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Dissertation

Coloring of Signed Graphs

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Abstract

The study on signed graphs have been one of the hot research fields in the past few years. Theories on ordinary graphs have been generalized to signed graphs in many major aspects, such as the areas of flows, circuit covers, homomorphisms and so on. Graph colorings theory, which is strongly related to these aspects, has a central position in discrete mathematics. However, there are very few knowledges known on colorings of signed graphs so far.

The thesis is devoted to generalize a series of concepts, results and methods on vertex colorings of graphs to signed graphs for the first time. In particular, we introduce the notions of circular colorings and related integer colorings and list colorings for signed graphs. Some fundamental results for each notion are proved. Analogues of some classical results like Brooks' Theorem and Hajós' Theorem for signed graphs are presented. Moreover, the relation between a signed graph and its underlying unsigned graph is investigated, especially for chromatic numbers and list-chromatic numbers. Some exclusive features for signed graphs such as the chromatic spectrum are studied. The thesis concludes with a result on 3-coloring of unsigned planar graph.

Zusammenfassung

Signierte Graphen sind ein hochinteressantes und aktives Forschungsgebiet der Graphentheorie mit vielfältigen Anwendungen in anderen Disziplinen wie z.B. der Physik oder der Soziologie. Viele graphentheoretische Konzepte, wie z.B. Flüsse oder kürzeste Kreisüberdeckungen, wurden auf signierte Graphen verallgemeinert. Unter diesem Aspekt sind Färbungen signierter Graphen von besonderem mathematischen Interesse, da viele Konzepte, die für unsignierte Graphen äquivalent sind, dies für signierte Graphen nicht sind.

In dieser Arbeit werden vornehmlich Eckenfärbungen auf signierten Graphen studiert. Es wird das Konzept der zirkulären Färbung von signierte Graphen eingeführt und die darauf basierenden Parameter wie z.B. die zirkuläre chromatische Zahl, die chromatische Zahl und die listenchromatische Zahl werden studiert.

Klassische Ergebnisse der Graphnetheorie, wie die Sätze von Brooks und Hajós werden auf signierte Graphen verallgemeinert. Das chromatische Spektrum signierter Graphen wird bestimmt. Die Beziehung zwischen der chromatischen Zahl des signierten und der chromatischen Zahl des unterliegenden unsignierten Graphen studiert. Weiterhin werden die unterschiedlichen Färbungskonzepte verglichen. Die Arbeit schließt mit einem Ergebnis zu unsignierten 3-färbbaren planaren Graphen.

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To my parents,

to Ligang and our son Lois.

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Chapter 1

Introduction

1.1 Signed graphs, balances and switchings

The notion of signed graphs was first introduced by Harary [21] in 1953, as a mathematical model for certain problems in social psychology. The social structure of a group of persons is often represented by a graph, for which each vertex represents a person and two vertices are connected by an edge (i.e., adjacent) if and only if they know each other. The motivation for defining signed graphs arise from the fact that psychologists sometimes describe the relation between two persons as liking, disliking or indifference. A signed graph is a graph together with a sign of " + " or " − " on each edge, representing the emotion of liking or disliking. Hence, the notion of signed graphs is a generalization of ordinary graphs.

The standard definitions used in the theory of graphs may be found in [54]. We consider a graph to be finite and simple, i.e., with no loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and the edge set by $E(G)$. Let G be a graph and $\sigma\colon\ E(G) \to \{1, -1\}$ be a mapping. The pair (G, σ) is called a *signed graph*. We say that G is the *underlying graph* of (G, σ) and σ is a *signature* of G. The *sign* of an edge e is the value $\sigma(e)$. An edge is *positive* if it has a positive sign; otherwise, the edge is *negative*. The set $N_\sigma = \{e\colon\ \sigma(e) = -1\}$ is the set of negative edges of (G, σ) and $E(G) - N_\sigma$ the set of positive edges. A signature σ is *all-positive* (*all-negative*, respectively) if it has a positive sign (negative sign, respectively) on each edge. A graph G together with an all-positive signature is called an all-positive signed graph and denoted by $(G, +)$. Similarly, $(G, -)$ denotes an all-negative signed graph, that is, a signed graph for which the signature is all-negative. Throughout the paper, "a graph" is always regarded as an unsigned simple graph for the distinction from "a signed graph" and "a multigraph".

The balance of a signed graph has been an important topic since the very beginning when the notion of a signed graph was first introduced. A circuit C of a signed graph is balanced, if it contains an even number of negative edges; otherwise we say that C is unbalanced. A signed graph (G, σ) is unbalanced, if it contains an unbalanced circuit, otherwise we say that (G, σ) is balanced. A signed graph is antibalanced if every circuit contains an even number of positive edges.

The concept of switchings, as well as balances, is an exclusive feature for signed graphs. The generalization of notions of graphs to signed graphs are often required to respect switchings. Let (G, σ) be a signed graph. A *switching* at a vertex v of G defines a signed graph (G, σ') with $\sigma'(e) = -\sigma(e)$ if $e \in E(v)$, and $\sigma'(e) = \sigma(e)$ if $e \in E(G) \setminus E(v)$. Two signed graphs (G, σ) and (G, σ^*) are *switching equivalent* (briefly, *equivalent*) if they can be obtained from each other by a sequence of switchings. We also say that σ and σ^* are *equivalent signatures* of G.

It is well known (see e.g. [40]) that (G, σ) is balanced if and only if it is switching equivalent to an all-positive signed graph, and (G, σ) is antibalanced if and only if it is switching equivalent to an all-negative signed graph. The former result is one of the earliest results on signed graphs, first proved in [21]. Note, that a balanced bipartite graph is also antibalanced.

The theory of graphs have been generalized to signed graphs especially in the recent years in many aspects: the matroids of signed graphs [59], orientation of signed graphs [61], circular flows of signed graphs [40, 74], nowhere-zero flows of signed graphs [25, 26, 36, 42, 55], homomorphisms of signed graphs

[39], circuit covers of signed graphs [9, 34], the way signed graph arise from geometry [62] and so on. However, so far there are only a few knowledges known on colorings of signed graphs.

Graph coloring problems have gained more and more attention since the proposal of the four-color problem in 1852. The theory of graph colorings has a central position in discrete mathematics and it is closely related to other areas such as flows, homomorphism, time tabling, and scheduling problems. The thesis concentrates on the generalization of vertex colorings of graphs to signed graphs.

1.2 Colorings of signed graphs

In the 1980s Zaslavsky [57, 58, 60] started studying vertex colorings of signed graphs. The natural constraints for a coloring c of a signed graph (G, σ) are, that $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$, and that the colors can be inverted under switching, i.e. equivalent signed graphs have the same chromatic number. In order to guarantee these properties of a coloring, Zaslavsky [58] used the set $\{-k, \dots, 0, \dots, k\}$ of $2k + 1$ "signed colors" and studied the interplay between colorings and zero-free colorings through the chromatic polynomial.

Recently, Máčajová, Raspaud, and Škoviera [35] modified Zaslavsky's approach as follows. If $n = 2k + 1$, then let $M_n = \{0, \pm 1, \dots, \pm k\}$, and if $n = 2k$, then let $M_n = \{\pm 1, \dots, \pm k\}$. A mapping c from $V(G)$ to M_n is an "n-coloring" of (G, σ), if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. They defined $\chi_{\pm}((G, \sigma))$ to be the smallest number n such that (G, σ) has an n-coloring and call it the "chromatic number" for signed graphs. This allows them to study the behaviour of colourings of individual signed graphs. For example, they proved an analogue of the Brooks' theorem for signed graphs. However, so far there are still few results on the invariant $\chi_{\pm}((G, \sigma))$.

Later on, another version of "chromatic number" of signed graphs, defined by homomorphisms, was proposed by Naserasr, Rollová and Sopena in [39]. Unfortunately, there is no further discussion on this definition.

A definition of a "chromatic number" for signed graphs strongly depends on properties of the colors, as those of the "signed colors" in the definitions of Zaslavsky and of Máčajová et al. Since every element of an additive abelian group has an inverse element, it is natural to choose the elements of an additive abelian group as colors for a coloring of signed graphs. The self-inverse elements of the group play a crucial role in such colorings, since the color classes which are induced by self-inverse elements are independent sets. Hence, the following statement is true.

Proposition 1.1. *Let G be a graph and $\chi(G) = k$. If $\mathcal{C}$ is a set of k pairwise different self-inverse elements of an abelian group (e.g. of $\mathbb{Z}_2^n$ $(k \leq 2^n)$), then every k-coloring of G with colors from $\mathcal{C}$ is a k-coloring of (G, σ), for every signature σ of G. In particular, the chromatic number of (G, σ) with respect to colorings with colors of $\mathcal{C}$ is k.*

This proposition shows that a coloring of a signed graph with colors from a k-element abelian group is reduced to a k-coloring of its underlying graph if each element of the abelian group is self-inverse.

A coloring parameter of an unsigned graph, where the colors are also the elements of an abelian group, namely the cyclic group of integers modulo n, and where the coloring properties are defined by using operations within the group, is the circular chromatic number. This parameter was introduced by Vince [48] in 1988, as "the star-chromatic number". In 1992, Zhu [63] gave an alternate definition of circular chromatic number by circular colorings instead of (k, d)-colorings. The circular chromatic number is a natural generalization of the chromatic number for unsigned graphs. For more details on the circular chromatic number for unsigned graphs, we refer the readers to [63, 64, 65, 66, 68, 69, 71, 72, 73].

We combine these two approaches to define the circular chromatic number of a signed graph, which extends its definition for unsigned graphs. Moreover, this implies a new notion of vertex colourings of signed graphs and the corresponding chromatic number of signed graphs.

For $x \in \mathbb{R}$ and a positive real number r, we denote by $[x]_r$ the remainder of x divided by r, and define $|x|_r = \min\{[x]_r, [-x]_r\}$. Clearly, $[x]_r \in [0, r)$ and $|x|_r = |-x|_r$.

Definition 1.2. *Let $\mathbb{Z}_n$ denote the cyclic group of integers modulo n, $\mathbb{Z}/n\mathbb{Z}$. Let k and d be positive integers such that $k \geq 2d$. A (k,d)-*coloring *of a signed graph (G, σ) is a mapping $c : V(G) \to \mathbb{Z}_k$ such that for each edge $e = vw$, $d \leq |c(v) - \sigma(e)c(w)|_k$. The circular chromatic number $\chi_c((G, \sigma))$ of a signed graph (G, σ) is $\inf\{\frac{k}{d} :\ (G, \sigma)$ has a (k, d)-coloring$\}$. For a signed graph, a $(k, 1)$-coloring is also called a k-coloring. The minimum k such that a signed (G, σ) has a k-coloring is called the chromatic number of (G, σ) and denoted by $\chi((G, \sigma))$.*

To be distinct, we call the invariant $\chi_{\pm}((G, \sigma))$ and corresponding n-colorings defined in [35] the signed chromatic number and the signed n-colorings, respectively.

If we ask the signature σ to be all-positive, then the definitions of (k, d)-colorings, circular chromatic number and chromatic number for signed graphs in Definition 1.2 are reduced to ones for unsigned graphs, respectively.

With these new notions, we are able to start building the theory for circular coloring of signed graphs, which will be presented in Chapter 2. Some fundamental results are generalized to signed graphs. In particular, we first improve the circular chromatic number as a infimum by definition to the one expressed as a minimum. To be precise, we prove that for a signed graph (G, σ) on n vertices,

$$\chi_c((G, \sigma)) = \min\{\frac{k}{d} :\ (G, \sigma) \text{ has a } (k, d)\text{-coloring and } k \leq 4n\}.$$

Therefore, the circular chromatic numbers are always rational, and to calculate $\chi_c((G, \sigma))$, it is enough to consider those pairs of integers k and d such that $2d \leq k \leq 4n$. If $\chi_c((G, \sigma)) = \frac{k}{d}$ with k and d being positive integers, then the signed graph $\chi_c((G, \sigma))$ has a (k, d)-coloring.

Next, we generalize the concept of circular r-colorings of graphs, first introduced by Zhu [63], to signed graphs.

Definition 1.3. *Let (G, σ) be a signed graph and r be a real number at least 1. A circular r-coloring of (G, σ) is a function $f : V(G) \to [0, r)$ such that for any edge e with $e = xy$: if $\sigma(e) = 1$ then $1 \leq |f(x) - f(y)| \leq r - 1$, and if $\sigma(e) = -1$ then $1 \leq |f(x) + f(y) - r| \leq r - 1$.*

We prove the equivalence between (k, d)-colorings and circular r-colorings in the context of signed graphs. Hence, the circular chromatic number of a signed graph can be equivalently defined by circular r-colorings:

$$\chi_c((G, \sigma)) = \min\{r\colon\ (G, \sigma) \text{ has a circular } r\text{-coloring}\}.$$

We may choose different definitions for the convenience of proof when we are studying on this invariant in the thesis.

At last, we provide a relation between the circular chromatic number and the chromatic number: for every signed graph (G, σ), we have

$$\chi((G, \sigma)) - 1 \leq \chi_c((G, \sigma)) \leq \chi((G, \sigma)).$$

In contrast to the unsigned case, for which we know from [68] that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, signed graphs (G, σ) such that $\chi((G, \sigma)) - 1 = \chi_c((G, \sigma))$ may have special meaning. We construct examples of such signed graphs for every possible value of the chromatic number. Hence, the lower bound is sharp. However, if the lower bound is not attained, then it can be improved to: for every signed graph (G, σ) with $\chi((G, \sigma)) - 1 \neq \chi_c((G, \sigma))$, we have $(\chi((G, \sigma)) - 1)(1 + \dfrac{1}{4n - 1}) \leq \chi_c((G, \sigma))$, and in particular, $\chi((G, \sigma)) - 1 + \frac{1}{2n} < \chi_c((G, \sigma))$.

We go on with the discussion on the chromatic number of signed graphs in Chapter 3. An advantage of the signed chromatic number $\chi_{\pm}((G, \sigma))$, i.e. the definition by Máčajová et al for the "chromatic number" of signed graphs, is the generalization of Brooks' Theorem to signed graphs. Here for the chromatic number $\chi((G, \sigma))$, we generalize not only the Brooks' theorem but also the Hajós' theorem to signed graphs. We then prove a relation between the chromatic number of a signed graph and the one of its underlying graph: for every signed graph (G, σ), we have

$$\chi((G, \sigma)) \leq 2\chi(G) - 2,$$

and the bound is sharp. Moreover, the chromatic spectrum is introduced. For a given graph G, we show that possible values of $\chi((G, \sigma))$ among all the signature σ of G form an interval. So the chromatic spectrum of a graph is always an interval.

We also take a look at the signed chromatic number $\chi_{\pm}((G, \sigma))$ of signed graphs, for which there are only a few results known so far. The results we obtained on this invariant will be addressed in Chapter 4. We prove that the signed chromatic spectrum of a graph is always an interval and we also generalize the Hajós' theorem to signed graphs for $\chi_{\pm}((G, \sigma))$ by similar arguments as what we apply for $\chi((G, \sigma))$. The relation between these two non-equivalent invariants $\chi((G, \sigma))$ and $\chi_{\pm}((G, \sigma))$ are investigated. We show that for every signed graph, either they are equal or the difference between them is precisely 1.

Besides the circular chromatic number, the list-chromatic number is another major extension of the chromatic number for a graph. List colorings and corresponding list-chromatic number (also named the choice number) of graphs were first introduced by Erdős, Rubin and Taylor [14] in 1980. Compared to the ordinary k-colorings of a graph, for which the color set for each vertex is a uniform set of k distinct colors, list-colorings preassign a flexible color set for each vertice not only on colors but also on the length of color list. List colorings or the choosability of graphs have been extensively studied and become one of the main topics in the theory of graph colorings.

In Chapter 5, we generalize the concepts of list-colorings and corresponding list-chromatic number of unsigned graphs to signed graphs. As we talked before, there are several non-equivalent definitions for "the chromatic number" of signed graph, all of them extend the chromatic number for unsigned graphs in a certain sense. It seems easier for us to define a list-chromatic number for signed graphs.

Definition 1.4. *Given a signed graph (G, σ), a list-assignment of (G, σ) is a function L defined on $V(G)$ such that $\emptyset \neq L(v) \subseteq \mathbb{Z}$ for each $v \in V(G)$. An L-coloring of (G, σ) is a proper coloring c of (G, σ) such that $c(v) \in L(v)$ for each $v \in V(G)$. A list-assignment L is called a k-list-assignment if $|L(v)| = k$ for each $v \in V(G)$. We say (G, σ) is k-choosable if it admits an L-coloring for every k-list-assignment L. The list-chromatic number or choice number $\chi_l((G, \sigma))$ of (G, σ) is the minimum number k such that (G, σ) is k-choosable.*

Notice that the difference between concepts of $\chi((G, \sigma))$ and $\chi_{\pm}((G, \sigma))$ arises from the different choice of the color set, $\mathbb{Z}_n$ for the former and M_n for the latter. Since a k-choosable signed graph asks for the existence of a proper coloring for any k-list assignment which has only restriction on the length of the color list, we can see that the list-chromatic number $\chi_l((G, \sigma))$ we define here extends both invariants $\chi((G, \sigma))$ and $\chi_{\pm}((G, \sigma))$. Clearly, for every signed graph (G, σ), we have $\min\{\chi((G, \sigma)), \chi_{\pm}((G, \sigma))\} \leq \chi_l((G, \sigma))$. We first provide an upper bound for the list-chromatic number of a signed graph in terms of the list-chromatic number of its underlying graph: for every signed graph (G, σ), we have

$$\chi_l((G, \sigma)) \leq 2\chi_l(G).$$

Then we concentrate on the class of signed planar graphs. We generalize the results of [15, 31, 45, 46, 47, 52] to signed graphs. In particular, it is true that every signed planar graph is 5-choosable. Other results we obtained are concerned about sufficient conditions for 3- or 4-choosability. Moreover, we construct some signed planar graphs that show the sharpness of these sufficient conditions and on the other hand, that have different values of $\chi_l((G, \sigma))$ and $\chi_l(G)$.

Chapter 6 focuses on a particular problem on colorings of planar graphs – the Steinberg conjecture. The conjecture was proposed by Steinberg [44] in 1993. It states that every planar graph without cycles of length 4 or 5 is 3-colorable. A series of partial results to this conjecture were obtained mainly following Erdös' suggestion that forbids more kinds of cycles in the condition. Though Steinberg Conjecture was disproved [10] very recently, it motived the proposal of some related questions that are still open, so the study on this topic is going on. In this chapter, we prove a result related to Steinberg Conjecture: if the planar graph additionally has no cycles of length 8, then it is 3-colorable. This result improves on some earlier results.

Some parts of our results in the thesis have been published already. The results of

- Sections 2.1, 2.2 and 2.3 except Theorems 2.24 and 2.26 are published in

  [29] Y. Kang and E. Steffen. Circular coloring of signed graphs. J. Graph Theory. 2017; 00, 1-14. https://doi.org/10.1002/jgt.22147
- Sections 3.2 and 4.2 are published in

  [28] Y. Kang and E. Steffen. The chromatic spectrum of signed graphs. Discrete Math. **339** (2016) 2660-2663.
- Sections 3.4 and 4.3 are published in

  [27] Y. Kang. Hajós-like theorem for signed graphs. European J. Combin. **67** (2018) 199-207.
- Chapter 5 except Theorem 5.4 are published in

  [23] L. Jin, Y. Kang and E. Steffen. Choosability in signed planar graphs. European J. Combin. **52** (2016) 234-243.
- Chapter 6 are published in

  [22] L. Jin, Y. Kang, M. Schubert and Y. Wang. Plane graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable. SIAM J. Discrete Math. **31**-3 (2017) 1836-1847.

Chapter 2

Circular chromatic number χ_c of signed graphs

In this chapter, we generalize (k, d)-colorings, circular r-colorings and the circular chromatic number from unsigned graphs to signed graphs. We establish some fundamental results on these new concepts. Moreover, we define the chromatic number of signed graphs from the viewpoint of circular colorings. The relation between circular chromatic number and chromatic number for signed graphs is studied.

The structure of this chapter is arranged as follows. In Section 2.1, we introduce (k, d)-colourings of a signed graph and define the circular chromatic number as an infimum on (k, d)-colourings. Some basic facts on (k, d)-colourings are presented. Furthermore, the circular chromatic number is improved to be a minimum; i.e. if $\chi_c((G, \sigma)) = \frac{k}{d}$ then there exists a (k, d)-coloring of (G, σ). In Section 2.2, we introduce circular colourings of a signed graph and give an alternate definition of the circular chromatic number of signed graphs by circular colorings. The relation between the circular chromatic number and its related chromatic number for a signed graph is studied in Section 2.3. In particular, we show that the difference between these two parameters is at most 1. Indeed, there are signed graphs where the difference is 1. On the other hand, for a signed graph on n vertices, if the difference is smaller than 1, then there exists $\epsilon_n > 0$ such that the difference is at most $1 - \epsilon_n$. Finally, we introduce the interval chromatic number of signed graphs and relate it to both the chromatic number and the circular chromatic number.

2.1 (k, d)-colorings of a signed graph

The results of this section have already been published in [29].

Let us first give the definitions of (k, d)-colorings, the circular chromatic number and its related chromatic number for a signed graph.

For $x \in \mathbb{R}$ and a positive real number r, we denote by $[x]_r$, the remainder of x divided by r, and define $|x|_r = \min\{[x]_r, [-x]_r\}$. Hence, $[x]_r \in [0, r)$ and $|x|_r = |-x|_r$.

Definition 2.1. *Let $\mathbb{Z}_n$ denote the cyclic group of integers modulo n, $\mathbb{Z}/n\mathbb{Z}$. Let k and d be positive integers such that $k \geq 2d$. A (k, d)-*coloring *of a signed graph (G, σ) is a mapping $c : V(G) \to \mathbb{Z}_k$ such that for each edge $e = vw$, $d \leq |c(v) - \sigma(e)c(w)|_k$. The circular chromatic number $\chi_c((G, \sigma))$ of a signed graph (G, σ) is $\inf\{\frac{k}{d}\colon\ (G, \sigma)$ has a (k, d)-coloring$\}$. The minimum k such that (G, σ) has a $(k, 1)$-coloring is the chromatic number of (G, σ) and it is denoted by $\chi((G, \sigma))$.*

One of the fundamental concepts on signed graphs is switching, by which signed graphs can be classified into equivalent groups. In the following proposition, we study (k, d)-colorings under switching and show that two equivalent signed graphs have the same circular chromatic number.

Proposition 2.2. *Let k, d be positive integers, (G, σ) be a signed graph and c be a (k, d)-coloring of (G, σ). If (G, σ) and (G, σ') are equivalent, then there is a (k, d)-coloring c' of (G, σ'). In particular, $\chi_c((G, \sigma)) = \chi_c((G, \sigma'))$.*

Proof. Let $x \in V(G)$ and (G, σ') be obtained from (G, σ) by a switching at x. Define $c' : V(G) \to \mathbb{Z}_k$ with $c'(v) = c(v)$, if $v \neq x$, and $c'(x) = -c(x)$. For every edge e with $e = uw$: If $x \notin \{u, w\}$, then $|c(u) - \sigma(e)c(w)|_k = |c'(u) - \sigma'(e)c'(w)|_k$, and if $x \in \{u, w\}$, say $x = w$, then $|c'(u) - \sigma'(e)c'(w)|_k =$

$|c(u) - (-\sigma(e))(-c(w))|_k = |c(u) - \sigma(e)c(w)|_k$. Hence, c' is a (k, d)-coloring of (G, σ'), and therefore, $\chi_c((G, \sigma)) = \chi_c((G, \sigma'))$. □

Note, that if (G, σ) has a (k, d)-coloring, then by switching we can obtain an equivalent signed graph (G, σ') and a (k, d)-coloring c' on (G, σ') such that $c'(v) \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$ for each $v \in V(G)$.

Next, we determine the circular chromatic number χ_c and the related chromatic number χ of some specific graphs. For $n \geq 3$, let C_n denote the circuit with n vertices.

Proposition 2.3. *Let k be a positive integer.*

1. *If (C_{2k+1}, σ) is balanced, then $\chi_c((C_{2k+1}, \sigma)) = 2 + \frac{1}{k}$; otherwise, $\chi_c((C_{2k+1}, \sigma)) = 2$. Furthermore, $\chi((C_{2k+1}, \sigma)) = 3$.*
2. *$\chi((G, \sigma)) = 2$ if and only if G is bipartite. Furthermore, $\chi((G, \sigma)) = \chi_c((G, \sigma))$ if G is bipartite.*

Proof. **1.** If (C_{2k+1}, σ) is balanced, then (C_{2k+1}, σ) is switching equivalent to $(C_{2k+1}, +)$, hence, $\chi_c((C_{2k+1}, \sigma)) = 2 + \dfrac{1}{k}$. If (C_{2k+1}, σ) is unbalanced, then (C_{2k+1}, σ) is switching equivalent to C_{2k+1} which has one negative edge say, uv. Thus, we can assign to vertex u and v color 1, and to other vertices colors 3 and 1 alternatively. We thereby get a $(4, 2)$-coloring of (C_{2k+1}, σ), i.e., $\chi_c((C_{2k+1}, \sigma)) = 2$. And it is easy to check (C_{2k+1}, σ) has a $(3, 1)$-coloring, but can not be colored properly by two colors, therefore, $\chi((C_{2k+1}, \sigma)) = 3$.

2. If G is bipartite, then it can be colored with colors 0 and 1 and therefore, $\chi((G, \sigma)) = 2$. If $\chi((G, \sigma)) = 2$, then, since both colors are self-inverse in $\mathbb{Z}_2$, both color classes are independent sets. Hence, G is bipartite. Since $\chi((G, \sigma)), \chi_c((G, \sigma)) \geq 2$, it follows that $\chi((G, \sigma)) = \chi_c((G, \sigma))$ if G is bipartite. □

2.1.1 A recoloring technique: Updating

We introduce and study a recoloring technique, namely updating, applied on (k, d)-colorings of a signed graph. This technique will play a crucial role in several proofs in the next subsection.

Definition 2.4. *Let c be a (k,d)-coloring of a signed graph (G,σ) in which colors x_0 and its inverse $k-x_0$ are missing. Updating c at x_0 is defined as follows: if v is a vertex of color $[x_0+d]_k$, then recolor v by $[x_0+d-1]_k$; and meanwhile, if u is a vertex of color $[k-x_0-d]_k$, then recolor u by $[k-x_0-d+1]_k$. Let r be a positive integer. Updating c at a sequence of colors $x_0, [x_0+d]_k, \ldots, [x_0+(r-1)d]_k$ is called* updating c from x_0 by r steps. *We also say that a function c' is obtained from c by updating at x_0 (in r steps) if c' is the final function from $V(G)$ to $\mathbb{Z}_k$ in this process.*

Definition 2.5. *Let k,d be two positive integers. We define*

$$P(k,d) = \{\frac{1}{2}(k-2d+1), \frac{1}{2}(k-d+1), \frac{1}{2}(2k-d+1)\}.$$

Clearly, if both k and d are even, then $\mathbb{Z}_k \cap P(k,d) = \emptyset$; otherwise, $|\mathbb{Z}_k \cap P(k,d)| = 2$.

Lemma 2.6. *Let (G,σ) be a signed graph, c be a (k,d)-coloring of (G,σ), and let c' be obtained from c by updating at x_0. The following statements are equivalent:*

(1) *Either $x_0 \notin P(k,d)$ or both colors $[x_0+d]_k$ and $[k-x_0-d]_k$ are not used in c.*

(2) *c' is a (k,d)-coloring of (G,σ) in which the colors $x_0, [x_0+d]_k, [k-x_0]_k$ and $[k-x_0-d]_k$ are not used.*

Proof. ((**1**) $\Rightarrow$ (**2**)) If both $[x_0+d]_k$ and $[k-x_0-d]_k$ are not used in c, then it follows that c' is the same coloring as c since nothing happens in the updating process. So we are done in this case.

Let $x_0 \notin P(k,d)$, and suppose to the contrary that c' is not a (k,d)-coloring of (G,σ). Then there exists an edge e with two end-points u and v such that $|c'(u)-\sigma(e)c'(v)|_k < d$. Since c is a (k,d)-coloring of (G,σ), it follows that $|c(u)-\sigma(e)c(v)|_k \geq d$. Hence, the distance between the colors of u and v has been decreased in the updating process. The distance can be decreased by at most 2. Hence, we distinguish two cases.

Case a: The distance between the colors of u and v decreases by 2. In this case, both u and v have been recolored, say $c(u) = [x_0 + d]_k$ and $c(v) = [k - x_0 - d]_k$; and moreover, $[c(u) - \sigma(e)c(v)]_k \in \{d, d+1\}$. It follows that $\sigma(e) = 1$ and furthermore, $[c(u) - \sigma(e)c(v)]_k = d + 1$ since otherwise $c(u)$ and $c(v)$ are in fact the colors $[k - x_0]_k$ and x_0 which are missing in c. By simplification of this equality, we get $[2(x_0 + d) - k]_k = d + 1$ and thus, $x_0 \in \{\frac{k-d+1}{2}, \frac{2k-d+1}{2}\}$, contradicting the assumption that $x_0 \notin P(k,d)$.

