

**INFINITE-DIMENSIONAL
SUPERMANIFOLDS, LIE SUPERGROUPS
AND THE SUPERGROUP OF
SUPERDIFFEOMORPHISMS**

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- *This thesis is typeset with \LaTeX .*

*Für meinen Opa,
der die Mathematik
wie kaum ein anderer
zu schätzen weiß.*

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Abstract

In this thesis, we provide an accessible introduction to the theory of locally convex supermanifolds in the categorical approach with a focus on Lie supergroups and the supergroup of superdiffeomorphisms. In this setting, a supermanifold is a functor $\mathcal{M}: \mathbf{Gr} \rightarrow \mathbf{Man}$ from the category of Grassmann algebras to the category of locally convex manifolds that has certain local models, forming something akin to an atlas. We give a mostly self-contained, concrete definition of supermanifolds along these lines, closing several gaps in the literature on the way. If $\Lambda_n \in \mathbf{Gr}$ is the Grassmann algebra with n generators, we show that \mathcal{M}_{Λ_n} has the structure of a so called multilinear bundle over the base manifold $\mathcal{M}_{\mathbb{R}}$. We use this fact to show that the projective limit $\varprojlim_n \mathcal{M}_{\Lambda_n}$ exists in the category of manifolds. In fact, this gives us a faithful functor $\varprojlim: \mathbf{SMan} \rightarrow \mathbf{Man}$ from the category of supermanifolds to the category of manifolds. This functor respects products, commutes with the respective tangent functor and retains the respective Hausdorff property. In this way, supermanifolds can be seen as a particular kind of infinite-dimensional fiber bundles.

For Lie supergroups, we use similar techniques to show several useful trivializations. For a Lie supergroup \mathcal{G} , it holds $\mathcal{G}_{\Lambda_n} \cong \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) \rtimes \mathcal{G}_{\mathbb{R}}$ as Lie groups, where $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ is a so called polynomial group. Moreover, we construct a canonical decomposition of \mathcal{G} into a purely even and a purely odd part. Using this, we are able to generalize the classical equivalence between Lie supergroups and super Harish-Chandra pairs to the case of arbitrary locally convex Lie supergroups.

The supergroup of superdiffeomorphisms of \mathcal{M} is a certain functor $\mathrm{SDiff}(\mathcal{M}): \mathbf{Gr} \rightarrow \mathbf{Set}$ that captures even and odd aspects of supersmooth transformations of \mathcal{M} . As a tool for our study of superdiffeomorphisms, we introduce spaces of sections of super vector bundles, and in particular super vector fields, turning them into suitable locally convex spaces. We show that $\mathrm{SDiff}(\mathcal{M})$ has essentially the same decompositions as a Lie supergroup for an arbitrary supermanifold \mathcal{M} and we discuss the respective components in detail. If \mathcal{M} is a Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional and σ -compact, we are able to turn the supergroup $\mathrm{SDiff}_c(\mathcal{M})$ of superdiffeomorphisms with compact support into a Lie supergroup.

German translation: In dieser Arbeit stellen wir eine zugängliche Einführung in die Theorie lokalkonvexer Supermannigfaltigkeiten im Rahmen des kategoriellen Ansatzes vor. Hierbei wird ein besonderer Schwerpunkt auf Lie-Supergruppen und die Supergruppe der Superdiffeomorphismen gelegt. In diesem Zugang ist eine Supermannigfaltigkeit ein Funktor $\mathcal{M}: \mathbf{Gr} \rightarrow \mathbf{Man}$ von der Kategorie der Grassmann-Algebren in die Kategorie der lokalkonvexen Mannigfaltigkeiten, der bestimmte lokale Modelle besitzt, die etwas wie einen Atlas bilden. Wir geben eine, im wesentlichen in sich geschlossene, konkrete Definition von Supermannigfaltigkeiten, wobei wir einige Lücken in der Literatur schließen. Wir zeigen, dass \mathcal{M}_{Λ_n} ein sogenanntes multilineares Bündel über der Basis $\mathcal{M}_{\mathbb{R}}$ ist, wenn $\Lambda_n \in \mathbf{Gr}$ die Grassmann-Algebra mit n Erzeugern ist. Wir nutzen dies aus um zu zeigen, dass der projektive Limes $\varprojlim_n \mathcal{M}_{\Lambda_n}$ in der Kategorie der Mannigfaltigkeiten existiert. Dies liefert uns einen treuen Funktor $\varprojlim: \mathbf{SMan} \rightarrow \mathbf{Man}$ von der Kategorie der Supermannigfaltigkeiten in die Kategorie der Mannigfaltigkeiten. Dieser Funktor erhält Produkte, vertauscht mit dem jeweiligen Tangentialfunktor und erhält die jeweilige Hausdorff Eigenschaft. Auf diese Weise können wir Supermannigfaltigkeiten als eine besondere Art von unendlich-dimensionalen Faserbündeln betrachten.

Mittels ähnlicher Techniken erhalten wir einige nützliche Trivialisierungen von Lie-Supergruppen. Für jede Lie-Supergruppe \mathcal{G} gilt $\mathcal{G}_{\Lambda_n} \cong \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) \rtimes \mathcal{G}_{\mathbb{R}}$ als Lie Gruppe. Hierbei ist $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ eine sogenannte polynomielle Gruppe. Darüber hinaus konstruieren wir eine kanonische Zerlegung von \mathcal{G} in einen rein geraden und einen rein ungeraden Teil. Dies erlaubt uns die klassische Äquivalenz zwischen Lie-Supergruppen und Super-Harish-Chandra-Paaren auf den Fall lokalkonvexer Lie-Supergruppen zu verallgemeinern.

Die Supergruppe der Superdiffeomorphismen von \mathcal{M} ist ein bestimmter Funktor $\mathrm{SDiff}(\mathcal{M}): \mathbf{Gr} \rightarrow \mathbf{Man}$, der gewisse Aspekte gerader und ungerader Transformationen von \mathcal{M} beschreibt. Als Hilfsmittel zur Untersuchung von Superdiffeomorphismen führen wir Räume von Schnitten von Supervektorbündeln, und insbesondere Supervektorfelder, ein und geben ihnen die Struktur geeigneter lokalkonvexer Räume. Wir zeigen, dass $\mathrm{SDiff}(\mathcal{M})$ sich im Wesentlichen genau wie eine Lie-Supergruppe zerlegen lässt und untersuchen die Bestandteile im Detail. Falls \mathcal{M} eine Banach-Supermannigfaltigkeit mit σ -kompakter, endlich-dimensionaler Basis $\mathcal{M}_{\mathbb{R}}$ ist, gelingt es uns, der Supergruppe $\mathrm{SDiff}_c(\mathcal{M})$ der kompakt getragenen Superdiffeomorphismen die Struktur einer Lie-Supergruppe zu geben.

Introduction

In this thesis, we aim to provide an accessible introduction to the theory of infinite-dimensional supermanifolds as defined by Molotkov in [38]. Beyond generalizing his results and closing gaps in the literature, we attempt to lay the foundations of a general structure theory for locally convex supermanifolds by discussing their inherent bundle structure. Applied to Lie supergroups, this enables us to understand an arbitrary locally convex Lie supergroup in terms of its Lie superalgebra and the action of its base Lie group thereon (i.e. in terms of its super Harish-Chandra pair). Similar techniques let us describe the supergroup of superdiffeomorphisms of a, not necessarily finite-dimensional, supermanifold in some detail, culminating in the construction of a Lie supergroup structure for the superdiffeomorphisms of an appropriate class of supermanifolds.

Supermanifolds were developed in the early 1970's to provide a framework for a geometry combining commuting and anticommuting coordinates, with the original motivation coming from particle physics. There have been various, not all equivalent, approaches to achieve this. The first, and most commonly used, rigorous definition of a supermanifold is as a ringed space due to Berezin and Leites [9] (see also [8]). The basic idea is to enlarge the structure sheaf of a manifold to a sheaf of superrings that is locally isomorphic to the sheaf of functions with values in an exterior algebra. In the case of real supermanifolds, an equivalent approach using Hopf algebras was proposed shortly thereafter by Kostant in [32]. In an attempt to make the language of supermanifolds more accessible to physicists, DeWitt [15] and Rogers [43] introduced a definition of supermanifolds mirroring the one of ordinary manifolds, which we will call the concrete approach. Simply put, from this point of view supermanifolds are modelled locally on a certain exterior algebra such that the transition functions satisfy suitable conditions. More recent works in this regard include [53] and [44]. A comparison between the sheaf theoretic and the concrete approach can be found in [4].

In many situations, infinite-dimensional objects arise naturally that one would like to endow with an appropriate “super” structure. For example, it is well-known that the even and odd vector fields of a finite-dimensional compact supermanifold form a Fréchet super vector space. Other examples include mapping spaces between supermanifolds or supergroups of gauge transformations. However, all the approaches mentioned are restricted to the finite-dimensional case. In fact, not even all infinite-dimensional ordinary manifolds can be described by their sheaf of functions (see [57]) and for supermanifolds additional obstacles appear (compare [1, p.587]). These problems notwithstanding, in the case of analytic supermanifolds, Schmitt [48] was able to extend the sheaf theoretic approach to include infinite-dimensional supermanifolds. More generally, following Kostant's approach via Hopf algebras and an idea of Batchelor from [7], a definition of \mathbb{R} -Fréchet su-

permanifolds using coalgebras was devised in [30]. With regard to the concrete approach, possible topological problems which limit its applicability to infinite-dimensions were suggested in [1].

The first rigorous definition of infinite-dimensional supermanifolds, and also the one we will use in this work, is the categorical approach suggested by Molotkov in [39].¹ In this approach supermanifolds are defined to be functors from the category of finitely generated Grassmann algebras \mathbf{Gr} to the category of manifolds \mathbf{Man} with additional local information contained in an ‘atlas’ consisting of certain natural transformations. Let us briefly relate this to the sheaf theoretic approach. In the latter, the functor of points (i.e., the Yoneda embedding) has long been known to be a useful tool (see for example [34]). Moreover, to fully understand the functor of points, it suffices to consider supermanifolds whose base manifold is a single point, the so called *superpoints*. The superpoints are parametrized by the Grassmann algebras and for every superpoint \mathcal{P} the set of morphisms $\mathrm{Hom}_{\mathbf{sMan}}(\mathcal{P}, \mathcal{M})$ to a given supermanifold \mathcal{M} can be turned into a smooth manifold. In this way one obtains a functor $\mathbf{Gr} \rightarrow \mathbf{Man}$. Shvarts [52] and Voronov [54] had the idea to use such functors to *define* finite-dimensional supermanifolds and Molotkov extended this definition to infinite-dimensional supermanifolds.² We call this the categorical approach.

Because of its close relation to the functor of points, some of the intuition from the finite-dimensional situation carries over to the infinite-dimensional setting. For example, the definition of an internal Hom and the related superdiffeomorphisms are obtained quite easily in this way (see [40, 8.2, p.415 and 8.4, p.417]). Using this, Hanisch [27] was able to endow the inner Hom object of two finite-dimensional supermanifolds with a supermanifold structure in Molotkov’s framework. Another nice feature of the categorical approach is that the definition of finite-dimensional and infinite-dimensional supermanifolds, along with their morphisms and their tangent bundles, is exactly the same. No special topological considerations are necessary. Similarly, as has been shown in [1], it lends itself to easy generalization beyond the real or complex case. What is more, many constructions and calculations can essentially be done pointwise, i.e., for every Grassmann algebra. This means that for finite-dimensional supermanifolds one often only has to deal with finite-dimensional ordinary manifolds.

Despite these advantages, the categorical approach has rarely been used and even where it appears, it is usually only applied half-heartedly. For instance, when superspaces of morphisms between supermanifolds are considered, the morphisms are usually expressed in the sheaf theoretic language (see for example [47], [27] and [12]). The reason for this lack of interest appears to be twofold. For one, the categorical language of natural transformations, Grothendieck topologies, sheaves in categories and so on is rather abstract and not part of the usual toolbox employed in the field of analysis. This is then exacerbated by the fact that Molotkov’s foundational article [39] (resp. [40]) contains almost no proofs. While some proofs

¹Throughout this work, we will cite the more readily available and slightly updated article [40].

² Most statements in [40] are made for Banach supermanifolds but many can be easily transferred to Fréchet or locally convex supermanifolds (compare [40, 8.5, p.418]).

for Molotkov’s statements were subsequently offered by Sachse in [45] and [46], he often falls back to the sheaf theoretic approach so that the statements are not shown in their original generality and one obtains little intuition for the categorical approach.

We attempt to remedy both of these problems in this thesis. On the one hand, we give a complete definition of infinite-dimensional supermanifolds and their morphisms, proving all statements that we use (with the rare exception where the proof in the literature can directly be applied to our situation and is relatively straightforward). On the other hand, we simplify the categorical language as much as possible. As it turns out, one can develop the categorical approach in fairly concrete terms closely resembling the definition of ordinary manifolds. In this way, we completely avoid dealing with more involved questions like representability.

Remarkably, this concrete point of view leads to a canonical faithful functor from the category of supermanifolds to the category of manifolds. This functor has good properties such as respecting products (i.e., mapping Lie supergroups to Lie groups), commuting with the respective tangent functor and retaining the respective Hausdorff property. It can be turned into an equivalence of categories if one considers a specific type of fiber bundles on the right-hand side. In other words, *we may consider supermanifolds as ordinary manifolds with a particular kind of atlas in a canonical, well-behaved way.* All non-trivial supermanifolds are at best mapped to Fréchet manifolds and one wonders whether techniques of infinite-dimensional analysis could prove useful in finite-dimensional superanalysis.

To streamline our work, we only consider supermanifolds over the base field \mathbb{R} . However, many of our constructions derive from [1] and [10], where much more general fields and even rings are considered. We have consciously formulated our proofs in such a way that they can easily be generalized where possible. The only noteworthy obstacles to such generalizations are Batchelor’s Theorem (which necessitates a partition of unity) and combinatorial formulas which do not allow for base rings with positive characteristic. For the latter, we indicate ways around the problem.

Many standard constructions for supermanifolds and Lie supergroups are beyond the scope of this thesis, but we hope to have provided the reader with the tools to rectify this with relative ease. While equivalences between certain categories of supermanifolds in the sheaf theoretic, the concrete and the categorical approach have been discussed in some detail in [1], it is not immediately obvious how objects like vector fields can be translated between the different points of view. More work to this effect will be critical to enable one to pick and choose effectively which approach is most suitable for the problem at hand. One final drawback of our work that should not go unmentioned is that in trying to be as concrete as possible, we lose some of the intuition offered by the functor of points approach. Thus, a close reading of [40] is still advisable.

This thesis is organized in three parts. In Chapter 1, we summarize several key concepts that are standard in their respective fields but are possibly not generally well-known. This includes the locally convex differential calculus, functor categories, algebraic structures in categories and some multilinear super algebra.

Chapter 2 discusses supermanifolds and super vector bundles. Lie supergroups are dealt with in Chapter 3. Finally, in Chapter 4, we deal with the supergroup of superdiffeomorphisms. To this end, we also introduce and topologize spaces of sections of super vector bundles and discuss the group of automorphisms of a supermanifold.

Supermanifolds

A Grassmann algebra is a free associative \mathbb{R} -algebra $\Lambda_n := \mathbb{R}[\lambda_1, \dots, \lambda_n]$, where the generators satisfy the relation $\lambda_i \lambda_j = -\lambda_j \lambda_i$. There exists a natural grading $\Lambda_n = \Lambda_{n,0} \oplus \Lambda_{n,1}$ and the set of objects $\{\mathbb{R}, \Lambda_1, \Lambda_2, \dots\}$ together with the graded morphisms form the category **Gr** of Grassmann algebras. Generators of Grassmann algebras behave infinitesimally in the sense that $\lambda_i^2 = 0$ and we will see that for this reason (together with functoriality) the structure of supermanifolds has many similarities to the structure of higher tangent bundles. This enables us to make heavy use of the techniques developed by Bertram in [10] for dealing with higher tangent bundles, higher tangent groups and higher order diffeomorphism groups.

As mentioned, we want to define supermanifolds as functors from the category of Grassmann algebras to the category of manifolds with certain local models. In analogy to ordinary manifolds, we begin by describing the differential calculus on the model space:

1. Instead of a vector space, the model space of a supermanifold is a functor of the form

$$\overline{E}: \mathbf{Gr} \rightarrow \mathbf{Top}, \quad \Lambda \mapsto \overline{E}_\Lambda := (E_0 \otimes \Lambda_{\overline{0}}) \oplus (E_1 \otimes \Lambda_{\overline{1}}),$$

where $E = E_0 \oplus E_1$ is a \mathbb{Z}_2 -graded Hausdorff locally convex vector space and \overline{E}_Λ is given the obvious product topology. Then \overline{E}_Λ is a $\Lambda_{\overline{0}}$ -module and the functor \overline{E} has the structure of a so called $\overline{\mathbb{R}}$ -module in the category $\mathbf{Top}^{\mathbf{Gr}}$.

2. Open subsets of the model space correspond to open subfunctors, i.e. functors

$$\mathcal{U}: \mathbf{Gr} \rightarrow \mathbf{Top}$$

such that $\mathcal{U}_\Lambda \subseteq \overline{E}_\Lambda$ is open for all $\Lambda \in \mathbf{Gr}$ and the inclusion is a natural transformation. We call such functors *super domains*. One can show that superdomains have the form

$$\mathcal{U}_\Lambda = \mathcal{U}_{\mathbb{R}} \times (E_0 \otimes \Lambda_{\overline{0}}^+) \times (E_1 \otimes \Lambda_{\overline{1}}),$$

where $\Lambda_{\overline{0}}^+$ is the nilpotent part of $\Lambda_{\overline{0}}$.

3. Smooth functions correspond to supersmooth morphisms, i.e. natural transformations

$$f: \mathcal{U} \rightarrow \overline{F}$$

such that f_Λ is smooth for all $\Lambda \in \mathbf{Gr}$ and

$$df_\Lambda: \mathcal{U}_\Lambda \times \overline{E}_\Lambda \rightarrow \overline{F}_\Lambda$$

is $\Lambda_{\overline{0}}$ -linear in the second component.

Using the infinitesimal behavior of the generators, one obtains an “exact Taylor expansion” for supersmooth morphisms. This can then be used to identify a supersmooth morphism $f: \mathcal{U} \rightarrow \overline{F}$ with its *skeleton*, i.e., a family $(f_k)_{k \in \mathbb{N}_0}$ of maps $f_k: \mathcal{U}_{\mathbb{R}} \rightarrow \mathcal{A}lt^k(E_1, F_{k \bmod 2})$ that are smooth in an appropriate sense. Skeletons are of utmost importance for many proofs and the description of spaces of supersmooth morphisms.

A supermanifold is defined to be a functor $\mathcal{M}: \mathbf{Gr} \rightarrow \mathbf{Man}$, $\Lambda \mapsto \mathcal{M}_\Lambda$ such that there exists an atlas of natural transformations $\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}$ from superdomains \mathcal{U}^α to \mathcal{M} for which any change of charts is supersmooth. If $\varepsilon_{\Lambda_n}: \Lambda_n \rightarrow \mathbb{R}$ denotes the natural projection, we show that $\mathcal{M}_{\varepsilon_{\Lambda_n}}: \mathcal{M}_{\Lambda_n} \rightarrow \mathcal{M}_{\mathbb{R}}$ gives \mathcal{M}_{Λ_n} the structure of a so called multilinear bundle of degree n over the base manifold $\mathcal{M}_{\mathbb{R}}$ (compare [10]). What is more, we show in Theorem 2.3.11 that the family $(\mathcal{M}_\Lambda)_{\Lambda \in \mathbf{Gr}}$ gives one an inverse system of such bundles and that the limit $\varprojlim_n \mathcal{M}_{\Lambda_n}$ exists in the category of manifolds. This provides us with the functor

$$\varprojlim: \mathbf{SMan} \rightarrow \mathbf{Man}$$

from the category of supermanifolds to the category of manifolds mentioned above. Multilinear bundles and their limits are discussed in Appendix B.

In the sheaf theoretic approach every manifold together with its sheaf of functions is clearly a supermanifold. For us the situation is a bit more complicated since a manifold is not a functor $\mathbf{Gr} \rightarrow \mathbf{Man}$. However, there exists a natural embedding

$$\iota: \mathbf{Man} \rightarrow \mathbf{SMan}$$

introduced by Molotkov in [38]. In Proposition 2.3.16, we give a description of $\iota(M)$ via higher tangent bundles of the manifold M , which is particularly useful for understanding Lie supergroups. Similarly, Molotkov constructed a faithful functor

$$\iota_\infty^1: \mathbf{VBun} \rightarrow \mathbf{SMan}$$

from the category of vector bundles to the category of supermanifolds. He showed in [39] that any supermanifold whose base manifold allows a partition of unity is (non-canonically) isomorphic to a supermanifold that comes from a vector bundle. Since this result, generally known as Batchelor’s Theorem, is important for us and [39] is rather difficult to find, we briefly summarize its proof.

Lie Supergroups

A Lie supergroup is simply a group object in the category \mathbf{SMan} . In particular, if \mathcal{G} is a Lie supergroup then \mathcal{G}_Λ is a Lie group for every $\Lambda \in \mathbf{Gr}$. Taking the

Lie algebra of each Lie group \mathcal{G}_Λ leads in a natural way to a Lie superalgebra \mathfrak{g} functorially associated to \mathcal{G} . We get a short exact sequence of Lie groups

$$1 \rightarrow \ker(\mathcal{G}_{\varepsilon_\Lambda}) \rightarrow \mathcal{G}_\Lambda \rightarrow \mathcal{G}_\mathbb{R} \rightarrow 1$$

that splits along $\mathcal{G}_{\eta_\Lambda}$, with $\eta_\Lambda: \mathbb{R} \rightarrow \Lambda$ the natural embedding. Since the group operations are morphisms of multilinear bundles, we see that $\ker(\mathcal{G}_{\varepsilon_\Lambda})$ is a polynomial Lie group (see Appendix C and compare [10]), which provides us with an exponential map

$$\exp_\Lambda^\mathcal{G}: \overline{\mathfrak{g}}_{\Lambda+} \rightarrow \ker(\mathcal{G}_{\varepsilon_\Lambda})$$

that is a diffeomorphism even in the locally convex setting. Together with the action of $\mathcal{G}_\mathbb{R}$ on $\ker(\mathcal{G}_{\varepsilon_\Lambda})$, this effectively describes the Lie group structure of \mathcal{G}_Λ , but the supersmooth structure is better understood with the trivialization

$$\iota(\mathcal{G}_\mathbb{R})_\Lambda \times \overline{\mathfrak{g}}_{1\Lambda} \rightarrow \mathcal{G}_\Lambda, \quad (g, v) \mapsto g \cdot \exp_\Lambda^\mathcal{G}(v).$$

Combining both, we generalize the classical correspondence between super Harish-Chandra pairs and Lie supergroups to the case of arbitrary locally convex Lie supergroups in Theorem 3.3.8. One consequence is that every Lie supergroup \mathcal{G} is completely determined by \mathcal{G}_{Λ_2} . A brief discussion of classical Lie supergroups finishes the chapter.

The Supergroup of Superdiffeomorphisms

We start by discussing spaces of sections of super vector bundles. While we are able to turn such spaces into locally convex vector spaces for arbitrary super vector bundles, in the special case of super vector bundles whose fiber is Banach over a Banach supermanifold with finite dimensional base, we introduce another topology that is more suitable to our needs.

Next, we examine the structure of the group $\text{Aut}(\mathcal{M})$ of automorphisms of a supermanifold \mathcal{M} . One has a short exact sequence

$$1 \rightarrow \text{Aut}_{\text{id}}(\mathcal{M}) \rightarrow \text{Aut}(\mathcal{M}) \rightarrow \text{Aut}(\mathcal{M}_{\Lambda_1}) \rightarrow 1,$$

where $\text{Aut}_{\text{id}}(\mathcal{M})$ is the group of automorphisms that are the identity on \mathcal{M}_{Λ_1} and $\text{Aut}(\mathcal{M}_{\Lambda_1})$ is just the group of vector bundle automorphisms of \mathcal{M}_{Λ_1} . The former is a so called pro-polynomial group (see Appendix C) and when Batchelor's Theorem applies, the sequence splits. This enables us to turn the group $\text{Aut}_c(\mathcal{M})$ of compactly supported automorphisms³ of a σ -compact Banach supermanifold with finite-dimensional base into a Lie group. The Lie group structure of $\text{Aut}_c(\mathcal{M}_{\Lambda_1})$ is discussed in Appendix D. This generalizes results by Wockel and Sachse from [47], where automorphisms of compact finite-dimensional supermanifolds were considered.

³Given the bundle structure of supermanifolds, “compactly supported” generally means compactly supported on the base.

As mentioned, the categorical approach allows for an easy definition of the supergroup $\text{SDiff}(\mathcal{M})$ of superdiffeomorphisms of a supermanifold \mathcal{M} . Even if \mathcal{M} is infinite-dimensional, $\text{SDiff}(\mathcal{M})$ shows appropriate behavior as a functor $\mathbf{Gr} \rightarrow \mathbf{Set}$: It is a supergroup (i.e. a group object in the category $\mathbf{Set}^{\mathbf{Gr}}$) and $\text{SDiff}(\mathcal{M})_{\mathbb{R}} = \text{Aut}(\mathcal{M})$. Further, every supersmooth action of a Lie supergroup on a supermanifold \mathcal{M} factors through a natural action of the supergroup of superdiffeomorphisms of \mathcal{M} (see [40, Proposition 8.4.2, p.417]). Like with Lie supergroups, we have a split short exact sequence

$$1 \rightarrow \ker(\text{SDiff}(\mathcal{M})_{\varepsilon_{\Lambda}}) \rightarrow \text{SDiff}(\mathcal{M}) \rightarrow \text{Aut}(\mathcal{M}) \rightarrow 1,$$

where $\ker(\text{SDiff}(\mathcal{M})_{\varepsilon_{\Lambda}})$ is a polynomial group. Sachse and Wockel [47] attempted to use this splitting to define a Lie supergroup structure on $\text{SDiff}(\mathcal{M})$ in the case of \mathcal{M} being a compact finite-dimensional supermanifold. However, as already discussed for Lie supergroups, this splitting does not explain the supersmooth structure very well and the attempt failed (see Remark 4.4.15).⁴ Instead, like for Lie supergroups, we use a trivialization of the form

$$\text{SDiff}(\mathcal{M})_{\bar{0}} \times \overline{\mathcal{X}(\mathcal{M})_{\bar{1}}} \rightarrow \text{SDiff}(\mathcal{M}),$$

where $\text{SDiff}(\mathcal{M})_{\bar{0}}$ is the supergroup of purely even superdiffeomorphisms and $\mathcal{X}(\mathcal{M})_{\bar{1}}$ is the space of odd vector fields. If the structure of a Lie supergroup on $\text{SDiff}(\mathcal{M})$ exists, then $\iota(\text{Aut}(\mathcal{M})) \cong \text{SDiff}(\mathcal{M})_{\bar{0}}$ as Lie supergroups must hold. Indeed, in the case of a σ -compact finite-dimensional manifold M , we see that

$$\iota(\text{Diff}_c(M)) \cong \text{SDiff}_c(\iota(M)).$$

More generally, if \mathcal{M} is a σ -compact Banach supermanifold with finite-dimensional base, we are able to turn the compactly supported superdiffeomorphisms $\text{SDiff}_c(\mathcal{M})$ into a Lie supergroup in this way. For arbitrary \mathcal{M} , we use the higher order tangent groups studied by Bertram in [10] as a substitute of the functor ι to describe $\text{SDiff}(\mathcal{M})_{\bar{0}}$ in some detail. The necessary constructions are discussed in Appendix E.

⁴I am thankful to C. Wockel and T. Ohrmann for pointing out that the change of charts obtained in this way is not supersmooth.

1. Preliminaries and Notation

We set $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, respectively. Let $k \in \mathbb{N}_0$. Throughout this work, we will write $\bar{k} := k \bmod 2 \in \{0, 1\}$. We denote by \mathfrak{S}_k the symmetrical group of order k and let $\text{sgn}(\sigma) \in \{1, -1\}$ be the *sign* of a permutation $\sigma \in \mathfrak{S}_k$. If R is a unitary commutative ring and E_1, \dots, E_k , E and F are R -modules, we let $L_R^k(E_1, \dots, E_k; F)$ be the R -module of R - k -multilinear maps

$$f: E_1 \times \dots \times E_k \rightarrow F.$$

On $L_R^k(E; F) := L_R^k(E, \dots, E; F)$, \mathfrak{S}_k acts from the left via

$$f \circ \sigma(v) := f(v^\sigma) := f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

for $f \in L_R^k(E; F)$, $\sigma \in \mathfrak{S}_k$ and $v = (v_1, \dots, v_k) \in E^k$. We denote by $\text{Alt}_R^k(E; F) \subseteq L_R^k(E; F)$ the space of alternating R - k -multilinear maps.

If R is additionally a topological ring and the modules are topological R -modules, we denote by $\mathcal{L}_R^k(E_1, \dots, E_k; F)$ and $\mathcal{L}_R^k(E; F)$ the respective R -module of continuous R - k -multilinear maps and let $\mathcal{A}lt_R^k(E; F)$ be the subspace of continuous alternating maps.

We let $\mathcal{L}_R^0(E; F) = \mathcal{A}lt_R^0(E; F) := F$. If $R = \mathbb{R}$, we simply write $L^k(E; F)$, $L^k(E_1, \dots, E_k; F)$, $\mathcal{L}^k(E_1, \dots, E_k; F)$, $\mathcal{L}^k(E; F)$, $\text{Alt}^k(E; F)$ and $\mathcal{A}lt^k(E; F)$. In this case, we define the projection

$$\mathfrak{A}^k: L^k(E; F) \rightarrow \text{Alt}^k(E; F), \quad f \mapsto \sum_{\sigma \in \mathfrak{S}_k} \frac{\text{sgn}(\sigma)}{k!} f \circ \sigma,$$

which clearly also defines a projection $\mathfrak{A}^k: \mathcal{L}^k(E; F) \rightarrow \mathcal{A}lt^k(E; F)$.

1.1. Partitions

We largely use the notation of [10] for partitions. Let A be a finite set. A *partition* of A is a subset $\nu = \{\nu_1, \dots, \nu_\ell\}$ of the power set $\mathcal{P}(A)$ of A such that the sets ν_i , $1 \leq i \leq \ell$, are non-empty, pairwise disjoint and their union is A . In this situation, we call A the *total set* of ν and let $\underline{\nu} := A$. We define the *length* of the partition ν as $\ell(\nu) := |\nu|$. Furthermore, we denote by $\mathcal{P}(A)$ the set of all partitions of A and by $\mathcal{P}_\ell(A)$ the set of all partitions of A of length ℓ . If $|A|$ is even, then we define $\mathcal{P}(A)_{\bar{0}}$ as those partitions which only contain sets of even cardinality and $\mathcal{P}_\ell(A)_{\bar{0}}$ as the partitions from $\mathcal{P}(A)_{\bar{0}}$ of length ℓ .

For $k \in \mathbb{N}$, we define $\mathcal{P}^k := \mathcal{P}(\{1, \dots, k\})$ and $\mathcal{P}_+^k := \mathcal{P}^k \setminus \{\emptyset\}$. Occasionally, it will

be convenient to consider only subsets of even, resp. odd, cardinality and we define $\mathcal{P}_0^k := \{A \in \mathcal{P}^k : |A| \text{ even}\}$, $\mathcal{P}_1^k := \{A \in \mathcal{P}^k : |A| \text{ odd}\}$ as well as $\mathcal{P}_{0,+}^k := \mathcal{P}_0^k \setminus \{\emptyset\}$. As a convention, $\{i_1, \dots, i_r\} \subseteq \{1, \dots, k\}$ is understood to imply $i_1 < \dots < i_r$. With this, the lexicographic order induces a total order on the power set \mathcal{P}^k and every partition ν can be viewed as an ordered $\ell(\nu)$ -tuple, which we will do in the sequel (compare [10, MA.4, p.170]). There is another total order on \mathcal{P}^k that will be useful for us: On \mathcal{P}_0^k and \mathcal{P}_1^k , we use the order induced by \mathcal{P}^k but for all $B \in \mathcal{P}_0^k$ and all $C \in \mathcal{P}_1^k$, we let $B < C$. We will specify whenever we want to use this order which we will call the *graded lexicographic order*. For a partition $\nu = (\nu_1, \dots, \nu_\ell)$, we define $e(\nu)$, resp. $o(\nu)$, as the number of sets in ν with even, resp. odd, cardinality. In other words, in the graded lexicographic order, we have

$$\underbrace{\nu_1 < \dots < \nu_{e(\nu)}}_{\text{even cardinality}} < \underbrace{\nu_{e(\nu)+1} < \dots < \nu_{e(\nu)+o(\nu)}}_{\text{odd cardinality}}.$$

Let A be a finite set and $\nu, \omega \in \mathcal{P}(A)$. We call ν a *refinement* of ω , or ω *coarser* than ν , and write $\omega \preceq \nu$ if for every set $L \in \nu$ there exists a set $O \in \omega$ such that $L \subseteq O$. For $\omega \preceq \nu$ and $O \in \omega$, we define the ν -*induced partition* of O by

$$O|_\nu := \{L \in \nu \mid L \subseteq O\} \in \mathcal{P}(O).$$

In this situation, $\{\omega_1|_\nu, \dots, \omega_{\ell(\omega)}|_\nu\}$ is a partition of the finite set ν . One easily checks that this defines a one-to-one correspondence between partitions that are coarser than ν and $\mathcal{P}(\nu)$.

1.2. The Category of Grassmann Algebras

For any $k \in \mathbb{N}_0$, we let $\Lambda_k := \mathbb{R}[\lambda_1, \dots, \lambda_k]$ be the unital associative algebra freely generated by the generators λ_i with the relation $\lambda_i \lambda_j = -\lambda_j \lambda_i$ for all $i, j \in \mathbb{N}$. Note that this implies $\lambda_i \lambda_i = 0$. For $I = \{i_1, \dots, i_\ell\} \subseteq \mathbb{N}$ with $1 \leq i_1 < \dots < i_\ell \leq k$, we set $\lambda_I := \lambda_{i_1} \cdots \lambda_{i_\ell}$. These so called *Grassmann algebras* have a natural \mathbb{Z}_2 -grading given by $\Lambda_{k, \bar{0}} := \bigoplus_{I \in \mathcal{P}_0^k} \lambda_I \mathbb{R}$ and $\Lambda_{k, \bar{1}} := \bigoplus_{I \in \mathcal{P}_1^k} \lambda_I \mathbb{R}$ which, with the product topology, turns them into topological \mathbb{R} -algebras. A morphism $\varphi: \Lambda \rightarrow \Lambda'$ between two Grassmann algebras is a morphism of unital \mathbb{R} -algebras that is even in the sense that

$$\varphi(\Lambda_{\bar{i}}) \subseteq \Lambda'_{\bar{i}} \quad \text{for } i \in \{0, 1\}.$$

We denote by \mathbf{Gr} the category of Grassmann algebras, and for every $n \in \mathbb{N}_0$, we let $\mathbf{Gr}^{(n)}$ be the full subcategory containing only the objects $\Lambda_0, \dots, \Lambda_n$. For the sake of convenience, we let $\mathbf{Gr}^{(\infty)} := \mathbf{Gr}$.

We denote the subalgebra of nilpotent elements of Λ by Λ^+ and set $\Lambda_{\bar{1}}^+ := \Lambda_{\bar{1}}$ and $\Lambda_{\bar{0}}^+ := \Lambda^+ \cap \Lambda_{\bar{0}}$. For every $m \geq n \geq 0$, we fix morphisms $\varepsilon_{m,n}: \Lambda_m \rightarrow \Lambda_n$ and

$\eta_{n,m}: \Lambda_n \rightarrow \Lambda_m$ by setting

$$\varepsilon_{m,n}(\lambda_k) := \begin{cases} \lambda_k & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

and $\eta_{n,m}(\lambda_k) = \lambda_k$ for $1 \leq k \leq n$. In the special case $n = 0$, we let $\varepsilon_{\Lambda_m} := \varepsilon_{m,0}: \Lambda_m \rightarrow \mathbb{R}$ and $\eta_{\Lambda_m} := \eta_{0,m}: \mathbb{R} \rightarrow \Lambda_m$.

1.3. Locally Convex Manifolds

All locally convex vector spaces in this thesis are meant to be Hausdorff locally convex \mathbb{R} -vector spaces.

1.3.1. Differential calculus in locally convex spaces

A very general differential calculus for topological modules was developed in [11].¹ We follow this approach but restrict ourselves to the case of Hausdorff locally convex \mathbb{R} -vector spaces. In this situation, the \mathcal{C}^k -maps coincide with the classical \mathcal{C}^k -maps in the sense of Bastiani [5] (also known as Keller's \mathcal{C}_c^k -maps, see [31]). However, it is useful to keep the more general setting in mind since large parts of this work can be easily generalized without substantial changes. See also [10, Chapter I, p.14ff.] for a concise overview.

Definition 1.3.1. Let E, F be locally convex spaces, $U \subseteq E$ be open and $f: U \rightarrow F$ continuous. We define the open set $U^{[1]} := \{(x, v, t): x \in U, x + tv \in U\} \subseteq U \times E \times \mathbb{R}$ and say that f is \mathcal{C}^1 if there exists a continuous map

$$f^{[1]}: U^{[1]} \rightarrow F$$

such that

$$f(x + tv) - f(x) = t \cdot f^{[1]}(x, v, t)$$

for $(x, v, t) \in U^{[1]}$. The *differential of f at $x \in U$* is then defined as

$$df(x): E \rightarrow F, \quad v \mapsto df(x)(v) := f^{[1]}(x, v, 0).$$

We also use the notation $df(x, v) := df(x)(v)$. Inductively, we say f is \mathcal{C}^{k+1} if $f^{[1]}$ is \mathcal{C}^k for $k \in \mathbb{N}$. If f is \mathcal{C}^k for every $k \in \mathbb{N}$, we call f *smooth* or \mathcal{C}^∞ .

The usual rules for differentials apply and we sum them up and fix our notation in the following remark.

Remark 1.3.2. In the situation of the definition, the map $f^{[1]}$ is unique and $df(x)(v)$ is linear in v . If f is \mathcal{C}^2 , then for every $v \in E$ the partial map $\partial_v f := df(\bullet, v)$ is \mathcal{C}^1 and we define $d^k f(x)(v_1, \dots, v_k) := \partial_{v_1} \cdots \partial_{v_k} f(x)$ if f is \mathcal{C}^k . The

¹To be precise: Hausdorff topological modules over commutative Hausdorff topological rings whose unit group is dense.

map $d^k f(x): E^k \rightarrow F$ is continuous, \mathbb{R} - k -multilinear and symmetric. In particular Schwarz's theorem holds in this setting. If $V \subseteq U$ is open and f is \mathcal{C}^k , then the restriction $f|_V$ is so. If g and f are \mathcal{C}^k and composable, then $g \circ f$ is \mathcal{C}^k and we have the *chain rule*

$$d(g \circ f)(x, v) = dg(f(x), df(x, v)).$$

If $h: U_1 \times U_2 \rightarrow F$ is \mathcal{C}^1 we define $d_1 h(x_1, x_2)(v_1) := dh(x_1, x_2)(v_1, 0)$ and $d_2 h(x_1, x_2)(v_2) := dh(x_1, x_2)(0, v_2)$ and we have the *rule of partial differentials*

$$dh(x_1, x_2)(v_1, v_2) = d_1 h(x_1, x_2)(v_1) + d_2 h(x_1, x_2)(v_2).$$

If f is of the form (f_1, f_2) then f is \mathcal{C}^k if and only if f_1 and f_2 are \mathcal{C}^k and it holds that $df = (df_1, df_2)$.

The following lemma is well-known. As it is instrumental for the rest of the work, we give a quick proof nevertheless. Clearly, the proof works in the most general setting as well.

Lemma 1.3.3. *Let $n \in \mathbb{N}$ and E_1, \dots, E_n and F be locally convex spaces. Each continuous \mathbb{R} - n -multilinear map $f: E_1 \times \dots \times E_n \rightarrow F$ is automatically \mathcal{C}^1 and thus smooth by induction. In this case, we have*

$$df(x)(v) = \sum_{i=1}^n f(x_1, \dots, x_{i-1}, v_i, x_{i+1}, \dots, x_n)$$

for $x = (x_1, \dots, x_n), v = (v_1, \dots, v_n) \in E_1 \times \dots \times E_n$.

Proof. Let ${}_0 y_i := x_i$ and ${}_1 y_i = v_i$ for $1 \leq i \leq n$. With this we calculate

$$f(x + tv) - f(x) = t \cdot \underbrace{\sum_{j \in \{0,1\}^n, \ell_j \geq 1} t^{\ell_j - 1} f({}_{j_1} y_1, \dots, {}_{j_n} y_n)}_{f^{[1]}(x, v, t) :=}$$

where $\ell_j := j_1 + \dots + j_n$. As $f^{[1]}$ is continuous, the statement follows. \square

Corollary 1.3.4. *Let E, F, E' and F' be locally convex spaces, $U \subseteq E$, $U' \subseteq E'$ be open and $f: U \rightarrow F$ and $g: U' \rightarrow F'$ be smooth maps. Moreover, let $\alpha: F \rightarrow F'$ and $\beta: E' \rightarrow E$ be continuous linear maps such that $\beta(U) \subseteq U'$ and $\alpha \circ f = g \circ \beta|_U$. Then we have*

$$\alpha \circ d^n f = d^n g \circ (\beta|_U \times \beta^n)$$

for all $n \in \mathbb{N}_0$.

Proof. This follows from applying the chain rule and Lemma 1.3.3 to

$$d^n(\alpha \circ f) = d^n(g \circ \beta|_U).$$

\square

1.3.2. Manifolds

With the above, the definition of manifolds over locally convex spaces is analogous to the finite-dimensional case (see also [11, Section 8, p.253] or [10, Section 2, p.20]). We fix a locally convex space E and let M be a topological space. A set $\mathcal{A} := \{\varphi_\alpha: U_\alpha \rightarrow V_\alpha: \alpha \in A\}$ such that $U_\alpha \subseteq M$ and $V_\alpha \subseteq E$ are open, φ_α is a homeomorphism, $\bigcup_{\alpha \in A} U_\alpha = M$ and

$$\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)}: \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

and its inverse $\varphi_{\beta\alpha}$ are smooth is called a (*smooth*) *atlas* of M and the elements of \mathcal{A} are called *charts* of M .² Two atlases of M are equivalent if and only if their union is again an atlas. Together with an equivalence class of atlases, M is called a (*smooth*) *manifold modelled on E* and E is the *model space* of M . We usually only mention a representative atlas of the equivalence class. Moreover, we will generally assume manifolds to be Hausdorff.³ A manifold is called *paracompact*, resp. *σ -compact*, if it is so as a topological space and *finite-dimensional* if its model space is finite-dimensional. If M and N are manifolds with the atlases $\{\varphi_\alpha: \alpha \in A\}$ and $\{\psi_\beta: \beta \in B\}$, then $\{\varphi_\alpha \times \psi_\beta: \alpha \in A, \beta \in B\}$ is an atlas of $M \times N$.

A continuous map $f: M \rightarrow N$ between two manifolds is a *morphism of (smooth) manifolds* if for any charts $\varphi: U \rightarrow \varphi(U)$ of M and $\psi: W \rightarrow \psi(W)$ of N the map

$$\psi \circ f \circ \varphi^{-1}: \varphi(U \cap f^{-1}(W)) \rightarrow \psi(W)$$

is smooth. This property is independent of the choice of atlases. If M and N are manifolds, we denote by $\mathcal{C}^\infty(M, N)$ the set of all smooth maps $M \rightarrow N$ and we denote by **Man** the category of Hausdorff manifolds and their morphisms.

The definition of *vector bundles* or more general *fiber bundles*, their morphisms and their products is similar to the finite-dimensional case and for it, we refer to [11, p.255] or [10, Section 3, p.22]. The particular charts of a bundle are called *bundle charts* and they are elements of a *bundle atlas*. We write **VBun** for the category of vector bundles.

The definition of the *tangent bundle* $\pi_M: TM \rightarrow M$ of a manifold M via equivalence classes of smooth curves works as in the finite-dimensional case. For locally convex \mathbb{R} -vector spaces, this is equivalent to the more general definition in [11, p.254] and [10, Section 3, p.22] (see [23]). For the elements of the tangent bundle, we occasionally write $[t \mapsto v_t] \in T_{v_0}M$, where $t \mapsto v_t$ denotes some curve in M . In this notation one has

$$Tf[t \mapsto v_t] := [t \mapsto f(v_t)] \in T_{f(v_0)}N,$$

if $f: M \rightarrow N$ is a smooth map between manifolds. If F is a locally convex space, one has a natural isomorphism $TF \cong F \times F$ and if $g: M \rightarrow F$ is smooth, we also write $dg: TM \rightarrow F$ for $\text{pr}_2 \circ Tg$ with the projection $\text{pr}_2: F \times F \rightarrow F$ onto the

²The index set A is just to simplify our notation, it does not belong to the data of the atlas \mathcal{A} .

³This assumption is necessary to guarantee the existence of smooth partitions of unity for finite-dimensional σ -compact manifolds.

second component. Like in the finite-dimensional case, the above defines a functor $T: \mathbf{Man} \rightarrow \mathbf{VBun}$ and considering \mathbf{VBun} as a subcategory of \mathbf{Man} , we define $T^0 = \text{id}_{\mathbf{Man}}$ and $T^n := T \circ T^{n-1}: \mathbf{Man} \rightarrow \mathbf{Man}$ for $n \in \mathbb{N}$. Finally, if $\{\varphi_\alpha: \alpha \in A\}$ is an atlas of M , then $\{T\varphi_\alpha: \alpha \in A\}$ is a bundle atlas of TM and there is a natural isomorphism of vector bundles $T(M \times N) \cong TM \times TN$.

Smooth partitions of unity

A *smooth partition of unity* of a manifold M is an open covering $(U_i)_{i \in I}$ of M together with smooth maps $h_i: M \rightarrow \mathbb{R}$, such that

- (a) For all $x \in M$, we have $h_i(x) \geq 0$.
- (b) The support of h_i is contained in U_i for all $i \in I$.
- (c) The covering is locally finite.
- (d) For each $x \in M$, we have $\sum_{i \in I} h_i(x) = 1$.

In this situation, we say that $(h_i)_{i \in I}$ is a partition of unity that is *subordinate* to $(U_i)_{i \in I}$. We say that a manifold M *admits partitions of unity* if it is paracompact and for every locally finite open cover (U_i) of M , we find smooth maps $h_i: M \rightarrow \mathbb{R}$ that constitute a partition of unity subordinate to (U_i) (see [33, p.34]). Paracompact (and in particular σ -compact) finite-dimensional manifolds always admit partitions of unity (compare [33, Corollary 3.8, p.38]).

Vector fields

Let M be a manifold modelled on a locally convex space E . A *vector field* is a smooth map $X: M \rightarrow TM$ such that $\pi_M \circ X = \text{id}_M$. We denote by $\mathfrak{X}(M)$ the \mathbb{R} -vector space of vector fields. Let $X, Y \in \mathfrak{X}(M)$. If $\varphi: U_\varphi \rightarrow V_\varphi$ is a chart of M , we define the local representation X^φ of X by

$$X^\varphi := d\varphi \circ X \circ \varphi^{-1}: V_\varphi \rightarrow E.$$

The space of vector fields is a Lie algebra with the Lie bracket locally given by

$$[X, Y]^\varphi(x) = dX^\varphi(x, Y^\varphi(x)) - dY^\varphi(x, X^\varphi(x))^4$$

(see [10, Theorem 4.2, p.25]). If M is finite-dimensional, we define the *support* of X , $\text{supp}(X)$, as the smallest closed subset $K \subseteq M$ such that $X|_{M \setminus K} = 0$. With this, we define the subspace of *compactly supported vector fields*

$$\mathfrak{X}_c(M) := \{X \in \mathfrak{X}(M): \text{supp}(X) \text{ is compact}\}.$$

⁴ We choose this order of X and Y to stay consistent with [10]. Traditionally, in the literature the reverse order is used.

Then $\mathfrak{X}_c(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$. If M is σ -compact with an atlas \mathcal{A} that is countable and such that the family $(U_\varphi)_{\varphi \in \mathcal{A}}$ is a covering of M by relatively compact open sets U_φ , then giving $\mathcal{X}_c(M)$ the topology that turns

$$\mathcal{X}_c(M) \rightarrow \bigoplus_{\varphi \in \mathcal{A}} \mathcal{C}^\infty(V_\varphi, E), \quad X \mapsto (X^\varphi)_{\varphi \in \mathcal{A}}$$

into an embedding makes $\mathcal{X}_c(M)$ a locally convex Lie algebra. The induced topology does not depend on the choice of \mathcal{A} (see Lemma 4.1.9 and Lemma 4.1.17, cf. [18]).

1.4. Categories

We follow [49] in the standard definitions. Let us give a brief overview to fix our notations. Throughout, we fix a universe \mathcal{U} (see [49, 3.2.1, p.17]) that contains the natural numbers \mathbb{N} as an element. *Sets* are then elements of \mathcal{U} and *classes* are subsets of \mathcal{U} . A category \mathcal{C} consists of a class of objects $|\mathcal{C}|$ and a set of morphisms $\text{Hom}_{\mathcal{C}}(A, B)$ for any objects A, B such that we have a composition map

$$\text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C), \quad (f, g) \mapsto f \circ g$$

(where $C \in \mathcal{C}$) that satisfies the usual conditions. In particular, we have a unique identity morphism $\text{id}_A \in \text{Hom}_{\mathcal{C}}(A, A)$. For $f \in \text{Hom}_{\mathcal{C}}(A, B)$ we also write $f: A \rightarrow B$ and we call f an isomorphism if there exists $f^{-1} \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$. As a shorthand, we write $A \in \mathcal{C}$ instead of $A \in |\mathcal{C}|$. A *small category* is a category whose objects form a set.

We denote by **Set** the category whose objects are sets and whose morphisms are maps between sets. The category **Top** has topological spaces as objects and continuous maps between them as morphisms.⁵

1.4.1. Functors and Functor Categories

Let \mathcal{C} and \mathcal{D} be categories. A *functor* $T: \mathcal{C} \rightarrow \mathcal{D}$ assigns to each $A \in \mathcal{C}$ an object $T(A) \in \mathcal{D}$ and to each morphism $f \in \text{Hom}_{\mathcal{C}}(A, B)$ a morphism $T(f) \in \text{Hom}_{\mathcal{D}}(T(A), T(B))$ such that $T(\text{id}_A) = \text{id}_{T(A)}$ and $T(f \circ g) = T(f) \circ T(g)$ hold for all $A, B, C \in \mathcal{C}$ and all $f \in \text{Hom}_{\mathcal{C}}(B, C)$, $g \in \text{Hom}_{\mathcal{C}}(A, B)$. Let $S: \mathcal{C} \rightarrow \mathcal{D}$ be another functor. A *natural transformation* $\alpha: S \rightarrow T$ consists of morphisms $\alpha_A: S(A) \rightarrow T(A)$ for every $A \in \mathcal{C}$ such that for every $f \in \text{Hom}_{\mathcal{C}}(A, B)$, we have $T(f) \circ \alpha_A = \alpha_B \circ S(f)$. We always have the natural transformation $\text{id}_T: T \rightarrow T$ defined by $(\text{id}_T)_A = \text{id}_{T(A)}$ and if $U: \mathcal{C} \rightarrow \mathcal{D}$ is another functor and $\beta: T \rightarrow U$ is a natural transformation, then the object-wise composition $\beta_A \circ \alpha_A$ defines a natural transformation $\beta \circ \alpha: S \rightarrow U$.

If \mathcal{C} is a small category, then the functors $\mathcal{C} \rightarrow \mathcal{D}$ are the objects and the natural transformations are the morphisms of a category which we denote by $\mathcal{D}^{\mathcal{C}}$ (see [49,

⁵Schubert, [49], uses the notations $[A, B]_{\mathcal{C}}$ for $\text{Hom}_{\mathcal{C}}(A, B)$, 1_A for id_A and *Ens* for **Set**.

Proposition 3.4.3, p.19]).

1.4.2. Algebraic Structures in Functor Categories

In any category \mathcal{D} with finite products and a terminal object $Z \in \mathcal{D}$ (see [49, 5.4.1, p.35 and Section 7.3, p.49f.] for these notions), one can define a given algebraic structure⁶ on an object $A \in \mathcal{D}$ by encoding the structure in certain commutative diagrams. Let us use the example of groups to illustrate this general principle. As multiplication, we now have a morphism $\mu: A \times A \rightarrow A$,⁷ the inversion is described by a morphism $i: A \rightarrow A$ and the neutral element corresponds to a morphism $e: Z \rightarrow A$. Denote by $*$: $A \rightarrow Z$ the unique morphism $A \rightarrow Z$. Then the relation between inversion and multiplication is given by

$$\begin{array}{ccc}
 A & \xrightarrow{(id_A, i)} & A \times A \\
 \downarrow (i, id_A) & \searrow * & \downarrow \mu \\
 & Z & \downarrow e \\
 A \times A & \xrightarrow{\mu} & A
 \end{array}$$

It is not difficult to see how one can also describe the properties of the neutral element and associativity in this way. For details see [49, Section 11.1, p.96ff.]. In this situation, we call (A, μ, i, e) a *group in the category \mathcal{D}* or a *group object*.

This approach can be extended to structures where one object operates on another, like modules, algebras and Lie algebras over a commutative ring. Morphisms can also be defined using appropriate commutative diagrams and one obtains new categories of objects with a given structure. For details see [49, Chapter 11, p.96ff.]. Let $R \in \mathcal{D}$ be a commutative ring. We denote by $\mathbf{Mod}_R(\mathcal{D})$, $\mathbf{Alg}_R(\mathcal{D})$ and $\mathbf{LAlg}_R(\mathcal{D})$ the categories of modules, algebras and Lie algebras over R in \mathcal{D} .⁸ If R is an ordinary ring then $\mathbf{Mod}_R(\mathbf{Set})$, $\mathbf{Alg}_R(\mathbf{Set})$ and $\mathbf{LAlg}_R(\mathbf{Set})$ are just the categories of ordinary R -modules, R -algebras and R -Lie algebras, respectively. If R is a topological ring, one obtains with $\mathbf{Mod}_R(\mathbf{Top})$, $\mathbf{Alg}_R(\mathbf{Top})$ and $\mathbf{LAlg}_R(\mathbf{Top})$ the usual topological modules, algebras and Lie algebras over R .

If \mathcal{C} is a small category, then the functor category $\mathcal{D}^{\mathcal{C}}$ also has finite products and a terminal object. A functor $T \in \mathcal{D}^{\mathcal{C}}$ having an algebraic structure is then equivalent to $T(A)$ having that structure in \mathcal{D} for all $A \in \mathcal{C}$ and $T(f)$ being a morphism of that structure for all $A, B \in \mathcal{C}$ and all $f \in \text{Hom}_{\mathcal{C}}(A, B)$ (see [49, Proposition 11.4.1, p.102]).⁹ Likewise, natural transformations are morphisms if

⁶One caveat: This means only algebraic structures where the operations are defined everywhere.

For example fields are excluded because the multiplicative inversion is only defined on a subset.

⁷Schubert uses the notation $A \sqcap A$ instead of $A \times A$ for the product.

⁸In [49, 11.3.7, p.101f.], one can clearly restrict oneself to objects over the same ring and restrict morphisms to those which are the identity on the ring to obtain these categories.

⁹It is easy to generalize the proposition to the situation where one has an operation of another object with an algebraic structure. In this case morphisms need to include morphisms on

and only if they are so object-wise.

We are mainly interested in the case of the categories $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$ or $\mathbf{Top}^{\mathbf{Gr}^{(k)}}$ with the structures of modules, algebras and Lie algebras. With the above in mind, we make the following definitions.

Definition 1.4.1. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A group object in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$ (resp. $\mathbf{Top}^{\mathbf{Gr}^{(k)}}$ etc.) is called a *supergroup*. Concretely, (\mathcal{G}, μ, i, e) is a supergroup if and only if $(\mathcal{G}(\Lambda), \mu_\Lambda, i_\Lambda, e_\Lambda)$ is a group for each $\Lambda \in \mathbf{Gr}^{(k)}$ and $\mathcal{G}(\varrho)$ is a morphism of groups for each $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. We call a supergroup \mathcal{H} a *sub-supergroup* of a supergroup \mathcal{G} if $\mathcal{H}(\Lambda) \subseteq \mathcal{G}(\Lambda)$ is a subgroup for each $\Lambda \in \mathbf{Gr}$. A *morphism of supergroups* is a natural transformation $f: \mathcal{G} \rightarrow \mathcal{G}'$ between supergroups \mathcal{G} and \mathcal{G}' , such that $f_\Lambda: \mathcal{G}(\Lambda) \rightarrow \mathcal{G}'(\Lambda)$ is a morphism of groups for each $\Lambda \in \mathbf{Gr}$.

Definition 1.4.2. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A (unitary) *ring in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$* is a functor $R: \mathbf{Gr}^{(k)} \rightarrow \mathbf{Set}$ such that $R(\Lambda)$ has a fixed (unitary) ring structure for all $\Lambda \in \mathbf{Gr}^{(k)}$ and $R(\varrho)$ is a morphism of rings for all $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. We call R *commutative* if every $R(\Lambda)$ is a commutative ring. Let R be a unitary ring in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$. A (left) *R -module in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$* is a functor $M: \mathbf{Gr}^{(k)} \rightarrow \mathbf{Set}$ such that $M(\Lambda)$ is a (left) $R(\Lambda)$ -module for every $\Lambda \in \mathbf{Gr}^{(k)}$ and $M(\varrho): M(\Lambda) \rightarrow M(\Lambda')$ is a morphism of additive groups with

$$M(\varrho)(gm) = R(\varrho)(g)M(\varrho)(m)$$

for all $g \in R(\Lambda), m \in M(\Lambda)$ and $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. The object-wise module operations then define a natural transformation in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$. A natural transformation $\alpha: M \rightarrow N$ between two R -modules in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$ is a *morphism of R -modules* if α_Λ is a morphism of $R(\Lambda)$ -modules for all $\Lambda \in \mathbf{Gr}^{(k)}$. Then $\mathbf{Mod}_R(\mathbf{Set}^{\mathbf{Gr}^{(k)}})$ is the category of R -modules in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$.

An *R -algebra in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$* over a commutative ring R in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$ is an R -module A in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$ such that $A(\Lambda)$ is an algebra over $R(\Lambda)$ for every $\Lambda \in \mathbf{Gr}^{(k)}$ and such that

$$A(\varrho)(\mu_\Lambda(a, b)) = \mu_{\Lambda'}(A(\varrho)(a), A(\varrho)(b))$$

for all $a, b \in A(\Lambda)$ and $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$, where $(\mu_\Lambda)_{\Lambda \in \mathbf{Gr}^{(k)}}$ denotes the algebra multiplication. Then $(\mu_\Lambda)_{\Lambda \in \mathbf{Gr}^{(k)}}$ is a natural transformation. We call an R -algebra A *commutative* (resp. *associative*, or *Lie*) if every $A(\Lambda)$ is commutative (resp. an associative algebra, or a Lie algebra). A natural transformation $\beta: A \rightarrow B$ between R -algebras in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$ is a *morphism of R -algebras* if β_Λ is a morphism of algebras over $R(\Lambda)$ for all $\Lambda \in \mathbf{Gr}^{(k)}$. Then $\mathbf{Alg}_R(\mathbf{Set}^{\mathbf{Gr}^{(k)}})$ is the category of R -algebras and $\mathbf{LAlg}_R(\mathbf{Set}^{\mathbf{Gr}^{(k)}})$ is the category of R -Lie algebras in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$.

The definitions of rings, modules, algebras and Lie algebras in $\mathbf{Top}^{\mathbf{Gr}^{(k)}}$ is completely analogous.

both structures, e.g. morphisms of modules with a change of rings (see [49, 11.3.7, p.101f.]).

Definition 1.4.3. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, let R be a unitary commutative ring in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$ and $M_1, \dots, M_n, M \in \mathbf{Mod}_R(\mathbf{Set}^{\mathbf{Gr}^{(k)}})$. We say that a natural transformation $f: M_1 \times \dots \times M_n \rightarrow M$ is R - n -multilinear if f_Λ is $R(\Lambda)$ - n -multilinear for every $\Lambda \in \mathbf{Gr}^{(k)}$ and let

$$L_R^n(M_1, \dots, M_n; M) := \{f: M_1 \times \dots \times M_n \rightarrow M : f \text{ is } R\text{-}n\text{-multilinear}\}.$$

We define $\mathcal{L}_R^n(M_1, \dots, M_n; M)$ analogously for the case of $\mathbf{Top}^{\mathbf{Gr}^{(k)}}$ if R is a unitary commutative ring in $\mathbf{Top}^{\mathbf{Gr}^{(k)}}$.

Lemma 1.4.4. *In the situation of Definition 1.4.3, setting*

$$r \cdot f + g := (R(\eta_\Lambda)(r) \cdot f_\Lambda + g_\Lambda)_{\Lambda \in \mathbf{Gr}^{(k)}}$$

for $f, g \in L_R^n(M_1, \dots, M_n; M)$ and $r \in R(\mathbb{R})$ turns $L_R^n(M_1, \dots, M_n; M)$ into an $R(\mathbb{R})$ -module in \mathbf{Set} . On $L_R^n(M_1, \dots, M_1; M)$, \mathfrak{S}_n acts from the left via $f \circ \sigma := (f_\Lambda \circ \sigma)_{\Lambda \in \mathbf{Gr}^{(k)}}$. The same is true for $\mathcal{L}_R^n(M_1, \dots, M_n; M)$, resp. $\mathcal{L}_R^n(M_1, \dots, M_1; M)$.

Proof. The first statement is easily seen because for every $\varrho \in \mathbf{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$, we have that $M(\varrho)$ is additive and

$$M(\varrho)(R(\eta_\Lambda)(r) \cdot m) = R(\varrho)(R(\eta_\Lambda)(r)) \cdot M(\varrho)(m) = R(\eta_{\Lambda'})(r) \cdot M(\varrho)(m)$$

holds for all $r \in R(\mathbb{R})$ and $m \in M(\Lambda)$. The second statement is obvious and the same arguments work in the case of $\mathcal{L}_R^n(M_1, \dots, M_n; M)$, resp. $\mathcal{L}_R^n(M_1, \dots, M_1; M)$. \square

See also [40, Chapter 1, p.378ff.] for a more general approach.

1.5. Linear Superspaces and Superalgebras

For any multilinear algebraic structure like rings, modules or algebras, one can define a corresponding “superalgebraic” structure. For this, generally speaking, one fixes a \mathbb{Z}_2 -grading such that the operations obey the principle “even times even is even, even times odd is odd and odd times odd is even.” If a rule for permutating elements is involved, one additionally has “permutating with even elements does not involve an additional sign, permutating two odd elements does.”

Definition 1.5.1. We call a ring R a *superring* if it is \mathbb{Z}_2 -graded, i.e., decomposes into additive subgroups $R_0 \oplus R_1$ such that

$$R_i \cdot R_j \subseteq R_{i+j}$$

holds for $i, j \in \{0, 1\}$. If, additionally, R is a topological ring such that $R \cong R_0 \oplus R_1$ holds as topological groups, we call R a *topological superring*.

A (left) module with a \mathbb{Z}_2 -grading $M = M_0 \oplus M_1$ over a superring R with a unit element is called an *R-supermodule* if

$$R_i \cdot M_j \subseteq M_{i+j}$$

holds for $i, j \in \{0, 1\}$. If, additionally, R is a topological superring and M is a topological R -module such that $M \cong M_0 \oplus M_1$ holds as topological groups, we call M a *topological R-supermodule*. The part M_0 is called *even* and M_1 is called *odd*. A supermodule of the form $M_0 \oplus \{0\}$, resp. $\{0\} \oplus M_1$, is called *purely even*, resp. *purely odd*. A *morphism* $f: M \rightarrow N$ of *R-supermodules* is a morphism of R -modules that preserves the grading, i.e., $f(M_i) \subseteq N_i$ for $i \in \{0, 1\}$. We denote by \mathbf{SMod}_R the category of R -supermodules. A morphism of topological supermodules shall be additionally continuous and we denote by $\mathbf{TopSMod}_R$ the category of topological R -supermodules. The product $M \times N$ of two R -supermodules M, N is the R -supermodule $(M_0 \times N_0) \oplus (M_1 \times N_1)$.

Every ring R can be considered as a purely even superring $R \oplus \{0\}$.

Definition 1.5.2. Let R be a superring and M be an R -supermodule. An element $m \in M$ is called *homogeneous* if $m \in M_0$ or $m \in M_1$. The *parity* $p(m)$ of an homogeneous element is defined as

$$p(m) := \begin{cases} 0 & \text{if } m \in M_0, \\ 1 & \text{if } m \in M_1. \end{cases}$$

Definition 1.5.3. Let R be a unitary commutative superring, $n \in \mathbb{N}$ and let M and ${}_i M = {}_i M_0 \oplus {}_i M_1$ for $1 \leq i \leq n$ be R -supermodules. An *R-n-multilinear morphism*

$$f: {}_1 M \times \cdots \times {}_n M \rightarrow M$$

is called *even* if

$$f({}_1 M_{j_1}, \dots, {}_n M_{j_n}) \subseteq M_{\overline{j_1 + \cdots + j_n}}$$

holds for $j_1, \dots, j_n \in \{0, 1\}$. We denote by ${}_0 L_R^n({}_1 M, \dots, {}_n M; M)$ the space of even R - n -multilinear morphisms ${}_1 M \times \cdots \times {}_n M \rightarrow M$. This space is obviously an R_0 -module. If R is a topological superring and all modules are topological R -supermodules, then we denote by ${}_0 \mathcal{L}_R^n({}_1 M, \dots, {}_n M; M)$ the R_0 -module of respective continuous even R - n -multilinear maps.

The symmetrical group \mathfrak{S}_n acts from the left on ${}_0 L_R^n({}_1 M, \dots, {}_1 M; M)$ via

$$(f \cdot \sigma_j)({}_1 v, \dots, {}_n v) := (-1)^{p({}_1 v)p({}_n v)} f({}_1 v, \dots, {}_{j+1} v, {}_j v, \dots, {}_n v)$$

for $f \in {}_0 L_R^n({}_1 M, \dots, {}_1 M; M)$, homogeneous ${}_1 v, \dots, {}_n v \in {}_1 M$ and any transposition $\sigma_j := (j, j+1)$.

Of course, one can also define odd R - n -multilinear morphisms and turn the space of all R - n -multilinear morphisms into an R -supermodule. See [40, Section 1.7, p383ff.] for this.

Definition 1.5.4. A supermodule over a field \mathbb{K} is called \mathbb{K} -*super vector space*. For $p, q \in \mathbb{N}_0$, we define the super vector space $\mathbb{R}^{p|q} := \mathbb{R}^p \oplus \mathbb{R}^q$ and if $\dim E_0 = p$ and $\dim E_1 = q$, we call $(p|q)$ the *dimension* of E .

We let $\mathbf{SVec} := \mathbf{SMod}_{\mathbb{R}}$ denote the category of \mathbb{R} -super vector spaces. Moreover, we denote by \mathbf{SVec}_{lc} the category of Hausdorff locally convex \mathbb{R} -super vector spaces and their continuous morphisms.

Definition 1.5.5. Let \mathbb{K} be a field. A \mathbb{K} -*superalgebra* is a \mathbb{K} -super vector space $A = A_0 \oplus A_1$ that is an algebra with the multiplication $\mu: A \times A \rightarrow A$ such that

$$\mu(A_i, A_j) \subseteq A_{i+j} \text{ for } i, j \in \{0, 1\}.$$

A super algebra A is called *associative*, resp. *unital*, resp. *topological* if it is so as an algebra. We say it is *supercommutative* if we have

$$\mu(a, b) = (-1)^{p(a)p(b)} \mu(b, a)$$

for all homogeneous elements $a, b \in A$. A *morphism of superalgebras* is a morphism of algebras that is a morphism of super vector spaces. We denote by $\mathbf{SAlg}_{\mathbb{K}}$ the category of \mathbb{K} -superalgebras. A morphism of topological superalgebras shall additionally be continuous and we let $\mathbf{TopSAlg}_{\mathbb{K}}$ be the category of topological \mathbb{K} -superalgebras if \mathbb{K} is a topological field.

Example 1.5.6. Every Grassmann algebra $\Lambda \in \mathbf{Gr}$ is a topological \mathbb{R} -superalgebra that is associative, unital and supercommutative.

One can of course define superalgebras over superrings, but for our purposes the above is sufficient.

Definition 1.5.7. Let \mathbb{K} be a field. We call a \mathbb{K} -superalgebra L a \mathbb{K} -*Lie superalgebra* if its multiplication $[\cdot, \cdot]: L \times L \rightarrow L$ is

(1) *super antisymmetric*, i.e., $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ for all homogeneous elements $a, b \in L$ and

(2) satisfies the *super Jacobi identity*, i.e.,

$$[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)}[b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)}[c, [a, b]] = 0$$

for all homogeneous elements $a, b, c \in L$.

