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Dissertation

Covers and Cores of r-graphs

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Abstract

There are many hard problems in graph theory which can be solved in the general case if they are solvable for cubic graphs. Possible minimal counterexamples for most of the problems are asked to be cyclically 4-edge-connected cubic class 2 graphs with girth at least 5. Such graphs were called snarks by Gardner. One major difficulty in proving theorems for snarks is to find appropriate structural parameters for a proof. An approach is to study invariants that "measure" how far a cubic graph is from being 3-edge-colorable. These invariants are called measures of edge-uncolorability in the literature.

Recently, Steffen introduced the core of a cubic graph as a structural parameter, and define a measure μ_3 by cores. By the study on cores and μ_3, he proved a couple of new or further results to some hard problems such as Berge-Fulkerson conjecture, Fan-Raspaud conjecture and problems on cycle-cover.

In this thesis, we first develop the theory of cores of cubic graphs and prove further results to Fan-Raspaud conjecture. Surprising to us, Fan-Raspaud conjecture is shown equivalent to a seemly weaker conjecture that every bridgeless cubic graph has a bipartite core. Moreover, we verify Fan-Raspaud conjecture for cubic graphs with relatively small value of μ_3, which improves a former result of Steffen. It is known that the Petersen coloring conjecture implies both Berge-Fulkerson conjecture and the cycle double cover conjecture. We prove a result to the Petersen coloring conjecture formulated by μ_3. This improves some earlier results of Bílková.

We relate μ_3 to some other measures, in particular, to the oddness ω. We prove that $\omega(G) \leq \frac{2}{3}\mu_3(G)$ for every bridgeless cubic graph G. Moreover, we introduce two more measures γ_2 and r_f, which are defined by 1-factors and by 4-flows, respectively. Relations among all these measures are given.

Secondly, we extend the theory of cores to weak cores, for cubic graphs. This allows to furnish the 5-line Fano-flow conjecture with several statements by weak cores, and also to define another new measure μ_3'. Analogously, $\frac{2}{3}\mu_3'$ is an upper bound for the weak oddness ω'.

Thirdly, we extend the theory of cores for cubic graph to r-graphs. One benefit is to pose the generalized Fan-Raspaud conjecture: every r-graph has an Eulerian core. This conjecture can be interpreted in the form of empty intersection of 1-factors, the same as Fan-Raspaud conjecture. It is a reasonable generalization of Fan-Raspaud conjecture because of its natural reflection to cores and of being implied by the generalized Berge-Fulkerson conjecture. Another benefit is to define a measure μ_3^r of edge-uncolorability for r-graphs. It is the first measure particularly for r-graphs, so far as we know.

As an approach to the solution to the generalized Berge-Fulkerson conjecture, we consider the union of 1-factors and for every integers $k \geq 1$ and $r \geq 3$, we prove a constant lower bound for the fraction of edges covered by k 1-factors in an r-graph. For the particular case $r = 3$, we obtain the result of Kaiser, Král and Norine, and of Mazzuoccolo.

Besides r-graphs, planar graphs are under discussion as well. We introduce two parameters "average face degree" and "local average face degree" for planar graphs and use them to characterize planar critical graphs G with $\Delta(G) \leq 6$. In particular, our result offers a characterization on the structure of possible minimal counterexamples to Vizing's planar graph conjecture.

Zusammenfassung

Viele schwere Probleme in der Graphentheorie können auf kubische Graphen reduziert werden. Für die meisten Probleme sind mögliche minimale Gegenbeispiele zyklisch 4-fach kantenzusammenhängende kubische Klasse 2 Graphen mit Taillenweite von mindestens 5. Diese Graphen wurden von Gardner als Snarks bezeichnet. Eine hauptsächliche Schwierigkeit bei den Beweisen von Theoremen für Snarks ist das Finden von geeigneten Strukturparametern für den Beweis. Eine Herangehensweise ist es Invarianten zu studieren, die ,,messen" wie weit ein kubischer Graph davon entfernt ist 3-kantenfärbbar zu sein; solche Invarianten werden auch Unfärbbarkeitsparameter genannt.

Kürzlich führte Steffen den Kern von kubischen Graphen als einen Strukturparameter ein und definierte den Parameter μ_3 durch Kerne. Durch das

Studium von Kernen und von μ_3 bewies er einige neue oder weitere Ergebnisse zu einigen schwierigen Problemen, wie der Berge-Fulkerson Vermutung, der Fan-Raspaud Vermutung und Problemen über Kreisüberdeckungen.

In dieser Arbeit entwickeln wir als erstes die Theorie von Kernen von kubischen Graphen weiter und beweisen weitere Ergebnisse zu der Fan-Raspaud Vermutung. Es ist überraschend, dass die Fan-Raspaud Vermutung äquivalent zu der scheinbar schwächeren Vermutung ist, dass jeder brückenlose kubische Graph einen bipartiten Kern hat. Außerdem verifizieren wir die Fan-Raspaud Vermutung für kubische Graphen mit $\mu_3 \leq 9$, was ein früheres Ergebnis von Steffen verbessert. Es ist bekannt, dass die Petersen-Färbungs Vermutung die Berge-Fulkerson Vermutung und die doppelte Kreisüberdeckungsvermutung impliziert. Wir studieren partielle Petersen-Färbungen, um das Problem zu approximieren.

Wir vergleichen μ_3 mit einigen anderen Parametern, welche die „Unfärbbarkeit" kubischer Graphen messen, z. B. die Ungeradheit ω. Wir beweisen $\omega(G) \leq \frac{2}{3}\mu_3(G)$ für jeden brückenlosen kubischen Graphen G. Zudem führen wir zwei weitere Maße γ_2 und r_f ein und setzen sie in Beziehung zu anderen Parametern.

Zweitens erweiterten wir die Theorie von Kernen zu schwachen Kernen von kubischen Graphen. Dies erlaubt eine äquivalente Formulierung der 5-Linien Fano-Fluss Vermutung durch schwache Kerne, und außerdem die Definition von einem weiteren neuen Maß μ_3'. Analog ist $\frac{2}{3}\mu_3'$ eine obere Grenze für die schwache Ungeradheit ω'.

Drittens erweitern wir die Theorie von Kernen für kubische Graphen zu r-Graphen, was eine kanonische Verallgemeinerung der Fan-Raspaud Vermutung ermöglicht: Jeder r-Graph hat einen Eulerschen Kern. Diese Vermutung kann in Form von leeren Schnittmengen von 1-Faktoren interpretiert werden, ebenso wie die Fan-Raspaud Vermutung. Es ist eine sinnvolle Verallgemeinerung der Fan-Raspaud Vermutung durch ihren natürlichen Bezug zu Kernen und dadurch, dass sie durch die Verallgemeinerung der Berge-Fulkerson Vermutung impliziert ist. Ein weiterer Nutzen ist es, ein Maß μ_3^r von Kanten-

Unfärbbarkeit für r-Graphen zu definieren. Soweit es uns bekannt ist, ist es das erste Maß speziell für r-Graphen.

Als Lösungsansatz zu der verallgemeinerten Berge-Fulkerson Vermutung betrachten wir die Vereinigung von 1-Faktoren; und wir beweisen für alle ganzzahligen $k \geq 1$ und $r \geq 3$ eine konstante untere Schranke für den Anteil von Kanten, die von k 1-Faktoren in einem r-Graphen überdeckt werden. Für den speziellen Fall $r = 3$ erhalten wir die Ergebnisse von Kaiser, Král und Norine und von Mazzuoccolo.

Neben r-Graphen werden Kantenfärbung von planaren Graphen untersucht. Wir führen zwei Parameter, ,,durchschnittlicher Grad einer Fläche" und ,,lokaler durchschnittlicher Grad einer Fläche", für planare Graphen ein und benutzen diese zur Charakterisierung planarer kritischer Graphen G mit $\Delta(G) \leq 6$. Im Speziellen ermöglicht unser Ergebnis eine Charakterisierung der Struktur von möglichen minimalen Gegenbeispielen für Vizing's Vermutung über planare Graphen.

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To my parents,

to Yingli and our son Lois.

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Chapter 1

Introduction

1.1 Graphs: notations and terminologies

For the notations and terminologies not mentioned in the thesis, we follow the ones used in [75]. A *graph* G consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge joins two vertices, which are not necessarily distinct. Denote by $|S|$ the cardinality of a set S. The values $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of G, respectively. Denote by xy an edge between two vertices x and y, which are called the two *ends* of xy. A *loop* is an edge whose ends are the same vertex. Two or more edges having the same two ends are called *multiple edges*. A graph is *simple* if it contains no loops or multiple edges. A *multigraph* is a graph that has no loops but may have multiple edges. In this thesis, we consider finite multigraphs.

Let G be a graph. Two vertices are *adjacent* if there exists an edge between them. Two edges are *adjacent* if they share a common end. A vertex and an edge are *incident* if the vertex is an end of the edge. If two vertices are adjacent, then one is called a *neighbor* of the other. For a vertex u of G, let $N(u)$ be the set of neighbors of u; and for $S \subseteq V(G)$, let $N(S) = \bigcup_{v \in S} N(v)$. We write $N(x, y)$ short for $N(\{x, y\})$. Denote by $E(v)$ the set of edges incident with v. The value $|E(v)|$ is called the *degree* of v, denoted by $d_G(v)$, except that each loop in $E(v)$ counts twice. The *maximum degree* $\Delta(G)$ of a graph G

is defined as $\Delta(G) = \max\{d_G(v)\colon\ v \in V(G)\}$. If it is clear from the context, then $d(v)$ and Δ are frequently used.

A *circuit* is a close walk with no repetition of vertices and edges. A circuit of vertices $u_1, u_2, \ldots, u_k$ located in cyclic order is written as $[u_1u_2\ldots u_k]$. The union of pairwise disjoint circuits is called a *cycle*.

Let G be a graph. If either $S \subseteq V(G)$ or $S \subseteq E(G)$, then $G[S]$ denotes the subgraph of G induced by S. For $T \subseteq V(G)$, the set of edges of G with precisely one end in T is denoted by ∂T. Let H be a subgraph of G. We write $\partial(H)$ short for $\partial V(H)$. The *complement* $\overline{H}$ of H is a subgraph of G induced by the edges not in H, that is, $\overline{H} = G[E(G) \setminus E(H)]$.

Let G be a graph that may be unconnected. An *odd component* of G is a component of G containing odd number of vertices. Denote by $c(G)$ the number of components of G, and by $o(G)$ the number of odd components of G.

1.2 Background, contribution and structure of the thesis

In mathematics, one of the most well-known problems is the four color problem, which states that given any separation of a plane into contiguous regions, four colors are enough to color all the regions so that no two adjacent regions receive the same color. This problem was first proposed by Guthrie in 1852, and was solved by Appel and Haken in 1976 with assistance of computer. So now it is a theorem. In the language of graph theory, the Four Color Theorem (briefly, the 4CT) simply states that every loopless planar graph admits a 4-vertex-coloring.

The early attempts at proving the 4CT, though all failed, bring many significant results and useful techniques to graph theory. The following equivalence is due to Tait in 1880.

Theorem 1.1 ([67]). *The 4CT is equivalent to the statement that every bridgeless cubic planar graph admits a 3-edge-coloring.*

This theorem initiates the study on several aspects of graph theory, such as edge-colorings, snarks, factors, flows, and so on. The thesis contributes to most of these aspects.

Compared with vertex coloring, the theory of edge coloring has received less attention until relatively recently. However, edge coloring has strong connections to many other research fields, such as matching theory, factorization theory, Latin squares and scheduling theory. The chromatic index $\chi'(G)$ of a graph G is the minimum integer k such that G has a k-edge-coloring. A fundamental result on edge coloring is Vizing's bound for the chromatic index of a graph. By Vizing's theorems [69, 70], if G is a simple graph, then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$; and if G is a multigraph, then $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum multiplicity of an edge of G. Due to these results, a graph G is class 1 if $\chi'(G) = \Delta(G)$, and is class 2 otherwise.

1.2.1 Planar graphs

For planar graphs, Vizing [69] showed for each $k \in \{2, 3, 4, 5\}$ that there is a planar class 2 graph G with $\Delta(G) = k$. Hence, for these values of k, a k-critical graph does exist. Moreover, he proved that every planar graph G with $\Delta(G) \geq 8$ is class 1, and conjectured that every planar graph G with $\Delta(G) \in \{6, 7\}$ is class 1. This conjecture is called Vizing's planar graph conjecture. The case $\Delta = 7$ has been confirmed true [17, 58, 79], but the case $\Delta = 6$ is still open.

In Chapter 2, we introduce new parameters "local average face-degree" and "average face-degree" of a plane graph, where the former depends on the embedding of the planar graph but the latter does not. By these two parameters, we define "local average face-degree bound b_k^*" and "average face-degree bound $\overline{b}_k$" for a k-critical planar graph. We prove both upper bound and lower bound for each of $\overline{b}_k$ and b_k^*, and propose the question asking for the precise values of $\overline{b}_k$ and b_k^*. Beyond face-degree, 3-faces are also used to characterize planar critical graphs. We give short proofs to the following two statements: (1) every 5-critical plane graph has a 3-face adjacent to a 3-face or to a 4-face; (2) every 6-critical plane graph, if exists, has a vertex incident to at least four 3-faces. A significant longer proof of the statement (2) is given in [73], where the statement is formulated for all plane graphs. However, we point out that their proof works for critical graphs only. In particular, our results offer structural properties for possible minimal counterexamples to Vizing's planar graph conjecture.

1.2.2 Cubic graphs

Many hard problems in graph theory can be solved in general case if they are solvable for cubic graphs. Examples of such problems are the 4-color-problem and problems on cycle-covers, matching-covers and flows of graphs. We consider cubic graphs in Chapters 3, 4, 5 and 6.

For a cubic graph G that admits a 3-edge-coloring, each color class induces a 1-factor. So the edge set $E(G)$ can be covered by three 1-factors, and we say that G is 1-factorable. The study on factors and factorization starts from Petersen in 19th century, who proved two important theorems: (1) every even regular graph is 2-factorable; (2) every 2-connected 3-regular graph has a 1-factor. Later on König's theorem follows: every bipartite regular graph is 1-factorable. In Chapters 3, we review on 1-factors and on some long-standing conjectures concerning 1-factor covers. We obtain some new results on graphs having 1-factors with certain property.

A snark is a bridgeless cubic graph that is not 3-edge-colorable. To avoid trivial cases, a snark is often restricted to be cyclically 4-edge-connected and to have girth at least 5. By Tait's theorem, the 4CT asserts that there are no planar snarks. However, non-planar snarks do exist. The first known example is the Petersen graph, discovered in 1898. It is also the smallest snark. "The hunting of the snark", a name borrowed from a poem by Carroll, starts since then, regarding the essential role of snarks as possible minimal counterexamples to many hard problems in graph theory.

Throughout Chapters 4 and 5, snarks with specific properties are constructed to deal with problems mainly related to two well-known conjectures: (1) Berge-Fulkerson conjecture [13]: every bridgeless cubic graph G has six 1-factors such that each edge of G is contained in precisely two of them; (2) Fan-Raspaud conjecture [10]: every bridgeless cubic graph has three 1-factors M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

In Chapter 4, we follow the very recent introduction by Steffen [64] of the concept "cores of cubic graph", which provides a different but very promising approach to treat on these two conjectures, as well as some other hard conjectures. A core can be defined as follows. Let S_3 be a list of three 1-factors of a cubic graph G. For $0 \leq i \leq 3$, let E_i be the set of edges that are contained in precisely i elements of S_3. The core of G with respect to S_3 is the subgraph G_c of G induced by $E_0 \cup E_2 \cup E_3$. A k-core is a core with $|E_0| = k$.

We develop the theory of cores and furnish for Fan-Raspaud conjecture (equivalently, for the 4-line Fano-flow conjecture) several equivalent statements by cores. It is straightforward to restate Fan-Raspaud conjecture as: every bridgeless cubic graph has a cyclic core. Surprising to us, Fan-Raspaud conjecture is shown equivalent to a seemly weaker conjecture that every bridgeless cubic graph has a bipartite core. Moreover, we extend the theory of cores to weak cores, for cubic graphs. This allows to analogously furnish the 5-line Fano-flow conjecture with several equivalent statements by weak cores. Finally, we disprove a conjecture of Mazzuoccolo [49] that is stronger than Fan-Raspaud conjecture, and we reformulate this conjecture under a stronger connectivity assumption.

Invariants that could measure how far a cubic graph is from being class 1 is called measures of edge-uncolorability in the literature. Measures are used to prove partial results to some hard conjectures. In Chapter 5, we first review some well known measures such as oddness, resistance and so on. We mainly study the measure μ_3, which is defined by cores. We relate μ_3 to some other measures, in particular, to the oddness ω. We prove that $\omega(G) \leq \frac{2}{3}\mu_3(G)$ for every bridgeless cubic graph G. The bound is achieved by a family of snarks. For the equality case, every core has a specific structure and is called a Petersen core. However, the difference between them can be arbitrarily big, even if we additionally fix the oddness. Moreover, we introduce two more measures γ_2 and r_f, which is defined by the intersection of two 1-factors and by the support of 4-flows. Relations among all these measures and some other known measures are given. It turns out that μ_3 bounds all other measures mentioned. Finally, we verify Fan-Raspaud conjecture for 3-edge-connected cubic graphs G having no nontrivial 3-edge-cuts such that $\mu_3(G) \leq 9$. This improves a result of Steffen [64], where G is asked to satisfy $\mu_3(G) \leq 6$ instead.

Chapter 6 focuses on Jaeger's Petersen coloring conjecture [29], which s-tates that every bridgeless cubic graph has a Petersen coloring. This conjecture is stronger than Berge-Fulkerson conjecture, and also implies some other conjectures, such as cycle double cover conjecture. There are several equivalent statements to the Petersen coloring conjecture, one of them is that every bridgeless cubic graph has a normal 5-edge-coloring. However, only few results on this conjecture is known. Here, we follow Šámal's new approach [71] that might leads to a solution to this conjecture. For a given bridgeless cubic graph, we look for a 5-edge-coloring yielding normal edges as much as possible. In other words, we color the graph "as normal as possible" while the conjecture asserts that we can color the graph completely normal. The result of Bílková [1] targets some classes of cubic graphs and shows that, we can color a generalized prism so that $\frac{2}{3}$ of the edges are normal, and we can color a cubic graph of large girth so that almost $\frac{1}{2}$ of the edges are normal. Our result shows that every bridgeless cubic graph G has a proper 5-edge-coloring such that at least $|E(G)| - \mu_3(G)$ edges are normal, which improves these former results.

1.2.3 r-graphs

In Chapter 7, we discuss on r-graphs, in which field there are not much results either. An r-regular multigraph G is an r-graph if $|\partial(X)| \geq r$ for each odd $X \subseteq V(G)$. The class of r-graphs is a special class of r-regular graphs maintaining certain property of those that are r-edge-colorable. Moreover, the concept of

r-graph is an generalization of bridgeless cubic graph. Both the facts gain interest on r-graphs.

In 1979, Seymour [61] presented some basic results and proposed several conjectures on r-graphs. Vizing's bound for the chromatic index of simple graphs is conjectured to be true for all r-graphs. That is, if G is an r-graph then $\chi'(G) \leq r+1$. This conjecture, namely the r-graph conjecture, is proposed by Seymour. It is one of the central conjectures in the theory of edge coloring, and is true for $r \leq 15$. Furthermore, Seymour proposed a generalization of Berge-Fulkerson conjecture for r-graphs: every r-graph has $2r$ 1-factors such that each edge is contained in precisely two of them. He proved partial results to it. Deep results on r-graphs were obtained by Rizzi in 1999 [57], where he constructed r-graphs with specific properties in terms of 1-factors to disprove some conjectures of Seymour.

Here, we consider the union of 1-factors of r-graphs, targeting the generalized Berge-Fulkerson conjecture. Analogous to the cubic case, Mazzuoccolo [50] proved that the generalized Berge-Fulkerson conjecture is equivalent to the generalized Berge conjecture, which states that every r-graph has $2r-1$ 1-factors such that each edge is contained in at least one of them. We prove a constant lower bound for the maximum proportion of covered edges by k 1-factors for all r-graphs. This lower bound depends on k and r only. In particular, we obtain partial result to the generalized Berge conjecture, which asserts that the maximum proportion is one hundred percent. For the particular case $r = 3$, we obtain the result of Kaiser, Král and Norine [36] and of Mazzuoccolo [48].

Furthermore, we extend the theory of cores for cubic graphs to r-graphs, which provides a new perspective to deal with the problems on r-graphs. It is known that Fan-Raspaud conjecture can be easily restated as: every bridgeless cubic graph has a cyclic core. From this point of view, we propose the generalized Fan-Raspaud conjecture in the language of cores as well: every r-graph has an Eulerian core. We further interpret this conjecture in the normal form: every r-graph has r 1-factors $M_1, M_2, \ldots, M_r$ such that any $\lfloor \frac{r}{2} \rfloor_o + 2$

of them have empty intersection. By taking $r = 3$, this conjecture reduces to Fan-Raspaud conjecture. It is a reasonable generalization of Fan-Raspaud conjecture because of its natural reflection to cores and of being implied by generalized Berge-Fulkerson conjecture.

Moreover, we define a measure μ_3^r of edge-uncolorability for r-graphs which, as far as we know, is the first measure particularly for r-graphs. The basic question on r-graphs is to determine which r-graph is r-edge-colorable. Regarding the difficulty on answering this question directly, it is important to study measures for r-graphs, which determine how far an r-graph is from being r-edge-colorable.

Some parts of our results in the thesis have been published already. The results of

- Sections 2.2 and 2.5 are published in

  [32] L. Jin, Y. Kang and E. Steffen. Face-degree bounds for planar critical graphs. Electron. J. Combin. **23**(3) (2016) #P3.21.

- Section 2.4 are published in

  [33] L. Jin, Y. Kang and E. Steffen. Remarks on planar edge-chromatic critical graphs. Discrete Applied Math. **200** (2016) 200-202.

- Chapter 4 and Section 5.6 are published in

  [34] L. Jin, G. Mazzuoccolo and E. Steffen. Cores, joins and the Fano-Flow conjectures. To appear in Discuss. Math. Graph. arXiv:1601.05762 (2016).

- Sections 5.2–5.4 are published in

  [35] L. Jin and E. Steffen. Petersen cores and the oddness of cubic graphs. J. Graph Theory **84** (2017) 109-120.

- Section 7.2 are published in

  [31] L. Jin. Unions of 1-factors in r-graphs. arXiv:1509.01823 (2015).

Chapter 2

Matchings and edge-colorings

2.1 Preliminary

Let k be a positive integer. A *k-edge-coloring* of a graph G is a mapping $\phi\colon\ E(G) \to \{1, 2, \cdots, k\}$ such that $\phi(e_1) \neq \phi(e_2)$ for any two adjacent edges e_1 and e_2. In other words, a *k-edge-coloring* assigns each edge a color from $\{1, 2, \cdots, k\}$ so that no two adjacent edges receive the same color. The *edge-chromatic number* or *chromatic index* $\chi'(G)$ of a graph G is the minimum k such that G admits a k-edge-coloring.

For a k-edge-coloring of a graph G, a set of all the edges receiving one same color is called a *color class*. A *matching* of a graph G is a set of pairwise nonadjacent edges of G. Clearly, every color class is a matching. Hence, there is a one-to-one correspondence between a k-edge-coloring and a partition of the edge set into k pairwise disjoint matchings. A fundamental result on edge coloring is due to Vizing [69].

Theorem 2.1 ([69])**.** *If G is a simple graph, then $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$.*

By this theorem, we can divide simple graphs into two classes. The graph G is *class 1* if $\chi'(G) = \Delta(G)$, and *class 2* if $\chi'(G) = \Delta(G) + 1$. For more informations on edge coloring, we refer the readers to [65].

We conclude this part with further notations and terminologies needed for the rest of this chapter.

Let G be a graph. A vertex u of G is a *k-vertex* or a *k^+-vertex* or a *k^--vertex* if $d_G(u) = k$ or $d_G(u) \geq k$ or $d_G(u) \leq k$, respectively. If v is a neighbor of u, then we further call u a *k-neighbor* or a *k^+-neighbor* or a *k^--neighbor* of v, respectively.

A graph is *planar* if it is embeddable into the Euclidean plane. A *plane graph* (G, Σ) is a planar graph G together with an embedding Σ of G into the Euclidean plane. Let (G, Σ) be a plane graph. Denote by $F((G, \Sigma))$ the *face set* of (G, Σ). The *degree* $d_{(G,\Sigma)}(f)$ of a face f is the length of its facial circuit. If there is no harm of confusion, we also write $d_G(f)$ or $d(f)$ for short. Let k be a positive integer. A *k-face* or a *k^+-face* or a *k^--face* is a face of degree k or at least k or at most k, respectively. A vertex or an edge is *incident* with a face if it is contained in the facial circuit.

2.1.1 Critical graphs

Let k be a positive integer. A *k-critical* graph is a class 2 graph G with $\Delta(G) = k$ and $\chi'(H) < \chi'(G)$ for each proper subgraph H of G. We will collect necessary facts on critical graphs for the proof of our results demonstrated in Sections 2.4 and 2.5.

Lemma 2.2. *Let G be a critical graph and $e \in E(G)$. If $e = xy$, then $d_G(x) \geq 2$, and $d_G(x) + d_G(y) \geq \Delta(G) + 2$.*

Lemma 2.3 (Vizing's Adjacency Lemma [69])**.** *Let G be a critical graph. If $e = xy \in E(G)$, then at least $\Delta(G) - d_G(y) + 1$ vertices in $N(x) \setminus \{y\}$ have degree $\Delta(G)$.*

Lemma 2.4 ([79])**.** *Let G be a critical graph and $xy \in E(G)$. If $d(x)+d(y) = \Delta(G) + 2$, then the following three statements hold true.*

(1) Every vertex of $N(x, y) \setminus \{x, y\}$ is a $\Delta(G)$-vertex.

(2) Every vertex in $N(N(x, y))\setminus\{x, y\}$ has degree at least $\Delta(G) - 1$.

(3) If $d(x) < \Delta(G)$ and $d(y) < \Delta(G)$, then every vertex in $N(N(x, y))\setminus\{x, y\}$ has degree $\Delta(G)$.

Lemma 2.5 ([58])**.** *No critical graph has pairwise distinct vertices x, y, z, such that x is adjacent to y and z, $d(z) < 2\Delta(G) - d(x) - d(y) + 2$, and xz is in at least $d(x) + d(y) - \Delta(G) - 2$ triangles not containing y.*

Theorem 2.6 ([30])**.** *If G is a 3-critical graph, then $|E(G)| \geq \frac{4}{3}|V(G)|$.*

Theorem 2.7 ([76])**.** *Let G be a k-critical graph. If $k = 4$, then $|E(G)| \geq \frac{12}{7}|V(G)|$; and if $k = 5$, then $|E(G)| \geq \frac{15}{7}|V(G)|$.*

Theorem 2.8 ([43])**.** *If G is a 6-critical graph, then $|E(G)| \geq \frac{1}{2}(5|V(G)|+3)$.*

2.2 (Local) average face-degree

The results of this section have already been published in [32]. In this section, we introduce two new parameters of a planar graph: average face-degree and local average face-degree. Both parameters are defined in terms of faces. The former one is related to the parameter average degree, which is defined to be $\frac{\sum_{v \in V(G)} d_G(v)}{|V(G)|}$ for a graph G. It gives globe information on the structure of a planar graph, while the latter one carries information around each vertex for a plane graph. These two parameters will be used to characterize the structure of a planar critical graph G with $\Delta(G) \leq 6$.

Let (G, Σ) be a 2-connected plane graph and $F(G)$ be the set of faces of (G, Σ). The *average face-degree* $\overline{F}(G)$ of G is defined as

$$\overline{F}(G) = \frac{1}{|F(G)|} \sum_{f \in F(G)} d_{(G,\Sigma)}(f).$$

By applying Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ together with the fact $\sum_{f \in F(G)} d_{(G,\Sigma)}(f) = 2|E(G)|$, we can deduce that $\overline{F}(G) = \frac{2|E(G)|}{|E(G)|-|V(G)|+2}$, which is independent on the embedding Σ. Hence, it is reasonable to say the average face-degree for a planar graph. For convenience of using, we repeat the following fact.

Proposition 2.9. *If G is a planar graph, then $\overline{F}(G) = \frac{2|E(G)|}{|E(G)|-|V(G)|+2}$.*

Let u be a vertex of a plane graph (G,Σ) of degree k. Thus, u is incident to k pairwise distinct faces, say $f_1,\dots,f_k$. Let $F_{(G,\Sigma)}(u)=\frac{1}{k}(d_{(G,\Sigma)}(f_1)+\dots+d_{(G,\Sigma)}(f_k))$ and $F((G,\Sigma))=\min\{F_{(G,\Sigma)}(v)\colon\ v\in V(G)\}$. Clearly, $F((G,\Sigma))\geq 3$ since every face is of degree at least 3. As Figure 2.1 shows, $F((G,\Sigma))$ depends on the embedding Σ. The *local average face-degree* of a 2-connected planar graph G is defined as

$$F^*(G)=\max\{F((G,\Sigma))\colon\ (G,\Sigma) \text{ is a plane graph}\}.$$

This parameter is independent from the embeddings of G, and $F^*(G)\geq 3$ for all planar graphs.

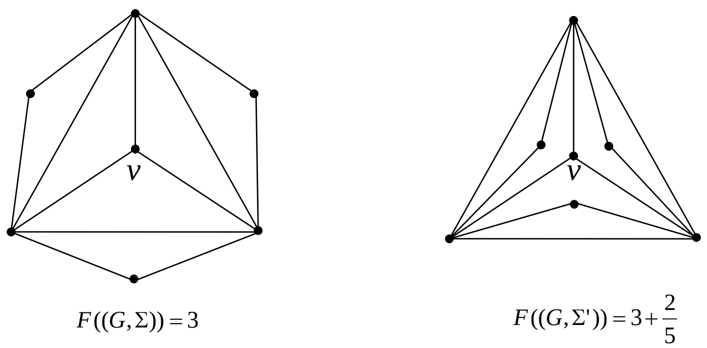

Figure 2.1: Graph G has two embeddings Σ, Σ' such that $F((G,\Sigma))\neq F((G,\Sigma'))$.

2.3 Vizing's planar graph conjecture

Vizing [69] showed for each $k\in\{2,3,4,5\}$ that there is a planar class 2 graph G with $\Delta(G)=k$. He proved that every planar graph with $\Delta\geq 8$ is a class 1 graph, and proposed the following conjecture.

Conjecture 2.10 (Vizing's planar graph conjecture [69]). *Every planar graph G with $\Delta(G)\in\{6,7\}$ is a class 1 graph.*

Vizing's conjecture is proved for planar graphs with $\Delta = 7$ by Grünewald [17], by Sanders and Zhao [58], and by Zhang [79] independently. However, the case $\Delta = 6$ is still open.

Zhou [80] proved for each $k \in \{3, 4, 5\}$ that if G is a planar graph with $\Delta(G) = 6$ and G does not contain a circuit of length k, then G is a class 1 graph. Vizing's conjecture is confirmed also for some other classes of planar graphs where some specific circuits are forbidden [3, 72, 73].

The next two sections devote to build structural properties for critical planar graphs of maximum degree at most 6. The main technique applied for the proofs is the Discharging Method, whose most famous application is the proof of 4CT. For a guide to this technique, we refer the readers to [6].

2.4 Characterization by 3-faces

The results of this section have already been published in [33]. This section provides short proofs for the following two theorems.