Case b: The distance between the colors of u and v decreases by 1. In this case, exactly one of u and v has been recolored, say u; and moreover, $|c(u) - \sigma(e)c(v)|_k = d$. Without loss of generality, we may assume $c(u) = [x_0 + d]_k$. It follows that $c(v) = x_0$, contradicting the fact that x_0 is not used in c.

We obtain contradictions in both cases. Hence, c' is a (k,d)-coloring of (G,σ). Moreover, If the colors $[x_0 + d]_k$ and $[k - x_0 - d]_k$ occur in c', then they have been recolored by each other, which can happen in the only case that k is odd and $[x_0 + d]_k = x_0 + d = \frac{k+1}{2}$. However, this case is impossible since $x_0 \notin P(k,d)$. Finally, suppose to the contrary that the colors x_0 and $[k - x_0]_k$ occur in c'. Since they are not used in c, they have been reused in the updating process. Thus, $[x_0 + d - 1]_k = [k - x_0]_k$ and so $x_0 \in \{\frac{k-d+1}{2}, \frac{2k-d+1}{2}\}$, a contradiction.

((**1**) $\Leftarrow$ (**2**)) Suppose to the contrary that $x_0 \in P(k,d)$ and at least one of $[x_0 + d]_k$ and $[k - x_0 - d]_k$ are used in c. Without loss of generality, say $[x_0 + d]_k$ is used. We distinguish two cases according to the value of x_0.

Case 1: $x_0 \in \{\frac{k-d+1}{2}, \frac{2k-d+1}{2}\}$. Thus, $[x_0 + d - 1]_k = [k - x_0]_k$, which implies that the color $[k - x_0]_k$ has been reused in the updating process, a contradiction.

Case 2: $x_0 = \frac{k-2d+1}{2}$. Thus, $[x_0 + d]_k = [k - x_0 - d]_k + 1$, which implies that the colors $[x_0 + d]_k$ and $[k - x_0 - d]_k$ have been exchanged, a contradiction. □

2.1.2 Basic properties of (k,d)-colorings

(k,d)-colorings with smaller $\frac{k}{d}$

We first consider such a problem: given a signed graph (G,σ), look for a (k,d)-coloring of $((G,\sigma))$ with minimum $\frac{k}{d}$ and subjecting to that, k is minimum. The results we obtain to this problem will be used to prove Theorem 2.16, which is one of the main theorems on circular coloring of signed graphs.

We will need Updating technique which is introduced in section 2.1.1 and a lemma which follows from the rearrangement theorem of group theory, (see e.g. [13], section 1.4).

Theorem 2.7 (Rearrangement theorem of group theory)**.** *Each row and each column in the group multiplication table lists each of the group elements once and only once.*

From this, it follows that no two elements may be in the identical location in two rows or two columns. Thus, each row and each column is a rearranged list of the group elements. By the rearrangement theorem of group theory, we have the following lemma.

Lemma 2.8. *Let k,d and x be three integers with $k,d>0$ and $gcd(k,d)=1$. If $A=\{0,1,\ldots,k-1\}$ and $B=\{[x+id]_k: \ i\in A\}$, then $A=B$.*

Now, we are ready to prove some basic properties on (k,d)-colorings of signed graphs.

Theorem 2.9. *Let d,k,t be positive integers and $t\geq 3$, and let (G,σ) be a signed graph. If (G,σ) has a (tk,td)-coloring, then it has a $(tk-2k,td-2d)$-coloring.*

Proof. For $i\in\{0,1,\cdots,t-1\}$, let $A_i=\{i,i+t,i+2t,\cdots,i+(k-1)t\}$. Clearly, $A_0,\ldots,A_{t-1}$ are t pairwise disjoint sets of colors whose union is exactly the color set $\mathbb{Z}_{tk}$. We shall recolor each color in both sets A_1 and A_{t-1} by a color of A_0 as follows: for $i\in A_1$, recolor i by $i-1$, and for $i\in A_{t-1}$, recolor i by $i+1$. We obtain a new (tk,td)-coloring of (G,σ) in which no vertex receives a color from $A_1 \cup A_{t-1}$. Let $k' = tk - 2k$. Since the colors of the set $A_1 \cup A_{t-1}$ are not used, we define a new coloring by renaming colors by elements of $\mathbb{Z}_{k'}$. Change color x (from $\mathbb{Z}_{tk}$) to $x - |\{y\colon\ y \in A_1 \cup A_{t-1} \text{ and } y < x\}|$ (interpreted as element in $\mathbb{Z}_{k'}$) to obtain a mapping $\phi' : V(G) \to \mathbb{Z}_{k'}$. Let $d' = td - 2d$. We claim that ϕ' is a (k',d')-coloring of (G,σ). Denote by I_j the set $\{j, j+1, \ldots, j+td-1\}$ which is an interval of $\mathbb{Z}_{tk}$. Each interval I_j contains exactly $2d$ elements of $A_1 \cup A_{t-1}$, and any pair of mutually inverse elements of $\mathbb{Z}_{tk}$ has been recolored by a pair of mutually inverse elements of $\mathbb{Z}_{k'}$. It follows that ϕ' is a (k',d')-coloring of (G,σ), as required. □

Theorem 2.10. *Let (G,σ) be a signed graph on n vertices that has a $(2k,2d)$-coloring and $gcd(k,d) = 1$. If $k > 2n$, then (G,σ) has a (k,d)-coloring.*

Proof. Let c be a $(2k,2d)$-coloring of (G,σ). Since $k > 2n$, we may assume that there is an odd x_0, such that x_0 and $k - x_0$ are not used in c. Update c from x_0 by k steps to obtain a function c'. Denote by A the set of odd elements of $\mathbb{Z}_{2k}$. Since both $2k$ and $2d$ are even it follows with Lemma 2.8 that the colors of $A \cap \{c(v) : v \in V(G)\}$ have been recolored by colors of $\mathbb{Z}_{2k} \setminus A$ in the updating process. Hence, $A \cap \{c'(v) : v \in V(G)\} = \emptyset$, and by Lemma 2.6, c' is a $(2k,2d)$-coloring of (G,σ). Thus, $\phi : V(G) \to \mathbb{Z}_k$ with $\phi(v) = \frac{1}{2}c'(v)$ is a coloring of (G,σ). Let $I_j = \{j, j+1, \ldots, j+2d-1\}$ which is an interval of $\mathbb{Z}_{2k}$. Each interval I_j contains exactly d elements of A. Moreover, any pair of mutually inverse elements of $\mathbb{Z}_{2k}$ has been recolored by a pair of mutually inverse elements of $\mathbb{Z}_k$. Hence, ϕ is a (k,d)-coloring of (G,σ), as required. □

Theorem 2.11. *If (G,σ) is a signed graph on n vertices that has a (k,d)-coloring with $gcd(k,d) = 1$ and $k > 4n$, then (G,σ) has a (k',d')-coloring with $k' < k$ and $\frac{k'}{d'} < \frac{k}{d}$.*

Proof. Since $gcd(k,d) = 1$, we may assume that $P(k,d) \cap \mathbb{Z}_k = \{p,q\}$ and $p < q$.

Let $f\colon\ \mathbb{Z}_k \to \mathbb{Z}_k$ such that $x \equiv f(x)d \pmod{k}$. Lemma 2.8 implies that f is a bijection. Further, x and y are mutually inverse elements of $\mathbb{Z}_k$ if and only if $f(x)$ and $f(y)$ are mutually inverse ones, and $|f(p) - f(q)|_k = \lfloor \frac{k}{2} \rfloor$.

Let c be a (k,d)-coloring of (G,σ). Since $k > 4n$ we may assume that $x_0 \in \mathbb{Z}_k$ such that x_0 and $k - x_0$ are not used in c, and that $f(q), f(x_0)$ and $f(p)$ are in clockwise order in $\mathbb{Z}_k$.

Hence, c can be updated from x_0 by $[f(p) - f(x_0)]_k$ steps to obtain a (k,d)-coloring c' of (G,σ) in which colors p and $k-p$ are not used. Let $r = \min\{[f(p) - f(k-p)]_k, [f(q) - f(k-p)]_k\}$, i.e., r is the minimum positive integer such that

(∗) either $k - p + rd \equiv p \pmod{k}$ or $k - p + rd \equiv q \pmod{k}$.

Updating c' from $k-p$ by r steps, we obtain a function c'', which is a (k,d)-coloring c'' of (G,σ) by Lemma 2.6. We will show that no colors are reused in this updating process.

Let $A = \{[k-p+id]_k\colon\ 0 \le i \le r\}$ and $B = \{k-a\colon\ a \in A\}$. By simplifying the congruence expressions, we reformulate the minimality of r as: r is the minimum positive integer such that

(1) either $(r+1)d \equiv 1 \pmod{k}$ or $(2r+2)d \equiv 2 \pmod{k}$, if k is even;

(2) either $(r+2)d \equiv 1 \pmod{k}$ or $(2r+3)d \equiv 2 \pmod{k}$, if k is odd.

Claim. *No element of $A \cup B$ is used in coloring c''.*

Proof of the Claim. Suppose to the contrary that $A \cup B$ has a color α with $\alpha \in \{[k-p+r_1d]_k, k - [k-p+r_1d]_k\}$ appearing in c''. Considering that the color α is missing in the resulting coloring after exactly r_1 steps in the updating process, its appearance in c'' yields that it has been reused in some r_2 step with $r > r_2 > r_1$. It follows that either $k - p + r_2d \equiv k - p + r_1d + 1 \pmod{k}$ or $k - p + r_2d \equiv -(k - p + r_1d) + 1 \pmod{k}$.

In the former case, the congruence expression can be simplified as $(r_2 - r_1)d \equiv 1 \pmod{k}$. Note that $0 < r_2 - r_1 < r + 1$. A contradiction is obtained by the minimality of r.

In the latter case, the congruence expression can be simplified as $(r_1 + r_2 + 1)d \equiv 2 \pmod{k}$ if k is even and $(r_1 + r_2 + 2)d \equiv 2 \pmod{k}$ if k is odd. But then $r_1 + r_2 < 2r + 1$ which is a contradiction to the minimality of r. This completes the proof of the claim.

We continue the proof of the theorem. We distinguish the following four cases first by the parity of k and then by the condition $(*)$ for the minimality of r. In each case, we will define a (k', d')-coloring of (G, σ) with $\frac{k'}{d'} < \frac{k}{d}$ and $k' < k$, as desired.

Case 1: k is even. In this case, $p = \frac{1}{2}(k - d + 1)$ and $q = \frac{1}{2}(2k - d + 1)$.

Case 1.a: $k - p + rd \equiv p$ (mod k). The colors $[k-p+id]_k$ and $[k-p+(r-i)d]_k$ are mutually inverse, for $0 \le i \le r$. Thus, the set A consists of $\lceil \frac{r+1}{2} \rceil$ pairs of mutually inverse elements of $\mathbb{Z}_k$ and $\{0, \frac{k}{2}\} \nsubseteq A = B$. Since the colors of $A \cup B$ are not used in c'' by the claim, we rename the other colors: if $0 \notin A$, then change color x to $x - |\{y\colon\ y \in A \text{ and } y < x\}|$; otherwise, change color x to $x - |\{y\colon\ y \in A \text{ and } y < x\}| - \lfloor \frac{k-|A|}{2} \rfloor$. Define $k' = k - r - 1$. We thereby obtain a mapping $\phi' : V(G) \to \mathbb{Z}_{k'}$. Denote by I_j the set $\{j, j+1, \ldots, j+d-1\}$, $j \neq p$, which is an interval of $\mathbb{Z}_k$. Each interval I_j contains at most $\frac{rd+d-1}{k}$ elements of A. Define $d' = d - \frac{rd+d-1}{k}$. Moreover, any pair of mutually inverse colors of $\mathbb{Z}_k$ has been recolored to mutually inverse colors of $\mathbb{Z}_k$ and then has been renamed to be mutually inverse colors of $\mathbb{Z}_{k'}$. Hence, ϕ' is a (k', d')-coloring of (G, σ), and $\frac{k'}{d'} = \frac{k(k-r-1)}{d(k-r-1)+1} < \frac{k}{d}$.

Case 1.b: $k - p + rd \equiv q$ (mod k). We have that either $0 < f(q), f(k-p) < \frac{k}{2}$ or $\frac{k}{2} < f(q), f(k-p) < k$. Since $|f(p) - f(q)|_k = \frac{k}{2}$, it follows that neither $\{f(a) : a \in A\}$ nor A contains any pair of mutually inverse colors. Thus, $A \cup B$ consists of $r + 1$ pairs of mutually inverse colors and $0, \frac{k}{2} \notin A \cup B$. Define $k' = k - 2(r + 1)$. Since the colors in the set $A \cup B$ are not used in c'' by the claim, we may rename the other colors, changing color x to $x - |\{y\colon\ y \in A \cup B \text{ and } y < x\}|$, thereby obtain a mapping $\phi' : V(G) \to \mathbb{Z}_{k'}$. Denote by I_j the set $\{j, j + 1, \ldots, j + d - 1\}$ which is an interval of $\mathbb{Z}_k$. Each interval I_j contains at most $\frac{2rd+2d-2}{k}$ elements of A. Define $d' = d - \frac{2rd+2d-2}{k} = \frac{(k-2r-2)d+2}{k}$. By repeating the argument as in Case 1.a, we get a (k', d')-coloring of (G, σ). Furthermore, $\frac{k'}{d'} = \frac{k(k-2r-2)}{d(k-2r-2)+2} < \frac{k}{d}$.

Case 2: k is odd. In this case, $p = \frac{1}{2}(k - 2d + 1)$, and $q = \frac{1}{2}(k - d + 1)$ when d is even and $q = \frac{1}{2}(2k - d + 1)$ when d is odd.

Case 2.a: $k-p+rd \equiv p$ (mod k). The colors $[p+id]_k$ and $[p+(r-i)d]_k$ are mutually inverse for $0 \le i \le r$. Thus, $A = B$ and A consists of $\lceil \frac{r+1}{2} \rceil$ pairs of mutually inverse colors of $\mathbb{Z}_k$. Since the colors of $A \cup B$ are not used in c'' by the claim, we may rename the other colors: if $0 \notin A$, then change x to $x - |\{y\colon\ y \in A \text{ and } y < x\}|$ for each $x \le \lfloor \frac{k}{2} \rfloor$ and to $x - |\{y\colon\ y \in A \text{ and } y < x\}| - 1$ for each $x > \lfloor \frac{k}{2} \rfloor$; otherwise, change x to $x - |\{y\colon\ y \in A \text{ and } y < x\}| - \frac{k-|A|}{2} + 1$ for each $x \le \lfloor \frac{k}{2} \rfloor$ and to $x - |\{y\colon\ y \in A \text{ and } y < x\}| - \frac{k-|A|}{2}$ for each $x > \lfloor \frac{k}{2} \rfloor$. The mutually inverse colors $\frac{k-1}{2}$ and $\frac{k+1}{2}$ of $\mathbb{Z}_k$ are not in A and they have been renamed into the same color. Define $k' = k - r - 2$. We thereby obtain a mapping $\phi' : V(G) \to \mathbb{Z}_{k'}$. Denote by I_j the set $\{j, j+1, \ldots, j+d-1\}$ which is an interval of $\mathbb{Z}_k$. Define $d^* = \frac{1}{k}(rd + 2d - 1)$. For each interval I_j, if both colors $\frac{k-1}{2}$ and $\frac{k+1}{2}$ belong to I_j, then I_j contains at most $d^* - 1$ elements of A; otherwise, I_j contains at most d^* elements of A. Define $d' = d - d^*$. Moreover, any pair of mutually inverse colors of $\mathbb{Z}_k$ has been recolored to be mutually inverse colors of $\mathbb{Z}_k$ and then has been renamed to be mutually inverse colors of $\mathbb{Z}_{k'}$. Hence, ϕ' is a (k', d')-coloring of (G, σ), and $\frac{k'}{d'} = \frac{k(k-r-2)}{d(k-r-2)+1} < \frac{k}{d}$.

Case 2.b: $k-p+rd \equiv q$ (mod k). By similar argument as in Case 1.b, we may assume that A contains no mutually inverse colors of $\mathbb{Z}_k$. Thus, $A \cup B$ consists of $r+1$ pairs of mutually inverse colors and $0 \notin A \cup B$. Since the colors of $A \cup B$ are not used in c'' by the claim, we may rename the other colors: change x to $x - |\{y\colon\ y \in A \text{ and } y < x\}|$ for each $x \le \lfloor \frac{k}{2} \rfloor$ and to $x - |\{y\colon\ y \in A \text{ and } y < x\}| - 1$ for each $x > \lfloor \frac{k}{2} \rfloor$. The mutually inverse colors $\frac{k-1}{2}$ and $\frac{k+1}{2}$ of $\mathbb{Z}_k$ are not contained in the set A and have been renamed into the same color. Define $k' = k - 2r - 3$. We thereby obtain a mapping $\phi' : V(G) \to \mathbb{Z}_{k'}$. Denote by I_j the set $\{j, j+1, \ldots, j+d-1\}$ which is an interval of $\mathbb{Z}_k$. Define $d^* = \frac{1}{k}(2rd + 3d - 2)$. Clearly, d^* is a positive integer because of the assumption of Case 2.b. For each interval I_j, if both colors $\frac{k-1}{2}$ and $\frac{k+1}{2}$ belong to I_j, then I_j contains at most $d^* - 1$ elements of A; otherwise, I_j contains at most d^* elements of A. Define $d' = d - d^*$. Any pair of mutually inverse colors of $\mathbb{Z}_k$ has been recolored to be mutually inverse colors of $\mathbb{Z}_k$ and then has been renamed to be mutually inverse colors of $\mathbb{Z}_{k'}$. Hence, ϕ' is a (k',d')-coloring of (G,σ), and $\frac{k'}{d'} = \frac{k(k-2r-3)}{d(k-2r-3)+2} < \frac{k}{d}$. □

(k,d)-colorings with larger $\frac{k}{d}$

Lemma 2.12. *If a signed graph (G,σ) has a (k,d)-coloring, then for any positive integer t, (G,σ) has a (tk,td)-coloring.*

Proof. Let c be a (k,d)-coloring of (G,σ). Define a (tk,td)-coloring c' of (G,σ) by

$$c'(x) = tc(x), \text{ for all } x \in V(G).$$

□

Lemma 2.13. *If a signed graph (G,σ) has a (k,d)-coloring and $k' > k$, where k' is a positive integer, then (G,σ) has a (k',d)-coloring.*

Proof. Let c be a (k,d)-coloring of (G,σ). Define the mapping $c' : V(G) \to \mathbb{Z}_{k'}$ by

$$c'(x) = \begin{cases} c(x) & \text{if } c(x) \leq \lfloor \frac{k}{2} \rfloor, \\ c(x) + k' - k & \text{otherwise,} \end{cases}$$

for all $x \in V(G)$. It is easy to check that c' is a (k',d)-coloring of (G,σ). □

Theorem 2.14. *If a signed graph (G,σ) has a (k,d)-coloring, and k' and d' are two positive integers such that $\frac{k}{d} < \frac{k'}{d'}$, then (G,σ) has a (k',d')-coloring.*

Proof. By Lemma 2.12, (G,σ) has a (kd',dd')-coloring. Since $\frac{k}{d} < \frac{k'}{d'}$, Lemma 2.13 implies that (G,σ) has a $(k'd-1,dd')$-coloring and a $(k'd,dd')$-coloring as well. If d is odd, then by Theorem 2.9, a $(k'd,dd')$-coloring of (G,σ) yields a (k',d')-coloring of (G,σ) and we are done. Let d be even and c'' be a $(k'd-1,dd')$-coloring of (G,σ). Define the mapping $c : V(G) \to \{1-\frac{d}{2}, 2-$

$\frac{d}{2}, \ldots, k'd - 1 - \frac{d}{2}\}$ as follows. For $x \in V(G)$ let

$$c(x) = \begin{cases} c''(x) - (k'd - 1), & \text{if } c''(x) > k'd - 1 - \frac{d}{2}, \\ c''(x), & \text{otherwise.} \end{cases}$$

Define the mapping $c' : V(G) \to \mathbb{Z}_{k'}$ by

$$c'(x) = \lfloor \frac{c(x)}{d} + \frac{1}{2} \rfloor, \text{ for all } x \in V(G).$$

We will show that c' is a (k', d')-coloring of (G, σ).

Consider an edge uv. First assume that $\sigma(uv) = 1$. Without loss of generality, let $c(u) > c(v)$. Note that $1 \leq c(u) - c(v) \leq k'd - 2$. Since c'' is a $(k'd - 1, dd')$-coloring of (G, σ),

$$dd' \leq c(u) - c(v) \leq k'd - 1 - dd'.$$

Therefore,

$$\begin{aligned} c'(u) - c'(v) &= \lfloor \frac{c(u)}{d} + \frac{1}{2} \rfloor - \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &\leq \lfloor k' - d' + \frac{c(v) - 1}{d} + \frac{1}{2} \rfloor - \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &\leq k' - d', \end{aligned}$$

and

$$\begin{aligned} c'(u) - c'(v) &= \lfloor \frac{c(u)}{d} + \frac{1}{2} \rfloor - \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &\geq \lfloor d' + \frac{c(v)}{d} + \frac{1}{2} \rfloor - \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &= d'. \end{aligned}$$

Next assume that $\sigma(uv) = -1$. Note that $2 - d \leq c(u) + c(v) \leq 2(k'd - 1) - d$. Since c'' is a $(k'd-1, dd')$-coloring of (G, σ), either

$$dd' \leq c(u) + c(v) \leq k'd - 1 - dd'$$

or

$$k'd - 1 + dd' \leq c(u) + c(v) \leq 2(k'd - 1) - dd'.$$

In the former case,

$$\begin{aligned} c'(u) + c'(v) &= \lfloor \frac{c(u)}{d} + \frac{1}{2} \rfloor + \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &\leq \lfloor k' - d' - \frac{c(v)+1}{d} + \frac{1}{2} \rfloor - \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &\leq \lfloor k' - d' - \frac{1}{d} + 1 \rfloor. \\ &= k' - d', \end{aligned}$$

and

$$\begin{aligned} c'(u) + c'(v) &= \lfloor \frac{c(u)}{d} + \frac{1}{2} \rfloor + \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &\geq \lfloor d' - \frac{c(v)}{d} + \frac{1}{2} \rfloor + \lfloor \frac{c(v)}{d} + \frac{1}{2} \rfloor \\ &= d'. \end{aligned}$$

In the latter case, by a similar calculation, we deduce

$$k' + d' \leq c'(u) + c'(v) \leq 2k' - d'.$$

Therefore, c' is a (k', d')-coloring of (G, σ). □

(k,d)-colorings with the same $\frac{k}{d}$

Proposition 2.15. *If a signed graph (G,σ) has a (k,d)-coloring with d odd, and k' and d' are two positive integers such that $\frac{k}{d} = \frac{k'}{d'}$, then (G,σ) has a (k', d')-coloring.*

Proof. By Lemma 2.12, (G,σ) has a (kd',dd')-coloring, i.e., a $(k'd,dd')$-coloring since $\frac{k}{d}=\frac{k'}{d'}$. Since d is odd, by Theorem 2.9, (G,σ) has a (k',d')-coloring of (G,σ). □

Proposition 2.15 is not true if d is assumed to be even instead. For example, an antibalanced triangle has a $(4,2)$-coloring, but it does not have a $(2,1)$-coloring.

2.1.3 $\chi_c((G,\sigma))$: from infimum to minimum

Recall that the circular chromatic number $\chi_c((G,\sigma))$ of a signed graph (G,σ) is $\inf\{\frac{k}{d}\colon (G,\sigma)$ has a (k,d)-coloring$\}$. We will show that it is further a minimum, i.e., if $\chi_c((G,\sigma))=\frac{k}{d}$, then there exists a (k,d)-coloring of (G,σ). This is one of the fundamental theorems on circular colorings of signed graphs..

Theorem 2.16. *If (G,σ) is a signed graph on n vertices, then*

$$\chi_c((G,\sigma))=\min\{\frac{k}{d}\colon\ (G,\sigma) \textit{ has a } (k,d)\textit{-coloring and } k\le 4n\}.$$

Proof. By Theorems 2.9, 2.10 and 2.11, if (G,σ) has a (k,d)-coloring then it has a (k',d')-coloring with $k'\le 4n$ and $\frac{k'}{d'}\le\frac{k}{d}$. Therefore,

$$\chi_c((G,\sigma))=\inf\{\frac{k}{d}\colon\ (G,\sigma) \text{ has a } (k,d)\text{-coloring and } k\le 4n\}.$$

Since the set $\{\frac{k}{d}\colon\ (G,\sigma)$ has a (k,d)-coloring and $k\le 4n\}$ is finite, the infimum can be replaced by a minimum. □

2.2 Circular r-colorings of a signed graph

The results of this section have already been published in [29].

The name "circular coloring" was introduced by Zhu [63], and motivated by the equivalence of (k,d)-colorings to r-colorings. For more information, we refer the readers to [68, 73]. In this section we will show that this is also true in the context of signed graphs. Hence, the circular chromatic number can be equivalently defined by circular r-colorings.

Definition 2.17. *Let (G, σ) be a signed graph and r be a real number at least 1. A circular r-coloring of (G, σ) is a function $f : V(G) \to [0, r)$ such that for any edge e with $e = xy$: if $\sigma(e) = 1$ then $1 \leq |f(x) - f(y)| \leq r - 1$, and if $\sigma(e) = -1$ then $1 \leq |f(x) + f(y) - r| \leq r - 1$.*

Clearly, if we identify 0 and r of the interval $[0, r]$ into a single point, then we obtain a circle with perimeter r. Let S^r be this circle. The colors are the points on S^r, and the distance between two points a, b of S^r is the shorter arc of S^r connecting a and b, which is $|a - b|_r$. For $a \in S^r$ let $r - a$ be the inverse element of a. By this notation, Definition 2.17 can be written as

Definition 2.18. *Let (G, σ) be a signed graph and r be a real number at least 1. A circular r-coloring of (G, σ) is a function $f\colon\ V(G) \to S^r$ such that $|f(x) - \sigma(e)f(y)|_r \geq 1$ for each edge e with $e = xy$.*

Note, that this definition also respects switchings. Let f be an r-coloring of (G, σ) and let (G, σ') be obtained from (G, σ) by a switching at $v \in V(G)$. Then f' with $f'(x) = f(x)$ if $x \in V(G) \setminus \{v\}$ and $f'(v) = r - f(v)$ is an r-coloring of (G, σ'). As above we deduce that there is always a coloring on an equivalent graph of (G, σ), which only uses colors in the interval $[0, \frac{r}{2}]$.

Lemma 2.19. *Let r and r' be two real numbers and $r' > r$. If a signed graph (G, σ) has an r-coloring, then (G, σ) has an r'-coloring.*

Proof. Let c be an r-coloring of (G, σ) and $\epsilon = r' - r$. Define a function $c'\colon\ V(G) \to [0, r')$ as for each $x \in V(G)$,

$$c'(x) = c(x) + \frac{\epsilon}{2}.$$

We will show that c' is a circular r'-coloring of (G, σ). Let $e = uv$ be an edge of (G, σ). Then

$$|c'(u) - c'(v)| = |(c(u) + \frac{\epsilon}{2}) - (c(v) + \frac{\epsilon}{2})| = |c(u) - c(v)|$$

and

$$|c'(u) + c'(v) - r'| = |(c(u) + \frac{\epsilon}{2}) + (c(v) + \frac{\epsilon}{2}) - r'| = |c(u) + c(v) - r|.$$

Since c is a circular r-coloring of (G, σ), by Definition 2.17, if $\sigma(e) = 1$ then $1 \leq |c(u) - c(v)| \leq r - 1$, and if $\sigma(e) = -1$ then $1 \leq |c(u) + c(v) - r| \leq r - 1$. Since $r < r'$, we can deduce that if $\sigma(e) = 1$ then $1 \leq |c'(u) - c'(v)| < r' - 1$, and if $\sigma(e) = -1$ then $1 \leq |c(u) + c(v) - r| < r' - 1$. Therefore, c' is a circular r'-coloring of (G, σ), as required. □

The following theorem gives the equivalent relation between (k, d)-colorings and circular r-colorings for a given signed graph.

Theorem 2.20. *Let (G, σ) be a signed graph and k, d be positive integers with $2d \leq k$. (G, σ) has a $(2k, 2d)$-coloring if and only if (G, σ) has a circular $\frac{k}{d}$-coloring.*

Proof. We give an analogous proof to the one for unsigned graphs (see Theorem 1 in [63]).

Suppose that $c : V(G) \to \mathbb{Z}_{2k}$ is a $(2k, 2d)$-coloring of (G, σ). For each $v \in V(G)$ set $f(v) = \frac{c(v)}{2d}$. It is easy to verify that f is a circular $\frac{k}{d}$-coloring of (G, σ).

