of the Stokes operator under homogeneous Dirichlet boundary conditions in the solenoidal subspace $L^2_\sigma(\Omega)$ of $L^2(\Omega;\mathbb{R}^N)$.
If $\kappa = \mu = 0$ (and $f \equiv 0$), this model is an instance of the one for which the existence of global weak solutions in $\Omega = \mathbb{R}^2$ was shown in [62]. The existence of global classical solutions

in two-dimensional bounded convex domains was discovered in [114]. Global weak solutions on $\Omega = \mathbb{R}^2$ have been treated in [128] under weaker conditions on the initial data. In the setting of [114], the convergence of solutions to the stationary state was proven in [118]; its rate was given in [126]. Upon neglection of the nonlinear fluid term $(U \cdot \nabla)U$, that is upon consideration of Stokes flow instead of a Navier-Stokes governed fluid, global weak solutions can also be found in bounded three-dimensional domains ([114]). (The results of [114, 118] have been extended to non-convex domains in [40].) For the three-dimensional setting (of bounded convex domains) with full Navier-Stokes-fluid and large initial data only recently the existence of weak solutions has been demonstrated by Winkler ([114]). He furthermore showed that every eventual energy solution becomes smooth after some waiting time, and converges as $t \to \infty$ ([107]).
Other variants of the model that are commonly treated include nonlinear (porous medium type) diffusion of bacteria, where Δu is replaced by Δu^m for some $m > 1$ (see [96, 97, 20, 17, 127]), thereby improving chances for finding bounded solutions, or, exchanging $\chi\nabla \cdot (u\nabla v)$ for $\nabla \cdot (uS(u, v, x)\nabla v)$, more complex sensitivity functions S ([119, 104, 103, 37, 14]), which may be matrix-valued, thus introducing new mathematical challenges by destroying the natural energy structure of the system and, seen from the biological viewpoint, taking care of more complicated swimming behaviour of bacteria (cf. [19, 84, 124]).
In contrast to (6.1), in the classical Keller-Segel system the chemoattractant is produced by the bacteria themselves and not consumed (accounting for terms $+u-v$ in place of $-uv$ in the second equation of (6.1)), and models of Keller-Segel-Stokes type have also been considered ([104, 11]). In $\Omega = \mathbb{R}^3$, mild solutions to a system encompassing both mechanisms at the same time were proven to exist under a smallness condition on initial data ([46]).
Chemotaxis fluid models including logistic growth ($\kappa, \mu > 0$) have been treated in [102, 99, 98, 108, 12].
In [102], a result on the existence of weak solutions for (6.1) is given, and for the case of sufficiently nonlinear cell diffusion, attractors are considered. In a Keller-Segel-Navier-Stokes system with logistic source ($\mu > 0$, $\kappa \geq 0$) in two-dimensional bounded domains global classical solutions have been detected in [99], which furthermore converge to 0 if $\kappa = 0$. Under the assumptions of a Stokes fluid and sufficiently large μ (explicitly: $\mu > 23$), in [98] these results have been achieved for three-dimensional bounded domains as well. In [108], for $\mu > \frac{1}{4}\sqrt{\kappa_+}\chi$ in bounded convex domains $\Omega \subset \mathbb{R}^3$ generalized solutions are constructed, which then are shown to converge to the homogeneous steady state with respect to the topology of $L^1(\Omega) \times L^p(\Omega) \times L^2(\Omega)$ for $p \in [1, 6)$, if certain conditions on f are satisfied.
It is the main goal of the present chapter to achieve similar results for the consumption-chemotaxis-fluid model (6.1). Having to deal with a consumption instead of production term in the v-equation seems more beneficial for proving boundedness of solutions and encourages us to hope that the solutions remain bounded and thus exist globally without any further largeness condition on μ except positivity and that the convergence takes place with respect to stronger topologies than in [108]. This is indeed what we will prove. Moreover, we will shed light on asymptotic regularity properties of the solutions we are going to construct.

Let us state the main results in detail: Posing the condition

$$\begin{cases} f \in L^2((0,\infty); L^{\frac{6}{5}}(\Omega; \mathbb{R}^3)) \cap L^\infty(\Omega \times (0,\infty); \mathbb{R}^3) \cap C^{\beta, \frac{\beta}{2}}(\overline{\Omega} \times [0,\infty); \mathbb{R}^3), \\ \|f(\cdot, t)\|_{L^{\frac{3}{2}}(\Omega)} \to 0 \quad \text{as } t \to \infty \qquad \text{for some } \beta > 0 \end{cases} \tag{6.3}$$

on the external force on the fluid, we will first (re-)derive the following theorem on global existence of weak solutions:

Theorem 6.1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain and let $\chi, \kappa \geq 0$, $\mu > 0$. Let u_0, v_0, U_0 be as in* (6.2) *with some $q > 3$ and $\alpha \in (\frac{3}{4}, 1)$, let $\Phi \in C^{1+\beta}(\overline{\Omega})$ for some $\beta > 0$, and let f satisfy* (6.3) *for some $\beta \in (0,1)$. Then there is a weak solution (in the sense of Definition 6.2.16 below) to* (6.1)*, which can be approximated by a sequence of solutions $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$ to* (6.4) *in a pointwise manner (and moreover with respect to the topologies indicated in Proposition 6.2.17).*

(For weak solutions to (6.1) with $f \equiv 0$ see also [102, Thm. 4.1] or, for a setting with different boundary conditions, [12].) The solutions $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$ to the approximate system (6.4) that are mentioned in Theorem 6.1.1 (but do not appear in [102]) will serve as essential tool also in the proof of our second theorem, which is concerned with the asymptotic behaviour and eventual regularity of solutions.

Theorem 6.1.2. *Let the assumptions of Theorem 6.1.1 be satisfied. Then there are $T > 0$ and $\gamma \in (0,1)$ such that the solution (u, v, U) given by Theorem 6.1.1 satisfies*

$$u, v \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\overline{\Omega} \times [T, \infty)), \quad U \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\overline{\Omega} \times [T, \infty); \mathbb{R}^3).$$

Moreover,

$$u(\cdot, t) \to \frac{\kappa}{\mu}, \quad v(\cdot, t) \to 0, \quad U(\cdot, t) \to 0 \qquad \text{as } t \to \infty,$$

where the convergence takes place with respect to the norm of $C^1(\overline{\Omega})$ and $C^1(\overline{\Omega}; \mathbb{R}^3)$, respectively.

As to the proofs, we will first turn our attention to Theorem 6.1.1: In Section 6.2, namely, we will be concerned with solutions to the approximate problem (6.4) (see Lemma 6.2.1) and with the derivation of estimates that allow for compactness arguments in constructing solutions to (6.1) (Proposition 6.2.17). The foundation for the acquisition of these estimates will be an examination of the derivative of

$$\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\chi}{2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + K\chi \int_\Omega |U_\varepsilon|^2$$

for suitable $K > 0$ (see Lemma 6.2.10). In contrast to a system without logistic source terms in the equation for u, mass conservation of the bacteria is not guaranteed in (6.1). We begin Section 6.3 by finding a suitable substitute, and then, relying on this, prove convergence of $\int_t^{t+1} \int_\Omega v_\varepsilon$ and of $\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$ as $t \to \infty$ (Lemma 6.3.4 and Lemma 6.3.6, respectively). In Lemma 6.3.8 we derive a differential inequality for $\int_\Omega \frac{u_\varepsilon^p}{(\eta - v_\varepsilon)^\theta}$ for appropriate parameters η, θ, finally yielding L^p-bounds on n whenever the second solution component is small. Using these bounds, we then prove eventual Hölder regularity of U_ε (Lemma 6.3.13), v_ε (Lemma 6.3.14), and u_ε (Lemma 6.3.15), which can be transferred to u, v, U and turned into higher regularity (Lemma 6.3.17). For convergence as $t \to \infty$, we finally draw upon uniform Hölder bounds (Corollary 6.3.16) and the compact embedding $C^{1+\gamma, \frac{\gamma}{2}}(\overline{\Omega} \times [t, t+1]) \hookrightarrow C^{1,0}(\overline{\Omega} \times [t, t+1])$ as well as some of the properties collected during the course of Section 6.3; concerning n, for example, Lemma 6.3.2 (and thus, indirectly, Lemma 6.2.5) will once more be important.

Notation. We will refer to the partial derivative with respect to the last argument by $\frac{d}{dt} w$. The symbol $\overset{cpt}{\hookrightarrow}$ will be used to indicate compact embeddings. For vectors $V, W \in \mathbb{R}^3$ we let $V \otimes W$ denote the matrix $(V_i W_j)_{i,j=1,2,3}$. Finally, $\mathcal{P} \colon L^p(\Omega; \mathbb{R}^3) \to L^p_\sigma(\Omega)$ stands for the Helmholtz projection in $L^p(\Omega)$.

6.2 Existence of weak solutions

We will start by considering an approximate problem, namely

$$\begin{cases} u_{\varepsilon t} + U_\varepsilon \cdot \nabla u_\varepsilon = \Delta u_\varepsilon - \chi \nabla \cdot \left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \nabla v_\varepsilon \right) + \kappa u_\varepsilon - \mu u_\varepsilon^2 & (6.4a) \\ v_{\varepsilon t} + U_\varepsilon \cdot \nabla v_\varepsilon = \Delta v_\varepsilon - v_\varepsilon \frac{1}{\varepsilon} \ln(1+\varepsilon u_\varepsilon) & (6.4b) \\ U_{\varepsilon t} + (Y_\varepsilon U_\varepsilon \cdot \nabla) U_\varepsilon = \Delta U_\varepsilon + \nabla P_\varepsilon + u_\varepsilon \nabla \Phi + f(x,t) & (6.4c) \\ \partial_\nu u_\varepsilon\big|_{\partial\Omega} = \partial_\nu v_\varepsilon\big|_{\partial\Omega} = 0, \qquad U_\varepsilon\big|_{\partial\Omega} = 0 & (6.4d) \\ u_\varepsilon(\cdot,0) = u_0, \quad v_\varepsilon(\cdot,0) = v_0, \quad U_\varepsilon(\cdot,0) = U_0 & (6.4e) \end{cases}$$

where $Y_\varepsilon = (1+\varepsilon A)^{-1}$, and provide estimates for its solutions. In Proposition 6.2.17, these estimates will enable us to construct a solution to (6.1) by a limiting process. An approximation in this way was also employed in [114], [107], [108].

6.2.1 Local existence and basic properties

First, let us recall that locally these solutions actually exist. Because the reasoning is well-established (and not central to later parts of the chapter), we shall only briefly hint at the proofs, both here and in Lemma 6.2.12, where their global existence is indicated.

Lemma 6.2.1. *Let $q > 3$, $\alpha \in (\frac{3}{4}, 1)$, $\beta \in (0,1)$, $\kappa, \chi \geq 0$, $\mu > 0$, $\Phi \in C^{1+\beta}(\overline{\Omega})$ and f as in* (6.3)*, let u_0, v_0, U_0 satisfy* (6.2) *and let $\varepsilon > 0$. Then there are $T_{max,\varepsilon}$ and uniquely determined functions*

$$\begin{aligned} u_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})), \\ v_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})) \cap L^\infty((0, T_{max,\varepsilon}); W^{1,q}(\Omega)), \\ U_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{max,\varepsilon}); \mathbb{R}^3) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon}); \mathbb{R}^3) \end{aligned}$$

which together with some $P_\varepsilon \in C^{1,0}(\overline{\Omega} \times (0, T_{max,\varepsilon}))$ solve (6.4) *classically, and satisfy $T_{max,\varepsilon} = \infty$ or*

$$\limsup_{t \nearrow T_{max,\varepsilon}} \Big(\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \Big) = \infty. \tag{6.5}$$

Proof. The proof follows the reasoning of [114, Lemma 2.1] if some of the adaptions necessary in [114, Lemma 2.2] and [108, Lemma 3.1] are taken into account.
Banach's fixed point theorem applied in a closed ball in $L^\infty((0,T); C^0(\overline{\Omega}) \times W^{1,q}(\Omega) \times \mathcal{D}(A^\alpha))$ to a mapping whose fixed points are mild solutions to the system establishes the existence of such solutions on a time interval $[0,T)$, where T depends on the norms featured in (6.5) only. By an invocation of standard regularity theory for parabolic equations and the Stokes semigroup these solutions turn out to be classical solutions. □

For the rest of the chapter let us fix parameters $\chi, \kappa \geq 0$, $\mu > 0$, $\alpha \in (\frac{3}{4}, 1)$, $q > 3$ and $\beta \in (0,1)$, let f be as in (6.3), $\Phi \in C^{1+\beta}(\overline{\Omega})$, initial data u_0, v_0, U_0 satisfying (6.2) and, given $\varepsilon > 0$, let us denote by $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$ the corresponding solution to (6.4).

Lemma 6.2.2. *For any $\varepsilon > 0$, $x \in \Omega$ and $t \in (0, T_{max,\varepsilon})$ we have $u_\varepsilon(x,t) \geq 0$ and $v_\varepsilon(x,t) \geq 0$.*

Proof. An application of the parabolic comparison principle to the subsolution 0 of (6.4a) or (6.4b), respectively, immediately results in the claimed nonnegativity. □

Similarly, we obtain boundedness of v_ε.

Lemma 6.2.3. *There is $C > 0$ such that for any $\varepsilon > 0$ and for any $t \in (0, T_{max,\varepsilon})$,*

$$\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C \tag{6.6}$$

and, for any $\varepsilon > 0$, $t \mapsto \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$ is nonincreasing on $(0, T_{max,\varepsilon})$.

Proof. With $C = \|v_0\|_{L^\infty(\Omega)}$, both assertions are a consequence of the parabolic comparison principle. □

Another quantity whose boundedness, in this case in $L^2(\Omega)$, quickly results from the second equation is the gradient of v_ε:

Lemma 6.2.4. *There is $C > 0$ such that for any $\varepsilon > 0$,*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega |\nabla v_\varepsilon|^2 \le C. \tag{6.7}$$

Proof. Let $\varepsilon > 0$. Upon multiplication by v_ε, integration over Ω and integration by parts, (6.4b) results in

$$\int_\Omega v_\varepsilon v_{\varepsilon t} = -\int_\Omega |\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^2 \frac{1}{\varepsilon} \ln(1 + \varepsilon u_\varepsilon) + \int_\Omega v_\varepsilon U_\varepsilon \cdot \nabla v_\varepsilon \quad \text{on } (0, T_{max,\varepsilon}),$$

where the last term vanishes by $\int_\Omega v_\varepsilon U_\varepsilon \cdot \nabla v_\varepsilon = \frac{1}{2}\int_\Omega U_\varepsilon \cdot \nabla(v_\varepsilon^2) = -\frac{1}{2}\int_\Omega v_\varepsilon^2 \nabla \cdot U_\varepsilon = 0$ due to $\nabla \cdot U_\varepsilon(\cdot, t) = 0$ for all $t \in (0, T_{max,\varepsilon})$, and integration with respect to time entails

$$\frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot, t) + \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 \le \frac{1}{2}\int_\Omega v_{0,\varepsilon}^2 \qquad \text{for any } t \in (0, T_{max,\varepsilon}),$$

so that we may conclude (6.7) by taking $t \nearrow T_{max,\varepsilon}$. □

In contrast to the situation without source terms, we cannot hope for mass conservation in the first component. Nevertheless, the following inequality still holds:

Lemma 6.2.5. *There is $C > 0$ such that for any $\varepsilon > 0$,*

$$\int_\Omega u_\varepsilon(\cdot, t) \le C \quad \textit{for all } t \in (0, T_{max,\varepsilon}) \qquad \textit{and} \qquad \int_t^{t+\tau} \int_\Omega u_\varepsilon^2 \le C \quad \textit{for any } t \in (0, T_{max,\varepsilon} - \tau),$$

where $\tau := \min\{1, \frac{1}{2} T_{max,\varepsilon}\}$.