We call $[\cdot, \cdot]$ the *Lie superbracket* of L . We denote by $\mathbf{LSAlg}_{\mathbb{K}}$ the category of \mathbb{K} -Lie superalgebras and by $\mathbf{TopLSAlg}_{\mathbb{K}}$ the category of topological \mathbb{K} -Lie superalgebras if \mathbb{K} is a topological field.

The even part L_0 of a Lie superalgebra L together with the restricted Lie superbracket is clearly an ordinary Lie algebra.

Remark 1.5.8. One can express the various additional properties of superalgebras with the action of the symmetric group from Definition 1.5.3. Let A be a \mathbb{K} -superalgebra with the multiplication $\mu: A \times A \rightarrow A$. Then supercommutativity is equivalent to $\mu.(1, 2) = \mu$, super antisymmetry means $\mu.(1, 2) = -\mu$ and the super Jacobi identity can be expressed by

$$\sum_{\sigma \in \mathfrak{S}_3, \text{sgn}(\sigma)=1} \mu(\bullet, \mu(\bullet, \bullet)).\sigma = 0.$$

In fact, Molotkov defines multilinear algebraic structures of a given super type by simply substituting the usual action of the symmetric group on multilinear maps with this graded version in [40, Section 1.9, p.385].

Superalgebraic structures can, of course, also be defined for general categories. This is done in [40, Section 1.7, p.383f.].

2. Supermanifolds

2.1. Open Subfunctors

Open subfunctors of functors from $\mathbf{Top}^{\mathbf{Gr}}$ will play the same role as open subsets of topological spaces in ordinary differential geometry. The following definitions of intersections, restrictions, open covers and so on are intuitive and even provide one with a Grothendieck topology on $\mathbf{Top}^{\mathbf{Gr}}$ (see [1, Definition 3.17, p.591f.]).

Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{F} \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}$. For $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$, we set $\mathcal{F}_\Lambda := \mathcal{F}(\Lambda)$ and $\mathcal{F}_\varrho := \mathcal{F}(\varrho)$.

Definition 2.1.1. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{F}, \mathcal{F}' \in \mathbf{Set}^{\mathbf{Gr}^{(k)}}$. We call \mathcal{F}' a *subfunctor* of \mathcal{F} if for every $\Lambda \in \mathbf{Gr}^{(k)}$, we have $\mathcal{F}'_\Lambda \subseteq \mathcal{F}_\Lambda$ and these inclusions define a natural transformation $\mathcal{F}' \rightarrow \mathcal{F}$. In this situation, we write $\mathcal{F}' \subseteq \mathcal{F}$. For $\mathcal{F}, \mathcal{F}' \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}$ (or $\mathcal{F}, \mathcal{F}' \in \mathbf{Man}^{\mathbf{Gr}^{(k)}}$), we define subfunctors analogously. In this situation a subfunctor \mathcal{F}' of \mathcal{F} is called *open* if every \mathcal{F}'_Λ is open in \mathcal{F}_Λ .

Lemma/Definition 2.1.2. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{F} \in \mathbf{Set}^{\mathbf{Gr}^{(k)}}$. For a subset $U \subseteq \mathcal{F}_\mathbb{R}$, we define the *restriction* $\mathcal{F}|_U$ by setting

$$\mathcal{F}|_U(\Lambda) := (\mathcal{F}_{\varepsilon_\Lambda})^{-1}(U) \quad \text{for } \Lambda \in \mathbf{Gr}^{(k)}$$

and $\mathcal{F}|_U(\varrho) := \mathcal{F}_\varrho|_{\mathcal{F}|_U(\Lambda)}$ for morphisms $\varrho: \Lambda \rightarrow \Lambda'$. For functors $\mathcal{F} \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}$ (or $\mathcal{F} \in \mathbf{Man}^{\mathbf{Gr}^{(k)}}$), we define the restriction analogously for open subsets $U \subseteq \mathcal{F}_\mathbb{R}$. Then $\mathcal{F}|_U$ is an open subfunctor of \mathcal{F} .

Proof. Let $x \in \mathcal{F}|_U(\Lambda)$ and $\Lambda \in \mathbf{Gr}^{(k)}$. Then $\mathcal{F}_{\varepsilon_{\Lambda'}} \circ \mathcal{F}_\varrho(x) = \mathcal{F}_{\varepsilon_{\Lambda'} \circ \varrho}(x) = \mathcal{F}_{\varepsilon_\Lambda}(x) \in U$ holds for all morphisms $\varrho: \Lambda \rightarrow \Lambda'$ since $\varepsilon_{\Lambda'} \circ \varrho = \varepsilon_\Lambda$. In the topological case, we have that $\mathcal{F}_{\varepsilon_\Lambda}^{-1}(U)$ is open because $\mathcal{F}_{\varepsilon_\Lambda}$ is continuous. \square

Lemma/Definition 2.1.3. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{F} \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}$ and $\mathcal{F}', \mathcal{F}''$ be open subfunctors of \mathcal{F} . Then $(\mathcal{F}' \cap \mathcal{F}'')_\Lambda := \mathcal{F}'_\Lambda \cap \mathcal{F}''_\Lambda$ and $(\mathcal{F}' \cap \mathcal{F}'')_\varrho := \mathcal{F}_\varrho|_{(\mathcal{F}' \cap \mathcal{F}'')_\Lambda}$ for $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$ defines an open subfunctor $\mathcal{F}' \cap \mathcal{F}'' \subseteq \mathcal{F}$.

Proof. By definition $\mathcal{F}'_\Lambda \cap \mathcal{F}''_\Lambda$ is open in \mathcal{F}_Λ . If $x \in \mathcal{F}'_\Lambda \cap \mathcal{F}''_\Lambda$, then by functoriality $\mathcal{F}_\varrho(x) \in \mathcal{F}'_{\Lambda'} \cap \mathcal{F}''_{\Lambda'}$, which shows that $\mathcal{F}' \cap \mathcal{F}''$ is a functor and that the inclusion is a natural transformation. \square

Definition 2.1.4. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{F}, \mathcal{F}' \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}$. A natural transformation $f: \mathcal{F}' \rightarrow \mathcal{F}$ is called an *open embedding* if $f_\Lambda: \mathcal{F}'_\Lambda \rightarrow \mathcal{F}_\Lambda$ is an open embedding for every $\Lambda \in \mathbf{Gr}^{(k)}$.

Lemma/Definition 2.1.5. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{F}, \mathcal{F}' \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}$ and $f: \mathcal{F}' \rightarrow \mathcal{F}$ be a natural transformation. Let $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$ be arbitrary.

- (a) Let $\mathcal{V} \subseteq \mathcal{F}$ be an open subfunctor. Setting $f^{-1}\mathcal{V}_\Lambda := f_\Lambda^{-1}(\mathcal{V}_\Lambda)$ and $f^{-1}\mathcal{V}_\varrho := \mathcal{F}'_\varrho|_{f^{-1}\mathcal{V}_\Lambda}$ defines an open subfunctor $f^{-1}\mathcal{V} \subseteq \mathcal{F}'$.
- (b) If f is an open embedding, then $f(\mathcal{F}')_\Lambda := f_\Lambda(\mathcal{F}'_\Lambda)$ and $f(\mathcal{F}')_\varrho := \mathcal{F}_\varrho|_{f(\mathcal{F}')_\Lambda}$ define an open subfunctor $f(\mathcal{F}') \subseteq \mathcal{F}$.
- (c) Let $\mathcal{U} \subseteq \mathcal{F}'$ be an open subfunctor. Then $f|_{\mathcal{U}}(\Lambda) := f_\Lambda|_{\mathcal{U}_\Lambda}$ defines a natural transformation $f|_{\mathcal{U}}: \mathcal{U} \rightarrow \mathcal{F}$.

Proof. (a) Because f_Λ is continuous, $f^{-1}\mathcal{V}_\Lambda$ is open. For $x \in f^{-1}\mathcal{V}_\Lambda$, naturality of f implies $f_{\Lambda'}(\mathcal{F}'_\varrho(x)) = \mathcal{V}_\varrho(f_\Lambda(x))$ and therefore $f^{-1}\mathcal{V}_\varrho(x) \in f^{-1}\mathcal{F}_{\Lambda'}$.

(b) Because f is an open embedding, $f(\mathcal{F}')_\Lambda$ is open. For $x \in f(\mathcal{F}')_\Lambda$, naturality of f implies $f(\mathcal{F}')_\varrho(x) \in f(\mathcal{F}')_{\Lambda'}$.

(c) This is obvious. □

Definition 2.1.6. We call a set $\{f^\alpha: \mathcal{F}^\alpha \rightarrow \mathcal{F}: \alpha \in A\}^1$ of open embeddings a *covering* if $\bigcup_{\alpha \in A} f_\Lambda^\alpha(\mathcal{F}_\Lambda^\alpha) = \mathcal{F}_\Lambda$ holds for all $\Lambda \in \mathbf{Gr}^{(k)}$. In this situation, we define for all pairs $\alpha, \beta \in A$ an open subfunctor $\mathcal{F}^{\alpha\beta} \subseteq \mathcal{F}^\alpha$ by $\mathcal{F}_\Lambda^{\alpha\beta} := (f_\Lambda^\alpha)^{-1}(f_\Lambda^\alpha(\mathcal{F}_\Lambda^\alpha) \cap f_\Lambda^\beta(\mathcal{F}_\Lambda^\beta))$ and $\mathcal{F}_\varrho^{\alpha\beta} := \mathcal{F}_\varrho^\alpha|_{\mathcal{F}_\Lambda^{\alpha\beta}}$ as well as natural transformations $f^{\alpha\beta}: \mathcal{F}^{\alpha\beta} \rightarrow \mathcal{F}^{\beta\alpha}$ by $f_\Lambda^{\alpha\beta} := (f_\Lambda^\beta)^{-1} \circ f_\Lambda^\alpha|_{\mathcal{F}_\Lambda^{\alpha\beta}}$ for all $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and all morphisms $\varrho: \Lambda \rightarrow \Lambda'$.

Definition 2.1.7. For $k \in \mathbb{N}_0 \cup \{\infty\}$ a functor $\mathcal{F} \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}$ is called *Hausdorff* if \mathcal{F}_Λ is Hausdorff for every $\Lambda \in \mathbf{Gr}^{(k)}$.

2.2. Superdomains

Superdomains take the role of open subsets of vector spaces in ordinary analysis. Together with appropriately defined supersmooth morphisms between them, they enable us to define supermanifolds from local data much in the same way as for manifolds. The main result in this section is the description of supersmooth morphisms through so called skeletons in Proposition 2.2.13. Since skeletons will be our main tool for concrete calculations, other important results are a formula for the composition (see Proposition 2.2.16) and a formula for the inversion (see Lemma 2.2.18) in terms of skeletons. We follow [1] in this section, with only small additions to accommodate k -superdomains (i.e., certain functors $\mathbf{Gr}^{(k)} \rightarrow \mathbf{Top}$). With the exception of a concrete inversion formula, these results have already been stated in [38].

At the end, we briefly discuss the correspondence between multilinear algebraic structures of a given super type in **Set** (resp. **Top**) and algebraic structures of the respective ordinary type in **Set**^{Gr} (resp. **Top**^{Gr}).

¹As with atlases before, the index set A is just used for the sake of an easier notation and not part of the data of a covering.

Lemma/Definition 2.2.1. For every $E \in \mathbf{SVec}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, we get a functor $\overline{E}^{(k)} : \mathbf{Gr}^{(k)} \rightarrow \mathbf{Set}$ by setting

$$\overline{E}_\Lambda^{(k)} := \overline{E}^{(k)}(\Lambda) := (E_0 \otimes \Lambda_{\bar{0}}) \oplus (E_1 \otimes \Lambda_{\bar{1}})$$

on objects $\Lambda \in \mathbf{Gr}^{(k)}$ and $\overline{E}_\varrho^{(k)} := \overline{E}^{(k)}(\varrho) := (\text{id}_E \otimes \varrho)|_{\overline{E}_\Lambda^{(k)}}$ on morphisms $\varrho : \Lambda \rightarrow \Lambda'$ of Grassmann algebras. We abbreviate $\overline{E} := \overline{E}^{(\infty)}$, $\overline{E}_0^{(k)} := \overline{E}_0 \oplus \{0\}^{(k)}$ and $\overline{E}_1^{(k)} := \{0\} \oplus \overline{E}_1^{(k)}$ and let $\overline{\mathbb{R}}^{(k)} := \overline{\mathbb{R}} \oplus \{0\}^{(k)}$, i.e., $\overline{\mathbb{R}}_\Lambda^{(k)} = \Lambda_{\bar{0}}$. Then $\overline{E}^{(k)}$ is an $\overline{\mathbb{R}}^{(k)}$ -module in $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$. For $E \in \mathbf{SVec}_{lc}$, we have a functor

$$\overline{E}^{(k)} : \mathbf{Gr}^{(k)} \rightarrow \mathbf{Top},$$

by giving $\overline{E}_\Lambda^{(k)}$ the product topology. Then $\overline{E}_\Lambda^{(k)}$ is a locally convex space and an $\overline{\mathbb{R}}^{(k)}$ -module in $\mathbf{Top}^{\mathbf{Gr}^{(k)}}$.

If $\Delta \subseteq \Lambda$ is an \mathbb{R} -vector subspace, we set $\overline{E}_\Delta^{(k)} := (E_0 \otimes \Delta_{\bar{0}}) \oplus (E_1 \otimes \Delta_{\bar{1}})$, where $\Delta_{\bar{0}} := \Delta \cap \Lambda_{\bar{0}}$ and $\Delta_{\bar{1}} := \Delta \cap \Lambda_{\bar{1}}$. For $n \leq m \leq k$, we will always consider the natural embedding $\overline{E}_{\Lambda_n}^{(k)} \subseteq \overline{E}_{\Lambda_m}^{(k)}$ via $\overline{E}_{\eta_{n,m}}^{(k)}$.

Proof. It is easy to see that $\overline{E}^{(k)}$ is a functor. In the locally convex case, we give $E_0 \otimes \Lambda_{k,\bar{0}} = \prod_{I \in \mathcal{P}_0^k} \lambda_I E_0$, resp. $E_1 \otimes \Lambda_{k,\bar{1}} = \prod_{I \in \mathcal{P}_1^k} \lambda_I E_1$, the natural locally convex vector space structure. Since every $\Lambda \in \mathbf{Gr}$ is a $\Lambda_{\bar{0}}$ -algebra, $\overline{E}_\Lambda^{(k)}$ is a $\Lambda_{\bar{0}}$ -module with the obvious multiplication. This multiplication is continuous in the locally convex case because its components are simply finite linear combinations. For the same reason, $\overline{E}_\varrho^{(k)}$ is continuous and linear. That we have $\overline{\mathbb{R}}_\varrho^{(k)}(x) \cdot \overline{E}_\varrho^{(k)}(v) = \overline{E}_\varrho^{(k)}(x \cdot v)$ for all $x \in \overline{\mathbb{R}}_\Lambda^{(k)}$ and $v \in \overline{E}_\Lambda^{(k)}$, follows directly from the definition of the multiplication. \square

In the definition of super manifolds the functors \overline{E} will play the same role as vector spaces do for regular manifolds. Accordingly, we need a notion of open subfunctors and appropriate “smooth morphisms” between open subfunctors. All open subfunctors of $\mathcal{U} \subseteq \overline{E}$ for $E \in \mathbf{SVec}_{lc}$ are uniquely determined by $\mathcal{U}_{\mathbb{R}}$.

Lemma 2.2.2. Let $k \in \mathbb{N} \cup \{\infty\}$ and $E \in \mathbf{SVec}_{lc}$. Recall the restriction from Lemma/Definition 2.1.2. Every open subfunctor $\mathcal{U} \subseteq \overline{E}^{(k)}$ arises as such a restriction, i.e., we have $\overline{E}^{(k)}|_{\mathcal{U}_{\mathbb{R}}} = \mathcal{U}$.

Proof. For $k = \infty$ this is just [45, Proposition 3.5.8, p. 61]. The same proof holds for $k \in \mathbb{N}_0$ if one only considers $\Lambda \in \mathbf{Gr}^{(k)}$ (see also [40, Section 3.1, p.388 f.]). \square

Definition 2.2.3. Let $E, F \in \mathbf{SVec}_{lc}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. We call an open subfunctor $\mathcal{U} \subseteq \overline{E}^{(k)}$ a *k-superdomain*. In the case of $k = \infty$ we simply call it a *superdomain*. A natural transformation $f : \mathcal{U} \rightarrow \mathcal{V}$ of *k-superdomains* $\mathcal{U} \subseteq \overline{E}^{(k)}$

and $\mathcal{V} \subseteq \overline{F}^{(k)}$ is called *supersmooth* if for all $\Lambda \in \mathbf{Gr}^{(k)}$ the map $f_\Lambda: \mathcal{U}_\Lambda \rightarrow \mathcal{V}_\Lambda$ is smooth and the derivative

$$df_\Lambda: \mathcal{U}_\Lambda \times \overline{E}_\Lambda^{(k)} \rightarrow \overline{F}_\Lambda^{(k)}$$

is $\Lambda_{\bar{0}}$ -linear in the second component, i.e., for any $x \in \mathcal{U}_\Lambda$, the map

$$df_\Lambda(x, \bullet): \overline{E}_\Lambda^{(k)} \rightarrow \overline{F}_\Lambda^{(k)}, \quad v \mapsto df_\Lambda(x)(v)$$

is $\Lambda_{\bar{0}}$ -linear. We denote by $\mathcal{SC}^\infty(\mathcal{U}, \mathcal{V})$ the set of all supersmooth morphisms $f: \mathcal{U} \rightarrow \mathcal{V}$.

It is obvious from the usual chain rule that the k -superdomains together with the supersmooth natural transformations form a category, which we denote by $\mathbf{SDom}^{(k)}$. In the case of $k = \infty$, we also use the notation \mathbf{SDom} .

Note that for \mathbb{R} -linear maps, it suffices to check $\Lambda_{\bar{0}}$ -linearity on the generators: For $E, F \in \mathbf{SVec}_{lc}$ and an \mathbb{R} -linear map $L: \overline{E}_{\Lambda_n} \rightarrow \overline{F}_{\Lambda_n}$ with $L(\lambda_I x) = \lambda_I L(x)$ for all $x \in \overline{E}_\Lambda$ and $\lambda_I \in \Lambda_{n, \bar{0}}$, we have

$$L(t \cdot x) = L\left(\sum_{I \in \mathcal{P}_0^n} \lambda_I t_I \cdot x\right) = \sum_{I \in \mathcal{P}_0^n} \lambda_I t_I \cdot L(x) = t \cdot L(x),$$

where $t = \sum_{I \in \mathcal{P}_0^n} \lambda_I t_I \in \Lambda_{n, \bar{0}}$, $t_I \in \mathbb{R}$. As it turns out, even natural transformations that are merely “smooth” already have very convenient properties.

Lemma 2.2.4. *Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor and $f: \mathcal{U} \rightarrow \overline{F}^{(k)}$ be a natural transformation such that f_Λ is smooth for all $\Lambda \in \mathbf{Gr}^{(k)}$. Then for all $n \in \mathbb{N}_0$, the maps $d^n f_\Lambda$ define a natural transformation*

$$d^n f: \mathcal{U} \times \overline{E}^{(k)} \times \cdots \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}.$$

Proof. Let $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and let $\varrho: \Lambda \rightarrow \Lambda'$ be a morphism. Because we have $\overline{F}_\varrho^{(k)} \circ f_\Lambda = f_{\Lambda'} \circ \mathcal{U}_\varrho$ and $\overline{E}_\varrho^{(k)}|_{\mathcal{U}_\Lambda} = \mathcal{U}_{\varrho'}$, Corollary 1.3.4 implies that

$$\overline{F}_\varrho^{(k)} \circ d^n f_\Lambda = d^n f_{\Lambda'} \circ (\mathcal{U}_\varrho \times \overline{E}_\varrho^{(k)} \times \cdots \times \overline{E}_\varrho^{(k)}).$$

Thus $d^n f$ is a natural transformation. Compare also [42, Lemma 3.6.5, p.812f.] and [1, Lemma 2.15, p.577]. \square

In the situation of the lemma, we write df for the natural transformation defined by df_Λ .

Lemma 2.2.5. *Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor and $f: \mathcal{U} \rightarrow \overline{F}^{(k)}$ be a natural transformation such that f_Λ is smooth for all $\Lambda \in \mathbf{Gr}^{(k)}$. For $n, m \in \mathbb{N}$, $\Lambda_m \in \mathbf{Gr}^{(k)}$ let $x \in \mathcal{U}_{\mathbb{R}} \subseteq \mathcal{U}_{\Lambda_m}$ and $y_i \in \lambda_{I_i} E_{|I_i|} \subseteq \overline{E}_{\Lambda_m}^{(k)}$, where $I_i \in \mathcal{P}_+^m$ and $1 \leq i \leq n$. Then, we have*

$$d^n f_{\Lambda_m}(x)(y_1, \dots, y_n) \in \lambda_{I_1} \cdots \lambda_{I_n} F_{\bar{\ell}} \subseteq \overline{F}_{\Lambda_m}^{(k)},$$

for $\ell := |\bigcup_{i=1}^n I_i|$. If the sets I_i are not pairwise disjoint, we have $d^n f_{\Lambda_m}(x)(y_1, \dots, y_n) = 0$.

Proof. Consider $d^n f_{\Lambda_m}$ as a map into $\prod_{I \in \mathcal{P}^m} \lambda_I F_{|I|}$. Let $I := \bigcup_{i=1}^n I_i$, $p \in I$ and define $\varrho: \Lambda_m \rightarrow \Lambda_m$ by $\varrho(\lambda_p) = 0$ and $\varrho(\lambda_j) = \lambda_j$ for $j \neq p$. By Lemma 2.2.4, we have

$$0 = d^n f_{\Lambda_m}(\mathcal{U}_{\varrho}(x))(\overline{E}_{\varrho}^{(k)}(y_1), \dots, \overline{E}_{\varrho}^{(k)}(y_n)) = \overline{F}_{\varrho}^{(k)}(d^n f_{\Lambda_m}(x)(y_1, \dots, y_n)).$$

In other words, all components that do not contain λ_p are zero. Conversely, let $p' \notin I$ and let $\varrho': \Lambda_m \rightarrow \Lambda_m$ be a morphism given by $\varrho'(\lambda_{p'}) = 0$ and $\varrho'(\lambda_j) = \lambda_j$ for $j \neq p'$. Then, we have

$$d^n f_{\Lambda_m}(\mathcal{U}_{\varrho'}(x))(\overline{E}_{\varrho'}^{(k)}(y_1), \dots, \overline{E}_{\varrho'}^{(k)}(y_n)) = d^n f_{\Lambda_m}(x)(y_1, \dots, y_n),$$

but all components of $\overline{F}_{\varrho'}^{(k)}(d^n f_{\Lambda_m}(x)(y_1, \dots, y_n))$ that contain $\lambda_{p'}$ vanish. It follows that $d^n f_{\Lambda_m}(x)(y_1, \dots, y_n) \in \lambda_{I_1} \cdots \lambda_{I_n} F_{\ell}$. Finally, assume that the sets I_i are not pairwise disjoint, for instance let p'' occur in $r > 1$ sets. For $c \in \mathbb{R}$, we define a morphism $\varrho'': \Lambda_m \rightarrow \Lambda_m$ by $\varrho''(\lambda_{p''}) := c\lambda_{p''}$ and $\varrho''(\lambda_j) := \lambda_j$ for $j \neq p''$. We have

$$d^n f_{\Lambda_m}(\mathcal{U}_{\varrho''}(x))(\overline{E}_{\varrho''}^{(k)}(y_1), \dots, \overline{E}_{\varrho''}^{(k)}(y_n)) = c^r d^n f_{\Lambda_m}(x)(y_1, \dots, y_n).$$

But we also have $(\overline{F}_{\varrho''}^{(k)}(d^n f_{\Lambda_m}(x)(y_1, \dots, y_n)))_I = c(d^n f_{\Lambda_m}(x)(y_1, \dots, y_n))_I$, which implies $d^n f_{\Lambda_m}(x)(y_1, \dots, y_n) = 0$. \square

The next lemma, a variation of [1, Proposition 2.16, p.578], is one of the rare cases where the proof for superdomains does not automatically translate to k -superdomains. In a sense, it shows the infinitesimal character of the generators λ_i .

Lemma 2.2.6. *Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor and $f: \mathcal{U} \rightarrow \overline{F}^{(k)}$ be a natural transformation such that f_{Λ} is smooth for all $\Lambda \in \mathbf{Gr}^{(k)}$. Let $1 \leq p \leq k$, $x \in \mathcal{U}_{\Lambda} \setminus \overline{E}_{\lambda_p \Lambda}^{(k)}$ and $y \in \overline{E}_{\lambda_p \Lambda}^{(k)}$. Then, we have*

$$f_{\Lambda}(x + y) = f_{\Lambda}(x) + df_{\Lambda}(x)(y).$$

Proof. Let $c \in \mathbb{R}$. We define a morphism $\varrho_c: \Lambda \rightarrow \Lambda$ by $\varrho_c(\lambda_p) := c\lambda_p$ and $\varrho_c(\lambda_i) := \lambda_i$ for $i \neq p$. Then $\mathcal{U}_{\varrho_c}(x) = x$ and $\overline{E}_{\varrho_c}^{(k)}(y) = cy$. Therefore, we have

$$f_{\Lambda}(\overline{E}_{\varrho_0}^{(k)}(x + y)) - f_{\Lambda}(\overline{E}_{\varrho_0}^{(k)}(x)) = 0 = \overline{F}_{\varrho_0}^{(k)}(f_{\Lambda}(x + y) - f_{\Lambda}(x))$$

and thus $f_{\Lambda}(x + y) - f_{\Lambda}(x) \in \overline{F}_{\lambda_p \Lambda}^{(k)}$. It follows that

$$\begin{aligned} c \cdot f_{\Lambda}^{[1]}(x, y, c) &= f_{\Lambda}(\overline{E}_{\varrho_c}^{(k)}(x + y)) - f_{\Lambda}(\overline{E}_{\varrho_c}^{(k)}(x)) \\ &= \overline{F}_{\varrho_c}^{(k)}(f_{\Lambda}(x + y) - f_{\Lambda}(x)) = c \cdot f_{\Lambda}^{[1]}(x, y, 1). \end{aligned}$$

Taking the limit $c \rightarrow 0$, we see that $f_\Lambda^{[1]}(x, y, 0) = f_\Lambda^{[1]}(x, y, 1)$ or in other words $f_\Lambda(x + y) - f_\Lambda(x) = df_\Lambda(x, y)$ (compare [1, Proposition 2.16, p.578]). \square

Accordingly, we get the following variation of [1, Corollary 2.17, p.579].

Proposition 2.2.7. *Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor and $f: \mathcal{U} \rightarrow \overline{F}^{(k)}$ be a natural transformation such that f_Λ is smooth for all $\Lambda \in \mathbf{Gr}^{(k)}$. For $x := x_0 + \sum_{I \in \mathcal{P}_+^n} x_I \in \mathcal{U}_{\Lambda_n}$, where $n \leq k$, $x_0 \in \mathcal{U}_{\mathbb{R}}$ and $x_I \in \lambda_I E_{\overline{[I]}}$, we have*

$$f_{\Lambda_n}(x) = f_{\Lambda_n}(x_0) + \sum_{I \in \mathcal{P}_+^n} \sum_{\omega \in \mathcal{P}(I)} d^{\ell(\omega)} f_{\Lambda_n}(x_0)(x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}}).$$

Proof. We first define a suitable partition of \mathcal{P}_+^n . Let $\mathcal{I}_1 := \{\{1\}\}$ and $\mathcal{I}_j := \mathcal{P}_+^j \setminus \mathcal{P}_+^{j-1}$ for $1 < j \leq n$, i.e., \mathcal{I}_j contains all subsets that contain j but no larger index. Set $x_{\mathcal{I}_j} := \sum_{I \in \mathcal{I}_j} x_I$; then we can write $x = x_0 + \sum_{j=1}^n x_{\mathcal{I}_j}$. We prove the proposition by induction on the largest index of an odd generator appearing in x . Lemma 2.2.6 gives us the induction basis. Assume that the formula holds for $1 \leq m < n$, i.e., assume that

$$f_{\Lambda_n}(x_0 + \sum_{j=1}^m x_{\mathcal{I}_j}) = f_{\Lambda_n}(x_0) + \sum_{I \in \mathcal{P}_+^m} \sum_{\omega \in \mathcal{P}(I)} d^{\ell(\omega)} f_{\Lambda_n}(x_0)(x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}}).$$

With this, differentiating in the direction of $x_{\mathcal{I}_{m+1}}$ gives us

$$\begin{aligned} df_{\Lambda_n}(x_0 + \sum_{j=1}^m x_{\mathcal{I}_j})(x_{\mathcal{I}_{m+1}}) &= df_{\Lambda_n}(x_0)(x_{\mathcal{I}_{m+1}}) + \\ &\quad \sum_{I \in \mathcal{P}_+^m} \sum_{\omega \in \mathcal{P}(I)} d^{\ell(\omega)+1} f_{\Lambda_n}(x_0)(x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}}, x_{\mathcal{I}_{m+1}}) \\ &= \sum_{I \in \mathcal{I}_{m+1}} \sum_{\omega \in \mathcal{P}(I)} d^{\ell(\omega)} f_{\Lambda_n}(x_0)(x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}}). \end{aligned}$$

It follows from Lemma 2.2.6 that the addition of both equations results in the desired formula for $f_{\Lambda_n}(x_0 + \sum_{j=1}^{m+1} x_{\mathcal{I}_j})$ (compare [40, Section 10.2, p.421]). \square

The proposition can be rewritten in the following way.

Lemma 2.2.8. *Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor and $f: \mathcal{U} \rightarrow \overline{F}^{(k)}$ be a natural transformation such that f_Λ is smooth for all $\Lambda \in \mathbf{Gr}^{(k)}$. For $\Lambda \in \mathbf{Gr}^{(k)}$ fix $x \in \mathcal{U}_{\mathbb{R}}$, $n_0 \in \overline{E}_{\Lambda_+}^{(k)}$ and $n_1 \in \overline{E}_{\Lambda_-}^{(k)}$. Then*

$$\begin{aligned} f_\Lambda(x + n_0 + n_1) &= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \cdot d^{m+l} f_\Lambda(x) \underbrace{(n_0, \dots, n_0)}_{m \text{ times}} \underbrace{(n_1, \dots, n_1)}_{l \text{ times}} \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} \cdot d^i f_\Lambda(x)(n_0 + n_1, \dots, n_0 + n_1). \end{aligned}$$

Proof. Let $\Lambda = \Lambda_n$. By Lemma 2.2.5 the sums are finite and after multilinear expansion we only need to consider the summands that consist of partitions. For any partition $\mathcal{I} \in \mathcal{P}(I)$, $I \in \mathcal{P}_+^n$ in graded lexicographic order containing m even and l odd sets, there appear exactly $m!!l!$ copies of the term

$$d^{m+l}f_\Lambda(x)(n_{0,\mathcal{I}_1}, \dots, n_{0,\mathcal{I}_m}, n_{1,\mathcal{I}_{m+1}}, \dots, n_{1,\mathcal{I}_{m+l}})$$

in the first sum because we have to consider all permutations of $n_{0,\mathcal{I}_1}, \dots, n_{0,\mathcal{I}_m}$, resp. of $n_{1,\mathcal{I}_{m+1}}, \dots, n_{1,\mathcal{I}_{m+l}}$. The first equality follows then from Proposition 2.2.7 (see also [1, Proposition 2.21, p.582]). The second equality holds because multilinear expansion of $d^{m+l}f_\Lambda(x)(n_0 + n_1, \dots, n_0 + n_1)$ leads to $\binom{m+l}{l}$ copies of $d^{m+l}f_\Lambda(x)(n_0, \dots, n_0, n_1, \dots, n_1)$ (m times n_0 and l times n_1) and $\binom{m+l}{l} \cdot \frac{1}{(m+l)!} = \frac{1}{m!!l!}$. \square

Corollary 2.2.9. *Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor and $f: \mathcal{U} \rightarrow \overline{F}^{(k)}$ be a natural transformation such that f_Λ is smooth for all $\Lambda \in \mathbf{Gr}^{(k)}$. If additionally $df_\Lambda(x_0): \overline{E}_\Lambda^{(k)} \rightarrow \overline{F}_\Lambda^{(k)}$ is $\Lambda_{\overline{0}}$ -linear for all $x_0 \in \mathcal{U}_{\mathbb{R}}$, then f is supersmooth.*

Proof. Let $\Lambda = \Lambda_n$. Because $df_\Lambda(x_0)$ is $\Lambda_{\overline{0}}$ -linear it follows by symmetry of the higher derivatives that $d^m f_\Lambda(x_0): \overline{E}_\Lambda^{(k)} \times \dots \times \overline{E}_\Lambda^{(k)} \rightarrow \overline{F}_\Lambda^{(k)}$ is $\Lambda_{\overline{0}}$ - m -multilinear for all $m \in \mathbb{N}$. Let $x = x_0 + \sum_{I \in \mathcal{P}_+^n} x_I$ and $y = y_0 + \sum_{I \in \mathcal{P}_+^n} y_I$ where $x_I, y_I \in \lambda_I E_{|I|}$ and $t \in \Lambda_{\overline{0}}$. With Proposition 2.2.7, we calculate

$$\begin{aligned} df_\Lambda(x)(ty) &= d\left(f_\Lambda(x_0) + \sum_{I \in \mathcal{P}_+^n} \sum_{\omega \in \mathcal{P}(I)} d^{\ell(\omega)} f_\Lambda(x_0)(x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}})\right)(ty) \\ &= df_\Lambda(x_0)(ty) + \sum_{I \in \mathcal{P}_+^n} \sum_{\omega \in \mathcal{P}(I)} d^{\ell(\omega)+1} f_\Lambda(x_0)(x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}}, ty) \\ &= t\left(df_\Lambda(x_0)(y) + \sum_{I \in \mathcal{P}_+^n} \sum_{\omega \in \mathcal{P}(I)} d^{\ell(\omega)+1} f_\Lambda(x_0)(x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}}, y)\right). \quad \square \end{aligned}$$

This was already stated in [40, Theorem 3.3.2, p.391] without proof. The corollary simplifies some calculations considerably. A small example is the next lemma.

Lemma 2.2.10. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E, F \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor. If $f: \mathcal{U} \rightarrow \overline{F}^{(k)}$ is supersmooth, then $df: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ is supersmooth as well.*

Proof. By Corollary 2.2.9, it suffices to calculate

$$\begin{aligned} d(df_\Lambda(x_0, y_0))(t \cdot u, t \cdot v) &= d^2 f_\Lambda(x_0)(y_0, t \cdot u) + df_\Lambda(x_0)(t \cdot v) \\ &= t \cdot d^2 f_\Lambda(x_0)(y_0, u) + t \cdot df_\Lambda(x_0)(v) = t \cdot (d(df_\Lambda(x_0, y_0))(v, u)), \end{aligned}$$

for $(x_0, y_0) \in \mathcal{U}_{\mathbb{R}} \times \overline{E}_{\mathbb{R}}^{(k)}$, $(v, u) \in \overline{E}_\Lambda^{(k)} \times \overline{E}_\Lambda^{(k)}$ and $t \in \Lambda_{\overline{0}}$. \square

By induction, it follows that all higher derivatives of supersmooth maps are supersmooth again. A more general but also more involved version was proved in [1, Proposition 2.18, p.580].

We will now give an explicit description of supersmooth morphisms as so called skeletons, which is essential for almost all applications. It was already stated in [40, Proposition 3.3.3, p.391] and proofs can be found in [46, Theorem 4.11, p.20] or in higher generality in [1, Proposition 3.4, p.584].

Definition 2.2.11. Let $n \in \mathbb{N}$, let E_0, \dots, E_n and F be locally convex spaces and $U \subseteq E_0$ open. Denote by $\mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F))$ the set of maps $f: U \rightarrow \mathcal{L}^n(E_1, \dots, E_n; F)$ such that

$$f^\wedge: U \times (E_1 \times \dots \times E_n) \rightarrow F, \quad f^\wedge(x, v) := f(x)(v)$$

is smooth. In this situation, we define

$$d^m f(x)(w, v) := \partial_{(w_m, 0)} \dots \partial_{(w_1, 0)} f^\wedge(x, v),$$

for $m \in \mathbb{N}$, $x \in U$, $v \in E_1 \times \dots \times E_n$ and $w = (w_1, \dots, w_m) \in E_0^m$. Analogously, we define $\mathcal{C}^\infty(U, \mathcal{A}ltn(E_1; F))$ as the set of maps $f: U \rightarrow \mathcal{A}ltn(E_1; F)$ that are smooth in the above sense.

Definition 2.2.12. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E, F \in \mathbf{SVec}_{lc}$ and $U \subseteq E_0$ open. A $(k-)$ skeleton is a family of maps $(f_n)_{0 \leq n < k+1}$ such that $f_n \in \mathcal{C}^\infty(U, \mathcal{A}ltn^n(E_1; F_n))$. It will be convenient to set $d^0 f_n := f_n$ and let $d^0 f_n(x)(w_1, \dots, w_m, v) := d^0 f_n(x)(v)$ as well as $d^m f_0(x)(w, v) := d^m f_0(x)(w)$ for $x \in U$, $w = (w_1, \dots, w_m) \in E_0^m$ and $v \in E_1^n$.

Proposition 2.2.13 ([1, Proposition 3.4, p.584]). *Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$, $\mathcal{V} \subseteq \overline{F}^{(k)}$ be open subfunctors and $f \in \mathcal{SC}^\infty(\mathcal{U}, \mathcal{V})$. Then the equation*

$$f_{\Lambda_k}(x + \sum_{l=1}^k \lambda_l y_l) = f_0(x) + \sum_{l=1}^k \sum_{\{i_1, \dots, i_l\} \in \mathcal{P}^k} \lambda_l f_l(x)(y_{i_1}, \dots, y_{i_l}),$$

where $x \in \mathcal{U}_{\mathbb{R}}$ and $y_l \in E_1$, defines a k -skeleton $(f_n)_n$. For this skeleton, we have

$$f_{\Lambda_N}(x + n_0 + n_1) = \sum_{m, l=0}^{\infty} \frac{1}{m!l!} \cdot d^m f_l(x)(\underbrace{n_0, \dots, n_0}_{m \text{ times}}, \underbrace{n_1, \dots, n_1}_{l \text{ times}}), \quad (2.1)$$

where $x \in \mathcal{U}_{\mathbb{R}}$, $n_0 \in \overline{E}_{\Lambda_{N,0}^+}^{(k)}$, $n_1 \in \overline{E}_{\Lambda_{N,1}}^{(k)}$ and $N \leq k$. Here it is understood that

$$d^m f_l(x)(\lambda_{I_1} v_1, \dots, \lambda_{I_{l+m}} v_{l+m}) = \lambda_{I_1} \dots \lambda_{I_{l+m}} d^m f_l(x)(v_1, \dots, v_{l+m})$$

for $v_1, \dots, v_m \in E_0$, $v_{m+1}, \dots, v_{m+l} \in E_1$ and $|I_j|$ even if $1 \leq j \leq m$ and odd if $m+1 \leq j \leq m+l$. Conversely, every k -skeleton defines a supersmooth map via formula (2.1) and the skeleton of this map is the original one.

Proof. Using Lemma 2.2.8 instead of [1, Proposition 2.21, p.582], the proof follows in the same way as [1, Proposition 3.4, p.584]. For the reader's convenience, we will sketch the steps using our notation. Let $N \leq k$. By Proposition 2.2.7 we have

$$f_{\Lambda_N}(x + \sum_{l=1}^N \lambda_l y_l) = \sum_{l=0}^N \sum_{\{i_1, \dots, i_l\} \in \mathcal{P}^N} d^l f_{\Lambda_N}(x)(\lambda_{i_1} y_{i_1}, \dots, \lambda_{i_l} y_{i_l}).$$

The maps on the right-hand side are symmetric in $\lambda_{i_j} y_{i_j}$ but swapping two odd generators leads to a sign change by the natural transformation property. With Lemma 2.2.5 one sees that this determines alternating maps in y_{i_j} , where it is understood that the odd generators can be pulled out in order of their appearance. Now, one applies Proposition 2.2.7 to derive formula (2.1). Note that by supersmoothness, the alternating maps defined above determine $d^{m+l} f_{\Lambda_N}$ completely.

To see that the right-hand side of formula (2.1) defines a natural transformation for a given skeleton is straightforward and supersmoothness then follows directly or with Corollary 2.2.9. This supermooth map has the original skeleton by a combinatorial argument similar to the one used for Lemma 2.2.8. \square

Remark 2.2.14. In the situation of the proposition above, we can use Proposition 2.2.7 instead of Lemma 2.2.8 to get

$$f_{\Lambda_N}(x + n_0 + n_1) = \sum_{I \in \mathcal{P}_+^N} \sum_{\omega \in \mathcal{P}(I)} \lambda_\omega d^{(e(\omega))} f_{o(\omega)}(x)(n_\omega),$$

where the partitions ω are in graded lexicographic order, $\lambda_\omega = \lambda_{\omega_1} \cdots \lambda_{\omega_{\ell(\omega)}}$ and $n_\omega := (n_{0,\omega_1}, \dots, n_{0,\omega_{\ell(\omega)}}, n_{1,\omega_{\ell(\omega)+1}}, \dots, n_{1,\ell(\omega)})$.

Remark 2.2.15. Let $f: \mathcal{U} \rightarrow \mathcal{V}$ be as in Proposition 2.2.13. We have already seen that $df: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ is supersmooth. For $\Lambda \in \mathbf{Gr}^{(k)}$, $x \in \mathcal{U}_{\mathbb{R}}$, $y \in E_0$ and $x_i, y_i \in E_i \otimes \Lambda_i^+$ set $u := x + x_0 + x_1$ and $v := y + y_0 + y_1$. Then use the proposition to calculate

$$\begin{aligned} df_\Lambda(u)(v) &= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \cdot \left(d^{m+1} f_l(x)(y, x_0, \dots, x_0, x_1, \dots, x_1) \right. \\ &\quad \left. + m \cdot d^m f_l(x)(y_0, x_0, \dots, x_0, x_1, \dots, x_1) \right. \\ &\quad \left. + l \cdot d^m f_l(x)(x_0, \dots, x_0, y_1, x_1, \dots, x_1) \right) \\ &= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \cdot \left(d^{m+1} f_l(x)(y + y_0, x_0, \dots, x_0, x_1, \dots, x_1) \right) + \\ &\quad \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \cdot \left(d^m f_{l+1}(x)(x_0, \dots, x_0, y_1, x_1, \dots, x_1) \right). \end{aligned}$$

We see that the skeleton of df is given by

$$(df)_n = df_n(\text{pr}_{\mathcal{U}_{\mathbb{R}}}, \text{pr}_{E_0})(\text{pr}_1, \dots, \text{pr}_1) + n \cdot \mathfrak{A}^n f_n(\text{pr}_{\mathcal{U}_{\mathbb{R}}})(\text{pr}_2, \text{pr}_1, \dots, \text{pr}_1),$$

with the projections $\text{pr}_{\mathcal{U}_{\mathbb{R}}}: \mathcal{U}_{\mathbb{R}} \times E_0 \rightarrow \mathcal{U}_{\mathbb{R}}$, $\text{pr}_{E_0}: \mathcal{U}_{\mathbb{R}} \times E_0 \rightarrow E_0$, the projection to the first component $\text{pr}_1: E_1 \times E_1 \rightarrow E_1$ and the projection to the second argument $\text{pr}_2: E_1 \times E_1 \rightarrow E_1$.

In the sequel, we will not differentiate between supersmooth morphisms and their skeletons. In other words, if $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are k -superdomains and $f \in \mathcal{SC}^\infty(\mathcal{U}, \mathcal{V})$ has the skeleton $(f_n)_n$, we will write $(f_n)_n: \mathcal{U} \rightarrow \mathcal{V}$. If additionally $g \in \mathcal{SC}^\infty(\mathcal{V}, \mathcal{W})$ has the skeleton $(g_n)_n$ we let $(g_n)_n \circ (f_n)_n$ be the skeleton of $g \circ f$. For this composition the concrete formula is given as follows.

Proposition 2.2.16 (compare [1, Proposition 3.7, p.586]). *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E \in \mathbf{SVec}_{lc}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor and $\mathcal{V}, \mathcal{W} \in \mathbf{SDom}^{(k)}$. For two supersmooth morphisms $(f_r)_r: \mathcal{U} \rightarrow \mathcal{V}$ and $(g_r)_r: \mathcal{V} \rightarrow \mathcal{W}$ the skeleton $(h_n)_n := (g_r)_r \circ (f_r)_r$ is given by $h_0 := g_0 \circ f_0$ for $n = 0$ and otherwise by*

$$h_n(x)(v) = \sum_{\substack{m, l; \sigma \in \mathfrak{S}_n, \\ (\alpha, \beta) \in I_{m, l}^n}} \frac{\text{sgn}(\sigma)}{m! l! \alpha! \beta!} d^m g_l(f_0(x))((f_\alpha \times f_\beta)(x)(v^\sigma)) \quad (2.2)$$

for $x \in \mathcal{U}_{\mathbb{R}}$ and $v = (v_1, \dots, v_n) \in E_1^n$, where $v^\sigma := (v_{\sigma(1)}, \dots, v_{\sigma(n)})$,

$$\begin{aligned} I_{m, l}^n &:= \{(\alpha, \beta) \in (2\mathbb{N})^m \times (2\mathbb{N}_0 + 1)^l \mid |\alpha| + |\beta| = n\}, \\ f_\alpha &:= f_{\alpha_1} \times \dots \times f_{\alpha_m}, \quad f_\beta := f_{\beta_1} \times \dots \times f_{\beta_l} \text{ and} \\ \alpha! &= \alpha_1! \dots \alpha_m!, \quad \beta! = \beta_1! \dots \beta_l!. \end{aligned}$$

Proof. By Proposition 2.2.13 $(h_n)_n$ is defined by

$$g_\Lambda(f_\Lambda(x + y)) = \sum_{l=0}^{\infty} \frac{1}{l!} h_l(x)(y, \dots, y) \text{ for all } \Lambda = \Lambda_n \in \mathbf{Gr}^{(k)},$$

where $x \in \mathcal{U}_{\mathbb{R}}$, and $y = \sum_{j=1}^n \lambda_j y_j \in \overline{E}_\Lambda^{(k)}$. For $i \in \{0, 1\}$, we let

$$n_i := \sum_{l \in 2\mathbb{N}-i} \frac{1}{l!} f_l(x)(y, \dots, y).$$

Together with Proposition 2.2.13, this implies

$$g_\Lambda(f_\Lambda(x + y)) = \sum_{m, l=0}^{\infty} \frac{1}{m! l!} d^m g_l(f_0(x))(n_0, \dots, n_0, n_1, \dots, n_1). \quad (2.3)$$

Since in formula (2.2), h_n only depends on $(f_r)_{r \leq n}$ and $(g_r)_{r \leq n}$, it suffices to compare the component containing all odd generators of $\Lambda = \Lambda_n$, i.e., the component $I := \{1, \dots, n\}$. The formula follows then by trivial induction. Multilinear expansion of the n_i in formula (2.3) shows that exactly those summands contribute, where the indices of all occurring f_i add up to n . In other words exactly those containing $f_\alpha \times f_\beta$ with $(\alpha, \beta) \in I_{m, l}^n$. Applying multilin-

ear expansion to y , we see that for every $(\alpha, \beta) \in I_{m,l}^n$ exactly all permutations $\lambda_{\sigma(1)} \cdots \lambda_{\sigma(n)} \frac{1}{\alpha! \beta!} (f_\alpha \times f_\beta)(y_{\sigma(1)}, \dots, y_{\sigma(n)})$ for $\sigma \in \mathfrak{S}_n$ appear in formula (2.3) since equal indices cancel each other. The sign in the formula is explained by $\lambda_{\sigma(1)} \cdots \lambda_{\sigma(n)} = \text{sgn}(\sigma) \lambda_I$. \square

Remark 2.2.17. Formula (2.2) was already stated in [40, Proposition 3.3.3, p.91 f.] but the first proof in the literature was [1, Proposition 3.7, p.586]. Unfortunately, the proof is incomplete and there is a small mistake in the formula (the original one in [40] is correct), which is why we decided to give the proof in its entirety. To see that our formula differs from the one proposed in [1], consider that in the situation of Proposition 2.2.16 the latter leads to

$$\sum_{\sigma \in \mathfrak{S}_2} \frac{1}{2} dg_1(f_0(x))(f_2(x)(\cdot^\sigma), f_1(x)(\cdot))(v_1, v_2, v_3) = 0,$$

while in general

$$\sum_{\sigma \in \mathfrak{S}_3} \frac{\text{sgn}(\sigma)}{2} dg_1(f_0(x))(f_2(x)(\cdot), f_1(x)(\cdot))(v_1, v_2, v_3)^\sigma \neq 0.$$

Lemma 2.2.18. *Let $k \in \mathbb{N} \cup \{\infty\}$, $E, F \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$, $\mathcal{V} \subseteq \overline{F}^{(k)}$ be open subfunctors. A supersmooth morphism $f: \mathcal{U} \rightarrow \mathcal{V}$ is an isomorphism in $\mathbf{SDom}^{(k)}$ if and only if $f_{\Lambda_1}: \mathcal{U}_{\Lambda_1} \rightarrow \mathcal{V}_{\Lambda_1}$ is a diffeomorphism. In this case, using the same notation as in formula (2.2), the inverse g has the skeleton*

$$\begin{aligned} g_0: \mathcal{V}_{\mathbb{R}} &\rightarrow \mathcal{U}_{\mathbb{R}}, & g_0(x') &:= f_0^{-1}(x'), \\ g_1: \mathcal{V}_{\mathbb{R}} &\rightarrow \mathcal{Alt}^1(F_1; E_1), & g_1(x') &:= f_1(g_0(x'))^{-1} \quad \text{and} \\ g_n: \mathcal{V}_{\mathbb{R}} &\rightarrow \mathcal{Alt}^n(F_1; E_1), \\ g_n(x')(v') &:= - \sum_{\substack{m, l < n, (\alpha, \beta) \in I_{m,l}^n, \\ \sigma \in \mathfrak{S}_n}} \frac{\text{sgn}(\sigma)}{m! l! \alpha! \beta!} d^m g_l(x')((f_\alpha \times f_\beta)(g_0(x'))(v^\sigma)), \end{aligned}$$

where $n > 1$, $v' = (v'_1, \dots, v'_n) \in F_1^n$ and $v := (g_1(x')(v'_1), \dots, g_1(x')(v'_n)) \in E_1^n$.

Proof. If a supersmooth morphism $f: \mathcal{U} \rightarrow \mathcal{V}$ is invertible, then clearly f_Λ is a diffeomorphism for every $\Lambda \in \mathbf{Gr}^{(k)}$. Conversely, let f_{Λ_1} be a diffeomorphism. Then $f_{\Lambda_1}(x + \lambda_1 v) = f_0(x) + \lambda_1 f_1(x)(v)$ for all $x \in \mathcal{U}_{\mathbb{R}}$ and $v \in E_1$. A direct calculation shows that $g_{\Lambda_1}(x' + \lambda_1 v') := g_0(x') + \lambda_1 g_1(x')(v')$ is the inverse of f_{Λ_1} . With the supersmooth morphism $(g_n)_n: \mathcal{V} \rightarrow \mathcal{U}$, we calculate

$$\begin{aligned} ((g_r)_r \circ (f_r)_r)_n(x)(v) &= \sum_{\substack{m, l < n, (\alpha, \beta) \in I_{m,l}^n, \\ \sigma \in \mathfrak{S}_n}} \frac{\text{sgn}(\sigma)}{m! l! \alpha! \beta!} d^m g_l(f_0(x))((f_\alpha \times f_\beta)(x)(v^\sigma)) \\ &+ \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{n!} g_n(f_0(x))((f_1 \times \cdots \times f_1)(x)(v^\sigma)), \end{aligned}$$

for $n > 1$, $x \in \mathcal{U}_{\mathbb{R}}$, $v \in E_1^n$. Note that in the second summand the sum over \mathfrak{S}_n together with the factor $\frac{1}{n!}$ can be omitted because the expression is already alternating. With $f_0(x) := x'$ and $(f_1(x)(v_1), \dots, f_1(x)(v)) := v'$ it follows from the definition of g_n that $((g_r)_r \circ (f_r)_r)_n = 0$. This implies $(g_r)_r \circ (f_r)_r = (\text{id}_{\mathcal{U}_{\mathbb{R}}}, c_{\text{id}_{E_1}}, 0, 0, \dots)$, which is the skeleton of the identity $\text{id}_{\mathcal{U}}: \mathcal{U} \rightarrow \mathcal{U}$. Thus, $(f_n)_n$ has a left inverse. Since the same construction also works for $(g_n)_n$, the left inverse of $(f_n)_n$ also has a left inverse. Therefore, $(g_n)_n$ is the inverse of $(f_n)_n$ and f is invertible in $\mathbf{SDom}^{(k)}$. \square

In general, it is quite difficult to check that smooth bijective maps between locally convex spaces are diffeomorphisms. However, if the map has the form of f_{Λ_1} in the above lemma, a result of Hamilton ([26, Theorem 5.3.1, p.102]) can be directly generalized to the locally convex case. We do not need this result in the sequel but since it might be of interest for inverting supersmooth maps, we state it nevertheless.

Lemma 2.2.19 ([24, Lemma 2.3, p.11]). *Let E_0, E_1 and F_1 be locally convex spaces, $U \subseteq E_0$ open and $f: U \times E_1 \rightarrow F_1$ be smooth such that $f_x := f(x, \bullet): E_1 \rightarrow F_1$ is linear for all $x \in U$. If f_x is invertible for all $x \in U$ and $g: U \times F_1 \rightarrow E_1$, $(x, v) \mapsto f_x^{-1}(v)$ is continuous, then g is smooth. Moreover, we have*

$$d_1 g(x, v)(u) = -g\left(x, d_1 f(x, g(x, v))(u)\right)$$

for $x \in U$, $v \in F_1$ and $u \in E_0$.

It is easy to generalize this to the situation where additionally a diffeomorphism $f_0: U \rightarrow V$ between open sets of locally convex spaces is involved.

2.2.1. Supersmooth multilinear algebra

Lemma/Definition 2.2.20. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $n \in \mathbb{N}$, ${}_i F \in \mathbf{SVec}$ for $1 \leq i \leq n$, $E \in \mathbf{SVec}$ and $f: {}_1 F \times \dots \times {}_n F \rightarrow E$ be an even n -multilinear map. We define

$$\overline{f}^{(k)}: \overline{{}_1 F}^{(k)} \times \dots \times \overline{{}_n F}^{(k)} \rightarrow \overline{E}^{(k)}$$

via multilinear expansion of $\overline{f}_{\Lambda}^{(k)}(\lambda_{I_1} v_1, \dots, \lambda_{I_n} v_n) := \lambda_{I_1} \dots \lambda_{I_n} f(v_1, \dots, v_n)$ for $\Lambda \in \mathbf{Gr}^{(k)}$, $\lambda_{I_1}, \dots, \lambda_{I_n} \in \Lambda$ and $v_i \in {}_i F_{|\overline{I_i}|}$. Then $\overline{f}^{(k)}$ is a natural transformation. If no confusion is possible, we also write \overline{f} instead of $\overline{f}^{(k)}$. If ${}_i F \in \mathbf{SVec}_{lc}$ for $1 \leq i \leq n$, $E \in \mathbf{SVec}_{lc}$ and f is continuous, then $\overline{f}^{(k)}$ is supersmooth.

Proof. That \overline{f} is a natural transformation follows immediately from the definition. Likewise, for continuous f it is obvious that \overline{f}_{Λ} is a continuous Λ_0 - n -multilinear map for every $\Lambda \in \mathbf{Gr}^{(k)}$ and thus supersmooth. \square

Proposition 2.2.21 ([40, Proposition 2.1.1, p.387]). *Let $n \in \mathbb{N}$ and ${}_1 F, \dots, {}_n F, E \in \mathbf{SVec}$.*

(a) For $k \in \mathbb{N} \cup \{\infty\}$, $k \geq n$ the map

$${}_0L_{\mathbb{R}}^n({}_1F, \dots, {}_nF; E) \rightarrow L_{\mathbb{R}}^n(\overline{{}_1F}^{(k)}, \dots, \overline{{}_nF}^{(k)}; \overline{E}^{(k)}), \quad f \mapsto \overline{f}$$

is an isomorphism of \mathbb{R} -modules, where the \mathbb{R} -module structure on the right-hand side is as in Lemma 1.4.4.

(b) If ${}_1F, \dots, {}_nF, E$ are topological super vector spaces, then

$${}_0\mathcal{L}_{\mathbb{R}}^n({}_1F, \dots, {}_nF; E) \rightarrow \mathcal{L}_{\mathbb{R}}^n(\overline{{}_1F}^{(k)}, \dots, \overline{{}_nF}^{(k)}; \overline{E}^{(k)}), \quad f \mapsto \overline{f}$$

is also an isomorphism of \mathbb{R} -modules.

(c) If g_1, \dots, g_n are even \mathbb{R} -multilinear maps such that for $1 \leq i \leq n$ the codomain of g_i is ${}_iF$, then for any $f \in {}_0L_{\mathbb{R}}^n({}_1F, \dots, {}_nF; E)$, we have

$$\overline{f \circ (g_1, \dots, g_n)} = \overline{f} \circ (\overline{g_1}, \dots, \overline{g_n}).$$

(d) If $f \in {}_0L_{\mathbb{R}}^n({}_1F, \dots, {}_nF; E)$ and $\sigma \in \mathfrak{S}_n$, then we have

$$\overline{f \cdot \sigma} = \overline{f} \circ \sigma,$$

with the left-hand side as in Definition 1.5.3 and the right-hand side as in Lemma 1.4.4.

Proof. The proof of (a) for the case $k = \infty$ given in [46, Proposition 3.1, p.10] can be easily transferred to the case of $k \in \mathbb{N}$, $k \geq n$.

In the situation of (b), it is obvious from the definition that $\overline{f} \in \mathcal{L}_{\mathbb{R}}^n(\overline{{}_1F}^{(k)}, \dots, \overline{{}_nF}^{(k)}; \overline{E}^{(k)})$. Conversely, the map

$${}_1F_{i_1} \times \dots \times {}_nF_{i_n} \rightarrow \lambda_{\{1, \dots, \ell\}} E_{\overline{E}}, \quad ({}_1v, \dots, {}_nv) \mapsto \lambda_{\{1, \dots, \ell\}} f({}_1v, \dots, {}_nv)$$

is continuous for all $i_1, \dots, i_n \in \{0, 1\}$, where $0 \leq \ell \leq n$ denotes the number of odd i_j . It follows that f is continuous.

Statement (c) is obvious. For (d) let ${}_iv \in {}_iF$, $1 \leq i \leq n$ be homogeneous and let $I_i \in \mathcal{P}_{p({}_iv)}^k$. Then, we have

$$\begin{aligned} \overline{f}_{\Lambda_k}(\lambda_{I_1} {}_1v, \dots, \lambda_{I_j} {}_jv, \lambda_{I_{j+1}} {}_{j+1}v, \dots, \lambda_{I_n} {}_nv) = \\ (-1)^{p({}_jv)p({}_{j+1}v)} \lambda_{I_1} \dots \lambda_n f({}_1v, \dots, {}_jv, {}_{j+1}v, \dots, {}_nv). \end{aligned}$$

This shows that $\overline{f \cdot (j, j+1)} = \overline{f} \circ (j, j+1)$ holds for any transposition $(j, j+1) \in \mathfrak{S}_n$. The general statement follows because the transpositions generate \mathfrak{S}_n . \square

Corollary 2.2.22 (c.f. [40, Corollary 2.1.2, p.388]). *The assignment $E \rightarrow \overline{E}^{(k)}$ for objects and $f \rightarrow \overline{f}$ for morphisms defines fully faithful functors*

$$\overline{\cdot}: \mathbf{SMod}_{\mathbb{R}} \rightarrow \mathbf{Mod}_{\mathbb{R}^{(k)}}(\mathbf{Set}^{\mathbf{Gr}^{(k)}}),$$

$$\bar{\cdot} : \mathbf{TopSMod}_{\mathbb{R}} \rightarrow \mathbf{Mod}_{\overline{\mathbb{R}}^{(k)}}(\mathbf{Top}^{\mathbf{Gr}^{(k)}}),$$

for $1 \leq k$, fully faithful functors

$$\bar{\cdot} : \mathbf{SAlg}_{\mathbb{R}} \rightarrow \mathbf{Alg}_{\overline{\mathbb{R}}^{(k)}}(\mathbf{Set}^{\mathbf{Gr}^{(k)}}),$$

$$\bar{\cdot} : \mathbf{TopSAlg}_{\mathbb{R}} \rightarrow \mathbf{Alg}_{\overline{\mathbb{R}}^{(k)}}(\mathbf{Top}^{\mathbf{Gr}^{(k)}}),$$

for $2 \leq k$ and fully faithful functors

$$\bar{\cdot} : \mathbf{SLAlg}_{\mathbb{R}} \rightarrow \mathbf{LAlg}_{\overline{\mathbb{R}}^{(k)}}(\mathbf{Set}^{\mathbf{Gr}^{(k)}}),$$

$$\bar{\cdot} : \mathbf{TopLSAlg}_{\mathbb{R}} \rightarrow \mathbf{LAlg}_{\overline{\mathbb{R}}^{(k)}}(\mathbf{Top}^{\mathbf{Gr}^{(k)}}),$$

for $k \geq 3$. If $A \in \mathbf{SAlg}_{\mathbb{R}}$ is associative, resp. unital, resp. supercommutative, then $\overline{A}^{(k)}$ is associative, resp. unital, resp. commutative.

Proof. Let us first consider the non-topological cases. That the functors are fully faithful follows from Proposition 2.2.21(a). In view of Remark 1.5.8, we see that Proposition 2.2.21(d) implies that an algebraic structures of a super type is mapped to the respective normal algebraic structure. For the same reason, supercommutative superalgebras map to commutative algebras. Note that $k \geq 3$ is necessary for applying the proposition to the super Jacobi identity.

Proposition 2.2.21(c) shows functoriality and that associativity is retained. Finally, if $A \in \mathbf{SAlg}_{\mathbb{R}}$ has the unit element 1_A and \mathbf{p} is a terminal object of $\mathbf{Set}^{\mathbf{Gr}^{(k)}}$, then $z : \mathbf{p} \rightarrow \overline{A}^{(k)}$, $z_{\Lambda}(\mathbf{p}_{\Lambda}) := \overline{A}_{\eta_{\Lambda}}^{(k)}(1_A)$ clearly defines a neutral element of $\overline{A}^{(k)}$. With Proposition 2.2.21(b), the topological case follows. Compare also [46, Corollary 3.2 and Corollary 3.3, p.12]. \square

In particular, if one restricts the objects on the right-hand side to $\overline{\mathbb{R}}^{(k)}$ -modules isomorphic to modules of the form $\overline{E}^{(k)}$, so called superrepresentable $\overline{\mathbb{R}}^{(k)}$ -modules, these functors establish equivalences of categories. More generally, it can be shown that one gets an equivalence of categories from any category of multilinear “super-algebraic” structures over \mathbb{R} in \mathbf{Man} to the respective category of multilinear algebraic structures over $\overline{\mathbb{R}}$ in \mathbf{SMan} (see [40, Corollary 4.4.2, p.397]).²

2.2.2. Generalizations

One obvious generalization is to consider a differential calculus for other base fields (or even rings) than \mathbb{R} . A robust framework for this is provided by [11] and then further developed for the super case in [1]. In the most general case, one has a unital commutative Hausdorff topological ring R such that the group of units R^{\times} is dense, i.e., integers need not necessarily be invertible. For simplicity’s sake, we formulated our results over \mathbb{R} but we made a conscious effort to make them easily adaptable to more general situations.

²It is not immediately clear whether this result holds beyond Banach supermanifolds.

In this way Lemma 2.2.4 through Proposition 2.2.7 can easily be shown to hold in the most general case. While Corollary 2.2.9 and Lemma 2.2.10 also translate, our definition of supersmoothness just means \mathcal{C}_{MS}^1 (together with smoothness over \mathbb{R}) in the terminology of [1]. Note however that \mathcal{C}_{MS}^1 is equivalent to \mathcal{C}_{MS}^∞ if R is an \mathbb{Q} -algebra and one has smoothness over R (see [1, Proposition 2.18, p.580]). In this case Lemma 2.2.8, Proposition 2.2.13, Proposition 2.2.16 and Lemma 2.2.18 carry over as well.

It should be noted that Remark 2.2.14 enables us to show an analog to Proposition 2.2.13 if not all integers are invertible in R , i.e., supersmooth maps are given by something like skeletons even in the most general case. The resulting analog to the composition formula from Proposition 2.2.16 can be obtained with general results about multilinear bundles (compare Remark B.1.3) and a similar induction as in Lemma 2.2.18 leads to an inversion formula (compare [10, Theorem MA.6(2), p.172]).

The second apparent generalization is to define morphisms of finite differentiability order $n \in \mathbb{N}_0$. Given only k -superdomains with $k \leq n$, one can simply define k -skeletons where the differentiability class of the components is appropriately chosen. For a more detailed discussion see [40, 10.1, p.420f.].

2.3. Supermanifolds

The construction of supermanifolds from superdomains is conceptually very close to the respective construction of manifolds. In the categorical approach proposed by Molotkov in [40], one defines a Grothendieck topology on $\mathbf{Top}^{\mathbf{Gr}}$ that takes the same role as the usual topology in the manifold case. As model space one uses functors of the form \bar{E} for $E \in \mathbf{SVec}_{lc}$ with open subfunctors \mathcal{U} as the open subsets (respectively functors isomorphic to such functors). A supermanifold is then a functor $\mathcal{M} \in \mathbf{Man}^{\mathbf{Gr}}$ together with an atlas consisting of natural transformations $\varphi: \mathcal{U} \rightarrow \mathcal{M}$, such that the change of charts is supersmooth. Here a technical problem arises. In this approach, the intersection of two chart domains in \mathcal{M} is defined as a fiber product in the category $\mathbf{Man}^{\mathbf{Gr}}$, which is not guaranteed to be a superdomain. This has to be demanded in the definition. We avoid this and other technicalities by using concrete definitions of the model spaces. For a concise version of the categorical approach see [1, p.591 ff.].

We introduce k -supermanifolds in the same way as supermanifolds by considering functors $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ and obtain respective categories $\mathbf{SMan}^{(k)}$ for $k \in \mathbb{N}_0 \cup \{\infty\}$. One has the obvious restriction functors $\pi_n^m: \mathbf{SMan}^{(m)} \rightarrow \mathbf{SMan}^{(n)}$ for $n \leq m$ and the embeddings $\iota_k^0: \mathbf{SMan}^{(0)} \rightarrow \mathbf{SMan}^{(k)}$ and $\iota_k^1: \mathbf{SMan}^{(1)} \rightarrow \mathbf{SMan}^{(k)}$, which play an important part in understanding the structure of supermanifolds. Note in particular that $\mathbf{SMan}^{(0)} \cong \mathbf{Man}$ and $\mathbf{SMan}^{(1)} \cong \mathbf{VBun}$.

These statements are not particularly difficult to prove and were already stated in [38]. Noteworthy new results include the following. For any supermanifold \mathcal{M} , we show that \mathcal{M}_{Λ_n} has the natural structure of a so called multilinear bundle of degree n over $\mathcal{M}_{\mathbb{R}}$. What is more, $(\mathcal{M}_{\Lambda_n})_{n \in \mathbb{N}_0}$ forms an inverse system of

multilinear bundles which enables us to obtain a functor

$$\mathbf{SMan} \rightarrow \mathbf{Man}, \quad \mathcal{M} \mapsto \varprojlim_n \mathcal{M}_{\Lambda_n}$$

in Theorem 2.3.11. As already mentioned, this functor has good properties such as respecting products. Another important result is the characterization of purely even supermanifolds in terms of higher tangent bundles of the base manifold in Proposition 2.3.16.

Definition 2.3.1. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E \in \mathbf{SVec}_{lc}$ and $\mathcal{M} \in \mathbf{Top}^{\mathbf{Gr}^{(k)}}_{\text{Hausdorff}}$. Recall Definition 2.1.6. We call a covering $\mathcal{A} := \{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ of \mathcal{M} such that all \mathcal{U}^α are open subfunctors of $\overline{E}^{(k)}$ an *atlas* of \mathcal{M} if the natural transformations

$$\varphi^{\alpha\beta} := (\varphi^\beta)^{-1} \circ \varphi^\alpha|_{\mathcal{U}^{\alpha\beta}}: \mathcal{U}^{\alpha\beta} \rightarrow \mathcal{U}^{\beta\alpha}$$

are supersmooth for all $\alpha, \beta \in A$. Two atlases \mathcal{A} and \mathcal{B} are called *equivalent* if their union $\mathcal{A} \cup \mathcal{B}$ is again an atlas. As with ordinary manifolds, this clearly defines an equivalence relation and we call the pair $(\mathcal{M}, [\mathcal{A}])$ a *k-supermanifold modelled on E*. If $k = \infty$ we also simply call \mathcal{M} a *supermanifold*. We will usually omit $[\mathcal{A}]$ from our notation and if we talk about an atlas of a supermanifold, it is meant to belong to this equivalence class. An element of any of the equivalent atlases will be called a *chart* of \mathcal{M} . For any two charts φ^α and φ^β , we call $\varphi^{\alpha\beta}$ the *change of charts*.

