Global solutions of some chemotaxis systems

Am Institut für Mathematik der Universität Paderborn unter der Betreuung durch Herrn Univ. Prof. Dr. Michael Winkler erarbeitete und der Fakultät für Elektrotechnik, Informatik und Mathematik der Universität Paderborn im Oktober 2017 vorgelegte Dissertation

Xinru Cao

Zusammenfassung

In dieser Arbeit werden die globale Existenz und das Langzeitverhalten der Lösungen in Chemotaxis-Systemen betrachtet. Zuerst konzentrieren wir uns auf das parabolisch-parabolische Keller-Segel-Modell und untersuchen eine hinreichende Bedingung für die Existenz globaler Lösungen. Auch die Beschränktheit und globale Existenz der Lösungen eines Chemotaxis-Haptotaxis-Modells werden unter geeigneten Annahmen an die Parameter demonstriert. Weiterhin wird das Langzeitverhalten in einem Keller-Segel-Modell mit logistischer Quelle bewiesen. Für den speziellen Fall, dass das logistische Keller-Segel-Modell ohne Wachstumsterm betrachtet wird und mit einem zusätzlichen Konvektionsterm gekoppelt ist, wird eine optimale Konvergenzabschätzung bewiesen. Schließlich wird die Existenz klassischer Lösungen eines Chemotaxis-Navier-Stokes-Modells im zwei- und dreidimensionalen Fall unter geeigneten Kleinheitsbedingungen an die Anfangsdaten erhalten.

Abstract

In this work, global existence and large time behavior of solutions in chemotaxis systems are considered. We first focus on the fully parabolic Keller-Segel model and investigate a sufficient condition for the existence of global solutions. The boundedness and global existence of solutions in a chemotaxis-haptotaxis model are also demonstrated under suitable assumptions on the parameters. Similarly, the long time behavior in a Keller-Segel model with logistic dampening is identified. Particularly, when the logistic Keller-Segel model is without growth term and is coupled with an additional convection term, an optimal decay estimate is given. In addition, the existence of classical solutions of a chemotaxis-Navier-Stokes model in the two- and three-dimensional cases is obtained under suitable smallness conditions on the initial data.

Acknowledgement

At this moment, I would like to thank my supervisor Michael Winkler, without whom this thesis would not be possible. When I first met him in 2012, I was a little backward, and really hesitated about my mathematical career. He offered me an opportunity and lead me to the field of chemotaxis models from which I have found so much fun.

Not to mention his brilliant mathematical thinkings, he is an excellent advisor, who is always willing to listen, to motivate, and to comment, which turn out to be significant. In China there is a saying, it is better to teach a man fishing than give him fish. I think Michael prefers to teach and encourage me to make my own fishing nets. During these years, he let me know what is a real mathematician, not by words but by actions. His passion and concentration for mathematics and attitude for work are the best lecture for me. I thank him for offering me a very comfortable working place in Paderborn, where I feel free and enjoy mathematics all the time.

Many thanks go to Dr. Sachiko Ishida. When we started work together, she had already been a very successful young mathematician while I was a beginner. I always cherish this experience as one of the most important because what it brings me is much more than a joint paper. Many appreciation also goes to my colleagues in Paderborn: Johannes, who also contributes partly in Chapter 6 and always patiently reads my preprints, gives important comments, I thank him for his professionality and selflessness; Tobias and Marcel, who form a nice atmosphere around and build a very high level in our weakly seminar.

I am very grateful to Professor Yuan Lou, who is very open mind and always encouraging and advising me. With his support I am able to return to Paderborn for a long period. I also appreciate Professor Sining Zheng, who was my supervisor in Dalian. I decided to study Partial Differential Equations when I was an undergraduate student in his PDE lecture, which opened a new world to me. Many thanks to old friends: Professor Youshan Tao, for his kindness and support in many different situations; Dr. Xueli Bai, with whom I enjoy discusses not only for math but also at personal level; Professor Tomomi Yokota, who gives many suggestions about my research when we meet in conferences.

Additionally, I would like to mention many other colleagues from whom I have learnt: Dongmei, Bingran, Yulan, Christian, Jakob, Elio, Tomesz, Kentarou, Masaaki, Shunsuke, ...

Last but not least, I thank all my family members and friends, especially my parents for their endless love, understanding and unconditional support since I was born.

Contents

1. Introduction

In recent decades, mathematical biology has been rapidly developed as an interdisciplinary scientific subject receiving attention from both mathematicians and biologists. It aims at mathematical modeling, analysis and simulation of biological processes by using mathematical methods and techniques.
An important mathematical treatment is to convert biological processes into systems which are composed of several partial differential equations (PDE for short) linked together. Therefore, a study on the resulting PDE systems may contribute to a better understanding of these biological processes; not only by possibly explaining evident experimental observations but also by possibly predicting some properties beyond.
Especially since rather few meaningful differential equations could have explicit solutions due to their complexity, mathematical tools can help us to gain some qualitative analysis on the properties of solutions; some basic questions concerning these are: existence, uniqueness and stability of the solutions.
In the present thesis, we are going to study a class of second order semilinear parabolic systems, which arise in biological mathematics and are usually called chemotaxis models.

1.1. Taxis models

Taxis is the ability of organisms to motivate their movement in response to an external stimulus. A celebrated taxis model called Keller-Segel model describes the evolution of cell populations and their movement partly directed by a chemical signal produced by themselves. It was introduced in 1970 by Keller and Segel in the style of the following initial-boundary value problem [45]:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v), & (x,t) \in \Omega \times (0,T), \\ \tau v_t = \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\ \nabla u \cdot \nu = \nabla u \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary and ν is the outer normal vector on $\partial\Omega$, $\chi \in \mathbb{R}$ and $\tau \geq 0$ are constants, and (u_0, v_0) is a pair of nonnegative functions. The unknown functions $u = u(x,t)$ and $v = v(x,t)$ denote the density of the cell population and the concentration of chemical substance, respectively. In the first equation, Δu indicates that the cells diffuse randomly, and $-\nabla \cdot (\chi u \nabla v)$ reflects the hypothesis that cells move towards higher densities of the signal, where the number χ measures the sensitivity of the chemotactic response to the chemical gradients. The second equation models the assumptions that the chemical is produced by the cells and degrades and also diffuses by itself.
A very special feature in this system is the appearance of the term $-\nabla \cdot (\chi u \nabla v)$, which in contrast to diffusion $\Delta u = \nabla \cdot (\nabla u)$, is usually called cross-diffusion. Since this cross-diffusion term models a process which may result in the aggregation of cells, the occurrence of a certain blow-up phenomenon can be detected, namely, u becomes unbounded in respect to the spatial L^∞-norm.

The goal of this work is to develop mathematical methods of analyzing global bounded solutions in several taxis models by ruling out such blow up phenomena. A proper estimate on the cross-diffusion term $-\chi\nabla\cdot(u\nabla v)$ which is coupled with u and ∇v seems critical in the analysis. One main technique we rely on is a result on maximal Sobolev regularity which links Δv to u such that u and ∇v can partly be decoupled; when attempting to study $\frac{d}{dt}\int_\Omega u^p$ ($p\in\mathbb{R}$), ∇v would appear, and then the maximal Sobolev regularity may help to control this term by u through certain spatio-temporal integrals such that we can arrive at an inequality only containing u (see Lemmata 2.4.1, 3.3.1 and 4.3.1 below for example). Compared with many previous works where instead $\frac{d}{dt}\int_\Omega|\nabla v|^q$ ($q\in\mathbb{R}$) is additionally considered, this idea apparently provides a more efficient way to estimate the L^p-norm of u. Since the equation for v already offers a degrading structure, the maximal Sobolev regularity result for the second equation of (1.1) also involves a time potential function, which turns out to be crucial to prove a temporarily uniform estimate. This special version of maximal Sobolev regularity will be applied in different situations in this work and will be first proven in the next part of this chapter.

In the main part of this thesis, chemotatic cross-diffusion will be treated in several particular contexts. In Chapter 2, we give a sufficient condition for the existence of global and bounded solutions of (1.1), which improves previous knowledge in this issue. In particular, the outcome strongly relies on an interpolation inequality for equi-integrable functions which is an improvement of a special case of the well-known Gagliardo-Nirenberg inequality.
If we consider a larger time scale, it is reasonable to include the effect of spontaneous proliferation of cells, which is commonly given in the form of a logistic source $g(u):=\kappa u-\mu u^2$ with $\kappa\geq 0$, $\mu>0$. Thus the first equation is replaced by

$$u_t=\Delta u-\nabla\cdot(\chi u\nabla v)+g(u). \tag{1.2}$$

Such logistic sources may be expected to restrain ultimate growth and to thus exert an opposite effect compared to the cross-diffusion term. The competition within these two effects usually results in colorful types of behaviors.
Chapter 3 is devoted to study the large time behavior of solutions in a parabolic-parabolic logistic Keller-Segel model, that is, (1.1) with the first equation replaced by (1.2). We show that if the ratio $\frac{\chi}{\mu}$ is sufficiently small, the solution (u,v) converges to $(\frac{\kappa}{\mu},\frac{\kappa}{\mu})$ in the large time limit. The approach depends on a result on maximal Sobolev regularity involving a potential function.
A similar idea is also used in Chapter 4 to prove boundedness of solutions in a chemotaxis-haptotaxis model

$$\begin{cases} u_t=\Delta u-\chi\nabla\cdot(u\nabla v)-\xi\nabla\cdot(u\nabla w)+\mu u(1-u-w), & (x,t)\in\Omega\times(0,T),\\ \tau v_t=\Delta v-v+u, & (x,t)\in\Omega\times(0,T),\\ w_t=-vw+\eta w(1-u-w), & (x,t)\in\Omega\times(0,T), \end{cases} \tag{1.3}$$

where $\xi\in\mathbb{R}$ and $\eta>0$. In addition to chemotaxis, here a new cross-diffusion term $-\xi\nabla\cdot(u\nabla w)$ appears in the first equation, meaning that the cells also orient their movement toward another chemical whose density is denoted by w. This taxis mechanism is referred to as haptotaxis because w is non-diffusible. The main result in this chapter demonstrates that chemotaxis dominates the solutions behavior in certain parameter regions. Namely, the boundedness of solutions is guaranteed when $\frac{\chi}{\mu}$ is small and it is not depending on the haptotaxis sensitivity ξ.
In Chapter 5, we furthermore analyze the qualitative behavior of bounded solutions in a logistic Keller-Segel model in a liquid environment,

$$\begin{cases} u_t+U\cdot\nabla u = \Delta u-\chi\nabla\cdot(u\nabla v)-\mu u^2, & (x,t)\in\Omega\times(0,T),\\ v_t+U\cdot\nabla v = \Delta v-v+u, & (x,t)\in\Omega\times(0,T), \end{cases} \tag{1.4}$$

where U is a given fluid velocity which influences the migration of cells by means of transport. We obtain an optimal decay rate for all bounded solutions in the sense that both upper and lower estimates are given by the same rate.
A more comprehensive variant of (1.4) will also include a gravitational effect of cells in a liquid environment. Thus the fluid is described by the full Navier-Stokes equation with an external force $u\nabla\Phi$,

$$\begin{cases} u_t = \Delta u - \nabla\cdot(uS(x,u,v)\cdot\nabla v) - U\cdot\nabla u, & (x,t)\in\Omega\times(0,T),\\ v_t = \Delta v - uv - U\cdot\nabla v, & (x,t)\in\Omega\times(0,T),\\ U_t = \Delta U - \kappa(U\cdot\nabla)U + \nabla P + u\nabla\Phi, & (x,t)\in\Omega\times(0,T),\\ \nabla\cdot U = 0, & (x,t)\in\Omega\times(0,T). \end{cases} \tag{1.5}$$

Since in the complicated fluid environment, some interactions between cells swimming speed and direction have been detected experimentally, a matrix valued function $S(x,u,v)$ is introduced to represent a rotational effect while cells trying to tend to the signal. This brings significant difficulties in mathematical analysis because mathematically useful gradient-like structural properties, well-known as favorite features of e.g. (1.1), seem to be lacking for general choices of S. In Chapter 6, we investigate a smallness condition on the initial data such that (1.5) with $N=3$ admits a global classical solution which approaches a constant steady state.
Chapter 7 considers the same problem in the two-dimensional case and $\kappa=0$ in the three-dimensional setting. Via a certain conditional functional approach, the previous results can partly be improved.

1.2. A result from maximal Sobolev regularity theory

As we have announced in the last section that a version of maximal Sobolev regularity plays a central role in dealing with the chemotactic cross-diffusion term, let us first introduce the well known maximal Sobolev regularity for Laplacian associated with Neumann boundary condition, which is an application of [34, Theorem 2.1]. Before going into details, we prepare the following Ehrling type lemma.

Lemma 1.2.1. *Let $q>1$ and $s\in(0,q)$. For any $\varepsilon>0$, we can find $C>0$ such that*

$$\|\psi\|_{L^q(\Omega)} \le \varepsilon\|\Delta\psi\|_{L^q(\Omega)} + C\|\psi\|_{L^s(\Omega)} \text{ for all } \psi\in W^{2,q}(\Omega) \text{ satisfying } \nabla\psi\cdot\nu=0 \text{ on } \partial\Omega. \tag{1.6}$$

Proof. First we know from [28, Theorem19.1] that there is a constant $c_1>0$ such that

$$\|\psi\|_{L^q(\Omega)} + \|\nabla\psi\|_{L^q(\Omega)} + \|D^2\psi\|_{L^q(\Omega)} \le c_1(\|\Delta\psi\|_{L^q(\Omega)} + \|\psi\|_{L^q(\Omega)}) \tag{1.7}$$

for all $\psi\in W^{2,q}(\Omega)$ satisfying $\nabla\psi\cdot\nu=0$ on $\partial\Omega$. Noting that $W^{2,q}(\Omega)\hookrightarrow\hookrightarrow W^{1,q}(\Omega)\hookrightarrow L^s(\Omega)$, we can apply Ehrling's Lemma; given any $\varepsilon>0$, let $\varepsilon' := \frac{\varepsilon}{c_1+c_1\varepsilon}\in(0,\frac{1}{c_1})$, there is a constant $C_{\varepsilon'}>0$ such that

$$\begin{aligned}\|\psi\|_{L^q(\Omega)} + \|\nabla\psi\|_{L^q(\Omega)} &\le \varepsilon'\left(\|\psi\|_{L^q(\Omega)} + \|\nabla\psi\|_{L^q(\Omega)} + \|D^2\psi\|_{L^q(\Omega)}\right) + C_{\varepsilon'}\|\psi\|_{L^s(\Omega)}\\ &\le c_1\varepsilon'(\|\Delta\psi\|_{L^q(\Omega)} + \|\psi\|_{L^q(\Omega)}) + C_{\varepsilon'}\|\psi\|_{L^s(\Omega)}.\end{aligned}$$

This implies that

$$\begin{aligned}\|\psi\|_{L^q(\Omega)} &\le \frac{c_1\varepsilon'}{1-c_1\varepsilon'}\|\Delta\psi\|_{L^q(\Omega)} + C_{\varepsilon'}\|\psi\|_{L^s(\Omega)}\\ &= \varepsilon\|\Delta\psi\|_{L^q(\Omega)} + C_\varepsilon\|\psi\|_{L^s(\Omega)}.\end{aligned}$$

Thus we complete the proof. □

Lemma 1.2.2. *Let $q, r \in (1,\infty)$. There exists $C = C(q,r) > 0$ with the property that for all $T > 0$, if $f \in C^0(\overline{\Omega} \times [0,T])$ and $v \in C^{2,1}(\overline{\Omega} \times (0,T)) \cap C^0(\overline{\Omega} \times [0,T))$ is a classical solution to the evolution problem*

$$\begin{cases} v_t = \Delta v - \frac{1}{2}v + f, & (x,t) \in \Omega \times (0,T), \\ \nabla v \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ v(x,0) = 0, & x \in \Omega, \end{cases} \tag{1.8}$$

then we have

$$\int_0^T \|\Delta v(\cdot,t)\|_{L^q(\Omega)}^r dt \le C \int_0^T \|f(\cdot,t)\|_{L^q(\Omega)}^r dt. \tag{1.9}$$

Proof. Letting $A := \Delta - \frac{1}{2}$ and applying [34, Lemma 2.1], we obtain a constant $c_1 > 0$ such that

$$\int_0^T \|(\Delta - \frac{1}{2})v(\cdot,t)\|_{L^q(\Omega)}^r dt \le c_1 \int_0^T \|f(\cdot,t)\|_{L^q(\Omega)}^r dt. \tag{1.10}$$

Moreover, integrating the first equation over Ω implies that

$$\frac{d}{dt}\|v(\cdot,t)\|_{L^1(\Omega)} + \frac{1}{2}\|v(\cdot,t)\|_{L^1(\Omega)} \le \|f(\cdot,t)\|_{L^1(\Omega)} \qquad \text{for all } t \in (0,T). \tag{1.11}$$

Testing $r\|v(\cdot,t)\|_{L^1(\Omega)}^{r-1}$ to the above inequality, using Young's inequality and then integrating over $(0,T)$, we obtain $c_2 > 0$ and $c_3 > 0$ such that

$$\int_0^T \|v(\cdot,t)\|_{L^1(\Omega)}^r dt \le c_2 \int_0^T \|f(\cdot,t)\|_{L^1(\Omega)}^r dt \le c_3 \int_0^T \|f(\cdot,t)\|_{L^q(\Omega)}^r dt. \tag{1.12}$$

According to Lemma 1.2.1, we conclude the existence of $c_4 > 0$ such that

$$\|v(\cdot,t)\|_{L^q(\Omega)}^r \le \|\Delta v(\cdot,t)\|_{L^q(\Omega)}^r + c_4\|v(\cdot,t)\|_{L^1(\Omega)}^r \qquad \text{for all } t \in (0,T). \tag{1.13}$$

Due to the fact that $\Delta v = (\Delta - \frac{1}{2})v + \frac{1}{2}v$ and (1.13), we see that

$$\begin{aligned} &\int_0^T \|\Delta v(\cdot,t)\|_{L^q(\Omega)}^r dt \\ \le\ & 2^{r-1}\int_0^T \|\frac{1}{2}v(\cdot,t)\|_{L^q(\Omega)}^r dt + 2^{r-1}\int_0^T \|(\Delta - \frac{1}{2})v(\cdot,t)\|_{L^q(\Omega)}^r dt \\ \le\ & \frac{1}{2}\int_0^T \|\Delta v(\cdot,t)\|_{L^q(\Omega)}^r + \frac{1}{2}c_4 \int_0^T \|v(\cdot,t)\|_{L^1(\Omega)}^r dt + 2^{r-1}c_1 \int_0^T \|f(\cdot,t)\|_{L^q(\Omega)}^r dt \\ \le\ & \frac{1}{2}\int_0^T \|\Delta v(\cdot,t)\|_{L^q(\Omega)}^r + (\frac{1}{2}c_4c_3 + 2^{r-1}c_1)\int_0^T \|f(\cdot,t)\|_{L^q(\Omega)}^r dt, \end{aligned} \tag{1.14}$$

which leads to (1.9) if we let $C := \frac{1}{2}c_4c_3 + 2^{r-1}c_1$. □

Now we adapt the above result to derive the following statement that will be an indispensable ingredient i.e. for Lemmata 2.4.1, 3.3.1 and 4.3.1 below.

Lemma 1.2.3. *Let $\tau > 0$, $q, r \in (1, \infty)$. There exists $C = C(q, r) > 0$ with the following property: For all $T > 0$, if $f \in C^0(\overline{\Omega} \times [0, T])$ and $v \in C^{2,1}(\overline{\Omega} \times (0, T)) \cap C^0(\overline{\Omega} \times [0, T))$ is a classical solution to*

$$\begin{cases} \tau v_t = \Delta v - v + f, & (x,t) \in \Omega \times (0,T), \\ \nabla v \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.15}$$

then for any $t_0 \in (0, T)$, we have

$$\int_{t_0}^{T} e^{\frac{r}{2\tau}t} \|\Delta v(\cdot,t)\|^r_{L^q(\Omega)} dt \le C \int_{t_0}^{T} e^{\frac{r}{2\tau}t} \|f(\cdot,t)\|^r_{L^q(\Omega)} dt + C\tau e^{\frac{r}{2\tau}t_0} \|v(\cdot,t_0)\|^r_{W^{2,q}(\Omega)}. \tag{1.16}$$

Proof. For given $t_0 \in (0, T)$, we know that $\partial_\nu v(\cdot, t_0) = 0$ on $\partial\Omega$. Let $d := \min\{\frac{T-t_0}{4\tau}, 1\}$ and let $\chi \in C_0^\infty([0, \infty))$ be a cut-off function satisfying

$$\begin{cases} \chi(s) = 1, & s = 0, \\ \chi(s) \le 1, & 0 < s < d, \\ \chi(s) = 0, & s \ge d. \end{cases} \tag{1.17}$$

Moreover, $|\chi'(s)| \le \frac{2}{d}$ for all $s \in [0, \infty)$. Let $w(x, s) := e^{\frac{1}{2}s} v(x, \tau s + t_0) - \chi(s) v(x, t_0)$ for $(x, s) \in \Omega \times [0, \frac{T-t_0}{\tau})$. We see that w solves the following equation

$$\begin{cases} w_s(x,s) = (\Delta - \frac{1}{2}) w(x,s) + e^{\frac{1}{2}s} f(x, \tau s + t_0) + g(x,s), & (x,s) \in \Omega \times (0, \frac{T-t_0}{\tau}), \\ \nabla w \cdot \nu = 0, & (x,s) \in \partial\Omega \times [0, \frac{T-t_0}{\tau}), \\ w(x,0) = 0, & x \in \Omega, \end{cases} \tag{1.18}$$

where $g(x, s) := \chi(s)\Delta v(x, t_0) - \chi'(s) v(x, t_0) - \frac{1}{2}\chi(s) v(x, t_0)$ in $\Omega \times [0, \frac{T-t_0}{\tau})$.
An application of the maximal Sobolev regularity result from Lemma 1.2.2 implies the existence of $C_{q,r} > 0$ such that

$$\begin{aligned} &\int_0^{\frac{T-t_0}{\tau}} \|\Delta w(\cdot,s)\|^r_{L^q(\Omega)} ds \\ \le & C_{q,r} \int_0^{\frac{T-t_0}{\tau}} \|e^{\frac{1}{2}s} f(x, \tau s + t_0)\|^r_{L^q(\Omega)} ds \\ &\qquad + C_{q,r} \int_0^{\frac{T-t_0}{\tau}} \|\chi(s)\Delta v(x,t_0) - \chi'(s) v(x,t_0) - \frac{1}{2}\chi(s) v(x,t_0)\|^r_{L^q(\Omega)} ds \\ \le & C_{q,r} \int_0^{\frac{T-t_0}{\tau}} \|e^{\frac{1}{2}s} f(x, \tau s + t_0)\|^r_{L^q(\Omega)} ds + 3^{r-1} C_{q,r} d (\frac{2}{d} + \frac{3}{2}) \|v(\cdot, t_0)\|^r_{W^{2,q}(\Omega)} \\ \le & C_{q,r} \int_0^{\frac{T-t_0}{\tau}} \|e^{\frac{1}{2}s} f(x, \tau s + t_0)\|^r_{L^q(\Omega)} ds + 4^r C_{q,r} \|v(\cdot, t_0)\|^r_{W^{2,q}(\Omega)}. \end{aligned}$$

Since $e^{\frac{1}{2}s}\Delta v(x, \tau s + t_0) = \Delta w(x, s) + \chi(s)\Delta v(x, t_0)$, we have

$$\int_0^{\frac{T-t_0}{\tau}} e^{\frac{rs}{2}} \|\Delta v(\cdot, \tau s + t_0)\|^r_{L^q(\Omega)} ds$$

$$
\begin{aligned}
&\leq 2^{r-1}\int_0^{\frac{T-t_0}{\tau}} \|\Delta w(\cdot,s)\|^r_{L^q(\Omega)}ds + 2^{r-1}\int_0^{\frac{T-t_0}{\tau}} \|\chi(s)\Delta v(\cdot,t_0)\|^r_{L^q(\Omega)} \\
&\leq 2^{r-1}C_{q,r}\int_0^{\frac{T-t_0}{\tau}} \|e^{\frac{1}{2}s}f(x,\tau s+t_0)\|^r_{L^q(\Omega)}ds + 2^{r-1}(4^rC_{q,r}+1)\|v(\cdot,t_0)\|^r_{W^{2,q}(\Omega)}.
\end{aligned}
$$

Upon changing variables, we obtain that

$$
\begin{aligned}
&\frac{1}{\tau}\int_{t_0}^T e^{\frac{r}{2\tau}(t-t_0)}\|\Delta v(\cdot,t)\|^r_{L^q(\Omega)}dt \\
\leq &\frac{1}{\tau}2^{r-1}C_{q,r}\int_{t_0}^T e^{\frac{r}{2\tau}(t-t_0)}\|f(\cdot,t)\|^r_{L^q(\Omega)}dt + (8^rC_{q,r}+2^{r-1})\|v(\cdot,t_0)\|^r_{W^{2,q}(\Omega)}, \qquad (1.19)
\end{aligned}
$$

where (1.16) follows by multiplying (1.19) by $\tau e^{\frac{r}{2\tau}t_0}$ and choosing $C := 8^rC_{q,r} + 2^{r-1}$. □

1.3. Previous publications

Chapter 2:
[10] X. Cao. An interpolation inequality and its application in the Keller–Segel model, preprint, 2017.

Chapter 3:
[11]: X. Cao. Large time behavior in the logistic Keller-Segel model via maximal Sobolev regularity. *Discrete and Continuous Dynamical Systems, Series B*, 22(9), 3369–3378, 2017.

Chapter 4:
[8]: X. Cao. Boundedness in a three-dimensional chemotaxis–haptotaxis model. *Zeitschrift für angewandte Mathematik und Physik*, 67(1):1–13, 2016.

Chapter 5:
[14]: X. Cao and M. Winkler. Sharp decay estimates in a bioconvection model with quadratic degradation in bounded domains. *Proceedings of the Royal Society of Edinburgh: Section A*, in press.

Chapter 6:
[13]: X. Cao and J. Lankeit. Global classical small-data solutions for a three-dimensional chemotaxis Navier-Stokes system involving matrix-valued sensitivities. *Calc. Var. Partial Differential Equations*, 55(4):Paper No. 107, 39, 2016.

Chapter 7:
[9]: X. Cao. Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term. *J. Differential Equations*, 261(12): 6883-6914, 2016.

1.4. Statement of contributions

Chapter 1 : I wrote the introductory chapter.

Chapter 2 : I wrote the whole chapter.

Chapter 3 : I wrote the whole chapter.

Chapter 4 : I wrote the whole chapter.

Chapter 5 : This chapter is based on a joint work [14] with Michael Winkler, each author contributed equally.

Chapter 6 : This chapter is based on a joint work [13] with Johannes Lankeit, each author contributed equally.

Chapter 7 : I wrote the whole chapter.

2. A refined criterion for boundedness in the classical Keller-Segel model

2.1. Introduction

In this chapter, we study the classical Keller-Segel model [46] to model chemotatic migration

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\ \nabla u \cdot \nu = \nabla u \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded smooth domain, $T \in (0, \infty]$, and ν denotes the outer normal vector on $\partial\Omega$. Let (u_0, v_0) be a nonnegative function pair, u and v denote the density of cells and chemical concentration, respectively. As introduced in Chapter 1, the system (2.1) describes an interaction between the cells and the chemical signal. This biological model plays an important role in numerous biological processes such as wound healing, cancer invasion. It also draws interests from many mathematicians, for surveys in this area we refer to [4, 39, 37] and the references therein.

A striking feature of this model is the occurrence of a blow-up phenomenon caused by the aggregation of cells, related research can be found in [36, 40, 66, 65, 109, 64]. The spatial dimension seems crucial in the mathematical analysis of detecting blow-up. In the one dimensional setting, blow-up never happens; all solutions are global and bounded. However, considering the two-dimensional case, one can prove the existence of radial blow-up solutions if the initial data (u_0, v_0) exceed the critical mass: $\int_\Omega u_0 > 8\pi$ [64]; otherwise, the solution always remains bounded [67]. In higher dimensions, whether a solution blows up does not depend on the total mass any more; blow-up solutions are constructed with any small mass [109]. On the other hand, looking for a sufficient condition which can prevent blow-up may be of some interest, especially in two or higher dimensions.

Throughout this chapter, we consider the classical solution (u, v) of (2.1) on $\Omega \times [0, T_{\max})$ emanating from the nonnegative initial pair $(u_0, v_0) \in C^0(\overline{\Omega}) \times W^{1,q}(\Omega)$ with $q > N$, where $T_{\max} \in (0, \infty]$ denotes the maximal existence time of the solution. The local existence theory concerning this issue is the following lemma. The proof can be found in many previous works (see e.g. [4, Lemma 3.1]).

Lemma 2.1.1. *Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary and that the initial data (u_0, v_0) are nonnegative and satisfy $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ with $q > N$. There exists $T_{\max} \in (0, \infty]$ with the property such that the problem* (2.1) *possesses a unique nonnegative classical solution (u, v) satisfying*

$$\begin{aligned} &u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ &v \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}([0, T_{\max}); W^{1,q}(\Omega)). \end{aligned}$$

Moreover, if $T_{\max} < \infty$*, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty, \quad as\ t \to T_{\max}.$$

Beyond this, a well known sufficient condition for global solutions is the following [4, Lemma 3.2]:

Proposition 2.1.2. *Let* $N \geq 1$ *and* $p > \frac{N}{2}$*. Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with smooth boundary and* (u,v) *is a nonnegative classical solution of* (2.1) *in* $\Omega \times (0,T_{\max})$ *with maximal existence time* $T_{\max} \in (0,\infty]$*. If*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty, \tag{2.2}$$

then

$$\sup_{t\in(0,T_{\max})} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)}\right) < \infty.$$

The proof is carried out either by using Neumann heat semigroup estimates or by studying a coupled energy evolution of $\int_\Omega u^p$ and $\int_\Omega |\nabla v|^{2q}$ with p,q sufficiently large [87, 27]. Generally, the condition in the above proposition can not reach the borderline value $p = \frac{N}{2}$. In the special case when $N = 2$ and thus $\frac{N}{2} = 1$, we already mentioned that blow-up can happen even though $\int_\Omega u(\cdot,t) = \int_\Omega u_0$ is bounded [64]. Therefore, we cannot expect that boundedness of $\|u(\cdot,t)\|_{L^{\frac{N}{2}}(\Omega)}$ can prevent blow-up. However, if we require a little more, namely that $\{u^{\frac{N}{2}}(\cdot,t)\}_{t\in(0,T_{\max})}$ is not only bounded with respect to the spatial L^1-norm, but also enjoys an additional equi-integrability property, we will be able to show global existence and boundedness for the system. Accordingly, the main result in the chapter reads as follows:

Theorem 2.1.3. *Assume that* $\Omega \subset \mathbb{R}^N$ *(*$N \geq 2$*) is a bounded domain with smooth boundary, and that the nonnegative initial data* (u_0, v_0) *satisfy* $u_0 \in C^0(\overline{\Omega})$ *and* $v_0 \in W^{1,q}(\Omega)$ *(*$q > N$*). Let* (u,v) *be a nonnegative classical solution of* (2.1) *on* $\Omega \times (0,T_{\max})$ *with maximal existence time* $T_{\max} \in (0,\infty]$*. If*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^{\frac{N}{2}}(\Omega)} < \infty, \tag{2.3}$$

$$and\ \{u(\cdot,t)^{\frac{N}{2}}\}_{t\in(0,T_{\max})}\ is\ equi\text{-}integrable, \tag{2.4}$$

then (u,v) *is global and bounded.*

Recalling the De la Vallée-Poussin Theorem, we obtain the following equivalent extension criterion:

Corollary 2.1.4. *Assume that* (u,v) *be a nonnegative classical solution of* (2.1) *on* $\Omega\times(0,T_{\max})$ *with* $T_{\max} \in (0,\infty]$*. Let* $f : [0,\infty) \to [0,\infty)$ *be continuous and such that*

$$\lim_{s\to\infty} \frac{f(s)}{s^{\frac{N}{2}}} = \infty.$$

If we have

$$\sup_{t\in(0,T_{\max})} \int_\Omega f(u(\cdot,t)) < \infty, \tag{2.5}$$

then (u,v) *is global and bounded.*

The above corollary inter alia shows that the boundedness of $\int_\Omega u^{\frac{N}{2}} \log u$ is sufficient for our conclusion, which is obviously not covered by Proposition 2.1.2.
On the other hand, Corollary 2.1.4 also improves the previous knowledge in the two-dimensional Keller-Segel model; it is known that the boundedness of $\int_\Omega u \log u$ and $\int_\Omega |\nabla v|^2$ can exclude blow up [4, Lemma 3.3]. Now we can immediately remove the requirement on $\int_\Omega |\nabla v|^2$. Actually, in the simplified parabolic-elliptic system where the second equation in (2.1) is replaced by $\Delta v - v + u = 0$, a crucial elliptic estimate shows that the boundedness of $\int_\Omega |\nabla v|^2$ already results from the boundedness of $\int_\Omega u \ln u$ [93, Lemma A.4]. Thus we know the solution is bounded only if $\int_\Omega u \ln u$ is bounded without applying the current result. However, since a corresponding estimate for $\int_\Omega |\nabla v|^2$ in a parabolic equation appears to be lacking, the outcome of the above corollary seems not trivial in the fully parabolic model. Moreover, the condition can be weakened to the boundedness of the L^1-norm of essentially any superlinear functional of u, e.g. $\int_\Omega u \log\log(u+e)$.
Additionally, by virtue of an equivalent definition of equi-integrability, Theorem 2.1.3 can be rephrased in the following way:

Corollary 2.1.5. *Let (u, v) be a classical solution of (2.1) on $\Omega \times (0, T_{\max})$. For all $\varepsilon > 0$ there is $\delta > 0$ such that for any measurable set $E \subset \Omega$ with $|E| < \delta$, if we have*

$$\sup_{t\in(0,T_{\max})} \int_E u^{\frac{N}{2}}(\cdot, t) < \varepsilon,$$

then

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

We note that this property resembles the feature of ε-regularity derived in [81] for a porous medium type parabolic-elliptic Keller-Segel model in the whole space. This analogy is further underlined in the following consequence describing the behavior of unbounded solutions.

Theorem 2.1.6. *Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary. Let (u, v) be a classical solution of (2.1) on $\Omega \times (0, T_{\max})$ with $T_{\max} \in (0, \infty]$. Suppose that*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Then $\{u^{\frac{N}{2}}(\cdot, t)\}_{t\in(0,T_{\max})}$ is not equi-integrable. In other words, there are $\varepsilon_0 > 0$, and $x_0 \in \Omega$ such that for all $\rho > 0$,

$$\sup_{t\in(0,T_{\max})} \int_{B_\rho(x_0)\cap\Omega} u^{\frac{N}{2}}(\cdot, t) > \varepsilon_0.$$

2.2. An interpolation inequality

In the analysis of chemotaxis models, the Gagliardo-Nirenberg inequality is frequently used, especially in the style of the following form

$$\|\varphi\|_{L^q(\Omega)} \leq C_1 \|\nabla\varphi\|^a_{L^r(\Omega)} \|\varphi\|^{1-a}_{L^p(\Omega)} + C_2 \|\varphi\|_{L^p(\Omega)} \text{ for all } \varphi \in W^{1,r}(\Omega), \tag{2.6}$$

where $a = \frac{\frac{N}{p} - \frac{N}{q}}{1 - \frac{N}{r} + \frac{N}{p}} \in (0, 1)$ [28, Theorem 10.1]. Here the constant $C_1 > 0$ depends on p, q, r and Ω. When applying the Gargliardo-Nirenberg inequality, we usually require the exponent a to be strictly less than a given power in order to control a target term. One can imagine that if $C_1 > 0$ could be chosen arbitrarily small, we would be able to deal with more subtle critical cases [5].

The purpose of this section is to investigate a kind of interpolation inequality with the aforementioned ambition that the constant C_1 can be arbitrarily small. However, this is not generally true. Following the idea from [60, Lemma 5.1], we actually show that such an interpolation inequality holds for the class of equi-integrable functions. This is similar to that of [5, Theorem 3] and [60, Lemma 5.1].

Lemma 2.2.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded with smooth boundary. Let $r \geq 1$, $0 < q < \frac{Nr}{(N-r)_+}$. For any $0 < \theta < q$, we define*

$$p := \begin{cases} N(\frac{q}{r} - 1), & \text{if } q > r, \\ \theta, & \text{if } q \leq r, \end{cases} \qquad q_0 := \begin{cases} q, & \text{if } q > r, \\ r(1 + \frac{p}{N}), & \text{if } q \leq r. \end{cases} \tag{2.7}$$

$$a := \frac{\frac{N}{p} - \frac{N}{q}}{1 - \frac{N}{r} + \frac{N}{p}} \in (0,1), \quad b := \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q_0}} \in (0,1].$$

Let $\delta : (0,1) \to (0,\infty)$ be nondecreasing. Then for each $\varepsilon > 0$, we can find $C_\varepsilon > 0$ such that

$$\begin{aligned} \|\varphi\|_{L^q(\Omega)} \leq \varepsilon \|\nabla\varphi\|^a_{L^r(\Omega)} \|\varphi\|^{1-b}_{L^p(\Omega)} + C_\varepsilon \|\varphi\|_{L^p(\Omega)}^{(1-\frac{N}{r}+\frac{N+r}{q_0})b+(1-b)} \\ + C_\varepsilon \|\varphi\|_{L^p(\Omega)} + C_\varepsilon \|\varphi\|^{1-b}_{L^p(\Omega)}. \end{aligned} \tag{2.8}$$

is valid for any

$$\varphi \in \mathcal{F}_\delta := \left\{ \psi \in W^{1,r}(\Omega) \;\middle|\; \text{For all } \varepsilon' \in (0,1), \text{ we have } \int_E |\psi|^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega \text{ with } |E| < \delta(\varepsilon') \right\}. \tag{2.9}$$

Proof. We first consider the case $q > r$, hence $\frac{q}{r} - 1 > 0$. We abbreviate $s := \frac{Nr}{N+r} < \min\{N, r\}$. Then according to the Sobolev embedding $W_0^{1,s}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, there is a constant $c_1 > 0$ such that

$$\|\psi\|^r_{L^r(\mathbb{R}^N)} \leq c_1 \|\nabla\psi\|^r_{L^s(\mathbb{R}^N)} \tag{2.10}$$

for all $\psi \in W_0^{1,s}(\mathbb{R}^N)$. Let Ω' be a bounded open set such that $\Omega \subseteq \Omega'$. In light of Theorem 2.A.1 in section 2.A, we can find $c_2 > 0$ and extend $\varphi \in W^{1,r}(\Omega)$ to $\widetilde{\varphi} \in W_0^{1,r}(\mathbb{R}^N)$ in such a way that

$$\begin{aligned} &\widetilde{\varphi} = \varphi \text{ a.e. in } \Omega, \quad \operatorname{supp} \widetilde{\varphi} \subset \Omega', \\ &\|\widetilde{\varphi}\|_{L^q(\Omega')} \leq c_2 \|\varphi\|_{L^q(\Omega)}, \quad \|\nabla\widetilde{\varphi}\|^r_{L^r(\Omega')} \leq c_2 \|\nabla\varphi\|^r_{L^r(\Omega)}, \end{aligned} \tag{2.11}$$

and that there is a nondecreasing function $\widetilde{\delta} : (0,1) \to (0,\infty)$ such that

$$\widetilde{\varphi} \in \mathcal{F}_{\widetilde{\delta}} := \left\{ \psi \in W^{1,r}(\Omega') \;\middle|\; \text{For all } \varepsilon' \in (0,1), \text{ we have } \int_E |\psi|^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega' \text{ with } |E| < \widetilde{\delta}(\varepsilon') \right\}. \tag{2.12}$$

Given $\varepsilon > 0$, let $\varepsilon' := \left(\frac{\varepsilon^q}{2^r (\frac{q}{r})^r c_1 c_2} \right)^{\frac{N}{r}}$ and let $\delta := \widetilde{\delta}(\varepsilon') > 0$. We have

$$\int_B |\widetilde{\varphi}|^p < \varepsilon' \tag{2.13}$$

for any ball $B \subset \Omega'$ and with radius no bigger than $\eta := \left(\frac{\delta}{w_N}\right)^{\frac{1}{N}}$, where w_N denotes the volume of the unit ball in $\mathbb{R}^N$.
Since Ω is bounded, we can find a family of finite balls $\{B_j\}_{1\le j\le M}$ with radius not lager than η to cover $\overline{\Omega}$ with $\overline{\Omega} \subset \underset{1\le j\le M}{\cup} B_j \subseteq \Omega'$. Moreover, there exist $c_3 > 0$ and a smooth partition of unity for $\underset{1\le j\le M}{\cup} B_j$ given by a family of nonnegative functions $\{\zeta_j\}_{1\le j\le M}$ satisfying

$$\operatorname{supp}\zeta_j \subset B_j, \quad |\nabla\zeta_j^{\frac{1}{r}}| < \frac{c_3}{\eta} \text{ for all } 1 \le j \le M, \text{ and } \sum_{j=1}^{j=M}\zeta_j = 1. \tag{2.14}$$

We can invoke (2.10), and the elementary inequality

$$(a+b)^s \le 2^{s-1}a^s + 2^{s-1}b^s \text{ for all } s > 1 \text{ and } a, b > 0,$$

to obtain that

$$\begin{aligned}\int_{\Omega'}\widetilde{\varphi}^q\zeta_j &= \|\widetilde{\varphi}^{\frac{q}{r}}\zeta_j^{\frac{1}{r}}\|^r_{L^r(B_j)}\\ &\le c_1\|\nabla(\widetilde{\varphi}^{\frac{q}{r}}\zeta_j^{\frac{1}{r}})\|^r_{L^s(B_j)}\\ &\le c_1\|\frac{q}{r}\widetilde{\varphi}^{\frac{q}{r}-1}\zeta_j^{\frac{1}{r}}\nabla\widetilde{\varphi} + \widetilde{\varphi}^{\frac{q}{r}}\nabla\zeta_j^{\frac{1}{r}}\|^r_{L^s(B_j)}\\ &\le c_1 2^{r-1}(\frac{q}{r})^r\left(\int_{B_j}|\widetilde{\varphi}^{\frac{q}{r}-1}\zeta_j^{\frac{1}{r}}\nabla\widetilde{\varphi}|^s\right)^{\frac{r}{s}} + c_1 2^{r-1}(\frac{c_3}{\eta})^r\left(\int_{B_j}\widetilde{\varphi}^{s\frac{q}{r}}\right)^{\frac{r}{s}}.\end{aligned} \tag{2.15}$$

On applying Hölder's inequality and (2.13), the first term on the right-hand side of (2.15) can be estimated as

$$\begin{aligned}c_1 2^{r-1}(\frac{q}{r})^r\left(\int_{B_j}|\widetilde{\varphi}^{\frac{q}{r}-1}\zeta_j^{\frac{1}{r}}\nabla\widetilde{\varphi}|^s\right)^{\frac{r}{s}} &\le c_1 2^{r-1}(\frac{q}{r})^r\left(\int_{B_j}|\widetilde{\varphi}|^{s(\frac{q}{r}-1)\frac{r}{r-s}}\right)^{\frac{r}{s}-1}\left(\int_{\Omega'}\zeta_j|\nabla\widetilde{\varphi}|^r\right)\\ &= c_1 2^{r-1}(\frac{q}{r})^r\left(\int_{B_j}|\widetilde{\varphi}|^{N(\frac{q}{r}-1)}\right)^{\frac{r}{s}-1}\int_{\Omega'}\zeta_j|\nabla\widetilde{\varphi}|^r\\ &= c_1 2^{r-1}(\frac{q}{r})^r\left(\int_{B_j}|\widetilde{\varphi}|^p\right)^{\frac{r}{N}}\int_{\Omega'}\zeta_j|\nabla\widetilde{\varphi}|^r\\ &\le c_1 2^{r-1}(\frac{q}{r})^r(\varepsilon')^{\frac{r}{N}}\int_{\Omega'}\zeta_j|\nabla\widetilde{\varphi}|^r \le \frac{\varepsilon^q}{2c_2}\int_{\Omega'}\zeta_j|\nabla\widetilde{\varphi}|^r.\end{aligned} \tag{2.16}$$

Now we claim that for all $r < q < \frac{Nr}{(N-r)_+}$, there are positive constants c_ε, c_4 and c_5 such that

$$\begin{aligned}&c_1 2^{r-1}(\frac{c_3}{\eta})^r\left(\int_{B_j}|\widetilde{\varphi}|^{s\frac{q}{r}}\right)^{\frac{r}{s}}\\ &\le \frac{\varepsilon^q}{2c_2 M}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r + c_\varepsilon\|\widetilde{\varphi}\|^{q-\frac{Nq}{r}+N+r}_{L^{N(\frac{q}{r}-1)}(\Omega')} + c_4\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')} + c_5\varepsilon^{\frac{q^2}{q-r}}.\end{aligned} \tag{2.17}$$

If $r < q < \frac{r(N+r)}{N}$, let $d = \frac{\frac{1}{\frac{q}{r}-1} - \frac{Nr}{sq}}{1-\frac{N}{r}+\frac{1}{\frac{q}{r}-1}}$, hence $d \in (0,1)$. Moreover, since $s < r$, we know that $dq < r$. The Gagliardo-Nirenberg inequality thus implies the existence of $c_4 > 0$ and $c_\varepsilon > 0$ such that

$$\begin{aligned}
2^{r-1}c_1(\frac{c_3}{\eta})^r\left(\int_{B_j}|\widetilde{\varphi}|^{\frac{sq}{r}}\right)^{\frac{r}{s}} &\le 2^{r-1}c_1(\frac{c_3}{\eta})^r\|\widetilde{\varphi}\|^q_{L^{\frac{sq}{r}}(\Omega')}\\
&\le c_4\|\nabla\widetilde{\varphi}\|^{dq}_{L^r(\Omega')}\|\widetilde{\varphi}\|^{(1-d)q}_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_4\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')}\\
&= c_4\left(\int_{\Omega'}|\nabla\widetilde{\varphi}|^r\right)^{\frac{dq}{r}}\left(\int_{\Omega'}\widetilde{\varphi}^{N(\frac{q}{r}-1)}\right)^{\frac{(1-d)q}{N(\frac{q}{r}-1)}}+c_4\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')}\\
&\le \frac{\varepsilon^q}{2c_2M}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r+c_\varepsilon\|\widetilde{\varphi}\|^{q-\frac{Nq}{r}+N+r}_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_4\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')}. \qquad (2.18)
\end{aligned}$$

If $\frac{r(N+r)}{N}\le q<\frac{Nr}{(N-r)_+}$, hence $\frac{sq}{r}\le N(\frac{q}{r}-1)$. We can simply use Hölder's inequality to obtain a constant $c_5>0$ fulfilling

$$2^{r-1}c_1(\frac{c_3}{\eta})^r\left(\int_{B_j}|\widetilde{\varphi}|^{s\frac{q}{r}}\right)^{\frac{r}{s}}\le 2^{r-1}c_1(\frac{c_3}{\eta})^r|\Omega'|^{1-\frac{sq}{pr}}\left(\int_{B_j}|\widetilde{\varphi}|^p\right)^{\frac{q}{p}}\le c_5\varepsilon^{\frac{q^2}{q-r}}.$$

Hence (2.17) holds for all $r<q<\frac{Nr}{(N-r)_+}$. Since (2.15), (2.16) and (2.17) and $\widetilde{\varphi}\in W^{1,r}(\mathbb{R}^N)$ in particular entail that for each $1\le j\le M$, $\widetilde{\varphi}^{\frac{q}{r}}\zeta_j^{\frac{1}{r}}\in W_0^{1,s}(B_j)$, we can invoke (2.10) and combining it with (2.15-2.17), we see that for each $1\le j\le M$,

$$\begin{aligned}
\int_{\Omega'}|\widetilde{\varphi}|^q\zeta_j = \|\widetilde{\varphi}^{\frac{q}{r}}\zeta_j^{\frac{1}{r}}\|^r_{L^r(B_j)} &\le c_1\|\nabla(\widetilde{\varphi}^{\frac{q}{r}}\zeta_j^{\frac{1}{r}})\|^r_{L^s(B_j)}\\
&\le \frac{\varepsilon^q}{2c_2}\int_{\Omega'}\zeta_j|\nabla\widetilde{\varphi}|^r+\frac{\varepsilon^q}{2c_2M}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r\\
&\qquad +c_\varepsilon\|\widetilde{\varphi}\|^{q-\frac{Nq}{r}+N+r}_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_4\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_5\varepsilon^{\frac{q^2}{q-r}}. \qquad (2.19)
\end{aligned}$$

Finally, we obtain from (2.19) and (2.14) that

$$\begin{aligned}
&\|\varphi\|^q_{L^q(\Omega)}\\
&\le \|\widetilde{\varphi}\|^q_{L^q(\Omega')}=\int_{\Omega'}|\widetilde{\varphi}|^q\left(\sum_{j=1}^{j=M}\zeta_j\right)=\sum_{j=1}^{j=M}\int_{\Omega'}|\widetilde{\varphi}|^q\zeta_j\\
&\le \sum_{j=1}^{j=M}\left(\frac{\varepsilon^q}{2c_2}\int_{\Omega'}\zeta_j|\nabla\widetilde{\varphi}|^r+\frac{\varepsilon^q}{2c_2M}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r+c_\varepsilon\|\widetilde{\varphi}\|^{q-\frac{Nq}{r}+N+r}_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_4\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_5\varepsilon^{\frac{q^2}{q-r}}\right)\\
&\le \frac{\varepsilon^q}{2c_2}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r\left(\sum_{j=1}^{j=M}\zeta_j\right)\\
&\qquad +M\left(\frac{\varepsilon^q}{2c_2M}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r+c_\varepsilon\|\widetilde{\varphi}\|^{q-\frac{Nq}{r}+N+r}_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_4\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_5\varepsilon^{\frac{q^2}{q-r}}\right)\\
&\le \frac{\varepsilon^q}{2c_2}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r+\frac{\varepsilon^q}{2c_2}\int_{\Omega'}|\nabla\widetilde{\varphi}|^r+c_\varepsilon M\|\widetilde{\varphi}\|^{q-\frac{Nq}{r}+N+r}_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_4M\|\widetilde{\varphi}\|^q_{L^{N(\frac{q}{r}-1)}(\Omega')}+c_5M\varepsilon^{\frac{q^2}{q-r}}\\
&\le \varepsilon^q\int_\Omega|\nabla\varphi|^r+c_6\|\varphi\|^{q-\frac{Nq}{r}+N+r}_{L^{N(\frac{q}{r}-1)}(\Omega)}+c_6\|\varphi\|^q_{L^{N(\frac{q}{r}-1)}(\Omega)}+c_6
\end{aligned}$$

with some constant $c_6 > 0$. Note that $b = 1$ if $q > r$, taking the q-th root on both sides leads to (2.8) for the case $q > r$.
If $q \le r$, we see that $q_0 > r \ge q > \theta$. The Hölder inequality with $b = \frac{\frac{1}{\theta} - \frac{1}{q}}{\frac{1}{\theta} - \frac{1}{q_0}}$ shows that

$$\|\varphi\|_{L^q(\Omega)} \le \|\varphi\|^b_{L^{q_0}(\Omega)} \|\varphi\|^{1-b}_{L^\theta(\Omega)}. \tag{2.20}$$

Since $q_0 > r$, we have already proven that for all $\varepsilon > 0$, there is $c_6 > 0$ so that

$$\begin{aligned}\|\varphi\|_{L^{q_0}(\Omega)} &\le \left(\varepsilon^{\frac{q_0}{b}} \|\nabla\varphi\|^r_{L^r(\Omega)} + c_6\|\varphi\|^{q_0 - \frac{Nq_0}{r} + N + r}_{L^\theta(\Omega)} + c_6\|\varphi\|^{q_0}_{L^\theta(\Omega)} + c_6\right)^{\frac{1}{q_0}} \\ &\le \varepsilon^{\frac{1}{b}} \|\nabla\varphi\|^{\frac{r}{q_0}}_{L^r(\Omega)} + c_6\|\varphi\|^{1 - \frac{N}{r} + \frac{N+r}{q_0}}_{L^\theta(\Omega)} + c_6\|\varphi\|_{L^\theta(\Omega)} + c_6,\end{aligned}$$

which combined with the previous interpolation inequality (2.20) yields that

$$\begin{aligned}\|\varphi\|_{L^q(\Omega)} &\le \left(\varepsilon^{\frac{1}{b}} \|\nabla\varphi\|^{\frac{r}{q_0}}_{L^r(\Omega)} + c_6\|\varphi\|^{1 - \frac{N}{r} + \frac{N+r}{q_0}}_{L^\theta(\Omega)} + c_6\|\varphi\|_{L^\theta(\Omega)} + c_6\right)^b \|\varphi\|^{1-b}_{L^\theta(\Omega)}. \\ &\le \varepsilon\|\nabla\varphi\|^{b\cdot\frac{r}{q_0}}_{L^r(\Omega)} \|\varphi\|^{1-b}_{L^\theta(\Omega)} + c_6\|\varphi\|^{(1-\frac{N}{r}+\frac{N+r}{q_0})b+1-b}_{L^\theta(\Omega)} + c_6\|\varphi\|_{L^\theta(\Omega)} + c_6\|\varphi\|^{1-b}_{L^\theta(\Omega)}\end{aligned}$$

We easily check that $b \cdot \frac{r}{q_0} = \frac{\frac{N}{\theta} - \frac{N}{q}}{1 - \frac{N}{r} + \frac{N}{\theta}}$, thus (2.8) is valid for $q \le r$ as well. □

Remark 2.2.2. *The exponent a in* (2.8) *is exactly the one from the Gagliardo-Nirenberg inequality*

$$\|\varphi\|_{L^q(\Omega)} \le C\|\nabla\varphi\|^a_{L^r(\Omega)} \|\varphi\|^{1-a}_{L^p(\Omega)} + C\|\varphi\|_{L^p(\Omega)} \text{ for all } \varphi \in W^{1,r}(\Omega).$$

However $1 - b \ne 1 - a$. In fact, following the proof we can find $a + 1 - b < 1$.

Remark 2.2.3. *Given a family of functions $\{f_j\}_{j\in\mathbb{N}}$ such that $\{f_j^p\}_{j\in\mathbb{N}}$ is equi-integrable, there exists $\delta : (0,1) \to (0,\infty)$ nondecreasing such that $f_j \in \mathcal{F}_\delta$, where $\mathcal{F}_\delta$ is defined in* (2.9)*. Therefore, we can apply Lemma 2.2.1 to a family of functions enjoying equi-integrability.*

2.3. Preliminary

Before proving our main result, some basic knowledge on the Keller-Segel system is prepared. The following properties can be easily checked by integrating.

Lemma 2.3.1. *We have*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \text{ and} \tag{2.21}$$

$$\int_\Omega v(\cdot,t) \le \max\left\{\int_\Omega v_0, \int_\Omega u_0\right\} \text{ for all } t \in (0, T_{\max}). \tag{2.22}$$

Before going into details, let us first prepare the following embedding lemma.

Lemma 2.3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and let $\alpha \in (1, N)$. For all $s \in (0, \infty]$, there is $C > 0$ such that*

$$\|\nabla\varphi\|_{L^{\frac{N\alpha}{N-\alpha}}(\Omega)} \le C\|\Delta\varphi\|_{L^\alpha(\Omega)} + C\|\varphi\|_{L^s(\Omega)} \tag{2.23}$$

for all $\varphi \in W^{2,\alpha}(\Omega)$ with $\nabla\varphi \cdot \nu = 0$ on $\partial\Omega$.