Proof. Integration of (6.4a) and an application of Hölder's inequality yield that for any $\varepsilon > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon = \kappa \int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^2 \le \kappa \int_\Omega u_\varepsilon - \frac{\mu}{|\Omega|}\left(\int_\Omega u_\varepsilon\right)^2 \qquad \text{on } (0, T_{max,\varepsilon}),$$

so that an ODE comparison argument gives boundedness of $(0, T_{max,\varepsilon}) \ni t \mapsto \int_\Omega u_\varepsilon(\cdot, t)$ and integration with respect to time allows to conclude the existence of a bound on $(0, T_{max,\varepsilon} - \tau) \ni t \mapsto \int_t^{t+\tau}\int_\Omega u_\varepsilon^2$ by means of the equality $\mu \int_t^{t+\tau}\int_\Omega u_\varepsilon^2 = \kappa \int_t^{t+\tau}\int_\Omega u_\varepsilon + \int_\Omega u_\varepsilon(\cdot, t) - \int_\Omega u_\varepsilon(t + \tau)$. □

6.2.2 A priori estimates implied by an energy type inequality

We want to derive a (quasi-)energy inequality for the function

$$\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\chi}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + K\chi \int_\Omega |U_\varepsilon|^2. \tag{6.8}$$

As preparation, we first deal with the derivatives of the summands separately:

Lemma 6.2.6. *There is $C > 0$ such that for any $\varepsilon > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\mu}{2}\int_\Omega u_\varepsilon^2 \ln u_\varepsilon + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le \chi \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{1+\varepsilon u_\varepsilon} + C \qquad \text{on } (0, T_{max,\varepsilon}).$$

Proof. First we observe that $s \mapsto \kappa s - \mu s^2, s \in [0,\infty)$, and $s \mapsto (\kappa s - \frac{\mu}{2}s^2)\ln s, s \in (0,\infty)$, are bounded from above by some constant C_1. Using these estimates and (6.4a), from integration by parts we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon \ln u_\varepsilon &= \int_\Omega u_{\varepsilon t}\ln u_\varepsilon + \int_\Omega u_{\varepsilon t} \\
&= \int_\Omega \Delta u_\varepsilon (\ln u_\varepsilon) - \chi \int_\Omega \ln u_\varepsilon \nabla \cdot \left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon\right) - \int_\Omega U_\varepsilon \cdot \nabla u_\varepsilon \ln u_\varepsilon \\
&\quad + \kappa \int_\Omega u_\varepsilon \ln u_\varepsilon - \mu \int_\Omega u_\varepsilon^2 \ln u_\varepsilon + \kappa \int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^2 \\
&\le - \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \chi \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{\mu}{2}\int_\Omega u_\varepsilon^2 \ln u_\varepsilon + 2C_1 \qquad \text{on } (0, T_{max,\varepsilon})
\end{aligned}$$

for any $\varepsilon > 0$, so that the claim results with $C = 2C_1$. □

In the next lemma we will collect statements that will enable us to deal with terms arising from differentiation of $\int_\Omega |\nabla \sqrt{v_\varepsilon}|^2$. In particular, it is this lemma that will render any convexity condition on the domain unnecessary. The proofs are either contained in or adapted from the articles [68, 38, 114]. The first use of this strategy for removal of the convexity condition in the context of chemotaxis-fluid models can be found in [40].

Lemma 6.2.7. *i) Let $\psi \in C^2(\overline{\Omega})$ satisfy $\partial_\nu \psi = 0$ on $\partial\Omega$. Then*

$$\partial_\nu |\nabla \psi|^2 \le \mathfrak{K} |\nabla \psi|^2,$$

where $\mathfrak{K}$ is an upper bound on the curvature of $\partial\Omega$.
ii) Furthermore, for any $\eta > 0$ there is $C(\eta) > 0$ such that every $\psi \in C^2(\overline{\Omega})$ with $\partial_\nu\psi = 0$ on $\partial\Omega$ fulfils

$$\|\nabla \psi\|_{L^2(\partial\Omega)} \le \eta \|\Delta\psi\|_{L^2(\Omega)} + C(\eta)\|\psi\|_{L^2(\Omega)}.$$

iii) For any positive $\psi \in C^2(\overline{\Omega})$

$$\left\|\Delta\sqrt{\psi}\right\|_{L^2(\Omega)} \le \frac{1}{2}\left\|\sqrt{\psi}\Delta \ln \psi\right\|_{L^2(\Omega)} + \frac{1}{4}\left\|\psi^{-\frac{3}{2}}|\nabla\psi|^2\right\|_{L^2(\Omega)}. \tag{6.9}$$

iv) For any positive $\psi \in C^2(\overline{\Omega})$ with $\partial_\nu\psi = 0$ on $\partial\Omega$ we have

$$-2\int_\Omega \frac{|\Delta\psi|^2}{\psi} + \int_\Omega \frac{|\nabla\psi|^2\Delta\psi}{\psi^2} = -2\int_\Omega \psi |D^2 \ln \psi|^2 + \int_{\partial\Omega} \frac{1}{\psi}\partial_\nu |\nabla\psi|^2.$$

v) There is $k > 0$ such that for all positive $\psi \in C^2(\overline{\Omega})$ with $\partial_\nu \psi = 0$ on $\partial\Omega$ the inequality

$$\int_\Omega \psi |D^2 \ln \psi|^2 \ge k \int_\Omega \frac{|\nabla\psi|^4}{\psi^3}$$

holds.
vi) There are $C > 0$ and $k > 0$ such that every positive $\psi \in C^2(\overline{\Omega})$ fulfilling $\partial_\nu \psi = 0$ on $\partial\Omega$ satisfies

$$-2\int_\Omega \frac{|\Delta\psi|^2}{\psi} + \int_\Omega \frac{|\nabla\psi|^2\Delta\psi}{\psi^2} \le -k\int_\Omega \psi|D^2\ln\psi|^2 - k\int_\Omega \frac{|\nabla\psi|^4}{\psi^3} + C\int_\Omega \psi. \tag{6.10}$$

Proof. i) This is [68, Lemma 4.10].
ii) Let us fix $r \in (0, \frac{1}{2})$. Thanks to the boundedness of the trace operator $tr\colon W^{r+\frac{1}{2},2}(\Omega) \to W^{r,2}(\partial\Omega)$ (cf. [29, Thm. 4.24]) and the embedding $W^{r,2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ (see [29, Prop. 4.22(ii)]) there is $k_1 > 0$ such that $\|\phi\|_{L^2(\partial\Omega)} \le k_1 \|\phi\|_{W^{r+\frac{1}{2},2}(\Omega)}$ for all $\phi \in W^{r+\frac{1}{2},2}(\Omega)$. If we let $\theta := \frac{3}{4} + \frac{r}{2}$, the interpolation inequality [60, Prop. 2.3] guarantees the existence of $k_2 > 0$ such that $\|\phi\|_{W^{r+\frac{3}{2},2}(\Omega)} \le k_2 \|\phi\|_{W^{2,2}(\Omega)}^\theta \|\phi\|_{L^2(\Omega)}^{1-\theta}$ for all $\phi \in W^{2,2}(\Omega)$. Furthermore, according to e.g. [23, Thm. 19.1], there is $k_3 > 0$ such that $\|\phi\|_{W^{2,2}(\Omega)} \le k_3 \|\Delta\phi\|_{L^2(\Omega)} + k_3 \|\phi\|_{L^2(\Omega)}$ for all $\phi \in W^{2,2}(\Omega)$ with $\partial_\nu\phi = 0$ on $\partial\Omega$. Moreover, for any $\eta > 0$, Young's inequality provides us with $k_4 = k_4(\eta)$ such that for any $a, b \in [0,\infty)$ we have $a^\theta b^{1-\theta} \le \frac{\eta}{k_1k_2k_3} a + k_4(\eta) b$, since the choice of r implies $\theta \in (0,1)$. With these constants, for any $\psi \in W^{2,2}(\Omega)$ satisfying $\partial_\nu\psi = 0$ on $\partial\Omega$ we obtain

$$\begin{aligned}
\|\nabla\psi\|_{L^2(\partial\Omega)} \le& k_1 \|\nabla\psi\|_{W^{r+\frac{1}{2},2}(\Omega)} \le k_1 \|\psi\|_{W^{r+\frac{3}{2},2}(\Omega)} \le k_1k_2 \|\psi\|_{W^{2,2}(\Omega)}^\theta \|\psi\|_{L^2(\Omega)}^{1-\theta}\\
\le& k_1k_2k_3 \|\Delta\psi\|_{L^2(\Omega)}^\theta \|\psi\|_{L^2(\Omega)}^{1-\theta} + k_1k_2k_3 \|\psi\|_{L^2(\Omega)}^{\theta+1-\theta}\\
\le& \eta \|\Delta\psi\|_{L^2(\Omega)} + (k_1k_2k_3 + k_4(\eta)) \|\psi\|_{L^2(\Omega)}.
\end{aligned}$$

iii) Let $\psi \in C^2(\overline{\Omega})$ be positive. The pointwise equalities

$$\Delta\sqrt{\psi} = \nabla\cdot\left(\frac{1}{2\sqrt{\psi}}\nabla\psi\right) = \frac{\Delta\psi}{2\sqrt{\psi}} + \frac{1}{2}\nabla\psi\cdot\nabla\left(\psi^{-\frac{1}{2}}\right) = \frac{\Delta\psi}{2\sqrt{\psi}} - \frac{|\nabla\psi|^2}{4\psi^{\frac{3}{2}}}$$

and

$$\Delta\ln\psi = \nabla\cdot(\nabla\ln\psi) = \nabla\cdot\left(\frac{\nabla\psi}{\psi}\right) = \frac{\Delta\psi}{\psi} - \frac{|\nabla\psi|^2}{\psi^2}$$

immediately entail

$$\Delta\sqrt{\psi} = \frac{1}{2}\sqrt{\psi}\Delta\ln\psi + \frac{1}{4}\frac{|\nabla\psi|^2}{\psi^{\frac{3}{2}}}$$

and thus (6.9).
iv) Being a special case of assertions from [18], this can be found as Lemma 4.3.4 i) in Chapter 4.
v) This was proven as [114, Lemma 3.3], cf Lemma 4.3.5.
vi) Let $\eta > 0$. Part i) and Young's inequality in combination with ii) and iii), respectively, can be employed to yield $C > 0$ such that

$$\int_{\partial\Omega} \frac{1}{\psi}\partial_\nu|\nabla\psi|^2 \le \mathfrak{K}\int_{\partial\Omega}\frac{|\nabla\psi|^2}{\psi} = 4\mathfrak{K}\left\|\nabla\sqrt{\psi}\right\|_{L^2(\partial\Omega)}^2 \le \eta\left\|\Delta\sqrt{\psi}\right\|_{L^2(\Omega)}^2 + C\left\|\sqrt{\psi}\right\|_{L^2(\Omega)}^2$$

$$\leq \eta \left\| \sqrt{\psi} \Delta \ln \psi \right\|_{L^2(\Omega)}^2 + \eta \left\| \psi^{-\frac{3}{2}} |\nabla \psi|^2 \right\|_{L^2(\Omega)}^2 + C \left\| \sqrt{\psi} \right\|_{L^2(\Omega)}^2$$

for all positive $\psi \in C^2(\overline{\Omega})$ satisfying $\partial_\nu \psi\big|_{\partial\Omega} = 0$. Thus, for any such ψ,

$$\int_{\partial\Omega} \frac{1}{\psi} \partial_\nu |\nabla \psi|^2 \leq \eta \int_\Omega \psi |\Delta \ln \psi|^2 + \eta \int_\Omega \frac{|\nabla \psi|^4}{\psi^3} + C \int_\Omega \psi.$$

Taking into account iv) and v), we readily obtain (6.10). □

We can take these estimates to their use in the next proof, which is concerned with the derivatives of the second summand in (6.8).

Lemma 6.2.8. *There are $K > 0, C > 0$ and $k > 0$ such that for every $\varepsilon > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + k \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + k \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq C + K \int_\Omega |\nabla U_\varepsilon|^2 - 2 \int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla u_\varepsilon}{1 + \varepsilon u_\varepsilon}$$

holds true on $(0, T_{max,\varepsilon})$.

Proof. We begin by computing $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}$: For any $\varepsilon > 0$, on $(0, T_{max,\varepsilon})$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} &= 2 \int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla v_{\varepsilon t}}{v_\varepsilon} - \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_{\varepsilon t} \\
&= -2 \int_\Omega \frac{\Delta v_\varepsilon v_{\varepsilon t}}{v_\varepsilon} + 2 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_{\varepsilon t} - \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_{\varepsilon t} \\
&= -2 \int_\Omega \frac{\Delta v_\varepsilon v_{\varepsilon t}}{v_\varepsilon} + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_{\varepsilon t} \\
&= -2 \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + 2 \int_\Omega \frac{\Delta v_\varepsilon \frac{1}{\varepsilon} v_\varepsilon \ln(1 + \varepsilon u_\varepsilon)}{v_\varepsilon} + 2 \int_\Omega \frac{\Delta v_\varepsilon}{v_\varepsilon} U_\varepsilon \cdot \nabla v_\varepsilon \\
&+ \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \Delta v_\varepsilon - \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_\varepsilon \frac{1}{\varepsilon} \ln(1 + \varepsilon u_\varepsilon) - \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} U_\varepsilon \cdot \nabla v_\varepsilon. \qquad (6.11)
\end{aligned}$$

From Lemma 6.2.7 vi), we obtain $k_1 > 0, k_2 > 0$ such that for any $\varepsilon > 0$ we may estimate

$$-2 \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \Delta v_\varepsilon \leq -k_1 \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 - k_1 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + k_2 \int_\Omega v_\varepsilon \quad \text{on } (0, T_{max,\varepsilon}). \qquad (6.12)$$

As to the terms containing U_ε, we note that for all $\varepsilon > 0$,

$$2 \int_\Omega \frac{\Delta v_\varepsilon}{v_\varepsilon} (U_\varepsilon \cdot \nabla v_\varepsilon) = 2 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} U_\varepsilon \cdot \nabla v_\varepsilon - 2 \int_\Omega \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot (\nabla U_\varepsilon \nabla v_\varepsilon) - 2 \int_\Omega \frac{1}{v_\varepsilon} U_\varepsilon \cdot D^2 v_\varepsilon \nabla v_\varepsilon,$$

and

$$\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} U_\varepsilon \cdot \nabla v_\varepsilon = -\int_\Omega \nabla \left(\frac{1}{v_\varepsilon} \right) \cdot U_\varepsilon |\nabla v_\varepsilon|^2 = \int_\Omega \frac{1}{v_\varepsilon} U_\varepsilon \cdot \nabla |\nabla v_\varepsilon|^2 = 2 \int_\Omega \frac{1}{v_\varepsilon} U_\varepsilon \cdot D^2 v_\varepsilon \nabla v_\varepsilon$$

hold on $(0, T_{max,\varepsilon})$, so that for any $\varepsilon > 0$,

$$2 \int_\Omega \frac{\Delta v_\varepsilon}{v_\varepsilon} U_\varepsilon \cdot \nabla v_\varepsilon - \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} U_\varepsilon \cdot \nabla v_\varepsilon$$

$$\begin{aligned}
&= \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} U_\varepsilon \cdot \nabla v_\varepsilon - 2\int_\Omega \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot (\nabla U_\varepsilon \nabla v_\varepsilon) - 2\int_\Omega \frac{1}{v_\varepsilon} U_\varepsilon \cdot D^2 v_\varepsilon \nabla v_\varepsilon \\
&= -2\int_\Omega \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot (\nabla U_\varepsilon \nabla v_\varepsilon) \quad \text{ on } (0, T_{max,\varepsilon}), \qquad (6.13)
\end{aligned}$$

where we can estimate

$$2\int_\Omega \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot (\nabla U_\varepsilon \nabla v_\varepsilon) \le \frac{k_1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + k_3 \int_\Omega v_\varepsilon |\nabla U_\varepsilon|^2 \quad \text{ on } (0, T_{max,\varepsilon}) \qquad (6.14)$$

with some $k_3 > 0$ courtesy of Young's inequality. Moreover,

$$-\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_\varepsilon \frac{1}{\varepsilon} \ln(1 + \varepsilon u_\varepsilon) \le 0 \qquad \text{ on } (0, T_{max,\varepsilon}) \text{ and for all } \varepsilon > 0 \qquad (6.15)$$

and an integration by parts shows

$$2\int_\Omega \frac{\Delta v_\varepsilon \frac{1}{\varepsilon} v_\varepsilon \ln(1 + \varepsilon u_\varepsilon)}{v_\varepsilon} = -2\int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla u_\varepsilon}{1 + \varepsilon u_\varepsilon} \quad \text{ on } (0, T_{max,\varepsilon}) \text{ for all } \varepsilon > 0, \qquad (6.16)$$

so that, for any $\varepsilon > 0$, using (6.12), (6.13), (6.14), (6.15), (6.16) turns (6.11) into

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + k_1 \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{k_1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le k_2 \int_\Omega v_\varepsilon + k_3 \int_\Omega v_\varepsilon |\nabla U_\varepsilon|^2 - 2\int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla u_\varepsilon}{1 + \varepsilon u_\varepsilon}$$

on $(0, T_{max,\varepsilon})$, and finally inserting the uniform bound on v_ε provided by Lemma 6.2.3 gives the assertion. □

Finally, we turn our attention to the last term in (6.8).