A *morphism* $f: \mathcal{M} \rightarrow \mathcal{N}$ of k -supermanifolds \mathcal{M} and \mathcal{N} is a natural transformation $f: \mathcal{M} \rightarrow \mathcal{N}$ such that for any chart $\varphi: \mathcal{U} \rightarrow \mathcal{M}$ and any chart $\psi: \mathcal{V} \rightarrow \mathcal{N}$

$$\psi^{-1} \circ f \circ \varphi|_{(f \circ \varphi)^{-1}(\psi(\mathcal{V}))}: (f \circ \varphi)^{-1}(\psi(\mathcal{V})) \rightarrow \mathcal{V}$$

is supersmooth.

Note that the definition of morphisms between k -supermanifolds is independent of the atlases, because change of charts satisfies the cocycle condition. As with ordinary manifolds, one sees that the composition of two morphisms of supermanifolds is again a morphism by inserting charts between them. Thus, we get for every $k \in \mathbb{N}_0 \cup \{\infty\}$ the category $\mathbf{SMan}^{(k)}$ of k -supermanifolds. As always, we set $\mathbf{SMan} := \mathbf{SMan}^{(\infty)}$. For two k -supermanifolds \mathcal{M}, \mathcal{N} , we denote by $\mathcal{SC}^\infty(\mathcal{M}, \mathcal{N})$ the set of supersmooth morphisms $f: \mathcal{M} \rightarrow \mathcal{N}$.

Definition 2.3.2. A k -supermanifold \mathcal{M} modelled on $E \in \mathbf{SVec}_{lc}$ is a *finite-dimensional*, *Banach* or *Fréchet* k -supermanifold if E is so. If $E_1 = \{0\}$, then \mathcal{M} is *purely even* and if $E_0 = \{0\}$, then \mathcal{M} is *purely odd*. We call $\mathcal{M}_{\mathbb{R}}$ the *base manifold* of \mathcal{M} and say that \mathcal{M} is *σ -compact* if $\mathcal{M}_{\mathbb{R}}$ is σ -compact.

Remark 2.3.3. If one allows non-Hausdorff supermanifolds in the definition, it is easily seen that a supermanifold \mathcal{M} is Hausdorff if and only if its base manifold is Hausdorff. In fact, this follows because \mathcal{M}_{Λ} is a fiber bundle over $\mathcal{M}_{\mathbb{R}}$ whose typical fiber is Hausdorff by Theorem 2.3.11 below.

To get some intuition for supermanifolds, we start with several simple observations.

Lemma 2.3.4. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ with the atlas $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$.*

- (a) *For every $\Lambda \in \mathbf{Gr}^{(k)}$, $\{(\varphi_\Lambda^\alpha|_{\mathcal{U}_\Lambda^\alpha}^{\varphi_\Lambda^\alpha(\mathcal{U}_\Lambda^\alpha)})^{-1}: \varphi_\Lambda^\alpha(\mathcal{U}_\Lambda^\alpha) \rightarrow \mathcal{U}_\Lambda^\alpha: \alpha \in A\}$ is an atlas of \mathcal{M}_Λ .*
- (b) *For $n \leq m < k + 1$, the inclusions $\mathcal{M}_{\eta_{n,m}}: \mathcal{M}_{\Lambda_n} \rightarrow \mathcal{M}_{\Lambda_m}$ are topological embeddings and \mathcal{M}_{Λ_n} is a closed submanifold of \mathcal{M}_{Λ_m} .*
- (c) *For $n \leq m < k + 1$, the projections $\mathcal{M}_{\varepsilon_{m,n}}: \mathcal{M}_{\Lambda_m} \rightarrow \mathcal{M}_{\Lambda_n}$ are surjective.*

Proof. (a) This is obvious from the definition of a supermanifold, since the sets $\varphi_\Lambda^\alpha(\mathcal{U}_\Lambda^\alpha)$ form an open cover of \mathcal{M}_Λ , $\varphi_\Lambda^\alpha|_{\mathcal{U}_\Lambda^\alpha}^{\varphi_\Lambda^\alpha(\mathcal{U}_\Lambda^\alpha)}$ is a homeomorphism and the change of charts is smooth.

(b) Let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$. In the charts defined by $\varphi_{\Lambda_n}^\alpha$ and $\varphi_{\Lambda_m}^\alpha$ as in (a), the map $\mathcal{M}_{\eta_{n,m}}$ has the form $\mathcal{U}_{\eta_{n,m}}$ and we have $\mathcal{U}_{\Lambda_n}^\alpha \cong \mathcal{U}_{\eta_{n,m}}^\alpha(\mathcal{U}_{\Lambda_n}^\alpha) = \mathcal{U}_{\Lambda_m}^\alpha \cap \overline{E}_{\Lambda_n}^{(k)}$. By naturality, we have $\varphi_{\Lambda_m}^\alpha(\mathcal{U}_{\eta_{n,m}}^\alpha(\mathcal{U}_{\Lambda_n}^\alpha)) = \mathcal{M}_{\Lambda_m} \cap \mathcal{M}_{\eta_{n,m}}(\varphi_{\Lambda_n}^\alpha(\mathcal{U}_{\Lambda_n}^\alpha))$.

(c) In the charts defined by $\varphi_{\Lambda_n}^\alpha$ and $\varphi_{\Lambda_m}^\alpha$ as in (a), the map $\mathcal{M}_{\varepsilon_{m,n}}$ has the form $\mathcal{U}_{\varepsilon_{m,n}}$ which clearly defines a surjective map. \square

Part (c) of this lemma already suggests that \mathcal{M}_{Λ_m} is some kind of fiber bundle over \mathcal{M}_{Λ_n} . As we discuss below, this fiber bundle structure can be accurately described via multilinear bundles. Like ordinary manifolds, supermanifolds and morphisms thereof arise from local data.

Proposition 2.3.5 (see [1, Proposition 3.23, p.593]). *Let $k \in \mathbb{N}_0 \cup \{0\}$, $E \in \mathbf{SVec}_{lc}$ and let $(\mathcal{U}^\alpha)_{\alpha \in A}$ be a family of open subfunctors of $\overline{E}^{(k)}$ and $\mathcal{U}^{\alpha\alpha'} \subseteq \mathcal{U}^\alpha$ be open subfunctors for $\alpha, \alpha' \in A$ such that $\mathcal{U}^{\alpha\alpha} = \mathcal{U}^\alpha$. Further, let $\varphi^{\alpha\alpha'}: \mathcal{U}^{\alpha\alpha'} \rightarrow \mathcal{U}^{\alpha'\alpha}$ be isomorphisms in $\mathbf{SDom}^{(k)}$ such that we have $\varphi^{\alpha\alpha} = \text{id}_{\mathcal{U}^\alpha}$ and $\varphi^{\alpha\alpha''} = \varphi^{\alpha'\alpha''} \circ \varphi^{\alpha\alpha'}$ on $\mathcal{U}^{\alpha\alpha'} \cap \mathcal{U}^{\alpha\alpha''}$ for all $\alpha, \alpha', \alpha'' \in A$. Finally, for all $\alpha, \beta \in A$ and any two points $x \in \mathcal{U}_\mathbb{R}^\alpha$, $y \in \mathcal{U}_\mathbb{R}^\beta$ such that $x \notin \mathcal{U}_\mathbb{R}^{\alpha\beta}$ or $\varphi_\mathbb{R}^{\alpha\beta}(x) \neq y$, let there exist open neighbourhoods $V \subseteq \mathcal{U}_\mathbb{R}^\alpha$ of x and $V' \subseteq \mathcal{U}_\mathbb{R}^\beta$ of y , such that $\varphi_\mathbb{R}^{\alpha\beta}(\mathcal{U}_\mathbb{R}^{\alpha\beta} \cap V) \cap V' = \emptyset$. Then there exists a, up to unique isomorphism, unique k -supermanifold \mathcal{M} with an atlas $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ such that the change of charts coincides with the $\varphi^{\alpha\alpha'}$ defined above.*

Moreover, let $\mathcal{N} \in \mathbf{SMan}^{(k)}$ have the atlas $\{\psi^\beta: \mathcal{V}^\beta \rightarrow \mathcal{N}: \beta \in B\}$ and let $\tilde{\mathcal{U}}^{\alpha\beta} \subseteq \mathcal{U}^\alpha$ for $\alpha \in A$ and $\beta \in B$ such that $\bigcup_{\beta \in B} \tilde{\mathcal{U}}_\mathbb{R}^{\alpha\beta} = \mathcal{U}_\mathbb{R}^\alpha$. If $f^{\alpha\beta}: \tilde{\mathcal{U}}^{\alpha\beta} \rightarrow \mathcal{V}^\beta$ is a family of supersmooth maps such that $\psi^{\beta\beta'} \circ f^{\alpha\beta} \circ \varphi^{\alpha'\alpha} = f^{\alpha'\beta'}$ on $(\varphi^{\alpha'\alpha})^{-1}(\tilde{\mathcal{U}}^{\alpha\beta}) \cap (f^{\alpha'\beta'})^{-1}(\mathcal{V}^{\beta'})$ for all $\alpha, \alpha' \in A$, $\beta, \beta' \in B$, then there exists a unique supersmooth morphism $f: \mathcal{M} \rightarrow \mathcal{N}$ with $f^{\alpha\beta} = (\psi^\beta)^{-1} \circ f \circ \varphi^\alpha|_{\tilde{\mathcal{U}}^{\alpha\beta}}$.

Proof. This follows exactly as in [1, Proposition 3.23, p.593]. Essentially, we use the well-known equivalent statement for ordinary manifolds for every $\Lambda \in \mathbf{Gr}^{(k)}$ to construct \mathcal{M}_Λ , resp. f_Λ , and the rest follows from naturality. Note that

$\bigcup_{\beta \in B} \tilde{\mathcal{U}}_{\mathbb{R}}^{\alpha\beta} = \mathcal{U}_{\mathbb{R}}^{\alpha}$ implies $\bigcup_{\beta \in B} \tilde{\mathcal{U}}_{\Lambda}^{\alpha\beta} = \mathcal{U}_{\Lambda}^{\alpha}$ for all $\Lambda \in \mathbf{Gr}^{(k)}$. Moreover, if $\mathcal{M}_{\mathbb{R}}$ is Hausdorff then \mathcal{M}_{Λ} is Hausdorff for all $\Lambda \in \mathbf{Gr}$ (compare Remark 2.3.3). \square

Lemma 2.3.6 ([40, Corollary 6.2.2, p.409]). *Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{M}, \mathcal{N} \in \mathbf{SMan}^{(k)}$. A supersmooth morphism $f: \mathcal{M} \rightarrow \mathcal{N}$ is an isomorphism in $\mathbf{SMan}^{(k)}$ if and only if $f_{\Lambda_1}: \mathcal{M}_{\Lambda_1} \rightarrow \mathcal{N}_{\Lambda_1}$ is a diffeomorphism.*

Proof. Clearly, $f: \mathcal{M} \rightarrow \mathcal{N}$ is an isomorphism if and only if $f_{\Lambda}: \mathcal{M}_{\Lambda} \rightarrow \mathcal{N}_{\Lambda}$ is bijective and the maps f_{Λ}^{-1} define a supersmooth natural transformation for every $\Lambda \in \mathbf{Gr}^{(k)}$. In particular, f_{Λ_1} is a diffeomorphism in this situation. Let $\{\varphi^{\alpha}: \mathcal{U}^{\alpha} \rightarrow \mathcal{M}: \alpha \in A\}$ be an atlas of \mathcal{M} and $\{\psi^{\beta}: \mathcal{V}^{\beta} \rightarrow \mathcal{N}: \beta \in B\}$ be an atlas of \mathcal{N} . Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be supersmooth such that f_{Λ_1} is a diffeomorphism. For all $\alpha \in A$ and $\beta \in B$ we define $\tilde{\mathcal{U}}^{\alpha\beta} := (f \circ \varphi^{\alpha})^{-1}(\psi^{\beta}(\mathcal{V}^{\beta})) \subseteq \mathcal{U}^{\alpha}$ and $\tilde{\mathcal{V}}^{\beta\alpha} := f^{\alpha\beta}((f \circ \varphi^{\alpha})^{-1}(\psi^{\beta}(\mathcal{V}^{\beta}))) = (\psi^{\beta})^{-1}(f(\varphi^{\alpha}(\mathcal{U}^{\alpha}))) \subseteq \mathcal{V}^{\beta}$ and let

$$f^{\alpha\beta} := (\psi^{\beta})^{-1} \circ f \circ \varphi^{\alpha}|_{\tilde{\mathcal{U}}^{\alpha\beta}}: \tilde{\mathcal{U}}^{\alpha\beta} \rightarrow \tilde{\mathcal{V}}^{\beta\alpha}.$$

Since $f_{\mathbb{R}}$ is also a diffeomorphism, the sets $\tilde{\mathcal{V}}_{\mathbb{R}}^{\beta\alpha}$ cover $\mathcal{N}_{\mathbb{R}}$ and because every $f_{\Lambda_1}^{\alpha\beta}$ is a diffeomorphism, there exist unique supersmooth inverse morphisms $(f^{\alpha\beta})^{-1}: \tilde{\mathcal{V}}^{\beta\alpha} \rightarrow \tilde{\mathcal{U}}^{\alpha\beta}$ by Lemma 2.2.18. For every $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$, we have $(\psi^{\beta'\beta})^{-1} \circ f^{\alpha'\beta'} \circ \varphi^{\alpha\alpha'} = f^{\alpha\beta}$ on $\tilde{\mathcal{U}}^{\alpha\beta} \cap (\varphi^{\alpha\alpha'})^{-1}(\tilde{\mathcal{U}}^{\alpha'\beta'})$. Therefore, $(f^{\alpha\beta})^{-1} = (\varphi^{\alpha\alpha'})^{-1} \circ (f^{\alpha'\beta'})^{-1} \circ \psi^{\beta'\beta}$ on $\tilde{\mathcal{V}}^{\beta\alpha} \cap (\psi^{\beta'\beta})^{-1}(\tilde{\mathcal{V}}^{\beta'\alpha'})$ and the morphisms lead to a unique supersmooth morphism $f^{-1}: \mathcal{N} \rightarrow \mathcal{M}$ by Proposition 2.3.5. That it is inverse to f follows from the local description of $f^{-1} \circ f$ and $f \circ f^{-1}$. \square

Definition 2.3.7. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be modelled on $E \in \mathbf{SVec}_{lc}$. A subfunctor \mathcal{N} of \mathcal{M} is called a *sub-supermanifold* of \mathcal{M} if there exist sequentially closed vector subspaces $F_0 \subseteq E_0$ and $F_1 \subseteq E_1$ such that for every $x \in \mathcal{N}_{\mathbb{R}}$ there exists a chart $\varphi^{\alpha}: \mathcal{U}^{\alpha} \rightarrow \mathcal{M}$ of \mathcal{M} with $x \in \varphi_{\mathbb{R}}^{\alpha}(\mathcal{U}_{\mathbb{R}}^{\alpha})$ such that $\varphi^{\alpha}(\mathcal{U}^{\alpha} \cap \overline{F}^{(k)}) = \varphi^{\alpha}(\mathcal{U}^{\alpha}) \cap \mathcal{N}$, where $F := F_0 \oplus F_1 \in \mathbf{SVec}_{lc}$.

We call $\varphi^{\alpha}|_{\mathcal{U}^{\alpha} \cap \overline{F}^{(k)}}$ a *sub-supermanifold chart* of \mathcal{N} . Taking all sub-supermanifold charts of \mathcal{N} as the atlas turns \mathcal{N} into a supermanifold and we always give \mathcal{N} this structure.

Lemma 2.3.8. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{M} \in \mathbf{SMan}^{(k)}$ and \mathcal{N} be a sub-supermanifold of \mathcal{M} . Then the inclusion $i: \mathcal{N} \rightarrow \mathcal{M}$ is supersmooth.*

Proof. By definition of a subfunctor, the inclusion is a natural transformation. Let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$, \mathcal{N} be modelled on $F \subseteq E$ and $\{\varphi^{\alpha}: \mathcal{U}^{\alpha} \rightarrow \mathcal{M}: \alpha \in A\}$ be a collection of charts such that $\{\varphi^{\alpha}|_{\mathcal{U}^{\alpha} \cap \overline{F}^{(k)}}: \alpha \in A\}$ is an atlas of \mathcal{N} . In these charts the inclusion is just the inclusion $\mathcal{U}^{\alpha} \cap \overline{F}^{(k)} \rightarrow \mathcal{U}^{\alpha}$, which is obviously supersmooth. \square

Lemma/Definition 2.3.9. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$. For every open subfunctor of $\mathcal{U} \subseteq \mathcal{M}$, we have $\mathcal{U} = \mathcal{M}|_{\mathcal{U}_{\mathbb{R}}}$. In this case \mathcal{U} is a sub-supermanifold of \mathcal{M} and if $f: \mathcal{M} \rightarrow \mathcal{N}$ is a supersmooth morphism to $\mathcal{N} \in \mathbf{SMan}^{(k)}$, then so is $f|_{\mathcal{U}}: \mathcal{U} \rightarrow \mathcal{N}$. We call such sub-supermanifolds *open sub-supermanifolds*.

Proof. That $\mathcal{U} = \mathcal{M}|_{\mathcal{U}_{\mathbb{R}}}$ holds for $k = \infty$ follows directly from [45, Corollary 3.5.9, p. 62] and the same proof works for $k \in \mathbb{N}_0$ if one only considers $\Lambda \in \mathbf{Gr}^{(k)}$. Let $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ be an atlas of \mathcal{M} . Then $\{\varphi^\alpha|_{(\varphi^\alpha)^{-1}(\varphi^\alpha(\mathcal{U}^\alpha) \cap \mathcal{U})}: \alpha \in A\}$ is an atlas of \mathcal{U} . With these charts, the supersmoothness of $f|_{\mathcal{U}}$ is obvious. \square

Definition 2.3.10. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M}, \mathcal{N} \in \mathbf{SMan}^{(k)}$ be modelled on $E, F \in \mathbf{SVec}_{lc}$ with atlases $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ and $\{\psi^\beta: \mathcal{V}^\beta \rightarrow \mathcal{N}: \beta \in B\}$. We define the *product* $\mathcal{M} \times \mathcal{N}$ of \mathcal{M} and \mathcal{N} as the functor $\Lambda \mapsto \mathcal{M}_\Lambda \times \mathcal{N}_\Lambda$, resp. $\varrho \mapsto \mathcal{M}_\varrho \times \mathcal{N}_\varrho$, for $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. We will always give $\mathcal{M} \times \mathcal{N}$ the structure of a k -supermanifold modelled on $E \times F$ defined by the atlas $\{\varphi^\alpha \times \psi^\beta: \mathcal{U}^\alpha \times \mathcal{V}^\beta \rightarrow \mathcal{M} \times \mathcal{N}: (\alpha, \beta) \in A \times B\}$.

Clearly, the projections $\pi_{\mathcal{M}}: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M}$ and $\pi_{\mathcal{N}}: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{N}$ are super-smooth morphisms.

Recall the definition of multilinear bundles and inverse systems of multilinear bundles from Appendix B. The following theorem shows that for a supermanifold \mathcal{M} , the manifolds \mathcal{M}_{Λ_n} are multilinear bundles of degree n over $\mathcal{M}_{\mathbb{R}}$ and that $(\mathcal{M}_{\Lambda_m}, \mathcal{M}_{\varepsilon_{m,n}})$ is an inverse system of multilinear bundles. This lets us consider supermanifolds as ordinary manifolds.

Theorem 2.3.11. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{M}, \mathcal{N} \in \mathbf{SMan}^{(k)}$ and $f: \mathcal{M} \rightarrow \mathcal{N}$ be supersmooth. If \mathcal{M} is modelled on $E \in \mathbf{SVec}_{lc}$ with the atlas $\{\varphi^\alpha: \alpha \in A\}$, then \mathcal{M}_{Λ_n} is a multilinear bundle of degree n over $\mathcal{M}_{\mathbb{R}}$ with the fiber $\overline{E}_{\Lambda_n^+}$ and the bundle atlas $\{\varphi_{\Lambda_n}^\alpha: \alpha \in A\}$ for every $\Lambda_n \in \mathbf{Gr}^{(k)}$. Moreover, $f_{\Lambda_n}: \mathcal{M}_{\Lambda_n} \rightarrow \mathcal{N}_{\Lambda_n}$ is a morphism of multilinear bundles of degree n . With this, we obtain a faithful functor

$$\mathbf{SMan}^{(k)} \rightarrow \mathbf{MBun}^{(k)},$$

defined by $\mathcal{M} \mapsto \mathcal{M}_{\Lambda_k}$ and $f \mapsto f_{\Lambda_k}$ for $k \in \mathbb{N}_0$. Furthermore, if $k = \infty$, then $(\mathcal{M}_{\Lambda_m}, \mathcal{M}_{\varepsilon_{m,n}})$ is an inverse system of multilinear bundles with the adapted atlas $\{(\varphi_{\Lambda_n}^\alpha)^{-1}: n \in \mathbb{N}_0, \alpha \in A\}$ and

$$\varprojlim: \mathbf{SMan} \rightarrow \mathbf{MBun}^{(\infty)},$$

defined by $\mathcal{M} \mapsto \varprojlim_n \mathcal{M}_{\Lambda_n}$ and $f \mapsto \varprojlim_n f_{\Lambda_n}$, is a faithful functor. Along the forgetful functor, we have thus constructed faithful functors

$$\mathbf{SMan}^{(k)} \rightarrow \mathbf{Man}$$

for $k \in \mathbb{N}_0 \cup \{\infty\}$. All these functors respect products.

Proof. Let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$. We start by showing that $\{\varphi_{\Lambda_n}^\alpha: \mathcal{U}_{\Lambda_n}^\alpha \rightarrow \mathcal{M}_{\Lambda_n}: \alpha \in A\}$ is indeed a bundle atlas of a multilinear bundles of degree n . Let the change of charts $\varphi^{\alpha\beta}$ be defined by the skeleton $(\varphi_n^{\alpha\beta})$. We consider $\mathcal{U}_{\Lambda_n}^\alpha = \mathcal{U}_{\mathbb{R}}^\alpha \times \prod_{I \in \mathcal{P}_+^n} \lambda_I E_{|I|}$ as a trivial multilinear bundle over the n -multilinear space (E_I) with $E_I := \lambda_I E_{|I|}$. By naturality, we have $(\varphi_{\Lambda_n}^\alpha)^{-1}(\mathcal{M}_{\varepsilon_{\Lambda_n}}^{-1}(\{x\})) = (\mathcal{U}_{\Lambda_n}^\alpha)^{-1}(\varphi_{\mathbb{R}}^{-1}(\{x\}))$ for all $x \in \varphi_{\mathbb{R}}^\alpha(\mathcal{U}_{\mathbb{R}}^\alpha)$. In other words, the projection $\mathcal{M}_{\varepsilon_{\Lambda_n}}: \mathcal{M}_{\Lambda_n} \rightarrow \mathcal{M}_{\mathbb{R}}$ turns \mathcal{M}_{Λ_n} into a fiber bundle with typical fiber $\overline{E}_{\Lambda_n^+}$. Recall the sign of a partition defined in

Remark B.1.6. Then $\lambda_{\omega_1} \cdots \lambda_{\omega_{\ell(\omega)}} = \text{sgn}(\omega) \lambda_I$ for all $I \in \mathcal{P}_+^n$ and $\omega \in \mathcal{P}(I)$. With this, we use Remark 2.2.14 to calculate

$$\varphi^{\alpha\beta}(x + \sum_{I \in \mathcal{P}_+^n} \lambda_I x_I) = \varphi_0^{\alpha\beta}(x) + \sum_{I \in \mathcal{P}_+^n} \sum_{\omega \in \mathcal{P}(I)} \lambda_I \text{sgn}(\omega) d^{(e(\omega))} \varphi_{o(\omega)}^{\alpha\beta}(x)(x_\omega)$$

for ω in graded lexicographic order $x_I \in E_{|\overline{I}|}$ for $I \in \mathcal{P}_+^n$ and

$$x_\omega := (\lambda_{\omega_1} x_{\omega_1}, \dots, \lambda_{\omega_{\ell(\omega)}} x_{\omega_{\ell(\omega)}}).$$

In the notation of multilinear bundles, the change of chart is thus given by the sum of maps of the form $(\varphi^{\alpha\beta})_x^\omega := \text{sgn}(\omega) d^{(e(\omega))} \varphi_{o(\omega)}^{\alpha\beta}(x)$, which define an isomorphism of n -multilinear spaces for every $x \in \mathcal{U}_{\mathbb{R}}^{\alpha\beta}$. Thus, \mathcal{M}_{Λ_n} is a multilinear bundle over $\mathcal{M}_{\mathbb{R}}$ of degree n with typical fiber (E_I) .

For a morphism $f: \mathcal{M} \rightarrow \mathcal{N}$, we first note that by naturality $f_{\mathbb{R}} \circ \mathcal{M}_{\varepsilon_{\Lambda_n}} = \mathcal{N}_{\varepsilon_{\Lambda_n}} \circ f_{\Lambda_n}$ and therefore f_{Λ_n} is a fiber bundle morphism over $f_{\mathbb{R}}$. In bundle charts, we can make the exact same argument as above to see that f_{Λ_n} is a morphism of multilinear bundles.

It follows that we have a functor $\mathbf{SMan}^{(k)} \rightarrow \mathbf{MBun}^{(k)}$ as described in the theorem for $k \in \mathbb{N}_0$. Next, we show that $(\mathcal{M}_{\Lambda_m}, \mathcal{M}_{\varepsilon_{m,n}})$ defines an inverse system of multilinear bundles if $\mathcal{M} \in \mathbf{SMan}$. We have $\mathcal{M}_{\varepsilon_{m,n}} \circ \varphi_{\Lambda_m}^\alpha = \varphi_{\Lambda_n}^\alpha \circ \mathcal{U}_{\varepsilon_{m,n}}^\alpha$ for all $n \leq m$ and therefore $\mathcal{U}_{\varepsilon_{km,n}}^\alpha$ is the chart representation of $\mathcal{M}_{\varepsilon_{m,n}}$. Hence, in terms of multilinear bundles, $\mathcal{M}_{\varepsilon_{m,n}}$ is exactly the projection defined in Lemma B.3.1. It follows that $\mathcal{M}_{\Lambda_m}|_{\mathcal{P}_+^n} = \mathcal{M}_{\varepsilon_{m,n}}(\mathcal{M}_{\Lambda_m}) = \mathcal{M}_{\Lambda_n}$ and $\varphi_{\Lambda_m}^\alpha|_{\mathcal{P}_+^n} = \varphi_{\Lambda_n}^\alpha \circ \mathcal{U}_{\eta_{n,m}}^\alpha = \mathcal{M}_{\eta_{n,m}} \circ \varphi_{\Lambda_n}^\alpha$ shows that $\{(\varphi_{\Lambda_m}^\alpha)^{-1}: m \in \mathbb{N}_0, \alpha \in \mathcal{A}\}$ is indeed an adapted atlas.

On morphisms $f: \mathcal{M} \rightarrow \mathcal{N}$, $\mathcal{N} \in \mathbf{SMan}$, we have likewise $f_{\Lambda_n} \circ \mathcal{M}_{\varepsilon_{m,n}} = \mathcal{N}_{\varepsilon_{m,n}} \circ f_{\Lambda_m}$ which shows that $(f_{\Lambda_m})_{m \in \mathbb{N}_0}$ is a morphism of inverse systems of multilinear bundles.

It is clear from the definitions that products of supermanifolds correspond to products of inverse systems. \square

Remark 2.3.12. In [40, Remark 3.3.1, p.392] Molotkov constructs a functor $\mathbf{Man}^{\mathbf{Gr}} \rightarrow \mathbf{Man}$ by taking the disjoint union of the \mathcal{M}_Λ for $\mathcal{M} \in \mathbf{Man}^{\mathbf{Gr}}$ and $\Lambda \in \mathbf{Gr}$. He also considers this as a functor $\mathbf{SMan} \rightarrow \mathbf{Man}$ along the forgetful functor. For one, this functor relies on a more general definition of manifolds where the model spaces of different connected components may be non-isomorphic. More critically, this functor does not respect products, leading Molotkov to state that “Lie supergroups (groups of the category \mathbf{SMan}) are not groups at all (considered in \mathbf{Set} ” [40, Ibid.]. We hope to have convinced the reader with the above theorem that Lie supergroups can be seen not only as groups but even as Lie groups in a natural way.

Remark 2.3.13. Consider the following type of fiber bundles. For $E \in \mathbf{SVec}_{lc}$ let the base manifold M be modelled on E_0 , let the typical fiber be $\varprojlim_n \overline{E}_{\Lambda_n^+}$ and let the transition functions come from the limit of skeletons as in Theorem 2.3.11. The morphisms of such bundles shall locally also come from limits of skeletons. Obviously, these bundles are elements of $\mathbf{MBun}^{(\infty)}$ and restricting to

this subcategory turns the functor \varprojlim into an equivalence of categories. If E is finite-dimensional, we have that $\varprojlim_n \overline{E}_{\Lambda_n^+}$ is a Fréchet space. Consequently, non-trivial finite-dimensional supermanifolds are mapped to Fréchet manifolds under \varprojlim .

One can reconstruct the original supermanifold \mathcal{M} from $\varprojlim \mathcal{M}$ if one keeps track of any atlas of $\varprojlim \mathcal{M}$ coming from the limit of an atlas of \mathcal{M} . An interesting problem is whether one can at least recover the isomorphism class of a supermanifold without a specific atlas.

Problem. Is the functor $\varprojlim: \mathbf{SMan} \rightarrow \mathbf{MBun}^{(\infty)}$ injective on isomorphism classes, i.e., do we have $\varprojlim \mathcal{M} \cong \varprojlim \mathcal{N}$ in $\mathbf{MBun}^{(\infty)}$ if and only if we have $\mathcal{M} \cong \mathcal{N}$ in \mathbf{SMan} ?

If $\mathcal{M}_{\mathbb{R}}$ admits a smooth partition of unity, then it follows from Batchelor's Theorem 2.4.1 below that this is the case because $\varprojlim \mathcal{M} \cong \varprojlim \mathcal{N}$ in $\mathbf{MBun}^{(\infty)}$ implies $\mathcal{M}_{\Lambda_1} \cong \mathcal{N}_{\Lambda_1}$ in \mathbf{VBun} . Note that the functor $\varprojlim: \mathbf{SMan} \rightarrow \mathbf{Man}$ is not injective on isomorphism classes. For example

$$\varprojlim \overline{\mathbb{R}^{1|0}} \cong \prod_{\substack{I \subseteq \mathbb{N}, |I| < \infty, \\ |I| \text{ even}}} \mathbb{R} \cong \prod_{n \in \mathbb{N}} \mathbb{R} \cong \mathbb{R} \times \prod_{\substack{I \subseteq \mathbb{N}, |I| < \infty, \\ |I| \text{ odd}}} \mathbb{R} \cong \varprojlim \overline{\mathbb{R}^{0|1}}$$

in the category \mathbf{Man} .

We have seen in Theorem 2.3.11 how to embed the category of supermanifolds into the category of manifolds. Conversely, one can also embed the category \mathbf{Man} into the category \mathbf{SMan} . For this, let \mathbf{Dom} denote the category consisting of pairs (U, E_0) where E_0 is a Hausdorff locally convex space and $U \subseteq E_0$ is open and where the morphisms are smooth maps between the open subsets.

Proposition 2.3.14 ([40, cf. Proposition 4.2.1, p.396]). *Let $k \in \mathbb{N}_0 \cup \{\infty\}$. We define a functor*

$$\iota_k^0: \mathbf{Dom} \rightarrow \mathbf{SDom}^{(k)}$$

by setting $\iota_k^0(U) := \overline{E}^{(k)}|_U$ and $\iota_k^0(f_0) := (f_0, 0, 0, \dots)$ for $(U, E_0) \in \mathbf{Dom}$ and $E := E_0 \oplus \{0\} \in \mathbf{SVec}_{lc}$. This functor extends to a fully faithful functor

$$\iota_k^0: \mathbf{Man} \rightarrow \mathbf{SMan}^{(k)}.$$

In case of $k = \infty$ we also write $\iota: \mathbf{Man} \rightarrow \mathbf{SMan}$. The functor $\iota_0^0: \mathbf{Man} \rightarrow \mathbf{SMan}^{(0)}$ is an equivalence of categories. All of these functors respect products.

Proof. It follows from the composition formula in Proposition 2.2.16 that $\iota_k^0: \mathbf{Dom} \rightarrow \mathbf{SDom}^{(k)}$ is a functor. Let M be a manifold modelled on E_0 with atlas $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha: \alpha \in A\}$. Applying this functor to the change of charts $\varphi_{\alpha\beta}: U_{\alpha\beta} \rightarrow U_{\beta\alpha}$, defines an (up to unique isomorphism) unique supermanifold \mathcal{M} modelled on $E_0 \oplus \{0\}$ with the atlas $\{\iota_k^0((\varphi_\alpha)^{-1}): \iota_k^0(U_\alpha) \rightarrow \mathcal{M}: \alpha \in A\}$ by Proposition 2.3.5. If $N \in \mathbf{Man}$ has the atlas $\{\psi_\beta: V'_\beta \rightarrow U'_\beta: \beta \in B\}$ and $f: M \rightarrow N$ is

a smooth map then the same proposition applied to $f_{\alpha\beta} = \psi_\beta \circ f \circ (\varphi_\alpha)^{-1}$ leads to a unique morphism $\iota_k^0(f): \iota_k^0(M) \rightarrow \iota_k^0(N)$ such that $(\iota_k^0(f))^{\alpha\beta} = \iota_k^0(f_{\alpha\beta})$. Functoriality follows again by the local definition of the composition of supersmooth morphisms.

The uniqueness of this construction shows that ι_k^0 is faithful. On the other hand, every supersmooth map $g: \iota_k^0(M) \rightarrow \iota_k^0(N)$ is determined by its local chart descriptions $g^{\alpha\beta}$, whose skeletons have the form $(g_0^{\alpha\beta}, 0, 0, \dots)$ since $\iota_k^0(M)$ is purely even. Clearly, the maps $g_0^{\alpha\beta}$ define a unique smooth map $M \rightarrow N$ whose image under ι_k^0 is g . We already know from Theorem 2.3.11 that $\mathcal{M} \mapsto \mathcal{M}_{\mathbb{R}}$ and $f \mapsto f_{\mathbb{R}}$ defines a functor $\pi_0^0: \mathbf{SMan}^{(0)} \rightarrow \mathbf{Man}$ and the above shows that $\pi_0^0 \circ \iota_0^0 \cong \text{id}_{\mathbf{Man}}$ and that $\iota_0^0 \circ \pi_0^0 \cong \text{id}_{\mathbf{SMan}}$.

It is obvious that the functor $\iota_k^0: \mathbf{Dom} \rightarrow \mathbf{SDom}^{(k)}$ preserves products and from this it follows immediately that $\iota_k^0: \mathbf{Man} \rightarrow \mathbf{SMan}^{(k)}$ also preserves products. \square

Lemma 2.3.15. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For every supermanifold $\mathcal{M} \in \mathbf{SMan}^{(k)}$, we have that $\iota_k^0(\mathcal{M}_{\mathbb{R}})$ is a sub-supermanifold of \mathcal{M} . If \mathcal{M} is purely even, we have $\iota_k^0(\mathcal{M}_{\mathbb{R}}) \cong \mathcal{M}$.*

Proof. Let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$ and let $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ be an atlas of \mathcal{M} . If the changes of charts $\varphi^{\alpha\beta}$ have the skeletons $(\varphi_n^{\alpha\beta})$, then the skeletons $(\varphi_0^{\alpha\beta}, 0, \dots)$ define $\iota_0^k(\mathcal{M}_{\mathbb{R}})$ by Proposition 2.3.14. Because $\varphi^{\alpha\beta}|_{\overline{E_0 \oplus \{0\}}^{(k)}}$ has the skeleton $(\varphi_0^{\alpha\beta}, 0, \dots)$, it follows that $\iota_k^0(\mathcal{M}_{\mathbb{R}})$ is a sub-supermanifold of \mathcal{M} . If \mathcal{M} is purely even, then changes of charts have the form $(\varphi_0^{\alpha\beta}, 0, \dots)$ to begin with and it follows $\iota_k^0(\mathcal{M}_{\mathbb{R}}) \cong \mathcal{M}$ by Proposition 2.3.5. \square

Purely even supermanifolds \mathcal{M} can be described in terms of higher tangent bundles of $\mathcal{M}_{\mathbb{R}}$. This will be particularly important for the theory of Lie supergroups.

Proposition 2.3.16. *Let M be a manifold. Recall Example B.3.4. Using Lemma B.2.9 and Theorem 2.3.11, there are isomorphisms*

$$\Gamma_n^k: \iota_k^0(M)_{\Lambda_n} \rightarrow T^k M|_{\mathcal{P}_{0,+}^n}^-$$

of multilinear bundles of degree n for every $n \leq k < \infty$. These isomorphisms are natural in k and n in the sense that

$$(\iota_k^0(M)_{\Lambda_k}, \iota_k^0(M)_{\varepsilon_{k,n}}) \cong (T^k M|_{\mathcal{P}_{0,+}^k}^-, \pi_n^k|_{\mathcal{P}_{0,+}^k}^-)$$

holds as inverse systems of multilinear bundles. It follows that $\Lambda_k \mapsto T^k M|_{\mathcal{P}_{0,+}^k}^-$ can be made into a supermanifold isomorphic to $\iota(M)$.

Proof. To show that $\iota_0^k(M)_{\Lambda_n} \cong T^n M|_{\mathcal{P}_{0,+}^n}^-$ holds, we simply compare the change of charts. Let M be modelled on E_0 and $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha: \alpha \in A\}$ be an atlas of M . For a change of charts $\varphi^{\alpha\beta}: U_{\alpha\beta} \rightarrow U_{\beta\alpha}$, we have $\iota_0^k(\varphi^{\alpha\beta}) = (\varphi^{\alpha\beta}, 0, 0, \dots)$ and thus

$$\iota_0^k(\varphi^{\alpha\beta})_{\Lambda_n}(x + \sum_{I \in \mathcal{P}_{0,+}^n} \lambda_I x_I) = \varphi^{\alpha\beta}(x) + \sum_{I \in \mathcal{P}_{0,+}^n} \sum_{\omega \in \mathcal{P}_0(I)} \lambda_I \text{sgn}(\omega) d^{(e(\omega))} \varphi_{o(\omega)}^{\alpha\beta}(x)(x_\omega),$$

where $x \in U_{\alpha\beta}$ and $x_\omega = (x_{\omega_1}, \dots, x_{\omega_{\ell(\omega)}}) \in E_0^{e(\omega)}$. On the other hand, we know from Example B.2.7(b) and the Lemma B.2.9 that the change of charts for $T^k M|_{\mathcal{P}_{0,+}^k}$ is given by

$$T^k \varphi^{\alpha\beta}|_{\mathcal{P}_{0,+}^k} \left(x + \sum_{I \in \mathcal{P}_{0,+}^n} \varepsilon_I x_I \right) = \varphi^{\alpha\beta}(x) + \sum_{I \in \mathcal{P}_{0,+}^n} \sum_{\omega \in \mathcal{P}_0(I)} \varepsilon_I \text{sgn}(\omega) d^{(e(\omega))} \varphi_{o(\omega)}^{\alpha\beta}(x)(x_\omega)$$

for the same $x \in U_{\alpha\beta}$ and $x_\omega \in E_0^{e(\omega)}$. Therefore, there exists an isomorphism $\Gamma_n^k: \iota_0^k(M)_{\Lambda_n} \rightarrow T^k M|_{\mathcal{P}_{0,+}^k}$ such that $(\Gamma_n^k)^\alpha := T^k \varphi^\alpha|_{\mathcal{P}_{0,+}^k} \circ \Gamma_n^k \circ \iota_0^k(\varphi^\alpha)^{-1}_{\Lambda_n}$ is given by the obvious isomorphisms of trivial k -multilinear bundles

$$(\Gamma_n^k)^\alpha: \iota_0^k(U_\alpha)_{\Lambda_n} = U_\alpha \times \prod_{I \in \mathcal{P}_{0,+}^n} \lambda_I E_0 \rightarrow U_\alpha \times \prod_{I \in \mathcal{P}_{0,+}^n} \varepsilon_I E_0 = T^k|_{\mathcal{P}_{0,+}^n}(U_\alpha)^-.$$

Note that for any k -multilinear bundle F and $n \leq k$, we have $(F|_{\mathcal{P}_{0,+}^k})|_{\mathcal{P}_{0,+}^n} = F|_{\mathcal{P}_{0,+}^n}$ and it follows from the local description in Lemma B.3.1 that $(q_n^k)^-: F|_{\mathcal{P}_{0,+}^k}^- \rightarrow F|_{\mathcal{P}_{0,+}^n}^-$ is just the respective projection of $F|_{\mathcal{P}_{0,+}^k}^-$. This shows that $(T^k M|_{\mathcal{P}_{0,+}^k}^-, \pi_n^k|_{\mathcal{P}_{0,+}^k}^-)$ is indeed an inverse system of multilinear bundles. It is clear from the local description that $\Gamma_n^k \circ \iota(M)_{\varepsilon_{k,n}} = \pi_n^k|_{\mathcal{P}_{0,+}^k}^- \circ \Gamma_k^k$, which shows $(\iota(M)_{\Lambda_k}, \iota(M)_{\varepsilon_{k,n}}) \cong (T^k M|_{\mathcal{P}_{0,+}^k}^-, \pi_n^k|_{\mathcal{P}_{0,+}^k}^-)$.

To turn $\Lambda_k \mapsto T^k M|_{\mathcal{P}_{0,+}^k}^-$ into a supermanifold, one simply defines $(T^k M|_{\mathcal{P}_{0,+}^k}^-)_{\varrho}: T^k M|_{\mathcal{P}_{0,+}^k}^- \rightarrow T^n M|_{\mathcal{P}_{0,+}^n}^-$ for every morphism $\varrho: \Lambda_k \rightarrow \Lambda_n$ via $\iota(M)_{\varrho}: \iota(M)_{\Lambda_k} \rightarrow \iota(M)_{\Lambda_n}$ and the above isomorphisms. The charts are then given by $(\Gamma_k^k \circ \iota(\varphi_\alpha^{-1})_{\Lambda_k})_{\Lambda_k}$. \square

Proposition 2.3.17 ([40, cf. Proposition 4.2.1, p.396]). *Let $k \in \mathbb{N} \cup \{\infty\}$. There is a faithful functor*

$$\iota_k^1: \mathbf{VBun} \rightarrow \mathbf{SMan}^{(k)}.$$

The functor $\iota_1^1: \mathbf{VBun} \rightarrow \mathbf{SMan}^{(1)}$ is an equivalence of categories. All these functors respect products.

Proof. The proof is very similar to the proof of Proposition 2.3.14. Let $\pi: F \rightarrow M$ be a vector bundle with typical fiber E_1 and bundle atlas $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha \times E_1: \alpha \in A\}$. The change of bundle charts $\varphi_{\alpha\beta}: U_{\alpha\beta} \times E_1 \rightarrow U_{\beta\alpha} \times E_1$ has the form $(\varphi_0^{\alpha\beta}, \varphi_1^{\alpha\beta})$, where $\varphi_0^{\alpha\beta}: U_{\alpha\beta} \rightarrow U_{\beta\alpha}$ is a smooth and $\varphi_1^{\alpha\beta}: U_{\alpha\beta} \times E_1 \rightarrow E_1$ is smooth and linear in the second component. Note that there exists an atlas of M such that the change of charts is given by $\varphi_0^{\alpha\beta}$. Let M be modelled on E_0 . We define the super vector space $E := E_0 \oplus E_1$ and let $\iota_1^1(U_\alpha \times E_1) := \mathcal{U}^\alpha := \overline{E}^{(k)}|_{U_\alpha}$, as well as $\mathcal{U}^{\alpha\beta} := \overline{E}^{(k)}|_{U_{\alpha\beta}}$, for all $\alpha, \beta \in A$. Then $i_1^1(\varphi^{\alpha\beta}) := (\varphi_0^{\alpha\beta}, \varphi_1^{\alpha\beta}, 0, 0, \dots): \mathcal{U}^{\alpha\beta} \rightarrow \mathcal{U}^{\beta\alpha}$ defines isomorphisms that satisfy the conditions of Proposition 2.3.5 because by the composition formula from Proposition 2.2.16, we have

$$(\varphi_0^{\beta\gamma}, \varphi_1^{\beta\gamma}, 0, 0, \dots) \circ (\varphi_0^{\alpha\beta}, \varphi_1^{\alpha\beta}, 0, 0, \dots) = (\varphi_0^{\beta\gamma} \circ \varphi_0^{\alpha\beta}, \varphi_1^{\beta\gamma} \circ \varphi_1^{\alpha\beta}(\varphi_1^{\alpha\beta}), 0, 0, \dots)$$

$$= (\varphi_0^{\alpha\gamma}, \varphi_1^{\alpha\gamma}, 0, 0, \dots) = \iota_1^1(\varphi^{\alpha\gamma})$$

where defined. We let $\iota_k^1(F)$ be the supermanifold \mathcal{M} defined in Proposition 2.3.5 by the given change of charts.

Morphisms $f: F \rightarrow F'$ of vector bundles have again the local form $(f_0^{\alpha\beta}, f_1^{\alpha\beta})$, where $f_1^{\alpha\beta}$ is linear in the second component and define skeletons $(f_0^{\alpha\beta}, f_1^{\alpha\beta}, 0, 0, \dots)$ that satisfy Proposition 2.3.5. In this way, we obtain a unique supersmooth morphism $\iota_1^1(f): \iota_1^1(F) \rightarrow \iota_1^1(F')$. By the same argument as above, this construction is functorial and, by uniqueness, the resulting functor is faithful. \square

Lemma 2.3.18 ([40, cf. Proposition 4.2.1, p.396]). *Let $k, n \in \mathbb{N}_0 \cup \{\infty\}$ and $n \leq k$. The restriction of functors $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ to functors $\mathbf{Man}^{\mathbf{Gr}^{(n)}}$ leads to functors*

$$\pi_n^k: \mathbf{SMan}^{(k)} \rightarrow \mathbf{SMan}^{(n)}, \quad \pi_n^k(\mathcal{M}) =: \mathcal{M}^{(n)}.$$

On morphisms, we write $\pi_n^k(f) =: f^{(n)}$. These functors respect products and $\pi_m^k = \pi_n^k \circ \pi_m^n$ holds for all $m \leq n$. Identifying $\mathbf{SMan}^{(0)}$ with \mathbf{Man} and $\mathbf{SMan}^{(1)}$ with \mathbf{VBun} via Proposition 2.3.14 and Proposition 2.3.17, we have $\pi_0^k \circ \iota_k^0 \cong \text{id}_{\mathbf{Man}}$ and $\pi_1^k \circ \iota_k^1 \cong \text{id}_{\mathbf{VBun}}$ if $k > 0$.

Proof. Let $\mathcal{M}, \mathcal{N} \in \mathbf{SMan}^{(k)}$. It follows directly from the definition that $\Lambda \rightarrow \mathcal{M}_\Lambda$ for $\Lambda \in \mathbf{Gr}^{(n)}$ defines an n -supermanifold $\mathcal{M}^{(n)}$ with the obvious restricted atlas. Likewise, for morphisms $f: \mathcal{M} \rightarrow \mathcal{N}$, one defines $f^{(n)}$ by $f_\Lambda^{(n)} := f_\Lambda$ for $\Lambda \in \mathbf{Gr}^{(n)}$. This construction is clearly functorial, respects products and satisfies $\pi_m^k = \pi_n^k \circ \pi_m^n$ for all $m \leq n$. To see $\pi_0^k \circ \iota_k^0 \cong \text{id}_{\mathbf{Man}}$ and $\pi_1^k \circ \iota_k^1 \cong \text{id}_{\mathbf{VBun}}$, one simply checks that on the level of skeletons this composition does not change anything. \square

One can understand the projections $\pi_n^k: \mathbf{SMan}^{(k)} \rightarrow \mathbf{SMan}^{(n)}$ and the embeddings $\iota_k^0: \mathbf{Man} \rightarrow \mathbf{SMan}^{(k)}$ and $\iota_k^1: \mathbf{VBun} \rightarrow \mathbf{SMan}^{(k)}$ completely in terms of skeletons. The former simply cuts skeletons (f_0, \dots) down to (f_0, \dots, f_n) . The latter two extend skeletons (f_0) , resp. (f_0, f_1) , to $(f_0, 0, \dots)$, resp. $(f_0, f_1, 0, \dots)$. Proposition 2.2.16 ensures that the composition of two such skeletons is again of this form, which is why these embeddings are well-defined.

A natural question is now whether two k -supermanifolds $\mathcal{M}^{(k)}$ and $\mathcal{N}^{(k)}$ such that $\mathcal{M}^{(n)} \cong \mathcal{N}^{(n)}$ holds for $1 < n < k$ are automatically isomorphic as well. In other words, whether a supersmooth isomorphism $f^{(n)}: \mathcal{M}^{(n)} \rightarrow \mathcal{N}^{(n)}$ can be lifted to an isomorphism $f^{(k)}: \mathcal{M}^{(k)} \rightarrow \mathcal{N}^{(k)}$. This will be discussed in the following section on Batchelor's Theorem.

Definition 2.3.19. We denote by \mathbf{p} the supermanifold modelled on $\{0\} \oplus \{0\}$ that consists for every $\Lambda \in \mathbf{Gr}$ of a single point. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A *point* of a k -supermanifold \mathcal{M} is a morphism $x: \mathbf{p}^{(k)} \rightarrow \mathcal{M}$. We also write $x_\Lambda := x_\Lambda(\mathbf{p}_\Lambda^{(k)})$.

Lemma 2.3.20. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$. For every point $x: \mathbf{p}^{(k)} \rightarrow \mathcal{M}$ and every $\Lambda \in \mathbf{Gr}^{(k)}$, we have $x_\Lambda = \mathcal{M}_{\eta_\Lambda}(x_\mathbb{R})$. Conversely, for every $x_\mathbb{R} \in \mathcal{M}_\mathbb{R}$ the assignment $x_\Lambda := \mathcal{M}_{\eta_\Lambda}(x_\mathbb{R})$ defines a point.*

Proof. For every $\Lambda \in \mathbf{Gr}^{(k)}$, we have

$$\mathcal{M}_{\eta_\Lambda}(x_\mathbb{R}) = x_\Lambda \circ \mathbf{p}_{\eta_\Lambda}^{(k)}(\mathbf{p}_\mathbb{R}^{(k)}) = x_\Lambda.$$

Conversely, let $x_\mathbb{R} \in \mathcal{M}_\mathbb{R}$ be given, $x_\Lambda := \mathcal{M}_{\eta_\Lambda}(x_\mathbb{R})$ and $\varrho \in \mathrm{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. Then $\varrho \circ \eta_\Lambda = \eta_{\Lambda'}$ and therefore $\mathcal{M}_\varrho(x_\Lambda) = \mathcal{M}_{\eta_{\Lambda'}}(x_\mathbb{R}) = x_{\Lambda'}$. \square

Hence, the points of a supermanifold can be identified with the usual points of the base manifold.

2.3.1. Connection to the Sheaf Theoretic Approach

The full subcategory of finite-dimensional supermanifolds in the categorical approach is equivalent to the category of supermanifolds in the sheaf theoretic approach. This was already discussed in [54] and [38] but a more thorough and general proof can be found in [1]. Let us briefly sketch the idea behind the equivalence.

Let $p, q \in \mathbb{N}_0$ and $\mathcal{U} \subseteq \overline{\mathbb{R}^{p|q}}$ be an open subfunctor. In terms of skeletons, we have

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{\mathbb{R}^{1|1}}) = \mathcal{C}^\infty\left(\mathcal{U}_\mathbb{R}, \bigoplus_{i=0}^q \mathrm{Alt}^i(\mathbb{R}^q; \mathbb{R})\right) \cong \mathcal{C}^\infty(\mathcal{U}_\mathbb{R}, \mathbb{R}) \otimes \Lambda_q.$$

Therefore, for any supermanifold \mathcal{M} modelled on $\mathbb{R}^{p|q}$, the sheaf

$$U \mapsto \mathcal{SC}^\infty(\mathcal{M}|_U, \overline{\mathbb{R}^{1|1}}), \quad U \subseteq \mathcal{M}_\mathbb{R} \text{ open},$$

is locally isomorphic to the sheaf $\mathcal{C}_{\mathbb{R}^p}^\infty \otimes \Lambda_q$ as needed. One then checks that morphisms of supermanifolds lead to appropriate morphisms of these sheaves along the pullback.

2.3.2. Generalizations

Many of the generalizations for k -superdomains mentioned in 2.2.2 can be applied to supermanifolds without much difficulty such that the results in this section carry over. As already mentioned, one can consider non-Hausdorff supermanifolds by simply extending the category **Man** to non-Hausdorff manifolds. Analytic supermanifolds can be defined by demanding that the skeletons are analytic in an appropriate sense.

One should also note that many structural results do not rely on supersmoothness. In view of Proposition 2.2.7 and Lemma 2.2.5, one can define a subcategory of $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ of functors locally isomorphic to some $\overline{E}^{(k)}$, $E \in \mathbf{SVec}_{lc}$ where the changes of charts are simply natural transformations such that every component is smooth. Then an analog to Theorem 2.3.11 still holds and one obtains a geometry combining commuting and anticommuting coordinates with less stringent symmetry conditions than for supermanifolds.

2.4. Batchelor's Theorem

The classical version of Batchelor's Theorem (see [6]) states that any supermanifold, defined as a sheaf (M, \mathcal{O}_M) , is isomorphic to the supermanifold $(M, \Gamma(\bigwedge F))$, where $\bigwedge F$ is the exterior bundle of a vector bundle F that is determined by \mathcal{O}_M . The isomorphism is not canonical because its construction involves a partition of unity.

Molotkov transfers this result to supermanifolds in our sense and generalizes it to infinite-dimensional supermanifolds \mathcal{M} in [39]. In his version, the vector bundle \mathcal{M}_{Λ_1} takes the role of F in the classical version. Molotkov only considers Banach supermanifolds, but as we will see, his methods generalize to locally convex supermanifolds. It appears that [39] is not well-known and since it is not readily available, we describe his arguments in detail below. A closer look is also worthwhile because the techniques employed are close to the ones used in [6] and might make it easier to translate between the sheaf theoretic and the categorical approach. In Remark 4.2.7 below, we sketch an alternative proof of Batchelor's Theorem that relies only on our description of the automorphism group of a supermanifold.

Theorem 2.4.1 ([39, Corollary 4, p.279]). *Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be such that $\mathcal{M}_{\mathbb{R}}$ admits smooth partitions of unity. If $\mathcal{M}' \in \mathbf{SMan}^{(k)}$ is a k -supermanifold such that $\mathcal{M}^{(1)}$ and $\mathcal{M}'^{(1)}$ are isomorphic, then \mathcal{M} is isomorphic to \mathcal{M}' . In particular $\mathcal{M} \cong \iota_k^1(\mathcal{M}^{(1)})$.*

In other words, if one restricts the categories \mathbf{VBun} and $\mathbf{SMan}^{(k)}$ to the respective subcategories over finite-dimensional paracompact bases, the restricted functor ι_k^1 from Proposition 2.3.17 becomes essentially surjective.

Definition 2.4.2. Let $k \in \mathbb{N}$. We call k -supermanifolds of the form $\iota_k^1(\mathcal{M}^{(1)})$, where $\mathcal{M}^{(1)}$ is a vector bundle, *supermanifolds of Batchelor type*. An isomorphism $f: \mathcal{N} \rightarrow \iota_k^1(\mathcal{M}^{(1)})$ is called a *Batchelor model of \mathcal{N}* . We say an atlas $\mathcal{A} := \{\varphi^\alpha: \alpha \in A\}$ of a supermanifold is of *Batchelor type* if all changes of charts have the form $\varphi^{\alpha\beta} = (\varphi_0^{\alpha\beta}, \varphi_1^{\alpha\beta}, 0, \dots)$.

Remark 2.4.3. It follows from Proposition 2.3.5 that a supermanifold is of Batchelor type if and only if it has an atlas of Batchelor type. For a supermanifold of Batchelor type, the union of two atlases of Batchelor type is again of Batchelor type because the atlases define the same vector bundle. This does not need to be the case for arbitrary supermanifolds, which implies that there is no canonical choice of a Batchelor model in general.

One can reformulate this result as follows. In the situation of the theorem, any isomorphism $f^{(n)}: \mathcal{M}^{(n)} \rightarrow \mathcal{M}'^{(n)}$ can be lifted to an isomorphism $f^{(n+1)}: \mathcal{M}^{(n+1)} \rightarrow \mathcal{M}'^{(n+1)}$ for $1 \leq n < k$ (see [39, Theorem 1(a), p.273]). It is not difficult to see that one may assume $\mathcal{M}' = \mathcal{M}$, which we will do in the sequel to simplify our explanations (compare [39, Proposition 2, p.277]).

Let us introduce some notation for this section. For $\mathcal{M} \in \mathbf{SMan}^{(k)}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$ consider the group $\text{Aut}_{\text{id}_{\mathbb{R}}}(\mathcal{M})$ of automorphisms $f: \mathcal{M} \rightarrow \mathcal{M}$ such

that $f_{\mathbb{R}} = \text{id}_{\mathcal{M}_{\mathbb{R}}}$. We denote by $\mathcal{O}(\mathcal{M})$ the sheaf of groups over $\mathcal{M}_{\mathbb{R}}$ defined by $U \mapsto \text{Aut}_{\text{id}_{\mathbb{R}}}(\mathcal{M}|_U)$ for every open $U \subseteq \mathcal{M}_{\mathbb{R}}$. The restriction morphisms are given in the obvious way by Lemma/Definition 2.1.5(c). The restrictions are morphisms of groups because we only consider automorphisms over the identity on the base manifold. The functor $\pi_m^n: \mathbf{SMan}^{(n)} \rightarrow \mathbf{SMan}^{(m)}$ from Lemma 2.3.18 leads to morphisms $\varphi_m^n: \mathcal{O}(\mathcal{M}^{(n)}) \rightarrow \mathcal{O}(\mathcal{M}^{(m)})$ of sheaves of groups for $m \leq n < k+1$. Locally, $(\varphi_m^n)_U: \text{Aut}_{\text{id}_{\mathbb{R}}}(\mathcal{M}^{(n)}|_U) \rightarrow \text{Aut}_{\text{id}_{\mathbb{R}}}(\mathcal{M}^{(m)}|_U)$ just maps skeletons $(\text{id}, f_1, \dots, f_n)$ to $(\text{id}, f_1, \dots, f_m)$. We define

$$\mathcal{O}_m^n(\mathcal{M}) := \ker \varphi_m^n.$$

The elements of $\mathcal{O}_m^n(\mathcal{M})$ are exactly those which locally have the form $(\text{id}, c_{\text{id}}, 0, \dots, 0, f_{m+1}, \dots, f_n)$. In particular, we get a short exact sequence of sheaves of groups

$$1 \rightarrow \mathcal{O}_n^{n+1}(\mathcal{M}) \hookrightarrow \mathcal{O}_0^{n+1}(\mathcal{M}) \rightarrow \mathcal{O}_0^n(\mathcal{M}) \rightarrow 1$$

(see [39, Theorem 1, p.273f.]). Note that $\mathcal{O}_0^{n+1}(\mathcal{M}) = \mathcal{O}(\mathcal{M}^{(n+1)})$. We sum up the most relevant results from [39] about the structure of $\mathcal{O}_n^{n+1}(\mathcal{M})$ in the next lemma and give a sketch of the proof.

Lemma 2.4.4 (compare [39, Theorem 1(d), p.274]). *Let $k \in \mathbb{N} \cup \{\infty\}$, $n < k$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$. Then $\mathcal{O}_n^{n+1}(\mathcal{M})$ is a sheaf of abelian groups and a $\mathcal{C}_{\mathcal{M}_{\mathbb{R}}}^{\infty}$ -module. If \mathcal{M} is a Banach supermanifold, then there exist canonical isomorphisms of $\mathcal{C}_{\mathcal{M}_{\mathbb{R}}}^{\infty}$ -modules*

$$\mathcal{O}_n^{n+1}(\mathcal{M}) \cong \Gamma(\text{Alt}^{n+1}(\mathcal{M}_{\Lambda_1}; T\mathcal{M}_{\mathbb{R}})) \quad \text{if } n+1 \text{ is even and}$$

$$\mathcal{O}_n^{n+1}(\mathcal{M}) \cong \Gamma(\text{Alt}^{n+1}(\mathcal{M}_{\Lambda_1}; \mathcal{M}_{\Lambda_1})) \quad \text{if } n+1 \text{ is odd.}$$

Proof. Let $\{\varphi^{\alpha}: \mathcal{U}^{\alpha} \rightarrow \mathcal{M}^{(n+1)}: \alpha \in A\}$ be an atlas of $\mathcal{M}^{(n+1)}$ and $U \subseteq \mathcal{M}_{\mathbb{R}}$ be open. For $f \in \mathcal{O}_n^{n+1}(\mathcal{M})_U$, we set $f^{\alpha} := (\varphi^{\alpha})^{-1} \circ f \circ \varphi^{\alpha}$, where we may assume after restriction that φ^{α} is a chart of $\mathcal{M}^{(n+1)}|_U$. Locally, f has the form $f^{\alpha} = (\text{id}_{\mathcal{U}_{\mathbb{R}}^{\alpha}}, c_{\text{id}}, 0, \dots, 0, f_{n+1}^{\alpha})$. For $g \in \mathcal{O}_n^{n+1}(\mathcal{M})_U$, we use Proposition 2.2.16 to calculate

$$(f^{\alpha} \circ g^{\alpha}) = (f \circ g)^{\alpha} = (\text{id}_{\mathcal{U}_{\mathbb{R}}^{\alpha}}, c_{\text{id}}, 0, \dots, 0, f_{n+1}^{\alpha} + g_{n+1}^{\alpha}).$$

Thus, $\mathcal{O}_n^{n+1}(\mathcal{M})$ is a sheaf of abelian groups. Let $\varphi^{\beta\alpha}$ be a change of charts of $\mathcal{M}^{(n+1)}|_U$. We again use Proposition 2.2.16 to get

$$(\varphi^{\beta\alpha})^{-1} \circ f^{\alpha} \circ \varphi^{\beta\alpha} = (\text{id}_{\mathcal{U}_{\mathbb{R}}^{\beta\alpha}}, c_{\text{id}}, 0, \dots, 0, f_{n+1}^{\beta}),$$

where

$$f_{n+1}^{\beta}(\varphi_0^{\alpha\beta}(x))(\bullet) = d\varphi_0^{\alpha\beta}(x)(f_{k+1}^{\alpha}(x)(\varphi_1^{\beta\alpha}(x)(\bullet), \dots, \varphi_1^{\beta\alpha}(x)(\bullet)))$$

for $n+1$ even, $x \in \mathcal{U}_{\mathbb{R}}^{\beta\alpha}$ and

$$f_{n+1}^{\beta}(\varphi_0^{\alpha\beta}(x))(\bullet) = \varphi_1^{\alpha\beta}(x)(f_{k+1}^{\alpha}(x)(\varphi_1^{\beta\alpha}(x)(\bullet), \dots, \varphi_1^{\beta\alpha}(x)(\bullet)))$$

for $n + 1$ odd, $x \in \mathcal{U}_{\mathbb{R}}^{\beta\alpha}$. If \mathcal{M}_{Λ_1} is a Banach vector bundle, this describes exactly the change of charts for a section $f_{n+1}: \mathcal{M}_{\mathbb{R}}|_U \rightarrow \mathcal{A}lt^{n+1}(\mathcal{M}_{\Lambda_1}|_U; T\mathcal{M}_{\mathbb{R}}|_U)$, resp. $f_{n+1}: \mathcal{M}_{\mathbb{R}}|_U \rightarrow \mathcal{A}lt^{n+1}(\mathcal{M}_{\Lambda_1}|_U; \mathcal{M}_{\Lambda_1}|_U)$.

It is easy to see from these formulas that for a smooth map $h: \mathcal{M}_{\mathbb{R}}|_U \rightarrow \mathbb{R}$ with the local description $h^\alpha := h \circ \varphi_0^\alpha$, the multiplication $h \cdot f$, defined by

$$(h \cdot f)^\alpha = (\text{id}_{\mathcal{U}_{\mathbb{R}}^\alpha}, c_{\text{id}}, 0, \dots, 0, h^\alpha \cdot f_{n+1}^\alpha),$$

leads to a $\mathcal{C}_{\mathcal{M}_{\mathbb{R}}}^\infty$ -module structure on $\mathcal{O}_n^{n+1}(\mathcal{M})$. This structure corresponds to the $\mathcal{C}_{\mathcal{M}_{\mathbb{R}}}^\infty$ -module structure of the sheaves of sections defined above in the Banach case. \square

We will now return to finding a lift for an isomorphism $f^{(n)}: \mathcal{M}^{(n)} \rightarrow \mathcal{M}^{(n)}$ to an isomorphism $f^{(n+1)}: \mathcal{M}^{(n+1)} \rightarrow \mathcal{M}^{(n+1)}$. Let $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be modelled on $E \in \mathbf{SVec}_{lc}$ with atlas $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ and $n < k$. Locally, we have isomorphisms $f^{(n),\alpha}: \mathcal{U}^{\alpha,(n)} \rightarrow \mathcal{U}^{\alpha,(n)}$ of the form $(\text{id}_{\mathcal{U}_{\mathbb{R}}^\alpha}, f_1^{(n),\alpha}, \dots, f_n^{(n),\alpha})$. By Lemma 2.2.18, these can be lifted to isomorphisms $\tilde{f}^{\alpha,(n+1)}: \mathcal{U}^{\alpha,(n+1)} \rightarrow \mathcal{U}^{\alpha,(n+1)}$ in $\mathbf{SDom}^{(n+1)}$ of the form $(\text{id}_{\mathcal{U}_{\mathbb{R}}^\alpha}, f_1^{(n),\alpha}, \dots, f_n^{(n),\alpha}, \tilde{f}_{n+1}^{(n+1),\alpha})$, where $\tilde{f}_{n+1}^{(n+1),\alpha}: \mathcal{U}_{\mathbb{R}}^\alpha \rightarrow \mathcal{A}lt^{n+1}(E_1; E_{n+1}^-)$ is an arbitrary map which is smooth in the sense of skeletons.

The morphisms $\tilde{f}^{\beta,(n+1)}$ and $\varphi^{\alpha\beta,(n+1)} \circ \tilde{f}^{\alpha,(n+1)} \circ \varphi^{\beta\alpha,(n+1)}$ differ on $\mathcal{U}^{\beta\alpha,(n+1)}$ only in the $(n + 1)$ -th components of their skeletons because higher components do not affect the composition of any lower components. The difference is given by a unique element $h^{\beta\alpha} \in \mathcal{O}_n^{n+1}(\mathcal{U}^{\beta\alpha})$ such that

$$\tilde{f}^{\beta,(n+1)} = \varphi^{\alpha\beta,(n+1)} \circ \tilde{f}^{\alpha,(n+1)} \circ \varphi^{\beta\alpha,(n+1)} \circ h^{\beta\alpha}$$

and one checks that these $h^{\beta\alpha}$ define a cocycle in $\mathcal{O}_n^{n+1}(\mathcal{M})$ via $\tilde{h}^{\beta\alpha} := \varphi^{\beta,(n+1)} \circ h^{\beta\alpha} \circ (\varphi^{\beta,(n+1)})^{-1}$ on $\varphi^{\beta,(n+1)}(\mathcal{U}^{\beta\alpha,(n+1)})$. This cocycle describes the obstacle to lifting $f^{(n)}$ to $f^{(n+1)}$ (see [39, Theorem 3, p.277]). If $\mathcal{M}_{\mathbb{R}}$ admits smooth partitions of unity, then $\mathcal{O}_n^{n+1}(\mathcal{M})$ is a fine sheaf by Lemma 2.4.4 and thus acyclic. Therefore, the cocycle constructed above vanishes and a lift exists.

Remark 2.4.5. Mirroring the proof of the fact that for fine sheaves the higher Čech cohomologies vanish, one can directly construct the lift via a partition of unity. In the situation above, we assume that $(\varphi_{\mathbb{R}}^\alpha(\mathcal{U}_{\mathbb{R}}^\alpha))_{\alpha \in A}$ is a locally finite cover of $\mathcal{M}_{\mathbb{R}}$ and that $(\rho_\alpha)_{\alpha \in A}$ is a partition of unity subordinate to this cover. With the module structure from Lemma 2.4.4, we define

$$f_{n+1}^{(n+1),\alpha} := \left((\varphi^\alpha)^{-1} \circ \left(\sum_{\beta \in A} \rho_\beta \cdot \tilde{h}^{\beta\alpha} \right) \circ \varphi^\alpha \right)_{n+1},$$

where $\rho_\beta \cdot \tilde{h}^{\beta\alpha}$ is continued to $\varphi^{\alpha,(n+1)}(\mathcal{U}^{\alpha,(n+1)})$ by zero. It is elementary to check that the change of charts is well-defined for the resulting local descriptions of $f^{(n+1)}$.