Proof. Using the fact that with some $c_1 > 0$, the estimates

$$\|\varphi\|_{W^{2,\alpha}(\Omega)} \le c_1(\|\varphi\|_{L^\alpha(\Omega)} + \|\Delta\varphi\|_{L^\alpha(\Omega)})$$

holds for all $\varphi \in W^{2,\alpha}(\Omega)$ with $\nabla\varphi \cdot \nu|_{\partial\Omega} = 0$ [28, Theorem 19.1], we obtain a constant $c_2 > 0$ from the embedding $W^{2,\alpha}(\Omega) \hookrightarrow W^{1,\frac{N\alpha}{N-\alpha}}(\Omega)$ that

$$\|\nabla\varphi\|_{L^{\frac{N\alpha}{N-\alpha}}(\Omega)} \le c_2(\|\Delta\varphi\|_{L^\alpha(\Omega)} + \|\varphi\|_{L^\alpha(\Omega)}). \tag{2.24}$$

If $s < \alpha$, let $b = \frac{\frac{N}{s} - \frac{N}{\alpha}}{2 + \frac{N}{s} - \frac{N}{\alpha}} \in (0,1)$. The Gagliardo-Nirenberg inequality together with Poincaré's inequality and Young's inequality implies

$$\begin{aligned}\|\varphi\|_{L^\alpha(\Omega)} &\le c_3\|\nabla\varphi\|^b_{L^{\frac{N\alpha}{N-\alpha}}(\Omega)}\|\varphi\|^{1-b}_{L^s(\Omega)} + c_3\|\varphi\|_{L^s(\Omega)} \\ &\le \frac{1}{2c_2}\|\nabla\varphi\|_{L^{\frac{N\alpha}{N-\alpha}}(\Omega)} + c_4\|\varphi\|_{L^s(\Omega)}\end{aligned} \tag{2.25}$$

with some constant $c_3, c_4 > 0$ for all $\varphi \in W^{2,\alpha}(\Omega)$ with $\nabla\varphi \cdot \nu|_{\partial\Omega} = 0$. If $s \ge \alpha$, we use Hölder's inequality

$$\|\varphi\|_{L^\alpha(\Omega)} \le |\Omega|^{1-\frac{\alpha}{s}}\|\varphi\|_{L^s(\Omega)} \tag{2.26}$$

instead of (2.25). Collecting (2.24-2.26) together yields (2.23). □

2.4. Proof of Theorem 2.1.3

Now we are in a position to proceed the proof of our main ingredient. Having in hand Proposition 2.1.2, we see that it is sufficient to show that (2.2) holds for some $p > \frac{N}{2}$.

Lemma 2.4.1. *Assume that $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) is a bounded domain with smooth boundary. Let (u, v) be a classical solution of (2.1) on $\Omega \times (0, T_{\max})$ with $T_{\max} \in (0, \infty]$ and let $p \in (\frac{N}{2}, N)$. If*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^{\frac{N}{2}}(\Omega)} < \infty \tag{2.27}$$

$$\text{and } \{u^{\frac{N}{2}}(\cdot,t)\}_{t\in(0,T_{\max})} \text{ is equi-integrable,} \tag{2.28}$$

then

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty. \tag{2.29}$$

Proof. Let $p \in (\frac{N}{2}, N)$. Let $\theta \in (1,\infty)$ satisfy $\frac{1}{\theta} = 1 + \frac{2}{N} - \frac{2}{p} \in (0,1)$, and θ' be such that $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. We test the first equation in (2.1) with pu^{p-1} to obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 &= p(p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v \\ &\le \frac{p(p-1)}{4}\int_\Omega u^{p-2}|\nabla u|^2 + p(p-1)\int_\Omega u^p|\nabla v|^2\end{aligned}$$

for all $t \in (0, T_{\max})$. Applying Hölder's inequality, we get

$$\frac{d}{dt}\int_\Omega u^p + \frac{3(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le p(p-1)\int_\Omega u^p|\nabla v|^2$$

$$\leq p(p-1)\left(\int_\Omega u^{p\theta}\right)^{\frac{1}{\theta}}\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} \tag{2.30}$$

for all $t \in (0, T_{\max})$. Let $a := \frac{p-\frac{N}{2\theta}}{1-\frac{N}{2}+p} \in (0,1)$, and abbreviate $\frac{1}{1-a} =: \lambda > 1$. The Gagliardo-Nirenberg inequality implies the existence of $c_1 > 0$ such that

$$p(p-1)\left(\int_\Omega u^{p\theta}\right)^{\frac{1}{\theta}} = p(p-1)\|u^{\frac{p}{2}}\|^2_{L^{2\theta}(\Omega)} \leq c_1\|\nabla u^{\frac{p}{2}}\|^{2a}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2(1-a)}_{L^{\frac{N}{p}}(\Omega)} + c_1\|u^{\frac{p}{2}}\|^2_{L^{\frac{N}{p}}(\Omega)}$$

$t \in (0, T_{\max})$. Using Young's inequality and the assumption (2.27), we find some constant $c_2 > 0$ such that the right-hand side of (2.30) is estimated as

$$\begin{aligned}p(p-1)\left(\int_\Omega u^{p\theta}\right)^{\frac{1}{\theta}}\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} &\leq \left(c_1\|\nabla u^{\frac{p}{2}}\|^{2a}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2(1-a)}_{L^{\frac{N}{p}}(\Omega)} + c_1\|u^{\frac{p}{2}}\|^2_{L^{\frac{N}{p}}(\Omega)}\right)\|\nabla v\|^2_{L^{2\theta'}(\Omega)} \\ &\leq \frac{p-1}{p}\|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} + c_2\|\nabla v\|^{2\lambda}_{L^{2\theta'}(\Omega)} + c_2 \end{aligned} \tag{2.31}$$

$t \in (0, T_{\max})$. Due to the choices of θ and θ', we know that $p \in (1, N)$ and $2\theta' = \frac{Np}{N-p}$, hence an application of Lemma 2.3.2 yields $c_3 > 0$ such that

$$c_2\|\nabla v\|^{2\lambda}_{L^{2\theta'}(\Omega)} \leq c_3\|\Delta v\|^{2\lambda}_{L^p(\Omega)} + c_3\|v\|^{2\lambda}_{L^1(\Omega)} \text{ for all } t \in (0, T_{\max}). \tag{2.32}$$

We also recall from the Gagliardo-Nirenberg inequality that there is $c_4 > 0$ fulfilling

$$\frac{p-1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \geq \lambda\int_\Omega u^p - c_4 \text{ for all } t \in (0, T_{\max}). \tag{2.33}$$

Thus we conclude from the previous estimates (2.30-2.33) and Lemma 2.3.1 that

$$\frac{d}{dt}\int_\Omega u^p + \lambda\int_\Omega u^p + \frac{(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \leq c_3\|\Delta v\|^{2\lambda}_{L^p(\Omega)} + c_2 + c_4 + c_3\|v\|^{2\lambda}_{L^1(\Omega)} \tag{2.34}$$

for all $t \in (0, T_{\max})$. Letting $t_0 \in (0, T_{\max})$ and applying the variation-of-constants formula to the above inequality, we find a constant $c_5 > 0$ such that

$$\begin{aligned}\int_\Omega u^p(\cdot,t) \leq e^{-\lambda(t-t_0)}\int_\Omega u^p(\cdot,t_0) - \frac{(p-1)}{p}\int_{t_0}^t e^{-\lambda(t-s)}\int_\Omega |\nabla u^{\frac{p}{2}}(\cdot,s)|^2 ds \\ + c_3\int_{t_0}^t e^{-\lambda(t-s)}\|\Delta v(\cdot,s)\|^{2\lambda}_{L^p(\Omega)}ds + c_5\end{aligned} \tag{2.35}$$

for all $t \in (t_0, T_{\max})$. The maximal regularity result from Lemma 1.2.3 provides a constant $c_6 > 0$ satisfying

$$c_3\int_{t_0}^t e^{-\lambda(t-s)}\|\Delta v\|^{2\lambda}_{L^p(\Omega)}ds \leq c_6\int_{t_0}^t e^{-\lambda(t-s)}\|u\|^{2\lambda}_{L^p(\Omega)}ds + c_6 \tag{2.36}$$

for all $t \in (t_0, T_{\max})$.
Let $d = \frac{p-\frac{N}{2}}{1-\frac{N}{2}+p}$ and $b = \frac{\frac{p}{N}-\frac{1}{2}}{\frac{p}{N}-\frac{p}{2p+2}}$. We can easily check that $\frac{4\lambda}{p}d = 2$. Since $\{u^{\frac{N}{2}}(\cdot,t)\}_{t\in(0,T_{\max})}$ is uniformly integrable, there exists nondecreasing $\delta : (0,1) \to (0,\infty)$ such that $\{u^{\frac{N}{2}}(\cdot,t)\}_{t\in(0,T_{\max})}$

belongs to $\mathcal{F}_\delta$ defined in (2.9) (with $p = \frac{N}{p}$). Since (2.27),

$$\varepsilon := \frac{\frac{p-1}{p}}{\sup_{t\in(0,T_{\max})} \|u^{\frac{p}{2}}\|_{L^{\frac{N}{p}}(\Omega)}^{\frac{4\lambda}{p}(1-b)}} > 0.$$

Applying Lemma 2.2.1 (in the case $q = r = 2$, and with $\theta = \frac{N}{p} < q$ by virtue of $p > \frac{N}{2}$), we can find $c_\varepsilon > 0$ such that

$$\begin{aligned} c_6\|u\|_{L^p(\Omega)}^{2\lambda} &= c_6\|u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{4\lambda}{p}} \\ &\le \varepsilon c_6\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{4\lambda}{p}d}\|u^{\frac{p}{2}}\|_{L^{\frac{N}{p}}(\Omega)}^{\frac{4\lambda}{p}(1-b)} + c_\varepsilon \le \frac{(p-1)}{p}\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + c_\varepsilon \end{aligned} \tag{2.37}$$

for all $t \in (0, T_{\max})$, which leads to

$$c_3\int_{t_0}^t e^{-\lambda(t-s)}\|\Delta v(\cdot,s)\|_{L^p(\Omega)}^{2\lambda}ds \le \frac{(p-1)}{p}\int_{t_0}^t e^{-\lambda(t-s)}\int_\Omega |\nabla u^{\frac{p}{2}}(\cdot,s)|^2 ds + c_\varepsilon + c_6 \tag{2.38}$$

for all $t \in (t_0, T_{\max})$. Adding this to (2.35) shows that

$$\int_\Omega u^p(\cdot,t) \le e^{-\lambda(t-t_0)}\int_\Omega u^p(\cdot,t_0) + c_5 + c_6 + c_\varepsilon \le \int_\Omega u^p(\cdot,t_0) + c_5 + c_6 + c_\varepsilon$$

for all $t \in (t_0, T_{\max})$. Since $\sup_{t\in(0,t_0]} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty$ due to the local existence theory from Lemma 2.1.1, this shows (2.29). ☐

Proof of Theorem 2.1.3. Employing Lemma 2.4.1 and Proposition 2.1.2 proves

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty,$$

which combined with Lemma 2.1.1 implies that $T_{\max} = \infty$. Thus the solution is global and bounded. ☐

2.5. Blow-up behavior

From another aspect, the extension criterion in Theorem 2.1.3 also gives a characterization of blow-up solutions.

Proof of Theorem 2.1.6. Suppose on contrary that $\{u^{\frac{N}{2}}(\cdot,t)\}_{t\in(0,T_{\max})}$ is equi-integrable with $T_{\max} \in (0,\infty]$. We can apply Theorem 2.1.3 to show that there is a constant $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C,$$

for all $t \in (0, T_{\max})$, which is a contradiction. ☐

2.A. Appendix

We claim a basic property of extension functions which we have used in the proof of Lemma 2.2.1. Namely, the extension function $\widetilde{\varphi} \in W^{1,r}(\Omega')$ is equi-integrable with respect to some power in Ω' provided φ has the same property in Ω. Since we can not find this precise result in any reference, we also give a brief proof here.

Theorem 2.A.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and that $r > 1$, $1 \leq q < \frac{Nr}{(N-r)_+}$. Let Ω' be a bounded smooth domain with $\Omega \subset \Omega'$. Then there is $C > 0$ and for any nondecreasing function $\delta : (0,1) \to (0,\infty)$, we can find $\tilde{\delta} : (0,1) \to (0,\infty)$ nondecreasing such that we can extend any function $\varphi \in W^{1,r}(\Omega)$ to a function $\widetilde{\varphi} \in W_0^{1,r}(\mathbb{R}^N)$ in such a way that*

$$\widetilde{\varphi} = \varphi \text{ a.e. in } \Omega, \quad \operatorname{supp} \widetilde{\varphi} \subset \Omega', \tag{2.39}$$

$$\|\nabla \widetilde{\varphi}\|^r_{W^{1,r}(\Omega')} \leq C \|\nabla \varphi\|^r_{W^{1,r}(\Omega)}, \tag{2.40}$$

$$\|\widetilde{\varphi}\|_{L^q(\Omega')} \leq C \|\varphi\|_{L^q(\Omega)}. \tag{2.41}$$

Moreover, if $\varphi \in \mathcal{F}_\delta$ with

$$\mathcal{F}_\delta := \left\{ \psi \in W^{1,r}(\Omega) \;\middle|\; \text{For all } \varepsilon' \in (0,1), \text{ we have } \int_E |\psi|^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega \text{ with } |E| < \delta(\varepsilon') \right\}, \tag{2.42}$$

then $\widetilde{\varphi} \in \mathcal{F}_{\widetilde{\delta}}$ with

$$\mathcal{F}_{\widetilde{\delta}} := \left\{ \psi \in W^{1,r}(\Omega') \;\middle|\; \text{For all } \varepsilon' \in (0,1), \text{ we have } \int_E |\psi|^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega' \text{ with } |E| < \widetilde{\delta}(\varepsilon') \right\}. \tag{2.43}$$

Proof. First, (2.39) and (2.40) are precisely proven in [26, Theorem 5.4.1]. Now we recall the construction of the extension function in the proof to show the remaining properties. Since $\partial\Omega$ is compact, we can find finitely many points $\{x_i\}_{1 \leq i \leq K} \subset \partial\Omega$ and open sets $\{W_i\}_{1 \leq i \leq K} \subset \Omega'$ with $x_i \in W_i$ and $W_0 \subset \Omega$ such that $\partial\Omega \subset \underset{1 \leq i \leq K}{\cup} W_i$ and $\Omega \subset W_0 \cup (\underset{1 \leq i \leq K}{\cup} W_i) \subset \Omega'$. There exist C^1 diffeomorphisms $\Phi_i : W_i \to \mathbb{R}^N$ $(1 \leq i \leq K)$ which flatten out $\partial\Omega$ near x_i; namely, if we let $B_i := \Phi_i(W_i)$ be a ball, it satisfies $B_i^- = \Phi_i(W_i \cap \Omega^c) = \{y = (y_1, ..., y_N) |\ y_N < 0\}$, $B_i^+ = \Phi_i(W_i \cap \Omega) = \{y = (y_1, ..., y_N) |\ y_N > 0\}$. Now we define linear transformations

$$Y_1 : (y_1, ..., y_N) \in B_i^- \to (y_1, ..., y_{N-1}, -y_N) \in B_i^+,$$
$$Y_2 : (y_1, ..., y_N) \in B_i^- \to (y_1, ..., y_{N-1}, -\frac{1}{2} y_N) \in B_i^+.$$

Let $\varphi_i'(y) = \varphi(\Phi_i^{-1}(y))$ $(y \in B_i+, x = \Phi_i^{-1}(y) \in W_i \cap \Omega)$. A first order reflection of $\varphi_i'(y)$ is given by

$$\widetilde{\varphi}_i'(y) := \begin{cases} -3\varphi_i'(Y_1(y)) + 4\varphi_i'(Y_2(y)), & y \in B_i^-, \\ \varphi_i'(y), & y \in B_i^+ \text{ and } y_n = 0. \end{cases} \tag{2.44}$$

If we let $\{\zeta_i\}_{0\le i\le K}$ be a partition of unity subordinate to $\{W_i\}_{0\le i\le K}$, the associated extension $\widetilde{\varphi}:\Omega'\to\mathbb{R}^N$ of φ is defined by converting $\widetilde{\varphi}_i'$ back to W_i

$$\widetilde{\varphi}(x) := \begin{cases} \varphi(x), & x\in\Omega=\bigcup_{0\le i\le K} W_i^+,\\ \sum_{i=0}^{i=K}\zeta_i(x)\left\{-3\varphi(\Phi_i^{-1}(Y_1(\Phi_i(x)))) + 4\varphi(\Phi_i^{-1}(Y_2(\Phi_i(x))))\right\}, & x\in\bigcup_{1\le i\le K} W_i^-,\\ 0, & x\in\Omega'\setminus\bigcup_{0\le i\le K} W_i, \end{cases} \tag{2.45}$$

where $W_i^+ := \Phi_i^{-1}(B_i^+)$, $W_i^- := \Phi_i^{-1}(B_i^-)$. Since the mappings Φ_i, Φ_i^{-1} $(1\le i\le K)$, Y_j $(j=\{1,2\})$ are C^1, we can find a constant $c_1>0$ such that $|\Phi_i^{-1}(Y_i(\Phi_i(U)))|\le c_1|U|$ for all $U\subset W_i^-$ $(1\le i\le K)$. For any measurable subset $E'\subset\Omega'$, let $E_i := E'\cap W_i^-$, We note that $\Phi_i^{-1}(Y_2(\Phi_i(E_i)))\subset\Phi_i^{-1}(Y_1(\Phi_i(E_i)))\subset\Phi_i^{-1}(B_i^+)\subset\Omega$. By changing variables, for each $1\le i\le K$, we have

$$\begin{aligned}
\int_{E_i}|\widetilde{\varphi}(x)|^p dx &= \int_{E_i}|-3\varphi(\Phi_i^{-1}(Y_1(\Phi_i(x)))) + 4\varphi(\Phi_i^{-1}(Y_2(\Phi_i(x))))|^p dx\\
&= \int_{\Phi_i(E_i)}|-3\varphi(\Phi_i^{-1}(Y_1(y))) + 4\varphi(\Phi_i^{-1}(Y_2(y)))|^p|\det(D\Phi_i^{-1}(y))|dy\\
&= \int_{\Phi_i(E_i)}|-3\varphi_i'(y_1,...,y_{N-1},-y_n) + 4\varphi_i'(y_1,...y_{n-1},-\frac{1}{2}y_n)|^p|\det(D\Phi_i^{-1}(y))|dy\\
&\le 2^{p-1}\int_{Y_1(\Phi_i(E_i))}3^p|\varphi_i'(y)|^p|\det(D\Phi_i^{-1}(y))|dy\\
&\qquad + 2^{p-1}\int_{Y_2(\Phi_i(E_i))}4^p\frac{1}{2}|\varphi_i'(y)|^p|\det(D\Phi_i^{-1}(y))|dy\\
&\le 6^p\int_{\Phi_i^{-1}(Y_1(\Phi_i(E_i)))}|\varphi(x)|^p dx + 8^p\int_{\Phi_i^{-1}(Y_2(\Phi_i(E_i)))}|\varphi(x)|^p dx
\end{aligned}$$

According to (2.42), given $\varepsilon'>0$, we have that $\delta(\varepsilon')>0$ such that $\int_E\varphi^p < \frac{\varepsilon'}{8^p(3K)}$ for all $E\subset\Omega$ with $|E|\le\delta(\varepsilon')$. We let $\widetilde{\delta} := \frac{1}{c_1}\delta$ such that if $|E'|<\min\{\widetilde{\delta},\delta\}$, then $|\Phi_i^{-1}(Y_1(\Phi_i(E_i)))|, |\Phi_i^{-1}(Y_2(\Phi_i(E_i)))|<\delta$ for all $1\le i\le K$, hence

$$\begin{aligned}
&\int_{E'}|\widetilde{\varphi}(x)|^p dx\\
&= \int_{E'\cap\Omega}|\varphi(x)|^p dx + \int_{E'\cap\Omega^c}|\widetilde{\varphi}(x)|^p dx\\
&\le \int_{E'\cap\Omega}|\varphi(x)|^p dx + \sum_{i=1}^{i=K}\int_{E_i}|\varphi(x)|^p dx\\
&\le \int_{E'\cap\Omega}|\varphi(x)|^p dx + \sum_{i=1}^{i=K}\left(6^p\int_{\Phi_i^{-1}(Y_1(\Phi_i(E_i)))}|\varphi(x)|^p dx + 8^p\int_{\Phi_i^{-1}(Y_2(\Phi_i(E_i)))}|\varphi(x)|^p dx\right)\\
&\le \frac{\varepsilon'}{8^p3K} + K(\frac{6^p\varepsilon'}{8^p3K} + \frac{8^p\varepsilon'}{8^p3K}) < \varepsilon'.
\end{aligned}$$

Therefore, $\widetilde{\varphi}\in\mathcal{F}_{\widetilde{\delta}}$ is shown. Using $\int_{\Omega'}|\widetilde{\varphi}|^q = \sum_{0\le i\le K}\int_{E_i}|\widetilde{\varphi}|^q$, (2.41) can be proven in a similar way. □

3. Global solutions in a Keller-Segel model with logistic source and their asymptotic behavior

3.1. Introduction

In this chapter, we consider the following parabolic system

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + \kappa u - \mu u^2, & (x,t) \in \Omega \times (0,T), \\ \tau v_t = \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $\kappa > 0$, $\mu > 0$, $\chi > 0$ and $\tau > 0$, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary and ν denotes the outward normal vector on $\partial\Omega$. The initial distribution (u_0, v_0) is a pair of nonnegative functions satisfying

$$u_0 \in C^0(\overline{\Omega}) \text{ with } u_0 \not\equiv 0, \quad v_0 \in W^{1,q}(\Omega) \text{ with } q > N. \tag{3.2}$$

In contrast to the classical Keller-Segel system, a logistic source has been included in (3.1) if $\kappa \geq 0$, $\mu > 0$. One may expect that the interplay between diffusion, cross-diffusion and logistic growth restriction can result in colorful dynamics [111, 52]. As far as we know, only few results concerning finite time blow-up has been found except for that in [107], where $N \geq 5$ is required. It is also shown that the logistic source can prevent blow-up whenever $N \leq 2$, or μ is sufficiently large [96, 68, 106].
Going beyond the boundedness results, the study of global dynamics is a natural continuation [110], we refer to [97, 80, 6] for Keller-Segel models including multiple species. We note that (3.1) can be seen as a subsystem in a multiple species model. In the case $\tau = 0$, the results from [97, 80, 6] can be summarized as follows: If the quotient $\frac{\chi}{\mu}$ is suitably small, (3.1) admits a global classical solution and it converges to $(\frac{\kappa}{\mu}, \frac{\kappa}{\mu})$.
Considering the fully parabolic system, that is $\tau > 0$ in (3.1), [110] proves the same conclusion under the restrictions that $\tau = 1$ and Ω is convex, which are quite critical in the proof. Under these assumptions, the combination $y(x,t) = u + \frac{\chi}{2}|\nabla v|^2$ satisfies a scalar parabolic inequality

$$y_t \leq \Delta y - y + \frac{C}{\mu} \tag{3.3}$$

with some $C > 0$ for all $t > 0$ [106]. The comparison principle immediately yields that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \limsup_{t\to\infty} y(x,t) \leq \frac{C}{\mu}. \tag{3.4}$$

With this information, one can finally show convergence on the basis of estimates for the Neuman heat semigroup. However, if $\tau \neq 1$, the first step already fails; we can not find any combination like $y(x,t)$ satisfying a single parabolic inequality on its own. In a recent paper [3], the authors develop a functional approach to prove convergence for global bounded solutions if $\frac{\chi^2}{\mu}$ is small. This approach also works for $\tau \neq 1$.
It is our purpose in this chapter to investigate how the size of the quotient $\frac{\chi}{\mu}$ affects the global dynamics for any choice of $\tau > 0$ and for a general domain Ω. We find a replacement of (3.4):

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)} \leq \frac{C}{\mu} \tag{3.5}$$

with sufficiently large p and for some $C > 0$, which is sufficient for the conclusion in [110]. Our main result reads as follows:

Theorem 3.1.1. *Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Then there exists $\theta_0 > 0$ with the property that if $\chi > 0$, $\mu > 0$, $\kappa > 0$ and satisfy*

$$\frac{\chi}{\mu} < \theta_0, \tag{3.6}$$

then for all initial data (u_0, v_0) fulfilling (3.2)*, the system* (3.1) *possesses a global classical solution*

$$u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),$$
$$v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,q}(\Omega)).$$

Moreover, (u,v) satisfies

$$\|u(\cdot,t) - \frac{\kappa}{\mu}\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot,t) - \frac{\kappa}{\mu}\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty. \tag{3.7}$$

3.2. Preliminaries

Before going into details, we introduce the local existence result for (3.1). Compared with (2.1), an additional logistic term appears in (3.1), however, the following lemma is in the spirit of Lemma 2.1.1.

Lemma 3.2.1. *Suppose $\Omega \subset \mathbb{R}^N$ with $N \geq 1$, is a bounded domain with smooth boundary, $\mu > 0$ and $\chi > 0$, and $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ (with some $q > N$) both are nonnegative. Then there exist $T_{\max} \in (0,\infty]$ with the property that the problem* (3.1) *possesses a unique classical solution (u,v) satisfying*

$$u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})),$$
$$v \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \cap L^\infty_{loc}([0,T_{\max}); W^{1,q}(\Omega)).$$

Moreover, if $T_{\max} < \infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty, \text{ as } t \to T_{\max}. \tag{3.8}$$

3.3. Boundedness and large time behavior of L^p-norm

As already mentioned in the introduction, our first and the most important goal is to identify the large time behavior of $\|u(\cdot,t)\|_{L^p(\Omega)}$. The proof is very similar to that of Lemma 3.1 in [122]. We have

Lemma 3.3.1. *Let (u_0, v_0) satisfy (3.2). For all $p \in (1,\infty)$, there exist $\theta_1(p) > 0$ and $C(\kappa,\tau,p) > 0$ such that if χ, μ are positive constants and satisfy $\frac{\chi}{\mu} < \theta_1$, then*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C(\kappa,\tau,p), \text{ for all } t \in (0, T_{\max}). \tag{3.9}$$

Moreover, if $T_{\max} = \infty$, we have

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)} \le \frac{C(\kappa,\tau,p)}{\mu}. \tag{3.10}$$

Proof. First we see that for any $a, b > 0$, Young's inequality provides $k_p > 0$ such that

$$ab \le \frac{1}{4} a^{\frac{p+1}{p}} + k_p b^{p+1}. \tag{3.11}$$

Let $C_{p+1} := C(p+1, p+1)$ denote the constant from Lemma 1.2.3 for $p \in (1,\infty)$. Now we can find $\theta_1 > 0$ small enough such that

$$C_{p+1} k_p \theta^{p+1} < \frac{1}{2} \text{ for all } \theta < \theta_1. \tag{3.12}$$

We multiply the first equation in (3.1) by u^{p-1} and integrate over Ω to obtain that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2 \\
&= (p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla v + \kappa \int_\Omega u^p - \mu \int_\Omega u^{p+1} \\
&= -\frac{p-1}{p}\chi \int_\Omega u^p \Delta v + \kappa \int_\Omega u^p - \mu \int_\Omega u^{p+1} \\
&= -\frac{p+1}{2\tau p}\int_\Omega u^p - \frac{p-1}{p}\chi \int_\Omega u^p \Delta v + \left(\kappa + \frac{p+1}{2\tau p}\right)\int_\Omega u^p - \mu \int_\Omega u^{p+1}
\end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max})$. Now (3.11) implies that

$$\left(\kappa + \frac{p+1}{2\tau p}\right)\int_\Omega u^p \le \frac{\mu}{4}\int_\Omega u^{p+1} + k_p \mu^{-p}\left(\kappa + \frac{p+1}{2\tau p}\right)^{p+1} |\Omega|, \tag{3.14}$$

$$-\frac{p-1}{p}\chi \int_\Omega u^p \Delta v \le \frac{\mu}{4}\int_\Omega u^{p+1} + k_p \mu^{-p}\chi^{p+1}\int_\Omega |\Delta v|^{p+1}. \tag{3.15}$$

We see that (3.13)-(3.15) imply

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p \\
&\le -\frac{p+1}{2\tau}\int_\Omega u^p - \frac{\mu p}{2}\int_\Omega u^{p+1} + k_p p \mu^{-p}\chi^{p+1}\int_\Omega |\Delta v|^{p+1} + k_p p \mu^{-p}\left(\kappa + \frac{p+1}{2\tau p}\right)^{p+1}|\Omega|
\end{aligned}$$

for all $t \in (0, T_{\max})$. Let $t_0 \in (0, T_{\max})$. Applying Gronwall's inequality to the above inequality, we obtain that

$$\int_\Omega u^p(\cdot,t) \le e^{-\frac{p+1}{2\tau}(t-t_0)}\int_\Omega u^p(\cdot,t_0) - \frac{\mu p}{2}\int_{t_0}^t e^{-\frac{p+1}{2\tau}(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds$$

$$+ k_p p\mu^{-p}\chi^{p+1}\int_{t_0}^{t} e^{-\frac{p+1}{2\tau}(t-s)}\int_\Omega |\Delta v(\cdot,s)|^{p+1}ds$$
$$+ k_p p\mu^{-p}\left(\kappa+\frac{p+1}{2\tau p}\right)^{p+1}|\Omega|\int_{t_0}^{t} e^{-\frac{p+1}{2\tau}(t-s)}ds$$

for all $t\in(t_0,T_{\max})$. An application of Lemma 1.2.3 implies $C_{p+1}>0$ fulfilling

$$\begin{aligned}
&k_p p\mu^{-p}\chi^{p+1}\int_{t_0}^{t} e^{-\frac{p+1}{2\tau}(t-s)}\int_\Omega |\Delta v(\cdot,s)|^{p+1}ds\\
&\le C_{p+1}k_p p\mu^{-p}\chi^{p+1}\int_{t_0}^{t} e^{-\frac{p+1}{2\tau}(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\\
&\qquad + C_{p+1}\tau k_p p\mu^{-p}\chi^{p+1}e^{-\frac{p+1}{2\tau}(t-t_0)}\|v(\cdot,t_0)\|^{p+1}_{W^{2,p+1}(\Omega)}.
\end{aligned}$$

We therefore derive that

$$\begin{aligned}
\int_\Omega u^p(\cdot,t) &\le e^{-\frac{p+1}{2\tau}(t-t_0)}\int_\Omega u^p(\cdot,t_0) + C_{p+1}\tau k_p p\mu^{-p}\chi^{p+1}e^{-\frac{p+1}{2\tau}(t-t_0)}\|v(\cdot,t_0)\|^{p+1}_{W^{2,p+1}(\Omega)}\\
&\quad - p\mu\left(\frac{1}{2}-C_{p+1}k_p\left(\frac{\chi}{\mu}\right)^{p+1}\right)\int_{t_0}^{t} e^{-\frac{p+1}{2\tau}(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\\
&\quad + k_p p\mu^{-p}\left(\kappa+\frac{p+1}{2\tau p}\right)^{p+1}|\Omega|\int_{t_0}^{t} e^{-\frac{p+1}{2\tau}(t-s)}ds
\end{aligned}$$

for all $t\in(t_0,T_{\max})$. In view of (3.12) and the assumption that $\frac{\chi}{\mu}<\theta_1$, we have

$$-p\mu\left(\frac{1}{2}-C_{p+1}k_p(\frac{\chi}{\mu})^{p+1}\right)\int_{t_0}^{t} e^{-\frac{p+1}{2\tau}(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\le 0.$$

Thus

$$\begin{aligned}
\int_\Omega u^p(\cdot,t) &\le e^{-\frac{p+1}{2\tau}(t-t_0)}\int_\Omega u^p(\cdot,t_0) + C_{p+1}\tau k_p p\mu^{-p}\chi^{p+1}e^{-\frac{p+1}{2\tau}(t-t_0)}\|v(\cdot,t_0)\|^{p+1}_{W^{2,p+1}(\Omega)}\\
&\quad + k_p p\mu^{-p}\left(\kappa+\frac{p+1}{2\tau p}\right)^{p+1}|\Omega|\int_{t_0}^{t} e^{-\frac{p+1}{\tau}(t-s)}ds\\
&\le e^{-\frac{p+1}{2\tau}(t-t_0)}\int_\Omega u^p(\cdot,t_0) + C_{p+1}\tau k_p p\mu^{-p}\chi^{p+1}e^{-\frac{p+1}{2\tau}(t-t_0)}\|v(\cdot,t_0)\|^{p+1}_{W^{2,p+1}(\Omega)}\\
&\quad + k_p p\mu^{-p}\left(\kappa+\frac{p+1}{2\tau p}\right)^{p+1}|\Omega|\frac{p+1}{2\tau}\int_0^{\frac{p+1}{2\tau}(t-t_0)} e^{-\sigma}d\sigma\\
&\le e^{-\frac{p+1}{2\tau}(t-t_0)}\int_\Omega u^p(\cdot,t_0) + C_{p+1}\tau k_p p\mu^{-p}\chi^{p+1}e^{-\frac{p+1}{2\tau}(t-t_0)}\|v(\cdot,t_0)\|^{p+1}_{W^{2,p+1}(\Omega)}\\
&\quad + k_p p\mu^{-p}\left(\kappa+\frac{p+1}{2\tau p}\right)^{p+1}|\Omega|\frac{p+1}{2\tau}c_1(p,\tau)
\end{aligned}$$

for all $t\in(t_0,T_{\max})$, where $c_1(p,\tau):=\int_0^\infty e^{-\sigma}d\sigma$. This implies (3.9). Suppose that $T_{\max}=\infty$. Letting $t\to\infty$, we obtain that

$$\limsup_{t\to\infty}\int_\Omega u^p(\cdot,t)\le\frac{C}{\mu^p}$$

with $C:=k_p p\left(\kappa+\frac{p+1}{2\tau p}\right)^{p+1}|\Omega|\frac{p+1}{2\tau}c_1(p,\tau)$. Taking the p-th root on both sides, we finish the proof. □

3.4. Large time behavior of L^∞-norm

Applying the variation of constants formula to the second equation in (3.1) and the L^p-L^q estimate for the Neumann semigroup from Lemma 6.2.1, we readily have the following:

Lemma 3.4.1. *Let $p \geq 1$. Assume that*

$$\begin{cases} r < \frac{Np}{(N-p)_+}, & p \leq N, \\ r = \infty, & p > N. \end{cases} \tag{3.16}$$

Then for all $K > 0$, there exists $C(K,p,r) > 0$ such that for all (u_0, v_0) satisfying (3.2) and all $\chi, \mu, \kappa > 0$, if $T_{\max} = \infty$, and

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)} \leq \frac{K}{\mu}, \tag{3.17}$$

then

$$\limsup_{t\to\infty} \|\nabla v(\cdot,t)\|_{L^r(\Omega)} \leq \frac{C(K,p,r)}{\mu}. \tag{3.18}$$

Proof. Let $p \geq 1$ and suppose $T_{\max} = \infty$. For all $K > 0$, we can find $t_0 > t$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq \frac{2K}{\mu}, \text{ for all } t > t_0.$$

Due to the choice of r, we know that $c_1(p,r) := \int_0^\infty \sigma^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})} e^{-\sigma} d\sigma < \infty$. According to variation of constants formula for v,

$$v(\cdot,t) = e^{\frac{t-t_0}{\tau}(\Delta-1)} v(\cdot,t_0) + \int_{t_0}^t e^{\frac{t-s}{\tau}(\Delta-1)} \frac{1}{\tau} u(\cdot,s) ds$$

for all $t \in (t_0, \infty)$. We apply the L^p-L^q estimate for the Neumann heat semigroup from Lemma 6.2.1 to find $c_2 > 0$ such that

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^r(\Omega)} &\leq \|\nabla e^{\frac{t-t_0}{\tau}(\Delta-1)} v(\cdot,t_0)\|_{L^r(\Omega)} + \int_{t_0}^t \|\nabla e^{\frac{t-s}{\tau}(\Delta-1)} \frac{1}{\tau} u(\cdot,s)\|_{L^r(\Omega)} ds \\
&\leq c_2 e^{-\frac{t-t_0}{\tau}} \|\nabla v(\cdot,t_0)\|_{L^r} + \int_{t_0}^t c_2 (\frac{t-s}{\tau})^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})} e^{-\frac{t-s}{\tau}} \frac{1}{\tau} \|u(\cdot,s)\|_{L^p(\Omega)} ds \\
&\leq c_2 e^{-\frac{t-t_0}{\tau}} \|\nabla v(\cdot,t_0)\|_{L^r} + c_2 \frac{2K}{\mu} \frac{1}{\tau} \int_{t_0}^t (\frac{t-s}{\tau})^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})} e^{-\frac{t-s}{\tau}} ds \\
&\leq c_2 e^{-\frac{t-t_0}{\tau}} \|\nabla v(\cdot,t_0)\|_{L^r} + c_2 \frac{2K}{\mu} \int_0^\infty \sigma^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})} e^{-\sigma} d\sigma \\
&\leq c_2 e^{-\frac{t-t_0}{\tau}} \|\nabla v(\cdot,t_0)\|_{L^r} + c_2 c_1(p,r) \frac{2K}{\mu}
\end{aligned}$$

for all $t \in (t_0, \infty)$. Letting $t \to \infty$, we obtained the desired estimate by choosing $C(K,p,r) := 2Kc_1c_2$. ☐

Lemma 3.4.2. *Let $p > \frac{N}{2}$ and (u_0, v_0) satisfy (3.2). Suppose that $T_{\max} = \infty$. For all $K > 0$, there exists $C(K,p,\kappa,\theta) > 0$ such that if*

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)} \leq \frac{K}{\mu}, \tag{3.19}$$

then

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{C(K,p,\kappa,\theta)}{\mu}, \tag{3.20}$$

where $\theta := \frac{\chi}{\mu}$.

Proof. Assume that $p \in (\frac{N}{2}, N)$ without loss of generality. First we fix $r \in (N, \frac{Np}{N-p})$, $c_1(K) > 0$ and $t_0 > 0$ such that

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le \frac{c_1(K)}{\mu},$$
$$\|u(\cdot,t)\|_{L^p(\Omega)} \le \frac{c_1(K)}{\mu},$$
$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le \frac{c_1(K)}{\mu}$$

for all $t > t_0$. Let $s_0 \in (t_0, \infty)$. Using the variation of constants formula for the first equation in (3.1), we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{(t-s_0)\Delta}u(\cdot,s_0)\|_{L^\infty(\Omega)} &+ \chi\int_{s_0}^t \|e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &+ \int_{s_0}^t \|e^{(t-s)\Delta}(\kappa u - \mu u^2)(\cdot,s)\|_{L^\infty(\Omega)}ds \end{aligned} \tag{3.21}$$

for all $t \in (s_0, s_0+2)$. We begin with

$$\begin{aligned} \int_{s_0}^t \|e^{(t-s)\Delta}(\kappa u - \mu u^2)(\cdot,s)\|_{L^\infty(\Omega)}ds &\le \int_{s_0}^t \sup_{u\ge 0}\left(\kappa u - \mu u^2\right)_+ ds \\ &\le \int_{s_0}^t \frac{\kappa^2}{4\mu}ds \le \frac{\kappa^2}{2\mu} \end{aligned}$$

for all $t \in (s_0, s_0+2)$. By the L^p-L^q estimate for the Neumann heat semigroup from Lemma 6.2.1 (i), there exists a constant $c_2 > 0$ fulfilling

$$\|e^{(t-s_0)\Delta}u(\cdot,s_0)\|_{L^\infty(\Omega)} \le c_2(t-s_0)^{-\frac{N}{2p}}\|u(\cdot,s_0)\|_{L^p(\Omega)} \tag{3.22}$$

for all $t \in (s_0, s_0+2)$. Let q satisfy $\frac{1}{q} \in (\frac{1}{r}, \frac{1}{N})$, we can find $r' > q$ such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{r'}$, and $a = 1 - \frac{1}{r'} \in (0,1)$. Let $M(t) = (t-s_0)^{\frac{N}{2p}}\|u(\cdot,t)\|_{L^\infty(\Omega)}$. Using the L^p-L^q estimate for the Neumann heat semigroup, the Hölder inequality and the interpolation inequality $\|u\|_{L^{r'}(\Omega)} \le \|u\|^a_{L^\infty(\Omega)}\|u\|^{1-a}_{L^1(\Omega)}$, we obtain $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned} &\chi\int_{s_0}^t \|e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &\le c_3\chi\int_{s_0}^t (t-s)^{-\frac{1}{2}-\frac{N}{2q}}\|u\nabla v(\cdot,s)\|_{L^q(\Omega)}ds \\ &\le c_3\chi\int_{s_0}^t (t-s)^{-\frac{1}{2}-\frac{N}{2q}}\|u(\cdot,s)\|_{L^{r'}(\Omega)}\|\nabla v(\cdot,s)\|_{L^r(\Omega)}ds \end{aligned}$$

$$
\begin{aligned}
&\leq c_3\chi \int_{s_0}^{t} (t-s)^{-\frac{1}{2}-\frac{N}{2q}} \|u(\cdot,s)\|^{a}_{L^\infty(\Omega)} \|u(\cdot,s)\|^{1-a}_{L^1(\Omega)} \|\nabla v(\cdot,s)\|_{L^r(\Omega)} ds \\
&\leq c_3\chi \int_{s_0}^{t} (t-s)^{-\frac{1}{2}-\frac{N}{2q}} M^a(s)(s-s_0)^{-a\frac{N}{2p}} \left(\frac{c_1}{\mu}\right)^{1-a} \left(\frac{c_1}{\mu}\right) ds \\
&\leq c_1c_3 \left(\frac{\chi}{\mu}\right)\left(\frac{c_1}{\mu}\right)^{1-a} \left\{\sup_{t\in(s_0,s_0+2)} M(t)\right\}^a \int_0^{t-s_0} (t-s_0-\sigma)^{-\frac{1}{2}-\frac{N}{2q}} \sigma^{-a\frac{N}{2p}} d\sigma \\
&\leq c_1c_3 \left(\frac{\chi}{\mu}\right)\left(\frac{c_1}{\mu}\right)^{1-a} \left\{\sup_{t\in(s_0,s_0+2)} M(t)\right\}^a \cdot c_4(t-s_0)^{\frac{1}{2}-\frac{N}{2q}-\frac{N}{2p}a} \qquad (3.23)
\end{aligned}
$$

for all $t \in (s_0, s_0+2)$. Now we collect the above estimates (3.21-3.23) to see that

$$
M(t) \leq c_2\left(\frac{c_1}{\mu}\right) + (t-s_0)^{\frac{N}{2p}}\frac{\kappa^2}{2\mu} + c_1c_3c_4(t-s_0)^{\frac{1}{2}-\frac{N}{2q}+\frac{N}{2p}(1-a)} \left(\frac{\chi}{\mu}\right)\left(\frac{c_1}{\mu}\right)^{1-a} \left\{\sup_{t\in(s_0,s_0+2)} M(t)\right\}^a
$$

Let $\widetilde{M}(s_0) := \sup_{t\in(s_0,s_0+2)} M(t)$ and $\theta = \frac{\chi}{\mu}$. We take the supremum on both sides of the above inequality to obtain that

$$
\widetilde{M}(s_0) \leq c_2\left(\frac{c_1}{\mu}\right) + 2^{\frac{N}{2p}}\frac{\kappa^2}{2\mu} + c_1c_3c_4 2^{\frac{N}{2p}+1}\theta\left(\frac{c_1}{\mu}\right)^{1-a}\left(\widetilde{M}(s_0)\right)^a \text{ for all } s_0 > t_0.
$$

Since $a < 1$, this implies the existence of $c_5(K, p, \kappa, \theta) > 0$ such that

$$
\widetilde{M}(s_0) \leq \frac{c_5}{\mu} \text{ for all } s_0 > t_0.
$$

It also holds that

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{c_5}{\mu}
$$

for all $t \in (s_0+1, s_0+2)$. According to the choice of s_0, we conclude the assertion. □

For convenience, we introduce

$$
\mathcal{U} = u - \frac{\kappa}{\mu}, \;\; \mathcal{V} = v - \frac{\kappa}{\mu}. \qquad (3.24)
$$

It is easy to see that (U, V) satisfies

$$
\begin{cases}
\mathcal{U}_t = \Delta\mathcal{U} - \chi\nabla\cdot(u\nabla\mathcal{V}) - \kappa\mathcal{U} - \mu\mathcal{U}^2, & (x,t)\in\Omega\times(0,T),\\
\tau\mathcal{V}_t = \Delta\mathcal{V} - \mathcal{V} + \mathcal{U}, & (x,t)\in\Omega\times(0,T),\\
\nabla\mathcal{U}\cdot\nu = \nabla\mathcal{V}\cdot\nu = 0, & (x,t)\in\partial\Omega\times(0,T),\\
\mathcal{U}(x,0) = u_0(x) - \frac{\kappa}{\mu},\; \mathcal{V}(x,0) = v_0(x) - \frac{\kappa}{\mu}, & x\in\Omega.
\end{cases} \qquad (3.25)
$$

Here we note that $(\mathcal{U}_0, \mathcal{V}_0)$ is not necessarily nonnegative.
We fix $\lambda \in (0, \min\{1, \kappa\})$, and let $A = A_p$ be the realization of $-\Delta + \lambda$ under Neumann homogeneous boundary conditions. We know that A is sectorial in $L^p(\Omega)$ and possesses fractional powers A^α for $\alpha > 0$. The domain $D(A^\alpha)$ satisfies the embedding

$$
D(A^\alpha) \hookrightarrow W^{2,\infty}(\Omega), \text{ if } 2\alpha - \frac{N}{p} > 2. \qquad (3.26)
$$

Moreover, A generates an analytic semigroup $(e^{-tA})_{t\geq 0}$ and for all $\alpha > 0$ there is $c(p,\alpha) > 0$ such that

$$\|A^{\alpha} e^{-tA}\varphi\|_{L^p(\Omega)} \leq c(p,\alpha) t^{-\alpha} \|\varphi\|_{L^p(\Omega)} \tag{3.27}$$

for all $t > 0$ and $\varphi \in L^p(\Omega)$. We now follow Lemmata 4.1, 4.2 and 5.1 in [110] to prove that:

Lemma 3.4.3. *Let (u_0, v_0) satisfy (3.2). Let $p > \frac{N}{2}$ and $\theta_1(p)$ be defined as in Lemma 3.3.1. There exists $C(\kappa,\tau,\theta_1) > 0$ such that if χ, μ, κ are positive constants and satisfy $\frac{\chi}{\mu} < \theta_1(p)$, then $T_{\max} = \infty$ and*

$$\limsup_{t\to\infty} \|\Delta v(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{C(\kappa,\tau,\theta_1)}{\mu}. \tag{3.28}$$

Proof. In view of the hypothesis, Lemma 3.3.1 implies that $\|u(\cdot,t)\|_{L^p(\Omega)}$ $(p > \frac{N}{2})$ is bounded. Thus we infer that (u, v) is bounded and $T_{\max} = \infty$ [3, Lemma 2.6]. Moreover, Lemmata 3.3.1, 3.4.1 and 3.4.2 imply that there exist $c_1 > 0$ and $t_0 > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{c_1(\kappa,\tau,\theta_1)}{\mu}, \tag{3.29}$$

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{c_1(\kappa,\tau,\theta_1)}{\mu} \tag{3.30}$$

for all $t > t_0$. Now we fix $\eta \in (1, \frac{3}{2})$, then choose $\beta \in (\eta - 1, \frac{1}{2})$ and

$$p > \frac{N}{2(\eta - 1)}. \tag{3.31}$$

Applying the variantion of constants formula to the first equation in (3.25), invoking (3.29) and (3.30), and employing the same argument used in [110, Lemma 4.2], we show that

$$\limsup_{t\to\infty} \|A^{\beta}\mathcal{U}(\cdot,t)\|_{L^p(\Omega)} \leq \frac{c_2}{\mu} \tag{3.32}$$

with some $c_2(\kappa,\tau,\theta_1) > 0$. We again follow the idea of [110, Lemma 5.1] to find that

$$\limsup_{t\to\infty} \|A^{\eta}\mathcal{V}(\cdot,t)\|_{L^p(\Omega)} \leq \frac{c_3}{\mu} \tag{3.33}$$

with some $c_3(\kappa,\tau,\theta_1) > 0$. By the embedding theorem (3.26), there is $c_4 > 0$ such that

$$\|\Delta v\|_{L^\infty(\Omega)} = \|\Delta \mathcal{V}\|_{L^\infty(\Omega)} \leq \|\mathcal{V}\|_{W^{2,\infty}(\Omega)} \leq c_4 \|A^{\eta}\mathcal{V}\|_{L^p(\Omega)}. \tag{3.34}$$

The proof is complete. $\square$

3.5. Refined estimate for u

In this section, we show that after suitably large time, u lies in a neighborhood of $\frac{\kappa}{\mu}$ whose radius is measured by $\frac{\chi}{\mu}$. We can prove it by using maximum principle and the pointwise bound of Δv.

Lemma 3.5.1. *Let (u, v) be a global classical solution of (3.1) and $(\mathcal{U}, \mathcal{V})$ be defined in (3.24). For all $K > 0$ there exists $C(K) > 0$ such that if it holds that*

$$\limsup_{t\to\infty} \|\Delta v(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{K}{\mu} \tag{3.35}$$

then

$$\limsup_{t\to\infty} \|\mathcal{U}(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{C(K)\theta}{\mu}, \tag{3.36}$$

where $\theta := \frac{\chi}{\mu}$.

Proof. According to the assumption, we can find $t_0 > 0$ fulfilling

$$\|\Delta v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2K}{\mu} \text{ for all } t \ge t_0. \tag{3.37}$$

We use (3.37) and the first equation in (3.1) to estimate that

$$\begin{aligned} u_t &= \Delta u - \chi\nabla u\cdot\nabla v - \chi u\Delta v + \kappa u - \mu u^2 \\ &\le \Delta u - \chi\nabla u\cdot\nabla v + \chi u\frac{2K}{\mu} + \kappa u - \mu u^2 \\ &\le \Delta u - \chi\nabla u\cdot\nabla v + u\left(2K\theta + \kappa - \mu u\right) \end{aligned} \tag{3.38}$$

for all $x \in \Omega$ and $t > t_0$, where we use $\theta = \frac{\chi}{\mu}$. Let $z := z(t)$ be the solution to

$$\begin{cases} z'(t) = z(t)\big(2K\theta + \kappa - \mu z(t)\big),\ t > t_0, \\ z(t_0) = \sup\limits_{x\in\Omega} u(x,t_0). \end{cases} \tag{3.39}$$

It is easy to see that $z(t_0) > 0$ by the strong maximum principle. The comparison principle implies

$$u(x,t) \le z(t) \text{ for all } x\in\Omega, t > t_0. \tag{3.40}$$

Thus we can derive that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \lim_{t\to\infty} z(t) = \frac{2K\theta}{\mu} + \frac{\kappa}{\mu}.$$

This leads to

$$\limsup_{t\to\infty} \|\mathcal{U}_+(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2K\theta}{\mu}. \tag{3.41}$$

Similarly (see also in [110, Lemma 6.1]), using the lower bound of Δv in the first equation in (3.1) and letting $y(t) \in C^1([t_0,\infty))$ solve the following equation

$$\begin{cases} y'(t) = y(t)\big(-2K\theta + \kappa - \mu y(t)\big),\ t > t_0, \\ y(t_0) = \inf\limits_{x\in\Omega} u(x,t_0) > 0, \end{cases} \tag{3.42}$$

we see that

$$u(x,t) \ge y(t) \text{ for all } x\in\Omega, t > t_0,$$

which implies that

$$\limsup_{t\to\infty} \|\mathcal{U}_-(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2K\theta}{\mu}. \tag{3.43}$$

Combining (3.41) and (3.43), we establish (3.36) by choosing $C(K) := 2K$. □

3.6. Decay of $(\mathcal{U}, \mathcal{V})$

In the last section, we prove that $\mathcal{U}$ is in a neighborhood of 0 after suitably large time. This enables us to show that $\mathcal{U}$ in fact decays in the large time limit if θ is sufficiently small. At the same time, the decay of $\mathcal{V}$ is also obtained. Letting λ_1 be the first non-zero eigenvalue of $-\Delta$ associated with Neumann boundary conditions, we have the following:

Lemma 3.6.1. *Suppose that* $\kappa > 0$. *Let* (u_0, v_0) *satisfy* (3.2) *and* $(\mathcal{U}, \mathcal{V})$ *be defined as in* (3.24). *Let* $0 < \zeta < \min\{\frac{1}{\tau}, \lambda_1, 1\}$. *For all* $K > 0$, *there exists* $\theta_2 > 0$ *and* $C > 0$ *such that if* $\chi > 0$ *and* $\mu > 0$ *satisfy* $\theta := \frac{\chi}{\mu} < \theta_2$ *and*

$$\limsup_{t\to\infty} \|\mathcal{U}(\cdot, t)\|_{L^\infty(\Omega)} \le \frac{K\theta}{\mu}, \tag{3.44}$$

then

$$\|\mathcal{U}(\cdot, t)\|_{L^\infty(\Omega)} \le Ce^{-\zeta t} \text{ and} \tag{3.45}$$

$$\|\mathcal{V}(\cdot, t)\|_{L^\infty(\Omega)} \le Ce^{-\zeta t} \tag{3.46}$$

for all $t \ge 0$.

The proof follows from [110, Lemma 7.1 and the proof of Theorem 1.1].