On the other hand, suppose that f is a circular r-coloring of (G, σ) with $r = \frac{k}{d}$ and $gcd(k, d) = 1$. Let $S = \{f(v) \colon \ v \in V(G)\}$. The cardinality of S is finite since G is a finite graph. We first show that we can assume that all elements of S are rational numbers. We will show that each irrational color can be shifted to a rational color without creating a new pair of colors with distance less than 1. Let $s \in S$ and suppose that s is not a rational number. Let $P = P_1, \ldots, P_n$ be the longest sequence of pairwise distinct points in $[0, r)$ which satisfies the following constraints:

- $s \in P$, and
- $\{P_i, r - P_i\} \cap S \neq \emptyset$ and $P_{i+1} = [P_i + 1]_r$, where P_i is the element of P in the i place.

Define Q to be the sequence consisting of the opposite points of P. More precisely, $Q_i = r - P_i$. Let $\overline{P} = S \cap P$ and $\overline{Q} = S \cap Q$. Let ε be a positive real number such that $s + \varepsilon$ is rational. We shift the colors in $\overline{P}$ together by distance ε clockwise, and the ones in $\overline{Q}$ together by the same distance anticlockwise. Choose ε to be small enough. It is easy to see that this shift is the one required if we can show that the sequences $\overline{P}$ and $\overline{Q}$ contains no common colors. If α is a common color of $\overline{P}$ and $\overline{Q}$, then $s - \alpha$ is an integer and so does $r - s - \alpha$. It follows that $r - 2s$ is an integer, contradicting the fact that r is a rational number but s not.

Let m be a common denominator of all the colors in S. Then the mapping $f'\colon\ V(G) \mapsto \mathbb{Z}_{mk}$ defined as $f'(v) = f(v)md$ is a (mk, md)-coloring of (G, σ). Since m can be chosen to be even it follows with Theorem 2.9 that there is $(2k, 2d)$-coloring of (G, σ). □

The "$(2k, 2d)$-coloring" in the previous theorem can not be replaced by "(k, d)-coloring" since otherwise there exist counterexamples. An unbalanced triangle is one of the signed graphs that has a circular 2-coloring but it has no (2,1)-colorings.

Theorem 2.21. *If (G, σ) is a signed graph, then*

$$\chi_c((G, \sigma)) = \min\{r\colon\ (G, \sigma) \text{ has a circular } r\text{-coloring}\}.$$

Proof. Since χ_c must be a rational number by Theorem 2.16, let us assume $\chi_c((G, \sigma)) = k/d$, where k and d are integers. Let $R = \{r : (G, \sigma) \text{ has a circular } r\text{-coloring}\}$. Then the theorem states that $\chi_c = \min R$. What we have to show is that χ_c is the minimal element of R. Since (G, σ) has a (k, d)-coloring, by Lemma 2.12, it has a $(2k, 2d)$-coloring. So (G, σ) has a circular k/d-coloring by Theorem 2.20. Therefore, χ_c is an element of R.

Suppose that there is $r \in R$ with $r < k/d$. Then there is a rational number r' such that $r < r' < k/d$. Let $r' = k'/d'$, where k' and d' are integers. By Lemma 2.19, (G, σ) has a circular r'-coloring. Now, it follows with Theorem 2.20 that (G, σ) has a $(2k', 2d')$-coloring and $\frac{2k'}{2d'} < \chi_c((G, \sigma))$, a contradiction. □

This theorem shows that for signed graphs, the circular chromatic number χ_c can be equivalently defined by circular r-colorings and χ_c is again a minimum.

The circular chromatic number and circular r-colorings seem to be very natural notions for the coloring of signed graphs. Different from the color set $\mathbb{Z}_n$ used by a (k, d)-coloring, the color set S^r used by a circular r-coloring has always two self-inverse elements, namely 0 and $\frac{r}{2}$.

2.3 Relation between $\chi_c((G, \sigma))$ and $\chi((G, \sigma))$

In this section, we will relate χ and χ_c to each other. We prove that $\chi((G, \sigma)) - 1 \leq \chi_c((G, \sigma)) \leq \chi((G, \sigma))$ for every signed graph (G, σ). In contrast to the corresponding result for unsigned graphs we show, that for each even k there are signed graphs with circular chromatic number k and chromatic number $k + 1$, i.e. they do not have a $(k, 1)$-coloring. On the other hand, for a signed graph on n vertices, if the difference between these parameters is smaller than 1, then there exists $\epsilon_n > 0$, such that the difference is at most $1 - \epsilon_n$. Finally, the concepts of interval colorings and corresponding interval chromatic number χ^I of signed graphs are introduced, and it is proved that $\chi^I((G, \sigma)) = \chi((G, \sigma))$ for every signed graph (G, σ).

The results of this section except Theorems 2.24 and 2.26 have already been published in [29].

2.3.1 Relations in general

Theorem 2.22. *If (G, σ) is a signed graph, then $\chi((G, \sigma)) - 1 \leq \chi_c((G, \sigma)) \leq \chi((G, \sigma))$.*

Proof. By the definitions, we have $\chi_c((G, \sigma)) \leq \chi((G, \sigma))$. On the other hand, suppose to the contrary that $\chi_c((G, \sigma)) < \chi((G, \sigma)) - 1$. Theorem 2.16 implies that $\chi_c((G,\sigma))$ is a rational number. We may assume (G,σ) has a (k,d)-coloring with $\chi_c((G,\sigma)) = \frac{k}{d}$. By Theorem 2.14, (G,σ) has a $(\chi((G,\sigma))-1,1)$-coloring, a contradiction. □

If G is an unsigned graph, then $\chi(G)-1 < \chi_c(G) \leq \chi(G)$, see [48]. We will show that there are signed graphs (G,σ) with $\chi((G,\sigma))-1 = \chi_c((G,\sigma))$. An example for the case $\chi((G,\sigma))-1 = \chi_c((G,\sigma)) = 2$ is any antibalanced triangle, which has no $(2,1)$-colorings but has a $(4,2)$-coloring. Figure 2.1 shows a $(3,1)$-coloring (left) and a $(4,2)$-coloring (right) of an antibalanced triangle.

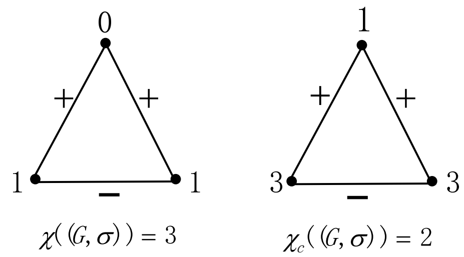

Figure 2.1: A signed graph (G,σ) with $\chi_c((G,\sigma)) = \chi((G,\sigma))-1$.

2.3.2 Signed graphs for which $\chi_c = \chi - 1$

First we give a sufficient and necessary condition for a signed graph (G,σ) having difference 1 between $\chi((G,\sigma))$ and $\chi_c((G,\sigma))$.

Theorem 2.23. *Let (G,σ) be a signed graph with $\chi((G,\sigma)) = t+1$ and t be a positive integer. Then $\chi_c((G,\sigma)) = t$ if and only if (G,σ) has a $(2t,2)$-coloring.*

Proof. (only if) Let $\chi_c((G,\sigma)) = t$. For each (k,d)-coloring of (G,σ) with $\frac{k}{d} = t$ it follows that $d > 1$ since otherwise we would get a $(t,1)$-coloring. If d is odd, then Theorem 2.9 implies that there is $(t,1)$-coloring, a contradiction. Hence, d is even and therefore, k as well. Again with Theorem 2.9 it follows that there is a $(2t,2)$-coloring.

(if) Since (G,σ) does not have a $(t,1)$-coloring but it has a $(2t,2)$-coloring, it follows that $\chi_c((G,\sigma)) = t = \chi((G,\sigma))-1$. □

The following statement is a consequence of Theorem 2.23.

Theorem 2.24. *Let (G,σ) be a signed graph. Then $\chi_c((G,\sigma)) = \chi((G,\sigma))-1$ if and only if there exists an integer k such that (G,σ) has a $(2k,2)$-coloring but has no $(k,1)$-colorings.*

Proof.

(only if) A direct consequence of Theorem 2.23 and the definition of χ.

(if) Assume that such k exists. Thus, $\chi_c((G,\sigma)) \leq k$ and $\chi((G,\sigma)) \geq k+1$. Since $\chi((G,\sigma)) - 1 \leq \chi_c((G,\sigma))$ by Theorem 2.22, we have $\chi((G,\sigma)) - 1 = \chi_c((G,\sigma)) = k$. □

Note that for every unsigned graph G, a $(2k,2)$-coloring of G yields a $(k,1)$-coloring of G. Hence, if σ is equivalent to an all-positive signature of G, then $\chi_c((G,\sigma)) > \chi((G,\sigma)) - 1$ for every G.

Theorem 2.25. *The following two statements hold true.*

1. *If (G,σ) is antibalanced and not bipartite, then $\chi((G,\sigma)) - 1 = \chi_c((G,\sigma)) = 2$.*
2. *For every even $n \geq 2$, there is a signed graph (G,σ) with $\chi((G,\sigma))-1 = \chi_c((G,\sigma)) = n$.*

Proof.

1. The mapping c from $V(G)$ to $\mathbb{Z}_3$ with $c(v) = 1$ is a 3-coloring of (G,σ). Hence, $\chi((G,\sigma)) \leq 3$. Since (G,σ) is not bipartite, $\chi((G,\sigma)) > 2$. Therefore, $\chi((G,\sigma)) = 3$. If we consider c as a mapping from $V(G)$ to $\mathbb{Z}_4$, then c is a $(4,2)$-coloring of (G,σ). By Theorem 2.23 , $\chi_c((G,\sigma)) = 2$.

2. For $i \in \{1,\dots,n\}$ let (G_i,σ^i) be a connected simple signed graph with at least one edge and all edges negative. Take $(G_1,\sigma^1),\dots,(G_n,\sigma^n)$, and for every $j \in \{1,\dots,n\}$ and every $v \in V(G_j)$ connect v to every vertex of $(\bigcup_{i=1}^n V(G_i)) - V(G_j)$ by a positive edge. The resulting graph is denoted by (K_n^*,σ_n), see Figure 2.2. We complete the proof of Statement 2 by the claim below.

Claim. *If n is even, then $\chi((K_n^*,\sigma_n)) = n+1$ and $\chi_c((K_n^*,\sigma_n)) = n$. If n is odd, then $\chi((K_n^*,\sigma_n)) = n+2$ and $\chi_c((K_n^*,\sigma_n)) = n+1$.*

Proof of the claim. Clearly, the all positive subgraph $K_n^* - N_{\sigma_n}$ has chromatic number n. Since for all $i \in \{1, \ldots, n\}$ the signed subgraph (G_i, σ^i) has only negative edges, and G_i has at least one edge, it follows that all used colors are not self-inverse. Since n is even, it follows that $\chi((K_n^*, \sigma_n)) = n + 1$. Furthermore $c : V(K_n^*) \longrightarrow \mathbb{Z}_{2n}$ with $c(v) = 2i - 1$ if $v \in V(G_i)$ is a $(2n, 2)$-coloring of (K_n^*, σ_n). Hence, $\chi_c((K_n^*, \sigma_n)) = n$.

If n is odd, the statement can be proved analogously, and the claim is proved. □

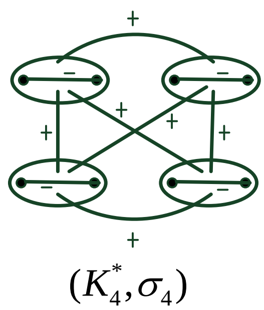

Figure 2.2: $\chi((K_4^*, \sigma_4)) - 1 = \chi_c((K_4^*, \sigma_4)) = 4$.

Note, that Theorem 2.10 does not apply to the graphs of Theorem 2.25 since the cardinality of the set of colors is smaller than the order of the graphs. It would be of interest to know whether a statement like Theorem 2.25 2. is also true for odd k. Furthermore, is there a non-trivial characterization of the signed graphs with $\chi((G,\sigma)) - 1 = \chi_c((G,\sigma))$? These two questions were addressed by Steffen and the author in [29]. Next we give a positive answer to the first question.

Theorem 2.26. *For every odd $k \geq 3$, there is a signed graph (G,σ) with $\chi((G,\sigma)) - 1 = \chi_c((G,\sigma)) = k$.*

Proof. Let n be a positive integer. Take two copies (H^1, σ^1) and (H^1, σ^2) of the signed graph (K_n^*, σ_n) that is defined in the proof of Theorem 2.25. Take any bipartite signed graph (T, σ_T) having at least one edge. Denote by A^1 and A^2 the two parts of T. For $i \in \{1, 2\}$, connect every vertex of A^i to every vertex of H^i by a positive edge. The resulting graph is denoted by (Q_n^*, σ_n^*), see Figure 2.3. We complete the proof of the theorem by the claim below.

Claim. *If n is even, then $\chi((Q_n^*, \sigma_n^*)) = n + 2$ and $\chi_c((Q_n^*, \sigma_n^*)) = n + 1$.*

Proof of the claim. Clearly, (K_n^*, σ_n) has a $(n + 1)$-coloring c such that no vertices receive the self-inverse color 0. Give c to each (H^i, σ^i). Then for each vertex u of H^1, rename its color $c(u)$ by $2c(u)$; and for each vertex v of H^2, rename its color $c(v)$ by $[2c(v) + n + 1]_{2n+2}$. Finally, since T is bipartite, we can properly color T by assigning the color 0 to each vertex of A^1 and the color $n + 1$ to each vertex of A^2. We can see that the resulting coloring is a $(2n + 2, 2)$-coloring of (Q_n^*, σ_n^*). Hence, $\chi_c((Q_n^*, \sigma_n^*)) \leq n + 1$.

Since $\chi((Q_n^*, \sigma_n^*)) - 1 \leq \chi_c((Q_n^*, \sigma_n^*))$ by Theorem 2.22, to complete the proof of the claim, it suffices to show that $\chi((Q_n^*, \sigma_n^*)) \geq n + 2$. Suppose to the contrary that (Q_n^*, σ_n^*) has a $(n + 1)$-coloring ϕ. Clearly, the all-positive subgraph $Q_n^* - N_{\sigma_n^*} - E(T)$ consists of two disjoint complete $(n + 1)$-partite graphs. It follows that both A^1 and A^2 receive the self-inverse color 0, since for any other part, its induced subgraph of Q_n^* has an edge. However, Q_n^* has an edge between A^1 and A^2. So the coloring ϕ is not proper, a contradiction. □

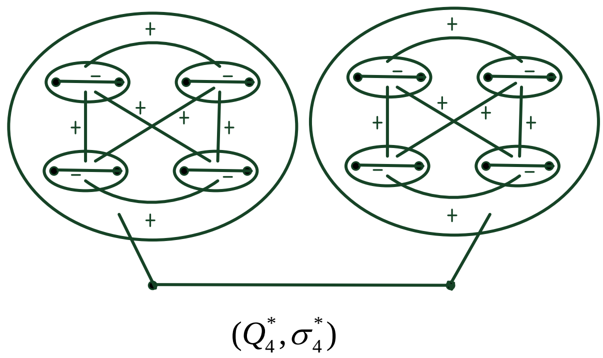

Figure 2.3: $\chi((Q_4^*, \sigma_4^*)) - 1 = \chi_c((Q_4^*, \sigma_4^*)) = 5$.

2.3.3 Improved lower bound

The next theorem shows that if the lower bound in Theorem 2.22 is not attained, then it can be improved.

Theorem 2.27. *Let (G,σ) be a signed graph on n vertices, then either $\chi((G,\sigma)) - 1 = \chi_c((G,\sigma))$ or $(\chi((G,\sigma)) - 1)(1 + \frac{1}{4n-1}) \leq \chi_c((G,\sigma)) \leq \chi((G,\sigma))$. In particular, if $\chi((G,\sigma)) - 1 \neq \chi_c((G,\sigma))$, then $\chi((G,\sigma)) - \chi_c((G,\sigma)) < 1 - \frac{1}{2n}$.*

Proof. By Theorem 2.22, it suffices to show, that if $\chi((G,\sigma)) - 1 \neq \chi_c((G,\sigma))$ then $(\chi((G,\sigma)) - 1)(1 + \frac{1}{4n-1}) \leq \chi_c((G,\sigma))$. By Theorem 2.16, we may assume that $\chi_c((G,\sigma)) = \frac{p}{q}$, where p and q are coprime integers and $p \leq 4n$. Then

$$\chi_c((G,\sigma)) - (\chi((G,\sigma)) - 1) \geq \frac{1}{q} = \frac{\chi_c((G,\sigma))}{p} \geq \frac{\chi_c((G,\sigma))}{4n}. \tag{2.1}$$

By simplifying the inequality, we get

$$(\chi((G,\sigma)) - 1)(1 + \frac{1}{4n-1}) \leq \chi_c((G,\sigma)).$$

Since $2q < p$, it follows with the first inequality of Equation (2.1) that $\chi((G,\sigma)) - \chi_c((G,\sigma)) < 1 - \frac{1}{2n}$. □

2.4 Signed graphs for which $\chi_c = \chi$

It is of particular interests to construct signed graphs (G,σ) for which $\chi_c((G,\sigma)) = \chi((G,\sigma))$. For the unsigned case, this is one of the problems posed by Vince [48] and investigated in many articles. It was shown by Guichard [19] that it is NP-hard to determine whether or not an arbitrary graph G satisfies $\chi_c(G) = \chi(G)$. Recall that $(G,+)$ is a graph G together with an all-positive signature. It is clear that, for every graph G with $\chi_c(G) = \chi(G)$, we have $\chi_c((G,+)) = \chi((G,+))$.

In this section, we generalize the concept of interval colorings by Zhu [63] from unsigned graphs to signed graphs and use it for the construction of signed graphs having the same value of χ and χ_c.

2.4.1 Interval colorings of signed graphs

Denote by $C(S^r)$ the set of all open arcs of unit length on the circle S^r. Two arcs of $C(S^r)$ are pairwise inverse if their middle points are pairwise inverse on S^r. The inverse of an arc a is written as $-a$. With these notions, the definition of a circular r-coloring of a signed graph (i.e., Definition 2.18) restates as follows.

Definition 2.28. *Let (G, σ) be a signed graph and r be a real number at least 1. A circular r-coloring of (G, σ) is a function $f : V(G) \to C(S^r)$ such that $f(x) \cap \sigma(e)f(y) = \emptyset$ for each edge e with $e = xy$.*

Let I^r be obtained from the circle S^r by cutting at the two points $\pm\frac{1}{2}$. Clearly, I^r consists of two intervals, one of length 1 and the other of length $r - 1$. Analogously, denote by $C(I^r)$ the set of all open arcs of unit length on I^r. If we replace the circle S^r in the definition of circular r-colorings of a signed graph by I^r, we define an interval-coloring of a signed graph, and by analogy, its interval-chromatic number. The concept of interval-colorings for an unsigned graph was introduced by Zhu [63], and it sheds new light on the relation between the circular chromatic number and the ordinary chromatic number for unsigned graphs. We follow this approach to study the relation of χ_c and χ for signed graphs.

Definition 2.29. *Let (G, σ) be a signed graph and r be a real number at least 1. An r-interval coloring of (G, σ) is a function $f : V(G) \to C(I^r)$ such that $f(x) \cap \sigma(e)f(y) = \emptyset$ for each edge e with $e = xy$. The interval-chromatic number $\chi^I((G, \sigma))$ of (G, σ) is $\inf\{r\colon (G, \sigma)$ has an r-interval coloring$\}$.*

It is well-known [16] that for every unsigned graph G, the chromatic number of G is the least real number r such that there is an r-interval coloring of G. We show that this is also true in the context of signed graphs.

Theorem 2.30. *For every signed graph (G, σ), we have $\chi^I((G, \sigma)) = \chi((G, \sigma))$.*

Proof. Let $\chi((G,\sigma)) = k$ and let ϕ be a $(k,1)$-coloring of (G,σ). By replacing each color x by the interval $(x - \frac{1}{2}, x + \frac{1}{2})$ and cutting S^k at the two points $\pm\frac{1}{2}$, we obtain a k-interval coloring of (G,σ) from the coloring ϕ. Hence, $\chi^I((G,\sigma)) \leq \chi((G,\sigma))$.

On the other hand, let $\chi^I((G,\sigma)) = r$ and let c be an r-interval coloring of (G,σ). We will transform c into an $(\lfloor r \rfloor, 1)$-coloring of (G,σ) by constituting integers for open unit length arcs. Denote by I_1 and I_2 the two intervals of I^r so that I_1 has length 1. Rename I_2 as $(\frac{1-r+\lfloor r \rfloor}{2}, r - \frac{1-r+\lfloor r \rfloor}{2})$. For each arc s of I_2, since s is of unit length, it covers at most one integer point. If s covers precisely one integer point, then constitute this integer for s; otherwise, both ends of s are integers and we take the one closer to the point 0 to constitute for s. Since the ends of the interval I_2 are greater than 0, the color 0 has not been used yet. For each arc s of I_1, s is exactly I_1 itself. We constitute the color 0 for s. Now c is transformed into an integer coloring, say c', using colors from $\{0, 1, \ldots, \lfloor r \rfloor\}$. Moreover, we can see that pairwise inverse arcs of c are transformed into pairwise inverse integers, and no two disjoint arcs of c are transformed into the same integer. It follows that c' is an $(\lfloor r \rfloor, 1)$-coloring. Thus, $\chi((G,\sigma)) \leq \chi^I((G,\sigma))$. □

2.4.2 Construction

Now we are ready to give a sufficient condition for a signed graph having the same chromatic number and circular chromatic number. We will need the results on interval colorings of signed graphs for the proof. A vertex of a graph G is universal if it is adjacent to every other vertices of G.

Theorem 2.31. *Let (G,σ) be a signed graph having an universal vertex u. Let $\chi_c((G,\sigma)) = r$. If (G,σ) has a circular r-coloring for which u receives the color 0, then $\chi_c((G,\sigma)) = \chi((G,\sigma))$.*

Proof. Since $\chi_c((G,\sigma)) \leq \chi((G,\sigma))$ by Theorem 2.22, it suffices to show that $\chi_c((G,\sigma)) \geq \chi((G,\sigma))$. By Definition 2.28 and the condition of the theorem, (G,σ) has a circular r-coloring c with colors from $C(S^r)$ for which u receives the arc $(-\frac{1}{2}, +\frac{1}{2})$. Since u is an universal vertex, both points $\pm\frac{1}{2}$ are not covered by any other arc of c. Hence, we can cut S^r at the points $\pm\frac{1}{2}$ to obtain I^r and meanwhile, the coloring c is transformed into an r-interval coloring of (G,σ). Therefore, $\chi^I((G,\sigma)) \leq r$, that is, $\chi((G,\sigma)) \leq \chi_c((G,\sigma))$ by Theorem 2.30. □

Chapter 3

Chromatic number χ of signed graphs

In Chapter 2 we introduced and studied circular colorings and related integer colorings of signed graphs. In this chapter, we take a further look at integer colorings and the corresponding chromatic number for signed graphs. Results on the chromatic spectrum are presented. Moreover, we generalize some classical results on chromatic number of unsigned graphs to signed graphs, including the Brooks' theorem and the Hajós theorem.

3.1 Some basic properties

We recall the definitions of a k-coloring and the chromatic number χ of a signed graph as follows.

Definition 3.1. *Let $\mathbb{Z}_k$ denote the cyclic group of integers modulo k, and the inverse of an element x is denoted by $-x$. A function $c : V(G) \to \mathbb{Z}_k$ is a k-coloring of (G, σ) if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. The chromatic number $\chi((G, \sigma))$ of a signed graph (G, σ) is the smallest k such that (G, σ) has a k-coloring.*

We say a signed graph (G, σ) is *k-chromatic*, if (G, σ) has chromatic number k.

Let S be a set of vertices of a signed graph (G, σ). Recall that S is an independent set if there is no edge between any two vertices of S; S is an antibalanced set if S induces an antibalanced subgraph.

By the previous definition, we can see that each nonempty color class together with its inverse color class is an antibalanced set and in particular, an independent set if they receive the self-inverse color. Hence, we have the following statement.

Proposition 3.2. *Let k be a positive integer and (G, σ) be a signed graph.*

1. *For even k, (G, σ) is k-chromatic if and only if $V(G)$ can be divided into two independent sets and $\frac{k-2}{2}$ antibalanced sets, where all these sets may be empty.*
2. *For odd k, (G, σ) is k-chromatic if and only if $V(G)$ can be divided into one independent set and $\frac{k-1}{2}$ antibalanced sets, where all these sets may be empty.*

Now we investigate the relation between the chromatic number of a signed graph and the chromatic number of its underlying graph.

Theorem 3.3. *For every loopless signed graph (G, σ), we have $\chi((G, \sigma)) \leq 2\chi(G) - 2$. Furthermore, for every integer $n \geq 2$, there exists a signed graph (G, σ) such that $\chi((G, \sigma)) = 2\chi(G) - 2$ and $\chi(G) = n$. Hence, the bound is sharp.*

Proof. Let c be an n-coloring of a graph G with colors from $\{0, 1, 2, ..., n-1\}$. For any signature σ, we can see that c is a $(2n-2)$-coloring of the signed graph (G, σ). Hence, the required inequality holds.

In what follows, we will prove the sharpness of the bound, that is, we will construct an infinite family of signed graphs (G_n, σ_n) such that $\chi((G_n, \sigma_n)) = 2\chi(G_n) - 2 = 2n - 2$. To construct (G_n, σ_n), we take one copy of the all-positive signed complete graph $(K_n, +)$, say (H_0, σ_0), and $n-2$ copies of the all-negative signed complete graph $(K_n, -)$, say $(H_1, \sigma_1), \ldots, (H_{n-2}, \sigma_{n-2})$. For $i \in \{0, 1, \ldots, n-2\}$, denote the vertices of H_i by $v_{i,0}, v_{i,1}, ..., v_{i,n-1}$. We say that any two vertices $v_{i,k}$ and $v_{j,k}$ with $i \neq j$ are corresponding. Now insert a positive edge between any pair of non-corresponding vertices from distinct copies, see figure 3.1.

Since every two corresponding vertices is not adjacent, the color assignment $v_{i,j} \mapsto j$ for each $i \in \{0, 1, \ldots, n-2\}$ and $j \in \{0, 1, \ldots, n-1\}$ defines an n-coloring of G_n. Thus, $\chi(G_n) \leq n$. Moreover, G_n contains a copy of the complete graph K_n as a subgraph, which implies $\chi(G_n) \geq n$. Therefore, $\chi(G_n) = n$.

We next prove that $\chi((G_n, \sigma_n)) = 2n-2$. By the inequality of the theorem, $\chi((G_n, \sigma_n)) \leq 2\chi(G_n) - 2 = 2n - 2$. Hence, we may suppose to the contrary that $\chi((G_n, \sigma_n)) \leq 2n - 3$. Let c be a $(2n-3)$-coloring of (G_n, σ_n). Since (H_0, σ_0) is a copy of $(K_n, +)$, n distinct colors from $\mathbb{Z}_{2n-3}$, say $a_0, a_2, \ldots, a_{n-1}$, have to be used for the vertices of H_0. For each $i \in \{1, \ldots, n-2\}$, since (H_i, σ_i) is a copy of $(K_n, -)$, there exist two vertices u_i and v_i of H_i receiving the same color, say b_i. Since each vertex of H_0 is connected to at least one of u_i and v_i by a positive edge, $b_i \notin \{a_0, \ldots, a_{n-1}\}$. Moreover, for each $j \in \{1, \ldots, i-1\}$, u_j is connected to at least one of u_i and v_i by a positive edge. Thus, $b_i \notin \{b_1, \ldots, b_{i-1}\}$. Now we can conclude that $a_0, \ldots, a_{n-1}, b_1, \ldots, b_{n-2}$ are $2n-2$ pairwise distinct colors of $\mathbb{Z}_{2n-3}$, a contradiction. □

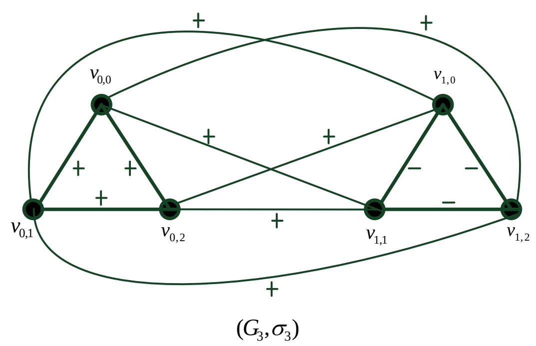

Figure 3.1: A construction of (G_3, σ_3).

A similar relation between the signed chromatic number of a signed graph and the chromatic number of its underlying graph was proved in [35].