2.5. Super Vector Bundles

In analogy to our definition of supermanifolds, we give a definition of super vector bundles as supermanifolds with a particular kind of bundle atlas. In this, we follow [40, Definition 5.1, p.29]. See also [47]. While a bit cumbersome, it will be useful to describe the change of bundle charts and the local form of bundle morphisms in terms of skeletons.

Definition 2.5.1 (compare [40, Subsection 1.3, p.5]). Let $E, F, H \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{U} \subseteq \overline{H}^{(k)}$ be an open subfunctor. A supersmooth morphism $f: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ is called a \mathcal{U} -family of \mathbb{R} -linear morphisms if for every $\Lambda \in \mathbf{Gr}^{(k)}$ and every $u \in \mathcal{U}_\Lambda$, the map

$$f_\Lambda(u, \bullet): \overline{E}_\Lambda^{(k)} \rightarrow \overline{F}_\Lambda^{(k)}$$

is $\Lambda_{\overline{0}}$ -linear.

Lemma 2.5.2. *In the situation of Definition 2.5.1, let $f: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ be a supersmooth morphism. Then f is a \mathcal{U} -family of \mathbb{R} -linear morphisms if and only if for all $\Lambda \in \mathbf{Gr}^{(k)}$, we have*

$$df_\Lambda((u, 0))((0, v)) = f_\Lambda(u, v) \text{ for } u \in \mathcal{U}_\Lambda, v \in \overline{E}_\Lambda^{(k)}.$$

Proof. If the equality holds, then f is an \mathcal{U} -family of \mathbb{R} -linear morphisms because the derivative df_Λ is $\Lambda_{\overline{0}}$ -linear at every $u \in \mathcal{U}_\Lambda$. The converse is true because any $\Lambda_{\overline{0}}$ -linear map is in particular \mathbb{R} -linear and thus the derivative of such a map is the map itself. \square

Lemma 2.5.3. *In the situation of Definition 2.5.1, let $f: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ be a supersmooth morphism with the skeleton $(f_n)_n$. We set $U := \mathcal{U}_{\mathbb{R}}$. Then f is an \mathcal{U} -family of \mathbb{R} -linear morphisms if and only if every f_n has the form*

$$\begin{aligned} f_n = & f_n(\text{pr}_U, \text{pr}_{E_0})((\text{pr}_1, 0), \dots, (\text{pr}_1, 0)) \\ & + n \cdot \mathfrak{A}^n f_n(\text{pr}_U, 0)((0, \text{pr}_2), (\text{pr}_1, 0), \dots, (\text{pr}_1, 0)), \end{aligned}$$

with $f_n(\text{pr}_U, \text{pr}_{E_0})((\text{pr}_1, 0), \dots, (\text{pr}_1, 0))$ linear in the second component and where $\text{pr}_U: U \times E_0 \rightarrow U$, $\text{pr}_{E_0}: U \times E_0 \rightarrow E_0$, $\text{pr}_1: H_1 \times E_1 \rightarrow H_1$ and $\text{pr}_2: H_1 \times E_1 \rightarrow E_1$ are the respective projections.

Proof. Let $f: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ be an \mathcal{U} -family of \mathbb{R} -linear morphisms. Choosing $u := x + \sum_{i=1}^k \lambda_i y_i$, $x \in U$ and $y_i \in H_1$, Proposition 2.2.13 implies that $f_n(x, \bullet)((y_1, \bullet), \dots, (y_n, \bullet))$ must be linear in $E_0 \oplus E_1 \oplus \dots \oplus E_1$. We use the multilinearity of $f_n(x, v)(\bullet)$ to calculate

$$\begin{aligned} f_n(x, v)((y_1, w_1), \dots, (y_n, w_n)) = \\ f_n(x, v)((y_1, 0), \dots, (y_n, 0)) + \sum_{i=1}^n f_n(x, v)((y_1, 0), \dots, (0, w_i), \dots, (y_n, 0)) = \end{aligned}$$

$$(f_n(x, v)((\text{pr}_1, 0), \dots, (\text{pr}_1, 0)) + \\ n \cdot \mathfrak{A}^n f_n(x, 0)((0, \text{pr}_2), (\text{pr}_1, 0), \dots, (\text{pr}_1, 0))((y_1, w_1), \dots, (y_n, w_n))$$

for $v \in E_0$ and $w_i \in E_1$. The second equality follows because for $v' \in E_0$, we have

$$f_n(x, v + v')((y_1, 0), \dots, (0, w_i), \dots, (y_n, 0)) = \\ f_n(x, v)((y_1, 0), \dots, (0, w_i), \dots, (y_n, 0)) + f_n(x, v')((y_1, 0), \dots, (0, 0), \dots, (y_n, 0)).$$

Conversely, let $(f_n)_n$ have the aforementioned form. Let $\Lambda \in \mathbf{Gr}^{(k)}$, $(x, y) \in U \times E_0$ and $(x_i, y_i) \in (H_i \oplus E_i) \otimes \Lambda_i^+$, $i \in \{0, 1\}$. To simplify our notation, we consider $\overline{H}_\Lambda^{(k)} \subseteq \overline{H} \oplus \overline{E}_\Lambda^{(k)}$ and $\overline{E}_\Lambda^{(k)} \subseteq \overline{H} \oplus \overline{E}_\Lambda^{(k)}$ in the obvious way. One sees

$$d^m f_l(x, y)((x_0, y_0), \dots, (x_0, y_0), (x_1, y_1), \dots, (x_1, y_1)) = \\ d^m f_l(x, y)(x_0, \dots, x_0, x_1, \dots, x_1) + m \cdot d^{m-1} f_l(x, y_0)(x_0, \dots, x_0, x_1, \dots, x_1) \\ + l \cdot d^m f_l(x)(x_0, \dots, x_0, y_1, x_1 \dots x_1),$$

where the last two summands are understood to be zero for $m = 0$ and $l = 0$, respectively. For $u = x + x_0 + x_1$ and $v = y + y_0 + y_1$, we use Remark 2.2.15 to get

$$df_\Lambda(u)(v) = \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \cdot \left(d^m f_l(x, y)(x_0, \dots, x_0, x_1, \dots, x_1) \right. \\ \left. + m \cdot d^{m-1} f_l(x, y_0)(x_0, \dots, x_0, x_1, \dots, x_1) \right. \\ \left. + l \cdot d^m f_l(x)(x_0, \dots, x_0, y_1, x_1 \dots, x_1) \right). \quad (2.4)$$

Comparing the terms, Proposition 2.2.13 implies that $df_\Lambda(u)(v) = f_\Lambda(u, v)$ and the result follows from Lemma 2.5.2. \square

Definition 2.5.4. Let $E, F \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N}_0 \cup \{\infty\}$ and let $\mathcal{E}, \mathcal{M} \in \mathbf{SMan}^{(k)}$ be such that \mathcal{E} is modelled on $E \oplus F$ and \mathcal{M} is modelled on E together with a supersmooth morphism $\pi: \mathcal{E} \rightarrow \mathcal{M}$ such that $\pi_\Lambda: \mathcal{E}_\Lambda \rightarrow \mathcal{M}_\Lambda$ is a vector bundle with fiber $\overline{F}_\Lambda^{(k)}$ for all $\Lambda \in \mathbf{Gr}^{(k)}$. A *bundle atlas* of \mathcal{E} is an atlas $\mathcal{A} := \{\Psi^\alpha: \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E}: \mathcal{U}^\alpha \subseteq \overline{E}^{(k)}, \alpha \in A\}$ such that $\{\Psi_\Lambda^\alpha: \alpha \in A\}$ is a bundle atlas of \mathcal{E}_Λ and the change of two charts Ψ^α and Ψ^β has the form $\Psi^{\alpha\beta}: \mathcal{U}^{\alpha\beta} \times \overline{F}^{(k)} \rightarrow \mathcal{U}^{\beta\alpha} \times \overline{F}^{(k)}$ with $\Psi^{\alpha\beta} = (\phi^{\alpha\beta}, \psi^{\alpha\beta})$, where

$$(i) \quad \phi^{\alpha\beta}: \mathcal{U}^{\alpha\beta} \rightarrow \mathcal{U}^{\beta\alpha} \text{ and}$$

$$(ii) \quad \psi^{\alpha\beta}: \mathcal{U}^{\alpha\beta} \times \overline{F}^{(k)} \rightarrow \overline{F}^{(k)} \text{ is an } \mathcal{U}^{\alpha\beta}\text{-family of } \mathbb{R}\text{-linear maps.}$$

The elements of \mathcal{A} are called *bundle charts*. Two bundle atlases are *equivalent* if their union is again a bundle atlas. We call $\pi: \mathcal{E} \rightarrow \mathcal{M}$ together with an equivalence class of bundle atlases a *k-super vector bundle over the base \mathcal{M} with typical fiber F* . The morphism π is called the *projection to the base*.

Let \mathcal{E}' be another k -super vector bundle with typical fiber $F' \in \mathbf{SVec}_{lc}$ and base \mathcal{N} . A supersmooth morphism $f: \mathcal{E} \rightarrow \mathcal{E}'$ is a *morphism of super vector bundles*, if in bundle charts Ψ^α of \mathcal{E} and $\Psi'^{\alpha'}$ of \mathcal{E}' , it has the form $(h^{\alpha\alpha'}, g^{\alpha\alpha'})$, where

$$(i) \quad h^{\alpha\alpha'}: \mathcal{U}^{\alpha\alpha'} \rightarrow \mathcal{U}^{\alpha'}, \mathcal{U}^{\alpha\alpha'} \subseteq \mathcal{U}^\alpha \text{ and}$$

$$(ii) \quad g^{\alpha\alpha'}: \mathcal{U}^{\alpha\alpha'} \times \overline{F}^{(k)} \rightarrow \overline{F'}^{(k)} \text{ is a } \mathcal{U}^{\alpha\alpha'}\text{-family of } \overline{\mathbb{R}}\text{-linear maps.}$$

Clearly, the $h^{\alpha\alpha'}$ define a supersmooth morphism $h: \mathcal{M} \rightarrow \mathcal{N}$ such that $\pi_{\mathcal{N}} \circ f = h \circ \pi_{\mathcal{M}}$. We say that f is a *morphism over h* . The k -super vector bundles and their morphisms form a category, which we denote by $\mathbf{SVBun}^{(k)}$, resp. \mathbf{SVBun} if $k = \infty$.

By this definition, a k -super vector bundle can be seen as a functor $\mathbf{Gr}^{(k)} \rightarrow \mathbf{VBun}$. This point of view is taken by Molotkov in [40].

Remark 2.5.5. It follows from Proposition 2.3.5 that, if one has a collection of change of charts that satisfy the conditions of Definition 2.5.4, then one gets a (up to unique isomorphism) unique super vector bundle. In the notation of the definition, the $\phi^{\alpha\beta}$ then define the base supermanifold \mathcal{M} and the bundle projection is locally given by

$$(\phi^\alpha)^{-1} \circ \pi \circ \Psi^\alpha := \text{pr}_{\mathcal{U}^\alpha}: \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{U}^\alpha.$$

In the same way, morphisms of super vector bundles are determined by their local description.

Lemma/Definition 2.5.6. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, let \mathcal{E} be a k -super vector bundle with typical fiber $F \in \mathbf{SVec}_{lc}$ over \mathcal{M} modelled on $E \in \mathbf{SVec}_{lc}$ and let $x: \mathbf{p}^{(k)} \rightarrow \mathcal{M}$ be a point of \mathcal{M} . We define \mathcal{E}_x , the *fiber of \mathcal{E} at x* , by setting $(\mathcal{E}_x)_\Lambda := (\pi_{\mathcal{M}})^{-1}(\{x_\Lambda\})$ for every $\Lambda \in \mathbf{Gr}^{(k)}$. Then \mathcal{E}_x is a sub-supermanifold of \mathcal{E} and \mathcal{E}_x is, in a canonical way, an $\overline{\mathbb{R}}$ -module such that $\mathcal{E}_x \cong \overline{E}^{(k)}$.

Proof. Let $\{\Psi^\alpha: \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E}: \alpha \in A\}$ be a bundle atlas of \mathcal{E} with corresponding atlas $\{\phi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ of \mathcal{M} . Let $\phi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}$ be such that $x_{\mathbb{R}} \in \phi_{\mathbb{R}}^\alpha(\mathcal{U}_{\mathbb{R}}^\alpha)$. We may assume that $0 \in \mathcal{U}_{\mathbb{R}}^\alpha$ and that $\phi_{\mathbb{R}}^\alpha(0) = x_{\mathbb{R}}$, because the translation defined by $\overline{E}_\Lambda^{(k)} \rightarrow \overline{E}_\Lambda^{(k)}$, $v \mapsto v - (\varphi_{\mathbb{R}}^\alpha)^{-1}(x_{\mathbb{R}})$ is clearly an isomorphism in $\mathbf{SMan}^{(k)}$. Let $\Lambda \in \mathbf{Gr}^{(k)}$. We have $y_\Lambda \in (\mathcal{E}_x)_\Lambda$ if and only if $(\pi_{\mathcal{M}})_\Lambda(y_\Lambda) = x_\Lambda$ holds and thus if and only if $y_\Lambda \in \Psi_\Lambda^\alpha(\overline{\{0\}}_\Lambda^{(k)} \times \overline{F}_\Lambda^{(k)})$ holds. Therefore, \mathcal{E}_x is a sub-supermanifold of \mathcal{E} . We define an $\overline{\mathbb{R}}$ -module structure on \mathcal{E}_x via the isomorphism

$$\Psi^\alpha \circ (0, \text{id}_{\overline{F}^{(k)}}): \overline{F}^{(k)} \rightarrow \mathcal{E}_x.$$

The $\overline{\mathbb{R}}$ -module structure on \mathcal{E}_x does not depend on Ψ^α because a change of bundle charts leads to an isomorphism of $\overline{\mathbb{R}}$ -modules in the second component. \square

Lemma 2.5.7. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let \mathcal{E} and \mathcal{E}' be k -super vector bundles over \mathcal{M} and \mathcal{N} . If $f: \mathcal{E} \rightarrow \mathcal{E}'$ is a morphism of k -super vector bundles over $g: \mathcal{M} \rightarrow \mathcal{N}$, then for every point x of \mathcal{M} the morphism*

$$f|_{\mathcal{E}_x}: \mathcal{E}_x \rightarrow \mathcal{E}'_{g \circ x}$$

is a well-defined morphism of $\overline{\mathbb{R}}$ -modules (and in particular supersmooth).

Proof. Let $\Lambda \in \mathbf{Gr}^{(k)}$ and $y_\Lambda \in (\mathcal{E}_x)_\Lambda$. We have

$$\begin{array}{ccc} y_\Lambda & \xrightarrow{f_\Lambda} & (\mathcal{E}')_\Lambda \\ (\pi_{\mathcal{M}})_\Lambda \downarrow & & \downarrow (\pi_{\mathcal{N}})_\Lambda \\ x_\Lambda & \xrightarrow{g_\Lambda} & g_\Lambda(x_\Lambda) \end{array}$$

and $g_\Lambda(x_\Lambda) = (g \circ x)_\Lambda$ implies that the morphism is well-defined. In charts the second component of f is $\overline{\mathbb{R}}$ -linear. Thus, $f|_{\mathcal{E}_x}$ is a morphism of $\overline{\mathbb{R}}$ -modules. \square

Lemma 2.5.8. *The functors ι_k^0 , ι_k^1 and π_n^k from Proposition 2.3.14, Proposition 2.3.17 and Lemma 2.3.18 extend to functors*

$$\begin{aligned} \iota_k^0: \mathbf{SVBun}^{(0)} &\rightarrow \mathbf{SVBun}^{(k)} \text{ for } k \in \mathbb{N}_0 \cup \{\infty\}, \\ \iota_k^1: \mathbf{SVBun}^{(1)} &\rightarrow \mathbf{SVBun}^{(k)} \text{ for } k \in \mathbb{N} \cup \{\infty\} \text{ and} \\ \pi_n^k: \mathbf{SVBun}^{(k)} &\rightarrow \mathbf{SVBun}^{(n)} \text{ for } k \in \mathbb{N}_0 \cup \{\infty\}, \ 0 \leq n \leq k. \end{aligned}$$

With these functors, we have $\pi_0^k \circ \iota_k^0 \cong \text{id}_{\mathbf{SVBun}^{(0)}}$ and $\pi_1^k \circ \iota_k^1 \cong \text{id}_{\mathbf{SVBun}^{(1)}}$.

Proof. Let us consider ι_k^0 and ι_k^1 as functors $\mathbf{SMan}^{(0)} \rightarrow \mathbf{SMan}^{(k)}$ and $\mathbf{SMan}^{(1)} \rightarrow \mathbf{SMan}^{(k)}$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E \in \mathbf{SVec}_{lc}$, $\mathcal{U} \subseteq \overline{E}^{(k)}$. We see from the concrete description in Lemma 2.5.3 that every \mathcal{U} -family of $\overline{\mathbb{R}}$ -linear morphisms is mapped to an \mathcal{U} -family of $\overline{\mathbb{R}}$ -linear morphisms under the original functors. From this, the result follows. \square

Note also that for any super vector bundle \mathcal{E} with base \mathcal{M} and typical fiber F , the inverse limit $\varprojlim \mathcal{E}$ is in a natural way a vector bundle over $\varprojlim \mathcal{M}$ with typical fiber $\varprojlim \overline{F}$.

2.5.1. The Change of Parity Functor

The space of sections of a super vector bundle can be turned into a vector space, as we will see below. However, in a sense this describes only the even sections. To incorporate odd sections and obtain a super vector space of sections, we need the so called change of parity functor. On super vector spaces this functor simply swaps the even and odd parts. Doing this fiberwise, one gets the change of parity functor for super vector bundles. As before, it will be useful to express this functor in terms of skeletons.

Definition 2.5.9. Let $E, F \in \mathbf{SVec}_{lc}$ and $f: E \rightarrow F$ be a morphism. We define a functor $\Pi: \mathbf{SVec}_{lc} \rightarrow \mathbf{SVec}_{lc}$ by setting $\Pi(E)_i := E_{\overline{i+1}}$ and $\Pi(f)_i := f_{\overline{i+1}}$ for $i \in \{0, 1\}$. Now, let $\Lambda = \Lambda_n \in \mathbf{Gr}$, $n \in \mathbb{N}$ and $g: \overline{E}_\Lambda \rightarrow \overline{F}_\Lambda$ be \mathbb{R} -linear such that there exist linear maps

$$g_{(0)}: E_0 \rightarrow \overline{F}_\Lambda \quad \text{and} \quad g_{(1)}: E_1 \rightarrow \overline{\Pi(F)}_\Lambda$$

with $g(\lambda_I v_I) = \lambda_I g_{(i)}(v_I)$ for $I \in \mathcal{P}_i^n$, $v_I \in E_i$, $i \in \{0, 1\}$. We call such a map g *parity changeable*. We define a parity changeable map

$$\overline{\Pi}_\Lambda(g): \overline{\Pi(E)}_\Lambda \rightarrow \overline{\Pi(F)}_\Lambda$$

by setting $\overline{\Pi}_\Lambda(g)_{(i)} := g_{\overline{i+1}}$ for $i \in \{0, 1\}$.

Note that in the above situation g is automatically $\Lambda_{\overline{0}}$ -linear and we have $\overline{\Pi}_\Lambda(\overline{\Pi}_\Lambda(g)) = g$. What is more, with $f_{(0)} := f_0$ and $f_{(1)} := f_1$, we see that \overline{f}_Λ is parity changeable and it follows $\overline{\Pi}_\Lambda(\overline{f}_\Lambda) = \overline{\Pi(f)}_\Lambda$.

Lemma 2.5.10. Let $E, F, H \in \mathbf{SVec}_{lc}$, $\Lambda = \Lambda_n \in \mathbf{Gr}$ with $n \in \mathbb{N}$ and let $f: E_\Lambda \rightarrow F_\Lambda$, $g: F_\Lambda \rightarrow H_\Lambda$ be parity changeable. Then $g \circ f$ is also parity changeable and we have

$$\overline{\Pi}_\Lambda(g \circ f) = \overline{\Pi}_\Lambda(g) \circ \overline{\Pi}_\Lambda(f).$$

Proof. Let $f_{(0)}, f_{(1)}$ and $g_{(0)}, g_{(1)}$ be as in Definition 2.5.9. For $I \in \mathcal{P}_i^n$, $v \in E_i$, $i \in \{0, 1\}$ let

$$f(\lambda_I v) = \lambda_I f_{(i)}(v) = \lambda_I \sum_{J \in \mathcal{P}^n} \lambda_J w_J,$$

where $w_J \in F_{\overline{|I|+|J|}}$. It follows that

$$(g \circ f)(\lambda_I v) = \sum_{J \in \mathcal{P}^n} \lambda_I \lambda_J g_{\overline{(|I|+|J|)}}(w_J).$$

This implies that $g \circ f$ is parity changeable with $(g \circ f)_{(0)} = g \circ f_{(0)}$ and $(g \circ f)_{(1)} = \overline{\Pi}_\Lambda(g) \circ f_{(1)}$. Thus, $\overline{\Pi}_\Lambda(g \circ f)_{(0)} = \overline{\Pi}_\Lambda(g) \circ f_{(1)}$ and $\overline{\Pi}_\Lambda(g \circ f)_{(1)} = g \circ f_{(0)}$. Applying this to $\overline{\Pi}_\Lambda(g) \circ \overline{\Pi}_\Lambda(f)$, we get

$$(\overline{\Pi}_\Lambda(g) \circ \overline{\Pi}_\Lambda(f))_{(0)} = \overline{\Pi}_\Lambda(g) \circ \overline{\Pi}_\Lambda(f)_{(0)} = \overline{\Pi}_\Lambda(g) \circ f_{(1)}$$

and

$$(\overline{\Pi}_\Lambda(g) \circ \overline{\Pi}_\Lambda(f))_{(1)} = \overline{\Pi}_\Lambda(\overline{\Pi}_\Lambda(g)) \circ \overline{\Pi}_\Lambda(f)_{(1)} = g \circ f_{(0)}$$

and therefore

$$\overline{\Pi}_\Lambda(g \circ f) = \overline{\Pi}_\Lambda(g) \circ \overline{\Pi}_\Lambda(f). \quad \square$$

Lemma 2.5.11. Let $E, F, H \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{U} \subseteq \overline{H}^{(k)}$ be an open subfunctor and let $f: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ be an \mathcal{U} -family of \mathbb{R} -linear morphisms. For $n \in \mathbb{N}$, $\Lambda = \Lambda_n \in \mathbf{Gr}^{(k)}$ and $u \in \mathcal{U}_\Lambda$, the map $f_\Lambda(u, \bullet): \overline{E}_\Lambda^{(k)} \rightarrow \overline{F}_\Lambda^{(k)}$ is parity

changeable. Defining $(\bar{\Pi}(f))_\Lambda(u, \bullet) := \bar{\Pi}_\Lambda(f_\Lambda(u, \bullet))$ leads to an \mathcal{U} -family of $\bar{\mathbb{R}}$ -linear morphisms

$$\bar{\Pi}(f): \mathcal{U} \times \bar{\Pi}(E)^{(k)} \rightarrow \bar{\Pi}(F)^{(k)}.$$

The skeleton of $\bar{\Pi}(f)$ has the components

$$\begin{aligned} \tilde{f}_0 &= f_1(\text{pr}_U)(\text{pr}_{\Pi(E)_0}) \quad \text{and} \\ \tilde{f}_l &= f_{l+1}(\text{pr}_U)(\text{pr}_{\Pi(E)_0}, \text{pr}_1, \dots, \text{pr}_l) + l \cdot \mathfrak{A}^l f_{l-1}(\text{pr}_U, \text{pr}_2)(\text{pr}_1, \dots, \text{pr}_l) \end{aligned}$$

for $l > 0$, where we consider

$$\begin{aligned} \mathfrak{A}^l f_{l-1}(\text{pr}_U, \text{pr}_2)(\text{pr}_1, \dots, \text{pr}_l): U \times \Pi(E)_0 &\rightarrow \mathcal{A}l^l(H_1 \oplus \Pi(E)_1; \Pi(F)_{\bar{k}}) \quad \text{and} \\ f_{l+1}(\text{pr}_U)(\text{pr}_{\Pi(E)_0}, \text{pr}_1, \dots, \text{pr}_l): U \times \Pi(E)_0 &\rightarrow \mathcal{A}l^l(H_1 \oplus \Pi(E)_1; \Pi(F)_{\bar{k}}) \end{aligned}$$

with the projections $\text{pr}_U: U \times \Pi(E)_0 \rightarrow U$, $\text{pr}_{\Pi(E)_0}: U \times \Pi(E)_0 \rightarrow \Pi(E)_0$, $\text{pr}_1: H_1 \times \Pi(E)_1 \rightarrow H_1$ and $\text{pr}_2: H_1 \times \Pi(E)_1 \rightarrow \Pi(E)_1$.

Proof. Let $U := \mathcal{U}_{\mathbb{R}}$. We set $\bar{\Pi}(f)_{\mathbb{R}} := \bar{\Pi}(f)_{\Lambda_1}|_{U \times E_1}: \mathcal{U}_{\mathbb{R}} \times \bar{\Pi}(E)_{\mathbb{R}}^{(k)} \rightarrow \bar{\Pi}(F)_{\mathbb{R}}^{(k)}$ so that $\bar{\Pi}(f)_\Lambda$ is defined for all $\Lambda \in \mathbf{Gr}^{(k)}$. To simplify our notation, we consider $\bar{H}_\Lambda^{(k)} \subseteq \bar{H} \oplus \bar{E}_\Lambda^{(k)}$ and $\bar{E}_\Lambda^{(k)} \subseteq \bar{H} \oplus \bar{E}_\Lambda^{(k)}$ in the obvious way. Let $x \in U$, $x_0 \in \bar{H}_{0\Lambda^+}^{(k)}$, $x_1 \in \bar{H}_{1\Lambda}^{(k)}$ and $y_0 \in \bar{E}_{0\Lambda}^{(k)}$, $y_1 \in \bar{E}_{1\Lambda}^{(k)}$. For $u = x + x_0 + x_1$ and $v = y_0 + y_1$, we use formula (2.4) to get

$$\begin{aligned} f_\Lambda(u, v) &= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} d^{m+1} f_l(x)(y_0, x_0, \dots, x_0, x_1, \dots, x_1) + \\ &\quad \sum_{m,l=0}^{\infty} \frac{1}{m!l!} d^m f_{l+1}(x)(x_0, \dots, x_0, y_1, x_1, \dots, x_1). \end{aligned}$$

Therefore, $f_\Lambda(u, \bullet)$ is parity changeable with

$$\begin{aligned} (f_\Lambda(u, \bullet))_{(0)} &= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} d^{m+1} f_l(x)(\bullet, x_0, \dots, x_0, x_1, \dots, x_1) \quad \text{and} \\ (f_\Lambda(u, \bullet))_{(1)} &= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} d^m f_{l+1}(x)(x_0, \dots, x_0, \bullet, x_1, \dots, x_1). \end{aligned}$$

In the next step, we show that $(\tilde{f}_n)_n$ is the skeleton of $\Pi(f)$. Let $\tilde{y} \in \Pi(E)_0$, $\tilde{y}_0 \in \bar{\Pi}(E)_{0\Lambda^+}^{(k)}$ and $\tilde{y}_1 \in \bar{\Pi}(E)_{1\Lambda}^{(k)}$. We calculate

$$\begin{aligned} d^m \tilde{f}_l(x, \tilde{y})((x_0, \tilde{y}_0), \dots, (x_0, \tilde{y}_0), (x_1, \tilde{y}_1), \dots, (x_1, \tilde{y}_1)) &= \\ d^m f_{l+1}(x)(x_0, \dots, x_0, \tilde{y}, x_1, \dots, x_1) + m \cdot d^{m-1} f_{l+1}(x)(x_0, \dots, x_0, \tilde{y}_0, x_1, \dots, x_1) \\ &\quad + l \cdot d^m f_{l-1}(x, \tilde{y}_1)(x_0, \dots, x_0, x_1, \dots, x_1), \end{aligned}$$

where the last two summands are zero for $m = 0$ and $l = 0$, respectively. Note

that

$$l \cdot d^m f_{l-1}(x, \tilde{y}_1)(x_0, \dots, x_0, x_1, \dots, x_1) = l \cdot d^{m+1} f_{l-1}(x)(\tilde{y}_1, x_0, \dots, x_0, x_1, \dots, x_1)$$

holds because of Lemma 2.5.3. If $\tilde{f}: \mathcal{U} \times \overline{\Pi(E)}^{(k)} \rightarrow \overline{\Pi(F)}^{(k)}$ is the morphism defined by $(\tilde{f}_n)_n$, then it follows

$$\begin{aligned} \tilde{f}_\Lambda(u, \tilde{v}) &= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} d^m f_{l+1}(x)(x_0, \dots, x_0, \tilde{y} + \tilde{y}_0, x_1, \dots, x_1) + \\ &\quad \sum_{m,l=0}^{\infty} \frac{1}{m!l!} d^{m+1} f_l(x)(\tilde{y}_1, x_0, \dots, x_0, x_1, \dots, x_1) \end{aligned}$$

for $\tilde{v} = \tilde{y} + \tilde{y}_0 + \tilde{y}_1$. This is exactly $(\overline{\Pi}(f))_\Lambda(u, \tilde{v})$. \square

Corollary 2.5.12. *Let $E, E', F, F', H, H' \in \mathbf{SVec}_{lc}$, $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{U} \subseteq \overline{H}^{(k)}$, $\mathcal{V} \subseteq \overline{H'}^{(k)}$ be open subfunctors. Moreover, let $f: \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ be an \mathcal{U} -family of \mathbb{R} -linear morphisms, $g: \mathcal{V} \times \overline{E'}^{(k)} \rightarrow \overline{F'}^{(k)}$ be an \mathcal{V} -family of \mathbb{R} -linear morphisms and $h: \mathcal{U} \rightarrow \mathcal{V}$ be supersmooth. Then $g \circ (h, f): \mathcal{U} \times \overline{E}^{(k)} \rightarrow \overline{F'}^{(k)}$ is an \mathcal{U} -family of \mathbb{R} -linear morphisms and we have*

$$\overline{\Pi}(g \circ (h, f)) = \overline{\Pi}(g) \circ (h, \overline{\Pi}(f)).$$

In addition, we have $\overline{\Pi}(\overline{\Pi}(f)) = f$.

Proof. This follows from the pointwise definition of $\overline{\Pi}$ in Lemma 2.5.11 and Lemma 2.5.10. \square

Proposition 2.5.13. *For $k \in \mathbb{N} \cup \{\infty\}$ let $\pi: \mathcal{E} \rightarrow \mathcal{M}$ be a k -super vector bundle with typical fiber $F \in \mathbf{SVec}_{lc}$, bundle atlas $\{\Psi^\alpha: \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E}: \alpha \in A\}$ and the respective change of charts $\Psi^{\alpha\beta} = (\phi^{\alpha\beta}, \psi^{\alpha\beta})$, $\alpha, \beta \in A$. Then the morphisms $(\phi^{\alpha\beta}, \overline{\Pi}(\psi^{\alpha\beta}))$ define a k -super vector bundle $\overline{\Pi}(\mathcal{E})$ over \mathcal{M} with typical fiber $\Pi(F)$.*

Let $\pi': \mathcal{E}' \rightarrow \mathcal{M}'$ be another k -vector bundle and $f: \mathcal{E} \rightarrow \mathcal{E}'$ be a morphism of k -super vector bundles over $h: \mathcal{M} \rightarrow \mathcal{M}'$. If f has the local form $(g^{\alpha\alpha'}, \varphi^{\alpha\alpha'})$, then $(g^{\alpha\alpha'}, \overline{\Pi}(\varphi^{\alpha\alpha'}))$ defines a morphism $\overline{\Pi}(f): \overline{\Pi}(\mathcal{E}) \rightarrow \overline{\Pi}(\mathcal{E}')$ over h . This construction is functorial and defines an equivalence of categories

$$\overline{\Pi}: \mathbf{SVBun}^{(k)} \rightarrow \mathbf{SVBun}^{(k)}.$$

Proof. In light of Remark 2.5.5, it follows from Corollary 2.5.12 that the morphisms $(\phi^{\alpha\beta}, \overline{\Pi}(\psi^{\alpha\beta}))$ define a super vector bundle. That $\overline{\Pi}$ is well-defined on morphisms and functorial follows by the same argument. The corollary also implies that $\overline{\Pi}(\overline{\Pi}(\mathcal{E})) \cong \mathcal{E}$ and $\overline{\Pi}(\overline{\Pi}(f)) \cong f$ hold under this identification, which shows that $\overline{\Pi}$ is an equivalence of categories. \square

2.6. The Tangent Bundle of a Supermanifold

In this section, we expand on the definition of the tangent functor \mathcal{T} for supermanifolds given by Molotkov (see [40, Section 5.3, p. 404f.]).

Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{Man}^{\mathbf{Gr}^{(k)}}$. We define a functor $\mathcal{T}\mathcal{M} \in \mathbf{Man}^{\mathbf{Gr}^{(k)}}$ by setting $(\mathcal{T}\mathcal{M})_\Lambda = \mathcal{T}\mathcal{M}_\Lambda := T\mathcal{M}_\Lambda$ for all $\Lambda \in \mathbf{Gr}^{(k)}$ and $(\mathcal{T}\mathcal{M})_\varrho = \mathcal{T}\mathcal{M}_\varrho := T\mathcal{M}_\varrho: \mathcal{T}\mathcal{M}_\Lambda \rightarrow \mathcal{T}\mathcal{M}_{\Lambda'}$ for all $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. It follows from the functoriality of $T: \mathbf{Man} \rightarrow \mathbf{Man}$ that $\mathcal{T}\mathcal{M}$ is indeed a functor. By the same argument, the bundle projections $\pi_\Lambda^{\mathcal{T}\mathcal{M}}: \mathcal{T}\mathcal{M}_\Lambda \rightarrow \mathcal{M}_\Lambda$ define a natural transformation $\pi^{\mathcal{T}\mathcal{M}}: \mathcal{T}\mathcal{M} \rightarrow \mathcal{M}$.

If $\mathcal{N} \in \mathbf{Man}^{\mathbf{Gr}^{(k)}}$ and $f: \mathcal{M} \rightarrow \mathcal{N}$ is a natural transformation, then it is easy to see that setting $\mathcal{T}f_\Lambda := Tf_\Lambda: \mathcal{T}\mathcal{M}_\Lambda \rightarrow \mathcal{T}\mathcal{N}_\Lambda$ for all $\Lambda \in \mathbf{Gr}^{(k)}$ defines a natural transformation $\mathcal{T}f: \mathcal{T}\mathcal{M} \rightarrow \mathcal{T}\mathcal{N}$ and that this gives us a functor $\mathcal{T}: \mathbf{Man}^{\mathbf{Gr}^{(k)}} \rightarrow \mathbf{Man}^{\mathbf{Gr}^{(k)}}$. We obviously have $\pi^{\mathcal{T}\mathcal{N}} \circ \mathcal{T}f = f \circ \pi^{\mathcal{T}\mathcal{M}}$.

Lemma 2.6.1. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be modelled on $E \in \mathbf{SVec}_{lc}$ with the atlas $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$. Then $\mathcal{T}\mathcal{M}$ is a k -super vector bundle over \mathcal{M} with typical fiber E , the bundle atlas $\{\mathcal{T}\varphi^\alpha: \mathcal{T}\mathcal{U}^\alpha \rightarrow \mathcal{T}\mathcal{M}: \alpha \in A\}$ and the projection $\pi^{\mathcal{T}\mathcal{M}}$. If $f: \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of k -supermanifolds, then $\mathcal{T}f: \mathcal{T}\mathcal{M} \rightarrow \mathcal{T}\mathcal{N}$ is a morphism of k -super vector bundles and the above defines a functor*

$$\mathcal{T}: \mathbf{SMan}^{(k)} \rightarrow \mathbf{SVBun}^{(k)}.$$

Proof. That $\{\mathcal{T}\varphi^\alpha: \mathcal{T}\mathcal{U}^\alpha \rightarrow \mathcal{T}\mathcal{M}: \alpha \in A\}$ is a covering is obvious. By functoriality, we have

$$\mathcal{T}(\varphi^\beta)^{-1} \circ \mathcal{T}\varphi^\alpha = \mathcal{T}\varphi^{\alpha\beta}$$

on $\mathcal{T}\mathcal{U}^{\alpha\beta} = \mathcal{U}^{\alpha\beta} \times \overline{E}^{(k)}$ for all $\alpha, \beta \in A$ and by definition, we have

$$\mathcal{T}\varphi^{\alpha\beta} = (\varphi^{\alpha\beta}, d\varphi^{\alpha\beta}): \mathcal{U}^{\alpha\beta} \times \overline{E}^{(k)} \rightarrow \mathcal{U}^{\beta\alpha} \times \overline{E}^{(k)},$$

which is a supersmooth morphism because of Lemma 2.2.10. Clearly, each $\pi_\Lambda^{\mathcal{T}\mathcal{M}}: \mathcal{T}\mathcal{M}_\Lambda \rightarrow \mathcal{M}_\Lambda$ is a vector bundle and we have that

$$(\varphi^\alpha)^{-1} \circ \pi^{\mathcal{T}\mathcal{M}} \circ \mathcal{T}\varphi^\alpha: \mathcal{U}^\alpha \times \overline{E}^{(k)} \rightarrow \mathcal{U}^\alpha$$

is simply the projection and thus supersmooth. Since, by definition, $d\varphi^{\alpha\beta}$ is an $\mathcal{U}^{\alpha\beta}$ -family of \mathbb{R} -linear morphisms, the above atlas is indeed a bundle atlas for $\mathcal{T}\mathcal{M}$. In such charts, $\mathcal{T}f$ has locally the form $(f^{\alpha\beta'}, df^{\alpha\beta'})$ and therefore is a morphism of k -super vector bundles for the same reason. Functoriality follows from the functoriality of \mathcal{T} as a functor $\mathbf{Man}^{\mathbf{Gr}^{(k)}} \rightarrow \mathbf{Man}^{\mathbf{Gr}^{(k)}}$. \square

In the situation of the lemma, we call $\mathcal{T}\mathcal{M}$ the *tangent bundle* of \mathcal{M} . We will write $\pi^{\mathcal{T}\mathcal{M}}: \mathcal{T}\mathcal{M} \rightarrow \mathcal{M}$ for the bundle projection and $\mathcal{T}_x\mathcal{M}$ instead of $(\mathcal{T}\mathcal{M})_x$ for the fiber of $\mathcal{T}\mathcal{M}$ at a point x of \mathcal{M} .

Lemma 2.6.2. *For every $\mathcal{M} \in \mathbf{SMan}$, we have*

$$\varprojlim \mathcal{T}\mathcal{M} \cong T \varprojlim \mathcal{M}$$

in $\mathbf{MBun}^{(\infty)}$ with the functor \varprojlim from Theorem 2.3.11. Moreover, $\varprojlim \pi^{\mathcal{T}\mathcal{M}} = \pi^{T \varprojlim \mathcal{M}}$ holds for the bundle projections $\pi^{\mathcal{T}\mathcal{M}}: \mathcal{T}\mathcal{M} \rightarrow \mathcal{M}$ and $\pi^{T \varprojlim \mathcal{M}}: T \varprojlim \mathcal{M} \rightarrow \varprojlim \mathcal{M}$. For morphisms $f: \mathcal{M} \rightarrow \mathcal{N}$ of supermanifolds, we have

$$\varprojlim \mathcal{T}f = T \varprojlim f$$

under the above identification.

Proof. This follows from the definition of $\mathcal{T}\mathcal{M}$, Lemma B.3.5 and the definition of \varprojlim in Theorem 2.3.11. \square

Remark 2.6.3. In view of Lemma 2.6.2, it seems likely that one can describe higher tangent bundles, higher jet bundles and higher tangent Lie supergroups analogously to [10].

Lemma 2.6.4. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$. With the functors from Lemma 2.5.8, we have $\mathcal{T}\iota_k^n(\mathcal{M}) \cong \iota_k^n(\mathcal{T}\mathcal{M})$ for $n \in \{0, 1\}, n \leq k$ in $\mathbf{SVBun}^{(k)}$ and $\mathcal{T}\pi_n^k(\mathcal{M}) \cong \pi_n^k(\mathcal{T}\mathcal{M})$ for $0 \leq n \leq k$ in $\mathbf{SVBun}^{(n)}$.*

Proof. With any atlas $\mathcal{A} := \{\varphi^\alpha: \alpha \in A\}$ of \mathcal{M} it is obvious that applying $\mathcal{T} \circ \iota_k^n$ and $\iota_k^n \circ \mathcal{T}$ to a change of charts leads to the same morphism. The same is true for $\mathcal{T} \circ \pi_n^k$ and $\pi_n^k \circ \mathcal{T}$. \square

3. Lie Supergroups

Recall supergroups as defined in Definition 1.4.1.

Definition 3.0.1. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A k -Lie supergroup is a group object in the category $\mathbf{SMan}^{(k)}$. Denote by $\mathbf{LSGrp}^{(k)}$ the category that has Lie supergroups as objects and supersmooth morphisms of supergroups as morphisms. We call ∞ -Lie supergroups *Lie supergroups* and set $\mathbf{LSGrp} := \mathbf{LSGrp}^{(\infty)}$.

In other words, for $\mathcal{G} \in \mathbf{LSGrp}^{(k)}$ there exists a supersmooth multiplication $\mu: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$, a supersmooth inversion $i: \mathcal{G} \rightarrow \mathcal{G}$ and a neutral element e such that the usual commutative diagrams defining a group commute. Here $e: \mathbf{p} \rightarrow \mathcal{G}$ is a point of \mathcal{G} . We also write $\mathcal{G} = (\mathcal{G}, \mu, i, e)$. In particular, $(\mathcal{G}_\Lambda, \mu_\Lambda, i_\Lambda, e_\Lambda)$ is a Lie group for every $\Lambda \in \mathbf{Gr}^{(k)}$. It follows immediately from naturality that $\mathcal{G}_\varrho: \mathcal{G}_\Lambda \rightarrow \mathcal{G}_{\Lambda'}$ is a morphism of Lie groups for every $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. A morphism of k -Lie supergroups is a supersmooth morphism $f: \mathcal{G} \rightarrow \mathcal{H}$ between k -Lie supergroups \mathcal{G} and \mathcal{H} such that $f_\Lambda: \mathcal{G}_\Lambda \rightarrow \mathcal{H}_\Lambda$ is a morphism of Lie groups for every $\Lambda \in \mathbf{Gr}^{(k)}$.

Note that Proposition 2.3.14 shows that the category of 0-supermanifolds is equivalent to the category of ordinary Lie groups. Similarly, by Proposition 2.3.17, $\mathbf{LSGrp}^{(1)}$ is equivalent to the category of group objects in the category of vector bundles.

We will see that every Lie supergroup has an associated Lie superalgebra which contains a lot of information about the Lie supergroup: For every k -Lie supergroup \mathcal{G} and every $\Lambda \in \mathbf{Gr}^{(k)}$, we have a split short exact sequence of Lie groups

$$1 \rightarrow \ker \mathcal{G}_{\varepsilon_\Lambda} \rightarrow \mathcal{G}_\Lambda \rightarrow \mathcal{G}_{\mathbb{R}} \rightarrow 1,$$

where $\ker \mathcal{G}_{\varepsilon_\Lambda}$ is a nilpotent Lie group that is completely determined by the Lie superalgebra of \mathcal{G} . Moreover, the action of $\mathcal{G}_{\mathbb{R}}$ on $\ker \mathcal{G}_{\varepsilon_\Lambda}$ is determined by a representation of the Lie superalgebra, which leads to the so called super Harish-Chandra pair. It is a classical result that the categories of finite-dimensional Lie supergroups and finite-dimensional super Harish-Chandra pairs are equivalent and we show in Theorem 3.3.8 that this also holds for arbitrary locally convex supermanifolds.

It is not difficult to construct an appropriate Lie group for every $\Lambda \in \mathbf{Gr}$ starting from a super Harish-Chandra pair. Supersmoothness is more problematic. One issue is that the Lie groups involved need not have an exponential map that is a diffeomorphism in a neighborhood of zero. This can be circumvented because $\ker \mathcal{G}_{\varepsilon_\Lambda}$ is a polynomial Lie group (see Appendix C) and thus has an exponential map that is a global chart. We use this exponential map to get a canonical

isomorphism

$$\iota(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1} \cong \mathcal{G},$$

where \mathfrak{g}_1 is the odd part of the Lie superalgebra of \mathcal{G} . This enables us to concretely construct a Lie supergroup starting from a super Harish-Chandra pair.

The equivalence between k -Lie supergroups and super Harish-Chandra pairs holds for $k \geq 3$ and it follows that all the categories $\mathbf{LSGrp}^{(k)}$ are equivalent for $k \geq 3$. We finish this chapter by discussing some classical examples of Lie supergroups.

Definition 3.0.2. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup. A *centered chart* of \mathcal{G} is a chart $\varphi: \mathcal{U} \rightarrow \mathcal{G}$ such that $0 \in \mathcal{U}_{\mathbb{R}}$ and $\varphi_{\mathbb{R}}(0) = e_{\mathbb{R}}$.

For $E \in \mathbf{SVec}_{lc}$ translation by elements $x \in E_0$ defines a supersmooth morphism and therefore centered charts always exist. There are many results from standard Lie theory that can be transferred to Lie supergroups with relative ease. For now, we just mention the following.

Lemma/Definition 3.0.3. Let \mathcal{H} and \mathcal{N} be Lie supergroups and $\alpha: \mathcal{H} \times \mathcal{N} \rightarrow \mathcal{N}$ a supersmooth morphism such that α_{Λ} is a group action by automorphisms for all $\Lambda \in \mathbf{Gr}$. Then $\Lambda \mapsto (\mathcal{N}_{\Lambda} \rtimes_{\alpha_{\Lambda}} \mathcal{H}_{\Lambda})$ defines a Lie supergroup $\mathcal{N} \rtimes_{\alpha} \mathcal{H}$ which we call *the semidirect product of \mathcal{N} and \mathcal{H} (with respect to α)*.

Proof. For all $\Lambda \in \mathbf{Gr}$, we have the multiplication

$$(\mathcal{N}_{\Lambda} \times \mathcal{H}_{\Lambda}) \times (\mathcal{N}_{\Lambda} \times \mathcal{H}_{\Lambda}) \rightarrow \mathcal{N}_{\Lambda} \times \mathcal{H}_{\Lambda}, \quad (n, h) \cdot (n', h') := (n\alpha_{\Lambda}(h, n'), hh')$$

and the inversion

$$\mathcal{N}_{\Lambda} \times \mathcal{H}_{\Lambda} \rightarrow \mathcal{N}_{\Lambda} \times \mathcal{H}_{\Lambda}, \quad (n, h)^{-1} := (\alpha_{\Lambda}(h^{-1}, n^{-1}), h^{-1})$$

which define supersmooth morphisms. □

3.1. The Lie Superalgebra of a Lie Supergroup

Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{G} \in \mathbf{LSGrp}^{(k)}$. For every $\Lambda \in \mathbf{Gr}^{(k)}$, we have the Lie algebra $L(\mathcal{G}_{\Lambda}) = T_{e_{\Lambda}}\mathcal{G}_{\Lambda}$ with the bracket $[\cdot, \cdot]_{\Lambda}: L(\mathcal{G}_{\Lambda}) \times L(\mathcal{G}_{\Lambda}) \rightarrow L(\mathcal{G}_{\Lambda})$. Since \mathcal{G}_{ϱ} is a morphism of Lie groups for every $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$, $T_{e_{\Lambda}}\mathcal{G}_{\varrho}$ is a morphism of Lie algebras and thus $[\cdot, \cdot]: \mathcal{T}_e\mathcal{G} \times \mathcal{T}_e\mathcal{G} \rightarrow \mathcal{T}_e\mathcal{G}$ defines a natural transformation.

Lemma/Definition 3.1.1. Let $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup and let $\mathfrak{g} := T_{e_{\mathbb{R}}}\mathcal{G}_{\mathbb{R}} \oplus T_{e_{\Lambda_1}}\ker(\mathcal{G}_{\varepsilon_{\Lambda_1}}) = \mathcal{T}_e\mathcal{G}_{\Lambda_1} \in \mathbf{SVec}_{lc}$. There exists a canonical isomorphism of $\overline{\mathbb{R}}^{(k)}$ -modules $\overline{\mathfrak{g}}^{(k)} \cong \mathcal{T}_e\mathcal{G}$. Together with the induced Lie bracket, $\overline{\mathfrak{g}}^{(k)}$ is a Lie algebra over $\overline{\mathbb{R}}^{(k)}$ and we call it the *Lie algebra of \mathcal{G}* . We set $L(\mathcal{G}) := \overline{\mathfrak{g}}$. For any morphism $f: \mathcal{G} \rightarrow \mathcal{H}$ of Lie supergroups, we have a morphism

$$L(f) := \overline{\mathcal{T}_e f_{\Lambda_1}}: L(\mathcal{G}) \rightarrow L(\mathcal{H})$$

of Lie algebras over $\overline{\mathbb{R}}^{(k)}$ and this defines a functor

$$L: \mathbf{LSGrp}^{(k)} \rightarrow \mathbf{LAlg}_{\overline{\mathbb{R}}^{(k)}}(\mathbf{Top}^{\mathbf{Gr}^{(k)}}).$$

Proof. Let \mathcal{G} be modelled on $E \in \mathbf{SVec}_{lc}$ and $\varphi: \mathcal{U} \rightarrow \mathcal{G}$ be a centered chart. We subsequently identify $\mathcal{T}_0 \overline{E}^{(k)} \cong \overline{E}^{(k)}$. On the one hand, we have the isomorphism of $\overline{\mathbb{R}}^{(k)}$ -modules $\mathcal{T}_0 \varphi: \overline{E}^{(k)} \rightarrow \mathcal{T}_e \mathcal{G}$ from Lemma/Definition 2.5.6. On the other hand, we have an isomorphism of $\overline{\mathbb{R}}^{(k)}$ -modules $\overline{\mathcal{T}_0 \varphi_{\Lambda_1}}: \overline{E}^{(k)} \rightarrow L(\mathcal{G})$. Thus $\mathcal{T}_0 \varphi \circ (\overline{\mathcal{T}_0 \varphi_{\Lambda_1}})^{-1}: L(\mathcal{G}) \rightarrow \mathcal{T}_e \mathcal{G}$ is an isomorphism. If $\psi: \mathcal{V} \rightarrow \mathcal{G}$ is another centered chart, then

$$\mathcal{T}_e \psi^{-1} \circ \mathcal{T}_0 \varphi = \mathcal{T}_0(\psi^{-1} \circ \varphi): \overline{E}^{(k)} \rightarrow \overline{E}^{(k)}$$

is a linear supersmooth map and therefore

$$\mathcal{T}_0 \psi^{-1} \circ \mathcal{T}_0 \varphi = \overline{\mathcal{T}_0(\psi^{-1} \circ \varphi)_{\Lambda_1}} = \overline{\mathcal{T}_0 \psi_{\Lambda_1}^{-1}} \circ \overline{\mathcal{T}_0 \varphi_{\Lambda_1}} = (\overline{\mathcal{T}_0 \psi_{\Lambda_1}})^{-1} \circ \overline{\mathcal{T}_0 \varphi_{\Lambda_1}}$$

holds by Corollary 2.2.22. Hence, the isomorphism does not depend on the chart. We check the supersmoothness of the Lie bracket in local coordinates. An open subfunctor $\mathcal{U}' \subseteq \mathcal{U}$ exists such that $0 \in \mathcal{U}'_{\mathbb{R}}$ and $\mu_{\mathbb{R}}(\varphi_{\mathbb{R}}(\mathcal{U}'_{\mathbb{R}}), \varphi_{\mathbb{R}}(\mathcal{U}'_{\mathbb{R}})) \subseteq \varphi_{\mathbb{R}}(\mathcal{U}_{\mathbb{R}})$. We define the local multiplication

$$m: \mathcal{U}' \times \mathcal{U}' \rightarrow \mathcal{U}, \quad m := \varphi^{-1} \circ \mu \circ (\varphi|_{\mathcal{U}'} \times \varphi|_{\mathcal{U}'})$$

and write $v' \in \overline{E}^{(k)}_{\Lambda}$ for $v = \mathcal{T}_0 \varphi_{\Lambda}(v') \in \mathcal{T}_e \mathcal{G}_{\Lambda}$. Then we have

$$([v, w]_{\Lambda})' = d^{(2)} m_{\Lambda}((0, 0), (0, w'), (v', 0)) - d^{(2)} m_{\Lambda}((0, 0), (0, v'), (w', 0))$$

for all $v, w \in \mathcal{T}_e \mathcal{G}_{\Lambda}$ (see the proof of the smoothness of the Lie algebra of a Lie group from [23]). Supersmoothness follows now from Lemma 2.2.10.

Finally, let $f: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of Lie supergroups. Then $\mathcal{T}_e f_{\Lambda}$ is a morphism of Lie algebras for every $\Lambda \in \mathbf{Gr}^{(k)}$. In respective centered charts, we can repeat the argument from above to see that the induced morphism is $\overline{\mathcal{T}_e f_{\Lambda_1}}: L(\mathcal{G}) \rightarrow L(\mathcal{H})$. Functoriality is obvious. \square

In particular, any k -Lie supergroup \mathcal{G} can be modelled on $L(\mathcal{G}) \in \mathbf{SVec}_{lc}$. Of course, one could also directly define $\mathcal{T}_e \mathcal{G}$ as the Lie algebra of \mathcal{G} , which is how it is generally done (see for example [40, Section 7.2, p.412]). This has some conceptual advantages but the definition we chose can be used more directly in our applications.

Corollary 3.1.2. *Let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 3$ and $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup. Then the Lie algebra $L(\mathcal{G})$ induces a canonical Lie superalgebra structure on $\mathfrak{g} := \mathcal{T}_e \mathcal{G}_{\Lambda_1} = T_{e_{\mathbb{R}}} \mathcal{G}_{\mathbb{R}} \oplus T_{e_{\Lambda_1}} \ker(\mathcal{G}_{\varepsilon_{\Lambda_1}})$. Moreover, if $f: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of k -Lie supergroups then $\mathcal{T}_e f_{\Lambda_1}$ is a morphism of Lie superalgebras and we obtain a functor*

$$sL: \mathbf{LSGrp}^{(k)} \rightarrow \mathbf{TopLSAlg}_{\mathbb{R}}.$$

Proof. With $L(\mathcal{G}) = \bar{\mathfrak{g}}^{(k)}$, we obtain the unique structure of a topological Lie superalgebra on \mathfrak{g} from Corollary 2.2.22. The same corollary applies to morphisms and functoriality follows. \square

3.2. Trivializations and the Exponential Map

We will show that every Lie supergroup is a simple supermanifold in the following sense.

Definition 3.2.1 ([40, p. 396]). Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ modelled on $E \in \mathbf{SVec}_{lc}$. We call \mathcal{M} *simple* if there exists an isomorphism

$$\mathcal{M} \cong \iota_k^0(\mathcal{M}_{\mathbb{R}}) \times \overline{E_1}^{(k)}.$$

In particular every simple supermanifold is isomorphic to a supermanifold of Batchelor type defined by a trivial vector bundle.

Lemma 3.2.2. Let $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ a k -Lie supergroup modelled on $E \in \mathbf{SVec}_{lc}$ and $f: \overline{E_1}^{(k)} \rightarrow \mathcal{G}$ supersmooth such that f_{Λ_1} is a diffeomorphism onto $\ker(\mathcal{G}_{\varepsilon_{\Lambda_1}})$. Then,

$$\Psi_{\Lambda}: (\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{E_1}^{(k)})_{\Lambda} \rightarrow \mathcal{G}_{\Lambda}, \quad (x, v) \mapsto \mu_{\Lambda}(x, f_{\Lambda}(v))$$

for $\Lambda \in \mathbf{Gr}^{(k)}$, defines an isomorphism of k -supermanifolds $\Psi: \iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{E_1}^{(k)} \rightarrow \mathcal{G}$.

Proof. Clearly, Ψ is supersmooth and by Lemma 2.3.6 it is enough to show that Ψ_{Λ_1} is a diffeomorphism. For $g \in \mathcal{G}_{\Lambda_1}$, we have $\mathcal{G}_{\varepsilon_{\Lambda_1}}(g)^{-1} \cdot g \in \ker(\mathcal{G}_{\varepsilon_{\Lambda_1}})$ and the inverse of Ψ_{Λ_1} is given by

$$\mathcal{G}_{\Lambda_1} \rightarrow (\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{E_1}^{(k)})_{\Lambda_1}, \quad g \mapsto (\mathcal{G}_{\varepsilon_{\Lambda_1}}(g), f_{\Lambda_1}^{-1}(\mathcal{G}_{\varepsilon_{\Lambda_1}}(g)^{-1} \cdot g)),$$

which is smooth. \square

Proposition 3.2.3. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Every k -Lie supergroup is a simple k -supermanifold.

Proof. The case $k = 0$ is trivial. Let $k \geq 1$ and let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup modelled on $E \in \mathbf{SVec}_{lc}$ with the centered chart $\varphi: \mathcal{U} \rightarrow \mathcal{G}$. By functoriality, we have $\varphi_{\Lambda_1}(\overline{E_{\Lambda_1^+}}^{(k)}) = \ker(\mathcal{G}_{\varepsilon_{\Lambda_1}})$. Because φ_{Λ_1} is a diffeomorphism, it follows that $\varphi|_{\overline{E_1}^{(k)}}$ satisfies the conditions of Lemma 3.2.2. \square

This was already stated without proof in [40, Proposition 7.4.1, p.413] for Banach Lie supergroups. As Molotov used the exponential map, our result is a generalization. We will see in Corollary 3.2.7 that there is a chart independent way to construct such a trivialization. Proposition 3.2.3 gives us a good idea of the supersmooth structure of Lie supergroups. However, the group structure is better captured by a different trivialization.

For a k -Lie supergroup \mathcal{G} one immediately obtains for each $0 \leq m < n \leq k$ a short exact sequence of Lie groups along the morphism $\mathcal{G}_{\varepsilon_{n,m}}$ that splits along $\mathcal{G}_{\eta_{m,n}}$. The most interesting case is $m = 0$ (compare [40, Section 7.4, p.413]).

Lemma 3.2.4. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup. For each $n \leq k$, one has a short exact sequence of Lie groups*

$$1 \longrightarrow \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) \hookrightarrow \mathcal{G}_{\Lambda_n} \xrightarrow{\mathcal{G}_{\varepsilon_{\Lambda_n}}} \mathcal{G}_{\mathbb{R}} \longrightarrow 1$$

that splits along $\mathcal{G}_{\eta_{\Lambda_n}}$. The group $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ is a closed Lie subgroup of \mathcal{G}_{Λ_n} and can be given the structure of a polynomial group of degree at most n . If $f: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of Lie supergroups, we have the commutative diagram

$$\begin{array}{ccccc} \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) & \longrightarrow & \mathcal{G}_{\Lambda_n} & \longrightarrow & \mathcal{G}_{\mathbb{R}} \\ f_{\Lambda_n}|_{\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})} \downarrow & & \downarrow f_{\Lambda_n} & & \downarrow f_{\mathbb{R}} \\ \ker(\mathcal{H}_{\varepsilon_{\Lambda_n}}) & \longrightarrow & \mathcal{H}_{\Lambda_n} & \longrightarrow & \mathcal{H}_{\mathbb{R}} \end{array}$$

and $f_{\Lambda_n}|_{\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})}$ is a morphism of polynomial Lie groups.

Proof. It is obvious by functoriality of \mathcal{G} that the exact sequence splits as claimed. Let \mathcal{G} be modelled on $E \in \mathbf{SVec}_{lc}$ and let $\varphi: \mathcal{U} \rightarrow \mathcal{G}$ be a centered chart of \mathcal{G} . By naturality, we have $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) = \varphi_{\Lambda_n}(\overline{E}_{\Lambda_n^+}^{(k)})$, thus $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ is a closed Lie subgroup of \mathcal{G}_{Λ_n} . In this global chart, the multiplication and inversion of $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ are morphisms of n -multilinear spaces by Theorem 2.3.11 and therefore polynomial of degree at most n by Example C.1.3(a). The same holds true for the iterated multiplications. As a result, $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ is a polynomial group of degree at most n .

Since f_{Λ_n} is a morphism of multilinear bundles over $f_{\mathbb{R}}$, commutativity of the diagram is obvious and in charts of the form $\varphi_{\Lambda_n}|_{\overline{E}_{\Lambda_n^+}}$, it is apparent that $f_{\Lambda_n}|_{\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})}$ is a polynomial morphism. \square

Lemma 3.2.5. *Let \mathcal{G} be a Lie supergroup. There exists a split short exact sequence of Lie groups*

$$1 \longrightarrow \varprojlim_n \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) \hookrightarrow \varprojlim_n \mathcal{G}_{\Lambda_n} \longrightarrow \mathcal{G}_{\mathbb{R}} \longrightarrow 1,$$

where $\mathcal{G}_+^\infty := \varprojlim_n \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ is a pro-polynomial closed Lie subgroup of $\varprojlim_n \mathcal{G}_{\Lambda_n}$. For a morphism $f: \mathcal{G} \rightarrow \mathcal{H}$ of Lie supergroups the diagram

$$\begin{array}{ccccc} \varprojlim_n \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) & \longrightarrow & \varprojlim_n \mathcal{G}_{\Lambda_n} & \longrightarrow & \mathcal{G}_{\mathbb{R}} \\ \varprojlim_n f_{\Lambda_n}|_{\mathcal{G}_+^\infty} \downarrow & & \downarrow \varprojlim_n f_{\Lambda_n} & & \downarrow f_{\mathbb{R}} \\ \varprojlim_n \ker(\mathcal{H}_{\varepsilon_{\Lambda_n}}) & \longrightarrow & \varprojlim_n \mathcal{H}_{\Lambda_n} & \longrightarrow & \mathcal{H}_{\mathbb{R}} \end{array}$$

commutes.

Proof. Note that by Theorem 2.3.11 $\varprojlim_n \mathcal{G}_{\Lambda_n}$ is indeed a Lie group. Let \mathcal{G} be modelled on $E \in \mathbf{SVec}_{lc}$ and let $\varphi: \mathcal{U} \rightarrow \mathcal{G}$ be a centered chart of \mathcal{G} . Let $m \leq n$.

By naturality, the projection maps

$$\mathcal{G}_{\varepsilon_{n,m}}|_{\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})} : \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) \rightarrow \ker(\mathcal{G}_{\varepsilon_{\Lambda_m}})$$

have the form $\overline{E}_{\varepsilon_{n,m}}|_{\overline{E}_{\Lambda_n}^+}$ in the chart φ and are thus polynomial. Interpreting $\ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ as a multilinear bundle with base $\{e_{\mathbb{R}}\}$, it follows that $\varprojlim_n \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ is a pro-polynomial Lie group. It is closed in $\varprojlim \mathcal{G}$ because $\varprojlim_n \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}}) = \varprojlim_n \varphi_{\Lambda_n}(\overline{E}_{\Lambda_n}^+)$.

Let $\psi: \mathcal{V} \rightarrow \mathcal{G}$ be a chart. In the charts $\psi_{\mathbb{R}}^{-1}$ and $\varprojlim_n \psi_{\Lambda_n}$, the projection $\varprojlim_n \mathcal{G}_{\Lambda_n} \rightarrow \mathcal{G}_{\mathbb{R}}$ is given by the projection $\varprojlim_n \mathcal{V}_{\Lambda_n} \rightarrow \mathcal{V}_{\mathbb{R}}$ and is therefore smooth. Conversely, the embeddings $\mathcal{G}_{\eta_{\Lambda_n}}: \mathcal{G}_{\mathbb{R}} \rightarrow \mathcal{G}_{\Lambda_n}$ define an embedding $\mathcal{G}_{\mathbb{R}} \rightarrow \varprojlim_n \mathcal{G}_{\Lambda_n}$, which in the above charts is simply the embedding $\mathcal{V}_{\mathbb{R}} \hookrightarrow \varprojlim_n \mathcal{V}_{\Lambda_n}$ and thus smooth. This embedding obviously defines a splitting.

In view of Lemma 3.2.4, the commutativity of the diagram is obvious. \square

Proposition 3.2.6. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup. For every $\Lambda \in \mathbf{Gr}^{(k)}$ there exists a unique exponential map*

$$\exp_{\Lambda}^{\mathcal{G}}: L(\mathcal{G})_{\Lambda^+} \rightarrow \ker(\mathcal{G}_{\varepsilon_{\Lambda}})$$

mapping 0 to e_{Λ} such that

- (a) $\exp_{\Lambda}^{\mathcal{G}}$ is a diffeomorphism,
- (b) the induced multiplication on $L(\mathcal{G})_{\Lambda^+}$ is the BCH multiplication with respect to the restriction of the Lie bracket $[\cdot, \cdot]_{\Lambda}$ of $L(\mathcal{G})_{\Lambda}$,
- (c) $(\exp_{\Lambda}^{\mathcal{G}})_{\Lambda \in \mathbf{Gr}^{(k)}}$ is a natural transformation in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ and
- (d) identifying $L(\mathcal{G})_{\Lambda} \cong \mathcal{T}_e \mathcal{G}_{\Lambda}$ as in Lemma/Definition 3.1.1, we have that $T_0 \exp_{\Lambda}^{\mathcal{G}} = \text{id}_{L(\mathcal{G})_{\Lambda^+}}$. In particular $T_0 \exp_{\Lambda}^{\mathcal{G}}$ is Λ_0 -linear.

If $f: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of k -Lie supergroups, then

$$f_{\Lambda} \circ \exp_{\Lambda}^{\mathcal{G}} = \exp_{\Lambda}^{\mathcal{H}} \circ L(f)_{\Lambda}|_{L(\mathcal{G})_{\Lambda^+}} \text{ for every } \Lambda \in \mathbf{Gr}^{(k)}.$$

Proof. We assume without loss of generality that \mathcal{G} is modelled on $L(\mathcal{G})_{\Lambda_1}$. Let $\varphi: \mathcal{U} \rightarrow \mathcal{G}$ be a centered chart with $\mathcal{U} \subseteq L(\mathcal{G})$ and let $\Lambda \in \mathbf{Gr}^{(k)}$. We have shown in Lemma 3.2.4 that $\varphi_{\Lambda}|_{L(\mathcal{G})_{\Lambda^+}}: L(\mathcal{G})_{\Lambda^+} \rightarrow \ker(\mathcal{G}_{\varepsilon_{\Lambda}})$ is a diffeomorphism that turns $L(\mathcal{G})_{\Lambda^+}$ into a polynomial group. Denote by $\exp_{\Lambda}^{\varphi}: L(\mathcal{G})_{\Lambda^+} \rightarrow L(\mathcal{G})_{\Lambda^+}$ the chart dependent exponential map of this polynomial group. Identifying $T_0 L(\mathcal{G})_{\Lambda_1} \cong L(\mathcal{G})_{\Lambda_1}$ in the usual way, we define

$$\exp_{\Lambda}^{\mathcal{G}} := \varphi_{\Lambda} \circ \exp_{\Lambda}^{\varphi} \circ \overline{\mathcal{T}_0 \varphi_{\Lambda_1}^{-1}}|_{L(\mathcal{G})_{\Lambda^+}}.$$

Let $\psi: \mathcal{V} \rightarrow \mathcal{G}$ be another centered chart. Then $\psi_{\Lambda}^{-1} \circ \varphi_{\Lambda}|_{L(\mathcal{G})_{\Lambda^+}}: L(\mathcal{G})_{\Lambda^+} \rightarrow L(\mathcal{G})_{\Lambda^+}$ is a polynomial isomorphism of the respective induced polynomial groups. We have $\overline{\mathcal{T}_0 \varphi_{\Lambda_1}^{-1}} \circ (\overline{\mathcal{T}_0 \psi_{\Lambda_1}^{-1}})^{-1} = \overline{\mathcal{T}_0(\varphi^{-1} \circ \psi)_{\Lambda_1}}$ and therefore Lemma C.2.2 implies that

$$\psi_{\Lambda}^{-1} \circ \varphi_{\Lambda} \circ \exp_{\Lambda}^{\varphi} \circ \overline{\mathcal{T}_0(\varphi^{-1} \circ \psi)_{\Lambda_1 \Lambda}}|_{L(\mathcal{G})_{\Lambda^+}} = \exp_{\Lambda}^{\psi}$$

holds, which shows that $\exp_\Lambda^\mathcal{G}$ is well-defined. By definition, $\exp_\Lambda^\mathcal{G}$ is a diffeomorphism and $\overline{\mathcal{T}_0\varphi_{\Lambda_1\Lambda}}$ is a Lie algebra isomorphism from $L(\mathcal{G})_\Lambda$ considered as the chart dependent Lie algebra to the usual Lie algebra of $L(\mathcal{G})_\Lambda$. Thus, the multiplication in terms of $\exp_\Lambda^\mathcal{G}$ is the BCH multiplication with respect to the usual (restricted) Lie bracket. Let $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$ then the naturality of φ implies that $L(\mathcal{G})_\varrho$ restricts to a linear morphism of the chart dependent polynomial groups $L(\mathcal{G})_{L(\mathcal{G})_{\Lambda^+}} \rightarrow L(\mathcal{G})_{L(\mathcal{G})_{\Lambda'^+}}$. It follows again from Lemma C.2.2 that $(\exp_\Lambda^\varphi)_{\Lambda \in \mathbf{Gr}^{(k)}}$ is a natural transformation in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ and thus $(\exp_\Lambda^\mathcal{G})_{\Lambda \in \mathbf{Gr}^{(k)}}$ is also a natural transformation in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$. By Theorem C.2.1, we have $T_0 \exp_L^\varphi = \text{id}_{L(\mathcal{G})_{\Lambda^+}}$ and with the identifications of (d), $T_0 \exp_\Lambda^\mathcal{G} = \text{id}_{L(\mathcal{G})_{\Lambda^+}}$ follows from the chain rule. That $\exp_\Lambda^\mathcal{G}$ is unique is obvious because the Lie algebra morphisms $\text{id}_{L(\mathcal{G})_{\Lambda^+}}$ corresponds to the group morphism $\text{id}_{\ker(\mathcal{G}_{\varepsilon_\Lambda})}$.