Proof of Theorem 3.1.1. Let $p > \frac{N}{2}$, and $\theta_1 := \theta_1(p)$ and θ_2 be defined as in Lemmata 3.3.1 and 3.6.1, respectively. Let $\theta_0 = \min\{\theta_1, \theta_2\}$. The condition that $\theta < \theta_0$ implies the boundedness and global existence of (u, v), thus $T_{\max} = \infty$. We obtain (3.7) directly by Lemmata 3.4.3, 3.5.1 and 3.6.1. □

4. Boundedness in a chemotaxis-haptotaxis model

4.1. Introduction

In Chapter 2 and 3, we have introduced the classical Keller-Segel model and the Keller-Segel model with logistic source. Apart from those, a large number of variants has been proposed to describe taxis phenomena in mathematical biology. Among them, a model for tumor invasion mechanism was introduced by Chaplain and Lolas [18]. In this model, tumor cells are assumed to produce a diffusible chemical substance, the so-called matrix-degrading enzyme (MDE), which decays non-diffusible static healthy tissue (ECM). It is observed that both the enzyme and the healthy tissue can attract the cancer cells in the sense that the cancer cells bias their movement along the gradients of the concentrations of both ECM and MDE, where the former of these processes, namely taxis toward a non-diffusible quantity, is usually referred as haptotaxis.
Additionally, the cancer cells compete for space with ECM, and at the considered time scales moreover logistic-type cell kinetics need to be taken into account. If furthermore the ability of ECM to spontaneously renew is included, the Chaplain-Lolas model becomes

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w), & (x,t)\in\Omega\times(0,T), \\ \tau v_t = \Delta v - v + u, & (x,t)\in\Omega\times(0,T), \\ w_t = -vw + \eta w(1-u-w), & (x,t)\in\Omega\times(0,T), \end{cases} \tag{4.1}$$

where u, v and w denote the density of cells, the concentration of MDE and the density of ECM, respectively, where the parameters ξ, χ, μ, η are positive constants and $\tau \geq 0$, and where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, denotes the physical domain under consideration.
Assuming $w \equiv 0$, (4.1) is reduced to the classical Keller-Segel system with logistic source, which has extensively been studied during the past 20 years. Compared with the pure chemotaxis system mentioned above, one may expect the logistic source and, especially, death terms to enhance the possibility of bounded solutions. In fact, Tello and Winkler [96] proved that if $\tau = 0$ and

$$\mu > \frac{(N-2)_+}{N}\chi, \tag{4.2}$$

then for any regular initial data, the logistic Keller-Segel system a unique global classical solution which is bounded. In the case $\tau = 1$, it is known that bounded solutions exist in lower dimensions ($N = 1, 2$) for any $\mu > 0$ [68], and that the same result holds for $\mu > \mu_0$ with some $\mu_0(\chi) > 0$ in higher dimensions [106]. More precisely, a careful inspection of the proofs therein shows that in fact large values of the ratio $\frac{\mu}{\chi^2}$ are sufficient to exclude blow up in either finite time or infinite time.
Concerning (4.1) with possibly nontrivial w, the strong coupling between remodeling and chemotaxis substantially complicates the situation, and accordingly the knowledge on this topic is quite incomplete so far. To the best of our knowledge, global existence of weak solution is

obtained in [79] for $N \leq 3$, where (4.1) is included as a subsystem. And global solvability of classical solutions in this full system is known only when $\tau = 0$ and $N = 2$ [93]. Disregarding the chemotaxis effect, the haptotaxis-only version with $\chi = 0$, $\tau = 1$ was studied in [84].
In real situations, the ECM degrades much faster than it renews, thus the remodeling effect can be neglected, that is, we may assume $\eta = 0$. Under this hypothesis, the corresponding parabolic-elliptic simplification $\tau = 0$ has been studied by Tao and Winkler in [92], where it has been proved that solutions stay bounded under the same condition as in the case $w \equiv 0$, that is, when (4.2) holds. This shows that in this situation the haptotaxis term does not affect the boundedness of solutions, and that accordingly the chemotaxis process essentially dominates the whole system.
A natural question is whether a similar conclusion holds in the fully parabolic system obtained on letting $\tau = 1$. In [85], Tao gives a partially positive answer in this direction by proving that when $N = 2$, solutions remain bounded for any $\mu > 0$, which thus parallels known results both for $\tau = 0$, and also for $\tau = 1$ when $w \equiv 0$. As far as we can tell, however, despite a result on global existence established in [86], the question of boundedness of solutions is completely open in higher dimensions. It is the purpose of this work to furthermore establish a corresponding parallel result for the three-dimensional parabolic-parabolic-ODE chemotaxis-haptotaxis model in this direction.
Accordingly, we deal with the system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w), & (x,t)\in\Omega\times(0,T), \\ v_t = \Delta v - v + u, & (x,t)\in\Omega\times(0,T), \\ w_t = -vw, & (x,t)\in\Omega\times(0,T), \\ (\nabla u - \chi u\nabla v - \xi u\nabla w)\cdot\nu = \nabla v\cdot\nu = 0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x),\ \ w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{4.3}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is bounded with smooth boundary and $\chi, \xi, \mu > 0$. We assume that the initial data are regular enough and satisfy a standard compatibility condition in the sense that

$$\begin{cases} u_0 \in C^0(\overline{\Omega}),\ \ v_0 \in W^{1,q}(\Omega) \text{ with } q > N,\ \ w_0 \in C^{2,\alpha}(\overline{\Omega})\,(\alpha\in(0,1)), \\ \nabla w_0\cdot\nu = 0. \end{cases} \tag{4.4}$$

Then our main result says the following.

Theorem 4.1.1. *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. There exists $\theta_0 > 0$ such that whenever $\chi > 0$, $\mu > 0$ and $\xi > 0$ are such that $\frac{\chi}{\mu} < \theta_0$, for any initial data (u_0, v_0, w_0) fulfilling* (4.4)*, there exists a unique global classical solution (u, v, w) satisfying*

$$\begin{aligned} &u \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \\ &v \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,q}(\Omega)), \\ &w \in C^{2,1}(\overline{\Omega}\times[0,\infty)). \end{aligned}$$

Moreover, it is bounded in $\Omega\times(0,\infty)$.

We see that although our hypothesis on the parameters is not as explicit as (4.2) obtained for the parabolic-elliptic counterpart, it still shows that again boundedness of solutions is enforced by a condition merely referring to the interplay between chemotaxis and quadratic degradation in logistic source.
Apart from this, we find it worth mentioning that our approach even shows a new result for

the pure fully parabolic chemotaxis system with logistic source in the sense that when $w \equiv 0$, $N \geq 3$, the system admits a classical bounded solution if $\frac{\mu}{\chi}$ is sufficiently large. Compared with a similar conclusion under the alternative assumption that $\frac{\mu}{\chi^2}$ be large [106], our result seems more consistent with (4.2) for the parabolic-elliptic system where the linear ratio $\frac{\mu}{\chi}$ is found to determine the boundedness of solution.

4.2. Preliminaries

Although a haptotaxis term is included in (4.3), the local existence theory is in a similar spirit as Lemma 2.1.1. The proof can be derived based on that in [91, Lemma 2.1].

Lemma 4.2.1. *Let $N \geq 3$, $\chi > 0$, $\xi > 0$ and $\mu > 0$. For (u_0, v_0, w_0) satisfying (4.4), there is $T_{\max} \in (0, \infty]$ such that (4.3) admits a unique classical solution*

$$\begin{aligned} &u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ &v \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}([0, T_{\max}); W^{1,q}(\Omega)), \\ &w \in C^{2,1}(\overline{\Omega} \times [0, T_{\max})), \end{aligned}$$

such that

$$u \geq 0, \quad v \geq 0 \quad \text{and} \quad 0 < w \leq \|w_0\|_{L^\infty(\Omega)} \quad \text{for all} \quad t \in [0, T_{\max}). \tag{4.5}$$

Moreover, if $T_{\max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty, \text{ as } t \to T_{\max}.$$

According to the above existence theory, we know that if we fix any $t_0 \in (0, T_{\max})$, then there exists $M > 0$ such that

$$\sup_{s \in [0, t_0]} \|u(\cdot, s)\|_{L^\infty(\Omega)} + \sup_{s \in [0, t_0]} \|v(\cdot, s)\|_{W^{2,\infty}(\Omega)} + \|w(\cdot, t_0)\|_{W^{2,\infty}(\Omega)} < M. \tag{4.6}$$

Observing that w can be represented by v and $w(x, t_0)$, we can compute Δw in a convenient way. Upon a slight adaptation of [92, Lemma 2.2], we can prove a one-sided pointwise estimate for Δw as follows.

Lemma 4.2.2. *Let (u_0, v_0, w_0) satisfy (4.4) and (u, v, w) solve (4.3). We have*

$$\begin{aligned} \Delta w(x, t) \geq\ & \Delta w(x, t_0) \cdot e^{-\int_{t_0}^t v(x,s)ds} - 2e^{-\int_{t_0}^t v(x,s)ds} \nabla w(x, t_0) \cdot \int_{t_0}^t \nabla v(x, s) ds \\ & - \frac{1}{e} w(x, t_0) - w(x, t_0) v(x, t) e^{-\int_{t_0}^t v(x,s)ds} \end{aligned} \tag{4.7}$$

for all $x \in \Omega$ and all $t \in (t_0, T_{\max})$.

Proof. Representing $w(x, t)$ according to

$$w(x, t) = e^{-\int_{t_0}^t v(x,s)ds} w(x, t_0) \tag{4.8}$$

for all $x \in \Omega$ and $t \in (t_0, T_{\max})$, we directly compute that

$$\Delta w(x, t) = \Delta w(x, t_0) e^{-\int_{t_0}^t v(x,s)ds} - 2e^{-\int_{t_0}^t v(x,s)ds} \nabla w(x, t_0) \cdot \int_{t_0}^t \nabla v(x, s) ds$$

$$+ w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds}|\int_{t_0}^t \nabla v(x,s)ds|^2 - w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds}\int_{t_0}^t \Delta v(x,s)ds.$$

Since $ze^{-z} \le \frac{1}{e}$ for all $z \ge 0$, by dropping some nonnegative terms, we obtain that

$$\begin{aligned}
&\Delta w(x,t)\\
&\ge \Delta w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds} - 2e^{-\int_{t_0}^t v(x,s)ds}\nabla w(x,t_0)\cdot\int_{t_0}^t \nabla v(x,s)ds\\
&\quad - w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds}\int_{t_0}^t (v_s(x,s)+v(x,s)-u(x,s))\\
&\ge \Delta w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds} - 2e^{-\int_{t_0}^t v(x,s)ds}\nabla w(x,t_0)\cdot\int_{t_0}^t \nabla v(x,s)ds\\
&\quad - w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds}(v(x,t)-v(x,t_0)) - w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds}\int_{t_0}^t v(x,s)ds\\
&\ge \Delta w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds} - 2e^{-\int_{t_0}^t v(x,s)ds}\nabla w(x,t_0)\cdot\int_{t_0}^t \nabla v(x,s)ds\\
&\quad - w(x,t_0)v(x,t)e^{-\int_{t_0}^t v(x,s)ds} - \frac{1}{e}w(x,t_0)
\end{aligned}$$

for all $t \in (t_0, T_{\max})$. Thus the proof is complete. □

With the aid of Lemma 4.2.2, we can furthermore prepare a preliminary estimate of an integral related to the haptotactic interaction. This estimate will be used in different ways later on.

Lemma 4.2.3. *Let $\chi > 0$, $\xi > 0$, and assume that* (4.4) *holds. Then for any $p > 1$, the solution of (4.3) satisfies*

$$(p-1)\xi\int_\Omega u^{p-1}\nabla u\cdot\nabla w \le (3M\xi + \frac{1}{e}M\xi)\int_\Omega u^p + M\xi\int_\Omega u^p v + 2M(p-1)\xi\int_\Omega u^{p-1}|\nabla u| \quad (4.9)$$

for all $t \in (t_0, T_{\max})$, where $M > 0$ is as in (4.6).

Proof. Integration by parts and an application of Lemma 4.2.2 yield that

$$\begin{aligned}
&(p-1)\xi\int_\Omega u^{p-1}\nabla u\cdot\nabla w\\
&= -\frac{p-1}{p}\xi\int_\Omega u^p\Delta w\\
&\le -\frac{p-1}{p}\xi\int_\Omega u^p\Big(\Delta w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds} - 2e^{-\int_{t_0}^t v(x,s)ds}\nabla w(x,t_0)\cdot\int_{t_0}^t \nabla v(x,s)ds\\
&\quad - \frac{1}{e}w(x,t_0) - w(x,t_0)v(x,t)e^{-\int_{t_0}^t v(x,s)ds}\Big)dx\\
&\le (M\xi + \frac{1}{e}M\xi)\int_\Omega u^p + M\xi\int_\Omega u^p v - 2\frac{p-1}{p}\xi\int_\Omega u^p\nabla w(x,t_0)\cdot\nabla e^{-\int_{t_0}^t v(x,s)ds}dx\\
&= (M\xi + \frac{1}{e}M\xi)\int_\Omega u^p + M\xi\int_\Omega u^p v + 2\frac{p-1}{p}\xi\int_\Omega u^p\Delta w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds}dx\\
&\qquad + 2\frac{p-1}{p}\xi\int_\Omega \nabla u^p\cdot\nabla w(x,t_0)e^{-\int_{t_0}^t v(x,s)ds}dx
\end{aligned}$$

$$\leq (3M\xi + \frac{1}{e}M\xi) \int_\Omega u^p + M\xi \int_\Omega u^p v + 2M(p-1)\xi \int_\Omega u^{p-1}|\nabla u|$$

for all $t \in (t_0, T_{\max})$. □

Lemma 4.2.4. *Let $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume that (4.4) holds. Then there exists $C := C(|\Omega|) > 0$ such that*

$$\int_\Omega u(\cdot,t) < C, \quad \int_\Omega v(\cdot,t) < C \tag{4.10}$$

for all $t \in (0, T_{\max})$.

Proof. The first inequality can be proved by simply integrating the first equation in (4.3) on Ω and using that $(\int_\Omega u)^2 \leq |\Omega|(\int_\Omega u^2)$ due to the Cauchy-Schwarz inequality. The estimate of $\int_\Omega v$ can be obtained in a similar way and with the aid of the first inequality. □

4.3. An L^p estimate for u

In this section, we derive the claimed boundedness result via combining the above result on maximum Sobolev regularity with a Moser-type iteration. We first estimate u in some appropriate Lebesgue space, from which a certain suitable estimate of ∇v will follow. This approach will be carried out to ensure that ∇v is bounded in $L^\infty(\Omega)$. Thereupon we can establish a series of inequalities based on which a Moser iteration is performed to finally achieve boundedness of u in $L^\infty(\Omega)$. An immediate consequence of Lemma 4.3.1 is that ∇v is bounded with respect to the norm in $L^\infty(\Omega)$. Let us first provide an important ingredient for the estimate of $\|u(\cdot,t)\|_{L^p(\Omega)}$ with $p \in (1,\infty)$.

Lemma 4.3.1. *Let (u_0, v_0, w_0) satisfy (4.4). For all $p \in (1,\infty)$, there exist constants $\theta_p > 0$ and $C > 0$ such that if $\chi, \xi > 0, \mu > 0$ are positive constants and satisfy $\frac{\chi}{\mu} < \theta_p$, then*

$$\int_\Omega u^p(\cdot,t) \leq C \quad \text{for all } t \in (t_0, T_{\max}). \tag{4.11}$$

Proof. Let $p \in (1,\infty)$. First we see that for any $a, b > 0$, Young's inequality provides $k_p > 0$ such that

$$ab \leq \frac{1}{8}a^{\frac{p+1}{p}} + k_p b^{p+1}. \tag{4.12}$$

Let C_{p+1} denote the constant from Lemma 1.2.3. Now we can find $\theta_p > 0$ small enough such that

$$C_{p+1}k_p\theta^{p+1} < \frac{1}{2} \text{ for all } \theta < \theta_p. \tag{4.13}$$

Testing the first equation in (4.3) with u^{p-1} ($p > 1$) and integrating by parts imply

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2$$
$$= (p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla v + (p-1)\xi \int_\Omega u^{p-1}\nabla u \cdot \nabla w$$

$$
\begin{aligned}
&+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1}-\mu\int_\Omega u^p w\\
&\le \frac{p-1}{p}\chi\int_\Omega \nabla u^p\cdot\nabla v+(p-1)\xi\int_\Omega u^{p-1}\nabla u\cdot\nabla w+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1}\\
&\le -\frac{p-1}{p}\chi\int_\Omega u^p\Delta v+(p-1)\xi\int_\Omega u^{p-1}\nabla u\cdot\nabla w+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1} \qquad (4.14)
\end{aligned}
$$

for all $t\in(t_0,T_{\max})$. We see that (4.9) and (4.12) entail the existence of $c_3(p,M)>0$ (M is as in (4.6)) satisfying

$$
\begin{aligned}
&(p-1)\xi\int_\Omega u^{p-1}\nabla u\cdot\nabla w\\
&\le c_3\xi\int_\Omega u^p+c_3\xi\int_\Omega u^p v+c_3p\xi\int_\Omega u^{p-1}|\nabla u|\\
&\le c_3\xi\int_\Omega u^p+\frac{\mu}{8}\int_\Omega u^{p+1}+k_pc_3^{p+1}\mu^{-p}\xi^{p+1}\int_\Omega v^{p+1}+\frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2+\frac{c_3^2\xi^2p^2}{2(p-1)}\int_\Omega u^p\\
&\le \Big(c_3\xi+\frac{c_3^2\xi^2p^2}{2(p-1)}\Big)\int_\Omega u^p+\frac{\mu}{8}\int_\Omega u^{p+1}+\frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2+k_pc_3^{p+1}\mu^{-p}\xi^{p+1}\int_\Omega v^{p+1} \qquad (4.15)
\end{aligned}
$$

for all $t\in(t_0,T_{\max})$. From (4.12), we estimate that for all $t\in(t_0,T_{\max})$,

$$
-\frac{p-1}{p}\chi\int_\Omega u^p\Delta v\le\chi\int_\Omega u^p|\Delta v|\le\frac{\mu}{8}\int_\Omega u^{p+1}+k_p\chi^{p+1}\mu^{-p}\int_\Omega|\Delta v|^{p+1}. \qquad (4.16)
$$

Inserting (4.15) and (4.16) into (4.14) and some rearrangement yield

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p+\frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2\\
&\le -\frac{3}{4}\mu\int_\Omega u^{p+1}+\left(c_3\xi+\frac{c_3^2\xi^2p^2}{2(p-1)}+\mu\right)\int_\Omega u^p\\
&\qquad +k_p\mu^{-p}\chi^{p+1}\int_\Omega|\Delta v|^{p+1}+k_p\mu^{-p}\xi^{p+1}\int_\Omega v^{p+1}\\
&= -\frac{p+1}{2p}\int_\Omega u^p+\left(\frac{p+1}{2p}+c_3\xi+\frac{c_3^2\xi^2p^2}{2(p-1)}+\mu\right)\int_\Omega u^p-\frac{3}{4}\mu\int_\Omega u^{p+1}\\
&\qquad +k_p\chi^{p+1}\mu^{-p}\int_\Omega|\Delta v|^{p+1}+k_pc_3^{p+1}\mu^{-p}\xi^{p+1}\int_\Omega v^{p+1} \qquad (4.17)
\end{aligned}
$$

for all $t\in(t_0,T_{\max})$. We again apply Young's inequality to obtain that

$$
\left(\frac{p+1}{2p}+c_3\xi+\frac{c_3^2\xi^2p^2}{2(p-1)}+\mu\right)\int_\Omega u^p\le\frac{\mu}{4}\int_\Omega u^{p+1}+c_4(\mu,\xi,p,M), \qquad (4.18)
$$

where $c_4(\mu,\xi,p,M)>0$. According to the assumption $\frac{\chi}{\mu}<\theta_p$ and (4.13), we know that $\frac{1}{2}-C_{p+1}k_p(\frac{\chi}{\mu})^{p+1}>0$. Let $\varepsilon\in(0,\frac{1}{2C_{p+1}k_p(\frac{\chi}{\mu})^{p+1}}-k_p)$. Lemma 1.2.1 implies a constant $c_5>0$ such that

$$
k_pc_3^{p+1}\mu^{-p}\xi^{p+1}\int_\Omega v^{p+1}\le\varepsilon\mu^{-p}\chi^{p+1}\int_\Omega|\Delta v|^{p+1}+c_5\|v\|^{p+1}_{L^1(\Omega)}. \qquad (4.19)
$$

Upon (4.17)-(4.19), we infer that

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{p+1}{2}\int_\Omega u^p - \frac{\mu p}{2}\int_\Omega u^{p+1} + (\varepsilon + k_p)p\chi^{p+1}\mu^{-p}\int_\Omega |\Delta v|^{p+1} + c_4(\mu,\xi,p)p + c_5 p\|v\|^{p+1}_{L^1(\Omega)}$$

for all $t \in (t_0, T_{\max})$. Applying the Gronwall inequality to the above inequality shows that

$$\begin{aligned}\int_\Omega u^p(\cdot,t) &\le e^{-(\frac{p+1}{2})(t-t_0)}\int_\Omega u^p(\cdot,t_0) - \frac{\mu}{2}p\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\\
&\quad + (\varepsilon + k_p)p\chi^{p+1}\mu^{-p}\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega |\Delta v(\cdot,s)|^{p+1}ds\\
&\quad + \int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}(c_4(\mu,\xi,p)p + c_5 p\|v(\cdot,s)\|^{p+1}_{L^1(\Omega)})ds\\
&\le e^{-(\frac{p+1}{2})(t-t_0)}\int_\Omega u^p(\cdot,t_0) - \frac{\mu}{2}p\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\\
&\quad + (\varepsilon + k_p)p\chi^{p+1}\mu^{-p}\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega |\Delta v(\cdot,s)|^{p+1}ds\\
&\quad + \frac{2}{p+1}\left(c_4(\mu,\xi,p)p + c_5 p\sup_{t\in(0,T_{\max})}\|v(\cdot,t)\|^{p+1}_{L^1(\Omega)}\right)
\end{aligned}\tag{4.20}$$

for all $t \in (t_0, T_{\max})$. In order to estimate the third term therein, let us note that an application of Lemma 1.2.3 results in

$$\begin{aligned}&(\varepsilon + k_p)p\chi^{p+1}\mu^{-p}\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega |\Delta v(\cdot,s)|^{p+1}ds\\
&\le (\varepsilon + k_p)pC_{p+1}\chi^{p+1}\mu^{-p}\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\\
&\qquad + C_{p+1}(\varepsilon + k_p)p\chi^{p+1}\mu^{-p}e^{-(\frac{p+1}{2})(t-t_0)}\|v(\cdot,t_0)\|^{p+1}_{W^{2,p+1}(\Omega)}\\
&= (\varepsilon + k_p)pC_{p+1}\chi^{p+1}\mu^{-p}\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\\
&\qquad + C_{p+1}(\varepsilon + k_p)p\chi^{p+1}\mu^{-p}e^{-(\frac{p+1}{2})(t-t_0)}M^{p+1}
\end{aligned}\tag{4.21}$$

for all $t \in (t_0, T_{\max})$ and M as in (4.6). Combining (4.20) and (4.21), we finally arrive at

$$\begin{aligned}&\int_\Omega u^p(\cdot,t)\\
&\le e^{-(\frac{p+1}{2})(t-t_0)}\int_\Omega u^p(\cdot,t_0) - \mu p\left(\frac{1}{2} - (\varepsilon + k_p)C_{p+1}\left(\frac{\chi}{\mu}\right)^{p+1}\right)\int_{t_0}^t e^{-(\frac{p+1}{2})(t-s)}\int_\Omega u^{p+1}(\cdot,s)ds\\
&\quad + C_{p+1}(\varepsilon + k_p)p\chi^{p+1}\mu^{-p}e^{-(\frac{p+1}{2})(t-t_0)}M^{p+1}\\
&\qquad + \frac{2}{p+1}\left(c_4(\mu,\xi,p) + c_5\sup_{t\in(0,T_{\max})}\|v(\cdot,t)\|^{p+1}_{L^1(\Omega)}\right)
\end{aligned}\tag{4.22}$$

for all $t \in (t_0, T_{\max})$. We see from Lemma 4.2.4 and the choice of ε that

$$\int_\Omega u^p(\cdot, t) \leq C(\mu, \chi, \xi, p, M) \tag{4.23}$$

for all $t \in (t_0, T_{\max})$ upon an obvious choice of $C(\mu, \chi, \xi, p, M) > 0$. Thus the assertion is derived. □

Lemma 4.3.2. *Let (u_0, v_0, w_0) satisfy (4.4). Then there exist $\theta_0 > 0$ and $C > 0$ such that if $\chi, \xi > 0, \mu > 0$ are positive constants and satisfy $\frac{\chi}{\mu} < \theta_0$, then*

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} < C,$$
$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} < C \quad \textit{for all } t \in (t_0, T_{\max}).$$

Proof. Let $p_1 > N$ and $\theta_0 = \theta_{p_1}$ be defined as in Lemma 4.3.1. Since $\frac{\chi}{\mu} < \theta_0$, an application of Lemma 4.3.1 implies a constant $c_1 > 0$ such that $\|u(\cdot, t)\|_{L^{p_1}(\Omega)} \leq c_1$ for all $t \in (t_0, T_{\max})$. Let

$$c_2 := \int_0^\infty (1 + \sigma^{-\frac{1}{2} - \frac{N}{2p_1}}) e^{-\sigma} d\sigma.$$

Using the variation-of-constants formula for v, we note that a standard estimate for the Neumann semigroup provides $c_3 > 0$ such that

$$\begin{aligned}
\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} &\leq \|\nabla e^{t(\Delta-1)} v(\cdot, t_0)\|_{L^\infty(\Omega)} + \int_{t_0}^t \|\nabla e^{(t-s)(\Delta-1)} u(\cdot, s)\|_{L^\infty(\Omega)} \\
&\leq e^{-t} \|\nabla v(\cdot, t_0)\|_{L^\infty(\Omega)} + \int_{t_0}^t c_3 (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2p_1}}) e^{-(t-s)} \|u(\cdot, s)\|_{L^{p_1}(\Omega)} ds \\
&\leq M + c_1 c_3 \int_0^{t-t_0} (1 + \sigma^{-\frac{1}{2} - \frac{N}{2p_1}}) e^{-\sigma} d\sigma \\
&\leq M + c_1 c_2 c_3
\end{aligned}$$

for all $t \in (t_0, T_{\max})$. Similarly, we can find $c_4 > 0$ and $c_5 > 0$ such that

$$\begin{aligned}
\|v(\cdot, t)\|_{L^\infty(\Omega)} &\leq \|e^{t(\Delta-1)} v(\cdot, t_0)\|_{L^\infty(\Omega)} + \int_{t_0}^t \|e^{(t-s)(\Delta-1)} u(\cdot, s)\|_{L^\infty(\Omega)} ds \\
&\leq e^{-t} \|v(\cdot, t_0)\|_{L^\infty(\Omega)} + \int_{t_0}^t c_4 (1 + (t-s)^{-\frac{N}{2p_1}}) e^{-(t-s)} \|u(\cdot, s)\|_{L^{p_1}(\Omega)} ds \\
&\leq M + c_1 c_4 c_5
\end{aligned}$$

for all $t \in (t_0, T_{\max})$. Therefore, the proof is complete. □

4.4. Boundedness of u

In the last section, we have already gained L^p-estimate for u with $p > N$. Since the estimate of ∇w still depends on time, it is not convenient to apply the Neumann heat semigroup estimates to study the boundedness of u. Here we use the well-developed Moser iteration procedure to show that u is bounded in $L^\infty(\Omega)$.

Lemma 4.4.1. *Let (u_0, v_0, w_0) satisfy (4.4) and $\theta_0 > 0$ be defined as in Lemma 4.3.2. There exists $C > 0$ such that if $\chi, \xi > 0, \mu > 0$ are positive constants and satisfy $\frac{\chi}{\mu} < \theta_0$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \in (t_0, T_{\max}). \tag{4.24}$$

Proof. We first see Lemma 4.3.2 implies the existence of $c_1 > 0$ such that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} < c_1 \tag{4.25}$$

for all $t \in (t_0, T_{\max})$. Testing the first equation in (4.3) with u^{p-1} $(p > 1)$, using (4.9), (4.25) and Young's inequality, we can find constants $c_2, c_3 > 0$ such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2 \\
&= (p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v + (p-1)\xi\int_\Omega u^{p-1}\nabla u\cdot\nabla w + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1} - \mu\int_\Omega u^p w \\
&\le (p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v + c_2\xi\int_\Omega u^p + c_2\xi\int_\Omega u^p v + c_2\xi(p-1)\int_\Omega u^{p-1}|\nabla u| + \mu\int_\Omega u^p \\
&\le \frac{p-1}{4}\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)\chi^2\int_\Omega u^p|\nabla v|^2 + c_2\xi\int_\Omega u^p + c_1c_2\xi\int_\Omega u^p \\
&\qquad + \frac{p-1}{4}\int_\Omega u^{p-2}|\nabla u|^2 + c_2^2\xi^2 p\int_\Omega u^p + \mu\int_\Omega u^p \\
&\le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + c_3 p\int_\Omega u^p
\end{aligned}$$

for all $t \in (t_0, T_{\max})$, where c_3 is independent of p. An obvious rearrangement implies

$$\frac{d}{dt}\int_\Omega u^p + c_4\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le c_5 p^2\int_\Omega u^p \tag{4.26}$$

for all $t \in (t_0, T_{\max})$, where $c_4, c_5 > 0$ are independent of p.
Next, we use (4.26) to perform the classical Moser iteration procedure ([1]) to obtain the boundedness of u.
Let $p_k = 2^k$, $k \in \mathbb{N}$ and $M_k := \sup_{t\in(t_0,T_{\max})} \int_\Omega u^{p_k}(\cdot,t) < \infty$ for all $k \in \mathbb{N}$. Since $p_k \ge 1$, it is easy to find $c_6 > 0$ such that

$$\frac{d}{dt}\int_\Omega u^{p_k} + \int_\Omega u^{p_k} + c_4\int_\Omega |\nabla u^{\frac{p_k}{2}}|^2 \le c_5 {p_k}^2\int_\Omega u^{p_k} + \int_\Omega u^{p_k} \le c_6 {p_k}^2\int_\Omega u^{p_k} \tag{4.27}$$

for all $t \in (t_0, T_{\max})$ and $k \in \mathbb{N}$. By means of the Gagliardo-Nirenberg inequality, we see that

$$\int_\Omega u^{p_k} = \|u^{\frac{p_k}{2}}\|^2_{L^2(\Omega)} \le c_7\|\nabla u^{\frac{p_k}{2}}\|^{2a}_{L^2(\Omega)}\|u^{\frac{p_k}{2}}\|^{2(1-a)}_{L^1(\Omega)} + c_7\|u^{\frac{p_k}{2}}\|^2_{L^1(\Omega)} \text{ for all } k \in \mathbb{N},$$

where $a = \frac{\frac{N}{2}}{1+\frac{N}{2}} \in (0,1)$ and $c_7 > 0$ is independent of k. Young's inequality and the definition of p_k ensure that there are $c_8 > 0$ and $b > 0$ satisfying

$$\begin{aligned}
c_6 p_k^2\int_\Omega u^{p_k} &\le c_4\int_\Omega |\nabla u^{\frac{p_k}{2}}|^2 + c_8\left(p_k^2\right)^{\frac{1}{1-a}}\left(\int_\Omega u^{p_{k-1}}\right)^2 + c_6c_7p_k^2\left(\int_\Omega u^{p_{k-1}}\right)^2 \\
&\le c_4\int_\Omega |\nabla u^{\frac{p_k}{2}}|^2 + b^k M_{k-1}^2 \text{ for all } k \in \mathbb{N}.
\end{aligned} \tag{4.28}$$

Combining (4.27-4.28) we find that

$$\frac{d}{dt}\int_\Omega u^{p_k} + \int_\Omega u^{p_k} \le b^k M_{k-1}^2$$

for all $t \in (t_0, T_{\max})$ and for all $k \in \mathbb{N}$. The comparison theorem for the above ODE yields

$$M_k \le \max\{b^k M_{k-1}^2, \int_\Omega u^{p_k}(\cdot, t_0)\} \text{ for all } k \in \mathbb{N}.$$

If $b^k M_{k-1}^2 < \int_\Omega u^{p_k}(\cdot, t_0)$ is valid for infinitely many k, (4.24) is already derived. Otherwise, we can find a constant $h > b$ such that

$$M_k \le h^k M_{k-1}^2 \text{ for all } k \in \mathbb{N}.$$

Hence a direct induction entails

$$M_k \le h^{\sum_{j=0}^{j=k-1} 2^j (k-j)} M_0^{2^k}.$$

Taking 2^k-th root on both sides leads to the assertion. □

Now we are ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. First we see that the boundedness of u and v follow from Lemma 4.4.1, Lemma 4.3.2 and (4.6). Thereupon the assertion of Theorem 4.1.1 is immediately obtained from Lemma 4.2.1. □

5. Sharp decay estimates of bounded solutions in a bioconvection einviroment

5.1. Introduction

In this chapter, we consider nonnegative solutions of the boundary value problem

$$\begin{cases} u_t + U\cdot\nabla u &= \Delta u - \chi\nabla\cdot(u\nabla v) - \mu u^2, \qquad x\in\Omega,\ t>0, \\ v_t + U\cdot\nabla v &= \Delta v - v + u, \qquad x\in\Omega,\ t>0, \\ \nabla\cdot U &= 0, \qquad x\in\Omega,\ t>0, \\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0,\ U=0, & \qquad x\in\partial\Omega,\ t>0, \end{cases} \tag{5.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^N$ with smooth boundary, where $N\geq 1$, where $\chi>0$ and μ are positive parameters, and where $U:\Omega\times(0,\infty)\to\mathbb{R}^n$ is a prescribed solenoidal vector field. Systems of this type arise in the macroscopic modeling of chemotactic migration under the influence of a liquid environment by transport through a given fluid, and in presence of quadratic degradation such as appearing in logistic-type cell kinetics. Here we focus on situations in which cell proliferation, in logistic models represented by linear production terms, can either be neglected on the considered time scales, or is absent in principle. A prototypical example for the latter arises in the context of coral broadcast spawning processes ([20], [48]) during which eggs release a chemical signal, with concentration denoted by $v=v(x,t)$, that attracts sperms, where both eggs and sperms jointly consitute a population with density $u=u(x,t)$, and where the transporting incompressible ocean flow is represented through its velocity field $U=U(x,t)$.

Already in the fluid-free case when $U\equiv 0$ a variety of previous results indicates quite a substantial effect of the cross-diffusive mechanism in (5.1), going far beyond well-established knowledge on the ability of the classical Keller-Segel system obtained on letting $\mu=0$, that is, of

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{5.2}$$

to generate singularities in the sense of finite-time blow-up of some solutions in two- and higher-dimensional settings ([36], [109]). Indeed, also in situations when $\mu>0$ in

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \mu u^2, \\ v_t = \Delta v - v + u, \end{cases} \tag{5.3}$$

and related systems, the destabilizing action of cross-diffusion may still enforce quite a complex solution behavior in comparison to the respective scalar absorptive parabolic equation, as indicated by numerical experiments ([71]) and rigorously confirmed by results on spontaneous emergence of large population densities at intermediate time scales ([116]; cf. also [111] and [52] for similar findings on associated parabolic-elliptic simplifications). In fact, even the drastic phenomenon of finite-time blow-up has been shown to be suppressed by the presence of quadratic degradation only when either $N \leq 2$ ([68], [69]) or $N \geq 3$ and μ is suitably large ([106]; see also [96] for a precedent). The question how far such systems at all are globally solvable when $N \geq 3$ and $\mu > 0$ is small has only been partially been answered so far by a statement on global existence of weak solutions, possibly unbounded but at least in the case $N = 3$ eventually bounded and smooth and asymptotically decay in both components ([53]). Strong cross-diffusive effects become manifest also in an example of blow-up despite certain subquadratic but yet superlinear degradation terms in some appropriately high-dimensional chemotaxis systems ([107]).

In light of these premises, for the investigation of common large-scale qualitative features of solutions to (5.1) in general N-dimensional frameworks it seems adequate to explicitly resort to situations when solutions are globally regular. Upon a time shift if necessary this will in fact cover widely arbitrary solutions to (5.3) in all physically relevant cases $N \leq 3$, but this will furthermore also capture more complex frameworks in which the fluid evolution itself is unknown, affected e.g. by the cell population, and governed by appropriate equations from fluid mechanics (cf. [4] for corresponding modeling aspects), at least in situations when the respectively obtained chemotaxis-fluid system is globally solvable by suitably regular functions ([94], [95]). Accordingly, the purpose of this work consists in describing the large time behavior of arbitrary global bounded solutions to (5.1) in bounded domains for any $N \geq 1$, thus ignoring the question under which particular assumptions on supposedly prescribed initial data $(u_0, v_0) \equiv (u(\cdot,0), v(\cdot,0))$ such solutions exist. Hence assuming to be given a sufficiently smooth vector field U and a nontrivial global bounded classical solution (u, v) of (5.1), we will more precisely focus on deriving optimal estimates for the decay rate of $u(\cdot,t)$ with respect to the norms both in $L^\infty(\Omega)$ and in $L^1(\Omega)$, bearing in mind the particular biological relevance of the latter as representing the total mass of the considered population.

Previous work in this direction addresses the Cauchy problem in $\Omega = \mathbb{R}^2$ for a simplified parabolic-ellitpic variant of (5.1) which can be rewritten in form of the scalar nonlocal parabolic equation

$$u_t + U \cdot \nabla u = \Delta u + \chi \nabla \cdot (u \nabla (\Delta)^{-1} u) - \mu u^q \tag{5.4}$$

with the additional parameter $q \geq 2$. For this problem with initial condition $u(\cdot,0) = u_0 \in L^1(\mathbb{R}^2)$, in the case $q > 2$ any sufficiently regular nonnegative global solution u is known to satisfy $\int_{\mathbb{R}^2} u(\cdot,t) \to m_\infty(\chi, u_0, U)$ as $t \to \infty$ with some $m_\infty(\chi, u_0, U) > 0$ fulfilling $m_\infty(\chi, u_0, U) \to 0$ as $\chi \to \infty$ ([48]). In the critical case $q = 2$, an influence of chemotaxis on the evolution of the total mass functional, which then decays to zero in both cases $\chi > 0$ and $\chi = 0$, has been shown to exist but to be of more subtle character, mainly relevant on finite time intervals ([47]).

Main results. It will turn out that in the presently considered framework of bounded domains, unlike in the latter Cauchy problem the solution behavior in (5.1) is essentially unaffected by chemotaxis at least on large time scales. Indeed, throughout the sequel assuming for simplicity that

$$\begin{aligned} & U \in C^{1,0}(\overline{\Omega} \times [0,\infty); \mathbb{R}^N) \cap L^\infty(\Omega \times (0,\infty); \mathbb{R}^N) \\ & \text{is such that} \quad \nabla \cdot U \equiv 0 \quad \text{in } \Omega \times (0,\infty), \text{ and } \quad U \equiv 0 \text{ on } \partial\Omega \times (0,\infty), \end{aligned} \tag{5.5}$$

we shall see that for any given nontrivial and sufficiently regular bounded solution of (5.1), with respect to the norms in either $X := L^1(\Omega)$ or in $L^\infty(\Omega)$ the quantity $\|u(\cdot,t)\|_X$ can be estimated from above and below by positive multiples, possibly depending on the solution e.g. through its norm in $L^\infty(\Omega \times (0,\infty))$, of $\frac{1}{t+1}$. More precisely, our main results read as follows.

Theorem 5.1.1. *Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, assume that $\mu > 0$ and that U satisfies (5.5), and suppose that $(u,v) \in (C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2$ is a classical solution of (5.1) for which both u and v are nonnegative, and which is bounded in the sense that u belongs to $L^\infty(\Omega \times (0,\infty))$.*

i) There exists $C_1 > 0$ with the property that

$$\frac{1}{|\Omega|}\|u(\cdot,t)\|_{L^1(\Omega)} \leq \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{C_1}{t+1} \qquad \text{for all } t > 0. \tag{5.6}$$

ii) If furthermore $u \not\equiv 0$, then one can find $C_2 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \geq \frac{1}{|\Omega|} \cdot \|u(\cdot,t)\|_{L^1(\Omega)} \geq \frac{C_2}{t+1} \qquad \text{for all } t > 0. \tag{5.7}$$

We remark that we do not pursue here the question how the constants appearing in the above statements depend on χ and μ, nor on the function U, thus leaving open whether chemotactic cross-diffusion possibly influences a fine structure in the large time asymptotics of solutions.
In corresponding chemotaxis-fluid systems in which the fluid evolution itself is affected by the presence of the other quantities e.g. through buoyant forces, the above results can directly be applied to solutions which are a priori known to enjoy the above regularity and boundedness properties; for two- and three-dimensional examples of situations when the latter in fact is guaranteed for all reasonably regular initial data we refer to [95] and [94]. However, Theorem 5.1.1 is actually more general by considering widely arbitrary fluid fields not necessarily receiving any feedback from the taxis components.

The main idea underlying our approach is directly motivated by the result to be finally achieved: The goal pursued in our analysis consists in showing appropriate negligibility of the cross-diffusive action in (5.1) in comparison to the further mechanisms therein. After establishing a preliminary but fundamental decay information on solutions in $L^1(\Omega) \times L^1(\Omega)$ in Section 5.2, this will be accomplished in Section 5.3 on the basis of the latter by means of a series of arguments relying on the smoothing action of the heat semigroup in the second equation in (5.1). A first exploitation of the outcome thereby achieved will yield the estimate from Theorem 5.1.1 i) in Section 5.4, whereafter a second application thereof will show in Section 5.5 that also in the inequality

$$\frac{d}{dt}\int_\Omega \ln u \geq -\frac{\chi^2}{4}\int_\Omega |\nabla v|^2 - \mu\int_\Omega u, \qquad t > 0,$$

constituting the key step in or proof of Theorem 5.1.1 ii), the summand originating from the taxis term in (5.1) decays suitably fast so as to become asymptotically irrelevant.

5.2. Upper decay estimates for u and v in $L^1(\Omega)$

The following basic one-sided decay estimates for the spatial L^1 norms of both solution components can be gained in quite an elementary way, and similar observations have previously been made in [94, Lemma 5.1] already. Since they will be fundamental to our subsequent analysis, and since in particular they already underline the difference between the case of bounded Ω and the case $\Omega = \mathbb{R}^N$ in a quantitative manner, we include a short proof here.

Lemma 5.2.1. *Let (u,v) be a nonnegative global classical solution of (5.1). Then*

$$\int_\Omega u(\cdot,t) \leq \frac{|\Omega|}{\mu} \cdot \frac{1}{t+\gamma} \qquad \text{for all } t > 0 \tag{5.8}$$

and

$$\int_\Omega v(\cdot,t) \le \frac{K}{t+2} \qquad \textit{for all } t>0, \tag{5.9}$$

where

$$\gamma := \frac{|\Omega|}{\mu\cdot\int_\Omega u(\cdot,0)} \tag{5.10}$$

and

$$K := \max\Big\{2\int_\Omega v(\cdot,0)\,,\, 4\int_\Omega u(\cdot,0)\,,\, \frac{2|\Omega|}{\mu}\Big\}. \tag{5.11}$$

Proof. We only need to consider the case when $u(\cdot,0)\not\equiv 0$, in which according to (5.1) and the Cauchy-Schwarz inequality,

$$\frac{d}{dt}\int_\Omega u = -\mu\int_\Omega u^2 \le -\frac{\mu}{|\Omega|}\Big\{\int_\Omega u\Big\}^2 \qquad \text{for all } t>0,$$

which on integration readily implies (5.8) with γ as in (5.10).
Since from (5.1) we moreover see that

$$\frac{d}{dt}\int_\Omega v = -\int_\Omega v + \int_\Omega u \qquad \text{for all } t>0,$$

we therefore obtain that

$$\frac{d}{dt}\int_\Omega v \le -\int_\Omega v + \frac{|\Omega|}{\mu(t+\gamma)} \qquad \text{for all } t>0. \tag{5.12}$$

Now with K as given by (5.11), $\overline{y}(t) := \frac{K}{t+2}$, $t\ge 0$, satisfies $\overline{y}(0)=\frac{K}{2}\ge\int_\Omega v(\cdot,0)$ by (5.11) and therefore

$$\begin{aligned}
\overline{y}'(t)+\overline{y}(t)-\frac{|\Omega|}{\mu(t+\gamma)} &= -\frac{K}{(t+2)^2}+\frac{K}{t+2}-\frac{|\Omega|}{\mu(t+\gamma)}\\
&= \frac{K}{t+2}\cdot\Big\{1-\frac{1}{t+2}-\frac{|\Omega|}{K\mu}\cdot\frac{t+2}{t+\gamma}\Big\}\\
&\ge \frac{K}{t+2}\cdot\Big\{1-\frac{1}{2}-\frac{|\Omega|}{K\mu}\cdot\max\Big\{\frac{2}{\gamma},1\Big\}\Big\}\\
&= \frac{K}{2(t+2)}\cdot\Big\{1-\frac{1}{K}\cdot\max\Big\{4\int_\Omega u(\cdot,0)\,,\,\frac{2|\Omega|}{\mu}\Big\}\Big\}\\
&\ge 0 \qquad \text{for all } t>0
\end{aligned}$$

due to (5.10) and the second and third restrictions contained in (5.11). By an ODE comparison, we thus conclude from (5.12) that $\int_\Omega v(\cdot,t)\le\overline{y}(t)$ for all $t>0$, and that hence indeed (5.9) holds. □

5.3. Boundedness and decay properties of ∇v

A crucial step toward both parts of Theorem 5.1.1 will consist in adequately identifying the cross-diffusive term in (5.1) as asymptotically negligible relative to the diffusive action therein, which basically amounts to deriving appropriate quantitative bounds for the chemotactic gradient ∇v. This will be achieved in this section by firstly making use of the L^1 decay poperty of u

from Lemma 5.2.1 in order to obtain decay of ∇v at an apparently optimal rate but in a yet unfavorable topology, and by secondly investing our assumption on boundedness of u to establish boundedness of v in certain higher norms but without any decay information. Interpolating these two extremal results will finally yield a decay result for ∇v in arbitrary L^p spaces at a rate which is probably far from optimal but sufficient for our purposes.

For what follows, let us recall that for $p \in (1,\infty)$, the realization $A = A_p$ of $-\Delta + 1$ under homogeneous Neumann boundary conditions, that is, the operator defined by letting $A_p\varphi := -\Delta\varphi + \varphi$ for $\varphi \in D(A_p) := \{\varphi \in W^{2,p}(\Omega) \,|\nabla\varphi \cdot \nu = 0 \text{ on } \partial\Omega\}$, is sectorial in the space $L^p(\Omega)$, with its spectrum contained in the half-line $[1,\infty)$. Accordingly, A possesses closed and densely defined fractional powers A^β for all $\beta \in \mathbb{R}$, and A^β is bounded whenever $\beta < 0$ ([35, Theorem 1.4.2]).

Now the space $L^1(\Omega)$ is continuously embedded into suitable among the correspondingly obtained spaces $D(A^{-\beta})$, an explicit definition of which is actually not necessary and thus omitted here, keeping the focus rather on an associated embedding inequality:

Lemma 5.3.1. *Let $p > 1$ and $\beta > \frac{N(p-1)}{2p}$. Then there exists $C > 0$ such that*

$$\|A^{-\beta}\varphi\|_{L^p(\Omega)} \le C\|\varphi\|_{L^1(\Omega)} \qquad \textit{for all } \varphi \in L^1(\Omega). \tag{5.13}$$

Proof. Since $\beta > \frac{N(p-1)}{2p}$ implies that $p' := \frac{p}{p-1}$ satisfies $2\beta - \frac{N}{p'} > 0$, it follows from known embedding results ([35, Theorem 1.6.1]) that $D(A^\beta_{p'}) \hookrightarrow L^\infty(\Omega)$, whence there exists $c_1 > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)} \le c_1\|A^\beta\phi\|_{L^{p'}(\Omega)} \qquad \text{for all } \phi \in D(A^\beta_{p'}). \tag{5.14}$$

Thus, given any $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(\Omega)$, using the self-adjointness of $A^{-\beta}$ in $L^2(\Omega)$ we can estimate

$$\left|\int_\Omega A^{-\beta}\varphi\cdot\psi\right| = \left|\int_\Omega \varphi\cdot A^{-\beta}\psi\right| \le \|\varphi\|_{L^1(\Omega)}\|A^{-\beta}\psi\|_{L^\infty(\Omega)} \le c_1\|\varphi\|_{L^1(\Omega)}\|\psi\|_{L^{p'}(\Omega)}.$$

Therefore,

$$\|A^{-\beta}\varphi\|_{L^p(\Omega)} = \sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\psi\|_{L^{p'}(\Omega)}\le 1}} \left|\int_\Omega A^{-\beta}\varphi\cdot\psi\right| \le c_1\|\varphi\|_{L^1(\Omega)},$$

as claimed. □

By appropriately making use of the latter in the course of an argument based on a variation-of-constants representation of v, we see that with respect to the norm in $L^p(\Omega)$ for suitably small $p > 1$, ∇v inherits the decay rate of the mass functional $\int_\Omega u$ from Lemma 5.2.1.

Lemma 5.3.2. *Let (u,v) be a nonnegative global classical solution of (5.1). Then for all $p \in (1, \frac{N}{N-1})$ one can find $C(p) > 0$ such that*

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{C(p)}{t} \qquad \textit{for all } t \ge 2. \tag{5.15}$$

Proof. Since $\frac{N}{N-2(1-\alpha)} \to \frac{N}{N-1} > p$ as $\alpha \searrow \frac{1}{2}$, it is possible to fix $\alpha \in (\frac{1}{2}, 1)$ such that $p < \frac{N}{N-2(1-\alpha)}$, which means that

$$\alpha + \frac{N}{2}\Big(1 - \frac{1}{p}\Big) < 1. \tag{5.16}$$

We thereupon choose an arbitrary $\epsilon \in (0, \alpha - \frac{1}{2})$ and pick $\beta > \frac{N(p-1)}{2p}$, so that since $D(A_p^{\frac{1}{2}+\epsilon}) \hookrightarrow W^{1,p}(\Omega)$ ([35, Theorem 1.6.1]), employing a well-known interpolation argument ([28, Theorem 14.1]) we can find $c_1 > 0$ and $c_2 > 0$ such that

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le c_1 \|A^{\frac{1}{2}+\epsilon} v(\cdot,t)\|_{L^p(\Omega)} \le c_2 \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}^a \|A^{-\beta} v(\cdot,t)\|_{L^p(\Omega)}^{1-a} \tag{5.17}$$

for all $t > 0$, where

$$a := \frac{\frac{1}{2} + \epsilon + \beta}{\alpha + \beta} \in (0,1).$$

Here the fact that $\beta > \frac{N(p-1)}{2p}$ enables us to invoke Lemma 5.3.1 and thereafter apply Lemma 5.2.1 to find $c_3 > 0$ and $c_4 > 0$ such that

$$\|A^{-\beta} v(\cdot,t)\|_{L^p(\Omega)} \le c_3 \|v(\cdot,t)\|_{L^1(\Omega)} \le \frac{c_3 c_4}{t} \qquad \text{for all } t > 0. \tag{5.18}$$

Now in order to derive (5.15), by means of a variation-of-constants representation of v we write

$$v(\cdot,t) = e^{-A} v(\cdot,t-1) + \int_{t-1}^t e^{-(t-s)A} u(\cdot,s) ds - \int_{t-1}^t e^{-(t-s)A} U(\cdot,s) \cdot \nabla v(\cdot,s) ds, \qquad t \ge 1,$$

and apply A^α to both sides to see that

$$\begin{aligned}
\left\|A^\alpha v(\cdot,t)\right\|_{L^p(\Omega)} \le{} & \left\|A^\alpha e^{-A} v(\cdot,t-1)\right\|_{L^p(\Omega)} \\
& + \int_{t-1}^t \left\|A^\alpha e^{-(t-s)A} u(\cdot,s)\right\|_{L^p(\Omega)} ds \\
& + \int_{t-1}^t \left\|A^\alpha e^{-(t-s)A} U(\cdot,s) \cdot \nabla v(\cdot,s)\right\|_{L^p(\Omega)} ds \ \text{ for all } t \ge 1.
\end{aligned} \tag{5.19}$$

Here according to known smoothing properties of $(e^{-\tau A})_{\tau \ge 0}$ and Lemma 5.2.1, there exist $c_5 > 0$ and $c_6 > 0$ fulfilling

$$\left\|A^\alpha e^{-A} v(\cdot,t-1)\right\|_{L^p(\Omega)} \le c_5 \|v(\cdot,t-1)\|_{L^1(\Omega)} \le \frac{c_6}{t-1} \qquad \text{for all } t \ge 2, \tag{5.20}$$

and making use of Lemma 5.2.1 and (5.16), once more by a standard semigroup estimate we can find $c_7 > 0$ and $c_8 > 0$ such that

$$\begin{aligned}
\int_{t-1}^t \left\|A^\alpha e^{-(t-s)A} u(\cdot,s)\right\|_{L^p(\Omega)} ds &\le c_7 \int_{t-1}^t (t-s)^{-\alpha - \frac{N}{2}(1-\frac{1}{p})} \|u(\cdot,s)\|_{L^1(\Omega)} ds \\
&\le c_8 \int_{t-1}^t (t-s)^{-\alpha - \frac{N}{2}(1-\frac{1}{p})} \cdot \frac{1}{s} ds \\
&\le c_8 \cdot \frac{1}{t-1} \cdot \int_{t-1}^t (t-s)^{-\alpha - \frac{N}{2}(1-\frac{1}{p})} ds \\
&= \frac{c_8}{1 - \alpha - \frac{N}{2}(1-\frac{1}{p})} \cdot \frac{1}{t-1} \ \text{ for all } t \ge 2.
\end{aligned} \tag{5.21}$$

To finally treat the last summand in (5.19) appropriately, let us introduce the numbers

$$M(T) := \sup_{t \in (1,T)} \Big\{ t \cdot \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \Big\}, \qquad T > 2,$$

which are all finite due to our overall assumption that $v \in C^{2,1}(\overline{\Omega} \times (0,\infty))$. In terms of $M(T)$, by boundedness of U, (5.17) and (5.18), with some $c_9 > 0$, $c_{10} > 0$ and $c_{11} > 0$ the integral in question can be estimated according to

$$\begin{aligned}
\int_{t-1}^{t} \left\| A^\alpha e^{-(t-s)A} U(\cdot,s) \cdot \nabla v(\cdot,s) \right\|_{L^p(\Omega)} ds &\le c_9 \int_{t-1}^{t} (t-s)^{-\alpha} \left\| U(\cdot,s) \cdot \nabla v(\cdot,s) \right\|_{L^p(\Omega)} ds \\
&\le c_{10} \int_{t-1}^{t} (t-s)^{-\alpha} \|\nabla v(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le c_2 c_{10} \int_{t-1}^{t} (t-s)^{-\alpha} \cdot \left\{ \frac{M(T)}{s} \right\}^a \cdot \left\{ \frac{c_3 c_4}{s} \right\}^{1-a} ds \\
&\le c_{11} M^a(T) \int_{t-1}^{t} (t-s)^{-\alpha} \cdot \frac{1}{s} ds \\
&\le c_{11} M^a(T) \cdot \frac{1}{t-1} \cdot \int_{t-1}^{t} (t-s)^{-\alpha} ds \\
&= \frac{c_{11}}{1-\alpha} M^a(T) \cdot \frac{1}{t-1} \qquad \text{for all } t \in [2,T].
\end{aligned}$$

Combined with (5.19)-(5.21), in view of the fact that $\frac{1}{t-1} \le \frac{2}{t}$ for all $t \ge 2$ this shows that there exists $c_{12} > 0$ such that for each $T > 2$,

$$t \cdot \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \le c_{12} + c_{12} M^a(T) \qquad \text{for all } t \in [2,T],$$

and that hence with the number

$$c_{13} := \max \left\{ c_{12} \, , \, \sup_{t \in (1,2)} \left\{ t \cdot \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \right\} \right\},$$

finite again by the inclusion $v \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ and the fact that $\alpha < 1$, we have

$$M(T) \le c_{13} + c_{13} M^a(T) \qquad \text{for all } T > 2.$$

As $a < 1$, by an elementary argument this implies that

$$M(T) \le c_{14} := \max \left\{ 1 \, , \, (2c_{13})^{\frac{1}{1-a}} \right\} \qquad \text{for all } T > 2$$

and thereby proves (5.15), because e.g. once more by (5.17) and (5.18) this yields the inequality

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le c_2 \cdot \left\{ \frac{c_{14}}{t} \right\}^a \cdot \left\{ \frac{c_3 c_4}{t} \right\}^{1-a}$$

for arbitrary $t \ge 1$. □

We next modify the above argument but make use of different ingredients, in particular of the boundedness of u, to derive the following higher-order boundedness property of v.