Lemma 6.2.9. *i) There is $C > 0$ such that for any $\zeta \in \mathbb{R}$ and any $\varepsilon > 0$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |U_\varepsilon|^2 + \int_\Omega |\nabla U_\varepsilon|^2 \le C\int_\Omega (u_\varepsilon - \zeta)^2 + C\left(\int_\Omega |f|^{\frac{6}{5}}\right)^{\frac{5}{3}} \qquad \text{on } (0, T_{max,\varepsilon}). \qquad (6.17)$$

ii) Moreover, for any $\eta > 0$ there is $C_\eta > 0$ such that for any $\varepsilon > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |U_\varepsilon|^2 + \int_\Omega |\nabla U_\varepsilon|^2 \le \eta \int_\Omega u_\varepsilon^2 \ln u_\varepsilon + C_\eta \left(\int_\Omega |f|^{\frac{6}{5}}\right)^{\frac{5}{3}} + C_\eta \qquad \text{on } (0, T_{max,\varepsilon}). \qquad (6.18)$$

Proof. If for any $\varepsilon > 0$ we multiply (6.4c) by U_ε, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |U_\varepsilon|^2 &= \int_\Omega U_\varepsilon \cdot U_{\varepsilon t} \\
&= \int_\Omega U_\varepsilon \cdot \Delta U_\varepsilon + \int_\Omega U_\varepsilon \cdot \nabla P - \int_\Omega (Y_\varepsilon U_\varepsilon \cdot \nabla) U_\varepsilon \cdot U_\varepsilon + \int_\Omega u_\varepsilon \nabla \Phi \cdot U_\varepsilon + \int_\Omega U_\varepsilon \cdot f \\
&= -\int_\Omega |\nabla U_\varepsilon|^2 + \int_\Omega u_\varepsilon \nabla \Phi \cdot U_\varepsilon + \int_\Omega U_\varepsilon \cdot f, \quad \text{ on } (0, T_{max,\varepsilon}), \qquad (6.19)
\end{aligned}$$

where we have used that $\nabla \cdot U_\varepsilon = 0$, so that for any $\varepsilon > 0$ on $(0, T_{max,\varepsilon})$, we have

$$\int_\Omega (Y_\varepsilon U_\varepsilon \cdot \nabla) U_\varepsilon \cdot U_\varepsilon = -\int_\Omega \nabla \cdot (Y_\varepsilon U_\varepsilon)|U_\varepsilon|^2 - \frac{1}{2}\int_\Omega Y_\varepsilon U_\varepsilon \cdot \nabla |U_\varepsilon|^2 = \frac{1}{2}\int_\Omega (\nabla \cdot Y_\varepsilon U_\varepsilon)|U_\varepsilon|^2 = 0.$$

That U_ε is divergence-free also shows $\int_\Omega u_\varepsilon(\cdot,t)\nabla\Phi\cdot U_\varepsilon(\cdot,t) = \int_\Omega (u_\varepsilon(\cdot,t)-\zeta)\,\nabla\Phi\cdot U_\varepsilon(\cdot,t)$ for any $t\in(0,T_{max,\varepsilon})$. Young's inequality in combination with Poincaré's inequality and the boundedness of $\nabla\Phi$ enables us to find $k_1>0$ such that

$$\int_\Omega |(u_\varepsilon(\cdot,t)-\zeta)\nabla\Phi\cdot U_\varepsilon(\cdot,t)| \le k_1\int_\Omega (u_\varepsilon(\cdot,t)-\zeta)^2 + \frac{1}{4}\int_\Omega |\nabla U_\varepsilon(\cdot,t)|^2 \tag{6.20}$$

holds for any $t\in(0,T_{max,\varepsilon})$ for any $\varepsilon>0$.
From the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ we can obtain a constant $k_2>0$ such that

$$\|\phi\|_{L^6(\Omega)} \le k_2\,\|\nabla\phi\|_{L^2(\Omega)} \quad \text{for all } \phi\in W_0^{1,2}(\Omega)$$

and hence Hölder's and Young's inequalities allow us to estimate

$$\begin{aligned}\int_\Omega U_\varepsilon\cdot f &\le \left(\int_\Omega |U_\varepsilon|^6\right)^{\frac16}\left(\int_\Omega |f|^{\frac65}\right)^{\frac56} \le k_2\,\|\nabla U_\varepsilon\|_{L^2(\Omega)}\left(\int_\Omega |f|^{\frac65}\right)^{\frac56}\\ &\le \frac14\int_\Omega |\nabla U_\varepsilon|^2 + k_3\left(\int_\Omega |f|^{\frac65}\right)^{\frac53} \quad \text{on } (0,T_{max,\varepsilon}) \text{ for all } \varepsilon>0\end{aligned} \tag{6.21}$$

with some $k_3>0$. Adding (6.19), (6.21) and (6.20) results in (6.17). If we furthermore use that $\int_\Omega \phi^2 \le a\int_\Omega \phi^2\ln\phi + |\Omega|\mathrm{e}^{\frac1a}$ for any positive function ϕ and any $a>0$, for each $\eta>0$ we can find $C_\eta>0$ such that

$$\int_\Omega |u_\varepsilon U_\varepsilon\cdot\nabla\Phi| \le \eta\int_\Omega u_\varepsilon^2\ln u_\varepsilon + \frac14\int_\Omega |\nabla U_\varepsilon|^2 + C_\eta \quad \text{on } (0,T_{max,\varepsilon}) \tag{6.22}$$

holds for any $\varepsilon>0$, thus establishing (6.18). □

If we now amalgamate Lemma 6.2.6, Lemma 6.2.8 and Lemma 6.2.9, we end up with

Lemma 6.2.10. *There are $C>0$, $k_0>0$, and $K>0$ such that*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega u_\varepsilon\ln u_\varepsilon + \frac{\chi}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + K\chi\int_\Omega |U_\varepsilon|^2\right]\\ &\quad + \frac{\mu}{4}\int_\Omega u_\varepsilon^2\ln u_\varepsilon + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + k_0\int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 + k_0\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + k_0\int_\Omega |\nabla U_\varepsilon|^2\\ &\le C\left(\int_\Omega |f|^{\frac65}\right)^{\frac53} + C\end{aligned}$$

on $(0,T_{max,\varepsilon})$ for all $\varepsilon>0$.

Proof. We fix K and k as in Lemma 6.2.8, apply Lemma 6.2.9 with $\eta=\frac{\mu}{4K\chi}$ and add the inequality given by Lemma 6.2.6 to the $\frac{\chi}{2}$-multiple of that from Lemma 6.2.8 and $K\chi$ times the inequality from Lemma 6.2.9 ii). With $k_0:=\frac{\chi}{2}\min\{K,k\}$, Lemma 6.2.10 results immediately. □

We collect the bounds this quasi-energy inequality gives rise to:

Lemma 6.2.11. *There is $C>0$ such that for any $\varepsilon>0$ the estimates*

$$\int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) + \int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} + \int_\Omega |U_\varepsilon(\cdot,t)|^2 \le C \quad \textit{hold for all } t\in(0,T_{max,\varepsilon}) \tag{6.23}$$

and such that we may estimate

$$\int_t^{t+\tau}\int_\Omega u_\varepsilon^2 \ln u_\varepsilon + \int_t^{t+\tau}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_t^{t+\tau}\int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \le C \tag{6.24}$$

$$\int_t^{t+\tau}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_t^{t+\tau}\int_\Omega |\nabla U_\varepsilon|^2 \le C \tag{6.25}$$

$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^{\frac{4}{3}} + \int_\Omega |\nabla v_\varepsilon|^2 + \int_t^{t+\tau}\int_\Omega |\nabla v_\varepsilon|^4 + \int_t^{t+\tau}\int_\Omega u_\varepsilon^2 \le C \tag{6.26}$$

for any $\varepsilon > 0$ and any $t \in [0, T_{max,\varepsilon} - \tau)$, where $\tau = \min\{1, \frac{1}{2}T_{max,\varepsilon}\}$.

Proof. We let $\mathcal{F}_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) + \frac{\chi}{2}\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} + K\chi\int_\Omega |U_\varepsilon(\cdot,t)|^2$, note that each of the summands is bounded from below, that $s\ln s \le \frac{1}{2e} + s^2\ln s$ for any $s > 0$, $\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} \le \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^4}{v_\varepsilon^3(\cdot,t)} + |\Omega|$ for any $t \in (0, T_{max,\varepsilon})$ and that there is $C_p > 0$ such that $\int_\Omega |U_\varepsilon(\cdot,t)|^2 \le C_p \int_\Omega |\nabla U_\varepsilon(\cdot,t)|^2$ for any $t \in (0, T_{max,\varepsilon})$. Hence (and by (6.3)), $\mathcal{F}_\varepsilon$ satisfies an ODI of the form $\mathcal{F}' + k_1\mathcal{F} \le k_2$ with some $k_1 > 0$ and $k_2 > 0$, and we may conclude the validity of (6.23). The estimates in (6.24) and (6.25) then directly result from Lemma 6.2.10 upon integration. For (6.26), we observe that the bound on $\int_t^{t+\tau}\int_\Omega u_\varepsilon^2$ results from Lemma 6.2.5, and that by Young's inequality

$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^{\frac{4}{3}} = \int_t^{t+\tau}\int_\Omega \frac{|\nabla u_\varepsilon|^{\frac{4}{3}}}{u_\varepsilon^{\frac{2}{3}}} u_\varepsilon^{\frac{2}{3}} \le \frac{2}{3}\int_t^{t+\tau}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{1}{3}\int_t^{t+\tau}\int_\Omega u_\varepsilon^2.$$

for all $t \in [0, T_{max,\varepsilon} - \tau)$. Furthermore, by Lemma 6.2.3 for any $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 &\le \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} \\ &\le \|v_0\|_{L^\infty(\Omega)}\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} \quad \text{for any } t \in (0, T_{max,\varepsilon}),\end{aligned}$$

which is bounded due to (6.23). The integral $\int_t^{t+\tau}\int_\Omega |\nabla v_\varepsilon|^4$ can be treated similarly, invoking (6.25). □

A first consequence of these bounds is that the approximate solutions are global and we may a posteriori ignore any condition of the type $t < T_{max,\varepsilon}$ in the previous lemmata.

Lemma 6.2.12. *For any $\varepsilon > 0$, $T_{max,\varepsilon} = \infty$.*

Proof. Under the assumption that $T_{max,\varepsilon} < \infty$, for any $\varepsilon > 0$, Lemma 6.2.11 would provide us with $C > 0$ such that

$$\int_0^{T_{max,\varepsilon}}\int_\Omega |\nabla v_\varepsilon|^4 \le C \quad \text{and} \quad \int_\Omega |U_\varepsilon|^2 \le C \quad \text{on } (0, T_{max,\varepsilon}).$$

With this as starting point, we could follow the reasoning of [114, Lemma 3.9] to derive a contradiction to (6.5). There differential inequalities for $\int_\Omega u_\varepsilon^4$ and $\int_\Omega |A^{\frac{1}{2}}U_\varepsilon|^2$ first yielded bounds for these quantities on $[0, T_{max,\varepsilon})$, then smoothing estimates for the Stokes semigroup (if combined with an embedding for the domains of fractional powers of A) and for the Neumann heat semigroup successively led to estimates for $\|U_\varepsilon\|_{L^\infty(\Omega\times(0,T_{max,\varepsilon}))}$, $\|\nabla v_\varepsilon\|_{L^\infty((0,T_{max,\varepsilon});L^4(\Omega))}$ and $\|u_\varepsilon\|_{L^\infty(\Omega\times(0,T_{max,\varepsilon}))}$. □

6.2.3 Time regularity

In preparation of an Aubin-Lions type compactness argument, we intend to supplement Lemma 6.2.11 with bounds on time derivatives. This will be the purpose of the following three lemmata.

Lemma 6.2.13. *For any $T > 0$ there is $C > 0$ such that for every $\varepsilon > 0$,*

$$\|u_{\varepsilon t}\|_{L^1((0,T);(W_0^{2,4}(\Omega))^*)} \leq C.$$

Proof. Let $T > 0$. We recall that $\left(L^1((0,T);(W_0^{2,4}(\Omega))^*)\right)^* = L^\infty((0,T),W_0^{2,4}(\Omega))$ and that hence for all $\phi \in L^1((0,T);(W_0^{2,4}(\Omega))^*)$

$$\|\phi\|_{L^1((0,T);(W_0^{2,4}(\Omega))^*)} = \sup\left\{\int_0^T\int_\Omega \phi\varphi;\ \varphi \in L^\infty((0,T),W_0^{2,4}(\Omega)), \|\varphi\|_{L^\infty((0,T),W_0^{2,4}(\Omega))} \leq 1\right\},$$

and introduce $k_1 > 0$ such that

$$\begin{aligned}\max\Big\{&\|\phi\|_{L^4((0,T);W^{1,4}(\Omega))}, \|\phi\|_{L^2((0,T);L^2(\Omega))}, \|\phi\|_{L^2((0,T);W^{1,2}(\Omega))}, \|\phi\|_{L^\infty((0,T);W^{1,\infty}(\Omega))}\Big\}\\ &\leq k_1\|\phi\|_{L^\infty((0,T),W_0^{2,4}(\Omega))} \qquad \text{for all } \phi \in L^\infty((0,T),W_0^{2,4}(\Omega)),\end{aligned}$$

which is guaranteed to exist by the continuous embeddings of $W^{2,4}(\Omega)$ into the spaces $W^{1,4}(\Omega)$, $L^2(\Omega)$, and $W^{1,\infty}(\Omega)$. We then pick an arbitrary function $\varphi \in L^\infty((0,T),W_0^{2,4}(\Omega))$ having norm $\|\varphi\|_{L^\infty((0,T),W_0^{2,4}(\Omega))} \leq 1$ and test (6.4a) by φ, so that we obtain

$$\begin{aligned}\int_0^T\int_\Omega u_{\varepsilon t}\varphi &= \int_0^T\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi - \chi\int_0^T\int_\Omega \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon\cdot\nabla\varphi\\ &\quad + \kappa\int_0^T\int_\Omega u_\varepsilon\varphi - \mu\int_0^T\int_\Omega u_\varepsilon^2\varphi - \int_0^T\int_\Omega u_\varepsilon U_\varepsilon\cdot\nabla\varphi\\ &\leq \|\nabla\varphi\|_{L^4((0,T);L^4(\Omega))}\left(\int_0^T\int_\Omega|\nabla u_\varepsilon|^{\frac{4}{3}}\right)^{\frac{3}{4}}\\ &\quad + \chi\|\nabla\varphi\|_{L^2((0,T);L^2(\Omega))}\left(\int_0^T\int_\Omega u_\varepsilon^2\right)^{\frac{1}{2}}\sup_{t\in(0,T)}\left(\int_\Omega|\nabla v_\varepsilon(\cdot,t)|^2\right)^{\frac{1}{2}}\\ &\quad + \kappa\|\varphi\|_{L^2((0,T);L^2(\Omega))}\left(\int_0^T\int_\Omega u_\varepsilon^2\right)^{\frac{1}{2}} + \mu\|\varphi\|_{L^\infty((0,T);L^\infty(\Omega))}\int_0^T\int_\Omega u_\varepsilon^2\\ &\quad + \|\nabla\varphi\|_{L^\infty((0,T);L^\infty(\Omega))}\left(\int_0^T\int_\Omega|U_\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_0^T\int_\Omega u_\varepsilon^2\right)^{\frac{1}{2}} \qquad \text{for all } \varepsilon > 0.\end{aligned}$$

If we let C be as in Lemma 6.2.11, we obtain

$$\int_0^T\int_\Omega u_{\varepsilon t}\varphi \leq k_1(CT)^{\frac{3}{4}} + \chi k_1(CT)^{\frac{1}{2}}C^{\frac{1}{2}} + \kappa k_1 C^{\frac{1}{2}} + \mu k_1 CT + k_1 CT \qquad \text{for all } \varepsilon > 0$$

and thus conclude the proof. □

We continue with a similar statement concerning the second component of the solution.

Lemma 6.2.14. *For all $T > 0$ there is $C > 0$ such that*

$$\|v_{\varepsilon t}\|_{L^2((0,T);(W_0^{1,2}(\Omega))^*)} \le C$$

for all $\varepsilon > 0$.