Theorem 3.4 ([35], Theorem 2.1)**.** *For every loopless signed graph (G, σ) we have $\chi_{\pm}((G, \sigma)) \leq 2\chi(G) - 1$. Furthermore, this bound is sharp.*

3.2 Chromatic spectrum of a graph

In this section, we consider possible values of the chromatic number of a signed graph (G,σ) where the underlying graph G is fixed. So, we introduce the chromatic spectrum of a graph and show that the chromatic spectrum is always an interval.

The results of this section have already been published in [28].

Definition 3.5. *Let G be a graph and $\Sigma(G)$ be the set of pairwise non-equivalent signatures on G. The chromatic spectrum of G is the set $\{\chi((G,\sigma))\colon\ \sigma\in\Sigma(G)\}$, which is denoted by $\Sigma_\chi(G)$.*

Define that $M_\chi(G)=\max\{\chi((G,\sigma))\colon\ \sigma\in\Sigma(G)\}$ and $m_\chi(G)=\min\{\chi((G,\sigma))\colon\ \sigma\in\Sigma(G)\}$. The following theorem is the main result in this section.

Theorem 3.6. *If G is a graph, then $\Sigma_\chi(G)=\{k\colon\ m_\chi(G)\le k\le M_\chi(G)\}$.*

This theorem shows that the chromatic spectrum $\Sigma_\chi(G)$ is an interval of integers for any graph G. Similarly, we can define the circular chromatic spectrum $\Sigma_{\chi_c}(G)$ of a graph G as $\Sigma_{\chi_c}(G)=\{\chi_c((G,\sigma))\colon\ \sigma\in\Sigma(G)\}$. For a bipartite graph G, Proposition 2.3 2. shows that the chromatic spectrum of G starts from 2, and so does the circular chromatic spectrum of G. Therefore, it is an interesting problem to figure out the circular chromatic spectrum of graphs.

3.2.1 Determination of $m_\chi(G)$

Proposition 3.7. *Let G be a nonempty graph. The following statements hold.*

1. *$\Sigma_\chi(G)=\{1\}$ if and only if $m_\chi(G)=1$ if and only if $E(G)=\emptyset$.*
2. *if $E(G)\neq\emptyset$, then $\Sigma_\chi(G)=\{2\}$ if and only if $m_\chi(G)=2$ if and only if G is bipartite.*
3. *If G is not bipartite, then $m_\chi(G)=3$.*

Proof. Statements 1 and 2 are obvious. For Statement 3, consider (G,σ) where σ is the signature with all edges negative. Then $c\colon\ V(G)\to\mathbb{Z}_3$ with

$c(v) = 1$ is a 3-coloring of G. Since G is not bipartite, the statement follows with Statements 1 and 2. □

3.2.2 Chromatic critical signed graphs

Definition 3.8. *If (G, σ) is a signed graph and $u \in V(G)$, then σ_u denotes the restriction of σ to $G - u$. A k-chromatic graph (G, σ) is k-chromatic critical if $\chi((G - u, \sigma_u)) < k$, for every $u \in V(G)$.*

Chromatic critical graphs are of particular interests for the studies of the chromatic number of graphs, since that they possess additional properties by the criticality and that it is often sufficient to consider chromatic critical graphs for proofs. In this subsection, we will give some basic facts on k-chromatic critical signed graphs. The complete graph on n vertices is denoted by K_n.

Proposition 3.9. *Let (G, σ) be a signed graph.*

1. *(G, σ) is 1-critical if and only if $G = K_1$*

2. *(G, σ) is 2-critical if and only if $G = K_2$.*

3. *(G, σ) is 3-critical if and only if G is an odd circuit.*

Proof. Statements 1 and 2 are obvious. An odd circuit with any signature is 3-critical. For the other direction let G be a 3-critical graph. By Proposition 3.7, we have: $(*)$ $G - u$ is bipartite for every $u \in V(G)$. Since G is not bipartite it follows that every vertex of G is contained in all odd circuits of G, and by $(*)$ every odd circuit C is hamiltonian. C cannot contain a chord, since for otherwise G contains a non-hamiltonian odd circuit, a contradiction. Hence, G is an odd circuit. □

Lemma 3.10. *Let $k \geq 1$ be an integer. If (G, σ) is k-chromatic, then $\chi((G - u, \sigma_u)) \in \{k, k-1\}$, for every $u \in V(G)$. In particular, if (G, σ) is k-critical, then $\chi((G - u, \sigma_u)) = k - 1$.*

Proof. For $k \in \{1, 2\}$, the statement follows with Proposition 3.7. Hence, we may assume that $k \geq 3$. Clearly, $\chi((G - u, \sigma_u)) \leq \chi((G, \sigma)) = k$. Suppose to the contrary that $\chi((G - u, \sigma_u)) \leq k - 2$, and let ϕ be a $(k-2)$-coloring of $(G-u,\sigma_u)$. We extend ϕ to a $(k-1)$-coloring of (G,σ). If k is odd, then change color x to $x+1$ for each $x \geq \frac{k-1}{2}$ and assign color $\frac{k-1}{2}$ to vertex u, and we are done. If k is even, then change color x to $x+1$ for each $x \geq \frac{k}{2}$, and assign color $\frac{k}{2}$ to vertex u. If $\phi(v) = \frac{k-2}{2}$ for a vertex v and $\sigma(uv) = -1$, then recolor v with color $\frac{k}{2}$ to obtain a $(k-1)$-coloring of (G,σ). Hence $\chi((G,\sigma)) \leq k-1 < k$, a contradiction. Clearly, if (G,σ) is k-critical, then $\chi((G-u,\sigma_u)) = k-1$. □

The following theorem is a direct consequence of Lemma 3.10.

Theorem 3.11. *Let (G,σ) be a signed graph and $k \geq 1$. If $\chi((G,\sigma)) = k$, then (G,σ) contains an induced i-critical subgraph for each $i \in \{1,\ldots,k\}$.*

3.2.3 Proof of Theorem 3.6

Lemma 3.12. *Let $k \geq 3$ be an integer and H be an induced subgraph of a graph G. If $k \in \Sigma_\chi(H)$, then $k \in \Sigma_\chi(G)$.*

Proof. If $k \in \Sigma_\chi(H)$, then there is a signature σ of H such that $\chi((H,\sigma)) = k$. Let ϕ be a k-coloring of (H,σ). Define a signature σ' of G as follows. Let $e \in E(G)$ with $e = uv$.

If $e \in E(H)$, then $\sigma'(e) = \sigma(e)$;

If $u, v \notin V(H)$ or if $u \in V(H), v \notin V(H)$ and $\phi(u) = 1$, then $\sigma'(e) = -1$;

If $u \in V(H), v \notin V(H)$ and $\phi(u) \neq 1$, then $\sigma'(e) = 1$.

It follows that ϕ can be extended to a k-coloring of (G,σ') by assigning color 1 to each vertex of $V(G) \setminus V(H)$. Thus $\chi((G,\sigma')) \leq k$. Moreover, (G,σ') has (H,σ) as a subgraph with chromatic number k, hence, $\chi((G,\sigma')) \geq k$. Therefore, $\chi((G,\sigma')) = k$ and thus, $k \in \Sigma_\chi(G)$. □

Theorem 3.13. *Let $k \geq 4$ be an integer and G be a graph. If $k \in \Sigma_\chi(G)$, then $k-1 \in \Sigma_\chi(G)$.*

Proof. By Theorem 3.11, (G,σ) contains an induced k-critical subgraph (H,σ'), where σ' is the restriction of σ to H. Since $k \geq 4$, it follows that $|V(H)| > 3$. Hence, there is $u \in V(H)$ such that $\chi(H-u,\sigma'_u) = k-1$.

Furthermore, $H - u$ is an induced subgraph of G. Thus, $k - 1 \in \Sigma_\chi(H - u)$, and hence, $k - 1 \in \Sigma_\chi(G)$ by Lemma 3.12.

Note that if $k = 3$, then by Proposition 3.7, G is not a bipartite graph and thus k can not be decreased to 2. □

Theorem 3.6 follows directly from Proposition 3.7 and Theorem 3.13.

3.3 An analogue of Brooks' Theorem for signed graphs

This section is devoted to state and prove a signed version of one of the most fundamental results on graph colourings, the famous Brooks' theorem [8]. In [35], the authors addressed a signed version of Brooks' Theorem with respect to the signed chromatic number $\chi_\pm$. Later on, a list version of this result by characterizing degree choosable signed graphs was proved in [43], following our definition on list colorings of signed graphs given in Chapter 5. The result we present here is an analogue of Brooks' Theorem for signed graphs with respect to the chromatic number χ.

Theorem 3.14. *Let (G, σ) be a simple connected signed graph. If (G, σ) is not a balanced complete graph or an odd circuit, then $\chi((G, \sigma)) \leq \Delta(G)$.*

The proof of this theorem follows a method from [11] and [35]. However, we apply new arguments for some cases in the proof. We will use the following four lemmas for the proof.

Lemma 3.15. *If (G, σ) is a signed complete graph on n-vertices, $n \geq 4$, then $\chi((G, \sigma)) \leq n$. Furthermore, $\chi((G, \sigma)) = n$ if and only if (G, σ) is balanced.*

Proof. It is easy to see that if (G, σ) is balanced, then $\chi((G, \sigma)) = n$. So it is sufficient to show that, if (G, σ) is assumed to be unbalanced, then $\chi((G, \sigma)) \leq n - 1$. We distinguish two cases by the parity of n.

Case 1: assume that n is even. The proof for this case is done by induction. For $n = 4$, since (G, σ) is unbalanced and complete, there is an unbalanced triangle. Since an unbalanced triangle is antibalanced as well, we may use the colors 1 and 2 of $\mathbb{Z}_3$ to properly color the triangle and the color 0 to color the remaining vertex, obtaining a 3-coloring of (G, σ). Hence, the conclusion holds. Now we proceed to the induction process. Since (G, σ) is unbalanced, there is an unbalanced triangle T. Pick any two vertices x and y of (G, σ) that do not belong to T, and switch the signature of (G, σ), if neccessary, to make the edge xy is negative. Hence, the signed graph $(G, \sigma) - \{x, y\}$ is an unbalanced signed complete graph on $n - 2$ vertices. By the induction hypothesis, $(G, \sigma) - \{x, y\}$ has a $(n - 3)$-coloring, that is, its vertices can be divided into one independent set and $\frac{n-4}{2}$ many antibalanced sets. Since $\{x, y\}$ is an antibalanced set of (G, σ), it follows that the vertices of (G, σ) can be divided into one independent set and $\frac{n-2}{2}$ antibalanced sets, that is, (G, σ) has a $(n - 1)$-coloring. Therefore, $\chi((G, \sigma)) \leq n - 1$, the conclusion holds.

Case 2: assume that n is odd. Since a switching does not change the chromatic number, we may assume that (G, σ) has a vertex v incident with negative edges only. We remove v and $(G, \sigma) - v$ is still unbalanced and complete. By the conclusion of Case 1, $(G, \sigma) - v$ has a $(n - 2)$-coloring, that is, its vertices can be divided into one independent set and $\frac{n-3}{2}$ many antibalanced sets. Take v as an independent set of G, thus the vertices of (G, σ) can be divided into two independent sets and $\frac{n-3}{2}$ antibalanced sets, that is, (G, σ) has a $(n - 1)$-coloring. Therefore, $\chi((G, \sigma)) \leq n - 1$, the conclusion holds. □

The following lemma is a standard tool for coloring graphs greedily.

Lemma 3.16. *The vertices of every connected graph G can be ordered in a sequence $x_1, x_2, \ldots, x_n$ so that x_n is any preassigned vertex of G and for each $i < n$ the vertex x_i has a neighbour among $x_{i+1}, x_{i+2}, \ldots, x_n$.*

The following lemma is due to Lovász and was crucial in his short proof of Brooks' theorem [32]. Its proof can also be found in [11] by Cranston and Rabern.

Lemma 3.17. *Let G be a 2-connected graph with $\Delta(G) \geq 3$. If G is not complete, then G contains a pair of vertices a and b at distance 2 such that the graph $G - \{a, b\}$ is connected.*

Lemma 3.18. *Let (G, σ) be a connected signed graph. If G is not regular, then $\chi((G, \sigma)) \leq \Delta(G)$.*

Proof. Let u be a vertex of G with degree less than $\Delta(G)$. Take an ordering $x_1, x_2, \ldots, x_n$ of the vertices of G as in Lemma 3.16 with $x_n = u$. We now start coloring $x_1, x_2, \ldots, x_n$ in the given order greedily with colors from $\mathbb{Z}_\Delta$. For $i < n$, each x_i has a neighbor among its successors, so x_i has at most $\Delta - 1$ neighbors previously colored. This is also true for x_n since x_n has degree less than $\Delta(G)$. Each colored neighbor forbids one color for x_i, so $\mathbb{Z}_\Delta$ still has a color available for x_i and finally, the same for x_n. □

Now we are ready to prove Theorem 3.14.

Proof of Theorem 3.14. If the signed graph (G, σ) is an unbalanced complete graph, then the conclusion follows from Lemma 3.15; and if G is not regular, then the conclusion follows from Lemma 3.18. The conclusion is also correct whenever (G, σ) is a path or an even circuit. Thus we may assume that (G, σ) is a simple connected signed graph of order n with maximum degree $\Delta \geq 3$ which is regular but not complete. We distinguish two cases.

Case 1: The signed graph (G, σ) is 2-connected. By Lemma 3.17 , (G, σ) contains a path axb such that a is not adjacent to b and $G - \{a, b\}$ is connected. Switch at a and b if necessary so that both the edges ax and bx are positive. Denote by n the order of G. Next, we take an ordering $x_1, x_2, \ldots, x_{n-2}$ of the vertices of $G - \{a, b\}$ as in Lemma 3.16 with $x_{n-2} = x$. We now start coloring (G, σ) with colors from $\mathbb{Z}_\Delta$ by assigning color 1 to both a and b. Then we color $x_1, x_2, \ldots, x_{n-2}$ in the given order greedily. Since each $x_i \neq x$ has a neighbor among its successors in $G - \{a, b\}$, x_i has at most $\Delta - 1$ neighbors previously colored. Hence, there is at least one color from $\mathbb{Z}_\Delta$ available for x_i, and we proceed up to x_{n-3}. For the vertex x_{n-2} (equivalently, x), since both a and

b forbid the color 1 for x, $\mathbb{Z}_\Delta$ still has a color available for x. This completes the proof of Case 1.

Case 2: The signed graph (G, σ) has a cut-vertex. Take a cut-vertex u of G such that $G - u$ has a component of minimum order. Denote by (H, σ_H) this component. By switching if necessary, we may assume that u is incident with positive edges only. By Lemma 3.18, each component of $(G, \sigma) - u$ has a Δ-coloring. To obtain a Δ-coloring of G, we will recolor the component H and take the coloring of all other components so that finally there is still a color available for u.

Denote by $S = \{v_1, \ldots, v_t\}$ the set of neighbors of u in H. If $t = 1$, then v_1 is a cut-vertex of G whose removal from G yields a component of smaller order than H, contradicting with the choice of u. Hence, $t \geq 2$. If every vertex in S is a cut-vertex of H, then there exists an element of S whose removal from H yields a component containing no other elements of S. It follows that this element of S is a cut-vertex of G that will contradict with the choice of u. Hence, S has a vertex that is not a cut-vertex of H. Without loss of generality, let v_1 be such a vertex. Thus, the signed graph $(H, \sigma_H) - v_1$ is connected. Let r be the order of H. We can choose an ordering $x_1, \ldots, x_{r-1}$ of the vertices of $H - v_1$ as in Lemma 3.16 with $x_{r-1} = v_t$. Let w be a neighbor of u not in H. We now start recoloring H with colors from $\mathbb{Z}_\Delta$. First assign the color of w to v_1. Then we colour the vertices $x_1, x_2, ..., x_{r-1}$ greedily in the given order. For each $i \in \{1, \ldots, r-2\}$, x_i has a neighbour among its successors in $H - v_1$, so x_i has at most $\Delta - 1$ neighbours previously coloured. Thus, we can properly color x_i. For the vertex x_{r-1}, since it has a uncolored neighbour u, we can properly color x_{r-1}. Finally, since u has two neighbours v_1 and w of the same color, we can properly color u. This completes the proof of Case 2. □

3.4 First Hajós-like theorem for signed graphs

This section addresses an analogue of a well-known theorem of Hajós for signed graphs. Throughout this section, "a graph" is always regarded as an unsigned simple graph for the distinction from "a signed graph" and "a multigraph".

The results of this section have already been published in [27].

3.4.1 Introduction

In 1961, Hajós proved a result on the chromatic number of graphs, which is one of the classical results in the field of graph colorings. This result has several equivalent formulations, one of them states as the following two theorems.

Theorem 3.19 ([20]). *The class of all graphs that are not q-colorable is closed under the following three operations:*

(1) Adding vertices or edges.

(2) Identifying two nonadjacent vertices.

(3) Given two vertex-disjoint graphs G_1 and G_2 with $a_1b_1 \in E(G_1)$ and $a_2b_2 \in E(G_2)$, construct a graph G from $G_1 \cup G_2$ by removing a_1b_1 and a_2b_2, identifying a_1 with a_2, and adding a new edge between b_1 and b_2 (see Figure 3.2).

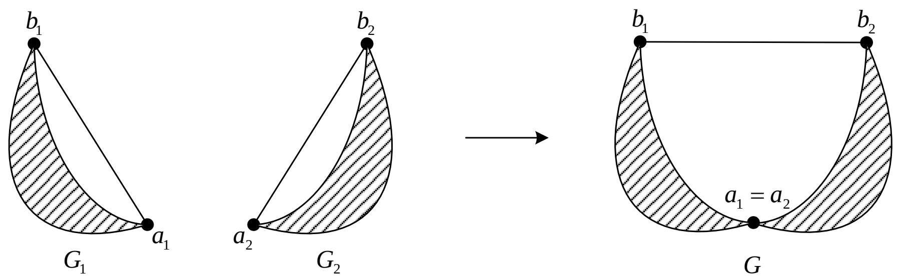

Figure 3.2: Operation (3)

Operation (3) is known as the Hajós construction in the literature.

Theorem 3.20 (Hajós' Theorem [20]). *Every non-q-colorable graph can be obtained by Operations (1)-(3) from the complete graph K_{q+1}.*

The Hajós theorem has been generalized in several different ways. The analogues of the Hajós theorem were proved for list-colorings by Gravier [17], for circular colorings by Zhu [67, 70], for colorings of edge-weighted graphs by Mohar [37], for group colorings by An and Wu [2], and for weighted colorings by Araujo and Sales [3].

It is shown in this section that the Hajós theorem has a very natural and simple generalization in the case of signed graphs for the chromatic number χ. We call this generalization the Hajós-like theorem of signed graphs (briefly, the Hajós-like theorem). Whereas the Hajós theorem needs three operations, the Hajós-like theorem keeps the first two, uses an operation more general than the third one, contains the operation of switchings for signed graphs, and needs one more additional operation. Moreover, these operations of the Hajós-like theorem enable us to construct all signed graphs (G, Σ) with $\chi((G, \Sigma)) \geq q$ from copies of the all-positive complete signed graph $(K_q, +)$ of order q, the same graph we start from to construct all graphs G with $\chi(G) \geq q$ by the Hajós theorem.

We will prove the Hajós-like theorem for signed multigraphs rather than signed graphs (i.e., signed simple graphs). For vertex colorings of signed multigraphs, it suffices to consider signed bi-graphs, a subclass of signed multigraphs in which no two edges of the same sign locate between same two vertices. Clearly, signed bi-graphs contain signed graphs as a subclass. Hence, the Hajós-like theorem holds particularly for signed graphs.

The structure of the rest of this section is arranged as follows. In Subsection 3.4.2, we give the definition of signed bi-graphs. In Subsection 3.4.3, we design five operations on signed bi-graphs and show that these operations are closed in the class of non-q-colorable signed bi-graphs for any given positive integer q. In Subsection 3.4.4, we establish some lemmas necessary for the proof of the Hajós-like theorem. In Section 3.4.5, we propose the Hajós-like theorem by using the operations that we defined before, and we address the proof of the theorem.

3.4.2 Signed bi-graphs

A *bi-graph* is a multigraph having no loops and having at most two edges between any two distinct vertices. Let G be a bi-graph, u and v be two distinct vertices of G. Denote by $E(u,v)$ the set of edges connecting u to v, and let $m(u,v) = |E(u,v)|$. Clearly, $0 \leq m(u,v) \leq 2$. A bi-graph G is *bi-complete* if $m(x,y) = 2$ for any $x, y \in V(G)$, is *complete* if $m(x,y) \geq 1$ for any $x, y \in V(G)$, and is *just-complete* if $m(x,y) = 1$ for any $x, y \in V(G)$.

A *signed bi-graph* (G, σ) is a bi-graph G together with a signature σ of G such that any two multiple edges have distinct signs. A bi-complete signed bi-graph of order n is denoted by $(K_n, \pm)$. It is not hard to calculate that $\chi((K_n, \pm)) = 2n - 2$. The concepts of k-colorings, the chromatic number and switchings of signed graphs are naturally extended to signed bi-graphs, working in the same way, and the related notations are inherited.

Let (G, σ) be a signed multigraph. Replace multiple edges of the same sign by a single edge of this sign. We thereby obtain a signed bi-graph (G', σ'). Clearly, G and G' have the same vertex set. We can see that c is a k-coloring of (G, σ) if and only if it is a k-coloring of (G', σ'). Therefore, for vertex colorings of signed multigraphs, it suffices to consider signed bi-graphs.

3.4.3 Graph operations on signed bi-graphs

Let k be a nonnegative integer. A signed bi-graph is *k-thin* if it can be obtained from a bi-complete signed bi-graph by removing at most k pairwise vertex-disjoint edges. Clearly, A k-thin signed bi-graph is complete, and a signed bi-graph is 0-thin if and only if it is bi-complete.

Theorem 3.21. *The class of all signed bi-graphs that are not q-colorable is closed under the following operations:*

($sb1$) *Adding vertices or signed edges.*

($sb2$) *Identifying two nonadjacent vertices.*

($sb3$) *Given two vertex-disjoint signed bi-graphs (G_1, σ_1) and (G_2, σ_2), a vertex v of G_1 and a positive edge e of G_2 with ends x and y, construct a signed* *bi-graph* (G, σ) *from* (G_1, σ_1) *and* (G_2, σ_2) *by splitting* v *into two new vertices* v_1 *and* v_2*, removing* e *and identifying* v_1 *with* x *and* v_2 *with* y *(see Figure 3.3).*

$(sb4)$ *Switching at a vertex.*

$(sb5)$ *When* q *is even, remove a vertex that has at most* $\frac{q}{2}$ *neighbors. When* q *is odd, remove a negative single edge, identify its two ends, and add signed edges (if needed) so that the resulting bi-signed graph is* $\frac{q-3}{2}$*-thin.*

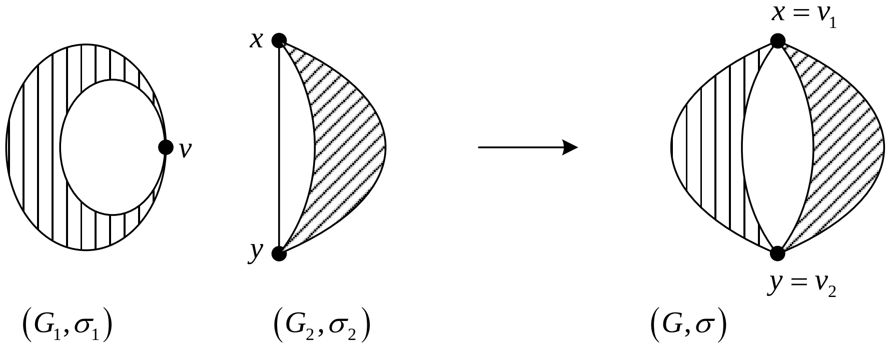

Figure 3.3: Operation $(sb3)$

Proof. Since Operations $(sb1), (sb2), (sb4)$ neither make loops nor decrease the chromatic number, it follows that the class of non-q-colorable signed bi-graphs is closed under these operations.

For Operation $(sb3)$, suppose to the contrary that (G, σ) is q-colorable. Let c be a q-coloring of (G, σ). Denote by x' and y' the vertices of G obtained from x and y, respectively. If $c(x') = c(y')$, then the restriction of c into G_1, where v is assigned with the same color as x' and y', gives a q-coloring of (G_1, σ_1), contradicting with the fact that (G_1, σ_1) is not q-colorable. Hence, we may assume that $c(x') \neq c(y')$. Note that e is a positive edge of (G_2, σ_2). Thus the restriction of c into G_2 gives a q-coloring of (G_2, σ_2), contradicting the fact that (G_2, σ_2) is not q-colorable. Therefore, the statement holds true for Operation $(sb3)$.

It remains to verify the theorem for Operation $(sb5)$. For q even, suppose to the contrary that the removal of a vertex u from a non-q-colorable signed bi-graph (G, σ) yields the q-colorability. Let $S = \{0, \pm 1, \ldots, \pm(\frac{q}{2} - 1), \frac{q}{2}\}$

and let ϕ be a q-coloring of $(G, \sigma) - u$ using colors from S. Notice that each neighbor of u makes at most two colors unavailable for u. Since u has at most $\frac{q}{2}$ neighbors, S still has a color available for u. Hence, we can extend ϕ to a q-coloring of (G, σ), a contradiction.

For the case that q is odd, let (H', σ'_H) be obtained from a non-q-colorable signed bi-graph (H, σ_H) by applying this operation to a negative edge e, and suppose to the contrary that (H', σ'_H) is q-colorable. Let ψ be a q-coloring of (H', σ'_H) using colors from the set $\{0, \pm 1, \dots, \pm(\frac{q-1}{2})\}$. Denote by x and y the two ends of e and by z the resulting vertex from them. If $\psi(z) \neq 0$, then by assigning x and y with the color $\psi(z)$, we complete a q-coloring of the original bi-graph (H, σ_H), a contradiction. Hence, we may assume that $\psi(z) = 0$. For $0 \leq i \leq \frac{q-1}{2}$, let $V_i = \{v \in V(G)\colon\ |\psi(v)| = i\}$. Clearly, each V_i induces an antibalanced signed graph and in particular, V_0 is an independent set. Since (H, σ_H) is $\frac{q-3}{2}$-thin, we can deduce that there exists $p \in \{1, \dots, \frac{q-1}{2}\}$ such that $|V_p| = 1$. Exchange the colors between V_0 and V_p, and then assign x' and y' with the same color as z, we thereby obtain a q-coloring of (H, σ_H) from ψ, a contradiction. □

3.4.4 Useful lemmas

Operation ($sb3$) can be extended from unsigned graphs to signed bi-graphs as follows.

($sb3'$) Let (G_1, σ_1) and (G_2, σ_2) be two vertex-disjoint signed bi-graphs. For each $i \in \{1, 2\}$, let e_i be an edge of (G_i, σ_i) with ends x_i and y_i. Make a graph (G, σ) from $G_1 \cup G_2$ by removing e_1 and e_2, identifying x_1 with x_2, and adding a new signed edge e between y_1 and y_2 such that $\sigma(e) = \sigma_1(e_1)\sigma_2(e_2)$.

Lemma 3.22. *Operation ($sb3'$) is a combination of Operations ($sb3$) and ($sb4$).*

Proof. We use the notations in the statement of Operation ($sb3'$). First assume that at least one of e_1 and e_2 is a positive edge. With loss of generality, say

e_1 is positive. We apply Operation ($sb3$) to (G_1, σ_1) and (G_2, σ_2) where e_1 is removed, x_2 is split into two new vertices x_2' and x_2'' with y_2 as the neighbor of x_2' and with all other neighbors of x_2 as the neighbors of x_2'', and then x_2' is identified with y_1 and x_2'' is identified with x_1. The resulting signed bi-graph is exactly (G, σ), we are done. Hence, we may next assume that both e_1 and e_2 are negative edges. Switch at x_1 in (G_1, σ_1) and at x_2 in (G_2, σ_2). Since e_1 is positive in the resulting signed bi-graph, we may apply Operation ($sb3$) similarly as above, obtaining a signed bi-graph, which leads to (G, σ) by switching again at x_1 (equivalently, at x_2). □

Let (G, σ) be a signed graph. The *sign product* $sp(H)$ of a subgraph H is defined as $sp(H) = \prod_{e \in E(H)} \sigma(e)$.

Lemma 3.23. *A just-complete signed bi-graph is antibalanced if and only if the sign product of each triangle is* -1*, and it is balanced if and only if the sign product of each triangle is* 1.

Proof. For the first statement, since a just-complete signed bi-graph (G, σ) is exactly a complete signed graph, (G, σ) is antibalanced if and only if the sign product of each circuit of length k is $(-1)^k$. Hence, the proof for the necessity is trivial. Let us proceed to the sufficiency, which will be proved by induction on k.

Clearly, the statement holds for $k = 3$ because of the assumption of the lemma. Assume that $k \geq 4$. Let C be a circuit of length k. Since G is complete, C has a chord e, which divides C into two paths, together with e forming two circuits C_1 and C_2 of length k_1 and k_2, respectively. Thus, $k = k_1 + k_2 - 2$. By applying the induction hypothesis, we have $sp(C_i) = (-1)^{k_i}$. It follows that $sp(C) = sp(C_1)sp(C_2) = (-1)^k$, the statement also holds.