Let \mathcal{H} be a Lie supergroup, $\phi: \mathcal{W} \rightarrow \mathcal{H}$ be a centered chart with $\mathcal{W} \subseteq L(\mathcal{H})$ and define the chart dependent exponential map \exp_Λ^ϕ of $\ker(\mathcal{H}_{\varepsilon_\Lambda})$ as above. If $f: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of Lie supergroups then $f_\Lambda^{\phi\varphi} := \phi_\Lambda^{-1} \circ f_\Lambda \circ \varphi_\Lambda|_{L(\mathcal{G})_{\Lambda^+}}$ is a polynomial morphism of polynomial groups and thus $f_\Lambda^{\phi\varphi} \circ \exp_\Lambda^\varphi = \exp_\Lambda^\phi \circ T_0 f_\Lambda^{\phi\varphi}$ holds by Lemma C.2.2. Using the isomorphism $\overline{\mathcal{T}_0\phi_{\Lambda_1}} \circ \mathcal{T}_0\phi^{-1}: \mathcal{T}_e\mathcal{H} \rightarrow L(\mathcal{H})$, the definition of the exponential map now implies the equality

$$f_\Lambda \circ \exp_\Lambda^\mathcal{G} = \exp_\Lambda^\mathcal{H} \circ L(f)_\Lambda|_{L(\mathcal{G})_{\Lambda^+}}.$$

□

Corollary 3.2.7. *Let $k \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup, $\mathfrak{g} \in \mathbf{SVec}_{lc}$ with $\overline{\mathfrak{g}}^{(k)} = L(\mathcal{G})$ and $\exp^\mathcal{G} := (\exp_\Lambda^\mathcal{G})_{\Lambda \in \mathbf{Gr}^{(k)}}$ the natural transformation from Proposition 3.2.6. Then,*

$$\Phi^\mathcal{G}: \iota_k^0(\mathcal{G}_\mathbb{R}) \times \overline{\mathfrak{g}}_1^{(k)} \rightarrow \mathcal{G}, \quad \mu \circ (\text{id}_{\iota_k^0(\mathcal{G}_\mathbb{R})} \times \exp^\mathcal{G}|_{\overline{\mathfrak{g}}_1^{(k)}})$$

is an isomorphism of k -supermanifolds.

Proof. We only need to check that $\exp^\mathcal{G}|_{\overline{\mathfrak{g}}_1^{(k)}}: \overline{\mathfrak{g}}_1^{(k)} \rightarrow \mathcal{G}$ satisfies the conditions of Lemma 3.2.2. Because of Corollary 2.2.9, supersmoothness of $\exp^\mathcal{G}|_{\overline{\mathfrak{g}}_1^{(k)}}: \overline{\mathfrak{g}}_1^{(k)} \rightarrow \mathcal{G}$ needs only to be checked at 0, where it holds by Proposition 3.2.6. It follows from the same proposition that $\exp^\mathcal{G}|_{\overline{\mathfrak{g}}_1^{(k)}}: \overline{\mathfrak{g}}_1^{(k)} \rightarrow \ker(\mathcal{G}_{\varepsilon_{\Lambda_1}})$ is a diffeomorphism. □

Corollary 3.2.8. *Let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 3$, let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$, \mathcal{H} be k -Lie supergroups and let $f: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of k -Lie supergroups. Then f is completely determined by $f_\mathbb{R}$ and $\mathcal{T}_e f_{\Lambda_1}$.*

Proof. By Lemma 3.2.4 f is completely determined by $f_\mathbb{R}$ and $f_\Lambda|_{\ker(\mathcal{G}_{\varepsilon_\Lambda})}$ for all $\Lambda \in \mathbf{Gr}^{(k)}$. Proposition 3.2.6 implies that $f_\Lambda|_{\ker(\mathcal{G}_{\varepsilon_\Lambda})}$ is determined by the Lie algebra morphism $\mathcal{T}_e f_\Lambda$, which in turn is determined by the Lie superalgebra morphism $\mathcal{T}_e f_{\Lambda_1}$ because of Corollary 3.1.2. □

Proposition 3.2.9. *Let $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup and $\mathfrak{g} := \text{sL}(\mathcal{G})$. For every $\Lambda \in \mathbf{Gr}^{(k)}$ and every $g \in \mathcal{G}_\mathbb{R}$, we define the conjugation*

$c_{g,\Lambda}: \mathcal{G}_\Lambda \rightarrow \mathcal{G}_\Lambda$, $h \mapsto \mathcal{G}_{\eta_\Lambda}(g) \cdot h \cdot \mathcal{G}_{\eta_\Lambda}(g^{-1})$. Then $(c_{g,\Lambda})_{\Lambda \in \mathbf{Gr}^{(k)}}$ is an isomorphism of Lie supergroups. With this, we define $\text{Ad}_g := L(c_g): \bar{\mathfrak{g}}^{(k)} \rightarrow \bar{\mathfrak{g}}^{(k)}$. For every $\Lambda \in \mathbf{Gr}^{(k)}$, we have that

(a) the map $\text{Ad}_{g,\Lambda}: \bar{\mathfrak{g}}_\Lambda^{(k)} \rightarrow \bar{\mathfrak{g}}_\Lambda^{(k)}$ is an isomorphism of Lie algebras,

(b) for $v \in \bar{\mathfrak{g}}_{\Lambda^+}^{(k)}$ the equality

$$c_{g,\Lambda} \circ \exp_\Lambda^{\mathcal{G}}(v) = \exp_\Lambda^{\mathcal{G}} \circ \text{Ad}_{g,\Lambda}(v)$$

holds and

(c) the map $G \times \bar{\mathfrak{g}}_\Lambda^{(k)} \rightarrow \bar{\mathfrak{g}}_\Lambda^{(k)}$, $(g, v) \mapsto \text{Ad}_{g,\Lambda}(v)$ is a smooth group action such that $(d\text{Ad}_{\bullet,\Lambda}(v))(w) = \overline{[w, v]}_\Lambda$ holds for $w \in T_{e_{\mathbb{R}}} \mathcal{G}_{\mathbb{R}}$, $v \in \bar{\mathfrak{g}}_\Lambda^{(k)}$. Here $\text{Ad}_{\bullet,\Lambda}(v)$ denotes the map $\mathcal{G}_{\mathbb{R}} \rightarrow \bar{\mathfrak{g}}_\Lambda^{(k)}$, $g \mapsto \text{Ad}_g(v)$ and $[\cdot, \cdot]$ is the Lie superbracket of \mathfrak{g} .

Proof. The conjugation map $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ defined by $\mathcal{G}_\Lambda \times \mathcal{G}_\Lambda \rightarrow \mathcal{G}_\Lambda$, $(g, h) \mapsto g \cdot h \cdot g^{-1}$ is clearly supersmooth. Therefore, it suffices to see that c_g is a natural transformation. This follows because $\mathcal{G}_\varrho \circ \mathcal{G}_{\eta_\Lambda} = \mathcal{G}_{\eta_{\Lambda'}}$ holds for all $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. That each $c_{g,\Lambda}$ is an isomorphism of groups is obvious and by supersmoothness it maps zero to zero in any centered chart and restricts to a polynomial map $\bar{\mathfrak{g}}_{\Lambda^+}^{(k)} \rightarrow \bar{\mathfrak{g}}_{\Lambda^+}^{(k)}$.

With Lemma/Definition 3.1.1, (a) is obvious and (b) follows from Proposition 3.2.6. To see (c), let $\mathcal{U} \subseteq \bar{\mathfrak{g}}^{(k)}$ be an open subfunctor such that $0 \in \mathcal{U}_{\mathbb{R}}$ and $\varphi: \mathcal{U} \rightarrow \mathcal{G}$ be a centered chart. We set $\mathcal{U}_g := \mathcal{U}|_{\varphi_{\mathbb{R}}^{-1}(c_{g,\mathbb{R}}^{-1}(\varphi_{\mathbb{R}}(\mathcal{U}_{\mathbb{R}})))}$ and define $c_{g,\Lambda}^\varphi := \varphi_\Lambda^{-1} \circ c_{g,\Lambda} \circ \varphi_\Lambda|_{\mathcal{U}_g}$. That $G \times \bar{\mathfrak{g}}_\Lambda^{(k)} \rightarrow \bar{\mathfrak{g}}_\Lambda^{(k)}$, $(g, v) \mapsto \text{Ad}_{g,\Lambda}(v)$ is a smooth group action is obvious because $\mathcal{G}_{\mathbb{R}} \times \mathcal{G}_{\Lambda_1} \rightarrow \mathcal{G}_{\Lambda_1}$, $(g, h) \mapsto c_{g,\Lambda_1}(h)$ is so. Denote by $[\cdot, \cdot]^\varphi$ the chart dependent Lie bracket on $\bar{\mathfrak{g}}^{(k)}$ and let $\text{Ad}_g^\varphi := \mathcal{T}_e c_g^\varphi$. Then, $(d\text{Ad}_{\bullet,\Lambda}(\tilde{v}))(\tilde{w}) = \overline{[\tilde{w}, \tilde{v}]}_\Lambda$ holds for $\tilde{v} \in \mathfrak{g}_0$ and $\tilde{w} \in \bar{\mathfrak{g}}_\Lambda^{(k)}$ by the respective result for ordinary Lie groups (see Proposition A.3.2). Applying the Lie algebra isomorphism $\overline{\mathcal{T}_0 \varphi_{\Lambda_1}}: (\bar{\mathfrak{g}}^{(k)}, [\cdot, \cdot]^\varphi) \rightarrow (\bar{\mathfrak{g}}^{(k)}, [\cdot, \cdot])$ to this, yields (c). \square

In the situation of the proposition, we call $\text{Ad}_{\bullet,\Lambda}$ the *adjoint action* of $\mathcal{G}_{\mathbb{R}}$ on $\bar{\mathfrak{g}}_\Lambda$. By Corollary 2.2.22 the adjoint action leads to an action of $\mathcal{G}_{\mathbb{R}}$ on $\text{sL}(\mathcal{G})$.

Remark 3.2.10. By taking inverse limits one can easily transfer the statements of Proposition 3.2.6 and Proposition 3.2.9 to $\varprojlim_n \ker(\mathcal{G}_{\varepsilon_{\Lambda_n}})$ to understand the structure of the Lie group $\varprojlim_n \mathcal{G}_{\Lambda_n}$.

The next lemma describes morphisms of Lie supergroups in terms of the trivialization from Corollary 3.2.7.

Lemma 3.2.11. Let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 3$, let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$, \mathcal{H} be k -Lie supergroups with $\text{sL}(\mathcal{G}) := \mathfrak{g}$, $\text{sL}(\mathcal{H}) := \mathfrak{h}$ and let $\Phi^{\mathcal{G}}: \iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \bar{\mathfrak{g}}_1^{(k)} \rightarrow \mathcal{G}$, $\Phi^{\mathcal{H}}: \iota_k^0(\mathcal{H}_{\mathbb{R}}) \times \bar{\mathfrak{h}}_1^{(k)} \rightarrow \mathcal{H}$ be the isomorphisms from Corollary 3.2.7. Then, a supersmooth morphism $f: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of k -Lie supergroups if and only if $(\Phi^{\mathcal{H}})^{-1} \circ f \circ \Phi^{\mathcal{G}} = \iota_k^0(f_0) \times \bar{f}_1$, where

- (a) $f_0: \mathcal{G}_{\mathbb{R}} \rightarrow \mathcal{H}_{\mathbb{R}}$ is a morphism of Lie groups,
- (b) $f_1: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$ is a continuous linear map,
- (c) $\mathrm{sL}(\iota_k^0(f_0)) \oplus f_1: \mathfrak{g} \rightarrow \mathfrak{h}$ is a morphism of Lie superalgebras and
- (d) the equality

$$\overline{\mathrm{sL}(\iota_k^0(f_0)) \oplus f_{1\Lambda}} \circ \mathrm{Ad}_{g,\Lambda}^{\mathcal{G}} = \mathrm{Ad}_{f_0(g),\Lambda}^{\mathcal{H}} \circ \overline{\mathrm{sL}(\iota_k^0(f_0)) \oplus f_{1\Lambda}}$$

holds for all $g \in \mathcal{G}_{\mathbb{R}}$ and $\Lambda \in \mathbf{Gr}^{(k)}$, where $\mathrm{Ad}^{\mathcal{G}}$ and $\mathrm{Ad}^{\mathcal{H}}$ are defined as in Proposition 3.2.9.

In this situation, we have $f_1 = \mathrm{sL}(f)|_{\mathfrak{g}_1}$ and $f_0 = f_{\mathbb{R}}$.

Proof. Note that indeed $\mathrm{sL}(\iota_k^0(\mathcal{G}_{\mathbb{R}})) = \mathfrak{g}_0$. We give $\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)}$ and $\iota_k^0(\mathcal{H}_{\mathbb{R}}) \times \overline{\mathfrak{h}_1}^{(k)}$ the induced Lie supergroup structures. By Proposition 3.2.6, we have $T_0 \exp_{\Lambda}^{\mathcal{G}} = \mathrm{id}_{L(\mathcal{G})_{\Lambda+}}$. Moreover, since $\Phi^{\mathcal{G}} = \mu \circ (\mathrm{id}_{\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \exp^{\mathcal{G}}|_{\overline{\mathfrak{g}_1}^{(k)}}})$ holds, it follows that $L(\Phi^{\mathcal{G}}): \overline{\mathfrak{g}_0} \oplus \overline{\mathfrak{g}_1} \rightarrow \overline{\mathfrak{g}}$ is just the addition, i.e., $L(\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)}) = \overline{\mathfrak{g}}^{(k)}$. Because we use the induced Lie supergroup structure, we have $\exp^{\mathcal{G}} = \Phi^{\mathcal{G}} \circ \exp^{\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)}}$ with this identification, which implies $\exp^{\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)}}|_{\overline{\mathfrak{g}_1}^{(k)}} = \mathrm{id}_{\overline{\mathfrak{g}_1}^{(k)}}$.

Let $f: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of k -Lie supergroups. Since $L(f)$ is $\overline{\mathbb{R}}^{(k)}$ -linear, we can use Proposition 3.2.6 to get

$$L(f)|_{\overline{\mathfrak{g}_1}^{(k)}} = \exp^{\iota_k^0(\mathcal{H}_{\mathbb{R}}) \times \overline{\mathfrak{h}_1}^{(k)}} \circ L(f)|_{\overline{\mathfrak{g}_1}^{(k)}} = f \circ \exp^{\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)}}|_{\overline{\mathfrak{g}_1}^{(k)}} = f|_{\overline{\mathfrak{g}_1}^{(k)}}.$$

We set $\overline{f_1} := L(f)|_{\overline{\mathfrak{g}_1}^{(k)}}$. In other words, we have $f_1 = \mathrm{sL}(f)|_{\mathfrak{g}_1}$. Since a morphism on purely even supermanifolds is determined by the morphism on the base manifold, we have $f|_{\iota_k^0(\mathcal{G}_{\mathbb{R}})} = \iota_k^0(f_{\mathbb{R}})$ and we set $f_0 := f_{\mathbb{R}}$. Then $L(f) = \mathrm{sL}(\iota(f_0)) \oplus f_1$ and property (d) follows from Lemma 3.2.4 together with Proposition 3.2.9. Moreover, for $g \in \iota_k^0(\mathcal{G}_{\mathbb{R}})_{\Lambda}$ and $v \in \overline{\mathfrak{g}_{1\Lambda}}^{(k)}$, we have

$$f_{\Lambda}(g \cdot \exp_{\Lambda}^{\mathcal{G}}(v)) = f_{\Lambda}(g) \cdot f_{\Lambda}(\exp_{\Lambda}^{\mathcal{G}}(v)) = \iota_k^0(f_0)_{\Lambda}(g) \cdot \exp_{\Lambda}^{\mathcal{H}}(\overline{f_1}_{\Lambda}(v))$$

and therefore $(\Phi^{\mathcal{H}})^{-1} \circ f \circ \Phi^{\mathcal{G}} = \iota_k^0(f_0) \times \overline{f_1}$.

Conversely, assume that f_0 and f_1 have the properties (a) to (d). With the usual identifications $\mathcal{T}_e \mathcal{G} \cong L(\mathcal{G})$ and $\mathcal{T}_e \mathcal{H} \cong L(\mathcal{H})$, it follows from Lemma C.2.2 that

$$f_{\Lambda+} := \exp_{\Lambda}^{\mathcal{H}} \circ \overline{\mathrm{sL}(\iota(f_0)) \oplus f_{1\Lambda}}|_{\overline{\mathfrak{g}_{1\Lambda}}^{(k)}} \circ (\exp_{\Lambda}^{\mathcal{G}})^{-1}: \ker(\mathcal{G}_{\varepsilon_{\Lambda}}) \rightarrow \ker(\mathcal{H}_{\varepsilon_{\Lambda}})$$

is a morphism of Lie groups with $T_0 f_{\Lambda+} = \overline{\mathrm{sL}(\iota(f_0)) \oplus f_{1\Lambda}}|_{\overline{\mathfrak{g}_{1\Lambda}}^{(k)}}$. This implies that $f_{\Lambda+}|_{\ker(\iota_k^0(\mathcal{G}_{\mathbb{R}})_{\varepsilon_{\Lambda}})} = \iota_k^0(f_0)_{\Lambda}|_{\ker(\iota_k^0(\mathcal{G}_{\mathbb{R}})_{\varepsilon_{\Lambda}})}$ and that $(\exp_{\Lambda}^{\mathcal{H}})^{-1} \circ f_{\Lambda+} \circ \exp_{\Lambda}^{\mathcal{G}}|_{\overline{\mathfrak{g}_{1\Lambda}}^{(k)}} = \overline{f_1}_{\Lambda}$. For $g \in \ker(\iota_k^0(\mathcal{G}_{\mathbb{R}})_{\varepsilon_{\Lambda}})$ and $v \in \overline{\mathfrak{g}_{1\Lambda}}^{(k)}$, the equality

$$(\Phi_{\Lambda}^{\mathcal{H}})^{-1} \circ f_{\Lambda+} \circ \Phi_{\Lambda}^{\mathcal{G}}(g, v) = (\iota(f_0)_{\Lambda}(g), \overline{f_1}_{\Lambda}(v))$$

follows, which means that $\iota_k^0(f_0)_\Lambda \times \overline{f_{1\Lambda}}|_{\ker(\iota_k^0(\mathcal{G}_\mathbb{R})_{\varepsilon_\Lambda}) \times \overline{\mathfrak{g}_{1\Lambda}}}$ is a morphism of groups. By definition, the actions of $\mathcal{G}_\mathbb{R}$ on $\overline{\mathfrak{g}_{\Lambda+}^{(k)}}$ induced by the action of $\mathcal{G}_\mathbb{R}$ on $\ker(\mathcal{G}_{\varepsilon_\Lambda})$ and $\ker(\iota_k^0(\mathcal{G}_\mathbb{R})_{\varepsilon_\Lambda}) \times \overline{\mathfrak{g}_{1\Lambda}^{(k)}}$ are the same. Because of (d), Proposition 3.2.9 together with Lemma A.3.4 show that

$$((\iota_k^0(f_0)_\Lambda \times \overline{f_{1\Lambda}}|_{\ker(\iota_k^0(\mathcal{G}_\mathbb{R})_{\varepsilon_\Lambda}) \times \overline{\mathfrak{g}_{1\Lambda}^{(k)}}) \times f_0):$$

$$(\ker(\iota_k^0(\mathcal{G}_\mathbb{R})_{\varepsilon_\Lambda}) \times \overline{\mathfrak{g}_{1\Lambda}^{(k)}}) \rtimes \mathcal{G}_\mathbb{R} \rightarrow (\ker(\iota_k^0(\mathcal{H}_\mathbb{R})_{\varepsilon_\Lambda}) \times \overline{\mathfrak{h}_{1\Lambda}^{(k)}}) \rtimes \mathcal{H}_\mathbb{R}$$

is a morphism of groups. But this map corresponds to $\iota_k^0(f_0)_\Lambda \times \overline{f_{1\Lambda}}$ under the splitting from Lemma 3.2.4. \square

3.3. Super Harish-Chandra Pairs

We have seen that for any Lie supergroup \mathcal{G} , the Lie groups \mathcal{G}_Λ are completely determined by $\mathfrak{sL}(\mathcal{G})$ and the action of $\mathcal{G}_\mathbb{R}$ on $\mathfrak{sL}(\mathcal{G})$ induced by the adjoint action. The pair $(\mathcal{G}_\mathbb{R}, \mathfrak{sL}(\mathcal{G}))$ forms a so called super Harish-Chandra pair. Conversely, we will show that in fact all super Harish-Chandra pairs define a Lie supergroup and that one obtains an equivalence of categories in this way. For finite-dimensional Lie supergroups this is a classical result by Kostant [32] (for a sheaf theoretic treatment see for example [14]). Neeb and Salmasian, [42], generalized this to the case of infinite-dimensional Lie supergroups \mathcal{G} modelled on Mackey complete super vector spaces such that $\mathcal{G}_\mathbb{R}$ has a smooth exponential map. For this, they used techniques not dissimilar to the classical proof via the universal enveloping superalgebra.

In our setting, this result was stated for Banach Lie supergroups without proof by Molotkov in [40]. It appears likely that Molotkov used the exponential map as a chart to show this. In contrast, our proof holds for arbitrary Lie supergroups and uses the preceding trivializations to construct concrete quasi-inverse functors.

Definition 3.3.1. The pair (G, \mathfrak{g}) of a Lie group G and a locally convex \mathbb{R} -Lie superalgebra \mathfrak{g} together with a morphism of groups $\text{Ad}_G: G \rightarrow \text{Aut}(\mathfrak{g})$ is called a *super Harish-Chandra pair* if

(1) $\mathfrak{g}_0 = \mathfrak{L}(G)$,

(2) the map

$$G \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad (g, v) \mapsto \text{Ad}_G(g)(v)$$

is smooth and

(3) if we set $c_v: G \rightarrow \mathfrak{g}$, $g \mapsto \text{Ad}_G(g)(v)$ for $v \in \mathfrak{g}$, then $d_w c_v(e) = [w, v]$ holds for every $w \in \mathfrak{g}_0$, where e denotes the identity element of G and $[\cdot, \cdot]$ the Lie superbracket of \mathfrak{g} .

Let (H, \mathfrak{h}) be another super Harish-Chandra pair with the morphism $\text{Ad}_H: H \times \mathfrak{h} \rightarrow \mathfrak{h}$. A *morphism* between (G, \mathfrak{g}) and (H, \mathfrak{h}) is a pair (f_0, f) such that

- (a) $f_0: G \rightarrow H$ is a morphism of Lie groups,
- (b) $f: \mathfrak{g} \rightarrow \mathfrak{h}$ is a continuous morphism of Lie superalgebras,
- (c) we have $f|_{\mathfrak{g}_0} = T_e f_0$ and
- (d) $\text{Ad}_H(f_0(g)) \circ f = f \circ \text{Ad}_G(g)$ holds for all $g \in G$.

The composition of morphisms is defined in the obvious way and super Harish-Chandra pairs and their morphisms form a category which we denote by **SHCP**.

Lemma 3.3.2. *Let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 3$ and let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a k -Lie supergroup. For $g \in \mathcal{G}_{\mathbb{R}}$, let c_{g, Λ_1} be as in Proposition 3.2.9. The action $\text{Ad}_{\mathcal{G}}: \mathcal{G}_{\mathbb{R}} \times \mathcal{T}_e \mathcal{G}_{\Lambda_1} \rightarrow \mathcal{T}_e \mathcal{G}_{\Lambda_1}$, $(g, v) \mapsto \mathcal{T}_e c_{g, \Lambda_1}(v)$ defines a super Harish-Chandra pair $(\mathcal{G}_{\mathbb{R}}, \text{sL}(\mathcal{G}))$. If $f: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of Lie supergroups then $(f_{\mathbb{R}}, \text{sL}(f))$ is a morphism of super Harish-Chandra pairs and in this way we get a functor*

$$\mathcal{H}^{(k)}: \mathbf{LSGrp}^{(k)} \rightarrow \mathbf{SHCP}.$$

Proof. That $(\mathcal{G}_{\mathbb{R}}, \text{sL}(\mathcal{G}))$ is a super Harish-Chandra pair follows from Proposition 3.2.9 together with Corollary 2.2.22.

Let $f: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of Lie supergroups. We already know from Corollary 3.1.2 that $\text{sL}(f) = \mathcal{T}_e f_{\Lambda_1}$ is a morphism of Lie superalgebras and that this assignment is functorial. Consider $\mathcal{G}_{\mathbb{R}} \subseteq \mathcal{G}_{\Lambda_1}$ and $\mathcal{H}_{\mathbb{R}} \subseteq \mathcal{H}_{\Lambda_1}$ via $\mathcal{G}_{\eta_{\Lambda_1}}$ and $\mathcal{H}_{\eta_{\Lambda_1}}$. Clearly, we have $T_{e_{\mathbb{R}}} f_{\mathbb{R}} = \mathcal{T}_e f_{\Lambda_1}|_{\mathcal{T}_e \mathcal{G}_{\mathbb{R}}}$. For $h \in \mathcal{H}_{\mathbb{R}}$, define $c_h^{\mathcal{H}}: \mathcal{H}_{\Lambda_1} \rightarrow \mathcal{H}_{\Lambda_1}$, $h' \mapsto hh'h^{-1}$. It follows from Lemma 3.2.4 that $f_{\Lambda_1} \circ c_g = c_{f_{\mathbb{R}}(g)}^{\mathcal{H}} \circ f_{\Lambda_1}$. Taking derivatives at e_{Λ_1} now shows $\mathcal{T}_e f_{\Lambda_1} \circ \text{Ad}_{\mathcal{G}}(g) = \text{Ad}_{\mathcal{H}}(f_{\mathbb{R}}(g)) \circ \mathcal{T}_e f_{\Lambda_1}$. \square

Our objective is to show that this correspondence establishes an equivalence of categories $\mathbf{LSGrp}^{(k)} \rightarrow \mathbf{SHCP}$ for $k \geq 3$. Let us sketch how the quasi-inverse functor $\mathbf{SHCP} \rightarrow \mathbf{LSGrp}^{(k)}$ is constructed. Given a super Harish-Chandra pair (G, \mathfrak{g}) , where $[\cdot, \cdot]$ is the Lie superbracket of \mathfrak{g} , we can define a nilpotent group \mathcal{N}_{Λ} for every $\Lambda \in \mathbf{Gr}^{(k)}$ by considering $\bar{\mathfrak{g}}_{\Lambda+}$ together with the BCH multiplication of the nilpotent Lie algebra $[\cdot, \cdot]_{\Lambda}|_{\bar{\mathfrak{g}}_{\Lambda+} \times \bar{\mathfrak{g}}_{\Lambda+}}$. Then G acts on \mathcal{N}_{Λ} via the induced action $\overline{\text{Ad}}_G: G \times \bar{\mathfrak{g}}_{\Lambda+} \rightarrow \bar{\mathfrak{g}}_{\Lambda+}$. Setting $\mathcal{G}_{\Lambda} := \mathcal{N}_{\Lambda} \rtimes G$, we use Corollary 3.2.7 to get an appropriate supersmooth structure $\iota_k^0(G) \times \bar{\mathfrak{g}}_1^{(k)} \cong \mathcal{G}$. The proof is complicated by the fact that the Lie group G does necessarily have an exponential map. However, we have seen that $\ker(\mathcal{G}_{\varepsilon_{\Lambda}})$ has an exponential map and we will show in the following lemmas that this exponential map has sufficiently good properties for our needs.

Definition 3.3.3. For $k \in \mathbb{N} \cup \{\infty\}$ and $E \in \mathbf{SVec}_{lc}$, we define a subfunctor $\overline{E}^{(k),+}: \mathbf{Gr}^{(k)} \rightarrow \mathbf{Top}$ of $\overline{E}^{(k)}$ by

$$\overline{E}_{\Lambda}^{(k),+} := \overline{E}_{\Lambda+}^{(k)}$$

for $\Lambda \in \mathbf{Gr}^{(k)}$ and

$$\overline{E}_{\varrho}^{(k),+} := \overline{E}_{\varrho}^{(k)}|_{\overline{E}_{\Lambda+}^{(k)}}$$

for $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$.

Lemma 3.3.4. *Let $k \in \mathbb{N} \cup \{\infty\}$, $E, F \in \mathbf{SVec}_{lc}$ and $f: \overline{E}^{(k),+} \rightarrow \overline{F}^{(k),+}$ a morphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ such that f_Λ is a diffeomorphism for all $\Lambda \in \mathbf{Gr}^{(k)}$. Then, f is an isomorphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$. If additionally the derivative of f_Λ is $\Lambda_{\overline{0}}$ -linear at every point, then so is the derivative of f_Λ^{-1} .*

Proof. Let $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$. For $\varrho \in \text{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$, we have

$$f_{\Lambda'} \circ \overline{E}_\varrho^{(k),+} \circ f_\Lambda^{-1} = \overline{E}_\varrho^{(k),+} \circ f_\Lambda \circ f_\Lambda^{-1} = \overline{E}_\varrho^{(k),+},$$

which implies that $f_{\Lambda'}^{-1} \circ \overline{E}_\varrho^{(k),+} = \overline{E}_\varrho^{(k),+} \circ f_\Lambda^{-1}$. Similarly, let the derivative of f_Λ be $\Lambda_{\overline{0}}$ -linear at every point. For $x, v \in \overline{E}_\Lambda^{(k),+}$ and $\lambda_I \in \Lambda_{\overline{0}}$, we calculate

$$df_\Lambda(f_\Lambda^{-1}(x), \lambda_I df_\Lambda^{-1}(x, v)) = \lambda_I v = df_\Lambda(f_\Lambda^{-1}(x), df_\Lambda^{-1}(x, \lambda_I v)),$$

from which $df_\Lambda^{-1}(x, \lambda_I v) = \lambda_I df_\Lambda^{-1}(x, v)$ follows. \square

Lemma 3.3.5. *Let \mathfrak{g} be a locally convex \mathbb{R} -Lie superalgebra with the bracket $[\cdot, \cdot]$. For $k \in \mathbb{N} \cup \{\infty\}$ and $\Lambda \in \mathbf{Gr}^{(k)}$, let $*_\Lambda: \overline{\mathfrak{g}}_\Lambda^{(k),+} \times \overline{\mathfrak{g}}_\Lambda^{(k),+} \rightarrow \overline{\mathfrak{g}}_\Lambda^{(k),+}$ denote the BCH multiplication with respect to the nilpotent Lie bracket $[\cdot, \cdot]_\Lambda^+ := [\cdot, \cdot]_\Lambda|_{\overline{\mathfrak{g}}_\Lambda^{(k),+} \times \overline{\mathfrak{g}}_\Lambda^{(k),+}}$. Then $(*_\Lambda)_{\Lambda \in \mathbf{Gr}^{(k)}}$ defines a natural transformation $*$: $\mathbf{Gr}^{(k)} \rightarrow \mathbf{Man}$ and the derivative of $*_\Lambda$ is $\Lambda_{\overline{0}}$ -linear at every point. Moreover, the natural transformation Ψ defined by*

$$\Psi_\Lambda := *_\Lambda|_{\overline{\mathfrak{g}}_{0\Lambda}^{(k),+} \times \overline{\mathfrak{g}}_{1\Lambda}^{(k)}}: \overline{\mathfrak{g}}_{0\Lambda}^{(k),+} \times \overline{\mathfrak{g}}_{1\Lambda}^{(k)} \rightarrow \overline{\mathfrak{g}}_\Lambda^{(k),+}, \quad (v_0, v_1) \mapsto v_0 *_\Lambda v_1,$$

is an isomorphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ and the derivative of Ψ_Λ is $\Lambda_{\overline{0}}$ -linear at every point for every $\Lambda \in \mathbf{Gr}^{(k)}$.

Proof. Clearly, $\Lambda \mapsto [\cdot, \cdot]_\Lambda^+$ defines a natural transformation. From this, it easily follows that $*$ is a morphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$. The finite sum over combinations of $[\cdot, \cdot]_\Lambda^+$ that defines $*_\Lambda$ is just the restriction of the same sum using $[\cdot, \cdot]_\Lambda$ instead, which is supersmooth. Therefore, the derivative of $*_\Lambda$ is $\Lambda_{\overline{0}}$ -linear at all points.

Interpreting $\overline{\mathfrak{g}}_\Lambda^+$ as a multilinear space, one sees that Ψ_Λ is a morphism of multilinear spaces whose linear part is the identity. Therefore, Ψ_Λ is invertible by Theorem B.1.2. The claim follows now from Lemma 3.3.4. \square

Lemma 3.3.6. *Let $k \in \mathbb{N} \cup \{\infty\}$, $E, F, H \in \mathbf{SVec}_{lc}$, $f: \overline{E}^{(k)} \rightarrow \overline{F}^{(k)}$ be supersmooth and $g: \overline{F}^{(k),+} \rightarrow \overline{H}^{(k),+}$ be a morphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$. If, for all $\Lambda \in \mathbf{Gr}^{(k)}$, we have $f_\Lambda(\overline{E}_\Lambda^{(k)}) \subseteq \overline{F}_\Lambda^{(k),+}$ and the derivative of g_Λ is $\Lambda_{\overline{0}}$ -linear at every point, then $g \circ f: \overline{E} \rightarrow \overline{H}$ is supersmooth. If E is purely odd, then it suffices that the derivative of g_Λ is $\Lambda_{\overline{0}}$ -linear at zero.*

Proof. The inclusion $\overline{H}^{(k),+} \rightarrow \overline{H}^{(k)}$ is a morphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ such that the derivative of its Λ -components is $\Lambda_{\overline{0}}$ -linear at all points for all $\Lambda \in \mathbf{Gr}^{(k)}$. Thus,

$g \circ f: \overline{E}^{(k)} \rightarrow \overline{H}^{(k)}$ is a morphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$ and the derivative of $(g \circ f)_\Lambda$ is $\Lambda_{\overline{0}}$ -linear at all points by the chain rule. If E is purely odd, then we have $f_\Lambda(0) = 0$ and the claim follows from Corollary 2.2.9. \square

Proposition 3.3.7. *Let (G, \mathfrak{g}) together with Ad_G be a super Harish-Chandra pair. Let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 3$. Then a canonical Lie supergroup structure exists on $\iota_k^0(G) \times \overline{\mathfrak{g}}_1^{(k)}$ such that $\text{sL}(\iota_k^0(G) \times \overline{\mathfrak{g}}_1^{(k)}) \cong \mathfrak{g}$. If (f_0, f) is a morphism between the super Harish-Chandra pairs (G, \mathfrak{g}) and (H, \mathfrak{h}) then $\iota_k^0(f_0) \times \overline{f|_{\mathfrak{g}_1}}$ is a morphism of the respective k -Lie supergroups and this defines a functor*

$$\mathcal{H}^{(k)}: \mathbf{SHCP} \rightarrow \mathbf{LSGrp}^{(k)}$$

such that $\mathcal{H}^{(k)} \circ \mathcal{H}^{(k)} \cong \text{id}_{\mathbf{SHCP}}$.

Proof. Let $e \in G$ be the neutral element and $[\cdot, \cdot]$ be the Lie superbracket of \mathfrak{g} . For $\Lambda \in \mathbf{Gr}^{(k)}$, we define $\mathfrak{n}_\Lambda^G := \overline{\mathfrak{g}}_{\Lambda^+}^{(k)}$ together with the continuous nilpotent Lie bracket $[\cdot, \cdot]_{\Lambda^+}$ and give $\mathcal{N}_\Lambda^G := \mathfrak{n}_\Lambda^G$ the structure of a polynomial Lie group with the BCH multiplication. Clearly $\mathfrak{n}_{\Lambda_{\overline{0}}}^G := \overline{\mathfrak{g}}_{0\Lambda^+}^{(k)}$ is a closed Lie subalgebra and we let $\mathcal{N}_{\Lambda_{\overline{0}}}^G$ be the respective closed Lie subgroup. By Corollary 2.2.22, $\overline{\text{Ad}_G(g)}_\Lambda: \overline{\mathfrak{g}}_\Lambda^{(k)} \rightarrow \overline{\mathfrak{g}}_\Lambda^{(k)}$ is a continuous morphism of Lie algebras for every $g \in G$ and this induces a smooth group action by automorphisms $\overline{\text{Ad}_G}: G \times \mathcal{N}_\Lambda^G \rightarrow \mathcal{N}_\Lambda^G$, $(g, v) \mapsto \overline{\text{Ad}_G(g)}(v)$. Therefore, we have the Lie groups $\mathcal{G}_\Lambda := \mathcal{N}_\Lambda^G \rtimes G$ for every $\Lambda \in \mathbf{Gr}^{(k)}$ and because $\text{Ad}_G(g)$ is an even map, this group has the closed Lie subgroup $\mathcal{G}_{\Lambda_{\overline{0}}} := \mathcal{N}_{\Lambda_{\overline{0}}}^G \rtimes G$. Since \mathfrak{g}_0 is just the Lie algebra of G , we obtain an isomorphism of Lie groups $\ker(\iota_k^0(G)_{\varepsilon_\Lambda}) \rightarrow \mathcal{N}_{\Lambda_{\overline{0}}}^G$ via

$$\begin{array}{ccc} \overline{\mathfrak{g}}_{\Lambda^+}^{(k)} & \xrightarrow{\exp \iota_k^0(G)} & \ker(\iota_k^0(G)_{\varepsilon_\Lambda}) \\ \downarrow \text{id} & & \downarrow \\ \mathfrak{n}_{\Lambda_{\overline{0}}}^G & \xrightarrow[\exp \mathcal{N}_{\Lambda_{\overline{0}}}^G]{} & \mathcal{N}_{\Lambda_{\overline{0}}}^G \end{array}$$

where $\exp \mathcal{N}_{\Lambda_{\overline{0}}}^G = \text{id}_{\mathfrak{n}_{\Lambda_{\overline{0}}}^G}$ is the exponential map of $\mathcal{N}_{\Lambda_{\overline{0}}}^G$. Moreover, by Proposition 3.2.9, the action of $G = \iota_k^0(G)_{\mathbb{R}}$ with respect to the respective exponential map is in both cases the same. Thus, we will subsequently identify $\iota_k^0(G)_\Lambda \cong \mathcal{G}_{\Lambda_{\overline{0}}}$ for all $\Lambda \in \mathbf{Gr}^{(k)}$. It follows from Lemma 3.3.5 and the definition of the semidirect product that

$$\Phi_\Lambda: \iota_k^0(G)_\Lambda \times \overline{\mathfrak{g}}_{1\Lambda}^{(k)} \rightarrow \mathcal{G}_\Lambda, \quad (g, v) \mapsto g \cdot v$$

is bijective for every $\Lambda \in \mathbf{Gr}^{(k)}$. We will show that $\Phi := (\Phi_\Lambda)_{\Lambda \in \mathbf{Gr}^{(k)}}$ defines a Lie supergroup structure on \mathcal{G} . For this, we first calculate the supersmoothness of the conjugation defined by

$$\sigma_\Lambda: \iota_k^0(G)_\Lambda \times \overline{\mathfrak{g}}_{1\Lambda}^{(k)} \rightarrow \overline{\mathfrak{g}}_{1\Lambda}^{(k)}, \quad (g, v_1) \mapsto g \cdot v_1 \cdot g^{-1}.$$

Writing $g = g_0 \cdot v_0$ with $g_0 \in G$ and $v_0 \in \mathcal{N}_\Lambda$ and letting $\Lambda = \Lambda_n$, we use Lemma

C.2.7 to calculate

$$g_0 \cdot v_0 \cdot v_1 \cdot v_0^{-1} \cdot g_0^{-1} = \sum_{m=0}^{\infty} \frac{1}{m!} \overline{\left[\text{Ad}_G(g_0)_\Lambda(v_0), \text{Ad}_G(g_0)_\Lambda(v_1) \right]_{\Lambda, m}},$$

where $\overline{[v_0, v_1]_{\Lambda, m}} := \overline{[v_0, \overline{[v_0, v_1]_{\Lambda, m-1}}]_\Lambda}$ for $m > 0$ and $\overline{[v_0, v_1]_{\Lambda, 0}} := v_1 \in \overline{\mathfrak{g}_{1\Lambda}}^{(k)}$. Note that $\overline{[v_0, v_1]_{\Lambda, m}} = 0$ for $m > n$. It follows that σ_Λ is well-defined. By Corollary 2.2.9, it is enough to check supersmoothness at points of the form $(\iota_k^0(G)_{\eta_\Lambda}(g_0), 0)$, $g_0 \in G_{\mathbb{R}}$. But, we have $d_1\sigma_\Lambda(\iota_k^0(G)_{\eta_\Lambda}(g_0), 0) = 0$ and $d_2\sigma_\Lambda(\iota_k^0(G)_{\eta_\Lambda}(g_0), 0) = \sigma_\Lambda(\iota_k^0(G)_{\eta_\Lambda}(g_0), \bullet)$, which are both $\Lambda_{\bar{0}}$ -linear.

With this, we can show the supersmoothness of the group operations. Let $i: \iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)} \rightarrow \iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)}$ be the inversion. The restriction $i|_{\iota_k^0(G)}$ is just the inversion in $\iota_k^0(G)$ and therefore supersmooth. The inverse of $v_1 \in \overline{\mathfrak{g}_{1\Lambda}}^{(k)}$ is given by $-v_1$ thus $i|_{\overline{\mathfrak{g}_1}^{(k)}}$ is also supersmooth. Let $(g, v_1) \in \iota_k^0(G)_\Lambda \times \overline{\mathfrak{g}_{1\Lambda}}^{(k)}$. Then, we have

$$(g \cdot v_1)^{-1} = v_1^{-1} \cdot g^{-1} = g^{-1} \cdot g \cdot v_1^{-1} \cdot g^{-1} = \underbrace{g^{-1}}_{\in \iota_k^0(G)_\Lambda} \cdot \underbrace{\sigma_\Lambda(g, -v_1)}_{\in \overline{\mathfrak{g}_{1\Lambda}}^{(k)}}$$

and hence, the inversion is supersmooth. Similarly, for the multiplication $\mu: (\iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)})^2 \rightarrow \iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)}$, the restriction $\mu|_{\iota_k^0(G) \times \iota_k^0(G)}$ is the multiplication in $\iota_k^0(G)$ and therefore supersmooth. For the restriction $\mu|_{\overline{\mathfrak{g}_1}^{(k)} \times \overline{\mathfrak{g}_1}^{(k)}}: (\overline{\mathfrak{g}_1}^{(k)})^2 \rightarrow \iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)}$, we use the isomorphism Ψ from Lemma 3.3.5 to obtain

$$v_1 \cdot v'_1 = \underbrace{\exp_\Lambda^{\iota_k^0(G)} \circ \text{pr}_{0,\Lambda}(\Psi_\Lambda^{-1}(v_1 \cdot v'_1))}_{\in \iota_k^0(G)_\Lambda} \cdot \underbrace{\text{pr}_{1,\Lambda}(\Psi_\Lambda^{-1}(v_1 \cdot v'_1))}_{\in \overline{\mathfrak{g}_{1\Lambda}}^{(k)}},$$

where $v_1, v'_1 \in \overline{\mathfrak{g}_{1\Lambda}}^{(k)}$ and $\text{pr}_0: \iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)} \rightarrow \iota_k^0(G)$, $\text{pr}_1: \iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)} \rightarrow \overline{\mathfrak{g}_1}^{(k)}$ are the projections. With Lemma 3.3.5 and Lemma 3.3.6, it follows that $\mu|_{\overline{\mathfrak{g}_1}^{(k)} \times \overline{\mathfrak{g}_1}^{(k)}}$ is supersmooth. If additionally $g, g' \in \iota_k^0(G)_\Lambda$, then we have

$$g \cdot v_1 \cdot g' \cdot v'_1 = g \cdot g' \cdot \sigma_\Lambda(g'^{-1}, v_1) \cdot v'_1,$$

which together with the above shows supersmoothness of μ .

By Remark A.3.3, we have $L(\mathcal{G}_\Lambda) = \overline{\mathfrak{g}_\Lambda}^{(k)}$. Let $\iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)}$ have the Lie supergroup structure induced by Φ . Since Φ_Λ is the restriction of the multiplication, it follows that $L(\Phi_\Lambda)$ is just the addition $\overline{\mathfrak{g}_{0\Lambda}}^{(k)} \times \overline{\mathfrak{g}_{1\Lambda}}^{(k)} \rightarrow \overline{\mathfrak{g}_\Lambda}^{(k)}$, $(v_0, v_1) \mapsto v_0 + v_1$. Thus, $L(\mathcal{G}) = \overline{\mathfrak{g}}^{(k)}$ holds and it follows $\text{sL}(\iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)}) \cong \mathfrak{g}$. Identifying $T_0\mathcal{N}_\Lambda^G \cong \mathfrak{n}_\Lambda^G$, the adjoint action of G on $L(\mathcal{G})_\Lambda$ is by definition $\overline{\text{Ad}_G}: G \times \overline{\mathfrak{g}}^{(k)} \rightarrow \overline{\mathfrak{g}}^{(k)}$ and therefore, we have $\mathcal{H}^{(k)}(\iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)}) \cong (G, \mathfrak{g})$.

Finally, let (H, \mathfrak{h}) together with Ad_H be another super Harish-Chandra pair and $(f_0, f): (G, \mathfrak{g}) \rightarrow (H, \mathfrak{h})$ be a morphism of super Harish-Chandra pairs. Setting $f_1 := f|_{\mathfrak{g}_1}$, we have that f_0 and f_1 satisfy the conditions of Lemma 3.2.11. Thus, $\iota_k^0(f_0) \times \overline{f_1}: \iota_k^0(G) \times \overline{\mathfrak{g}_1}^{(k)} \rightarrow \iota_k^0(H) \times \overline{\mathfrak{h}_1}^{(k)}$ is a morphism of Lie supergroups.

The functoriality of $\mathcal{K}^{(k)}$ is obvious. Under the identifications above, we have $(\iota_k^0(f_0) \times \overline{f_1|_{\mathfrak{g}_1}})_{\mathbb{R}} = f_0$ and $\mathrm{sL}(\iota_k^0(f_0) \times \overline{f_1}) = f$ and therefore $\mathcal{H}^{(k)} \circ \mathcal{K}^{(k)}(f_0, f) = (f_0, f)$. \square

Theorem 3.3.8. *For $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 3$, the functors $\mathcal{H}^{(k)}: \mathbf{LSGrp}^{(k)} \rightarrow \mathbf{SHCP}$ from Lemma 3.3.2 and $\mathcal{K}^{(k)}: \mathbf{SHCP} \rightarrow \mathbf{LSGrp}^{(k)}$ from Proposition 3.3.7 are quasi-inverse to each other, i.e., establish an equivalence of categories*

$$\mathbf{LSGrp}^{(k)} \cong \mathbf{SHCP}.$$

Proof. We have already shown $\mathcal{H}^{(k)} \circ \mathcal{K}^{(k)} \cong \mathrm{id}_{\mathbf{SHCP}}$ in Proposition 3.3.7. Conversely, let $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ be a Lie supergroup with $\mathfrak{g} := \mathrm{sL}(\mathcal{G})$ and let $\Phi^{\mathcal{G}}: \iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1} \rightarrow \mathcal{G}$ be the isomorphism from Corollary 3.2.7. We equip $\iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)}$ with the induced k -Lie supergroup structure. On the other hand, we have $\mathcal{K}^{(k)} \circ \mathcal{H}^{(k)}(\mathcal{G}) = \mathcal{K}^{(k)}(\mathcal{G}_{\mathbb{R}}, \mathrm{sL}(\mathcal{G})) = \iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)}$ and we have seen in Proposition 3.3.7 that the adjoint action is the same in both cases. Therefore, it follows from Lemma 3.2.11 that $\iota_k^0(\mathrm{id}_{\mathcal{G}_{\mathbb{R}}}) \times \overline{\mathfrak{g}_1}^{(k)}: \iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)} \rightarrow \mathcal{K}^{(k)} \circ \mathcal{H}^{(k)}(\mathcal{G})$ is an isomorphism of k -Lie supergroups.

Let \mathcal{H} be another k -Lie supergroup and $f: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of k -Lie supergroups. Setting $f_1 := \mathrm{sL}(f)|_{\mathfrak{g}_1}$, we have $\mathcal{K}^{(k)} \circ \mathcal{H}^{(k)}(f) = \iota_k^0(f_{\mathbb{R}}) \times \overline{f_1}$, which is exactly the morphism $\iota_k^0(f_{\mathbb{R}}) \times \overline{f_1}: \iota_k^0(\mathcal{G}_{\mathbb{R}}) \times \overline{\mathfrak{g}_1}^{(k)} \rightarrow \iota_k^0(\mathcal{H}_{\mathbb{R}}) \times \overline{\mathfrak{h}_1}^{(k)}$ obtained in Lemma 3.2.11. \square

Corollary 3.3.9. *For $n, k \in \mathbb{N} \cup \{\infty\}$, $3 \leq n \leq k$, we have*

$$\mathbf{LSGrp}^{(k)} \cong \mathbf{LSGrp}^{(n)}.$$

The projection functor $\pi_n^k: \mathbf{LSGrp}^{(k)} \rightarrow \mathbf{LSGrp}^{(n)}$ obtained from Lemma 2.3.18 is fully faithful and essentially surjective.

Proof. This follows from Theorem 3.3.8 because the projection functor does not change the associated super Harish-Chandra pair. \square

In the Banach case, this was already stated in [40, Proposition 7.7.1, p.414]. One can easily deal with $\mathbf{LSGrp}^{(0)}$ and $\mathbf{LSGrp}^{(1)}$ by embedding these categories into \mathbf{LSGrp} via Proposition 2.3.14 and Proposition 2.3.17. We leave the special case of $\mathbf{LSGrp}^{(2)}$ for future work. Here, Corollary 2.2.22 does not apply and the Lie algebra of a 2-Lie supergroup is not necessarily related to a Lie superalgebra in the required manner. It should be noted that for $k \geq 3$, any k -Lie supergroup \mathcal{G} is nevertheless completely determined by $\pi_2^k(\mathcal{G})$, because the Lie algebra $L(\mathcal{G}_{\Lambda_2})$ already determines the Lie superalgebra of \mathcal{G} .

3.3.1. Generalizations

Any of the generalizations for supermanifolds discussed in 2.3.2, leads to a generalization of Lie supergroups, and structure results like Proposition 3.2.3 and Lemma 3.2.4 hold even in the most general setting. The biggest problem regarding the

super Harish-Chandra pair construction is the existence of a suitable exponential map, which cannot be assumed if the characteristic of the base ring is not zero. It seems likely that one can instead use an analog of the right and left trivializations introduced in [10, Section 24, p.116 ff.] to obtain a correspondence to super Harish-Chandra pairs in this case (see E.1 for an overview).

3.4. Lie Supergroups with a Smooth Exponential Map

Definition 3.4.1. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. We say a k -Lie supergroup $\mathcal{G} = (\mathcal{G}, \mu, i, e)$ has a smooth exponential map $\exp^{\mathcal{G}}$ if \mathcal{G}_{Λ} has a smooth exponential map $\exp_{\Lambda}^{\mathcal{G}}: \mathcal{T}_e \mathcal{G}_{\Lambda} \rightarrow \mathcal{G}_{\Lambda}$ for every $\Lambda \in \mathbf{Gr}^{(k)}$.

In particular, every Banach k -Lie supergroup has a smooth exponential map. We will usually consider the exponential map as a morphism $\exp^{\mathcal{G}}: L(\mathcal{G}) \rightarrow \mathcal{G}$ via Lemma/Definition 3.1.1.

Lemma 3.4.2. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let \mathcal{G} be a k -Lie supergroup that has a smooth exponential map $\exp^{\mathcal{G}}$. Then $\exp^{\mathcal{G}}: L(\mathcal{G}) \rightarrow \mathcal{G}$ is a well-defined morphism in $\mathbf{Man}^{\mathbf{Gr}^{(k)}}$.

Proof. Let $\Lambda, \Lambda' \in \mathbf{Gr}^{(k)}$ and $\varrho \in \mathrm{Hom}_{\mathbf{Gr}^{(k)}}(\Lambda, \Lambda')$. Then, by naturality of the ordinary exponential map, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{G}_{\Lambda} & \xrightarrow{\mathcal{G}_{\varrho}} & \mathcal{G}_{\Lambda'} \\ \exp_{\Lambda}^{\mathcal{G}} \uparrow & & \uparrow \exp_{\Lambda'}^{\mathcal{G}} \\ L(\mathcal{G})_{\Lambda} & \xrightarrow{L(\mathcal{G})_{\varrho}} & L(\mathcal{G})_{\Lambda'}. \end{array}$$

Therefore, $\exp^{\mathcal{G}}$ is a natural transformation. □

If \mathcal{G} has a smooth exponential map $\exp^{\mathcal{G}}$ then, by uniqueness, the restriction $\exp_{\Lambda}^{\mathcal{G}}|_{L(\mathcal{G})_{\Lambda+}}$ corresponds to the exponential map defined in Proposition 3.2.6, which is why we use the same notation in both cases.

In [42] a Lie supergroup \mathcal{G} is defined as a Lie supergroup in our sense such that $\mathcal{G}_{\mathbb{R}}$ has a smooth exponential map and the model space is Mackey complete (see [42, Section 4 and Definition 4.1, p.813]). It is then shown in [42, Proposition 4.3.2, p.815] that in this case \mathcal{G} automatically has a smooth exponential map as defined above.

Lemma 3.4.3. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, let (\mathcal{G}, μ, i, e) be a Banach k -Lie supergroup and let $\mathfrak{g} \in \mathbf{SVec}_{lc}$ such that $\bar{\mathfrak{g}}^{(k)} = L(\mathcal{G})$. Then an open subfunctor $\mathcal{U} \subseteq \bar{\mathfrak{g}}^{(k)}$ exists such that the restriction $\exp^{\mathcal{G}}|_{\mathcal{U}}: \mathcal{U} \rightarrow \mathcal{G}$ of the exponential map of \mathcal{G} is a chart of \mathcal{G} .

Proof. Let $U \subseteq \mathfrak{g}_0$ be open such that $\exp_{\mathbb{R}}^{\mathcal{G}}|_U: U \rightarrow \exp_{\mathbb{R}}^{\mathcal{G}}(U)$ is a diffeomorphism and set $\mathcal{U} := \overline{\mathfrak{g}}|_U$. Then $\exp_0 := \iota_k^0(\exp_{\mathbb{R}}^{\mathcal{G}}): \iota_k^0(U) \rightarrow \iota_k^0(\mathcal{G}_{\mathbb{R}}) \subseteq \mathcal{G}$ is a chart. It follows from Corollary 3.2.7 that $\widetilde{\exp} := \mu \circ (\exp_0 \times \exp_1): \mathcal{U} \rightarrow \mathcal{G}$ is a chart and by functoriality, we have $\exp_{\Lambda}^{\mathcal{G}}(\mathcal{U}_{\Lambda}) \subseteq \widetilde{\exp}_{\Lambda}(\mathcal{U}_{\Lambda})$ for all $\Lambda \in \mathbf{Gr}$. Next, we check that $\exp_{\Lambda}^{\mathcal{G}}|_{\mathcal{U}_{\Lambda}}: \mathcal{U}_{\Lambda} \rightarrow \widetilde{\exp}_{\Lambda}(\mathcal{U}_{\Lambda})$ is bijective. There exists an open zero-neighborhood $U' \subseteq \mathcal{U}_{\Lambda_1}$ such that $\exp_{\Lambda_1}^{\mathcal{G}}|_{U'}$ is a chart of \mathcal{G}_{Λ_1} . After shrinking, we may assume $U' = U \times U_1 \subseteq \mathfrak{g}_0 \times \lambda_1 \mathfrak{g}_1$. In a chart, it follows from Proposition 2.2.7 and Lemma 2.2.5 that $\exp_{\Lambda_n}^{\mathcal{G}}(x, \bullet)$ is a morphism of n -multilinear bundles for every $x \in U$. In particular, $\exp_{\Lambda_1}^{\mathcal{G}}(x, \bullet)$ is a linear map to the fiber $\mathcal{G}_{\varepsilon_{\Lambda_1}}^{-1}(\{x\})$. It follows that $\exp_{\Lambda_1}^{\mathcal{G}}|_{\mathcal{U}_{\Lambda_1}}$ is bijective, which implies that $\exp_{\Lambda}^{\mathcal{G}}|_{\mathcal{U}_{\Lambda}}$ is bijective by Theorem B.1.2.

It now suffices to check that $(\exp^{\mathcal{G}})^{-1} \circ \widetilde{\exp}$ is supersmooth. For $\Lambda \in \mathbf{Gr}^{(k)}$ let $X \in \iota_k^0(U)_{\Lambda}$ and $Y \in \overline{\mathfrak{g}}_{1\Lambda}^{(k)}$. Then, we have that $(\exp_0)_{\Lambda}(X) = \exp_{\Lambda}^{\mathcal{G}}(X)$ and $(\exp_1)_{\Lambda}(Y) = \exp_{\Lambda}^{\mathcal{G}}(Y)$. Therefore, for X, Y close enough to zero, $(\exp_{\Lambda}^{\mathcal{G}})^{-1} \circ \widetilde{\exp}_{\Lambda}(X, Y)$ is given by the BCH series with respect to the Lie bracket $[\cdot, \cdot]_{\Lambda}$ of $L(\mathcal{G})_{\Lambda}$. After shrinking U again, we may assume that this is the case for $X \in U$, $Y = 0$. Thus, for $\tilde{X} \in \overline{\mathfrak{g}}_{0\Lambda}^{(k)}$ and $\tilde{Y} \in \overline{\mathfrak{g}}_{1\Lambda}^{(k)}$, we have

$$\partial_{(\tilde{X}, \tilde{Y})}((\exp^{\mathcal{G}})_{\Lambda}^{-1} \circ \widetilde{\exp}_{\Lambda})(X, 0) = \tilde{X} + \tilde{Y} + \sum_{i=1}^{\infty} P_i \cdot \underbrace{[X, [X, \dots, [X, \tilde{Y}]_{\Lambda} \dots]_{\Lambda}}_{i \text{ times}}_{\Lambda},$$

where P_i is a rational number coming from the factors of the BCH series. This expression is clearly Λ_0 -linear and thus supersmoothness follows from Corollary 2.2.9. \square

Lemma 3.4.4. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let \mathcal{G} be a k -Lie supergroup with a smooth exponential map $\exp^{\mathcal{G}}$. If an open subfunctor $\mathcal{U} \subseteq L(\mathcal{G})$ with $0 \in \mathcal{U}_{\mathbb{R}}$ exists such that $\exp^{\mathcal{G}}|_{\mathcal{U}}$ is supersmooth, then $\exp^{\mathcal{G}}$ is supersmooth.*

Proof. For $n \in \mathbb{N}$, we define an open subfunctor $n\mathcal{U} \subseteq L(\mathcal{G})$ by $(n\mathcal{U})_{\Lambda} := n \cdot \mathcal{U}_{\Lambda}$. We have $L(\mathcal{G})_{\Lambda} = \bigcup_{n \in \mathbb{N}} n\mathcal{U}_{\Lambda}$ for all $\Lambda \in \mathbf{Gr}$ and therefore it suffices to show supersmoothness of $\exp^{\mathcal{G}}|_{n\mathcal{U}}$. Because multiplication with scalars is supersmooth, so is $\exp_{\Lambda}^{\mathcal{G}}|_{n\mathcal{U}_{\Lambda}} = (\exp_{\Lambda}^{\mathcal{G}}|_{\mathcal{U}_{\Lambda}}(\frac{1}{n}\bullet))^n$ (compare [23]). \square

Corollary 3.4.5. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let \mathcal{G} be a Banach k -Lie supergroup. Then \mathcal{G} has a supersmooth exponential map $\exp^{\mathcal{G}}: L(\mathcal{G}) \rightarrow \mathcal{G}$.*

This was already stated without proof in [40, Proposition 7.3.1, p.413]. It is possible to prove automatic supersmoothness of the exponential map beyond the Banach case. We leave this for future work.

3.5. Examples

One classical Lie supergroup comes from the unit group of a superalgebra. In our setting this was already mentioned in [40, Example 7.1.1, p.411] (see also [46, 4.7, p.27f.]), but since it is very instructive let us discuss a slightly more general version here.

Definition 3.5.1. A *continuous inverse algebra* is a locally convex unital associative \mathbb{R} -algebra A such that the group of units A^\times is open in A and the inversion $j: A^\times \rightarrow A$ is continuous.

Proposition 3.5.2 ([18, Proposition 2.2, p.15]). *If A is a continuous inverse algebra, then A^\times considered as an open submanifold of A is a Lie group.*

Let A be a topological \mathbb{R} -superalgebra such that A_0 is a continuous inverse algebra with the unit element 1_A . Since the multiplication $\mu: A \times A \rightarrow A$ is a continuous even bilinear map, it follows from Corollary 2.2.22 that $\bar{\mu}: \bar{A} \rightarrow \bar{A}$ is a supersmooth morphism that turns \bar{A}_Λ into a topological algebra with the unit element $\bar{A}_{\eta_\Lambda}(1_A) = 1_A$ for all $\Lambda \in \mathbf{Gr}$.

Proposition 3.5.3. *Let A be a topological \mathbb{R} -superalgebra such that A_0 is a continuous inverse algebra with the unit element 1_A . Define $\mathcal{A} := \bar{A}$ and $\mathcal{A}^\times := \bar{A}|_{A_0^\times}$. Then, for every $\Lambda \in \mathbf{Gr}$, we have $(\mathcal{A}_\Lambda)^\times = \mathcal{A}_\Lambda^\times$ and the inversion $i: \mathcal{A}^\times \rightarrow \mathcal{A}$ is a supersmooth morphism. In particular, \mathcal{A}^\times is a Lie supergroup.*

Proof. Let μ be the multiplication of A . We have

$$\bar{\mu}_{\Lambda_n}((\lambda_I X_I, \lambda_I Y_I)_{I \in \mathcal{P}^n}) = \sum_{I, J \in \mathcal{P}^n} \lambda_I \lambda_J X_I Y_J = \sum_{I, J \in \mathcal{P}^n, I \cap J = \emptyset} \lambda_I \lambda_J X_I Y_J,$$

where $X_I, Y_I \in A_{\bar{I}}$. Thus, $\bar{\mu}$ has the skeleton $(\mu_0, \mu_1, \mu_2, 0, \dots)$ with

$$\begin{aligned} \mu_0: A_0^2 &\rightarrow A_0, & (X_0, Y_0) &\mapsto X_0 Y_0, \\ \mu_1: A_0^2 &\rightarrow \mathcal{A}_1^1(A_1^2; A_1), & \mu_1(X_0, Y_0)(X_1, Y_1) &= X_0 Y_1 + X_1 Y_0, \\ \mu_2: A_0^2 &\rightarrow \mathcal{A}_1^2(A_1^2; A_0), & \mu_2(X_0, Y_0)((X_1, Y_1), (X_2, Y_2)) &= X_1 Y_2 + X_2 Y_1. \end{aligned}$$

We construct the inversion inductively. Let $j: A_0^\times \rightarrow A_0$ be the inversion of A_0^\times . We define $i^{(0)} := j$. For $k \in \mathbb{N}_0$ assume $i^{(k)}$ has the skeleton (i_0, \dots, i_k) . We set $\tilde{i}^{(k+1)} := (i_0, \dots, i_k, 0)$. Then, we have

$$\bar{\mu}^{(k+1)} \circ (\text{id}_{\mathcal{A}^{(k+1)}}, \tilde{i}^{(k+1)}) = (c_{1_A}, 0, \dots, 0, g_{k+1})$$

for some $g_{k+1}: A_0 \rightarrow \mathcal{A}^{k+1}(A_1; A_k)$. Define $i_{k+1} := \mu(i_0(\cdot), -g_{k+1}(\cdot)): A_0 \rightarrow \mathcal{A}^{k+1}(A_1; A_k)$ and let $i^{(k+1)} = (i_0, \dots, i_k, i_{k+1})$. With this, we calculate $\bar{\mu}^{(k+1)} \circ (\text{id}_{\mathcal{A}^{(k+1)}}, i^{(k+1)}) = (c_{1_A}, 0, \dots, 0)$. Then $i := (i_0, i_1, \dots): \mathcal{A}^\times \rightarrow \mathcal{A}$ defined in this way is the supersmooth inverse.

We have shown that every $X \in \mathcal{A}_\Lambda^\times$ is invertible. Conversely, if $X \in \mathcal{A}_\Lambda$ is invertible, then so is $\mathcal{A}_{\varepsilon_\Lambda}(X) \in A_0$, which implies $X \in \mathcal{A}_\Lambda^\times$. \square

The Lie algebra of \mathcal{A} is then given by $\overline{[X, Y]}_\Lambda = \bar{\mu}_\Lambda(X, Y) - \bar{\mu}_\Lambda(Y, X)$ for every $\Lambda \in \mathbf{Gr}$ and $X, Y \in \mathcal{A}_\Lambda$ and the Lie superbracket $[\cdot, \cdot]$ of \mathcal{A} is

$$[X_0 + X_1, Y_0 + Y_1] = (X_0 Y_0 - Y_0 X_0) + (X_0 Y_1 - Y_1 X_0 + X_1 Y_0 - Y_0 X_1) + (X_1 Y_1 + Y_1 X_1)$$

for $X_0, Y_0 \in A_0$ and $X_1, Y_1 \in A_1$. In other words, $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$ for homogeneous elements $X, Y \in A$ and p as in Definition 1.5.2.

3.5.1. The General Linear Lie Supergroup

As one application of Proposition 3.5.3, we get the classical *general linear supergroup*. Let $p, q \in \mathbb{N}_0$. We define a super vector space $M_{p|q}(\mathbb{R})$ by setting

$$\begin{aligned} M_{p|q}(\mathbb{R})_0 &:= \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \in \mathbb{R}^{p+q \times p+q} : A \in \mathbb{R}^{p \times p}, D \in \mathbb{R}^{q \times q} \right\} \quad \text{and} \\ M_{p|q}(\mathbb{R})_1 &:= \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \in \mathbb{R}^{p+q \times p+q} : B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{q \times p} \right\}. \end{aligned}$$

The even matrices $M_{p|q}(\mathbb{R})_0$, resp. the odd matrices $M_{p|q}(\mathbb{R})_1$, can be understood as even, resp. odd, endomorphisms $\mathbb{R}^{p|q} \rightarrow \mathbb{R}^{p|q}$. Clearly, $M_{p|q}(\mathbb{R})$ together with the normal matrix multiplication is a topological \mathbb{R} -superalgebra and $(M_{p|q}(\mathbb{R})_0)^\times = \{M \in M_{p|q}(\mathbb{R})_0 : M \text{ invertible}\}$ is a continuous inverse algebra with unity. Thus, $\mathcal{GL}_{p|q}(\mathbb{R}) := \overline{M_{p|q}(\mathbb{R})}|_{(M_{p|q}(\mathbb{R})_0)^\times}$ is a Lie supergroup. Elements $M \in \mathcal{GL}_{p|q}(\mathbb{R})_{\Lambda_n}$ can be written as

$$M = \left(\begin{array}{c|c} A_0 & 0 \\ \hline 0 & D_0 \end{array} \right) + \sum_{I \in \mathcal{P}_{0,+}^n} \lambda_I \left(\begin{array}{c|c} A_I & 0 \\ \hline 0 & D_I \end{array} \right) + \sum_{J \in \mathcal{P}_1^n} \lambda_J \left(\begin{array}{c|c} 0 & B_J \\ \hline C_J & 0 \end{array} \right),$$

where A_0 and D_0 are invertible. The multiplication is the ordinary multiplication in $M_{p|q}(\mathbb{R}) \otimes \Lambda_n$ and the Lie superalgebra is $\mathfrak{gl}_{p|q}(\mathbb{R}) := M_{p|q}(\mathbb{R})$ with the Lie superbracket given as $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$ for homogeneous elements $X, Y \in \mathfrak{gl}_{p|q}(\mathbb{R})$ and p as in Definition 1.5.2.