Proof. Let $T > 0$. Employing Hölder's inequality and using that for any $\varepsilon > 0$ and $s > 0$ apparently $\frac{\ln(1+\varepsilon s)}{\varepsilon} \le s$, we see that for any $\varphi \in L^2((0,T); W_0^{1,2}(\Omega))$ satisfying $\|\varphi\|_{L^2((0,T);W^{1,2}(\Omega))} \le 1$ we have

$$\begin{aligned}
&\int_0^T \int_\Omega v_{\varepsilon t}\varphi \\
&= -\int_0^T \int_\Omega \nabla v_\varepsilon \cdot \nabla\varphi - \int_0^T \int_\Omega v_\varepsilon \frac{1}{\varepsilon}\ln(1+\varepsilon u_\varepsilon)\varphi + \int_0^T \int_\Omega v_\varepsilon U_\varepsilon \cdot \nabla\varphi \\
&\le \left(\int_0^T \int_\Omega |\nabla v_\varepsilon|^2\right)^{\frac12} \|\nabla\varphi\|_{L^2((0,T);L^2(\Omega))} + \|v_\varepsilon\|_{L^\infty(\Omega\times(0,T))} \left(\int_0^T \int_\Omega u_\varepsilon^2\right)^{\frac12} \|\varphi\|_{L^2((0,T);L^2(\Omega))} \\
&\quad + \|v_\varepsilon\|_{L^\infty(\Omega\times(0,T))} \left(\int_0^T \int_\Omega |U_\varepsilon|^2\right)^{\frac12} \|\nabla\varphi\|_{L^2((0,T);L^2(\Omega))},
\end{aligned}$$

again concluding the proof with the aid of Lemma 6.2.11. ☐

Lemma 6.2.15. *For all $T > 0$ there is $C > 0$ such that*

$$\|U_{\varepsilon t}\|_{L^2((0,T);(W_\sigma^{1,3}(\Omega))^*)} \le C \tag{6.27}$$

holds for any $\varepsilon > 0$.

Proof. Let $\female \in L^2(0,T; W_\sigma^{1,3}(\Omega))$ with $\|\female\|_{L^2(0,T;W_\sigma^{1,3}(\Omega))} = 1$. Then

$$\begin{aligned}
&\int_0^T\int_\Omega U_{\varepsilon t} \cdot \female \le \left| -\int_0^T\int_\Omega \nabla U_\varepsilon \cdot \nabla\female + \int_0^T\int_\Omega Y_\varepsilon U_\varepsilon \otimes U_\varepsilon \nabla\female + \int_0^T\int_\Omega u_\varepsilon \nabla\Phi \cdot \female + \int_0^T\int_\Omega f \cdot \female \right| \\
&\le \|\nabla U_\varepsilon\|_{L^2((0,T);L^2(\Omega))} \|\nabla\female\|_{L^2((0,T);L^2(\Omega))} \\
&\quad + \|Y_\varepsilon U_\varepsilon\|_{L^2((0,T);L^6(\Omega))} \|U_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} \|\nabla\female\|_{L^2((0,T);L^3(\Omega))} \\
&\quad + \|\nabla\Phi\|_{L^\infty(\Omega)} \|u_\varepsilon\|_{L^2((0,T);L^2(\Omega))} \|\female\|_{L^2((0,T);L^2(\Omega))} \\
&\quad + \|f\|_{L^2((0,T);L^{\frac65}(\Omega))} \|\female\|_{L^2((0,T);L^6(\Omega))}.
\end{aligned} \tag{6.28}$$

Here we can use that by the embedding $W_\sigma^{1,2}(\Omega) \hookrightarrow L^6(\Omega;\mathbb{R}^3)$ and nonexpansiveness of Y_ε on $L_\sigma^2(\Omega)$ (see e.g. [87, (II.3.4.6)]) there is $k > 0$ such that for any $\varepsilon > 0$

$$\begin{aligned}
\|Y_\varepsilon U_\varepsilon\|_{L^6(\Omega)} &\le k\|\nabla Y_\varepsilon U_\varepsilon\|_{L^2(\Omega)} = k\left\|A^{\frac12} Y_\varepsilon U_\varepsilon\right\|_{L^2(\Omega)} = k\left\|Y_\varepsilon A^{\frac12} U_\varepsilon\right\|_{L^2(\Omega)} \\
&\le k\left\|A^{\frac12} U_\varepsilon\right\|_{L^2(\Omega)} = k\|\nabla U_\varepsilon\|_{L^2(\Omega)}.
\end{aligned}$$

Thus, the bounds on $\|\nabla U_\varepsilon\|_{L^2((0,T);L^2(\Omega))}$, $\|U_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))}$, $\|u_\varepsilon\|_{L^2((0,T);L^2(\Omega))}$ from Lemma 6.2.11 entail boundedness of the expression in (6.28), so that (6.27) results. ☐

6.2.4 Passing to the limit. Proof of Theorem 6.1.1

With these lemmata we have collected sufficiently many estimates to construct weak solutions by compactness arguments. Before doing so, let us define what a weak solution is supposed to be:

Definition 6.2.16. *A weak solution of* (6.1) *is a triple* (u, v, U) *of functions such that*

$$\begin{aligned}
&u \in L^2_{loc}([0,\infty); L^2(\Omega)) \cap L^{\frac{4}{3}}_{loc}([0,\infty); W^{1,\frac{4}{3}}(\Omega)),\\
&v \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \text{ and}\\
&U \in L^2_{loc}([0,\infty); W^{1,2}_{0,\sigma}(\Omega)),
\end{aligned}$$

and that

$$\begin{aligned}
-\int_0^T\!\!\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) - \int_0^T\!\!\int_\Omega uU\cdot\nabla\varphi &= -\int_0^T\!\!\int_\Omega \nabla u\cdot\nabla\varphi + \chi\int_0^T\!\!\int_\Omega u\nabla v\cdot\nabla\varphi\\
&\quad + \kappa\int_0^T\!\!\int_\Omega u\varphi - \mu\int_0^T\!\!\int_\Omega u^2\varphi,\\
-\int_0^T\!\!\int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0) - \int_0^T\!\!\int_\Omega vU\cdot\nabla\varphi &= -\int_0^T\int_\Omega \nabla v\cdot\nabla\varphi - \int_0^T\int_\Omega uv\varphi \text{ and}\\
-\int_0^T\!\!\int_\Omega U\cdot\psi_t - \int_\Omega U_0\cdot\psi(\cdot,0) - \int_0^T\!\!\int_\Omega U\otimes U\cdot\nabla\psi &= -\int_0^T\!\!\int_\Omega \nabla U\cdot\nabla\psi + \int_0^T\!\!\int_\Omega u\nabla\psi\nabla\Phi + \int_0^T\!\!\int_\Omega f\cdot\psi
\end{aligned}$$

hold for any $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ *and any* $\psi \in C^\infty_{0,\sigma}(\Omega\times[0,\infty))$*, respectively.*

Such weak solutions do exist:

Proposition 6.2.17. *There exist a sequence* $(\varepsilon_j)_{j\in\mathbb{N}} \searrow 0$ *and functions* u, v, U *such that* $u \in L^2_{loc}([0,\infty); L^2(\Omega)) \cap L^{\frac{4}{3}}_{loc}([0,\infty); W^{1,\frac{4}{3}}(\Omega))$, $v \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$, $U \in L^2_{loc}([0,\infty); W^{1,2}_{0,\sigma}(\Omega))$ *and that*

$$\begin{aligned}
u_\varepsilon &\to u && \text{in } L^{\frac{4}{3}}_{loc}([0,\infty); L^p(\Omega)) && \text{for all } p\in[1,\tfrac{12}{5}) \text{ and a.e. in } \Omega\times(0,\infty), && (6.29)\\
v_\varepsilon &\to v && \text{in } C^0_{loc}([0,\infty); L^p(\Omega)) && \text{for all } p\in[1,6) \text{ and a.e. in } \Omega\times(0,\infty), && (6.30)\\
v_\varepsilon &\overset{*}{\rightharpoonup} v && \text{in } L^\infty(\Omega\times(t,t+1)) && \text{for all } t\ge 0, && (6.31)\\
U_\varepsilon &\to U && \text{in } L^2_{loc}([0,\infty); L^p(\Omega;\mathbb{R}^3)) && \text{for all } p\in[1,6) \text{ and a.e. in } \Omega\times(0,\infty), && (6.32)\\
\nabla u_\varepsilon &\rightharpoonup \nabla u && \text{in } L^{\frac{4}{3}}_{loc}([0,\infty); L^{\frac{4}{3}}(\Omega;\mathbb{R}^3)), && && (6.33)\\
\nabla v_\varepsilon &\overset{*}{\rightharpoonup} \nabla v && \text{in } L^\infty_{loc}([0,\infty); L^2(\Omega;\mathbb{R}^3)), && && (6.34)\\
\nabla U_\varepsilon &\rightharpoonup \nabla U && \text{in } L^2_{loc}([0,\infty); L^2(\Omega;\mathbb{R}^{3\times3})), && && (6.35)\\
Y_\varepsilon U_\varepsilon &\to U && \text{in } L^2_{loc}([0,\infty); L^2(\Omega;\mathbb{R}^3)) && \text{and} && (6.36)\\
u_\varepsilon &\to u && \text{in } L^2_{loc}([0,\infty); L^2(\Omega)), && && (6.37)
\end{aligned}$$

as $\varepsilon = \varepsilon_j \searrow 0$*, and such that* (u, v, U) *form a weak solution to* (6.1) *in the sense of Definition 6.2.16.*

Proof. For any $p \in [1, \frac{12}{5})$, $W^{1,\frac{4}{3}}(\Omega) \overset{cpt}{\hookrightarrow} L^p(\Omega) \hookrightarrow (W_0^{2,4}(\Omega))^*$, so that for any $T > 0$ the bound on $\|u_\varepsilon\|_{L^{\frac{4}{3}}([0,T);W^{1,\frac{4}{3}}(\Omega))}$ from Lemma 6.2.11, which was independent of ε, together with Lemma

6.2.13 and [86, Corollary 4] shows relative compactness of $\{u_\varepsilon; \varepsilon > 0\}$ in $L^{\frac{4}{3}}((0,T); L^p(\Omega))$ and thus ensures the existence of a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ satisfying (6.29). Because for any $T > 0$ there is a uniform bound on $\int_0^T \int_\Omega \Psi(u_\varepsilon^2)$ for $\Psi(x) = \frac{x}{2}\ln(x)$, $\{u_\varepsilon^2; \varepsilon > 0\}$ is weakly relatively precompact in $L^1(\Omega \times (0,T))$ by the Dunford-Pettis theorem (cf. [21, Thm. IV.8.9]) and hence, along a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$, $u_\varepsilon^2 \rightharpoonup z$ in $L^1(\Omega \times (0,T))$ for some $z \in L^1(\Omega \times (0,T))$, where z has to coincide with u^2 due to (6.29). In particular, $\int_0^T \int_\Omega u_\varepsilon^2 \to \int_0^T \int_\Omega u^2$ as $\varepsilon_j \to 0$. Since moreover, along a further subsequence, $u_\varepsilon \rightharpoonup u$ in $L^2_{loc}([0,\infty); L^2(\Omega))$ due to (6.26), we obtain (6.37). Similarly, bounds on v_ε with respect to the norm of $L^\infty([0,T); W^{1,2}(\Omega))$ and on $v_{\varepsilon t}$ in $L^2([0,T); (W^{1,2}(\Omega))^*)$ as obtained in Lemma 6.2.11 and Lemma 6.2.14 and the embedding $W^{1,2}(\Omega) \overset{cpt}{\hookrightarrow} L^p(\Omega) \hookrightarrow (W_0^{1,2}(\Omega))^*$ for all $p \in [1,6)$ allow for an application of [86, Corollary 4], which yields (6.30) along a suitable subsequence of the sequence previously found. Similar reasoning for U, combining bounds on U_ε in $L^2([0,T); W_\sigma^{1,2}(\Omega))$ and on $U_{\varepsilon t}$ in $L^2([0,T); (W_\sigma^{1,3}(\Omega))^*)$, results in (6.32). Due to (6.32), also for almost every $t > 0$ we have $U_\varepsilon(\cdot,t) \to U(\cdot,t)$ and taking into account [87, II.(3.4.6)] and [87, II.(3.4.8)] shows that $\|Y_\varepsilon U_\varepsilon - U\|_{L^2(\Omega)} \le \|Y_\varepsilon U_\varepsilon - Y_\varepsilon U + Y_\varepsilon U - YU\|_{L^2(\Omega)} \le \|U_\varepsilon - U\|_{L^2(\Omega)} + \|(Y_\varepsilon - Y)U\|_{L^2(\Omega)} \to 0$ for a.e. $t > 0$. Since $\|Y_\varepsilon U_\varepsilon\|_{L^2(\Omega)} \le \|U_\varepsilon\|_{L^2(\Omega)}$ and $\|U_\varepsilon\|_{L^2(\Omega)}$ converges in $L^2((0,T))$, a version of Lebesgue's theorem ensures the validity of (6.36). Convergence of the gradients along further subsequences, as asserted in (6.33), (6.34) and (6.35), is easily obtained from the bounds given in Lemma 6.2.11. The convergence properties asserted in (6.30), (6.32), (6.33), (6.34),(6.35), (6.36) and (6.37) finally, are sufficient to pass to the limit in each integral making up a weak formulation of system (6.4), so that (u, v, U) is a weak solution to (6.1). □

The most important consequence of this proposition is that the existence theorem is proven:

Proof of Theorem 6.1.1. The theorem is part of the statement proven by Proposition 6.2.17. □

6.3 Eventual smoothness and asymptotics

6.3.1 Lower bound for the bacterial mass

Although we already know an upper bound for $\int_\Omega u$, we are still lacking a corresponding estimate from below, which was crucial in the derivation of the convergence of v in [107]. Consideration of the function

$$\mathcal{G}_{\varepsilon,B}(t) := \int_\Omega u_\varepsilon(\cdot,t) - \frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_\varepsilon(\cdot,t)}{\kappa} + \frac{B}{2}\int_\Omega v_\varepsilon^2(\cdot,t), \qquad t \in (0,\infty)$$

will help us to recover this lower bound. At the same time, we will obtain another cornerstone for the proof of convergence of u (see Lemma 6.3.9 and Lemma 6.3.18).

In [108], a similar functional has been employed to obtain convergence of u and v to a constant equilibrium. The model considered there contains the Keller-Segel equation as second equation and due to the contributions of the production term $+u$ therein, whose influence is increased with increasing values of B, it was not possible to choose B arbitrarily large there, which in the end resulted in a largeness condition on μ ([108, (8.3)]). Thanks to the consumption term in (6.4a), all terms obtained from this equation work in favour of our estimate and we do not need a corresponding condition on μ and can choose B in such a way that $\mathcal{G}_{\varepsilon,B}$ becomes an energy functional.

Lemma 6.3.1. *There is B_0 such that for any $B > B_0$ and any $\varepsilon > 0$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_{\varepsilon,B}(t) + \mu\int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 + \frac{\kappa}{2\mu}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \le 0 \qquad \text{on } (0,\infty). \tag{6.38}$$

Proof. Let $B_0 := \frac{\kappa\chi^2}{2\mu}$ and $B > B_0$. The derivative of $\mathcal{G}_{\varepsilon,B}$ then satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_{\varepsilon,B} &= \int_\Omega u_{\varepsilon t} - \frac{\kappa}{\mu}\int_\Omega \frac{u_{\varepsilon t}}{u_\varepsilon} + B\int_\Omega v_\varepsilon v_{\varepsilon t}\\ &= \kappa\int_\Omega u_\varepsilon - \mu\int_\Omega u_\varepsilon^2 - \frac{\kappa}{\mu}\int_\Omega \frac{\Delta u_\varepsilon}{u_\varepsilon} + \frac{\kappa}{\mu}\chi\int_\Omega \frac{\nabla\cdot\left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon\right)}{u_\varepsilon} - \frac{\kappa^2}{\mu}\int_\Omega 1 + \kappa\int_\Omega u_\varepsilon\\ &+ \frac{\kappa}{\mu}\int_\Omega U_\varepsilon\cdot\frac{\nabla u_\varepsilon}{u_\varepsilon} + B\int_\Omega v_\varepsilon\Delta v_\varepsilon - B\int_\Omega v_\varepsilon^2\frac{1}{\varepsilon}\ln(1+\varepsilon u_\varepsilon) - B\int_\Omega v_\varepsilon\nabla v_\varepsilon\cdot U_\varepsilon\end{aligned}$$

on $(0,\infty)$ and for all $\varepsilon > 0$. Here we can use that U_ε is divergence-free and hence integration by parts shows that $\int_\Omega \nabla(\frac{1}{2}v_\varepsilon^2)\cdot U_\varepsilon$ vanishes as well as $\int_\Omega U_\varepsilon\cdot\nabla\ln u_\varepsilon$. Furthermore we can summarize the terms without derivatives according to

$$-\mu\int_\Omega u_\varepsilon^2 - \frac{\kappa^2}{\mu}\int_\Omega 1 + 2\kappa\int_\Omega u_\varepsilon = -\mu\int_\Omega\left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 \qquad \text{on } (0,\infty) \text{ and for all } \varepsilon > 0$$

so that for all $\varepsilon > 0$ we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_{\varepsilon,B} = -\mu\int_\Omega\left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 - \frac{\kappa}{\mu}\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{\kappa\chi}{\mu}\int_\Omega \frac{u_\varepsilon\nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)u_\varepsilon^2}\cdot\nabla u_\varepsilon\\ - B\int_\Omega|\nabla v_\varepsilon|^2 - B\int_\Omega v_\varepsilon^2\frac{1}{\varepsilon}\ln(1+\varepsilon u_\varepsilon)\end{aligned}$$

on $(0,\infty)$. Nonnegativity of $\int_\Omega v_\varepsilon^2\frac{1}{\varepsilon}\ln(1+\varepsilon u_\varepsilon)$ and an application of Young's inequality together with the trivial estimate $\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)u_\varepsilon} \le 1$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_{\varepsilon,B} \le -\mu\int_\Omega\left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 - \frac{\kappa}{\mu}\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{\kappa}{2\mu}\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{\kappa\chi^2}{2\mu}\int_\Omega|\nabla v_\varepsilon|^2 - B\int_\Omega|\nabla v_\varepsilon|^2$$

on $(0,\infty)$ for any $\varepsilon > 0$, so that we finally arrive at (6.38). □