Theorem 2.11. *There is no 6-critical plane graph (G, Σ), such that every vertex of G is incident to at most three 3-faces.*

Proof. Suppose to the contrary that there is a counterexample to the statement. Then there is a 6-critical graph G which has an embedding Σ such that every vertex of G is incident to at most three 3-faces. By Euler's formula and Lemma 2.8 we deduce that $\sum_{f \in F(G)}(d_G(f) - 4) = 2|E(G)| - 4|F(G)| = 2|E(G)| - 4(|E(G)| + 2 - |V(G)|) \leq -|V(G)| - 11$. Therefore, $|V(G)| + \sum_{f \in F(G)}(d_G(f) - 4) \leq -11$.

Give initial charge 1 to each $v \in V(G)$ and charge $d_G(f) - 4$ to each $f \in F(G)$. Discharge the elements of $V(G) \cup F(G)$ according to the following rules:

R1: Every vertex sends $\frac{1}{3}$ to its incident 3-faces.

The rule only moves the charge around and does not affect the sum. Furthermore, the finial charge of every vertex and face is at least 0. Therefore,

$0 \leq \sum_{v \in V(G)} 1 + \sum_{f \in F(G)} (d_G(f) - 4) = |V(G)| + \sum_{f \in F(G)} (d_G(f) - 4) \leq -11$, a contradiction. □

If Vizing's conjecture is not true, then every 6-critical graph has the following property as a direct consequence of Theorem 2.11.

Corollary 2.12. *Let (G, Σ) be a plane graph. If G is 6-critical, then there is a vertex of G which is incident to at least four 3-faces.*

Theorem 2.13. *Let (G, Σ) be a plane graph. If G is 5-critical, then (G, Σ) has a 3-face which is adjacent to a 3-face or to a 4-face.*

Proof. Suppose to the contrary that there is a counterexample to the statement. Then there is a 5-critical graph G which has an embedding Σ such that every 3-face is adjacent to 5^+-faces only. Hence, every vertex of G is incident to at most two 3-faces, and every vertex which is incident to a 3-face is also incident to a 5^+-face. By Lemma 2.7, we have $\sum_{f \in F(G)} (d_G(f) - 4) \leq -\frac{2}{7}|V(G)| - 8$. Therefore, $\frac{2}{7}|V(G)| + \sum_{f \in F(G)} (d_G(f) - 4) \leq -8$.

Give initial charge of $\frac{2}{7}$ to each vertex and $d_G(f) - 4$ to each face of G. Discharge the elements of $V(G) \cup F(G)$ according to the following rules:
R1: Every vertex sends $\frac{1}{3}$ to its incident 3-faces.
R2: Every 5^+-face sends $\frac{d_G(f)-4}{d_G(f)}$ to its incident vertices.

Denote the finial charge by ch^*. Rules R1 and R2 imply that $ch^*(f) \geq 0$ for every $f \in F(G)$. Let $n \leq 2$ and v be a vertex which is incident to n 3-faces. If $n = 0$, then $ch^*(v) \geq \frac{2}{7} > 0$. If $n = 1$, then v is incident to at least one 5^+-face, and therefore, $ch^*(v) \geq \frac{2}{7} + \frac{1}{5} - \frac{1}{3} > 0$ by rule R2. If $n = 2$, then v is incident to at least two 5^+-faces, and therefore $ch^*(v) \geq \frac{2}{7} + 2 \times \frac{1}{5} - 2 \times \frac{1}{3} = \frac{2}{105} > 0$, by rule R2. Hence, $0 \leq \sum_{v \in V(G)} \frac{2}{7} + \sum_{f \in F(G)} (d_G(f) - 4) \leq -8$, a contradiction. □

A significant longer proof of Theorem 2.11 is given in [73], but the statement is formulated for plane graphs. However, the proof works for critical graphs only. The assumption that a minimal counterexample is critical is wrong. It might be that a subgraph of this minimal counterexample G does not fulfill the pre-condition of the statement. For example, if G has a triangle

$[vxyv]$ and a bivalent vertex u such that u is the unique vertex inside $[vxyv]$ and u is adjacent to x and y, then the removal of u increases the number of 3-faces containing v (see Figure 2.2).

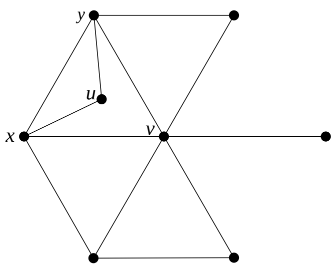

Figure 2.2: An example

2.5 Characterization by average face degree

The results of this section have already been published in [32].

Let k be a positive integer. Let $\bar{b}_k = \sup\{\overline{F}(G)\colon\ G$ is a k-critical planar graph$\}$ and $b_k^* = \sup\{F^*(G)\colon\ G$ is a k-critical planar graph$\}$. We call $\bar{b}_k$ the *average face-degree bound*, and b_k^* the *local average face-degree bound* for k-critical planar graphs. If $k = 1$ or $k \geq 7$, then every planar graph with $\Delta = k$ is a class 1 graph and therefore, $\{\overline{F}(G)\colon\ G$ is a k-critical planar graph$\} = \{F^*(G)\colon\ G$ is a k-critical planar graph$\} = \emptyset$. Hence, $\bar{b}_k$ and b_k^* do not exist in these cases. Therefore, we focus on these two parameters with index $k \in \{2, 3, 4, 5, 6\}$.

2.5.1 Lower bounds for $\overline{F}(G)$ and $F^*(G)$

Lemma 2.14. *Let t be a positive integer and $\epsilon > 0$.*

1. *For $k \in \{2, 3, 4\}$ there is a k-critical planar graph G and $F^*(G) > t$.*

2. *There is a 2-critical planar graph G with $\overline{F}(G) > t$.*

3. *There is a 3-critical planar graph G such that $6 - \epsilon < \overline{F}(G) < 6$.*

4. *There is a 4-critical planar graph G such that $4 - \epsilon < \overline{F}(G) < 4$.*

5. *There is a 5-critical planar graph G such that $3 + \frac{1}{3} - \epsilon < \overline{F}(G) < 3 + \frac{1}{3}$ and $F^*(G) \geq 3 + \frac{1}{5}$.*

Proof. The odd circuits are the only 2-critical graphs. Hence, the second statement and the first statement for $k = 2$ are proved. Let X and Y be two circuits of length $n \geq 3$, with $V(X) = \{x_i\colon\ 0 \leq i \leq n-1\}$, $V(Y) = \{y_i\colon\ 0 \leq i \leq n-1\}$ and edges x_ix_{i+1} and y_iy_{i+1}, where the indices are added modulo n. Consider an embedding, where Y is inside X. Add edges x_iy_i to obtain a planar cubic graph G with $F^*(G) = \frac{1}{3}(n+8)$. Add edges x_iy_{i+1} in G to obtain a 4-regular planar graph H with $F^*(H) = \frac{1}{4}(n+9)$. Subdividing one edge in G and one in H yields a critical planar graph G_n with $\Delta(G_n) = 3$, and a critical planar graph H_n with $\Delta(H_n) = 4$. If $n \geq 4t$, then $F^*(G_n) > t$ and $F^*(H_n) > t$. The proof that G_n and H_n are critical will be given in the last paragraph.

Since $|F(G_n)| = n + 2$, and $\sum_{f \in F(G_n)} d_{G_n}(f) = 6n + 2$, it follows that $\overline{F}(G_n) = 6 - \frac{10}{n+2}$. Analogously, we have $|F(H_n)| = 2n + 2$ and $\sum_{f \in F(H_n)} d_{H_n}(f) = 8n + 2$ and therefore, $\overline{F}(H_n) = 4 - \frac{3}{n+1}$. Now, the statements for 3-critical and 4-critical graphs follow. Examples of these graphs are given in Figure 2.3.

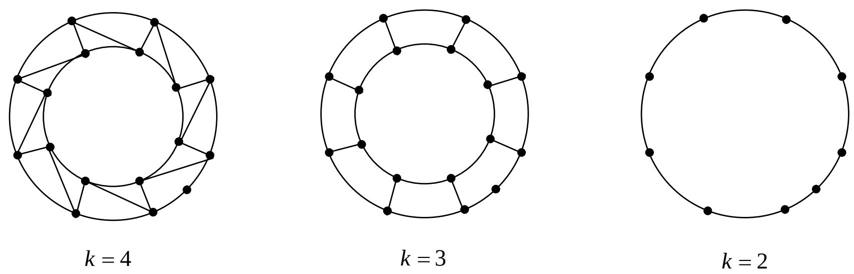

Figure 2.3: Examples for $k \in \{2, 3, 4\}$

Let $m \geq 4$ be an integer. Let $C_i = [c_{i,1}c_{i,2}\cdots c_{i,4}]$ be a circuit of length 4 for $i \in \{1, m\}$, and $C_i = [c_{i,1}c_{i,2}\cdots c_{i,8}]$ be a circuit of length 8 for $i \in \{2, \ldots, m-1\}$. Consider an embedding, where C_i is inside C_{i+1} for $i \in \{1, \ldots, m-1\}$. Add edges $c_{1,j}c_{2,2j-1}$, $c_{1,j}c_{2,2j}$, $c_{1,j}c_{2,2j+1}$ for $j \in \{1, \ldots, 4\}$, edges $c_{i,j}c_{i+1,j}$ for $i \in \{2, \ldots, m-2\}$ and $j \in \{1, \ldots, 8\}$, edges $c_{i,j}c_{i+1,j+1}$ for $i \in \{2, \ldots, m-2\}$

and $j \in \{2,4,6,8\}$, and edges $c_{m-1,2j-2}c_{m,j}$, $c_{m-1,2j-1}c_{m,j}$ and $c_{m-1,2j}c_{m,j}$ for $j \in \{1,\dots,4\}$ to obtain a 5-regular planar graph T (the indices are added modulo 8). Subdividing the edge $c_{m,1}c_{m,2}$ in T yields a critical planar graph T_m with $\Delta(T_m) = 5$ (Figure 2.4 illustrates T_6).

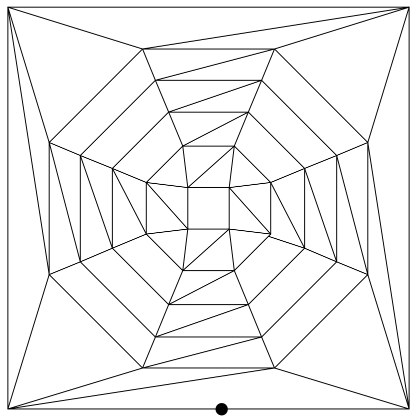

Figure 2.4: The plane graph (T_6, Σ_6)

Since $|F(T_m)| = 12m - 10$ and $\sum_{f \in F(T_m)} d_{T_m}(f) = 40m - 38$, it follows that $\overline{F}(T_m) = \frac{10}{3} - \frac{7}{18m-15}$. Furthermore, for the embedding Σ_m of T_m as indicated in Figure 2.4 (for $m = 6$) we calculate that $F((T_m, \Sigma_m)) = 3 + \frac{1}{5}$ and therefore, $F^*(T_m) \geq 3 + \frac{1}{5}$.

It remains to prove that G_n, H_n and T_m are critical. For G_n and H_n we proceed by induction on n. It is easy to verify the truth for $3 \leq n \leq 6$. We proceed to induction step. We argue first on G_n. Let u be the vertex of degree 2. Since $n \geq 7$, for any edge e of G_n, there exists some k such that no vertex of the circuit C is incident with e or adjacent to u, where $C = [x_{k+1}y_{k+1}y_{k+2}x_{k+2}]$. Reduce G_n to G_{n-2} by removing the edges $x_{k+1}y_{k+1}$ and $x_{k+2}y_{k+2}$ and suppressing their ends. Let G' be the resulting graph and e' be the resulting edge from e. By the induction hypothesis, G' is critical. Hence, $G' - e'$ has a 3-edge-coloring, say ϕ. Assign $\phi(x_k x_{k+3})$ to $x_k x_{k+1}$ and $x_{k+2}x_{k+3}$, and $\phi(y_k y_{k+3})$ to $y_k y_{k+1}$ and $y_{k+2}y_{k+3}$, and consequently, the edges of C can be properly colored. Now a 3-edge-coloring of $G_n - e$ is completed and so, $G_n - e$ is class 1. Moreover, since G_n is overfull, this graph is class 2. Therefore, G_n is critical. The argument on H_n is analogous.

For any T_m, recall that T is the graph obtained from T_m by suppressing the bivalent vertex. Consider T. Since each circuit C_i has even length, their edges can be decomposed into two perfect matchings M_1 and M_2, so that M_1 contains $c_{i,1}c_{i,2}$ for $i \in \{1, m\}$ and $c_{i,2}c_{i,3}$ for $2 \leq i \leq m-1$. Let $M_3 = \{c_{1,j}c_{2,2j+1}\colon\ 1 \leq j \leq 4\} \cup \{c_{i,2j}c_{i+1,2j+1}\colon\ 2 \leq i \leq m-2, 1 \leq j \leq 4\} \cup \{c_{m-1,2j-2}c_{m,j}\colon\ 1 \leq j \leq 4\}$. Clearly, M_3 is a perfect matching disjoint with M_1 and M_2. We can see that $E(G) \setminus (M_1 \cup M_2 \cup M_3)$ induces even circuits and hence, their edges can be decomposed into two perfect matchings M_4 and M_5, so that M_4 contains $c_{1,j}c_{2,2j}$ for $1 \leq j \leq 4$. Clearly, $M_1, \ldots, M_5$ constitute a decomposition of $E(T)$.

Let $e_i = c_{m,i}c_{m,i+1}$ for $1 \leq i \leq 4$. Let $M_2' = M_2 \cup \{e_1, e_3\} \setminus \{e_2, e_4\}$. Define $A_1 = M_1 \cup M_3, A_2 = M_2' \cup M_4, A_3 = M_2' \cup M_5$.

Let h_m be an edge of T_m. Since T_m is overfull, to prove that T_m is critical, it suffices to show that $T_m - h_m$ is a 5-edge-colorable.

Let h be the edge of T that corresponds to h_m. We can see that $A_1 \cup A_2 \cup A_3 = E(T) \setminus \{e_2, e_4\}$ and $e_1 \in A_1 \cap A_2 \cap A_3$. Hence, if $h \notin \{e_2, e_4\}$ then there exists $A \in \{A_1, A_2, A_3\}$ such that $e_1, h \in A$. Note that e_1 is the edge subdivided to get T_m from T, and that A induces a circuit of T. It follows that this circuit corresponds to a path P of $T_m - h_m$. Moreover, note that the edges of $T - A$ can be decomposed into 3 perfect matchings, and thus the same to the edges of $T_m - h_m - E(P)$. Therefore, $T_m - h_m$ is 5-edge-colorable.

If $h \in \{e_2, e_4\}$ then C_m corresponds to a path of $T_m - h_m$. Note that $E(C_m) \subseteq M_1 \cup M_2$ and that $M_1, \ldots, M_5$ constitute a decomposition of $E(T)$. Similarly, we can argue that $T_m - h_m$ is 5-edge-colorable in this case. □

2.5.2 Upper bounds for $\overline{F}(G)$ and $F^*(G)$

Proposition 2.15. *Let G be a k-critical planar graph.*

1. *If $k = 3$, then $\overline{F}(G) < 8$.*
2. *If $k = 4$, then $\overline{F}(G) < 4 + \frac{4}{5}$.*
3. *If $k = 5$, then $\overline{F}(G) < 3 + \frac{3}{4}$.*

4. If $k = 6$, then $\overline{F}(G) < 3 + \frac{1}{3}$.

Proof. Let $k = 3$ and suppose to the contrary that $\overline{F}(G) \geq 8$. With Lemma 2.9 and Theorem 2.6 we deduce $\frac{4}{3}|V(G)| \leq |E(G)| \leq \frac{4}{3}(|V(G)| - 2)$, a contradiction.

The other statements follow analogously from Proposition 2.9 and Theorem 2.7 ($k \in \{4, 5\}$) and Theorem 2.8 ($k = 6$). □

Theorem 2.16. *If G is a planar 5-critical graph, then $F^*(G) \leq 7 + \frac{1}{2}$.*

Proof. Suppose to the contrary that $F^*(G) = r > 7 + \frac{1}{2}$. Let Σ be an embedding of G into the Euclidean plane such that $F^*(G) = F((G, \Sigma))$. Let $V = V(G)$, $E = E(G)$, and $F = F((G, \Sigma))$. We proceed by a discharging argument in G and eventually deduce a contradiction. Define the initial charge ch in G as $ch(x) = d_G(x) - 4$ for $x \in V \cup F$. Euler's formula $|V| - |E| + |F| = 2$ can be rewritten as:

$$\sum_{x \in V \cup F} ch(x) = \sum_{x \in V \cup F} (d_G(x) - 4) = -8.$$

We define suitable discharging rules to change the initial charge function ch to the final charge function ch^* on $V \cup F$ such that $\sum_{x \in V \cup F} ch^*(x) \geq 0$ for all $x \in V \cup F$. Thus,

$$-8 = \sum_{x \in V \cup F} ch(x) = \sum_{x \in V \cup F} ch^*(x) \geq 0,$$

which is the desired contradiction.

Note that if a face f sends charge $-\frac{1}{3}$ to a vertex y, then this can also be considered as f receives charge $\frac{1}{3}$ from y. The discharging rules are defined as follows.

R1: Every 3^+-face f sends $\frac{d_G(f)-4}{d_G(f)}$ to each incident vertex.

R2: Let y be a 5-vertex of G.

R2.1: If z is a 2-neighbor of y, then y sends $\frac{2}{3} + \frac{2}{\lceil 2r \rceil - 3}$ to z.

R2.2: If z is a 3-neighbor of y, then y sends charge to z as follows:

R2.2.1: if z has a 4-neighbor, then y sends $\frac{1}{3} + \frac{2}{\lceil 3r \rceil - 6}$ to z;

R2.2.2: if z has no 4-neighbor, then y sends $\frac{2}{9}+\frac{4}{3(\lceil 3r\rceil-6)}$ to z.

R2.3: If z is a 4-neighbor of y and z is adjacent to n 5-vertices ($2 \le n \le 4$), then y sends $\frac{4}{n(\lceil 4r\rceil-9)}$ to z.

R2.4: If y is adjacent to five 4^+-vertices, then y sends $\frac{1}{3}(\frac{4}{\lceil 5r\rceil-12}+\frac{2}{\lceil 2r\rceil-3})$ to each 5-neighbor which is adjacent to a 2-vertex.

Claim 2.16.1. *If u is a k-vertex, then u receives at least $\frac{4-k}{3}-\frac{4}{\lceil rk\rceil-3k+3}$ in total from its incident faces by R1. In particular, if u is incident with at most two triangles, then u receives at least $\frac{1}{3}-\frac{4}{\lceil rk\rceil-4k+6}$ in total from its incident faces.*

Proof. Note that if a and b are integers and $2 \le a \le b$, then

$$\frac{1}{a-1}+\frac{1}{b+1} \ge \frac{1}{a}+\frac{1}{b}. \tag{2.1}$$

Let u be a k-vertex which is incident with faces $f_1, f_2, \cdots, f_k$. According to rule R1, u totally receives charge $S=\sum_{i=1}^{k}\frac{d_G(f_i)-4}{d_G(f_i)} = k-4\sum_{i=1}^{k}\frac{1}{d_G(f_i)}$ from its incident faces. The supposition $r \ge \frac{15}{2}$ implies that not all of $f_1,\ldots,f_k$ are triangles. It follows by formula 2.1 that $\sum_{i=1}^{k}\frac{1}{d_G(f_i)}$ reaches its maximum when all of $f_1,\ldots,f_k$ are triangles except one. Since $\sum_{i=1}^{k} d_G(f_i) \ge \lceil rk\rceil$, we have $S \ge k-4(\frac{1}{3}(k-1)+\frac{1}{\lceil rk\rceil-3(k-1)}) = \frac{4-k}{3}-\frac{4}{\lceil rk\rceil-3k+3}$. In particular, if u is incident with at most two triangles, then we have $S \ge k-4(\frac{2}{3}+\frac{1}{4}(k-3)+\frac{1}{\lceil rk\rceil-6-4(k-3)}) = \frac{1}{3}-\frac{4}{\lceil rk\rceil-4k+6}$. □

Claim 2.16.2. *The charge that a 5-vertex sends to a 4-neighbor by R2.3 is smaller than or equal to the charge that a 5-vertex sends to a 5-neighbor which is adjacent to a 2-vertex by R2.4, that is, $\frac{4}{n(\lceil 4r\rceil-9)} \le \frac{1}{3}(\frac{4}{\lceil 5r\rceil-12}+\frac{2}{\lceil 2r\rceil-3})$.*

Proof. Since $\frac{4}{n(\lceil 4r\rceil-9)} \le \frac{2}{\lceil 4r\rceil-9} \le \frac{2}{4r-9}$ and $\frac{1}{3}(\frac{4}{5r+1-12}+\frac{2}{2r+1-3}) \le \frac{1}{3}(\frac{4}{\lceil 5r\rceil-12}+\frac{2}{\lceil 2r\rceil-3})$, we only need to prove that $\frac{2}{4r-9} \le \frac{1}{3}(\frac{4}{5r+1-12}+\frac{2}{2r+1-3})$, which is equivalent to $2r^2-15r+23 \ge 0$ by simplification. Clearly, this inequality is true for every $r \ge 5+\frac{2}{5}$. □

It remains to check the final charge for all $x \in V \cup F$.

Let $f \in F$, then $ch^*(f) \geq d_G(f) - 4 - d_G(f)\frac{d_G(f)-4}{d_G(f)} = 0$ by R1.

Let $v \in V$. If $d_G(v) = 2$, then v receives at least $\frac{2}{3} - \frac{4}{\lceil 2r\rceil-3}$ in total from its incident faces by Claim 2.16.1. By Lemma 2.2, v has two 5-neighbors. Thus, v receives $\frac{2}{3} + \frac{2}{\lceil 2r\rceil-3}$ from each of them by R2.1. So we have $ch^*(v) \geq d_G(v) - 4 + (\frac{2}{3} - \frac{4}{\lceil 2r\rceil-3}) + 2(\frac{2}{3} + \frac{2}{\lceil 2r\rceil-3}) = 0$.

If $d_G(v) = 3$, then v receives at least $\frac{1}{3} - \frac{4}{\lceil 3r\rceil-6}$ in total from its incident faces by Claim 2.16.1. By Lemmas 2.2 and 2.3, v has three 4^+-neighbors, and two of them have degree 5. If v has a 4-neighbor, then by R2.2.1, $ch^*(v) \geq d_G(v) - 4 + (\frac{1}{3} - \frac{4}{\lceil 3r\rceil-6}) + 2(\frac{1}{3} + \frac{2}{\lceil 3r\rceil-6}) = 0$. Otherwise, by R2.2.2, $ch^*(v) \geq d_G(v) - 4 + (\frac{1}{3} - \frac{4}{\lceil 3r\rceil-6}) + 3(\frac{2}{9} + \frac{4}{3(\lceil 3r\rceil-6)}) = 0$.

If $d_G(v) = 4$, then v receives at least $-\frac{4}{\lceil 4r\rceil-9}$ in total from its incident faces by Claim 2.16.1. Say v has precisely n 5-neighbors. By Lemma 2.2, we have $2 \leq n \leq 4$. By R2.3, each of these 5-neighbors send $\frac{4}{n(\lceil 4r\rceil-9)}$ to v. Therefore, $ch^*(v) \geq d_G(v) - 4 - \frac{4}{\lceil 4r\rceil-9} + n\frac{4}{n(\lceil 4r\rceil-9)} = 0$.

If $d_G(v) = 5$, then v receives at least $-\frac{1}{3} - \frac{4}{\lceil 5r\rceil-12}$ in total from its incident faces by Claim 2.16.1. First assume v has a 2-neighbor, then by Lemma 2.4, v has four 5-neighbors and at least three of them are adjacent to no 3^--vertex. Hence, by R2.1 and R2.4, $ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} + \frac{4}{\lceil 5r\rceil-12}) - (\frac{2}{3} + \frac{2}{\lceil 2r\rceil-3}) + 3(\frac{1}{3}(\frac{4}{\lceil 5r\rceil-12} + \frac{2}{\lceil 2r\rceil-3})) = 0$.

Next assume that v has a 3-neighbor u, then by Lemma 2.3, v has at least three 5-neighbors. In this case, v sends nothing to each 5-neighbor. Let w be the remaining neighbor of v. Then $d_G(w) \in \{3, 4, 5\}$.

If $d_G(w) = 3$, then $uw \notin E(G)$ by Lemma 2.2. Furthermore, Lemma 2.5 implies that neither vw nor uv is contained in a triangle. It follows that v is incident with at most two triangles. Thus, by Claim 2.16.1, v receives a charge of at least $\frac{1}{3} - \frac{4}{\lceil 5r\rceil-14}$ in total from its incident faces. Moreover, both u and w have no 4^--neighbors. Suppose to the contrary that t is a 4^--neighbor of u (analogously of w). By Lemma 2.2, we have $d_G(t) = 4$. By applying Lemma 2.4 to ut, we have $d_G(w) \geq 4$, a contradiction. Hence, v sends $\frac{2}{9} + \frac{4}{3(\lceil 3r\rceil-6)}$ to each of u and w by rule R2.2.2, yielding $ch^*(v) \geq d_G(v) - 4 + (\frac{1}{3} - \frac{4}{\lceil 5r\rceil-14}) - 2(\frac{2}{9} + \frac{4}{3(\lceil 3r\rceil-6)}) = \frac{8}{9} - \frac{4}{\lceil 5r\rceil-14} - \frac{8}{3(\lceil 3r\rceil-6)}$.

If $d_G(w) = 4$, and if u is adjacent to w, then by Lemma 2.4, w has three 5-neighbors. Hence, by R2.2 and R2.3, $ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} + \frac{4}{\lceil 5r\rceil - 12}) - (\frac{1}{3} + \frac{2}{\lceil 3r\rceil - 6}) - \frac{4}{3(\lceil 4r\rceil - 9)} = \frac{1}{3} - \frac{2}{\lceil 3r\rceil - 6} - \frac{4}{3(\lceil 4r\rceil - 9)} - \frac{4}{\lceil 5r\rceil - 12}$. If u is not adjacent to w, then for any neighbor t of u, we have $d_G(t) \geq 4$ by Lemma 2.2. If $d_G(t) = 4$, then by applying Lemma 2.4 to ut we have $d_G(w) = 5$, a contradiction. Hence, $d_G(t) = 5$. This means all neighbors of u are of degree 5. By R2.2.2, $ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} + \frac{4}{\lceil 5r\rceil - 12}) - (\frac{2}{9} + \frac{4}{3(\lceil 3r\rceil - 6)}) - \frac{2}{\lceil 4r\rceil - 9} = \frac{4}{9} - \frac{4}{3(\lceil 3r\rceil - 6)} - \frac{2}{\lceil 4r\rceil - 9} - \frac{4}{\lceil 5r\rceil - 12}$.

If $d_G(w) = 5$, then v sends charge only to u. Hence, $ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} + \frac{4}{\lceil 5r\rceil - 12}) - (\frac{1}{3} + \frac{2}{\lceil 3r\rceil - 6}) = \frac{1}{3} - \frac{2}{\lceil 3r\rceil - 6} - \frac{4}{\lceil 5r\rceil - 12}$.

It remains to consider the case when v has five 4^+-neighbors. In this case it follows with Claim 2.16.2 that $ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} + \frac{4}{\lceil 5r\rceil - 12}) - 5(\frac{1}{3}(\frac{4}{\lceil 5r\rceil - 12} + \frac{2}{\lceil 2r\rceil - 3})) = \frac{2}{3} - \frac{32}{3(\lceil 5r\rceil - 12)} - \frac{10}{3(\lceil 2r\rceil - 3)}$.

Since $r > 7 + \frac{1}{2}$ it follows that $ch^*(x) \geq 0$ for all $x \in V \cup F$. $\square$

Theorem 2.17. *If G is a planar 6-critical graph, then $F^*(G) \leq 3 + \frac{2}{5}$.*

Proof. Suppose to the contrary that $F^*(G) > 3 + \frac{2}{5}$. Let Σ be an embedding of G into the Euclidean plane and $F^*(G) = F((G, \Sigma))$. We have

$$\begin{aligned}\sum_{f \in F(G)} (2d_G(f) - 6) &= 4|E(G)| - 6|F(G)| \\ &= 4|E(G)| - 6(|E(G)| + 2 - |V(G)|) \quad \text{(by Euler's formula)} \\ &= 6|V(G)| - 2|E(G)| - 12 \\ &\leq |V(G)| - 15 \qquad \text{(by Theorem 2.8)}\end{aligned}$$

and therefore,

$$-|V(G)| + \sum_{f \in F(G)} (2d_G(f) - 6) \leq -15. \tag{2.2}$$

Define the initial charge $ch(x)$ for each $x \in V(G) \cup F(G)$ as follows: $ch(v) = -1$ for every $v \in V(G)$ and $ch(f) = 2d_G(f) - 6$ for every $f \in F(G)$. It follows from inequality 2.2 that $\sum_{x \in V(G) \cup F(G)} ch(x) \leq -15$.

A vertex v is called heavy if $d_G(v) \in \{5, 6\}$ and v is incident with a face of length 4 or 5. A vertex v is called light if $2 \leq d_G(v) \leq 4$ and v is incident with no 6^+-face and with at most one 4^+-face. We say a light vertex v is bad-light if v has a neighbor u such that $d_G(u) + d_G(v) = 8$, and good-light otherwise.

Discharge the elements of $V(G) \cup F(G)$ according to following rules.

$R1$: every 4^+-face f sends $\frac{2d_G(f)-6}{d_G(f)}$ to each incident vertex.

$R2$: every heavy vertex sends $\frac{3}{10}$ to each bad-light neighbor, and $\frac{1}{10}$ to each good-light neighbor.

Let $ch^*(x)$ denote the final charge of each $x \in V(G) \cup F(G)$ after discharging. On one hand, the sum of charge over all elements of $V(G) \cup F(G)$ is unchanged. Hence, we have $\sum_{x \in V(G) \cup F(G)} ch^*(x) \leq -15$. On the other hand, we show that $ch^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$ and hence, this obvious contradiction completes the proof.

It remains to show that $ch^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$.

Let $f \in F(G)$. If $d_G(f) = 3$, then no rule is applied for f. Thus, $ch^*(f) = ch(f) = 0$.

If $d_G(f) \geq 4$, then by R1 we have $ch^*(f) = ch(f) - d_G(f)\frac{2d_G(f)-6}{d_G(f)} = 0$.

Let $v \in V(G)$. Firstly, we consider the case when v is heavy. On one hand, since $F((G, \Sigma)) > 3 + \frac{2}{5}$, it follows that either v is incident with a 5^+-face and another 4^+-face or v is incident with at least three 4-faces. In both cases, v receives at least $\frac{13}{10}$ in total from its incident faces by R1. On the other hand, we claim that v sends at most $\frac{3}{10}$ out in total. If v is adjacent to a bad-light vertex u, then all other neighbors of v have degree at least 5 by Lemma 2.4. Hence, v sends $\frac{3}{10}$ to u by R2 and nothing else to its other neighbors. If v is adjacent to no bad-light vertex, then v has at most three good-light neighbors by Lemma 2.3. Hence, v sends $\frac{1}{10}$ to each good-light neighbor by R2 and nothing else to its other neighbors. Therefore, $ch^*(v) \geq ch(v) + \frac{13}{10} - \frac{3}{10} = 0$.