Lemma 5.3.3. *Let (u,v) be a nonnegative global classical solution of (5.1) with the property that u is bounded in $\Omega \times (0,\infty)$. Then for all $p > 1$ and each $\alpha \in (\frac{1}{2},1)$ there exists $C(p,\alpha) > 0$ such that*

$$\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \le C(p,\alpha) \qquad \textit{for all } t \ge 1. \tag{5.22}$$

Proof. Following a variant of the strategy pursued in Lemma 5.3.2, we let

$$M(T) := \sup_{t\in(1,T)} \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}, \qquad T>2,$$

and note that since $\alpha<1$, the inclusion $v\in C^{2,1}(\overline{\Omega}\times(0,\infty))$ again warrants that $M(T)<\infty$ for all $T>2$.
To prepare an adequate estimation of $M(T)$ on the basis of variation-of-constants representation associated with the second equation in (5.1), we once more invoke standard smoothing estimates for $(e^{-\tau A})_{\tau\ge 0}$ to find $c_1>0$ and $c_2>0$ such that

$$\left\|A^\alpha e^{-A} v(\cdot,t-1)\right\|_{L^p(\Omega)} \le c_1\|v(\cdot,t-1)\|_{L^1(\Omega)} \le c_2 \qquad \text{for all } t\ge 1, \tag{5.23}$$

for Lemma 5.2.1 in particular warrants that $(v(\cdot,t))_{t\ge0}$ is bounded in $L^1(\Omega)$. Next, as u is assumed to be bounded in $\Omega\times(0,\infty)$, there exist $c_3>0$ and $c_4>0$ such that

$$\begin{aligned}
\int_{t-1}^t \left\|A^\alpha e^{-(t-s)A}u(\cdot,s)\right\|_{L^p(\Omega)} ds &\le c_3\int_{t-1}^t (t-s)^{-\alpha}\|u(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le c_3c_4\int_{t-1}^t (t-s)^{-\alpha}ds \\
&= \frac{c_3c_4}{1-\alpha} \qquad \text{for all } t\ge1,
\end{aligned} \tag{5.24}$$

because $\alpha<1$. Moreover, once more fixing any $\epsilon\in(0,\alpha-\frac12)$ and $\beta>\frac{N(p-1)}{2p}$ we may apply known embedding and interpolation estimates along with Lemma 5.3.1 to gain positive constants c_5,c_6,c_7,c_8 and c_9 such that with $a:=\frac{\frac12+\epsilon+\beta}{\alpha+\beta}\in(0,1)$ we have

$$\begin{aligned}
\int_{t-1}^t \|A^\alpha e^{-(t-s)A}U\cdot\nabla v(\cdot,s)\|_{L^p(\Omega)}ds &\le c_5\int_{t-1}^t (t-s)^{-\alpha}\|U(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla v(\cdot,t)\|_{L^p(\Omega)}ds \\
&\le c_6\int_{t-1}^t (t-s)^{-\alpha}\|\nabla v(\cdot,t)\|_{L^p(\Omega)}ds \\
&\le c_7\int_{t-1}^t (t-s)^{-\alpha}\|A^{\frac12+\epsilon}v(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le c_8\int_{t-1}^t (t-s)^{-\alpha}\|A^\alpha v(\cdot,s)\|^a_{L^p(\Omega)}\|A^{-\beta}v(\cdot,s)\|^{1-a}_{L^p(\Omega)}ds \\
&\le c_9\int_{t-1}^t (t-s)^{-\alpha}M^a(T)\|v(\cdot,s)\|^{1-a}_{L^1(\Omega)}ds \\
&\le c_{10}M^a(T)\int_{t-1}^t (t-s)^{-\alpha}ds \\
&\le \frac{c_{10}}{1-a}M^a(T) \qquad \text{for all } t\in[1,T],
\end{aligned} \tag{5.25}$$

again due to the fact that v belongs to $L^\infty((0,\infty);L^1(\Omega))$ by Lemma 5.2.1.

Now using (5.23)-(5.25), we can estimate

$$\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \le \left\|A^\alpha e^{-A}v(\cdot,t-1)\right\|_{L^p(\Omega)}$$

$$
\begin{aligned}
&+ \int_{t-1}^{t} \Big\| A^\alpha e^{-(t-s)A} u(\cdot,s) \Big\|_{L^p(\Omega)} ds \\
&+ \int_{t-1}^{t} \Big\| A^\alpha e^{-(t-s)A} U(\cdot,s) \cdot \nabla v(\cdot,s) \Big\|_{L^p(\Omega)} ds \\
&\le c_2 + \frac{c_3 c_4}{1-\alpha} + \frac{c_{10}}{1-a} M^a(T) \qquad \text{for all } t \in [2,T],
\end{aligned}
$$

so that with the evidently finite constant

$$
c_{11} := \max\left\{ c_2 + \frac{c_3 c_4}{1-\alpha} \, , \, \frac{c_{10}}{1-\alpha} \, , \, \sup_{t\in(1,2)} \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \right\}
$$

we have

$$
M(T) \le c_{11} + c_{11} M^a(T) \qquad \text{for all } T > 1
$$

and therefore

$$
M(T) \le \max\left\{ 1 \, , \, (2c_{11})^{\frac{1}{1-a}} \right\} \qquad \text{for all } T > 1,
$$

which proves the lemma. □

A straightforward interpolation shows that the above two lemmata imply decay of ∇v in Lebesgue spaces with high summability powers, but at rates slower than that in Lemma 5.3.2. The following statement on this will be applied to some large value of p and $\kappa := 0$ in proving the upper estimate claimed in Theorem 5.1.1 i), and to $p := 2$ with some $\kappa > \frac{1}{2}$ in Corollary 5.5.1 preparing the proof of the lower bound for $\int_\Omega u$ in Theorem 5.1.1 ii).

Lemma 5.3.4. *Let (u,v) be a nonnegative global classical solution of (5.1) such that u is bounded, and let $p > 1$. Then for all $\kappa < \min\{1, \frac{N}{(N-1)p}\}$ there exists $C(p,\kappa) > 0$ such that*

$$
\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{C(p,\kappa)}{t^\kappa} \qquad \textit{for all } t \ge 2. \tag{5.26}
$$

Proof. If $p < \frac{N}{N-1}$, the claim immediately results from Lemma 5.3.2. In the case $p \ge \frac{N}{N-1}$, our assumption ensures that $\kappa < \frac{N}{(N-1)p}$, so that we can fix $r \in [1, \frac{N}{N-1})$ such that still $\kappa < \frac{r}{p}$, whence writing

$$
q := \frac{(1-\kappa)pr}{r - p\kappa},
$$

we can easily verify that $q > p > r$, and that

$$
\frac{\frac{1}{r} - \frac{1}{p}}{\frac{1}{r} - \frac{1}{q}} = 1 - \kappa.
$$

Therefore, the Hölder inequality says that

$$
\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \|\nabla v(\cdot,t)\|_{L^q(\Omega)}^{1-\kappa} \|\nabla v(\cdot,t)\|_{L^r(\Omega)}^{\kappa} \qquad \text{for all } t > 0, \tag{5.27}
$$

where picking any $\alpha \in (\frac{1}{2}, 1)$ we infer from the continuity of the embedding $D(A_q^\alpha) \hookrightarrow W^{1,q}(\Omega)$ ([35]) and from Lemma 5.3.3 that

$$
\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1 \|A^\alpha v(\cdot,t)\|_{L^q(\Omega)} \le c_2 \qquad \text{for all } t \ge 2
$$

with some $c_1 > 0$ and $c_2 > 0$. As moreover the inequality $r < \frac{N}{N-1}$ along with Lemma 5.3.2 yields $c_3 > 0$ fulfilling

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le \frac{c_3}{t} \qquad \text{for all } t \ge 2,$$

from (5.27) we readily derive (5.26). □

5.4. Upper bound for u in $L^\infty(\Omega)$. Proof of Theorem 5.1.1 i)

On the basis of a Duhamel formula now associated with the first equation in (5.1), knowing that cross-diffusive gradient ∇v is bounded in $L^\infty((0,\infty);L^p(\Omega))$ for any finite $p > 1$ we can then turn the L^1 decay information on u from Lemma 5.2.1 into a corresponding estimate in $L^\infty(\Omega)$.

Prooof Theorem 5.1.1 i). We fix an arbitrary $p > N$ and recall that then by standard regularization properties of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge0}$ on Ω ([105]) one can pick $c_1 > 0$ and $c_2 > 0$ such that for all $\tau \in (0,1)$ we have

$$\|e^{\tau\Delta}\varphi\|_{L^\infty(\Omega)} \le c_1\tau^{-\frac{N}{2}}\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in L^1(\Omega) \tag{5.28}$$

and

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_2\tau^{-\frac{1}{2}-\frac{N}{2p}}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \ \varphi \in C^1(\overline{\Omega};\mathbb{R}^n) \text{ such that } \varphi\cdot\nu = 0 \text{ on } \partial\Omega. \tag{5.29}$$

Now in order to estimate the numbers

$$M(T) := \sup_{t\in(0,T)}\Big\{(t+1)\cdot\|u(\cdot,t)\|_{L^\infty(\Omega)}\Big\}, \qquad T > 2,$$

we use that $\nabla\cdot U \equiv 0$ in representing $u(\cdot,t)$ according to

$$\begin{aligned} u(\cdot,t) &= e^{\Delta}u(\cdot,t-1) - \chi\int_{t-1}^t e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)ds \\ &\quad - \mu\int_{t-1}^t e^{(t-s)\Delta}u^2(\cdot,s)ds - \int_{t-1}^t e^{(t-s)\Delta}\nabla\cdot\Big(U(\cdot,s)u(\cdot,s)\Big)ds \qquad \text{for all } t \ge 1. \end{aligned}$$

Since $e^{(t-s)\Delta}u^2(\cdot,s)$ is nonnegative in Ω for all $t > 0$ and $s \in (0,t)$ due to the maximum principle, by nonnegativity of u we therefore see that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{\Delta}u(\cdot,t-1)\|_{L^\infty(\Omega)} \\ &\quad + \chi\int_{t-1}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \\ &\quad + \int_{t-1}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(U(\cdot,s)u(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \qquad \text{for all } t \ge 1, \end{aligned} \tag{5.30}$$

where combining (5.28) with Lemma 5.2.1 we can find $c_3 > 0$ such that

$$\|e^{\Delta}u(\cdot,t-1)\|_{L^\infty(\Omega)} \le c_1\|u(\cdot,t-1)\|_{L^1(\Omega)} \le \frac{c_3}{t-1} \le \frac{2c_3}{t} \qquad \text{for all } t \ge 2. \tag{5.31}$$

To relate the two rightmost integrals in (5.30) to $M(T)$, we first invoke (5.29) to obtain

$$\chi\int_{t-1}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \le c_2\chi\int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{N}{2p}}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)}ds,$$

for all $t \geq 1$ and then twice use the Hölder inequality to infer that again due to Lemma 5.2.1, and as a consequence of the boundedness of ∇v in $\Omega \times (1,2)$ and Lemma 5.3.4 when applied to $\kappa := 0$, with some $c_4 > 0$ and $c_5 > 0$ and $a := 1 - \frac{1}{2p}$ we have

$$\begin{aligned}
\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)} &\leq \|u(\cdot,s)\|_{L^{2p}(\Omega)} \|\nabla v(\cdot,s)\|_{L^{2p}(\Omega)} \\
&\leq \|u(\cdot,s)\|^a_{L^\infty(\Omega)} \|u(\cdot,s)\|^{1-a}_{L^1(\Omega)} \|\nabla v(\cdot,s)\|_{L^{2p}(\Omega)} \\
&\leq \left\{ \frac{M(T)}{s+1} \right\}^a \cdot \left\{ \frac{c_4}{s+1} \right\}^{1-a} \cdot c_5 \\
&= c_4^{1-a} c_5 M^a(T) \cdot \frac{1}{s+1} \qquad \text{for all } s \in (1,T)
\end{aligned}$$

and hence

$$\begin{aligned}
&\chi \int_{t-1}^t \left\| e^{(t-s)\Delta} \nabla \cdot \Big(u(\cdot,s)\nabla v(\cdot,s) \Big) \right\|_{L^\infty(\Omega)} ds \\
&\leq c_2 c_4^{1-a} c_5 \chi M^a(T) \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{N}{2p}} \cdot \frac{1}{s+1} ds \\
&\leq c_2 c_4^{1-a} c_5 \chi M^a(T) \cdot \frac{1}{t} \cdot \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{N}{2p}} ds \\
&= \frac{c_2 c_4^{1-a} c_5 \chi}{\frac{1}{2} - \frac{N}{2p}} M^a(T) \cdot \frac{1}{t} \qquad \text{for all } t \in [2,T], \qquad (5.32)
\end{aligned}$$

because $p > N$.

Likewise, combining (5.29) with the boundedness of U we obtain $c_6 > 0$ such that

$$\begin{aligned}
\int_{t-1}^t \left\| e^{(t-s)\Delta} \nabla \cdot \Big(U(\cdot,s) u(\cdot,s) \Big) \right\|_{L^\infty(\Omega)} ds &\leq c_2 \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{N}{2p}} \|U(\cdot,s)u(\cdot,s)\|_{L^p(\Omega)} ds \\
&\leq c_6 \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{N}{2p}} \|u(\cdot,s)\|_{L^p(\Omega)} ds, \qquad (5.33)
\end{aligned}$$

for all $t \geq 1$, where again by the Hölder inequality and Lemma 5.2.1, there exists $c_7 > 0$ such that

$$\begin{aligned}
\|u(\cdot,s)\|_{L^p(\Omega)} &\leq \|u(\cdot,s)\|^b_{L^\infty(\Omega)} \|u(\cdot,s)\|^{1-b}_{L^1(\Omega)} \\
&\leq \left\{ \frac{M(T)}{s+1} \right\}^b \cdot \left\{ \frac{c_7}{s+1} \right\}^{1-b} \\
&= c_7^{1-b} M^b(T) \cdot \frac{1}{s+1} \qquad \text{for all } s \in (1,T)
\end{aligned}$$

with $b := 1 - \frac{1}{p}$. Therefore, (5.33) implies that

$$\begin{aligned}
\int_{t-1}^t \left\| e^{(t-s)\Delta} \nabla \cdot \Big(U(\cdot,s) u(\cdot,s) \Big) \right\|_{L^\infty(\Omega)} ds &\leq c_6 c_7^{1-b} M^b(T) \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{N}{2p}} \cdot \frac{1}{s+1} ds \\
&\leq c_6 c_7^{1-b} M^b(T) \cdot \frac{1}{t} \cdot \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{N}{2p}} ds \\
&= \frac{c_6 c_7^{1-b}}{\frac{1}{2} - \frac{N}{2p}} M^b(T) \cdot \frac{1}{t} \qquad \text{for all } t \in [2,T],
\end{aligned}$$

so that summarizing (5.30), (5.31) and (5.32) and using Young's inequality yields $c_8 > 0$ and $c_9 > 0$ such that

$$\begin{aligned} t \cdot \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le c_8 + c_8 M^a(T) + c_8 M^b(T) \\ &\le c_9 + c_9 M^a(T) \qquad \text{for all } t \in [2,T], \end{aligned}$$

because $b < a$. Since u is bounded in $\Omega \times (0,2)$, this entails that for some $c_{10} > 0$ we have

$$M(T) \le c_{10} + c_{10} M^a(T) \qquad \text{for all } T > 2$$

and thus

$$M(T) \le \max\left\{1\,,\,(2c_{10})^{\frac{1}{1-a}}\right\} \qquad \text{for all } T > 2,$$

which readily yields (5.6), for $T > 2$ was arbitrary. □

5.5. Lower bound for u in $L^1(\Omega)$. Proof of Theorem 5.1.1 ii)

In deriving the lower bound for $\int_\Omega u$ claimed in Theorem 5.1.1 ii), we will make essential use of the following consequence of Lemma 5.3.4 which strongly relies on the fact that the decay exponent κ appearing therein can be chosen favorably large at least in the particular case $p := 2$.

Corollary 5.5.1. *There exist $\lambda > 1$ and $C > 0$ such that*

$$\int_\Omega |\nabla v(\cdot,t)|^2 \le \frac{C}{t^\lambda} \qquad \textit{for all } t \ge 2. \tag{5.34}$$

Proof. This immediately results from an application of Lemma 5.3.4 to any $\kappa > \frac{1}{2}$ fulfilling $\kappa < \min\{1, \frac{N}{2(N-1)}\}$. □

Now the fact that the function on the right of (5.34) is integrable over $t \in (2,\infty)$ enables us to make sure that the taxis term in (5.1) becomes asymptotically negligible in the framework of the following testing procedure.

Lemma 5.5.2. *There exists $C > 0$ such that*

$$\int_\Omega \ln u(\cdot,t) \ge -|\Omega| \ln(t+\gamma) - C \qquad \textit{for all } t \ge 2, \tag{5.35}$$

where $\gamma > 0$ is the constant defined in (5.10).

Proof. As u is positive in $\overline{\Omega} \times (0,\infty)$ according to the strong maximum principle, we may test the first equation in (5.1) against $\frac{1}{u}$ so as to see that

$$\begin{aligned} \frac{d}{dt}\int_\Omega \ln u &= \int_\Omega \frac{1}{u} u_t \\ &= \int_\Omega \frac{1}{u}\Delta u - \chi \int_\Omega \frac{1}{u}\nabla\cdot(u\nabla v) - \mu \int_\Omega u \\ &= \int_\Omega \frac{|\nabla u|^2}{u^2} - \chi \int_\Omega \frac{\nabla u}{u}\cdot\nabla v - \mu\int_\Omega u \qquad \text{for all } t > 0, \end{aligned} \tag{5.36}$$

where by Young's inequality,

$$-\chi \int_\Omega \frac{\nabla u}{u} \cdot \nabla v \geq -\int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{\chi^2}{4} \int_\Omega |\nabla v|^2 \qquad \text{for all } t > 0. \tag{5.37}$$

Now from Lemma 5.2.1 we know that

$$\mu \int_\Omega u \leq \frac{|\Omega|}{t+\gamma} \qquad \text{for all } t > 0,$$

whereas Corollary 5.5.1 provides $\lambda > 1$ and $c_1 > 0$ satisfying

$$\frac{\chi^2}{4} \int_\Omega |\nabla v|^2 \leq \frac{c_1}{t^\lambda} \qquad \text{for all } t \geq 2.$$

From (5.36) and (5.37) we therefore obtain the inequality

$$\frac{d}{dt} \int_\Omega \ln u \geq -\frac{|\Omega|}{t+\gamma} - \frac{c_1}{t^\lambda} \qquad \text{for all } t \geq 2,$$

which on direct integration shows that

$$\begin{aligned} \int_\Omega \ln u(\cdot,t) - \int_\Omega \ln u(\cdot,2) &\geq -|\Omega| \int_2^t \frac{ds}{s+\gamma} - c_1 \int_2^t \frac{ds}{s^\lambda} \\ &= -|\Omega| \ln(t+\gamma) + |\Omega| \ln(2+\gamma) - \frac{c_1}{2^{\lambda-1}(\lambda-1)} + \frac{c_1}{(\lambda-1)t^{\lambda-1}} \\ &\geq -|\Omega| \ln(t+\gamma) - \frac{c_1}{2^{\lambda-1}(\lambda-1)} \qquad \text{for all } t \geq 2. \end{aligned}$$

As $\int_\Omega \ln u(\cdot,2)$ is finite by strict positivity of $u(\cdot,2)$ throughout $\overline{\Omega}$, this establishes (5.35). □

Thanks to the precise information on the multiple of $\ln(t+\gamma)$ appearing in (5.35), upon a simple application of Jensen's inequality we can turn this into a lower estimate for $\int_\Omega u$ involving exactly the desired decay rate.

Lemma 5.5.3. *There exists $C > 0$ such that*

$$\int_\Omega u(\cdot,t) \geq \frac{C}{t+1} \qquad \textit{for all } t > 0. \tag{5.38}$$

Proof. From Lemma 5.5.2 we know that with $\gamma > 0$ taken from (5.10), for some $c_1 > 0$ we have

$$\int_\Omega \ln u \geq -|\Omega| \ln(t+\gamma) - c_1 \qquad \text{for all } t \geq 2.$$

Since by Jensen's inequality we can estimate

$$\int_\Omega \ln u = |\Omega| \cdot \left\{ \frac{1}{|\Omega|} \int_\Omega \ln u \right\} \leq |\Omega| \cdot \ln \left\{ \frac{1}{|\Omega|} \int_\Omega u \right\} = |\Omega| \cdot \ln \left\{ \int_\Omega u \right\} - |\Omega| \ln |\Omega|$$

for all $t > 0$, this implies that

$$\int_\Omega u \geq |\Omega| \cdot e^{\frac{1}{|\Omega|} \int_\Omega \ln u}$$

$$
\begin{aligned}
&\geq |\Omega| \cdot e^{\frac{1}{|\Omega|} \cdot \left\{ -|\Omega| \ln(t+\gamma) - c_1 \right\}} \\
&= |\Omega| e^{-\frac{c_1}{|\Omega|}} \cdot \frac{1}{t+\gamma} \\
&\geq |\Omega| e^{-\frac{c_1}{|\Omega|}} \cdot \min\left\{ \frac{1}{\gamma}, 1 \right\} \cdot \frac{1}{t+1} \qquad \text{for all } t \geq 2.
\end{aligned}
$$

Therefore, the proof is completed upon the observation that $\min_{t\in[0,2]} \left\{ (t+1) \int_\Omega u(\cdot,t) \right\}$ must be positive by continuity of u and the fact that $u \not\equiv 0$. □

We can thereby complete the proof of our main results.

Proof of Theorem 5.1.1 ii). For appropriately small $C > 0$, the second inequality in (5.7) is precisely asserted by Lemma 5.5.3, whereas the first is obvious. □

6. A 3D Chemotaxis-Navier-Stokes Model

6.1. Introduction

Even simple life-forms, like certain species of bacteria, can exhibit a complex collective behavior. One particular biological mechanism responsible for some instances of such demeanour is that of chemotaxis, where the bacteria adapt their movement according to the concentration gradient of a particular chemical in their neighborhood. If this process takes place in a liquid environment, it is not unreasonable to take into account interactions with the surrounding fluid as well. Indeed, as description for colonies of *bacillus subtilis*, chemotactic bacteria that are known to display organized swimming and bioconvection patterns in a fluid habitat [38, 61, 77, 23], the following model has been suggested in [99]:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(v) u \nabla v) - U \cdot \nabla u, \\ v_t = \Delta v - uk(v) - U \cdot \nabla v, \\ U_t = \Delta U - (U \cdot \nabla) U + \nabla P + u \nabla \Phi, \\ \nabla \cdot U = 0, \end{cases} \tag{6.1}$$

where a prototypical choice for the functions χ and k is $\chi(v) = const = \chi$ and $k(v) = v$. Herein, u denotes the unknown population density of bacteria that move in part randomly and in part as directed by chemotactic effects, and are transported by the surrounding fluid; v denotes the concentration of oxygen, which again diffuses and is transported by the fluid, but at the same time is consumed by the bacteria. The evolution of the velocity field U of the fluid, finally, is governed by the incompressible Navier-Stokes equations, where the bacteria exert influence by means of bouyant forces due to different densities of water with a high concentration of cells versus low concentration. Using the Boussinesq approximation, this effect is incorporated into the model via the gravitational potential $\nabla\Phi$, $\Phi \in C^{1+\delta}(\overline{\Omega})$ for some $\delta \in (0, 1)$ being a given function. The usual boundary conditions posed along with initial conditions to complement (6.1) are

$$\nabla u \cdot \nu = \nabla v \cdot \nu = 0, \qquad U = 0 \qquad \text{on } \partial\Omega.$$

Let us remark that in this model the chemoattractant (oxygen) is consumed and not supplied by the bacteria, which is in contrast to the celebrated Keller-Segel system of chemotaxis [45] and its variants constituting the center of extensive mathematical investigations since the 1970s, see e.g. the surveys [37, 39, 4] and references therein.

Since its introduction and first analytical results (asserting the local existence of weak solutions in [59]), also the chemotaxis-fluid system has inspired several works addressing mainly the question of existence of classical or weak solutions (the works mentioned below) and long-term behaviour of solutions ([21, 16, 82, 112, 44, 124]).

Due to the difficulties associated with the Navier-Stokes equations in three-dimensional domains, many of these works focus on the two-dimensional case ([112, 108, 89, 126, 124]) or more favorable variants of the model, for example by resorting to the Stokes equation upon neglection of the nonlinear convective term ([108, 25]) or by considering nonlinear instead of linear diffusion of the

bacteria ([58, 21, 89, 90, 25, 19, 100]) and consider the three-dimensional case under smallness conditions on the initial data ([24, 82, 123, 15]). Also in [49], where existence and uniqueness of mild solutions to a model including (6.1) as a submodel in addition to Keller-Segel-type chemotaxis, are proven for the full space in arbitrary dimensions, a smallness assumption (in this case, in the scaling invariant space) is required.
Only recently, the existence of global weak solutions to the system (6.1) with large initial data has been demonstrated for bounded three-dimensional domains in [117], see also [125] for even milder diffusion effects, followed by studies of the long-term behaviour of any such "eventual energy solution" [118], which, namely, become smooth on some interval $[T, \infty)$ and uniformly converge in the large-time-limit.

With this model one further peculiar effect is still unaccounted for that can be observed in colonies of *Proteus mirabilis*. Colonies of these bacteria form spiralling streams that always wind counterclockwise [120]. A reason underlying this behaviour is that the swimming of the bacteria, like that of the similar species *E. coli*, is biased, when they are close to a surface (cf. [54, 22]). This can be reflected in chemotaxis equations by allowing for a more general, tensor-valued and spatially inhomogeneous chemotactic sensitivity, so that the model reads

$$\begin{cases} u_t = \Delta u - \nabla \cdot (uS(x,u,v) \cdot \nabla v) - U \cdot \nabla u, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v - uv - U \cdot \nabla v, & (x,t) \in \Omega \times (0,T), \\ U_t = \Delta U - (U \cdot \nabla) U + \nabla P + u \nabla \Phi, & (x,t) \in \Omega \times (0,T), \\ \nabla \cdot U = 0, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{6.2}$$

where the sensitivity $S(x,u,v) = (s_{i,j})_{N \times N}$ is a matrix-valued function. Indeed, when in [121] a macroscale model for chemotaxis is derived from a velocity jump process rooted in a cell based model incorporating a minimal description of signal transduction in single cells and accounting for this swimming bias, in the chemotaxis term a contribution perpendicular to the concentration gradient appears ([121, (5.26)]). (For tensor-valued sensitivities arising in chemotaxis equations see also [70, sec. 4.2.1] or [119, eq.(3.3)].)
Mathematically, the introduction of these general sensitivities has the disadvantage that it destroys the natural energy structure coming with (6.1). In point of fact, many results concerning global existence of solutions to (6.1) rely on the use of an energy inequality featuring an upper estimate of

$$\frac{d}{dt}\left[\int_\Omega u \log u + \frac{1}{2}\int_\Omega \frac{\chi(v)}{k(v)}|\nabla v|^2\right] + \int_\Omega \frac{|\nabla u|^2}{u} + \frac{1}{4}\int_\Omega \frac{k(v)}{\chi(v)}|D^2 \rho(v)|^2 \tag{6.3}$$

or very similar quantities, where ρ denotes a primitive of $\frac{\chi}{k}$, see [108, Formula (3.11)], [112, (2.15)], [24, (3.11)] or [118, (1.11)] or [15, (3.8)]. For the derivation of appropriate estimates, more precisely for certain cancellations of contributions of the first and the second term in the brackets to occur, it seems to be essential that the functions k and χ satisfy conditions like those given in [112, (1.8)-(1.10)], [108, (1.7)-(1.9)], [24, (A)(iii)], [118, (1.7)] or even [15, (AA), (B)]. There is next to no hope of transferring such delicate cancellations to the case of functions χ that are no longer scalar-valued.
Nevertheless, for some instances of such a system including a rotational sensitivity, the existence of solutions could be shown: The fluid-free system, obtained from (6.2) upon setting $U \equiv 0$, possesses global classical solutions for even more general equations modeling the consumption of oxygen if posed in two-dimensional domains and under a smallness condition on initial data v_0. In this case, furthermore, these solutions converge to spatially homogeneous equilibria as $t \to \infty$

([55]). Also in the case of degenerate diffusion the existence of global bounded weak solutions was obtained for the two-dimensional fluid-free case in [12]. For large initial data and higher spatial dimensions, generalized solutions have been shown to exist in [115].
In the presence of a Stokes-governed fluid in two-dimensional domains, global generalized solutions that become smooth eventually and stabilize were constructed in [113]. The existence of global weak solutions with bounded u-component for the full model including Navier-Stokes equations for the fluid in two-dimensional domains is asserted in [42] under the assumption of porous-medium-type diffusion with exponent $m > 1$ for the bacteria.
In three dimensions, the existence of a global classical solution to the model with a Stokes-governed fluid was proven in [101] under the hypothesis that $|S| \le C(1+u)^{-\alpha}$, with some $C > 0$ and $\alpha > \frac{1}{6}$. A similar decay assumption on S, here with $\alpha > 0$, made it possible to obtain global existence and boundedness of classical solutions for the same model with the second equation replaced by $v_t = \Delta v - u + v - U \cdot \nabla v$ in two-dimensional domains [102].
In [17], the chemotactic sensitivity and the diffusion coefficient for the bacterial motion, both being u- and x-dependent, were even assumed to vanish for $N = 1$. By a semi-discretization procedure, the existence of weak solutions was established for bounded domains of dimension up to four and in the presence of either Navier-Stokes- or Stokes-fluid.
An alternative assumption prompting the existence of weak solutions in the 3D-Stokes-setting is that of nonlinear diffusion of bacteria, that is, with Δu replaced by $\nabla \cdot (u^{m-1}\nabla u)$, with an exponent $m > \frac{7}{6}$, [114]. Also the long-term behaviour of solutions is examined there: they converge to the semi-trivial steady state.
In the present article we consider (6.2) without decay assumptions on S and with Navier-Stokes fluid in three-dimensional domains. The boundary conditions posed will be

$$\nabla v \cdot \nu = (\nabla u - uS(x,u,v) \cdot \nabla v) \cdot \nu = 0, \;\; U = 0, \qquad\qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{6.4}$$

where ν denotes the outer unit normal. We concentrate on classical solutions and therefore pose a smallness condition on the initial data. We then obtain global existence of classical solutions and exponential convergence to a constant steady state. Unlike the study of mild solutions to a Keller-Segel-Navier-Stokes system in [49], we are concerned with bounded domains and admit non-scalar sensitivities.
The consideration of convergence rates seems to be new in the context of tensor valued (and space-dependent) sensitivities, although convergence rates for solutions of the chemotaxis-fluid model (6.1) in the full space have been reported in [24] and [82] and in [123] and also, for Stokes fluid, in [16]. The only corresponding result for bounded domains, and thus the only one giving exponential decay, is the recent work [124], where two-dimensional bounded domains are considered. In the derivation of decay estimates in [124], it was possible to rely on the already established existence ([108]) and convergence ([112]) of solutions. Contrasting this, in the present work we additionally have to ensure global existence of the solutions we are working with and will do so by using a continuation argument that has been used in a similar fluid-free context in [105]. Moreover, our proof will entail an improvement of the convergence rate of the fluid component if compared to [124].
For these tools and the local existence result to be employable, we will first have to restrict our course of action to the case of S vanishing on the boundary. Only in a later step will we approximate fully general sensitivity functions. With regards to this step, we will give more detailed proofs, which have not been contained in any previous works concerned with rotational sensitivities. We will focus on the three-dimensional case. However, since it is possible without further labour, we will perform all calculations and state all results for $N \in \{2,3\}$. The only assumption we place on the domain $\Omega \subset \mathbb{R}^N$ is that it be bounded with smooth boundary. Results concerning bounded domains often include a convexity assumption (see e.g. [108]),

which is used to cope with boundary terms stemming from integration by parts when dealing with an energy functional. By arguments relying on estimates from [43] or [62], it has become possible to remove this assumption (cf. [44] or also [102, 101, 42]). Since our approach does not involve such functionals, these terms will not arise in the first place.
In order to formulate our main result, let us briefly introduce the remaining necessary part of the technical framework: On the sensitivity function S we will impose the conditions

$$S \in C^2\left(\overline{\Omega} \times [0,\infty) \times [0,\infty), \mathbb{R}^{N\times N}\right) \quad \text{and} \quad |S(x,u,v)| \le C_S \tag{6.5}$$

for any $(x,u,v) \in \overline{\Omega} \times [0,\infty) \times [0,\infty)$, where C_S is a given positive constant. The initial data are assumed to satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), \quad u_0 \ge 0 \text{ on } \overline{\Omega}, \\ v_0 \in W^{1,q_0}(\Omega), \quad v_0 > 0 \text{ on } \overline{\Omega}, \\ U_0 \in D(A^\beta), \end{cases} \tag{6.6}$$

for some $\beta \in (\frac{N}{4}, 1)$ and $q_0 > N$, where A denotes the (L^2-realization of the) Stokes operator under Dirichlet boundary conditions in Ω.
Here and in the following, we will denote the first eigenvalue of A by λ_1', and by λ_1 the first nonzero eigenvalue of $-\Delta$ on Ω under Neumann boundary conditions. (For more details on notation and the precise choice of q_0 and β we refer to Sections 6.2 and 6.3 as well as Theorem 6.1.1.)
For $T \in (0,\infty]$ and initial data with the smoothness indicated in (6.6), a classical solution of (6.2), (6.4) on $[0,T)$ is a quadruple of functions (u,v,U,P) satisfying (6.2) and (6.4) in a pointwise sense as well as $u(\cdot,0) = u_0$, $v(\cdot,0) = v_0$, $U(\cdot,0) = U_0$ and exhibiting the following regularity properties:

$$\begin{cases} u \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right), \\ v \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap L^\infty\left((0,T); W^{1,q_0}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right), \\ U \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap L^\infty\left((0,T); D(A^\beta)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right), \\ P \in C^{1,0}\left(\overline{\Omega} \times (0,T)\right) \end{cases} \tag{6.7}$$

It is called global solution if $T = \infty$. The main result will be the following:

Theorem 6.1.1. *Let $N \in \{2,3\}$, $p_0 \in (\frac{N}{2},\infty)$, $q_0 \in (N,\infty)$ and $\beta \in (\frac{N}{4},1)$. Let $m > 0$, $C_S > 0$ and $\Phi \in C^{1+\delta}(\overline{\Omega})$ with some $\delta > 0$. Then for any $\alpha_1 \in (0, \min\{m,\lambda_1\})$ and $\alpha_2 \in (0,\min\{\alpha_1,\lambda_1'\})$ there are $\epsilon > 0$, $C > 0$ such that for any initial data (u_0,v_0,U_0) fulfilling* (6.6) *and*

$$\overline{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 = m, \quad \|u_0 - \overline{u}_0\|_{L^{p_0}(\Omega)} \le \epsilon, \quad \|v_0\|_{L^\infty(\Omega)} \le \epsilon, \quad \|U_0\|_{L^N(\Omega)} \le \epsilon \tag{6.8}$$

and any function S satisfying (6.5)*, system* (6.2) *with boundary condition* (6.4) *and initial data (u_0,v_0,U_0) has a global classical solution, which moreover satisfies*

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\alpha_1 t}, \qquad \|v(\cdot,t)\|_{W^{1,q_0}(\Omega)} \le Ce^{-\alpha_1 t}, \qquad \|U(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\alpha_2 t}$$

for any $t > 0$.

Condition (6.8) in Theorem 6.1.1 could be replaced by

$$\overline{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 = m, \quad \|u_0\|_{L^{p_0}(\Omega)} \le \epsilon, \quad \|\nabla v_0\|_{L^N(\Omega)} \le \epsilon, \quad \|U_0\|_{L^N(\Omega)} \le \epsilon. \tag{6.9}$$

without affecting the validity of the Theorem, thus exchanging conditions asking for the smallness of oxygen concentration and some kind of uniformity in the distribution of bacteria by conditions that indicate smallness of the bacterial concentration and a somewhat homogeneous dispersion of oxygen. Let us state this alternative variant:

Theorem 6.1.2. *Let* $N \in \{2,3\}$*,* $p_0 \in (\frac{N}{2}, N)$*,* $q_0 \in (N, (\frac{1}{p_0} - \frac{1}{N})^{-1})$*, and* $\beta \in (\frac{N}{4}, 1)$*. Let* $M > 0$*,* $C_S > 0$ *and* $\Phi \in C^{1+\delta}(\overline{\Omega})$ *with some* $\delta > 0$*. Then there exist* $\epsilon > 0$ *and* $m_0 < \epsilon|\Omega|^{-\frac{1}{p_0}}$ *such that for any* $m > m_0$*, any* $\alpha_1 \in (0, \min\{m, \lambda_1\})$ *and* $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1'\})$ *there is* $C > 0$ *such that for any initial data* (u_0, v_0, U_0) *fulfilling* (6.6), (6.9) *and* $\|v_0\|_{L^\infty(\Omega)} = M$ *and any function* S *satisfying* (6.5)*, system* (6.2) *with boundary condition* (6.4) *and initial data* (u_0, v_0, U_0) *has a global classical solution, which moreover satisfies*

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\alpha_1 t}, \qquad \|v(\cdot,t)\|_{W^{1,q_0}(\Omega)} \le Ce^{-\alpha_1 t}, \qquad \|U(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\alpha_2 t}$$

for any $t > 0$*.*

Remark 6.1.3. *The condition* $m_0 < \epsilon|\Omega|^{-\frac{1}{p_0}}$ *ensures the existence of initial data to which the theorem is applicable. For* $m > \epsilon|\Omega|^{-\frac{1}{p_0}}$ *the conditions in* (6.9) *cannot be satisfied simultaneously.*

We will not give a separate proof for Theorem 6.1.2 in detail, since it is very similar to that of Theorem 6.1.1. In Remark 6.4.11 at the end of Section 6.4 we will indicate the necessary changes in the proof; an appropriately adapted version of Lemma 6.3.1 will be given in the Appendix.
In order to derive these theorems, we will begin in Section 6.2 by recalling or providing a local existence result and some useful estimates. In Section 6.3, we will then ensure the applicability of these estimates and fix constants and parameters that will make it possible to prove Proposition 6.4.1, which is Theorem 6.1.1 for $S = 0$ on the boundary. The basic approach employed in Section 6.4 partially parallels that from [29] and is moreover closely related to that of [105]. In Section 6.5 we ensure sufficient boundedness in appropriate spaces to pass to the limit in an approximation procedure for more general sensitivity functions so that the last part of that section, finally, can be devoted to the proof of Theorem 6.1.1.

6.2. Preliminaries

The purpose of this section is to provide the ground for estimates needed in the global existence proof. Due to the central importance of semigroups in this undertaking, we next recall L^p-L^q estimates for the Neumann heat semigroup as given in [105, Lemma 1.3]. In fact, we include a small improvement on the statements in part (iii) and (iv). Here and in the following, by λ_1 we will denote the first nonzero eigenvalue of $-\Delta$ on Ω under Neumann boundary conditions and by $(e^{t\Delta})_{t>0}$ we will denote the Neumann heat semigroup in the domain Ω.

Lemma 6.2.1. *There exist* $k_1, k_2, k_3, k_4 > 0$ *which only depend on* Ω *and which have the following properties:*
(i) If $1 \le q \le p \le \infty$*, then*

$$\|e^{t\Delta}w\|_{L^p(\Omega)} \le k_1 \left(1 + t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \text{ for all } t > 0 \tag{6.10}$$

holds for all $w \in L^q(\Omega)$ *with* $\int_\Omega w = 0$*.*
(ii) If $1 \le q \le p \le \infty$*, then*

$$\|\nabla e^{t\Delta}w\|_{L^p(\Omega)} \le k_2 \left(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \text{ for all } t > 0 \tag{6.11}$$

holds for each $w \in L^q(\Omega)$.
(iii) If $2 \le q \le p \le \infty$*, then*

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le k_3 \left(1 + t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) e^{-\lambda_1 t} \|\nabla w\|_{L^q(\Omega)} \text{ for all } t > 0 \tag{6.12}$$

is true for all $w \in W^{1,p}(\Omega)$.
(iv) Let $1 < q \le p < \infty$ *or* $1 < q < \infty$ *and* $p = \infty$*, then*

$$\|e^{t\Delta} \nabla \cdot w\|_{L^p(\Omega)} \le k_4 \left(1 + t^{-\frac{1}{2}-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \text{ for all } t > 0 \tag{6.13}$$

is valid for any $w \in (L^q(\Omega))^N$.

Proof. This is [105, Lemma 1.3]. The parts of Cases (iii) and (iv) which are missing there, are proven in [7, Lemma 2.1]. □

Because of the third equation in (6.2), the Neumann Laplacian is not the only operator generating a semigroup which is important for analyzing the solutions of (6.2). Before introducing the Stokes operator and recalling estimates for the corresponding semigroup, however, let us briefly familiarize ourselves with the appropriate spaces.
For $p \in (1, \infty)$ the spaces of solenoidal vector fields are defined as the L^p-closure of the set of divergence-free smooth vector fields:

$$L^p_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega, \mathbb{R}^N)}^{\|\cdot\|_{L^p(\Omega)}} = \overline{\{\varphi \in C^\infty_0(\Omega, \mathbb{R}^N); \nabla \cdot \varphi = 0\}}^{\|\cdot\|_{L^p(\Omega)}}.$$

Indeed, the space $L^p(\Omega, \mathbb{R}^N)$ is the direct sum of this solenoidal space and a space $\{\nabla\varphi; \varphi \in W^{1,p}(\Omega)\}$ consisting of gradients and there exists a projection from $L^p(\Omega, \mathbb{R}^N)$ onto $L^p_\sigma(\Omega)$, the so-called Helmholtz projection $\mathscr{P}$. More precisely, we have the following:

Lemma 6.2.2. *The Helmholtz projection* $\mathscr{P}$ *defines a bounded linear operator* $\mathscr{P}\colon L^p(\Omega, \mathbb{R}^N) \to L^p_\sigma(\Omega)$*; in particular, for any* $p \in (1, \infty)$ *there is* $k_5(p) > 0$ *such that*

$$\|\mathscr{P} w\|_{L^p(\Omega)} \le k_5(p) \|w\|_{L^p(\Omega)}$$

for every $w \in (L^p(\Omega))^N$.

Proof. See [30, Thm. 1 and Thm. 2]. □

The Stokes operator on $L^p_\sigma(\Omega)$ is defined as $A_p = -\mathscr{P}\Delta$ with domain $D(A_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega)$. Since A_{p_1} and A_{p_2} coincide on the intersection of their domains for $p_1, p_2 \in (1, \infty)$, we will drop the index p in the following without fearing confusion. This operator generates a semigroup for which estimates similar to the previous ones hold:

Lemma 6.2.3. *The Stokes operator* A *generates the analytic semigroup* $(e^{-tA})_{t>0}$ *in* $L^r_\sigma(\Omega)$*. Its spectrum satisfies* $\lambda'_1 := \inf Re\, \sigma(A) > 0$ *and we fix* $\mu \in (0, \lambda'_1)$*. For any such* μ*, the following holds:*
(i) For any $p \in (1, \infty)$ *and* $\gamma \ge 0$ *there is* $k_6(p, \gamma) > 0$ *such that*

$$\|A^\gamma e^{-tA} \phi\|_{L^p(\Omega)} \le k_6(p, \gamma) t^{-\gamma} e^{-\mu t} \|\phi\|_{L^p(\Omega)} \tag{6.14}$$

holds for all $t > 0$ *and all* $\phi \in L^p_\sigma(\Omega)$.
(ii) For p, q *satisfying* $1 < p \le q < \infty$ *there exists* $k_7(p, q) > 0$ *such that*

$$\|e^{-tA} \phi\|_{L^q(\Omega)} \le k_7(p, q) t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-\mu t} \|\phi\|_{L^p(\Omega)} \tag{6.15}$$

holds for all $t > 0$ *and all* $\phi \in L^p_\sigma(\Omega)$.
(iii) For any p, q *with* $1 < p \le q < \infty$ *there is* $k_8(p,q) > 0$ *such that for all* $t > 0$ *and* $\phi \in L^p_\sigma(\Omega)$

$$\|\nabla e^{-tA}\phi\|_{L^q(\Omega)} \le k_8(p,q) t^{-\frac{1}{2}-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-\mu t}\|\phi\|_{L^p(\Omega)}. \tag{6.16}$$

(iv) If $\gamma \ge 0$ *and* $1 < q < p < \infty$ *satisfy* $2\gamma - \frac{N}{q} \ge 1 - \frac{N}{p}$*, then there is* $k_9(\gamma, p, q)$ *such that for all* $\phi \in D(A_q^\gamma)$

$$\|\phi\|_{W^{1,p}(\Omega)} \le k_9(\gamma,p,q)\|A^\gamma \phi\|_{L^q(\Omega)}. \tag{6.17}$$

Proof. That A generates an analytic semigroup in $L^r_\sigma(\Omega)$ was shown in [31]. The estimate in (i) for its fractional powers is a consequence of this fact, see [35, Def. 1.4.7 and Theorem 1.4.3]. Estimates like those in (ii) and (iii) constitute another well-known property of the Stokes semigroup, see e.g. [103, Chapter 6]. They can be proven by combining the Sobolev type embedding theorem and an embedding result for domains of fractional powers of A with estimates as in (i). Namely, according to [33, Prop. 1.4], $D(A_r^\gamma) \hookrightarrow H_r^{2\gamma}$ for any $\gamma \ge 0$, where $H_r^{2\gamma} = F_{r,2}^{2\gamma}$ is a Bessel potential space. Such spaces are covered by the embedding theorem [98, Thm. 3.3.1 (ii)], which states that $F^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow F^{s_1}_{p_1,q_1}(\Omega)$, if $s_0 - \frac{N}{p_0} \ge s_1 - \frac{N}{p_1}$, $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 \le \infty$, $0 < q_1 \le \infty$ and $-\infty < s_1 < s_0 < \infty$. In particular,

$$D(A_p^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}) \hookrightarrow H_p^{N(\frac{1}{p}-\frac{1}{q})}(\Omega) = F_{p,2}^{N(\frac{1}{p}-\frac{1}{q})}(\Omega) \hookrightarrow F^0_{q,2}(\Omega) = L^q(\Omega)$$

and analogously $D(A^{\frac{1}{2}+\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}) \hookrightarrow W^{1,q}(\Omega)$, so that an application of (i) yields (ii) and (iii), respectively. The same embedding results also readily ensure the validity of (iv). □

The following lemma, giving elementary estimates for integrals that arise in calculations involving semigroup representations of solutions, will find frequent use in the proof of Proposition 6.4.1.

Lemma 6.2.4. *For all* $\eta > 0$ *there is* $C = C(\eta) > 0$ *such that for all* $\alpha \in [0, 1-\eta]$, $\beta \in [\eta, 1-\eta]$, $\gamma, \delta \in \mathbb{R}$ *satisfying* $\frac{1}{\eta} \ge \gamma - \delta \ge \eta$ *and for all* $t > 0$*, we have*

$$\int_0^t \left(1 + s^{-\alpha}\right)\left(1 + (t-s)^{-\beta}\right) e^{-\gamma s} e^{-\delta(t-s)} ds \le C(\eta) e^{-\min\{\gamma,\delta\}t}\left(1 + t^{\min\{0,1-\alpha-\beta\}}\right).$$

Proof. Since the statement is a minimally sharpened version of [105, Lemma 1.2], it is not surprisig that its proof can be performed along the same lines as in [105, Lemma 1.2]. We include a proof in the appendix of this chapter. □

Remark 6.2.5. *The roles of* δ *and* γ *can of course be exchanged if those of* α *and* β *are. The constant* $C(\eta)$ *becomes unbounded as* $\eta \to 0^+$.

In cases where the previous lemma yields another than the desired exponent, the following elementary fact may be of use:

Lemma 6.2.6. *Let* $0 \ge a \ge b$ *and* $t > 0$*. Then* $(1 + t^a) \le 2(1 + t^b)$.

Proof. If $t > 1$, then $1 + t^a \le 2 \le 2 + 2t^b$. If $t \le 1$, by the nonnegativity of $a - b$ the inequality $t^{a-b} \le 1^{a-b}$ holds and hence $1 + t^a \le 1 + t^b = 1 + t^b t^{a-b} < 2(1 + t^b)$. □

Another similarly elementary observation is the following:

Lemma 6.2.7. *Let either* $a, b \ge 0$ *or* $a, b \le 0$*. Then for any* $t > 0$*, the inequality* $(1+t^a)(1+t^b) \le 3(1 + t^{a+b})$ *holds.*

Proof. If $a, b \geq 0$, for $t \geq 1$, we have $t^a \leq t^{a+b} \leq 1 + t^{a+b}$, whereas for $t \leq 1$, $t^a \leq 1 \leq 1 + t^{a+b}$. The same estimates hold for t^b, and thus $(1+t^a)(1+t^b) = 1 + t^a + t^b + t^{a+b} \leq 3(1+t^{a+b})$. For $a, b < 0$, one has to exchange the cases $t \geq 1$ and $t \leq 1$. □

As final preparatory step, we include the following result on local existence of solutions:

Lemma 6.2.8. *Let $N \in \{2,3\}$, $q > N$, $\beta \in (\frac{N}{4}, 1)$ and $C_S > 0$, and let S be a function satisfying* (6.5)*. In addition assume that there exists a compact set $K \subset \Omega$ such that*

$$S(x,u,v) = 0 \quad \text{for any } u \geq 0, v \geq 0, x \in \Omega \setminus K. \tag{6.18}$$

Assume that (u_0, v_0, U_0) satisfy (6.6)*.*
(i) There exist

$$\begin{aligned}\tau =& \tau(q, \beta, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{W^{1,q}(\Omega)}, \|A^\beta U_0\|_{L^2(\Omega)}, C_S) > 0 \qquad \text{and} \\ \Gamma =& \Gamma(q, \beta, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{W^{1,q}(\Omega)}, \|A^\beta U_0\|_{L^2(\Omega)}, C_S) > 0\end{aligned}$$

(where for fixed β and q the value of Γ is nondecreasing in the arguments $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{W^{1,q}(\Omega)}$, $\|A^\beta U_0\|_{L^2(\Omega)}$, C_S, and τ is nonincreasing with respect to them) and a classical solution (u, v, U, P) of (6.2), (6.4) *on $[0, \tau]$ with initial data (u_0, v_0, U_0) which satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} + \|U(\cdot,t)\|_{D(A^\beta)} \leq \Gamma \text{ for every } t \in [0,\tau].$$

(ii) This solution can be extended to a maximal time interval, more precisely: There are $T_{\max} > 0$ and a classical solution (u, v, U, P) of (6.2) *in $\Omega \times [0, T_{\max})$ such that*

$$\text{if } T_{\max} < \infty, \text{then } \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\beta U(\cdot,t)\|_{L^2(\Omega)} \to \infty \text{ as } t \nearrow T_{\max}. \tag{6.19}$$

Moreover, we have $u > 0$ and $v > 0$ on $\Omega \times (0, T_{\max})$. For any $T \in (0, T_{max})$, this solution is unique among all functions satisfying (6.7)*, up to addition of functions $\widehat{p}$, such that $\widehat{p}(\cdot, t)$ is constant for any $t \in (0,T)$ to P.*

Proof. Condition (6.18) removes any nonlinearity or inhomogeneity from the boundary condition (6.4). Thus, a proof for a very similar system can be found in [108, Lemma 2.1, p. 324-328], where this is shown by means of a Banach fixed-point argument. Differences mainly stem from the presence of S, which can be estimated in the Frobenius norm by C_S whenever necessary, so that the reasoning there can almost word by word be applied to the current setting. □

6.3. Constants and parameters

Given m, N, p_0, q_0, β, α_1 and α_2 as in Theorem 6.1.1, in this section we shall, mainly by application of Lemma 6.2.4, produce constants $C_1, \ldots, C_8$ (which, accordingly, will only depend on $m > 0$, N, p_0, q_0, β and α_1, α_2) to be used in the continuation argument in the proof of Proposition 6.4.1. We let $k_1, \ldots, k_9$ denote the constants appearing in the estimates of Lemma 6.2.1, Lemma 6.2.2 and Lemma 6.2.3. As stated before, λ_1' and λ_1 will be used to refer to the smallest positive eigenvalues of the Stokes operator or the Neumann Laplacian in Ω. As in Proposition 6.4.1 (or Theorem 6.1.1), we will rely on

$$m > 0, \tag{6.20}$$

$$N \in \{2,3\}, \tag{6.21}$$

$$\frac{N}{2} < p_0 < N, \tag{6.22}$$

$$q_0 > N \text{ and } \frac{1}{q_0} > \frac{1}{p_0} - \frac{1}{N}, \tag{6.23}$$

$$\frac{N}{4} < \beta < 1, \tag{6.24}$$

$$\alpha_1 \in (0, \min\{m, \lambda_1\}), \tag{6.25}$$

$$\alpha_2 \in (0, \min\{\alpha_1, \lambda_1'\}) \tag{6.26}$$

being satisfied, where we have included upper bounds on p_0 and q_0 in (6.22) and (6.23) that will be used during Section 6.4. We pick $\mu \in (\alpha_1, \lambda_1')$ and will henceforth apply Lemma 6.2.3 with this value of μ only.

We first note some elementary consequences of these choices that are nevertheless important as they make it possible to use Lemma 6.2.4. Because $\alpha_2 < \min\{\alpha_1, \mu\}$ and $-\frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0}) \in (-\frac{1}{2}, 0)$ due to (6.23), Lemma 6.2.4 ensures the existence of $C_1 > 0$ such that for all $t > 0$

$$\int_0^t (1 + s^{-\frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0})}) e^{-\mu(t-s)} e^{-\alpha_1 s} ds \le C_1 e^{-\alpha_2 t}. \tag{6.27}$$

Since $-\frac{1}{2} \in (-1, 0)$, $-1 + \frac{N}{2q_0} \in (-1, 0)$ and $1 - \frac{1}{2} - 1 + \frac{N}{2q_0} = -\frac{1}{2} + \frac{N}{2q_0} < 0$, Lemma 6.2.4 also provides us with $C_2 > 0$ such that

$$\int_0^t (t-s)^{-\frac{1}{2}} (1 + s^{-1+\frac{N}{2q_0}}) e^{-\mu(t-s)} e^{-\alpha_2 s} ds \le C_2 (1 + t^{-\frac{1}{2} + \frac{N}{2q_0}}) e^{-\alpha_2 t} \quad \text{for all } t > 0. \tag{6.28}$$

Because $-\frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0}) \in (-\frac{1}{2}, 0)$ by (6.23) and $1 - \frac{1}{2} - \frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0}) > 0 > -\frac{1}{2}$, Lemma 6.2.4 in combination with Lemma 6.2.6 yields $C_3 > 0$ satisfying

$$\int_0^t (t-s)^{-\frac{1}{2}} (1 + s^{-\frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0})}) e^{-\mu(t-s)} e^{-\alpha_1 s} ds \le C_3 (1 + t^{-\frac{1}{2}}) e^{-\alpha_2 t} \quad \text{for all } t > 0. \tag{6.29}$$

As $-\frac{1}{2} - \frac{N}{2q_0} \in (-1, 0)$ due to the choice of q_0, $-1 + \frac{N}{2q_0} \in (-1, 0)$ and $1 - \frac{1}{2} - \frac{N}{2q_0} - 1 + \frac{N}{2q_0} = -\frac{1}{2}$, Lemma 6.2.4 makes it possible to find $C_4 > 0$ such that for all $t > 0$

$$\int_0^t e^{-\mu(t-s)} (t-s)^{-\frac{1}{2} - \frac{N}{2q_0}} (1 + s^{-1+\frac{N}{2q_0}}) e^{-2\alpha_2 s} ds \le C_4 (1 + t^{-\frac{1}{2}}) e^{-\alpha_2 t}. \tag{6.30}$$

Since $-\frac{N}{2p_0} \in (-1, 0)$ and $1 - \frac{1}{2} - \frac{N}{2p_0} \ge -\frac{1}{2}$, Lemmata 6.2.4 and 6.2.6 warrant the existence of $C_5 > 0$ such that for any $q \ge q_0$ and any $t > 0$ we have

$$\int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} (1 + s^{-\frac{N}{2p_0}}) e^{-\alpha_1 s} ds \le C_5 (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t}. \tag{6.31}$$

Moreover, $-\frac{1}{2} - \frac{N}{2q_0} \in (-1, 0)$ since $q_0 > N$, and $1 - \frac{1}{2} - \frac{N}{2q_0} - 1 + \frac{N}{2q_0} = -\frac{1}{2}$. Hence it is possible to find $C_6 > 0$ such that for all $t > 0$,

$$\int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2q_0}}) e^{-\lambda_1(t-s)} (1 + s^{-1+\frac{N}{2q_0}}) e^{-\alpha_1 s} ds \le C_6 (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t}. \tag{6.32}$$

Finally, for $\theta \geq q_0$, $-\frac{1}{2} - \frac{N}{2}(\frac{1}{q_0} - \frac{1}{\theta}) \in (-\frac{1}{2} - \frac{N}{2q_0}, -\frac{1}{2}) \subset (-1, 0)$; by (6.23) also $-\frac{1}{2} - \frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0}) \in (-1, 0)$, and $1 - \frac{1}{2} - \frac{N}{2}(\frac{1}{q_0} - \frac{1}{\theta}) - \frac{1}{2} - \frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0}) = -\frac{N}{2}(\frac{1}{p_0} - \frac{1}{\theta})$. Thus Lemma 6.2.4 provides $C_7 > 0$ such that for any $\theta \geq q_0$

$$\begin{aligned}&\int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{q_0} - \frac{1}{\theta})})e^{-\lambda_1(t-s)}(1 + s^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{p_0} - \frac{1}{q_0})})e^{-\alpha_1 s} ds \\ \leq & C_7(1 + t^{-\frac{N}{2}(\frac{1}{p_0} - \frac{1}{\theta})})e^{-\alpha_1 t} \text{ for all } t > 0. \end{aligned} \tag{6.33}$$

Let

$$\sigma := \int_0^\infty (1 + s^{-\frac{N}{2p_0}})e^{-\alpha_1 s} ds \tag{6.34}$$

and observe that, by the condition (6.22) on p_0, this is finite.