We collect the estimates implicitly contained in Lemma 6.3.1:

Lemma 6.3.2. *There are $k > 0$ and $C > 0$ such that for any $\varepsilon > 0$*

$$\int_\Omega u_\varepsilon(\cdot,t) > k \qquad \text{for all } t > 0, \text{ and} \tag{6.39}$$

$$\int_0^\infty\int_\Omega\left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 \le C. \tag{6.40}$$

Proof. Let $B > B_0$ with B_0 as in Lemma 6.3.1 and $\varepsilon > 0$. For any $t > 0$, integration of (6.38) on $(0,t)$ yields

$$\int_\Omega u_\varepsilon(\cdot,t) - \frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_\varepsilon(\cdot,t)}{\kappa} + \frac{B}{2}\int_\Omega v_\varepsilon^2(\cdot,t) + \mu\int_0^t\int_\Omega\left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 + \frac{\kappa}{2\mu}\int_0^t\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2}$$

$$\leq \int_\Omega u_0 - \frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_0}{\kappa} + \frac{B}{2}\int_\Omega v_0^2.$$

In particular, for all $t > 0$,

$$\begin{aligned}\frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_\varepsilon(\cdot,t)}{\kappa} &\geq \int_\Omega u_\varepsilon(\cdot,t) - \int_\Omega u_0 + \frac{B}{2}\int_\Omega v_\varepsilon^2(\cdot,t) - \frac{B}{2}\int_\Omega v_0^2 + \frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_0}{\kappa} \\ &\quad + \mu\int_0^t\int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 + \frac{\kappa}{2\mu}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \qquad (6.41) \\ &\geq -\int_\Omega u_0 - \frac{B}{2}\int_\Omega v_0^2 + \frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_0}{\kappa} + \mu\int_0^t\int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 + \frac{\kappa}{2\mu}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2}.\end{aligned}$$

Since $\frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_\varepsilon(\cdot,t)}{\kappa} \leq \frac{\kappa}{\mu}\int_\Omega \frac{\mu u_\varepsilon(\cdot,t)}{\kappa} = \int_\Omega u_\varepsilon(\cdot,t)$ is bounded according to Lemma 6.2.5, this entails (6.40). The estimate in (6.41) also shows that

$$\int_\Omega \ln\frac{\mu u_\varepsilon(\cdot,t)}{\kappa}\frac{1}{|\Omega|} \geq \frac{\mu}{|\Omega|\kappa}\left[-\int_\Omega u_0 - \frac{B}{2}\int_\Omega v_0^2 + \frac{\kappa}{\mu}\int_\Omega \ln\frac{\mu u_0}{\kappa}\right] =: k_1 \qquad \text{for all } t > 0,$$

so that Jensen's inequality implies

$$\frac{\mu}{\kappa}\int_\Omega u_\varepsilon(\cdot,t)\frac{1}{|\Omega|} \geq \mathrm{e}^{\int_\Omega \ln\frac{\mu u_\varepsilon(\cdot,t)}{\kappa}\frac{1}{|\Omega|}} \geq \mathrm{e}^{k_1} \qquad \text{for all } t > 0$$

and hence (6.39). □

For later reference, we state an immediate consequence of (6.40).

Corollary 6.3.3. *The function u obtained in Proposition 6.2.17 satisfies*

$$\int_0^\infty \int_\Omega \left(u - \frac{\kappa}{\mu}\right)^2 < \infty.$$

Proof. With C as in (6.40), for any $T \in (0,\infty)$,

$$\int_0^T\int_\Omega \left(u - \frac{\kappa}{\mu}\right)^2 = \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 \leq C$$

by (6.37). □

6.3.2 Decay of oxygen

With the lower bound on the bacterial mass from Lemma 6.3.2 we are well-equipped for the derivation of decay of v by means of (6.4b). Smallness of v will play an important role in Section 6.3.3, when we derive bounds on u_ε in higher L^p-norms via a differential inequality holding for small values of v_ε only. For turning such bounds into information on u, it will be crucial that the validity of the ODI does not hinge on ε too much, i.e. that the decay of v_ε be uniform in ε. In pursuance of this uniformity, in the following lemma we will consider v instead of v_ε and afterwards carry back the decay information to the v_ε (which, due to their differentiability, are much better suited for making an appearance in ODIs like that in the proof of Lemma 6.3.8). The idea of the proof of boundedness of v_ε is taken from [107, Sec. 4].

Lemma 6.3.4. *For any $\eta > 0$ there is $T > 0$ such that for any $t > T$*

$$0 \le \int_t^{t+1}\int_\Omega v < \eta.$$

Proof. Integrating (6.4b) shows that

$$\int_\Omega v_0 \ge \int_0^t \int_\Omega v_\varepsilon \frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon} \qquad \text{for any } t > 0 \quad \text{and any } \varepsilon > 0,$$

which in light of (6.30), (6.29) asserts that

$$\int_\Omega v_0 \ge \int_0^t \int_\Omega uv \qquad \text{for all } t > 0$$

and hence in particular

$$\int_t^{t+1}\int_\Omega uv \to 0 \qquad \text{as } t \to \infty. \tag{6.42}$$

Denoting the average value $\frac{1}{|\Omega|}\int_\Omega v(\cdot,t)$ of $v(\cdot,t)$ by $\bar{v}(\cdot,t)$ we observe that

$$\left|\int_t^{t+1}\int_\Omega uv\right| = \int_t^{t+1}\int_\Omega u(v-\bar{v}) + \int_t^{t+1} \bar{v}\int_\Omega u \qquad \text{for all } t > 0, \tag{6.43}$$

where

$$\int_t^{t+1}\int_\Omega u(v-\bar{v}) \le \left(\int_t^{t+1}\int_\Omega u^2\right)^{\frac{1}{2}} \left(\int_t^{t+1}\int_\Omega (v-\bar{v})^2\right)^{\frac{1}{2}} \le k_1^{\frac{1}{2}}\left(c_p\int_t^{t+1}\int_\Omega |\nabla v|^2\right)^{\frac{1}{2}}$$

with c_p, k_1 being constants obtained from Poincaré's inequality and Lemma 6.2.5 in combination with (6.37). We use k_2 to denote the positive lower bound for $\int_\Omega u$, which is guaranteed to exist by Lemma 6.3.2 and (6.29). Since $\nabla v \in L^2(\Omega\times(0,\infty);\mathbb{R}^3)$ due to Lemma 6.2.4, $\int_t^{t+1}\int_\Omega u(v-\bar{v}) \to 0$ as $t \to \infty$ and taking (6.42) and (6.43) into account, we see that

$$0 \le \frac{1}{|\Omega|}\int_t^{t+1}\int_\Omega vk_2 \le \int_t^{t+1}\bar{v}\int_\Omega u \to 0 \qquad \text{as } t \to \infty. \qquad \square$$

We transfer this information back to the functions v_ε:

Corollary 6.3.5. *For any $\eta > 0$ there is $T > 0$ such that for all $t \ge T$ there is $\varepsilon_0 = \varepsilon_0(t) > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$*

$$\int_t^{t+1}\int_\Omega v_\varepsilon < \eta.$$

Proof. This directly results from Lemma 6.3.4 and (6.30). $\square$

Lemma 6.3.6. *For any $\eta > 0$ there are $T > 0$ and $\varepsilon_0 > 0$ such that for every $t > T$ and every $\varepsilon \in (0,\varepsilon_0)$ we have*

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < \eta.$$

Proof. The Gagliardo-Nirenberg inequality asserts the existence of $k_1 > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)} \le k_1 \left(\|\nabla\phi\|_{L^4(\Omega)}^{\frac{12}{13}} \|\phi\|_{L^1(\Omega)}^{\frac{1}{13}} + \|\phi\|_{L^1(\Omega)} \right) \qquad \text{for all } \phi \in W^{1,4}(\Omega) \tag{6.44}$$

and according to Lemma 6.2.11 there is $k_2 > 0$ such that

$$\int_t^{t+1}\int_\Omega |\nabla v_\varepsilon|^4 \le k_2 \qquad \text{for all } t > 0, \varepsilon > 0. \tag{6.45}$$

Let $\eta > 0$. Let $\delta > 0$ be such that $k_1 k_2^{\frac{3}{52}} \delta^{\frac{1}{13}} + k_1\delta < \eta$. Due to Corollary 6.3.5 there are $T_0 > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have $\int_{T_0}^{T_0+1}\int_\Omega v_\varepsilon < \delta$. Invoking (6.44) and (6.45), we see that

$$\begin{aligned}\int_{T_0}^{T_0+1} \|v_\varepsilon\|_{L^\infty(\Omega)} \le& k_1 \int_{T_0}^{T_0+1} \|\nabla v_\varepsilon\|_{L^4(\Omega)}^{\frac{12}{13}} \|v_\varepsilon\|_{L^1(\Omega)}^{\frac{1}{13}} + k_1 \int_{T_0}^{T_0+1} \|v_\varepsilon\|_{L^1(\Omega)} \\ \le& k_1 \left(\int_{T_0}^{T_0+1} \|\nabla v_\varepsilon\|_{L^4(\Omega)}^4 \right)^{\frac{3}{13}} \left(\int_{T_0}^{T_0+1} \|v_\varepsilon\|_{L^1(\Omega)} \right)^{\frac{1}{13}} \cdot 1^{\frac{9}{13}} + k_1\delta \\ \le& k_1 k_2^{\frac{3}{52}} \delta^{\frac{1}{13}} + k_1\delta \le \eta \qquad \text{for any } \varepsilon \in (0, \varepsilon_0).\end{aligned}$$

In particular, for every $\varepsilon \in (0, \varepsilon_0)$, there is at least one $t_0 \in [T_0, T_0+1]$ such that $\|v_\varepsilon(\cdot, t_0)\|_{L^\infty(\Omega)} \le \eta$ and thus, due to monotonicity of v_ε (Lemma 6.2.3), for all $\varepsilon \in (0, \varepsilon_0)$ and all $t > T := T_0 + 1$,

$$\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le \eta.$$

□

Corollary 6.3.7. *The function v obtained in Proposition 6.2.17 satisfies*

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty.$$

Proof. Combining Lemma 6.3.6 with (6.31) this convergence statement results immediately. □

6.3.3 Boundedness of u

In obtaining eventual smoothness and convergence of the solutions constructed in Proposition 6.2.17, we will heavily rely on estimates for higher norms of u. We can achieve those for large times in Lemma 6.3.8 and prepare this by deriving a differential inequality for $y_\varepsilon(t) := \int_\Omega \frac{u_\varepsilon^p}{(\eta - v_\varepsilon)^\theta}$, which holds for small values of v_ε. Fortunately, we already have established that $\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$ converges to 0.

The same quantity has proven useful in the derivation of estimates for $\|u(\cdot, t)\|_{L^p(\Omega)}$ for large t already in [118, Sec. 5] and [107, Sec. 5]. Note, however, that there (that is: in the setting without logistic source) the analogue of (6.49) below would read

$$y_\varepsilon' + \left(\frac{[2p\theta + \chi p(p-1)\eta]^2}{4(\theta(1+\theta) - \chi p\theta\eta)} - p(p-1) \right) \int_\Omega \frac{u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2}{(\eta - v_\varepsilon)^\theta} \le 0,$$

so that the right hand side already equalled zero, and hence at the same time bounds on the expression $\int_t^{t+1}\int_\Omega \frac{u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2}{(\eta - v_\varepsilon)^\theta}$ could be obtained.

The fact that y is defined on intervals (T, ∞) for large T only, raises the problem that the initial values $y_\varepsilon(T)$ are unknown and differ for varying ε. Fortunately, the nonlinear absorptive term

allows for comparison with solutions 'starting from initial data ∞' (see (6.50)), so that the bound on y_ε does not depend on $\int_\Omega u_\varepsilon^p(T)$ and hence not on ε.
Also, it is important to note that T may (and will) depend on p, but is independent of ε due to the uniformity of the decay of v_ε asserted by Lemma 6.3.6. This will be decisive when transferring the bounds on $\|u_\varepsilon\|_{L^p(\Omega)}$ to $\|u\|_{L^p(\Omega)}$.

Lemma 6.3.8. *For any $p \in (1,\infty)$ there are $T^\star > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C$$

for all $t > T^\star$ and all $\varepsilon \in (0,\varepsilon_0)$.

Proof. Let $p > 1$. First fix $\theta > 0$ so small that

$$4p^2\theta + 4p^2\theta\chi(p-1) + \chi^2p^2(p-1)^2\theta < 2p(p-1)$$

and let $0 < \eta < \min\{1, \theta, \frac{1}{2p\chi}\}$. Then

$$4p^2\theta + 4p^2\theta\chi(p-1)\frac{\eta}{\theta} + \chi^2p^2(p-1)^2\theta\frac{\eta^2}{\theta^2} < 4p(p-1)[1+\theta-\chi p\eta]$$

and hence

$$4p^2\theta^2 + 4p^2\theta\chi(p-1)\eta + \chi^2p^2(p-1)^2\eta^2 < 4p(p-1)\theta[1+\theta-\chi p\eta],$$

that is,

$$(2p\theta + \chi p(p-1)\eta)^2 < p(p-1)4\theta(1+\theta-\chi p\eta). \tag{6.46}$$

We use Lemma 6.3.6 to fix $T > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, $t > T$, we have

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\eta}{2}. \tag{6.47}$$

Let $\varepsilon \in (0,\varepsilon_0)$. Then

$$y_\varepsilon(t) := \int_\Omega \frac{u_\varepsilon^p}{(\eta - v_\varepsilon)^\theta}, \qquad t \ge T,$$

is well-defined, and on (T,∞) we can compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{u_\varepsilon^p}{(\eta-v_\varepsilon)^\theta} =& p\int_\Omega \frac{u_\varepsilon^{p-1}u_{\varepsilon t}}{(\eta-v_\varepsilon)^\theta} + \theta\int_\Omega \frac{u_\varepsilon^p}{(\eta-v_\varepsilon)^{1+\theta}}v_{\varepsilon t}\\
=& \, p\int_\Omega \frac{u_\varepsilon^{p-1}\Delta u_\varepsilon}{(\eta-v_\varepsilon)^\theta} - \chi p\int_\Omega \frac{u_\varepsilon^{p-1}\nabla\cdot(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon)}{(\eta-v_\varepsilon)^\theta} - p\int_\Omega \frac{u_\varepsilon^{p-1}U_\varepsilon\cdot\nabla u_\varepsilon}{(\eta-v_\varepsilon)^\theta}\\
&+ p\kappa\int_\Omega \frac{u_\varepsilon^p}{(\eta-v_\varepsilon)^\theta} - p\mu\int_\Omega \frac{u_\varepsilon^{p+1}}{(\eta-v_\varepsilon)^\theta}\\
&+ \theta\int_\Omega \frac{u_\varepsilon^p\Delta v_\varepsilon}{(\eta-v_\varepsilon)^{\theta+1}} - \theta\int_\Omega \frac{u_\varepsilon^p\frac{v_\varepsilon}{\varepsilon}\ln(1+\varepsilon u_\varepsilon)}{(\eta-v_\varepsilon)^{1+\theta}} - \theta\int_\Omega \frac{u_\varepsilon^pU_\varepsilon\cdot\nabla v_\varepsilon}{(\eta-v_\varepsilon)^{1+\theta}}.
\end{aligned} \tag{6.48}$$

Here we use that, since U_ε is divergence-free, we have

$$-p\int_\Omega \frac{u_\varepsilon^{p-1}U_\varepsilon\cdot\nabla u_\varepsilon}{(\eta-v_\varepsilon)^\theta} - \theta\int_\Omega \frac{u_\varepsilon^pU_\varepsilon\cdot\nabla v_\varepsilon}{(\eta-v_\varepsilon)^{1+\theta}} = -\int_\Omega U_\varepsilon\cdot\nabla\left(\frac{u_\varepsilon^p}{(\eta-v_\varepsilon)^\theta}\right) = 0 \qquad \text{on } (T,\infty).$$