More generally, let A be a topological supercommutative \mathbb{R} -superalgebra such that A_0 is a continuous inverse algebra. One can define $M_{p|q}(A)$ analogously to above. Note that a matrix $M \in M_{p|q}(A)_i$ now has the form

$$M = \left(\begin{array}{c|c} S_1 & S_2 \\ \hline S_3 & S_4 \end{array} \right),$$

where $i \in \{0, 1\}$, $S_1 \in A_i^{p \times p}$, $S_2 \in A_{i+1}^{p \times q}$, $S_3 \in A_{i+1}^{q \times p}$ and $S_4 \in A_i^{q \times q}$ (see [14, Section 1.4, p.10ff.]). Interestingly, $M \in M_{p|q}(A)_0$ is invertible if and only if S_1 and S_4 are invertible. Indeed, we have

$$\left(\begin{array}{c|c} S_1^{-1} & 0 \\ \hline 0 & S_4^{-1} \end{array} \right) \cdot \left(\begin{array}{c|c} S_1 & S_2 \\ \hline S_3 & S_4 \end{array} \right) = \text{id} + \left(\begin{array}{c|c} 0 & S_1^{-1}S_2 \\ \hline S_4^{-1}S_3 & 0 \end{array} \right) =: \text{id} + N,$$

where N is nilpotent since A is supercommutative. The inverse of $\text{id} + N$ is thus given by the (finite) Neumann series. It is elementary to check that $\mathcal{GL}_{p|q}(A) := \overline{M_{p|q}(A)}|_{(M_{p|q}(A)_0)^\times}$ is again a Lie supergroup.

Typical Lie sub-supergroups like the special linear Lie supergroup (see for example [14, Examples 11.1.13, p.206]) can be defined using the same approach. However, if they do not arise from open subsets of a super vector space, the super-smooth structure is more complicated. If A is a Banach superalgebra, a restriction of the exponential map is a chart. Otherwise, one can use Corollary 3.2.7.

4. Superdiffeomorphisms

In this chapter, we discuss the supergroup of superdiffeomorphisms. For this, we first turn to vector fields and automorphisms of supermanifolds.

4.1. Spaces of Sections of Super Vector Bundles

While only even and odd vector fields of supermanifolds are necessary to deal with automorphisms and superdiffeomorphisms, we consider also general sections of super vector bundles. For an arbitrary super vector bundle $\pi: \mathcal{E} \rightarrow \mathcal{M}$, we turn the space of sections $\Gamma(\mathcal{E})$ into a locally convex space. If \mathcal{E} is a Banach supermanifold and $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, we introduce another topology on $\Gamma(\mathcal{E})$, which is more suitable for the Lie group structures we are interested in. We also define respective spaces of compactly supported sections $\Gamma_c(\mathcal{E})$.

4.1.1. Spaces of Supersmooth Morphisms

Recall Definition 2.2.12. For $n \in \mathbb{N}_0$, $E, F \in \mathbf{SVec}_{lc}$ and $U \subseteq E_0$ open, we equip $\mathcal{C}^\infty(U, \mathcal{A}lt^n(E_1; F_{\bar{n}}))$ with the Hausdorff locally convex topology that turns

$$\mathcal{C}^\infty(U, \mathcal{A}lt^n(E_1; F_{\bar{n}})) \rightarrow \mathcal{C}^\infty(U \times E_1^n, F_{\bar{n}}), \quad f \mapsto f^\wedge$$

into an embedding and denote this space by $\mathcal{C}^\infty(U, \mathcal{A}lt^n(E_1; F_{\bar{n}}))_c$. For more details on the topology of $\mathcal{C}^\infty(U \times E_1^n, F_{\bar{n}})$ see A.2. If E and F are Banach super vector spaces, we denote by $\mathcal{A}lt^n(E_1; F_{\bar{n}})_b$ the space $\mathcal{A}lt^n(E_1; F_{\bar{n}})$ together with the topology of bounded convergence, which is a Banach space. If additionally E_0 is finite-dimensional, we have that

$$\mathcal{C}^\infty(U, \mathcal{A}lt^n(E_1; F_{\bar{n}})) \rightarrow \mathcal{C}^\infty(U, \mathcal{A}lt^n(E_1; F_{\bar{n}})_b), \quad f \mapsto f$$

is an isomorphism by Lemma A.2.12. We write $\mathcal{C}^\infty(U, \mathcal{A}lt^n(E_1; F_{\bar{n}}))_b$ for the space $\mathcal{C}^\infty(U, \mathcal{A}lt^n(E_1; F_{\bar{n}}))$ equipped with the induced topology.

Definition 4.1.1. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E, F \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor. We turn $\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})$ into a Hausdorff locally convex vector space such that

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)}) \cong \prod_{n=0}^k \mathcal{C}^\infty(\mathcal{U}_{\mathbb{R}}, \mathcal{A}lt^n(E_1; F_{\bar{n}}))_c,$$

and denote this topological vector space by $\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})_c$. Likewise, if E_0 is finite-dimensional and E_1, F are Banach spaces, we turn $\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})$ into a Hausdorff

locally convex vector space such that

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)}) \cong \prod_{n=0}^k \mathcal{C}^\infty(\mathcal{U}_{\mathbb{R}}, \mathcal{A}l t^n(E_1; F_{\overline{n}}))_b,$$

and denote this topological vector space by $\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})_b$.

Lemma 4.1.2. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E, F \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor. If $\mathcal{V} \subseteq \mathcal{U}$ is an open subfunctor, then the restriction*

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})_c \rightarrow \mathcal{SC}^\infty(\mathcal{V}, \overline{F}^{(k)})_c, \quad f \mapsto f|_{\mathcal{V}}$$

is continuous and linear. If E_0 is finite dimensional and E_1, F are Banach spaces, then

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})_b \rightarrow \mathcal{SC}^\infty(\mathcal{V}, \overline{F}^{(k)})_b, \quad f \mapsto f|_{\mathcal{V}}$$

is also continuous and linear.

Proof. With Lemma A.1.2, both claims follow from Lemma A.2.4. \square

Lemma 4.1.3. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E, F, H \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor. If $f: \mathcal{U} \times \overline{F}^{(k)} \rightarrow \overline{H}^{(k)}$ is supersmooth, then*

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})_c \rightarrow \mathcal{SC}^\infty(\mathcal{U}, \overline{H}^{(k)})_c, \quad \gamma \mapsto f \circ (\text{id}_{\mathcal{U}}, \gamma)$$

is smooth. If E_0 is finite-dimensional and E_1, F, H are Banach spaces, then

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)})_b \rightarrow \mathcal{SC}^\infty(\mathcal{U}, \overline{H}^{(k)})_b, \quad \gamma \mapsto f \circ (\text{id}_{\mathcal{U}}, \gamma)$$

is also smooth.

Proof. Using the concatenation formula for skeletons (2.2) and Lemma A.1.2, this follows from and Proposition A.2.1 in the first case and Lemma A.2.14 and Proposition A.2.1 in the second case. \square

Lemma 4.1.4. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E, F, H \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$, $\mathcal{V} \subseteq \overline{F}^{(k)}$ be open subfunctors. If $f: \mathcal{U} \rightarrow \mathcal{V}$ is supersmooth, then the “pullback”*

$$\mathcal{SC}^\infty(\mathcal{V}, \overline{H}^{(k)})_c \rightarrow \mathcal{SC}^\infty(\mathcal{U}, \overline{H}^{(k)})_c, \quad \gamma \mapsto \gamma \circ f$$

is a continuous linear map. If E_0 is finite-dimensional and E_1, F, H are Banach spaces, then

$$\mathcal{SC}^\infty(\mathcal{V}, \overline{H}^{(k)})_b \rightarrow \mathcal{SC}^\infty(\mathcal{U}, \overline{H}^{(k)})_b, \quad \gamma \mapsto \gamma \circ f$$

is also continuous and linear.

Proof. It follows from the concatenation formula for skeletons (2.2) that both maps are linear. Using Lemma A.1.2, continuity follows from Lemma A.2.2 in the first case and Lemma A.2.14 and Lemma A.2.2 in the second case. \square

4.1.2. The vector space structure of the space of sections

Definition 4.1.5. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\pi: \mathcal{E} \rightarrow \mathcal{M}$ be a k -super vector bundle. A *section* of \mathcal{E} is a supersmooth morphism $\sigma: \mathcal{M} \rightarrow \mathcal{E}$ such that $\pi \circ \sigma = \text{id}_{\mathcal{M}}$. We denote the set of sections by $\Gamma(\mathcal{E})$.

In the situation of the definition, let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$ and let \mathcal{E} have the typical fiber $F \in \mathbf{SVec}_{lc}$. Let further $\{\Psi^\alpha: \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E}: \alpha \in A\}$ be a bundle atlas of \mathcal{E} and $\{\phi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ the corresponding atlas of \mathcal{M} . Any section $\sigma: \mathcal{M} \rightarrow \mathcal{E}$ has then the local form $(\Psi^\alpha)^{-1} \circ \sigma \circ \phi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{U}^\alpha \times \overline{F}^{(k)}$. Because $\pi \circ \sigma = \text{id}_{\mathcal{M}}$, we see that the first component of this morphism is $\text{id}_{\mathcal{U}^\alpha}$. Thus, we define

$$\sigma^\alpha := \text{pr}_{\overline{F}^{(k)}} \circ (\Psi^\alpha)^{-1} \circ \sigma \circ \phi^\alpha: \mathcal{U}^\alpha \rightarrow \overline{F}^{(k)}.$$

We arrive at the injective map

$$\Theta: \Gamma(\mathcal{E}) \rightarrow \prod_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)}), \quad \sigma \mapsto \sigma^\alpha. \quad (4.1)$$

Let the change of charts $\Psi^{\alpha\beta} = (\phi^{\alpha\beta}, \psi^{\alpha\beta})$ be as in Definition 2.5.4. Calculating the change of charts for σ , we get

$$\psi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, \sigma^\alpha)|_{\mathcal{U}^{\alpha\beta}} = \sigma^\beta \circ \phi^{\alpha\beta}.$$

On the other hand, by Proposition 2.3.5 a family of morphisms with this property uniquely defines a section. It follows

$$\begin{aligned} \text{im}(\Theta) = \left\{ \sigma^\alpha \in \prod_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)}): \alpha, \beta \in A, \right. \\ \left. (\psi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, \sigma^\alpha)) = (\sigma^\beta \circ \phi^{\alpha\beta}) \text{ on } \mathcal{U}^{\alpha\beta} \right\}. \end{aligned} \quad (4.2)$$

The right-hand side is clearly linear in $(\sigma_n^\beta)_n \in \prod_n C^\infty(\mathcal{U}_{\mathbb{R}}^\beta, \text{Alt}^n(E_1; F_{\overline{n}}))$. For the left-hand side, we use Lemma 2.5.3 to calculate

$$\begin{aligned} (\psi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, \sigma^\alpha))_n = n! \cdot \mathfrak{A}^n \left(\sum_{n \geq l \text{ odd}} \frac{1}{(n-l)!l!} \psi_{n-l+1}^{\alpha\beta}(\text{id}_{\mathcal{U}_{\mathbb{R}}^\alpha}, 0)(\sigma_l^\alpha, \bullet) \right. \\ \left. + \sum_{n \geq l \text{ even}} \frac{1}{(n-l)!l!} \psi_{n-l}^{\alpha\beta}(\text{id}_{\mathcal{U}_{\mathbb{R}}^\alpha}, \sigma_l^\alpha)(\bullet) \right) \end{aligned} \quad (4.3)$$

on $\mathcal{U}_{\mathbb{R}}^{\alpha\beta}$. This expression is also linear in $(\sigma_n^\alpha)_n \in \prod_n C^\infty(\mathcal{U}_{\mathbb{R}}^\alpha, \text{Alt}^n(E_1; F_{\overline{n}}))$ and therefore $\text{im}(\Theta)$ is a subspace of $\prod_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})$.

Lemma 4.1.6. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let $\pi: \mathcal{E} \rightarrow \mathcal{M}$ be a k -super vector bundle with typical fiber $F \in \mathbf{SVec}_{lc}$ over a k -supermanifold \mathcal{M} modelled on $E \in \mathbf{SVec}_{lc}$.

For a bundle atlas $\mathcal{A} := \{\Psi^\alpha: \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E}: \alpha \in A\}$ the image of the injection

$$\Theta_c: \Gamma(\mathcal{E}) \rightarrow \prod_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c, \quad \sigma \mapsto \sigma^\alpha,$$

is a closed vector subspace. If E_0 is finite-dimensional and E_1, F are Banach spaces then the same holds for

$$\Theta_b: \Gamma(\mathcal{E}) \rightarrow \prod_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_b, \quad \sigma \mapsto \sigma^\alpha.$$

In both cases, the topology induced on $\Gamma(\mathcal{E})$ only depend on the equivalence class of \mathcal{A} .

Proof. The proof works essentially like the one for vector bundles (see [22, Lemma 3.7 and Lemma 3.9, p.10 f.]). By Lemma 4.1.2 and Lemma 4.1.3, the map

$$\sigma^\alpha \mapsto \psi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^{\alpha\beta}}, \sigma^\alpha|_{\mathcal{U}^{\alpha\beta}})$$

is smooth. Likewise, it follows Lemma 4.1.2 and Lemma 4.1.4 that

$$\sigma^\beta \mapsto \sigma^\beta|_{\mathcal{U}^{\beta\alpha}} \circ \phi^{\alpha\beta}$$

is continuous and linear. This implies that $\text{im}(\Theta_c)$, resp. $\text{im}(\Theta_b)$, is closed because equalizers of continuous maps are closed in the case of Hausdorff spaces.

Let $\mathcal{B} = \{\Psi^\beta: \beta \in B\} \in [\mathcal{A}]$ be another bundle atlas. Because $\mathcal{B} \cup \mathcal{A}$ is an equivalent atlas, it suffices to assume $\mathcal{A} \subseteq \mathcal{B}$. For $\beta \in B$ define

$$\theta_\beta: \Gamma(\mathcal{E}) \rightarrow \mathcal{SC}^\infty(\mathcal{U}^\beta, \overline{F}^{(k)})_c, \quad \sigma \mapsto \sigma^\beta.$$

The initial topology with respect to $(\theta_\alpha)_{\alpha \in A}$ (the topology induced by Θ_c) is clearly coarser than the one with respect to $(\theta_\beta)_{\beta \in B}$. It remains to be seen that θ_β is continuous for every $\beta \in B$. We have that $\{\mathcal{U}_\mathbb{R}^{\beta\alpha}: \alpha \in A\}$ is an open cover of $\mathcal{U}_\mathbb{R}^\beta$. By Lemma A.2.4 it suffices to see that the map $\sigma \mapsto \theta_\beta(\sigma)|_{\mathcal{U}_\mathbb{R}^{\beta\alpha}}$ is continuous for all $\alpha \in A$. But, we have

$$\theta_\beta(\sigma)|_{\mathcal{U}_\mathbb{R}^{\beta\alpha}} = (\psi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^{\alpha\beta}}, \theta_\alpha(\sigma)|_{\mathcal{U}^{\alpha\beta}})) \circ \phi^{\beta\alpha}$$

and therefore θ_β is continuous by the first part of the proof. The same arguments work for $\Gamma(\mathcal{E})_b$. \square

In the situation of the lemma, we denote by $\Gamma(\mathcal{E})_c$, resp. $\Gamma(\mathcal{E})_b$, the vector space $\Gamma(\mathcal{E})$ together with the topology induced by Θ_c , resp. Θ_b .

Definition 4.1.7. Let $k \in \mathbb{N}_0$ and $\pi: \mathcal{E} \rightarrow \mathcal{M}$ be a k -super vector bundle with typical fiber $F \in \mathbf{SVec}_{lc}$ with bundle atlas $\{\Psi^\alpha: \mathcal{U}_\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E}: \alpha \in A\}$ and corresponding atlas $\{\phi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ of \mathcal{M} . For an open sub-supermanifold

\mathcal{U} of \mathcal{M} , we define the *restricted bundle* $\mathcal{E}|_{\mathcal{U}} := \mathcal{E}|_{(\pi|_{\mathbb{R}})^{-1}(\mathcal{U}_{\mathbb{R}})}$. The atlas

$$\{\Psi^\alpha|_{(\phi^\alpha)^{-1}(\phi^\alpha(\mathcal{U}^\alpha) \cap \mathcal{U}) \times \overline{F}^{(k)}} : \alpha \in A\}$$

turns $\mathcal{E}|_{\mathcal{U}}$ into a k -super vector bundle over \mathcal{U} . For a section $\sigma \in \Gamma(\mathcal{E})$, one easily sees that the restriction $\sigma|_{\mathcal{U}}$ defines a section $\sigma|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{E}|_{\mathcal{U}}$.

Definition 4.1.8. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let $\pi : \mathcal{E} \rightarrow \mathcal{M}$ be a k -super vector bundle. Let $\sigma \in \Gamma(\mathcal{E})$. We define $\text{supp}(\sigma)$, the *support* of σ , as the smallest closed set $K \subseteq \mathcal{M}_{\mathbb{R}}$ such that $\sigma|_{\mathcal{M} \setminus K} = 0$. The *compactly supported sections* of \mathcal{E} are defined as

$$\Gamma_c(\mathcal{E}) := \{\sigma \in \Gamma(\mathcal{E}) : \text{supp}(\sigma) \text{ compact}\}.$$

Clearly, $\Gamma_K(\mathcal{E})$ and $\Gamma_c(\mathcal{E})$ are subspaces of $\Gamma(\mathcal{E})$.

Lemma 4.1.9. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let $\pi : \mathcal{E} \rightarrow \mathcal{M}$ be a k -super vector bundle with typical fiber F over a σ -compact supermanifold \mathcal{M} modelled on $E \in \mathbf{SVec}_{lc}$. Let $\mathcal{A} := \{\Psi^\alpha : \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E} : \alpha \in A\}$ be an atlas of \mathcal{E} such that A is countable and such that for the corresponding atlas $\{\phi^\alpha : \mathcal{U}^\alpha \rightarrow \mathcal{M} : \alpha \in A\}$ of \mathcal{M} it holds that $(\phi_{\mathbb{R}}^\alpha(\mathcal{U}_{\mathbb{R}}^\alpha))_{\alpha \in A}$ is a locally finite covering of $\mathcal{M}_{\mathbb{R}}$ by relatively compact sets. Then

$$\Omega_c^A : \Gamma_c(\mathcal{E}) \rightarrow \bigoplus_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c, \quad \sigma \mapsto \sigma^\alpha$$

is a well-defined injective map with closed image. If additionally E_1, F are Banach spaces then the same holds for

$$\Omega_b^A : \Gamma_c(\mathcal{E}) \rightarrow \bigoplus_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_b, \quad \sigma \mapsto \sigma^\alpha.$$

In both cases all equivalent bundle atlases that satisfy the above conditions induce the same topology on $\Gamma_c(\mathcal{E})$.

Proof. For $\sigma \in \Gamma_c(\mathcal{E})$ it follows from (4.2) that almost all σ^α , $\alpha \in A$ are zero in this situation. Therefore Ω_c^A , resp. Ω_b^A , is well-defined and obviously injective. If \mathcal{B} is a bundle atlas equivalent to \mathcal{A} that satisfies the conditions of the lemma, then so is $\mathcal{B} \cup \mathcal{A}$. Thus, we only need to show that the topology induced on $\Gamma_c(\mathcal{E})$ is the same if $\mathcal{A} \subseteq \mathcal{B}$. Let $\mathcal{B} := \{\Psi^\beta : \mathcal{U}^\beta \times \overline{F}^{(k)} \rightarrow \mathcal{E} : \beta \in B\}$ be such an atlas with corresponding atlas $\{\phi^\beta : \mathcal{U}^\beta \rightarrow \mathcal{M} : \beta \in B\}$. If we can find smooth maps Φ and Σ such that the following diagram commutes, the proof is finished

$$\begin{array}{ccc} & \Gamma_c(\mathcal{E}) & \\ \Omega^{\mathcal{A}} \swarrow & & \searrow \Omega^{\mathcal{B}} \\ \bigoplus_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c & \xrightleftharpoons[\Phi]{\Sigma} & \bigoplus_{\beta \in B} \mathcal{SC}^\infty(\mathcal{U}^\beta, \overline{F}^{(k)})_c. \end{array}$$

Defining

$$\Phi: \bigoplus_{\beta \in B} \mathcal{SC}^\infty(\mathcal{U}^\beta, \overline{F}^{(k)})_c \rightarrow \bigoplus_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c, \quad (\sigma^\beta)_{\beta \in B} \mapsto (\sigma^\alpha)_{\alpha \in A}$$

clearly leads to a commutative diagram and it is easy to see with Lemma A.1.4 that this map is smooth. Let now $(h'_\alpha)_{\alpha \in A}$ be a smooth partition of unity on $\mathcal{M}_\mathbb{R}$ subordinate to $(\phi_\mathbb{R}^\alpha(\mathcal{U}_\mathbb{R}^\alpha))_{\alpha \in A}$ and set $h_\alpha := h'_\alpha \circ \phi_\mathbb{R}^\alpha$. For $\beta \in B$ define the finite set $A_\beta := \{\alpha \in A: \phi_\mathbb{R}^\alpha(\mathcal{U}_\mathbb{R}^\alpha) \cap \phi_\mathbb{R}^\beta(\mathcal{U}_\mathbb{R}^\beta) \neq \emptyset\}$ and let

$$\begin{aligned} \Sigma_\beta: \bigoplus_{\alpha \in A_\beta} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c &\rightarrow \mathcal{SC}^\infty(\mathcal{U}^\beta, \overline{F}^{(k)})_c, \\ (\sigma^\alpha)_{\alpha \in A_\beta} &\mapsto \sum_{\alpha \in A_\beta} (\psi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, h_\alpha \cdot \sigma^\alpha) \circ \phi^{\beta\alpha}|_{\mathcal{U}_\mathbb{R}^{\beta\alpha}}), \end{aligned}$$

where the tilde indicates that the respective skeletons are continued by zero from $\mathcal{U}_\mathbb{R}^{\beta\alpha}$ to $\mathcal{U}_\mathbb{R}^\beta$. It follows from Lemma A.2.6, Lemma 4.1.2, Lemma 4.1.3 and Lemma 4.1.4 that Σ_β is smooth. Because the respective coverings are locally finite, the sets A_β satisfy the conditions of Lemma A.1.4 and we see that

$$\Sigma: \bigoplus_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c \rightarrow \bigoplus_{\beta \in B} \mathcal{SC}^\infty(\mathcal{U}^\beta, \overline{F}^{(k)})_c, \quad (\sigma^\alpha)_{\alpha \in A} \mapsto (\Sigma_\beta((\sigma^\alpha)_{\alpha \in A_\beta}))_{\beta \in B}$$

is smooth. Because $\psi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, h_\alpha \cdot \sigma^\alpha) \circ \phi^{\beta\alpha}|_{\mathcal{U}_\mathbb{R}^{\beta\alpha}} = ((h'_\alpha \circ \phi_\mathbb{R}^\beta) \cdot \sigma^\beta)|_{\mathcal{U}_\mathbb{R}^{\beta\alpha}}$, it follows that

$$\Sigma_\beta((\sigma^\alpha)_{\alpha \in A_\beta}) = \sum_{\alpha \in A_\beta} (((h'_\alpha \circ \phi_\mathbb{R}^\beta) \cdot \sigma^\beta)|_{\mathcal{U}_\mathbb{R}^{\beta\alpha}})^\sim = \sigma^\beta$$

for all $\sigma \in \Gamma_c(\mathcal{E})$, hence Σ makes the above diagram commutative. To see that $\text{im}(\Omega_c^A)$ is closed, simply let $\mathcal{A} = \mathcal{B}$ in the construction above. Then Σ is a smooth left inverse of the inclusion $\text{im}(\Omega_c^A) \hookrightarrow \bigoplus_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c$. The same arguments apply in the case of Ω_b^A . \square

In the situation of the lemma, we denote by $\Gamma_c(\mathcal{E})_c$, resp. by $\Gamma_c(\mathcal{E})_b$, the vector space $\Gamma_c(\mathcal{E})$ equipped with the topology induced by Ω_c^A , resp. Ω_b^A .

Definition 4.1.10. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor. For $n \leq k$, we define

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})^{\geq n} := \{f \in \mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)}): f = (0, \dots, 0, f_n, \dots)\}.$$

With the induced topology $\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})^{\geq n}$ is a closed subspace of $\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})$ for which we write $\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})_c^{\geq n}$. If E and F are Banach spaces and E_0 is finite-dimensional, we write $\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})_b^{\geq n}$ for the respective closed subspace of $\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})_b$.

Lemma/Definition 4.1.11. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let \mathcal{E} be a k -super vector bundle with typical fiber $F \in \mathbf{SVec}_{lc}$ and bundle atlas $\mathcal{A} = \{\Psi^\alpha: \mathcal{U}^\alpha \times \overline{F}^{(k)} \rightarrow \mathcal{E}: \alpha \in A\}$, where $\mathcal{U}^\alpha \subseteq \overline{E}^{(k)}$ for $E \in \mathbf{SVec}_{lc}$. For $0 \leq n \leq k$ define

$$\Gamma(\mathcal{E})^{\geq n} := \{\sigma \in \Gamma(\mathcal{E}): \sigma^\alpha \in \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})^{\geq n} \text{ for all } \alpha \in A\}.$$

Then $\Gamma(\mathcal{E})^{\geq n}$ is a closed subspace of $\Gamma(\mathcal{E})_c$. In the situation of Lemma 4.1.9 we set $\Gamma_c(\mathcal{E})^{\geq n} := \Gamma(\mathcal{E})^{\geq n} \cap \Gamma_c(\mathcal{E})$. Then $\Gamma_c(\mathcal{E})^{\geq n}$ is a closed subspace of $\Gamma_c(\mathcal{E})_c$. We write $\Gamma_c(\mathcal{E})^{\geq n}$ for this space equipped with the induced topology. If additionally E_1, F are Banach spaces, then we define the spaces $\Gamma(\mathcal{E})_b^{\geq n}$ and $\Gamma_c(\mathcal{E})_b^{\geq n}$ analogously and see that they are closed in $\Gamma(\mathcal{E})_b$ and $\Gamma_c(\mathcal{E})_b$ respectively.

Proof. With Proposition 2.2.16 and formula (4.3), one sees that $\Gamma(\mathcal{E})^{\geq n}$ and $\Gamma_c(\mathcal{E})^{\geq n}$ are subspaces of $\Gamma(\mathcal{E})$ because only components of lower or equal degree contribute to the respective component in the change of charts. The respective subspaces are closed because $\mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})^{\geq n}_c \subseteq \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_c$ and $\mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})^{\geq n}_b \subseteq \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{F}^{(k)})_b$ are closed subspaces in the respective situation. \square

Lemma 4.1.12. Let $n \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{E} \in \mathbf{SVBun}^{(n)}$. The functors ι_k^0, ι_k^1 and π_k^n define continuous linear maps

$$\begin{aligned} \iota_k^n: \Gamma(\mathcal{E})_c &\rightarrow \Gamma(\iota_k^n(\mathcal{E}))_c, & \sigma &\mapsto \iota_k^n(\sigma) & \text{for } n \in \{0, 1\}, n \leq k \leq \infty \text{ and} \\ \pi_k^n: \Gamma(\mathcal{E})_c &\rightarrow \Gamma(\pi_k^n(\mathcal{E}))_c, & \sigma &\mapsto \pi_k^n(\sigma) & \text{for } 0 \leq k \leq n. \end{aligned}$$

When defined, one obtains analogously continuous linear maps for $\Gamma_c(\mathcal{E})_c, \Gamma(\mathcal{E})_b$ and $\Gamma_c(\mathcal{E})_b$.

Proof. Let \mathcal{E} have the base \mathcal{M} . From the local definition of the projection $\pi_{\mathcal{M}}: \mathcal{E} \rightarrow \mathcal{M}$ and Lemma 2.5.8 it is clear that $\iota_k^n(\pi_{\mathcal{M}}) = \pi_{\iota_k^n(\mathcal{M})}: \iota_k^n(\mathcal{E}) \rightarrow \iota_k^n(\mathcal{M})$ and $\pi_k^n(\pi_{\mathcal{M}}) = \pi_{\pi_k^n(\mathcal{M})}: \pi_k^n(\mathcal{E}) \rightarrow \pi_k^n(\mathcal{M})$, respectively. By functoriality, it follows that sections are indeed mapped to sections. In a trivialization linearity and continuity are obvious. Thus, by the definition of the vector space topology on the spaces of sections, it follows from Lemma A.1.2 and Lemma A.1.4 that the above maps are continuous and linear. \square

Lemma 4.1.13. Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\pi: \mathcal{E} \rightarrow \mathcal{M}$ and $\pi': \mathcal{E}' \rightarrow \mathcal{M}'$ be k -super vector bundles. If $f: \mathcal{E} \rightarrow \mathcal{E}'$ is an isomorphism of k -super vector bundles over $h: \mathcal{M} \rightarrow \mathcal{M}'$, then we have linear isomorphisms

$$\begin{aligned} \Gamma(\mathcal{E})^{\geq n} &\rightarrow \Gamma(\mathcal{E}')^{\geq n}, & \sigma &\mapsto f \circ \sigma \circ h^{-1} & \text{and} \\ \Gamma_c(\mathcal{E})^{\geq n} &\rightarrow \Gamma_c(\mathcal{E}')^{\geq n}, & \sigma &\mapsto f \circ \sigma \circ h^{-1} \end{aligned}$$

for $0 \leq n \leq k$.

Proof. Because

$$\pi' \circ f \circ X \circ h^{-1} = h \circ \pi \circ X \circ h^{-1} = h \circ h^{-1} = \text{id}_{\mathcal{M}'}$$

holds, we have indeed an morphism $\Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E}')$ with the inverse defined by $\sigma' \mapsto f^{-1} \circ \sigma' \circ h$. That this morphism is linear follows locally along the lines of formula (4.3). With the same formula one also sees that the degree of the non-zero part of the skeleton of σ cannot decrease and thus we have isomorphisms $\Gamma(\mathcal{E})^{\geq n} \rightarrow \Gamma(\mathcal{E}')^{\geq n}$. It is easy to see that $\text{supp}(f \circ \sigma \circ h^{-1}) = h_{\mathbb{R}}(\text{supp}(\sigma))$ and therefore compactly supported sections are mapped to compactly supported sections. \square

4.1.3. Vector fields

Definition 4.1.14. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$. We denote by $\mathcal{X}(\mathcal{M})_{\bar{0}} := \Gamma(\mathcal{TM})$ the *even vector fields of \mathcal{M}* and by $\mathcal{X}_c(\mathcal{M})_{\bar{0}} := \Gamma_c(\mathcal{TM})$ the *even vector fields of \mathcal{M} with compact support*. If $k > 0$, we define the *odd vector fields of \mathcal{M}* by $\mathcal{X}(\mathcal{M})_{\bar{1}} := \Gamma(\bar{\Pi}\mathcal{TM})$ and the *odd vector fields of \mathcal{M} with compact support* by $\mathcal{X}_c(\mathcal{M})_{\bar{1}} := \Gamma_c(\bar{\Pi}\mathcal{TM})$. In this case, we let $\mathcal{X}(\mathcal{M}) := \mathcal{X}(\mathcal{M})_{\bar{0}} \oplus \mathcal{X}(\mathcal{M})_{\bar{1}} \in \mathbf{SVec}_{lc}$ and $\mathcal{X}_c(\mathcal{M}) := \mathcal{X}_c(\mathcal{M})_{\bar{0}} \oplus \mathcal{X}_c(\mathcal{M})_{\bar{1}} \in \mathbf{SVec}_{lc}$. If $\mathcal{A} := \{\varphi^\alpha : \alpha \in A\}$ is an atlas of \mathcal{M} , we denote by X^α the trivialization of X as in (4.1), with respect to the bundle atlas $\{\mathcal{T}\varphi^\alpha : \alpha \in A\}$ if $X \in \mathcal{X}(\mathcal{M})_{\bar{0}}$ and with respect to the bundle atlas $\{\bar{\Pi}(\mathcal{T}\varphi^\alpha) : \alpha \in A\}$ if $X \in \mathcal{X}(\mathcal{M})_{\bar{1}}$.

Remark 4.1.15. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be modelled on $E \in \mathbf{SVec}_{lc}$ with the atlas $\mathcal{A} = \{\varphi^\alpha : \mathcal{U}^\alpha \rightarrow \mathcal{M} : \alpha \in A\}$. We see that for $X \in \mathcal{X}(\mathcal{M})_{\bar{0}}$ and $\alpha, \beta \in A$, we have

$$d\varphi^{\alpha\beta}(\text{id}_{\mathcal{U}^\alpha}, X^\alpha)|_{\mathcal{U}^{\alpha\beta}} = X^\beta \circ \varphi^{\alpha\beta}$$

and with Remark 2.2.15 and formula (4.3), we calculate

$$\begin{aligned} (d\varphi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, X^\alpha)|_{\mathcal{U}^{\alpha\beta}})_n &= \mathfrak{A}^n \left(\sum_{n \geq l \text{ even}} \frac{n!}{(n-l)!l!} d\varphi_{n-l}^{\alpha\beta}(\bullet)(X_l^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, \text{id}_{E_1}, \dots, \text{id}_{E_1}) \right. \\ &\quad \left. + \sum_{n \geq l \text{ odd}} \frac{n!}{(n-l)!l!} \varphi_{n-l+1}^{\alpha\beta}(\bullet)(X_l^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, \text{id}_{E_1}, \dots, \text{id}_{E_1}) \right). \end{aligned}$$

If \mathcal{A} is an atlas of Batchelor type, the formula simplifies to

$$\begin{aligned} (d\varphi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, X^\alpha)|_{\mathcal{U}^{\alpha\beta}})_n &= \\ n \cdot \mathfrak{A}^n(d\varphi_1^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, X_{n-1}^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(\text{id}_{E_1})) &+ \varphi_1^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(X_n^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}) \\ &= X_n^\beta(\varphi_0^{\alpha\beta})(\varphi_1^{\alpha\beta}, \dots, \varphi_1^{\alpha\beta}) = (X^\beta \circ \varphi^{\alpha\beta})_n \end{aligned} \quad (4.4)$$

if n is odd and

$$\begin{aligned} (d\varphi^{\alpha\beta} \circ (\text{id}_{\mathcal{U}^\alpha}, X^\alpha)|_{\mathcal{U}^{\alpha\beta}})_n &= d\varphi_0^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, X_n^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}) = \\ X_n^\beta(\varphi_0^{\alpha\beta})(\varphi_1^{\alpha\beta}, \dots, \varphi_1^{\alpha\beta}) &= (X^\beta \circ \varphi^{\alpha\beta})_n \end{aligned} \quad (4.5)$$

if n is even. In particular, formula (4.4) and (4.5) imply that if \mathcal{A} is an atlas of Batchelor type, then the even vector fields of the local form $X^\alpha = (0, 0, \dots, X_n^\alpha, X_{n+1}^\alpha, 0, \dots)$ form a closed subspace of $\mathcal{X}(\mathcal{M})_{\bar{0}}$ for every even n with

$0 \leq n \leq k$. Let us also describe the odd vector fields. First of all, we see from Lemma 2.5.11 that

$$\begin{aligned} (\bar{\Pi}(d\varphi^{\alpha\beta}))_n &= \varphi_{n+1}^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(\text{pr}_{\Pi(E)_0}, \text{pr}_1, \dots, \text{pr}_1) \\ &\quad + n \cdot \mathfrak{A}^n d\varphi_{n-1}^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, \text{pr}_{\Pi(E)_1})(\text{pr}_1, \dots, \text{pr}_1) \end{aligned}$$

for $n > 0$ and $(\bar{\Pi}(d\varphi^{\alpha\beta}))_0 = \varphi_1^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(\text{pr}_{\Pi(E)_0})$. Therefore, we get

$$\begin{aligned} &(\bar{\Pi}(d\varphi^{\alpha\beta}) \circ (\text{id}_{\mathcal{U}^\alpha}, Y^\alpha)|_{\mathcal{U}^{\alpha\beta}})_n = \\ &n! \cdot \mathfrak{A}^n \left(\sum_{n \geq l \text{ even}} \frac{1}{(n-l)!l!} \varphi_{n+1-l}^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(Y_l^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, \text{id}_{E_1}, \dots, \text{id}_{E_1}) \right. \\ &\quad \left. + \sum_{n \geq l \text{ odd}} \frac{1}{(n-l)!l!} d\varphi_{n-l}^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, Y_l^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(\text{id}_{E_1}, \dots, \text{id}_{E_1}) \right) \\ &= (Y^\beta \circ \varphi^{\alpha\beta})_n \end{aligned}$$

for $Y \in \mathcal{X}(\mathcal{M})_{\bar{1}}$ and $n > 0$. For $n = 0$, we have

$$(\bar{\Pi}(d\varphi^{\alpha\beta}) \circ (\text{id}_{\mathcal{U}^\alpha}, Y^\alpha)|_{\mathcal{U}^{\alpha\beta}})_0 = \varphi_1^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(Y_0^\alpha|_{\mathcal{U}^{\alpha\beta}_{\mathbb{R}}}) = Y_0^\beta \circ \varphi_0^{\alpha\beta}.$$

If \mathcal{A} is an atlas of Batchelor type, this simplifies to

$$\begin{aligned} &(\bar{\Pi}(d\varphi^{\alpha\beta}) \circ (\text{id}_{\mathcal{U}^\alpha}, Y^\alpha)|_{\mathcal{U}^{\alpha\beta}})_n = \varphi_1^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(Y_n^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}) \\ &= Y_n^\beta(\varphi_0^{\alpha\beta})(\varphi_1^{\alpha\beta}, \dots, \varphi_1^{\alpha\beta}) = (Y^\beta \circ \varphi^{\alpha\beta})_n \end{aligned}$$

for n even and

$$\begin{aligned} &(\bar{\Pi}(d\varphi^{\alpha\beta}) \circ (\text{id}_{\mathcal{U}^\alpha}, Y^\alpha)|_{\mathcal{U}^{\alpha\beta}})_n = \\ &d\varphi_0^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, Y_n^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}) + n \cdot \mathfrak{A}^n d\varphi_1^{\alpha\beta}(\text{pr}_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}}, Y_{n-1}^\alpha|_{\mathcal{U}_{\mathbb{R}}^{\alpha\beta}})(\text{id}_{E_1}) \\ &= Y_n^\beta(\varphi_0^{\alpha\beta})(\varphi_1^{\alpha\beta}, \dots, \varphi_1^{\alpha\beta}) = (Y^\beta \circ \varphi^{\alpha\beta})_n \end{aligned}$$

for n odd.

Lemma 4.1.16. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ with atlas $\mathcal{A} := \{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$. Then $\mathcal{X}(\mathcal{M})_{\bar{0}}$ is a Lie algebra with the Lie bracket $[\cdot, \cdot]$ given by*

$$[X, Y]^\alpha = dX^\alpha \circ (\text{id}_{\mathcal{U}^\alpha}, Y^\alpha) - dY^\alpha \circ (\text{id}_{\mathcal{U}^\alpha}, X^\alpha)$$

for $\alpha \in A$ and $X, Y \in \mathcal{X}(\mathcal{M})_{\bar{0}}$. For every $0 \leq n \leq k$, the subspace $\mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq n}$ is a Lie subalgebra of $\mathcal{X}(\mathcal{M})_{\bar{0}}$. Moreover, if $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then $\mathcal{X}_c(\mathcal{M})_{\bar{0}}$ and $\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq n}$ are also Lie subalgebras of $\mathcal{X}(\mathcal{M})_{\bar{0}}$.

Proof. We start with the case $k = \infty$. By Lemma 2.6.2, every vector field $X: \mathcal{M} \rightarrow \mathcal{TM}$ maps to a vector field $\tilde{X} := \varprojlim X: \varprojlim \mathcal{M} \rightarrow T\varprojlim \mathcal{M}$. With the atlas $\{\tilde{\varphi}^\alpha := \varprojlim \varphi^\alpha: \alpha \in A\}$, we know from [10, Theorem 4.2, p.25] that the Lie bracket

of \tilde{X} and \tilde{Y} is locally given by

$$d\tilde{X}^{\tilde{\varphi}^\alpha} \circ (\text{id}_{\varprojlim \mathcal{U}^\alpha}, \tilde{Y}^{\tilde{\varphi}^\alpha}) - d\tilde{Y}^{\tilde{\varphi}^\alpha} \circ (\text{id}_{\varprojlim \mathcal{U}^\alpha}, \tilde{X}^{\tilde{\varphi}^\alpha}).$$

But since we have

$$d\tilde{X}^{\tilde{\varphi}^\alpha} \circ (\text{id}_{\varprojlim \mathcal{U}^\alpha}, \tilde{Y}^{\tilde{\varphi}^\alpha}) = \varprojlim dX^\alpha \circ (\text{id}_{\mathcal{U}^\alpha}, Y^\alpha),$$

it follows that $[X, Y]^\alpha$ as defined in the lemma gives us a unique vector field $[X, Y]: \mathcal{M} \rightarrow \mathcal{TM}$. The calculations in Remark 4.1.15 show that the degree of the resulting alternating maps does not decrease. Therefore, $\mathcal{X}(\mathcal{M})_0^{\geq n}$ is closed under the Lie bracket. From the local definition of the bracket, it follows that $\text{supp}([X, Y]) \subseteq \text{supp}(X) \cap \text{supp}(Y)$. Hence, $\mathcal{X}_c(\mathcal{M})_{\bar{0}}$ and $\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq n}$ are also Lie subalgebras in case of finite-dimensional $\mathcal{M}_{\mathbb{R}}$.

For $k \in \mathbb{N}_0$, we can repeat the same arguments with the vector field $X_{\Lambda_k}: \mathcal{M}_{\Lambda_k} \rightarrow T\mathcal{M}_{\Lambda_k}$. \square

Lemma 4.1.17. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. Then $\mathcal{X}(\mathcal{M})_{\bar{0},b}$ is a topological Lie algebra. If $\mathcal{M}_{\mathbb{R}}$ is σ -compact, then $\mathcal{X}_c(\mathcal{M})_{\bar{0},b}$ is a topological Lie algebra as well.*

Proof. Let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \bar{E}^{(k)}$. By definition of the topology of $\mathcal{X}(\mathcal{M})_{\bar{0},b}$, resp. $\mathcal{X}_c(\mathcal{M})_{\bar{0},b}$, and the local form of the Lie bracket, it suffices to see that

$$\mathcal{SC}^\infty(\mathcal{U}, \bar{E}^{(k)})_b \times \mathcal{SC}^\infty(\mathcal{U}, \bar{E}^{(k)})_b \rightarrow \mathcal{SC}^\infty(\mathcal{U}, \bar{E}^{(k)})_b, \quad (X, Y) \mapsto dX \circ (\text{id}_{\mathcal{U}}, Y)$$

is smooth. This follows from Lemma A.2.14. \square

Lemma 4.1.18. *Let $n \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(n)}$. The linear maps*

$$\begin{aligned} \iota_k^n: \mathcal{X}(\mathcal{M})_{\bar{0}} &\rightarrow \mathcal{X}(\iota_k^n(\mathcal{M}))_{\bar{0}}, & X &\mapsto \iota_k^n(X) & \text{for } n \in \{0, 1\}, \ n \leq k \leq \infty \text{ and} \\ \pi_k^n: \mathcal{X}(\mathcal{M})_{\bar{0}} &\rightarrow \mathcal{X}(\pi_k^n(\mathcal{M}))_{\bar{0}}, & X &\mapsto \pi_k^n(X) & \text{for } 0 \leq k \leq n \end{aligned}$$

given by Lemma 4.1.12 together with Lemma 2.6.4 are Lie algebra morphisms. The same holds true for the respective maps for $\mathcal{X}_c(\mathcal{M})_{\bar{0}}$ if $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional.

Proof. This follows immediately by applying the functors to the local definition of the Lie bracket. \square

4.2. The Automorphism Group of a Supermanifold

In a sense, the group of automorphisms of a supermanifold is completely contained in the supergroup of superdiffeomorphisms. However, an examination of the former provides one with valuable insights for the latter, which has a much more complicated structure.

Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$. We denote by

$$\mathrm{Aut}(\mathcal{M}) := \{f \in \mathcal{SC}^\infty(\mathcal{M}, \mathcal{M}) : f \text{ invertible}\}$$

the group of automorphisms of \mathcal{M} . Let $k > 0$ and recall the functor $\pi_1^k : \mathbf{SMan}^{(k)} \rightarrow \mathbf{SMan}^{(1)}$ from Lemma 2.3.18. We have a short exact sequence of groups

$$1 \rightarrow \ker(\pi_1^k) \longrightarrow \mathrm{Aut}(\mathcal{M}) \xrightarrow{\pi_1^k} \mathrm{Aut}(\mathcal{M}^{(1)}) \rightarrow 1$$

and it follows immediately from the definition of π_1^k that

$$\mathrm{Aut}_{\mathrm{id}}(\mathcal{M}) := \{f \in \mathrm{Aut}(\mathcal{M}) : f_{\Lambda_1} = \mathrm{id}_{\mathcal{M}_{\Lambda_1}}\} = \ker(\pi_1^k).$$

We will consider $\mathcal{M}^{(1)}$ as a vector bundle and $\mathrm{Aut}(\mathcal{M}^{(1)})$ as the group of vector bundle automorphisms via Proposition 2.3.17. If \mathcal{M} is a supermanifold of Batchelor type, then the above sequence splits along $\iota_1^1 : \mathrm{Aut}(\mathcal{M}^{(1)}) \rightarrow \mathrm{Aut}(\mathcal{M})$ because of Lemma 2.3.18.

If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, we define the group of automorphisms of \mathcal{M} with compact support by

$$\mathrm{Aut}_c(\mathcal{M}) := \{f \in \mathrm{Aut}(\mathcal{M}) : \exists K \subseteq \mathcal{M}_{\mathbb{R}} \text{ compact with } f|_{\mathcal{M}|_{\mathcal{M}_{\mathbb{R}} \setminus K}} = \mathrm{id}_{\mathcal{M}|_{\mathcal{M}_{\mathbb{R}} \setminus K}}\}.$$

Clearly, $\mathrm{Aut}_c(\mathcal{M})$ is a subgroup of $\mathrm{Aut}(\mathcal{M})$ and with $\mathrm{Aut}_{\mathrm{id}}^c(\mathcal{M}) := \mathrm{Aut}_{\mathrm{id}}(\mathcal{M}) \cap \mathrm{Aut}_c(\mathcal{M})$, we again get a short exact sequence

$$1 \rightarrow \mathrm{Aut}_{\mathrm{id}}^c(\mathcal{M}) \longrightarrow \mathrm{Aut}_c(\mathcal{M}) \xrightarrow{\pi_1^k} \mathrm{Aut}_c(\mathcal{M}^{(1)}) \rightarrow 1.$$

Note that under the identification from Proposition 2.3.17, $\mathrm{Aut}_c(\mathcal{M}^{(1)})$ corresponds to the vector bundle automorphisms of $\mathcal{M}^{(1)}$ with compact support. In the Batchelor case, the sequence splits as before.

As it turns out, $\mathrm{Aut}_{\mathrm{id}}(\mathcal{M})$ is a so called pro-polynomial group that can be described by the vector fields $\mathcal{X}(\mathcal{M})_0^{\geq 2}$. In the Batchelor case, the action of $\mathrm{Aut}(\mathcal{M}^{(1)})$ on $\mathrm{Aut}_{\mathrm{id}}(\mathcal{M})$ can be understood as the pullback of the vector fields. To turn $\mathrm{Aut}(\mathcal{M})$ into a Lie group, one then just needs that $\mathcal{X}(\mathcal{M})_0^{\geq 2}$ is a continuous Lie algebra and that $\mathrm{Aut}(\mathcal{M}^{(1)})$ can be turned into a Lie group that acts smoothly on $\mathcal{X}(\mathcal{M})_0^{\geq 2}$. This approach is the same as the one taken in [47] for finite-dimensional compact supermanifolds, but our language is completely different and our results are much more general.

Lemma 4.2.1. *Let \mathcal{M} be a supermanifold. Then $(\mathrm{Aut}(\mathcal{M}^{(m)})_{m \in \mathbb{N}_0}, (\pi_n^m)_{n \leq m})$ and $(\mathrm{Aut}_{\mathrm{id}}(\mathcal{M}^{(m)})_{m \in \mathbb{N}}, (\pi_n^m)_{n \leq m})$ are inverse systems of groups and we have*

$$\varprojlim_m \mathrm{Aut}(\mathcal{M}^{(m)}) = \mathrm{Aut}(\mathcal{M}) \quad \text{and} \quad \varprojlim_m \mathrm{Aut}_{\mathrm{id}}(\mathcal{M}^{(m)}) = \mathrm{Aut}_{\mathrm{id}}(\mathcal{M})$$

as groups.

Proof. That we have inverse systems of groups follows from Lemma 2.3.18 and that the equalities hold is obvious from the definition of supersmooth morphisms. \square

4.2.1. Local considerations

Lemma 4.2.2. *Let $k \in \mathbb{N} \cup \{\infty\}$, $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor. We give $\text{Aut}_{\text{id}}(\mathcal{U})$ the vector space structure induced by the bijection*

$$\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})^{\geq 2} \rightarrow \text{Aut}_{\text{id}}(\mathcal{U}), \quad (0, 0, f_2, \dots) \mapsto (\text{id}_{\mathcal{U}_{\mathbb{R}}}, c_{\text{id}_{E_1}}, f_2, \dots).$$

If $k \in \mathbb{N}$ or E_1 is finite-dimensional, then $\text{Aut}_{\text{id}}(\mathcal{U})$ is a polynomial group of degree at most $\min\{k, \dim(E_1)\}$. If $k = \infty$ and E_1 is infinite-dimensional, then $\text{Aut}_{\text{id}}(\mathcal{U})$ is a pro-polynomial group. The Lie bracket is given by

$$[X, Y] = dX \circ (\text{id}_{\mathcal{U}}, Y) - dY \circ (\text{id}_{\mathcal{U}}, X)$$

in both cases.

Proof. Let $k \in \mathbb{N}$, $f, g \in \text{Aut}_{\text{id}}(\mathcal{U})$ and recall formula (2.2) from Proposition 2.2.16 to see

$$(g \circ f)_n(\bullet) = \sum_{\substack{m, l, \\ (\alpha, \beta) \in I_{m, l}^n}} \frac{n!}{m!l!\alpha!\beta!} \cdot \mathfrak{A}^n d^m g_l(\bullet)((f_\alpha \times f_\beta)(\bullet)).$$

Because the other cases are trivial, we may assume $\min\{k, \dim(E_1)\} \geq n > 1$. This expression is polynomial in $(g_r)_{r>1}, (f_r)_{r>1}$ and the number of g_r, f_r with $r > 1$ that can appear in a summand is bounded by n . The same argument applies to the iterated products. Because of Lemma 2.2.18 the inversion is a polynomial map. Let $g = \text{id}_{\mathcal{U}} + X$ and $f = \text{id}_{\mathcal{U}} + Y$ with $X, Y \in \mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})^{\geq 2}$. Then the part of the composition linear in both X and Y has the n -th component

$$\begin{aligned} (dX \circ (\text{id}_{\mathcal{U}}, Y))_n = \mathfrak{A}^n \Big(& \sum_{n \geq l \text{ even}} \frac{n!}{(n-l)!l!} dX_{n-l}(\text{id}_{\mathcal{U}_{\mathbb{R}}})(Y_l, \text{id}_{E_1}, \dots, \text{id}_{E_1}) + \\ & \sum_{n \geq l \text{ odd}} \frac{n!}{(n-l)!l!} X_{n-l+1}(\text{id}_{\mathcal{U}_{\mathbb{R}}})(Y_l, \text{id}_{E_1}, \dots, \text{id}_{E_1}) \Big), \end{aligned}$$

where the equality follows from Remark 4.1.15. Therefore, the Lie bracket is as claimed by C.2.

Let now $k = \infty$ and $1 \leq n \leq n < \infty$. The morphism of groups $\pi_n^m: \text{Aut}_{\text{id}}(\mathcal{U}^{(m)}) \rightarrow \text{Aut}_{\text{id}}(\mathcal{U}^{(n)})$ is linear and thus $\text{Aut}_{\text{id}}(\mathcal{U}) = \varprojlim_n \text{Aut}_{\text{id}}(\mathcal{U}^{(n)})$ is pro-polynomial. That the Lie bracket is correct also follows from the case of finite k by taking the limit. \square

In the situation above, we denote by $\exp_{\mathcal{U}}: \mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})^{\geq 2} \rightarrow \text{Aut}_{\text{id}}(\mathcal{U})$ the exponential map of the polynomial, resp. pro-polynomial, group.

Lemma 4.2.3. *Let $k \in \mathbb{N} \cup \{\infty\}$ and $E \in \mathbf{SVec}_{lc}$ such that E_0 is finite-dimensional and E_1 is a Banach space. Then the global chart*

$$\exp_{\mathcal{U}}: \mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})_b^{\geq 2} \rightarrow \text{Aut}_{\text{id}}(\mathcal{U})$$

turns $\text{Aut}_{\text{id}}(\mathcal{U})$ into a Lie group for any open subfunctor $\mathcal{U} \subseteq \overline{E}^{(k)}$.

Proof. By Lemma 4.1.17 the Lie bracket of $\mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})_b^{\geq 2}$ is continuous. Thus, the claim follows from Lemma C.2.5 and Lemma C.3.1. \square

The next two lemmas are the main tools to go from the local to the global situation.

Lemma 4.2.4. *Let $k \in \mathbb{N} \cup \{\infty\}$, $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}^{(k)}$ be an open subfunctor. If $\mathcal{V} \subseteq \mathcal{U}$ is an open subfunctor, then we have*

$$\exp_{\mathcal{U}}(X)|_{\mathcal{V}} = \exp_{\mathcal{V}}(X|_{\mathcal{V}})$$

for all $X \in \mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})^{\geq 2}$.

Proof. For $k \in \mathbb{N}$ this follows immediately from Lemma C.2.2 because

$$\text{Aut}_{\text{id}}(\mathcal{U}) \rightarrow \text{Aut}_{\text{id}}(\mathcal{V}), \quad f \mapsto f|_{\mathcal{V}}$$

is a linear group morphism. The result follows then for $k = \infty$ by taking the inverse limit because $\pi_n^m(X|_{\mathcal{V}^{(m)}}) = \pi_n^m(X)|_{\mathcal{V}^{(n)}}$ for all $X \in \mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(m)})$ and $1 \leq n \leq m < \infty$. \square

Lemma 4.2.5. *Let $k \in \mathbb{N} \cup \{\infty\}$, $E \in \mathbf{SVec}_{lc}$ and let $\mathcal{U} \subseteq \overline{E}^{(k)}$ and $\mathcal{V} \subseteq \overline{E}^{(k)}$ be open subfunctors. For any invertible supersmooth morphism $\varphi: \mathcal{U} \rightarrow \mathcal{V}$, the morphism*

$$\text{Ad}_\varphi: \text{Aut}_{\text{id}}(\mathcal{U}) \rightarrow \text{Aut}_{\text{id}}(\mathcal{V}), \quad f \mapsto \varphi \circ f \circ \varphi^{-1}$$

is a polynomial isomorphism of groups if $k \in \mathbb{N}$ or $\dim(E_1) < \infty$. If $k = \infty$ and E_1 is infinite-dimensional, then Ad_φ is the limit of polynomial isomorphisms of groups. Moreover, in both cases, we have

$$\varphi \circ \exp_{\mathcal{U}}(X) \circ \varphi^{-1} = \exp_{\mathcal{V}}(d\varphi \circ (\varphi^{-1}, X \circ \varphi^{-1}))$$

for all $X \in \mathcal{SC}^\infty(\mathcal{U}, \overline{E}^{(k)})^{\geq 2}$.

Proof. Let $k \in \mathbb{N}$, $\varphi = (\varphi_0, \dots, \varphi_k)$ and $\varphi^{-1} = (\varphi_0^{-1}, \dots, \varphi_k^{-1})$. For $f \in \text{Aut}_{\text{id}}(\mathcal{U})$, we have $f \circ \varphi^{-1} = \varphi^{-1} + X$ where $X \in \mathcal{SC}^\infty(\mathcal{V}, \overline{E}^{(k)})^{\geq 2}$ and the sum is taken in $\mathcal{SC}^\infty(\mathcal{V}, \overline{E}^{(k)})$. Clearly, X depends linearly on f . We only need to consider $1 < n \leq \min\{k, \dim(E_1)\}$ and in this case we calculate

$$(\varphi \circ (\varphi^{-1} + X))_n = \mathfrak{A}^n \sum_{m, l; (\alpha, \beta) \in I_{m, l}^n} \frac{n!}{m! l! \alpha! \beta!} d^m \varphi_l(\varphi_0^{-1}) ((\varphi^{-1} + X)_\alpha \times (\varphi^{-1} + X)_\beta).$$

After multilinear expansion, we see that the summands depend multilinearly on X . Thus Ad_φ is polynomial. Note that the summands where only terms of the form $\varphi_\alpha^{-1} \times \varphi_\beta^{-1}$ appear add up to zero.

The linear term of Ad_φ consists of those summands that only contain one term depending on X . In other words, we have

$$(L(\text{Ad}_\varphi)(f))_n = \mathfrak{A}^n \sum_{m+l>0; (\alpha, \beta) \in I_{m,l}^n} \frac{n!}{m!l!\alpha!\beta!} \sum_{i=1}^{m+l} d^m \varphi_l(\varphi_0^{-1})(\widehat{\varphi_\alpha^{-1} \times_i \varphi_\beta^{-1}}),$$

where $\widehat{\varphi_\alpha^{-1} \times_i \varphi_\beta^{-1}}$ results from substituting φ_{α_i} with X_{α_i} for $1 \leq i \leq m$ and $\varphi_{\beta_{i-m}}$ with $X_{\beta_{i-m}}$ for $m < i \leq l+m$. Using Remark 2.2.15, we calculate

$$\begin{aligned} & (L(\text{Ad}_\varphi)(X))_n \\ &= \mathfrak{A}^n \sum_{\substack{m,l>0; \\ (\alpha, \beta) \in I_{m,l}^n}} \frac{n!}{m!l!\alpha!\beta!} l \cdot d^m \varphi_l(\varphi_0^{-1})(\varphi_\alpha^{-1} \times (X_{\beta_1}, \varphi_{\beta_2}^{-1}, \dots, \varphi_{\beta_l}^{-1})) \\ &+ \mathfrak{A}^n \sum_{\substack{m,l,m>0; \\ (\alpha, \beta) \in I_{m,l}^n}} \frac{n!}{m!l!\alpha!\beta!} m \cdot d^m \varphi_n(\varphi_0^{-1})((X_{\alpha_1}, \varphi_{\alpha_2}^{-1}, \dots, \varphi_{\alpha_m}^{-1}) \times \varphi_\beta^{-1}) \\ &= (d\varphi \circ (\varphi^{-1}, X))_n \end{aligned}$$

and the claim follows from Lemma C.2.2. Consider now infinite-dimensional E_1 and $k = \infty$. Let $1 \leq n \leq m < \infty$. By functoriality of π_n^m , we have

$$\pi_n^m(\text{Ad}_{\varphi^{(m)}}(f)) = \text{Ad}_{\pi_n^m(\varphi^{(m)})}(\pi_n^m(f))$$

for all $f \in \text{Aut}_{\text{id}}(\mathcal{U}^{(m)})$. This implies $\text{Ad}_\varphi = \varprojlim_n \text{Ad}_{\varphi^{(n)}}$. For $X \in \mathcal{SC}^\infty(\mathcal{U}, \overline{E})^{\geq 2}$, we have $\pi_n^m(\exp_{\mathcal{U}^{(m)}}(X^{(m)})) = \exp_{\mathcal{U}^{(n)}}(\pi_n^m(X^{(m)}))$ because $\pi_n^m: \text{Aut}_{\text{id}}(\mathcal{U}^{(m)}) \rightarrow \text{Aut}_{\text{id}}(\mathcal{U}^{(n)})$ is a linear morphism of groups. With Remark 2.2.15, one easily sees that $\pi_n^m(dX^{(m)}) = d\pi_n^m(X^{(m)})$. Overall, it follows

$$\varphi \circ \exp_{\mathcal{U}}(X) \circ \varphi^{-1} = \exp_{\mathcal{V}}(d\varphi \circ (\varphi^{-1}, X \circ \varphi^{-1}))$$

from the first part of the proof. \square

4.2.2. The global case

Proposition 4.2.6. *Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be modelled on $E \in \mathbf{SVec}_{ic}$ with atlas $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$. For the bundle atlas $\{\mathcal{T}\varphi^\alpha: \alpha \in A\}$ of $\mathcal{T}\mathcal{M}$, recall from Lemma 4.1.6 (using Lemma/Definition 4.1.11) the embedding*

$$\Theta: \mathcal{X}(\mathcal{M})_{\geq 0}^{\geq 2} \rightarrow \prod_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{E}^{(k)})^{\geq 2}, \quad X \mapsto \prod_{\alpha \in A} X^\alpha$$

and consider the injective morphism of groups

$$\Psi: \text{Aut}_{\text{id}}(\mathcal{M}) \rightarrow \prod_{\alpha \in A} \text{Aut}_{\text{id}}(\mathcal{U}^\alpha), \quad f \mapsto (\varphi^\alpha)^{-1} \circ f \circ \varphi^\alpha.$$

Then there exists a unique bijective map $\exp_{\mathcal{M}}: \mathcal{X}(\mathcal{M})_0^{\geq 2} \rightarrow \text{Aut}_{\text{id}}(\mathcal{M})$ such that the diagram

$$\begin{array}{ccc} \mathcal{X}(\mathcal{M})_0^{\geq 2} & \xrightarrow{\exp_{\mathcal{M}}} & \text{Aut}_{\text{id}}(\mathcal{M}) \\ \downarrow \Theta & & \downarrow \Psi \\ \prod_{\alpha \in A} \mathcal{SC}^{\infty}(\mathcal{U}^{\alpha}, \overline{E}^{(k)})^{\geq 2} & \xrightarrow{\prod_{\alpha} \exp_{\mathcal{U}^{\alpha}}} & \prod_{\alpha \in A} \text{Aut}_{\text{id}}(\mathcal{U}^{\alpha}) \end{array}$$

commutes. If $k \in \mathbb{N}$ and $\dim(E_1) < \infty$, then this bijection turns $\text{Aut}_{\text{id}}(\mathcal{M})$ into a polynomial group of degree at most $\min\{k, \dim(E_1)\}$. Otherwise, $\text{Aut}_{\text{id}}(\mathcal{M})$ becomes a pro-polynomial group. Neither $\exp_{\mathcal{M}}$, nor the topology induced on $\text{Aut}_{\text{id}}(\mathcal{M})$ by the chosen topology of $\mathcal{X}(\mathcal{M})_0^{\geq 2}$, depend on the atlas. The Lie algebra of $\text{Aut}_{\text{id}}(\mathcal{M})$ is given by $\mathcal{X}(\mathcal{M})_0^{\geq 2}$.

Proof. To see that $\exp_{\mathcal{M}}$ is well-defined and bijective, it suffices to show $\prod_{\alpha} \exp_{\mathcal{U}^{\alpha}}(\text{im}(\Theta)) = \text{im}(\Psi)$ since the maps $\exp_{\mathcal{U}^{\alpha}}$ are bijective for all $\alpha \in A$. Like always, let $\varphi^{\alpha\beta}: \mathcal{U}^{\alpha\beta} \rightarrow \mathcal{U}^{\beta\alpha}$ be the change of charts for any two charts φ^{α} and φ^{β} . We have $\prod_{\alpha \in A} X^{\alpha} \in \text{im}(\Theta)$ if and only if $d(\varphi^{\beta\alpha})^{-1} \circ (\varphi^{\beta\alpha}, X^{\alpha}|_{\mathcal{U}^{\alpha\beta}} \circ \varphi^{\beta\alpha}) = X^{\beta}|_{\mathcal{U}^{\beta\alpha}}$ for all $\alpha, \beta \in A$. By Lemma 4.2.5 this is true if and only if

$$(\varphi^{\beta\alpha})^{-1} \circ \exp_{\mathcal{U}^{\alpha\beta}}(X^{\alpha}|_{\mathcal{U}^{\alpha\beta}}) \circ \varphi^{\beta\alpha} = \exp_{\mathcal{U}^{\beta\alpha}}(X^{\beta}|_{\mathcal{U}^{\beta\alpha}})$$

holds for all $\alpha, \beta \in A$. It follows from Lemma 4.2.4 that this is exactly the condition for $\prod_{\alpha \in A} \exp_{\mathcal{U}^{\alpha}}(X^{\alpha})$ to be in $\text{im}(\Psi)$. Thus, $\prod_{\alpha} \exp_{\mathcal{U}^{\alpha}}(\text{im}(\Theta)) \subseteq \text{im}(\Psi)$. For $(f^{\alpha})_{\alpha \in A} \in \text{im}(\Psi)$, we repeat the same argument in reverse to get the other inclusion. Because the vector space structure of $\mathcal{X}(\mathcal{M})_0^{\geq 2}$ is given by the vector space structure of $\prod_{\alpha \in A} \mathcal{SC}^{\infty}(\mathcal{U}^{\alpha}, \overline{E}^{(k)})^{\geq 2}$, it follows from Lemma 4.2.2 that $\text{Aut}_{\text{id}}(\mathcal{M})$ has the described structure of a polynomial, resp. pro-polynomial, group. Proposition 2.3.5 shows that $\exp_{\mathcal{M}}(X)$ is already uniquely determined by $((\varphi^{\alpha})^{-1} \circ \exp(X) \circ \varphi^{\alpha})_{\alpha \in A}$ for all $X \in \mathcal{X}(\mathcal{M})_0^{\geq 2}$. By Lemma 4.1.6 the topology of $\mathcal{X}(\mathcal{M})_0^{\geq 2}$ does not depend on the atlas and thus neither does the topology of $\text{Aut}_{\text{id}}(\mathcal{M})$. By the local definition of the Lie bracket together with Lemma 4.2.2, it is obvious that $\mathcal{X}(\mathcal{M})_0^{\geq 2}$ is the Lie algebra of $\text{Aut}_{\text{id}}(\mathcal{M})$. \square

Remark 4.2.7. Proposition 4.2.6 enables us to give an alternative proof of Batchelor's Theorem. Let $k \in \mathbb{N} \cup \{\infty\}$ and let \mathcal{M} be a k -supermanifold. Then an isomorphism $f^{(n)}: \mathcal{M}^{(n)} \rightarrow \mathcal{M}^{(n)}$ with $f_{\Lambda_1} = \text{id}_{\mathcal{M}_{\Lambda_1}}$ and $1 \leq n < k$ can be lifted to an isomorphism $f^{(n+1)}: \mathcal{M}^{(n+1)} \rightarrow \mathcal{M}^{(n+1)}$ if and only if the vector field $X^{(n)} \in \mathcal{X}(\mathcal{M}^{(n)})_0^{\geq 2}$ with $\exp_{\mathcal{M}^{(n)}}(X^{(n)}) = f^{(n)}$ can be lifted to a vector field $X^{(n+1)} \in \mathcal{X}(\mathcal{M}^{(n+1)})^{\geq 2}$. If $\mathcal{M}_{\mathbb{R}}$ admits smooth partitions of unity, the lift of X can be constructed with standard arguments.

Corollary 4.2.8. Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be a σ -compact k -supermanifold modelled on $E \in \mathbf{SVec}_{\text{lc}}$. Then the restriction

$$\exp_{\mathcal{M}}^c := \exp_{\mathcal{M}}|_{\mathcal{X}_c(\mathcal{M})_0^{\geq 2}}: \mathcal{X}_c(\mathcal{M})_0^{\geq 2} \rightarrow \text{Aut}_{\text{id}}^c(\mathcal{M})$$

is a well-defined bijection that turns $\text{Aut}_{\text{id}}^c(\mathcal{M})$ into a polynomial group if $\min\{k, \dim(E_1)\} < \infty$ and into a pro-polynomial group otherwise. The Lie algebra of $\text{Aut}_{\text{id}}^c(\mathcal{M})$ is $\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}$.