Secondly, we consider the case when v is not heavy. In this case, v sends no charge out. If v is incident with a 6^+-face, then v receives at least 1 from this 6^+-face by R1. This gives $ch^*(v) = ch(v) + 1 = 0$. If v is incident with at least two 4^+-faces, then v receives at least $\frac{1}{2}$ from each of them by R1. This gives $ch^*(v) = ch(v) + \frac{1}{2} + \frac{1}{2} = 0$. We are done in both cases above. Hence, we may assume that v is incident with no 6^+-face and with at most one 4^+-face. From $F((G,\Sigma)) > 3 + \frac{2}{5}$ it follows that v is incident to a face f_v such that $d_G(f_v) \in \{4,5\}$. Since v is not heavy, $2 \leq d(v) \leq 4$. Hence, v is light by definition. We distinguish two cases by the length of f_v.

If $d_G(f_v) = 4$, then by the fact $F^*(G) \geq 3 + \frac{2}{5}$, we have $d_G(v) = 2$. By Lemma 2.2, both neighbors of v are heavy and v is bad-light. Thus, v receives $\frac{1}{2}$ from f_v by R1 and $\frac{3}{10}$ from each neighbor by R2, yielding $ch^*(v) = ch(v) + \frac{1}{2} + \frac{3}{10} + \frac{3}{10} > 0$.

If $d_G(f_v) = 5$, then v receives $\frac{4}{5}$ from f_v. If v is not a bad-light 4-vertex, then Lemma 2.2 implies that each neighbor of v has degree 5 or 6. Hence, both of the two neighbors of v contained in f_v are heavy. By R2, each of them sends charge at least $\frac{1}{10}$ to v, and therefore, $ch^*(v) \geq ch(v) + \frac{4}{5} + \frac{1}{10} + \frac{1}{10} = 0$. If v is a bad-light 4-vertex, then Lemma 2.3 implies that at least one of the two neighbors of v contained in f_v is heavy. Thus, this heavy neighbor sends charge $\frac{3}{10}$ to v, and therefore, $ch^*(v) \geq ch(v) + \frac{4}{5} + \frac{3}{10} > 0$. □

2.5.3 Bounds for $\bar{b}_k$ and b_k^*

The main results in this chapter are the following two theorems.

Theorem 2.18. *Let $k \geq 2$ be an integer.*

- *If $k = 2$, then $\bar{b}_k = \infty$.*
- *If $k = 3$, then $6 \leq \bar{b}_k \leq 8$.*
- *If $k = 4$, then $4 \leq \bar{b}_k \leq 4 + \frac{4}{5}$*
- *If $k = 5$, then $3 + \frac{1}{3} \leq \bar{b}_k \leq 3 + \frac{3}{4}$.*
- *If $k = 6$ and $\bar{b}_k$ exists, then $\bar{b}_k \leq 3 + \frac{1}{3}$.*

Proof. The statement for $k = 2$ and the lower bounds for $\bar{b}_k$ if $k \in \{3,4,5\}$ follow from Lemma 2.14. The other statements of Theorem 2.18 are implied by Proposition 2.15. □

Theorem 2.19. *Let $k \geq 2$ be an integer.*

- *If $k \in \{2, 3, 4\}$, then $b_k^* = \infty$.*
- *If $k = 5$, then $3 + \frac{1}{5} \leq b_k^* \leq 7 + \frac{1}{2}$.*
- *If $k = 6$ and b_k^* exists, then $b_k^* \leq 3 + \frac{2}{5}$.*

Proof. The statement for $k \in \{2, 3, 4\}$ and the lower bound for b_k^* follow from Lemma 2.14. The results for b_5^* and for b_6^* are implied by Theorem 2.16 and 2.17, respectively. □

Vizing [70] proved that a class 2 graph contains k-critical subgraph for each $k \in \{2, \ldots, \Delta\}$. Hence a smallest counterexample to Vizing's conjecture is critical and thus, our results for $k = 6$ partially characterize smallest counterexamples to this conjecture. For $k \leq 5$, they provide insight into the structure of planar critical graphs.

A graph G is *overfull* if G is of odd order and $|E(G)| \geq \Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor + 1$. Seymour's exact conjecture [65] says that every critical planar graph is overfull. If this conjecture is true for $k \in \{3, 4, 5\}$, then $\overline{b}_k$ is equal to the lower bound given in Theorem 2.18. It is not clear whether $\overline{b}_k$ and b_k^* are related to each other, or $\overline{F}(G)$ and $F^*(G)$ are. Furthermore, the precise values of $\overline{b}_k$ and b_k^* are also unknown.

Problem 2.20. *What are the precise values of $\overline{b}_k$ and b_k^*?*

2.5.4 Concluding remarks

Recently, Cranston and Rabern [5] improved Jakobsen's result (Theorem 2.6) on the lower bound on the number of edges in a 3-critical graph. They gave a computer-aided proof of the following statement.

Theorem 2.21 ([5])**.** *Every 3-critical graph G, other than the Petersen graph with a vertex deleted, has $|E(G)| \geq \frac{50}{37}|V(G)|$.*

Hence, $|E(G)| \geq \frac{50}{37}|V(G)|$ for every planar 3-critical graph. By a similar argument as in the proof of Proposition 2.15, this result improves the bound of $\overline{b}_3$ from $6 \leq \overline{b}_3 < 8$ to $6 \leq \overline{b}_3 < \frac{100}{13}$.

By Proposition 2.15, $\overline{F}(G)$ has an upper bound for every critical planar graph G. However, this is not always true for class 2 planar graphs. Similarly, Theorems 2.16 and 2.17 can not be generalized to class 2 planar graphs.

Chapter 3

1-factors and 1-factor covers

3.1 1-factor covers and cycle covers

Let k be a positive integer. A *k-regular* graph is a graph where each vertex has degree k. Let G be a graph. A subgraph of G is *spanning* if it has the same vertex set as G. A *k-factor* of G is a spanning k-regular subgraph of G. Thus, every perfect matching induces a 1-factor. A *1-factor cover* of G is a list of 1-factors whose union is $E(G)$, and a *1-factorization* of G is a partition of $E(G)$ into disjoint 1-factors. Thus, every 1-factorization is a particular 1-factor cover. If a graph has a 1-factorization, then it must be a regular graph. However, not all regular graphs have a 1-factorization. A k-regular graph has a 1-factorization if and only if it is k-edge-colorable.

Since the complement of a 1-factor of a 3-regular graph is a 2-factor, covers by 1-factors are closely related to covers by cycles. A cycle cover of a graph is a list $\mathcal{C}$ of cycles such that every edge of G is contained in at least one of them. It is a cycle double cover if each edge is contained in precisely two cycles, and is a k-cycle double cover if $\mathcal{C}$ consists of at most k cycles. Celmins and Preissmann independently formulated the 5-cycle double cover conjecture (briefly, 5CDCC) which is a stronger version of the cycle double conjecture (briefly, CDCC) of Seymour and Szekeres. An exhaustive survey on cycle (double) covers of graphs and related topics is given by Zhang [78].

Conjecture 3.1 (5-cycle double cover conjecture, see [77]). *Every bridgeless graph has a 5-cycle double cover.*

In this thesis, we focus on 1-factor covers other than cycle covers.

A *join* of a graph G is a parity subgraph of G, that is, a subgraph H where each vertex has the same parity of its degree in H and in G. Hence, in a 3-regular graph, every 1-factor is a join, and the complement of a join is a cycle.

3.2 Cubic graphs and snarks

A 3-regular graph is also called a *cubic graph.* There are many hard problems in graph theory for which it suffices to solve it for cubic graphs. Examples of such problems are the four color problem (now a theorem), problems concerning cycle covers or 1-factor covers, flow problems, and so on. For a cubic multigraph, remove two multiple edges, identify their two ends, and suppress the resulting bivalent vertex. Repeat this operation until we obtain a new cubic graph that is simple. Usually, if these problems can be solved for the new simple graph, then they are solvable for the original graph as well. Therefore, in the rest of the thesis, if not particularly indicated, cubic graphs are always assumed to be simple.

By Vizing's theorem, a cubic graph has chromatic index either 3 or 4, so it is class 1 or class 2, repectively. A *snark* is a class 2 cubic graph that is cyclically 4-edge-connected and of girth at least 5. Sometime, snarks are defined to be more relaxed in the literature: class 2 cubic graphs. Throughout the thesis, we follow the former definition, to avoid some trivial cases. Snarks were so named by Gardner [14] in 1976. Most of the problems on cubic graphs can be easily solved for class 1 cubic graphs. For class 2 cubic graphs that are not snarks, they easily reduce to smaller ones, see c.f. [4, 14, 16, 26, 54, 62]. Thus, possible minimal counterexamples to those problems are snarks.

The Petersen graph is the first known snark, discovered in 1898. It is also the smallest snark and serves as a useful example or counterexample for many problems in graph theory. Tutte conjectured that every snark has the Petersen graph as a minor, that is, every snark can be obtained from the Petersen graph by deleting edges and vertices and by contracting edges. As we notice, for lots of conjectures and theorems in graph theory, if there is a bound, then it can be achieved by the Petersen graph. All these make the Petersen graph play a very important role in graph theory.

There were only four snarks known until Isaacs [26] constructed infinite families of snarks in 1975. Later on, stronger criteria of non-triviality, reduction and constructions of snarks are considered. We next collect some well known snarks necessary for the thesis.

(1) Petersen graph. Given any 5-element set. Take 2-element subsets as vertices and put an edge between two vertices if and only if their corresponding sets are disjoint. We thereby obtain a graph with 10 vertices and 15 edges, which is called Petersen graph (see Figure 3.1) .

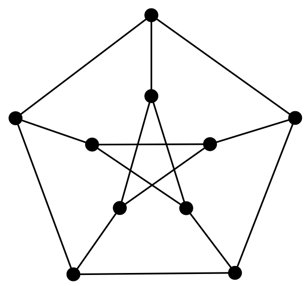

Figure 3.1: Petersen graph

(2) Flower snarks. Let n be an odd integer with $n \geq 5$. The flower snark J_n (see Figure 3.2) can be constructed in such a way: take n copies of $K_{1,3}$ where the vertex u is connected to three other vertices x, y, z; construct the circuits $[x_1 x_2 \ldots x_n]$ and $[y_1 \ldots y_n z_1 \ldots z_n]$. A flower snark J_n has girth 5 if $n = 5$, and girth 6 otherwise.

(3) Goldberg snarks. Let k be an odd integer with $k \geq 5$. The Goldberg snark G_k is formed from k copies $B_1, \ldots, B_k$ of the graph B (see Figure 3.3) and the edges of $\{a_i a_{i+1}, c_i b_{i+1}, e_i d_{i+1}\}$ for all $i \in \{1, \ldots, k\}$, where the indices are added modulo k.

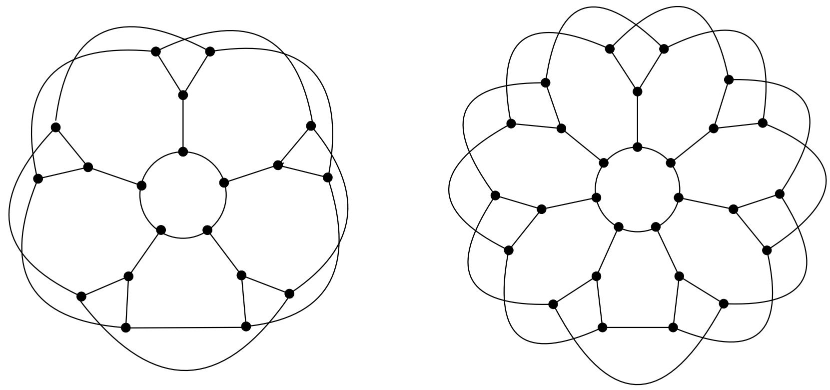

Figure 3.2: The Flower snarks J_5 (left) and J_7 (right)

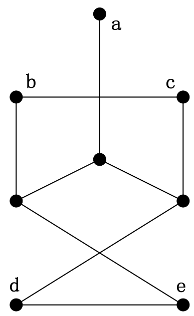

Figure 3.3: The graph B for constructing Goldberg snarks

3.3 Berge-Fulkerson conjecture and Berge conjecture

The following celebrated conjecture, often referred to as Berge-Fulkerson conjecture, is due to Fulkerson and appears first in [13]:

Conjecture 3.2 (Berge-Fulkerson conjecture [13]). *Every bridgeless cubic graph G has six 1-factors such that each edge of G is contained in precisely two of them.*

A set of such six 1-factors in the conjecture is called a *Fulkerson cover* of G. This conjecture trivially holds true for 3-edge-colorable cubic graphs. Thus a possible minimum counterexample to the conjecture is a snark. The conjecture has been verified for some families of snarks, see e.g. [22, 38, 44].

It is straightforward that Berge-Fulkerson conjecture implies the existence of five 1-factors whose union is the edge-set of the graph G. This naturally raises a seemly weaker conjecture, attributed to Berge (unpublished, see e.g. [77]).

Conjecture 3.3 (Berge conjecture). *Every bridgeless cubic graph G has five 1-factors such that each edge of G is contained in at least one of them.*

A set of such five 1-factors in the conjecture is called a *Berge cover* of G. Recently, Mazzuoccolo [47] proved that the previous two conjectures are equivalent. Note that this equivalence is referred to the class of bridgeless cubic graphs. However, it is still unclear whether the equivalence holds for every bridgeless cubic graph, that is, does a graph having a Berge cover always have a Fulkerson cover?

3.4 Fan-Raspaud conjecture and Fano flows

In 1994, the following statement was conjectured to be true by Fan and Raspaud [10].

Conjecture 3.4 (Fan-Raspaud conjecture [10]). *Every bridgeless cubic graph has three 1-factors M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

We remark that this conjecture is implied by Berge-Fulkerson conjecture. However, with regards to the structure of a possible minimal counterexample, this conjecture seems to be more difficult to treat on than Berge-Fulkerson conjecture. So far it is known that a possible minimal counterexample to Fan-Raspaud conjecture has girth at least 5 [45], but there are no constraints on the cyclic connectivity known.

The study of Fan-Raspaud conjecture leads to a deep analysis of Fano-flows on graphs. Consider the Fano plane $\mathcal{F}_7$ that has 7 points and 7 lines, where each point lies in 3 lines and each line touches 3 points, see Figure 3.4. A *Fano-coloring* of a cubic graph G is a mapping from $E(G)$ to the points of $\mathcal{F}_7$ such that any three edges of G around a vertex are mapped to three vertices of $\mathcal{F}_7$ that lie in a line.

As we can see in Figure 3.4, the points of the Fano plane can be labelled with the non-zero elements of $\mathbb{Z}_2^3$ so that the values of the points in a line sum up to zero. Inversely, for any three non-zero values from $\mathbb{Z}_2^3$ summing

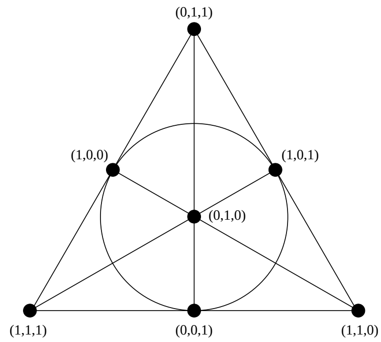

Figure 3.4: Fano plane $\mathcal{F}_7$

up to zero, they lie in a line of Fano plane. Hence, there is a one-to-one correspondence between a Fano-coloring and a nowhere-zero $\mathbb{Z}_2^3$-flow for a cubic graph. By this reason, a Fano-coloring is also called a *Fano-flow.*

By Jaeger's 8-flow Theorem [27], every bridgeless cubic graph has a nowhere-zero $\mathbb{Z}_2^3$-flow and hence a Fano-flow. However, it is possible that not all combinations of three non-zero elements of $\mathbb{Z}_2^3$ appear at a vertex of G, that is, not all the 7 lines are necessarily needed in a Fano-flow. For $k \leq 7$, a *k-line Fano-flow* of a cubic graph G is a Fano-flow of G where at most k lines of $\mathcal{F}_7$ appear as flow values at the vertices of G. Clearly, a cubic graph that is class 1 has a 1-line Fano-flow. Máčajová and Škoviera [45] proved that every Fano-flow of a bridgeless cubic class 2 graph needs all 7 points and at least 4 lines of the Fano plane. Furthermore, they proved that every bridgeless cubic graph has a 6-line Fano-flow, and conjectured that 4 lines are sufficient.

Conjecture 3.5 (4-line Fano-flow conjecture [45])**.** *Every bridgeless cubic graph has a 4-line Fano-flow.*

A natural relaxation of this conjecture is the following statement, namely the 5-line Fano-flow conjecture, still unsolved so far.

Conjecture 3.6 (5-line Fano-flow conjecture [45]). *Every bridgeless cubic graph has a 5-line Fano-flow.*

Conjectures 3.5 and 3.6 have surprisingly counterparts in terms of 1-factors. Máčajová and Škoviera [45] proved the equivalence between Fan-Raspaud conjecture and the 4-line Fano-flow conjecture. Some other equivalences were revealed in [40]. In particular, it was proved there that the 5-line Fano-flow conjecture is equivalent to Conjecture 3.7, and the 6-line Fano-flow theorem is equivalent to the statement that every bridgeless cubic graph has a 1-factor and two joins with empty intersection.

Conjecture 3.7. *Every bridgeless cubic graph has two 1-factors M_1, M_2 and a join J such that $M_1 \cap M_2 \cap J = \emptyset$.*

3.5 The existence of 1-factors

One of the earliest results in graph theory, Petersen's Theorem [55] from 1891, states that every bridgeless cubic graph has a 1-factor.

Theorem 3.8 (Petersen's Theorem [55]). *Every bridgeless cubic graph has a 1-factor.*

The first criterion for a graph to have a 1-factor was obtained by Tutte [68] in 1947. It is one of the most important results in factor theory, called the 1-Factor Theorem.

Theorem 3.9 (Tutte's 1-Factor Theorem [68]). *A graph G has a 1-factor if and only if*

$$o(G - S) \leq |S|, \text{ for every } S \subseteq V(G).$$

In modern textbooks, Petersen's theorem is covered as an application of Tutte's 1-Factor theorem. A generalization of Petersen's theorem appears in the same article of Petersen.

Theorem 3.10 ([55]). *Let G be a cubic graph. If there exists a path containing all the bridges of G, then G has a 1-factor.*

It is trivial that this theorem implies Petersen's Theorem. Here, we further extend this theorem from cubic graphs to k-regular graphs for every $k \geq 3$.

Theorem 3.11. *Let G be a k-regular graph of even order. If there exists a path containing an odd number of edges from each edge-cut of cardinality at most $k-2$, then G has a 1-factor.*

Proof. Take any $S \subseteq V(G)$. Denote by $H_1, \ldots, H_t$ all the odd components of $G - S$. Let P be the path mentioned in the condition of the theorem. Since P is a path, without loss of generality, we may assume that P intersects with each $\partial(H_i)$ on odd number of edges for $i \in \{1, 2\}$, and intersects with each $\partial(H_i)$ on even number of edges for $i \in \{3, \ldots, t\}$. So

$$|E(P) \cap \partial(H_1)| + |E(P) \cap \partial(H_2)| \geq 2. \tag{3.1}$$

For each $j \in \{3, \ldots, t\}$, it follows from the assumption on P that

$$|E(P) \cap \partial(H_j)| \geq k - 1. \tag{3.2}$$

Since the graph G is k-regular and the component H_i is odd, $|\partial(H_j)|$ has the same parity as k and thus, so does $|E(P) \cap \partial(H_j)|$. It follows from formula 3.2 that

$$|E(P) \cap \partial(H_j)| \geq k. \tag{3.3}$$

Now we calculate that

$$\begin{aligned} |\partial(S)| &\geq \sum_{i=1}^{t} |E(P) \cap \partial(H_i)| \\ &\geq k(t-2) + 2. \text{ (by formulas 3.1 and 3.3)} \end{aligned} \tag{3.4}$$

Moreover, since G is k-regular,

$$|\partial(S)| \leq k|S|. \tag{3.5}$$

Notice that $o(G-S)=t$. Combining formula 3.4 and 3.5 gives

$$o(G-S) \leq |S| + 2 - \frac{2}{r}. \tag{3.6}$$

Since G is of even order, $o(G-S)$ and $|S|$ have the same parity. Hence, we have from formula 3.6 that

$$o(G-S) \leq |S|. \tag{3.7}$$

By Tutte's 1-Factor Theorem, G has a 1-factor. □

3.5.1 1-factors containing certain edges

Petersen's Theorem can be strengthened so that the 1-factor contains an arbitrarily given edge. It is a result due to Schönberger [59].

Proposition 3.12 ([59]). *If e is an edge of a bridgeless cubic graph G, then G has a 1-factor containing e.*

Hence, it is of self-interest to explore sufficient conditions for a cubic graph to have a 1-factor such that the 1-factor contains more than one given edge. Following this direction, we present two results, one is due to Steffen [64] and the other is new.

Proposition 3.13 ([64]). *Let G be a bridgeless cubic graph having no non-trivial 3-edge-cut. Let M be a 1-factor of G and P be a path of length 3. If M and P have no common edge, then there is a 1-factor M' of G containing the two end-edges of P.*

We remark that this proposition was used to prove partial result to Berge conjecture in [64].

Theorem 3.14. *Let G be a bridgeless cubic bipartite graph. If e and f are two edges of G which are not contained in any 3-edge-cut of G, then G has a 1-factor containing both e and f.*

Proof. Since G is a bipartite cubic graph, the vertex set $V(G)$ can be divided into two independent sets A and B of the same order. Since e and f are not contained in any 3-edge-cut of G, they are disjoint. Assume that $e = a_1b_1$ and $f = a_2b_2$ with $\{a_1, a_2\} \subseteq A$ and $\{b_1, b_2\} \subseteq B$. Let G' be a graph obtained from G by removing e and f and adding a_1a_2 and b_1b_2, that is, $G' = G - e - f + a_1a_2 + b_1b_2$. Clearly, G' is cubic. Since G is bridgeless and since e and f are not contained in any 3-edge-cut of G, it follows that G' is bridgeless. Hence, by Proposition 3.12, the graph G' has a 1-factor M' containing a_1a_2. Notice that all the edges of G', except a_1a_2 and b_1b_2, connect vertices of A to vertices of B. It follows that M' contains b_1b_2 as well. Define M to be the 1-factor of G corresponding to M', that is, $M = (M' \setminus \{a_1a_2, b_1b_2\}) \cup \{e, f\}$. Hence, M is the 1-factor desired, we are done with the proof. □

3.5.2 1-factors avoiding certain edges

Now we consider the question about the existence of 1-factors avoiding certain edges for a cubic graph. Indeed, Proposition 3.12 can be reformulated as: if e and f are two adjacent edges of a bridgeless cubic graph G, then G has a 1-factor containing neither e nor f. Actually, the constrain "adjacent" is not necessary. Such an improvement holds true and has even been extended to k-regular graphs, due to Plesnik [56].

Theorem 3.15 ([56])**.** *Let G be a $(k-1)$-edge-connected k-regular graph of even order. For any $k-1$ many edges, G has a 1-factor containing none of them.*

We will discuss a variation of this theorem in Chapter 7. For the particular case $k = 3$, we have the following proposition.

Proposition 3.16. *Let G be a bridgeless cubic graph. For any two edges e and f of G, there exists a 1-factor of G containing neither e nor f.*

We are going to establish an analogous result but involving four edges. For doing this, we need a graph operation, namely "pushing".

Let G be a cubic graph, and let e_1 and e_2 be two disjoint edges of G with $e_i = u_i v_i$ for $i \in \{1, 2\}$. An operation that we delete e_1 and e_2 and then add two new vertex u, v and five new edges $uv, uu_1, uu_2, vv_1, vv_2$ is called *pushing* $\{e_1, e_2\}$ into $\{u, v\}$.

The proof for the following statement is straightforward.

Observation 3.17. *Let e and f be two disjoint edges of a graph G. Denote by G' the graph obtained from G by pushing $\{e, f\}$ into $\{u, v\}$. The following two statements hold true.*
(1) If F is a 1-factor of G' containing uv, then $F \setminus \{uv\}$ is a 1-factor of G containing neither e nor f.
(2) If C is an edge-cut of G' containing uv, then $(F \setminus \{uv\}) \cup \{e, f\}$ is an edge-cut of G.

Theorem 3.18. *Let G be a bridgeless cubic bipartite graph. If e, f, g, h are four edges of G such that there exists no 5-edge-cut of G containing all of them, then G has a 1-factor containing none of e, f, g, h.*

Proof. If there exist two edges in $\{e, f, g, h\}$ that are adjacent, without loss of generality, say e and f, then the edge shares the same end with e and f together with e, f, g, h forms a 5-edge-cut of G, a contradiction. Hence, the edges e, f, g, h are pairwise disjoint. By pushing $\{e, f\}$ into $\{u, v\}$ and $\{g, h\}$ into $\{x, y\}$, we obtain from G a new graph G'. Observation 3.17 (2) implies that G' has no 3-edge-cut containing both uv and xy, Hence, G' has a 1-factor F containing both uv and xy by Theorem 3.14. It follows from Observation 3.17 (1) that $F \setminus \{uv, xy\}$ is a 1-factor of the original graph G containing none of $\{e, f, g, h\}$. □

Chapter 4

Cores of cubic graphs

The results of this chapter have already been published in [34].

4.1 Definition and basic properties of cores

Cores were introduced by Steffen [64] very recently and were used to prove partial results on some hard conjectures which are related to 1-factors of cubic graphs, such as Berge conjecture, Fan-Raspaud conjecture, and conjectures on cycle cover and on cycle double cover.

Let G be a cubic graph and S_3 be a list of three 1-factors M_1, M_2, M_3 of G. For $0 \leq i \leq 3$, let E_i be the set of edges that are contained in precisely i elements of S_3. The edges of E_0 are called *uncovered edges*. Let $\mathcal{M} = E_2 \cup E_3$, $\mathcal{U} = E_0$ and $|\mathcal{U}| = k$. The *k-core* of G with respect to S_3 (or to M_1, M_2, M_3) is the subgraph G_c of G which is induced by $\mathcal{M} \cup \mathcal{U}$; that is, $G_c = G[\mathcal{M} \cup \mathcal{U}]$. If the value of k is irrelevant, then we say that G_c is a core of G. A core G_c is *proper* if $G_c \neq G$. Hence, G_c is not proper if and only if $M_1 = M_2 = M_3$. If G_c is a cycle, i.e., the union of pairwise disjoint circuits, then we call G_c a *cyclic core*. A *minimal core* of G is a k-core of G with minimum k. In other words, a minimal core has the least uncovered edges. Note that a cubic graph may have more than one minimal core. In [64] it is shown that every bridgeless cubic graph has a proper core and therefore, every minimal core is proper.

Some basic properties on the structure of a core and particularly of a minimal core were proposed in [64]. Here, we would like to mention two of them.

Lemma 4.1 ([64]). *Let k be a positive integer. If G_c is a k-core of a cubic graph G, then $k = |E_2| + 2|E_3|$.*

Lemma 4.2 ([64]). *Let G_c be a minimal core of a cubic graph G. If C is a circuit of G_c whose edges belong to $E_2 \cup E_3$ and to E_0 alternately along C, then for each $i \in \{1, 2, 3\}$, the circuit C has an edge $e \in E_2 \setminus M_i$.*

The proof of Lemma 4.1 is due to the adjacency between the edges from E_0 and from $\mathcal{M}$. Both of the lemmas will be used later for further exploration on the structure of cores.

4.2 Equivalent statements to Fan-Raspaud conjecture

As already mentioned in Section 3.4, Fan-Raspaud conjecture is equivalent to the 4-line Fano-flow conjecture. In this section, we restate Fan-Raspaud conjecture in the language of cores, and furnish for it more equivalent formulations. This offers insight into the structure of possible counterexamples to Fan-Ranspaud conjecture and inversely, into the structure of cores of bridgeless cubic graphs.

It is straightforward to reformulate Fan-Raspaud conjecture in terms of cyclic core. This equivalent conjecture was first addressed in [64].

Conjecture 4.3 ([64]). *Every bridgeless cubic graph has a cyclic core.*

Since every circuit in a cyclic core has even length, it follows that every cyclic core is bipartite. Steffen proposed the following seemingly weaker conjecture:

Conjecture 4.4 ([64]). *Every bridgeless cubic graph has a bipartite core.*

However, the inverse implication is not straightforward. There are many bipartite cores that are not cyclic, for example, a core consisting of one circuit $[u_0u_1\dots u_9]$ and two edges u_0u_5 and u_1u_4 from E_3. Here, we show that the inverse implication is also true and therefore, Conjectures 4.3 and 4.4 are equivalent. Besides, the following two conjectures are proposed and proved equivalent to Fan-Raspaud conjecture as well.

Conjecture 4.5. *Every bridgeless cubic graph has a triangle-free core.*

Conjecture 4.6. *Every bridgeless cubic graph has three 1-factors such that the complement of their union is an acyclic graph.*

Conjecture 4.6 can also be restated in language of core as: every bridgeless cubic graph has a core where the uncovered edges induce a forest. Moreover, the number "three" in this conjecture can not be lowered to "two", which will be proved in the last section of this chapter.

Let G_1 and G_2 be two bridgeless graphs, e_1 and e_2 be two edges such that $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. The *2-cut connection* on $\{e_1, e_2\}$ is a graph operation that consists of deleting edges e_1 and e_2 and adding two new edges u_1u_2 and v_1v_2. Clearly, the graph obtained from G_1 and G_2 by applying 2-cut connection is also bridgeless.

Now we are going to prove all these equivalences, as concluded in the following theorem.

Theorem 4.7. *The following four statements are equivalent:*

(1) (Conjecture 3.4) Every bridgeless cubic graph has three 1-factors M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

(2) (Conjecture 4.4) Every bridgeless cubic graph has a bipartite core.

(3) (Conjecture 4.5) Every bridgeless cubic graph has a triangle-free core.

(4) (Conjecture 4.6) Every bridgeless cubic graph has three 1-factors such that the complement of their union is an acyclic graph.

Proof. If statement (1) holds, then the core G_c of a bridgeless cubic graph G with respect to M_1, M_2, M_3 is cyclic. More precisely, each circuit in G_c contains edges from E_0 and E_2 alternate in cyclic order. Hence, the core G_c is bipartite and triangle-free, and $G[E_0]$ is an acyclic graph. Therefore, statement (1) implies all of the statements (2), (3) and (4).

Let G be a bridgeless cubic graph with edge set $\{e_1, \ldots, e_m\}$. Take m copies $T_1, \ldots, T_m$ of the complete graph K_4. For each $i \in \{1, \ldots, m\}$, apply 2-cut connection on e_i and an edge of T_i, and let e_i' and e_i'' be the two added new edges. Let G' be the resulting graph, which is bridgeless and cubic. Let H be a core of G' with respect to three 1-factors M_1, M_2, M_3. For every 1-factor F of G', since F contains either both of e_i' and e_i'' or none of them for each $i \in \{1, \ldots, m\}$, we can let $con(F) = \{e\colon\ e = e_i \in E(G)$, and $e_i', e_i'' \in F\}$. Clearly, $con(F)$ is a 1-factor of G. We claim that if H is either bipartite or triangle-free or if the complement of the union of M_1, M_2, M_3 is acyclic, then $con(M_1), con(M_2)$ and $con(M_3)$ have empty intersection. This claim completes the proof. Suppose to the contrary that G has an edge e_1 such that $e_1 \in con(M_1) \cap con(M_2) \cap con(M_3)$. It follows that $e_1', e_1'' \in M_1 \cap M_2 \cap M_3$. Hence, in the copy T_1, the core H contains triangles and $G[E_0]$ contains a circuit of length 4, a contradiction with the supposition of our claim. $\square$

To be concluded, it is worth mentioning one more conjecture, which is weaker than Conjecture 4.3 but still open.

Conjecture 4.8 ([64])**.** *Every bridgeless cubic graph has a bridgeless core.*

As proved in [64], every bipartite core is bridgeless. Hence, this conjecture is even weaker than Conjecture 4.4.