Furthermore, employing Hölder's inequality, we estimate

$$\int_\Omega \frac{u_\varepsilon^p}{(\eta - v_\varepsilon)^\theta} \le \left(\int_\Omega \frac{1^{p+1}}{(\eta - v_\varepsilon)^\theta}\right)^{\frac{1}{p+1}} \left(\int_\Omega \frac{u_\varepsilon^{p+1}}{(\eta - v_\varepsilon)^\theta}\right)^{\frac{p}{p+1}} \le \left(\frac{2^\theta|\Omega|}{\eta^\theta}\right)^{\frac{1}{p+1}} \left(\int_\Omega \frac{u_\varepsilon^{p+1}}{(\eta - v_\varepsilon)^\theta}\right)^{\frac{p}{p+1}}$$

on (T,∞) by (6.47), so that

$$-\mu p \int_\Omega \frac{u_\varepsilon^{p+1}}{(\eta - v_\varepsilon)^\theta} \le -p\mu \left(\frac{\eta^\theta}{2^\theta|\Omega|}\right)^{\frac{1}{p}} \left(\int_\Omega \frac{u_\varepsilon^p}{(\eta - v_\varepsilon)^\theta}\right)^{1+\frac{1}{p}} =: -k_1 y_\varepsilon^{1+\frac{1}{p}} \qquad \text{on } (T,\infty).$$

We infer from (6.48) by integration by parts that

$$\begin{aligned} y_\varepsilon' \le &-p(p-1)\int_\Omega \frac{u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2}{(\eta - v_\varepsilon)^\theta} - p\theta \int_\Omega \frac{u_\varepsilon^{p-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(\eta - v_\varepsilon)^{\theta+1}} + \chi p(p-1)\int_\Omega \frac{u_\varepsilon^{p-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)(\eta - v_\varepsilon)^\theta} \\ &+ \chi p\theta \int_\Omega \frac{u_\varepsilon^p |\nabla v_\varepsilon|^2}{(1+\varepsilon u_\varepsilon)(\eta - v_\varepsilon)^{1+\theta}} + \kappa p y_\varepsilon - k_1 y_\varepsilon^{1+\frac{1}{p}} \\ &- \theta p \int_\Omega \frac{u_\varepsilon^{p-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(\eta - v_\varepsilon)^{1+\theta}} - \theta(1+\theta)\int_\Omega \frac{u_\varepsilon^p |\nabla v_\varepsilon|^2}{(\eta - v_\varepsilon)^{2+\theta}} - \theta \int_\Omega \frac{u_\varepsilon^p \frac{v_\varepsilon}{\varepsilon}\ln(1+\varepsilon u_\varepsilon)}{(\eta - v_\varepsilon)^{1+\theta}} \qquad \text{on } (T,\infty). \end{aligned}$$

We use that $\frac{1}{1+\varepsilon u_\varepsilon} \le 1$ and (by (6.47)) $1 \le \frac{\eta}{\eta - v_\varepsilon} \le 2$ as well as nonpositivity of the last term and get

$$\begin{aligned} y_\varepsilon' \le &- p(p-1)\int_\Omega \frac{u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2}{(\eta - v_\varepsilon)^\theta} - [\theta(1+\theta) - \chi p\theta\eta]\int_\Omega \frac{u_\varepsilon^p|\nabla v_\varepsilon|^2}{(\eta - v_\varepsilon)^{2+\theta}} \\ &+ [p\theta + \chi p(p-1)\eta + \theta p]\int_\Omega \frac{u_\varepsilon^{p-1}|\nabla u_\varepsilon||\nabla v_\varepsilon|}{(\eta - v_\varepsilon)^{1+\theta}} + \kappa p y_\varepsilon - k_1 y_\varepsilon^{1+\frac{1}{p}}, \qquad \text{on } (T,\infty), \end{aligned}$$

where an application of Young's inequality reveals that for any $t > T$,

$$\begin{aligned} &[2p\theta + \chi p(p-1)\eta]\int_\Omega \frac{u_\varepsilon^{p-1}(\cdot,t)|\nabla u_\varepsilon(\cdot,t)||\nabla v_\varepsilon(\cdot,t)|}{(\eta - v_\varepsilon(\cdot,t))^{1+\theta}} \\ &\le \frac{(2p\theta + \chi p(p-1)\eta)^2}{4(\theta(1+\theta) - \chi p\theta\eta)}\int_\Omega \frac{u_\varepsilon^{p-2}(\cdot,t)|\nabla u_\varepsilon(\cdot,t)|^2}{(\eta - v_\varepsilon)^\theta} + (\theta(1+\theta) - \chi p\theta\eta)\int_\Omega \frac{u_\varepsilon^p(\cdot,t)|\nabla v_\varepsilon(\cdot,t)|^2}{(\eta - v_\varepsilon)^{2+\theta}} \end{aligned}$$

and thereby leads to

$$\begin{aligned} y_\varepsilon' \le &\left(\frac{[2p\theta + \chi p(p-1)\eta]^2}{4(\theta(1+\theta) - \chi p\theta\eta)} - p(p-1)\right)\int_\Omega \frac{u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2}{(\eta - v_\varepsilon)^\theta} + \kappa p y_\varepsilon - k_1 y_\varepsilon^{1+\frac{1}{p}} \\ \le &\kappa p y_\varepsilon - k_1 y_\varepsilon^{1+\frac{1}{p}} \qquad \text{on } (T,\infty) \end{aligned} \tag{6.49}$$

by (6.46). Because

$$z(t) := \left[\left(\frac{1}{y_\varepsilon(T)^{\frac{1}{p}}} - \frac{k_1}{\kappa p}\right)\mathrm{e}^{-\kappa(t-T)} + \frac{k_1}{\kappa p}\right]^{-p} \le \left[\frac{k_1}{\kappa p}\left(1 - \mathrm{e}^{-\kappa(t-T)}\right)\right]^{-p}, \qquad t > T, \tag{6.50}$$

solves $z' = \kappa p z - k_1 z^{1+\frac{1}{p}}$, $z(T) = y_\varepsilon(T)$, by a straightforward comparison argument we infer $y_\varepsilon \le z$ on (T,∞) and thus

$$\int_\Omega u_\varepsilon^p \le \int_\Omega (\eta - v_\varepsilon)^\theta \frac{u_\varepsilon^p}{(\eta - v_\varepsilon)^\theta} \le \eta^\theta \int_\Omega \frac{u_\varepsilon^p}{(\eta - v_\varepsilon)^\theta} = \eta^\theta y_\varepsilon(t) \le \eta^\theta \left[\frac{k_1}{\kappa p}(1 - \mathrm{e}^{-\kappa})\right]^{-p}.$$

for $t > T^\star := T + 1$. □

One particular consequence of this bound is the following:

Lemma 6.3.9. *For any $p > 1$ and any $\delta > 0$ there is $T > 0$ such that for any $t > T$ there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$*

$$\int_t^{t+3} \left\| u_\varepsilon - \frac{\kappa}{\mu} \right\|_{L^p(\Omega)} < \delta.$$

Proof. Let $\delta > 0$, let $p > 1$. Employing Lemma 6.3.8 we let $T_0 > 0$, $\varepsilon_\star > 0$ and $C > 0$ be such that $\int_\Omega u_\varepsilon^{2p}(\cdot, t) \le C^{2p}$ for all $t > T_0$, $\varepsilon \in (0, \varepsilon_\star)$. From (6.40) and (6.37) we infer that $\int_0^\infty \int_\Omega \left(u - \frac{\kappa}{\mu}\right)^2$ is finite. Hence, there is $T > T_0$ such that for all $t > T$ we have $\int_t^{t+3} \int_\Omega \left(u - \frac{\kappa}{\mu}\right)^2 \le \frac{1}{2} \delta^{2p-2} 3^{3-2p} (C + \frac{\kappa}{\mu} |\Omega|^{\frac{1}{2p}})^{4-2p}$. Due to (6.37), for all $t > T$ we can find $\varepsilon_t > 0$ such that for all $\varepsilon \in (0, \varepsilon_t)$ we have $\int_t^{t+3} \int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2 < \delta^{2p-2} 3^{3-2p} \left(C + \frac{\kappa}{\mu} |\Omega|^{\frac{1}{2p}}\right)^{4-2p}$. For any $t > T$, we let $\varepsilon_0 := \min\{\varepsilon_\star, \varepsilon_t\}$. By interpolation and Hölder's inequality, for any $t > T$ and any $\varepsilon \in (0, \varepsilon_0)$:

$$\begin{aligned}
\int_t^{t+3} \left\| u_\varepsilon - \frac{\kappa}{\mu} \right\|_{L^p(\Omega)} &\le \int_t^{t+3} \left\| u_\varepsilon - \frac{\kappa}{\mu} \right\|_{L^2(\Omega)}^{\frac{1}{p-1}} \left\| u_\varepsilon - \frac{\kappa}{\mu} \right\|_{L^{2p}(\Omega)}^{\frac{p-2}{p-1}} \\
&\le \left(\int_t^{t+3} \left\| u_\varepsilon - \frac{\kappa}{\mu} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2p-2}} \left(\int_t^{t+3} \left\| u_\varepsilon - \frac{\kappa}{\mu} \right\|_{L^{2p}(\Omega)}^{\frac{2p-4}{2p-3}} \right)^{\frac{2p-3}{2p-2}} \\
&\le \left(\int_t^{t+3} \int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu} \right)^2 \right)^{\frac{1}{2p-2}} \left(\int_t^{t+3} \left(\|u_\varepsilon\|_{L^{2p}(\Omega)} + \left\| \frac{\kappa}{\mu} \right\|_{L^{2p}(\Omega)} \right)^{\frac{2p-4}{2p-3}} \right)^{\frac{2p-3}{2p-2}} \\
&\le \left(\delta^{2p-2} 3^{3-2p} \left(C + \frac{\kappa}{\mu} |\Omega|^{\frac{1}{2p}} \right)^{4-2p} \right)^{\frac{1}{2p-2}} \left(3 \left(C + \frac{\kappa}{\mu} |\Omega|^{\frac{1}{2p}} \right)^{\frac{2p-4}{2p-3}} \right)^{\frac{2p-3}{2p-2}} \\
&= \delta
\end{aligned}$$

□

6.3.4 Convergence of U

As starting point for convergence and eventual smoothness of U we prove the following

Lemma 6.3.10. *For any $q \in [1, 6)$ and any $\eta > 0$ there is $T > 0$ such that for any $t > T$ one can find $\varepsilon_0 = \varepsilon_0(t) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\int_t^{t+1} \|U_\varepsilon\|_{L^q(\Omega)}^2 < \eta.$$

Proof. According to Lemma 6.2.9 applied to $\zeta = \frac{\kappa}{\mu}$, there is $C > 0$ such that for any $\varepsilon > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |U_\varepsilon|^2 + \int_\Omega |\nabla U_\varepsilon|^2 \le C \int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu} \right)^2 + C \left(\int_\Omega |f|^{\frac{6}{5}} \right)^{\frac{5}{3}} \qquad \text{on } (0, \infty).$$

Due to (6.3) and the uniform bound for $\int_0^\infty \int_\Omega \left(u_\varepsilon - \frac{\kappa}{\mu}\right)^2$ from (6.40), apparently there is $C > 0$ such that

$$\int_\Omega |U_\varepsilon(\cdot, t)|^2 - \int_\Omega U_0^2 + \int_0^t \int_\Omega |\nabla U_\varepsilon|^2 \le C$$

for all $\varepsilon > 0$ and any $t > 0$. Accordingly, due to (6.35),

$$\int_0^\infty \int_\Omega |\nabla U|^2 \le \int_\Omega U_0^2 + C \quad \text{and hence} \quad \lim_{t\to\infty} \int_t^{t+1} \int_\Omega |\nabla U|^2 = 0.$$

Because $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$, for any $\eta > 0$ there is $T > 0$ such that for any $t > T$ we have $\int_t^{t+1} \|U\|_{L^q(\Omega)}^2 < \frac{\eta}{2}$ and thus, by (6.32), for any $\eta > 0$ there is $T > 0$ such that for any $t > T$ there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\int_t^{t+1} \|U_\varepsilon\|_{L^q(\Omega)}^2 < \eta$. □

Lemma 6.3.11. *For any $p \in [6, \infty)$ and any $\delta > 0$ there is $T > 0$ such that for any $t > T$ there is $\varepsilon_0 = \varepsilon_0(t) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\|U_\varepsilon(\cdot, s)\|_{L^p(\Omega)} < \delta \qquad \textit{for any } s \in [t, t+1].$$

Proof. We let $p \ge 6$ and choose $q \in (3, 6)$ such that

$$\frac{1}{2} - \frac{3}{2p} - 3\left(\frac{1}{q} - \frac{1}{p}\right) = \frac{1}{2} + \frac{3}{2p} - \frac{3}{q} \ge 0 \tag{6.51}$$

and define $\gamma := \frac{3}{2}(\frac{1}{q} - \frac{1}{p})$. We use L^p-L^q-estimates for the Stokes semigroup (see e.g. [14, Lemma 2.3]) to choose constants k_1, k_2, k_3, k_4 and k_5 such that

$$\begin{aligned} \left\|e^{-tA}\mathcal{P}\varphi\right\|_{L^p(\Omega)} &\le k_1 t^{-\gamma} \|\varphi\|_{L^q(\Omega)} && \text{for all } \varphi \in L^q(\Omega; \mathbb{R}^3) \text{ and all } t > 0, \\ \left\|e^{-tA}\mathcal{P}\nabla\cdot\varphi\right\|_{L^p(\Omega)} &\le k_2 t^{-\frac{1}{2}-\frac{3}{2p}} \|\varphi\|_{L^{\frac{p}{2}}(\Omega)} && \text{for all } \varphi \in L^{\frac{p}{2}}(\Omega; \mathbb{R}^3) \text{ and all } t > 0 \text{ and} \\ \left\|e^{-tA}\mathcal{P}\varphi\right\|_{L^p(\Omega)} &\le k_3 \|\varphi\|_{L^p(\Omega)} && \text{for all } \varphi \in L^p(\Omega; \mathbb{R}^3) \text{ and all } t > 0 \end{aligned} \tag{6.52}$$

and that

$$\begin{aligned} \int_0^3 \left\|e^{-(3-s)A}\mathcal{P}\varphi(\cdot, s)\right\|_{L^p(\Omega)} ds &\le k_4 \int_0^3 (3-s)^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})} \|\mathcal{P}\varphi(\cdot, s)\|_{L^{\frac{3}{2}}(\Omega)} ds \\ &\le k_5 \sup_{s\in(0,3)} \|\phi(\cdot, s)\|_{L^{\frac{3}{2}}(\Omega)} \end{aligned} \tag{6.53}$$

for all $\varphi \in L^\infty((0,3); L^{\frac{3}{2}}(\Omega; \mathbb{R}^3))$, and pick $\delta_0 \in (0, \delta)$ such that

$$\delta_0 < \left(2k_2 \int_0^3 s^{-\frac{1}{2}-\frac{3}{2p}} s^{-2\gamma} ds\right)^{-1} \quad \text{and} \quad \delta_0 \le \left(2k_2 \int_0^3 s^{-\frac{1}{2}-\frac{3}{2p}} ds\right)^{-1}. \tag{6.54}$$

We then choose t_0 such that for every $t > t_0$ we can find $\varepsilon_t > 0$ such that for any $\varepsilon \in (0, \varepsilon_t)$

$$\int_t^{t+1} \|U_\varepsilon\|_{L^q(\Omega)} < \frac{\delta_0}{4k_1}, \qquad \int_t^{t+3} \left\|u_\varepsilon - \frac{\kappa}{\mu}\right\|_{L^p(\Omega)} < \frac{\delta_0}{3^\gamma 4k_3 \|\nabla\Phi\|_{L^\infty(\Omega)}},$$
$$\text{and} \quad \sup_{s\in(t,t+3)} \|f(\cdot, s)\|_{L^{\frac{3}{2}}(\Omega)} < \frac{\delta_0}{3^\gamma 4k_5},$$

which is possible due to Lemma 6.3.10 (applied to $\eta = \left(\frac{\delta_0}{4k_1}\right)^2$ and combined with Hölder's inequality), Lemma 6.3.9 and (6.3). We let $t_1 > t_0$ and $\varepsilon \in (0, \varepsilon_{t_1})$ and find $t_\star \in (t_1, t_1 + 1)$ such that $\|U_\varepsilon(t_\star)\|_{L^q(\Omega)} < \frac{\delta_0}{4k_1}$. We define $T = t_0 + 2$. In $X = \{V \colon (t_\star, t_\star + 3) \to$