Proof. Let $\{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ be an atlas of \mathcal{M} such that $(\varphi_{\mathbb{R}}^\alpha(\mathcal{U}_{\mathbb{R}}^\alpha))_\alpha$ is a locally finite cover of $\mathcal{M}_{\mathbb{R}}$ by relatively compact sets. Recall the maps Θ and Ψ from Proposition 4.2.6. It follows from the definition and our choice of the atlas that for $X \in \mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}$, we have $X \in \mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}$ if and only if $(\Theta(X))_\alpha = 0$ for almost all $\alpha \in A$. Likewise, for $f \in \text{Aut}_{\text{id}}(\mathcal{M})$, we have $f \in \text{Aut}_{\text{id}}^c(\mathcal{M})$ if and only if $(\Psi(f))_\alpha = (\text{id}_{\mathcal{U}^\alpha}, c_{\text{id}_{E_1}}, 0, \dots)$ for almost all $\alpha \in A$. The claim follows now from Proposition 4.2.6. \square

Remark 4.2.9. In the situation of the corollary above, we get the following commutative diagram corresponding to the diagram in Proposition 4.2.6:

$$\begin{array}{ccc} \mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2} & \xrightarrow{\exp_{\mathcal{M}}^c} & \text{Aut}_{\text{id}}^c(\mathcal{M}) \\ \downarrow \Theta|_{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}} & & \downarrow \Psi|_{\text{Aut}_{\text{id}}^c(\mathcal{M})} \\ \bigoplus_{\alpha \in A} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{E}^{(k)})_{\geq 2} & \xrightarrow{\bigoplus_{\alpha} \exp_{\mathcal{U}^\alpha}} & \prod_{\alpha \in A}^* \text{Aut}_{\text{id}}(\mathcal{U}^\alpha). \end{array}$$

Proposition 4.2.10. Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be modelled on $E \in \mathbf{SVec}_{lc}$ such that E_0 is finite-dimensional and E_1 is a Banach space. Then the global chart

$$\exp_{\mathcal{M}}: \mathcal{X}(\mathcal{M})_{\bar{0},b}^{\geq 2} \rightarrow \text{Aut}_{\text{id}}(\mathcal{M})$$

turns $\text{Aut}_{\text{id}}(\mathcal{M})$ into a Lie group with the Lie algebra $\mathcal{X}(\mathcal{M})_{\bar{0},b}^{\geq 2}$. If, in addition, $\mathcal{M}_{\mathbb{R}}$ is σ -compact, then $\text{Aut}_{\text{id}}^c(\mathcal{M})$ becomes a Lie group with the global chart

$$\exp_{\mathcal{M}}^c: \mathcal{X}_c(\mathcal{M})_{\bar{0},b}^{\geq 2} \rightarrow \text{Aut}_{\text{id}}^c(\mathcal{M})$$

and the Lie algebra $\mathcal{X}_c(\mathcal{M})_{\bar{0},b}^{\geq 2}$

Proof. In view of Proposition 4.2.6 and Remark 4.2.9, both results follow immediately from Lemma 4.2.3. \square

Lemma 4.2.11. Let $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{M} \in \mathbf{SMan}^{(k)}$, $f \in \text{Aut}(\mathcal{M})$ and $X \in \mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}$. Then we have

$$\exp_{\mathcal{M}}(\mathcal{T}f \circ X \circ f^{-1}) = f \circ \exp_{\mathcal{M}}(X) \circ f^{-1}.$$

Proof. Let $\mathcal{A} = \{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ be an atlas of \mathcal{M} . For a $\alpha, \beta \in A$ we define $\tilde{\mathcal{U}}^{\alpha\beta} := f^{-1}(\varphi^\alpha(\mathcal{U}^{\alpha\beta}))$ and $f^{\alpha\beta} := (\varphi^\beta)^{-1} \circ f \circ \varphi^\alpha|_{\tilde{\mathcal{U}}^{\alpha\beta}}$. Then we have

$$\mathcal{T}f^{\alpha\beta} \circ X^\alpha \circ (f^{\alpha\beta})^{-1} = \mathcal{T}(\varphi^\beta)^{-1} \circ \mathcal{T}f \circ X \circ f \circ \varphi^\beta|_{\widetilde{\mathcal{U}}^{\beta\alpha}}$$

and the result follows from Lemma 4.2.5. \square

Lemma 4.2.12. *Let $k \in \mathbb{N} \cup \{\infty\}$ and \mathcal{M} be a σ -compact Banach k -supermanifold of Batchelor type with finite-dimensional base. If we identify $\text{Aut}_c(\mathcal{M}^{(1)}) \cong \iota_\infty^1(\text{Aut}_c(\mathcal{M}^{(1)}))$, then the actions*

$$\beta: \text{Aut}_c(\mathcal{M}^{(1)}) \times \mathcal{X}(\mathcal{M})_{\bar{0},b} \rightarrow \mathcal{X}(\mathcal{M})_{\bar{0},b}, \quad (f, X) \mapsto \mathcal{T}f \circ X \circ f^{-1}$$

and

$$\beta_c: \text{Aut}_c(\mathcal{M}^{(1)}) \times \mathcal{X}_c(\mathcal{M})_{\bar{0},b} \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0},b}, \quad (f, X) \mapsto \mathcal{T}f \circ X \circ f^{-1}$$

are smooth. These actions restrict to smooth actions $\text{Aut}_c(\mathcal{M}^{(1)}) \times \mathcal{X}(\mathcal{M})_{\bar{0},b}^{\geq 2} \rightarrow \mathcal{X}(\mathcal{M})_{\bar{0},b}^{\geq 2}$ and $\text{Aut}_c(\mathcal{M}^{(1)}) \times \mathcal{X}_c(\mathcal{M})_{\bar{0},b}^{\geq 2} \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0},b}^{\geq 2}$. Moreover, the action

$$\text{Aut}_c(\mathcal{M}^{(1)}) \times \mathcal{X}_c(\mathcal{M})_{\bar{1},b} \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{1},b}, \quad (f, X) \mapsto \bar{\Pi}(\mathcal{T}f) \circ X \circ f^{-1}$$

is smooth.

Proof. We identify $\mathcal{M}^{(1)}$ with the vector bundle over $M := \mathcal{M}_{\mathbb{R}}$, $\text{Aut}_c(\mathcal{M}^{(1)})$ with the compactly supported vector bundle morphisms and use the notations from Chapter D. Let $\{\varphi^i: \mathcal{U}^i \rightarrow \mathcal{M}: i \in \mathbb{N}\}$ be the atlas defined by $\varphi_{\mathbb{R}}^i(\mathcal{U}_{\mathbb{R}}^i) = U_i$ and $\tau_i = \varphi_{\Lambda_1}^i \circ ((\varphi_{\mathbb{R}}^i)^{-1}, \text{id}_{E_1})$. Recall the covering $(W_i)_{i \in \mathbb{N}}$ of M from Remark D.2.1. We set $\tilde{U}_i := (\varphi_{\mathbb{R}}^i)^{-1}(U_i)$, $\tilde{V}_i := (\varphi_{\mathbb{R}}^i)^{-1}(V_i)$ and $\tilde{W}_i := (\varphi_{\mathbb{R}}^i)^{-1}(W_i)$. First, we show that for each $f \in \text{Aut}_c(\mathcal{M}^{(1)})$ the automorphism

$$\mathcal{X}_c(\mathcal{M})_{\bar{0},b} \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0},b}, \quad X \mapsto \mathcal{T}f \circ X \circ f^{-1}$$

is smooth. Let $X^{\varphi^i} := \text{pr}_2 \circ \mathcal{T}(\varphi^i)^{-1} \circ X \circ \varphi^i$ denote the usual trivialization. The set $\{f^{-1} \circ \varphi^i: i \in \mathbb{N}\}$ is also a locally finite atlas of \mathcal{M} because f is an automorphism of \mathcal{M} . Thus, the mapping

$$X \mapsto \bigoplus_{i \in \mathbb{N}} \text{pr}_2 \circ \mathcal{T}((\varphi^i)^{-1} \circ f) \circ X \circ f^{-1} \circ \varphi^i = \bigoplus_{i \in \mathbb{N}} X^{f^{-1} \circ \varphi^i}$$

is smooth by definition of the topology on $\mathcal{X}_c(\mathcal{M})_{\bar{0},b}$. Therefore, the mapping

$$X \mapsto \bigoplus_{i \in \mathbb{N}} \left(\mathcal{T}f \circ X \circ f^{-1} \right)^{\varphi^i}$$

is smooth as well, which implies the claim. For $X \in \mathcal{X}(\mathcal{M})_{\bar{0},b}$ the same argument works after substituting direct sums with products. By Lemma A.3.1, it now suffices to show that β_c is smooth on $O \times \mathcal{X}_c(\mathcal{M})_{\bar{0},b}$ for some open unity neighborhood $O \subseteq \text{Gau}_c(\mathcal{M}^{(1)}) \circ S(\mathcal{O}) \subseteq \text{Aut}_c(\mathcal{M}^{(1)})$. Thus, we may check smoothness separately for open subsets of $\text{Gau}_c(\mathcal{M}^{(1)})$ and $\text{Diff}_c(\mathcal{M}_{\mathbb{R}})$. For $f \in O$, we define the local descriptions $f^i = (f_0^i, f_1^i, 0, \dots) := (\varphi^i)^{-1} \circ f \circ \varphi^i|_{\mathcal{U}^i|_{\tilde{W}_i}}$ and $(f^i)^{-1} = ((f_0^i)^{-1}, (f_1^i)^{-1}, 0, \dots) := (\varphi^i)^{-1} \circ f^{-1} \circ \varphi^i|_{\mathcal{U}^i|_{\tilde{V}_i}}$. With this, the action on

$X \in \mathcal{X}(\mathcal{M})_{\bar{0},b}$ is given locally by

$$\text{pr}_2 \circ \mathcal{T}(\varphi^i)^{-1} \circ \mathcal{T}f \circ X \circ f^{-1} \circ \varphi^i|_{\mathcal{U}^i|_{V_i}} = df^i \circ (\text{id}_{\mathcal{U}^i}, X^{\varphi^i}) \circ (f^i)^{-1}.$$

More specifically, by Remark 2.2.15, $(df^i \circ (\text{id}_{\mathcal{U}^i}, X^{\varphi^i}) \circ (f^i)^{-1})_n$ equals

$$\begin{aligned} X_0^{f^i} &:= df_0^i((f_0^i)^{-1}, X_0^{\varphi^i}((f_0^i)^{-1})) \quad \text{for } n = 0, \\ X_n^{f^i} &:= df_0^i\left((f_0^i)^{-1}, X_n^{\varphi^i}((f_0^i)^{-1})((f_1^i)^{-1}, \dots, (f_1^i)^{-1})\right) \quad \text{for } n > 0 \text{ even} \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} X_1^{f^i} &:= f_1^i((f_0^i)^{-1})(X_1^{\varphi^i}((f_0^i)^{-1})((f_1^i)^{-1})) \quad \text{for } n = 1, \\ X_n^{f^i} &:= f_1^i((f_0^i)^{-1})(X_n^{\varphi^i}((f_0^i)^{-1})((f_1^i)^{-1}, \dots, (f_1^i)^{-1})) + \\ &\quad n \cdot \mathfrak{A}^n df_1^i((f_0^i)^{-1})(X_{n-1}^{\varphi^i}((f_0^i)^{-1})((f_1^i)^{-1}, \dots, (f_1^i)^{-1}), (f_1^i)^{-1}) \end{aligned} \quad (4.7)$$

for $n > 1$ odd. For $X \in \mathcal{X}_c(\mathcal{M})_{\bar{0},b}$ define the projections $p_{\tilde{U}_i}(X) := X^{\varphi^i}$, resp. $p_{\tilde{V}_i}(X) := X^{\varphi^i|_{\tilde{V}_i}}$. Furthermore, let $\Omega = \varphi_g^{-1}(\mathcal{O})$ and $\rho_i: \mathfrak{X}_c(M) \rightarrow \mathcal{C}^\infty(U_i, E_0)$ be like in Remark D.2.2.

If we find an open zero-neighborhood $\tilde{\Omega} \subseteq \Omega$ and smooth maps $\beta_c^i: \tilde{\Omega}_i \times \prod_{n=0} \mathcal{C}^\infty(\tilde{U}_i, \text{Alt}^n(E_1; E_{\bar{n}})_b) \rightarrow \prod_{n=0} \mathcal{C}^\infty(\tilde{V}_i, \text{Alt}^n(E_1; E_{\bar{n}})_b)$, where $\rho_i(\tilde{\Omega}) \subseteq \tilde{\Omega}_i \subseteq \mathcal{C}^\infty(U_i, E_0)$ open, such that for

$$\tilde{\beta}: \tilde{\Omega} \times \mathcal{X}_c(\mathcal{M})_{\bar{0},b} \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0},b}, \quad (Y, X) \mapsto \beta_c(S(\exp_g \circ Y), X),$$

we have $\beta_c^i(\rho_i(Y), p_{\tilde{U}_i}(X)) = p_{\tilde{V}_i}(\tilde{\beta}(Y, X))$ for all $Y \in \tilde{\Omega}$ and $X \in \mathcal{X}_c(\mathcal{M})_{\bar{0},b}$, then smoothness follows from Lemma A.1.4.

We construct open sets $\tilde{\Omega}_i$ and smooth maps $\zeta_i: \tilde{\Omega}_i \rightarrow \mathcal{O}$ such that $\zeta_i(Y)|_{W_i}$ depends only on $\rho_i(Y)|_{W_i}$ like in Remark D.2.2. Let β_c^i be defined by the local action of $f = S(\zeta_i(Y)) \in S(\zeta_i(\tilde{\Omega}_i))$ on $\prod_{n=0} \mathcal{C}^\infty(\tilde{U}_i, \text{Alt}^n(E_1; E_{\bar{n}})_b)$ as in formula (4.6) and (4.7). By Remark D.3.3, this action depends only on $\rho_i(Y)|_{W_i}$ if $Y \in \tilde{\Omega}_i$. By Proposition A.2.7 and Corollary A.2.13, it suffices to see that the map

$$((\beta_c^i)_n^\wedge)^\wedge: \tilde{\Omega}_i \times \prod_{n=0} \mathcal{C}^\infty(\tilde{U}_i, \text{Alt}^n(E_1; E_{\bar{n}})_b) \times \tilde{V}_i \times E_1^n \rightarrow E_0, \quad (Y, X, x, v) \mapsto X_n^{f^i}(x, v)$$

is smooth. But this is true because the map can be written using the smooth evaluations from Proposition A.2.8, Lemma D.3.5 and Remark D.3.6.

All that remains to be seen is that there exists an open unity neighborhood of $\text{Gau}_c(\mathcal{M}^{(1)})$ which acts smoothly on $\mathcal{X}_c(\mathcal{M})_{\bar{0},b}$. The argument is essentially the same as before. Note that for the embedding

$$\text{Gau}_c(\mathcal{M}^{(1)}) \hookrightarrow \prod_{i \in \mathbb{N}}^* \mathcal{C}^\infty(\bar{U}_i, \text{Gl}_{E_1}), \quad f \mapsto (f_i)_{i \in \mathbb{N}}$$

like in (D.2), the skeletons $(\text{id}_{\tilde{U}_i}, f_i \circ \varphi_0^i, 0, \dots)$ correspond to the local descriptions

above (we do not need to restrict to \widetilde{V}_i or \widetilde{W}_i because these maps are the identity on the base manifold). Let $P \subseteq \mathfrak{gl}_{E_1}$ be as in Remark D.1.1 and $Q_i := \mathcal{C}^\infty(\overline{U}_i, P)$. Then the mappings

$$Q_i \times \prod_{n=0} \mathcal{C}^\infty(\widetilde{U}_i, \mathcal{A}l t^n(E_1; E_{\overline{n}})) \rightarrow \prod_{n=0} \mathcal{C}^\infty(\widetilde{U}_i, \mathcal{A}l t^n(E_1; E_{\overline{n}}))$$

defined by the local action of $(\exp_{\mathrm{Gl}_{E_1}})^i(Q_i)$ as in formula (4.6) and (4.7) are smooth by Lemma D.3.4 because of the same arguments as before. Overall, smoothness follows again from Lemma A.1.4.

Checking that β is smooth on $O \times \mathcal{X}(\mathcal{M})_{\overline{0},b}$ is much simpler. We only need to see that the components (4.6) and (4.7) depend smoothly on X and f and this follows directly by using the same evaluations as above. That the actions restrict to actions on the closed subspaces $\mathcal{X}(\mathcal{M})_{\overline{0},b}^{\geq 2}$ and $\mathcal{X}_c(\mathcal{M})_{\overline{0},b}^{\geq 2}$ follows from Lemma 4.1.13.

In the case of $X \in \mathcal{X}_c(\mathcal{M})_{\overline{1},b}$, we set $X^{\varphi^i} := \mathrm{pr}_2 \circ \overline{\Pi}(\mathcal{T}\varphi^i) \circ X \circ (\varphi^i)^{-1}$. With this, the same arguments as above show that

$$\mathcal{X}_c(\mathcal{M})_{\overline{1},b} \rightarrow \mathcal{X}_c(\mathcal{M})_{\overline{1},b}, \quad X \mapsto \overline{\Pi}(\mathcal{T}f) \circ X \circ f^{-1}$$

is smooth for every $f \in \mathrm{Aut}_c(\mathcal{M}^{(1)})$. We again use Remark 4.1.15 to calculate $(\overline{\Pi}(f^i) \circ \mathrm{id}_{\mathcal{U}^i}, X^{\varphi^i}) \circ (\varphi^i)^{-1}$ as

$$f_1^i((f_0^i)^{-1})(X_n^{\varphi^i}((f_0^i)^{-1})((f_1^i)^{-1}, \dots, (f_1^i)^{-1}))$$

for even n and as

$$\begin{aligned} & df_0^i((f_0^i)^{-1}, X_n^{\varphi^i}((f_0^i)^{-1})((f_1^i)^{-1}, \dots, (f_1^i)^{-1})) + \\ & n \cdot \mathfrak{A}^n df_1^i(f_0^i)^{-1}(X_{n-1}^{\varphi^i}(f_0^i)^{-1}((f_1^i)^{-1}, \dots, (f_1^i)^{-1}), (f_1^i)^{-1}) \end{aligned}$$

for odd n . From here the same arguments as before show smoothness. \square

Proposition 4.2.13. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be a σ -compact Banach k -supermanifold of Batchelor type with finite-dimensional base. Then*

$$\mathrm{Aut}_c(\mathcal{M}) = \mathrm{Aut}_{\mathrm{id}}^c(\mathcal{M}) \rtimes \mathrm{Aut}_c(\mathcal{M}^{(1)})$$

is a Lie group with the Lie algebra $\mathcal{X}_c(\mathcal{M})_{\overline{0},b}$ and

$$\mathrm{Aut}_{\Pi,c}(\mathcal{M}) := \mathrm{Aut}_{\mathrm{id}}(\mathcal{M}) \rtimes \mathrm{Aut}_c(\mathcal{M}^{(1)})$$

is a Lie group with the Lie algebra $\mathcal{X}(\mathcal{M})_{\overline{0},b}^{\geq 2} \oplus \mathcal{X}_c(\mathcal{M}^{(1)})_{\overline{0},b}$.

Proof. Let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$. That $\mathrm{Aut}_c(\mathcal{M})$ and $\mathrm{Aut}_{\Pi,c}(\mathcal{M})$ are Lie groups follows from Proposition 4.2.10, Lemma 4.2.11 and Lemma 4.2.12. We already know that $\mathrm{Aut}_c(\mathcal{M}^{(1)})$ has the Lie bracket $\mathcal{X}_c(\mathcal{M}^{(1)})_{\overline{0},b}$ from Lemma D.3.9 and that $\mathrm{Aut}_{\mathrm{id}}(\mathcal{M})$ has the Lie bracket $\mathcal{X}_c(\mathcal{M})_{\overline{0},b}^{\geq 2}$ from Corollary 4.2.8. By Remark

A.3.3, we need to calculate the derivative of the map

$$\text{Aut}_c(\mathcal{M}^{(1)}) \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0},b}^{\geq 2}, \quad (f, X) \mapsto Tf \circ X \circ f^{-1}$$

for an arbitrary $X \in \mathcal{X}_c(\mathcal{M})_{\bar{0},b}^{\geq 2}$ at the identity. To this end, let $\text{ev}: \text{Aut}_c(\mathcal{M}^{(1)}) \times \mathcal{M}^{(1)} \rightarrow \mathcal{M}^{(1)}$ be the evaluation, let $\text{ev}_T: \text{Aut}_c(\mathcal{M}^{(1)}) \times T\mathcal{M}^{(1)} \rightarrow T\mathcal{M}^{(1)}$ be as in Remark D.3.6 and let $[t \mapsto {}_t f] \in T_{\text{id}}\text{Aut}_c(\mathcal{M}^{(1)})$ be identified with $Y \in \mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0},b}$ as in Lemma D.3.9. Note that the lemma implies that $[t \mapsto {}_t f^{-1}]$ corresponds to $-Y$. We use the same notation as in Lemma 4.2.12 for an atlas $\{\varphi^i: \mathcal{U}^i \rightarrow \mathcal{M}: i \in \mathbb{N}\}$ of Batchelor type and the localization ${}_t f^i$ (for t small enough this is well-defined). Because of Corollary A.2.13, we may simply calculate the derivative of the n -th component of the local description of $(\mathcal{T}_t f \circ X \circ {}_t f^{-1})^{\varphi^i}$ after evaluating in $x \in \tilde{U}_i$ and $v_1, \dots, v_n \in E_1$. To simplify our notation, identify φ^i with $(\varphi^i)^{(1)}$ which is possible because the atlas is of Batchelor type. Note that we have

$$[t \mapsto (\varphi^i)^{-1} \circ {}_t f \circ \varphi^i(x, v_1)] = T(\varphi^i)^{-1} \circ Y \circ \varphi^i(x, v_1) = Y^{\varphi^i}(x, v_1).$$

Recall the formulas (4.6) and (4.7) from the proof of Lemma 4.2.12. For even $n > 0$, we have to calculate the derivative of

$$t \mapsto d_{{}_t f_0^i} \left(({}_t f_0^i)^{-1}(x), X_n^{\varphi^i}(({}_t f_0^i)^{-1}(x))(({}_t f_1^i)^{-1}(x, v_1), \dots, ({}_t f_1^i)^{-1}(x, v_n)) \right)$$

at zero. With Lemma D.3.11 and the chain rule, the result is

$$\begin{aligned} & dY_0^{\varphi^i}(x, X_n^{\varphi^i}(x)(v_1, \dots, v_n)) + dX_n^{\varphi^i}(x, -Y_0^{\varphi^i}(x))(v_1, \dots, v_n) \\ & + \sum_{j=1}^n X_n^{\varphi^i}(x)(v_1, \dots, -Y_1^{\varphi^i}(x, v_j), \dots, v_n) \\ & = \left(dY_0^{\varphi^i}(x, X_n^{\varphi^i}(x)(\bullet, \dots, \bullet)) - dX_n^{\varphi^i}(x, Y_0^{\varphi^i}(x))(\bullet, \dots, \bullet) - \right. \\ & \quad \left. n \cdot \mathfrak{A}^n X_n^{\varphi^i}(x)(Y_1^{\varphi^i}(x)(\bullet, \bullet, \dots, \bullet)) \right)(v_1, \dots, v_n). \end{aligned}$$

Likewise, for odd $n > 1$, we consider the derivative of the map

$$\begin{aligned} t \mapsto & \left(n \cdot \mathfrak{A}^n d_{{}_t f_1^i}(({}_t f_0^i)^{-1})(X_{n-1}^{\varphi^i}(({}_t f_0^i)^{-1})(({}_t f_1^i)^{-1}, \dots, ({}_t f_1^i)^{-1}), ({}_t f_1^i)^{-1} \right) + \\ & {}_t f_1^i(({}_t f_0^i)^{-1})(X_n^{\varphi^i}(({}_t f_0^i)^{-1})(({}_t f_1^i)^{-1}, \dots, ({}_t f_1^i)^{-1}))(x, v_1, \dots, v_n) \end{aligned}$$

at zero. By the same arguments as before, this results in

$$\begin{aligned} & \left(n \cdot \mathfrak{A}^n d_1 Y_1^{\varphi^i}(x, X_{n-1}^{\varphi^i}(x)(\bullet, \dots, \bullet))(\bullet) + Y_1^{\varphi^i}(x)(X_n^{\varphi^i}(x)(\bullet, \dots, \bullet)) \right. \\ & \quad \left. - dX_n^{\varphi^i}(x, Y_0^{\varphi^i}(x))(\bullet, \dots, \bullet) - n \cdot \mathfrak{A}^n X_n^{\varphi^i}(x)(Y_1^{\varphi^i}(x)(\bullet, \bullet, \dots, \bullet)) \right)(v_1, \dots, v_n). \end{aligned}$$

Comparing the formulas with the calculations in Remark 4.1.15, it follows that the derivative of $t \mapsto T_t f \circ X \circ {}_t f^{-1}$ at zero is $[Y, X]$. The same calculations show

that $\text{Aut}_{\Pi,c}(\mathcal{M})$ has the Lie algebra $\mathcal{X}(\mathcal{M})_{\bar{0},b}^{\geq 2} \oplus \mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0},b}$. \square

Corollary 4.2.14. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be a σ -compact Banach k -supermanifold with finite-dimensional base. Let $g: \mathcal{M} \rightarrow \iota_k^1(\mathcal{M}^{(1)})$ be a Batchelor model of \mathcal{M} . Then we turn $\text{Aut}_c(\mathcal{M})$, resp. $\text{Aut}_{\Pi,c}(\mathcal{M})$, into a Lie group via the isomorphism of groups*

$$\begin{aligned} \Theta_g: \text{Aut}_c(\iota_k^1(\mathcal{M}^{(1)})) &\rightarrow \text{Aut}_c(\mathcal{M}), & f &\mapsto g \circ f \circ g^{-1}, & \text{resp.} \\ \Theta'_g: \text{Aut}_{\Pi,c}(\iota_k^1(\mathcal{M}^{(1)})) &\rightarrow \text{Aut}_{\Pi,c}(\mathcal{M}), & f &\mapsto g \circ f \circ g^{-1}. \end{aligned}$$

Both Lie group structures do not depend on the Batchelor model.

Proof. Let $g': \mathcal{M} \rightarrow \iota_k^1(\mathcal{M}^{(1)})$ be another Batchelor model and let $\text{Aut}_c(\mathcal{M})'$ denote the Lie group $\text{Aut}_c(\mathcal{M})$ with respect to the Batchelor model g' . The identity $\text{Aut}_c(\mathcal{M}) \rightarrow \text{Aut}_c(\mathcal{M})'$ is smooth if and only if $\Theta_{g'} \circ \Theta_g^{-1}: \text{Aut}_c(\iota_k^1(\mathcal{M}^{(1)})) \rightarrow \text{Aut}_c(\iota_k^1(\mathcal{M}^{(1)}))$ is smooth. But this map is just the conjugation with $g' \circ g^{-1}$, which is smooth. The proof for $\text{Aut}_{\Pi,c}(\mathcal{M})$ is the same. \square

Lemma 4.2.15. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{M} \in \mathbf{SMan}^{(k)}$ be a σ -compact Banach k -supermanifold with finite-dimensional base. Then the linear group actions*

$$\text{Aut}_c(\mathcal{M}) \times \mathcal{X}_c(\mathcal{M})_{\bar{0},b} \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0},b}, \quad (f, X) \mapsto \mathcal{T}f \circ X \circ f^{-1}$$

and

$$\text{Aut}_c(\mathcal{M}) \times \mathcal{X}_c(\mathcal{M})_{\bar{1},b} \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{1},b}, \quad (f, X) \mapsto \bar{\Pi}(\mathcal{T}f) \circ X \circ f^{-1}$$

are smooth.

Proof. Let \mathcal{M} be modelled on $E \in \mathbf{SVec}_{lc}$. We may assume that \mathcal{M} is of Batchelor type, that $\mathcal{A} := \{\varphi^\alpha: \mathcal{U}^\alpha: \mathcal{M}: \alpha \in A\}$ is an atlas of Batchelor type, A is countable and $(\varphi_{\mathbb{R}}^\alpha(\mathcal{U}_{\mathbb{R}}^\alpha))$ is a locally finite cover by relatively compact sets of $\mathcal{M}_{\mathbb{R}}$. Because of the way the smooth structure on $\text{Aut}_c(\mathcal{M})$ is defined, it suffices to check that the respective actions of $\text{Aut}_c(\mathcal{M}^{(1)})$ and $\text{Aut}_{\text{id}}^c(\mathcal{M})$ are smooth. That the former is smooth was shown in Lemma 4.2.12. By definition of the topologies and Lemma A.1.4, it suffices to calculate the latter action locally. The local version of the first action is given by

$$\begin{aligned} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{E}^{(k)})_b^{\geq 2} \times \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{E}^{(k)})_b &\rightarrow \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{E}^{(k)})_b, \\ (f, X) &\mapsto d(\text{id}_{\mathcal{U}^\alpha} + f) \circ (\text{id}_{\mathcal{U}^\alpha}, X) \circ (\text{id}_{\mathcal{U}^\alpha} + f)^{-1}. \end{aligned}$$

The smoothness of this map follows from the fact that the inversion in $\text{Aut}_{\text{id}}(\mathcal{U})$ is smooth and by applying Lemma A.2.14 to the formulas given in Proposition 2.2.16 and Remark 4.1.15. The local version of the second action is given by

$$\begin{aligned} \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{E}^{(k)})_b^{\geq 2} \times \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{\Pi}(\bar{E})^{(k)})_b &\rightarrow \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{\Pi}(\bar{E})^{(k)})_b, \\ (f, X) &\mapsto \bar{\Pi}(d(\text{id}_{\mathcal{U}^\alpha} + f)) \circ (\text{id}_{\mathcal{U}^\alpha}, X) \circ (\text{id}_{\mathcal{U}^\alpha} + f)^{-1}. \end{aligned}$$

With the formula from Lemma 2.5.11 together with Lemma A.2.2, we see that that

$$\mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{E}^{(k)})_b^{\geq 2} \rightarrow \mathcal{SC}^\infty(\mathcal{U}^\alpha \times \overline{\Pi}(\overline{E}^{(k)}), \overline{\Pi}(\overline{E}^{(k)}))_b^{\geq 2}, \quad f \mapsto \overline{\Pi}(\mathrm{d}(\mathrm{id}_{\mathcal{U}^\alpha} + f))$$

is smooth. Therefore, the second action is smooth by the same arguments as before. \square

4.3. The Functor of Supermorphisms and Superdiffeomorphisms

In a sense, the set of supersmooth morphisms $\mathcal{SC}^\infty(\mathcal{M}, \mathcal{N})$ between two supermanifolds \mathcal{M} and \mathcal{N} only consists of the even morphisms. To get the full picture, we need to define a functor $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N}): \mathbf{Gr} \rightarrow \mathbf{Set}$ that describes also higher points of the supersmooth morphisms. One has a composition law that turns $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})$ into a monoid in $\mathbf{Set}^{\mathbf{Gr}}$. The supergroup of superdiffeomorphisms is then defined as the subfunctor of $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})$ consisting of invertible elements.

For everything that follows, it will be integral to understand Grassmann algebras as purely odd supermanifolds. As mentioned in the introduction, in the sheaf theoretic approach this is trivial. However, in the categorical approach, this is more involved.

4.3.1. Superpoints

Consider $\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \dots \subseteq \bigoplus_{i \in \mathbb{N}} \mathbb{R}$. For the remainder of this work, we fix a basis $(\mathbf{v}_i)_{i \in \mathbb{N}}$ of $\bigoplus_{i \in \mathbb{N}} \mathbb{R}$ such that $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis of \mathbb{R}^n for all $n \in \mathbb{N}$. For a set $I = \{i_1, \dots, i_r\} \in \mathcal{P}^n$ with $i_1 < \dots < i_r$, we define $\mathbf{v}_I := \mathbf{v}_{i_1}^* \wedge \dots \wedge \mathbf{v}_{i_r}^* \in \mathcal{A}lt^r(\mathbb{R}^n; \mathbb{R})$. It is well-known that the wedge product turns $\bigwedge(\mathbb{R}^n)^* := \bigoplus_{0 \leq r \leq n} \mathcal{A}lt^r(\mathbb{R}^n; \mathbb{R})$ into an \mathbb{R} -superalgebra and that

$$\Lambda_n \rightarrow \bigoplus_{0 \leq r \leq n} \mathcal{A}lt^r(\mathbb{R}^n; \mathbb{R}), \quad (a_I \lambda_I)_{I \in \mathcal{P}^n} \mapsto \sum_{I \in \mathcal{P}^n} a_I \mathbf{v}_I \quad (4.8)$$

is an isomorphism of \mathbb{R} -superalgebras. We cannot avoid fixing a base because we need consistent isomorphisms (4.8) for all $n \in \mathbb{N}$. This should be understood as analogous to fixing the generators λ_i .

Definition 4.3.1. Let **SPoint** denote the full subcategory of **SMan** consisting of finite-dimensional purely odd supermanifolds whose base manifold consists of a single point.

Lemma 4.3.2. Let \mathbf{Gr}° be the dual category of \mathbf{Gr} . There exists an equivalence of categories

$$\mathcal{P}: \mathbf{Gr}^\circ \rightarrow \mathbf{SPoint}$$

such that $\mathcal{P}(\Lambda_n) = \overline{\mathbb{R}^{0|n}}$. To avoid clunky notation, we will also consider \mathcal{P} as a contravariant functor $\mathbf{Gr} \rightarrow \mathbf{SPoint}$.

Proof. This follows from [1, Theorem 3.13, p.589]. See also [40, Section 8.1, p.415] and [46, Proposition 2.8, p.8]. \square

Let us explain how \mathcal{P} acts on morphisms. We identify an element $x = (a_I \lambda_I)_{I \in \mathcal{P}^n} \in \Lambda_n$, where $a_I \in \mathbb{R}$, with the skeletons $(\sum_{I \in \mathcal{P}^n, |I|=l} a_I \mathbf{v}_I)_{0 \leq l \leq n}$, i.e., we can see x as a supersmooth morphism $x: \overline{\mathbb{R}^{0|n}} \rightarrow \overline{\mathbb{R}^{1|1}}$. For any supersmooth morphism $(f_l)_l: \overline{\mathbb{R}^{0|m}} \rightarrow \overline{\mathbb{R}^{0|n}}$, we then get a morphism of superalgebras $\varrho: \Lambda_n \rightarrow \Lambda_m$, $x \mapsto x \circ (f_l)_l$ (compare [1, Corollary 3.6, p.585]). Conversely, $\varrho(\lambda_i)$ corresponds to the skeleton defining the i -th component (with respect to the basis (\mathbf{v}_i)) of $(f_l)_l$.

4.3.2. Internal Hom objects and supermorphisms

Let \mathcal{C} be a category with products and $B \in \mathcal{C}$. An internal Hom functor is a functor $\underline{\text{Hom}}_{\mathcal{C}}(B, \bullet): \mathcal{C} \rightarrow \mathcal{C}$ that satisfies

$$\text{Hom}_{\mathcal{C}}(A, \underline{\text{Hom}}_{\mathcal{C}}(B, C)) \cong \text{Hom}_{\mathcal{C}}(A \times B, C) \text{ for all } A, C \in \mathcal{C}$$

(see for example [35, p.180]). Of course, in general inner Hom functors need not exist but for appropriate objects A, B and C one might at least find an internal Hom object $\underline{\text{Hom}}_{\mathcal{C}}(B, C) \in \mathcal{C}$ satisfying the above. In our situation, the following fact gives us necessary properties an internal Hom object needs to have.

Corollary 4.3.3 ([40, Corollary 8.1.2]). *For every $\mathcal{M} \in \mathbf{SMan}$ and every $\Lambda \in \mathbf{Gr}$ there exists an isomorphism of sets*

$$\mathcal{M}_{\Lambda} \cong \mathcal{SC}^{\infty}(\mathcal{P}(\Lambda), \mathcal{M})$$

natural both in \mathcal{M} and Λ .

Assuming that $\underline{\text{Hom}}_{\mathbf{SMan}}(\mathcal{M}, \mathcal{N})$ exists for $\mathcal{M}, \mathcal{N} \in \mathbf{SMan}$, it follows

$$\underline{\text{Hom}}_{\mathbf{SMan}}(\mathcal{M}, \mathcal{N})_{\Lambda} \cong \mathcal{SC}^{\infty}(\mathcal{P}(\Lambda), \underline{\text{Hom}}_{\mathbf{SMan}}(\mathcal{M}, \mathcal{N})) \cong \mathcal{SC}^{\infty}(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{N})$$

for all $\Lambda \in \mathbf{Gr}$. We simply take this as the definition of a functor

$$\widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N}): \mathbf{Gr} \rightarrow \mathbf{Set}, \quad \Lambda \mapsto \mathcal{SC}^{\infty}(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{N}).$$

For morphisms $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$, we define

$$\begin{aligned} \widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N})_{\varrho}: \widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N})_{\Lambda} &\rightarrow \widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N})_{\Lambda'}, \\ f &\mapsto f \circ (\mathcal{P}(\varrho) \times \text{id}_{\mathcal{M}}). \end{aligned}$$

Functoriality follows immediately from the properties of the contravariant functor \mathcal{P} . We call this functor the *functor of supermorphisms (of \mathcal{M} to \mathcal{N})*. Further motivation for the functor of supermorphisms can be found in [45, Section 3.3.3, p.54] and [40, Section 8.2, p.415].

This raises the question for which supermanifolds \mathcal{M}, \mathcal{N} the functor $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N})$ can be given the structure of a supermanifold. For finite-dimensional supermanifolds this has been achieved in [27]. See also [40, Section 8.5, p.418].

A nice feature of the functor of supermorphisms is that one has a natural composition. Let $\mathcal{M}, \mathcal{M}', \mathcal{M}'' \in \mathbf{SMan}$. For every $\Lambda \in \mathbf{Gr}$, we define a map

$$\begin{aligned} \circ_\Lambda: \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M}')_\Lambda \times \widehat{\mathcal{SC}}^\infty(\mathcal{M}', \mathcal{M}'')_\Lambda &\rightarrow \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M}'')_\Lambda, \\ (g, f) &\mapsto f \circ_\Lambda g := f \circ (\text{id}_{\mathcal{P}(\Lambda)}, g). \end{aligned}$$

This defines a natural transformation

$$\circ: \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M}') \times \widehat{\mathcal{SC}}^\infty(\mathcal{M}', \mathcal{M}'') \rightarrow \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M}'')$$

because for all $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda', \Lambda)$, $f \in \widehat{\mathcal{SC}}^\infty(\mathcal{M}', \mathcal{M}'')_\Lambda$ and $g \in \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M}')_\Lambda$, we have

$$\begin{aligned} f \circ (\text{id}_{\mathcal{P}(\Lambda)}, g) \circ (\mathcal{P}(\varrho) \times \text{id}_{\mathcal{M}}) &= f \circ (\mathcal{P}(\varrho), (g \circ (\mathcal{P}(\varrho), \text{id}_{\mathcal{M}}))) \\ &= (f \circ (\mathcal{P}(\varrho) \times \text{id}_{\mathcal{M}'})) \circ (\text{id}_{\mathcal{P}(\Lambda)}, g \circ (\mathcal{P}(\varrho) \times \text{id}_{\mathcal{M}})). \end{aligned} \quad (4.9)$$

This natural transformation is associative in the sense that for every $\Lambda \in \mathbf{Gr}$, we have

$$\begin{aligned} (f \circ_\Lambda (g \circ_\Lambda h)) &= f \circ (\text{id}_{\mathcal{P}(\Lambda)}, g \circ (\text{id}_{\mathcal{P}(\Lambda)}, h)) \\ &= (f \circ (\text{id}_{\mathcal{P}(\Lambda)}, g)) \circ (\text{id}_{\mathcal{P}(\Lambda)}, h) = (f \circ_\Lambda g) \circ_\Lambda h, \end{aligned}$$

for all $f \in \widehat{\mathcal{SC}}^\infty(\mathcal{M}', \mathcal{M}'')_\Lambda$, $g \in \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M}')_\Lambda$ and $h \in \widehat{\mathcal{SC}}^\infty(\mathcal{N}, \mathcal{M})_\Lambda$, where $\mathcal{N} \in \mathbf{SMan}$. If one just considers $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})$, then there exists a unit element $e_{\mathcal{M}}$ for the composition. Defining

$$e_{\mathcal{M}}: \mathbf{Gr} \rightarrow \mathbf{Set}, \quad \Lambda \mapsto \{\text{pr}_{\mathcal{M}}: \mathcal{P}(\Lambda) \times \mathcal{M} \rightarrow \mathcal{M}\},$$

we see that $e_{\mathcal{M}}$ is a subfunctor of $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})$ and for every $\Lambda \in \mathbf{Gr}$, we have that $e_{\mathcal{M}}(\Lambda)$ is the unit element of $\circ_\Lambda: \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})_\Lambda \times \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})_\Lambda \rightarrow \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})_\Lambda$. In other words, $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})$ is a monoid with the unit element $e_{\mathcal{M}}$ in the category $\mathbf{Set}^{\mathbf{Gr}}$.

Definition 4.3.4. Let $\mathcal{M}, \mathcal{N} \in \mathbf{SMan}$, $\Lambda \in \mathbf{Gr}$ and $f \in \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N})_\Lambda$. We say f is *invertible* if there exists a morphism $f^{-1} \in \widehat{\mathcal{SC}}^\infty(\mathcal{N}, \mathcal{M})_\Lambda$ such that

$$f \circ_\Lambda f^{-1} = e_{\mathcal{N}}(\Lambda) \quad \text{and} \quad f^{-1} \circ_\Lambda f = e_{\mathcal{M}}(\Lambda).$$

If an inverse exists it is unique.

The above discussion of supermorphisms is largely taken from [47, Section 5, p.301 ff.]. For a slightly different approach see also [40, Section 8.3, p.416 f.].

Remark 4.3.5. One can identify $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N})_\Lambda$ with a subset of $\mathcal{SC}^\infty(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{P}(\Lambda) \times \mathcal{N})$ by mapping

$$\Psi_\Lambda: \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N})_\Lambda \rightarrow \mathcal{SC}^\infty(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{P}(\Lambda) \times \mathcal{N}), \quad f \mapsto (\text{id}_{\mathcal{P}(\Lambda)}, f).$$

Restricting the map

$$\mathcal{SC}^\infty(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{P}(\Lambda) \times \mathcal{N}) \rightarrow \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N})_\Lambda, \quad f' \mapsto \text{pr}_\mathcal{M} \circ f',$$

to the image of Ψ_Λ gives us an inverse. If $\mathcal{N} = \mathcal{M}$, then Ψ_Λ is clearly a morphism of monoids. In particular, the unit element of $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})_\Lambda$ corresponds to $\text{id}_{\mathcal{P}(\Lambda) \times \mathcal{M}}$ and the inverse of an invertible morphism $f \in \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})_\Lambda$ corresponds to $(\text{id}_{\mathcal{P}(\Lambda)}, f)^{-1}$. Conversely, if $(\text{id}_{\mathcal{P}(\Lambda)}, f)$ is invertible, then f is invertible with $f^{-1} = \text{pr}_\mathcal{M} \circ (\text{id}_{\mathcal{P}(\Lambda)}, f)^{-1}$.

Lemma 4.3.6. *Let $E, F \in \mathbf{SVec}_{lc}$, $\mathcal{U} \subseteq \overline{E}$ and $\mathcal{V} \subseteq \overline{F}$ be open subfunctors. Then $f \in \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \mathcal{V})_{\Lambda_n}$ is invertible if and only if $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \mathcal{V})_{\varepsilon_{\Lambda_n}}(f)$ is an isomorphism. In this case $\tilde{f} := (\text{id}_{\mathcal{P}(\Lambda_n)}, f)$ is an isomorphism and the inverse g of f has the skeleton*

$$\begin{aligned} g_0: \mathcal{V}_\mathbb{R} &\rightarrow \mathcal{U}_\mathbb{R}, \quad g_0(x') := f_0^{-1}(x'), \\ g_1: \mathcal{V}_\mathbb{R} &\rightarrow \mathcal{Alt}^1(\mathbb{R}^n \oplus F_1; E_1), \quad g_1(x') := \text{pr}'_2 \circ \tilde{f}_1(g_0(x'))^{-1} \quad \text{and} \\ g_k: \mathcal{V}_\mathbb{R} &\rightarrow \mathcal{Alt}^k(\mathbb{R}^n \oplus F_1; E_1), \\ g_k(x')(v') &:= - \sum_{\substack{m, l < k; (\alpha, \beta) \in I_{m, l}^k, \\ \sigma \in \mathfrak{S}_k}} \frac{\text{sgn}(\sigma)}{m!l!\alpha!\beta!} d^m g_l(x')((\tilde{f}_\alpha \times \tilde{f}_\beta)(g_0(x'))(v^\sigma)), \end{aligned}$$

where $k > 1$, $\text{pr}'_2: \mathbb{R}^n \times F_1 \rightarrow F_1$ is the projection, $v' = (v'_1, \dots, v'_k) \in (\mathbb{R}^n \oplus F_1)^k$ and $v := (\tilde{f}_1(g_0(x'))^{-1}(v'_1), \dots, \tilde{f}_1(g_0(x'))^{-1}(v'_k)) \in (\mathbb{R}^n \oplus E_1)^k$.

Proof. Let $\Lambda := \Lambda_n$. Because $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \mathcal{V})_{\varepsilon_\Lambda}$ respects the composition, it maps invertible elements to invertible elements. Conversely, let $h := \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \mathcal{V})_{\varepsilon_\Lambda}(f)$ be an isomorphism. By Remark 4.3.5 and Lemma 2.2.18, f is invertible if and only if $\tilde{f}^{(1)}: \mathcal{P}(\Lambda) \times \mathcal{U} \rightarrow \mathcal{P}(\Lambda) \times \mathcal{V}$ is invertible. Note that $\tilde{f}_\mathbb{R} = f_\mathbb{R} = h_\mathbb{R}$ has the inverse $h_\mathbb{R}^{-1}$. In terms of skeletons, we have that $\tilde{f}_0 = f_0 = h_0: \mathcal{U}_\mathbb{R} \rightarrow \mathcal{V}_\mathbb{R}$ is a diffeomorphism and $f_1 \in \mathcal{C}^\infty(\mathcal{U}_\mathbb{R}, \mathcal{Alt}^1(\mathbb{R}^n \oplus E_1; F_1))$. In particular we have $f_{11} \in \mathcal{C}^\infty(\mathcal{U}_\mathbb{R}, \mathcal{Alt}^1(\mathbb{R}^n, F_1))$ and $f_{12} \in \mathcal{C}^\infty(\mathcal{U}_\mathbb{R}, \mathcal{Alt}^1(E_1, F_1))$ such that

$$f_1(x) = f_{11}(x)(\text{pr}_1) + f_{12}(x)(\text{pr}_2)$$

for $x \in \mathcal{U}_\mathbb{R}$ with the projections $\text{pr}_1: \mathbb{R}^n \times E_1 \rightarrow \mathbb{R}^n$ and $\text{pr}_2: \mathbb{R}^n \times E_1 \rightarrow E_1$. Because $h_{\Lambda_1} = (f \circ (\mathcal{P}(\varepsilon_\Lambda), \text{id}_\mathcal{U}))_{\Lambda_1}$, it follows $h_1 = f_{12}$ and therefore, we have $f_{12}^{-1} \in \mathcal{C}^\infty(\mathcal{V}_\mathbb{R}, \mathcal{Alt}^1(F_1, E_1))$ for

$$f_{12}^{-1}: \mathcal{V}_\mathbb{R} \rightarrow \mathcal{Alt}^1(F_1, E_1), \quad x' \mapsto (f_{12}(x'))^{-1}.$$

We define

$$\begin{aligned} f_{11}^{-1}: \mathcal{V}_{\mathbb{R}} &\rightarrow \mathcal{A}lt^1(\mathbb{R}^n, E_1), \quad x' \mapsto -f_{12}^{-1}(x') \circ f_{11}(f_0^{-1}(x')) \quad \text{and} \\ f_1^{-1} &:= f_{11}^{-1}(\cdot)(\text{pr}'_1) + f_{12}^{-1}(\cdot)(\text{pr}'_2), \end{aligned}$$

with the projections $\text{pr}'_1: \mathbb{R}^n \times F_1 \rightarrow \mathbb{R}^n$ and $\text{pr}'_2: \mathbb{R}^n \times F_1 \rightarrow F_1$. It follows

$$f_{12}^{-1}(f_0(x))(f_{12}(x)(v_2) + f_{11}(x)(v_1)) + f_{11}^{-1}(f_0(x))(v_1) = v_2$$

for all $x \in \mathcal{U}_{\mathbb{R}}$ and $v = (v_1, v_2) \in \mathbb{R}^n \times E_1$. This implies $(f_0^{-1}, f_1^{-1}) \circ \tilde{f}^{(1)} = \text{pr}_{\mathcal{U}}^{(1)}: \mathcal{P}(\Lambda)^{(1)} \times \mathcal{U}^{(1)} \rightarrow \mathcal{U}^{(1)}$. In the same way, one sees $(f_0, f_1) \circ (\text{id}_{\mathcal{P}(\Lambda)^{(1)}}, (f_0^{-1}, f_1^{-1})) = \text{pr}_{\mathcal{V}}^{(1)}$. Therefore, $\tilde{f}^{(1)}$ is invertible with $(\tilde{f}^{(1)})^{-1} = (\text{id}_{\mathcal{P}(\Lambda)^{(1)}}, (f_0^{-1}, f_1^{-1}))$. With this, the formula follows from Lemma 2.2.18. \square

Corollary 4.3.7. *Let $\mathcal{M}, \mathcal{N} \in \mathbf{SMan}$ and $\Lambda \in \mathbf{Gr}$. Then $f \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N})_{\Lambda}$ is invertible if and only if $\widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N})_{\varepsilon_{\Lambda}}(f)$ is an isomorphism.*

Proof. Because $\widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N})_{\varepsilon_{\Lambda}}$ respects the composition, it maps invertible elements to invertible elements. Conversely, let $\widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{N})_{\varepsilon_{\Lambda}}(f)$ be an isomorphism. By Remark 4.3.5 and Lemma 2.3.6, we just need to see is that $\tilde{f}^{(1)}$ is an isomorphism in $\mathbf{SMan}^{(1)}$ for

$$\tilde{f} := (\text{id}_{\mathcal{P}(\Lambda)}, f): \mathcal{P}(\Lambda) \times \mathcal{M} \rightarrow \mathcal{P}(\Lambda) \times \mathcal{N}.$$

Since $\tilde{f}^{(0)} = f^{(0)}$ is a diffeomorphism and $\tilde{f}^{(1)}$ is a morphism of vector bundles over $f^{(0)}$, it suffices to see this locally. Thus, the claim follows from Lemma 4.3.6. \square

4.3.3. The local structure of the functor of supermorphisms

To analyze the structure of the superdiffeomorphisms, it is important to have a good understanding of the local description of the functor of supermorphisms in terms of skeletons. As we will see, for $E, F \in \mathbf{SVec}_{lc}$ one has a decomposition

$$\mathcal{A}lt^l(\mathbb{R}^n \oplus E_1; F_l) \cong \bigoplus_{I \in \mathcal{P}_0^n, |I| \leq l} \mathbf{v}_I \wedge \mathcal{A}lt^{l-|I|}(E_1; F_l) \oplus \bigoplus_{I \in \mathcal{P}_1^n, |I| \leq l} \mathbf{v}_I \wedge \mathcal{A}lt^{l-|I|}(E_1; F_l).$$

If $\mathcal{U} \subseteq \overline{E}$ is an open subfunctor and one identifies λ_I and \mathbf{v}_I , this leads to

$$\widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{F})_{\Lambda_n} \cong \overline{\mathcal{SC}^{\infty}(\mathcal{U}, \overline{F}) \oplus \mathcal{SC}^{\infty}(\mathcal{U}, \Pi(\overline{F}))}_{\Lambda_n}$$

as $\Lambda_{n, \overline{0}}$ -modules (see [40, 10.6, p.426 f.]). We give a concrete description of the composition in these terms. Heuristically speaking, the \mathbf{v}_I can be pulled out of the composition formula (2.2). However, if $|I|$ is odd, additional signs appear.

For the following general constructions let E, F be \mathbb{R} -vector spaces and let $m, n, k \in \mathbb{N}_0$. We define a map

$$L^m(\mathbb{R}^k; \mathbb{R}) \times L^n(\mathbb{R}^k \oplus E; F) \rightarrow L^{m+n}(\mathbb{R}^k \oplus E; F), \quad (f, L) \mapsto f \cdot L,$$

where $f \cdot L(v) := f(\text{pr}_1(v_1), \dots, \text{pr}_1(v_m)) \cdot L(v_{m+1}, \dots, v_{m+n})$ for $v = (v_1, \dots, v_{m+n}) \in \mathbb{R}^k \oplus E^{m+n}$ and where $\text{pr}_1: \mathbb{R}^k \times E \rightarrow \mathbb{R}^k$ is the projection. We consider $\text{Alt}^m(E; F) \subseteq \text{Alt}^m(\mathbb{R}^k \oplus E; F)$ in the obvious way and define $\mathbf{v}_I \wedge L$ for $L \in \text{Alt}^m(E; F)$ as above. Note that with this, the wedge product can be written as

$$\mathbf{v}_I \wedge \mathbf{v}_J = \frac{(n+m)!}{m!n!} \mathfrak{A}^{n+m}(\mathbf{v}_I \cdot \mathbf{v}_J)$$

for $I, J \in \mathcal{P}^k$ with $|I| = n$ and $|J| = m$. Analogously, we define

$$\mathbf{v}_I \wedge L := \frac{(n+m)!}{n!m!} \mathfrak{A}^{n+m}(\mathbf{v}_I \cdot L),$$

for $I \in \mathcal{P}^k$ with $|I| = n$ and $L \in \text{Alt}^m(\mathbb{R}^k \oplus E; F)$.

If E, F are locally convex, U is an open subset of a locally convex space and $f \in \mathcal{C}^\infty(U, \text{Alt}^m(\mathbb{R}^k \oplus E; F))$, then we define $\mathbf{v}_I \wedge f$ pointwise. Clearly, one has $\mathbf{v}_I \wedge f \in \mathcal{C}^\infty(U, \text{Alt}^{n+m}(\mathbb{R}^k \oplus E; F))$. Note that with the inclusion $j_2: E \rightarrow \mathbb{R}^k \times E$, the map

$$\mathcal{C}^\infty(U, \text{Alt}^m(E; F))_c \rightarrow \mathcal{C}^\infty(U, \text{Alt}^m(\mathbb{R}^k \oplus E; F))_c, \quad f \mapsto f(\cdot)(j_2, \dots, j_2)$$

is continuous and has a continuous left-inverse given by $g \mapsto g(\cdot)(\text{pr}_2, \dots, \text{pr}_2)$ by Lemma A.2.2. Thus, $\mathcal{C}^\infty(U, \text{Alt}^m(E; F))_c$ can be considered as a closed subspace of $\mathcal{C}^\infty(U, \text{Alt}^m(\mathbb{R}^k \oplus E; F))_c$. If E and F are Banach spaces and U is a subset of a finite-dimensional space, the same is obviously true for $\mathcal{C}^\infty(U, \text{Alt}^m(E; F))_b$ and $\mathcal{C}^\infty(U, \text{Alt}^m(\mathbb{R}^k \oplus E; F))_b$.

Lemma 4.3.8 (compare [40, 10.6, p.426 f.]). *Let $E, F \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}$ be an open subfunctor. We turn $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})$ into a functor $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_c: \mathbf{Gr} \rightarrow \mathbf{Top}$ by setting $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_{c,\Lambda} := \mathcal{SC}^\infty(\mathcal{P}(\Lambda) \times \mathcal{U}, \overline{F})_{c,\Lambda}$. Defining $H := \mathcal{SC}^\infty(\mathcal{U}, \overline{F})_c \oplus \mathcal{SC}^\infty(\mathcal{U}, \overline{\Pi(F)})_c \in \mathbf{SVec}_{lc}$, we have $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_c \cong \overline{H}$ as topological \mathbb{R} -modules. If E_0 is finite-dimensional and E_1, F are Banach spaces, define $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_b$ in analogy to $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_c$. Then an analogous statement holds for $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_b$.*

Proof. Let $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda_n, \Lambda)$. Then

$$\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_\varrho: \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_{c,\Lambda_n} \rightarrow \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_{c,\Lambda}$$

is linear and continuous by Lemma 4.1.4. By Corollary A.2.16, we have

$$\begin{aligned} \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_{c,\Lambda_n} &\cong \prod_{l=0}^{\infty} \left(\bigoplus_{I \in \mathcal{P}_0^n} \mathbf{v}_I \wedge \mathcal{C}^\infty(\mathcal{U}_{\mathbb{R}}, \text{Alt}^l(E_1; F_l))_c \right. \\ &\quad \left. \oplus \bigoplus_{I \in \mathcal{P}_1^n} \mathbf{v}_I \wedge \mathcal{C}^\infty(\mathcal{U}_{\mathbb{R}}, \text{Alt}^l(E_1; F_{l+1}))_c \right) \\ &\cong \overline{H}_{\Lambda_n}, \end{aligned}$$

turning $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_{c, \Lambda_n} \cong \overline{H}_{\Lambda_n}$ into a topological $\Lambda_{n, \overline{0}}$ -module under the identification (4.8). The same arguments work for $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_{b, \Lambda_n}$ in the other case. What remains to be seen is that \overline{H}_ϱ equals $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_\varrho$ under this identification. Let $\mathcal{P}(\varrho)$ have the skeleton $(\varrho_l)_{0 \leq l \leq n}$. Note that $\varrho_l = 0$ for even l . Let $f \in \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_{\Lambda_n}$. Since $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F})_\varrho$ is linear, we may assume $f = (0, \dots, 0, \mathbf{v}_J \wedge_J f_m, 0, \dots)$ for some $J \in \mathcal{P}_i^n$, $i \in \{0, 1\}$ with $j := |J|$ and $_J f_m \in \mathcal{C}^\infty(\mathcal{U}_{\mathbb{R}}, \mathcal{Alt}^{m-j}(E_1; F_{\overline{m}}))$. We have to show that the skeleton of $f \circ (\mathcal{P}(\varrho) \times \text{id}_{\mathcal{U}})$ is given by $\varrho(\mathbf{v}_J) \wedge_J f_m$, where we identify $\bigwedge(\mathbb{R}^n)^*$ with Λ_n as in (4.8). We have

$$(\varrho(\mathbf{v}_J))_r = (\mathbf{v}_J \circ (\varrho_l)_l)_r = \sum_{\beta \in I_{0,j}^r, \sigma \in \mathfrak{S}_r} \frac{\text{sgn}(\sigma)}{j! \beta!} \mathbf{v}_J(\varrho_\beta)(\bullet^\sigma).$$

This yields

$$(\varrho(\mathbf{v}_J))_r \wedge_J f_m = \sum_{\beta \in I_{0,j}^r, \tau \in \mathfrak{S}_{r+m-j}} \frac{\text{sgn}(\tau)}{j! \beta! (m-j)!} \mathbf{v}_J(\varrho_\beta(\text{pr}_{\mathbb{R}^n}^{|\beta|})) \cdot _J f_m(\text{pr}_{E_1}^{m-j})(\bullet^\tau),$$

where $\text{pr}_{\mathbb{R}^n}: \mathbb{R}^n \times E_1 \rightarrow \mathbb{R}^n$ and $\text{pr}_{E_1}: \mathbb{R}^n \times E_1 \rightarrow E_1$ are the projections. Let $(\varrho'_l)_l$ denote the skeleton of $\mathcal{P}(\varrho) \times \text{id}_{\mathcal{U}}$. We have to show that the above sum equals $(f \circ (\varrho'_l)_l)_{r+m-j}$. By definition of the composition, we have

$$(f \circ (\varrho'_l)_l)_{r+m-j} = \sum_{\substack{\beta' \in I_{0,m}^{r+m-j}, \\ \sigma \in \mathfrak{S}_{r+m-j}}} \frac{\text{sgn}(\sigma)}{m! \beta'!} (\mathbf{v}_J \wedge_J f_m)(\varrho'_{\beta'})(\bullet^\sigma). \quad (4.10)$$

Recall that $\mathbf{v}_J \wedge_J f_m = \sum_{\tau \in \mathfrak{S}_m} \frac{\text{sgn}(\tau)}{j!(m-j)!} (\mathbf{v}_J(\text{pr}_{\mathbb{R}^n}^j)) \cdot (_J f_m(\text{pr}_{E_1}^{m-j}))(\bullet^\tau)$. For every $\tau \in \mathfrak{S}_m$ the contribution to (4.10) is zero unless $\beta'_{\tau(i)} = 1$ for all $\tau(i) > j$. Thus, the relevant β' are determined by $\beta'_{\tau(i)} = \beta_i$ for some $\beta \in I_{0,j}^r$ when $\tau(i) \leq j$. By Lemma A.2.17, we have the same contribution to the outer sum for every $\tau \in \mathfrak{S}_m$. Therefore, we may substitute $\mathbf{v}_J \wedge_J f_m$ in (4.10) with $\frac{m!}{j!(m-j)!} (\mathbf{v}_J(\text{pr}_{\mathbb{R}^n}^j)) \cdot (_J f_m(\text{pr}_{E_1}^{m-j}))$ while letting the sum run over $\beta' = (\beta_1, \dots, \beta_j, 1, \dots, 1)$ with $\beta \in I_{0,j}^r$. Because $\beta'! = \beta!$, this yields the proposed equality. \square

Lemma 4.3.9. *Let $n, k \in \mathbb{N}_0$, $E, F \in \mathbf{SVec}_{lc}$ and $U \subseteq E_0$ open. For $f \in \mathcal{C}^\infty(U, \mathcal{Alt}^n(\mathbb{R}^k \oplus E_1; F_{\overline{n}}))$ and $I \in \mathcal{P}^k$, we have*

$$d(\mathbf{v}_I \wedge f) = \mathbf{v}_I \wedge df \in \mathcal{C}^\infty(U \times E_0, \mathcal{Alt}^{n+|I|}(\mathbb{R}^k \oplus E_1; F_{\overline{n}})).$$

Proof. This is obvious from the definitions. \square

Corollary 4.3.10. *Let $n \in \mathbb{N}_0$, $E, F, H \in \mathbf{SVec}_{lc}$, $\mathcal{U} \subseteq \overline{E}$, $\mathcal{V} \subseteq \overline{F}$, $\mathcal{W} \subseteq \overline{H}$ be open subfunctors and $f \in \widehat{\mathcal{SC}}^\infty(\mathcal{V}, \mathcal{W})_{\Lambda_n}$, $g \in \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \mathcal{V})_{\Lambda_n}$. We decompose*

$$g_r = \sum_{0 \leq l \leq r} \sum_{I \in \mathcal{P}^n, |I|=l} \mathbf{v}_I \wedge_I g_r \quad \text{and} \quad f_r = \sum_{0 \leq l \leq r} \sum_{I \in \mathcal{P}^n, |I|=l} \mathbf{v}_I \wedge_I f_r$$

with ${}_I g_r \in \mathcal{C}^\infty(\mathcal{U}_{\mathbb{R}}, \mathcal{A}lt^{r-|I|}(E_1; F_{\overline{r}}))$ and ${}_I f_r \in \mathcal{C}^\infty(\mathcal{V}_{\mathbb{R}}, \mathcal{A}lt^{r-|I|}(F_1; H_{\overline{r}}))$. Then, we have

$$(f \circ (\text{id}_{\mathcal{P}(\Lambda_n)}, g))_r = \sum_{\substack{m, l, \\ B \subseteq \{1, \dots, n\}}} \sum_{\substack{I=(I_0, I_1, \dots, I_{m+l}) \\ \in \mathcal{P}_{m+l+1}(B), i_0 \leq l}} \sum_{(\alpha, \beta) \in I_{m, l-i_0}^{r-i_0}} \frac{(-1)^{N_{(\alpha, \beta)}^I} (L - s_{(\alpha, \beta)}^I)!}{m!(l-i_0)!\alpha_I! \beta_I!} \\ \mathbf{v}_{I_0} \wedge \mathbf{v}_{I_1} \wedge \dots \wedge \mathbf{v}_{I_{m+l}} \wedge \mathfrak{A}^{L-s_{(\alpha, \beta)}^I} d^m {}_{I_0} f_l({}_{\emptyset} g_0)({}_I g_\alpha \times {}_I g_\beta),$$

where $i_s := |I_s|$, $N_{(\alpha, \beta)}^I := \sum_{j=2}^{l+m} \sum_{t=1}^{j-1} i_j \cdot (\gamma_t - i_t)$ with $\gamma_t := \alpha_t$ for $t \leq m$ and $\gamma_t = \beta_{t-m}$ else, $L := (r - i_0 - \dots - i_{m+l})$, $\alpha_I := (\alpha_1 - i_1, \dots, \alpha_m - i_m)$, $\beta_I := (\beta_1 - i_{m+1}, \dots, \beta_l - i_{m+l})$ and ${}_I g_\alpha := ({}_{I_1} g_{\alpha_1} \times \dots \times {}_{I_m} g_{\alpha_m})$ as well as ${}_I g_\beta := ({}_{I_{m+1}} g_{\beta_1} \times \dots \times {}_{I_{m+l}} g_{\beta_l})$ and where $s_{(\alpha, \beta)}^I$ is the number of indices for which $\gamma_j - i_j = 0$, i.e., for which ${}_{I_j} g_{\gamma_j} \in \mathcal{C}^\infty(\mathcal{U}_{\mathbb{R}}, F_{\overline{\gamma_j}})$.

Proof. This is an immediate consequence of the ordinary formula for the composition of skeletons (2.2) together with Lemma 4.3.9, Lemma A.2.18 and Lemma A.2.19. \square

Lemma 4.3.11. Let $E, E', F, H \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}$, $\mathcal{V} \subseteq \overline{E'}$ be open subfunctors. Let further $f: \mathcal{U} \times \overline{F} \rightarrow \overline{H}$ be an \mathcal{U} -family of \mathbb{R} -linear maps and $g \in \widehat{\mathcal{SC}}^\infty(\mathcal{V}, \overline{F})_{\Lambda_n}$ be a supersmooth morphism for $n \in \mathbb{N}_0$. Then, we have

$$f \circ (\text{id}_{\mathcal{U}}, g) = \sum_{I \in \mathcal{P}_0^n} \mathbf{v}_I \wedge f \circ (\text{id}_{\mathcal{U}}, {}_I g) + \sum_{I \in \mathcal{P}_1^n} \mathbf{v}_I \wedge \overline{\Pi}(f) \circ (\text{id}_{\mathcal{U}}, {}_I g)$$

if we decompose $g = \sum_{I \in \mathcal{P}^n} \mathbf{v}_I \wedge {}_I g$ with ${}_I g \in \mathcal{SC}^\infty(\mathcal{V}, \overline{F})$ for $|I|$ even and ${}_I g \in \mathcal{SC}^\infty(\mathcal{V}, \overline{\Pi(F)})$ for $|I|$ odd.