4.3 Weak cores

In this section, we are going to generalize the concept of cores to weak cores in a natural way that is involved with three joins instead of three 1-factors.

The aim of doing this is to deal with the k-line Fano-flow problems for other values of k.

Let J be a join of a cubic graph G. Clearly, every vertex has degree either 1 or 3 in J. A *J-vertex* is a vertex of degree 3 in J. Let $n(J)$ denote the number of J-vertices.

Let S be a set of three joins J_1, J_2, J_3 of a cubic graph G. For each $i \in \{0, \ldots, 3\}$, let $E_i(S)$ (briefly, E_i) be the set of edges that are contained in precisely i elements of S. The *weak core* of G with respect to S (or to J_1, J_2 and J_3) is a subgraph G_c induced by the union of the sets E_0, E_2 and E_3, that is, $G_c = G[E_0 \cup E_2 \cup E_3]$. Let l be the precise number of elements of S that are not 1-factors and let $k = |E_0| + \frac{3}{2}\sum_{i=1}^{3} n(J_i)$. The weak core G_c is further called a *l-weak k-core*. Our particular choice for the value of k will be more clear in the proof of Theorem 5.25. We can see that, the name of core is short for 0-weak core.

A join J is *simple* if the graph induced by all the J-vertices contains no circuit. Clearly, every 1-factor of G is a simple join, and every join of G contains a simple join as a subgraph. When we ask for empty intersection of three joins, it suffices to restrict the joins to being simple. Thus, we will focus on simple joins. A *simple weak core* is a weak core with respect to three simple joins. A weak core is *cyclic* if it is a cycle.

Analogously, Conjecture 3.7 can be directly formulated as a statement on cyclic 1-weak cores and will be proved equivalent to a statement on triangular-free simple 1-weak cores. Both statements are shown as conjectures below.

Conjecture 4.9. *Every bridgeless cubic graph has a cyclic 1-weak core.*

Conjecture 4.10. *Every bridgeless cubic graph has a triangle-free simple 1-weak core.*

Fano-flows can be related to cyclic weak cores in general sense. As a substitution of the k-line Fano-flow problem, we ask the following question:

Problem 4.11. *What is the minimum k such that every bridgeless cubic graph has a cyclic k-weak core?*

So far we known that $k \leq 2$. Conjectures 4.9 and 4.3 assert that $k \leq 1$ and $k = 0$, respectively.

Now we classify the vertices of a cubic graph G. This will benefit the proof of the next two propositions, which gives us some basic informations on weak cores. Let J_1, J_2 and J_3 be three joins of a cubic graph G. We say that a vertex v of G has *type* (x, y, z) if the three edges incident to v are covered x, y and z times by $\{J_1, J_2, J_3\}$, respectively. We denote by a, b, c, d, e, f, g the number of vertices of type $(3,3,3), (3,2,2), (3,1,1), (2,2,1), (1,1,1), (2,1,0), (3,0,0)$, respectively (see also Figure 4.1). Clearly, every vertex has precisely one type. Note that vertices of type $(3,3,3)$, $(3,2,2)$, $(3,1,1)$ and $(2,2,1)$ are J_i-vertices for some i.

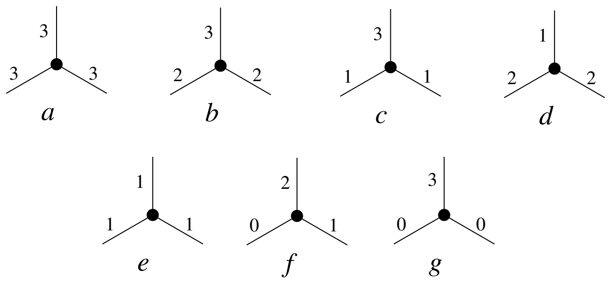

Figure 4.1: Vertex types

Proposition 4.12. *Let G be a cubic graph, and J_1, J_2, J_3 be three joins of G. We have*

$$|E_0| + \sum_{i=1}^{3} n(J_i) = |E_2| + 2|E_3|.$$

Proof. By type definitions, we have $\sum_i n(J_i) = 3a + 2b + c + d$, $|E_0| = \frac{f}{2} + g$, $|E_2| = b + d + \frac{f}{2}$ and $|E_3| = \frac{3a}{2} + \frac{b}{2} + \frac{c}{2} + \frac{g}{2}$. Hence, $\sum_i n(J_i) + |E_0| = 3a + 2b + c + d + \frac{f}{2} + g = |E_2| + 2|E_3|$ holds. □

Proposition 4.13. *If G_c is a weak core of a cubic graph G, then $G[E_0 \cup E_2]$ is either an empty graph or a cycle.*

Proof. By type definitions, it is easy to see that every vertex is incident with either none or precisely two edges of $E_0 \cup E_2$. Therefore, $G[E_0 \cup E_2]$ is either an empty graph or a cycle. □

Now we are ready to prove the main result on weak core.

Theorem 4.14. *The following four statements are equivalent:*

(1) (Conjecture 3.6) Every bridgeless cubic graph has a 5-line Fano-flow.

(2) (Conjecture 3.7) Every bridgeless cubic graph has a join J and two 1-factors M_1 and M_2 such that $J \cap M_1 \cap M_2 = \emptyset$.

(3) (Conjecture 4.9) Every bridgeless cubic graph has a cyclic 1-weak core.

(4) (Conjecture 4.10) Every bridgeless cubic graph has a triangle-free simple 1-weak core.

Proof. As already mentioned, the equivalence of statements (1) and (2) is proved in [40] (Theorem 3.1).

(2) $\rightarrow$ (3): By Proposition 4.13, the 1-weak core with respect to M_1, M_2 and J is cyclic. Therefore, statement (2) implies statement (3).

(3) $\rightarrow$ (4): Suppose to the contrary that there is a bridgeless cubic graph G that has no triangle-free simple 1-weak core. Let G_c be a cyclic 1-weak core of G with respect to two 1-factors M_1, M_2 and a join J such that $E(G_c)$ is minimum. We claim that G_c is simple. Otherwise, J is not simple, that is, G contains a circuit C such that each vertex of C is a J-vertex. Recall that G_c is cyclic, by type definitions according to M_1, M_2 and J, every vertex of C has type $(2, 2, 1)$. Let J_1 be the new join obtained from join J by removing all the edges of C. Thus J_1 is also a join of G. The 1-weak core with respect to M_1, M_2 and J_1 is cyclic and has fewer edges than G_c, a contradiction. This completes the proof of the claim.

By our supposition and the previous claim, G_c has a triangle $[xyz]$. It follows that two of vertices x, y and z have type $(2, 1, 0)$ and the last one has type $(2, 2, 1)$, which is the only possible case. Without loss of generality we assume that z is of type $(2, 2, 1)$. Set $J_2 = J \cup \{xy\} \setminus \{xz, yz\}$. Clearly, J_2 is a join of G. Now the 1-weak core with respect to M_1, M_2 and J_2 is cyclic and has fewer edges than G_c, a contradiction. Therefore, statement (3) implies statement (4).

(4) $\to$ (2): Let G be a bridgeless cubic graph with edge set $\{e_1, \dots, e_m\}$. Take m copies $T_1, \dots, T_m$ of the complete graph K_4. For each $i \in \{1, \dots, m\}$, apply the 2-cut connection on e_i and an edge of T_i, and let e'_i and e''_i be the two added new edges. The resulting graph G' is bridgeless and cubic. By (3), G' has a triangle-free simple 1-weak core H. Let H be with respect to two 1-factors M_1, M_2 and a simple join J. For every join F of G', since F contains either both of e'_i and e''_i or none of them for each $i \in \{1, \dots, m\}$, let $con(F) = \{e\colon\ e = e_i \in E(G)$, and $e'_i, e''_i \in F\}$. Clearly, $con(F)$ is a join of G and in particular, $con(F)$ is a 1-factor of G if F is a 1-factor of G'. We claim that $con(M_1) \cap con(M_2) \cap con(J) = \emptyset$ and hence, statement (1) holds. Suppose to the contrary that G has an edge e_1 contained in all of $con(M_1), con(M_2)$ and $con(J)$. It follows that $e'_1, e''_1 \in M_1 \cap M_2 \cap J$, and hence one can easily deduce that in copy T_1, the 1-weak core H contains either a triangle or a circuit of length 4 whose vertices are J-vertices, a contradiction. Therefore, statement (3) implies statement (1). □

4.4 Counterexample to a conjecture

If Fan-Raspaud Conjecture is true, then every bridgeless cubic graph has two 1-factors, say M_1 and M_2, with no odd edge-cut in their intersection; in particular, the complement of $M_1 \cup M_2$ is a bipartite graph which is union of paths and even circuits. One could ask if even circuits could be forbidden in such a bipartite graph. It is verified to be true for all snarks of order at most 34 and proposed as a conjecture in [49].

Conjecture 4.15 ([49])**.** *Every bridgeless cubic graph has two 1-factors such that the complement of their union is an acyclic graph.*

Note that this conjecture is formulated in the same way as Conjecture 4.6 but stronger than the latter. Here, we disprove Conjecture 4.15 by using the same technique already applied in the proof of Theorem 4.14.

Let P be the Petersen graph and let $\{e_1, \dots, e_{15}\}$ be its edge-set. Take 15 copies $T_1, \dots, T_{15}$ of the complete graph K_4. For each $i \in \{1, \dots, 15\}$, apply a 2-cut connection on e_i and an arbitrary edge of T_i. Denote by G the graph obtained. Let M_1 and M_2 be two 1-factors of G, and let $con(M_1)$ and $con(M_2)$ be the two corresponding 1-factors of P, respectively. Since every pair of 1-factors of P has exactly an edge in common, without loss of generality we can assume $\{e_1\} = con(M_1) \cap con(M_2)$. Hence, T_1 has an edge covered twice and a circuit of length four uncovered, that is, the complement of $M_1 \cup M_2$ is not acyclic. The disproof of Conjecture 4.15 is completed.

Even the previous conjecture is false in that general form, we would like to stress that the counterexample constructed above has a lot of 2-edge-cuts. So, we believe that the conjecture could be still true under stronger connectivity assumptions. In particular, we recall that it was verified true for all snarks, hence cyclically 4-edge-connected cubic graphs, of order at most 34 (see [49]).

More precisely, we wonder if every 3-connected (cyclically 4-edge-connected) cubic graph has two 1-factors such that the complement of their union is an acyclic graph.

4.5 Concluding remarks

We summarize in Figure 4.2 all the implications announced in this chapter.

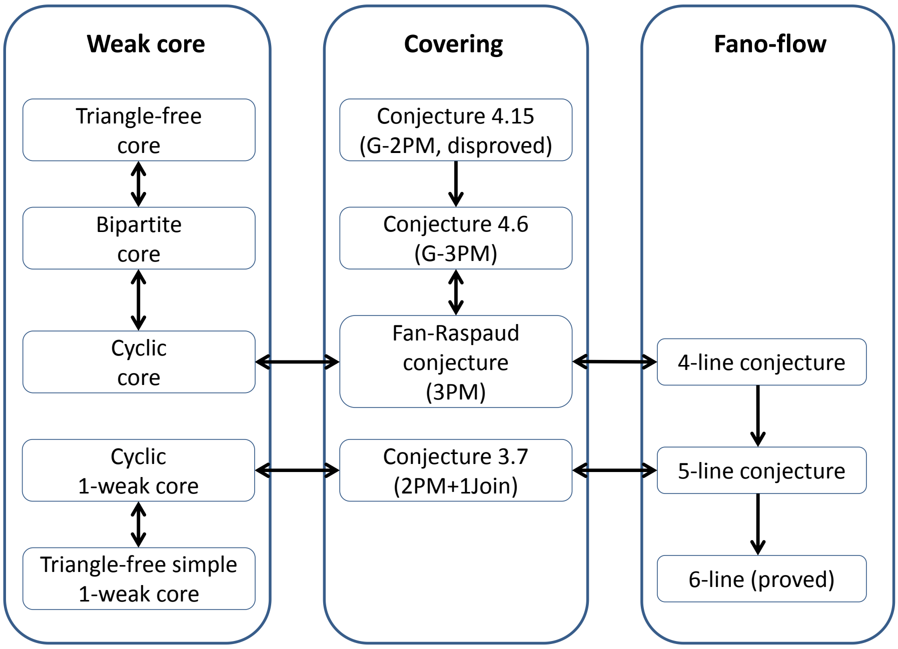

Figure 4.2: Conjectures related to Fan-Raspaud Conjecture

Chapter 5

Measures of edge-uncolorability

One major difficulty in proving theorems for snarks is to find/define appropriate structural parameters for a proof. Intuitively, a snark which is not reducible to a class 1 cubic graph seems to be more complicated than a snark which does. Another approach to define some structural property of class 2 cubic graph is to study invariants that "measure" how far the graph is from being class 1. Isaacs called cubic graphs uncolorable if they are class 2. Hence, these invariants are also called *measures* of edge-uncolorability in the literature.

5.1 Introduction to measures

5.1.1 Oddness ω

One major parameter measuring the complexity of a cubic graph G is its *oddness*, which is the minimum number of odd circuits in a 2-factor of G. It is denoted by $\omega(G)$. A cubic graph G is class 1 if and only if $\omega(G) = 0$. Cubic graphs with big oddness can be considered as more complicated than those with small oddness. Since every cubic graph has even order, its oddness must even. For instance, the Petersen graph has oddness 2. Snarks of oddness at least 4 were constructed in the literature with effort to minimize the order of the snark. So far, the best known result was given in [41], a snark of girth 5 with oddness 4 on 44 vertices (see Figure 5.1). For the construction of snarks with any larger oddness, see [20, 62].

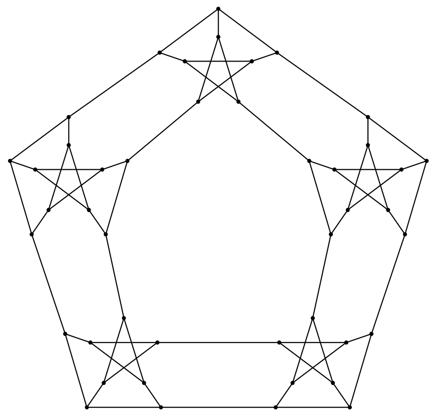

Figure 5.1: A snark with oddness 4

Indeed, many hard conjectures have been proved for cubic graphs with very small oddness. Máčajová and Škoviera [46] verified Fan-Raspaud conjecture for cubic graphs with oddness 2. This implies the truth of Conjecture 3.7 for these graphs as well. A proof of this particular result was given by Kaiser and Raspaud in [37]. Moreover, the 5-flow conjecture (every bridgeless graph has a nowhere-zero 5-flow) was verified for bridgeless cubic graphs with oddness at most 4 by Mazzuoccolo and Steffen [52] very recently.

The discussion on oddness will be continued later in this chapter. Besides the oddness, many other measures have been proposed and studied in the literature.

5.1.2 Resistance r

We follow the definition of the two measures r_3 and r_2 given in [63]. For $k \in \{2, 3\}$, let $c_k(G)$ be the maximum size of a k-colorable subgraph of a cubic graph G. Define $r_3(G) = |E(G)| - c_3(G)$ and $r_2(G) = \frac{2}{3}|E(G)| - c_2(G)$, and call r_3 the *resistance* of G. In other words, $r_3(G)$ is the minimum number of edges not covered by three matchings of G. In what follows, we take the notation of resistance in [53], written as r.

It is equivalent to say that: (1) G is class 1; (2) $r(G) = 0$; (3) $r_2(G) = 0$. For any 2-factor F of G, the subgraph obtained from G by removing an edge from each odd circuit of F is 3-edge-colorable and has size $|E(G)| - \omega(G)$. Hence, $r(G) \leq \omega(G)$. Moreover, since $c_2(G) \geq \frac{2}{3}c_3(G)$, it follows that $r_2(G) \leq r(G)$, where the equality holds if and only if G is class 1. Therefore, if G is a class 2 cubic graph, then $1 \leq r_2(G) < r(G) \leq \omega(G)$. This implies that $r(G) \geq 2$ for any cubic class 2 graph G. A tighter relation was given in [62]: $\frac{1}{2}r(G) \leq r_2(G) \leq \min\{\frac{2}{3}r(G), \frac{1}{2}\omega(G)\}$ holds true for every bridgeless class 2 cubic graph G and the bounds are attained. The following proposition is well known.

Proposition 5.1 (c.f. [62]). *If G is a bridgeless class 2 cubic graph, then $r_2(G) = 1 \Leftrightarrow r(G) = 2 \Leftrightarrow \omega(G) = 2$.*

However, the analogous proposition holds not true for larger values. The difference between r_2 and r can be arbitrarily big and the difference between r and ω either. For the construction of the graphs with these property and for more informations on the measures r_2 and r, we refer the readers to [63].

5.1.3 $r_v(G)$ of a graph G

We can restate the resistance r for general graphs: $r(G)$ is the minimum number of edges of a graph G that have to be removed from G to obtain a $\Delta(G)$-edge-colorable graph. An analogous measure $r'_v(G)$ is defined as the minimum number of vertices of a graph G that have to be removed from G to obtain a $\Delta(G)$-edge-colorable graph. A modification of $r'_v(G)$ is $r_v(G)$, which is the minimum number of vertices of a graph G that have to be removed from G to obtain a class 1 graph. These measures were introduced in [39, 53]. Clearly, $r'_v(G) \leq r_v(G) \leq r(G)$ for a graph G. While r'_v and r_v are different in the general case, they are of the same value for cubic graphs. In [62], it even shows that r_v and r are same for cubic graphs. Among these three equivalent parameters for cubic graphs, we will use only r. For a general graph G, $r(G)$ can be bounded by a function of $r'_v(G)$. It was proved in [53] that $r(G) \leq \lfloor \frac{\Delta(G)}{2} \rfloor r'_v(G)$, and the bound is best possible.

5.1.4 Weak oddness ω'

The weak oddness was introduced by Huck and Kochol in [25]. Recently, the question whether the oddness is always the same as weak oddness gains much attention. Let G be a bridgeless cubic graph. Recall that the oddness $\omega(G)$ of G is the minimum number of odd circuits of the complement of a 1-factor. Analogously, the *weak oddness* $\omega'(G)$ of G is defined to be the minimum number of odd components of the complement of a join.

The 5-CDCC was verified for bridgeless cubic graphs with oddness 2 by Huck and Kochol [25], and for bridgeless cubic graphs with oddness 4 by Huck [24] and independently by Häggkvist and McGuinness [19].

Let G be a bridgeless cubic graph. Same as the oddness, $\omega'(G)$ must be even. Let J be a join of G. Clearly, each component of the complement $\overline{J}$ is either an isolated vertex or a circuit. By removing from G a vertex of each odd component of $\overline{J}$, we obtain a subgraph of G which is 3-edge-colorable. Hence, $r(G) \leq \omega'(G)$. It is known that, for every bridgeless cubic graph G, if $r(G) = 2$ then $\omega'(G) = 2$. Furthermore, it was proved in [11] that, for every bridgeless cubic graph G, if $r(G) = 3$ then $\omega'(G) = 4$; and that, there exists a bridgeless cubic graph H such that $r(H) = 4$ and $\omega'(H) \geq 6$.

Since every 1-factor is a join, $\omega'(G) \leq \omega(G)$ by definition. The following known statement tells us that the equality always holds for G such that $\omega'(G) = 2$.

Proposition 5.2. *$\omega'(G) = 2$ if and only if $\omega(G) = 2$, for every bridgeless cubic graph G.*

Proof. It suffices to prove the direction "only if". Since $\omega'(G) = 2$, let J be a join whose complement has precisely two odd components, say H_1 and H_2. Since G is cubic, each odd component is either an isolated vertex or an odd circuit, and each even component is an even circuit. If both H_1 and H_2 are odd circuits, we have nothing to prove. Hence, we may assume that H_1 is an isolated vertex.

Case 1: assume that H_2 is an isolated vertex as well. Give the colors 1 and 2 to each even circuit of $\overline{J}$ alternately along the circuit, and the color 3 to the join J. Let K denote the subgraph of G induced by the edges of color 1 or 3. We can see that K is a subcubic graph having the vertices H_1 and H_2 of degree 3 and all the remaining vertices of degree 2. It follows that K has a path between H_1 and H_3. Swap the colors 1 and 3 on this path and consequently, the edges of color 1 induce a 1-factor of G, whose complement contains precisely two odd circuits. Therefore, $\omega(G) = 2$.

Case 2: assume that H_2 is an odd circuit. Give the colors 1 and 2 to each even circuit alternately along the circuit, and similarly to each H_i except one edge. Give the color 3 to the join J. Let K denote the subgraph of G induced by the edges of color 1 or 3. We can see that K is a subcubic graph having the vertice H_1 of degree 3, H_2 of degree 1 and all the remaining vertices of degree 2. It follows that K has a path between H_1 and H_3. Again, swap the colors 1 and 3 on this path and consequently, the edges of color 1 induce a 1-factor of G, whose complement contains precisely two odd circuits. Therefore, $\omega(G) = 2$. □

There was a long-standing discussion on the question whether $\omega(G) = \omega'(G)$ for all bridgeless cubic graphs G. However, recently, Lukot'ka and Mazák [42] gave a negative answer to this question by constructing a bridgeless cubic graph having $\omega'(G) = 12$ and $\omega(G) = 14$. This construction can be easily modified to obtain a bridgeless cubic graph with $k = \omega'(G) < \omega(G)$ for every even $k \geq 14$. Later on, for each $k \in \{6, 8, 10\}$, bridgeless cubic graphs with $k = \omega'(G) < \omega(G)$ are proposed [51]. Therefore, the case $\omega'(G) = 4$ is the only one unclear.

Problem 5.3. *Is it true that: $\omega'(G) = 4$ if and only if $\omega(G) = 4$, for every bridgeless cubic graph G?*

Concerning this problem, the proof of Proposition 5.2 might give a hint on the structure of a bridgeless cubic graph G with $\omega'(G) = 4$.

5.2 Further measures γ_2 and μ_3

The results of this section have already been published in [35]. In this section, we discuss on two further measures γ_2 and μ_3 of cubic graphs in terms of 1-factors, where μ_3 was first introduced in [64]. Let G be a cubic graph. Define that $\gamma_2(G) = \min\{|M_1 \cap M_2|\colon\ M_1$ and M_2 are 1-factors of $G\}$. A cubic graph G is class 1 if and only if $\gamma_2(G) = 0$. The Petersen graph has γ_2 equal to 1 since any two distinct 1-factors intersect on precisely one edge. Therefore, a class 2 cubic graph has no two disjoint 1-factors. The class of r-graphs is a generalization of bridgeless cubic graphs. Rizzi [57] constructed r-graphs having no two disjoint 1-factors for every $r \geq 3$, and call them poorly matchable r-graphs. Therefore, it is reasonable and of interest to define an analogous of γ_2 as a measure for r-graphs. We will study r-graphs in Chapter 7.

We relate γ_2 to ω and r_2. Let G be a cubic graph and let F_1 and F_2 be two 1-factors of G having precisely $\gamma_2(G)$ many common edges. We can easily see that the complement of F_1 contains at most $2\gamma_2(G)$ odd circuits. Hence, $\omega(G) \leq 2\gamma_2(G)$. Moreover, $F_1 \cup F_2$ induces a 2-edge-colorable subgraph. Since $|F_1 \cap F_2| + |F_1 \cup F_2| = \frac{2}{3}|E(G)|$, we can deduce that $r_2(G) \leq \gamma_2(G)$. On one hand, we can see here that γ_2 bounds both $\frac{1}{2}\omega$ and r_2, and the bounds can be achieved by the Petersen graph P, where $\gamma_2(P) = \frac{1}{2}\omega(P) = r_2(P) = 1$. On the other hand, for a cubic graph G with $\omega(G) = 2\gamma_2(G)$, it follows that $r_2(G) \leq \frac{1}{2}\omega(G)$, a bound much better than the general bound $r_2(G) < r(G) \leq \omega(G)$ mentioned in Section 5.1.2.

Let us proceed with the introduction of μ_k, a family of parameters which includes the measure μ_3 as a member. Let G be a cubic graph, $k \geq 1$, and S_k be a list of k 1-factors of G. By a list we mean a collection with possible repetition. For $i \in \{0, \dots, k\}$, let $E_i(S_k)$ be the set of edges which are in precisely i elements of S_k. We define $\mu_k(G) = \min\{|E_0(S_k)|\colon\ S_k$ is a list of k 1-factors of $G\}$. In other words, $\mu_k(G)$ is the minimum number of uncovered edges of G by k 1-factors. Proposition 3.12 implies that, for a bridgeless cubic graph G and for any positive integer k, $\mu_{k+1}(G) \leq \mu_k(G) - 1$.

Berge conjecture asserts that $\mu_5(G) = 0$ for every bridgeless cubic graph G.

If $\mu_4(G) = 0$, then the edges of G can be covered by four 1-factors. Esperet and Mazzuoccolo [9] showed that the problem whether $\mu_4(G) = 0$ for a given bridgeless cubic graph G is NP-complete. Snarks whose edges can be covered by four 1-factors are of particular interests. Some informations on this class were given in [12]. Moreover, it was proved by Hou, Lai and Zhang [23] and independently by Steffen [64] that every cubic graph G with $\mu_4(G) = 0$ has a 5-cycle double cover. In [64], it was even proved that those graphs have an even 4-cycle cover of length $\frac{4}{3}|E(G)|$, and that every cubic graph G with $\mu_4(G) \leq 3$ has a 4-cycle cover of length $\frac{4}{3}|E(G)|+4\mu_4(G)$. The Petersen graph has one uncovered edge by any 4 pairwise distinct 1-factors. Hence, it has a non-zero value of μ_4. Besides the Petersen graph, infinite families of snarks with non-zero value of μ_4 were constructed, see [9, 20].

μ_2 is strongly related to γ_2. By definition, we have $\mu_2(G) = \gamma_2(G) + \frac{1}{3}|E(G)|$.

The measure μ_3 was first introduced by Steffen in [64]. By definition, a $\mu_3(G)$-core and a minimal core have the same meaning for a cubic graph G. The following statement trivially holds true.

Proposition 5.4. *A cubic graph G is class 1 if and only if $\mu_3(G) = 0$.*

Thus, μ_3 can be taken as a measure of edge-uncolorability of cubic graphs, and a cubic graph with smaller value of μ_3 is regarded closer to being class 1.

Proposition 5.5 ([64]). *Let G be a loopless cubic graph. If $\mu_3(G) \neq 0$, then $\mu_3(G) \geq 3$.*

The lower bound 3 is sharp. It is easy to check that the Petersen graph P has three uncovered edges by any three given 1-factors. Thus, $\mu_3(P) = 3$, reaching the lower bound. Besides the Petersen graph, there are infinitely many snarks with this property, such as Goldberg snarks [16] and Isaacs flower snarks [26], which are two well-known families of snarks. The proof of the following proposition is not hard.

Proposition 5.6. *If G is a flower snark or a Goldberg snark, then $\mu_3(G) = 3$ and $\omega(G) = 2$.*

The following theorem tells us that μ_3 can serve as an upper bound for γ_2.

Theorem 5.7. *Let G be a bridgeless cubic graph. If G is not 3-edge-colorable, then $2\gamma_2(G) \leq \mu_3(G) - 1$. Furthermore, if G has a cyclic $\mu_3(G)$-core, then $\gamma_2(G) \leq \frac{1}{3}\mu_3(G)$.*

Proof. Let G_c be a $\mu_3(G)$-core of G. By the minimality of $\gamma_2(G)$, we have $3\gamma_2(G) \leq |E_2| + 3|E_3|$. Combining this inequality with $\mu_3(G) = |E_2| + 2|E_3|$ (Lemma 4.1) yields

$$2\gamma_2(G) \leq \mu_3(G) - \frac{1}{3}|E_2|. \tag{5.1}$$

Hence, the first statement is trivial if $\mu_3(G)$ is odd. If $\mu_3(G)$ is even, then it follows from the fact that $|E_2| \neq 0$, since G_c is a proper core of G.

Furthermore, if G_c is cyclic, then the inequality 5.1 implies that $\gamma_2(G) \leq \frac{1}{3}\mu_3(G)$. □

Clearly, the bound of this theorem is attained by every snark G with $\mu_3(G) = 3$. We will see that there are infinitely many snarks with this property. Theorem 5.7 yields that $\mu_3(G) \geq 2\gamma_2(G) + 1 \geq 3$, which also proves the truth of Proposition 5.5. Moreover, For every bridgeless cubic class 2 graph G, since $\omega(G) \leq 2\gamma_2(G)$, Theorem 5.7 provides an upper bound of the oddness in terms of μ_3: $\omega(G) \leq \mu_3(G) - 1$. If G additionally has a cyclic minimal core then $\omega(G) \leq \frac{2}{3}\mu_3(G)$. In the next section, we will show that this bound $\omega(G) \leq \frac{2}{3}\mu_3(G)$ actually holds true for all bridgeless cubic graphs.

5.3 μ_3 and ω

The results of this section have already been published in [35].

5.3.1 Bounds

A *girth* of a graph G is denoted by $girth(G)$. The following proposition is trivial but surprising to us, as proposed in [64].

Proposition 5.8 ([64]). *If G is a cubic graph, then $girth(G) \leq 2\mu_3(G)$.*

Hence, the girth of a snark G can be bounded by $\mu_3(G)$. We show that the oddness of G can be bounded by $\mu_3(G)$ as well. More precisely, we prove that $\omega(G) \leq \frac{2}{3}\mu_3(G)$ for every bridgeless cubic graph G.

Before the proof of this result, we give a necessary definition. Let G be a bridgeless cubic graph and G_c a core of G with respect to three 1-factors M_1, M_2, M_3. The core G_c is called a *Petersen* core if the following two conditions hold:

(1) G_c is cyclic;

(2) if P is a path of length 5 in G_c, then there exists no pair of edges e_1, e_2 of P and two integers i, j such that $e_1, e_2 \in M_i \cap M_j$ and $1 \leq i < j \leq 3$.

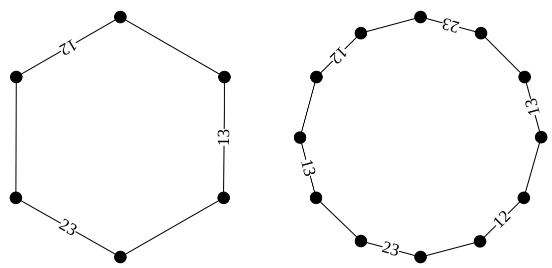

Figure 5.2: An example of Petersen core, where an edge labelled with ij belongs to $M_i \cap M_j$

Theorem 5.9. *Let G be a bridgeless cubic graph. If G_c is a k-core of G with respect to three 1-factors M_1, M_2, M_3, then $o(\overline{M_1}) + o(\overline{M_2}) + o(\overline{M_3}) \leq 2k$.*

Moreover, if G_c is a k-core such that the equality holds, then G_c is a Petersen core.