The second statement can be argued in the same way as for the first one. We only have to pay attention to the equivalence between that (G, σ) is balanced and that the sign product of each circuit of length k is 1. □

A signed bi-graph of order $3r$ is ∇-*complete* if it can be obtained from $(K_{3r}, \pm)$ by removing r pairwise vertex-disjoint all-positive triangles. Clearly, a ∇-complete signed bi-graph is complete.

Lemma 3.24. *The ∇-complete signed bi-graph of order $3r$ can be obtained from $(K_{2r+1}, +)$ by Operations $(sb1)$-$(sb5)$.*

Proof. Take $r + 1$ copies of $(K_{2r+1}, +)$, say $(H_i, +)$ of vertex set $\{v_i^0, \ldots, v_i^{2r}\}$ for $0 \leq i \leq r$. For each $j \in \{1, \ldots, r\}$, switch at v_0^j, and then apply Operation $(sb3')$ to H_0 and H_j so that $v_0^j v_0^{j+r}$ and $v_j^0 v_j^{2j}$ are removed and that v_0^j is identified with v_j^0, and finally identify v_0^j with v_0^{j+r} into a new vertex x^j. The resulting signed bi-graph is denoted by (G, σ). By Theorem 3.21, since $(K_{2r+1}, +)$ is not $2r$-colorable, (G, σ) is not $2r$-colorable either. Note that v_0^0 has precisely r neighbors in G. We can apply Operation $(sb5)$ to v_0^0, i.e., we remove v_0^0 from (G, σ). In the resulting signed bi-graph, for each $1 \leq k \leq 2r$, since $v_1^k, \ldots, v_r^k$ are pairwise nonadjacent, we can apply Operation $(sb2)$ to identify them into a new vertex y^k. Denote by (H, σ_H) the resulting signed bi-graph.

We can see that (H, σ_H) is of vertex set $\{x^1, \ldots, x^r, y^1, \ldots, y^{2r}\}$ and hence, it is of order $3r$. We can also see that (H, σ_H) is complete and more precisely, it has multiple edges between any two vertices from $\{x^1, \ldots, x^r\}$ and a single edge between any two other vertices of H. In particular, for $1 \leq j \leq r$, the set $\{x^j, y^{2j}, y^{2j-1}\}$ induces a just-complete triangle with the signs $-, +$ and $+$ on the edges $x^j y^{2j}, x^j y^{2j-1}$ and $y^{2j} y^{2j-1}$, respectively. Switch at vertices $y^1, y^3, \ldots, y^{2r-1}$ and add signed edges as many as possible but keeping all the triangles of the form $[x^j y^{2j} y^{2j-1}]$ just-complete. We thereby obtain the ∇-complete signed bi-graph of order $3r$ from (H, σ_H). □

Lemma 3.25. *The signed bi-graph $(K_r, \pm)$ can be obtained from $(K_{2r-2}, +)$ by Operations $(sb1)$-$(sb5)$.*

Proof. Let (G, σ) be a copy of $(K_{2r-2}, +)$ of vertices $v_1, \ldots, v_{2r-2}$. Since (G, σ) is not $(2r-3)$-colorable, switch at v_1 and then we can apply Operation $(sb5)$ to $v_1 v_2$ so that $v_3 v_4, v_5 v_6, \ldots, v_{2r-5} v_{2r-4}$ are single edges. Since the resulting signed bi-graph is $(r-3)$-thin, all other edges are multiple edges. For each $i \in \{2, 3, \ldots, r-2\}$, switch at v_{2i} and apply Operation $(sb5)$ to $v_{2i-1}v_{2i}$ so that no new signed edges are added. The resulting signed bi-graph is exactly $(K_r, \pm)$. □

3.4.5 Proof of the theorem

We will need the following definitions for the proof of the Hajós-like theorem.

Let (G, σ) be a signed bi-graph. An *antibalanced set* is a set of vertices that induce an antibalanced signed graph. Let c be a k-coloring of (G, σ). A set of all vertices v with the same value of $|c(v)|$ is called a *partite set* of (G, σ). Thus, every partite set is an antibalanced set. Let U and V be two partite sets. They are *completely adjacent* if $m(u, v) \geq 1$ for any $u \in U$ and $v \in V$, *bi-completely adjacent* if $m(u, v) = 2$ for any $u \in U$ and $v \in V$, and *just-completely adjacent* if $m(u, v) = 1$ for any $u \in U$ and $v \in V$.

Let (G, σ) be a signed bi-graph. A sequence (x, y, z) of three vertices of G is a *triple* if there exist three integers a, b, c satisfying the following three conditions:

(i) $a, b, c \in \{1, -1\}$,

(ii) $ab = c$,

(iii) $a \notin \{\sigma(e)\colon\ e \in E(x, y)\}$, $b \notin \{\sigma(e)\colon\ e \in E(x, z)\}$, and $c \in \{\sigma(e)\colon\ e \in E(y, z)\}$.

The sequence (a, b, c) is called a *code* of (x, y, z). Note that a triple may have more than one code.

Theorem 3.26 (Hajós-like theorem)**.** *Every signed bi-graph with chromatic number q can be obtained from $(K_q, +)$ by Operations $(sb1)$-$(sb5)$.*

Proof. Let (G, σ) be a counterexample with minimum $|V(G)|$ and subjecting to it, $|E(G)|$ is maximum.

Claim 1: (G, σ) is complete. Suppose to the contrary that G has two non-adjacent vertices x and y. Let (G_1, σ_1) and (G_2, σ_2) be obtained from a copy of (G, σ) by identifying x with y into a new vertex v and by adding a positive edge e between x and y, respectively. Since (G, σ) has chromatic number q, it follows by Theorem 3.21 that both (G_1, σ_1) and (G_2, σ_2) have chromatic number at least q. Note the fact that $(K_i, +)$ can be obtained from $(K_j, +)$ by Operation $(sb1)$ whenever $i > j$. Thus by the minimality of $|V(G)|$, the graph (G_1, σ_1) can be obtained from $(K_q, +)$ by Operations $(sb1)$-$(sb5)$, and by the maximality of $|E(G)|$, so does (G_2, σ_2). We next show that (G, σ) can be obtained from (G_1, σ_1) and (G_2, σ_2) by Operations $(sb2)$ and $(sb3)$, which contradicts the fact that (G, σ) is a counterexample. This contradiction completes the proof of the claim. Apply Operation $(sb3)$ to (G_1, σ_1) and (G_2, σ_2) so that e is removed and v is split into x and y. In the resulting graph, identify each pair of vertices that corresponds to the same vertex of G except x and y, we thereby obtain exactly (G, σ).

Claim 2: (G, σ) has no triples. The proof of this claim is analogous to Claim 1. Suppose to the contrary that (G, σ) has a triple, say (x, y, z). Let (a, b, c) be a code of (x, y, z). Take two copies of (G, σ). Add an edge e_1 with sign a into one copy between x and y, obtaining (G', σ'). Add an edge e_2 with sign b into the other copy between x and z, obtaining (G'', σ''). Clearly, both (G', σ') and (G'', σ'') have chromatic number at least q. By the maximality of $|E(G)|$, they can be obtained by Operations $(sb1)$-$(sb5)$ from $(K_q, +)$. To complete the proof of the claim, it remains to show that (G, σ) can be obtained from (G', σ') and (G'', σ'') by Operations $(sb1)$-$(sb5)$. Note that Operation $(sb3')$ is a combination of Operations $(sb3)$ and $(sb4)$ by Lemma 3.22. Apply Operation $(sb3')$ to (G', σ') and (G'', σ'') so that e_1 and e_2 are removed, x' is identified with x'', and an edge e is added between y' and z''. We have $\sigma(e) = \sigma(e_1)\sigma(e_2) = ab = c \in E(y, z)$. By applying Operation $(sb2)$ to each pair of vertices that are the copies of the same vertex of G except x, we obtain (G, σ).

We continue the proof of the theorem by distinguishing two cases according to the parity of q.

Case 1: Assume that q is odd. Since $\chi((G, \sigma)) = q$, the vertex set $V(G)$ can be divided into k partite sets $V_1, \ldots, V_k$, where $k = \frac{q+1}{2}$, so that V_1 is an independent set and all others are antibalanced sets but not independent and subjecting to it, $|V_1|$ is minimum. For each $i \in \{2, \ldots, k\}$, since V_i is not an independent set, $|V_i| \geq 2$. Moreover, since V_i is an antibalanced set and (G, σ) is complete by Claim 1, V_i induces a just-complete signed bi-graph, that is, a complete signed graph.

($*$) *We show that any two of* $V_2, \ldots, V_k$ *are bi-completely adjacent. Suppose to the contrary that there exists* $2 \leq j < l \leq k$ *such that* V_j *and* V_l *are not bi-completely adjacent. Notice that* $|V_j| + |V_l| \geq 3$. *If* V_j *and* V_l *are not just-completely adjacent, then there always exist three vertices* x, y, z, *without loss of generality, say* $x \in V_j$ *and* $y, z \in V_l$, *such that* $m(x, y) = 1$ *and* $m(x, z) = 2$. *Since* V_l *induces a complete signed graph,* $m(y, z) = 1$. *Thus,* (y, x, z) *is a triple of* (G, σ), *contradicting Claim 2. Hence,* V_j *and* V_l *are just-completely adjacent. Recall that both* V_j *and* V_l *induce complete signed graphs. Thus,* $V_j \cup V_l$ *induces a complete signed graph, say* (Q, σ_Q), *as well. By Claim 2, every triangle in* (Q, σ_Q) *has sign product* -1. *Thus,* (Q, σ_Q) *is antibalanced by Lemma 3.23, and so* $V_j \cup V_l$ *is an antibalanced set. The division of* $V(G)$, *obtained from* $V_1, \ldots, V_k$ *by constituting* $V_j \cup V_l$ *for* V_j *and* V_l, *yields that* $\chi((G, \sigma)) \leq q - 2$, *a contradiction.*

Recall that V_1 is an independent set. By Claim 1, $|V_1| \leq 1$. Hence, we distinguish two cases.

Subcase 1.1: Assume that $|V_1| = 0$. Thus, for each $i \in \{2, \ldots, k\}$, we have $|V_i| \geq 3$, since otherwise, the division of $V(G)$, obtained from $\{V_1, \ldots, V_k\}$ by removing V_1 and splitting V_i into two independent sets, yields $\chi((G, \sigma)) \leq q-1$, a contradiction. Take three vertices from each partite set except V_1, and denote by (H, σ_H) the signed bi-graph induced by all these vertices. Clearly, $|V(H)| = \frac{3(q-1)}{2}$. Recall that $V_2, \ldots, V_k$ induce just-complete signed bi-graphs and any two of them are bi-completely adjacent. Thus, (H, σ_H) is a ∇-complete signed bi-graph. By Lemma 3.24, (H, σ_H) can be obtained from $(K_q, +)$ by Operations $(sb1)$-$(sb5)$ and therefore, so does (G, σ), a contradiction.

Subcase 1.2: Assume that $|V_1| = 1$. We show that V_1 is bi-completely adjacent to each of $V_2, \ldots, V_k$, by applying the same argument as in ($*$), except that the final contradiction is obtained by the minimality of $|V_1|$ instead of the decrease of $\chi((G,\sigma))$. Take the vertex in V_1 and two arbitrary vertices from each of $V_2,\ldots,V_k$. Let (H,σ_H) be the signed bi-graph induced by all these vertices. Clearly, $|V(H)| = q$. As we already proved, any two of $V_1,\ldots,V_k$ are bi-completely adjacent and each of them induce a just-complete signed bi-graph. It follows that (H,σ_H) can be obtained from a bi-complete signed bi-graph by removing disjoint edges and thus, from $(K_q,+)$ by switching at vertices and adding signed edges. Therefore, (G,σ) can be obtained from $(K_q,+)$ by Operations $(sb1)$ and $(sb4)$, a contradiction.

Case 2: Assume that q is even. Since $\chi((G,\sigma)) = q$, the vertex set $V(G)$ can be divided into k non-empty partite sets $V_1,\ldots,V_k$, where $k = \frac{q+2}{2}$, so that at least two of them are independent sets, say V_1 and V_2. It follows by Claim 1 that $|V_1| = |V_2| = 1$.

Subcase 2.1: Assume that every two partite sets are bi-completely adjacent. Take a vertex from each partite set. Clearly, these vertices induce $(K_{\frac{q+2}{2}},\pm)$. Hence, (G,σ) can be obtained from $(K_{\frac{q+2}{2}},\pm)$ by Operation $(sb1)$. By Lemma 3.25, $(K_{\frac{q+2}{2}},\pm)$ can be obtained from $(K_q,+)$ by Operations $(sb1)$-$(sb5)$ and therefore, so does (G,σ), a contradiction.

Subcase 2.2: Assume that there exist two partite sets V_j and V_l that are not bi-completely adjacent.

By applying the same argument as in $(*)$, we arrive at the conclusion that $V_j \cup V_l$ is an antibalanced set. It follows that $\chi((G,\sigma)) \leq q-2$ when $|\{j,l\}\cap\{1,2\}| = 0$ and $\chi((G,\sigma)) \leq q-1$ when $|\{j,l\}\cap\{1,2\}| = 1$. Hence, $\{j,l\} = \{1,2\}$. This implies that every other two partite sets are bi-completely adjacent. Moreover, since $|V_1| = |V_2| = 1$, it follows that $V_1 \cup V_2$ induces two vertices together with a single edge between them and so, it is an antibalanced set. This implies that $|V_3|,\ldots,|V_k| \geq 2$ since otherwise, the division of $V(G)$, obtained from $V_1,\ldots,V_k$ by constituting $V_1 \cup V_2$ for V_1 and V_2, yields that $\chi((G,\sigma)) \leq q-1$, a contradiction.

Take a vertex from each of V_1 and V_2, and two vertices from each of the remaining partite sets. We can see that the signed bi-graph, induced by these vertices, can be obtained from $(K_q, \pm)$ by removing disjoint edges. Hence, it can be obtained from $(K_q, +)$ by adding signed edges and switching at vertices. Therefore, (G, σ) can be obtained from $(K_q, +)$ by Operations $(sb1)$-$(sb5)$, a contradiction. □

Corollary 3.27. *Every signed graph with chromatic number q can be obtained from $(K_q, +)$ by Operations $(sb1)$-$(sb5)$.*

Chapter 4

Signed chromatic number $\chi_\pm$ of signed graphs

In this chapter, we focus on signed chromatic number of signed graphs, first introduced by Máčajová, Raspaud and Škoviera [35]. This invariant is a non-equivalent notion to the chromatic number that we introduced and discussed in the previous two chapters. The relation between these two invariants are discussed. So for there is only few results on the signed chromatic number. We presents results on signed chromatic spectrum and Hajós' Theorem for this invariant, in a similar way as we treated on the chromatic number of signed graphs.

4.1 Preliminary

We first recall the definitions of signed k-colorings and the signed chromatic number $\chi_\pm$ of signed graph given in [35].

Definition 4.1. *Let (G, σ) be a signed graph. If $n = 2k + 1$, then let $M_n = \{0, \pm 1, \ldots, \pm k\}$, and if $n = 2k$, then let $M_n = \{\pm 1, \ldots, \pm k\}$. A mapping c from $V(G)$ to M_n is a signed n-coloring of (G, σ), if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. Define $\chi_\pm((G, \sigma))$ to be the smallest number n such that (G, σ)*

has a signed n-coloring, and call it the signed chromatic number of (G,σ). We also say that (G,σ) is signed n-chromatic.

So far, there are only a few results on $\chi_\pm((G,\sigma))$. In [35], the authors proved that $\chi_\pm((G,\sigma)) \leq 2\chi(G) - 1$ for every graph G, and they proved an extension of the Brooks' theorem to signed graphs: every signed graph (G,σ) satisfies $\chi_\pm((G,\sigma)) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum vertex degree of G. A further study on this extension was addressed in [43].

Following the way to study the chromatic number of signed graphs in Chapter 3, we establish analogous results for the signed chromatic number $\chi_\pm$. In Section 4.2, we show that the signed chromatic spectrum of a graph is always an interval. In Section 4.3, we prove an analogue of the Hajós' theorem with respect to the signed chromatic number for signed graphs. The difference between these two parameters, $\chi((G,\sigma))$ and $\chi_\pm((G,\sigma))$, is investigated in Section 4.4.

4.2 Signed chromatic spectrum of a graph

The results of this section have already been published in [28].

In Section 3.2, we studied the chromatic spectrum of a graph, which is related to the chromatic number of signed graph. Recall that the chromatic number and the signed chromatic number are two non-equivalent parameters on vertex colorings of singed graphs. Hence, it is natural to define the signed chromatic spectrum that is related to the signed chromatic number. We prove a similar result that the signed chromatic spectrum is always an interval.

Definition 4.2. *Let G be a graph and $\Sigma(G)$ be the set of pairwise non-equivalent signatures on G. The signed chromatic spectrum of G is the set $\{\chi_\pm((G,\sigma))\colon\ \sigma \in \Sigma(G)\}$, which is denoted by $\Sigma_{\chi_\pm}(G)$.*

Define that $M_{\chi_\pm}(G) = \max\{\chi_\pm((G,\sigma))\colon\ \sigma \in \Sigma(G)\}$ and $m_{\chi_\pm}(G) = \min\{\chi_\pm((G,\sigma))\colon\ \sigma \in \Sigma(G)\}$.

Proposition 4.3. *Let G be a nonempty graph. The following two statements hold true.*

1. $\Sigma_{\chi_{\pm}}(G) = \{1\}$ *if and only if* $E(G) = \emptyset$.

2. *if* $E(G) \neq \emptyset$, *then* $m_{\chi_{\pm}}(G) = 2$.

Proof. Statement 1 is obvious. For Statement 2, since G has at least one edge, G cannot be colored by using only one color. Hence, $m_{\chi_{\pm}}(G) \geq 2$. Moreover, let σ be the all-negative signature of G, we have $m_{\chi_{\pm}}(G) \leq m_{\chi_{\pm}}((G, \sigma)) = 2$. Therefore, $m_{\chi_{\pm}}(G) = 2$. □

The following theorem is the main result in this section.

Theorem 4.4. *If G is a graph, then* $\Sigma_{\chi_{\pm}}(G) = \{k : m_{\chi_{\pm}}(G) \leq k \leq M_{\pm}(G)\}$.

4.2.1 Signed chromatic critical graph

Definition 4.5. *A signed k-chromatic graph (G, σ) is signed k-chromatic critical if* $\chi_{\pm}((G - u, \sigma_u)) < k$, *for every* $u \in V(G)$.

In [43] Schweser and Stiebitz defined a graph (G, σ) to be critical with respect to $\chi_{\pm}$ if $\chi_{\pm}((H, \sigma')) < \chi_{\pm}((G, \sigma))$ for every proper signed subgraph (H, σ') of (G, σ), where σ' is the restriction of σ to $E(H)$. However, for trees and circuits the two definitions coincide. The analogue statement to Proposition 3.9 for signed colorings is due to Schweser and Stiebitz in [43].

Proposition 4.6 ([43]). *Let (G, σ) be a signed graph.*

1. (G, σ) *is signed 1-critical if and only if* $G = K_1$

2. (G, σ) *is signed 2-critical if and only if* $G = K_2$.

3. (G, σ) *is signed 3-critical if and only if G is a balanced odd circuit or an unbalanced even circuit.*

Lemma 4.7. *Let $k \geq 1$ be an integer. If (G, σ) is signed k-chromatic, then* $\chi_{\pm}((G - u, \sigma_u)) \in \{k, k-1\}$, *for every* $u \in V(G)$. *In particular, if (G, σ) is signed k-critical, then* $\chi_{\pm}((G - u, \sigma_u)) = k - 1$.

Proof. For $k \in \{1,2\}$, the statement follows with Proposition 4.3. Hence, we may assume that $k \geq 3$. Clearly, $\chi_\pm((G-u,\sigma_u)) \leq \chi_\pm((G,\sigma)) = k$. Suppose to the contrary that $\chi_\pm((G-u,\sigma_u)) \leq k-2$ and let ϕ be a $(k-2)$-coloring of $(G-u,\sigma_u)$. We shall extend ϕ to a $(k-1)$-coloring of (G,σ). If k is even, then assign color 0 to vertex u, we are done. If k is odd, then assign color $\frac{k-1}{2}$ to vertex u, and for each vertex v such that $\phi(v) = 0$ and $\sigma(uv) = -1$, recolor v with color $\frac{k-1}{2}$, and for each vertex v such that $\phi(v) = 0$ and $\sigma(uv) = 1$, recolor v with color $-\frac{k-1}{2}$ to obtain a $(k-1)$-coloring of (G,σ). Hence $\chi_\pm((G,\sigma)) \leq k-1 < k$, a contradiction. Clearly, if (G,σ) is signed k-critical, then $\chi_\pm((G-u,\sigma_u)) = k-1$. □

Theorem 4.8. *Let (G,σ) be a signed graph and $k \geq 1$. If $\chi_\pm((G,\sigma)) = k$, then (G,σ) contains an induced signed i-critical subgraph for each $i \in \{1,\dots,k\}$.*

4.2.2 Proof of Theorem 4.4

Lemma 4.9. *Let $k \geq 2$ be an integer and H be an induced subgraph of a graph G. If $k \in \Sigma_{\chi_\pm}(H)$, then $k \in \Sigma_{\chi_\pm}(G)$.*

The proof of this lemma is similar to the proof of Lemma 3.12.

Theorem 4.10. *Let $k \geq 3$ be an integer and G be a graph. If $k \in \Sigma_{\chi_\pm}(G)$, then $k-1 \in \Sigma_{\chi_\pm}(G)$.*

Proof. By Theorem 4.8, (G,σ) contains an induced signed k-critical subgraph (H,σ'), where σ' is the restriction of σ to H. Since $k \geq 3$, it follows that $|V(H)| \geq 3$. Hence, there is $u \in V(H)$ such that $\chi_\pm(H-u,\sigma'_u) = k-1$. Furthermore, $H-u$ is an induced subgraph of G. Thus, $k-1 \in \Sigma_{\chi_\pm}(H-u)$, and hence, $k-1 \in \Sigma_{\chi_\pm}(G)$ by Lemma 4.9. □

Theorem 4.4 follows from Proposition 4.3 and Theorem 4.10.

4.3 Second Hajós-like theorem for signed graphs

The results of this section have already been published in [27].

Section 3.4 is devoted to an analogue of Hajós' theorem for the chromatic number χ of a signed graph. In this section, we present an analogue of Hajós' theorem for the signed chromatic number $\chi_{\pm}$ of a signed graph, and call it the second Hajós-like theorem. The formulations and the proofs of these two analogues are similar. However, for the sake of completeness, we address below, for the latter analogue, the theorem together with its proof, emphasizing the difference from the former analogue. In the next theorem, we recall the Operations ($sb1$)-($sb4$) defined in Subsection 3.4.3 and define a new operation.

Theorem 4.11. *The class of all signed bi-graphs that are not signed q-colorable is closed under Operations ($sb1$)-($sb4$) and the following operation:*

($sb1$) *Adding vertices or signed edges.*

($sb2$) *Identifying two nonadjacent vertices.*

($sb3$) *Given two vertex-disjoint signed bi-graphs (G_1, σ_1) and (G_2, σ_2), a vertex v of G_1 and a positive edge e of G_2 with ends x and y, construct a signed bi-graph (G, σ) from (G_1, σ_1) and (G_2, σ_2) by splitting v into two new vertices v_1 and v_2, removing e and identifying v_1 with x and v_2 with y.*

($sb4$) *Switching at a vertex.*

($sb6$) *When q is even, remove a negative single edge and identify its two ends.*

Proof. The proof for Operations ($sb1$)-($sb4$) can be done in the same way as in the proof of Theorem 3.21. For Operation ($sb6$), let ϕ be a signed q-coloring of the resulting signed bi-graph and z be the resulting vertex by identifying two vertices x and y. Notice that the partite set with regard to ϕ that contains z is an antibalanced set but not necessarily an independent set. We could complete a signed q-coloring of the original signed bi-graph from ϕ by giving the color of z to x and y. □

Lemma 4.12. *The signed bi-graph $(K_r, \pm)$ can be obtained from $(K_{2r-1}, +)$ by Operations ($sb1$)-($sb4$) and ($sb6$).*

Proof. Let (G, σ) be a copy of $(K_{2r-1}, +)$ of vertices $v_1, \ldots, v_{2r-2}$. Since (G, σ) is not $(2r-2)$-colorable, switch at v_1 and then apply Operation ($sb6$) to v_1v_2. By Theorem 4.11, the resulting signed bi-graph remains signed non-$(2r-2)$-colorable. Hence, for each $i \in \{2, \ldots, r-1\}$, we could switch at v_{2i} and then apply Operation ($sb6$) to $v_{2i-1}v_{2i}$, finally obtaining a signed bi-graph that is exactly $(K_r, \pm)$. □

Theorem 4.13 (The second Hajós-like theorem)**.** *Every signed bi-graph with signed chromatic number q can be obtained from $(K_q, +)$ by Operations ($sb1$)-($sb4$) and ($sb6$).*

Proof. Let (G, σ) be a counterexample with minimum $|V(G)|$ and subjecting to it, $|E(G)|$ is maximum. Claims 1 and 2 in the proof of Theorem 3.26 still hold true by the same proofs.

Claim 1: (G, σ) is complete.

Claim 2: (G, σ) has no triples.

We distinguish two cases according to the parity of q.

Case 1: Assume that q is odd.

Since $\chi_{\pm}((G, \sigma)) = q$, the vertex set $V(G)$ can be divided into k nonempty partite sets $V_1, \ldots, V_k$, where $k = \frac{q+1}{2}$, so that V_1 is an independent set.

Since Claims 1 and 2 still hold true and since each partite set is non empty, if there exist two partite sets V_s and V_t that are not bi-completely adjacent, then we could apply the argument $(*)$ to V_s and V_t, arriving at the conclusion that $V_s \cup V_t$ is an antibalanced set. It follows that the division of $V(G)$, obtained from $V_1, \ldots, V_k$ by constituting $V_s \cup V_t$ for V_s and V_t, yields that $\chi_{\pm}((G, \sigma)) \leq q - 1$ when $1 \in \{s, t\}$ and $\chi_{\pm}((G, \sigma)) \leq q - 2$ when $1 \notin \{s, t\}$, a contradiction. Therefore, every two partite sets are bi-completely adjacent.

Take a vertex from each partite set. It follows that these vertices induce the bi-complete signed bi-graph $(K_{\frac{q+1}{2}}, \pm)$ of order $\frac{q+1}{2}$. By Lemma 4.12, $(K_{\frac{q+1}{2}}, \pm)$ can be obtained from $(K_q, +)$ by Operations ($sb1$)-($sb4$) and ($sb6$) and therefore, so does (G, σ), a contradiction.

Case 2: Assume that q is even. Since $\chi_{\pm}((G, \sigma)) = q$, the vertex set $V(G)$ can be divided into k partite sets $V_1, \ldots, V_k$ of cardinality at least 2, where $k = \frac{q}{2}$. Thus, by the argument $(*)$, any two partite sets are bi-completely adjacent. Take two vertices from each partite set and denote by (H, σ_H) the signed bi-graph induced by all these vertices. It follows that (H, σ_H) can be obtained from $(K_q, \pm)$ by removing k pairwise disjoint edges and hence, it can be obtained from $(K_q, +)$ by switchings and adding signed edges. Therefore, (G, σ) can be obtained from $(K_q, +)$ by Operations $(sb1)$ and $(sb4)$, a contradiction. □

Corollary 4.14. *Every signed graph with signed chromatic number q can be obtained from $(K_q, +)$ by Operations $(sb1)$-$(sb4)$ and $(sb6)$.*

4.4 Relation between χ and $\chi_{\pm}$

The results of this section have already been published in [29].

The following proposition describes the relation between the chromatic number and the signed chromatic number for signed graphs. The difference between these two parameters is shown to be at most 1.

Proposition 4.15. *If (G, σ) is a signed graph, then $\chi_{\pm}((G, \sigma)) - 1 \leq \chi((G, \sigma)) \leq \chi_{\pm}((G, \sigma)) + 1$.*

Proof. Let $\chi_{\pm}((G, \sigma)) = n$ and c be an n-coloring of (G, σ) with colors from M_n.