Lemma 6.3.1. *Given m, N, p_0, q_0, β, α_1 and α_2 as in Theorem 6.1.1, it is possible to choose $M_1, M_2, M_3, M_4 > 0$ and $\epsilon > 0$ such that*

$$\begin{aligned}&k_7(N, q_0) + k_5(q_0)k_7(q_0, q_0)(M_1 + k_1)C_1\|\nabla\Phi\|_{L^\infty(\Omega)} \\ &\qquad + 3k_7(\tfrac{N}{1+\frac{N}{q_0}}, q_0)k_5(\tfrac{N}{1+\frac{N}{q_0}})M_3M_4C_2\epsilon \leq \frac{M_3}{2},\end{aligned} \tag{6.35}$$

$$\begin{aligned}&k_8(N, N) + k_8(N, N)k_5(N)|\Omega|^{\frac{q_0 - N}{Nq_0}}(M_1 + k_1)C_3\|\nabla\Phi\|_{L^\infty(\Omega)} \\ &\qquad + 3k_8(\tfrac{1}{\frac{1}{q_0} + \frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0} + \frac{1}{N}})C_4M_3M_4\epsilon \leq \frac{M_4}{2},\end{aligned} \tag{6.36}$$

$$k_2 + C_5k_2(m + (M_1 + k_1)\epsilon)e^{(M_1 + k_1)\sigma\epsilon} + 3k_2M_2M_3C_6\epsilon \leq \frac{M_2}{2}, \tag{6.37}$$

$$3C_SC_7k_4M_2m|\Omega|^{\frac{1}{q_0}} + 3C_SC_7k_4M_2(M_1 + k_1)\epsilon + 3(M_1 + k_1)C_7k_4M_3\epsilon \leq \frac{M_1}{2} \tag{6.38}$$

hold.

Proof. First let $A > 0$ and $M_2 > 0$ be such that

$$k_2 + C_5k_2me^A < \frac{M_2}{4}. \tag{6.39}$$

Then we fix $M_1, M_3, M_4 > 0$ such that

$$\begin{cases} 3C_SC_7k_4M_2m|\Omega|^{\frac{1}{q_0}} < \frac{M_1}{4}, \\ k_7(N, q_0) + k_5(q_0)k_7(q_0, q_0)(M_1 + k_1)C_1\|\nabla\Phi\|_{L^\infty(\Omega)} < \frac{M_3}{4}, \\ k_8(N, N) + k_8(N, N)k_5(N)|\Omega|^{\frac{q_0 - N}{Nq_0}}(M_1 + k_1)C_3\|\nabla\Phi\|_{L^\infty(\Omega)} < \frac{M_4}{4}. \end{cases} \tag{6.40}$$

Finally, letting $\epsilon > 0$ small enough satisfying

$$\begin{aligned}\epsilon < \min\Bigg\{&\frac{A}{(M_1 + k_1)\sigma}, \frac{1}{12k_7(\frac{N}{1+\frac{N}{q_0}}, q_0)k_5(\frac{N}{1+\frac{N}{q_0}})M_4C_2}, \frac{1}{12k_8(\frac{1}{\frac{1}{q_0} + \frac{1}{N}}, N)k_5(\frac{1}{\frac{1}{q_0} + \frac{1}{N}})M_3C_4}, \\ &\frac{M_2}{4C_5k_2(M_1 + k_1)e^A + 12k_2M_2M_3C_6}, \frac{M_1}{12C_7k_4(M_1 + k_1)(C_SM_2 + M_3)}\Bigg\},\end{aligned}$$

we can easily check that (6.38), (6.37), (6.35) and (6.36) are true. □

6.4. Proof of a special case: Sensitivities vanishing near the boundary

This section contains the core of the proof of Theorem 6.1.1, concerning global existence and the convergence estimates both. Nevertheless, for the moment we will restrict ourselves to the situation that the sensitivity function S vanishes close to the boundary. That has the considerable advantage that the nonlinear boundary conditions posed in (6.4) reduce to classical homogeneous Neumann boundary conditions and the existence theorem (Lemma 6.2.8) and standard results concerning the heat semigroup (cf. Section 6.2) become applicable. The case of more general S will be dealt with in Section 6.5.

Let us first state what we are going to prove. The main difference between this proposition and Theorem 6.1.1 lies in the additional condition on S.

Proposition 6.4.1. *Let $N \in \{2,3\}$, $p_0 \in (\frac{N}{2}, N)$, $q_0 \in (N, (\frac{1}{p_0} - \frac{1}{N})^{-1})$, $q_1 \geq q_0$ and $\beta \in (\frac{N}{4}, 1)$. Let $C_S > 0$ and $\Phi \in C^{1+\delta}(\overline{\Omega})$ with some $\delta > 0$, $m > 0$. Then for any $\alpha_1 \in (0, \min\{m, \lambda_1\})$ and $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1'\})$ there are $C_8, C_9, C_{10}, C_{11} > 0$ such that, with the same choice of $\epsilon > 0$, $M_1, M_2, M_3, M_4 > 0$ as in Lemma 6.3.1, the following holds: For any initial data (u_0, v_0, U_0) fulfilling (6.6) as well as $v_0 \in W^{1,q_1}(\Omega)$ and*

$$\overline{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 = m, \quad \|u_0 - \overline{u}_0\|_{L^{p_0}(\Omega)} \leq \epsilon, \quad \|v_0\|_{L^\infty(\Omega)} \leq \epsilon, \quad \|U_0\|_{L^N(\Omega)} \leq \epsilon, \tag{6.41}$$

and any function S satisfying (6.5) and

$$S(x,u,v) = 0 \quad \text{for any } u \geq 0, v \geq 0, x \in \Omega \setminus K$$

for some compact set $K \subset \Omega$, system (6.2) with boundary condition (6.4) and initial data (u_0, v_0, U_0) has a global classical solution, which, for any $t > 0$, moreover satisfies

$$\begin{aligned}
\|u(\cdot,t) - e^{t\Delta}u_0\|_{L^\theta(\Omega)} <& M_1\epsilon\left(1 + t^{-\frac{N}{2}\left(\frac{1}{p_0}-\frac{1}{\theta}\right)}\right)e^{-\alpha_1 t} \quad \text{for all } \theta \in [q_0, \infty],\\
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leq& M_2\epsilon\left(1 + t^{-\frac{1}{2}}\right)e^{-\alpha_1 t},\\
\|U(\cdot,t)\|_{L^{q_0}(\Omega)} \leq& M_3\epsilon\left(1 + t^{-\frac{1}{2}+\frac{N}{2q_0}}\right)e^{-\alpha_2 t},\\
\|\nabla U(\cdot,t)\|_{L^N(\Omega)} \leq& M_4\epsilon\left(1 + t^{-\frac{1}{2}}\right)e^{-\alpha_2 t},
\end{aligned} \tag{6.42}$$

and

$$\begin{aligned}
\|A^\beta U(\cdot,t)\|_{L^2(\Omega)} \leq C_8 e^{-\alpha_2 t}, \qquad & \|U(\cdot,t)\|_{L^\infty(\Omega)} \leq C_9 e^{-\alpha_2 t},\\
\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \leq C_{10}e^{-\alpha_1 t}, \qquad & \|v(\cdot,t)\|_{W^{1,q_1}(\Omega)} \leq C_{11}e^{-\alpha_1 t}.
\end{aligned}$$

Lemma 6.2.8 asserts that there is a solution to (6.2), which is defined on some interval $[0, T_{\max})$. We will denote this solution by (u, v, U, P) in the following. Our main goal is to prove that $T_{\max} = \infty$. In order to show this and to achieve estimates (6.42), we define a number $T > 0$ as follows:

Definition 6.4.2. With $\epsilon > 0$, $M_1, M_2, M_3, M_4 > 0$, p_0, q_0, α_1 and α_2 as in Proposition 6.4.1, we let

$$T := \sup \left\{ \tilde{T} \in (0, T_{\max}) \;\middle|\; \begin{array}{l} \|u(\cdot,t) - e^{t\Delta}u_0\|_{L^\theta(\Omega)} \le M_1\epsilon\big(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\big)e^{-\alpha_1 t} \\ \qquad\qquad\qquad \text{for all } \theta \in [q_0,\infty], \\ \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le M_2\epsilon\big(1 + t^{-\frac{1}{2}}\big)e^{-\alpha_1 t}, \\ \|U(\cdot,t)\|_{L^{q_0}(\Omega)} \le M_3\epsilon\big(1 + t^{-\frac{1}{2}+\frac{N}{2q_0}}\big)e^{-\alpha_2 t}, \\ \|\nabla U(\cdot,t)\|_{L^N(\Omega)} \le M_4\epsilon\big(1 + t^{-\frac{1}{2}}\big)e^{-\alpha_2 t} \\ \qquad\qquad\qquad \text{for all } t \in [0,\tilde{T}) \end{array} \right\}. \tag{6.43}$$

By Lemma 6.2.8, T is well-defined and positive. Thus what we want to show is $T = \infty$. In doing so, we will proceed in several steps and at first derive estimates for the component u that are satisfied on $(0,T)$. We will then show that all of the estimates mentioned in (6.43) hold true with even smaller coefficients on the right hand side than appearing in (6.43) and finally conclude that $T = \infty$. The derivation of these estimates will mainly rely on Lemma 6.2.1, Lemma 6.2.2 and Lemma 6.2.3 by means of the estimates from Section 6.3 and on the fact that the classical solutions on $(0,T)$ can be represented as

$$u(\cdot,t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\big(\nabla\cdot(uS(\cdot,u,v)\nabla v) + U\cdot\nabla u\big)(\cdot,s)ds, \tag{6.44}$$

$$v(\cdot,t) = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}\big(uv + U\cdot\nabla v\big)(\cdot,s)ds, \tag{6.45}$$

$$U(\cdot,t) = e^{-tA}U_0 - \int_0^t e^{-(t-s)A}\mathscr{P}\big((U\cdot\nabla)U - u\nabla\Phi\big)(\cdot,s)ds, \tag{6.46}$$

for all $t \in (0,T_{\max})$ as per the variation-of-constants formula.

Lemma 6.4.3. *Under the assumptions of Proposition 6.4.1, for all $\theta \in [q_0,\infty]$ we have*

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\theta(\Omega)} \le (M_1 + k_1)\epsilon\left(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t} \quad \text{for all } t \in (0,T). \tag{6.47}$$

Proof. Since $\overline{u}_0$ is a constant, $e^{t\Delta}\overline{u}_0 = \overline{u}_0$ for all $t \in (0,T)$, and moreover due to $\int_\Omega(u_0 - \overline{u}_0) = 0$, Lemma 6.2.1(i), (6.43) and (6.41) show that

$$\begin{aligned} \|u(\cdot,t) - \overline{u}_0\|_{L^\theta(\Omega)} &\le \|u(\cdot,t) - e^{t\Delta}u_0\|_{L^\theta(\Omega)} + \|e^{t\Delta}(u_0 - \overline{u}_0)\|_{L^\theta(\Omega)} \\ &\le M_1\epsilon\left(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t} + k_1\left(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\lambda_1 t}\|u_0 - \overline{u}_0\|_{L^{p_0}(\Omega)} \\ &\le M_1\epsilon\left(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t} + k_1\left(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\lambda_1 t}\epsilon \\ &\le (M_1 + k_1)\epsilon\left(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t} \end{aligned}$$

for all $t \in (0,T)$ and $\theta \in [q_0,\infty]$. □

Lemma 6.4.4. *Under the assumptions of Proposition 6.4.1, the second component of the solution satisfies*

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le e^{(M_1+k_1)\sigma\epsilon}\epsilon e^{-\alpha_1 t} \qquad \text{for all } t \in (0,T) \tag{6.48}$$

with σ taken from (6.34).

Proof. We let $p \geq 1$, multiply the second equation of (6.2) by pv^{p-1} and integrate over Ω, so that we have

$$\frac{d}{dt}\int_\Omega v^p \leq -p\int_\Omega uv^p \qquad \text{on } (0,T). \tag{6.49}$$

By an obvious pointwise estimate and (6.47) with $\theta = \infty$,

$$-u(x,t) \leq \|u(\cdot,t)-\overline{u}_0\|_{L^\infty(\Omega)} - \overline{u}_0 \leq (M_1+k_1)\,\epsilon\left(1+t^{-\frac{N}{2p_0}}\right)e^{-\alpha_1 t} - \overline{u}_0 \tag{6.50}$$

for all $x \in \Omega, t \in (0,T)$. Due to the nonnegativity of pv^p, we infer that

$$\frac{d}{dt}\int_\Omega v^p \leq \left((M_1+k_1)\epsilon(1+t^{-\frac{N}{2p_0}})e^{-\alpha_1 t} - \overline{u}_0\right)p\int_\Omega v^p \tag{6.51}$$

for all $t \in (0,T)$. Thus we get

$$\int_\Omega v^p \leq \exp\left(p\int_0^t \left((M_1+k_1)\epsilon(1+s^{-\frac{N}{2p_0}})e^{-\alpha_1 s} - \overline{u}_0\right)ds\right)\int_\Omega v_0^p \qquad \text{for all } t \in (0,T). \tag{6.52}$$

Taking the p-th root on both sides, we are left with

$$\begin{aligned}\|v(\cdot,t)\|_{L^p(\Omega)} &\leq \|v_0\|_{L^p(\Omega)}e^{-\overline{u}_0 t}\exp\left(\epsilon(M_1+k_1)\int_0^t(1+s^{-\frac{N}{2p_0}})e^{-\alpha_1 s}ds\right)\\ &\leq \|v_0\|_{L^p(\Omega)}e^{-\overline{u}_0 t}e^{(M_1+k_1)\sigma\epsilon} \qquad \text{for all } t \in (0,T),\end{aligned}$$

which holds for arbitrary $p \geq 1$ and where σ is as defined in (6.34). In the limit $p \to \infty$, we therefore obtain

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)}e^{\sigma(M_1+k_1)\epsilon}e^{-\overline{u}_0 t} \tag{6.53}$$

for all $t \in (0,T)$ and may, due to (6.25), (6.41), conclude (6.48). □

Lemma 6.4.5. *Under the assumptions of Proposition 6.4.1, the component U of the solution satisfies*

$$\|U(\cdot,t)\|_{L^{q_0}(\Omega)} \leq \frac{M_3}{2}\epsilon\left(1+t^{-\frac{1}{2}+\frac{N}{2q_0}}\right)e^{-\alpha_2 t} \qquad \textit{for all } t \in (0,T). \tag{6.54}$$

Proof. If we use that $\mathscr{P}\nabla\Phi = 0$ and apply the triangle inequality in the variation-of-constants formula (6.46) for U, Lemma 6.2.2 and Lemma 6.2.3 (ii) yield

$$\begin{aligned}\|U(\cdot,t)\|_{L^{q_0}(\Omega)} \leq& k_7(N,q_0)t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{q_0})}e^{-\mu t}\|U_0\|_{L^N(\Omega)}\\ &+\int_0^t k_7(q_0,q_0)k_5(q_0)e^{-\mu(t-s)}\|u(\cdot,s)-\overline{u}_0\|_{L^{q_0}(\Omega)}\|\nabla\Phi\|_{L^\infty(\Omega)}ds\\ &+\int_0^t k_7(\tfrac{N}{1+\frac{N}{q_0}},q_0)(t-s)^{-\frac{N}{2}\left(\frac{1+\frac{N}{q_0}}{N}-\frac{1}{q_0}\right)}e^{-\mu(t-s)}\|\mathscr{P}(U\cdot\nabla U)(\cdot,s)\|_{L^{\frac{N}{1+\frac{N}{q_0}}}(\Omega)}ds\\ =:& k_7(N,q_0)t^{-\frac{1}{2}+\frac{N}{2q_0}}e^{-\mu t}\|U_0\|_{L^N(\Omega)} + I_1 + I_2\end{aligned}$$

for all $t \in (0,T)$. Here an application of estimate (6.47) for $\theta = q_0$ and (6.27) in the first integral shows that

$$I_1 \leq k_5(q_0)k_7(q_0,q_0)(M_1+k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}\int_0^t \epsilon\left(1+s^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0})}\right)e^{-\mu(t-s)}e^{-\alpha_1 s}ds$$

$$\le k_5(q_0)k_7(q_0,q_0)(M_1+k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}\epsilon C_1 e^{-\alpha_2 t}$$
$$\le k_5(q_0)k_7(q_0,q_0)(M_1+k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}C_1(1+t^{-\frac{1}{2}+\frac{N}{2q_0}})e^{-\alpha_2 t}\epsilon$$

for all $t\in(0,T)$. Hölder's inequality and Lemma 6.2.2 imply that

$$\|\mathscr{P}(U\cdot\nabla U)(\cdot,t)\|_{L^{\frac{N}{1+\frac{N}{q_0}}}(\Omega)} \le k_5(\tfrac{N}{1+\frac{N}{q_0}})\|U(\cdot,t)\|_{L^{q_0}(\Omega)}\|\nabla U(\cdot,t)\|_{L^N(\Omega)} \qquad \text{for all } t\in(0,T)$$

and the estimates for the latter two terms, which are valid by (6.43), give

$$\begin{aligned}
I_2 &\le k_7(\tfrac{N}{1+\frac{N}{q_0}},q_0)k_5(\tfrac{N}{1+\frac{N}{q_0}})\int_0^t (t-s)^{-\frac{1}{2}}M_3M_4\epsilon^2 e^{-\mu(t-s)}(1+s^{-\frac{1}{2}+\frac{N}{2q_0}})e^{-\alpha_2 s}(1+s^{-\frac{1}{2}})e^{-\alpha_2 s}ds\\
&\le k_7(\tfrac{N}{1+\frac{N}{q_0}},q_0)k_5(\tfrac{N}{1+\frac{N}{q_0}})M_3M_4\epsilon^2\int_0^t (t-s)^{-\frac{1}{2}}e^{-\mu(t-s)}3(1+s^{-1+\frac{N}{2q_0}})e^{-2\alpha_2 s}ds\\
&\le 3k_7(\tfrac{N}{1+\frac{N}{q_0}},q_0)k_5(\tfrac{N}{1+\frac{N}{q_0}})M_3M_4\epsilon^2 C_2\left(1+t^{-\frac{1}{2}+\frac{N}{2q_0}}\right)e^{-\alpha_2 t} \qquad \text{for all } t\in(0,T),
\end{aligned}$$

where we have also used Lemma 6.2.7 and (6.28). Hence,

$$\begin{aligned}
\|U(\cdot,t)\|_{L^{q_0}(\Omega)} \le& k_7(N,q_0)t^{-\frac{1}{2}+\frac{N}{2q_0}}e^{-\mu t}\epsilon\\
&+k_5(q_0)k_7(q_0,q_0)(M_1+k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}C_1\left(1+t^{-\frac{1}{2}+\frac{N}{2q_0}}\right)e^{-\alpha_2 t}\epsilon\\
&+3k_7(\tfrac{N}{1+\frac{N}{q_0}},q_0)k_5(\tfrac{N}{1+\frac{N}{q_0}})M_3M_4\epsilon^2 C_2\left(1+t^{-\frac{1}{2}+\frac{N}{2q_0}}\right)e^{-\alpha_2 t}\\
\le&\Bigg(k_7(N,q_0)+k_5(q_0)k_7(q_0,q_0)(M_1+k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}C_1\\
&+3k_7(\tfrac{N}{1+\frac{N}{q_0}},q_0)k_5(\tfrac{N}{1+\frac{N}{q_0}})M_3M_4C_2\epsilon\Bigg)\epsilon\left(1+t^{-\frac{1}{2}+\frac{N}{2q_0}}\right)e^{-\alpha_2 t}\\
\le&\frac{M_3}{2}\epsilon\left(1+t^{-\frac{1}{2}+\frac{N}{2q_0}}\right)e^{-\alpha_2 t}
\end{aligned}$$

for all $t\in(0,T)$, according to (6.35). □

Also the estimate for the gradient is preserved:

Lemma 6.4.6. *Under the assumptions of Proposition 6.4.1, we also have*

$$\|\nabla U(\cdot,t)\|_{L^N(\Omega)} \le \frac{\epsilon}{2}M_4\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_2 t}, \quad \textit{for all } t\in(0,T).$$

Proof. Starting from

$$\nabla U(\cdot,t)=\nabla e^{-tA}U_0+\int_0^t \nabla e^{-(t-s)A}\mathscr{P}\left((u(\cdot,s)-\overline{u}_0)\nabla\Phi\right)ds-\int_0^t \nabla e^{-(t-s)A}\mathscr{P}(U\cdot\nabla)U(\cdot,s)ds$$

for all $t\in(0,T)$, we obtain from Lemma 6.2.3 (iii), Hölder's inequality, Lemma 6.2.2 and (6.47) that

$$\begin{aligned}
&\|\nabla U(\cdot,t)\|_{L^N(\Omega)}\\
\le& \|\nabla e^{tA}U_0\|_{L^N(\Omega)}+\int_0^t k_8(N,N)(t-s)^{-\frac{1}{2}}e^{-\mu(t-s)}k_5(N)\|(u(\cdot,s)-\overline{u}_0)\nabla\Phi\|_{L^N(\Omega)}ds
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^t k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}+\frac{1}{N}-\frac{1}{N})} e^{-\mu(t-s)} k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}) \|(U\cdot\nabla)U(\cdot,s)\|_{L^{\frac{1}{\frac{1}{q_0}+\frac{1}{N}}}(\Omega)} ds \\
&\le k_8(N,N) t^{-\frac{1}{2}} e^{-\mu t}\|U_0\|_{L^N(\Omega)} \\
&\quad + k_8(N,N)k_5(N)\int_0^t (t-s)^{-\frac{1}{2}}|\Omega|^{\frac{q_0-N}{Nq_0}}\|u(\cdot,s)-\overline{u}_0\|_{L^{q_0}(\Omega)}\|\nabla\Phi\|_{L^\infty(\Omega)} e^{-\mu(t-s)}ds \\
&\quad + k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}})\int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2q_0}} e^{-\mu(t-s)}\|U(\cdot,s)\|_{L^{q_0}(\Omega)}\|\nabla U(\cdot,s)\|_{L^N(\Omega)}ds \\
&=: k_8(N,N)t^{-\frac{1}{2}}e^{-\mu t}\|U_0\|_{L^N(\Omega)} + I_3 + I_4 \qquad \text{for all } t\in(0,T).
\end{aligned}
$$

Here by (6.29), we have

$$
\begin{aligned}
I_3 &\le k_8(N,N)k_5(N)|\Omega|^{\frac{q_0-N}{Nq_0}}(M_1+k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}\epsilon \\
&\qquad \times \int_0^t (t-s)^{-\frac{1}{2}}(1+s^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0})})e^{-\mu(t-s)}e^{-\alpha_1 s}ds \\
&\le k_8(N,N)k_5(N)|\Omega|^{\frac{q_0-N}{Nq_0}}(M_1+k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}\epsilon C_3(1+t^{-\frac{1}{2}})e^{-\alpha_2 t} \qquad \text{for all } t\in(0,T).
\end{aligned}
$$

Furthermore, by Lemma 6.2.7 and (6.30),

$$
\begin{aligned}
I_4 &\le \epsilon^2 M_3M_4 k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}})\int_0^t e^{-\mu(t-s)}(t-s)^{-\frac{1}{2}-\frac{N}{2q_0}}(1+s^{-\frac{1}{2}+\frac{N}{2q_0}})(1+s^{-\frac{1}{2}})e^{-2\alpha_2 s}ds \\
&\le 3\epsilon^2 M_3M_4 k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}})\int_0^t e^{-\mu(t-s)}(t-s)^{-\frac{1}{2}-\frac{N}{2q_0}}(1+s^{-1+\frac{N}{2q_0}})e^{-2\alpha_2 s}ds \\
&\le 3\epsilon^2 M_3M_4 k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}})C_4\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_2 t} \qquad \text{for all } t\in(0,T).
\end{aligned}
$$

And thus finally, thanks to the above estimate and (6.36), we arrive at

$$
\begin{aligned}
\|\nabla U(\cdot,t)\|_N &\le k_8(N,N)t^{-\frac{1}{2}}e^{-\mu t}\epsilon + k_8(N,N)k_5(N)|\Omega|^{\frac{q_0-N}{Nq_0}}(M_1+k_1)\|\nabla\Phi\|_\infty \epsilon C_3\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_2 t} \\
&\quad + 3\epsilon^2 M_3M_4 k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}})C_4\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_2 t} \\
&\le \Bigg(k_8(N,N) + k_8(N,N)k_5(N)|\Omega|^{\frac{q_0-N}{Nq_0}}(M_1+k_1)C_3\|\nabla\Phi\|_{L^\infty(\Omega)} \\
&\qquad + 3k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}})C_4M_3M_4\epsilon\Bigg)\epsilon\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_2 t} \\
&\le \frac{\epsilon M_4}{2}\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_2 t}
\end{aligned}
$$

for all $t\in(0,T)$. □

Lemma 6.4.7. *Under the assumptions of Proposition 6.4.1, we have*

$$
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\epsilon M_2}{2}\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_1 t}
$$

for all $t\in(0,T)$.

Proof. If we use the variation-of-constants formula (6.45) for v, we obtain from Lemma 6.2.1(ii) that

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} &\le \|\nabla e^{t\Delta}v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}u(\cdot,s)v(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\qquad + \int_0^t \|\nabla e^{(t-s)\Delta}U(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\le k_2\left(1+t^{-\frac{1}{2}}\right)e^{-\lambda_1 t}\|v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}u(\cdot,s)v(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\qquad + \int_0^t \|\nabla e^{(t-s)\Delta}U(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&=: k_2\left(1+t^{-\frac{1}{2}}\right)e^{-\lambda_1 t}\|v_0\|_{L^\infty(\Omega)} + I_5 + I_6 \qquad \text{on } (0,T).
\end{aligned} \tag{6.55}$$

In the first integral we can again apply Lemma 6.2.1 (ii), which gives

$$\begin{aligned}
I_5 &\le \int_0^t k_2(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\|u(\cdot,s)v(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\le \int_0^t k_2(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|v(\cdot,s)\|_{L^\infty(\Omega)}ds
\end{aligned}$$

on $(0,T)$. At this point, Lemma 6.4.3, Lemma 6.4.4 and (6.31) lead to

$$\begin{aligned}
I_5 &\le \int_0^t k_2(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\big(\overline{u}_0+(M_1+k_1)\epsilon\big)(1+s^{-\frac{N}{2p_0}})\epsilon e^{\sigma(M_1+k_1)\epsilon}e^{-\alpha_1 s}ds \\
&\le C_5k_2\Big(\overline{u}_0+(M_1+k_1)\epsilon\Big)e^{(M_1+k_1)\sigma\epsilon}\epsilon\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_1 t}
\end{aligned}$$

for all $t\in(0,T)$.
Next, using Lemma 6.2.1 (ii) and Hölder's inequality, we derive that

$$\begin{aligned}
I_6 &\le \int_0^t k_2(1+(t-s)^{-\frac{1}{2}-\frac{N}{2q_0}})e^{-\lambda_1(t-s)}\|U(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\le \int_0^t k_2(1+(t-s)^{-\frac{1}{2}-\frac{N}{2q_0}})e^{-\lambda_1(t-s)}\|U(\cdot,s)\|_{L^{q_0}(\Omega)}\|\nabla v(\cdot,s)\|_{L^\infty(\Omega)}ds \qquad \text{for all } t\in(0,T).
\end{aligned}$$

If we insert estimates from (6.43) and employ Lemma 6.2.7 and (6.32), we see that

$$\begin{aligned}
I_6 &\le \int_0^t k_2(1+(t-s)^{-\frac{1}{2}-\frac{N}{2q_0}})e^{-\lambda_1(t-s)}M_3\epsilon(1+s^{-\frac{1}{2}+\frac{N}{2q_0}})e^{-\alpha_2 s}M_2\epsilon(1+s^{-\frac{1}{2}})e^{-\alpha_1 s}ds \\
&\le 3\int_0^t k_2(1+(t-s)^{-\frac{1}{2}-\frac{N}{2q_0}})e^{-\lambda_1(t-s)}M_3\epsilon(1+s^{-1+\frac{N}{2q_0}})M_2\epsilon e^{-\alpha_1 s}ds \\
&\le 3k_2M_2M_3\epsilon^2C_6\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_1 t}
\end{aligned}$$

for all $t\in(0,T)$. Combining the above inequalities, we obtain that

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} &\le \Big(k_2+C_5k_2(\overline{u}_0+(M_1+k_1)\epsilon)e^{(M_1+k_1)\sigma\epsilon}+3k_2M_2M_3\epsilon C_6\Big)\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_1 t}\epsilon \\
&\le \frac{M_2\epsilon}{2}\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_1 t}
\end{aligned} \tag{6.56}$$

holds for all $t\in(0,T)$ by (6.37). □

Having achieved these estimates for ∇v, we may re-examine the first solution component and sharpen the estimate from Lemma 6.4.3.

Lemma 6.4.8. *Under the assumptions of Proposition 6.4.1, finally also*

$$\|u(\cdot,t) - e^{t\Delta}u_0\|_{L^\theta(\Omega)} < \frac{M_1\epsilon}{2}\left(1 + t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha t}$$

is valid for all $t \in (0,T)$ and for all $\theta \in [q_0,\infty]$.

Proof. Let $\theta \in [q_0,\infty]$. Then

$$\begin{aligned}
&\|u(\cdot,t) - e^{t\Delta}u_0\|_{L^\theta(\Omega)} \\
\leq &\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(uS(\cdot,u,v)\cdot\nabla v)(\cdot,s)\|_{L^\theta(\Omega)}ds + \int_0^t \|e^{(t-s)\Delta}U(\cdot,s)\cdot\nabla u(\cdot,s)\|_{L^\theta(\Omega)}ds \\
=: &I_7 + I_8 \text{ for all } t \in [0,T].
\end{aligned}$$

According to Lemma 6.2.1 (iv) we have

$$\begin{aligned}
I_7 &\leq \int_0^t k_4(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}\|(uS(\cdot,u,v)\cdot\nabla v)(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\leq C_S\int_0^t k_4(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}\|u(\cdot,s)\|_{L^{q_0}(\Omega)}\|\nabla v(\cdot,s)\|_{L^\infty(\Omega)}ds.
\end{aligned}$$

Here we can employ the estimates provided by (6.47), (6.43) and Lemma 6.2.7 to gain

$$\begin{aligned}
I_7 &\leq C_S\int_0^t k_4(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)} \\
&\qquad\qquad \times (\overline{u}_0|\Omega|^{\frac{1}{q_0}} + (M_1+k_1)\epsilon)(1+s^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0})})M_2\epsilon(1+s^{-\frac{1}{2}})e^{-\alpha_1 s}ds \\
&\leq 3C_Sk_4M_2\left(\overline{u}_0|\Omega|^{\frac{1}{q_0}} + (M_1+k_1)\epsilon\right)\epsilon\int_0^t(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) \\
&\qquad\qquad \times e^{-\lambda_1(t-s)}(1+s^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0})})e^{-\alpha_1 s}ds \\
&\leq 3C_SC_7k_4M_2\left(m|\Omega|^{\frac{1}{q_0}} + (M_1+k_1)\epsilon\right)\epsilon\left(1+t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t}.
\end{aligned}$$

As $\overline{u}_0$ is constant and $\nabla\cdot U = 0$,

$$I_8 = \int_0^t \|e^{(t-s)\Delta}(U\cdot\nabla(u-\overline{u}_0))(\cdot,s)\|_{L^\theta(\Omega)}ds = \int_0^t \|e^{(t-s)\Delta}\nabla\cdot((u-\overline{u}_0)U)(\cdot,s)\|_{L^\theta(\Omega)}ds$$

and hence, treating this integral similarly as I_7 before, we obtain

$$\begin{aligned}
I_8 &\leq \int_0^t k_4(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}\|(u(\cdot,s)-\overline{u}_0)U(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\leq \int_0^t k_4(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}\|u(\cdot,s)-\overline{u}_0\|_{L^\infty(\Omega)}\|U(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\leq \int_0^t k_4(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}(M_1+k_1)\epsilon(1+s^{-\frac{N}{2p_0}})e^{-\alpha_1 s} \\
&\qquad\qquad \times M_3\epsilon(1+s^{-\frac{1}{2}+\frac{N}{2q_0}})e^{-\alpha_2 s}ds
\end{aligned}$$

$$\leq 3(M_1+k_1)k_4M_3\epsilon^2\int_0^t(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}(1+s^{-\frac{1}{2}-\frac{N}{2p_0}+\frac{N}{2q_0}})e^{-\alpha_1 s}ds$$
$$\leq 3(M_1+k_1)C_7k_4M_3\epsilon^2\left(1+t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t}.$$

Using the choice of ϵ and (6.38) we arrive at

$$\begin{aligned}&\|u(\cdot,t)-e^{t\Delta}u_0\|_{L^\theta(\Omega)}\\ \leq&\left(3C_SC_7k_4M_2m|\Omega|^{\frac{1}{q_0}}+3C_SC_7k_4M_2(M_1+k_1)\epsilon+3(M_1+k_1)C_7k_4M_3\epsilon\right)\\ &\quad\times\epsilon\left(1+t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t}\\ \leq&\frac{M_1\epsilon}{2}\left(1+t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{\theta})}\right)e^{-\alpha_1 t}\end{aligned}$$

for all $t\in(0,T)$. □

While we have obtained some estimates for u, one for $\|A^\beta U(\cdot,t)\|_{L^2}$ is not yet among them, although this is the quantity featured by the extensibility criterion in Lemma 6.2.8. We rectify this in the next lemma:

Lemma 6.4.9. *Given N, p_0, q_0, q_1, β, C_S, Φ, m, α_1, α_2, ϵ as in the statement of Proposition 6.4.1, it is possible to find $C_8>0$ with the property asserted there. In particular, for any $t\in(0,T)$, we have*

$$\|A^\beta U(\cdot,t)\|_{L^2(\Omega)}\leq C_8e^{-\alpha_2 t}. \tag{6.57}$$

Proof. We first define $M(t):=e^{\alpha_2 t}\|A^\beta U(\cdot,t)\|_{L^2(\Omega)}$ for $t\in(0,T)$. Moreover, let us pick $r>N$ such that

$$\frac{1}{q_0}+\frac{1}{N}>\frac{1}{r}\geq\frac{1}{N}+\frac{1}{2}-\frac{2\beta}{N},$$

which is evidently possible due to $\frac{2\beta}{N}>\frac{2}{N}\cdot\frac{N}{4}=\frac{1}{2}$. If we set $b:=\frac{1}{q_0}/(\frac{1}{q_0}+\frac{1}{N}-\frac{1}{r})$, we have $b\in(0,1)$ and the Gagliardo-Nirenberg inequality and Lemma 6.2.3 (iv) provide us with $c_1>0$ and $c_2=k_9(\beta,r,2)c_1$ such that

$$\|\varphi\|_{L^\infty(\Omega)}\leq c_1\|\varphi\|^b_{W^{1,r}(\Omega)}\|\varphi\|^{1-b}_{L^{q_0}(\Omega)}\leq c_2\|A^\beta\varphi\|^b_{L^2(\Omega)}\|\varphi\|^{1-b}_{L^{q_0}(\Omega)}$$

for all $\varphi\in L^{q_0}(\Omega)\cap W^{1,r}(\Omega)\cap L^2_\sigma(\Omega)$. In particular,

$$\begin{aligned}&\|(U\cdot\nabla)U(\cdot,s)\|_{L^2(\Omega)}\\ \leq&\|U(\cdot,s)\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{2}-\frac{1}{N}}\|\nabla U(\cdot,s)\|_{L^N(\Omega)}\\ \leq&c_2|\Omega|^{\frac{1}{2}-\frac{1}{N}}\|A^\beta U(\cdot,s)\|^b_{L^2(\Omega)}\|U(\cdot,s)\|^{1-b}_{L^{q_0}(\Omega)}\|\nabla U(\cdot,s)\|_{L^N(\Omega)},\qquad s\in(0,T).\end{aligned} \tag{6.58}$$

We set

$$t_0:=\tau(q_0,\beta,\epsilon,\epsilon,\epsilon,C_S),\qquad\Gamma:=\Gamma(q_0,\beta,\epsilon,\epsilon,\epsilon,C_S)$$

as provided by Lemma 6.2.8 and choose $c_3>0$ such that $\|\varphi\|_{L^{q_0}(\Omega)}\leq c_3\|A^\beta\varphi\|_{L^2(\Omega)}$ for all $\varphi\in D(A^\beta)$. If we use that $\|u\|_{W^{1,N}(\Omega)}\leq k_9(\beta,N,2)\|A^\beta u\|_{L^2(\Omega)}$ according to Lemma 6.2.3 (iv), (6.58) then shows that

$$\|(U\cdot\nabla)U(\cdot,s)\|_{L^2(\Omega)}\leq c_2|\Omega|^{\frac{1}{2}-\frac{1}{N}}\Gamma^bc_3\Gamma^{1-b}\|\nabla U(\cdot,s)\|_{L^N(\Omega)}$$

$$\leq c_2|\Omega|^{\frac{1}{2}-\frac{1}{N}}\Gamma^2 k_9(\beta, N, 2) =: c_4 \tag{6.59}$$

for $s \in (0, t_0)$, and that

$$\begin{aligned}&\|(U\cdot\nabla)U(\cdot,s)\|_{L^2(\Omega)}\\ \leq& c_2|\Omega|^{\frac{1}{2}-\frac{1}{N}}e^{-\alpha_2 bs}e^{-\alpha_2(1-b)s}M^b(s)\left(M_3\epsilon(1+({t_0}/{2})^{-\frac{1}{2}+\frac{N}{2q_0}})\right)\left(M_4\epsilon(1+({t_0}/{2})^{-\frac{1}{2}})\right)\\ =& c_5 e^{-\alpha_2 s}M^b(s)\end{aligned} \tag{6.60}$$

for all $s \in (\frac{t_0}{2}, T)$ for an obvious choice of $c_5 > 0$. For $t > t_0$ we now aim at estimating

$$\begin{aligned}\|A^\beta U(\cdot,t)\|_{L^2(\Omega)} \leq \|A^\beta e^{-tA}U_0\|_{L^2(\Omega)} &+ \int_0^t \|A^\beta e^{-(t-s)A}\mathscr{P}(u(\cdot,s)-\overline{u}_0)\nabla\Phi\|_{L^2(\Omega)}\\ &+\int_0^t \|A^\beta e^{-(t-s)A}\mathscr{P}(U\cdot\nabla U)(\cdot,s)\|_{L^2(\Omega)}ds\end{aligned} \tag{6.61}$$

and observe that

$$\|A^\beta e^{-tA}U_0\|_{L^2(\Omega)} \leq k_6(2,\beta)t^{-\beta}e^{-\mu t}\|U_0\|_{L^2(\Omega)} \leq k_6(2,\beta)t_0^{-\beta}|\Omega|^{\frac{N-2}{2N}}e^{-\alpha_2 t}\|U_0\|_{L^N(\Omega)} \tag{6.62}$$

for $t \in [t_0, T)$.
Since $\beta \in (0,1)$ and $-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0}) \in (-1,0)$ and $1-\beta-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0}) > -1$, Lemma 6.2.4 and Lemma 6.2.6 provide $c_6 > 0$ such that for all $t > 0$

$$\int_0^t (t-s)^{-\beta}e^{-\mu(t-s)}(1+s^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0})})e^{-\alpha_1 s}ds \leq c_6\left(1+t^{-1}\right)e^{-\alpha_1 t}. \tag{6.63}$$

From Lemma 6.2.3(i), Lemma 6.2.2, Lemma 6.2.4 and (6.47), we infer

$$\begin{aligned}&\int_0^t \|A^\beta e^{-(t-s)A}\mathscr{P}(u(\cdot,s)-\overline{u}_0)\nabla\Phi\|_{L^2(\Omega)}\\ &\leq k_6(2,\beta)k_5(2)\int_0^t e^{-\mu(t-s)}(t-s)^{-\beta}\|u(\cdot,s)-\overline{u}_0\|_{L^2(\Omega)}\|\nabla\Phi\|_{L^\infty(\Omega)}ds\\ &\leq k_6(2,\beta)k_5(2)\int_0^t e^{-\mu(t-s)}(t-s)^{-\beta}|\Omega|^{\frac{1}{2}-\frac{1}{q_0}}\|u(\cdot,s)-\overline{u}_0\|_{L^{q_0}(\Omega)}\|\nabla\Phi\|_{L^\infty(\Omega)}ds\\ &\leq k_6(2,\beta)k_5(2)\|\nabla\Phi\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{2}-\frac{1}{q_0}}\int_0^t e^{-\mu(t-s)}(t-s)^{-\beta}(M_1+k_1)\varepsilon(1+s^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q_0})})e^{-\alpha_1 s}ds\\ &\leq k_6(2,\beta)k_5(2)\|\nabla\Phi\|_{L^\infty(\Omega)}(M_1+k_1)c_6|\Omega|^{\frac{1}{2}-\frac{1}{q_0}}\varepsilon(1+t^{-1})e^{-\alpha_1 t}\\ &\leq k_6(2,\beta)k_5(2)\|\nabla\Phi\|_{L^\infty(\Omega)}(M_1+k_1)c_6|\Omega|^{\frac{1}{2}-\frac{1}{q_0}}\varepsilon(1+t_0^{-1})e^{-\alpha_1 t} \qquad \text{for all } t\in[t_0,T).\end{aligned} \tag{6.64}$$

Moreover, from $-\beta \in (-1,0)$ and $0 \geq \min\{0, 1-\beta-\frac{1}{2}\} > -1$, by means of Lemma 6.2.4 and Lemma 6.2.6 we conclude the existence of $c_7 > 0$ such that

$$\int_0^t (t-s)^{-\beta}e^{-\mu(t-s)}e^{-\alpha_2 s}ds \leq c_7(1+t^{-1})e^{-\alpha_2 t} \tag{6.65}$$

holds for any $t > 0$. Furthermore, for any $t \in [t_0, T)$ we have

$$\int_0^t \|A^\beta e^{-(t-s)A}(\mathscr{P}(U\cdot\nabla)U)(\cdot,s)\|_{L^2(\Omega)}ds$$

$$\leq k_6(2,\beta)k_5(2)\int_0^{\frac{t_0}{2}}(t-s)^{-\beta}e^{-\mu(t-s)}\|(U\cdot\nabla U)(\cdot,s)\|_{L^2(\Omega)}ds$$
$$+\,k_6(2,\beta)k_5(2)\int_{\frac{t_0}{2}}^{t}(t-s)^{-\beta}e^{-\mu(t-s)}\|(U\cdot\nabla U)(\cdot,s)\|_{L^2(\Omega)}ds,$$

where we can use (6.59) to estimate the first summand by

$$\int_0^{\frac{t_0}{2}}(t-s)^{-\beta}e^{-\mu(t-s)}\|(U\cdot\nabla U)(\cdot,s)\|_{L^2(\Omega)}ds \leq \int_0^{\frac{t_0}{2}}\left({}^{t_0}\!/_2\right)^{-\beta}e^{-\mu t}e^{\mu s}c_4ds$$
$$\leq c_4\left({}^{t_0}\!/_2\right)^{-\beta}\left(\frac{e^{\mu t_0/2}-1}{\mu}\right)e^{-\alpha_2 t}, \tag{6.66}$$

whereas the integral concerned with larger times by (6.60) can be controlled according to

$$\int_{\frac{t_0}{2}}^{t}(t-s)^{-\beta}e^{-\mu(t-s)}\|(U\cdot\nabla U)(\cdot,s)\|_{L^2(\Omega)}ds \leq \int_{\frac{t_0}{2}}^{t}(t-s)^{-\beta}e^{-\mu(t-s)}c_5e^{-\alpha_2 s}M^b(s)ds$$
$$\leq c_5\sup_{s\in(0,t)}M^b(s)\int_0^t(t-s)^{-\beta}e^{-\mu(t-s)}e^{-\alpha_2 s}ds$$
$$\leq c_5c_7(1+t_0^{-1})e^{-\alpha_2 t}\sup_{s\in(0,t)}M^b(s) \tag{6.67}$$

for all $t\in[t_0,T)$, due to (6.65). As to $t\in(0,t_0)$, we know from Lemma 6.2.8 that

$$\|A^\beta U(\cdot,t)\|_{L^2(\Omega)}\leq\Gamma\leq\Gamma e^{\alpha_2 t_0}e^{-\alpha_2 t}\text{ for all }t\in(0,t_0). \tag{6.68}$$

If we then insert (6.62), (6.64), (6.66) and (6.67) into (6.61) and take into account (6.68), we obtain some $c_8>0$ such that for all $t\in(0,T)$

$$\|A^\beta U(\cdot,t)\|_{L^2(\Omega)}\leq c_8e^{-\alpha_2 t}+c_8e^{-\alpha_2 t}\sup_{t\in(0,T)}M^b(t),$$

where multiplication by $e^{\alpha_2 t}$ shows that

$$M(t)\leq c_8+c_8\sup_{t\in(0,T)}M^b(t)\qquad\text{for all }t\in(0,T)$$

Due to $b<1$, we may hence infer the existence of $C_8>0$ such that

$$C_8\geq M(t)=e^{\alpha_2 t}\|A^\beta U(\cdot,t)\|_{L^2(\Omega)}\qquad\text{for all }t\in(0,T)$$

This entails (6.57). □

In order to infer the decay asserted in Proposition 6.4.1, we have to combine the estimates from Definition 6.4.2 with Lemma 6.2.8.

Lemma 6.4.10. *Given N, p_0, q_0, q_1, β, C_S, Φ, m, α_1, α_2, ϵ as in the statement of Proposition 6.4.1, it is possible to find there are $C_9>0$, $C_{10}>0$ and $C_{11}>0$ with the properties asserted there. In particular,*

$$\|U(\cdot,t)\|_{L^\infty(\Omega)}\leq C_9e^{-\alpha_2 t}, \tag{6.69}$$
$$\|u(\cdot,t)-\overline{u}_0\|_{L^\infty(\Omega)}\leq C_{10}e^{-\alpha_1 t} \tag{6.70}$$
$$\text{and}\quad\|v(\cdot,t)\|_{W^{1,q_1}(\Omega)}\leq C_{11}e^{-\alpha_1 t} \tag{6.71}$$

for all $t\in(0,T)$.

Proof. Since $D(A^\beta) \hookrightarrow L^\infty(\Omega)$ with $\beta \in (\frac{N}{4}, 1)$, we can conclude the existence of $C_9 > 0$ such that (6.69) holds from Lemma 6.4.9. If we set

$$t_0 := \tau(q_1, \beta, \epsilon, \epsilon, \epsilon, C_S), \qquad \Gamma := \Gamma(q_1, \beta, \epsilon, \epsilon, \epsilon, C_S)$$

as provided by Lemma 6.2.8, we see that Lemma 6.2.8 ensures $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \Gamma$ on $[0, t_0)$, and thus

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\overline{u}_0\|_{L^\infty(\Omega)} \le \Gamma + m \qquad \text{for } t \in (0, t_0),$$

that is

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le (\Gamma + m)e^{\alpha t_0} e^{-\alpha_1 t} \qquad \text{for } t \in [0, t_0).$$

At the same time, Lemma 6.4.3 asserts that

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le (M_1 + k_1)\Big(1 + t^{-\frac{N}{2p_0}}\Big)e^{-\alpha_1 t} \le (M_1 + k_1)\Big(1 + t_0^{-\frac{N}{2p_0}}\Big)e^{-\alpha_1 t}, \qquad \text{for } t \in (t_0, T)$$

so that with $C_{10} = \max\left\{(\Gamma + m)e^{\alpha t_0}, (M_1 + k_1)\Big(1 + t_0^{-\frac{N}{2p_0}}\Big)\right\}$, we have

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le C_{10} e^{-\alpha t} \qquad \text{for all } t > 0.$$

Lemma 6.2.8 also guarantees that $\|v(\cdot,t)\|_{W^{1,q_1}(\Omega)} \le \Gamma$, and hence $\|v(\cdot,t)\|_{W^{1,q_1}(\Omega)} \le \Gamma e^{\alpha_1 t_0} e^{-\alpha_1 t}$ for all $t \in [0, t_0)$. Combining this with Lemma 6.4.7 and Lemma 6.4.4, which show that

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\epsilon M_2}{2}\left(1 + t^{-\frac{1}{2}}\right)e^{-\alpha_1 t}, \quad \|v(\cdot,t)\|_{L^\infty(\Omega)} \le e^{(M_1+k_1)\sigma\epsilon}\epsilon e^{-\alpha_1 t}$$

for all $t > 0$, we can infer that

$$\|v(\cdot,t)\|_{W^{1,q_1}(\Omega)} \le C_{11} e^{-\alpha_1 t}, \text{ for all } t > 0.$$

where $C_{11} = \max\left\{\Gamma e^{\alpha t_0}, \epsilon M_2 |\Omega|^{\frac{1}{q_1}}\left(1 + t_0^{-\frac{1}{2}}\right), 2|\Omega|^{\frac{1}{q_1}} e^{(M_1+k_1)\sigma\epsilon}\epsilon\right\}$. □

Now we are ready to complete the proof Proposition 6.4.1.

Proof of Proposition 6.4.1. First we claim that the solution is global. In order to show this, we observe that if $T_{\max} < \infty$, then according to the blow-up criterion in (6.19), the inequalities required in the definition (6.43) of T, and Lemma 6.4.9, we have $T < T_{\max}$ and one of the following holds:

$$\begin{aligned}
\|u(\cdot,T) - e^{T\Delta}u_0\|_{L^\theta(\Omega)} &= M_1\epsilon\left(1 + T^{-\frac{N}{2}(\frac{1}{p_0} - \frac{1}{\theta})}\right)e^{-\alpha_1 T},\\
\|\nabla v(\cdot,T)\|_{L^\infty(\Omega)} &= M_2\epsilon\left(1 + T^{-\frac{1}{2}}\right)e^{-\alpha_1 T},\\
\|U(\cdot,T)\|_{L^{q_0}(\Omega)} &= M_3\epsilon\left(1 + T^{-\frac{1}{2} + \frac{N}{2q_0}}\right)e^{-\alpha_2 T},\\
\|\nabla U(\cdot,T)\|_{L^N(\Omega)} &= M_4\epsilon\left(1 + T^{-\frac{1}{2}}\right)e^{-\alpha_2 T},
\end{aligned}$$

for some $\theta \in [q_0, \infty]$. But these quantities continuously depend on t and hence each of these items would contradict Lemma 6.4.8, Lemma 6.4.7, Lemma 6.4.5 or Lemma 6.4.6, respectively. The same contradiction arises if $T_{\max} = \infty$ and $T < \infty$. Hence $T = \infty = T_{\max}$. The remaining estimates and assertions about convergence result from Definition 6.4.2 and Lemma 6.4.10. □

Remark 6.4.11. *After having shown Proposition 6.4.1, let us briefly indicate the changes that are necessary in order to prove Theorem 6.1.2 instead of Theorem 6.1.1. Indeed, these are confined to the proof of the counterpart of Proposition 6.4.1; the approximation procedure that is to follow in Section 6.5 remains unaffected. We note that*

$$m = \overline{u}_0 = \frac{1}{|\Omega|}\int u_0 \leq |\Omega|^{-\frac{1}{p_0}}\|u_0\|_{L^{p_0}(\Omega)} \leq |\Omega|^{-\frac{1}{p_0}}\epsilon \tag{6.72}$$

and hence, in particular, $\|\overline{u}_0\|_{L^{p_0}(\Omega)} \leq \epsilon$ *and* $\|u_0 - \overline{u}_0\|_{L^{p_0}(\Omega)} < 2\epsilon$ *so that in* (6.47) *(and by extension, in all of Sections 6.3 and 6.4), replacing* k_1 *by* $2k_1$ *is sufficient to retain the validity of Lemma 6.4.3 and its consequences. The only remaining - but most noticable - place which is affected by the change from* (6.8) *to* (6.9) *is Lemma 6.4.7. With the new condition, for the estimate of the first term in* (6.55)*, we invoke Lemma 6.2.1(iii) instead of Lemma 6.2.1(ii). In the estimate of* I_5*, we have to exchange a factor* ϵ *by* $\|v_0\|_{L^\infty(\Omega)} = M$*, but can, thanks to* (6.72)*, rely on the smallness of* $(\overline{u}_0 + (M_1 + 2k_1)\epsilon) \leq (|\Omega|^{-\frac{1}{p_0}} + M_1 + 2k_1)\epsilon$ *instead, so that* (6.56) *would read*

$$\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \\
&\leq \left(k_3 + C_5 k_2 \left(|\Omega|^{-\frac{1}{p_0}} + M_1 + 2k_1\right) M e^{(M_1+2k_1)\sigma\epsilon} + 3k_2 M_2 M_3 C_6 \epsilon\right)\left(1 + t^{-\frac{1}{2}}\right) e^{-\alpha_1 t}\epsilon \\
&\leq \frac{M_2}{2}\epsilon\left(1 + t^{-\frac{1}{2}}\right) e^{-\alpha_1 t}.
\end{aligned}$$

Of course, this mandates changes also in Lemma 6.3.1. We give an appropriately modified version in the appendix of this chapter (Lemma 6.A.2).