$L^p(\Omega;\mathbb{R}^3)$; $\sup_{s\in(0,3)} s^\gamma \|V(\cdot,t_\star+s)\|_{L^p(\Omega)} \le \delta_0\}$ we now consider the mapping $\Psi\colon X \to X$ given by

$$\Psi(V)(t) = \mathrm{e}^{-tA}U(\cdot,t_\star) + \int_{t_\star}^t \mathrm{e}^{(t-s)A}\mathcal{P}\big[-\nabla\cdot(Y_\varepsilon V\otimes V)(\cdot,s) + u_\varepsilon(\cdot,s)\nabla\Phi + f(\cdot,s)\big]\mathrm{d}s$$

for $t\in(t_\star,t_\star+3)$.
First, we verify that actually $\Psi(V)\in X$ for all $V\in X$. Taking into account (6.52) and (6.53), for any such V we may estimate

$$\begin{aligned}\|\Psi V(t)\|_{L^p(\Omega)} \le& k_1(t-t_\star)^{-\gamma}\|U_\varepsilon(\cdot,t_\star)\|_{L^q(\Omega)} + k_2\int_{t_\star}^t (t-s)^{-\frac12-\frac32(\frac2p-\frac1p)}\|(V\otimes V)(\cdot,s)\|_{L^{\frac p2}(\Omega)}\mathrm{d}s\\ &+ k_3\|\nabla\Phi\|_{L^\infty(\Omega)}\int_{t_\star}^{t_\star+3}\left\|u_\varepsilon(\cdot,s)-\frac{\kappa}{\mu}\right\|_{L^p(\Omega)}\mathrm{d}s + k_5\sup_{s\in(t_\star,t_\star+3)}\|f(\cdot,s)\|_{L^{\frac32}(\Omega)}\end{aligned}$$

for all $t\in(t_\star,t_\star+3)$. Thus, if we use the choice of $t_\star$ and ε, that Y_ε is contracting and that $\|V\otimes V\|_{L^{\frac p2}(\Omega)} \le \|V\|^2_{L^p(\Omega)}$ by Hölder's inequality, we see that for every $t\in(t_\star,t_\star+3)$ and every $V\in X$,

$$(t-t_\star)^\gamma\|\Psi(V)(t)\|_{L^p(\Omega)} \le \frac{\delta_0}{4} + \delta_0\left(\delta_0 3^\gamma k_2\int_{t_\star}^t (t-s)^{-\frac12-\frac{3}{2p}}(s-t_\star)^{-2\gamma}\mathrm{d}s\right) + \frac{\delta_0}{4} + \frac{\delta_0}{4} \le \delta_0, \tag{6.55}$$

where for the estimate $\int_{t_\star}^t (t-s)^{-\frac12-\frac{3}{2p}}(s-t_\star)^{-2\gamma}\mathrm{d}s \le \int_0^3 (t-s)^{-\frac12-\frac{3}{2p}}s^{-2\gamma}\mathrm{d}s$ we rely on (6.51) and where we take into account (6.54). Moreover, for any $V,W\in X$,

$$\begin{aligned}\|V\otimes V - W\otimes W\|_{L(\Omega)^{\frac p2}} &= \|V\otimes(V-W) + (V-W)\otimes W\|_{L^{\frac p2}(\Omega)}\\ &\le \left(\|V\|_{L^p(\Omega)} + \|W\|_{L^p(\Omega)}\right)\|V-W\|_{L^p(\Omega)} \le 2\delta_0\|V-W\|_{L^p(\Omega)}\end{aligned}$$

and hence for all $t\in(t_\star,t_\star+3)$,

$$\begin{aligned}\|\Psi(V)(t)-\Psi(W)(t)\|_{L^p(\Omega)} &\le k_2\int_{t_\star}^t (t-s)^{-\frac12-\frac{3}{2p}}\|V\otimes V - W\otimes W\|_{L^{\frac p2}(\Omega)}\mathrm{d}s\\ &\le 2k_2\delta_0\int_0^3 s^{-\frac12-\frac{3}{2p}}\mathrm{d}s\,\|V-W\|_{L^\infty((0,T);L^p(\Omega))},\end{aligned}$$

so that Ψ apparently is a contraction on X. Therefore, there is a unique fixed point of Ψ on X, which, due to the definition of Ψ, must coincide with the unique weak solution U_ε of (6.4c) on $(t_\star,t_\star+3)$ (cf. [87, Thm. V.2.5.1]). From (6.55) we may conclude that

$$\|U_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le \delta$$

for all $t\in(t_\star+1,t_\star+3)\supset(t_1+2,t_1+3)$. □

In the following lemmata, we will attempt to prove Hölder regularity of the components of a solution on intervals of the form (t_0,t_0+1) for $t_0>0$ by using that they satisfy certain PDEs. The estimates used for this purpose take into account initial data, that is, e.g., $U(t_0)$, about which we do not know much. Therefore, we introduce the following cut-off functions:

Definition 6.3.12. *Let $\xi_0\colon\mathbb{R}\to[0,1]$ be a smooth, monotone function, satisfying $\xi_0\equiv 0$ on $(-\infty,0]$ and $\xi_0\equiv 1$ on $(1,\infty)$ and for any $t_0\in\mathbb{R}$ we let $\xi_{t_0} := \xi_0(\cdot - t_0)$.*

We will employ this function in the proof of the following lemma on regularity of U_ε.

Lemma 6.3.13. *There are $\gamma \in (0,1)$, $T > 0$ and $C > 0$ such that for any $t > T$ one can find $\varepsilon_0 = \varepsilon_0(t) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the estimate*

$$\|U_\varepsilon\|_{C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1];\mathbb{R}^3)} \le C \tag{6.56}$$

holds true.

Proof. Let $s > 3$ and $s_1 > 2s$. Let $r > 1$ and let s_1' be such that $\frac{1}{s_1} + \frac{1}{s_1'} = 1$. According to Lemma 6.3.8 there are $C > 0$, $T_1 > 0$ satisfying that for any $t > T_1$ there is $\varepsilon_1 > 0$ such that

$$\int_t^{t+1} \|u_\varepsilon\|_{L^s(\Omega)} < C \qquad \text{for all } \varepsilon \in (0, \varepsilon_1) \tag{6.57}$$

and Lemma 6.3.11 makes it possible to find $T \ge T_1$, such that for all $t > T$ there is $\varepsilon_t \in (0, \varepsilon_1)$ such that

$$\|U_\varepsilon\|_{L^\infty((t,t+2);L^r(\Omega)} < C, \qquad \|U_\varepsilon\|_{L^\infty((t,t+2);L^s(\Omega))} < C, \qquad \|U_\varepsilon\|_{L^\infty((t,t+2);L^{s_1'}(\Omega))} < C \tag{6.58}$$

for every $\varepsilon \in (0, \varepsilon_t)$. Moreover, given any $t_0 > T$ we let $\xi := \xi_{t_0}$ as in Definition 6.3.12 and note that due to (6.57), (6.58) and (6.3) there is $k_0 > 0$ such that

$$\int_{t_0}^{t_0+2} \left\|\mathcal{P}\xi\left(u - \frac{\kappa}{\mu}\right)\nabla\Phi\right\|_{L^s(\Omega)}^s + \int_{t_0}^{t_0+2} \|\mathcal{P}\xi' U_\varepsilon\|_{L^s(\Omega)}^s + \int_{t_0}^{t_0+2} \|\mathcal{P}\xi f\|_{L^s(\Omega)}^s \le k_0 \tag{6.59}$$

for any $t_0 > T$ and $\varepsilon \in (0, \varepsilon_{t_0})$. We then for any $t_0 > 0$ let $\xi := \xi_{t_0}$ and observe that the function ξU_ε solves

$$\begin{cases} (\xi U_\varepsilon)_t = \Delta(\xi U_\varepsilon) - (Y_\varepsilon U_\varepsilon \cdot \nabla)\xi U_\varepsilon + \nabla(\xi P_\varepsilon) + \xi u_\varepsilon \nabla\Phi + \xi f + \xi' U_\varepsilon & \text{in } \Omega \times (t_0, \infty) \\ \nabla \cdot (\xi U_\varepsilon) = 0 \quad \text{in } \Omega \times (t_0, \infty) \\ (\xi U_\varepsilon)(\cdot, t_0) = 0 \quad \text{in } \Omega, \qquad (\xi U_\varepsilon) = 0 \quad \text{on } \partial\Omega \times (t_0, \infty) \end{cases}$$

and hence the known maximal Sobolev regularity estimate for the Stokes semigroup ([27], cf. Lemma 5.3.6 for the corresponding statement concerning the heat semigroup) yields a constant $k_1 > 0$ such that

$$\begin{aligned} &\int_{t_0}^{t_0+2} \|(\xi U_\varepsilon)_t\|_{L^s(\Omega)}^s + \int_{t_0}^{t_0+2} \left\|D^2(\xi U_\varepsilon)\right\|_{L^s(\Omega)}^s \\ &\quad \le k_1 \Bigg[0 + \int_{t_0}^{t_0+2} \|\mathcal{P}((\xi Y_\varepsilon U_\varepsilon \cdot \nabla) U_\varepsilon)\|_{L^s(\Omega)}^s + \int_{t_0}^{t_0+2} \left\|\mathcal{P}\xi\left(u - \frac{\kappa}{\mu}\right)\nabla\Phi\right\|_{L^s(\Omega)}^s \\ &\qquad + \int_{t_0}^{t_0+2} \|\mathcal{P}\xi' U_\varepsilon\|_{L^s(\Omega)}^s + \int_{t_0}^{t_0+2} \|\mathcal{P}\xi f\|_{L^s(\Omega)}^s \Bigg] \end{aligned} \tag{6.60}$$

From the boundedness of the Helmholtz projection in L^r-spaces and Hölder's inequality we obtain $k_2 > 0$ such that for any $t_0 > T$

$$\begin{aligned} \|\mathcal{P}(Y_\varepsilon U_\varepsilon \cdot \nabla)\xi U_\varepsilon(\cdot, t)\|_{L^s(\Omega)}^s &\le k_2 \|Y_\varepsilon U_\varepsilon(\cdot, t)\|_{L^{s_1'}(\Omega)}^s \|\nabla(\xi U_\varepsilon(\cdot, t))\|_{L^{s_1}(\Omega)}^s \\ &\le k_2 \|U_\varepsilon(\cdot, t)\|_{L^{s_1'}(\Omega)}^s \|\nabla(\xi U_\varepsilon)(\cdot, t)\|_{L^{s_1}(\Omega)}^s \end{aligned}$$

$$\leq k_2 C^s \|\nabla(\xi U_\varepsilon)(\cdot,t)\|^s_{L^{s_1}(\Omega)}$$

for all $t \in (t_0, t_0+2)$ and any $\varepsilon \in (0, \varepsilon_{t_0})$. We let $a = \frac{\frac{1}{3}-\frac{1}{s_1}+\frac{1}{r}}{\frac{2}{3}-\frac{1}{s}+\frac{1}{r}}$ and observe that $a \in (\frac{1}{2}, 1)$. Hence the Gagliardo-Nirenberg inequality (Lemma 3.4.3) provides us with $k_3 > 0$ such that for all $t_0 > T$

$$\begin{aligned}\|\nabla(\xi U_\varepsilon(\cdot,t))\|^s_{L^{s_1}(\Omega)} &\leq k_3 \left\|D^2(\xi U_\varepsilon)(\cdot,t)\right\|^{as}_{L^s(\Omega)} \|\xi U_\varepsilon(\cdot,t)\|^{(1-a)s}_{L^r(\Omega)} \\ &\leq k_3 C^{(1-a)s} \left\|D^2(\xi U_\varepsilon)(\cdot,t)\right\|^{as}_{L^s(\Omega)} \qquad \text{for all } t \in (t_0,t_0+2), \varepsilon \in (0,\varepsilon_{t_0}).\end{aligned}$$

Therefore, employing Young's inequality, we find $k_4 > 0$ such that for all $t_0 > T$

$$\begin{aligned}k_1 \int_{t_0}^{t_0+2} \|\mathcal{P}((Y_\varepsilon U_\varepsilon \cdot \nabla)(\xi U_\varepsilon))\|^s_{L^s(\Omega)} &\leq k_1 k_2 k_3 C^{(2-a)s} \int_{t_0}^{t_0+2} \left\|D^2(\xi U_\varepsilon)\right\|^{as}_{L^s(\Omega)} \\ &\leq \frac{1}{2} \int_{t_0}^{t_0+2} \left\|D^2(\xi U_\varepsilon)\right\|^s_{L^s(\Omega)} + 2k_4\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_{t_0})$. Combining this with (6.59), we thus can find $k_5 > 0$ such that (6.60) turns into

$$\int_{t_0}^{t_0+2} \|(\xi_{t_0} U_\varepsilon)_t\|^s_{L^s(\Omega)} + \frac{1}{2} \int_{t_0}^{t_0+2} \left\|D^2(\xi_{t_0} U_\varepsilon)\right\|_{L^s(\Omega)} \leq k_5$$

for any $t_0 > T$ and any $\varepsilon \in (0, \varepsilon_0)$. Accordingly, for any $s > 1$ there are $C > 0$, $T > 0$ such that for any $t > T$ there is $\varepsilon_0 > 0$ satisfying that for any $\varepsilon \in (0, \varepsilon_0)$

$$\|U_{\varepsilon t}\|_{L^s((t,t+1);L^s(\Omega))} + \|U_\varepsilon\|_{L^s((t,t+1);W^{2,s}(\Omega))} \leq C.$$

Finally, by [5, Thm. 1.1], this implies (6.56). □

6.3.5 Eventual smoothness of v

Applying a similar reasoning, concerning v we obtain bounds of the same kind.

Lemma 6.3.14. *For every $p \in (1, \infty)$ there are $C > 0$ and $T > 0$, such that for any $t > T$ there is $\varepsilon_0 > 0$ such that*

$$\int_t^{t+1} \|v_{\varepsilon t}\|_{L^p(\Omega)} + \int_t^{t+1} \|v_\varepsilon\|_{W^{2,p}(\Omega)} \leq C \tag{6.61}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and moreover there are $C > 0$, $T > 0$ and $\gamma \in (0,1)$ such that for any $t > T$ there is $\varepsilon_0 = \varepsilon_0(t) > 0$ such that

$$\|v_\varepsilon\|_{C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \leq C$$

for any $\varepsilon \in (0, \varepsilon_0)$.