If $n = 2k + 1$, then let $\phi : M_{2k+1} \to \mathbb{Z}_{2k+1}$ with $\phi(t) = t$ if $t \in \{0, \ldots, k\}$ and $\phi(t) = 2k + 1 + t$ if $t \in \{-k, \ldots, -1\}$. Then c is a $(2k + 1)$-coloring of (G, σ) with colors from M_{2k+1} if and only if $\phi \circ c$ is a $(2k + 1)$-coloring of (G, σ). Hence, $\chi((G, \sigma)) \leq \chi_{\pm}((G, \sigma))$. If $n = 2k$, then let $\phi' : M_{2k} \to \mathbb{Z}_{2k+1}$ with $\phi(t) = t$ if $t \in \{1, \ldots, k\}$ and $\phi(t) = 2k + 1 + t$ if $t \in \{-k, \ldots, -1\}$. Then $\phi' \circ c$ is a $(2k + 1)$-coloring of (G, σ). Hence, $\chi((G, \sigma)) \leq \chi_{\pm}((G, \sigma)) + 1$.

We analogously deduce that $\chi_{\pm}((G, \sigma)) \leq \chi((G, \sigma)) + 1$. □

The next proposition shows that there exist signed graphs for which $\chi((G, \sigma)) = \chi_{\pm}((G, \sigma)) + 1$ (see Figure 4.1 left) and signed graphs for which $\chi((G, \sigma)) = \chi_{\pm}((G, \sigma)) - 1$ (see Figure 4.1 right). Hence, the bounds of Proposition 4.15 cannot be improved.

Proposition 4.16. *Let (G, σ) be a connected signed graph with at least three vertices.*

1. *If* (G, σ) *is antibalanced and not bipartite, then* $\chi_{\pm}((G, \sigma)) = 2$ *and* $\chi((G, \sigma)) = 3$.

2. *If* (G, σ) *is bipartite but not antibalanced, then* $\chi_{\pm}((G, \sigma)) = 3$ *and* $\chi((G, \sigma)) = 2$.

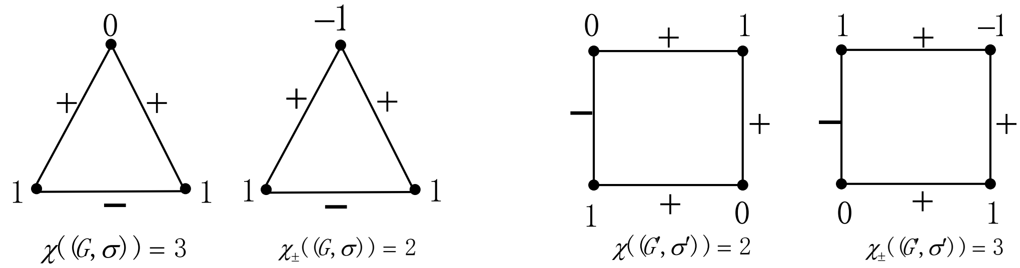

Figure 4.1: Two signed graphs with $|\chi - \chi_{\pm}| = 1$

Chapter 5

Choosability in signed graphs

In this chapter, we generalize the concept of list-colorings and list-chromatic number of unsigned graphs to signed graphs. It is known that for unsigned graphs, the list-chromatic number is an extension of the chromatic number. The list-chromatic number of signed graphs we define here extends both the chromatic number and the signed chromatic number. We provide bound for this new invariant in terms of the list-chromatic number of its underlying unsigned graph. We then focus on the choosability of signed planar graphs and generalizes the results of [15, 31, 45, 46, 47, 52] to signed graphs.

The results of this chapter except Theorem 5.4 have already been published in [23].

5.1 Definitions and basic properties

We combine the approaches of [14], [29] and [35] to define list colorings of signed graphs.

Definition 5.1. *Given a signed graph (G, σ), a list-assignment of (G, σ) is a function L defined on $V(G)$ such that $\emptyset \neq L(v) \subseteq \mathbb{Z}$ for each $v \in V(G)$. An L-coloring of (G, σ) is a proper coloring c of (G, σ) such that $c(v) \in L(v)$ for each $v \in V(G)$. A list-assignment L is called a k-list-assignment if $|L(v)| = k$ for each $v \in V(G)$. We say (G, σ) is k-choosable if it admits an L-coloring* *for every k-list-assignment L. The list-chromatic number or choice number $\chi_l((G,\sigma))$ of (G,σ) is the minimum number k such that (G,σ) is k-choosable.*

Clearly, if a signed graph is k-choosable, then it is also signed k-colorable.

Let (G,σ) be a signed graph, L be a list assignment of (G,σ), and c be an L-coloring of (G,σ). Let $X \subseteq V(G)$. We say σ', L' and c' are obtained from σ, L and c by a *switch* at X if

$$\sigma'(e) = \begin{cases} -\sigma(e), & \text{if } e \in \partial(X), \\ \sigma(e), & \text{if } e \in E(G) \setminus \partial(X); \end{cases}$$

$$L'(u) = \begin{cases} \{-\alpha\colon\ \alpha \in L(u)\}, & \text{if } u \in X, \\ L(u), & \text{if } u \in V(G) \setminus X; \end{cases}$$

$$c'(u) = \begin{cases} -c(u), & \text{if } u \in X, \\ c(u), & \text{if } u \in V(G) \setminus X. \end{cases}$$

Recall that two signed graphs (G,σ) and (G,σ^*) are equivalent if they can be obtained from each other by a switch at some subset of $V(G)$. The proof of the following proposition is trivial.

Proposition 5.2. *Let (G,σ) be a signed graph, L be a list-assignment of G and c be an L-coloring of (G,σ). If σ', L' and c' are obtained from σ, L and c by a switch at a subset of $V(G)$, then c' is an L'-coloring of (G,σ'). Furthermore, two equivalent signed graphs have the same chromatic number and the same choice number.*

Let G be a graph. By definition, G and $(G,+)$ have the same chromatic number and the same choice number. Hence, the following statement holds.

Corollary 5.3. *If (G,σ) is a balanced signed graph, then $\chi(G) = \chi((G,\sigma))$ and $\chi_l(G) = \chi_l((G,\sigma))$.*

Now we investigate the relation between the list-chromatic number of a signed graph and the list-chromatic number of its underlying graph.

Theorem 5.4. *For every signed graph (G, σ), we have $\chi_l((G, \sigma)) \leq 2\chi_l(G)$.*

Proof. Let $k = \chi_l(G)$ and L be an arbitrary $2k$-list-assignment of (G, σ). Since switching a vertex does not change the choice number of (G, σ) but change the list of colors for this vertex into inverse ones, we may assume that $L(v)$ contains at least k positive colors for each $v \in V(G)$. Let us take a k-list-assignment L' of G such that $L'(v) \subseteq (L(v) \cap \mathbb{Z}^+)$. Since G is k-choosable, G has an L'-coloring, say c. Now we assign the signature σ to G. Notice that c uses positive colors only. Thus, no matter which sign is assigned to an edge of G, the coloring c is still proper for the adjacency by this edge. Hence, c is also an L-coloring of (G, σ), giving $\chi_l((G, \sigma)) \leq 2k = 2\chi_l(G)$. □

5.2 Choosability in signed planar graphs

A planar graph is a graph that can be drawn in the Euclidean plane without crossings, that is, so that no two edges intersect geometrically except at a vertex. The coloring problems of planar graphs are one of the main topics in the theory of graph colorings. In particular, the choosability of planar graphs have been wildly discussed. In this section, we consider the choosability of signed planar graphs. We generalizes the results of [15, 31, 45, 46, 47, 52] to signed graphs.

The structure of this section is arranged as follows. Section 5.2.1 proves that every signed planar graph is 5-choosable. Furthermore, there is a signed planar graph (G, σ) which is not 4-choosable, but $(G, +)$ is 4-choosable. Section 5.2.2 proves for each $k \in \{3, 4, 5, 6\}$ that every signed planar graph without k-circuits is 4-choosable. The main theorem of this section is proved by discharging. See [12] for more details on this method. Section 5.2.3 proves that every signed planar graph with neither 3-circuits nor 4-circuits is 3-choosable. Furthermore, there exists a signed planar graph (G, σ) such that G has girth 4 and (G, σ) is not 3-choosable but $(G, +)$ is 3-choosable. These two constructions of signed graphs also show, that the choice number of a signed graph (G, σ) cannot be easily calculated from the choice number of G.

We give some necessary notations and terminologies for this section. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. We say a vertex u is a *neighbor* of another vertex v if $uv \in E(G)$. If $v \in V(G)$, then $d(v)$ denotes the degree of v and furthermore, v is called a *k-vertex* (or *k^+-vertex* or *k^--vertex*) if $d(v) = k$ (or $d(v) \geq k$ or $d(v) \leq k$). Similarly, a *k-circuit* (or *k^+-circuit* or *k^--circuit*) is a circuit of length k (or at least k or at most k), and if G is planar, then a *k-face* (or *k^+-face* or *k^--face*) is a face of size k (or at least k or at most k). Let $[x_1 \ldots x_k]$ denote a k-circuit with vertices $x_1, \ldots, x_k$ in cyclic order. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X, and $\partial(X)$ denotes the set of edges between X and $V(G) \setminus X$.

5.2.1 5-choosability

Theorem 5.5. *Every signed planar graph is 5-choosable.*

We use the method described in [45] to prove the following theorem which implies Theorem 5.5. A plane graph G is a *near triangulation* if the boundary of each bounded face of G is a triangle.

Theorem 5.6. *Let (G, σ) be a signed graph, where G is a near-triangulation. Let C be the boundary of the unbounded face of G and $C = [v_1 \ldots v_p]$. If L is a list-assignment of (G, σ) such that $L(v_1) = \{\alpha\}$, $L(v_2) = \{\beta\}$ and $\alpha \neq \beta\sigma(v_1v_2)$, and that $|L(v)| \geq 3$ for $v \in V(C) \setminus \{v_1, v_2\}$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus V(C)$, then (G, σ) has an L-coloring.*

Proof. Let us prove Theorem 5.6 by induction on $|V(G)|$.

If $|V(G)| = 3$, then $p = 3$ and $G = C$. Choose a color from $L(v_3) \setminus \{\alpha\sigma(v_1v_3), \beta\sigma(v_2v_3)\}$ for v_3. So we proceed to the induction step.

If C has a chord which divides G into two graphs G_1 and G_2, then we choose the notation so that G_1 contains v_1v_2, and we apply the induction hypothesis first to G_1 and then to G_2. Hence, we can assume that C has no chord.

Let $v_1, u_1, u_2, \ldots, u_m, v_{p-1}$ be the neighbors of v_p in cyclic order around v_p. Since the boundary of each bounded face of G is a triangle, G contains the path $P\colon v_1u_1\dots u_mv_{p-1}$. Since C has no chord, $P \cup (C - v_p)$ is a circuit C'. Let γ_1 and γ_2 be two distinct colors of $L(v_p) \setminus \{\alpha\sigma(v_1v_p)\}$. Define $L'(x) = L(x) \setminus \{\gamma_1\sigma(v_px), \gamma_2\sigma(v_px)\}$ for $x \in \{u_1,\dots,u_m\}$, and $L'(x) = L(x)$ for $x \in V(G) \setminus \{v_p, u_1, \dots, u_m\}$. Let σ' be the restriction of σ to $G - v_p$. By the induction hypothesis, signed graph $(G - v_p, \sigma')$ has an L'-coloring. Let c be the color vertex v_{p-1} receives. We choose a color from $\{\gamma_1, \gamma_2\} \setminus \{c\sigma(v_{p-1}v_p)\}$ for v_p, giving an L-coloring of (G, σ). □

Non-4-choosable examples

Voigt [49, 50] constructed two planar graphs which are not 4-choosable. By Corollary 5.3 these two examples generate two groups of signed planar graphs which are not 4-choosable. We extend this result to signed graphs.

Theorem 5.7. *There exists a signed planar graph (G, σ) such that (G, σ) is not 4-choosable but G is 4-choosable.*

Proof. We construct (G, σ) as follows. Take a copy G_1 of the complete graph K_4 and embed it into Euclidean plane. Insert a claw into each 3-face of G_1 and denote the resulting graph by G_2. Once again, insert a claw into each 3-face of G_2 and denote by G_3 the resulting graph. A vertex v of G_3 is called an initial-vertex if $v \in V(G_1)$, a solid-vertex if $v \in V(G_2) \setminus V(G_1)$ and a hollow-vertex if $v \in V(G_3) \setminus V(G_2)$ (Figure 5.1 illustrates graph G_3). A 3-face of G_3 is called a special 3-face if it contains an initial-vertex, a solid-vertex and a hollow-vertex. Clearly, G_3 has twenty-four special 3-faces, say $T_1, \dots, T_{24}$.

Let H be the plane graph as shown in Figure 5.2, which consists of a circuit $[xyz]$ and its interior. For $i \in \{1, \dots, 24\}$, replace T_i by a copy H_i of H such that x_i, y_i and z_i are identified with the solid-vertex, hollow-vertex and initial-vertex of T_i, respectively. Let G be the resulting graph. Clearly, G is planar.

Define a signature σ of G as follows: $\sigma(P_iQ_i) = -1$ for $i \in \{1, \dots, 24\}$ and $\sigma(e) = 1$ for $e \in E(G) \setminus \{P_iQ_i\colon\ i \in \{1, \dots, 24\}\}$.

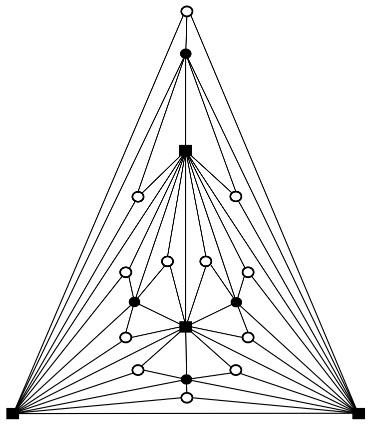

Figure 5.1: The graph G_3

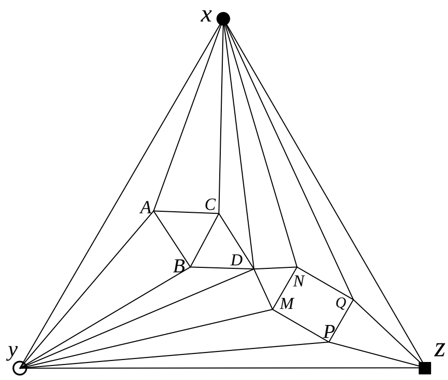

Figure 5.2: The graph H

Let L be a 4-list-assignment of signed graph (G, σ) defined as follows: $L(v) = \{1, 2, 3, 4\}$ for $v \in V(G_3)$, and $L(A_i) = \{1, 2, 6, 7\}$, $L(B_i) = \{2, 4, 6, 7\}$, $L(C_i) = \{1, 4, 6, 7\}$, $L(D_i) = \{1, 2, 4, 5\}$, $L(M_i) = \{2, 5, 6, -6\}$, $L(N_i) = \{1, 5, 6, -6\}$, $L(P_i) = \{2, 3, 6, -6\}$ and $L(Q_i) = \{1, 3, 6, -6\}$ for $i \in \{1, \ldots, 24\}$.

We claim that signed graph (G, σ) has no L-colorings. Suppose to the contrary that ϕ is an L-coloring of (G, σ). By the construction of G_3, precisely one of the special 3-faces of G_3 is assigned in ϕ color 1 to its solid-vertex, color 2 to its hollow-vertex and color 3 to its initial-vertex. Without loss of generality, let T_1 be such a special 3-face. Let us consider ϕ in H_1. Clearly, $\phi(x_1) = 1, \phi(y_1) = 2$ and $\phi(z_1) = 3$. It follows that $\phi(D_1) \in \{4, 5\}$. Notice that the odd circuit $[A_1B_1C_1]$ is balanced and the even circuit $[M_1N_1Q_1P_1]$ is unbalanced, and thus, both of them are not 2-choosable. It follows that if

$\phi(D_1) = 4$, then ϕ is not proper in $[A_1B_1C_1]$, and that if $\phi(D_1) = 5$, then ϕ is not proper in $[M_1N_1Q_1P_1]$. Therefore, (G, σ) has no L-colorings and thus, is not 4-choosable.

Let L' be any 4-list-assignment of G. By the construction, it is not hard to see that G_3 is 4-choosable. Let c be an L'-coloring of G_3. Clearly, for $i \in \{1, \ldots, 24\}$, each of vertices x_i, y_i and z_i receives a color in c. Let α and β be two distinct colors from $L(D_i) \setminus \{c(x_i), c(y_i)\}$. Choose a color from $L(C_i) \setminus \{\alpha, \beta, c(x_i)\}$ for C_i, and then vertices A_i, B_i and D_i can be list-colored by L' in turn. Since circuit $[M_iN_iQ_iP_i]$ is 2-choosable, it follows that vertices M_i, N_i, P_i and Q_i can also be list-colored by L'. Therefore, c can be extended to an L'-coloring of G. This completes the proof that G is 4-choosable. □

5.2.2 4-choosability

A graph G is *d-degenerate* if every subgraph H of G has a vertex of degree at most d in H. It is known that every $(d-1)$-degenerate graph is d-choosable. This proposition can be extended for signed graphs.

Theorem 5.8. *Let (G, σ) be a signed graph. If G is $(d-1)$-degenerate, then (G, σ) is d-choosable.*

Proof. (induction on $|V(G)|$) Let L be any d-list-assignment of G. The proof is trivial if $|V(G)| = 1$. For $|V(G)| \geq 2$, since G is $(d-1)$-degenerate, G has a vertex v of degree at most $d-1$ and moreover, graph $G-v$ is $(d-1)$-degenerate. Let σ' and L' be the restriction of σ and L to $G-v$, respectively. By applying the induction hypothesis to $(G - v, \sigma')$, we conclude that $(G - v, \sigma')$ is d-choosable and thus, it has an L'-coloring ϕ. Since v has degree at most $d-1$, we can choose a color α for v with $\alpha \in L(v) \setminus \{\phi(u)\sigma(uv) \colon\ uv \in E(G)\}$. We complete an L-coloring of (G, σ) with ϕ and α. □

It is an easy consequence of Euler's formula that every triangle-free planar graph contains a vertex of degree at most 3. Therefore, the following statement is true:

Lemma 5.9. *Planar graphs without 3-circuits are 3-degenerate.*

Moreover, we will use two more lemmas.

Lemma 5.10 ([52]). *Planar graphs without 5-circuits are 3-degenerate.*

Lemma 5.11 ([15]). *Planar graphs without 6-circuits are 3-degenerate.*

Theorem 5.12. *Let (G, σ) be a signed planar graph. For each $k \in \{3, 4, 5, 6\}$, if G has no k-circuits, then (G, σ) is 4-choosable.*

Proof. For $k \in \{3, 5, 6\}$ we deduce the statement from Theorem 5.8, together with Lemmas 5.9, 5.10 and 5.11, respectively. It remains to prove Theorem 5.12 for the case $k = 4$.

Suppose to the contrary that the statement is not true. Let (G, σ) be a counterexample of smallest order, and L be a 4-list-assignment of (G, σ) such that (G, σ) has no L-colorings. Clearly, G is connected by the minimality of (G, σ).

Claim 5.12.1. $\delta(G) \geq 4$.

Let u be a vertex of G of minimal degree. Suppose to the contrary that $d(u) < 4$. Let σ' and L' be the restriction of σ and L to $G - u$, respectively. By the minimality of (G, σ), the signed graph $(G - u, \sigma')$ has an L'-coloring c. Since every neighbor of u forbids one color for u no matter what the signature of the edge between them is, $L(u)$ still has a color left for coloring u. Therefore, c can be extended to an L-coloring of (G, σ), a contradiction.

Claim 5.12.2. *G has no 6-circuit C such that $C = [u_0 \dots u_5]$ and $u_0u_2 \in E(G)$, and $d(u_0) \leq 5$ and all other vertices of C are of degree 4.*

Suppose to the contrary that G has such a 6-circuit C. Since G has no 4-circuits, u_0u_2 is the only chord of C. There always exists a subset X of $V(C)$ such that all of the edges u_0u_2, u_1u_2 and u_2u_3 are positive after a switch at X. Let σ' and L' be obtained from σ and L by a switch at X, respectively. Proposition 5.2 implies that (G, σ') has no L'-coloring. Hence, (G, σ') is also a minimal counterexample. Let σ_1 and L_1 be the restriction of σ' and L' to $G - V(C)$, respectively. It follows that $(G - V(C), \sigma_1)$ has an L_1-coloring ϕ.

We obtain a contradiction by further extending ϕ to an L'-coloring of (G, σ') as follows. By the condition on the vertex degrees of C, there exists a list-assignment L_2 of $G[V(C)]$ such that $L_2(u) \subseteq L'(u) \setminus \{\phi(v)\sigma'(uv)\colon\ uv \in E(G) \textit{ and } v \notin V(C)\}$ for $u \in V(C)$, and $|L_2(u_2)| = 3$ and $|L_2(u)| = 2$ for $u \in V(C) \setminus \{u_2\}$. Let $L_2(u_2) = \{\alpha, \beta, \gamma\}$. Suppose that $L_2(u_2)$ has a color, say α, that does not appear in at least two of the lists $L_2(u_0), L_2(u_1)$ and $L_2(u_3)$. We color u_2 with α, and then all other vertices of C can be list-colored by L_2 in some order. For example, if α does not appear in $L_2(u_0)$ and $L_2(u_1)$, then we color $V(C)$ in the order $u_2, u_3, u_4, u_5, u_0, u_1$. Hence, we may assume that $L_2(u_0) = \{\alpha, \gamma\}$, $L_2(u_1) = \{\alpha, \beta\}$ and $L_2(u_3) = \{\beta, \gamma\}$. If $\beta \neq \gamma\sigma'(u_0u_1)$, then color u_0 with γ, u_1 with β, and u_2 with α, and the remaining vertices of C can be list-colored by L_2 in the order u_5, u_4, u_3. Hence, we may assume $\beta = \gamma\sigma'(u_0u_1)$. It follows that $\sigma'(u_0u_1) = -1$ and $\beta = -\gamma \neq 0$. If $\alpha \neq 0$, then color both u_0 and u_1 with α, and the remaining vertices of C can be list-colored by L_2 in the order u_5, u_4, u_3, u_2. Hence, we may assume $\alpha = 0$. Now the color 0 is included in $L_2(u_0)$ but not in $L_2(u_3)$. Thus, there exists an integer i with $i \in \{3, 4, 5\}$ such that $0 \in L_2(u_{i+1})$ and $0 \notin L_2(u_i)$ (index is added modular 6). We color u_{i+1} with 0, and then the remaining vertices of C can be list-colored by L_2 in cyclic order on C ending at u_i.

Claim 5.12.3. *G has no 10-circuit C such that $C = [u_0 \dots u_9]$ and $u_0u_8, u_2u_6, u_2u_7 \in E(G)$, and u_2 has degree 6 and all other vertices of C have degree 4.*

Suppose to the contrary that G has such a 10-circuit C. Let σ' and L' be the restriction of σ and L to graph $G - V(C)$, respectively. By the minimality of (G, σ), the signed graph $(G - V(C), \sigma')$ has an L'-coloring ϕ. A contradiction is obtained by further extending ϕ to an L-coloring of (G, σ) as follows. We shall list-color the vertices of C by L in the cyclic order $u_0, u_1, \dots, u_9$. For $i \in \{0, \dots, 9\}$, let $F_i = \{\phi(v)\sigma(u_iv)\colon\ u_iv \in E(G) \textit{ and } v \notin V(C)\}$. Clearly, F_i is the set of colors that are forbidden for u_i by its neighbors, which are not in $V(C)$. Since $d(u_0) = d(u_9) = 4$ and moreover, if there is any other chord of C, then the list F_i will not become longer, it follows that $|F_0| \leq 1$ and $|F_9| \leq 2$. Hence, we can let α and β be two distinct colors from $L(u_9) \setminus F_9$. Let $\gamma \in L(u_0) \setminus (F_0 \cup \{\alpha\sigma(u_0u_9), \beta\sigma(u_0u_9)\})$, and color vertex u_0 with γ. For $i \in \{1, \ldots, 8\}$, vertex u_i has at most three neighbors colored before u_i in this color-assigning process and thus, $L(u_i)$ still has a color available for u_i. Denote by ζ the color of vertex u_8. We complete the extending of ϕ by assigning a color from $\{\alpha, \beta\} \setminus \{\zeta\sigma(u_8u_9)\}$ to u_9.

Discharging

Consider an embedding of G into Euclidean plane. Let G denote the resulting plane graph. We say two faces are *adjacent* if they share an edge. Two adjacent faces are *normally adjacent* if they share an edge xy and no vertex other than x and y. Since G is a simple graph, the boundary of each 3- or 5-face is a circuit. Since G has no 4-circuits, we can deduce that if a 3-face and a 5-face are adjacent, then they are normally adjacent. A vertex is *bad* if it is of degree 4 and incident with two nonadjacent 3-faces. A *bad 3-face* is a 3-face containing three bad vertices. A 5-face f is *magic* if it is adjacent to five 3-faces, and if all the vertices of these six faces have degree 4 except one vertex of f.

We shall obtain a contradiction by applying discharging method. Let $V = V(G)$, $E = E(G)$, and F be the set of faces of G. Denote by $d(f)$ the size of a face f of G. Give initial charge $ch(x)$ to each element x of $V \cup F$, where $ch(v) = 3d(v) - 10$ for $v \in V$, and $ch(f) = 2d(f) - 10$ for $f \in F$. Discharge the elements of $V \cup F$ according to the following rules:

R1. Every vertex u sends each incident 3-face charge 1 if u is a bad vertex, and charge 2 otherwise.

R2. Every 5-vertex sends $\frac{1}{3}$ to each incident 5-face.

R3. Every 6-vertex sends each incident 5-face f charge 1 if f is magic, charge $\frac{2}{3}$ if f is not magic but contains four 4-vertices, charge $\frac{1}{3}$ if f contains at most three 4-vertices.

R4. Every 7^+-vertex sends 1 to each incident 5-face.

R5. Every 3-face sends $\frac{1}{3}$ to each adjacent 5-face if this 3-face contains at most one bad vertex.

R6. Every 5^+-face sends $\frac{k}{3}$ to each adjacent bad 3-face, where k is the number of common edges between them.

Let $ch^*(x)$ denote the final charge of each element x of $V \cup F$ when the discharging process is over. On one hand, by Euler's formula we deduce $\sum_{x \in V \cup F} ch(x) = -20$. Since the sum of charge over all elements of $V \cup F$ is unchanged, we have $\sum_{x \in V \cup F} ch^*(x) = -20$. On the other hand, we show that $ch^*(x) \geq 0$ for $x \in V \cup F$. Hence, this obvious contradiction completes the proof of Theorem 5.12.

It remains to show that $ch^*(x) \geq 0$ for $x \in V \cup F$.

Claim 5.12.4. *If $v \in V$, then $ch^*(v) \geq 0$.*

Let p be the number of 3-faces that contains v. Since G has no 4-circuits, $p \leq \lfloor \frac{d(v)}{2} \rfloor$. Moreover, $d(v) \geq 4$ by Claim 5.12.1.

Suppose $d(v) = 4$. We have $p \leq 2$. If $p = 2$, then v is a bad vertex and thus, $ch^*(v) = 3d(v) - 10 - p = 0$ by R1; otherwise, $ch^*(v) = 3d(v) - 10 - 2p \geq 0$ by R1 again.

If $d(v) = 5$, then $p \leq 2$ and thus, by R1 and R2, $ch^*(v) \geq 3d(v) - 10 - 2p - \frac{1}{3}(5 - p) \geq 0$.

Suppose that $d(v) = 6$. Thus, $p \leq 3$. By R1 and R3, if $p \leq 2$, then $ch^*(v) \geq 3d(v) - 10 - 2p - (6 - p) \geq 0$, and if v is incident with no magic 5-faces, then $ch^*(v) \geq 3d(v) - 10 - 2p - \frac{2}{3}(6 - p) \geq 0$. Hence, we may assume that $p = 3$ and that v is incident with a magic 5-face f. For any other 5^+-face f' containing v than f, Claim 5.12.3 implies that if f' has size 5, then it contains at most three 4-vertices, and thus, v sends at most $\frac{1}{3}$ to f' by R3. Hence, $ch^*(v) \geq 3d(v) - 10 - 2 \times 3 - 1 - \frac{1}{3} \times 2 > 0$.

It remains to suppose $d(v) \geq 7$. By R1 and R4, we have $ch^*(v) \geq 3d(v) - 10 - 2p - (d(v) - p) \geq 2d(v) - 10 - \lfloor \frac{d(v)}{2} \rfloor > 0$.

Claim 5.12.5. *If $f \in F$, then $ch^*(f) \geq 0$.*

Suppose $d(f) = 3$. Recall that in this case the boundary of f is a circuit. We have $ch^*(f) \geq 2d(f) - 10 + 2 + 2 + 1 - 3 \times \frac{1}{3} = 0$ by R1 and R5 when f has at most one bad vertex, and $ch^*(f) \geq 2d(f) - 10 + 2 + 1 + 1 = 0$ by R1 when f has precisely two bad vertices. Hence, for the remaining we can assume that f has precisely three bad vertices, that is, f is a bad 3-face. In this case, f receives charge 1 in total from adjacent faces by R6, and charge 3 in total from incident vertices by R1. Hence, $ch^*(f) \geq 2d(f) - 10 + 1 + 3 = 0$.