6.5. System with rotational flux (general S)

In this section, we deal with the more general model, where $S \in C^2(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{N\times N})$ is a more arbitrary matrix-valued function, without the requirement of being zero close to the boundary. In this case, we construct solutions by an approximation procedure. In order to make the previous result applicable, we introduce a family of smooth functions

$$\rho_\varepsilon \in C_0^\infty(\Omega) \text{ and } 0 \leq \rho_\varepsilon(x) \leq 1 \ \text{ for } \varepsilon \in (0,1), \ \ \rho_\varepsilon(x) \nearrow 1 \text{ as } \varepsilon \searrow 0 \tag{6.73}$$

and given any function S satisfying the assumptions of Theorem 6.1.1, we let

$$S_\varepsilon(x,u,v) = \rho_\varepsilon(x) S(x,u,v). \tag{6.74}$$

Using this definition, we regularize (6.2) as follows:

$$\begin{cases}
u_{\varepsilon t} = \Delta u_\varepsilon - \nabla\cdot(u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon) - U_\varepsilon\cdot\nabla u_\varepsilon, & (x,t)\in\Omega\times(0,T),\\
v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon - U_\varepsilon\cdot\nabla v_\varepsilon, & (x,t)\in\Omega\times(0,T),\\
U_{\varepsilon t} = \Delta U_\varepsilon - (U_\varepsilon\cdot\nabla)U_\varepsilon + \nabla P + u_\varepsilon\nabla\Phi, \quad \nabla\cdot U_\varepsilon = 0, & (x,t)\in\Omega\times(0,T),\\
\nabla u_\varepsilon\cdot\nu = \nabla v_\varepsilon\cdot\nu = 0, \ \ U_\varepsilon = 0, & (x,t)\in\partial\Omega\times(0,T),\\
u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), \quad U_\varepsilon(x,0) = U_0(x), & x\in\Omega.
\end{cases} \tag{6.75}$$

We have chosen S_ε in such a way that it satisfies the additional condition imposed in Proposition 6.4.1. Therefore the existence of solutions follows from the previous section:

Lemma 6.5.1. *Let $N \in \{2,3\}$, $p_0 \in (\frac{N}{2},\infty)$, $q_0 \in (N,\infty)$, and $\beta \in (\frac{N}{4},1)$. Let $C_S > 0$, $\Phi \in C^{1+\delta}(\overline{\Omega})$ with some $\delta > 0$, $m > 0$. Let $\alpha_1 \in (0,\min\{m,\lambda_1\})$ and $\alpha_2 \in (0,\min\{\alpha_1,\lambda_1'\})$. Let (u_0,v_0,U_0) satisfy (6.6) and (6.8). Then for any $\varepsilon \in (0,1)$ there is a global classical solution $(u_\varepsilon,v_\varepsilon,U_\varepsilon,P_\varepsilon)$ of (6.75) and there are constants $C_8,C_9,C_{10},C_{11} > 0$ such that for any $\varepsilon \in (0,1)$ the estimates*

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,q_0}(\Omega)} \le C_{11}e^{-\alpha_1 t}, \|u_\varepsilon(\cdot,t)-\overline{u}_0\|_{L^\infty(\Omega)} \le C_{10}e^{-\alpha_1 t}, \|U_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_9 e^{-\alpha_2 t} \quad (6.76)$$

hold for any $t > 0$ and such that moreover the solutions satisfy

$$\|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_8 e^{-\alpha_2 t} \quad (6.77)$$

for any $t > 0$ and any $\varepsilon \in (0,1)$. Moreover there is $C_{12} > 0$ such that for any $\varepsilon \in (0,1)$ and any $t > 0$

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{12}\left(1+t^{-\frac{1}{2}}\right)e^{-\alpha_1 t}. \quad (6.78)$$

Proof. These assertions are part of Proposition 6.4.1 if we set $C_{12} := \epsilon M_2$ in (6.42), at least for $p_0 < N$, $q_0 < \left(\frac{1}{p_0}-\frac{1}{N}\right)^{-1}$. For larger values of p_0 or q_0, (6.8) entails the validity of (6.8) for smaller p_0, q_0 if ϵ is adequately adjusted, and Lemma 6.5.1 still follows from Proposition 6.4.1, if q_0, p_0 and q_1 are suitably chosen therein. □

From this family of approximate solutions we aim to extract a convergent sequence. Already the frail manner of convergence of S_ε, however, puts us far from the immediate conclusion that the limiting object satisfies (6.2) in a pointwise sense. Accordingly, we will first ensure that it is a weak solution; afterwards we will show that it is sufficiently regular so as to be a classical solution. For this purpose, we require a definition of “weak solution”:

Definition 6.5.2. We say that (u,v,U) is a weak solution of (6.2) associated to initial data (u_0,v_0,U_0) which satisfy $(u_0,v_0,U_0) \in C^0(\overline{\Omega}) \times W^{1,q_0}(\Omega) \times D(A^\beta)$ for some $q_0 > N$ and $\beta \in (\frac{N}{4},1)$ as well as $u_0 \ge 0$ and $v_0 > 0$ in $\overline{\Omega}$ if

$$u,v \in L^2_{loc}([0,\infty),W^{1,2}(\Omega)), U \in L^2_{loc}([0,\infty),W^{1,2}_{0,\sigma}(\Omega)),$$

and for all $\psi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ and all $\Psi \in C_{0,\sigma}^\infty(\Omega\times[0,\infty))$ the following identities hold:

$$\begin{aligned}
-\int_0^\infty\int_\Omega u\psi_t - \int_\Omega u_0\psi(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\psi + \int_0^\infty\int_\Omega uS(x,u,v)\nabla v\cdot\nabla\psi \\
&\quad + \int_0^\infty\int_\Omega uU\cdot\nabla\psi, \\
-\int_0^\infty\int_\Omega v\psi_t - \int_\Omega v_0\psi(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla v\cdot\nabla\psi - \int_0^\infty\int_\Omega uv\psi + \int_0^\infty\int_\Omega vU\cdot\nabla\psi, \quad (6.79) \\
-\int_0^\infty\int_\Omega U\cdot\Psi_t - \int_\Omega U_0\cdot\Psi(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla U\cdot\nabla\Psi - \int_0^\infty\int_\Omega (U\cdot\nabla)U\cdot\Psi \\
&\quad + \int_0^\infty\int_\Omega u\nabla\Phi\cdot\Psi.
\end{aligned}$$

Within this framework, we shall show the sequence of solutions to (6.75) to have a limit. We begin the extraction of convergent subsequences with convergence of u and v in Hölder spaces in the following lemma:

Lemma 6.5.3. *There are $\gamma > 0$, a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ and $u, v \in C^{1+\gamma,\gamma}_{loc}(\overline{\Omega} \times (0,\infty))$ such that*

$$u_{\varepsilon_j} \to u \qquad \text{in } C^{\gamma,\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times (0,\infty)) \tag{6.80}$$

$$v_{\varepsilon_j} \to v \qquad \text{in } C^{\gamma,\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times (0,\infty)) \tag{6.81}$$

as $j \to \infty$.

Proof. For any $\varepsilon \in (0,1)$ the function u_ε is a bounded distributional solution of the parabolic equation

$$\widetilde{u}_t - div\, a(x,t,\widetilde{u},\nabla\widetilde{u}) = b(x,t,\widetilde{u},\nabla\widetilde{u}) \qquad \text{in } \Omega \times (0,\infty)$$

for the unknown function $\widetilde{u}$, with $a(x,t,\widetilde{u},\nabla\widetilde{u}) = \nabla\widetilde{u} - u_\varepsilon S_\varepsilon \nabla v_\varepsilon - U_\varepsilon u_\varepsilon$, and $b \equiv 0$, and $a(x,t,\widetilde{u},\nabla\widetilde{u}) \cdot \nu = 0$ on the boundary of the domain. Defining $\psi_0(x,t) = |u_\varepsilon(x,t)S_\varepsilon(x,u_\varepsilon(x,t),v_\varepsilon(x,t))\nabla v_\varepsilon(x,t)|^2 + |U_\varepsilon(x,t)u_\varepsilon(x,t)|^2$ and $\psi_1 = |u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon| + |U_\varepsilon u_\varepsilon|$ we see that $a(x,t,\widetilde{u},\nabla\widetilde{u})\nabla\widetilde{u} \geq \frac{1}{2}|\nabla\widetilde{u}|^2 - \psi_0$ and $|a(x,t,\widetilde{u},\nabla\widetilde{u})| \leq |\nabla\widetilde{u}| + \psi_1$. If we let $T > 0$ and $\tau \in (0,T)$, the regularity result [73, Thm 1.3] therefore asserts the existence of $\gamma_1 \in (0,1)$ and $c_1 > 1$ such that $\|u_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times(\tau,T))} \leq c_1$.
According to the aforementioned theorem, these numbers γ_1 and c_1 depend on $\|u_\varepsilon\|_{L^\infty(\Omega\times(\tau,T))}$ and the norms of ψ_0, ψ_1 in certain spaces $L^p((\tau,T),L^q(\Omega))$, where p and q must be sufficiently large, but need not be infinite. Such bounds have been asserted independently of ε in (6.76) and (6.78) in Lemma 6.5.1, so that we can conclude the existence of $\gamma_1 \in (0,1)$ and $c_1 > 0$ such that

$$\|u_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times[\tau,T])} \leq c_1 \qquad \text{for every } \varepsilon \in (0,1).$$

Moreover, since $b \equiv 0$, according to [73, Remark 1.3], γ_1 is independent of τ. By a similar reasoning applied to the second equation and again invoking [73, Thm 1.3], we can find $\gamma_2 \in (0,1)$ and $c_2 > 0$ such that

$$\|v_\varepsilon\|_{C^{\gamma_2,\frac{\gamma_2}{2}}(\overline{\Omega}\times[\tau,T])} \leq c_2 \qquad \text{for every } \varepsilon \in (0,1).$$

If we now pick $\gamma \in (0,\min\{\gamma_1,\gamma_2\})$, the compact embeddings $C^{\gamma_i,\frac{\gamma_i}{2}}(\overline{\Omega} \times [\tau,T]) \hookrightarrow\hookrightarrow C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times [\tau,T])$, $i \in \{1,2\}$, allow for extraction of a sequence such that (6.80) and (6.81) hold. □

In order to achieve convergence in the third component of the solutions, we will combine estimates we already have obtained with Theorem 2.8 of [34] and the embedding result [2, Thm 1.1], which asserts that for $\gamma \in (0,1)$ the set of functions with $\|U\|_{L^p(0,T;W^{2,p}(\Omega))}$ and $\|U_t\|_{L^p(0,T;L^p(\Omega))}$ being bounded is a compact subset of $C^\gamma(0,T;C^{1+\gamma}(\overline{\Omega}))$ if p is large. The latter is an argument employed also in [118, Cor. 7.7], the former also lies at the center of the proof of [118, Lemma 7.6], but is substantially easier here due to the estimates stated in Lemma 6.5.1.

Lemma 6.5.4. *There are $\gamma > 0$, a subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ of the sequence given in Lemma 6.5.3 and $U \in C^{1+\gamma,\gamma}_{loc}(\overline{\Omega} \times (0,\infty);\mathbb{R}^N)$ such that*

$$U_{\varepsilon_j} \to U \qquad \text{in } C^{1+\gamma,\gamma}_{loc}(\overline{\Omega} \times (0,\infty)) \tag{6.82}$$

as $j \to \infty$.

Proof. Let us fix $\tau \in (0,\infty)$. We introduce a smooth, nondecreasing function $\xi\colon \mathbb{R} \to \mathbb{R}$ which satisfies $\xi(t) = 0$ for $t \leq \tau$ and $\xi(t) = 1$ for $t \geq 2\tau$ and will consider the functions ξU_ε with $\varepsilon \in (0,1)$ in the following. Given $s \in (1,\infty)$, [34, Thm. 2.8] provides $c_1 = c_1(s,\Omega)$ such that, for any $\varepsilon \in (0,1)$, ξU_ε, being a solution of the Stokes equation with right-hand side $\mathscr{P}(\xi(U_\varepsilon \cdot \nabla)U_\varepsilon) + \mathscr{P}(\xi u_\varepsilon \nabla\Phi) + \mathscr{P}(\xi' U_\varepsilon)$ satisfies

$$\int_\tau^T \|(\xi U_\varepsilon)_t\|_{L^s(\Omega)}^s + \int_\tau^T \|D^2(\xi U_\varepsilon)\|_{L^s(\Omega)}^s$$
$$\leq c_1 \left(0 + \int_\tau^T \|\mathscr{P}(\xi U_\varepsilon \cdot \nabla)U_\varepsilon + \mathscr{P}\xi(u_\varepsilon - \overline{u}_0)\nabla\Phi + \mathscr{P}\xi' U_\varepsilon\|_{L^s(\Omega)}^s \right)$$

for any $T > \tau$. From the exponential decay of $\|u_\varepsilon - \overline{u}_0\|_{L^\infty(\Omega)}$ and of $\|U_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ as stated in (6.76) we obtain the existence of $c_2, c_3 > 0$ such that for any $\varepsilon \in (0,1)$

$$\int_\tau^T \|(\xi U_\varepsilon)_t\|_{L^s(\Omega)}^s + \int_\tau^T \|D^2(\xi U_\varepsilon)\|_{L^s(\Omega)}^s \leq c_2 + c_3 \int_\tau^T \|\nabla(\xi U_\varepsilon)\|_{L^s(\Omega)}^s \qquad \text{for any } T > \tau. \quad (6.83)$$

Let $s > N$ and fix $r \in (1,s)$, so that $\frac{1}{N} + \frac{1}{r} - \frac{1}{s} > 0$. Defining

$$a = \frac{\frac{1}{N} + \frac{1}{r} - \frac{1}{s}}{\frac{2}{N} + \frac{1}{r} - \frac{1}{s}},$$

we then observe that $a \in (\frac{1}{2}, 1)$ and hence the Gagliardo-Nirenberg inequality yields a constant $c_4 > 0$ such that

$$\|\nabla(\xi U_\varepsilon)(\cdot,t)\|_{L^s(\Omega)}^s \leq c_4 \|D^2(\xi U_\varepsilon)(\cdot,t)\|_{L^s(\Omega)}^{as} \|(\xi U_\varepsilon)(\cdot,t)\|_{L^r(\Omega)}^{(1-a)s} \qquad \text{for all } t \in (0,T)$$

and an application of this together with the L^∞-estimate for U_ε from (6.76) and Hölder's inequality in (6.83) shows that there is $c_5 > 0$ such that for any $T > \tau$ and any $\varepsilon \in (0,1)$

$$\int_\tau^T \|(\xi U_\varepsilon)_t\|_{L^s(\Omega)}^s + \int_\tau^T \|D^2(\xi U_\varepsilon)\|_{L^s(\Omega)}^s \leq c_2 + c_5 |T-\tau|^{1-a} \left(\int_\tau^T \|D^2(\xi U_\varepsilon)\|_{L^s(\Omega)}^s \right)^a,$$

and we can conclude boundedness of $\|D^2(\xi U_\varepsilon)\|_{L^s(\tau,T;L^s(\Omega))}$ and then of $\|(\xi U_\varepsilon)_t\|_{L^s(\tau,T;L^s(\Omega))}$ with bounds independent of ε.
All in all, for any $s > 1$ and any $T > 2\tau$, there is $c_6 > 0$ such that for any $t \in (2\tau, T)$ and any $\varepsilon \in (0,1)$

$$\|U_{\varepsilon t}\|_{L^s((t,T);L^s(\Omega))} + \|U_\varepsilon\|_{L^s((t,T);W^{2,s}(\Omega))} \leq c_6. \quad (6.84)$$

Now, letting $\gamma' \in (0,1)$, using appropriately large s and referring to [2, Thm 1.1], for any $T > 0$ we obtain a constant $c_7 > 0$ so that

$$\|U_\varepsilon\|_{C^{1+\gamma',\gamma'}(\overline{\Omega}\times(t,T))} = \|\xi U_\varepsilon\|_{C^{1+\gamma',\gamma'}(\overline{\Omega}\times(t,T))} \leq c_7 \qquad \text{for all } t \in (2\tau, T).$$

Therefore, for any $\tau > 0$, $T > 2\tau$ we can find a subsequence of the sequence from Lemma 6.5.3 such that $U_\varepsilon \to U$ and $\nabla U_\varepsilon \to \nabla U$ in $C^{\gamma,\gamma}(\overline{\Omega}\times(t,T))$ for some $\gamma < \gamma'$ and for any $t \in (2\tau, T)$. □

For U, this lemma already covers the convergence of first spatial derivatives. Also concerning u and v, at least some kind of convergence of these quantities seems desirable. For the fluid velocity field, in fact, slightly higher derivatives are of interest. We obtain convergence for these in the following lemma:

Lemma 6.5.5. *There exists a subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j\to\infty$ of the sequence from Lemma 6.5.4 such that*

$$\nabla v_\varepsilon \overset{\star}{\rightharpoonup} \nabla v \qquad \text{in } L^\infty((0,\infty),L^{q_0}(\Omega)), \tag{6.85}$$

$$\nabla v_\varepsilon \rightharpoonup \nabla v \qquad \text{in } L^2(\Omega\times(0,\infty)), \tag{6.86}$$

$$U_\varepsilon \overset{\star}{\rightharpoonup} U \qquad \text{in } L^\infty((0,\infty),D(A^\beta)), \tag{6.87}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \qquad \text{in } L^2(\Omega\times(0,\infty)), \tag{6.88}$$

$$u_\varepsilon S(\cdot,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon \rightharpoonup uS(\cdot,u,v)\nabla v \qquad \text{in } L^1_{loc}(\Omega\times(0,\infty)), \tag{6.89}$$

$$u_{\varepsilon t} \rightharpoonup u_t \qquad \text{in } L^2((0,\infty),(W_0^{1,2}(\Omega))^*), \tag{6.90}$$

$$v_{\varepsilon t} \rightharpoonup v_t \qquad \text{in } L^2((0,\infty),(W_0^{1,2}(\Omega))^*), \tag{6.91}$$

$$U_{\varepsilon t} \rightharpoonup U_t \qquad \text{in } L^2((0,\infty),(W_{0,\sigma}^{1,2}(\Omega))^*). \tag{6.92}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Proof. From (6.76) we know that there is $c_1>0$ such that for all $\varepsilon\in(0,1)$

$$\|\nabla v_\varepsilon\|_{L^\infty((0,\infty),L^{q_0}(\Omega))} \le c_1.$$

Therefore we may conclude the existence of a sequence satisfying (6.85); this also entails (6.86). By the same reasoning we can use the $\varepsilon-$independent bound on $\|U_\varepsilon\|_{L^\infty((0,\infty),D(A^\beta))}$ given by (6.77) to extract a subsequence satisfying (6.87).
Concerning convergence of ∇u_ε, we multiply the first equation of (6.75) by u_ε so as to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega u_\varepsilon S_\varepsilon \nabla v_\varepsilon\cdot\nabla u_\varepsilon \le \frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2 + \frac{1}{2}\|u_\varepsilon\|^2_{L^\infty((0,\infty)\times\Omega)} C_S^2 \int_\Omega |\nabla v_\varepsilon|^2.$$

for any $\varepsilon\in(0,1)$ and on the whole time-interval $(0,\infty)$. Integrating this with respect to time and taking into account the exponential bound on $\int_\Omega|\nabla v_\varepsilon|^2$ and the uniform L^∞-bound on u_ε from (6.76) , we establish that

$$\sup_{\varepsilon\in\{\varepsilon_j\}_{j\in\mathbb{N}}} \int_0^\infty\int_\Omega |\nabla u_\varepsilon|^2 < \infty \tag{6.93}$$

and hence can find a subsequence of the previously extracted sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ along which (6.88) holds.
Because by Lemma 6.5.3, $u_\varepsilon\to u$ and $S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\to S(\cdot,u,v)$ pointwise and u_ε and $S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)$ both are bounded uniformly in ε due to (6.76) and (6.5) combined with (6.73), from Lebesgue's dominated convergence theorem we conclude that $u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\to uS(\cdot,u,v)$ in $L^2_{loc}(\Omega\times(0,\infty))$. Combined with (6.86), this gives (6.89). Turning our attention to the time derivatives, we let $\psi\in C_0^\infty(\Omega)$ with $\|\psi\|_{W^{1,2}(\Omega)}\le 1$ and test the first equation of (6.75) with ψ. We obtain

$$\begin{aligned}&\left|\int_\Omega (u_\varepsilon)_t\psi\right|\\ &= \left|-\int_\Omega \nabla u_\varepsilon\cdot\nabla\psi + \int_\Omega u_\varepsilon S_\varepsilon\nabla v_\varepsilon\cdot\nabla\psi + \int_\Omega u_\varepsilon U_\varepsilon\cdot\nabla\psi\right|\\ &\le \left(\left(\int_\Omega|\nabla u_\varepsilon|^2\right)^{\frac12} + \|u_\varepsilon\|_{L^\infty(\Omega)}C_S\left(\int_\Omega|\nabla v_\varepsilon|^2\right)^{\frac12} + \|u_\varepsilon\|_{L^\infty(\Omega)}\left(\int_\Omega|U_\varepsilon|^2\right)^{\frac12}\right)\left(\int_\Omega|\nabla\psi|^2\right)^{\frac12}\end{aligned}$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$. From the definition of the norm in dual spaces and Young's inequality, we derive that

$$\int_0^\infty \|u_{\varepsilon t}\|^2_{(W_0^{1,2}(\Omega))^*} \le 3\int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 + 3\|u_\varepsilon\|^2_{L^\infty(\Omega\times(0,\infty))} C_S^2 \int_0^\infty \int_\Omega |\nabla v_\varepsilon|^2 \\ + 3\|u_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}{}^2 \int_0^\infty \int_\Omega |U_\varepsilon|^2$$

for all $\varepsilon \in (0,1)$. Taking into account (6.93) and (6.76), we thus obtain $c_2 > 0$ such that

$$\|u_{\varepsilon t}\|_{L^2((0,\infty),(W_0^{1,2}(\Omega))^*)} \le c_2, \qquad \text{for all } \varepsilon \in (0,1),$$

and may extract a further subsequence such that (6.90) holds. The same reasoning applied to the second equation of (6.75) leads to (6.91). As to the third equation, employing (6.76) and (6.77) and repeating the procedure with some $\psi \in C_0^\infty(\Omega)$, we easily obtain uniform boundedness of $\int_0^\infty \|U_{\varepsilon t}\|^2_{(W^{1,2}_{0,\sigma}(\Omega))^*}$ (where $W^{1,2}_{\sigma,0}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}$) and may conclude (6.92) along a subsequence. □

Lemma 6.5.6. *The functions u, v, U from Lemma 6.5.3 and Lemma 6.5.4 form a weak solution to* (6.2) *in the sense of Definition 6.5.2.*

Proof. The convergence properties exhibited in (6.80), (6.88), (6.89), (6.82), (6.81) and (6.86) enable us to pass to the limit in the integral identities (6.79) for $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$ for any $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$. □

Moreover, these weak solutions obey the desired decay estimates.

Lemma 6.5.7. *With C_8, C_9, C_{10} and C_{11} as in Lemma 6.5.1, the functions u, v, U obtained from Lemma 6.5.3 and Lemma 6.5.4 obey the estimates*

$$\|v(\cdot,t)\|_{W^{1,q_0}(\Omega)} \le 2C_{11}e^{-\alpha_1 t}, \qquad \textit{for almost every } t > 0, \tag{6.94}$$
$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le C_{10}e^{-\alpha_1 t}, \qquad \textit{for every } t > 0, \tag{6.95}$$
$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le C_9 e^{-\alpha_2 t}, \qquad \textit{for every } t > 0, \tag{6.96}$$
$$\|U(\cdot,t)\|_{D(A^\beta)} \le C_8 e^{-\alpha_2 t}, \qquad \textit{for almost every } t > 0. \tag{6.97}$$

Proof. The estimates (6.95), (6.96) and a corresponding estimate for $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ result from (6.76) and the pointwise convergence entailed by Lemma 6.5.3 and Lemma 6.5.4. For $t > 0$ we let $\chi_{[t,\infty)}$ denote the characteristic function of the interval $[t,\infty)$ and observe that due to (6.85) also $\chi_{[t,\infty)}\nabla v_\varepsilon \overset{\star}{\rightharpoonup} \chi_{[t,\infty)}\nabla v$ in $L^\infty((0,\infty), L^{q_0}(\Omega))$ as $\varepsilon = \varepsilon_j \searrow 0$, and therefore

$$\|\nabla v\|_{L^\infty([t,\infty),L^{q_0}(\Omega))} = \|\chi_{[t,\infty)}\nabla v\|_{L^\infty((0,\infty),L^{q_0}(\Omega))} \\ \le \liminf_{j\to\infty} \|\chi_{[t,\infty)}\nabla v_\varepsilon\|_{L^\infty((0,\infty),L^{q_0}(\Omega))} \le C_{11}e^{-\alpha t}$$

for all $t > 0$, so that (6.94) results. The estimate (6.97) follows from (6.87) and (6.77) by the same reasoning. □

Naturally, in our search for classical solutions we are much more interested in obtaining smoothness of higher order than in these boundedness assertions.

Lemma 6.5.8. *The functions u, v, U from the previous lemmata satisfy*

$$u \in C^{2+\gamma,1+\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times (0,\infty)), \tag{6.98}$$

$$v \in C^{2+\gamma,1+\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times (0,\infty)), \text{ and} \tag{6.99}$$

$$U \in C^{2+\gamma,1+\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times (0,\infty)) \tag{6.100}$$

for some $\gamma > 0$.

Proof. We fix $\tau > 0$ and $T > 3\tau$. Moreover we choose a smooth function $\xi \colon \mathbb{R} \to [0,1]$ such that $\xi(t) = 0$ for $t \le 2\tau$ and $\xi(t) = 1$ for all $t \ge 3\tau$. Then we consider the problem

$$\begin{cases} \mathcal{L}w = w_t - \Delta w + U \cdot \nabla w = -\xi v + \xi u + \xi_t v =: f \quad \text{on } (\tau, T) \\ w(\cdot,\tau) = 0, \qquad \partial_\nu(w)\big|_{\partial\Omega} = 0 \end{cases}$$

of which clearly $w = \xi v$ is a weak solution. The coefficients of the parabolic operator $\mathcal{L}$ are Hölder-continuous in $\overline{\Omega} \times [\tau, T]$ by Lemma 6.5.4 and so is f (by Lemma 6.5.3). If combined with the uniqueness result for weak solutions in [50, Thm. III.5.1], Theorem IV.5.3 of [50] therefore asserts that $\xi v \in C^{2+\gamma_1,1+\frac{\gamma_1}{2}}(\overline{\Omega}\times[\tau,T])$ for some $\gamma_1 > 0$ and we conclude that $v \in C^{2+\gamma_1,1+\frac{\gamma_1}{2}}(\overline{\Omega}\times[3\tau,T])$ and finally (6.99).
When attempting to apply the same theorem to u (or ξu, similar as before), however, we face the additional difficulty that it requires $C^{1+\gamma,\frac{1+\gamma}{2}}$-regularity of the boundary values, whereas at this point we cannot guarantee more than $C^{\gamma,\frac{\gamma}{2}}$-regularity because of the involvement of u in the argument of S in the boundary condition. We apply (6.88) and (6.95) to see that u has the regularity properties needed for an application of [56, Thm. 1.1], which then guarantees that $u \in C^{1+\gamma_2,\frac{1+\gamma_2}{2}}(\overline{\Omega} \times (0,T))$ for some $\gamma_2 > 0$ and with that we can use [50, Thm. IV.5.3] in the same way as before and conclude (6.98).
Turning our attention to the function U we observe that $\xi(U \cdot \nabla)U + \xi u \nabla\Phi + \xi' U \in C^{\gamma_3,\frac{\gamma_3}{2}}(\overline{\Omega} \times (0,T))$ for some $\gamma_3 > 0$ by Lemma 6.5.4 and (6.98) and hence the same holds true for $\mathscr{P}(\xi(U \cdot \nabla)U + \xi u\nabla\Phi + \xi' U)$ by Lemma 6.A.1. Therefore the Schauder estimates for Stokes' equation given in [78, Thm. 1.1], if combined with the uniqueness result in [76, Thm. V.1.5.1], assert that ξU, being a solution to $(\xi U)_t = \Delta(\xi U) + \mathscr{P}[\xi(U\cdot\nabla)U + \xi u\nabla\Phi + \xi' U]$, $\nabla \cdot (\xi U) = 0$, belongs to the space $C^{2+\gamma_3,1+\frac{\gamma_3}{2}}(\overline{\Omega} \times (0,T))$ for some $\gamma_3 > 0$ and hence $u \in C^{2+\gamma_3,1+\frac{\gamma_3}{2}}(\overline{\Omega} \times [3\tau,T])$, so that we finally arrive at (6.100). ☐

Having obtained this smoothness, we can quickly fill in the missing information to see that u, v, U are as regular as required of classical solutions.

Lemma 6.5.9. *The functions u, v, U satisfy*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap L^\infty((0,\infty); W^{1,q_0}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \text{ and} \\ U \in C^0(\overline{\Omega} \times [0,\infty)) \cap L^\infty((0,\infty); D(A^\beta)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)). \end{cases} \tag{6.101}$$

Proof. For each of the functions, $C^{2,1}$-regularity follows from Lemma 6.5.8. That $v \in L^\infty((0,\infty), W^{1,q_0}(\Omega))$ and $U \in L^\infty((0,\infty), D(A^\beta))$ is asserted by (6.94) and (6.97), respectively. Therefore we are left with the task of proving the continuity at $t = 0$. From (6.94) and (6.91) we know that for $T > 0$ we have $v \in L^\infty((0,T), W^{1,q_0}(\Omega))$ and $v_t \in L^2((0,T), (W^{1,2}_0(\Omega))^*)$, where $W^{1,q_0}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega}) \hookrightarrow (W^{1,2}_0(\Omega))^*$, so that a well-known embedding result (see e.g. [75, Cor.

8.4]) assures us that $v \in C^0(\overline{\Omega} \times [0,T])$. For U we observe that $D(A^\beta) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$ and (6.92) and (6.97) once more make [75, Cor. 8.4] applicable. In order to show continuity of u at $t = 0$, we note that according to (6.76), there is $c_1 > 0$ such that $\|u_\varepsilon S(\cdot, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon - U_\varepsilon u_\varepsilon\|_{L^{q_0}(\Omega)} \le c_1$ for any $\varepsilon \in (0,1)$ and any $t > 0$. Consequently, for any $\varepsilon \in (0,1)$ and any $t > 0$, we have

$$\begin{aligned}
&\|u_\varepsilon(\cdot,t) - e^{t\Delta}u_0\|_{L^\infty(\Omega)} \\
\le& \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon S(\cdot,u_\varepsilon(\cdot,s),v_\varepsilon(\cdot,s))\nabla v_\varepsilon(\cdot,s) + u_\varepsilon(\cdot,s)U_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds \\
\le& \int_0^t k_4(1+(t-s)^{-\frac{1}{2}-\frac{N}{2q_0}})e^{-\lambda_1(t-s)} \\
&\qquad \times \|u_\varepsilon(\cdot,s)S(\cdot,u_\varepsilon(\cdot,s),v_\varepsilon(\cdot,s))\nabla v_\varepsilon(\cdot,s) + u_\varepsilon(\cdot,s)U_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
\le& c_1k_4\left(t + \int_0^t s^{-\frac{1}{2}-\frac{N}{2q_0}}ds\right).
\end{aligned}$$

Given $\zeta > 0$ we then fix $\delta > 0$ such that $\|e^{t\Delta}u_0 - u_0\|_{L^\infty(\Omega)} \le \frac{\zeta}{3}$ and $t + \int_0^t s^{-\frac{1}{2}-\frac{N}{2q_0}}ds < \frac{\zeta}{3c_1k_4}$ for all $t \in (0,\delta)$. Then using the uniform convergence $u_{\varepsilon_j}(\cdot,t) \to u(\cdot,t)$ as $j \to \infty$ asserted by Lemma 6.5.3 we pick ε_j such that $\|u(\cdot,t) - u_{\varepsilon_j}(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\zeta}{3}$. Then

$$\|u(\cdot,t)-u_0\|_{L^\infty(\Omega)} \le \|u(\cdot,t)-u_{\varepsilon_j}(\cdot,t)\|_{L^\infty(\Omega)} + \|u_{\varepsilon_j}(\cdot,t)-e^{t\Delta}u_0\|_{L^\infty(\Omega)} + \|e^{t\Delta}u_0-u_0\|_{L^\infty(\Omega)} < \zeta$$

for all $t \in (0,\delta)$. Thus the proof is complete. □

In order to prove Theorem 6.1.1, we now only have to collect the results prepared during this section:

Proof of Theorem 6.1.1. Approximating S by functions S_ε as indicated in (6.74), Proposition 6.4.1 has ensured the existence of solutions $(u_\varepsilon, v_\varepsilon, U_\varepsilon, P_\varepsilon)$ with the properties asserted in Lemma 6.5.1. From the family of these approximate solutions, in Lemma 6.5.3, Lemma 6.5.4 and Lemma 6.5.5 we were able to extract a subsequence that converges to functions (u, v, U) in a suitable sense, which according to Lemma 6.5.6 form a global weak solution to (6.2) in the sense of Definition 6.5.2, according to Lemma 6.5.9 have all regularity properties required of a classical solution and by Lemma 6.5.7 exhibits the desired decay properties. The missing component P can be obtained from [76, Thm. V.1.8.1]. In light of the smoothness of U, u, Φ, the third equation of (6.2) asserts that $\nabla P \in C^0(\overline{\Omega} \times (0,T))$. □

6.A. Appendix

We have postponed the proof of Lemma 6.2.4, which mainly consists in elementary calculus, but is too central to the reasoning of the present work to be left unproven. We begin the Appendix by giving this proof. After that, we will take care of a result on the Helmholtz projection, which was used as tool in the proof of Lemma 6.5.8. Finally, this appendix contains a variant of Lemma 6.3.1 adapted to the needs of the proof of Theorem 6.1.2.

Proof of Lemma 6.2.4. The assertion can be proven similarly as in [105, Lemma 1.2]. A simple observation shows that for any $t \in [0,\infty)$

$$\int_0^t (1+s^{-\alpha})(1+(t-s)^{-\beta})e^{-\delta(t-s)}e^{-\gamma s}ds \le e^{-\delta t}\int_0^t e^{-(\gamma-\delta)s}ds + e^{-\delta t}\int_0^t s^{-\alpha}e^{-(\gamma-\delta)s}ds$$

$$+ e^{-\delta t}\int_0^t (t-s)^{-\beta} e^{-(\gamma-\delta)s} ds + e^{-\delta t}\int_0^t s^{-\alpha}(t-s)^{-\beta} e^{-(\gamma-\delta)s} ds. \tag{6.102}$$

In order to obtain estimates for the summands, independently of the values of $\alpha, \beta, \gamma, \delta$, we can start with the observation that

$$\int_0^t e^{-(\gamma-\delta)s} ds = \frac{1}{\gamma-\delta}[1-e^{-(\gamma-\delta)t}] \le \frac{1}{\eta}, \qquad t \in [0,\infty),$$

and continue by estimating

$$\int_0^t s^{-\alpha} e^{-(\gamma-\delta)s} ds \le \int_0^1 s^{-\alpha} ds + \int_1^\infty e^{-(\gamma-\delta)s} ds \le \frac{1}{1-\alpha} + \frac{1}{\gamma-\delta} \le \frac{2}{\eta} \quad \text{for } t \in [0,\infty).$$

Also in the third term on the right hand side of (6.102) we can split the integral and use the obvious estimates $(t-s)^{-\beta} \le 1$ for $s < t-1$ and $e^{-(\gamma-\delta)(t-\sigma)} \le e^{-(\gamma-\delta)(-\sigma)} \le e^{\gamma-\delta}$ for $\sigma \in (0,1)$ to obtain

$$\begin{aligned}\int_0^t (t-s)^{-\beta} e^{-(\gamma-\delta)s} ds &\le \int_0^t e^{-(\gamma-\delta)s} ds + \int_0^1 \sigma^{-\beta} e^{-(\gamma-\delta)(t-\sigma)} d\sigma \\ &\le \frac{1}{\gamma-\delta} + \frac{1}{1-\beta} e^{\gamma-\delta} \le \frac{1}{\eta} + \frac{1}{\eta} e^{\frac{1}{\eta}}\end{aligned}$$

for any $t \in [0,\infty)$. The last integral can be rewritten as

$$\int_0^t s^{-\alpha}(t-s)^{-\beta} e^{-(\gamma-\delta)s} ds = t^{1-\alpha-\beta}\int_0^1 \sigma^{-\alpha}(1-\sigma)^{-\beta} e^{-(\gamma-\delta)\sigma t} d\sigma, \quad t \in [0,\infty), \tag{6.103}$$

where we have

$$\begin{aligned}\int_0^1 \sigma^{-\alpha}(1-\sigma)^{-\beta} e^{-(\gamma-\delta)\sigma t} d\sigma &\le \int_0^1 \sigma^{-\alpha}(1-\sigma)^{-\beta} \\ &\le 2^\beta \int_0^{\frac{1}{2}} \sigma^{-\alpha} d\sigma + 2^\alpha \int_0^{\frac{1}{2}} \sigma^{-\beta} d\sigma \le \frac{2}{1-\alpha} + \frac{2}{1-\beta} \le \frac{4}{\eta},\end{aligned}$$

so that (6.103) yields the estimate we are aiming for if $1-\alpha-\beta \le 0$ or if $t < 1$ and $1-\alpha-\beta > 0$. As to $1-\alpha-\beta > 0$ and $t \ge 1$, we estimate

$$\begin{aligned}&\int_0^1 \sigma^{-\alpha}(1-\sigma)^{-\beta} e^{-(\gamma-\delta)\sigma t} d\sigma \\ &\le \int_0^{\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\alpha}}} \sigma^{-\alpha}(1-\sigma)^{-\beta} e^{-(\gamma-\delta)\sigma t} d\sigma + \int_{\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\alpha}}}^1 \sigma^{-\alpha}(1-\sigma)^{-\beta} e^{-(\gamma-\delta)\sigma t} d\sigma \\ &\le (1/2)^{-\beta}\int_0^{\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\alpha}}} \sigma^{-\alpha} d\sigma + \left(\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\alpha}}\right)^{-\alpha} e^{-(\gamma-\delta)\frac{1}{2}t^{1-\frac{1-\alpha-\beta}{1-\alpha}}} \int_{\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\alpha}}}^1 (1-\sigma)^{-\beta} d\sigma \\ &\le \frac{2^{\beta+\alpha-1}}{1-\alpha} t^{-(1-\alpha-\beta)} + \frac{2^\alpha}{1-\beta} t^{-(1-\alpha-\beta)} t^{1-\frac{\beta}{1-\alpha}} e^{-\frac{\gamma-\delta}{2} t^{\frac{\beta}{1-\alpha}}}.\end{aligned}$$

Here,

$$t^{1-\frac{\beta}{1-\alpha}} e^{-\frac{\gamma-\delta}{2} t^{\frac{\beta}{1-\alpha}}} \le 1 + t e^{-\frac{\gamma-\delta}{2} t^{\frac{\beta}{1-\alpha}}}, \qquad t \in [1,\infty),$$

where we have

$$\frac{\beta}{1-\alpha} \geq \beta, \qquad t^{\frac{\beta}{1-\alpha}} \geq t^{\beta} \geq t^{\eta},$$

because $t \geq 1$, and hence

$$te^{-\frac{\gamma-\delta}{2} t^{\frac{\beta}{1-\alpha}}} \leq te^{-\frac{\gamma-\delta}{2} t^{\beta}} \leq te^{-\frac{\eta}{2} t^{\eta}}, \qquad t \in [1, \infty),$$

which in combination with the finiteness of $\sup_{t>0} te^{-\frac{\eta}{2}t^{\eta}}$ implies the assertion. □

In order to obtain regularity of U, we have employed the following result in the proof of Lemma 6.5.8. Other than in [30], we are concerned with the impact of the Helmholtz projection on Hölder-continuous functions (instead of on functions belonging to some L^p-space only.)

Lemma 6.A.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega \in C^{1+\alpha}$ for some $\alpha > 0$, and let $T > 0$. Moreover let $g \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times [0,T])$. Then $g = h + w$, where $\nabla \cdot h = 0$ in Ω and $h \cdot \nu = 0$ on $\partial\Omega$ and $w = \nabla\Phi$ for some function Φ. Then $h \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times [0,T])$.*

Proof. We have to find a decomposition $g = h + w$ with $\nabla \cdot h = 0$ in Ω and $h \cdot \nu = 0$ on $\partial\Omega$ and $w = \nabla\Phi$ for some function Φ. We will construct w and conclude from its smoothness that $\mathscr{P}g = h = g - w \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times [0,T]; \mathbb{R}^N)$. As preparation let us consider the elliptic problem

$$\Delta\Phi = \nabla \cdot f, \qquad \nabla\Phi \cdot \nu\big|_{\partial\Omega} = f \cdot \nu\big|_{\partial\Omega}, \qquad \int_{\Omega} \Phi = 0. \tag{6.104}$$

Only assuming $f \in C^{\alpha}(\overline{\Omega})$, we fix $p > N$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then [74, Thm. 4.1], which mirrors the usual Lax-Milgram type result in the context of L^p-spaces also for $p \neq 2$, asserts the existence of a unique weak solution $\Phi \in \{\Phi \in W^{1,p}(\Omega), \int_{\Omega} \Phi = 0\}$ such that

$$\int_{\Omega} \nabla\Phi \cdot \nabla\varphi = \int_{\Omega} f \nabla\varphi \qquad \text{for all } \varphi \in W^{1,q}(\Omega).$$

Moreover, this solution satisfies

$$\begin{aligned} c_1 \|\Phi\|_{L^\infty(\Omega)} \leq & c_2 \|\Phi\|_{W^{1,p}(\Omega)} \leq \|\nabla\Phi\|_{L^p(\Omega)} \\ \leq & c_3 \sup\left\{ \frac{\left|\int_{\Omega} f\nabla\varphi\right|}{\|\nabla\varphi\|_{L^q(\Omega)}}; \varphi \in W^{1,q}(\Omega), \nabla\varphi \not\equiv 0 \right\} \leq c_3 \|f\|_{L^p(\Omega)} \leq c_4 \|f\|_{C^{\alpha}(\overline{\Omega})} \end{aligned} \tag{6.105}$$

with positive constants c_1, c_2, c_3 and c_4 that are guaranteed to exist by the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega)$, Poincaré inequality, [74, Thm. 4.1] and continuity of the embedding $C^{\alpha}(\Omega) \hookrightarrow L^p(\Omega)$, respectively. A standard elliptic regularity result (see [41, Thm. 2.8]) moreover asserts the existence of $c_5 > 0$ such that $C^{1+\alpha}$-solutions Φ of (6.104) satisfy

$$\|\Phi\|_{C^{1+\alpha}(\Omega)} \leq c_5 \left(\|f\|_{C^{\alpha}(\overline{\Omega})} + \|\Phi\|_{L^{\infty}(\Omega)} \right)$$

and thus, taking into account (6.105),

$$\|\Phi\|_{C^{1+\alpha}(\overline{\Omega})} \leq c_6 \|f\|_{C^{\alpha}(\overline{\Omega})}$$

with $c_6 := c_5(1 + \frac{c_4}{c_1})$.

Approximating $f \in C^{\alpha}(\overline{\Omega})$ by a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset C^{\infty}(\overline{\Omega})$ for which the existence of classical solutions $\Phi_n \in C^{2+\alpha}(\overline{\Omega})$ is asserted by well-known results ([51, Thm. 3.3.2]), we see that for $f \in C^\alpha(\overline{\Omega})$ problem (6.104) has a unique solution $\Phi \in C^{1+\alpha}(\overline{\Omega})$, which moreover satisfies

$$\|\Phi\|_{C^{1+\alpha}(\overline{\Omega})} \le c_6 \|f\|_{C^\alpha(\overline{\Omega})}. \tag{6.106}$$

For each t let $\Phi(\cdot,t)$ denote the solution of

$$\Delta\Phi(\cdot,t) = \nabla\cdot g(\cdot,t), \qquad \nabla\Phi(\cdot,t)\cdot\nu\big|_{\partial\Omega} = g(\cdot,t)\cdot\nu\big|_{\partial\Omega}, \qquad \int_\Omega \Phi = 0,$$

and define $w(\cdot,t) := \nabla\Phi(\cdot,t)$ and $h(\cdot,t) := g(\cdot,t) - w(\cdot,t)$, so that clearly $\nabla\cdot h = \nabla\cdot g - \nabla\cdot w = \nabla\cdot g - \Delta\Phi = 0$ in Ω and $h\cdot\nu = g\cdot\nu - w\cdot\nu = g\cdot\nu - \partial_\nu\Phi = 0$ on $\partial\Omega$. Concerning smoothness, we see that $\Phi(\cdot,t) \in C^{1+\alpha}(\overline{\Omega})$ entails $w(\cdot,t) \in C^\alpha(\overline{\Omega})$ and for $t_1,t_2 \in [0,T]$ we have that $\Phi(\cdot,t_2) - \Phi(\cdot,t_1) =: \Psi$ solves

$$\Delta\Psi = \nabla\cdot(g(\cdot,t_2) - g(\cdot,t_1)), \qquad \nabla\Psi\cdot\nu\big|_{\partial\Omega} = (g(\cdot,t_2) - g(\cdot,t_1))\cdot\nu, \qquad \int_\Omega \Psi = 0$$

so that by (6.106)

$$\|w(\cdot,t_2) - w(\cdot,t_1)\|_{C^\alpha(\overline{\Omega})} \le \|\Psi\|_{C^{1+\alpha}(\overline{\Omega})} \le c_6 \|g(\cdot,t_2) - g(\cdot,t_1)\|_{C^\alpha(\overline{\Omega})}.$$

By the known regularity of g, in conclusion we have $w \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])$ and thus $\mathscr{P}g = h = g - w \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[0,T];\mathbb{R}^N)$. $\square$

The last statement we have postponed to this appendix is concerned with the adaptions necessary for proving Theorem 6.1.2 instead of Theorem 6.1.1.

Lemma 6.A.2. *Given $M, N, p_0, q_0, \beta, C_S$ as in Theorem 1.2 and some $\delta > 0$, it is possible to choose M_1, M_2, M_3, M_4, $\epsilon > 0$, $m_0 < \epsilon|\Omega|^{-\frac{1}{p_0}}$ such that for all $m > m_0$, for all $\alpha_1 \in (\frac{m}{2}, \min\{m, \lambda_1 - \delta\})$ and $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1' - \delta\})$ the inequalities*

$$k_7(N,q_0) + k_5(q_0)k_7(q_0,q_0)(M_1 + 2k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}C_1 + 3k_7(\tfrac{N}{1+\frac{N}{q_0}}, q_0)k_5(\tfrac{N}{1+\frac{N}{q_0}})M_3M_4C_2\epsilon \le \frac{M_3}{2},$$

$$k_8(N,N) + k_8(N,N)k_5(N)|\Omega|^{\frac{q_0-N}{Nq_0}}(M_1 + 2k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}C_3$$
$$+ 3M_3M_4k_8(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\tfrac{1}{\frac{1}{q_0}+\frac{1}{N}})C_4\epsilon \le \frac{M_4}{2},$$

$$k_3 + C_5k_2(|\Omega|^{-\frac{1}{p_0}} + M_1 + 2k_1)Me^{(M_1+2k_1)\sigma\epsilon} + 3k_2M_2M_3C_6\epsilon \le \frac{M_2}{2} \quad \textit{and}$$

$$3C_SC_7k_4M_2\epsilon|\Omega|^{-\frac{1}{p_0}} + 3C_SC_7k_4M_2(M_1 + 2k_1)\epsilon + 3(M_1 + 2k_1)C_7k_4M_3\epsilon \le \frac{M_1}{2}$$

hold, where k_1, k_2, k_3, k_4, $k_5(\cdot)$, $k_7(\cdot,\cdot)$, $k_8(\cdot,\cdot)$ are taken from Lemmata 6.2.1, 6.2.2 and 6.2.3, and C_1, C_2, C_3, C_4, C_5, C_6, C_7 are the constants defined in Section 6.3.

Proof. The condition $m_0 < \epsilon|\Omega|^{-\frac{1}{p_0}}$ that is used to ensure the existence of initial data satisfying (6.9) compells us to choose m_0 at the end of this proof, quite in contrast to the situation in Lemma 6.3.1. Furthermore this makes it necessary to have the estimates during the proof hold regardless of the values of α_1, α_2, which depend on m. Fortunately, $C_1,\dots,C_7$ indeed do not depend on α_1, α_2 (and thus not on m), but – thanks to Lemma 6.2.4 – rather on (a lower bound for) the differences between μ and α_1, μ and α_2 or λ_1 and α_1. (This is the purpose δ has been introduced for.) The only remaining parameter is $\sigma = \sigma(\alpha_1) = \int_0^\infty (1 + s^{-\frac{N}{2p_0}})e^{-\alpha_1 s}ds$, which is decreasing with respect to α_1. If we decide to concentrate on relatively "large" values of α_1 only, namely $\alpha_1 > \frac{m}{2}$, (which is of no effect to the generality of Theorem 6.1.2), given $m > 0$, for any $\alpha_1 \in (\frac{m}{2}, \min\{m, \lambda_1 - \delta\})$, we may rely on

$$\sigma(\alpha_1) \leq \int_0^\infty \left(1 + s^{-\frac{N}{2p_0}}\right) e^{-\frac{m}{2}s}ds \leq 2\int_0^\infty e^{-\frac{m}{2}s}ds + \int_0^1 s^{-\frac{N}{2p_0}}ds \leq \frac{4}{m} + \frac{2p_0}{2p_0 - N}.$$

We pick arbitrary $M_1 > 0$ and $A > 1$ such that

$$A > (M_1 + 2k_1)\left(8|\Omega|^{\frac{1}{p_0}} + \frac{1}{1 - \frac{N}{2p_0}}\right). \tag{6.107}$$

Moreover, we can choose M_2 such that $k_3 + C_5 k_2(|\Omega|^{-\frac{1}{p_0}} + M_1 + 2k_1)Me^A A \leq \frac{M_2}{4}$ and M_3 such that $k_7(N, q_0) + k_5(q_0)k_7(q_0, q_0)(M_1 + 2k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}C_1 \leq \frac{M_3}{4}$, and we choose M_4 such that $k_8(N, N) + k_8(N, N)k_5(N)|\Omega|^{\frac{q_0 - N}{Nq_0}}(M_1 + 2k_1)\|\nabla\Phi\|_{L^\infty(\Omega)}C_3 \leq \frac{M_4}{4}$. Then we let

$$0 < \epsilon < \min\left\{A, \frac{1}{12k_2M_3C_6}, \frac{1}{12M_3k_8(\frac{1}{\frac{1}{q_0}+\frac{1}{N}}, N)k_5(\frac{1}{\frac{1}{q_0}+\frac{1}{N}})C_4}, \frac{1}{12k_7(\frac{N}{1+\frac{N}{q_0}}, q_0)k_5(\frac{N}{1+\frac{N}{q_0}})C_2M_4},\right.$$
$$\left.\frac{M_1}{2(3C_SC_7k_4M_2(|\Omega|^{-\frac{1}{p_0}} + M_1 + 2k_1) + 3(M_1 + 2k_1)C_7k_4M_3)}, 1\right\}$$

Finally, we want to choose $m_0 < \epsilon|\Omega|^{-\frac{1}{p_0}}$ such that $(M_1 + 2k_1)\sigma(\alpha_1)\epsilon < A$ for all $\alpha_1 \in (\frac{m}{2}, \min\{m, \lambda_1 - \delta\})$, for all $m > m_0$. This is indeed feasible, since $\sigma(\frac{\epsilon}{2}|\Omega|^{-\frac{1}{p_0}}) < \frac{A}{(M_1+2k_1)\epsilon}$ due to

$$\epsilon\sigma\left(\frac{\epsilon}{2}|\Omega|^{-\frac{1}{p_0}}\right) < \epsilon\left(\frac{8}{\epsilon|\Omega|^{-\frac{1}{p_0}}} + \frac{2p_0}{2p_0 - N}\right) \leq 8|\Omega|^{\frac{1}{p_0}} + \frac{2p_0}{2p_0 - N} < \frac{A}{M_1 + 2k_1}$$

and by continuity we can find $m_0 < \epsilon|\Omega|^{-\frac{1}{p_0}}$ so that $\sigma(\frac{m_0}{2}) < \frac{A}{(M_1+2k_1)\epsilon}$. With this choice, for all $\alpha_1 \in (\frac{m}{2}, \min\{m, \lambda_1 - \delta\})$, for all $m > m_0$, we have $\sigma(\alpha_1) < \sigma(\frac{m}{2}) < \sigma(\frac{m_0}{2}) < \frac{A}{(M_1+2k_1)\epsilon}$. $\square$

7. Boundedness enforced by small signal concentrations in chemotaxis-fluid models

7.1. Introduction

In this chapter, we continue to study the chemotaxis(-Navier)-Stokes system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (uS(x,u,v)\cdot \nabla v) - U \cdot \nabla u, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v - uv - U \cdot \nabla v, & (x,t) \in \Omega \times (0,T), \\ U_t = \Delta U - \kappa (U \cdot \nabla) U + \nabla P + u \nabla \Phi, & (x,t) \in \Omega \times (0,T), \\ \nabla \cdot U = 0, & (x,t) \in \Omega \times (0,T), \\ \nabla v \cdot \nu = (\nabla u - S(x,u,v)\nabla v)\cdot \nu = 0, U = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), U(x,0) = U_0(x), & x \in \Omega, \end{cases} \tag{7.1}$$

where $T \in (0,\infty]$, $\kappa = 0,1$, $\Omega \subset \mathbb{R}^N$ $(N = 2,3)$ is a bounded domain with smooth boundary and ν denotes the outward normal vector on $\partial\Omega$. Here $S(x,u,v) = (s_{ij}(x,u,v))_{i,j\in\{1,..,N\}}$ is a matrix-valued function and $\Phi \in C^{1+\delta}(\overline{\Omega})$ for some $\delta \in (0,1)$.

The purpose of the present chapter is to study this full chemotaxis-Navier-Stokes system with tensor-valued sensitivity in dimension 2 and the corresponding chemotaxis-Stokes system in dimension 3. When a natural Lyapunov functional (6.3) is lacking, we impose a smallness assumption on the initial data to obtain some uniform bound for the solution. Under this assumption, we can prove global existence of a classical solution and its large time behavior. Compared with Chapter 6([13]), the smallness condition here is only on $\|v_0\|_{L^\infty(\Omega)}$, meaning that small concentration of oxygen can enforce stability. This result coincides with the fluid-free system in [55]. The convexity of the physical domain is unnecessary in this paper since we use an approach different from that in many previous works, e.g. [108].