Proof. Let $p \in (1, \infty)$ and choose $q \in (1, p)$. We use Lemma 6.3.13 and Lemma 6.3.8 to choose $T_\star > 0$ such that there are $C_U > 0$ and $C_u > 0$, for any $t > T_\star$ allowing us to find $\varepsilon_t > 0$ such that

$$\|U_\varepsilon\|_{L^\infty(\Omega\times(t,t+2))} \leq C_U \quad \text{and} \quad \|u_\varepsilon\|_{L^\infty((T,\infty);L^p(\Omega))} \leq C_u \qquad \text{for all } \varepsilon \in (0, \varepsilon_t).$$

We then employ maximal Sobolev estimates for the Neumann heat semigroup ([27], see Lemma 5.3.6), which yield $k_1 > 0$ such that

$$\begin{aligned}\int_{t_0}^{t_0+2} \|(\xi v_\varepsilon)_t\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2} \|\Delta(\xi v_\varepsilon)\|_{L^p(\Omega)}^p \\ \le k_1 \left(0 + \int_{t_0}^{t_0+2} \|\xi U_\varepsilon \cdot \nabla v_\varepsilon\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2} \|\xi v_\varepsilon \ln(1+\varepsilon u_\varepsilon)\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2} \|\xi' v_\varepsilon\|_{L^p(\Omega)}^p\right) \\ \le k_1 C_u^p \int_{t_0}^{t_0+2} \|\nabla(\xi v_\varepsilon)\|_{L^p(\Omega)}^p + 2k_1 \|v_0\|_{L^\infty(\Omega)}^p \left(C_u^p + \|\xi'\|_{L^\infty(\mathbb{R})}\right)\end{aligned}$$

for any $t_0 > T$ and any $\varepsilon \in (0, \varepsilon_0)$. With $k_2 > 0$ being the constant featured by the Gagliardo-Nirenberg inequality (Lemma 3.4.3 with the same argument as in Lemma 5.3.4 ii) to replace D^2 by Δ), for any $t_0 > T$ we moreover have

$$\begin{aligned}k_1 C_U^p \|\nabla(\xi v_\varepsilon)(\cdot,t)\|_{L^p(\Omega)}^p \le& k_2 \|\Delta(\xi v_\varepsilon)(\cdot,t)\|_{L^p(\Omega)}^{ap} \|\xi v_\varepsilon(\cdot,t)\|_{L^q(\Omega)}^{(1-a)p} + k_2 \|\xi v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^p \\ \le& \frac{1}{2} \|\Delta(\xi v_\varepsilon)(\cdot,t)\|_{L^p(\Omega)}^p + k_3 \|\xi v_\varepsilon(\cdot,t)\|_{L^q(\Omega)}^p + k_2 \|v_0\|_{L^\infty(\Omega)}^p\end{aligned}$$

for any $\varepsilon \in (0, \varepsilon_{t_0})$ and $t \in (t_0, t_0+2)$, where $k_3 > 0$ is obtained from Young's inequality and $a := \frac{\frac{1}{3} - \frac{1}{p} + \frac{1}{q}}{\frac{2}{3} - \frac{1}{p} + \frac{1}{q}}$ satisfies $a \in (\frac{1}{2}, 1)$. In total, we have found $k_4 > 0$ such that for any $t_0 > T$,

$$\int_{t_0}^{t_0+2} \|(\xi v_\varepsilon)_t\|_{L^p(\Omega)}^p + \frac{1}{2} \int_{t_0}^{t_0+2} \|\Delta(\xi v_\varepsilon)\|_{L^p(\Omega)}^p \le k_4$$

holds for any $\varepsilon \in (0, \varepsilon_{t_0})$. Due to $\xi \equiv 1$ on (t_0+1, t_0+2), we in particular have shown that for any $p > 1$ there are $C > 0$, $T := T_\star + 1 > 0$ such that for any $t_0 > T$ we can find $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$\int_{t_0+1}^{t_0+2} \|v_{\varepsilon t}\|_{L^p(\Omega)} + \int_{t_0+1}^{t_0+2} \|v_\varepsilon\|_{W^{2,p}(\Omega)} \le C.$$

Using sufficiently high values of p, an application of the embedding result in [5] refines this into the assertion on Hölder continuity. □

6.3.6 Smoothness of u

Inter alia depending on Lemma 6.3.8 and (6.61), we can achieve the same for u_ε:

Lemma 6.3.15. *There are $\gamma \in (0,1)$, $C > 0$ and $T > 0$ such that for any $t > T$ there is $\varepsilon_0 = \varepsilon_0(t) > 0$ with any $\varepsilon \in (0, \varepsilon_0)$ satisfying*

$$\|u_\varepsilon\|_{C^{1+\gamma, \frac{\gamma}{2}}(\overline{\Omega} \times [t, t+1])} \le C.$$

Proof. Let $p \in (1, \infty)$, $q \in (1, p)$. Using Lemma 6.3.8, we fix $T_0 > 0$, $C_u > 0$ such that for any $\varepsilon > 0$, for any $t > T_0$,

$$\|u_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)} \le C_u \qquad \|u_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le C_u, \qquad \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C_u.$$

Aided by Lemma 6.3.14 and Lemma 6.3.11 we then choose $C_v > 0$, $C_U > 0$ and $T > T_0$ such that for any $t > T$ there is $\varepsilon_t > 0$ such that for any $\varepsilon \in (0, \varepsilon_t)$

$$\|\nabla v_\varepsilon\|_{L^\infty(\Omega \times (t,t+2))} \le C_v \qquad \int_t^{t+2} \|\Delta v_\varepsilon\|_{L^{2p}(\Omega)}^{2p} \le C_v$$

$$\|U_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \le C_U \qquad \text{for all } s\in(t,t+2).$$

From the Gagliardo-Nirenberg inequality (Lemma 3.4.3 combined with e.g. [23, Thm. 19.1] (cf. Lemma 5.3.4 ii)) for estimating $\|D^2\phi\|$ by $\|\Delta\phi\|$)) and Young's inequality we see that for any $\eta>0$ we can find $C_\eta>0$ such that

$$\|\nabla\phi\|^p_{L^p(\Omega)} \le k_1\|\Delta\phi\|^{ap}_{L^p(\Omega)}\|\phi\|^{(1-a)p}_{L^q(\Omega)} + k_1\|\phi\|^p_{L^q(\Omega)} \le \eta\|\Delta\phi\|^p_{L^p(\Omega)} + C_\eta\|\phi\|^p_{L^q(\Omega)} \tag{6.62}$$

for all $\phi\in W^{2,p}(\Omega)$ with $\partial_\nu\phi\big|_{\partial\Omega}=0$, where $a=\frac{\frac13+\frac1q-\frac1p}{\frac23+\frac1q-\frac1p}$ and $k_1>0$ is the constant obtained from the Gagliardo-Nirenberg inequality. For any $t>T$ and any $\varepsilon\in(0,\varepsilon_t)$ we may estimate

$$\begin{aligned}\left\|\xi\nabla\cdot\left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon\right)\right\|^p_{L^p(\Omega)} &= \left\|\xi\frac{\nabla u_\varepsilon\cdot\nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}+\xi\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\Delta v_\varepsilon\right\|^p_{L^p(\Omega)}\\ &\le 2^p\|\nabla(\xi u_\varepsilon)\cdot\nabla v_\varepsilon\|^p_{L^p(\Omega)} + 2^p\|u_\varepsilon\|^p_{L^{2p}(\Omega)}\|\Delta v_\varepsilon\|^p_{L^{2p}(\Omega)}\\ &\le 2^pC_v^p\|\nabla(\xi u_\varepsilon)\|^p_{L^p(\Omega)} + 2^pC_u^p\|\Delta v_\varepsilon\|^p_{L^{2p}(\Omega)}\end{aligned}$$

on $(t,t+2)$. The maximal Sobolev estimates for the heat semigroup ([27]) once more assert that with some $k_2>0$

$$\begin{aligned}&\int_t^{t+2}\|\xi u_{\varepsilon t}\|^p_{L^p(\Omega)} + \int_t^{t+2}\|\Delta\xi u_\varepsilon\|^p_{L^p(\Omega)}\\ &\quad\le k_2\int_t^{t+2}\|\xi U_\varepsilon\cdot\nabla u_\varepsilon\|^p_{L^p(\Omega)} + k_2\chi\int_t^{t+2}\left\|\xi\nabla\cdot\left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon\right)\right\|^p_{L^p(\Omega)}\\ &\quad+ k_2\kappa\int_t^{t+2}\|\xi u_\varepsilon\|^p_{L^p(\Omega)} + k_2\mu\int_t^{t+2}\|\xi u_\varepsilon^2\|^p_{L^p(\Omega)}\end{aligned}$$

for any $t>0$ and any $\varepsilon>0$. Taking into account the estimates prepared above, for $t>T$ and $\varepsilon\in(0,\varepsilon_t)$ we thus obtain

$$\begin{aligned}&\int_t^{t+2}\|\xi u_{\varepsilon t}\|^p_{L^p(\Omega)} + \int_t^{t+2}\|\Delta(\xi u_\varepsilon)\|^p_{L^p(\Omega)}\\ &\quad\le k_2(C_U^p+2^pC_v^p)\int_t^{t+2}\|\nabla(\xi u_\varepsilon)\|^p_{L^p(\Omega)} + 2^pC_u^p\int_t^{t+2}\|\Delta v_\varepsilon\|^p_{L^{2p}(\Omega)} + k_2(\kappa+\mu)C_u(t+1-t)\\ &\quad\le 2^pC_u^p\sqrt{C_v} + c(\kappa+\mu)C_u(t+1-t) + \frac12\int_t^{t+2}\|\Delta(\xi u_\varepsilon)\|_{L^p(\Omega)} + C_\eta\|u_\varepsilon\|^p_{L^q(\Omega)},\end{aligned}$$

where we have used (6.62) with $\eta=\frac{1}{2k_2(C_U^p+2^pC_v^p)}$. Finally, since $\xi\equiv1$ on $(t+1,t+2)$, we conclude

$$\int_{t+1}^{t+2}\|u_{\varepsilon t}\|^p_{L^p(\Omega)} + \frac12\int_{t+1}^{t+2}\|\Delta u_\varepsilon\|^p_{L^p(\Omega)} \le C_1$$

for any $t>T$ and any $\varepsilon\in(0,\varepsilon_t)$, where we have set $C_1=2^pC_u^p\sqrt{C_v}+k_2(\kappa+\mu)C_u(t+1-t)+C_\eta C_u$. □

6.3.7 Improved smoothness

Having found uniform Hölder bounds on u_ε, v_ε and U_ε for $\varepsilon>0$ in the previous three lemmata, also u, v and U share this regularity and these bounds.

Corollary 6.3.16. *There are $\gamma \in (0,1)$ and $T_0 > 0$ as well as a subsequence $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}}$ of the sequence from Proposition 6.2.17 such that for any $t > T_0$*

$$u_\varepsilon \to u, \quad v_\varepsilon \to v \quad \text{in} \quad C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1]), \qquad U_\varepsilon \to U \quad \text{in} \quad C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1];\mathbb{R}^3)$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, there is $C > 0$ such that for all $t > T_0$

$$\|u\|_{C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \le C, \quad \|v\|_{C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \le C, \quad \|U\|_{C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1];\mathbb{R}^3)} \le C. \tag{6.63}$$

Proof. This is an immediate consequence of Lemma 6.3.15, Lemma 6.3.14 and Lemma 6.3.13. □

Lemma 6.3.17. *There are $T > 0$ and $\gamma \in (0,1)$ such that*

$$u, v \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T,\infty)), \quad U \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T,\infty);\mathbb{R}^3).$$

Proof. With ξ as in Definition 6.3.12 (and with T as in the previous lemmata), the problem

$$\widetilde{v}_t = \Delta\widetilde{v} + g, \quad \widetilde{v}(T) = 0, \quad \partial_\nu \widetilde{v}\big|_{\partial\Omega} = 0, \tag{6.64}$$

for $g = -\xi uv - \xi U\nabla v + v\xi' \in C^\gamma(\overline{\Omega}\times(T,\infty))$ is solved by ξv, the solution of this problem is unique according to [48, III.5.1], and there is a solution belonging to $C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T+1,\infty))$. We conclude that $v \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T+1,\infty))$.
Moreover, ξu solves the following initial boundary value problem for $\widetilde{u}$:

$$\widetilde{u}_t = \Delta\widetilde{u} - a\cdot\nabla\widetilde{u} + b, \quad \widetilde{u}(T) = 0, \quad \partial_\nu\widetilde{u}\big|_{\partial\Omega} = 0, \tag{6.65}$$

where

$$a = \chi\nabla v + U, \qquad b = -\chi u\Delta v\xi + \kappa u\xi - \mu u^2\xi + \xi' u$$

satisfy $a, b \in C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times(T,\infty))$. Theorem [48, IV.5.3] guarantees the existence of a solution $\widetilde{u} \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T,\infty))$ to (6.65), and the uniqueness assertion in [48, III.5.1] for weak solutions of (6.65) shows that $\xi u = \widetilde{u}$. Due to $\xi \equiv 1$ on $[T+1,\infty)$, we conclude $u \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[T+1,\infty))$.
Finally, ξU solves

$$\begin{cases} (\xi U)_t = \Delta(\xi U) + \mathcal{P}(\xi' U - \xi(U\cdot\nabla)U + \xi u\nabla\Phi + \xi f), & \nabla\cdot(\xi U) = 0, \\ (\xi U)(\cdot, T-1) = 0, \\ (\xi U)\big|_{\partial\Omega} = 0, \end{cases}$$

where $\mathcal{P}(\xi' U - \xi(U\cdot\nabla)U + \xi u\nabla\Phi + \xi f)$ is Hölder continuous according to Corollary 6.3.16, (6.3) and [14, Lemma A.2]. The regularity assertion of [88, Thm. 1.1], if combined with the uniqueness result in [87, Thm. V.1.5.1], thus yields the desired smoothness of ξU on $[T-1,\infty)$ and hence of U on $[T,\infty)$. □

6.3.8 Convergence

Lemma 6.3.18. *The solution (u, v, U) of (6.1) constructed in Proposition 6.2.17 satisfies*

$$u(\cdot,t) \to \frac{\kappa}{\mu}, \qquad v(\cdot,t) \to 0, \qquad U(\cdot,t) \to 0 \qquad \text{as } t\to\infty$$

in $C^1(\overline{\Omega})$ and $C^1(\overline{\Omega};\mathbb{R}^3)$, respectively.

Proof. Assume $v(\cdot,t) \not\to 0$ in $C^1(\overline{\Omega})$ as $t \to \infty$. Then there are $\eta_0 > 0$ and $t_j \to \infty$ such that $\|v(\cdot,t_j)\|_{C^1(\overline{\Omega})} > \eta_0$ for all $j \in \mathbb{N}$. Due to (6.63) and by the compact embedding $C^{1+\gamma}(\overline{\Omega}) \overset{cpt}{\hookrightarrow} C^1(\overline{\Omega})$ there is some function $v_\infty \in C^1(\overline{\Omega})$ such that $v(\cdot,t_{j_k}) \to v_\infty$ along a subsequence $(t_{j_k})_{k\in\mathbb{N}}$ of $(t_j)_{j\in\mathbb{N}}$ and therefore $v(\cdot,t_{j_k}) \to v_\infty$ in $L^\infty(\Omega)$ as $t \to \infty$, which shows that, according to Corollary 6.3.7, $v_\infty = 0$. But $\|v(\cdot,t_{j_k})\|_{C^1(\overline{\Omega})} \to 0$ contradicts $\|v(\cdot,t_j)\|_{C^1(\overline{\Omega})} > \eta_0$ for all $j \in \mathbb{N}$.
We proceed similarly for U: Assuming $U \not\to 0$ in $C^1(\overline{\Omega};\mathbb{R}^3)$, we find $\eta_0 > 0$, $t_j \to \infty$ and $U_\infty \in C^1(\overline{\Omega};\mathbb{R}^3)$ such that $\|U(\cdot,t_{j_k})\|_{C^1(\overline{\Omega};\mathbb{R}^3)} > \eta_0$ and $U(\cdot,t_j) \to U_\infty$ in $C^1(\overline{\Omega};\mathbb{R}^3)$. If $U_\infty \not\equiv 0$, for some arbitrary $p > 4$ there are $\eta_1 > 0$ and a subsequence $(t_{j_k})_{k\in\mathbb{N}}$ of $(t_j)_{j\in\mathbb{N}}$ such that $\|U(\cdot,t_{j_k})\|_{L^p(\Omega)} > \eta_1$. With T_0 as in Corollary 6.3.16, we may use Lemma 6.3.11 to obtain $T > T_0$ such that for any $t > T$ there is $\varepsilon_t > 0$ such that for all $\varepsilon \in (0,\varepsilon_t)$ we have $\|U_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \frac{\eta_1}{2}$, and pick j such that $t_j > T$ and ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$, $\|U_\varepsilon(\cdot,t_j)\|_{L^p(\Omega)} < \frac{\eta_1}{2}$. We observe that thereby $\|U(\cdot,t_j) - U_\varepsilon(\cdot,t_j)\|_{L^p(\Omega)} > \eta_1 - \frac{\eta_1}{2} > 0$, which contradicts Corollary 6.3.16.
As to the convergence of u we define

$$u_j(x,s) := u(x,j+s), \quad x \in \overline{\Omega},\ s \in [0,1],$$

and claim that $u_j \to \frac{\kappa}{\mu}$ in $C^{1,0}(\overline{\Omega}\times[0,1])$ as $j \to \infty$. Were this not the case, we could find $\eta_0 > 0$ and a sequence $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, $j_k \to \infty$, such that $\left\|u_{j_k} - \frac{\kappa}{\mu}\right\|_{C^{1,0}(\overline{\Omega}\times[0,1])} > \eta_0$ for all $k \in \mathbb{N}$. Due to the bound on u in (6.63), $u_{j_{k_l}} \to u_\infty$ in $C^{1,0}(\overline{\Omega}\times[0,1])$ with some $u_\infty \in C^{1,0}(\overline{\Omega}\times[0,1])$. Because $\int_0^1\int_\Omega \left(u_j(x,s) - \frac{\kappa}{\mu}\right)^2 \mathrm{d}x\mathrm{d}s \to 0$ as $j \to \infty$ according to Corollary 6.3.3, $u_\infty \equiv \frac{\kappa}{\mu}$, contradicting either $u_{j_{k_l}} \to u_\infty$ or $\left\|u_{j_k} - \frac{\kappa}{\mu}\right\|_{C^{1,0}(\overline{\Omega}\times[0,1])} > \eta_0$. Hence $u_j \to \frac{\kappa}{\mu}$ in $C^1(\overline{\Omega}\times[0,1])$ as $j \to \infty$, and in particular $\sup_{s\in[0,1]}\left\|u_j(\cdot,s) - \frac{\kappa}{\mu}\right\|_{C^1(\overline{\Omega})} \to 0$ as $j \to \infty$ implies that $u(\cdot,t) \to \frac{\kappa}{\mu}$ as $t \to \infty$. □

6.3.9 Proof of Theorem 6.1.2

Proof of Theorem 6.1.2. The theorem immediately results from Lemmata 6.3.17 and 6.3.18. □

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