Suppose $d(f) = 5$. Recall in this case that the boundary of f is a circuit and that if f is adjacent to a 3-face, then they are normally adjacent. Let q be the number of bad 3-faces adjacent to f. Clearly, f sends charge only to adjacent bad 3-faces by R6, and possibly receives charge from incident 5^+-vertices and from adjacent 3-faces by rules from R2 to R5. Hence, $ch^*(f) \geq 2d(f) - 10 = 0$ if $q = 0$. Claim 5.12.2 implies that $q \leq 3$ and that f contains a 5^+-vertex u, which sends at least $\frac{1}{3}$ to f. Hence, $ch^*(f) \geq 2d(f) - 10 - \frac{1}{3} + \frac{1}{3} = 0$ if $q = 1$. First suppose $q = 2$. If f has a 5^+-vertex different from u, then we are done by $ch^*(f) \geq 2d(f) - 10 - 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = 0$. Hence, we may assume that f contains four 4-vertices. It follows that if $d(u) \geq 6$, then f receives at least $\frac{2}{3}$ from v by R3 or R4 and we are done. Hence, we may assume that $d(u) = 5$. Through the drawing of 3-faces adjacent to f, we can assume u is incident with a 3-face $[uvw]$ that is adjacent to f on edge uv. Claim 5.12.2 implies that $d(w) \geq 5$. Hence, f receives $\frac{1}{3}$ from face $[uvw]$ by R5, and we are done. Let us next suppose $q = 3$. We may assume $f = [uv'w'x'y']$ such that $v'w'$, $w'x'$ and $x'y'$ are the three common edges between f and bad 3-faces. Since both vertices v' and y' are bad, edges uv' and uy' are contained in 3-faces $[uv't']$ and $[uy'z']$, respectively. If $d(u) = 5$, then Claim 5.12.2 implies that $d(t'), d(z') \geq 5$, and thus, f receives $\frac{1}{3}$ from each of faces $[uv't']$ and $[uy'z']$ by R5, and we are done. If $d(u) \geq 7$, then f receives 1 from u and we are done. Hence, we may assume that $d(u) = 6$. If both t' and z' have degree 4, that is, f is a magic 5-face, then f receives 1 from u by R3; otherwise, f receives $\frac{2}{3}$ from u and $\frac{1}{3}$ from at least one of faces $[uv't']$ and $[uy'z']$ by R3 again. We are done in both cases.

It remains to suppose $d(f) \geq 6$. Recall that f has no charge moving in or out except that it sends $\frac{1}{3}d(f)$ in total to adjacent bad 3-faces by R6. Hence, $ch^*(f) \geq 2 \times d(f) - 10 - \frac{1}{3}d(f) \geq 0$.

The proof of Theorem 5.12 is completed. □

5.2.3 3-choosability

In 1995, Thomassen [46] proved that every planar graph of girth at least 5 is 3-choosable. And then in 2003, he [47] gave a shorter proof of this result. We find out that the argument used in [47] also works for signed graphs. Hence, the following statement is true. For the sake of completeness, we include the proof.

Theorem 5.13. *Every signed planar graph with neither 3-circuit nor 4-circuit is 3-choosable.*

We will prove the following theorem which is stronger than Theorem 5.13 but easier for us to prove.

Theorem 5.14. *Let (G, σ) be a signed plane graph of girth at least 5, and D be the outer face boundary of G. Let P be a path or circuit of G such that $|V(P)| \leq 6$ and $V(P) \subseteq V(D)$, and σ_p be the restriction of σ to P. Assume that (P, σ_p) has a 3-coloring c. Let L be a list-assignment of G such that $L(v) = \{c(v)\}$ if $v \in V(P)$, $|L(v)| \geq 2$ if $v \in V(D) \setminus V(P)$, and $|L(v)| \geq 3$ if $v \in V(G) \setminus V(D)$. Assume furthermore that there is no edge joining vertices whose lists have at most two colors except for the edges in P. Then c can be extended to an L-coloring of (G, σ).*

Proof. The proof will be done by induction on the number of vertices. We assume that (G, σ) is a smallest counterexample and shall get a contradiction.

Claim 5.14.1. *G is 2-connected and hence, D is a circuit.*

We may assume that G is connected, since otherwise we apply the induction hypothesis to every connected component of G. Similarly, G has no cutvertex in P. Moreover, G has no cutvertex at all. Suppose to the contrary that u is a cutvertex contained in an endblock B disjoint from P. We first apply the induction hypothesis to $G-(B-u)$. If B has vertices with only two available colors joined to u, then we color each such vertex. These colored vertices of B together with the edges joining them to u divide B into parts each of which has at most three colored vertices inducing a path. Now we apply the induction hypothesis to each of those parts. This contradiction proves Claim 5.14.1.

Claim 5.14.2. *For $e \in E(P)$, e is not a chord of D.*

If some edge e of P is a chord of G, then e divides G into two parts, and we apply the induction hypothesis to each of those two parts. This contradiction proves Claim 5.14.2.

By Claims 5.14.1 and 5.14.2, we may choose the notion such that $D = [v_1 \dots v_k]$ and $P = v_1 \dots v_q$.

Let X be a set of colored vertices of G. To save writing we just say "delete the product colors of X from G" instead of "for $v \in V(G) \setminus X$, delete all of the colors in $\{c(u)\sigma(uv)\colon\ u \in X \textit{ and } uv \in E(G)\}$ from the list of v".

Claim 5.14.3. *P is a path, and $q+3 \leq k$.*

If $P = D$, then we delete any vertex from D, and delete the product color of that vertex from G. If $P \neq D$ and $k < q+3$, then we color the vertices of D not in P, we delete them together with their product colors from G.

Now we apply the induction hypothesis to the resulting graph G', if possible. As G has grith at least 5, the vertices with precisely two available colors are independent. For the same reason, such a vertex cannot be joined to two vertices of P. However, such a vertex may be joined to precisely one vertex of P. We then color it. Now the colored vertices of G' divide G' into parts each of which has at most 6 precolored vertices inducing a path. We then apply induction hypothesis to each of those parts. This contradiction proves Claim 5.14.3.

Claim 5.14.4. *D has no chord.*

Suppose to the contrary that xy is a chord of D. Then xy divides G into two graphs G_1, G_2, say. We may choose the notation such that G_2 has no more vertices of P than G_1 has, and subject to that condition, $|V(G_2)|$ is minimum. We apply the induction hypothesis first to G_1. In particular, x and y receive a color. The minimality of G_2 implies that the outer cycle of G_2 is chordless. So G_2 has at most two vertices which have only two available colors and which are joined to one of x and y. We color any such vertex, and then we apply the induction hypothesis to G_2. This contradiction proves Claim 5.14.4.

Claim 5.14.5. *G has no path of the form v_iuv_j where u lies inside D, except possibly when $q = 6$ and the path is of the form v_4uv_7 or v_3uv_k. In particular, u has only two neighbors on D.*

We define G_1 and G_2 as in the proof of Claim 5.14.4. We apply the induction hypothesis first to G_1. Although u may be joined to several vertices with only two available colors, the minimality of G_2 implies that no such vertex is in $G_2 - \{u, v_i, v_j\}$. There may be one or two vertices in $G_2 - \{u, v_i, v_j\}$ that have only two available colors and which are joined to one of v_i and v_j. We color any such vertex, and then at most six vertices of G_2 are colored. If possible, we apply the induction hypothesis to G_2. This is possible unless the coloring of G_1 is not valid in G_2. This happens only if P has a vertex in G_2 joined to one of v_i and v_j. This happens only if we have one of the two exceptional cases described in Claim 5.14.5.

Claim 5.14.6. *G has no path of the form v_iuwv_j such that u and w lie inside D, and $|L(v_i)| = 2$. Also, G has no path v_iuwv_j such that u and w lie inside D, $|L(v_i)| = 3$, and $j \in \{1, q\}$.*

Repeating the arguments in Claims 5.14.4 and 5.14.5, we can easily get Claim 5.14.6.

Claim 5.14.7. *If C is a circuit of G distinct from D and of length at most 6, then the interior of C is empty.*

Otherwise, we can apply the induction hypothesis first to C and its exterior and then to C and its interior. This contradiction proves Claim 5.14.7.

If $|L(v_{q+2})| \geq 3$, then we complete the proof by deleting v_q and its product color from G, and apply the induction hypothesis to $G - v_q$ and obtain thereby a contradiction. So we assume $|L(v_{q+2})| \leq 2$. By Claim 5.14.3, $|L(v_{q+2})| = 2$ and thus $|L(v_{q+3})| \geq 3$. If $|L(v_{q+4})| \geq 3$, then we first color v_{q+2} and v_{q+1}, then we delete them and their product colors from G. We obtain a contradiction by applying the induction hypothesis to the resulting graph. By Claims 5.14.4 and 5.14.5 this is possible unless $q = 6$ and G has a vertex u inside D joined to both v_4 and v_7. In this case we color u and delete both v_5 and v_6 before we apply the induction hypothesis. Hence, we may assume that $|L(v_{q+4})| \leq 2$.

We give v_{q+3} a color not in $\{\alpha\sigma(v_{q+3}v_{q+4}) \colon \alpha \in L(v_{q+4})\}$ and then color v_{q+2} and v_{q+1}, and finally we delete v_i and the product color of v_i from G for $i \in \{q+1, q+2, q+3\}$. We obtain a contradiction by applying the induction hypothesis to the resulting graph. If $q = 6$ and G has a vertex u inside D joined to v_4 and v_7, then, as above, we color u and delete v_5 and v_6 before we use induction. If $q = 6, q + 3 = k$, and G has a vertex u' inside D joined to v_3 and v_k, then we also color u' and delete v_1 and v_2 before we use induction. Finally, there may be a path $v_{q+1}wzv_{q+3}$ where w and z lies inside D. By Claim 5.14.7, this path is unique. We color w and z and delete them together with their product colors from G before we use induction. Note that u and u' may also exist in this case. If there are vertices joined to two colored vertices, then we also color these vertices before we use induction.

The colored vertices divide G into parts, and we shall show that each part satisfies the induction hypothesis. By second statement of Claim 5.14.6, there are at most six precolored vertices in each part, and they induce a path. Claim 5.14.5 and the first statement of Claim 5.14.6 imply that there is no vertex with precisely two available colors on D which is joined to a vertex inside D whose list has only two available colors after the additional coloring. Since G has girth at least 5 and by Claim 5.14.7, there is no other possibility for two adjacent vertices z and z' to have only two available colors in their lists, as both z and z' must be adjacent to a vertex that has been colored and deleted.

This contradiction completes the proof. □

Theorem 5.15. *There exists a signed planar graph (G, σ) such that G has girth 4 and (G, σ) is not 3-choosable but G is 3-choosable.*

Proof. Let T be a plane graph consisting of two circuits $[ABCD]$ and $[MNPQ]$ of length 4 and four other edges AM, BN, CP and DQ, as shown in Figure 5.3. Take nine copies $T_0, \ldots, T_8$ of T, and identify $A_0, \ldots, A_8$ into a vertex A' and $C_0, \ldots, C_8$ into a vertex C'. Let G be the resulting graph. Clearly, G is planar and has girth 4.

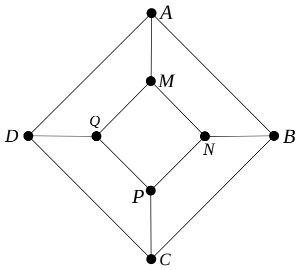

Figure 5.3: graph T

Define a signature σ of G as: $\sigma(e) = -1$ for $e \in \{M_iN_i : i \in \{0, \ldots, 8\}\}$, and $\sigma(e) = 1$ for $e \in E(G) \setminus \{M_iN_i : i \in \{0, \ldots, 8\}\}$.

For $i \in \{0, 1, 2\}$, let $a_i = i$ and $b_i = i + 3$. Define a 3-list-assignment L of G as follows: $L(A') = \{a_1, a_2, a_3\}, L(C') = \{b_1, b_2, b_3\}$; for $i, j \in \{0, 1, 2\}$, let $L(B_{3i+j}) = L(D_{3i+j}) = \{a_i, b_j, 6\}, L(N_{3i+j}) = L(Q_{3i+j}) = \{6, 7, -7\}, L(M_{3i+j}) = \{a_i, 7, -7\}$ and $L(P_{3i+j}) = \{b_j, 7, -7\}$.

We claim that signed graph (G, σ) has no L-colorings. Suppose to the contrary that c is an L-coloring of (G, σ). Let $c(A') = a_p$ and $c(C') = b_q$. Consider subgraph T_{3p+q}. It follows that $c(B_{3p+q}) = c(D_{3p+q}) = 6$. Furthermore, the circuit $[M_{3p+q}N_{3p+q}P_{3p+q}Q_{3p+q}]$ is unbalanced and thus not 2-choosable. Hence, T_{3p+q} is not properly colored in c, a contradiction. This proves that (G, σ) has no L-colorings and therefore, (G, σ) is not 3-choosable.

We claim that graph G is 3-choosable. For any 3-list-assignment of G, choose any color for vertices A' and C' from their color lists, respectively. Consider each subgraph T_i ($i \in \{0, \ldots, 8\}$). Both vertices B_i and D_i can be list colored. The 2-choosability of circuit $[M_iN_iP_iQ_i]$ yields a list coloring of T_i and hence a list coloring of G. This proves that G is 3-choosable. $\square$

Chapter 6

3-colorings of planar graphs

In this chapter, we consider the vertex coloring of unsigned graphs. We prove a result related to the famous Steinberg's conjecture from 1976, which states that every planar graph without cycles of length 4 or 5 is 3-colorable. Though Steinberg's Conjecture was disproved recently, we show that if the planar graph additionally has no cycles of length 8, then it is 3-colorable. This result improves on a series of earlier results.

The results of this chapter have already been published in [22].

6.1 Introduction

In the field of 3-colorings of planar graphs, one of the most active topics is a conjecture proposed by Steinberg in 1976: every planar graph without cycles of length 4 or 5 is 3-colorable. There had been no progress on this conjecture for a long time, until Erdös suggested a relaxation of it (see Problem 9.2 in [44]): does there exist a constant k such that every planar graph without cycles of length from 4 to k is 3-colorable? Abbott and Zhou [1] confirmed that such k exists and $k \leq 11$. This result was later improved to $k \leq 9$ independently by Borodin [4] and by Sanders and Zhao [41], and to $k \leq 7$ by Borodin, Glebov, Raspaud and Salavatipour [7].

Theorem 6.1 ([7]). *Planar graphs without cycles of length from 4 to 7 are 3-colorable.*

We remark that Steinberg's Conjecture was disproved in [10] recently, by constructing a counterexample to the conjecture. However, the question whether every planar graph without cycles of length from 4 to 6 is 3-colorable is still open. A partial result to this question was obtained in [53]. It was proved there that every planar graph without cycles of length from 4 to 6 can be decomposed into a matching and a 3-colorable graph.

Steinberg's Conjecture motivated a lot of work in the literature (e.g. [30]), in particular, the work on 3-colorability of planar graphs without cycles of certain lengths, besides the results on the relaxation by Erdös. Due to the theorem of Grötzsch [18] that planar graphs without triangles are 3-colorable, triangles are always allowed in further sufficient conditions. Several papers together contribute to the result below:

Theorem 6.2. *For any three integers i, j, k with $5 \leq i < j < k \leq 9$, it holds true that planar graphs having no cycles of length $4, i, j$ or k are 3-colorable.*

The following problem that strengthens this theorem was considered already, and partial results to this problem were obtained.

Problem 6.3. *What is $\mathcal{B}$: a set of pairs of integers (i, j) with $5 \leq i < j \leq 9$, such that planar graphs without cycles of length $4, i$ or j are 3-colorable?*

It was proved by Borodin et al. [5] and independently by Xu [56] that every planar graph having neither 5- and 7-cycles nor adjacent 3-cycles is 3-colorable. Hence, $(5, 7) \in \mathcal{B}$, which improves on Theorem 6.1. More elements of B were confirmed: $(6, 8) \in \mathcal{B}$ by Wang and Chen [51], $(7, 9) \in \mathcal{B}$ by Lu et al. [33], and $(6, 9) \in \mathcal{B}$ by Jin et al. [24]. The result $(6, 7) \in \mathcal{B}$ is implied in the following theorem, which reconfirms the results $(5, 7) \in \mathcal{B}$ and $(6, 8) \in \mathcal{B}$.

Theorem 6.4 ([6]). *Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable.*

In this chapter, we show that $(5,8) \in \mathcal{B}$, which leaves four pairs of integers $(5,6), (5,9), (7,8)$, $(8,9)$ unconfirmed as elements of $\mathcal{B}$.

Recently, Mondal gave a proof of the result $(5,8) \in \mathcal{B}$ in [38], where he actually proved a stronger theorem instead. Here we exhibit two couterexamples to that theorem. Let C be a cycle of length at most 12 in a plane graph. C is *terrible* if it is of length 9 or 12 and the area inside C has a partition into 3- and 6-cycles; otherwise, C is *nice*. Mondal formulated the follows statement.

Statement [Theorem 2 in [38]]

Let G be a graph without 4-, 5-, and 8-cycles. If D is a nice cycle of G, then every proper 3-coloring of D can be extended to a proper 3-coloring of the whole graph G.

Counterexamples to the statement. One is a plane graph G_1 consisting of a cycle C of length 12, say $C := [v_1 \dots v_{12}]$, and a vertex u inside C connected to all of v_1, v_2, v_6. The graph G_1 contradicts the statement since any proper 3-coloring of C, where v_1, v_2, v_6 receive pairwise distinct colors, cannot be extended to G_1. A second is a plane graph G_2 consisting of a cycle C of length 12 and a triangle T inside C, say $C := [v_1 \dots v_{12}]$ and $T := [u_1 u_2 u_3]$, and three more edges u_1v_1, u_2v_4, u_3v_7. The graph G_2 contradicts the statement since any proper 3-coloring of C where v_1, v_4, v_7 receive the same color cannot be extended to G_2 (see Figure 6.1).

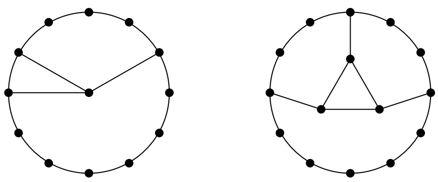

Figure 6.1: Two graphs as counterexamples to the statement

A problem that is more general than both Problem 6.3 and Steinberg's Conjecture was formulated in [33, 24].

Problem 6.5. *What is $\mathcal{A}$: a set of integers between 5 and 9, such that for $i \in \mathcal{A}$, every planar graph with cycles of length neither 4 nor i is 3-colorable?*

The disproof of Steinberg's Conjecture yields that $5 \notin \mathcal{A}$. So far no elements of $\mathcal{A}$ have been confirmed.

6.2 Notations and formulation of the main theorem

The graphs considered in this paper are finite and simple. A graph is planar if it is embeddable into the Euclidean plane. A plane graph (G, Σ) is a planar graph G together with an embedding Σ of G into the Euclidean plane, that is, (G, Σ) is a particular drawing of G in the Euclidean plane. In what follows, we will always say a plane graph G instead of (G, Σ), which causes no confusion since no two embeddings of the same graph G will be involved.

Let G be a plane graph and C be a cycle of G. By $Int(C)$ (or $Ext(C)$) we denote the subgraph of G induced by the vertices lying inside (or outside) C. The cycle C is *separating* if neither $Int(C)$ nor $Ext(C)$ is empty. By $\overline{Int}(C)$ (or $\overline{Ext}(C)$) we denote the subgraph of G consisting of C and its interior (or exterior). Two cycles are *adjacent* if they have at least one edge in common. The cycle C is *triangular* if it is adjacent to a triangle other than C; and C is *ext-triangular* if it is adjacent to a triangle T of $\overline{Ext}(C)$ other than C.

Now, we are ready to formulate the main result of this chapter.

Theorem 6.6. *Plane graphs with neither 4- and 5-cycles nor ext-triangular 7-cycles are 3-colorable.*

We remark that Theorem 6.6 reconfirms the known result that $(5, 7) \in \mathcal{B}$. Moreover, the following theorem is a direct consequence of Theorem 6.6.

Theorem 6.7. *Planar graphs without cycles of length 4, 5, 8 are 3-colorable, that is,* $(5, 8) \in \mathcal{B}$.

For the proof of Theorem 6.6, we will use the technique on coloring extension. We first give some necessary notations.

Let v be a vertex, P be a path, C be a cycle and f be a face of a plane graph G. The length of P is the number of edges of P. Denote by $d(v)$ the degree of v, by $|P|$ the length of P, by $|C|$ the length of C and by $d(f)$ the size of f. v is a *k-vertex* (or k^+-vertex or k^--vertex) if $d(v) = k$ (or $d(v) \geq k$ or $d(v) \leq k$). Similar notations are used for P, C and f with $|P|, |C|$ and $d(f)$ instead of $d(v)$, respectively.

Let C be a cycle of a plane graph G. If $S \subseteq V(G)$ or $S \subseteq E(G)$, then $G[S]$ denotes the subgraph of G induced by S. A *chord* of C is an edge of $\overline{Int}(C)$ that connects two nonconsecutive vertices on C. If $Int(C)$ has a vertex v with three neighbors v_1, v_2, v_3 on C, then $G[\{vv_1, vv_2, vv_3\}]$ is called a *claw* of C. If $Int(C)$ has two adjacent vertices u and v such that u has two neighbors u_1, u_2 on C and v has two neighbors v_1, v_2 on C, then $G[\{uv, uu_1, uu_2, vv_1, vv_2\}]$ is called a *biclaw* of C. If $Int(C)$ has three pairwise adjacent vertices u, v, w which has a neighbor u', v', w' on C, respectively, then $G[\{uv, vw, uw, uu', vv', ww'\}]$ is called a *triclaw* of C. If G has four vertices x, u, v, w inside C and four vertices x_1, x_2, v_1, w_1 on C such that $S = \{uv, vw, wu, ux, xx_1, xx_2, vv_1, ww_1\} \subseteq E(G)$, then $G[S]$ is called a *combclaw* of C (see Figure 6.2).

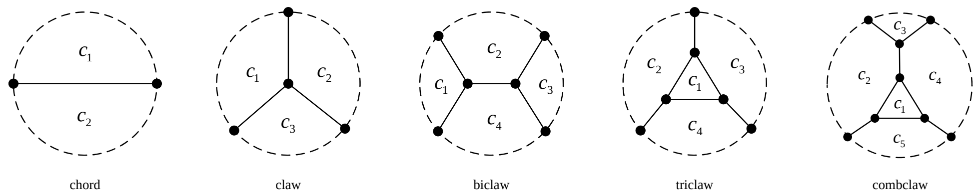

Figure 6.2: Chord, claw, biclaw, triclaw and combclaw of a cycle

A *good cycle* is an 11^--cycle that has no claws, biclaws, triclaws or combclaws. A *bad cycle* is an 11^--cycle that is not good.

Instead of Theorem 6.6, it is easier for us to prove the following stronger one.

Theorem 6.8. *Let G be a connected plane graph with neither 4- and 5-cycles nor ext-triangular 7-cycles. If D, the boundary of the exterior face of G, is a good cycle, then every proper 3-coloring of $G[V(D)]$ can be extended to a proper 3-coloring of G.*

Theorem 6.8 implies Theorem 6.6. Suppose to the contrary that Theorem 6.8 is true but Theorem 6.6 is not. Let G be a counterexample to Theorem

6.6 with minimum order of G. By the theorem of Grötzsch, G contains a triangle, say T. Take a proper 3-coloring ϕ of T. By the minimality of G, if T is a separating cycle, then ϕ can be extended respectively to proper 3-colorings of $\overline{Ext}(T)$ and of $\overline{Int}(T)$, which together yield a proper 3-coloring of G, a contradiction. Hence, either $Ext(T) = \emptyset$ or $Int(T) = \emptyset$. For the former case, T is the boundary of the exterior face of G. By Theorem 6.8, ϕ can be extended to a proper 3-coloring of G, a contradiction. For the latter case, we can reembed G into the Euclidean plane, obtaining a plane graph G', such that G' has neither 4- and 5-cycles nor ext-triangular 7-cycles and that T is the boundary of the exterior face of G'. Again by Theorem 6.8, ϕ can be extended to a proper 3-coloring of G' (equivalently, of G), a contradiction.

The proof of Theorem 6.8 will proceed by using the discharging method that will be given in the next section. For more information on the discharging method, we refer readers to [12]. The rest of this section provides other needed notations.

Let C be a cycle of a plane graph and T be a chord, a claw, a biclaw, a triclaw or a combclaw of C. We call the plane graph H consisting of C and T a *bad partition* of C. The boundary of each interior face of H is called a *cell* of H. In case of confusion, let us always order the cells $c_1, \cdots, c_t$ of H in the way as shown in Figure 6.2. Let k_i be the length of c_i. Then T is further called a (k_1, k_2)-chord, a (k_1, k_2, k_3)-claw, a (k_1, k_2, k_3, k_4)-biclaw, a (k_1, k_2, k_3, k_4)-triclaw or a $(k_1, k_2, k_3, k_4, k_5)$-combclaw, respectively.

A vertex is *external* if it lies on the exterior face; *internal* otherwise. A vertex (or an edge) is triangular if it is incident with a triangle. We say that a vertex is *bad* if it is an internal triangular 3-vertex; otherwise, it is a *good vertex.* A path is a *splitting path* of a cycle C if it has the two end-vertices on C and all other vertices inside C. A k-cycle with vertices $v_1, \ldots, v_k$ in cyclic order is denoted by $[v_1 \ldots v_k]$.

Let uvw be a path on the boundary of a face f of G with v internal. The vertex v is f-heavy if both uv and vw are triangular and $d(v) \geq 5$, and is f-Mlight if both uv and vw are triangular and $d(v) = 4$, and is f-Vlight if neither uv nor vw is triangular and v is triangular and of degree 4. A vertex is f-light if it is either f-Mlight or f-Vlight.

Denote by $\mathcal{G}$ the class of connected plane graphs with neither 4- and 5-cycles nor ext-triangular 7-cycles.

6.3 Proof of the main theorem

Suppose to the contrary that Theorem 6.8 is false. From now on, let G be a counterexample to Theorem 6.8 with fewest vertices. Thus, we may assume that the boundary D of the exterior face of G is a good cycle, and there exists a proper 3-coloring ϕ of $G[V(D)]$ which cannot be extended to a proper 3-coloring of G. By the minimality of G, we deduce that D has no chord.

6.3.1 Structural properties of the minimal counterexample G

Lemma 6.9. *Every internal vertex of G has degree at least 3.*

Proof. Suppose to the contrary that G has an internal vertex v with $d(v) \leq 2$. We can extend ϕ to $G - v$ by the minimality of G, and then to G by coloring v differently from its neighbors. □

Lemma 6.10. *G is 2-connected and therefore, the boundary of each face of G is a cycle.*

Proof. Otherwise, we can assume that G has a pendant block B with a cut vertex v such that $B-v$ does not intersect D. We first extend ϕ to $G-(B-v)$ by the minimality of G, and then 3-color B so that the color of v remains the same. □

Lemma 6.11. *G has no separating good cycle.*

Proof. Suppose to the contrary that G has a separating good cycle C. We extend ϕ to $G - Int(C)$ by the minimality of G. Furthermore, since C is a good cycle, again by the minimality of G, the coloring of C can be extended to its interior. □

By the definition of bad cycles, one can easily conclude the following lemma.

Lemma 6.12. *If C is a bad cycle of a graph in $\mathcal{G}$, then C has length either 9 or 11. Furthermore, if $|C| = 9$, then C has a (3,6,6)-claw or a (3,6,6,6)-triclaw; if $|C| = 11$, then C has a (3,6,8)-claw, or a (3,6,6,6)- or (6,3,6,6)-biclaw, or a (3,6,6,8)-triclaw, or a (3,6,6,6,6)-combclaw.*

Notice that all 3-, 6- and 8-cycles of G are facial. The following statement is a consequence of the previous lemma together with the fact that $G \in \mathcal{G}$.

Lemma 6.13. *G has neither bad cycles with a chord nor ext-triangular bad 9-cycles.*

Lemma 6.14. *Let P be a splitting path of D which divides D into two cycles D' and D''. The following four statements hold true.*

(1) *If $|P| = 2$, then at least one of D' and D'' is a triangle.*

(2) $|P| \neq 3$.

(3) *If $|P| = 4$, then at least one of D' and D'' is a 6- or 7-cycle.*

(4) *If $|P| = 5$, then at least one of D' and D'' is a 9^--cycle.*

Proof. Since D has length at most 11, we have $|D'| + |D''| = |D| + 2|P| \leq 11 + 2|P|$.