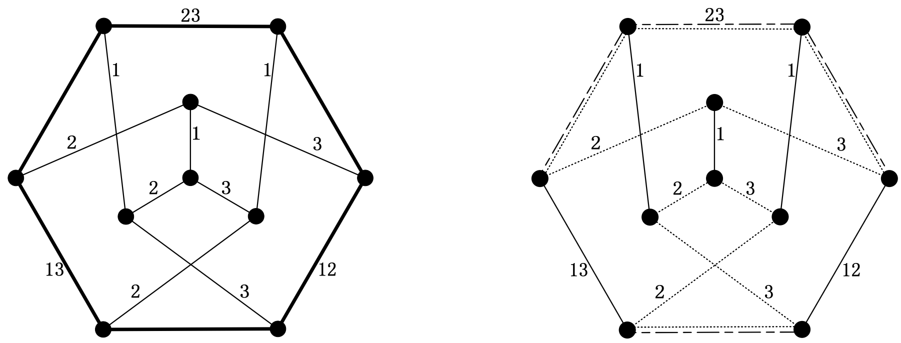

Figure 5.3: The left figure gives a 3-core (in bold line) of the Petersen graph where the equality holds, and the right figure gives $\overline{M_1}$ (in dotted line) and $\widehat{H_1}$ (in dashed line)

Proof. Let H be a subgraph of G_c which is induced by $E_0 \cup E_2$. Clearly, H consists of pairwise disjoint circuits. Let $E_{(i)} = E_1 \cap M_i$ for $i \in \{1, 2, 3\}$, and $E_{(i,j)} = E_2 \setminus M_l$ for $\{i, j, l\} = \{1, 2, 3\}$. We classify the components of H as follows: let D be a component of H. If D contains edges only from E_0, then D is of group 0. If D is not of group 0 and it contains no edge from M_i, then D is of group i, for $i \in \{1, 2, 3\}$. If D is not of group i for all $i \in \{0, 1, 2, 3\}$, then D is of group 4.

For $j \in \{0, 1, 2, 3, 4\}$ let Y_j be the graph consisting of components of H which are of group j.

Let $i \in \{1, 2, 3\}$. Let C be an odd circuit of $\overline{M_i}$. Then C has at least one uncovered edge, say e. Let H_i be a subgraph of H induced by $E(H) \setminus M_i$. Clearly, $e \in H_i$. Let P_e be the component of H_i containing e. Since C is a component of $\overline{M_i}$ and since H_i is a subgraph of $\overline{M_i}$, C contains P_e. Furthermore, P_e is either a path or an odd circuit. Let $\widehat{H_i}$ be the subgraph of H_i consisting of all the components of H_i each of which is either a path or an odd circuit. It follows that $o(\overline{M_i}) \le c(\widehat{H_i})$. Hence,

$$\sum_{i=1}^{3} o(\overline{M_i}) \le \sum_{i=1}^{3} c(\widehat{H_i}). \tag{5.2}$$

Let D be a component of H. If $E(D) \cap M_i = \emptyset$, then D is a component also of H_i; otherwise, the graph induced by $E(D) \setminus M_i$ consists of $|E(D) \cap M_i|$ many disjoint paths and each of these paths is a component of H_i. It follows that $c(\widehat{H_i}) = o(Y_0) + o(Y_i) + |E(H) \cap M_i| = o(Y_0) + o(Y_i) + |E_2 \cap M_i|$. Hence,

$$\sum_{i=1}^{3} c(\widehat{H_i}) = 3o(Y_0) + \sum_{i=1}^{3} o(Y_i) + \sum_{i=1}^{3} |E_2 \cap M_i|. \tag{5.3}$$

A vertex v of G is called a bad vertex if v is incident with two uncovered edges. Clearly, G has precisely $2|E_3|$ many bad vertices. Since every vertex of Y_0 is a bad vertex, Y_0 has at least $3o(Y_0)$ bad vertices. Let T be any odd component of Y_1. Since T is an odd circuit and every edge of T is either uncovered or from $E_{(2,3)}$, it follows that T has at least one pair of adjacent uncovered edges. Hence, T has at least one bad vertex. Thus, Y_1 has at least $o(Y_1)$ bad vertices. Similarly, for each $j \in \{2, 3\}$, Y_j has at least $o(Y_j)$ bad vertices. Since Y_0, Y_1, Y_2, Y_3 are pairwise disjoint subgraph of G, it follows that Y_0, Y_1, Y_2, Y_3 have at most $2|E_3|$ bad vertices in total. Thus,

$$3o(Y_0) + \sum_{i=1}^{3} o(Y_i) \leq 2|E_3|. \tag{5.4}$$

By combining inequalities 5.2–5.4 and the equality $\sum_{i=1}^{3} |E_2 \cap M_i| = 2|E_2|$, we conclude that $\sum_{i=1}^{3} o(\overline{M_i}) \leq 2|E_2| + 2|E_3|$. By Lemma 4.1 we have $k = |E_2| + 2|E_3|$ and therefore,

$$\sum_{i=1}^{3} o(\overline{M_i}) \leq 2k - 2|E_3| \leq 2k. \tag{5.5}$$

This completes the first part of the proof.

Now let G_c be a core such that $\sum\limits_{i=1}^{3} o(\overline{M_i}) = 2k$. By inequality 5.5, we have $|E_3| = 0$. Thus, G_c is a cyclic core. Furthermore, since $|E_3| = 0$, we deduce from (in-)equalities 5.2–5.4 that $2k = \sum_{i=1}^{3} o(\overline{M_i}) \leq \sum_{i=1}^{3} c(\widehat{H_i}) = 2|E_2|$ and from Lemma 4.1 that $k = |E_2|$. Therefore, $\sum_{i=1}^{3} o(\overline{M_i}) = \sum_{i=1}^{3} c(\widehat{H_i})$, that is, the inequality 5.2 becomes an equality.

A path P is bad if it is of odd length, and (a) there is $i \in \{1,2,3\}$ such that $M_i \cap E(P) = \emptyset$, and (b) the end-vertices of P are incident to an edge of $E_{(i,j)}$, for a $j \in \{1,2,3\} \setminus \{i\}$.

By definition, every bad path of G contains an uncovered edge.

We claim that G_c has no bad path. Suppose to the contrary that P is a bad path of G_c. Without loss of generality, suppose that $E(P) \cap M_1 = \emptyset$ and both end-vertices of P are incident with an edge from $E_{(1,2)}$. Thus P is a component of $\widehat{H_1}$. Let C be the circuit of $\overline{M_1}$ containing P. Since $\sum_{i=1}^{3} o(\overline{M_i}) = \sum_{i=1}^{3} c(\widehat{H_i})$, it follows that C is of odd length and contains no other component of $\widehat{H_1}$. This implies that $C - E(P)$ is a path of even length with edges from $E_{(2)}$ and from $E_{(3)}$ alternately. But then P has an end-vertex incident with an edge from $E_{(2)}$ and with an edge from $E_{(1,2)}$, a contradiction. This completes the proof of the claim.

It remains to show that G_c is a Petersen core. Suppose to the contrary that G_c is not a Petersen core. Then G_c violates the second part of the definition of a Petersen core. Without loss of generality, we may assume that $Q = uvwxyz$ is a path of length 5 in G_c and e_1, e_2 are two edges of Q such that $e_1, e_2 \in E_{(1,2)}$. It suffices to consider the following two cases.

Case 1: $e_1 = uv$ and $e_2 = wx$. Then vw is a bad path of G_c, a contradiction.

Case 2: $e_1 = uv$, $e_2 = yz$, and $wx \notin E_{(1,2)}$. Then $vwxy$ is a bad path of G_c, a contradiction.

This completes the proof. □

Theorem 5.10. *If G is a bridgeless cubic graph, then $\omega(G) \leq \frac{2}{3}\mu_3(G)$. Moreover, if $\omega(G) = \frac{2}{3}\mu_3(G)$, then $\omega(G) = 2\gamma_2(G)$ and every $\mu_3(G)$-core is a Petersen core.*

Proof. Let G_c be a $\mu_3(G)$-core of G with respect to three 1-factors M_1, M_2, M_3. By Theorem 5.9, we have $o(\overline{M_1}) + o(\overline{M_2}) + o(\overline{M_3}) \leq 2\mu_3(G)$. It follows that $\omega(G) \leq \frac{2}{3}\mu_3(G)$ by the minimality of $\omega(G)$.

If $\omega(G) = \frac{2}{3}\mu_3(G)$, then $o(\overline{M_1}) + o(\overline{M_2}) + o(\overline{M_3}) = 2\mu_3(G)$. Again by Theorem 5.9, G_c is a Petersen core. By Theorem 5.7, $\gamma_2(G) \leq \frac{1}{3}\mu_3(G)$. Therefore, $\omega(G) \leq 2\gamma_2(G) \leq \frac{2}{3}\mu_3(G) = \omega(G)$. Hence, $\omega(G) = 2\gamma_2(G)$. □

Theorem 5.10 implies that if a cubic graph G has a non-cyclic $\mu_3(G)$-core, then $\omega(G) < \frac{2}{3}\mu_3(G)$.

5.3.2 The equality case: Petersen core

We will construct an infinite family of snarks G with $\omega(G) = \frac{2}{3}\mu_3(G)$. Hence, the upper bound $\frac{2}{3}\mu_3(G)$ for $\omega(G)$ is best possible.

A *network* is an ordered pair (G, U) consisting of a graph G and a subset $U \subseteq V(G)$ whose elements are called *terminals*. A network with k terminals is a *k-pole*. We consider networks (G, U) with $d_G(v) = 1$ if v is a terminal and $d_G(v) = 3$ otherwise. A *terminal edge* is an edge which is incident to a terminal.

For $i \in \{1, 2\}$ let T_i be a network and u_i be a terminal of T_i. The *junction* of T_1 and T_2 on (u_1, u_2) is the network obtained from T_1 and T_2 by identifying u_1 and u_2 and suppressing the resulting bivalent vertex.

Theorem 5.11. *For every positive integer k, there is a cyclically 4-edge-connected cubic graph G_k of order $26k$ and $\omega(G_k) = r(G_k) = 2\gamma_2(G_k) = \frac{2}{3}\mu_3(G_k) = 2k$.*

Proof. We will construct graphs with these properties. Let B be a 4-pole with terminals a, b, c, d as shown in Figure 5.4. Take k copies $B_0, \ldots, B_{k-1}$ of B. Let G_k be the junction of $B_0, \ldots, B_{k-1}$ on (c_i, a_{i+1}) and (d_i, b_{i+1}) for $i \in \{0, \ldots, k-1\}$, where the indices are added modulo k (Figure 5.5 illustrates G_2 and a $\mu_3(G_2)$-core in bold line).

It is easy to check that $r(B) = 2$. Hence, we have $r(G_k) \geq 2k$. Furthermore, let M_i', M_i'', M_i''' be three matchings of B_i as shown in Figure 5.6 labeled with numbers $1, 2, 3$, respectively. Consider these matchings as matchings in G_k, where the edges with the suppressed bivalent vertices belong to M_1'. Let $M' = \bigcup_{i=0}^{k-1} M_i'$, $M'' = \bigcup_{i=0}^{k-1} M_i''$, $M''' = \bigcup_{i=0}^{k-1} M_i'''$. Then M', M'', M''' are

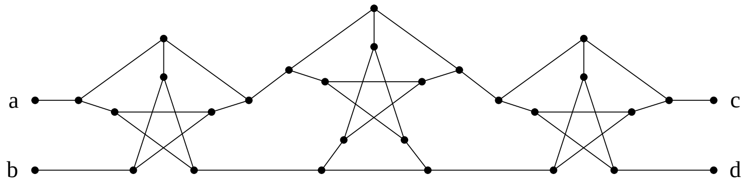

Figure 5.4: 4-pole B

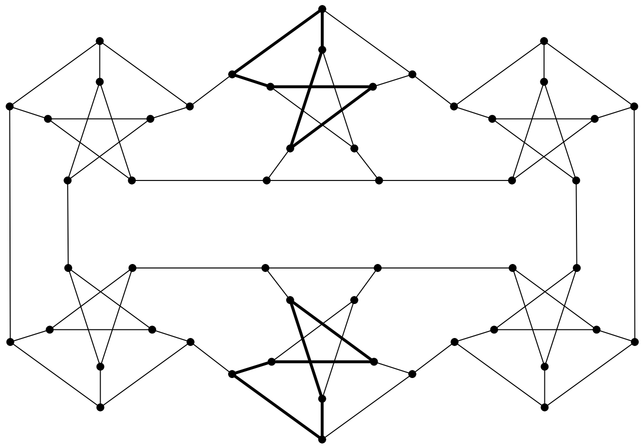

Figure 5.5: G_2 and a $\mu_3(G_2)$-core in bold line

three 1-factors of G_k, and G_k has precisely $3k$ edges contained in none of M', M'' and M'''. Hence, we have $\mu_3(G_k) \leq 3k$. Since $\omega(G_k) \leq \frac{2}{3}\mu_3(G_k)$ by Theorem 5.10, it follows that $2k \leq r(G_k) \leq \omega(G_k) \leq \frac{2}{3}\mu_3(G_k) \leq 2k$. Therefore, we have $\omega(G_k) = r(G_k) = \frac{2}{3}\mu_3(G_k) = 2k = 2\gamma_2(G_k)$, where the last equality follows by Theorem 5.10. □

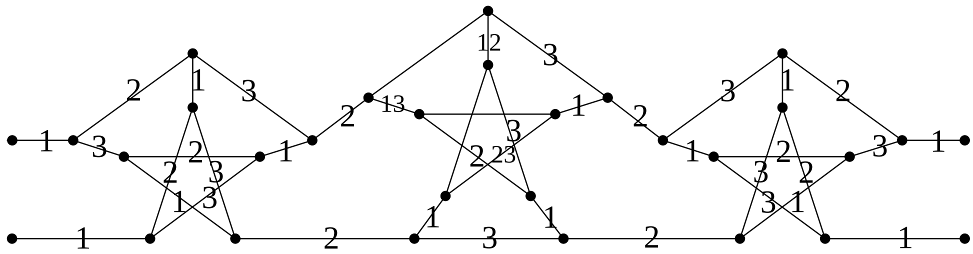

Figure 5.6: Three matchings of B_i

5.3.3 Difference

For $i \in \{1, 2\}$, let H_i be a cubic graph and u_i be a vertex of H_i of neighbors x_i, y_i, z_i. The *3-cut connection* on $\{u_1, u_2\}$ is a graph operation that consists of deleting vertices u_1 and u_2, and adding new edges x_1x_2, y_1y_2 and z_1z_2. We say that $\{x_1x_2, y_1y_2, z_1z_2\}$ is the *connection-cut* with respect to H_1 and H_2.

This subsection devotes to construct cubic graphs G with arbitrarily big difference between its oddness and $\frac{2}{3}\mu_3(G)$. We will use the following theorem which is a simple consequence of a result of Weiss.

Theorem 5.12 ([74])**.** *For every positive integer c there is a connected bipartite cubic graph H with $girth(H) \geq c$.*

Theorem 5.13. *For any positive integers k and c, there exists a bridgeless cubic graph G with $\omega(G) = 2k$ and $\mu_3(G) \geq c$.*

Proof. By Theorem 5.11 there is a cyclically 4-edge-connected cubic graph H with $\omega(H) = 2k = \frac{2}{3}\mu_3(G)$. Hence, we are done for $c \leq 3k$.

Let $V(H) = \{v_1, \ldots, v_n\}$. By Theorem 5.12, there is a connected bipartite cubic graph T with $girth(T) \geq 2c$. Since every bipartite cubic graph has no bridge, T is bridgeless. Take n copies $T_1, \ldots, T_n$ of T, and let u_i be a vertex of T_i. Let $H_0 = H$ and for $i \in \{1, \ldots, n\}$ let H_i be a graph obtained from H_{i-1} and T_i by applying 3-cut connection on (v_i, u_i), and let $G = H_n$.

We claim that $\omega(H_i) = \omega(H_{i-1})$. Let M be a 1-factor of H_{i-1} such that $\overline{M}$ has $\omega(H_{i-1})$ odd circuits. Precisely one edge of M is incident to v_i. Since T_i is bridgeless cubic and bipartite, it follows that $\overline{M}$ can be extended to a 2-factor of H_i that has $\omega(H_{i-1})$ many odd circuits. Hence, $\omega(H_i) \leq \omega(H_{i-1})$.

Let F be a 1-factor of H_i such that $\overline{F}$ has $\omega(H_i)$ many odd circuits. Let J be the connection-cut of H_i with respect to H_{i-1} and T_i. If F contains all edges of J, then every circuit of $\overline{F}$ lies either in $H_i - v_{i-1}$ or in $T_i - u_i$. Since the order of $H_i[V(T_i) \setminus \{u_i\}]$ is odd, it follows that $H_i[V(T_i) \setminus \{u_i\}]$ contains a circuit of odd length, contradicting the fact that T_i is bipartite. Hence, F contains precisely one edge of J. Then F can be transformed to a 1-factor of H_{i-1} by contracting T_i to a vertex. Since the complement of this 1-factor has at most $\omega(H_i)$ odd circuits, it follows that $\omega(H_{i-1}) \leq \omega(H_i)$. Therefore, $\omega(G) = \omega(H)$.

By construction we have $girth(G) \geq 2c$, and therefore, $\mu_3(G) \geq c$ by Proposition 5.8. $\square$

Since highly cyclically edge-connected snarks are of general interests, we prove the following statement, which tells as well that the difference between ω and μ_3 can be arbitrarily big, even we additionally fix the value of ω.

Theorem 5.14. *For every positive integer k, there is a cyclically 5-edge-connected cubic graph G^k with $\mu_3(G^k) = 2\omega(G^k) = 4k$.*

Proof. We will construct graphs with these properties.

Let D be a 5-pole with terminals u, v, w, x, y as shown in Figure 5.7. Let k be a positive integer. Take $2k$ copies $D_1, \ldots, D_{2k}$ of D, and denote by G^k the junction of $D_1, \ldots, D_{2k}$ on (x_i, u_{i+1}) and (y_i, v_{i+1}) for $i \in \{1, \ldots, 2k\}$ and on (w_i, w_{i+k}) for $i \in \{1, \ldots, k\}$ (Figure 5.8 illustrates G^2).

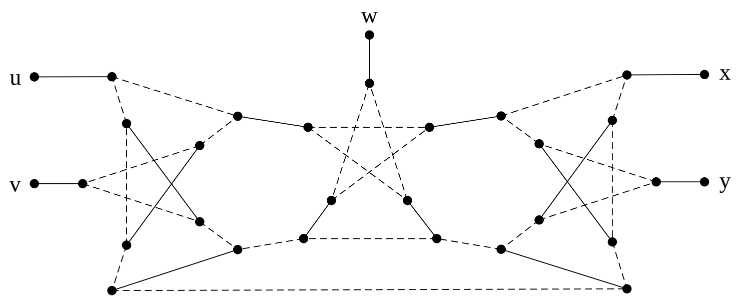

Figure 5.7: The 5-pole D and a 2-regular subgraph S of D in dotted line

We claim that G^k is a cyclically 5-edge-connected cubic graph such that $\omega(G^k) = 2k$ and $\mu_3(G^k) = 4k$.

Since D is not 3-edge-colorable, every cover by three matchings leaves at least one edge uncovered. Thus, $r(G^k) \geq 2k$ and $\omega(G^k) \geq 2k$.

Let S_i be a set of edges of D_i as shown in Figure 5.7 and let $F = \bigcup_{i=1}^{2k} S_i$. It is easy to see that F is a 2-factor of G^k that contains precisely $2k$ odd circuits. Thus, $\omega(G^k) \leq 2k$ and therefore, $\omega(G^k) = 2k$.

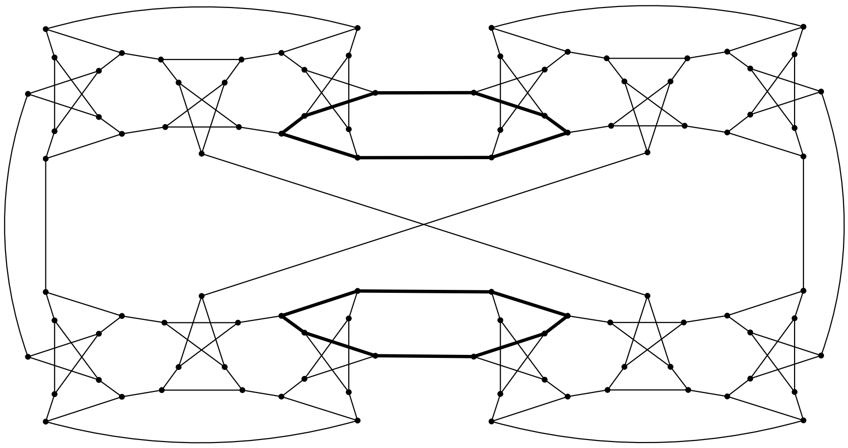

Figure 5.8: G^2 and a $\mu_3(G^2)$-core of G^2 in bold line

Let $\widetilde{D_i}$ be the junction of D_{2i-1} and D_{2i} on (x_{2i-1}, u_{2i}) and (y_{2i-1}, v_{2i}) $(i \in \{1, 2, \ldots, k\})$, and M_i', M_i'', M_i''' be three matchings of $\widetilde{D_i}$ as shown in Figure 5.9 labeled with numbers $1, 2, 3$, respectively. Let $M' = \bigcup_{i=1}^{k} M_i'$, $M'' = \bigcup_{i=1}^{k} M_i''$, $M''' = \bigcup_{i=1}^{k} M_i'''$. The three 1-factors M', M'', M''' cover all but $4k$ edges of G^k. Hence, $\mu_3(G^k) \leq 4k$. On the other hand, let G_c be a

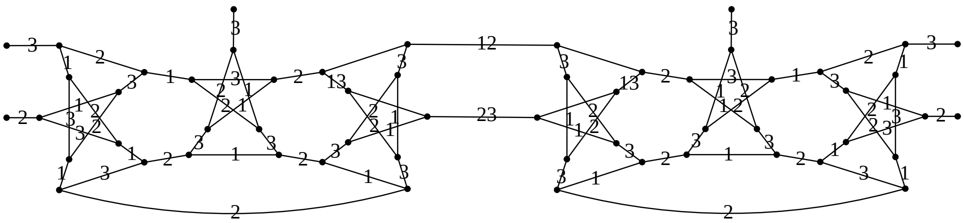

Figure 5.9: The 6-pole $\widetilde{D_i}$ and three matchings M_i', M_i'', M_i''' of $\widetilde{D_i}$ labeled with numbers 1,2,3, respectively.

$\mu_3(G^k)$-core of G^k. Since each D_i is not 3-edge-colorable, it has at least one uncovered edge of G_c, say e_i. Let C be any circuit of G_c containing precisely t members of $\{e_1, \ldots, e_{2k}\}$. First suppose that $t = 1$. Since the girth of G^k is at least 5, it follows that $|E(C)| \geq 5$. Next suppose that $t \geq 2$. Clearly, each path of D_i joining any two terminals of D_i is of length at least 3. Since C goes through t members of $\{D_1, \ldots, D_{2k}\}$, $|E(C)| \geq 4t$. In both cases we have $|E(C)| \geq 4t$ and thus, C contains at least $2t$ uncovered edges. Since each

e_i lies on precisely one circuit of G_c, it follows that G_c contains at least $4k$ uncovered edges. Thus, $\mu_3(G^k) \geq 4k$, and therefore, $\mu_3(G^k) = 4k$. □

5.3.4 Concluding remarks

We summarize the relations among the measures for edge-uncolorability of cubic graphs mentioned in this chapter as follows: for a bridgeless cubic class 2 graph G, $1 \leq r_2(G) < r(G) \leq \omega'(G) \leq \omega(G) \leq \frac{2}{3}\mu_3(G)$ and $\max\{r_2(G), \frac{1}{2}\omega(G)\} \leq \gamma_2(G) \leq \frac{1}{2}(\mu_3(G) - 1)$. Theorem 5.7 tells us that if G has a cyclic minimal core, then we further have $\gamma_2(G) \leq \frac{1}{3}\mu_3(G)$. We wonder whether $\gamma_2(G) \leq \frac{1}{3}\mu_3(G)$ holds true for all bridgeless cubic graphs G? If yes, then it implies the fact that $\omega(G) \leq \frac{2}{3}\mu_3(G)$ for all bridgeless cubic graphs G.

Let G be a cubic graph. Recall that it is equivalent to say: (1) $r_2(G) = 1$; (2) $r(G) = 2$; (3) $\omega'(G) = 2$; (4) $\omega(G) = 2$. However, they are not equivalent to the statement $\mu_3(G) = 3$, one of such examples is the first member G^1 of the family of graphs in Theorem 5.14 (also see Figure 5.10), for which $\omega(G^1) = 2$ and $\mu_3(G^1) = 4$.

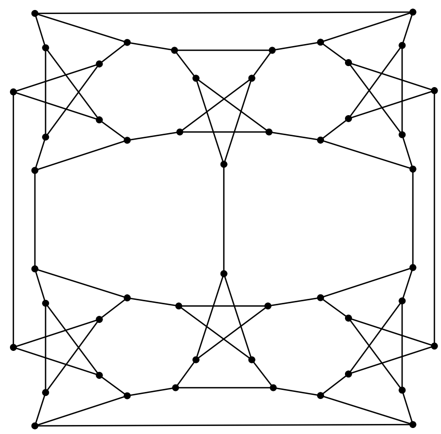

Figure 5.10: The graph G^1

The support $supp(\phi)$ of a flow ϕ of a graph G is defined as $supp(\phi) = \{e \in E(G)\colon\ \phi(e) \neq 0\}$. It is well known that a cubic graph G has a nowhere-zero 4-flow if and only if G is 3-edge-colorable. Hence, we introduce here a new measure r_f of edge uncolorability of a cubic graph G in terms of the support of 4-flows of G. The definition of r_f is as follows. Let G be a cubic graph. Define $r_f(G) = \min\{|E(G) - supp(\phi)|:\ \phi \text{ is a 4-flow of } G\}$. Clearly, G is class 1 if and only if $r_f(G) = 0$.

Theorem 5.15. *If G is a cubic graph, then $r_f(G) \leq \gamma_2(G)$.*

Proof. Let F_1 and F_2 be two 1-factors of G such that $|F_1 \cap F_2| = \gamma_2(G)$. Notice that each of $\overline{F_1}$ (the complement of F_1) and $\triangle(F_1, F_2)$ (the difference between F_1 and F_2) induces a cycle. For each circuit C of $\overline{F_1}$, fix a direction for C (either clockwise or anticlockwise) and assign each edge with flow value 1 and with direction same as C. Do the same to each circuit of $\triangle(F_1, F_2)$ except that we put flow value 2 instead of 1. We thereby obtain a 4-flow of G with support $|E(G)| - \gamma_2(G)$. By the minimality of $r_f(G)$, we have $r_f(G) \leq \gamma_2(G)$. □

Recall that $\max\{r_2(G), \frac{1}{2}\omega(G)\} \leq \gamma_2(G)$. It is of interests to relate r_f to r_2 or $\frac{1}{2}\omega$.

For the information on some other measures of edge-uncolorability for cubic graphs, we refer the readers to a recent survey paper [11].

5.4 The range of the value μ_3

The results of this section have already been published in [35]. Proposition 5.5 tells us that the integers 1 and 2 are unavailable to be the value $\mu_3(G)$ for some cubic graph G. One may raise such a natural question: are there more integers unavailable to be the value $\mu_3(G)$ for some cubic graph G? We give a negative answer to this question.

Theorems 5.11 and 5.14 already imply that for every positive integer k with $k \equiv 0$ (mod 3) there exists a cyclically 4-edge-connected cubic graph G with $\mu_3(G) = k$, and for every positive integer k with $k \equiv 0$ (mod 4) there exists a cyclically 5-edge-connected cubic graph G with $\mu_3(G) = k$. We will prove that for every $k \geq 3$ there is a bridgeless cubic graph G with $\mu_3(G) = k$. To construct such a graph, we need a graph operation, namely 2-junction.

Let G' and G'' be two bridgeless cubic graphs that are class 2. Let $e' = xy$ and $e'' = uv$ be an uncovered edge of a minimal core of G' and of G'', respectively. A *2-junction* of G' and G'' is the graph G with $V(G) = V(G') \cup V(G'')$ and $E(G) = E(G') \cup E(G'') \cup \{ux, vy\} \setminus \{e', e''\}$. The set $\{ux, vy\}$ is called a *2-junction-cut* of G (with respect to G' and G'').

Lemma 5.16. *Let G' and G'' be two bridgeless cubic graphs that are not 3-edge-colorable. If G is a 2-junction of G' and G'', then $\mu_3(G) = \mu_3(G') + \mu_3(G'')$.*

Proof. By construction, G has a k-core with $k \leq \mu_3(G') + \mu_3(G'')$. Hence, $\mu_3(G) \leq \mu_3(G') + \mu_3(G'')$.

Suppose to the contrary that $\mu_3(G) < \mu_3(G') + \mu_3(G'')$. Let ux, vy be the 2-junction-cut of G with respect to G' and G'', and $u, v \in V(G')$ and $x, y \in V(G'')$. Let G_c be a $\mu_3(G)$-core of G with respect to three 1-factors M_1, M_2, M_3. Then each M_i contains either none of ux and vy or both of them. Furthermore, M_i induces 1-factors F_i' and F_i'' in G' and G'', respectively. It follows that there is a k-core either in G' with $k < \mu_3(G')$ or in G'' with $k < \mu_3(G'')$, a contradiction. □

Theorem 5.17. *For every integer $k \geq 3$, there exists a bridgeless cubic graph G such that $\mu_3(G) = k$.*

Proof. Let us first consider the case $k \neq 5$. Then there exist two non-negative integers k' and k'' such that $k = 3k' + 4k''$. By Theorems 5.11 and 5.14, there is a cyclically 4-edge-connected cubic graph H' with $\mu_3(H') = 3k'$ and a cyclically 5-edge-connected cubic graph H'' with $\mu_3(H'') = 4k$. If $k' = 0$, then take $G = H''$ as desired. If $k'' = 0$, then take $G = H'$ as desired. Hence, we may next assume that $k', k'' > 0$. Let G be a 2-junction of H' and H''. By Lemma 5.16, $\mu_3(G) = \mu_3(H') + \mu_3(H'') = k$, we are done.

It remains to consider the case $k = 5$. Consider the flower snark J_7. Let J_c be a $\mu_3(J_7)$-core of J_7. Note that J_c is a circuit of length 6. Let u be a vertex of J_c and v, w, x be its three neighbors in J_7. Take two copies J', J'' of J_7. Apply 3-cut connection on $\{u, u''\}$, we obtain a graph G from J' and J''.

Let $\{v'v'', w'w'', x'x''\}$ be the connection-cut. This operation yields a core G_c of G that is a circuit of length 10. Hence, $\mu_3(G) \leq 5$.

On the other hand, let T be any $\mu_3(G)$-core of G. By the structure of T as a core, if $\{v'v'', w'w'', x'x''\} \cap (E_0 \cup E_2) = \emptyset$, then both J' and J'' contain a circuit of T. Since the girth of J_7 is 6, it follows that T has at least six uncovered edges, a contradiction. Hence, we may assume that $v'v'' \in E_0 \cup E_2$. Let C be the circuit of T containing $v'v''$. Clearly, C goes through both J' and J''. Since again the girth of J is 6, C is of length at least 10. It follows that T has at least five uncovered edges and thus, $\mu_3(G) \geq 5$. Therefore, $\mu_3(G) = 5$ and every $\mu_3(G)$-core of G is a circuit of length 10. □

5.5 Cubic graphs with small μ_3 or γ_2: towards conjectures

Some hard conjectures have been confirmed for snarks of small value of μ_3 in [64]. This gives us an insight into the structure of these snarks.

Theorem 5.18 ([64])**.** *Let G be a bridgeless cubic graph that has no nontrivial 3-edge-cut. If $\mu_3(G) \leq 4$, then G has a Berge-cover.*

Hence, Berge conjecture is true for snarks with μ_3 no larger than 4. Moreover, Fan-Raspaud conjecture is verified for snarks with μ_3 no larger than 6.