Throughout this chapter, as in Chapter 6 we assume that

$$s_{ij} \in C^2(\overline{\Omega} \times [0,\infty) \times [0,\infty)), \tag{7.2}$$

$$|S(x,u,v)| := \max_{i,j\in\{1,\dots,N\}} |s_{ij}(x,u,v)| \le S_0(v) \text{ for all } (x,u,v) \in \overline{\Omega} \times [0,\infty) \times [0,\infty), \tag{7.3}$$

where S_0 is a non-decreasing function on $[0,\infty)$. Again A denotes the Stokes operator under Dirichlet boundary conditions in Ω. The initial data are supposed to satisfy

$$\begin{cases} u_0 \in L^\infty(\Omega), \\ v_0 \in W^{1,q_0}(\Omega), \ q_0 > N, \\ U_0 \in D(A^\beta), \text{ for some } \beta \in (\frac{N}{4}, 1), \end{cases} \tag{7.4}$$

and

$$u_0 \geq 0, \;\; v_0 > 0 \text{ on } \Omega. \tag{7.5}$$

Under the above assumptions and notations, our main result is as follows:

Theorem 7.1.1. *Let $N \in \{2,3\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that S fulfills (7.2-7.3) and that one of the following conditions holds*
i) $N = 2$, $\kappa = 1$;
ii) $N = 3$, $\kappa = 0$.
Then there is $\delta_0 > 0$ with the following property: If the initial data fulfill (7.4-7.5) and are such that

$$\|v_0\|_{L^\infty(\Omega)} < \delta_0, \tag{7.6}$$

then (7.1) admits a global classical solution (u, v, U, P) with

$$\begin{cases} u \in C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right), \\ v \in C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap L^\infty\left((0,\infty); W^{1,q_0}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right), \\ U \in C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap L^\infty\left((0,\infty); D(A^\beta)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right), \\ P \in C^{1,0}\left(\overline{\Omega} \times (0,\infty)\right), \end{cases} \tag{7.7}$$

for which $u \geq 0$ and $v \geq 0$ in $\Omega \times (0,\infty)$. Moreover, one can find $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q_0}(\Omega)} + \|U(\cdot,t)\|_{D(A^\beta)} \leq C \text{ for all } t > 0.$$

Remark 7.1.2. *The uniqueness of classical solutions in the indicated class can be proved similarly as in [108].*

Apart from boundedness and global existence, we can also show the convergence of this clasical solution to the homogenous equilibrium.

Corollary 7.1.3. *Let the assumptions of Theorem 7.1.1 hold. Then (u, v, U) fulfills*

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot,t)\|_{W^{1,q_0}(\Omega)} \to 0, \text{ and } \|U(\cdot,t)\|_{L^\infty(\Omega)} \to 0$$

as $t \to \infty$.

Remark 7.1.4. *It is not difficult to show the convergence rates are exponential by using the uniqueness of the solutions and applying Theorem 6.1.1.*

We note that compared with the result in [112], Theorem 7.1.1 furthermore has restrictions on the size of initial data in the form of (7.6). As a subcase of (7.1), known results on the corresponding fluid-free version are not yet rich: Without assuming small data, the global generalized solutions constructed in [115] still possibly become unbounded at intermediate times; only additionally assuming $\|v_0\|_{L^\infty(\Omega)}$ small, global classical solutions are known to exist and blow-up is entirely ruled out [55]. When the system is coupled to fluid components, our results give the same condition which guarantee the global existence of smooth solutions.

The plan of this chapter is as follows: In Section 7.2, we approximate the problem by a system a priori known as globally well-posed (see (7.11) later). Sections 7.3-7.5 are devoted to studying the boundedness of solutions to this regularized problem, and we will see that the bounds are independent of the regularization parameter. Thus upon appropriate estimates, we can obtain limit functions of solutions to the regularized problems. This procedure is carried out in Section 7.6, and then also these limit functions are shown to be smooth enough and solve (7.1) classically for any positive time. In Section 7, the stabilization of the solution is given.

7.2. Approximation

Since again it is convenient to deal with the Neumann boundary conditions for both u and v, we employ the same approximation procedure as in Chapter 6 by fixing a family $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ of functions satisfying

$$\rho_\varepsilon \in C_0^\infty(\Omega) \quad \text{with} \quad 0 \le \rho_\varepsilon \le 1 \text{ in } \Omega \text{ and } \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon \searrow 0, \tag{7.8}$$

and defining

$$S_\varepsilon(x, u_\varepsilon, v_\varepsilon) = \rho_\varepsilon(x) S(x, u_\varepsilon, v_\varepsilon), \quad x \in \overline{\Omega}, u_\varepsilon > 0, v_\varepsilon > 0. \tag{7.9}$$

Then we have $S_\varepsilon(x, u_\varepsilon, v_\varepsilon) = 0$ on $\partial\Omega$ and, by (7.3)

$$|S_\varepsilon(x, u_\varepsilon, v_\varepsilon)| \le S_0(\|v_0\|_{L^\infty(\Omega)}) \quad \text{for all } x \in \Omega,\ u_\varepsilon > 0,\ v_\varepsilon > 0, \tag{7.10}$$

for $\varepsilon \in (0,1)$. Now we consider the following regularized problem

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon) - U_\varepsilon \cdot \nabla u_\varepsilon, & (x,t) \in \Omega \times (0, T_{\max,\varepsilon}), \\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon - U_\varepsilon \cdot \nabla v_\varepsilon, & (x,t) \in \Omega \times (0, T_{\max,\varepsilon}), \\ U_{\varepsilon t} = \Delta U_\varepsilon - \kappa (U_\varepsilon \cdot \nabla) U_\varepsilon + \nabla P_\varepsilon + u_\varepsilon \nabla \Phi, & (x,t) \in \Omega \times (0, T_{\max,\varepsilon}), \\ \nabla \cdot U_\varepsilon = 0, & (x,t) \in \Omega \times (0, T_{\max,\varepsilon}), \\ \nabla u_\varepsilon \cdot \nu = \nabla v_\varepsilon \cdot \nu = 0,\ U_\varepsilon = 0, & (x,t) \in \partial\Omega \times (0, T_{\max,\varepsilon}), \\ u_\varepsilon(x,0) = u_0(x), v_\varepsilon(x,0) = v_0(x), U_\varepsilon(x,0) = U_0(x), & x \in \Omega. \end{cases} \tag{7.11}$$

Without essential difficulty, we apply Lemma 6.2.8 to the above system to see its local solvability in the classical sense. For convenience, we summarize as follows:

Lemma 7.2.1. *Let $N \in \{2,3\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and $\kappa \in \mathbb{R}$. Assume that the initial data (u_0, v_0, U_0) satisfy (7.4) and (7.5), and that S fulfills (7.2-7.3). Then there exist $T_{\max,\varepsilon} \in (0,\infty]$ and a unique classical solution $(u_\varepsilon, v_\varepsilon, U_\varepsilon, P_\varepsilon)$ to (7.11) satisfying*

$$\begin{cases} u_\varepsilon \in C^0\left(\overline{\Omega} \times [0, T_{\max,\varepsilon})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{\max,\varepsilon})\right), \\ v_\varepsilon \in C^0\left(\overline{\Omega} \times [0, T_{\max,\varepsilon})\right) \cap L^\infty_{loc}\left([0, T_{\max,\varepsilon}); W^{1,q_0}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{\max,\varepsilon})\right), \\ U_\varepsilon \in C^0\left(\overline{\Omega} \times [0, T_{\max,\varepsilon})\right) \cap L^\infty_{loc}\left([0, T_{\max,\varepsilon}); D(A^\beta)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{\max,\varepsilon})\right), \\ P_\varepsilon \in C^{1,0}\left(\overline{\Omega} \times (0, T_{\max,\varepsilon})\right) \end{cases} \tag{7.12}$$

and $u_\varepsilon > 0$, $v_\varepsilon > 0$. Moreover, if $T_{\max,\varepsilon} < \infty$, then

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,q_0}(\Omega)} + \|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max,\varepsilon}. \tag{7.13}$$

In order to see the global existence and qualitative behavior of the solutions to the regularized problem, it is sufficient to show boundedness for each quantity in (7.13). The following lemma is obvious.

Lemma 7.2.2. *Let $(u_\varepsilon, v_\varepsilon, U_\varepsilon, P_\varepsilon)$ be a classical solution of (7.11). It follows that*

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{7.14}$$

$$\text{and } \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.15}$$

Proof. The mass conservation (7.14) is obtained by integrating the first equation of (7.11) on Ω and using the Neumann boundary condition. Since u_ε and v_ε are nonnegative, an application of the maximum principle to the second equation yields (7.15). □

We then obtain boundedness and global existence for the regularized problem (7.11).

Proposition 7.2.3. *Assume that S fulfills (7.2-7.3) and that one of the following conditions holds*

(i) $N = 2$, $\kappa = 1$*;*

(ii) $N = 3$, $\kappa = 0$.

There exists $\delta_0 > 0$ with the following property: If the initial data fulfill (7.4-7.5), and

$$\|v_0\|_{L^\infty(\Omega)} < \delta_0, \tag{7.16}$$

then (7.11) admits a global classical solution $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$. Moreover, there is $C > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C, \ \ \|v_\varepsilon(\cdot,t)\|_{W^{1,q_0}(\Omega)} \le C, \ \ \|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \tag{7.17}$$

for all $t \in (0,\infty)$ and all $\varepsilon \in (0,1)$.

We will prove boundedness for the 2-dimensional and 3-dimensional cases in Section 4 and Section 5, respectively. However, the $L^p(\Omega)$ estimate for u_ε derived in the next section will be applied to both.

7.3. An a priori estimate for u_ε

In this section, we obtain boundedness of u_ε in $L^p(\Omega)$ under the assumption that $\|v_0\|_{L^\infty(\Omega)}$ is suitably small. The approach is based on the weighted estimate of $\int_\Omega u_\varepsilon^p \varphi(v_\varepsilon)$ with appropriate choice of φ which has been developed in [104] and adapted to the consumed type signal in [83, 112].

Lemma 7.3.1. *Let $N \in \{2,3\}$ and $\kappa \in \mathbb{R}$. For any $p > 1$, there are $\delta_0 = \delta_0(p) > 0$ and $C = C(p) > 0$ with the following property: If the initial data satisfy (7.4)-(7.5) and*

$$\|v_0\|_{L^\infty(\Omega)} < \delta_0, \tag{7.18}$$

then for all $\varepsilon \in (0,1)$ we have

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \ \ \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{7.19}$$

$$\text{and} \ \int_0^{T_{\max,\varepsilon}} \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 \le C. \tag{7.20}$$

Proof. Let $p > 1$ and $0 < h < \frac{p-1}{12p}$. We then can find δ_0 satisfying

$$3p(p-1)\delta_0^2 S_0^2(\delta_0) \le h(h+1) \text{ and} \tag{7.21}$$

$$3p\delta_0 S_0(\delta_0) \le h+1, \tag{7.22}$$

where S_0 is the non-decreasing function introduced in (7.3). Under the assumption of (7.18), we can define $\varphi(v_\varepsilon) = (\delta_0 - v_\varepsilon)^{-h}$ according to (7.15), thus $\varphi(v_\varepsilon) > 0$. Elementary calculus shows that

$$\varphi'(v_\varepsilon) = h(\delta_0 - v_\varepsilon)^{-h-1} > 0, \tag{7.23}$$

$$\varphi''(v_\varepsilon) = h(h+1)(\delta_0 - v_\varepsilon)^{-h-2} > 0. \tag{7.24}$$

Using the first two equations in (7.11), upon integrating by parts we obtain that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p \varphi(v_\varepsilon) \\
&= \int_\Omega p u_\varepsilon^{p-1}\varphi(v_\varepsilon)(\Delta u_\varepsilon - \nabla\cdot(u_\varepsilon S_\varepsilon \cdot \nabla v_\varepsilon) - U_\varepsilon\cdot\nabla u_\varepsilon) + \int_\Omega u_\varepsilon^p \varphi'(v_\varepsilon)(\Delta v_\varepsilon - u_\varepsilon v_\varepsilon - U_\varepsilon \cdot \nabla v_\varepsilon) \\
&= -\int_\Omega \nabla u_\varepsilon \cdot (p(p-1)u_\varepsilon^{p-2}\varphi(v_\varepsilon)\nabla u_\varepsilon + p u_\varepsilon^{p-1}\varphi'(v_\varepsilon)\nabla v_\varepsilon) \\
&\quad + \int_\Omega u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon \cdot \big(p(p-1)\varphi(v_\varepsilon)u_\varepsilon^{p-2}\nabla u_\varepsilon + p u_\varepsilon^{p-1}\varphi'(v_\varepsilon)\nabla v_\varepsilon\big) \\
&\quad - \int_\Omega p u_\varepsilon^{p-1}\varphi(v_\varepsilon)U_\varepsilon\cdot\nabla u_\varepsilon - \int_\Omega \nabla v_\varepsilon\cdot(p u_\varepsilon^{p-1}\varphi'(v_\varepsilon)\nabla u_\varepsilon + u_\varepsilon^p\varphi''(v_\varepsilon)\nabla v_\varepsilon) \\
&\quad - \int_\Omega u_\varepsilon^p\varphi'(v_\varepsilon)U_\varepsilon\cdot\nabla v_\varepsilon - \int_\Omega u_\varepsilon^{p+1}v_\varepsilon\varphi'(v_\varepsilon) \\
&= -p(p-1)\int_\Omega u_\varepsilon^{p-2}\varphi(v_\varepsilon)|\nabla u_\varepsilon|^2 - p\int_\Omega u_\varepsilon^{p-1}\varphi'(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\quad + p(p-1)\int_\Omega u_\varepsilon^{p-1}\varphi(v_\varepsilon)S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\cdot\nabla u_\varepsilon + p\int_\Omega u_\varepsilon^p\varphi'(v_\varepsilon)S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\cdot\nabla v_\varepsilon \\
&\quad - p\int_\Omega u_\varepsilon^{p-1}\varphi'(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \int_\Omega u_\varepsilon^p\varphi''(v_\varepsilon)|\nabla v_\varepsilon|^2 - \int_\Omega u_\varepsilon^{p+1}\varphi'(v_\varepsilon)v_\varepsilon
\end{aligned} \tag{7.25}$$

for all $t \in (0, T_{\max,\varepsilon})$, where we have used the identity

$$\begin{aligned}
-p\int_\Omega u_\varepsilon^{p-1}\varphi(v_\varepsilon)U_\varepsilon\cdot\nabla u_\varepsilon - \int_\Omega u_\varepsilon^p\varphi'(v_\varepsilon)U_\varepsilon\cdot\nabla v_\varepsilon &= -\int_\Omega \varphi(v_\varepsilon)U_\varepsilon\cdot\nabla u_\varepsilon^p - \int_\Omega u_\varepsilon^p U_\varepsilon\cdot\nabla\varphi(v_\varepsilon) \\
&= \int_\Omega u_\varepsilon^p\varphi(v_\varepsilon)(\nabla\cdot U_\varepsilon) = 0 \text{ for all } t\in(0,T_{\max,\varepsilon}).
\end{aligned}$$

In light of (7.10), we find that for all $t \in (0, T_{\max,\varepsilon})$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p\varphi(v_\varepsilon) + p(p-1)\int_\Omega \varphi(v_\varepsilon)u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^p\varphi''(v_\varepsilon)|\nabla v_\varepsilon|^2 \\
&\le p(p-1)S_0(\|v_0\|_{L^\infty(\Omega)})\int_\Omega u_\varepsilon^{p-1}\varphi(v_\varepsilon)|\nabla u_\varepsilon||\nabla v_\varepsilon| + 2p\int_\Omega u_\varepsilon^{p-1}\varphi'(v_\varepsilon)|\nabla u_\varepsilon||\nabla v_\varepsilon| \\
&\qquad + pS_0(\|v_0\|_{L^\infty(\Omega)})\int_\Omega u_\varepsilon^p\varphi'(v_\varepsilon)|\nabla v_\varepsilon|^2.
\end{aligned} \tag{7.26}$$

Here Young's inequality yields that

$$\begin{aligned}
p(p-1)S_0(\|v_0\|_{L^\infty(\Omega)})\int_\Omega u_\varepsilon^{p-1}\varphi(v_\varepsilon)|\nabla u_\varepsilon||\nabla v_\varepsilon| &\le \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-2}\varphi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\quad + p(p-1)S_0^2(\|v_0\|_{L^\infty(\Omega)})\int_\Omega u_\varepsilon^p\varphi(v_\varepsilon)|\nabla v_\varepsilon|^2
\end{aligned} \tag{7.27}$$

and

$$2p\int_\Omega u_\varepsilon^{p-1}\varphi'(v_\varepsilon)|\nabla u_\varepsilon||\nabla v_\varepsilon| \le \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-2}\varphi(v_\varepsilon)|\nabla u_\varepsilon|^2 + \frac{4p}{p-1}\int_\Omega u_\varepsilon^p\frac{\varphi'^2(v_\varepsilon)}{\varphi(v_\varepsilon)}|\nabla v_\varepsilon|^2 \tag{7.28}$$

for all $t \in (0, T_{\max,\varepsilon})$. We see that (7.26)-(7.28) imply that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p \varphi(v_\varepsilon) + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-2}\varphi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&+ \int_\Omega u_\varepsilon^p |\nabla v_\varepsilon|^2 \left(\varphi''(v_\varepsilon) - \frac{4p}{p-1}\frac{\varphi'^2(v_\varepsilon)}{\varphi(v_\varepsilon)} - p(p-1)S_0^2(\|v_0\|_{L^\infty(\Omega)})\varphi(v_\varepsilon) - pS_0(\|v_0\|_{L^\infty(\Omega)})\varphi'(v_\varepsilon)\right) \\
&\le 0 \qquad (7.29)
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. Now using (7.21)-(7.22), and in view of the fact that $S_0(\delta)$ is non-decreasing, we obtain that

$$\begin{aligned}
\frac{4p}{p-1}\frac{\varphi'^2(v_\varepsilon)}{\varphi(v_\varepsilon)} &= \frac{4p}{p-1}h^2(\delta_0 - v_\varepsilon)^{-h-2} \le \frac{1}{3}\varphi''(v_\varepsilon), \\
p(p-1)S_0^2(\delta_0)\varphi(v_\varepsilon) &= p(p-1)S_0^2(\delta_0)(\delta_0 - v_\varepsilon)^{-h} \le \frac{1}{3}\varphi''(v_\varepsilon) \\
\text{and } pS_0(\delta_0)\varphi'(v_\varepsilon) &= hpS_0(\delta_0)(\delta_0 - v_\varepsilon)^{-h-1} \le \frac{1}{3}\varphi''(v_\varepsilon) \text{ in } \Omega \times (0, T_{\max,\varepsilon}).
\end{aligned}$$

As thus the term $\displaystyle\int_\Omega u_\varepsilon^p|\nabla v_\varepsilon|^2\left(\varphi''(v_\varepsilon) - 16\frac{\varphi'^2(v_\varepsilon)}{\varphi(v_\varepsilon)} - p(p-1)S_0^2(\delta_0)\varphi(v_\varepsilon) - pS_0(\delta_0)\varphi'(v_\varepsilon)\right)$ on the right hand side of (7.29) is nonnegative, we immediately deduce that for all $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p\varphi(v_\varepsilon) + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-2}\varphi(v_\varepsilon)|\nabla u_\varepsilon|^2 \le 0, \text{ for all } t \in (0, T_{\max}). \qquad (7.30)$$

Since

$$\delta_0^{-h} \le \varphi(v_\varepsilon) \le \left(\delta_0 - \|v_0\|_{L^\infty(\Omega)}\right)^{-h} \text{ in } \Omega \times (0, T_{\max,\varepsilon})$$

for all $\varepsilon \in (0,1)$, hence (7.19) and (7.20) result from the above inequality upon integrating on $(0, T_{\max,\varepsilon})$. □

7.4. Boundedness in the two-dimensional case ($N = 2$, $\kappa = 1$)

We expect that the $L^p(\Omega)$ estimate obtained in the last section guarantees boundedness of u_ε in $L^\infty(\Omega)$ as in the fluid-free system. However, this iteration procedure is much more delicate due to the appearance of the transport terms in the current case. Since the regularity of ∇v_ε is crucial, which is also associated to the regularity of U_ε, we will first derive some suitable regularity information on U_ε. More precisely, bounds for the $L^2(\Omega)$ norm of ∇U_ε imply boundedness of $\|U_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$ for any $p > 1$. This is sufficient to prove boundedness of $\|\nabla v_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)}$.

7.4.1. Boundedness of $\|\nabla U_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$

Lemma 7.4.1. *Let $N \in \{2,3\}$. Suppose that*

$$\sup_{\varepsilon\in(0,1)}\ \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \qquad (7.31)$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \qquad (7.32)$$

and

$$\int_k^{\min\{k+1,T_{\max,\varepsilon}\}}\int_\Omega |\nabla U_\varepsilon|^2 \le C \quad \text{for all } k \in \widetilde{\mathbb{N}} := \{s \in \mathbb{N}, s \le [T_{\max,\varepsilon}]\}. \qquad (7.33)$$

Proof. Testing the third equation with U_ε, integrating by parts and Young's inequality yield that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |U_\varepsilon|^2 + \int_\Omega |\nabla U_\varepsilon|^2 &= \int_\Omega u_\varepsilon \nabla\Phi \cdot U_\varepsilon \\ &\le \frac{\lambda_1'}{2}\int_\Omega |U_\varepsilon|^2 + \frac{1}{2\lambda_1'}\|\nabla\Phi\|^2_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^2 \qquad (7.34)\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. The Poincaré inequality combined with (7.31) implies the existence of $c_1 > 0$ such that

$$\frac{d}{dt}\int_\Omega |U_\varepsilon|^2 + \lambda_1'\int_\Omega |U_\varepsilon|^2 \le c_1 \qquad (7.35)$$

for all $t \in (0, T_{\max,\varepsilon})$. Thus, (7.32) is obtained by an ODE comparison theorem. Now we integrate (7.34) on $(k, k+1)$ ($k \in \widetilde{\mathbb{N}}$) to find that (7.33) holds due to (7.32). □

Based on (4.17) in [108], with the aid of (7.33) we can prove that $\|\nabla U_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ is bounded. The assumption $N = 2$ is crucial here.

Lemma 7.4.2. *Let $N = 2$. Suppose that*

$$\sup_{\varepsilon\in(0,1)}\ \sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \qquad (7.36)$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|\nabla U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \qquad (7.37)$$

Proof. First we apply Lemma 7.4.1 to obtain $c_1 > 0$ and $c_2 > 0$ such that

$$\|U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le c_1 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \qquad (7.38)$$

$$\text{and} \int_k^{\min\{k+1,T_{\max,\varepsilon}\}}\int_\Omega |\nabla U_\varepsilon|^2 \le c_2 \quad \text{for all } k \in \widetilde{\mathbb{N}} := \{s \in \mathbb{N}, s \le [T_{\max,\varepsilon}]\}. \qquad (7.39)$$

By the definition of A, we know that $\|A^{\frac{1}{2}}U_\varepsilon\|_{L^2(\Omega)} = \|\nabla U_\varepsilon\|_{L^2(\Omega)}$. Testing the third equation by AU_ε implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}U_\varepsilon|^2 + \int_\Omega |AU_\varepsilon|^2 &= \int_\Omega AU_\varepsilon(U_\varepsilon\cdot\nabla)U_\varepsilon - \int_\Omega u_\varepsilon\nabla\Phi\cdot AU_\varepsilon \\ &\le \frac{1}{4}\int_\Omega |AU_\varepsilon|^2 + \int_\Omega |U_\varepsilon|^2|\nabla U_\varepsilon|^2 + \frac{1}{4}\int_\Omega |AU_\varepsilon|^2 + \|\nabla\Phi\|^2_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^2 \\ &\le \int_\Omega |U_\varepsilon|^2|\nabla U_\varepsilon|^2 + \frac{1}{2}\int_\Omega |AU_\varepsilon|^2 + \|\nabla\Phi\|^2_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^2 \qquad (7.40)\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. By Young's inequality, an interpolation inequality for $\|U_\varepsilon\|_{L^4(\Omega)}$ and $\|\nabla U_\varepsilon\|_{L^4(\Omega)}$ (see also in [108, proof of Theorem 1.1]), and the equivalence between the norms $\|A(\cdot)\|_{L^2(\Omega)}$ and $\|\cdot\|_{W^{2,2}(\Omega)}$, we can find $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned}\int_\Omega |U_\varepsilon|^2|\nabla U_\varepsilon|^2 &\le (\int_\Omega |U_\varepsilon|^4)^{\frac{1}{2}}(\int_\Omega |\nabla U_\varepsilon|^4)^{\frac{1}{2}} \\ &\le c_3(\int_\Omega |\nabla U_\varepsilon|^2)^{\frac{1}{2}}(\int_\Omega |U_\varepsilon|^2)^{\frac{1}{2}}(\int_\Omega |AU_\varepsilon|^2)^{\frac{1}{2}}(\int_\Omega |\nabla U_\varepsilon|^2)^{\frac{1}{2}}\end{aligned}$$

$$\leq \frac{1}{2}\int_\Omega |AU_\varepsilon|^2 + c_4(\int_\Omega |U_\varepsilon|^2)(\int_\Omega |\nabla U_\varepsilon|^2)^2 \text{ for all } t \in (0, T_{\max,\varepsilon}). \quad (7.41)$$

We see that (7.40) and (7.41) in conjunction with our assumption and (7.38) imply that there is $c_5 > 0$ fulfilling

$$\frac{d}{dt}\int_\Omega |\nabla U_\varepsilon|^2 + \int_\Omega |AU_\varepsilon|^2 \leq c_5 \left(\int_\Omega |\nabla U_\varepsilon|^2 + 1\right)^2 \quad (7.42)$$

for all $t \in (0, T_{\max,\varepsilon})$. Letting $y(t) := \int_\Omega |\nabla U_\varepsilon(\cdot,t)|^2 + 1$, we thus see that $y(t)$ satisfies

$$y'(t) \leq c_5 y^2(t) \quad (7.43)$$

for all $t \in [k, \min\{k+1, T_{\max,\varepsilon}\})$.
If $T_{\max,\varepsilon} > 1$, for all $k \in \widetilde{\mathbb{N}}$, Lemma 7.4.1 warrants the existence of $c_6 > 0$ and $s_k \in [k, k+1]$ such that

$$y(s_k) \leq c_6 \text{ and } \int_k^{k+1} y(s)ds \leq c_6. \quad (7.44)$$

We deduce from (7.43-7.44) that

$$y(t) \leq e^{c_5 \int_{s_k}^t y(s)ds} y(s_k) \leq e^{c_5 \int_k^{\min\{k+2, T_{\max,\varepsilon}\}} y(s)ds} y(s_k) \leq e^{2c_5c_6} c_6 \quad (7.45)$$

for all $t \in [k+1, \min\{k+2, T_{\max,\varepsilon}\}] \subset [s_k, \min\{k+2, T_{\max,\varepsilon}\})$ ($k \in \widetilde{\mathbb{N}}$). Thus, (7.45) holds for all $t \in [1, T_{\max,\varepsilon})$. A similar reasoning gives

$$y(t) \leq e^{c_5 \int_0^1 y(s)ds} y(0) \leq e^{c_5c_6} y(0) \text{ for all } t \in [0,1]. \quad (7.46)$$

If $T_{\max,\varepsilon} < 1$, it is easy to see that the above estimate still holds for $t \in [0, T_{\max,\varepsilon})$. Thus, the proof is complete by letting $C := \max\{e^{2c_5c_6}c_6, e^{c_5c_6}\|\nabla U_0\|_{L^2(\Omega)}\}$. □

The following lemma is an immediate consequence of Lemma 7.4.2 and the Sobolev embedding theorem for dimension 2.

Lemma 7.4.3. *Let $N = 2$ and $p \in [1, \infty)$. Suppose that*

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \quad (7.47)$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|U_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}). \quad (7.48)$$

7.4.2. Boundedness of $\|\nabla v_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)}$

Now we are in a position to achieve higher regularity of ∇v_ε. The approach is carried out by a fixed-point type argument involving L^p-L^q estimates for semigroups combined with a typical integral estimate, which is again Lemma 6.2.4.

Lemma 7.4.4. *Let $N = 2$. Suppose that*

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \quad (7.49)$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}).$$

Proof. Let $\theta \in (1,2)$ and $\theta' \in (2,\infty)$ satisfy $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Testing the second equation in (7.11) with $-\Delta v_\varepsilon$, integrating by part and applying the Cauchy-Schwarz inequality and Hölder's inequality, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 \\
=& -\int_\Omega u_\varepsilon v_\varepsilon \Delta v_\varepsilon - \int_\Omega \Delta v_\varepsilon (U_\varepsilon \cdot \nabla v_\varepsilon) \\
\le& \frac{1}{4}\int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega u_\varepsilon^2 v_\varepsilon^2 + \frac{1}{4}\int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega |U_\varepsilon|^2 |\nabla v_\varepsilon|^2 \\
\le& \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega u_\varepsilon^2 v_\varepsilon^2 + \int_\Omega |U_\varepsilon|^2|\nabla v_\varepsilon|^2 \\
\le& \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 + \|u_\varepsilon\|^2_{L^2(\Omega)}\|v_\varepsilon\|^2_{L^\infty(\Omega)} + \|U_\varepsilon\|^2_{L^{2\theta}(\Omega)}\|\nabla v_\varepsilon\|^2_{L^{2\theta'}(\Omega)}
\end{aligned} \tag{7.50}$$

for all $t \in (0, T_{\max,\varepsilon})$. Let $a = \frac{1}{\theta} \in (\frac{1}{2}, 1)$. By applying Hölder's inequality, the Gagliardo-Nirenberg inequality and Young's inequality we can find $c_1 > 0$ and $c_2 > 0$ such that

$$\|\nabla v_\varepsilon\|^2_{L^2(\Omega)} \le c_1\|\nabla v_\varepsilon\|^2_{L^{2\theta'}(\Omega)} \le c_2\|\Delta v_\varepsilon\|^{2a}_{L^2(\Omega)}\|v_\varepsilon\|^{2(1-a)}_{L^\infty(\Omega)} + c_2\|v_\varepsilon\|^2_{L^\infty(\Omega)}. \tag{7.51}$$

Lemma 7.4.3 guarantees a constant $c_3 > 0$ such that for all $\varepsilon \in (0,1)$, $\|U_\varepsilon\|^2_{L^{2\theta}(\Omega)} \le c_3$ for all $t \in (0, T_{\max,\varepsilon})$. Therefore, we can fix $c_4 > 0$ such that

$$\begin{aligned}
\|U_\varepsilon\|^2_{L^{2\theta}(\Omega)}\|\nabla v_\varepsilon\|^2_{L^{2\theta'}(\Omega)} \le c_3\|\nabla v_\varepsilon\|^2_{L^{2\theta'}(\Omega)} &\le c_2c_3\|\Delta v_\varepsilon\|^{2a}_{L^2(\Omega)}\|v_\varepsilon\|^{2(1-a)}_{L^\infty(\Omega)} + c_2c_3\|v_\varepsilon\|_{L^\infty(\Omega)} \\
&\le \frac{1}{4}\int_\Omega |\Delta v_\varepsilon|^2 + c_4
\end{aligned} \tag{7.52}$$

for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. Similarly, according to (7.51), there exists $c_5 > 0$ fullfilling

$$\int_\Omega |\nabla v_\varepsilon|^2 \le \frac{1}{4}\int_\Omega |\Delta v_\varepsilon|^2 + c_5. \tag{7.53}$$

Collecting (7.50) (7.52) and (7.53), we obtain $c_6 > 0$ satisfying

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 \le c_6$$

for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$ due to (7.15) and (7.49). An application of ODE comparison implies the assertion. □

Lemma 7.4.5. *Let $N = 2$. Suppose that*

$$\sup_{\varepsilon\in(0,1)}\ \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty.$$

Then there is $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)} \le C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.54}$$

Proof. The variation-of-constants representation of v_ε implies that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)} \le \|\nabla e^{t\Delta}v_0\|_{L^{q_0}(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}ds$$

$$+\int_0^t \|\nabla e^{(t-s)\Delta}(U_\varepsilon(\cdot,s)\cdot\nabla v_\varepsilon(\cdot,s))\|_{L^{q_0}(\Omega)}ds \tag{7.55}$$

for all $t\in(0,T_{\max,\varepsilon})$ and all $\varepsilon\in(0,1)$. Recall that by the classical L^p-L^q estimates for the Neumann heat semigroup, there is $c_1>0$ such that

$$\|\nabla e^{t\Delta}v_0\|_{L^{q_0}(\Omega)}\le c_1\|\nabla v_0\|_{L^{q_0}(\Omega)} \tag{7.56}$$

for all $t\in(0,T_{\max,\varepsilon})$. Again an L^p-L^q estimate for the Neumann heat semigroup from Lemma 6.2.1, Lemma 6.2.4 and (7.15) imply constants $c_2>0$ and $c_3>0$ such that

$$\begin{aligned}
&\int_0^t \|\nabla e^{(t-s)\Delta}u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}ds\\
\le&\int_0^t c_1(1+(t-s)^{-\frac12-(\frac12-\frac1{q_0})})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds\\
\le&\int_0^t c_1(1+(t-s)^{-1+\frac1{q_0}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^2(\Omega)}\|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds\\
\le&\, c_2\int_0^t(1+(t-s)^{-1+\frac1{q_0}})e^{-\lambda_1(t-s)}ds\le c_2c_3
\end{aligned} \tag{7.57}$$

for all $t\in(0,T_{\max,\varepsilon})$ and all $\varepsilon\in(0,1)$. Next we fix $p_1\in(2,\infty)$ satisfying $\frac1{p_1}\in(\frac1{q_0},\frac12)$. Let $p_2\in(2,\infty)$ be such that $\frac12=\frac1{p_1}+\frac1{p_2}$ and $\theta=\frac{\frac12-\frac1{p_1}}{\frac12-\frac1{q_0}}\in(0,1)$. From Lemmata 7.4.4, 7.4.3 and 6.2.4 we thereby obtain $c_3>0$, $c_4>0$ and $c_5>0$ such that

$$\begin{aligned}
&\int_0^t \|\nabla e^{(t-s)\Delta}(U_\varepsilon(\cdot,t)\cdot\nabla v_\varepsilon(\cdot,t))\|_{L^{q_0}(\Omega)}ds\\
\le&\int_0^t c_1(1+(t-s)^{-\frac12-(\frac12-\frac1{q_0})})e^{-\lambda_1(t-s)}\|U_\varepsilon(\cdot,t)\cdot\nabla v_\varepsilon(\cdot,t)\|_{L^2(\Omega)}ds\\
\le&\int_0^t c_1(1+(t-s)^{-1+\frac1{q_0}})e^{-\lambda_1(t-s)}\|U_\varepsilon(\cdot,t)\|_{L^{p_2}(\Omega)}\|\nabla v_\varepsilon(\cdot,t)\|_{L^{p_1}(\Omega)}ds\\
\le&\int_0^t c_1(1+(t-s)^{-1+\frac1{q_0}})e^{-\lambda_1(t-s)}\|U_\varepsilon(\cdot,t)\|_{L^{p_2}(\Omega)}\|\nabla v_\varepsilon(\cdot,t)\|^\theta_{L^{q_0}(\Omega)}\|\nabla v_\varepsilon(\cdot,t)\|^{1-\theta}_{L^2(\Omega)}\\
\le&\sup_{s\in(0,t)}\|\nabla v_\varepsilon(\cdot,s)\|^\theta_{L^{q_0}(\Omega)}c_1c_3c_4\int_0^t(1+(t-s)^{-1+\frac1{q_0}})e^{-\lambda_1(t-s)}ds\\
\le&\, c_5\sup_{s\in(0,t)}\|\nabla v_\varepsilon(\cdot,s)\|^\theta_{L^{q_0}(\Omega)}
\end{aligned} \tag{7.58}$$

for all $t\in(0,T_{\max,\varepsilon})$ and all $\varepsilon\in(0,1)$. Let $T\in(0,T_{\max,\varepsilon})$ and $M(T):=\sup\limits_{t\in(0,T)}\|\nabla v_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)}$. Collecting (7.55)-(7.58), we thus obtain the existence of $c_6>0$ such that for all $\varepsilon\in(0,1)$,

$$M(T)\le c_6+c_6M^\theta(T)\ \text{for all}\ T\in(0,T_{\max,\varepsilon}).$$

Since $\theta<1$, (7.54) is obvious by Young's inequality. □

7.4.3. Boundedness of u_ε

Lemma 7.4.6. *Let $N=2$. Suppose that*

$$\sup_{\varepsilon\in(0,1)}\sup_{t\in(0,T_{\max,\varepsilon})}\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}<\infty.$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.59}$$

Proof. Following the variation-of-constants formula for u_ε, we see that

$$\begin{aligned}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} &+ \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &+ \int_0^t \|e^{(t-s)\Delta}U_\varepsilon(\cdot,s)\cdot\nabla u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \end{aligned} \tag{7.60}$$

for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. According to the maximum principle, the first term can be estimated as

$$\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.61}$$

Now we pick $p_0 \in (2, q_0)$ and $p_1 \in (p_0, \infty)$ such that $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{q_0}$. Let $a = 1 - \frac{1}{p_1} \in (0,1)$. Applying the L^p-L^q esatimates for the Neumann heat semigroup, Hölder's inequality and Lemmata 7.2.2, 7.4.5, Lemma 6.2.4, we obtain $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that

$$\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\le c_1\int_0^t (1+(t-s)^{-\frac12-\frac{1}{p_0}})e^{-\lambda_1(t-s)}\|(u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon)(\cdot,s)\|_{L^{p_0}(\Omega)}ds \\
&\le c_1 S_0(\|v_0\|_{L^\infty(\Omega)})\int_0^t (1+(t-s)^{-\frac12-\frac{1}{p_0}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^{p_1}(\Omega)}\|\nabla v_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\le c_1 S_0(\|v_0\|_{L^\infty(\Omega)})\int_0^t (1+(t-s)^{-\frac12-\frac{1}{p_0}})e^{-\lambda_1(t-s)} \\
&\qquad\qquad \times \|u_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)}\|u_\varepsilon(\cdot,s)\|^{1-a}_{L^1(\Omega)}\|\nabla v_\varepsilon(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\le c_2 S_0(\|v_0\|_{L^\infty(\Omega)})\sup_{s\in(0,t)}\|u_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)}\left(\int_0^t (1+(t-s)^{-\frac12-\frac{1}{p_0}})e^{-\lambda_1(t-s)}ds\right) \\
&\le c_3 \sup_{s\in(0,t)}\|u_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)}
\end{aligned} \tag{7.62}$$

for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. Noting that $U_\varepsilon \cdot \nabla u_\varepsilon = \nabla\cdot(u_\varepsilon U_\varepsilon)$, we pick $p \in (2,\infty)$ and $p_2, p' \in (p, \infty)$ such that $\frac{1}{p} = \frac{1}{p_2} + \frac{1}{p'}$. Let $b = 1 - \frac{1}{p_2} \in (0,1)$. A similar reasoning as in the above inequality shows that

$$\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}U_\varepsilon(\cdot,s)\cdot\nabla u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds = \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(\cdot,s)U_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds \\
&\le c_4\int_0^t (1+(t-s)^{-\frac12-\frac1p})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)U_\varepsilon(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le c_4\int_0^t (1+(t-s)^{-\frac12-\frac1p})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^{p_2}(\Omega)}\|U_\varepsilon(\cdot,s)\|_{L^{p'}(\Omega)}ds \\
&\le c_4\int_0^t (1+(t-s)^{-\frac12-\frac1p})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|^b_{L^\infty(\Omega)}\|u_\varepsilon(\cdot,s)\|^{1-b}_{L^1(\Omega)}\|U_\varepsilon(\cdot,s)\|_{L^{p'}(\Omega)}ds \\
&\le c_5\left(\int_0^t (1+(t-s)^{-\frac12-\frac1p})e^{-\lambda_1(t-s)}ds\right)\sup_{s\in(0,t)}\|u_\varepsilon(\cdot,s)\|^b_{L^\infty(\Omega)}
\end{aligned}$$

$$\leq c_6 \sup_{s\in(0,t)} \|u_\varepsilon(\cdot,s)\|^b_{L^\infty(\Omega)} \tag{7.63}$$

with some $c_4 > 0$, $c_5 > 0$ and $c_6 > 0$ for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$ due to (7.31) and (7.48). Letting $T \in (0, T_{\max,\varepsilon})$ and $M(T) = \sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$, we therefore obtain a constant $c_7 > 0$ such that

$$M(T) \leq c_7 M^a(T) + c_7 M^b(T) + c_7 \text{ for all } T \in (0, T_{\max,\varepsilon}).$$

Since $a, b \in (0,1)$, elementary argument implies a constant $c_8 > 0$ such that $M(T) \leq c_8$ for all $T \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$, and thereby proves the assertion. □

7.4.4. Proof of (i) in Proposition 7.2.3

In order to prove global existence of the solution, it is left to show boundedness of $\|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$.

Lemma 7.4.7. *Let $N = 2$. Suppose that*

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty, \quad \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty$$

$$\text{and} \ \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|\nabla U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty.$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{7.64}$$

where β is as in (7.4).

Proof. Let $a = \frac{N}{4\beta}$. From the Gagliardo-Nirenberg inequality and Lemma 6.2.3 we know that there is constant $c_1 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\|U_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 \|A^\beta U_\varepsilon(\cdot,t)\|^a_{L^2(\Omega)} \|U_\varepsilon(\cdot,t)\|^{1-a}_{L^2(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.65}$$

Applying A^β to both sides of the third equation in (7.11), for all $\varepsilon \in (0,1)$, we have

$$\begin{aligned} \|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq \|A^\beta e^{-tA} U_0\|_{L^2(\Omega)} &+ \int_0^t \|A^\beta e^{-(t-s)A} \mathscr{P}(U_\varepsilon\cdot\nabla)U_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \\ &+ \int_0^t \|A^\beta e^{-(t-s)A} \mathscr{P} u_\varepsilon(\cdot,s)\nabla\Phi\|_{L^2(\Omega)} ds \ \text{ for all } t \in (0, T_{\max,\varepsilon}). \end{aligned} \tag{7.66}$$

By an estimate from [76, (1.5.16)] and (6.15), we find $c_2 > 0$ such that

$$\|A^\beta e^{-tA} U_0\|_{L^2(\Omega)} = \|e^{-tA} A^\beta U_0\|_{L^2(\Omega)} \leq c_2 \|A^\beta U_0\|_{L^2(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.67}$$

In view of the assumption, we can fix $c_3 > 0$ such that $\|U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq c_3$, $\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq c_3$ and $\|\nabla U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq c_3$ hold for all $\varepsilon \in (0,1)$ and all $t \in (0, T_{\max,\varepsilon})$, which together with (6.14), (7.65) and Lemma 6.2.4 yields the existence of $c_4 > 0$, $c_5 > 0$ and $c_6 > 0$ such that

$$\int_0^t \|A^\beta e^{-(t-s)A} \mathscr{P}(U_\varepsilon\cdot\nabla)U_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds$$

$$
\begin{aligned}
&\leq \int_0^t c_4(t-s)^{-\beta} e^{-\lambda_1'(t-s)} \|(U_\varepsilon \cdot \nabla) U_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \\
&\leq \int_0^t c_4(t-s)^{-\beta} e^{-\lambda_1'(t-s)} \|U_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \|\nabla U_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \\
&\leq \int_0^t c_3 c_4(t-s)^{-\beta} e^{-\lambda_1'(t-s)} \|A^\beta U_\varepsilon(\cdot,s)\|^a_{L^2(\Omega)} \|U_\varepsilon(\cdot,s)\|^{1-a}_{L^2(\Omega)} ds \\
&\leq c_5 \sup_{t\in(0,T_{\max,\varepsilon})} \|A^\beta U_\varepsilon(\cdot,t)\|^a_{L^2(\Omega)} \int_0^t (t-s)^{-\beta} e^{-\lambda_1'(t-s)} ds \\
&\leq c_6 \sup_{s\in(0,t)} \|A^\beta U_\varepsilon(\cdot,s)\|^a_{L^2(\Omega)}
\end{aligned} \tag{7.68}
$$

for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. Furthermore, by Lemma 6.2.4 we can find $c_7 > 0$ such that for all $\varepsilon \in (0,1)$

$$
\begin{aligned}
&\int_0^t \|A^\beta e^{-(t-s)A} \mathscr{P} u_\varepsilon(\cdot,s) \nabla\Phi\|_{L^2(\Omega)} ds \\
&\leq \int_0^t c_3 \|\nabla\Phi\|_{L^\infty(\Omega)} (t-s)^{-\beta} e^{-\lambda_1'(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \\
&\leq c_3 c_4 \|\nabla\Phi\|_{L^\infty(\Omega)} \int_0^t (t-s)^{-\beta} e^{-\lambda_1'(t-s)} ds \\
&\leq c_7 \|\nabla\Phi\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}).
\end{aligned} \tag{7.69}
$$

Given $T \in (0, T_{\max,\varepsilon})$, we define $M(T) := \sup_{t\in(0,T)} \|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$. Taking supremum on both sides of (7.66) over $(0,T)$, by (7.67), (7.68) and (7.69) we find $c_8 > 0$ such that for all $\varepsilon \in (0,1)$, $M(T)$ satisfies

$$
M(T) \leq c_8 + c_6 M^a(T) \quad \text{for all } T \in (0, T_{\max,\varepsilon}). \tag{7.70}
$$

An application of Young's inequality to the above inequality leads to the assertion. □

Proof of Proposition 7.2.3 (i). Let $p_0 > 2$ and let $\delta_0 := \delta(p_0)$ be as defined in Lemma 7.3.1. We immediately see from Lemmata 7.4.1-7.4.6 that there is $C_1 > 0$ such that for all $\varepsilon \in (0,1)$, $\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_1$ for all $t \in (0, T_{\max,\varepsilon})$. Lemma 7.2.2 and Lemma 7.4.5 imply $C_2 > 0$ such that $\|v_\varepsilon(\cdot,t)\|_{W^{1,q_0}(\Omega)} \leq C_2$ for all $\varepsilon \in (0,1)$ for all $t \in (0, T_{\max,\varepsilon})$. Also, Lemma 7.4.7 implies $C_3 > 0$ with the property that $\|A^\beta U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_3$ for all $\varepsilon \in (0,1)$ and all $t \in (0, T_{\max,\varepsilon})$. According to Lemma 7.2.1, we deduce that $T_{\max,\varepsilon} = \infty$, thus the solution is global and bounded. □

7.5. Boundedness in the three-dimensional case ($\kappa = 0$)

In this section, we deal with the chemotaxis-Stokes system in the three-dimensional setting. We first give a sufficient condition for boundedness which in conjunction with Lemma 7.2.1 proves Theorem 7.1.1. In fact, since Lemma 7.3.1 provides an L^p estimate for u_ε for any $p > 1$, we can choose p sufficiently large to prove boundedness of u_ε in $L^\infty(\Omega)$. However, here we would like to give an optimal condition for this extension critearion.

Proposition 7.5.1. *Let $N = 3$ and $p > \frac{N}{2}$. Suppose that*

$$
\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \infty. \tag{7.71}
$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0, 1)$,

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} < C \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.72}$$

We will prove the proposition by several lemmata, which improve regularity for U_ε and ∇v_ε in suitable ways.

Lemma 7.5.2. *Let $N \in \{2, 3\}$. There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$,*

$$\int_0^t \int_\Omega |\nabla v_\varepsilon(\cdot, s)|^2 ds \le C \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

Proof. We test the second equation with v_ε to obtain that for all $\varepsilon \in (0, 1)$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 \le 0 \text{ for all } t \in (0, T_{\max,\varepsilon}). \tag{7.73}$$

Integrating over $(0, t)$ and using Lemma 7.2.2, we achieve our goal. □

Lemma 7.5.3. *Let $N = 3$, $p > \frac{N}{2}$ and $\beta' \in (\frac{N}{4}, \min\{1 - \frac{N}{2p} + \frac{N}{4}, 1\})$ such that $\beta' \le \beta$. Suppose that*

$$\sup_{\varepsilon \in (0,1)} \sup_{t \in (0, T_{\max,\varepsilon})} \|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} < \infty. \tag{7.74}$$

Then there exists $C > 0$ such that for any $\varepsilon \in (0, 1)$,

$$\|A^{\beta'} U_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \le C \text{ and} \tag{7.75}$$

$$\|U_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C \text{ for all } t \in (0, T_{\max,\varepsilon}). \tag{7.76}$$

Proof. The proof is similar to that of Lemma 7.4.7. We start with observing th at

$$\|A^{\beta'} U_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \le \|A^{\beta'} e^{-tA} U_0\|_{L^2(\Omega)} + \int_0^t \|A^{\beta'} e^{-(t-s)A} \mathscr{P} u_\varepsilon(\cdot, s) \nabla \Phi\|_{L^2(\Omega)} ds$$

and fixing $c_1, c_2 > 0$ such that

$$\|A^{\beta'} e^{-tA} U_0\|_{L^2(\Omega)} \le c_1 \|A^{\beta'} U_0\|_{L^2(\Omega)} \le c_2 \|A^\beta U_0\|_{L^2(\Omega)} \tag{7.77}$$

for all $t \in (0, T_{\max,\varepsilon})$. Lemmata 6.2.4, 7.3.1 and (6.14) imply the existence of $c_3, c_4, c_5, c_6 > 0$ such that

$$\begin{aligned}
&\int_0^t \|A^{\beta'} e^{(t-s)A} \mathscr{P} u_\varepsilon \nabla \Phi\|_{L^2(\Omega)} ds \\
\le &\int_0^t \|A^{\beta'} e^{\frac{(t-s)}{2}A} (e^{\frac{(t-s)}{2}A} \mathscr{P} u_\varepsilon(\cdot, s) \nabla \Phi)\|_{L^2(\Omega)} ds \\
\le &\int_0^t c_3 \left(\frac{t-s}{2}\right)^{-\beta'} e^{-\frac{\lambda_1'}{2}(t-s)} \|e^{\frac{(t-s)}{2}A} \mathscr{P} u_\varepsilon(\cdot, s) \nabla \Phi\|_{L^2(\Omega)} ds \\
\le &\int_0^t c_4 \left(\frac{t-s}{2}\right)^{-\beta'} e^{-\frac{\lambda_1'}{2}(t-s)} \left(\frac{t-s}{2}\right)^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{2})} \|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \|\nabla \Phi\|_{L^\infty(\Omega)} ds \\
\le &\, c_5 \int_0^t (t-s)^{-\beta' - \frac{N}{2p} + \frac{N}{4}} e^{-\frac{\lambda_1'}{2}(t-s)} \|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \|\nabla \Phi\|_{L^\infty(\Omega)} ds
\end{aligned}$$

$$\le c_6 \|\nabla \Phi\|_{L^\infty(\Omega)} \int_0^t (t-s)^{-\beta' - \frac{N}{2p} + \frac{N}{4}} e^{-\frac{\lambda_1'}{2}(t-s)} ds$$
$$\le c_7 \|\nabla \Phi\|_{L^\infty(\Omega)}$$

for all $t \in (0, T_{\max,\varepsilon})$. Therefore, we can find $c_8 > 0$ with the property that $\|A^{\beta'} U_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < c_8$ for all $t \in (0, T_{\max,\varepsilon})$ and for all $\varepsilon \in (0,1)$. Since $\beta' > \frac{N}{4}$, the embedding: $D(A^{\beta'}) \hookrightarrow L^\infty(\Omega)$ implies (7.76). Thus the proof is complete. □

Lemma 7.5.4. *Let $N = 3$, $p > \frac{N}{2}$ and $q \in (N, q_0]$ such that $q < \frac{Np}{(N-p)_+}$. Suppose that*

$$\sup_{\varepsilon \in (0,1)} \sup_{t \in (0, T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \infty. \tag{7.78}$$

Then there exists $C > 0$ such that $\varepsilon \in (0,1)$,

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{7.79}$$

Proof. The variation-of-constants of v_ε implies

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le \|\nabla e^{t\Delta} v_0\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}(u_\varepsilon v_\varepsilon)(\cdot,s)\|_{L^q(\Omega)} ds$$
$$+ \int_0^t \|\nabla e^{(t-s)\Delta}(U_\varepsilon \cdot \nabla v_\varepsilon)(\cdot,s)\|_{L^q(\Omega)} ds$$

for all $t \in (0, T_{\max,\varepsilon})$. From an L^p-L^q estimates for the Neumann heat semigroup and Hölder's inequality, we find some $c_1 > 0$ such that such that

$$\|\nabla e^{t\Delta} v_0\|_{L^q(\Omega)} \le c_1 e^{-\lambda_1 t} \|\nabla v_0\|_{L^{q_0}(\Omega)}$$

for all $t \in (0, T_{\max,\varepsilon})$. Since $1 < q < \frac{Np}{(N-p)_+}$, we know that $-\frac{1}{2} - \frac{N}{2}(\frac{1}{p} - \frac{1}{q}) > -1$, thus by an L^p-L^q estimate for the Neumann heat semigroup, our assumption on $\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$, and Lemmata 6.2.4 and 7.2.2, we can fix $c_2 > 0$ and $c_3 > 0$ such that

$$\int_0^t \|\nabla e^{(t-s)\Delta} u_\varepsilon(\cdot,s) v_\varepsilon(\cdot,s)\|_{L^q(\Omega)} ds$$
$$\le \int_0^t c_2 (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{p} - \frac{1}{q})}) e^{-\lambda_1 (t-s)} \|u_\varepsilon(\cdot,s) v_\varepsilon(\cdot,s)\|_{L^p(\Omega)} ds$$
$$\le \int_0^t c_2 (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{p} - \frac{1}{q})}) e^{-\lambda_1 (t-s)} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds \le c_3$$

for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. Now we fix $q' \in (2,q)$ satisfying $q' > \frac{1 + \frac{N}{2} - \frac{N}{q}}{\frac{1}{2} + \frac{N}{2q} - \frac{N}{q^2}}$. Let $a = \frac{\frac{1}{q'} - \frac{1}{q}}{\frac{1}{2} - \frac{1}{q}} \in (0,1)$. It can be easily checked that $\left(-\frac{1}{2} - \frac{N}{2}(\frac{1}{q'} - \frac{1}{q})\right) \frac{2}{2-a} > -1$. Moreover, the Hölder inequality $\|\nabla v_\varepsilon(\cdot,t)\|_{L^{q'}(\Omega)} \le \|\nabla v_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^a \|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)}^{1-a}$ holds for all $t \in (0, T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. Lemmata 6.2.1, 7.5.3, 6.2.4 and 7.5.2, Hölder's inequality yield the existence of $c_4 > 0$, $c_5 > 0$ and $c_6 > 0$ such that for all $\varepsilon \in (0,1)$

$$\int_0^t \|\nabla e^{(t-s)\Delta} U_\varepsilon(\cdot,s) \cdot \nabla v_\varepsilon(\cdot,s)\|_{L^q(\Omega)} ds$$

$$
\begin{aligned}
&\le \int_0^t c_2(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q'}-\frac{1}{q})})e^{-\lambda_1(t-s)}\|U_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s)\|_{L^{q'}(\Omega)}ds \\
&\le \int_0^t c_2(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q'}-\frac{1}{q})})e^{-\lambda_1(t-s)}\|\nabla v_\varepsilon(\cdot,s)\|_{L^{q'}(\Omega)}\|U_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\le \int_0^t c_2(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q'}-\frac{1}{q})})e^{-\lambda_1(t-s)}\|U_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla v_\varepsilon(\cdot,t)\|^a_{L^2(\Omega)}\|\nabla v_\varepsilon(\cdot,t)\|^{1-a}_{L^q(\Omega)}ds \\
&\le c_4 \sup_{s\in(0,t)}\|\nabla v_\varepsilon(\cdot,s)\|^{1-a}_{L^q(\Omega)}\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q'}-\frac{1}{q})}\right)e^{-\lambda_1(t-s)}\|\nabla v_\varepsilon(\cdot,s)\|^a_{L^2(\Omega)}ds \\
&\le c_4 \sup_{s\in(0,t)}\|\nabla v_\varepsilon(\cdot,s)\|^{1-a}_{L^q(\Omega)}\left(\int_0^t \|\nabla v_\varepsilon(\cdot,s)\|^2_{L^2(\Omega)}ds\right)^{\frac{a}{2}} \\
&\qquad\qquad \times\left(\int_0^t\left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q'}-\frac{1}{q})}\right)^{\frac{2}{2-a}}e^{-\frac{2}{2-a}\lambda_1(t-s)}ds\right)^{\frac{2-a}{2}} \\
&\le c_5 \sup_{s\in(0,t)}\|\nabla v_\varepsilon(\cdot,s)\|^{1-a}_{L^q(\Omega)}\left(\int_0^t\left(1+(t-s)^{\left(-\frac{1}{2}-\frac{N}{2}(\frac{1}{q'}-\frac{1}{q})\right)\frac{2}{2-a}}\right)e^{-\frac{2}{2-a}\lambda_1(t-s)}ds\right)^{\frac{2-a}{2}} \\
&\le c_6 \sup_{s\in(0,t)}\|\nabla v_\varepsilon(\cdot,s)\|^{1-a}_{L^q(\Omega)}
\end{aligned}
$$

for all $t\in(0,T_{\max,\varepsilon})$. Let $T\in(0,T_{\max,\varepsilon})$ and $M(T):=\sup_{t\in(0,T)}\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)}$. Combining the above estimates, we find $c_7>0$ such that $M(T)$ satisfies

$$M(T)\le c_7M^a(T)+c_7$$

for all $T\in(0,T_{\max,\varepsilon})$ and all $\varepsilon\in(0,1)$, which together with the fact that $a<1$ implies the assertion. □

Now we are ready to prove boundedness for u_ε.