(1) Let $P = xyz$. Suppose to the contrary that $|D'|, |D''| \geq 6$. By Lemma 6.9, y has a neighbor other than x and z, say y'. It follows that y' is internal since otherwise D is a bad cycle with a claw. Without loss of generality, let y' lie inside D'. Now D' is a separating cycle. By Lemma 6.11, D' is not good, i.e., either D' is bad or $|D'| \geq 12$. Since every bad cycle has length either 9 or 11 by Lemma 6.12, we have $|D'| \geq 9$. Recall that $|D'| + |D''| \leq 15$, thus $|D'| = 9$ and $|D''| = 6$. Now D' has either a (3,6,6)-claw or a (3,6,6,6)-triclaw by Lemma 6.12, which implies that D has a biclaw or a combclaw respectively, a contradiction.

(2) Suppose to the contrary that $|P| = 3$. Let $P = wxyz$. Clearly $|D'|, |D''| \geq 6$. Let x' and y' be a neighbor of x and y not on P, respectively. If both x' and y' are external, then D has a biclaw. Hence, we may

assume that x' lies inside D'. By Lemmas 6.11 and 6.12 and the inequality $|D'|+|D''| \le 17$, we deduce that D' is a bad cycle and D'' is a good 8^--cycle. If y' is internal, then y' lies inside D'. It follows with the specific interior of a bad cycle that $x' = y'$ and D' has either a claw or a biclaw, which implies that D has either a triclaw or a combclaw respectively, a contradiction. Hence, y' is external. Since every bad cycle as well as every 6^-- or 8-cycle contains no chords by Lemma 6.13, we deduce that yy' is a (3,6)-chord of D''. It follows that D' is an ext-triangular bad 9-cycle, contradicting Lemma 6.13.

(3) Let $P = vwxyz$. Suppose to the contrary that $|D'|, |D''| \ge 8$. Since $|D'| + |D''| \le 19$, we have $|D'|, |D''| \le 11$. Since G has no 4- and 5-cycles, if G has an edge e connecting two nonconsecutive vertices on P, then the cycle formed by e and P has to be a triangle, yielding a splitting 3-path of D, contradicting the statement (2). Therefore, no pair of nonconsecutive vertices on P are adjacent.

Let w', x', y' be a neighbor of w, x, y not on P, respectively. The statement (2) implies that x' is internal. Without loss of generality, let x' lie inside D'. Thus D' is a bad 9- or 11-cycle. If D' is a bad 11-cycle, then D'' is a facial 8-cycle, and thus both w' and y' lie in $\overline{Int}(D')$, which is impossible by the interior of a bad cycle. Hence, D' is a bad 9-cycle. By the statement (1), if $w' \in V(D'')$, then G contains the triangle $[vww']$, which makes D' ext-triangular, a contradiction. Hence, $w' \notin V(D'')$. Moreover, since D' has no chords by Lemma 6.13, $w' \notin V(D')$. Therefore, w' is internal. If w' lies inside D', then it gives the interior of D' no other choices than $w' = x'$ and D' has a $(3, 6, 6)$-claw, in which case D has a splitting 3-path, a contradiction. Hence, w' lies inside D''. Similarly, y' lies inside D'' as well. Notice that $|D''| \in \{8, 9, 10\}$, thus D'' is a bad 9-cycle but has to contain both w' and y' inside, which is impossible.

(4) Let $P = uvwxyz$. Suppose to the contrary that $|D'|, |D''| \ge 10$. Since $|D'| + |D''| \le 21$, we have $|D'|, |D''| \le 11$. By a similar argument as in the proof of statement (3), one can conclude that G has no edges connecting two

nonconsecutive vertices on P. Let v', w', x', y' be a neighbor of v, w, x, y not on P, respectively.

The statement (2) implies that both w' and x' are internal. Without loss of generality, we may assume that w' lies inside D'. So D' is a separating cycle and thus, it must be a bad 11-cycle by Lemma 6.11. It follows that D'' is a good 10-cycle and thus, it is not a separating cycle. So x' also lies inside D'. By Lemma 6.12 and the fact that all 3-, 6- and 8-cycles are facial, we can deduce that $x' = w'$ and D' has either a (3,6,8)-claw or a (3,6,6,6)-biclaw. It follows that $v', y' \in V(D'')$. By the statement (1), G has the two triangles $[uvv']$ and $[yy'z]$, at least one of them is adjacent to a 7-cycle of $\overline{Int}(D')$, a contradiction. □

Lemma 6.15. *Let G' be a connected plane graph obtained from G by deleting a set of internal vertices and identifying two other vertices. If we*

(a) identify no two vertices of D, and create no edges connecting two vertices of D, and

(b) create neither 6^--cycles nor ext-triangular 7-cycles,

then ϕ can be extended to a proper 3-coloring of G'.

Proof. The item (a) guarantees that D is unchanged and bounds G', and ϕ is a proper 3-coloring of $G'[V(D)]$. By item (b), the graph G' is simple and $G' \in \mathcal{G}$. Hence, to extend ϕ to G' by the minimality of G, it remains to show that D is a good cycle of G'.

Suppose to the contrary that D is a bad cycle. Thus it has a bad partition H in G'. By the specific structure of H, as depicted in Lemma 6.12, we can see that there exists a 6-cell C' of H such that the intersection of D and C' is a path $v_1 \dots v_k$ of length $k-1$ with $k \in \{4, 5\}$. Since we create no 6-cycles, C' corresponds to a 6-cycle C of the original graph G. Notice that only one pair of vertices are identified and the resulting vertex is not from $\{v_2, \dots, v_{k-1}\}$ since otherwise G has a 4-cycle. It follows that the intersection of D and C is a path P of form $v_1 \dots v_k$ or $v_1 \dots v_{k-1}$ or $v_2 \dots v_k$. Thus, $|P| \in \{2, 3, 4\}$. If $|P| \in \{3, 4\}$, then $C - E(P)$ contains a splitting 3- or 2-path of D in G, yielding a contradiction by Lemma 6.14. Hence, $|P| = 2$ and so $k = 4$. By the existence of C', H is not a $(3, 6, 6)$- or $(3, 6, 8)$-claw. If H is a $(3, 6, 6, 6)$- or $(6, 3, 6, 6)$-biclaw or a $(3, 6, 6, 6, 6)$-combiclaw, then we can choose some other C' whose corresponding value of k is 5. By the same argument above, we obtain a contradiction. Hence, H must be a $(3, 6, 6, 6)$- or $(3, 6, 6, 8)$-triclaw. Now, H contains three splitting 3-paths of D in G', at least one of which does not contain the resulting vertex by identifying two vertices. Hence, this splitting 3-path of D exists also in G, contradicting Lemma 6.14. □

Lemma 6.16. *G has no edge uv incident with a 6-face and a 3-face such that both u and v are internal 3-vertices and therefore, every bad cycle of G has either a (3,6,6)- or (3,6,8)-claw or a (3,6,6,6)-biclaw.*

Proof. Suppose to the contrary that such an edge uv exists. Denote by $[uvwxyz]$ and $[uvt]$ the facial cycles of the 6-face and the 3-face, respectively. Lemma 6.14 implies that not both w and z are external vertices. Without loss of generality, we may assume that w is internal. Let G' be the graph obtained from G by deleting u and v and identifying w with y. Clearly, G' is a plane graph on fewer vertices than G. We will show that both items of Lemma 6.15 are satisfied.

Since w is internal, we identify no two vertices on D. If we create an edge connecting two vertices on D, then w has a neighbor w_1 not adjacent to y and both y and w_1 are external. But now, Lemma 6.14 implies that x is external and thus, $[ww_1x]$ is a triangle which makes the 7-cycle $[utvwxyz]$ ext-triangular, a contradiction. Hence, item (a) holds.

Suppose we create a 6^--cycle or an ext-triangular 7-cycle C'. Thus the original graph G has a 7^--path P between w and y corresponding to C'. First assume that $x \in V(P)$. The vertex x divides P into two paths, say P_{wx} and P_{xy}, between w and x and between x and y, respectively. Without loss of generality, assume that $|P_{wx}| \leq |P_{xy}|$. Since $|P_{wx}| + |P_{xy}| \leq 7$, we have $|P_{wx}| \leq 3$. If $|P_{wx}| = 3$, then G has a 4-cycle formed by P_{wx} and wx, a contradiction. If $|P_{wx}| = 2$, then P_{wx} together with wx forms a triangle that makes $[utvwxyz]$ an ext-triangular 7-cycle of G, a contradiction. If $|P_{wx}| = 1$, then P_{wx} coincides with the edge wx, which implies that C' coincides with the cycle C_{xy} of G formed by P_{xy} and xy, contradicting the supposition that C' is created.

Hence, we may assume that $x \notin V(P)$. The paths P and wxy form a 9^--cycle, say C. By Lemma 6.9, $d(x) \geq 3$. Let x_1 be the neighbor of x other than y and w. If $x_1 \in V(P)$, then xx_1 is a chord of C. By Lemma 6.13, C is a good cycle. It follows that xx_1 is a (3,6)- or (3,8)-chord of C, which makes the 7-cycle $[utvwxyz]$ ext-triangular, a contradiction. Hence $x_1 \notin V(P)$. Now, we can see that C is a separating cycle and thus, it is a bad 9-cycle. By Lemma 6.13, C is not ext-triangular. It follows that C' is a 7-cycle of G' and is not ext-triangular, contradicting our supposition. Therefore, item (b) holds.

By Lemma 6.15, the precoloring ϕ can be extended to G'. Since z and w receive different colors, we can properly color v and u, completing the extension of ϕ to G. □

We follow the notations of M-faces and MM-faces in [7], and define weak tetrads. An M-face is an 8-face f containing no external vertices with boundary $[v_1 \dots v_8]$ such that the vertices $v_1, v_2, v_3, v_5, v_6, v_7$ are of degree 3 and the edges $v_1v_2, v_3v_4, v_4v_5, v_6v_7$ are triangular. An MM-face is an 8-face f containing no external vertices with boundary $[v_1 \dots v_8]$ such that v_2 and v_7 are of degree 4 and other six vertices on f are of degree 3, and the edges $v_1v_2, v_2v_3, v_4v_5, v_6v_7, v_7v_8$ are triangular. A weak tetrad is a path $v_1 \dots v_5$ on the boundary of a face f such that both the edges v_1v_2 and v_3v_4 are triangular, all of v_1, v_2, v_3, v_4 are internal 3-vertices, and v_5 is either of degree 3 or f-light.

Lemma 6.17. *G has no weak tetrad and therefore, every face of G contains no five consecutive bad vertices.*

Proof. Suppose to the contrary that G has a weak tetrad T following the notation used in the definition. Denote by v_0 the neighbor of v_1 on f other than v_2. Denote by x the common neighbor of v_1 and v_2, and by y the common neighbor of v_3 and v_4. If $x = v_0$, then v_1 is an internal 2-vertex, contradicting

Lemma 6.9. Hence, $x \neq v_0$ and similarly, $x \neq v_3$. Since G has no 4- and 5-cycles, $x \notin \{v_4, v_5\}$. Concluding above, $x \notin v_0 \cup V(T)$. Similarly, $y \notin v_0 \cup V(T)$. Moreover, $x \neq y$ since otherwise $[v_1v_2v_3x]$ is a 4-cycle. We delete $v_1, \ldots, v_4$ and identify v_0 with y, obtaining a plane graph G' on fewer vertices than G. We will show that both the items of Lemma 6.15 are satisfied.

Suppose that we create a 6^--cycle or an ext-triangular 7-cycle C'. Thus G has a 7^--path P between v_0 and y corresponding to C'. So, the paths P and $v_0v_1v_2v_3y$ form an 11^--cycle, say C. If $x \in V(P)$, then the cycle formed by P and v_0v_1x has length at least 6 and the cycle formed by P and xv_2v_3y has length at least 8, which gives $|P| \geq 9$, a contradiction. Hence, $x \notin V(P)$. Now, one of x and v_4 lies inside C and the other lies outside C. So C is a separating cycle and thus, it is a bad cycle. By Lemma 6.16, C has either a (3,6,6)- or (3,6,8)-claw or a (3,6,6,6)-biclaw. Notice that both the two faces incident with v_2v_3 have length at least 8. Thus, C has a bad partition owning an 8-cell no matter which one of x and v_4 lies inside C. It follows that C has a (3,6,8)-claw. If x lies inside C, then v_1x is incident with the 6-cell and a 3-face with $d(v_1) = d(x) = 3$, contradicting Lemma 6.16. Hence, v_4 lies inside C. In this case, f is the 8-cell, and the 6-cell contains the path yv_4v_5. Thus, we can deduce that v_5 is not f-light. Since T is a weak tetrad, $d(v_5) = 3$. We delete v_5 together with other vertices of T and repeat the argument above, obtaining a contradiction. Therefore, item (b) holds.

Suppose that we identify two vertices on D or create an edge connecting two vertices on D. Thus there is a splitting 4- or 5-path Q of D containing the path $v_0v_1v_2v_3y$. By Lemma 6.14, Q together with D forms a 9^--cycle which corresponds to a 5^--cycle in G'. Since we create no 6^--cycle, a contradiction follows. Hence, item (a) holds.

By Lemma 6.15, the precoloring ϕ can be extended to G'. We first properly color v_5 (if needed), v_4 and v_3 in turn. Since v_0 and v_3 receive different colors, we can properly color v_1 and v_2, completing the extension of ϕ to G. □

Lemma 6.18. *G has no M-faces.*

Proof. Suppose to the contrary that G has an M-face f following the notation used in the definition. For $(i,j) \in \{(1,2),(3,4),(4,5),(6,7)\}$, denote by t_{ij} the common neighbor of v_i and v_j. By similar argument as in the proof of the previous lemma, we deduce that the vertices $t_{12}, t_{34}, t_{45}, t_{67}$ are pairwise distinct and not incident with f. We delete $v_1, v_2, v_3, v_5, v_6, v_7$ and identify v_4 with v_8, obtaining a plane graph G' on fewer vertices than G. We will show that both items in Lemma 6.15 are satisfied.

Suppose that we create a 6^--cycle or an ext-triangular 7-cycle C'. Thus G has a 7^--path P between v_4 and v_8 corresponding to C'. By the symmetry of an M-face, we may assume that P together with the path $v_4v_5 \dots v_8$ forms an 11^--cycle C that contains v_1, v_2, v_3 inside. So C is a bad cycle containing at least three vertices inside, which is impossible by the interior structure of C indicated by Lemma 6.16. Therefore, item (b) holds.

The satisfaction of item (a) can be proved in a similar way as in the proof of the previous lemma. By Lemma 6.15, the pre-coloring ϕ can be extended to G'. We can properly color first v_3 and then v_1 and v_2 since v_3 is colored different from v_8. Similarly, we can properly color v_5, v_6 and v_7, completing the extension of ϕ to G. □

Lemma 6.19. *G has no MM-faces.*

Proof. Suppose to the contrary that G has an MM-face f following the notation used in the definition. For $(i,j) \in \{(1,2),(2,3),(4,5),(6,7),(7,8)\}$, denote by t_{ij} the common neighbor of v_i and v_j. Similarly to the proof of Lemma 6.18, we deduce that the vertices $t_{12}, t_{23}, t_{45}, t_{67}, t_{78}$ are pairwise distinct and not incident with f. We delete all the vertices of f and identify t_{12} with t_{67}, obtaining a plane graph G' on fewer vertices than G. To extend ϕ to G', it suffices to fulfill item (a) of Lemma 6.15, as we did in the previous lemma.

Suppose that we create a 6^--cycle or an ext-triangular 7-cycle C'. Thus G has a 7^--path P between t_{12} and t_{67} corresponding to C'. If $t_{78} \in V(P)$, then both the cycles formed by P and $t_{12}v_1v_8t_{78}$ and by P and $t_{78}v_7t_{67}$ have length at least 8, which gives $|P| \geq 11$, a contradiction. Hence, $t_{78} \notin V(P)$. The paths P and $t_{12}v_1v_8v_7t_{67}$ form an 11^--cycle, say C. It follows that C is a bad cycle containing either $v_2, \ldots, v_6$ or t_{78} inside. In the former case, C contains at least five vertices inside, a contradiction. In the latter case, G has either an ext-triangular bad 9-cycle (if $|P| = 5$) or an ext-triangular 7-cycle (if $|P| = 7$), a contradiction.

We further extend ϕ from G' to G as follows. Let α, β and γ be the three colors used in ϕ. First, disregarding the edge v_1v_8, we can properly color v_2, v_1, v_3 and v_7, v_8, v_6. If v_1 and v_8 receive different colors and so do v_3 and v_6, then v_4 and v_5 can be properly colored; we are done. Hence, we may assume that v_1 and v_8 receive the same color β (a similar argument as below works for the case that v_3 and v_6 receive the same color). Let α be the color assigned to t_{12} and t_{67}. Thus v_2 and v_7 are colored with γ and t_{78} is colored with α. We recolor v_8, v_7 and v_6 with γ, β and γ, respectively. Now, v_1 and v_8 receive different colors and so do v_3 and v_6. Again, v_4 and v_5 can be properly colored; we are done. □

6.3.2 Discharging in G

Let $V = V(G)$, $E = E(G)$, and F be the set of faces of G. Denote by f_0 the exterior face of G. Give initial charge $ch(x)$ to each element x of $V \cup F$, where $ch(f_0) = d(f_0) + 4$, $ch(v) = d(v) - 4$ for $v \in V$, and $ch(f) = d(f) - 4$ for $f \in F \setminus \{f_0\}$. Discharge the elements of $V \cup F$ according to the following rules:

$R1$. Every internal 3-face receives $\frac{1}{3}$ from each incident vertex.

$R2$. Every internal 6^+-face sends $\frac{2}{3}$ to each incident 2-vertex.

$R3$. Every internal 6^+-face sends each incident internal 3-vertex v charge $\frac{2}{3}$ if v is triangular, and charge $\frac{1}{3}$ otherwise.

$R4$. Every internal 6^+-face f sends $\frac{1}{3}$ to each f-light vertex, and receives $\frac{1}{3}$ from each f-heavy vertex.

$R5$. Every internal 6^+-face receives $\frac{1}{3}$ from each incident external 4^+-vertex.

$R6$. The exterior face f_0 sends $\frac{4}{3}$ to each incident vertex.

We remark that the discharging rules can be tracked back to the one used in [7].

Let $ch^*(x)$ denote the final charge of each element x of $V \cup F$ after discharging. On one hand, by Euler's formula we deduce $\sum_{x \in V \cup F} ch(x) = 0$. Since the sum of charges over all elements of $V \cup F$ is unchanged, it follows that $\sum_{x \in V \cup F} ch^*(x) = 0$. On the other hand, we will show that $ch^*(x) \geq 0$ for each $x \in V \cup F \setminus \{f_0\}$ and $ch^*(f_0) > 0$. Hence, this obvious contradiction completes the proof of Theorem 6.8. It remains to show that $ch^*(x) \geq 0$ for each $x \in V \cup F \setminus \{f_0\}$ and $ch^*(f_0) > 0$.

Lemma 6.20. *$ch^*(v) \geq 0$ for $v \in V$.*

Proof. First suppose that v is external. Since D is a cycle, $d(v) \geq 2$. If $d(v) = 2$, then since D has no chord, the internal face incident with v is not a triangle and sends $\frac{2}{3}$ to v by $R2$. Moreover, v receives $\frac{4}{3}$ from f_0 by $R6$, which gives $ch^*(v) = d(v) - 4 + \frac{2}{3} + \frac{4}{3} = 0$. If $d(v) = 3$, then v sends charge to at most one 3-face by $R1$ and thus $ch^*(v) \geq d(v) - 4 - \frac{1}{3} + \frac{4}{3} = 0$. If $d(v) \geq 4$, then v sends at most $\frac{1}{3}$ to each incident internal face by $R1$ and $R5$, yielding $ch^*(v) \geq d(v) - 4 - \frac{1}{3}(d(v) - 1) + \frac{4}{3} > 0$. Hence, we are done in any case.

It remains to suppose that v is internal. By Lemma 6.9, $d(v) \geq 3$. If $d(v) = 3$, then we have $ch^*(v) = d(v) - 4 - \frac{1}{3} + \frac{2}{3} \times 2 = 0$ by $R1$ and $R3$ when v is triangular, and $ch^*(v) = d(v) - 4 + \frac{1}{3} \times 3 = 0$ by $R3$ when v not. If $d(v) = 4$, then v is incident with k 3-faces with $k \leq 2$. By $R1$ and $R4$, we have $ch^*(v) = d(v) - 4 - \frac{1}{3} \times 2 + \frac{1}{3} \times 2 = 0$ when $k = 2$, $ch^*(v) = d(v) - 4 - \frac{1}{3} + \frac{1}{3} = 0$ when $k = 1$, and $ch^*(v) = d(v) - 4 = 0$ when $k = 0$. If $d(v) = 5$, then v sends charge to at most two 3-faces by $R1$ and to at most one 6^+-face by $R4$, which gives $ch^*(v) \geq d(v) - 4 - \frac{1}{3} \times 2 - \frac{1}{3} = 0$. Hence, we may next assume that $d(v) \geq 6$. Since v sends at most $\frac{1}{3}$ to each incident face by our rules, we get $ch^*(v) \geq d(v) - 4 - \frac{1}{3}d(v) \geq 0$. □

Lemma 6.21. *$ch^*(f_0) > 0$.*

Proof. Recall that $ch(f_0) = d(f_0) + 4$ and $d(f_0) \leq 11$. We have $ch^*(f_0) \geq d(f_0) + 4 - \frac{4}{3}d(f_0) > 0$ by $R6$. □

Lemma 6.22. *$ch^*(f) \geq 0$ for $f \in F \setminus \{f_0\}$.*

Proof. We distinguish cases according to the size of f. Since G has no 4- and 5-cycle, $d(f) \notin \{4, 5\}$.

If $d(f) = 3$, then f receives $\frac{1}{3}$ from each incident vertex by $R1$, which gives $ch^*(f) = d(f) - 4 + \frac{1}{3} \times 3 = 0$.

Let $d(f) = 6$. For any incident vertex v, by the rules, f sends to v charge $\frac{2}{3}$ if v is either of degree 2 or bad, and charge at most $\frac{1}{3}$ otherwise. Since G has no ext-triangular 7-cycles, f is adjacent to at most one 3-face. Furthermore, by Lemma 6.16, f contains at most one bad vertex. If f contains a 2-vertex, say u, we can deduce with Lemma 6.14 that u is the unique 2-vertex of f and the two neighbors of u on f are external 3^+-vertices which receive nothing from f. It follows that $ch^*(f) \geq d(f) - 4 - \frac{2}{3} - \frac{2}{3} - \frac{1}{3} \times 2 = 0$. Hence, we may assume that f contains no 2-vertices. If f has no bad vertices, then f sends each incident vertex at most $\frac{1}{3}$, which gives $ch^*(f) \geq d(f) - 4 - \frac{1}{3}d(f) = 0$. Hence, we may let x be a bad vertex of f. Denote by y the other common vertex between f and the triangle adjacent to f. By Lemma 6.16 again, y is not a bad vertex, i.e., y is either an internal 4^+-vertex or an external 3^+-vertex. By our rules, f sends nothing to y, yielding $ch^*(f) \geq d(f) - 4 - \frac{2}{3} - \frac{1}{3} \times 4 = 0$.

Let $d(f) = 7$. Since G has no ext-triangular 7-cycles, f contains no bad vertices. Moreover, by Lemma 6.14, we deduce that f has at most two 2-vertices. Thus, $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 2 - \frac{1}{3} \times 5 = 0$.

Let $d(f) \geq 8$. If f contains precisely one external vertex, say w, then $d(w) \geq 4$ and so f receives $\frac{1}{3}$ from w by $R5$. Furthermore, since f contains no weak tetrad by Lemma 6.17, f has a good vertex other than w and sends at most $\frac{1}{3}$ to it. Thus, $ch^*(f) \geq d(f) - 4 + \frac{1}{3} - \frac{1}{3} - \frac{2}{3}(d(f) - 2) \geq 0$; we are done. If f contains at least two external vertices, then at least two of them are of degree more than 2. Since f sends nothing to external 3^+-vertices, it follows that $ch^*(f) \geq d(f) - 4 - \frac{2}{3}(d(f) - 2) \geq 0$; we are done as well. Hence, we may assume that all the vertices of f are internal. We distinguish two cases.

Case 1: Assume that $d(f) = 8$. Denote by r the number of bad vertices of f. We have $ch^*(f) \geq d(f) - 4 - \frac{2}{3}r - \frac{1}{3}(d(f) - r) = \frac{4-r}{3} \geq 0$, provided by $r \leq 4$. Since f contains no weak tetrad, $r \leq 6$. Hence, we may assume that $r \in \{5, 6\}$. For $r = 5$, we claim that f has a vertex failing to take charge from f, which gives $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 5 - \frac{1}{3} \times 2 = 0$. Suppose to the contrary that no such vertex exists. Thus, the bad vertices of f can be paired so that any good vertex of the path of f between each pair is f-Mlight, contradicting the parity of r. For $r = 6$, since again f contains no five consecutive bad vertices, these six bad vertices of f are divided by the two good ones into cyclically either 3+3 or 2+4. We may assume that f has a good vertex that is either f-light or of degree 3, since otherwise we are done with $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 6 = 0$. Denote by u such a good vertex and by v the other one. By the drawing of u and of the 3-faces adjacent to f, we deduce that, for the case 3+3, f is an M-face, contradicting Lemma 6.18, and for the case 2+4, if u is f-Mlight then either f is an MM-face or v is f-heavy; otherwise f contains a weak tetrad. It follows with Lemmas 6.19 and 6.17 that v is f-heavy, which is the only possible case. Hence, f receives $\frac{1}{3}$ from v by $R4$, yielding $ch^*(f) \geq ch(f) - 4 - \frac{2}{3} \times 6 + \frac{1}{3} - \frac{1}{3} = 0$.

Case 2: Assume that $d(f) \geq 9$. By Lemma 6.17, we deduce that f contains at least two good vertices, each of them receives at most $\frac{1}{3}$ from f. Thus, $ch^*(f) \geq d(f) - 4 - \frac{2}{3}(d(f) - 2) - \frac{1}{3} \times 2 = \frac{d(f)-10}{3} \geq 0$, provided $d(f) \geq 10$. It remains to suppose $d(f) = 9$. If f has at most six bad vertices, then $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$. Hence, we may assume that f has precisely seven bad vertices. By the same argument as for the case $d(f) = 8$ and f has five bad vertices above, f has a vertex failing to take charge from f, which gives $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 7 - \frac{1}{3} = 0$. □

By the previous three lemmas, the proof of Theorem 6.8 is completed.

Chapter 7

Conclusion and future work

This thesis concentrates on vertex coloring of signed graphs. The circular coloring of graphs has been extensively studied in the literature. The thesis generalizes the concept of circular coloring of graphs to signed graphs. This is a very natural generalization, which we can see from the following points of view:

(1) The circular chromatic number of unsigned graphs has two basic definitions that are equivalent. One is defined by (k, d)-colorings and the other by circular r-colorings. We show that the circular chromatic number of signed graphs can also be equivalently defined by (k, d)-colorings and circular r-colorings, that is,

$$\chi_c((G,\sigma)) = \inf\{\frac{k}{d} : \ (G,\sigma) \text{ has a } (k,d)\text{-coloring}\}$$

The infimum in the definition is always attained, and hence can be replaced by the minimum, the same as the unsigned case.

(2) The circular coloring of signed graphs implies a new notion of chromatic number of signed graphs, for which we prove an analogue of the famous Brooks' theorem and Hajós' theorem.

(3) By the relation $\chi((G,\sigma)) - 1 \leq \chi_c((G,\sigma)) \leq \chi((G,\sigma))$ for every signed graph (G,σ), we can still take $\chi_c((G,\sigma))$ as a refinement of $\chi((G,\sigma))$ and inversely $\chi((G,\sigma))$ as an approximate of $\chi_c((G,\sigma))$.

Kang and Steffen propose a signed version of the four color theorem as a conjecture.

Conjecture 7.1. *Every signed planar graph is 4-colorable.*

This conjecture is slightly different from another version [35] that every signed planar graph is signed 4-colorable.

Besides the questions proposed before in the thesis, we can also consider the followings as the future work:

(1) Analogues of the Hajós' theorem for the circular chromatic number of graph were proved in [67, 70]. So as the next step, it is natural but challenging to extend the Hajós' theorem for the circular chromatic number of signed graphs.

(2) Different from the unsigned case that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, the lower bound can be attained for some signed graphs with any given χ. So it is of particular interest to give a characterization of signed graphs with $\chi((G,\sigma)) - 1 = \chi_c((G,\sigma))$. Moreover, it was proved by Guichard that it is NP-hard to decide whether a graph G attains the upper bound, i.e., satisfies $\chi_c(G) = \chi(G)$. There are some known sufficient conditions under which $\chi_c(G) = \chi(G)$. However, these conditions may not work for signed graphs. Hence, the question is to figure out sufficient conditions under which $\chi_c((G,\sigma)) = \chi((G,\sigma))$ and in particular, those conditions on the structure of G under which for any signature σ of G, $\chi_c((G,\sigma)) = \chi((G,\sigma))$.

(3) We introduce list coloring of signed graphs and then focus on the choosability of signed planar graphs. Hence there is still few knowledge on the choosability of non-planar signed graphs.

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