Theorem 5.19 ([64])**.** *Let G be a simple bridgeless cubic graph. If $\mu_3(G) \leq 6$, then G has a cyclic core. In particular, if G is triangle-free and $\mu_3(G) \leq 5$, then every $\mu_3(G)$-core is cyclic.*

The proof for this theorem given in [64] shows that if $\mu_3(G) \leq 6$ then $\omega(G) \leq 2$, and the proof is completed by using the result of E. Máčajová and M. Škoviera [46] that every bridgeless cubic graph with oddness 2 has a cyclic core. We will verify Fan-Raspaud conjecture for cubic graph G with $\mu_3(G) \leq 9$. Note that if $\mu_3(G) \leq 9$, then $\omega(G) \leq 6$ and in particular, both

$\omega(G) = 6$ and $\omega(G) = 4$ can be attained by some snarks. The proof given here avoids using the result of E. Máčajová and M. Škoviera and instead, we establish a lemma that plays a crucial role in the proof.

Lemma 5.20. *Let G be a 3-edge-connected cubic graph having no nontrivial 3-edge-cuts. For any two edges e and f of G, if G has a 1-factor F such that $e \in F$ and $f \notin F$, then G has another 1-factor M such that $e \notin M$ and $f \in M$;*

Proof. By Proposition 3.12, the lemma is true for the case that e and f are adjacent. Hence, we may assume that e and f are nonadjacent.

Replace e and f by two paths of length 3, say $u_1u_2u_3u_4$ and $v_1v_2v_3v_4$, respectively. Add two new edges u_2v_2 and u_3v_3. We thereby obtain a new graph G' from G. Since G is 3-edge-connected and has no nontrivial 3-edge-cuts, this graph operation yields the same for G'. Let $F' = (F \setminus \{e\}) \cup \{u_1u_2, u_3u_4, v_2v_3\}$. We can see that F' is a 1-factor of G' containing no edge of the path $v_1v_2u_2u_3$. By Proposition 3.13, G' has a 1-factor M' containing both v_1v_2 and u_2u_3. This yields that $v_3v_4 \in M'$. Now we obtain a 1-factor M of G from M' by removing u_2u_3 and replacing v_1v_2, v_3v_4 by f. Clearly, $e \notin M$ and $f \in M$, we are done with the proof of the lemma. □

Theorem 5.21. *Let G be a 3-edge-connected cubic graph having no nontrivial 3-edge-cuts. If $\mu_3(G) \leq 9$, then G has three 1-factors with empty intersection, i.e., G has a cyclic core.*

Proof. Let G_c be a $\mu_3(G)$-core of G with respect to three 1-factors M_1, M_2, M_3. By the proof of Steffen in [64], if $\mu_3(G) \leq 6$, then either G_c is already a cycle or for G_c we have $|E_3| = 1$ and $|E_2| \leq 4$. For the latter case, there exist $p \in \{1, 2, 3\}$ and $e \in E_0$ such that e is adjacent to all edges in $\bigcap_{p \neq i \in \{1,2,3\}} M_i$. Take a 1-factor M_4 of G containing e by Proposition 3.12 and therefore, $\{M_i\colon\ 1 \leq i \leq 4\} \setminus M_p$ is a list of three 1-factors with empty intersection, we are done. Hence we may assume that $\mu_3(G) \in \{7, 8, 9\}$. By Lemma 4.1, $|E_2| + 2|E_3| = \mu_3(G)$. It follows that $|E_3| \leq 4$. If $|E_3| = 0$, then G_c is a cycle, we are done. Hence, we may assume that $|E_3| \geq 1$. We distinguish four cases.

Case 1: assume that $|E_3| = 4$. Thus $|E_2| \leq 1$. Since every minimal core is proper, $|E_2| \geq 1$. Hence, $|E_2| = 1$. This implies that all the vertices of G_c has degree 3, except two of them which have degree 2. Hence, G has a 2-edge-cut whose removal leaves G_c as a component, contradicting the assumption that G is 3-edge-connected. Hence, this case is impossible.

Case 2: assume that $|E_3| = 3$. Thus $|E_2| \in \{1, 2, 3\}$. In this case, if C is a circuit of G_c having only edges from E_0, then $\partial(C) \subseteq E_3$, which implies that $\partial(C)$ is a bridge or a 2-edge-cut or a nontrivial 3-edge-cut of G, a contradiction. Hence, each circuit of $E_0 \cup E_2$ contains an edge from E_2. Let $E_0 \cup E_2$ consist of circuits $C_1, \ldots, C_k$. Since $|E_2| \leq 3$, we have $k \leq 3$.

Subcase 2.1: assume that $k = 1$. For writing convenience, we give some definitions. Give label a to both ends of an edge of E_3 and call this edge an (a, a)-edge. Analogously, we give labels b and c to ends of the remaining two edges of E_3 and call them (b, b)- and (c, c)-edges. The pattern of C_1 is a sequence of all labels on C_1 taken in clockwise order, regardless the starter. Let x and y be two labels (not necessary distinct) such that y is next to x in the pattern. A list of such sequences xy is called a segment of the pattern. The path of C_1 between x and y containing no other labels is called the (x, y)-path or the (y, x)-path.

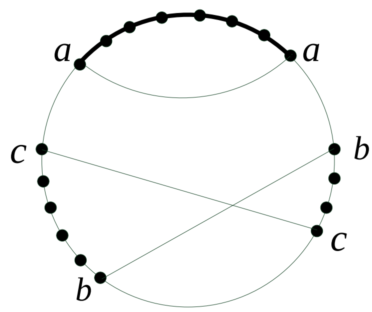

Figure 5.11: A pattern $aabcbc$ of C_1, where the (a, a)-path is the path in bold line.

If the pattern of C_1 has segment $\{aa\}$, then the (a, a)-path contains an edge from $E_2 \setminus M_i$ for each $i \in \{1, 2, 3\}$ by applying Lemma 4.2 to the circuit formed by the (a, a)-path and the (a, a)-edge, which implies that the (a, a)-

path has at least 3 edges from E_2. Similarly, if the pattern of C_1 has segment $\{ab, ab\}$, then the two (a, b)-paths have at least 3 edges from E_2 in total by applying Lemma 4.2 to the circuit formed by the two (a, b)-paths, the (a, a)-edge and the (b, b)-edge. The same result holds for the case that the pattern of C_1 has segment $\{ab, ba\}$. Since $|E_2| \leq 3$, at most one of the three segments above occurs and if it occurs, it does precisely one time. Hence, we can deduce that C_1 is of pattern $abcacb$, regardless the permutation of a, b, c. As already argued, the (b, c)-path and the (c, b)-path have precisely 3 edges from E_2 in total. Without loss of generality, say the (c, b)-path has an edge from E_2. We can apply Lemma 4.2 to the circuit formed by the three edges of E_3 and all the (x, y)-paths, where $(x, y) \in \{(b, a), (b, c), (a, c)\}$, yielding that the (b, a)-path, the (b, c)-path and the (a, c)-path have at least 3 edges from E_2 in total. Now we can conclude that C_1 has at least 4 edges from $|E_2|$, a contradiction. Hence, this subcase is impossible.

Subcase 2.2: assume that $k = 2$. Without loss of generality, let C_1 have chords in G_c no less than C_2 has. From the argument for Subcase 2.1, C_1 does not have three chords in G_c. If C_1 has two chords in G_c, whose ends divide C_1 into four paths, then at least three of the paths contain no end of the third edge of E_3. Hence, we can always apply Lemma 4.2 to some circuit formed by these three paths and the chords of C_1, yielding that C_1 has 3 edges from E_2, a contradiction. If C_1 has one chord in G_c, whose ends divide C_1 into two paths, then Lemma 4.2 implies that both paths contain an end of another edge of E_3. Let $E_3 = \{u_1u_2, v_1v_2, w_1w_2\}$ and let C_1 contain the vertices u_1, v_1, u_2, w_1. Now we can always apply Lemma 4.2 to at least two among the circuits formed by all the edges of E_3, one of the paths between v_2 and w_2 on C_2, and one of $P(u_1, v_1) \cup P(u_2, w_1)$ and $P(u_1, w_1) \cup P(u_2, v_1)$, where $P(x, y)$ denotes the path between x and y on C_1 that does not contain both u_1 and u_2, for $x, y \in \{u_1, v_1, u_2, w_1\}$. A contradiction follows. Hence, we can conclude that both C_1 and C_2 have no chord in G_c. The ends of the edges in E_3 divide C_i into 3 paths P_i^1, P_i^2, P_i^3 for each $i \in \{1, 2\}$. We may take the notation so that for each $x \in \{1, 2, 3\}$, the paths P_1^x and P_2^x have end-vertices from two same edges of E_3, all of them together form a circuit, say C^x. We can always apply Lemma 4.2 to at least two of C^1, C^2, C^3, yielding that G_c contains more than 3 edges from E_2, a contradiction. Hence, this subcase is impossible.

Subcase 2.3: assume that $k = 3$. For each $i \in \{1, 2, 3\}$, since $E_2 \leq 3$, the circuit C_i contains precisely one edge from E_2. It follows that C_i intersects with E_3, i.e., $|V(C_i) \cap V(E_3)| \geq 1$. Since G has no nontrivial 3-edge-cut, in particular, G has no triangles, we further have $|V(C_i) \cap V(E_3)| \geq 2$. Since $|E_3| = 3$, we can deduce that $|V(C_i) \cap V(E_3)| = 2$. Now the core G_c is specific and shown in Figure 5.12, from where we can see that G_c has a circuit of length 6 having edges from E_0 and E_3 alternately, contradicting with Lemma 4.2. Hence, this subcase is impossible.

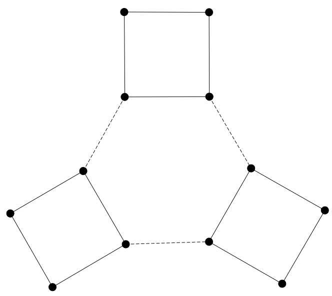

Figure 5.12: A core for Subcase 2.3, where dashed lines represent E_3

Now we conclude that Case 2 is impossible.

Case 3: assume that $|E_3| = 2$, say $E_3 = \{e_1, e_2\}$. Thus $|E_2| \in \{3, 4, 5\}$. For convenience, we give some definitions. Let x and y be two edges of $E_2 \cup E_3$ and z be an edge of E_0. x and y are U-connected at z if both of them are adjacent to z. Clearly, x, y, z induce a path rather than a star.

We claim that e_1 and e_2 are U-connected. Suppose to the contrary that they are not U-connected. For each $i \in \{1, 2\}$, e_i is U-connected to precisely four distinct edges of E_2 since otherwise, there is an edge of E_2 that is U-connected to e_i at two edges yielding either a triangle of G_c or a circuit of length 4 which we can apply Lemma 4.2 to. Since $|E_2| \leq 5$, there are at least three edges of E_2 that are U-connected to both e_1 and e_2. Among them there always exist two, say h' and h'', such that for each $i \in \{1, 2\}$, h'_i and h''_i are not adjacent, where h'_i (resp. h''_i) is the edge which e_i and h' (resp. h'') are U-connected at. Now G_c has a circuit of length 8 having edges from $\{e_1, e_2, h', h''\}$ and from E_0 alternately. We apply Lemma 4.2 to this circuit, obtaining a contradiction.

Since e_1 and e_2 are U-connected, let us say they are U-connected at e. Since $|E_2| \leq 5$, there exists $k \in \{1, 2, 3\}$ such that $|E_2 \setminus M_k| \leq 1$. By Proposition 3.8, G has a 1-factor containing e. If $|E_2 \setminus M_k| = 0$, then let M_4 be this 1-factor and it follows that $\{M_i\colon\ 1 \leq i \leq 4\} \setminus M_k$ is a list of three 1-factors that has empty intersection, we are done. Hence, we may next assume that $|E_2 \setminus M_k| = 1$, say $E_2 \setminus M_k = \{f\}$. Since any one of $\{M_i\colon\ 1 \leq i \leq 3\} \setminus M_k$ contains f but not e, by applying Lemma 5.20 to e and f, G has a 1-factor containing e but not f. Let M_4 be this 1-factor and again, $\{M_i\colon\ 1 \leq i \leq 4\} \setminus M_k$ is a list of three 1-factors that have empty intersection, we are done.

Case 4: assume that $|E_3| = 1$. Thus $|E_2| \in \{5, 6, 7\}$. Let $E_3 = \{e\}$. Without loss of generality, let $|E_2 \setminus M_1| \leq |E_2 \setminus M_2| \leq |E_2 \setminus M_3|$. Since $|E_2| \leq 7$, we have $|E_2 \setminus M_1| \leq 2$. If $|E_2 \setminus M_1| = 0$, then take a 1-factor of G not containing e, which together with M_2 and M_3 forms a list of three 1-factors with empty intersection, we are done. If $|E_2 \setminus M_1| = 1$, say $E_2 \setminus M_1 = \{f\}$, then we can apply Proposition 3.16 to e and f, obtaining a 1-factor of G containing neither e nor f. Again this 1-factor together with M_2 and M_3 forms a list of three 1-factors with empty intersection, we are done. It remains to assume that $|E_2 \setminus M_1| = 2$. Thus $|E_2 \setminus M_2| = 2$ and $|E_2 \setminus M_3| = 3$. As argued in Case 3, e is U-connected to precisely four distinct edges of E_2. Let g be one of these four edges such that $g \notin E_2 \setminus M_3$. Without loss of generality, say $g \in E_2 \setminus M_1$. Denote by g' the other edge of $E_2 \setminus M_1$ and h the edge that e and g' are U-connected at. We can apply Lemma 5.20 to h and g', obtaining a 1-factor M_4 of G containing h but not g'. It follows that $e, g \notin M_4$. Now M_2, M_3, M_4 are three 1-factors with empty intersection, we are done. □

We verify Fan-Raspaud conjecture also for cubic graphs with $\gamma_2 \leq 2$. The following theorem is a direct consequence of Proposition 3.16.

Theorem 5.22. *Let G be a bridgeless cubic graph. If $\gamma_2(G) \leq 2$, then G has three 1-factors with empty intersection, i.e., G has a cyclic core.*

5.5.1 Hypohamiltonian snarks

A graph G is *hypohamiltonian* if it is not hamiltonian but $G-v$ is hamiltonian for every vertex v of G. Since hamiltonian cubic graphs are 3-edge-colorable, hypohamiltonian snarks could be considered closest to being 3-edge-colorable.

Trivially, hypohamiltonian snarks are of weak oddness 2. Thus, by Proposition 5.2, they are of oddness 2 as well. Though not all snarks G with $\omega(G) = 2$ satisfy $\mu_3(G) = 3$, it was conjectured in [64] that the truth holds for this class of snarks.

Conjecture 5.23. *If G is a hypohamiltonian snark, then $\mu_3(G) = 3$.*

As already mentioned in the beginning of this section, the Petersen graph and the flower snarks are hypohamiltonian and have μ_3 equal to 3. Indeed, with the assistance of computer, Goedgebeur [15] verified this conjecture for hypohamiltonian snarks of relatively small order.

Observation 5.24 ([15]). *There are no counterexamples to Conjecture 5.23 among the hypohamiltonian snarks on at most 36 vertices, and also among the hypohamiltonian snarks on at most 44 vertices which are a dot product of two hypohamiltonian snarks.*

It is easy to see that if a cubic graph G has a vertex v such that $G - v$ is hamiltonian, then G has two 1-factors with one common edge. By Proposition 3.12, there is a third 1-factor avoiding this edge. Therefore, hypohamiltonian snarks satisfy Fan-Raspaud conjecture. Moreover, Sun [66] announced that Berge conjecture also holds true for hypohamiltonian snarks. Hence, it would be interesting to know whether Berge-Fulkerson conjecture holds true for hypohamiltonian snarks, as suggested by Häggkvist [18].

If Conjecture 5.23 is true, it would imply that every hypohamiltonian snark has a Berge cover by Theorem 5.18. Hence, the result of Sun gives a support for the truth of Conjecture 5.23.

5.6 A generalization of μ_3

The results of this section have already been published in [34].

Recall that a weak k-core with respect to three joins J_1, J_2, J_3 yields $k = |E_0| + \frac{3}{2}\sum_{i=1}^{3} n(J_i)$, as defined in Section 4.3. We define $\mu_3'(G) = \min\{k\colon\ G$ has a weak k-core$\}$. Clearly, $\mu_3'(G) \leq \mu_3(G)$ for a given cubic graph G. It is easy to see that a bridgeless cubic graph G is class 1 if and only if $\mu_3'(G) = 0$. Hence, μ_3' is also a measure of edge-uncolorability for cubic graphs.

We next relate μ_3' to the weak oddness ω' and show that the weak oddness of a bridgeless cubic graph can be bounded in terms of its weak cores.

Theorem 5.25. *Let G be a bridgeless cubic graph and G_c be a weak k-core with respect to three joins J_1, J_2 and J_3. Then $\sum_{i=1}^{3} o(\overline{J_i}) \leq 2k$.*

Proof. Each component of the complement of J_i is either an isolated vertex or a circuit. Any odd circuit of $\overline{J_i}$ contains either one edge from E_0 or a J_k-vertex with $k \neq i$. We call an odd circuit of $\overline{J_i}$ *bad* if it has no J_k-vertex for $k \neq i$. In what follows we distinguish elements of E_0 according to their behavior with respect to bad circuits. We define that, for $i \in \{1, 2, 3\}$,

$$X_i = \{e\colon\ e \text{ is the unique edge in } C \cap E_0, \text{ and } C \text{ is a bad circuit of } \overline{J_i}\},$$
$$Y_i = \{e\colon\ e \in E_0 \setminus X_i, \text{ and } e \in C \cap E_0, \text{ and } C \text{ is a bad circuit of } \overline{J_i}\}.$$

Set $x = |X_1| + |X_2| + |X_3|$ and $y = |Y_1| + |Y_2| + |Y_3|$. Since $X_i \cap Y_i = \emptyset$, it follows that

$$x + y \leq 3|E_0|. \tag{5.6}$$

Moreover, if $e \in X_i$, then $e \notin X_j$, and $e \notin X_k$ for $j, k \neq i$, that is

$$x \leq |E_0|. \tag{5.7}$$

Combining equations 5.6 and 5.7 implies

$$x + \frac{y}{2} \leq 2|E_0|. \tag{5.8}$$

Now, we are in position to prove our assertion. Since in an odd circuit of $\overline{J_i}$ there is either a J_k-vertex ($k \neq i$) or an edge of X_i or two edges of Y_i, the following relation holds:

$$o(\overline{J_i}) \leq |X_i| + \frac{|Y_i|}{2} + \sum_{i=1}^{3} n(J_i).$$

Therefore, by summing up for all three joins we deduce:

$$\sum_{i=1}^{3} o(\overline{J_i}) \leq x + \frac{y}{2} + 3\sum_{i=1}^{3} n(J_i) \leq 2|E_0| + 3\sum_{i=1}^{3} n(J_i) = 2k,$$

where the last inequality directly follows from the inequality (5.8). □

This result contains Theorem 5.9 as a particular case. Thus, the definition of weak k-core is the right generalization of k-core.

Theorem 5.26. *If G is a bridgeless cubic graph, then $\omega'(G) \leq \frac{2}{3}\mu_3'(G)$.*

Proof. Let G_c be a weak $\mu_3'(G)$-core of G with respect to three joins J_1, J_2 and J_3. By Theorem 5.25, we have $o(\overline{J_1}) + o(\overline{J_2}) + o(\overline{J_3}) \leq 2\mu_3'(G)$. By the minimality of the weak oddness $\omega'(G)$, it follows that $\omega'(G) \leq \frac{2}{3}\mu_3'(G)$. □

Chapter 6

Partially-normal 5-edge-colorings

6.1 Petersen coloring conjecture

Given graphs G and H, a mapping $\phi\colon\ E(G) \to E(H)$ is an H-coloring of G if any three mutually adjacent edges of G are mapped to three mutually adjacent edges of H. The mapping ϕ is called a *Petersen-coloring* if H is the Petersen graph.

Jaeger [29] posed the following conjecture which would imply both Berge-Fulkerson conjecture and the 5-CDCC.

Conjecture 6.1 (The Petersen coloring conjecture [29]). *Every bridgeless cubic graph has a Petersen-coloring.*

This section devotes to alternative formulations of the Petersen coloring conjecture.

Let G be a graph. A set of edges C is a *binary cycle* if C induces a subgraph of G where every vertex has even degree. DeVos, Nešetřil and Raspaud [7] defined that, given graphs G and H, a mapping $\phi\colon\ E(G) \to E(H)$ is *cycle-continuous* if the pre-image of each binary cycle of H is a binary cycle of G. When G and H are cubic and additionally H is cyclically 4-edge-connected, G

has a cycle-continuous mapping to H if and only if G has an H-coloring. This leads to the first alternate formulation of the Petersen coloring conjecture.

Theorem 6.2 (e.g. [1]). *A cubic graph has a Petersen-coloring if and only if it has a cycle-continuous mapping to the Petersen graph.*

However, the studies on cycle-continuous mapping make no progress on solving the Petersen coloring conjecture so far.

As already mentioned in Section 3.4, Fan-Raspaud conjecture is equivalent to the 4-line Fano-coloring conjecture. Surprisingly, all the three conjectures (the Petersen coloring conjecture, Berge-Fulkerson conjecture and 5-CDCC) can be reformulated in the form similar as Fano-coloring, proved in [40].

Consider Cremona-Richmond configuration G_{cr}, which has 15 points and 15 lines, as drawn in Figure 6.1. A *CR-coloring* of a graph G is a mapping from $E(G)$ to the points of G_{cr} such that any three mutually adjacent edges of G are mapped to three vertices of G_{cr} that lie in a line.

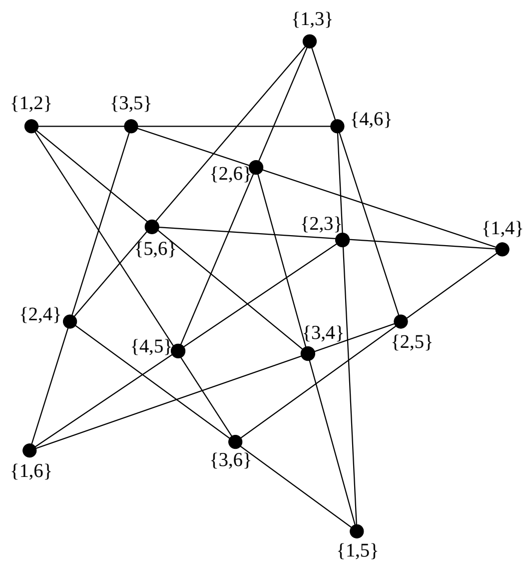

Figure 6.1: Cremona-Richmond configuration with $\{i, j\}$ labelling

Theorem 6.3 ([40]). *A cubic graph has a Berge-Fulkerson cover if and only if it has a CR-coloring.*

The truth of this theorem easily follows from a labelling of Cremona-Richmond configuration by $\{i, j\}$ with $1 \leq i < j \leq 6$, as shown in Figure 6.1. Here, we give another labelling of Cremona-Richmond configuration which yields that every CR-coloring is a CR-flow, that is, the flow values around a vertex sum up to zero. Such a labelling takes 15 non-zero elements of $\mathbb{Z}_2^4$, depicted in Figure 6.2.

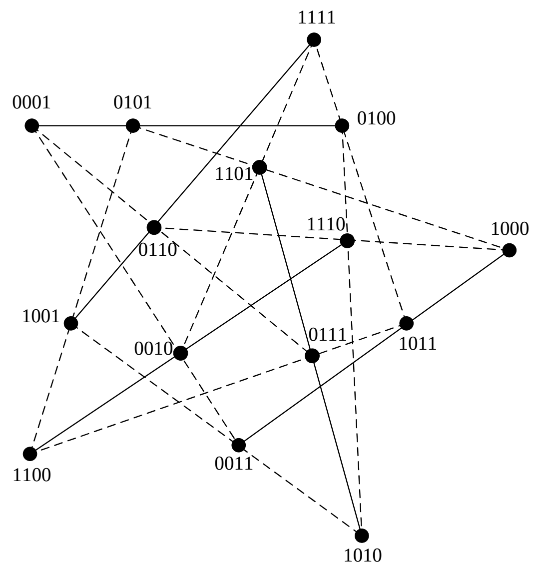

Figure 6.2: Cremona-Richmond configuration with $\mathbb{Z}_2^4$-labelling and with L_{cr} in dotted line

Let L_{cr} be a set of 10 lines obtained from the lines of G_{cr} by removing 5 pairwise disjoint lines. The dotted lines in Figure 6.2 indicate an example of L_{cr}.

Theorem 6.4 ([40]). *A cubic graph has a Petersen-coloring if and only if it has a CR-coloring using lines from L_{cr}.*

From the previous two theorems, it is easy to see again that the Petersen coloring conjecture implies Berge-Fulkerson conjecture.

A *Desargues-coloring* is defined in the same way as we define a CR-coloring except that Desargues configuration (see Figure 6.3) substitutes for Cremona-Richmond configuration.

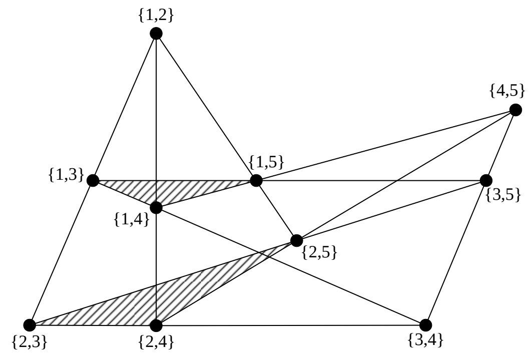

Figure 6.3: Desargues configuration

Theorem 6.5 ([40]). *A cubic graph has a 5-cycle double cover if and only if it has a Desargues-coloring.*

Unfortunately, the studies on CR-colorings make no progress on solving the Petersen coloring conjecture either. Here, we focus on the third alternative formulation of the Petersen coloring conjecture, in terms of normal 5-edge-coloring.

6.2 Normal 5-edge-coloring

Let G be a cubic graph and $\phi\colon\ E(G) \to \{1, 2, \ldots, 5\}$ be a proper 5-edge-coloring. An edge e is *poor* (or *rich*) if e together with its four adjacent edges uses precisely 3 (or 5) colors in total. An edge is *normal* if it is either rich or poor, and is *abnormal* otherwise. A *normal 5-edge-coloring* is a proper 5-edge-coloring such that each edge is normal. Jaeger showed the equivalence between Petersen-colorings and normal 5-edge-colorings of a cubic graph.

Theorem 6.6 ([28]). *A cubic graph has a Petersen-coloring if and only if it has a normal 5-edge-coloring.*

A possible minimal counterexample to the Petersen coloring conjecture is characterized in the literature. Jaeger [29] proved that it must be a cyclically 4-edge-connected snark. By learning normal 5-edge-coloring of cubic graphs, Hägglund and Steffen [21] showed that the minimal counterexample does not contain $K^*_{3,3}$ as a subgraph (see Figure 6.4 for $K^*_{3,3}$).

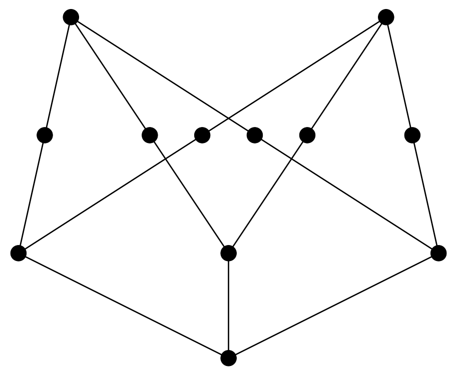

Figure 6.4: The graph $K^*_{3,3}$

A few classes of cubic graphs have been confirmed to have a normal 5-edge-coloring and thus a Petersen-coloring as well. In [21] it also showed that a cubic graph G has a normal 5-edge-coloring if G is a flower snark or a Goldberg snark or a generalized Blanuša snark of type 1 or 2. With the aid of computer, Brinkmann et al. [2] tested the Petersen coloring conjecture on cubic graphs of small order, and showed that every cubic graph of order no more than 36 has a normal 5-edge-coloring. However, no further results were obtained as far as we know.

6.3 Partially-normal 5-edge-coloring

Considering that a normal 5-edge-coloring requires each edge to be normal, Šámal [71] presented a weaker problem approximate to the Petersen coloring conjecture, that is, to search for a proper 5-edge-coloring such that the normal edges are as much as possible. Here, such a coloring is called a partially-normal 5-edge-coloring. Later on, Bílková proved that a generalized prism has a proper 5-edge-coloring with two third of the edges normal ([1], Theorem 2.3) and a cubic graph of large girth has a proper 5-edge-coloring with approximately half of the edges normal ([1], Theorem 3.6). In the rest of this section, we show that for every bridgeless cubic graph, there exists a proper 5-edge-coloring such that almost all the edges are normal. More precisely, we prove the following theorem.

Theorem 6.7. *Every bridgeless cubic graph G has a proper 5-edge-coloring such that at least $|E(G)| - \mu_3(G)$ many edges are normal.*

The proof of this theorem will be done by constructing such a proper 5-edge-coloring with the help of the structural properties on cores. First of all, we need some definitions and lemmas.

6.3.1 Useful definitions and lemmas

Let G be a cubic graph. If C is a circuit of G, then $\langle C \rangle$ denotes the set of edges not on C but having at least one end on C. Analogously, if P is a path of G with ends x and y, then $\langle P \rangle$ denotes the set of edges not on P but having at least one end on $P - x - y$. If H is a set of vertex-disjoint circuits or paths of G, then define that $\langle H \rangle = \bigcup_{h \in H} \langle h \rangle$.

Let G be a cubic graph and $X \subseteq E(G)$. Let $\psi\colon\ X \to \{1, \ldots, 5\}$ be a proper edge-coloring of $G[X]$. A circuit C of G is *ψ-extendable* if the following three items hold: (i) $E(C) \cap X = \emptyset$; (ii) $\psi(e) \in \{1, 2, 3\}$ for $e \in \langle C \rangle \cap X$; (iii) we can assign $E(C) \cup \langle C \rangle \setminus X$ with colors from $\{1, 2, 3\}$ so that the resulting coloring remains proper. Applying the third item is called *ψ-extending* C as well. We define a path P to be *ψ-extendable* and define *ψ-extending* P in exactly the same way (only with C replaced by P).

Let G be a cubic graph and $X \subseteq E(G)$. Let $\psi\colon\ X \to \{1, \ldots, 5\}$ be a proper edge-coloring of $G[X]$. Let v be an end of an edge e. Let $Y = \{\psi(h)\colon\ h \in X \cap E(v)\}$, i.e., Y is the set of colors around v. The edge e is *ψ-good* on v if either $Y = \{1, 2, 3\}$ or $Y = \{4, 5\}$ and $e \notin X$. Let H be a subgraph of G of minimum degree 2. Define $\mathcal{E}_\psi(H)$ to be the set of vertices v of H such that $d_H(v) = 2$ and the unique edge in $E(v) \setminus E(H)$ is not ψ-good on v. If ψ is clear from the context, we write $\mathcal{E}(H)$ for short.

Let G_c be a core of a cubic graph G with respect to three 1-factors M_1, M_2, M_3. The *major-coloring* of G with respect to M_1, M_2, M_3 (or to G_c) is a mapping $\phi\colon\ E(G) \setminus E(G_c) \to \{1, 2, 3\}$ defined as $\phi(e) = i$ for each $e \in (E(G) \setminus E(G_c)) \cap M_i$. A *string* P of G_c is a subgraph of G_c consisting of distinct odd circuits $C_0, C_1, \ldots, C_k$ of $G[E_0 \cup E_2]$ and edges $e_1, \ldots, e_k$ of

E_3 such that each e_i connects a vertex u_i of C_{i-1} to a vertex v_i of C_i. Such a string is denoted by $C_0e_1C_1 \ldots e_kC_k$ or $C_0(u_1v_1)C_1 \ldots (u_kv_k)C_k$. The two circuits C_0 and C_k are called *end-circuits* of P, and the remaining circuits are called *middle-circuits* of P.