Proof of Proposition 7.5.1. The representation formula for u_ε yields that

$$
\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} &+ \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&+\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(\cdot,s)U_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds
\end{aligned}
\tag{7.80}
$$

for all $t\in(0,T_{\max,\varepsilon})$. First, using the maximum principle, we have

$$\|e^{t\Delta}u_0\|_{L^\infty(\Omega)}\le\|u_0\|_{L^\infty(\Omega)} \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{7.81}$$

Let $N<q<\min\{q_0,\frac{NP}{(N-p)_+}\}$, we can find $p_0\in(N,q)$ and $q_1\in(1,\infty)$ such that $\frac{1}{p_0}=\frac{1}{q}+\frac{1}{q_1}$. Let $a=1-\frac{1}{q_1}\in(0,1)$. By an L^p-L^q estimates for the Neumann heat semigroup and the Hölder inequality, we obtain the existence of $c_1>0$ and $c_2>0$ such that for all $\varepsilon\in(0,1)$,

$$
\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon)(\cdot,s)\|_{L^\infty(\Omega)}ds \\
\le&\int_0^t c_1(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_0}})e^{-\lambda_1(t-s)}\|(u_\varepsilon S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon)(\cdot,s)\|_{L^{p_0}(\Omega)}ds
\end{aligned}
$$

$$\leq \int_0^t c_1 S_0(\|v_0\|_{L^\infty(\Omega)})(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_0}})e^{-\lambda_1(t-s)}\|\nabla v_\varepsilon(\cdot,s)\|_{L^q(\Omega)}\|u_\varepsilon(\cdot,s)\|_{L^{q_1}(\Omega)}ds$$
$$\leq \int_0^t c_1 S_0(\|v_0\|_{L^\infty(\Omega)})(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_0}})e^{-\lambda_1(t-s)}\|\nabla v_\varepsilon(\cdot,s)\|_{L^q(\Omega)}\|u_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)}\|u_\varepsilon(\cdot,s)\|^{1-a}_{L^1(\Omega)}ds$$
$$\leq c_2 \sup_{s\in(0,t)} \|u_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)} \text{ for all } t\in(0,T_{\max,\varepsilon}). \tag{7.82}$$

Now we pick $p_1 > N$, and let $b = 1 - \frac{1}{p_1} \in (0,1)$. An L^p-L^q estimate for the Neumann heat semigroup together with Lemma 6.2.4, 7.5.3 and 7.2.2 implies the existence of $c_3 > 0$ and $c_4 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(\cdot,s)U_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)}ds$$
$$\leq \int_0^t c_3(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_1}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)U_\varepsilon(\cdot,s)\|_{L^{p_1}(\Omega)}ds$$
$$\leq \int_0^t c_3(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_1}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^{p_1}(\Omega)}\|U_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds$$
$$\leq \int_0^t c_3(1+(t-s)^{-\frac{1}{2}-\frac{N}{2p_1}})e^{-\lambda_1(t-s)}\|U_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\|u_\varepsilon(\cdot,s)\|^b_{L^\infty(\Omega)}\|u_\varepsilon(\cdot,s)\|^{1-b}_{L^1(\Omega)}ds$$
$$\leq c_4 \sup_{s\in(0,t)} \|u_\varepsilon(\cdot,s)\|^b_{L^\infty(\Omega)} \text{ for all } t\in(0,T_{\max,\varepsilon}). \tag{7.83}$$

Let $T \in (0,T_{\max,\varepsilon})$ and $M(T) := \sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$. Finally, collecting (7.80)-(7.83), we conclude a constant $c_5 > 0$ such that for all $\varepsilon \in (0,1)$, $M(T)$ satisfies

$$M(T) \leq c_5 + c_5 M^a(T) + c_5 M^b(T)$$

for all $T \in (0,T_{\max,\varepsilon})$, which proves the assertion. □

7.5.1. Proof of Proposition 7.2.3 (ii)

Combining Proposition 7.5.1 and Lemma 7.3.1 proves Proposition 7.2.3.

Proof of Proposition 7.2.3 (ii). Let $p > \frac{3}{2}$ and let $\delta_0 := \delta_0(p)$ as defined in Lemma 7.3.1. We see that (7.16) implies a constant $C_1 > 0$ such that for all $\varepsilon \in (0,1)$, we have $\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C_1$ for all $t \in (0,T_{\max,\varepsilon})$, which combined with Proposition 7.5.1 yields a constant $C_2 > 0$ such that $\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_2$ for all $t \in (0,T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. Lemma 7.5.4 implies $C_3 > 0$ fulfills that $\|\nabla v_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)} \leq C_3$ for all $\varepsilon \in (0,1)$ and all $t \in (0,T_{\max,\varepsilon})$. Moreover, since now $\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty$, we apply Lemma 7.5.3 to find $C_4 > 0$ such that $\|A^\beta u(\cdot,t)\|_{L^2(\Omega)} \leq C_4$ for $t \in (0,T_{\max,\varepsilon})$ and all $\varepsilon \in (0,1)$. Therefore Lemma 7.2.1 implies that $T_{\max,\varepsilon} = \infty$ for all $\varepsilon \in (0,1)$, thus the solution is global. □

7.6. Passing to the limit

We now wish to obtain a solution of (7.1) by taking $\varepsilon \to 0$ in $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$. In order to achieve this, we shall first prepare some estimates for $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$ which are independent of ε. Since we cannot expect the regularity in $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times(0,\infty))$ to be uniform in ε due to the construction

of S_ε, we will first show that the triple of limit functions solves (7.1) in the sense of distributions, and then apply standard parabolic regularity result to conclude that it is actually a classical solution. This procedure is similar to that in Chapter 6.
Let us first recall the definition of a weak solution, which is in the spirit of Definition 6.5.2.

Definition 7.6.1. We say that (u, v, U) is a global weak solution of (7.1) associated to initial data (u_0, v_0, U_0) if

$$\begin{cases} u \in L^\infty((0,\infty)\times\Omega) \cap L^2_{loc}((0,\infty); W^{1,2}(\Omega)), \\ v \in L^\infty((0,\infty)\times\Omega) \cap L^2_{loc}((0,\infty); W^{1,2}(\Omega)), \\ U \in L^\infty((0,\infty)\times\Omega) \cap L^2_{loc}((0,\infty); W^{1,2}_{0,\sigma}(\Omega)) \end{cases} \tag{7.84}$$

and for all $\psi \in C_0^\infty(\overline{\Omega}\times[0,\infty);\mathbb{R})$ and all $\zeta \in C_{0,\sigma}^\infty(\Omega\times[0,\infty);\mathbb{R}^N)$ the following identities hold:

$$\begin{aligned} -\int_0^\infty\int_\Omega u\psi_t - \int_\Omega u_0\psi(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\psi + \int_0^\infty\int_\Omega uS(x,u,v)\cdot\nabla v\cdot\nabla\psi \\ &\quad + \int_0^\infty\int_\Omega uU\cdot\nabla\psi, \\ -\int_0^\infty\int_\Omega v\psi_t - \int_\Omega v_0\psi(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla v\cdot\nabla\psi - \int_0^\infty\int_\Omega uv\psi \\ &\quad + \int_0^\infty\int_\Omega vU\cdot\nabla\psi, \\ \text{and } -\int_0^\infty\int_\Omega U\cdot\zeta_t - \int_\Omega U_0\cdot\zeta(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla U\cdot\nabla\zeta - \kappa\int_0^\infty\int_\Omega (U\cdot\nabla)U\cdot\zeta \\ &\quad + \int_0^\infty\int_\Omega u\nabla\Phi\cdot\zeta. \end{aligned} \tag{7.85}$$

Next we prepare some estimates required to obtain the above identities. The idea is based on Section 6.5, but here we are going beyond to obtain some uniform Hölder estimates.

Lemma 7.6.2. *There exists $C > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\int_0^\infty\int_\Omega |\nabla v_\varepsilon|^2 \le C \quad \text{and} \tag{7.86}$$

$$\int_0^\infty\int_\Omega |\nabla u_\varepsilon|^2 \le C. \tag{7.87}$$

Proof. Since $T_{\max,\varepsilon} = \infty$, Lemma 7.5.2 implies (7.86). Testing the first equation in (7.11) with u_ε and using Young's inequality implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2 + \int_\Omega |\nabla u_\varepsilon|^2 &= \int_\Omega u_\varepsilon S_\varepsilon \nabla u_\varepsilon\cdot\nabla v_\varepsilon \\ &\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{2}\|u_\varepsilon\|^2_{L^\infty(\Omega)}(S_0(\|v_0\|_{L^\infty(\Omega)}))^2\int_\Omega |\nabla v_\varepsilon|^2, \end{aligned}$$

for all $t > 0$. Therefore, we establish (7.87) by (7.86) and Proposition 7.2.3. □

Lemma 7.6.3. *There are $\gamma \in (0,1)$ and a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that as $\varepsilon = \varepsilon_j \searrow 0$, it holds that*

$$u_\varepsilon \to u \text{ in } C^{\gamma,\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times (0,\infty)), \tag{7.88}$$

$$v_\varepsilon \to v \text{ in } C^{\gamma,\frac{\gamma}{2}}_{loc}(\overline{\Omega} \times (0,\infty)), \tag{7.89}$$

$$U_\varepsilon \to U \text{ in } C^{\gamma}_{loc}(\overline{\Omega} \times (0,\infty)), \tag{7.90}$$

$$\nabla U_\varepsilon \to \nabla U \text{ in } C^{\gamma}_{loc}(\overline{\Omega} \times (0,\infty)). \tag{7.91}$$

Moreover, there is $C > 0$ such that for all $s \in [1,\infty)$, we have

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \le C, \tag{7.92}$$

$$\|v\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \le C \text{ and} \tag{7.93}$$

$$\|U\|_{C^{1+\gamma,\gamma}(\overline{\Omega}\times[s,s+1])} \le C. \tag{7.94}$$

Proof. The first part (7.88)-(7.91) is precisely written in Lemma 6.5.3 and 6.5.4. In order to prove the remaining part, it is sufficient to show that there is $C > 0$ such that for all $\varepsilon \in (0,1)$ and $s \ge 1$,

$$\|u_\varepsilon\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \le C, \tag{7.95}$$

$$\|v_\varepsilon\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \le C \text{ and} \tag{7.96}$$

$$\|U_\varepsilon\|_{C^{1+\gamma,\gamma}(\overline{\Omega}\times[s,s+1])} \le C. \tag{7.97}$$

Let $s \ge 1$. Let $\xi_s \in C_0^\infty((0,\infty))$ satisfy $\xi = 0$ on $(0, s-\frac{1}{2}) \cup (s+\frac{3}{2},\infty)$ and $\xi = 1$ on $[s,s+1]$, and $\xi_s' \le 4$ for all $s \ge 1$. We see that ξu_ε is a weak solution of

$$(\xi u_\varepsilon)_t - \nabla\cdot(\nabla(\xi u_\varepsilon) - \xi u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon - u_\varepsilon U_\varepsilon) = \xi' u_\varepsilon, \quad t \in [0,\infty),$$

associated with Neumann boundary condition and $\xi u_\varepsilon(\cdot,0) = 0$. Since $(\nabla(\xi u_\varepsilon) - \xi u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon - u_\varepsilon U_\varepsilon)\cdot\nabla(\xi u_\varepsilon) > \frac{1}{2}|\nabla(\xi u_\varepsilon)|^2 - u_\varepsilon^2|S_\varepsilon|^2|\nabla v_\varepsilon|^2 - u_\varepsilon^2|U_\varepsilon|^2$, we see together with the fact guaranteed in Lemma 7.4.5 and Lemma 7.5.4 that the norms of $u_\varepsilon^2|S_\varepsilon|^2|\nabla v_\varepsilon|^2 + u_\varepsilon^2|U_\varepsilon|^2$ and $\xi' u_\varepsilon$ are bounded in $L^p((s-\frac{1}{2}, s+\frac{3}{2}); L^q(\Omega))$ for suitably large p or q and independent of s and ε, thus Theorem 1.3 in [73] implies there are $\gamma_1 \in (0,1)$ and $c_1 > 0$ such that for every $\varepsilon \in (0,1)$

$$\|u_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times[s,s+1])} \le \|\xi u_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times[s-\frac{1}{2},s+\frac{3}{2}])} \le c_1 \text{ for all } s \ge 1,$$

where c_1 depends on $\|\xi u_\varepsilon\|_{L^\infty(\Omega\times(s-\frac{1}{2},s+\frac{3}{2}))}$ and the norms of $u_\varepsilon^2|S_\varepsilon|^2|\nabla v_\varepsilon|^2 + u_\varepsilon^2|U_\varepsilon|^2$ in $L^p((s-\frac{1}{2}, s+\frac{3}{2}); L^q(\Omega))$. A similar reasoning yields some $\gamma_2 \in (0,1)$ and $c_2 > 0$ such that for each $\varepsilon \in (0,1)$

$$\|v_\varepsilon\|_{C^{\gamma_2,\frac{\gamma_2}{2}}(\overline{\Omega}\times[s,s+1])} \le c_2 \text{ for all } s \ge 1.$$

The derivation of the regularity of U_ε is similar to Lemma 6.5.4. We consider ξU_ε, which satisfies

$$(\xi U_\varepsilon)_t = \xi_t U_\varepsilon + \xi U_{\varepsilon t} = \Delta(\xi U_\varepsilon) - \kappa\xi(U_\varepsilon\cdot\nabla)U_\varepsilon + \xi u_\varepsilon\nabla\Phi + \xi' U_\varepsilon + \xi\nabla P, \text{ on } (s-\frac{1}{2}, s+\frac{3}{2}),$$

with $\xi U_\varepsilon(\cdot,0) = 0$ and $\xi U_\varepsilon = 0$ on $\partial\Omega$. Thus by an application of [34, Theorem 2.8], for any $r \in (1,\infty)$, we deduce the existence of a constant $c_3 > 0$ fulfilling

$$\int_0^\infty \|(\xi U_\varepsilon)_t\|^r_{L^r(\Omega)} + \int_0^\infty \|D^2(\xi U_\varepsilon)\|^r_{L^r(\Omega)}$$

$$\leq c_3\kappa \int_0^\infty \|\mathscr{P}\left((\xi U_\varepsilon \cdot \nabla)U_\varepsilon\right) + \mathscr{P}\left(\xi u_\varepsilon \nabla\Phi\right) + \mathscr{P}\left(\xi' U_\varepsilon\right)\|_{L^r(\Omega)}^r ds,$$

which, due to the definition of ξ and boundedness of U_ε, u_ε, ξ', for all $\varepsilon \in (0,1)$

$$\int_{s-\frac{1}{2}}^{s+\frac{3}{2}} \|(\xi U_\varepsilon)_t\|_{L^r(\Omega)}^r + \int_{s-\frac{1}{2}}^{s+\frac{3}{2}} \|D^2(\xi U_\varepsilon)\|_{L^r(\Omega)}^r \leq c_4\kappa \int_{s-\frac{1}{2}}^{s+\frac{3}{2}} \|\nabla(\xi U_\varepsilon)\|_{L^r(\Omega)}^r + c_5$$

for all $s \geq 1$ and for some $c_4 > 0$, $c_5 > 0$. Let $q \in (0,r)$ and $a = \frac{1-\frac{N}{r}+\frac{N}{q}}{2-\frac{N}{r}+\frac{N}{q}} \in (\frac{1}{2},1)$. Then the Gagliardo-Nirenberg inequality yields $c_6 > 0$ such that

$$\|\nabla(\xi U_\varepsilon)(\cdot,t)\|_{L^r(\Omega)}^r \leq c_6 \|D^2(\xi U_\varepsilon)(\cdot,t)\|_{L^r(\Omega)}^{ar} \|(\xi U_\varepsilon)(\cdot,t)\|_{L^q(\Omega)}^{(1-a)r} \text{ for all } t \geq 0.$$

Integrating the above inequality on $(s-\frac{1}{2}, s+\frac{3}{2})$ and using Young's inequality guarantee the existence of $c_7 > 0$ and $c_8 > 0$ fulfilling

$$\begin{aligned} \int_{s-\frac{1}{2}}^{s+\frac{3}{2}} \|\nabla(\xi U_\varepsilon)\|_{L^r(\Omega)}^r &\leq c_7 \int_{s-\frac{1}{2}}^{s+\frac{3}{2}} \|D^2(\xi U_\varepsilon)\|_{L^r(\Omega)}^{ar} \\ &\leq \int_{s-\frac{1}{2}}^{s+\frac{3}{2}} \left(\frac{1}{2}\|D^2(\xi U_\varepsilon)\|_{L^r(\Omega)}^r + c_8\right). \end{aligned} \tag{7.98}$$

for all $s \in [1,\infty)$ and for every $\varepsilon \in (0,1)$. Combining the above estimates we see that there is $c_9 > 0$ such that for all $s \geq 1$ and all $\varepsilon \in (0,1)$,

$$\int_s^{s+1} \|(U_\varepsilon)_t\|_{L^r(\Omega)}^r + \int_s^{s+1} \|D^2 U_\varepsilon\|_{L^r(\Omega)}^r \leq c_9.$$

Let $r \in (1,\infty)$ be sufficiently large, the Sobolev embedding theorem implies the existence of $\gamma_3 \in (0,1)$, $c_{10} > 0$ such that

$$\|U_\varepsilon\|_{C^{1+\gamma_3,\gamma_3}(\overline{\Omega}\times[s,s+1])} \leq c_{10}$$

for all $s \in [1,\infty)$ and for every $\varepsilon \in (0,1)$. Choosing $\gamma \in (0, \min\{\gamma_1,\gamma_2,\gamma_3\})$ we have proved (7.92)-(7.94).

□

Lemma 7.6.4. *There is $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that as $\varepsilon = \varepsilon_j \searrow 0$, it holds that*

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^2(\Omega \times (0,\infty)), \tag{7.99}$$

$$\nabla v_\varepsilon \rightharpoonup \nabla v \text{ in } L^2(\Omega \times (0,\infty)), \tag{7.100}$$

$$v_\varepsilon \overset{\star}{\rightharpoonup} v \text{ in } L^\infty((0,\infty); W^{1,q_0}(\Omega)), \tag{7.101}$$

$$U_\varepsilon \overset{\star}{\rightharpoonup} U \text{ in } L^\infty((0,\infty); D(A^\beta)), \tag{7.102}$$

$$S_\varepsilon(x, u_\varepsilon(x,t), v_\varepsilon(x,t)) \to S(x, u(x,t), v(x,t)) \text{ a.e. in } \Omega \times (0,\infty). \tag{7.103}$$

Proof. First (7.99) and (7.100) follow from Lemma 7.6.2. Proposition 7.2.3 implies (7.102) and (7.101). Due to the obtained convergence (7.88) and (7.89) and the continuity of S, we conclude that (7.103) holds. □

Lemma 7.6.5. *The functions u, v, U from Lemma 7.6.3 form a weak solution to* (7.1) *in the sense of Definition 7.6.1.*

Proof. We test (7.11) with ψ and ζ as specified in Definition 7.6.1. Lemma 7.6.4 allows us to take the limit in each integral, thus we obtain the weak formulation. □

Lemma 7.6.6. *The functions u, v, U from Lemma 7.6.4 satisfy*

$$u \in C^{2+\gamma,1+\frac{\gamma}{2}}_{loc}(\overline{\Omega}\times(0,\infty)),\ v \in C^{2+\gamma,1+\frac{\gamma}{2}}_{loc}(\overline{\Omega}\times(0,\infty)),\ U \in C^{2+\gamma,1+\frac{\gamma}{2}}_{loc}(\overline{\Omega}\times(0,\infty)) \tag{7.104}$$

for some $\gamma \in (0,1)$. Moreover, there is a constant $C > 0$ such that for all $s \geq 2$,

$$\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \leq C,\ \|v\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \leq C,\ \|U\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])} \leq C. \tag{7.105}$$

Proof. The property (7.104) is precisely proven in Lemma 6.5.8. It is left to show that (7.105) holds. Taking ξ_s as in Lemma 7.6.3, we see that ξv is a weak solution of $(\xi v)_t - \Delta(\xi v) + u(\xi v) + U\cdot\nabla(\xi v) - \xi' v = 0$ on $t \in (s-\frac{1}{2}, s+\frac{3}{2})$ associated with Neumann boundary condition and $\xi v(\cdot, s-\frac{1}{2}) = 0$. First [50, Thm IV.5.3] guarantees that $\xi v \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s-\frac{1}{2}, s+\frac{3}{2}])$. Furthermore, [57, Thm 4.9] can be applied to show that the norm $\|\xi v\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s-\frac{1}{2},s+\frac{3}{2}])}$ is controlled by the corresponding Hölder norms of u and U, which are bounded due to Lemma 7.6.3.
For the regularity of u, we improve it similarly as v with slight changes since its boundary condition also involves v. We first estimate the $C^{1+\gamma,\frac{1+\gamma}{2}}$-norm of ξu, and its $C^{2+\gamma,1+\frac{\gamma}{2}}$ norm by [57, Thm 4.8] and [57, Thm 4.9], respectively. Therefore, for all $s \geq 1$, $\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[s,s+1])}$ is uniformly bounded.
Again, we consider ξU, which satisfies $(\xi U)_t = \Delta(\xi U) - \kappa\xi(U\cdot\nabla)U + \xi u\nabla\Phi + \xi' U + \xi\nabla P$ with Dirichlet boundary condition. Lemma 7.6.3 already ensures the $C^{\gamma,\frac{\gamma}{2}}$ bound on the right hand side. Thus [78, Thm 1.1] together with the uniqueness guaranteed in [76, Thm.V.1.5.1] implies (7.105).

□

We have shown that the weak solution (u, v, U) of (7.1) also enjoys higher regularity as indicated in Lemma 7.6.6. Therefore, it is actually a classical solution.

Proof of Theorem 7.1.1. Lemma 7.6.5 shows that the function (u, v, U) obtained as the limit of $(u_\varepsilon, v_\varepsilon, U_\varepsilon)$ is a weak solution of (7.1) . Moreover, its smooth regularity is guaranteed by Lemma 7.6.6. Hence we know that it solves (7.1) classically. The boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ can be seen from Proposition 7.2.3 and (7.88). We see that (7.101) implies that $\|v(\cdot,t)\|_{W^{1,q_0}(\Omega)}$ is bounded for all $t > 0$. The associated component P is obtained by [76, Thm.V.1.8.1] and satisfies $\nabla P \in C^0(\overline{\Omega}\times(0,\infty))$ due to the smooth regularity of u, Φ. Moreover, $U \in L^\infty((0,T); D(A^\beta))$ is asserted by (7.102). The continuity up to the initial time can be proven similarly as [13, Lemma 5.8] (Lemma 6.5.9 in Chapter 6); first we prove that for $T > 0$, $u_t, v_t \in L^2((0,T); (W^{1,2}(\Omega))^*)$ and $U_t \in L^2((0,T); (W^{1,2}_{0,\sigma}(\Omega))^*)$. In conjunction with (7.101) and (7.102), we can conclude the assertions for v and U by the embedding [75, Cor. 8.4]. Using the continuity of u_ε and the uniform convergence (7.88), the continuity of u at $t = 0$ can be done similarly as in [Lemma 6.5.9, Chapter 6]. Hence we have proved Theorem 7.1.1. □

7.7. Stabilization

In this section, we prove the large time behavior for each component of the solution (u, v, U) obtained in the last section. Let us begin with the convergence of u, which will imply convergence for v and U later.

Lemma 7.7.1. *Let (u, v, U, P) be a bounded classical solution of (7.1). We have*

$$\int_0^\infty \int_\Omega |\nabla u|^2 < \infty.$$

Proof. The statement holds due to (7.87) and (7.99). □

Lemma 7.7.2. *Let (u, v, U, P) be a bounded classical solution of (7.1). We have*

$$\|u(\cdot, t) - \overline{u}_0\|_{L^\infty(\Omega)} \to 0, \ \text{as } t \to \infty. \tag{7.106}$$

Proof. Suppose on contrary that there are $c_1 > 0$ and a sequence $t_k \to \infty$ such that

$$\|u(\cdot, t_k) - \overline{u}_0\|_{L^\infty(\Omega)} > c_1 \quad \text{for } k \in \mathbb{N}. \tag{7.107}$$

Now we define

$$g_k(x, s) = u(x, s + t_k), \ (x, s) \in \overline{\Omega} \times [0, 1].$$

By the regularity guaranteed in Lemma 7.6.6, we see that there are $\alpha \in (0, 1)$ and $c_2 > 0$ such that for all $k \in \mathbb{N}$,

$$\|g_k\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0,1])} \le c_2.$$

The Arzelá-Ascoli theorem implies that $\{g_k\}_{k \in \mathbb{N}}$ is relatively compact in $C^1(\overline{\Omega} \times [0, 1])$. Thus we can find a subsequence $\{g_{k_j}\}_{j \in \mathbb{N}}$ and $u_\infty \in C^1(\overline{\Omega} \times [s, s + 1])$ such that

$$g_{k_j} \to u_\infty \in C^1(\overline{\Omega} \times [0, 1]), \ \text{as } j \to \infty. \tag{7.108}$$

It is left to show $u_\infty = \overline{u}_0$. From Lemma 7.7.1, we see that

$$\int_0^1 \int_\Omega |\nabla g_{k_j}|^2 \to 0 \text{ as } j \to \infty,$$

which combined with (7.108) implies that

$$\int_0^1 \int_\Omega |\nabla u_\infty|^2 = 0.$$

Since $u_\infty \in C^1(\overline{\Omega} \times [0, 1])$, we deduce that $u_\infty \equiv L$ with some $L \in \mathbb{R}$. Moreover, we have

$$|\Omega| \cdot L = \int_0^1 \int_\Omega u_\infty = \lim_{j \to \infty} \int_0^1 \int_\Omega g_{k_j} = \int_0^1 \int_\Omega u_0.$$

Thus we conclude $u_\infty \equiv \overline{u}_0$. This contradicts (7.107) by the definition of u_∞. □

Lemma 7.7.3. *Let (u, v, U, P) be a bounded classical solution of (7.1). For any $0 < \eta < \overline{u}_0$, there is $C > 0$ such that*

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le Ce^{-\eta t} \quad \textit{for all } t \ge 0. \tag{7.109}$$

Proof. For any $0 < \eta < \overline{u}_0$, we can find $T > 0$ such that

$$u(x,t) \geq \eta \quad \text{for all } (x,t) \in \Omega \times (T, \infty).$$

Thus the second equation of (7.1) can be written as

$$v_t \leq \Delta v - \eta v - U \cdot \nabla v \quad \text{for all } t \geq T.$$

The maximum principle yields that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v(\cdot,T)\|_{L^\infty(\Omega)} e^{-\eta t} \quad \text{for all } t \geq T.$$

Since v is bounded in $\Omega \times [0,T)$ by Theorem 7.1.1, an obvious choice of C completes the proof. □

Lemma 7.7.4. *Let (u, v, U, P) be a bounded classical solution of (7.1). Then we have*

$$\|U(\cdot,t)\|_{L^2(\Omega)} \to 0, \ \text{as } t \to \infty, \tag{7.110}$$
$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \to 0, \ \text{as } t \to \infty. \tag{7.111}$$

Proof. First we know that there is $C > 0$ such that

$$\|U(\cdot,t)\|_{L^2(\Omega)} \leq C, \tag{7.112}$$
$$\|A^\beta U(\cdot,t)\|_{L^2(\Omega)} \leq C \quad \text{for all } t > 0. \tag{7.113}$$

Since (7.65) together with (7.113) and (7.110) implies (7.111), it is sufficient to prove (7.110). Testing the third equation in (7.1) with U and integrating by part, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |U|^2 + \int_\Omega |\nabla U|^2 &= \int_\Omega u \nabla\Phi \cdot U = \int_\Omega (u - \overline{u}_0)\nabla\Phi \cdot U \\ &\leq \frac{\lambda_1'}{2}\int_\Omega |U|^2 + \frac{1}{2\lambda_1'}\|\nabla\Phi\|^2_{L^\infty(\Omega)}\int_\Omega |u - \overline{u}_0|^2\end{aligned}$$

for all $t \in (0, \infty)$. Using the Poincaré inequality, we obtain that

$$\frac{d}{dt}\int_\Omega |U|^2 + \lambda_1' \int_\Omega |U|^2 \leq \frac{1}{\lambda_1'}\|\nabla\Phi\|_{L^\infty(\Omega)}\|u - \overline{u}_0\|_{L^2(\Omega)} \quad \text{for all } t > 0. \tag{7.114}$$

Since the right hand side is bounded, using an ODE comparison principle, we conclude that $\|U(\cdot,t)\|_{L^2(\Omega)} < c_1$ for some $c_1 > 0$ and for all $t > 0$. Given $\epsilon > 0$, we apply Lemma 7.7.2 to find $t_0 > 0$ large enough satisfying

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} < \frac{\lambda_1' \epsilon}{\sqrt{2\|\nabla\Phi\|_{L^\infty(\Omega)}|\Omega|}} \ \text{for all } t > t_0.$$

Again, (7.114) with Gronwall's inequality implies that

$$\begin{aligned}\int_\Omega |U(\cdot,t)|^2 &\leq e^{-\lambda_1'(t-t_0)}\int_\Omega |U(\cdot,t_0)|^2 + \int_{t_0}^t e^{-\lambda_1'(t-s)}\frac{1}{\lambda_1'}\|\nabla\Phi\|_{L^\infty(\Omega)}|\Omega|\|u(\cdot,t) - \overline{u}_0\|^2_{L^\infty(\Omega)} ds \\ &\leq e^{-\lambda_1'(t-t_0)}c_1^2 + \frac{1}{(\lambda_1')^2}\|\nabla\Phi\|_{L^\infty(\Omega)}|\Omega| \sup_{t \in (t_0,\infty)} \|u(\cdot,t) - \overline{u}_0\|^2_{L^\infty(\Omega)} \\ &\leq \epsilon^2\end{aligned}$$

for all $t > \max\{t_0, \frac{1}{\lambda_1'}\ln e^{\lambda_1' t_0}\frac{2c_1^2}{\epsilon^2}\}$. Thus we have shown (7.110).

□

Lemma 7.7.5. *Let (u, v, U, P) be a bounded classical solution of (7.1). Then we have*

$$\|\nabla v(\cdot,t)\|_{L^{q_0}(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{7.115}$$

Proof. According to the L^p-L^q estimate for the Neumann semigroup, we can find $c_1 > 0$ such that

$$\|\nabla e^{t\Delta}\varphi\|_{L^{q_0}(\Omega)} \le c_1(1+t^{-\frac{1}{2}})e^{-\lambda_1 t}\|\varphi\|_{L^{q_0}(\Omega)} \tag{7.116}$$

for all $t > 0$ and $\varphi \in L^{q_0}(\Omega)$. Theorem 7.1.1 guarantees a constant $c_2 > 0$ such that

$$\|\nabla v(\cdot,s)\|_{L^{q_0}(\Omega)} \le c_2 \text{ for all } s > 0. \tag{7.117}$$

Let $\eta' \in \{0, \min\{\overline{u}_0, \lambda_1\}\}$. Moreover, from Lemma 6.2.4 we obtain a constant $c_3 > 0$ such that

$$\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-\eta s}e^{-\lambda_1(t-s)}ds \le c_3 e^{-\eta t} \text{ for all } \eta \in [0,\eta'], t > 0. \tag{7.118}$$

Given $\epsilon > 0$, Lemma 7.7.4 implies the existence of $t_0 > 0$ such that

$$\|U(\cdot,s)\|_{L^\infty(\Omega)} \le \frac{\epsilon}{4c_1c_2c_3} \text{ for all } s > t_0. \tag{7.119}$$

Since v is a classical solution, we invoke the variation-of-constants formula of v to see that

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^{q_0}(\Omega)} \le \|\nabla e^{(t-t_0)\Delta}v(\cdot,t_0)\|_{L^{q_0}(\Omega)} + \int_{t_0}^t \|\nabla e^{(t-s)\Delta}u(\cdot,s)v(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\ + \int_{t_0}^t \|\nabla e^{(t-s)\Delta}(U(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^{q_0}(\Omega)}ds\end{aligned} \tag{7.120}$$

for all $t > t_0$. Now (7.116) together with (7.117) implies $c_4 > 0$ such that

$$\|\nabla e^{t\Delta}v(\cdot,t_0)\|_{L^{q_0}(\Omega)} \le c_1(1+t^{-\frac{1}{2}})e^{-\lambda_1 t}\|\nabla v(\cdot,t_0)\|_{L^{q_0}(\Omega)} \le c_4 e^{-\lambda_1 t} \text{ for all } t > t_0.$$

Thanks to Theorem 7.1.1 and Lemma 7.7.3, there is $c_5 > 0$ such that $\|u(\cdot,s)\|_{L^{q_0}(\Omega)} \le c_5$ and $\|v(\cdot,s)\|_{L^\infty(\Omega)} \le c_5 e^{-\eta' s}$ for all $s > 0$. According to (7.116) and (7.118), we can find $c_6 > 0$ such that

$$\begin{aligned}&\int_{t_0}^t \|\nabla e^{(t-s)\Delta}u(\cdot,s)v(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\ &\le \int_{t_0}^t c_1(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\|u(\cdot,s)\|_{L^{q_0}(\Omega)}\|v(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &\le c_1c_5^2\int_{t_0}^t (1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}e^{-\eta' s}ds \\ &\le c_1c_5^2c_3e^{-\eta' t} \text{ for all } t > t_0.\end{aligned}$$

Collecting (7.116)-(7.119), we can estimate that

$$\int_{t_0}^t \|\nabla e^{(t-s)\Delta}(U(\cdot,s)\cdot\nabla v(\cdot,s))\|_{L^{q_0}(\Omega)}ds$$

$$
\begin{aligned}
&\le \int_{t_0}^{t} c_1(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\|(U\cdot\nabla v)(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\le \int_{t_0}^{t} c_1(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}\|U(\cdot,s)\|_{L^{\infty}(\Omega)}\|\nabla v(\cdot,t)\|_{L^{q_0}(\Omega)}ds \\
&\le \frac{\epsilon}{4c_1c_2c_3}c_1c_2\int_{t_0}^{t}(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}ds \\
&\le \frac{\epsilon}{4} \text{ for all } t>t_0.
\end{aligned}
$$

Therefore we conclude the existence of $c_6 > 0$ fulfilling

$$
\|\nabla v(\cdot,t)\|_{L^{q_0}(\Omega)} \le c_6 e^{-\eta' t} + \frac{\epsilon}{4}
$$

for all $t > t_0$. Let $T > \max\{t_0, \frac{\ln(\frac{4}{\epsilon}c_6)}{\eta'}\}$. We see that $\|\nabla v(\cdot,t)\|_{L^{q_0}(\Omega)} \le \epsilon$ for all $t > T$, which implies (7.115). □

Proof of Corollary 7.1.3. According to Theorem 7.1.1, we know that (7.1) admits a classical solution (u, v, U). Collecting Lemmata 7.7.2, 7.7.3, 7.7.4, and 7.7.5, we see that (u, v, U) enjoys all the convergence properties indicated in Corollary 7.1.3. Thus the proof is complete. □

Bibliography

[1] N. D. Alikakos. L^p-bounds of solutions of reaction-diffusion equations. *Comm. in Partial Differential Equations*, 4(8):827–868, 1979.

[2] H. Amann. Compact embeddings of vector-valued Sobolev and Besov spaces. *Glas. Mat. Ser. III*, 35(55)(1):161–177, 2000. Dedicated to the memory of Branko Najman.

[3] X. Bai and M. Winkler. Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics. *Indiana Univ. Math. J.*, 65(2):553–583, 2016.

[4] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler. Toward a mathematical theory of keller–segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.* , 25(09):1663–1763, 2015.

[5] P. Biler, W. Hebisch, and T. Nadzieja. The debye system: existence and large time behavior of solutions. *Nonlinear Anal.*, 23(9):1189–1209, 1994.

[6] T. Black, J. Lankeit, and M. Mizukami. On the weakly competitive case in a two-species chemotaxis model. *IMA J. Appl. Math.*, 81(5):860–876, 2016.

[7] X. Cao. Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces. *Discrete Contin. Dyn. Syst.*, 35:1891–1904, 2015.

[8] X. Cao. Boundedness in a three-dimensional chemotaxis–haptotaxis model. *Z. Angew. Math. Phys.*, 67(1):1–13, 2016.

[9] X. Cao. Global classical solutions in chemotaxis-(Navier)-Stokes system with rotational flux term. *J. Differential Equations*, 261(12):6883–6914, 2016.

[10] X. Cao. An interpolation inequality and its application in Keller-Segel model. *preprint*, 2017.

[11] X. Cao. Large time behavior in the logistic keller-segel model via maximal Sobolev regularity. *Discrete Contin. Dyn. Syst. Ser. B*, 22(9), 3369–3378, 2017.

[12] X. Cao and S. Ishida. Global-in-time bounded weak solutions to a degenerate quasilinear Keller-Segel system with rotation. *Nonlinearity*, 27(8):1899–1913, 2014.

[13] X. Cao and J. Lankeit. Global classical small-data solutions for a three-dimensional chemotaxis Navier-Stokes system involving matrix-valued sensitivities. *Calc. Var. Partial Differential Equations*, 55(4):Paper No. 107, 39, 2016.

[14] X. Cao and M. Winkler. Sharp decay estimates in a bioconvection model with quadratic degradation in bounded domains. *preprint*, 2016.

[15] M. Chae, K. Kang, and J. Lee. Existence of smooth solutions to coupled chemotaxis-fluid equations. *Discrete Contin. Dyn. Syst.*, 33(6):2271–2297, 2013.

[16] M. Chae, K. Kang, and J. Lee. Global existence and temporal decay in Keller-Segel models coupled to fluid equations. *Comm. Partial Differential Equations*, 39(7):1205–1235, 2014.

[17] G. Chamoun, M. Saad, and R. Talhouk. A coupled anisotropic chemotaxis-fluid model: the case of two-sidedly degenerate diffusion. *Comput. Math. Appl.*, 68(9):1052–1070, 2014.

[18] M. A. Chaplain and G. Lolas. Mathematical modelling of cancer invasion of tissue: dynamic heterogeneity. *NHM*, 1(3):399–439, 2006.

[19] Y.-S. Chung, K. Kang, and J. Kim. Global existence of weak solutions for a Keller-Segel-fluid model with nonlinear diffusion. *J. Korean Math. Soc.*, 51(3):635–654, 2014.

[20] J. P. Crimaldi, J. R. Hartford, and J. B. Weiss. Reaction enhancement of point sources due to vortex stirring. *Physical Review E*, 74(1):016307, 2006.

[21] M. Di Francesco, A. Lorz, and P. Markowich. Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior. *Discrete Contin. Dyn. Syst.*, 28(4):1437–1453, 2010.

[22] W. R. DiLuzio, L. Turner, M. Mayer, P. Garstecki, D. B. Weibel, H. C. Berg, and G. M. Whitesides. Escherichia coli swim on the right-hand side. *Nature*, 435(7046):1271–1274, 2005.

[23] C. Dombrowski, L. Cisneros, S. Chatkaew, R. E. Goldstein, and J. O. Kessler. Self-concentration and large-scale coherence in bacterial dynamics. *Phys. Rev. Lett.*, 93:098103, Aug 2004.

[24] R. Duan, A. Lorz, and P. Markowich. Global solutions to the coupled chemotaxis-fluid equations. *Comm. Partial Differential Equations*, 35(9):1635–1673, 2010.

[25] R. Duan and Z. Xiang. A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion. *Int. Math. Res. Not. IMRN*, (7), 1833–1852, 2012.

[26] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[27] M. Freitag. Blow-up profiles and refined extensibility criteria in quasilinear Keller-Segel systems. *preprint*, 2016.

[28] A. Friedman. *Partial differential equations*. Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969.

[29] H. Fujita and T. Kato. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.*, 16:269–315, 1964.

[30] D. Fujiwara and H. Morimoto. An L_r-theorem of the Helmholtz decomposition of vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 24(3):685–700, 1977.

[31] Y. Giga. Analyticity of the semigroup generated by the Stokes operator in L_r spaces. *Math. Z.*, 178(3):297–329, 1981.

[32] Y. Giga. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Differential Equations*, 62(2):186–212, 1986.

[33] Y. Giga and T. Miyakawa. Solutions in L_r of the Navier-Stokes initial value problem. *Arch. Rational Mech. Anal.*, 89(3):267–281, 1985.

[34] Y. Giga and H. Sohr. Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. *J. Funct. Anal.*, 102(1):72–94, 1991.

[35] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[36] M. A. Herrero and J. J. Velázquez. A blow-up mechanism for a chemotaxis model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 24(4):633–683, 1997.

[37] T. Hillen and K. J. Painter. A user's guide to PDE models for chemotaxis. *J. Math. Biol.*, 58(1-2):183–217, 2009.

[38] A. Hillesdon, T. Pedley, and J. Kessler. The development of concentration gradients in a suspension of chemotactic bacteria. *Bull. Math. Biol.*, 57(2):299–344, 1995.

[39] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.*, 105(3):103–165, 2003.

[40] D. Horstmann and G. Wang. Blow-up in a chemotaxis model without symmetry assumptions. *European J. Appl. Math.* , 12(02):159–177, 2001.

[41] B. Hu. *Blow-up theories for semilinear parabolic equations*, volume 2018 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[42] S. Ishida. Global existence and boundedness for chemotaxis-Navier-Stokes systems with position-dependent sensitivity in 2D bounded domains. *Discrete Contin. Dyn. Syst.*, 35(8):3463–3482, 2015.

[43] S. Ishida, K. Seki, and T. Yokota. Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains. *J. Differential Equations*, 256(8):2993–3010, 2014.

[44] J. Jiang, H. Wu, and S. Zheng. Global existence and asymptotic behavior of solutions to a chemotaxis-fluid system on general bounded domains. *Asymptot. Anal.*, 92(3-4):249–258, 2015.

[45] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theoret. Biol.*, 26(3):399 – 415, 1970.

[46] E. F. Keller and L. A. Segel. Model for chemotaxis. *J. Theoret. Biol.*, 30(2):225–234, 1971.

[47] A. Kiselev and L. Ryzhik. Biomixing by chemotaxis and efficiency of biological reactions: the critical reaction case. *J. Math. Phys.*, 53(11):115609, 9pp, 2012.

[48] A. Kiselev and L. Ryzhik. Biomixing by chemotaxis and enhancement of biological reactions. *Comm. Partial Differential Equations*, 37(2):298–318, 2012.

[49] H. Kozono, M. Miura, and Y. Sugiyama. Existence and uniqueness theorem on mild solutions to the Keller–Segel system coupled with the Navier–Stokes fluid. *J. Funct. Anal.* , 270(5):1663–1683, 2016.

[50] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[51] O. A. Ladyzhenskaya and N. N. Ural′tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.

[52] J. Lankeit. Chemotaxis can prevent thresholds on population density. *Discrete Contin. Dyn. Syst. Ser. B*, 20(5):1499–1527, 2015.

[53] J. Lankeit. Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source. *J. Differential Equations*, 258(4):1158–1191, 2015.

[54] E. Lauga, W. R. DiLuzio, G. M. Whitesides, and H. A. Stone. Swimming in circles: Motion of bacteria near solid boundaries. *Biophys. J.*, 90(2):400 – 412, 2006.

[55] T. Li, A. Suen, M. Winkler, and C. Xue. Global small-data solutions of a two-dimensional chemotaxis system with rotational flux terms. *Math. Models Methods Appl. Sci.*, 25(4):721–746, 2015.

[56] G. M. Lieberman. Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions. *Ann. Mat. Pura Appl. (4)*, 148:77–99, 1987.

[57] G. M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[58] J.-G. Liu and A. Lorz. A coupled chemotaxis-fluid model: global existence. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(5):643–652, 2011.

[59] A. Lorz. Coupled chemotaxis fluid model. *Math. Models Methods Appl. Sci.*, 20(6):987–1004, 2010.

[60] Y. Lou and M. Winkler. Global existence and uniform boundedness of smooth solutions to a cross-diffusion system with equal diffusion rates. *Comm. Partial Differential Equations*, 40(10):1905–1941, 2015.

[61] A. Metcalfe and T. Pedley. Bacterial bioconvection: weakly nonlinear theory for pattern selection. *J. Fluid Mech.*, 370:249–270, 1998.

[62] N. Mizoguchi and P. Souplet. Nondegeneracy of blow-up points for the parabolic Keller-Segel system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(4):851–875, 2014.

[63] N. Mizoguchi and M. Winkler. Blow-up in the two-dimensional parabolic Keller-Segel system. *preprint*, 2013.

[64] N. Mizoguchi and M. Winkler. Finite-time blow-up in the two-dimensional Keller-Segel system. *preprint*, 2013.

[65] T. Nagai. Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains. *J. Inequal*, 6:37–55, 2001.

[66] T. Nagai, T. Senba, T. Suzuki, et al. Chemotactic collapse in a parabolic system of mathematical biology. *Hiroshima Math. J.* , 30(3):463–449, 2000.

[67] T. Nagai, T. Senba, and K. Yoshida. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.*, 40(3):411–433, 1997.

[68] K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura. Exponential attractor for a chemotaxis-growth system of equations. *Nonlinear Anal.*, 51(1, Ser. A: Theory Methods):119–144, 2002.

[69] K. Osaki and A. Yagi. Finite dimensional attractor for one-dimensional Keller-Segel equations. *Funkcial. Ekvac.*, 44(3):441–469, 2001.

[70] H. G. Othmer and T. Hillen. The diffusion limit of transport equations ii: Chemotaxis equations. *SIAM J. Appl. Math.*, 62(4):1222–1250, 2002.

[71] K. J. Painter and T. Hillen. Spatio-temporal chaos in a chemotaxis model. *Physica D: Nonlinear Phenomena*, 240(4):363–375, 2011.

[72] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[73] M. M. Porzio and V. Vespri. Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations. *J. Differential Equations*, 103(1):146–178, 1993.

[74] C. G. Simader. The weak Dirichlet and Neumann problem for the Laplacian in L^q for bounded and exterior domains. Applications. In *Nonlinear analysis, function spaces and applications, Vol. 4 (Roudnice nad Labem, 1990)*, volume 119 of *Teubner-Texte Math.*, pages 180–223. Teubner, Leipzig, 1990.

[75] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[76] H. Sohr. *The Navier-Stokes equations*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2001. An elementary functional analytic approach.

[77] A. Sokolov, R. E. Goldstein, F. I. Feldchtein, and I. S. Aranson. Enhanced mixing and spatial instability in concentrated bacterial suspensions. *Phys. Rev. E*, 80:031903, Sep 2009.

[78] V. A. Solonnikov. Schauder estimates for the evolutionary generalized Stokes problem. In *Nonlinear equations and spectral theory*, volume 220 of *Amer. Math. Soc. Transl. Ser. 2*, pages 165–200. Amer. Math. Soc., Providence, RI, 2007.

[79] C. Stinner, C. Surulescu, and M. Winkler. Global weak solutions in a pde-ode system modeling multiscale cancer cell invasion. *SIAM Journal on Mathematical Analysis*, 46(3):1969–2007, 2014.

[80] C. Stinner, J. I. Tello, and M. Winkler. Competitive exclusion in a two-species chemotaxis model. *J. Math. Biol.*, 68(7):1607–1626, 2014.

[81] Y. Sugiyama. ε-regularity theorem and its application to the blow-up solutions of Keller-Segel systems in higher dimensions. *J. Math. Anal. Appl.*, 364(1):51–70, 2010.

[82] Z. Tan and X. Zhang. Decay estimates of the coupled chemotaxis-fluid equations in R^3. *J. Math. Anal. Appl.*, 410(1):27–38, 2014.

[83] Y. Tao. Boundedness in a chemotaxis model with oxygen consumption by bacteria. *J. Math. Anal. Appl.*, 381(2):521–529, 2011.

[84] Y. Tao. Global existence for a haptotaxis model of cancer invasion with tissue remodeling. *Nonlinear Anal. Real World Appl.*, 12(1):418–435, 2011.

[85] Y. Tao. Boundedness in a two-dimensional chemotaxis-haptotaxis system. *arXiv preprint arXiv:1407.7382*, 2014.

[86] Y. Tao and M. Wang. Global solution for a chemotactic–haptotactic model of cancer invasion. *Nonlinearity*, 21(10):2221, 2008.

[87] Y. Tao and M. Winkler. Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity. *J. Differential Equations*, 252(1):692–715, 2012.

[88] Y. Tao and M. Winkler. Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant. *J. Differential Equations*, 252(3):2520–2543, 2012.

[89] Y. Tao and M. Winkler. Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion. *Discrete Contin. Dyn. Syst.*, 32(5):1901–1914, 2012.

[90] Y. Tao and M. Winkler. Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(1):157–178, 2013.

[91] Y. Tao and M. Winkler. Boundedness and stabilization in a multi-dimensional chemotaxis—haptotaxis model. *Proc. Roy. Soc. Edinburgh Sect. A*, 144(05):1067–1084, 2014.

[92] Y. Tao and M. Winkler. Dominance of chemotaxis in a chemotaxis–haptotaxis model. *Nonlinearity*, 27(6):1225–1239, 2014.

[93] Y. Tao and M. Winkler. Energy-type estimates and global solvability in a two-dimensional chemotaxis–haptotaxis model with remodeling of non-diffusible attractant. *J. Differential Equations*, 257(3):784–815, 2014.

[94] Y. Tao and M. Winkler. Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system. *Z. Angew. Math. Phys.*, 66(5):2555–2573, 2015.

[95] Y. Tao and M. Winkler. Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system. *Z. Angew. Math. Phys.*, 67(6):Art. 138, 23, 2016.

[96] J. I. Tello and M. Winkler. A chemotaxis system with logistic source. *Comm. Partial Differential Equations*, 32(4-6):849–877, 2007.

[97] J. I. Tello and M. Winkler. Stabilization in a two-species chemotaxis system with a logistic source. *Nonlinearity*, 25(5):1413–1425, 2012.

[98] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[99] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, and R. E. Goldstein. Bacterial swimming and oxygen transport near contact lines. *Proc. Natl. Acad. Sci. USA*, 102(7):2277–2282, 2005.

[100] D. Vorotnikov. Weak solutions for a bioconvection model related to Bacillus subtilis. *Commun. Math. Sci.*, 12(3):545–563, 2014.

[101] Y. Wang and X. Cao. Global classical solutions of a 3D chemotaxis-Stokes system with rotation. *Discrete Contin. Dyn. Syst. Ser. B*, 204(9), 2015.

[102] Y. Wang and Z. Xiang. Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation. *J. Differential Equations*, 259(12):7578–7609, 2015.

[103] M. Wiegner. The Navier-Stokes equations–a neverending challenge? *Jahresber. Deutsch. Math.-Verein.*, 101(1):1–25, 1999.

[104] M. Winkler. Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity. *Math. Nachr.*, 283(11):1664–1673, 2010.

[105] M. Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional keller-segel model. *J. Differential Equations*, 248(12):2889–2905, 2010.

[106] M. Winkler. Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source. *Comm. Partial Differential Equations*, 35(8):1516–1537, 2010.

[107] M. Winkler. Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. *J. Math. Anal. Appl.*, 384(2):261–272, 2011.

[108] M. Winkler. Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. *Comm. Partial Differential Equations*, 37(2):319–351, 2012.

[109] M. Winkler. Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system. *Journal de Mathématiques Pures et Appliquées*, 100(5):748–767, 2013.

[110] M. Winkler. Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening. *J. Differential Equations*, 257(4):1056–1077, 2014.

[111] M. Winkler. How far can chemotactic cross-diffusion enforce exceeding carrying capacities? *J. Nonlinear Sci.*, 24(5):809–855, 2014.

[112] M. Winkler. Stabilization in a two-dimensional chemotaxis-Navier-Stokes system. *Arch. Ration. Mech. Anal.*, 211(2):455–487, 2014.

[113] M. Winkler. A two-dimensional chemotaxis-Stokes system with rotational flux: Global solvability, eventual smoothness and stabilization. 2014. preprint.

[114] M. Winkler. Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity. *Calc. Var. Partial Differential Equations*, 54(4):3789–3828, 2015.

[115] M. Winkler. Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities. *SIAM Journal on Mathematical Analysis*, 47(4):3092–3115, 2015.

[116] M. Winkler. Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems. *Discrete Contin. Dyn. Syst. Ser. B*, 22(7), 2777–2793, 2017.

[117] M. Winkler. Global weak solutions in a three-dimensional chemotaxis-navier-stokes system. *Ann. Inst. H. Poincaré Anal. Non Linéare*, 33(5), 1329–1352, 2016.

[118] M. Winkler. How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system? *Trans. Amer. Math. Soc.*, 369(5), 3067–3125, 2017.

[119] C. Xue. Macroscopic equations for bacterial chemotaxis: integration of detailed biochemistry of cell signaling. *J. Math. Biol.*, 70(1-2):1–44, 2015.

[120] C. Xue, E. O. Budrene, and H. G. Othmer. Radial and spiral stream formation in proteus mirabilis colonies. *PLoS Comput Biol*, 7(12):e1002332, 12 2011.

[121] C. Xue and H. G. Othmer. Multiscale models of taxis-driven patterning in bacterial populations. *SIAM J. Appl. Math.*, 70(1):133–167, 2009.

[122] C. Yang, X. Cao, Z. Jiang, and S. Zheng. Boundedness in a quasilinear fully parabolic keller–segel system of higher dimension with logistic source. *J. Math. Anal. Appl.*, 430(1):585–591, 2015.

[123] X. Ye. Existence and decay of global smooth solutions to the coupled chemotaxis-fluid model. *J. Math. Anal. Appl.*, 427(1):60–73, 2015.

[124] Q. Zhang and Y. Li. Convergence rates of solutions for a two-dimensional chemotaxis-Navier-Stokes system. *Discrete Contin. Dyn. Syst. Ser. B*, 20:2751–2759, 2015.

[125] Q. Zhang and Y. Li. Global weak solutions for the three-dimensional chemotaxis-Navier–Stokes system with nonlinear diffusion. *J. Differential Equations*, 2015.

[126] Q. Zhang and X. Zheng. Global well-posedness for the two-dimensional incompressible chemotaxis-Navier-Stokes equations. *SIAM J. Math. Anal.*, 46(4):3078–3105, 2014.