Let G_c be a core of a cubic graph G and ϕ_m be the major-coloring of G with respect to G_c. Let $P^1, \ldots, P^s$ be pairwise disjoint strings of G_c by notation $P^j = C_0^j(u_1^jv_1^j)C_1^j \ldots (u_{t_j}^jv_{t_j}^j)C_{t_j}^j$. Denote by H_3 the union of all the odd circuits of $G[E_0 \cup E_2]$ not contained in any of these strings. For each end-circuit C_i^j, denote by p_i^j the longest ϕ_m-extendable path of C_i^j with an end-vertex of notation either u_1^j or $v_{t_j}^j$. The union of the strings $P^1, \ldots, P^s$ is a *wave* if the following two items hold:

(1) Each middle-circuit C_i^j contains a ϕ_m-extendable path p_i^j between v_i^j and u_{i+1}^j.

(2) Let p consist of all the paths of notation p_i^j from each circuit of the strings, and let H_3' consist of H_3 and p. For any two distinct components q_1, q_2 of H_3', we have $\langle q_1 \rangle \cap \langle q_2 \rangle \cap E_3 = \emptyset$.

Such a wave is denoted by $P^1 + \ldots + P^s$.

Lemma 6.8. *Let G_c be a core of a bridgeless cubic graph G. If G_c has a string, then it has a wave.*

Proof. We construct such a wave W by an algorithm.

Let ϕ_m be the major-coloring of G with respect to G_c, and let $H = G[E_0 \cup E_2]$. Since G_c has a string, say s. Take any two consecutive circuits C_u and C_v of s. Denote by e an edge of the string that connects a vertex u of C_u to a vertex v of C_v. Initialize W to be a graph consisting of C_u, C_v and e. Let P be a set which will collect ϕ_m-extendable paths. Initialize P to be an empty set.

If there exists $u \neq w \in V(C_u)$ such that C_u has a ϕ_m-extendable path p between u and w, and that H has an odd circuit C_x not contained in W and a vertex x of C_x with $wx \in E_3$, then take such a vertex w so that the length of p

is minimum, let W include x and C_x, add p into P, and repeat the argument with x and C_x instead of u and C_u respectively until no such w exists any more.

Repeat the argument above with v and C_v instead of u and C_u, respectively. Now the first string of W is completed.

If G_c has a string disjoint with W, then by applying the same argument on this string as on s, we get the second string of W. Repeat this until G_c has no strings disjoint with W.

Now the construction of W is completed. Clearly, W consists of pairwise disjoint strings. From the algorithm itself, we can see that Property (1) of the wave definition holds for W. By the minimality of the length of each element of P, Property (2) holds for W as well. Therefore, W is a wave. □

Let G_c be a core of a cubic graph G. Let D be a circuit of $G[E_0 \cup E_2]$. Define $\sigma(D)$ to be the number of vertices of D incident with an edge from E_3. Note that $\sigma(D) \geq |\langle D\rangle \cap E_3|$. Define $\Omega(G_c) = \{C\colon\ C$ is a circuit of $G[E_0 \cup E_2]$ such that $\sigma(C) = 1$ and $|E(C)| \leq 5\}$. Let $C_1, C_2 \in \Omega(G_c)$. Clearly, C_1 and C_2 are vertex-disjoint. Let $X \subseteq E(G)$ and $\psi\colon\ X \to \{1, \dots, 5\}$ be a proper edge-coloring of $G[X]$. C_1 and C_2 are *ψ-connected* if G has a path $uvwx$ such that $\psi(uv), \psi(vw), \psi(wx) \in \{4, 5\}$ and that v and w are incident with the edge of $\langle C_1\rangle \cap E_3$ and the edge of $\langle C_2\rangle \cap E_3$, respectively. C_1 and C_2 are *ψ-adjacent* if there is an edge from E_1 connecting a vertex of C_1 to a vertex of C_2.

Let X be a set of edges of a cubic graph G, and let $\psi\colon\ X \to \{1, \dots, 5\}$ be a 5-edge-coloring of $G[X]$. An edge e of G is *ψ-inner* if e together with its adjacent edges belongs to X; otherwise, e is called *ψ-outer*. Let G_c be a core of G. Define $\theta_{G_c,\psi}$ as a function on $E(G)$ given by

$$\text{for each } \psi\text{-inner edge } e,\ \theta_{G_c,\psi}(e) = \begin{cases} 1 & \text{if } e \in E_0 \text{ and } e \text{ is normal,} \\ -1 & \text{if } e \notin E_0 \text{ and } e \text{ is abnormal,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\text{and for each } \psi\text{-outer edge } e,\ \theta_{G_c,\psi}(e) = \begin{cases} 0 & \text{if } e \in E_0, \\ -1 & \text{if } e \notin E_0. \end{cases}$$

If G_c and ψ are clear from the context, we write θ for short. Moreover, for $X \subseteq E(G)$, define $\theta(X) = \sum_{x \in X} \theta(x)$. We write $\theta(H)$ short for $\theta(E(H))$ for a subgraph H of G.

A direct consequence of the function θ is the following lemma.

Lemma 6.9. *Let G_c is a k-core of a cubic graph G. If $\psi\colon\ E(G) \to \{1, \ldots, 5\}$ is a 5-edge-coloring of G, then G has $k - \theta(G)$ abnormal edges.*

6.3.2 Proof of Theorem 6.7

Trivially, the theorem holds true for 3-edge-colorable cubic graphs. We may assume that G is not 3-edge-colorable. Let G_c be a $\mu_3(G)$-core of G with respect to three 1-factors M_1, M_2, M_3, and let ϕ_m be the major-coloring of G with respect to G_c. Let $H = G[E_0 \cup E_2]$ and denote by H_1 the graph consisting of all the even circuits of H. If G_c has a string, then it has a wave W by Lemma 6.8, denote by H_2 the graph consisting of all the odd circuits of H that are contained in W; otherwise, to be convenient, we say that W and H_2 are empty graphs. Let $H_3 = H - H_1 - H_2$. We will extend ϕ_m to a proper 5-edge-coloring ϕ'_m of G by coloring H_1, H_2, H_3 in order (meanwhile, some edges in E_3 may receive colors) and finally coloring all the uncolored edges in E_3. By Lemma 6.9, to show that the final coloring ϕ'_m yields at most $\mu_3(G)$ edges abnormal, it suffices to prove $\theta_{G_c,\phi'_m}(G) \geq 0$. In what follows, since G_c is fixed and we always consider the current coloring extended from ϕ_m, we write θ and $\mathcal{E}$, briefly. Let $\mathcal{K}$ be a set with initial value an empty set. We will use $\mathcal{K}$ to collect subgraphs of G which receive colors during the extension of ϕ_m.

For each circuit C of H_1, assign $E(C)$ with colors 4 and 5 alternately along C. For each $e \in E(C)$, by the definition of the function θ, if $e \in E_0$ then $\theta(e) \geq 0$. If $e \in E_2$, then e is adjacent to two edges of the same color from

$\{1, 2, 3\}$, so e is poor yielding $\theta(e) = 0$. Therefore, $\theta(C) \geq 0 = |\mathcal{E}(C)|$. Add C into the set $\mathcal{K}$.

To describe the structure of the wave W, we use the same notations as in the definition of a wave. By Property (1) in the definition of a wave, each component of p is ϕ_m-extendable; and by Property (2), there is no uncolored edge with ends in two distinct components of p. Moreover, since no two edges from E_3 are adjacent, there is no uncolored path of length 2 with ends in two distinct components of p. Therefore, we can ϕ_m-extend the components of p one by one. The remaining part of W are disjoint paths. We can properly color their edges with the colors 4 and 5 alternately along each path. Add each string of W into $\mathcal{K}$.

Claim 6.9.1. *For each string P^j of W, we have $\theta(P^j) \geq 2 = |\mathcal{E}(P^j)|$.*

Proof. Let $t_j = d$. Clearly, the paths $p_0^j, \ldots, p_d^j$ together with the edges $u_1^j v_1^j, \ldots, u_d^j v_d^j$ form a path, say Q'. Denote by a, b, c, a', b', c' the edges incident with an end of Q' such that $a, a' \in E(Q')$ and $b, b' \notin E(P^j)$ and $a, b, c \in E(C_0^j)$. From the coloring of P^j, we can easily see that $\mathcal{E}(P^j) = \{b, b'\}$.

Let $Q = P^j - E(Q') - c - c'$. For each $q \in E(Q)$, we have $q \in E_0 \cup E_2$. Again, by the definition of the function θ, if $q \in E_0$ then $\theta(q) \geq 0$; and if $q \in E_2$ then q is adjacent to two edges of the same color from $\{1, 2, 3\}$, so q is poor yielding $\theta(q) = 0$. Therefore, $\theta(Q) \geq 0$.

We will prove that, if $d \geq 1$ then $\theta(p_i^j) + \theta(u_i^j v_i^j) \geq 1$ for each $i \in \{1, \ldots, d-1\}$. Since each edge of p_i^j is either rich or poor, $\theta(p_i^j) = |E(p_i^j) \cap E_0|$. Since $\theta(u_i^j v_i^j) \geq -1$, the conclusion holds true, provided that $|E(p_i^j) \cap E_0| \geq 2$. Hence, we may next assume that $|E(p_i^j) \cap E_0| \leq 1$. By the existence of $u_i^j v_i^j$ and $u_{i+1}^j v_{i+1}^j$, the path p_i^j is just an edge from E_0. So we could choose the ϕ_m-extension of p_i^j so that $u_i^j v_i^j$ is poor. The conclusion holds as well.

We next show that $\theta(c) + \theta(p_0^j) \geq 1$, while the equality holds only if C_0^j is a triangle. To do so, we distinguish two cases.

Case 1: assume that c is not incident with u_1^j. By the length maximality of p_0^j given in the definition of a wave, all the colors $1, 2, 3$ appear on the adjacent edges of c, yielding that c is rich and $c \in E_0$. Thus, $\theta(c) = 1$. Moreover, since each edge of p_0^j is either rich or poor except the edge a, we have $\theta(p_0^j) = |E(p_0^j) \cap E_0| - 1$. Therefore, $\theta(c) + \theta(p_0^j) = |E(p_0^j) \cap E_0|$. By again the length maximality of p_0^j, we can deduce that $\langle p_0^j \cup c \rangle$ uses at least two kinds of colors. It follows that $|E(p_0^j) \cap E_0| \geq 2$ and so, $\theta(c) + \theta(p_0^j) \geq 2$.

Case 2: assume that c is incident with u_1^j. Now c and p_0^j together form the circuit C_0^j. Notice that both a and c might be neither rich nor poor. We have $\theta(c) + \theta(p_0^j) \geq |E(C_0^j) \cap E_0| - 2$. Hence, the conclusion holds, provided that $|E(C_0^j) \cap E_0| \geq 4$. We may next assume that $|E(C_0^j) \cap E_0| \leq 3$. It follows that C_0^j is of length either 5 or 3. If C_0^j is of length 5, then $|E(C_0^j) \cap E_0| = 3$. Without loss of generality, see Figure 6.5 for the coloring of $C_0^j \cup \langle C_0^j \rangle$, which yields $\theta(c) + \theta(p_0^j) = 2$, we are done. We may assume that C_0^j is of length 3,

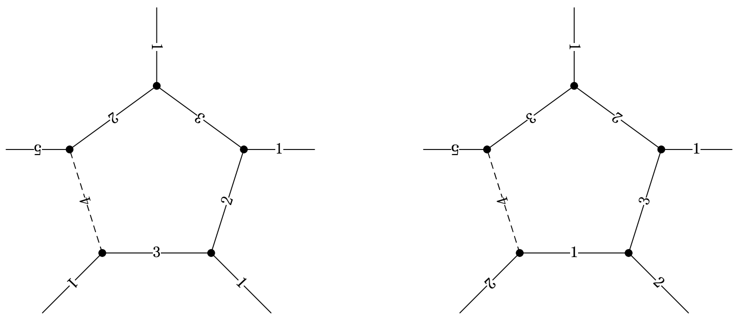

Figure 6.5: A coloring of $\langle C_0^j \rangle \cup E(C_0^j)$ in two cases. Case 1 (left): $\langle p_0^j \rangle$ uses one color; case 2 (right): $\langle p_0^j \rangle$ uses at least two colors.

i.e., it is a triangle. If $|E(C_0^j) \cap E_0| = 3$, then $\theta(c) + \theta(p_0^j) \geq 1$, we are done. Hence, we may assume that $|E(C_0^j) \cap E_0| = 2$. Without loss of generality, see Figure 6.6 for the coloring of $C_0^j \cup \langle C_0^j \rangle$, which yields $\theta(c) + \theta(p_0^j) = 1$, we are done as well.

Similarly, we can prove that $\theta(c') + \theta(p_d^j) \geq 1$, while the equality holds only if C_d^j is a triangle.

Now we are ready to calculate $\theta(P^j)$, given by $\theta(P^j) = \theta(Q) + \sum_{i=1}^{d-1}(\theta(p_i^j) + \theta(u_i^j v_i^j)) + (\theta(c) + \theta(p_0^j)) + (\theta(c') + \theta(p_d^j)) + \theta(u_d^j v_d^j) \geq 0 + (d-1) + 1 + 1 - 1 = d \geq 1$, while the equality holds only if $d = 1$ and both C_0^j and C_d^j are triangles. Hence, to prove $\theta(P^j) \geq 2$, it suffices to consider the equality case. Without loss of

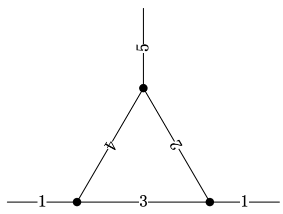

Figure 6.6: A coloring of C_0^j.

generality, see the coloring of C_0^j and C_d^j in Figure 6.7, from which we can calculate that $\theta(P^j) \geq 2$. We are done with the proof of the claim. □

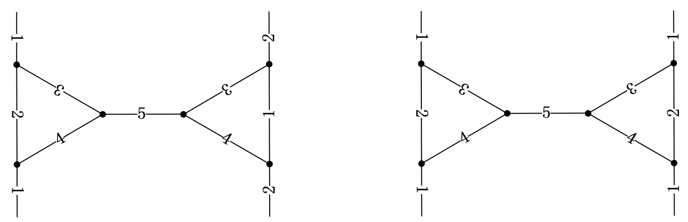

Figure 6.7: A coloring of P^j in equality case.

The circuits of H_3 can be divided into two parts H_3' and H_3'' so that $\Omega(G_c)$ contains all the circuits of H_3'' but none of H_3'. We will color H_3' and H_3'' in order.

For each circuit C of H_3', we add C into $\mathcal{K}$, and we will color $E(C)$ so that $\theta(C) \geq |\mathcal{E}(C)|$. Property (2) in the wave definition implies that all the edges of $\langle C \rangle \cap E_3$ are still uncolored. Thus, $\langle C \rangle$ uses only colors from $\{1, 2, 3\}$. If C is ϕ_m-extendable, then ϕ_m-extend C and consequently, $\theta(C) = |E(C) \cap E_0| \geq 0 = |\mathcal{E}(C)|$. Let us next assume that C is not ϕ_m-extendable. Since C is of odd length, $\sigma(C)$ is odd. Take the longest ϕ_m-extendable path q on C such that $E_3 \cap \langle q \rangle \neq \emptyset$. Denote by e_1 and e_2 the two end-edges of q and by e_i' the edge of $E(C) \setminus E(q)$ that is adjacent to e_i for $i \in \{1, 2\}$. Since C is not ϕ_m-extendable, $|E(C) \setminus E(q)| \geq 1$. We distinguish three cases.

Case 1: assume that $|E(C) \setminus E(q)| > 1$. We ϕ_m-extend q and assign $E(C) \setminus E(q)$ with colors 4 and 5 alternately. By the choice of q, all of the colors $1, 2, 3$ appear on the adjacent edges of e_1', yielding that e_1' is rich and belongs to E_0. Thus $\theta(e_1') = 1$. Similarly, we can deduce that $\theta(e_2') = 1$.

Moreover, since $E_3 \cap \langle q \rangle \neq \emptyset$, it follows that $|E(q) \cap E_0| \geq 2$. Hence, $\theta(C) \geq |E(q) \cap E_0| + \theta(e_1') + \theta(e_2') + \theta(e_1) + \theta(e_2) \geq 2 = |\mathcal{E}(C)|$.

Case 2: assume that $|E(C) \setminus E(q)| = 1$ and $|E(C) \cap E_0| \geq 5$. In this case, e_1' and e_2' are the same edge. We ϕ_m-extend q and assign e_1' with the color 4. So, $\theta(C) \geq |E(C) \cap E_0| - 3 \geq 2 = |\mathcal{E}(C)|$.

Case 3: assume that $|E(C) \setminus E(q)| = 1$ and $|E(C) \cap E_0| \leq 4$. It follows that $\sigma(C) \in \{1, 3\}$. If $\sigma(C) = 3$, then C is of length either 3 or 5, in both cases C is ϕ_m-extendable, a contradiction. Hence, $\sigma(C) = 1$. Let $E_3 \cap \langle C \rangle = \{f\}$. Recall that f is uncolored. Since $C \notin \Omega(G_c)$, we have $|E(C)| \geq 7$. Recall that $\frac{|E(C)|}{2} \leq |E(C) \cap E_0| \leq 4$ and that C is of odd length, thus $|E(C)| = 7$. We proceed in two subcases according to the colors $\langle C \rangle$ receives.

Subcase 3.1: assume that $\langle C \rangle$ uses at most two kinds of colors from $\{1, 2, 3\}$, say the colors 1 and 2. Assign f with the color 3 and its two adjacent edges on C with the colors 4 and 5. The remaining edges of C can be properly assigned with colors from $\{1, 2, 3\}$. One can directly calculate from the coloring that $\theta(C) \geq 2 = |\mathcal{E}(C)|$.

Subcase 3.2: assume that $\langle C \rangle$ uses all the colors $1, 2, 3$. Without loss of generality, see the left of Figure 6.8 for the coloring of $\langle C \rangle$. We extend the coloring to $E(C)$ and f as depicted in the right of Figure 6.8. By direct calculation, $\theta(C) \geq 2 = |\mathcal{E}(C)|$.

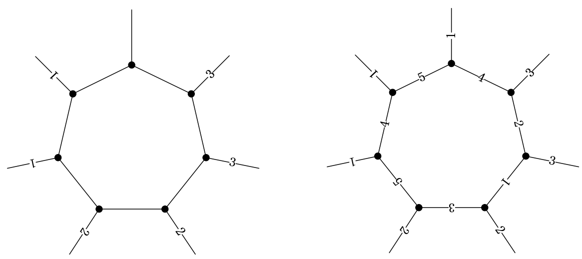

Figure 6.8: A coloring of the circuit C for subcase 3.2

To complete the coloring of H, it remains to color the edges of H_3''. Let ϕ_2 be the current coloring extended from ϕ_m. We will do it by coloring first all pairs of uncolored ϕ_2-connected circuits and then all pairs of uncolored ϕ_2-adjacent circuits and finally the remaining uncolored circuits of H_3''.

Let C' and C'' be a pair of uncolored ϕ_2-connected circuits of H_3''. Say that $C' = [u_1' \dots u_{k'}']$ and $C'' = [u_1'' \dots u_{k''}'']$ with $u_1'x', u_1''x'' \in E_3$. Clearly, $k', k'' \in \{3, 5\}$. Let y' and y'' be the third neighbors of u_2' and u_2'', respectively. By Property (2) of the wave definition, $u_1'x'$ and $u_1''x''$ are uncolored. Assign them with the color of $x'x''$. Choose $\alpha \in \{1, 2, 3\} \setminus \{\phi_m(u_2'y'), \phi_m(u_2''y'')\}$, and with the color α we assign $u_1'u_2'$ and $u_1''u_2''$ and reassign $x'x''$. Let ϕ_1 be the resulting coloring. Next, ϕ_1-extend the longest ϕ_1-extendable path on C' starting from u_2' and do the same to C''. Finally, properly assign the remaining edges of C' and of C'' with colors from $\{4, 5\}$.

If $x'x'' \in E(W)$, then let C_x be the string of W containing $x'x''$; otherwise, $x'x''$ is contained in a circuit of $H_1 \cup H_3'$, and let C_x be this circuit. Let $\mathcal{C}$ be the graph consisting of C_x, the circuits C' and C'', and the edges $u_1'x'$ and $u_1''x''$. We substitute C_x for $\mathcal{C}$ in the set $\mathcal{K}$ and will show that $\theta(\mathcal{C}) \geq |\mathcal{E}(\mathcal{C})|$.

We first prove that $\theta(C'), \theta(C'') \geq 1$. Recall that $k' \in \{3, 5\}$. If $k' = 3$, then without loss of generality, see Figure 6.9 for the coloring, which yields $\theta(C') \geq 1$. If $k' = 5$, then $\langle C' \rangle$ uses either one or two kinds of colors from $\{1, 2, 3\}$. Without loss of generality, see Figure 6.10 for the coloring in three cases. In each case, we can calculate that $\theta(C') \geq 1$. Similarly, we can prove $\theta(C'') \geq 1$.

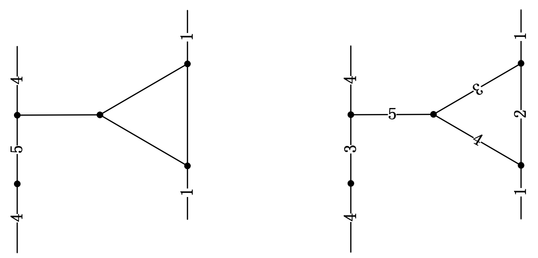

Figure 6.9: A coloring of C' of length 3

Denote by a and b the values of $\theta(C_x)$ and $|\mathcal{E}(C_x)|$ before $C' \cup C''$ receives colors, respectively. We already have the conclusion that $a \geq b$. Note that $x'u_1'$ and $x''u_1''$ are uncolored edges before $C' \cup C''$ receives colors. By the definition of the function θ, the coloring of $C' \cup C''$ does not decrease the value θ of $x'x''$ and of its two adjacent edges locating on C_x, and does make $x'u_1'$

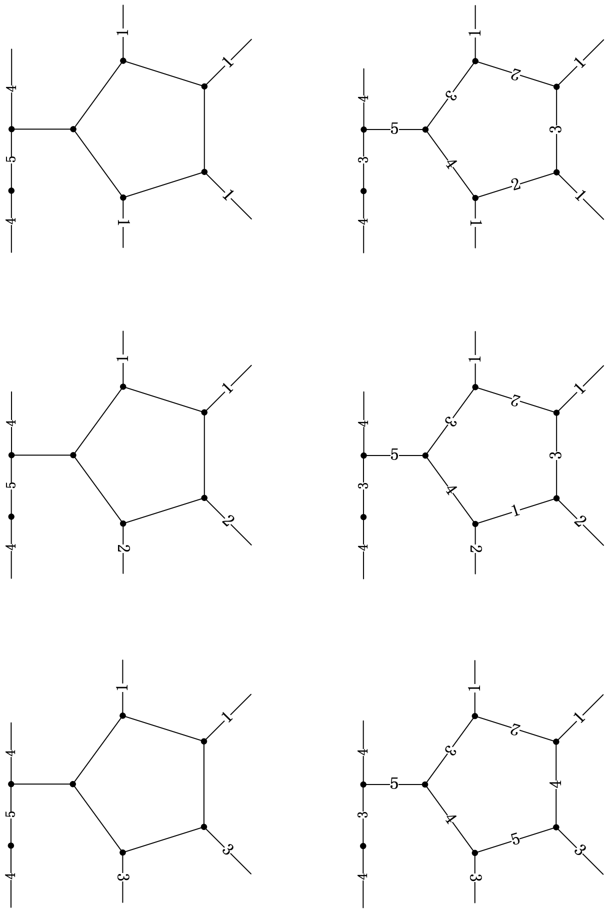

Figure 6.10: A coloring of C' of length 5

and $x''u''_1$ poor. It follows that $\theta(C_x) \geq a$ and $\theta(x'u'_1) = \theta(x''u''_1) = 0$. Thus, $\theta(\mathcal{C}) = \theta(C_x) + \theta(C') + \theta(C'') + \theta(x'u'_1) + \theta(x''u''_1) \geq a + 2$. Moreover, $\mathcal{E}(\mathcal{C})$ contains two more edges than $\mathcal{E}(C_x)$, one in $\langle C' \rangle$ and the other in $\langle C'' \rangle$. Hence, $|\mathcal{E}(\mathcal{C})| = b + 2$. Now we can see that $\theta(\mathcal{C}) \geq |\mathcal{E}(\mathcal{C})|$.

Let C' and C'' be a pair of uncolored ϕ_2-adjacent circuits of H_3. Choose an edge $e \in \langle C' \rangle \cap \langle C'' \rangle \cap E_1$. Let $\mathcal{C}$ be the graph consisting of C', C'' and e. Add $\mathcal{C}$ into $\mathcal{K}$. If there exists $\beta \in \{1, 2, 3\}$ such that β has not been used by $\langle C' \rangle \cap \langle C'' \rangle$, then reassign e with color β and consequently, C' and C'' are ϕ_2-extendable. So we ϕ_2-extend them, giving $\theta(\mathcal{C}) = |E(\mathcal{C}) \cap E_0| \geq 0 = |\mathcal{E}(\mathcal{C})|$. If such β does not exist, then we can deduce that both C' and C'' are of length 5. Reassign e with the color 4 and consequently, C' and C'' are ϕ_2-extendable. So we ϕ_2-extend them, giving $\theta(\mathcal{C}) \geq |E(\mathcal{C}) \cap E_0| - 5 = 1 > |\mathcal{E}(\mathcal{C})|$.

Let T be the remaining uncolored circuit of H_3''. To complete the coloring ϕ_m' of the whole graph G, we will first color all the uncolored edges in $E_3 \setminus \langle T \rangle$, and then color T and $E_3 \cap \langle T \rangle$.

For each uncolored edge e of $E_3 \setminus \langle T \rangle$, all the edges adjacent to e are already colored. We assign e with a color different from the colors of its adjacent edges.

Let $T' = T + \langle T \rangle$ and let $\overline{T'}$ be the complement of T' in G. We will show that $\theta(\overline{T'}) \geq 0$. We already get the conclusion that $\theta(k) \geq |\mathcal{E}(k)|$ for each element k of $\mathcal{K}$. Let K be the graph formed by the union of all the elements of $\mathcal{K}$. Since the elements of $\mathcal{K}$ are pairwise disjoint, $\theta(K) = \sum_{k \in \mathcal{K}} \theta(k)$. However, for $k_1, k_2 \in \mathcal{K}$, the sets $\mathcal{E}(k_1)$ and $\mathcal{E}(k_2)$ may have common elements. Hence, $|\mathcal{E}(K)| \leq \sum_{k \in \mathcal{K}} |\mathcal{E}(k)|$. Therefore, $\theta(K) \geq |\mathcal{E}(K)|$. Since the value θ of an edge is at least -1, it follows that $\theta(K) + \theta(\mathcal{E}(K)) \geq 0$.

Let e be an edge of $\overline{T'} - E(K) - \mathcal{E}(K)$. Let ϕ_3 be the current coloring extended from ϕ_m. Note that all the edges of color 4 or 5 locate on K. Since $e \notin E(K) \cup \mathcal{E}(K)$, the edge e is ϕ_3-good on both ends, yielding $\theta(e) \geq 0$. Hence, $\theta(\overline{T'} - E(K) - \mathcal{E}(K)) \geq 0$. Now we can get that $\theta(\overline{T'}) \geq 0$.

It remains to color T and $E_3 \cap \langle T \rangle$. For each circuit C of T, we will color C so that $\theta(\overline{T'}) + \theta(C) + \theta(\langle C \rangle) \geq 0$. Let ϕ_4 be the current coloring extended from ϕ_m. Say that C is of length k and of vertices $u_1, \ldots, u_k$ in cyclic order. For $1 \leq i \leq k$, denote by v_i the neighbor of u_i not on C. Since $C \in \Omega(G_c)$, $k \in \{3, 5\}$ and the set $\langle C \rangle \cap E_3$ contains exactly one edge, say $e = u_1v_1$. Let e_1 and e_2 be the other two edges incident with v_1. By Property (2) of the wave definition, u_1v_1 is uncolored, and e_1 and e_2 are of colors 4 and 5. If e_1 is adjacent to an uncolored edge e' rather than e, then $e' \in E_3 \cup \langle C' \rangle$ for some $C' \in T$, yielding that C and C' are ϕ_2-connected circuits of T, a contradiction. Hence, e is the only uncolored edge adjacent to e_1 or e_2. Let γ_i be the color making e_i either rich or poor if e receives it. Such γ_i always exists and $\gamma_i \in \{1, 2, 3\}$. Recall that $k \in \{3, 5\}$. We distinguish three cases.

Case 1: $k = 3$. Clearly, $\langle C \rangle$ uses one same color, say color 1.

Subcase 1.1: assume that at least one of γ_1 and γ_2 is not color 1, say $\gamma_1 = 2$. Assign the edges $e, u_1u_2, u_2u_3, u_3u_1$ with colors $2, 4, 3, 5$, respectively.

Since the coloring of e makes e_1 from a ϕ_4-outer edge to a rich edge, it increases the value $\theta(e_1)$ (and thus, the value $\theta(\overline{T'})$) by 1. Moreover, we can calculate that $\theta(C) + \theta(\langle C\rangle) \geq -1$. Therefore, $\theta(\overline{T'}) + \theta(C) + \theta(\langle C\rangle) \geq 0$.

Subcase 1.2: assume that $\gamma_1 = \gamma_2 = 1$.

Subcase 1.2.1: assume that not both v_2 and v_3 are incident with edges of color 2 and of color 3. Without loss of generality, let v_2 be incident with no edges of color 2. Reassign u_2v_2 with color 2 and consequently, we can ϕ_4-extend C. We can calculate that $\theta(C)+\theta(\langle C\rangle) = 2$. Let h_1 and h_2 be the edges other than u_2v_2 that are incident with v_2. Since T contains no ϕ_2-connected circuits, $h_1, h_2 \in \overline{T'}$. Hence, reassigning u_2v_2 decreases the value $\theta(h_1)+\theta(h_2)$ (and thus, the value $\theta(\overline{T'})$) by at most 2. Therefore, $\theta(\overline{T'})+\theta(C)+\theta(\langle C\rangle) \geq 0$.

Subcase 1.2.2: assume that both v_2 and v_3 are incident with edges of color 2 and of color 3. Reassign v_2u_2 and v_3u_3 with color 4 and color 5, and assign $u_1u_2, u_2u_3, u_1u_3, e$ with colors $5, 1, 4, 1$, respectively. We decrease by at most 1 the value θ of each of the other four edges adjacent to u_2v_2 or to u_3v_3, and increase by 1 the value θ of both e_1 and e_2. Moreover, $\theta(C) + \theta(\langle C\rangle) = 2$. Therefore, $\theta(\overline{T'}) + \theta(C) + \theta(\langle C\rangle) \geq 0$.

Case 2: $k = 5$ and $\langle C\rangle$ uses one same color, say color 1.

Subcase 2.1: assume at least one of γ_1 and γ_2 is not color 1, say $\gamma_1 = 2$. Assign $e, u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1$ with $2, 4, 3, 2, 3, 5$, respectively. By similar argument as subcase 1.1, we have $\theta(\overline{T'}) + \theta(C) + \theta(\langle C\rangle) \geq 0$.

Subcase 2.2: assume that $\gamma_1 = \gamma_2 = 1$.

Subcase 2.2.1: assume that not both u_2v_2 and u_5v_5 have colors 2 and 3 on its adjacent edges, say u_2v_2 does not have color 2 on its adjacent edges. Reassign u_2v_2 with color 2 and consequently, we can ϕ_4-extend C. By similar argument as subcase 1.2.1, we have $\theta(\overline{T'}) + \theta(C) + \theta(\langle C\rangle) \geq 0$.

Subcase 2.2.2: assume that both u_2v_2 and u_5v_5 have colors 2 and 3 on its adjacent edges. Reassign u_2v_2 with color 4 and u_1u_2 with color 5 and consequently, we can ϕ_4-extend C so that e receives color 1. By similar argument as subcase 1.2.2, we have $\theta(\overline{T'}) + \theta(C) + \theta(\langle C\rangle) \geq 0$.

Case 3: $k = 5$ and $\langle C \rangle$ uses two kinds of colors, say colors 1 and 2. Without loss of generality, let u_2v_2 be of color 1. Assign e, u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1 with colors $1, 4, 5, 4, 3, 5$, respectively. From the coloring, we can calculate that $\theta(C) + \theta(\langle C \rangle) \geq 0$. Therefore, $\theta(\overline{T'}) + \theta(C) + \theta(\langle C \rangle) \geq 0$.

Now we complete the coloring ϕ'_m of G so that $\theta_{\phi'_m}(G) \geq 0$. We are done with the proof of the theorem.

Chapter 7

r-graphs

7.1 1-factors and conjectures on r-graphs

Throughout this chapter, we consider r-regular multigraphs. The early study on r-graphs was proceeded by Seymour [61] in 1979. Let r be an integer with $r \geq 3$. It is defined that G is an *r-graph* if (1) G is an r-regular graph and, (2) $|\partial(X)| \geq r$ for each odd $X \subseteq V(G)$.

If an r-regular graph G is r-edge-colorable, then $E(G)$ can be divided into r pairwise disjoint 1-factors, in this case the structure of G is quite clear. However, it is a notoriously difficult problem to determine which r-regular graph is r-edge-colorable. By the definition of r-graphs, we can see that every r-edge-colorable r-regular graph is an r-graph. This raises interests on the study of r-graphs. Seymour [61] asked the inverse question that which r-graph is r-edge-colorable, and proposed several conjectures on it earlier or later. Here, we mention three of them.

Conjecture 7.1 ([60]). *Every planar r-graph is r-edge-colorable.*

Conjecture 7.2 (r-graph conjecture [61]). *If G is an r-graph, then $\chi'(G) \leq r+1$.*

Conjecture 7.3 (Generalized Berge-Fulkerson conjecture [61]). *Every r-graph has $2r$ 1-factors such that each edge is contained in precisely two of them.*

Conjecture 7.1 is verified for $r \leq 8$. By taking $r = 3$, this conjecture is exactly the Four Color Theorem, by Tait's result on the equivalence between 4-vertex-colorability and 3-edge-colorability for planar cubic graphs.

By Vizing's theorem, if G is a simple r-graph, then it has chromatic index at most $r + 1$. Conjecture 7.2 asserts that the truth holds even for all r-graphs. This conjecture is one of the central conjectures in the theory of edge colorings and closely related to other fundamental conjectures on edge-colorings of multigraphs. This fact motivates the research on the structure of r-graphs. The conjecture has been verified for $r \leq 15$, stepwise by several authors.

It is easy to see that a cubic graph is bridgeless if and only if it is a 3-graph. Conjecture 7.3 extends Berge-Fulkerson conjecture from 3-graph to r-graphs for all $r \geq 3$, and is called the generalized Berge-Fulkerson conjecture. Analogous to the cubic case, Mazzuoccolo [50] proved that this conjecture is equivalent to the following statement, namely generalized Berge conjecture here.

Conjecture 7.4 (Generalized Berge conjecture [50])**.** *Every r-graph has $2r-1$ 1-factors such that each edge is contained in at least one of them.*

The number "$2r - 1$" in the conjecture can not be lower, since otherwise there exists a counterexample, constructed in [50].

So far there are not many results on the structure of r-graphs. By the definition, an r-graph must have even order. Some basic properties were observed in [61].

Proposition 7.5 ([61])**.** *Every r-graph has a 1-factor.*

Theorem 7.6 ([8],[61])**.** *For any r-graph G, there is a positive integer p such that G has rp 1-factors and each edge is contained in precisely p of them.*

This theorem is a corollary of Edmonds' matching polytope theorem, and trivially implies the following proposition.

Proposition 7.7. *Let G be an r-graph. For any edge e of G, the graph G has a 1-factor containing e.*

This proposition implies Proposition 7.5. We show that they can be further strengthened. Proposition 7.7 can be easily reformulated as: if G is an r-graph and S is a list of $r-1$ edges of G that has a common end, then G has a 1-factor containing none of S. We show in the following theorem that the condition "that has a common end" is not necessary.

Theorem 7.8. *If G is an r-graph and S is a list of $r-1$ edges of G, then G has a 1-factor containing none of S.*

We remark that this theorem is an improvement of Theorem 3.15, and the proof for the former follows from the latter. For the sake of completeness, we give the proof as follows.

Proof of Theorem 7.8. Suppose to the contrary that every 1-factor of G intersects with S. Thus, $G-S$ has no 1-factor. Let $G' = G-S$. By Theorem 3.9, there exists $T \in V(G')$ such that $o(G'-T) > |T|$. Since every r-graph has even order, in particular for G, it follows that $o(G'-T)$ and $|T|$ have the same parity. Hence,

$$o(G'-T) \geq |T| + 2. \tag{7.1}$$

Let $O_1, \ldots, O_k$ be the odd components and $O_{k+1}, \ldots, O_{k+s}$ be the even components of $G'-T$. For each $i \in \{1, \ldots, k+s\}$, let a_i and b_i be the number of edges of S joining O_i respectively to T and to some other component O_j and let m_i be the number of edges of G' joining O_i to T.

The total number of edges going out of O_i is $a_i + b_i + m_i$. Since G is an r-graph, for $i \in \{1, \ldots, k\}$, we have $a_i + b_i + m_i \geq r$. Hence,

$$\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i + \sum_{i=1}^{k} m_i \geq kr. \tag{7.2}$$

Moreover, since S is of size $r-1$, we have

$$2\sum_{i=1}^{k+s} a_i + \sum_{i=1}^{k+s} b_i \leq 2(r-1). \tag{7.3}$$

Since the total number of edges going out of T must be at most $\sum_{v\in T} d_G(v)$, we have

$$\sum_{i=1}^{k+s} a_i + \sum_{i=1}^{k+s} m_i \leq \sum_{v\in T} d_G(v) = r|T|. \tag{7.4}$$

Add the inequalities 7.3 and 7.4, we thereby obtain

$$3\sum_{i=1}^{k+s} a_i + \sum_{i=1}^{k+s} b_i + \sum_{i=1}^{k+s} m_i \leq r(|T|+2) - 2. \tag{7.5}$$

Since the left of 7.2 is less than the left of 7.5, we have $kr < r(|T|+2) - 2 < r(|T|+2)$. This gives $k < |T|+2$, contradicting the inequality 7.1. □

Further results on r-graphs were obtained by Rizzi [57], who constructed r-graphs with specific properties in terms of 1-factors to disprove some conjectures of Seymour.

An approach to the solution to the generalized Berge conjecture (hence, to the generalized Berge-Fulkerson conjecture) is to look for the minimum constant c such that every r-graph has c 1-factors whose union is $E(G)$. However, it is even open whether such c exists.

Here, we follow another approach by asking such a question: at least how many edges we can cover by $2r-1$ 1-factors for every r-graph, and more general by k 1-factors for every k? This question will be treated on in the next section.

7.2 Union of 1-factors in r-graphs

The results of this section have already been published in [31].

Given an r-graph G, let $\mathcal{F}$ be the set of 1-factors in G. Fix a positive integer k. Define

$$m(r,k,G) = \max_{M_1,\ldots,M_k\in\mathcal{F}} \frac{|\bigcup_{i=1}^{k} M_i|}{|E(G)|},$$

and

$$m(r,k) = \inf_{G} m(r,k,G),$$

where the infimum is taken over all r-graphs. The parameter $m(r,k,G)$ is the maximum fraction of the edges covered by k 1-factors in an r-graph G. Clearly, $m(r,k) \leq m(r,k+1) \leq 1$. Conjecture 7.4 can be reformulated as follows:

Conjecture 7.9. *$m(r, 2r-1) = 1$ for every integer r with $r \geq 3$.*

By Theorem 7.6, the following lower bound for $m(r,k)$ can be easily obtained.

Theorem 7.10. *$m(r,k) \geq 1 - (\frac{r-1}{r})^k$ for every positive integers r and k with $r \geq 3$.*

Proof. The proof is by induction on k. Since every r-graph has a 1-factor, which covers fraction $\frac{1}{r}$ of the edges, the proof is trivial for $k = 1$. We proceed to the induction step. Let G be an r-graph and $E = E(G)$. By the induction hypothesis, G has $k-1$ many 1-factors $M_1, \dots, M_{k-1}$ such that

$$\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - (\frac{r-1}{r})^{k-1}. \tag{7.6}$$

Moreover, by Theorem 7.6, there exist a positive integer p such that G has rp 1-factors $F_1, \dots, F_{rp}$ and each edge is contained in precisely p of them. It follows that for every $X \subseteq E$, graph G has a 1-factor F among $F_1, \dots, F_{rp}$ such that $|F \cap X| \geq \frac{|X|}{r}$. In particular, let $X = E \setminus \bigcup_{i=1}^{k-1} M_i$ and $M_k = F$. Thus,

$$|M_k \cap (E \setminus \bigcup_{i=1}^{k-1} M_i)| \geq \frac{|E \setminus \bigcup_{i=1}^{k-1} M_i|}{r}. \tag{7.7}$$

Since the left side equals to $|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|$, dividing the inequality by $|E|$ yields

$$\frac{|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq \frac{1}{r}(1 - \frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|}) \geq \frac{1}{r}(\frac{r-1}{r})^{k-1}, \tag{7.8}$$

where the last inequality follows from inequality 7.6. Therefore, by summing up formulas 7.6 and 7.8, we obtain

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - (\frac{r-1}{r})^k,$$

and so $m(r,k) \geq 1 - (\frac{r-1}{r})^k$ by the choice of G. □

By a similar argument as for this theorem, one can deduce the following observation.

Observation 7.11. *If the generalized Berge-Fulkerson conjecture is true, then for every integers r and k with $r \geq 3$ and $1 \leq k \leq 2r-1$,*

$$m(r,k) \geq 1 - \prod_{i=1}^{k} \frac{2r-1-i}{2r+1-i}.$$

Now we are going to improve the lower bound of $m(r,k)$ given in Theorem 7.10. The following theorem is the main result in this chapter.

Theorem 7.12. *Let r and k be two positive integers with $r \geq 3$. The following two statements hold true:*
(1) if r is even, then

$$m(r,k) \geq 1 - \prod_{i=1}^{k} \frac{(r^2-3r+1)i - (r^2-5r+3)}{(r^2-2r-1)i - (r^2-4r-1)};$$

(2) if r is odd, then

$$m(r,k) \geq 1 - \prod_{i=1}^{k} \frac{(r^2-2r-1)i - (r^2-4r+1)}{(r^2-r-2)i - (r^2-3r-2)}.$$

For the particular case $r = 3$, we obtain the result of Kaiser, Král and Norine [36] and of Mazzuoccolo [48].

The following table partly lists the data calculated according to the formulas in Theorem 7.12 for instance of r and k.

	$r=3$	$r=4$	$r=5$
$m(r,2) \geq$	$\frac{3}{5} = 0.6$	$\frac{9}{20} = 0.45$	$\frac{13}{35} \approx 0.3714$
$m(r,3) \geq$	$\frac{27}{35} \approx 0.7714$	$\frac{3}{5} = 0.6$	$\frac{409}{805} \approx 0.5081$
$m(r,4) \geq$	$\frac{55}{63} \approx 0.873$	$\frac{103}{145} \approx 0.7103$	$\frac{793}{1288} \approx 0.6157$
$m(r,5) \geq$	$\frac{215}{231} \approx 0.9307$	$\frac{344}{435} \approx 0.7908$	$\frac{4621}{6601} \approx 0.7$
$m(r,6) \geq$	$\frac{413}{429} \approx 0.9627$	$\frac{15884}{18705} \approx 0.8492$	$\frac{25283}{33005} \approx 0.766$
$m(r,7) \geq$	$\frac{6307}{6435} \approx 0.9801$	$\frac{138949}{155875} \approx 0.8914$	$\frac{69221}{84665} \approx 0.8176$
$m(r,8) \geq$	$\frac{12027}{12155} \approx 0.9895$	$\frac{2730303}{2961625} \approx 0.9219$	$\frac{1234672}{1439305} \approx 0.8578$
$m(r,9) \geq$	$\frac{45933}{46189} \approx 0.9945$	$\frac{44725797}{47386000} \approx 0.9439$	$\frac{1791791}{2015027} \approx 0.8892$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 7.1: A lower bound for the parameter $m(r,k)$

7.2.1 The perfect matching polytope

Let G be a graph and w be a vector of $\mathbb{R}^{E(G)}$. The entry of w corresponding to an edge e is denoted by $w(e)$, and for $A \subseteq E$, we define $w(A) = \sum_{e \in A} w(e)$. The vector w is a *fractional 1-factor* if it satisfies

(i) $0 \leq w(e) \leq 1$ for every $e \in E(G)$,

(ii) $w(\partial\{v\}) = 1$ for every $v \in V(G)$, and

(iii) $w(\partial S) \geq 1$ for every $S \subseteq V(G)$ with odd cardinality.

Let $\mathcal{F}(G)$ denote the set of all fractional 1-factors of a graph G. If M is a 1-factor, then its characteristic vector χ^M is contained in $\mathcal{F}(G)$. Furthermore, if $w_1, \ldots, w_n \in \mathcal{F}(G)$, then any convex combination $\sum_{i=1}^{n} \alpha_i w_i$ (where $\alpha_1, \ldots, \alpha_n$ are nonnegative real numbers summing up to 1) also belongs to $\mathcal{F}(G)$. It follows that $\mathcal{F}(G)$ contains the convex hull of all the vectors χ^M where M is a 1-factor of G. The Perfect Matching Polytope Theorem asserts that the converse inclusion also holds:

Theorem 7.13 ([8])**.** *For any graph G, the set $\mathcal{F}(G)$ coincides with the convex hull of the characteristic vectors of all 1-factors of G.*

Besides this theorem, the following property on fractional 1-factors is also needed for the proof of Theorem 7.12.

Lemma 7.14 ([36]). *Let w be a fractional 1-factor of a graph G and $c \in \mathbb{R}^{E(G)}$. Then G has a 1-factor M such that $c \cdot \chi^M \geq c \cdot w$, where $\cdot$ denotes the scalar product, and $|M \cap C| = 1$ for each edge-cut C with odd cardinality and with $w(C) = 1$.*

7.2.2 Proof of Theorem 7.12

Instead of Theorem 7.12, we prove the following stronger one.

Theorem 7.15. *Let G be an r-graph with $V = V(G)$ and $E = E(G)$.*

(a) If r is even and $r \geq 4$, then for any positive integer k, graph G has k 1-factors $M_1, \ldots, M_k$ such that

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-3r+1)i - (r^2-5r+3)}{(r^2-2r-1)i - (r^2-4r-1)}$$

and $\sum_{i=1}^{k} \chi^{M_i}(C) \leq (r-1)k + 2$ for each $(r+1)$-edge-cut C.

(b) If r is odd and $r \geq 3$, then for any positive integer k, graph G has k 1-factors $M_1, \ldots, M_k$ such that

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-2r-1)i - (r^2-4r+1)}{(r^2-r-2)i - (r^2-3r-2)},$$

$\sum_{i=1}^{k} \chi^{M_i}(C) = k$ for each r-edge-cut C and $\sum_{i=1}^{k} \chi^{M_i}(D) \leq rk + 2$ for each $(r+2)$-edge-cut D.

Proof. (The proof is by induction on k).

Statement (a). The statement holds for $k = 1$, since the required M_1 can be an arbitrary 1-factor of G. Assume that $k \geq 2$. By the induction hypothesis, G has $k-1$ many 1-factors $M_1, \ldots, M_{k-1}$ such that

$$\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k-1} \frac{(r^2-3r+1)i - (r^2-5r+3)}{(r^2-2r-1)i - (r^2-4r-1)}$$

and

$$\sum_{i=1}^{k-1} \chi^{M_i}(C) \leq (r-1)(k-1)+2 \tag{7.9}$$

for each $(r+1)$-edge-cut C.

For $e \in E$, let $n(e)$ denote the number of 1-factors among $M_1, \ldots, M_{k-1}$ that contains e, and define

$$w_k(e) = \frac{(r-2)k-(r-4)-n(e)}{(r^2-2r-1)k-(r^2-4r-1)}.$$

We claim that w_k is a fractional 1-factor of G, that is, $w_k \in \mathcal{F}(G)$. Since $k \geq 2, r \geq 4$ and $0 \leq n(e) \leq k-1$, we can deduce that $\frac{1}{r+3} < w_k(e) < 1$. Moreover, note that for every $X \subseteq E$, the equality $\sum_{e \in X} n(e) = \sum_{i=1}^{k-1} \chi^{M_i}(X)$ always holds and so

$$w_k(X) = \sum_{e \in X} w_k(e) = \frac{((r-2)k-(r-4))|X| - \sum_{i=1}^{k-1} \chi^{M_i}(X)}{(r^2-2r-1)k-(r^2-4r-1)}. \tag{7.10}$$

Thus for $v \in V$, since $\sum_{i=1}^{k-1} \chi^{M_i}(\partial\{v\}) = k-1$, we have $w_k(\partial\{v\}) = \frac{((r-2)k-(r-4))r-(k-1)}{(r^2-2r-1)k-(r^2-4r-1)} = 1$. Finally, let $S \subseteq V$ with odd cardinality. Since G is an r-graph, we have $|\partial S| \geq r$. Recall that $w_k(e) > \frac{1}{r+3}$ for each edge e. So, $w_k(\partial S) > 1$, provided that $|\partial S| \geq r+3$. Hence, we may next assume that $|\partial S| \in \{r+1, r+2\}$. By parity, $|\partial S| = r+1$. Formula 7.9 implies $\sum_{i=1}^{k-1} \chi^{M_i}(\partial S) \leq (r-1)(k-1)+2$. With the help of formula 7.10, we deduce that $w_k(\partial S) \geq \frac{((r-2)k-(r-4))(r+1)-((r-1)(k-1)+2)}{(r^2-2r-1)k-(r^2-4r-1)} = 1$. This completes the proof of the claim.

By Lemma 7.14, graph G has a 1-factor M_k such that

$$(1-\chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot \chi^{M_k} \geq (1-\chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot w_k.$$

Since the left side is just $|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|$ and the right side equals to $\frac{(r-2)k-(r-4)}{(r^2-2r-1)k-(r^2-4r-1)}(|E| - |\bigcup_{i=1}^{k-1} M_i|)$, it follows that

$$|\bigcup_{i=1}^{k} M_i| \geq \frac{(r^2-3r+1)k-(r^2-5r+3)}{(r^2-2r-1)k-(r^2-4r-1)}|\bigcup_{i=1}^{k-1} M_i| + \frac{(r-2)k-(r-4)}{(r^2-2r-1)k-(r^2-4r-1)}|E|,$$

which leads to

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-3r+1)i-(r^2-5r+3)}{(r^2-2r-1)i-(r^2-4r-1)},$$

as desired.

Moreover, let C be an edge cut with cardinality $r+1$. Clearly, $\chi^{M_k}(C) \leq r+1$. Thus, if $\sum_{i=1}^{k-1} \chi^{M_i}(C) \leq (r-1)(k-1)$ then $\sum_{i=1}^{k} \chi^{M_i}(C) \leq (r-1)k+2$, as desired. Hence, we may assume that $\sum_{i=1}^{k} \chi^{M_i}(C) > (r-1)(k-1)$. By formula 7.9 and by parity, we have $\sum_{i=1}^{k-1} \chi^{M_i}(C) = (r-1)(k-1)+2$. We calculate from formula 7.10 that $w_k(C) = 1$. By Lemma 7.14, we have $\chi^{M_k}(C) = 1$, which yields $\sum_{i=1}^{k} \chi^{M_i}(C) = (r-1)k - r + 4 < (r-1)k+2$, as desired. This completes the proof of statement (a).

Statement (b). Let w_1 be a vector of $\mathbb{R}^E$ defined by $w_1(e) = \frac{1}{r}$ for $e \in E$. Clearly, $w_1 \in \mathcal{F}(G)$. By Lemma 7.14, G has a 1-factor M_1 such that $\chi^{M_1}(C) = 1$ for each edge cut C with odd cardinality and with $w_1(C) = 1$, that is, for each r-edge-cut C. Therefore, the statement is true for $k = 1$.

Assume $k \geq 2$. By the induction hypothesis, G has $k-1$ many 1-factors $M_1, \ldots, M_{k-1}$ such that

$$\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k-1} \frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)},$$

and for each r-edge-cut C

$$\sum_{i=1}^{k-1} \chi^{M_i}(C) = k-1, \tag{7.11}$$

and for each $(r+2)$-edge-cut D

$$\sum_{i=1}^{k-1} \chi^{M_i}(D) \leq r(k-1)+2. \tag{7.12}$$

For $e \in E$, let $n(e)$ denote the number of 1-factors among $M_1, \ldots, M_{k-1}$ that contains e, and define

$$w_k(e) = \frac{(r-1)k-(r-3)-2n(e)}{(r^2-r-2)k-(r^2-3r-2)}.$$

We claim that $w_k \in \mathcal{F}(G)$. Since $k \geq 2, r \geq 3$ and $0 \leq n(e) \leq k-1$, we can deduce that $0 < \frac{1}{r+4} < w_k(e) < 1$. Moreover, note that for every $X \subseteq E$, the equality $\sum_{e \in X} n(e) = \sum_{i=1}^{k-1} \chi^{M_i}(X)$ always holds and so

$$w_k(X) = \frac{((r-1)k-(r-3))|X| - 2\sum_{i=1}^{k-1} \chi^{M_i}(X)}{(r^2-r-2)k-(r^2-3r-2)}. \tag{7.13}$$

Thus for $v \in V$, since $\sum_{i=1}^{k-1} \chi^{M_i}(\partial\{v\}) = k-1$, we have $w_k(\partial\{v\}) = \frac{((r-1)k-(r-3))r-2(k-1)}{(r^2-r-2)k-(r^2-3r-2)} = 1$. Finally, let $S \subseteq V$ with odd cardinality. Since G is an r-graph, $|\partial S| \geq r$. On the other hand, by recalling that $w_k(e) > \frac{1}{r+4}$ for each edge e, we have $w_k(\partial S) > 1$, provided that $|\partial S| \geq r+4$. Hence, we may next assume that $r \leq |\partial S| \leq r+3$. By parity, either $|\partial S| = r$ or $|\partial S| = r+2$. In the former case, formula 7.11 implies $\sum_{i=1}^{k-1} \chi^{M_i}(\partial S) = k-1$, and thus we can calculate from formula 7.13 that $w_k(\partial S) = 1$. In the latter case, formula 7.12 implies $\sum_{i=1}^{k-1} \chi^{M_i}(\partial S) \leq r(k-1)+2$ and similarly, we get $w_k(\partial S) \geq \frac{((r-1)k-(r-3))(r+2)-2(r(k-1)+2)}{(r^2-r-2)k-(r^2-3r-2)} = 1$. This proves the claim.

By Lemma 7.14, graph G has a 1-factor M_k such that

$$(1 - \chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot \chi^{M_k} \geq (1 - \chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot w_k.$$

Since the left side is just $|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|$ and the right side equals to $\frac{(r-1)k-(r-3)}{(r^2-r-2)k-(r^2-3r-2)}(|E| - |\bigcup_{i=1}^{k-1} M_i|)$, it follows that

$$|\bigcup_{i=1}^{k} M_i| \geq \frac{(r-1)k-(r-3)}{(r^2-r-2)k-(r^2-3r-2)}|E| + \frac{(r^2-2r-1)k-(r^2-4r+1)}{(r^2-r-2)k-(r^2-3r-2)}|\bigcup_{i=1}^{k-1} M_i|,$$

which leads to

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)},$$

as desired.

Moreover, let C be an edge cut of cardinality r. Formula 7.11 implies $\sum_{i=1}^{k-1} \chi^{M_i}(C) = k-1$. On the other hand, We can calculate from formula 7.13 that $w_k(C) = 1$, and thus $\chi^{M_k}(C) = 1$ by Lemma 7.14. Therefore, $\sum_{i=1}^{k} \chi^{M_i}(C) = k$, as desired.

We next let D be an edge cut of cardinality $r+2$. Clearly, $\chi^{M_k}(D) \leq r+2$. Thus if $\sum_{i=1}^{k-1} \chi^{M_i}(D) \leq r(k-1)$, then $\sum_{i=1}^{k} \chi^{M_i}(D) \leq rk+2$, as desired. By formula 7.12 and by parity, we may next assume that $\sum_{i=1}^{k-1} \chi^{M_i}(D) = r(k-1)+2$. By calculation we can get $w_k(D) = 1$, and thus $\chi^{M_k}(D) = 1$ by Lemma 7.14, which also yields $\sum_{i=1}^{k} \chi^{M_i}(D) \leq rk+2$. This completes the proof of the theorem. □

7.3 Cores and measures of r-graphs

Recall that the basic question on r-graphs is to determine which r-graph is r-edge-colorable. Regarding the difficulty on answering this question directly, we consider the question in a more general sense—to determine how far an r-graph is from being r-edge-colorable?

We will extend the concept of cores of cubic graphs to r-graphs. Analogous to the cubic case, we define in terms of cores a parameter μ_3^r which measures how far an r-graph is from being r-edge-colorable. Such a parameter is also called a measure of edge-uncolorability (for r-graphs). As we can see from Chapter 5, many measures for cubic graphs are proposed in the literature, and cubic graphs with small value of measures are shown satisfy some hard conjectures. However, so far no such measures for all r-graphs are known. The study for r-graphs on cores and on μ_3^r gives us an insight into the structure of r-graphs, on which there are not many knowledges so far.

Throughout this chapter, we take the following definitions and notations. For a real number p, let $\lfloor p \rfloor_o$ (resp., $\lfloor p \rfloor_e$) denote the odd (resp., even) number of $\{\lfloor p \rfloor, \lfloor p \rfloor - 1\}$. An *Eulerian graph* is a graph where each vertex has even degree.

Let G be an r-graph and S_r be a list of r 1-factors $M_1, M_2, \ldots, M_r$ of G. For $0 \leq i \leq r$, let E_i be the set of edges that are contained in precisely i elements of S_r. Let $k = |E_0|$. The *k-core* of G with respect to S_r (or to $M_1, M_2, \ldots, M_r$) is the subgraph G_c of G which is induced by $E(G) \setminus (E_1 \cup E_3 \cup \cdots \cup E_{\lfloor \frac{r}{2} \rfloor_o})$. If the value of k is irrelevant, then we say that G_c is a *core* of G. If G_c is an Eulerian graph, then we call G_c an *Eulerian core*.

Proposition 7.16. *If G_c is a k-core of an r-graph G, then $E(G_c)$ can be divided into two parts E_{odd} and E_{even}, where $E_{even} = E_0 \cup E_2 \cdots \cup E_{\lfloor r \rfloor_e}$ and $E_{odd} = E_{\lfloor \frac{r}{2} \rfloor_o + 2} \cup E_{\lfloor \frac{r}{2} \rfloor_o + 4} \cup \cdots \cup E_{\lfloor r \rfloor_o}$, and the following statements hold:*
(1) $G[E_{even}]$ is an Eulerian graph,
(2) E_{odd} is either an empty set or a matching of G,
(3) $k = |E_2| + 2|E_3| + \cdots + (r-1)|E_r|$.

Proof. Let $v \in V(G)$. Denote that $E(v) = \{e \in E(G)\colon\ e \text{ is incident with } v\}$, $E^o(v) = \{e \in E(v)\colon\ e \in E_i \text{ and } i \text{ is odd}\}$, and $E^e(v) = E(v) \setminus E^o(v)$. Since every 1-factor touches v precisely once, we can deduce that $|E^o(v)|$ has the same parity as r. Since v has degree r, that is, $|E^e(v)| + |E^o(v)| = r$, it follows that $|E^e(v)|$ is an even number. Therefore, statement (1) holds true.

For statement (2), suppose to the contrary that there are two edges f and h of $E_{\lfloor \frac{r}{2} \rfloor_o + 2} \cup E_{\lfloor \frac{r}{2} \rfloor_o + 4} \cup \cdots \cup E_{\lfloor r \rfloor_o}$ sharing precisely one common end

v. Thus the r 1-factors with respect to G_c touch v at least $2(\lfloor \frac{r}{2} \rfloor_o + 2)$ times, contradicting with the fact that they touch v precise r times.

By counting the edges of the r 1-factors with respect to G_c with repetition in two different ways, we can get $\sum_{i=0}^{r} i|E_i| = r\frac{|V|}{2} = |E|$, where the last equality follows from the fact the G is r-regular. Thus, $|E_0| = \sum_{i=2}^{r}(i-1)|E_i|$, we are done with the proof. □

We propose the generalized Fan-Raspaud conjecture in terms of empty intersection of 1-factors, and then reformulate it in the language of cores of r-graphs.

Conjecture 7.17 (The generalized Fan-Raspaud conjecture)**.** *Every r-graph has r 1-factors $M_1, M_2, \ldots, M_r$ such that any $\lfloor \frac{r}{2} \rfloor_o + 2$ of them have empty intersection.*

By taking $r = 3$, this conjecture reduces to Fan-Raspaud conjecture. Moreover, any r 1-factors from a Berge-Fulkerson cover of an r-graph cover each edge at most twice and hence, they satisfy the property in Conjecture 7.17. This shows that the generalized Berge-Fulkerson conjecture implies the generalized Fan-Raspaud conjecture. The following conjecture is a reformulation of the generalized Fan-Raspaud conjecture in terms of cores.

Conjecture 7.18. *Every r-graph has an Eulerian core.*

Proposition 7.19. *The Conjectures 7.17 and 7.18 are equivalent.*

Proof. Let G_c be a core of an r-graph G with respect to r 1-factors $M_1, M_2, \ldots, M_r$. Denote by H_1 the subgraph of G induced by $E_0 \cup E_2 \cdots \cup E_{\lfloor r \rfloor_e}$, and by H_2 the subgraph of G induced by $E_{\lfloor \frac{r}{2} \rfloor_o + 2} \cup E_{\lfloor \frac{r}{2} \rfloor_o + 4} \cup \cdots \cup E_{\lfloor r \rfloor_o}$. Since H_1 is an Eulerian graph by Proposition 7.16 statement (1), it follows that G_c is an Eulerian core if and only if H_2 is an Eulerian graph. Since H_2 is a matching of G by Proposition 7.16 statement (2), the latter one is equivalent to that H_2 is an empty graph, that is, equivalent to that any $\lfloor \frac{r}{2} \rfloor_o + 2$ of $M_1, M_2, \ldots, M_r$ has empty intersection. □

Now we are ready to introduce a measure μ_3^r of edge-uncolorability of r-graphs. Let G be an r-graph. Define $\mu_3^r(G) = \min\{k\colon\ G \text{ has a } k\text{-core}\}$, that is, $\mu_3^r(G)$ is the minimum number of edges of G uncovered by r 1-factors. Clearly, an r-graph is r-edge-colorable if and only if it has zero value of μ_3^r. An r-graph with small value of μ_3^r is regarded close to being r-edge-colorable.

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