Proof. With $I \in \mathcal{P}^n$, $i := |I| \geq m$, we use Lemma A.2.18 and Lemma 2.5.3 (compare formula (4.3)) to calculate

$$\begin{aligned} & (f \circ (\text{id}_{\mathcal{U}}, \mathbf{v}_I \wedge {}_I g))_m \\ &= \mathfrak{A}^m \left(\sum_{\substack{m \geq l \text{ odd,} \\ l \geq i}} \frac{m!}{(m-l)!l!} f_{m-l+1}(\text{pr}_{\mathcal{U}_{\mathbb{R}}}, 0) (\mathbf{v}_I \wedge {}_I g_{l-i}(\text{pr}_{\mathcal{V}_{\mathbb{R}}})(\text{pr}_{23}^l), \text{pr}_1^{m-l}) \right. \\ & \quad \left. + \sum_{\substack{m \geq l \text{ even,} \\ l \geq i}} \frac{m!}{(m-l)!l!} f_{m-l}(\text{pr}_{\mathcal{U}_{\mathbb{R}}}, \mathbf{v}_I \wedge {}_I g_{l-i}(\text{pr}_{\mathcal{V}_{\mathbb{R}}})(\text{pr}_{23}^l)) (\text{pr}_1^{m-l}) \right) \\ &= \mathbf{v}_I \wedge \mathfrak{A}^{m-i} \left(\sum_{\substack{m \geq l \text{ odd,} \\ l \geq i}} \frac{(m-i)!}{(m-l+1)!(l-i)!} f_{m-l+1}(\text{pr}_{\mathcal{U}_{\mathbb{R}}}, 0) ({}_I g_{l-i}(\text{pr}_{\mathcal{V}_{\mathbb{R}}})(\text{pr}_3^{l-i}), \text{pr}_1^{m-l}) \right. \\ & \quad \left. + \sum_{\substack{m \geq l \text{ even,} \\ l \geq i}} \frac{(m-i)!}{(m-l)!(l-i)!} f_{m-l}(\text{pr}_{\mathcal{U}_{\mathbb{R}}}, {}_I g_{l-i}(\text{pr}_{\mathcal{V}_{\mathbb{R}}})(\text{pr}_3^{l-i})) (\text{pr}_1^{m-l}) \right) \\ &= \begin{cases} \mathbf{v}_I \wedge (f \circ (\text{id}_{\mathcal{U}}, {}_I g))_{m-i} & \text{if } I \in \mathcal{P}_0^n, \\ \mathbf{v}_I \wedge (\overline{\Pi}(f) \circ (\text{id}_{\mathcal{U}}, {}_I g))_{m-i} & \text{if } I \in \mathcal{P}_1^n, \end{cases} \end{aligned}$$

with the projections $\text{pr}_{\mathcal{U}_{\mathbb{R}}}: \mathcal{U}_{\mathbb{R}} \times \mathcal{V}_{\mathbb{R}} \rightarrow \mathcal{U}_{\mathbb{R}}$, $\text{pr}_{\mathcal{V}_{\mathbb{R}}}: \mathcal{U}_{\mathbb{R}} \times \mathcal{V}_{\mathbb{R}} \rightarrow \mathcal{V}_{\mathbb{R}}$, $\text{pr}_{23}: E_1 \times \mathbb{R}^n \times E'_1 \rightarrow \mathbb{R}^n \times E'_1$, $\text{pr}_3: E_1 \times \mathbb{R}^n \times E'_1 \rightarrow E'_1$ and $\text{pr}_1: E_1 \times \mathbb{R}^n \times E'_1 \rightarrow E_1$. The last equality follows from Lemma 2.5.3 for $I \in \mathcal{P}_0^n$ and from Lemma 2.5.11 for $I \in \mathcal{P}_1^n$, respectively. The claim follows now by linearity. \square

Remark 4.3.12. One nice application of Lemma 4.3.11 is that it enables us to decompose the so called supersections. Let $\pi: \mathcal{E} \rightarrow \mathcal{M}$ be a super vector bundle with typical fiber $F \in \mathbf{SVec}_{lc}$. As we have discussed, one can give the sections $\Gamma(\mathcal{E})$ the structure of a locally convex vector space. Like with supermorphisms, this only describes the even sections. To incorporate odd sections, one can proceed as follows. It is not difficult to see that for a supersmooth map $f: \mathcal{M} \rightarrow \mathcal{N}$ one has a canonical pullback super vector bundle $f^*\mathcal{E} \rightarrow \mathcal{N}$ by letting $f^*\mathcal{E}_{\Lambda} := f_{\Lambda}^*\mathcal{E}_{\Lambda}$ (compare [40, Section 5.2, p.403]). For every $\Lambda \in \mathbf{Gr}$ let $\text{pr}_{\mathcal{M},\Lambda}: \mathcal{P}(\Lambda) \times \mathcal{M} \rightarrow \mathcal{M}$ be the projection. We define the *supersections* as a functor $\widehat{\Gamma}(\mathcal{E})_c: \mathbf{Gr} \rightarrow \mathbf{Top}$ by letting

$$\widehat{\Gamma}(\mathcal{E})_{c,\Lambda} := \Gamma(\pi_{\Lambda,\mathcal{M}}^*(\mathcal{E}))_c.$$

The sections $\sigma: \mathcal{P}(\Lambda) \times \mathcal{M} \rightarrow \pi_{\Lambda,\mathcal{M}}^*(\mathcal{E})$ have the local form $\sigma^{\alpha} \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}^{\alpha}, \overline{F})_{c,\Lambda}$ and the topological $\Lambda_{\overline{0}}$ -module structure of $\widehat{\mathcal{SC}}^{\infty}(\mathcal{U}^{\alpha}, \overline{F})_{c,\Lambda}$ turns $\widehat{\Gamma}(\mathcal{E})$ into a topological $\overline{\mathbb{R}}$ -module. Applying Lemma 4.3.11 to the change of charts, one sees that $\widehat{\Gamma}(\mathcal{E})_c \cong \Gamma(\mathcal{E})_c \oplus \Gamma(\overline{\Pi}(\mathcal{E}))_c$ holds as topological $\overline{\mathbb{R}}$ -modules (for an abstract, non-topological version of this see [40, Section 10.7, p.428]). Of course, for an appropriate bundle \mathcal{E} , one achieves the same results for $\widehat{\Gamma}(\mathcal{E})_b$ and one can also consider compactly supported supersections. It will be convenient for us to directly work with $\Gamma(\mathcal{E})_c \oplus \Gamma(\overline{\Pi}(\mathcal{E}))_c$ instead of the supersections, which is why we leave the details to the reader. Nevertheless, we mention this fact as it greatly generalizes a long standing claim by Molotkov (see [40, Section 8.5, p.418]).

4.3.4. Superdiffeomorphisms

Definition 4.3.13. Let $\mathcal{M} \in \mathbf{SMan}$. For every $\Lambda \in \mathbf{Gr}$, we define

$$\text{SDiff}(\mathcal{M})_{\Lambda} := \{f \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{M})_{\Lambda} : f \text{ invertible}\}.$$

In view of Remark 4.3.5 this means that the group $\text{SDiff}(\mathcal{M})_{\Lambda}$ can be identified with the subgroup of $\text{Aut}(\mathcal{P}(\Lambda) \times \mathcal{M})$ consisting of isomorphisms $f: \mathcal{P}(\Lambda) \times \mathcal{M} \rightarrow \mathcal{P}(\Lambda) \times \mathcal{M}$ such that $\text{pr}_{\mathcal{P}(\Lambda)} \circ f = \text{id}_{\mathcal{P}(\Lambda)}$ for the projection $\text{pr}_{\mathcal{P}(\Lambda)}: \mathcal{P}(\Lambda) \times \mathcal{M} \rightarrow \mathcal{P}(\Lambda)$.

Proposition 4.3.14 (compare [47, Proposition 6.1, p.308]). *Let $\mathcal{M} \in \mathbf{SMan}$. The assignment $\Lambda \mapsto \text{SDiff}(\mathcal{M})_{\Lambda}$ defines a subfunctor of $\widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{M})$. This subfunctor is a supergroup, i.e., a group object in $\mathbf{Set}^{\mathbf{Gr}}$, which we call the supergroup of superdiffeomorphisms.*

Proof. By definition, every $\text{SDiff}(\mathcal{M})_{\Lambda}$ is a group. Since

$$\widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{M})_{\varrho}: \widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{M})_{\Lambda} \rightarrow \widehat{\mathcal{SC}}^{\infty}(\mathcal{M}, \mathcal{M})_{\Lambda'},$$

is a morphism of monoids for $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$, it follows that $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})_\varrho(f)$ is invertible if $f \in \text{SDiff}(\mathcal{M})_\Lambda$. Therefore,

$$\text{SDiff}(\mathcal{M})_\varrho := \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})_\varrho|_{\text{SDiff}(\mathcal{M})_\Lambda} : \text{SDiff}(\mathcal{M})_\Lambda \rightarrow \text{SDiff}(\mathcal{M})_{\Lambda'}$$

is a well-defined morphism of groups. This also shows that $\text{SDiff}(\mathcal{M})$ is a subfunctor of $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})$. \square

Lemma 4.3.15. *If $\mathcal{M} \in \mathbf{SMan}$, then we have*

$$\text{SDiff}(\mathcal{M}) = \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})|_{\text{Aut}(\mathcal{M})}.$$

Proof. This follows directly from Corollary 4.3.7. \square

An alternative proof can be found in [47, Theorem 6.1, p.310] but the proof only works for finite-dimensional supermanifolds \mathcal{M} because the description of $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{M})$ as morphisms of superalgebras in [47, Section 5.2, p.303] does not carry over to the infinite-dimensional case. Instead, we followed a remark by Molotkov in [40, Section 10.6, p.426].

4.4. The Supergroup of Superdiffeomorphisms

Concerning the supergroup of superdiffeomorphisms, we have two objectives. First, we want to describe the structure of this supergroup for arbitrary supermanifolds. Second, we want to find a class of supermanifolds for which the superdiffeomorphisms are a Lie supergroup.

Even for an arbitrary infinite-dimensional supermanifold \mathcal{M} , the superdiffeomorphisms $\text{SDiff}(\mathcal{M})$ display a lot of similarities to Lie supergroups. For every $\Lambda \in \mathbf{Gr}$, we have a short exact sequence of groups

$$1 \rightarrow \ker(\text{SDiff}(\mathcal{M})_{\varepsilon_\Lambda}) \rightarrow \text{SDiff}(\mathcal{M})_\Lambda \rightarrow \text{Aut}(\mathcal{M}) \rightarrow 1$$

that splits along $\text{SDiff}(\mathcal{M})_{\eta_\Lambda}$. We set

$$\text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda := \ker(\text{SDiff}(\mathcal{M})_{\varepsilon_\Lambda}) = \text{SDiff}(\mathcal{M})_{\varepsilon_\Lambda}^{-1}(\{\text{id}_\mathcal{M}\})$$

which clearly defines a subfunctor $\text{SDiff}_{\text{id}}(\mathcal{M}) : \mathbf{Gr} \rightarrow \mathbf{Set}$ of $\text{SDiff}(\mathcal{M})$. We show that $\text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda$ can be turned into a polynomial group that is isomorphic to $\mathcal{X}(\mathcal{M})_{\bar{0}} \oplus \mathcal{X}(\mathcal{M})_{\bar{1}\Lambda^+}$ equipped with the BCH multiplication and that the resulting exponential maps $\exp_{\mathcal{M}, \Lambda}$ define a natural transformation. Moreover, we are able to define a sub-supergroup $\text{SDiff}(\mathcal{M})_{\bar{0}}$ of purely even superdiffeomorphisms and obtain a natural trivialization

$$\text{SDiff}(\mathcal{M})_{\bar{0}, \Lambda} \times \overline{\mathcal{X}(\mathcal{M})_{\bar{1}\Lambda}} \rightarrow \text{SDiff}(\mathcal{M})_\Lambda, \quad (f, X) \mapsto f \circ_\Lambda \exp_{\mathcal{M}, \Lambda}(X).$$

The structure of $\text{SDiff}(\mathcal{M})_{\bar{0}}$ can be described in detail. One also has a short exact

sequence

$$1 \rightarrow \mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rightarrow \mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda} \rightarrow \mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \rightarrow 1,$$

where $\mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$ is a pro-polynomial group and $\mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$ can be expressed in terms of the higher order automorphisms of the vector bundle $\mathcal{M}^{(1)}$ (see E.3). This sequence corresponds to the decomposition

$$1 \rightarrow \mathrm{Aut}_{\mathrm{id}}(\mathcal{M}) \rightarrow \mathrm{Aut}(\mathcal{M}) \rightarrow \mathrm{Aut}(\mathcal{M}^{(1)}) \rightarrow 1$$

and it also splits for supermanifolds of Batchelor type.

The main difficulty in turning $\mathrm{SDiff}(\mathcal{M})$ into a Lie supergroup is providing $\mathrm{SDiff}(\mathcal{M})_{\mathbb{R}} = \mathrm{Aut}(\mathcal{M})$ with a Lie group structure. Consequently, if $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, we define the supergroup of compactly supported superdiffeomorphisms $\mathrm{SDiff}_c(\mathcal{M})$ and are able to give it the structure of a Lie supergroup if $\mathcal{M}_{\mathbb{R}}$ is additionally σ -compact and \mathcal{M} is a Banach supermanifold. For this, the above trivializations are crucial. For example, we have

$$\iota(\mathrm{Aut}_c(\mathcal{M})) = \iota(\mathrm{Aut}_{\mathrm{id}}^c(\mathcal{M})) \rtimes \iota(\mathrm{Aut}_c(\mathcal{M}^{(1)})) \cong \mathrm{SDiff}_c(\mathcal{M})_{\bar{0}}^{\geq 2} \rtimes \mathrm{SDiff}_c(\mathcal{M}^{(1)})$$

if $\mathrm{Aut}_c(\mathcal{M})$ is a Lie group.

As with the automorphism group, we start by directly showing that $\mathrm{SDiff}_{\mathrm{id}}(\mathcal{U})_{\Lambda}$ is a polynomial group for every superdomain \mathcal{U} and every $\Lambda \in \mathbf{Gr}$. Then the global case follows much in the same way.

4.4.1. The local structure of $\mathrm{SDiff}_{\mathrm{id}}(\mathcal{M})$

Let $n \in \mathbb{N}$, $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \bar{E}$ be an open subfunctor. We set

$$\widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \bar{E})_{\Lambda^+} := \sum_{I \in \mathcal{P}_{0,+}^n} \mathbf{v}_I \wedge \mathcal{SC}^{\infty}(\mathcal{U}, \bar{E}) + \sum_{I \in \mathcal{P}_1^n} \mathbf{v}_I \wedge \mathcal{SC}^{\infty}(\mathcal{U}, \overline{\Pi(E)})$$

and conclude from Lemma 4.3.8 that $\Lambda \mapsto \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \bar{E})_{\Lambda^+}$ defines a subfunctor of $\widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \bar{E})$.

Let \mathcal{M} be a supermanifold modelled on E and $\varphi: \mathcal{U} \rightarrow \mathcal{M}$ be a chart. By definition, for any $f \in \mathrm{SDiff}_{\mathrm{id}}(\mathcal{M})_{\Lambda_n}$ the chart representation $f^{\varphi} := \varphi^{-1} \circ f \circ (\mathrm{id}_{\mathcal{P}(\Lambda_n)} \times \varphi) \in \mathrm{SDiff}_{\mathrm{id}}(\mathcal{U})_{\Lambda_n} \subseteq \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \bar{E})_{\Lambda_n}$ decomposes as $\mathrm{pr}_{\mathcal{U}} + \sum_{I \in \mathcal{P}_+^n} \mathbf{v}_I \wedge_I f^{\varphi}$ with the projection $\mathrm{pr}_{\mathcal{U}}: \mathcal{P}(\Lambda) \times \mathcal{U} \rightarrow \mathcal{U}$, $_I f^{\varphi} \in \mathcal{SC}^{\infty}(\mathcal{U}, \bar{E})$ for $I \in \mathcal{P}_{0,+}^n$ and $_I f^{\varphi} \in \mathcal{SC}^{\infty}(\mathcal{U}, \overline{\Pi(E)})$ for $I \in \mathcal{P}_1^n$.

Lemma 4.4.1. *Let $n \in \mathbb{N}$, $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \bar{E}$ be an open subfunctor. Then $\mathrm{SDiff}_{\mathrm{id}}(\mathcal{U})_{\Lambda_n}$ is a polynomial group of degree at most n with the vector space structure given by the bijection*

$$\widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \bar{E})_{\Lambda_n^+} \rightarrow \mathrm{SDiff}_{\mathrm{id}}(\mathcal{U})_{\Lambda_n}, \quad (X_k)_k \mapsto \mathrm{pr}_{\mathcal{U}} + (X_k)_k,$$

with the projection $\text{pr}_{\mathcal{U}}: \mathcal{P}(\Lambda_n) \times \mathcal{U} \rightarrow \mathcal{U}$. The morphism of groups $\text{SDiff}_{\text{id}}(\mathcal{U})_{\varrho}$ is linear for each $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$.

Proof. Let $X, Y \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{E})_{\Lambda_n^+}$. By Corollary 4.3.10 each term that depends on X or Y and that appears in the composition formula $(\text{pr}_{\mathcal{U}} + X) \circ (\text{id}_{\mathcal{P}(\Lambda_n)}, Y)$, adds at least one \mathbf{v}_i in the outer wedge product. Thus, the degree of the composition is bounded by n and the same argument applies to the iterated product maps. Lemma 4.3.6 shows that the inversion is also polynomial and Lemma A.2.18 and Lemma A.2.19 show that the degree of this polynomial is bounded by n . Under our identification, $\text{SDiff}_{\text{id}}(\mathcal{U})_{\varrho}$ corresponds to $\widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{E})_{\varrho}|_{\widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{E})_{\Lambda^+}}$ which is linear. \square

We write

$$\exp_{\mathcal{U}, \Lambda}: \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{E})_{\Lambda^+} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{U})_{\Lambda}$$

for the exponential map of $\text{SDiff}_{\text{id}}(\mathcal{U})_{\Lambda}$.

Corollary 4.4.2. *Let $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}$ be an open subfunctor. Then $(\exp_{\mathcal{U}, \Lambda})_{\Lambda \in \mathbf{Gr}}$ is a morphism in $\mathbf{Set}^{\mathbf{Gr}}$.*

Proof. This follows immediately from Lemma C.2.2, because $\text{SDiff}_{\text{id}}(\mathcal{U})_{\varrho}$ is a linear morphism of polynomial groups for each $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$. \square

Lemma 4.4.3. *Let $E, F \in \mathbf{SVec}_{lc}$, $\mathcal{U} \subseteq \overline{E}$, $\mathcal{V} \subseteq \overline{F}$ be open subfunctors and $\Lambda \in \mathbf{Gr}$. If $\varphi: \mathcal{U} \rightarrow \mathcal{V}$ is an isomorphism, then*

$$\text{Ad}_{\varphi, \Lambda}: \text{SDiff}_{\text{id}}(\mathcal{U})_{\Lambda} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{V})_{\Lambda}, \quad f \mapsto \varphi \circ f \circ (\text{id}_{\mathcal{P}(\Lambda)} \times \varphi^{-1})$$

is a polynomial isomorphism of polynomial groups and we have

$$\varphi \circ \exp_{\mathcal{U}, \Lambda}(X) \circ \varphi^{-1} = \exp_{\mathcal{V}, \Lambda}(\text{d}\varphi \circ (\varphi^{-1}, X \circ (\text{id}_{\mathcal{P}(\Lambda)} \times \varphi^{-1})))$$

for each $X \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{E})_{\Lambda}$. This defines a natural transformation

$$\text{Ad}_{\varphi}: \text{SDiff}_{\text{id}}(\mathcal{U}) \rightarrow \text{SDiff}_{\text{id}}(\mathcal{V}).$$

Proof. It is obvious that $\text{Ad}_{\varphi, \Lambda}$ is an isomorphism of groups that defines a natural transformation. Let $X \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{E})_{\Lambda^+}$. Then $\tilde{X} := X \circ (\text{id}_{\mathcal{P}(\Lambda)} \times \varphi^{-1}) \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{V}, \overline{E})_{\Lambda^+}$ depends linearly on X and we have $(\text{pr}_{\mathcal{U}} + X) \circ (\text{id}_{\mathcal{P}(\Lambda)} \times \varphi^{-1}) = \varphi^{-1} \circ \text{pr}_{\mathcal{V}} + \tilde{X}$, with the projection $\text{pr}_{\mathcal{V}}: \mathcal{P}(\Lambda) \times \mathcal{V} \rightarrow \mathcal{V}$. The claim now follows from an analogous calculation to the one in Lemma 4.2.5. \square

Lemma 4.4.4. *Let $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}$, $\mathcal{V} \subseteq \mathcal{U}$ be open subfunctors. Then*

$$\exp_{\mathcal{U}, \Lambda}(X)|_{\mathcal{P}(\Lambda) \times \mathcal{V}} = \exp_{\mathcal{V}}(X|_{\mathcal{P}(\Lambda) \times \mathcal{V}})$$

holds for all $\Lambda \in \mathbf{Gr}$ and each $X \in \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}, \overline{E})_{\Lambda^+}$. In particular, the restrictions

$$\text{SDiff}_{\text{id}}(\mathcal{U})_{\Lambda} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{V})_{\Lambda}, \quad f \mapsto f|_{\mathcal{P}(\Lambda) \times \mathcal{V}}$$

are linear morphisms of polynomial groups that define a morphism in $\mathbf{Set}^{\mathbf{Gr}}$.

Proof. It is obvious that the restrictions are morphisms of groups and that they define a natural transformation. Let $X \in \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_\Lambda$. Then $(\text{pr}_\mathcal{U} + X)|_{\mathcal{P}(\Lambda) \times \mathcal{V}} = \text{pr}_\mathcal{V} + X|_{\mathcal{P}(\Lambda) \times \mathcal{V}}$ is linear. Thus, the claim follows from Lemma C.2.2. \square

Lemma 4.4.5. *Let $n \in \mathbb{N}$, $E \in \mathbf{SVec}_{lc}$ and $\mathcal{U} \subseteq \overline{E}$ be an open subfunctor. Under the bijection $\exp_{\mathcal{U}, \Lambda_n}: \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_{\Lambda_n^+} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{U})_{\Lambda_n}$, the Lie bracket of $\text{SDiff}_{\text{id}}(\mathcal{U})_{\Lambda_n}$ is given by*

$$[\mathbf{v}_I \wedge_I X, \mathbf{v}_J \wedge_J Y] = \begin{cases} \mathbf{v}_I \wedge \mathbf{v}_J \wedge (\text{d}_I X \circ (\text{id}_\mathcal{U}, JY) - \text{d}_J Y \circ (\text{id}_\mathcal{U}, IX)) & \text{if } I, J \in \mathcal{P}_0^n, \\ \mathbf{v}_I \wedge \mathbf{v}_J \wedge (\overline{\Pi}(\text{d}_I X) \circ (\text{id}_\mathcal{U}, JY) - \text{d}_J Y \circ (\text{id}_\mathcal{U}, IX)) & \text{if } I \in \mathcal{P}_0^n, J \in \mathcal{P}_1^n, \\ \mathbf{v}_I \wedge \mathbf{v}_J \wedge (\text{d}_I X \circ (\text{id}_\mathcal{U}, JY) - \overline{\Pi}(\text{d}_J Y) \circ (\text{id}_\mathcal{U}, IX)) & \text{if } I \in \mathcal{P}_1^n, J \in \mathcal{P}_0^n, \\ \mathbf{v}_I \wedge \mathbf{v}_J \wedge (\overline{\Pi}(\text{d}_I X) \circ (\text{id}_\mathcal{U}, JY) + \overline{\Pi}(\text{d}_J Y) \circ (\text{id}_\mathcal{U}, IX)) & \text{if } I, J \in \mathcal{P}_1^n, \end{cases}$$

for $_I X, _I Y \in \mathcal{SC}^\infty(\mathcal{U}, \overline{E})$ if $I \in \mathcal{P}_{0,+}^n$ and $_I X, _I Y \in \mathcal{SC}^\infty(\mathcal{U}, \overline{\Pi}(\overline{E}))$ if $I \in \mathcal{P}_1^n$.

Proof. Let $X := \sum_{I \in \mathcal{P}_+^n} \mathbf{v}_I \wedge_I X$ and $Y := \sum_{I \in \mathcal{P}_+^n} \mathbf{v}_I \wedge_I Y$. Given the vector space structure of $\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{E})_{\Lambda_n^+}$ and the definition of the Lie bracket in Section C.2, we have to calculate the part of

$$Y + X \circ (\text{id}_{\mathcal{P}(\Lambda_n)}, \text{pr}_\mathcal{U} + Y)$$

that is bilinear in X and Y . Obviously, we only need to consider $X \circ (\text{id}_{\mathcal{P}(\Lambda_n)}, \text{pr}_\mathcal{U} + Y)$ and, since this expression is already linear in X , we may assume $X = \mathbf{v}_I \wedge_I X$ for some $I \in \mathcal{P}_+^n$. By Lemma A.2.19, we have

$$(\mathbf{v}_I \wedge_I X \circ (\text{id}_{\mathcal{P}(\Lambda_n)}, (\text{pr}_\mathcal{U} + Y)))_{r+|I|} = \mathbf{v}_I \wedge (_I X \circ (\text{pr}_\mathcal{U} + Y))_r$$

for each $r \in \mathbb{N}_0$. Using Proposition 2.2.16, we directly calculate the part of $(_I X \circ (\text{pr}_\mathcal{U} + Y))_r$ that depends linearly on Y as

$$\begin{aligned} & \mathfrak{A}^r \sum_{\substack{m+l=r, \\ m \text{ even}}} \left(\frac{r!}{m!l!} d_I X_l(\bullet)(Y_m, \text{pr}_{E_1}, \dots, \text{pr}_{E_1}) \right. \\ & \quad \left. + \frac{r!}{(m+1)!l!} \sum_{i=1}^l _I X_l(\bullet)(\text{pr}_{E_1}, \dots, \underbrace{Y_{m+1}}_i, \dots, \text{pr}_{E_1}) \right) \\ &= \mathfrak{A}^r \sum_{\substack{m+l=r, \\ m \text{ even}}} \left(\frac{r!}{m!l!} d_I X_l(\bullet)(Y_m, \text{pr}_{E_1}, \dots, \text{pr}_{E_1}) \right. \\ & \quad \left. + \frac{r!}{(m+1)!(l-1)!} _I X_l(\bullet)(Y_{m+1}, \text{pr}_{E_1}, \dots, \text{pr}_{E_1}) \right) \\ &= (\text{d}_I X \circ (\text{id}_\mathcal{U}, Y))_r. \end{aligned}$$

Here $\text{pr}_{E_1} : \mathbb{R}^n \times E_1 \rightarrow E_1$ is the projection and the last equality is easily seen with Remark 2.2.15. With this, Lemma 4.3.11 implies

$$\text{d}_I X \circ (\text{id}_{\mathcal{U}}, Y) = \sum_{J \in \mathcal{P}_{0,+}^n} \mathbf{v}_J \wedge \text{d}_I X \circ (\text{id}_{\mathcal{U}}, {}_J Y) + \sum_{J \in \mathcal{P}_1^n} \mathbf{v}_J \wedge \bar{\Pi}(\text{d}_I X) \circ (\text{id}_{\mathcal{U}}, {}_J Y),$$

from which the claim follows. \square

Lemma 4.4.6. *Let $E \in \mathbf{SVec}_{lc}$ be such that E_0 is finite-dimensional and E_1 is a Banach space. For each open subfunctor $\mathcal{U} \subseteq \bar{E}$ and each $\Lambda \in \mathbf{Gr}$, the bijection*

$$\exp_{\mathcal{U}, \Lambda} : \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \bar{E})_{b, \Lambda^+} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{U})_\Lambda$$

turns $\text{SDiff}_{\text{id}}(\mathcal{U})_\Lambda$ into a Lie group.

Proof. By Lemma C.2.5 it suffices to see that the Lie bracket is smooth. But this is obvious by Lemma 4.4.5 and Lemma A.2.14. \square

4.4.2. The structure of $\text{SDiff}_{\text{id}}(\mathcal{M})$

Proposition 4.4.7. *Let $\Lambda \in \mathbf{Gr}$ and let \mathcal{M} be a supermanifold modelled on $E \in \mathbf{SVec}_{lc}$ with the atlas $\mathcal{A} := \{\varphi^\alpha : \mathcal{U}^\alpha \rightarrow \mathcal{M} : \alpha \in A\}$. We consider the linear injective map*

$$\Theta_\Lambda : \overline{\mathcal{X}(\mathcal{M})}_{\Lambda^+} \rightarrow \prod_{\alpha \in A} \widehat{\mathcal{SC}}^\infty(\mathcal{U}^\alpha, \bar{E})_{\Lambda^+}, \quad (\lambda_I {}_I X)_{I \in \mathcal{P}_+^n} \mapsto \left(\sum_{I \in \mathcal{P}_+^n} \mathbf{v}_I \wedge {}_I X^\alpha \right)_{\alpha \in A}$$

and the injective morphism of groups

$$\Psi_\Lambda : \text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda \rightarrow \prod_{\alpha \in A} \text{SDiff}_{\text{id}}(\mathcal{U}^\alpha)_\Lambda, \quad f \mapsto (\varphi^\alpha)^{-1} \circ f \circ (\text{id}_{\mathcal{P}(\Lambda)} \times \varphi^\alpha).$$

Then,

- (a) Θ_Λ is continuous as a map $\overline{\mathcal{X}(\mathcal{M})}_{c, \Lambda^+} \rightarrow \prod_{\alpha \in A} \widehat{\mathcal{SC}}^\infty(\mathcal{U}^\alpha, \bar{E})_{c, \Lambda^+}$ and $\text{im}(\Theta_\Lambda)$ is closed,
- (b) if \mathcal{M} is a Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, the same holds for Θ_Λ as a map $\overline{\mathcal{X}(\mathcal{M})}_{b, \Lambda^+} \rightarrow \prod_{\alpha \in A} \widehat{\mathcal{SC}}^\infty(\mathcal{U}^\alpha, \bar{E})_{b, \Lambda^+}$ and
- (c) there exists a unique bijective map $\exp_{\mathcal{M}, \Lambda} : \overline{\mathcal{X}(\mathcal{M})}_{\Lambda^+} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda$ such that the diagram

$$\begin{array}{ccc} \overline{\mathcal{X}(\mathcal{M})}_{\Lambda^+} & \xrightarrow{\exp_{\mathcal{M}, \Lambda}} & \text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda \\ \downarrow \Theta_\Lambda & & \downarrow \Psi_\Lambda \\ \prod_{\alpha \in A} \widehat{\mathcal{SC}}^\infty(\mathcal{U}^\alpha, \bar{E})_{\Lambda^+} & \xrightarrow{\prod_{\alpha} \exp_{\mathcal{U}^\alpha, \Lambda}} & \prod_{\alpha \in A} \text{SDiff}_{\text{id}}(\mathcal{U}^\alpha)_\Lambda \end{array}$$

commutes.

This turns $\text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda$ into a polynomial group of degree at most n for $\Lambda = \Lambda_n$. The family $(\exp_{\mathcal{M},\Lambda})_{\Lambda \in \mathbf{Gr}}$ defines a morphism in $\mathbf{Set}^{\mathbf{Gr}}$ and neither $\exp_{\mathcal{M},\Lambda}$, nor the topology induced by the chosen topology of $\mathcal{X}(\mathcal{M})$ on $\text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda$, depend on the atlas.

Proof. That Θ_Λ is continuous is obvious and that $\text{im}(\Theta_\Lambda)$ is closed in both cases follows from Lemma 4.1.6. The proof of the bijectivity of $\exp_{\mathcal{M},\Lambda}$ is analogous to the proof of Proposition 4.2.6. It again suffices to see $\prod_\alpha \exp_{\mathcal{U}^\alpha, \Lambda}(\text{im}(\Theta)) = \text{im}(\Psi)$. For $X \in \overline{\mathcal{X}(\mathcal{M})}_{\Lambda_n^+}$ and $\alpha \in A$, we decompose $\Theta(X)_\alpha$ into $(\mathbf{v}_I \wedge {}_IX^\alpha)_{I \in \mathcal{P}_+^n}$ with ${}_IX^\alpha \in \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{E})$ for $I \in \mathcal{P}_{0,+}^n$ and ${}_IX^\alpha \in \mathcal{SC}^\infty(\mathcal{U}^\alpha, \overline{\Pi E})$ for $I \in \mathcal{P}_1^n$. By Lemma 4.3.11 we have $(X^\alpha)_{\alpha \in A} \in \text{im}(\Theta_{\Lambda_n})$ if and only if for all $\alpha, \beta \in A$ it holds that

$$\begin{aligned} & d(\varphi^{\beta\alpha})^{-1} \circ (\varphi^{\beta\alpha}, X^\alpha \circ (\text{id}_{\mathcal{P}(\Lambda_n)} \times \varphi^{\beta\alpha})) \\ &= \sum_{I \in \mathcal{P}_+^n} d(\varphi^{\beta\alpha})^{-1} \circ (\varphi^{\beta\alpha}, \mathbf{v}_I \wedge {}_IX^\alpha \circ (\text{id}_{\mathcal{P}(\Lambda_n)} \times \varphi^{\beta\alpha})) \\ &= \sum_{I \in \mathcal{P}_+^n} \mathbf{v}_I \wedge {}_IX^\beta|_{\mathcal{U}^{\beta\alpha}} = X^\beta|_{\mathcal{P}(\Lambda_n) \times \mathcal{U}^{\beta\alpha}}. \end{aligned}$$

Then Lemma 4.4.4 and Lemma 4.4.3 show

$$(\varphi^{\beta\alpha})^{-1} \circ \exp_{\mathcal{U}^\alpha, \Lambda_n}(X^\alpha) \circ (\text{id}_{\mathcal{P}(\Lambda_n)} \times \varphi^{\beta\alpha}) = \exp_{\mathcal{U}^{\beta\alpha}}(X^\beta|_{\mathcal{P}(\Lambda_n) \times \mathcal{U}^{\beta\alpha}}),$$

which is exactly the condition for $(\exp(X^\alpha))_{\alpha \in A}$ to be in $\text{im}(\Psi_{\Lambda_n})$. Applying the same argument in reverse shows that $\exp_{\mathcal{M},\Lambda}$ is bijective. This also shows that $\exp_{\mathcal{M},\Lambda}$ and the induced topology are independent of the atlas as in Proposition 4.2.6. That $(\exp_{\mathcal{M},\Lambda})_{\Lambda \in \mathbf{Gr}}$ is a natural transformation follows from Corollary 4.4.2 and the fact that $(\Theta_\Lambda)_{\Lambda \in \mathbf{Gr}}$ is a natural transformation. The polynomial group structure of $\text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda$ is obtained from Lemma 4.4.1 as well as the linearity of Θ_Λ . \square

Corollary 4.4.8. *Let $\mathcal{M} \in \mathbf{SMan}$ be modelled on $E \in \mathbf{SVec}_{lc}$ with the atlas $\mathcal{A} := \{\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}: \alpha \in A\}$ and let $n \in \mathbb{N}$ be fixed. Under the bijection $\exp_{\mathcal{M},\Lambda_n}: \overline{\mathcal{X}(\mathcal{M})}_{\Lambda_n^+} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda_n}$, the Lie bracket of $\text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda_n}$ is given by*

$$([\lambda_I {}_IX, \lambda_J {}_JY]_{\Lambda_n})^\alpha = \begin{cases} \lambda_I \lambda_J (d {}_IX^\alpha \circ (\text{id}_{\mathcal{U}}, {}_JY^\alpha) - d {}_JY^\alpha \circ (\text{id}_{\mathcal{U}}, {}_IX^\alpha)) & \text{if } I, J \in \mathcal{P}_{0,+}^n, \\ \lambda_I \lambda_J (\overline{\Pi}(d {}_IX^\alpha) \circ (\text{id}_{\mathcal{U}}, {}_JY^\alpha) - d {}_JY^\alpha \circ (\text{id}_{\mathcal{U}}, {}_IX^\alpha)) & \text{if } I \in \mathcal{P}_{0,+}^n, J \in \mathcal{P}_1^n, \\ \lambda_I \lambda_J (d {}_IX^\alpha \circ (\text{id}_{\mathcal{U}}, {}_JY^\alpha) - \overline{\Pi}(d {}_JY^\alpha) \circ (\text{id}_{\mathcal{U}}, {}_IX^\alpha)) & \text{if } I \in \mathcal{P}_1^n, J \in \mathcal{P}_{0,+}^n, \\ \lambda_I \lambda_J (\overline{\Pi}(d {}_IX^\alpha) \circ (\text{id}_{\mathcal{U}}, {}_JY^\alpha) + \overline{\Pi}(d {}_JY^\alpha) \circ (\text{id}_{\mathcal{U}}, {}_IX^\alpha)) & \text{if } I, J \in \mathcal{P}_1^n, \end{cases}$$

for $X = \sum_{I \in \mathcal{P}_+^n} \lambda_I {}_IX, Y = \sum_{J \in \mathcal{P}_+^n} \lambda_J {}_JY \in \overline{\mathcal{X}(\mathcal{M})}_{\Lambda_n^+}$ and $\alpha \in A$. The family of Lie brackets $([\cdot, \cdot]_\Lambda)_{\Lambda \in \mathbf{Gr}}$ is a morphism in $\mathbf{Set}^{\mathbf{Gr}}$.

Proof. This is clear from Proposition 4.4.7 and Lemma 4.4.5. \square

Corollary 4.4.9. *Omitting λ_I and λ_J from the definition of the Lie bracket in Corollary 4.4.8 gives us a Lie superalgebra*

$$\mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \rightarrow \mathcal{X}(\mathcal{M}).$$

Proof. The claim follows from Corollary 2.2.22, once we have extended the Lie algebra $\mathcal{X}(\mathcal{M})_{\Lambda_3^+}$ to a Lie algebra $\mathcal{X}(\mathcal{M})_{\Lambda_3}$. For this, consider $\mathcal{X}(\mathcal{M})_{\Lambda_3^+} \oplus \lambda_{\{4,5\}} \mathcal{X}(\mathcal{M})_{\bar{0}}$, which is clearly a Lie subalgebra of $\overline{\mathcal{X}(\mathcal{M})_{\Lambda_5^+}}$. Identifying $\lambda_{\{4,5\}}$ and 1 turns it into a Lie algebra over $\overline{\mathbb{R}}^{(3)}$, so that Corollary 2.2.22 can be applied. \square

Corollary 4.4.10. *Let $\mathcal{M} \in \mathbf{SMan}$, $n \in \mathbb{N}$, $I \in \mathcal{P}_+^n$ and ${}_IX \in \mathcal{X}(\mathcal{M})_{|\overline{I}|}$. If $\varphi^\alpha: \mathcal{U}^\alpha \rightarrow \mathcal{M}$ is a chart of \mathcal{M} , then we have*

$$(\varphi_{\Lambda_n}^\alpha)^{-1} \circ \exp_{\mathcal{M}, \Lambda_n}(\lambda_I {}_IX) \circ (\text{id}_{\mathcal{P}(\Lambda_n)} \times \varphi_{\Lambda_n}^\alpha) = \text{pr}_{\mathcal{U}^\alpha} + \mathbf{v}_I \wedge {}_IX^\alpha.$$

Proof. This follows immediately from Proposition 4.4.7 because the linear term of $\exp_{\mathcal{U}^\alpha, \Lambda_n}$ is the identity. \square

Corollary 4.4.11. *Let \mathcal{M} be a Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. Then the global chart $\exp_{\mathcal{M}, \Lambda}: \overline{\mathcal{X}(\mathcal{M})_{b\Lambda^+}} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda^+}$ turns $\text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda}$ into a Lie group for each $\Lambda \in \mathbf{Gr}$.*

Proof. In view of Lemma A.1.2, this follows from Proposition 4.4.7 together with Corollary 4.4.6. \square

Lemma 4.4.12. *Let \mathcal{M} be a supermanifold and $n \in \mathbb{N}$ be fixed. Then, under the bijection $\exp_{\mathcal{M}, \Lambda_n}: \overline{\mathcal{X}(\mathcal{M})_{\Lambda_n}} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda_n}$, the group action*

$$\text{Aut}(\mathcal{M}) \times \text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda_n}, \quad (f, \Phi) \mapsto f \circ \Phi \circ (\text{id}_{\mathcal{P}(\Lambda_n)} \times f^{-1})$$

corresponds to the group action

$$\beta_{\Lambda_n}: \text{Aut}(\mathcal{M}) \times \overline{\mathcal{X}(\mathcal{M})_{\Lambda_n}} \rightarrow \overline{\mathcal{X}(\mathcal{M})_{\Lambda_n}}, \quad (f, (\lambda_I {}_IX)_{I \in \mathcal{P}_+^n}) \mapsto (\lambda_I {}_I\tilde{X})_{I \in \mathcal{P}_+^n},$$

where

$${}_I\tilde{X} := \begin{cases} \mathcal{T}f \circ {}_IX \circ f^{-1}, & \text{if } I \in \mathcal{P}_{0,+}^n, \\ \overline{\Pi}(\mathcal{T}f) \circ {}_IX \circ f^{-1}, & \text{if } I \in \mathcal{P}_1^n. \end{cases}$$

Proof. In analogy to Lemma 4.2.11, this follows by applying Lemma 4.4.3 in local coordinates and then using Lemma 4.3.11. \square

4.4.3. Superdiffeomorphisms with compact support

Let \mathcal{M} be a supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. For each $\Lambda \in \mathbf{Gr}$, we define

$$\text{SDiff}_c(\mathcal{M})_{\Lambda} := \{f \in \text{SDiff}(\mathcal{M})_{\Lambda} : \exists K \subseteq \mathcal{M}_{\mathbb{R}} \text{ compact with } f|_{\mathcal{P}(\Lambda) \times \mathcal{M}|_{\mathcal{M}_{\mathbb{R}} \setminus K}} = \text{pr}_{\mathcal{M}|_{\mathcal{M}_{\mathbb{R}} \setminus K}}\}.$$

This clearly defines a sub-group object $\text{SDiff}_c(\mathcal{M})$ of $\text{SDiff}(\mathcal{M})$, which we call the *superdiffeomorphisms with compact support*. By definition, we have $\text{SDiff}_c(\mathcal{M})_{\mathbb{R}} = \text{Aut}_c(\mathcal{M})$. Moreover, if we set $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda} := \text{SDiff}_c(\mathcal{M})_{\varepsilon_{\Lambda}}^{-1}(\{\text{id}_{\mathcal{M}}\})$, we get a sub-group object $\text{SDiff}_{\text{id}}^c(\mathcal{M})$ of $\text{SDiff}_{\text{id}}(\mathcal{M})$ and a split short exact sequence of groups

$$1 \rightarrow \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda} \rightarrow \text{SDiff}_c(\mathcal{M})_{\Lambda} \rightarrow \text{Aut}_c(\mathcal{M}) \rightarrow 1.$$

Lemma 4.4.13. *Let \mathcal{M} be a σ -compact supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. For each $\Lambda \in \mathbf{Gr}$, the exponential map $\exp_{\mathcal{M},\Lambda}$ from Proposition 4.4.7 restricts to a bijective map*

$$\exp_{\mathcal{M},\Lambda}^c: \overline{\mathcal{X}_c(\mathcal{M})}_{\Lambda+} \rightarrow \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda},$$

turning $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda}$ into a polynomial subgroup of $\text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda}$. If $\mathcal{A} := \{\varphi^{\alpha}: \mathcal{U}^{\alpha} \rightarrow \mathcal{M}: \alpha \in A\}$ is an atlas of \mathcal{M} such that $(\varphi_{\mathbb{R}}^{\alpha}(\mathcal{U}_{\mathbb{R}}^{\alpha}))_{\alpha \in A}$ is a locally finite cover of $\mathcal{M}_{\mathbb{R}}$, then restricting Θ_{Λ} and Ψ_{Λ} from the proposition leads to the commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{X}_c(\mathcal{M})}_{\Lambda+} & \xrightarrow{\exp_{\mathcal{M},\Lambda}} & \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda} \\ \downarrow \Theta_{\Lambda} & & \downarrow \Psi_{\Lambda} \\ \bigoplus_{\alpha \in A} \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}^{\alpha}, \overline{E})_{\Lambda+} & \xrightarrow{\bigoplus_{\alpha} \exp_{\mathcal{U}^{\alpha},\Lambda}} & \bigoplus_{\alpha \in A} \text{SDiff}_{\text{id}}(\mathcal{U}^{\alpha})_{\Lambda}, \end{array}$$

where each $\text{SDiff}_{\text{id}}(\mathcal{U}^{\alpha})_{\Lambda}$ is considered as a vector space. The map Θ_{Λ} is continuous as a map $\overline{\mathcal{X}_c(\mathcal{M})}_{c\Lambda+} \rightarrow \bigoplus_{\alpha \in A} \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}^{\alpha}, \overline{E})_{c\Lambda+}$ and its image is a closed subspace. The same holds for Θ_{Λ} as a map $\overline{\mathcal{X}_c(\mathcal{M})}_{b\Lambda+} \rightarrow \bigoplus_{\alpha \in A} \widehat{\mathcal{SC}}^{\infty}(\mathcal{U}^{\alpha}, \overline{E})_{b\Lambda+}$ if \mathcal{M} is a Banach supermanifold. The respective topology induced on $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda}$ does not depend on the atlas. The action from Lemma 4.4.12 restricts to an action

$$\beta_{\Lambda}: \text{Aut}_c(\mathcal{M}) \times \overline{\mathcal{X}_c(\mathcal{M})}_{\Lambda+} \rightarrow \overline{\mathcal{X}_c(\mathcal{M})}_{\Lambda+}.$$

Proof. By definition, an element $X \in \overline{\mathcal{X}_c(\mathcal{M})}_{\Lambda+}$ is an element of $\overline{\mathcal{X}_c(\mathcal{M})}_{\Lambda+}$ if and only if $(\Theta_{\Lambda}(X))_{\alpha} = 0$ holds for almost all $\alpha \in A$. Likewise, an element $f \in \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda}$ is an element of $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda}$ if and only if $(\Psi_{\Lambda}(f))_{\alpha} = \text{pr}_{\mathcal{U}^{\alpha}}$ holds for almost all $\alpha \in A$. It follows from Lemma 4.1.9 that Θ_{Λ} is continuous, that $\text{im}(\Theta_{\Lambda})$ is a closed subspace and that the induced topology does not depend on the atlas. The group action restricts as stated by Lemma 4.1.13. The lemma then follows from Proposition 4.4.7. \square

Proposition 4.4.14. *Let \mathcal{M} be a σ -compact Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. For each $\Lambda \in \mathbf{Gr}$, the global chart*

$$\exp_{\mathcal{M},\Lambda}^c: \overline{\mathcal{X}_c(\mathcal{M})}_{b\Lambda+} \rightarrow \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda}$$

turns $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda}$ into a Lie group. Moreover, the action of $\text{Aut}_c(\mathcal{M})$ on $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda}$ is smooth and therefore $\text{SDiff}_c(\mathcal{M})_{\Lambda}$ is a Lie group as well. For each $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$, the morphism $\text{SDiff}_c(\mathcal{M})_{\varrho}$ is a morphism of Lie groups.

Proof. That $\text{SDiff}_{\text{id}}^c(\mathcal{M})_\Lambda$ is a Lie group follows from Lemma 4.4.13 and Lemma 4.4.6 together with Proposition A.3.5. The smoothness of β_Λ follows from Lemma 4.2.15 and Lemma 4.4.12. Note that $\text{SDiff}_c(\mathcal{M})_\varrho(f) = f$ holds for $f \in \text{Aut}_c(\mathcal{M})$ if we identify $\text{Aut}_c(\mathcal{M})$ with $\text{SDiff}_c(\mathcal{M})_{\eta_\Lambda}(\text{Aut}_c(\mathcal{M}))$. Thus, since $\text{SDiff}_c(\mathcal{M})_\varrho$ is a morphism of groups and since $\text{SDiff}_c(\mathcal{M})_\Lambda$ and $\text{SDiff}_c(\mathcal{M})_{\Lambda'}$ arise as semidirect products with $\text{Aut}_c(\mathcal{M})$, it suffices to calculate the smoothness of $\text{SDiff}_c(\mathcal{M})_\varrho|_{\text{SDiff}_{\text{id}}^c(\mathcal{M})_\Lambda}^{\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda'}}$. But, in the exponential charts, this is just the restriction $\overline{\mathcal{X}_c(\mathcal{M})_{b_\varrho}}|_{\overline{\mathcal{X}_c(\mathcal{M})_{b_{\Lambda^+}}}}^{\overline{\mathcal{X}_c(\mathcal{M})_{b_{\Lambda'^+}}}}$, which is smooth. \square

Remark 4.4.15. In view of Proposition 4.4.14, it is enticing to simply define charts of $\text{SDiff}_c(\mathcal{M})$ by taking any chart of $\varphi: U \rightarrow V$ of $\text{Aut}_c(\mathcal{M})$ and then letting

$$\varphi_\Lambda((\lambda_I(IX))_{I \in \mathcal{P}^n}) := \exp_{\mathcal{M}, \Lambda}((\lambda_I(IX))_{I \in \mathcal{P}_+^n}) \circ \varphi^{-1}(\varnothing X)$$

for $(\lambda_I(IX))_{I \in \mathcal{P}^n} \in \overline{\mathcal{X}_c(\mathcal{M})_{b_\Lambda}}$ with $\varnothing X \in U$. Modulo notations, this is in fact the chart defined in [47, Section 7.6, p.321] for finite-dimensional compact supermanifolds. This defines certainly a natural transformation. However, the change of charts need not be supersmooth. The problem is that changing the chart of $\text{Aut}_c(\mathcal{M})$ does not affect the nilpotent part $\mathcal{X}_c(\mathcal{M})_{b_{\Lambda^+}}$, so that the derivative cannot be Λ_- -linear.

4.4.4. Purely even superdiffeomorphisms

As mentioned at the beginning of this section, there exists a natural definition of purely even superdiffeomorphisms $\text{SDiff}(\mathcal{M})_{\overline{0}}$ for an arbitrary supermanifold \mathcal{M} , such that we have a decomposition

$$\text{SDiff}(\mathcal{M})_{\overline{0}} \times \overline{\mathcal{X}(\mathcal{M})_{\overline{1}}} \rightarrow \text{SDiff}(\mathcal{M}).$$

Moreover, one has a short exact sequence

$$1 \rightarrow \text{SDiff}(\mathcal{M})_{\overline{0}, \Lambda}^{\geq 2} \rightarrow \text{SDiff}(\mathcal{M})_{\overline{0}, \Lambda} \rightarrow \text{SDiff}(\mathcal{M}^{(1)})_{\overline{0}, \Lambda} \rightarrow 1$$

that splits if \mathcal{M} is of Batchelor type. We discuss all these components in detail below, considering also compactly supported and topological versions. Once we have seen that $\iota(\text{Aut}_c(\mathcal{M})) \cong \text{SDiff}_c(\mathcal{M})_{\overline{0}}$ holds for σ -compact Banach supermanifolds with finite-dimensional base $\mathcal{M}_{\mathbb{R}}$, it will be easy to turn $\text{SDiff}_c(\mathcal{M})$ into a Lie supergroup. To emphasize naturality of this approach, let us also mention that for such $\mathcal{M}_{\mathbb{R}}$, we obtain an isomorphism

$$\iota(\text{Diff}_c(\mathcal{M}_{\mathbb{R}})) \cong \text{SDiff}_c(\iota(\mathcal{M}_{\mathbb{R}})).$$

Lemma/Definition 4.4.16. Let $\mathcal{M} \in \mathbf{SMan}$ be modelled on $E \in \mathbf{SVec}_{lc}$ and let $\Lambda \in \mathbf{Gr}$ be fixed. Then $\text{SDiff}_{\text{id}}(\mathcal{M})_{\overline{0}, \Lambda} := \exp_{\mathcal{M}, \Lambda}(\overline{\mathcal{X}(\mathcal{M})_{\overline{0}, \Lambda^+}})$ is a polynomial subgroup of $\text{SDiff}_{\text{id}}(\mathcal{M})_\Lambda$ and these subgroups define a sub-supergroup $\text{SDiff}_{\text{id}}(\mathcal{M})_{\overline{0}}$.

The action of $\text{Aut}(\mathcal{M})$ respects these groups and setting

$$\text{SDiff}(\mathcal{M})_{\bar{0},\Lambda} := \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda} \rtimes \text{Aut}(\mathcal{M})$$

defines a sub-supergroup $\text{SDiff}(\mathcal{M})_{\bar{0}}$ of $\text{SDiff}(\mathcal{M})$ that we will call the *purely even superdiffeomorphisms*. The elements $f \in \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda_n}$ are characterized by the property that each chart representation f^α of f decomposes as $f^\alpha = \text{pr}_{\mathcal{U}^\alpha} + \sum_{I \in \mathcal{P}_{0,+}^n} \mathbf{v}_I \wedge_I f^\alpha$ with $_I f^\alpha \in \mathcal{SC}^\infty(\mathcal{U}^\alpha, \bar{E})$.

If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then analogous statements hold true for $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\bar{0}}$ and $\text{SDiff}_c(\mathcal{M})_{\bar{0}}$ defined by $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\bar{0},\Lambda} := \exp_{\mathcal{M},\Lambda}(\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0},\Lambda+}})$ and $\text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda} := \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\bar{0},\Lambda} \rtimes \text{Aut}_c(\mathcal{M})$, respectively. If \mathcal{M} is additionally a σ -compact Banach supermanifold, then $\text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}$ is a closed Lie subgroup of $\text{SDiff}_c(\mathcal{M})_{\Lambda}$.

Proof. By Corollary 4.4.8 $\overline{\mathcal{X}(\mathcal{M})_{\bar{0},\Lambda}}$ is a Lie subalgebra of $\overline{\mathcal{X}(\mathcal{M})_{\Lambda}}$ and therefore $\overline{\mathcal{X}(\mathcal{M})_{\bar{0},\Lambda}}$ defines a polynomial subgroup under the exponential map. With Lemma 4.4.12, we see that the semi-direct product is well-defined. Note that

$$\prod_{I \in \mathcal{P}_{0,+}^n} \lambda_I \mathcal{X}(\mathcal{M})_{\bar{0}} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda_n}, \quad (\lambda_I X_I)_I \mapsto \prod_{I \in \mathcal{P}_{0,+}^n}^{\uparrow} \exp_{\mathcal{M},\Lambda}(\lambda_I X_I),$$

where the product is taken in ascending lexicographic order, is a morphism of multilinear spaces whose linear part is the identity. Hence, it is bijective by Theorem B.1.2. With Corollary 4.4.10 we can now make an induction analogous to the one in Remark E.2.1 to see that the elements of $\text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda_n}$ are characterized as claimed. That $\text{SDiff}(\mathcal{M})_{\varrho}$ restricts to a map $\text{SDiff}(\mathcal{M})_{\bar{0},\varrho} : \text{SDiff}(\mathcal{M})_{\bar{0},\Lambda} \rightarrow \text{SDiff}(\mathcal{M})_{\bar{0},\Lambda'}$ for each $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$ follows from the fact that ϱ is graded together with Lemma 4.3.8.

The same arguments and Lemma 4.4.13 show that the respective statements also hold in the case of compact support. That $\text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}$ is a closed Lie subgroup of $\text{SDiff}_c(\mathcal{M})_{\Lambda}$ in case of a σ -compact Banach supermanifold \mathcal{M} follows immediately because $\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0},b\Lambda}}$ is a closed subspace of $\overline{\mathcal{X}_c(\mathcal{M})_{b\Lambda}}$. \square

Lemma 4.4.17. *Let \mathcal{M} be a supermanifold. For each $\Lambda \in \mathbf{Gr}$, the map*

$$\text{SDiff}(\mathcal{M})_{\bar{0},\Lambda} \times \overline{\mathcal{X}(\mathcal{M})_{\bar{1},\Lambda}} \rightarrow \text{SDiff}(\mathcal{M})_{\Lambda}, \quad (f, X) \mapsto f \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}(X)$$

is a bijection. If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, this restricts to a bijection

$$\text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},\Lambda}} \rightarrow \text{SDiff}_c(\mathcal{M})_{\Lambda}.$$

Both bijections define natural transformations.

Proof. As a composition of natural transformations, both maps define natural transformations. Since $\text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}$ and $\text{SDiff}(\mathcal{M})_{\Lambda}$ are both defined as semidirect products with $\text{Aut}(\mathcal{M})$, it suffices to show that the restriction

$$\text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda} \times \overline{\mathcal{X}(\mathcal{M})_{\bar{1},\Lambda}} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_{\Lambda}$$

is bijective. But, this is a morphism of multilinear spaces if we identify both sides with $\overline{\mathcal{X}(\mathcal{M})}_{\Lambda^+}$ via the respective exponential map. The linear component of this morphism is then the identity. Thus, the morphism is bijective by Theorem B.1.2. The same arguments apply to the case of compact support. \square

Splitting the purely even superdiffeomorphisms

Let $\mathcal{M} \in \mathbf{SMan}$. By Lemma 4.1.18, we have a short exact sequence

$$0 \rightarrow \mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2} \hookrightarrow \mathcal{X}(\mathcal{M})_{\bar{0}} \xrightarrow{\iota_{\infty}^1 \circ \pi_1^{\infty}} \iota_{\infty}^1(\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}}) \rightarrow 0 \quad (4.11)$$

of Lie algebras, where $\iota_{\infty}^1(\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}})$ is the Lie subalgebra of $\mathcal{X}(\iota_{\infty}^1(\mathcal{M}^{(1)})_{\bar{0}})$ consisting of vector fields $X: \iota_{\infty}^1(\mathcal{M}^{(1)}) \rightarrow \mathcal{T}\iota_{\infty}^1(\mathcal{M}^{(1)})$ with the local form $X^{\alpha} = (X_0^{\alpha}, X_1^{\alpha}, 0, \dots)$ for any atlas of Batchelor type. The sequence obviously splits canonically if \mathcal{M} is of Batchelor type. If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, we obtain an analogous sequence

$$0 \rightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2} \hookrightarrow \mathcal{X}_c(\mathcal{M})_{\bar{0}} \xrightarrow{\iota_{\infty}^1 \circ \pi_1^{\infty}} \iota_{\infty}^1(\mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}}) \rightarrow 0.$$

If we give $\mathcal{X}(\mathcal{M})_{\bar{0}}$, resp. $\mathcal{X}_c(\mathcal{M})_{\bar{0}}$, a topology as in Lemma 4.1.6, resp. Lemma 4.1.9, all the Lie subalgebras are closed.

Lemma 4.4.18. *Let $\mathcal{M} \in \mathbf{SMan}$ and $\Lambda \in \mathbf{Gr}$. Setting*

$$\begin{aligned} \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2} &:= \exp_{\mathcal{M}, \Lambda}(\overline{\mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda^+}) \quad \text{and} \\ \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda} &:= \exp_{\iota_{\infty}^1(\mathcal{M}^{(1)})_{\bar{0}}, \Lambda}(\overline{\iota_{\infty}^1(\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}})}_{\Lambda^+}), \end{aligned}$$

leads to sub-group objects $\text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0}}^{\geq 2}$ of $\text{SDiff}_{\text{id}}(\mathcal{M})$ and $\text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0}}$ of $\text{SDiff}_{\text{id}}(\iota_{\infty}^1(\mathcal{M}^{(1)}))$, respectively. We have a short exact sequence of groups

$$1 \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0}, \Lambda} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda} \rightarrow 1$$

natural in $\Lambda \in \mathbf{Gr}$ that splits canonically if \mathcal{M} is of Batchelor type. If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then the same holds true for $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2} := \exp_{\mathcal{M}, \Lambda}^c(\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda^+})$ and $\text{SDiff}_{\text{id}}^c(\mathcal{M}^{(1)})_{\bar{0}, \Lambda} := \exp_{\iota_{\infty}^1(\mathcal{M}^{(1)})_{\bar{0}}, \Lambda}^c(\overline{\iota_{\infty}^1(\mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}})}_{\Lambda^+})$. If additionally \mathcal{M} is a σ -compact Banach supermanifold, this leads to a short exact sequence of Lie groups

$$1 \rightarrow \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2} \hookrightarrow \text{SDiff}_{\text{id}}^c(\mathcal{M})_{\Lambda} \rightarrow \text{SDiff}_{\text{id}}^c(\mathcal{M}^{(1)})_{\Lambda} \rightarrow 1$$

that also splits canonically if \mathcal{M} is of Batchelor type.

Proof. By Corollary 2.2.22, the exact sequence (4.11) leads to an exact sequence

$$0 \rightarrow \overline{\mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda^+} \hookrightarrow \overline{\mathcal{X}(\mathcal{M})_{\bar{0}}}_{\Lambda^+} \xrightarrow{\iota_{\infty}^1 \circ \pi_1^{\infty}} \overline{\iota_{\infty}^1(\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}})}_{\Lambda^+} \rightarrow 0$$

of Lie algebras. It follows from Lemma C.2.2 that this leads to an exact sequence

of groups that splits if the sequence of Lie algebras splits. The same argument also works in the case of compact support. Let \mathcal{M} be a σ -compact Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. Since $\mathcal{X}(\mathcal{M})_{\bar{0},b}^{\geq 2} \subseteq \mathcal{X}(\mathcal{M})_{\bar{0},b}$ and $\iota_{\infty}^1(\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0},b}) \subseteq \mathcal{X}_c(\iota_{\infty}^1(\mathcal{M}^{(1)}))_{\bar{0},b}$ are closed and since all the Lie algebra morphisms are continuous, we get a sequence of Lie groups as claimed. \square

Lemma 4.4.19. *Let $\mathcal{M} \in \mathbf{SMan}$.*

(a) *For each $\Lambda \in \mathbf{Gr}$, the semidirect products*

$$\begin{aligned} \mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} &:= \mathrm{SDiff}_{\mathrm{id}}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rtimes \mathrm{Aut}_{\mathrm{id}}(\mathcal{M}) \quad \text{and} \\ \mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0},\Lambda} &:= \mathrm{SDiff}_{\mathrm{id}}(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \rtimes \iota_{\infty}^1(\mathrm{Aut}(\mathcal{M}^{(1)})) \end{aligned}$$

are well-defined and lead to sub-group objects $\mathrm{SDiff}(\mathcal{M})_{\bar{0}}^{\geq 2}$ of $\mathrm{SDiff}(\mathcal{M})$ and $\mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0}}$ of $\mathrm{SDiff}(\iota_{\infty}^1(\mathcal{M}^{(1)}))$, respectively. There exists a short exact sequence of groups

$$1 \rightarrow \mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \hookrightarrow \mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda} \rightarrow \mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \rightarrow 1$$

natural in $\Lambda \in \mathbf{Gr}$ that splits canonically if \mathcal{M} is of Batchelor type.

(b) *If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then analogous statements to (a) hold for*

$$\begin{aligned} \mathrm{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} &:= \mathrm{SDiff}_{\mathrm{id}}^c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rtimes \mathrm{Aut}_{\mathrm{id}}^c(\mathcal{M}) \quad \text{and} \\ \mathrm{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0},\Lambda} &:= \mathrm{SDiff}_{\mathrm{id}}^c(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \rtimes \iota_{\infty}^1(\mathrm{Aut}_c(\mathcal{M}^{(1)})). \end{aligned}$$

(c) *If \mathcal{M} is a σ -compact Banach supermanifold with finite-dimensional base, we have a short exact sequence of Lie groups*

$$1 \rightarrow \mathrm{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \hookrightarrow \mathrm{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda} \rightarrow \mathrm{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \rightarrow 1$$

that splits canonically if \mathcal{M} is of Batchelor type.

Proof. (a) If $X \in \mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}$ and $f \in \mathrm{Aut}_{\mathrm{id}}(\mathcal{M})$ holds, then we have $\mathcal{T}f \circ X \circ f^{-1} \in \mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}$ by Lemma 4.1.13. Likewise, if $X \in \mathcal{X}(\mathcal{M}^{(1)})$ has the local form $(X_0^{\alpha}, X_1^{\alpha}, 0, \dots)$ in some atlas of Batchelor type, then $\mathcal{T}f \circ X \circ f^{-1}$ also has this form for $f \in \iota_{\infty}^1(\mathrm{Aut}(\mathcal{M}^{(1)}))$, because f has the local form $(f_0^{\alpha\beta}, f_1^{\alpha\beta}, 0, \dots)$. Thus, the semidirect products are well-defined. With Lemma/Definition 4.4.16, we can express each element of $\mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda}$ uniquely as $f \circ \exp_{\mathcal{M},\Lambda}(X)$ with $f \in \mathrm{Aut}(\mathcal{M})$ and $X \in \overline{\mathcal{X}(\mathcal{M})_{\bar{0},\Lambda+}}$. We define the projection $\mathrm{SDiff}(\mathcal{M})_{\bar{0},\Lambda} \rightarrow \mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$ via

$$f \circ \exp_{\mathcal{M},\Lambda}(X) \mapsto \iota_{\infty}^1 \circ \pi_1^{\infty}(f) \circ \exp_{\mathcal{M},\Lambda}(\overline{\iota_{\infty}^1 \circ \pi_1^{\infty}(X)}).$$

This is a morphism of groups because $\iota_{\infty}^1 \circ \pi_1^{\infty}$ is a functor, $\overline{\iota_{\infty}^1 \circ \pi_1^{\infty}}|_{\overline{\mathcal{X}(\mathcal{M})_{\bar{0},\Lambda+}}}$ is a morphism of Lie algebras and

$$\iota_{\infty}^1 \circ \pi_1^{\infty}(\mathcal{T}f \circ Y \circ f^{-1}) = (\mathcal{T}(\iota_{\infty}^1 \circ \pi_1^{\infty}(f))) \circ (\iota_{\infty}^1 \circ \pi_1^{\infty}(Y)) \circ (\iota_{\infty}^1 \circ \pi_1^{\infty}(f^{-1}))$$

holds for $f \in \text{Aut}(\mathcal{M})$ and $Y \in \mathcal{X}(\mathcal{M})_{\bar{0}}$. Naturality in Λ follows because for all $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$, we have $\text{SDiff}(\mathcal{M})_{\varrho}(f \circ \exp_{\mathcal{M}, \Lambda}(X)) = f \circ \exp_{\mathcal{M}, \Lambda'}(\overline{\mathcal{X}(\mathcal{M})_{\bar{0}_{\varrho}}}(X))$.

(b) The same arguments apply to the case of compact support.

(c) This leads to an exact sequence of Lie groups as claimed, because the involved Lie algebra morphisms are continuous and

$$\text{Aut}_c(\mathcal{M}) \mapsto \iota_{\infty}^1(\text{Aut}_c(\mathcal{M}^{(1)})), \quad f \mapsto \iota_{\infty}^1 \circ \pi_1^{\infty}(f)$$

is smooth by the definition of the Lie group structure on $\text{Aut}_c(\mathcal{M})$. \square

Lemma 4.4.20. *Let \mathcal{M} be a supermanifold*

(a) *For each $\Lambda \in \mathbf{Gr}$, the group $\text{SDiff}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}$ is pro-polynomial and admits an exponential map*

$$\exp_{\mathcal{M}, \Lambda}^{\geq 2}: \overline{\mathcal{X}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}} \rightarrow \text{SDiff}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}.$$

The exponential maps define a morphism in $\mathbf{Set}^{\mathbf{Gr}}$ and the restriction to $\mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}$, resp. to $\overline{\mathcal{X}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}}$, is just the exponential map from Proposition 4.2.6, resp. from Lemma 4.4.18.

(b) *If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then analogous statements to (a) hold for $\text{SDiff}_c(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}$ and*

$$\exp_{\mathcal{M}, \Lambda}^{c, \geq 2}: \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}} \rightarrow \text{SDiff}_c(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}.$$

(c) *If \mathcal{M} is a σ -compact Banach supermanifold with finite-dimensional base $\mathcal{M}_{\mathbb{R}}$, then $\exp_{\mathcal{M}, \Lambda}^{c, \geq 2}: \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}} \rightarrow \text{SDiff}_c(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}$ is a global chart for the Lie group structure defined in Lemma 4.4.19. This chart turns $\text{SDiff}_c(\mathcal{M})_{\bar{0}}^{\geq 2}$ into a Lie supergroup such that*

$$\text{SDiff}_c(\mathcal{M})_{\bar{0}}^{\geq 2} \cong \iota(\text{Aut}_{\text{id}}^c(\mathcal{M}))$$

holds as Lie supergroups.

Proof. (a) For all $0 < n \leq k \leq \infty$, the Lie algebra morphism $\pi_n^k: \mathcal{X}(\mathcal{M}^{(k)})_{\bar{0}} \rightarrow \mathcal{X}(\mathcal{M}^{(n)})_{\bar{0}}$ defines a morphism of nilpotent Lie algebras

$$\overline{\pi_n^k}: \overline{\mathcal{X}(\mathcal{M}^{(k)})_{\bar{0}, \Lambda}^{\geq 2}} \rightarrow \overline{\mathcal{X}(\mathcal{M}^{(n)})_{\bar{0}, \Lambda}^{\geq 2}}.$$

If we define $\text{SDiff}_{\text{id}}(\mathcal{M}^{(n)})_{\bar{0}, \Lambda}^{\geq 2}$ for $1 < n < \infty$ as $\overline{\mathcal{X}(\mathcal{M}^{(n)})_{\bar{0}, \Lambda}^{\geq 2}}$ equipped with the BCH multiplication, we see with Lemma 4.2.11 that the group action by automorphisms

$$\alpha: \text{Aut}_{\text{id}}(\mathcal{M}^{(n)}) \times \text{SDiff}_{\text{id}}(\mathcal{M}^{(n)})_{\bar{0}, \Lambda}^{\geq 2} \rightarrow \text{SDiff}_{\text{id}}(\mathcal{M}^{(n)})_{\bar{0}, \Lambda}^{\geq 2},$$

$$(f, (\lambda_I \mathcal{T}_I X)_I) \mapsto (\lambda_I \mathcal{T}_I f \circ \mathcal{T}_I X \circ f^{-1})_I$$

is in each component I given by the action of the polynomial group $\text{Aut}_{\text{id}}(\mathcal{M}^{(n)})$ on its own Lie algebra. Therefore, Lemma C.2.4 is satisfied and we obtain a polynomial group $\text{SDiff}_{\text{id}}(\mathcal{M}^{(n)})_{\bar{0},\Lambda}^{\geq 2} \rtimes \text{Aut}_{\text{id}}(\mathcal{M}^{(n)})$ whose Lie bracket is given by $\overline{[\cdot, \cdot]^{(n)}}_{\Lambda}$. Here $[\cdot, \cdot]^{(n)}$ denotes the Lie bracket of $\mathcal{X}(\mathcal{M}^{(n)})_{\bar{0}}^{\geq 2}$. Clearly, we have

$$\begin{aligned} & \varprojlim_n (\text{SDiff}_{\text{id}}(\mathcal{M}^{(n)})_{\bar{0},\Lambda}^{\geq 2} \rtimes \text{Aut}_{\text{id}}(\mathcal{M}^{(n)})) \\ & \cong \varprojlim_n \text{SDiff}_{\text{id}}(\mathcal{M}^{(n)})_{\bar{0},\Lambda}^{\geq 2} \rtimes \varprojlim_n \text{Aut}_{\text{id}}(\mathcal{M}^{(n)}) \\ & \cong \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rtimes \text{Aut}_{\text{id}}(\mathcal{M}) = \text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}. \end{aligned}$$

Moreover, since the Lie bracket of the limit is $\overline{[\cdot, \cdot]}_{\Lambda}$ (for $[\cdot, \cdot]$ the Lie bracket of $\mathcal{X}(\mathcal{M})_{\bar{0}}^{\geq 2}$) one easily sees that the exponential map restricts to the exponential map of $\text{Aut}_{\text{id}}(\mathcal{M})$ and $\text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$ as claimed. If we define

$$\begin{aligned} \text{SDiff}(\mathcal{M}^{(n)})_{\bar{0},\varrho}^{\geq 2} &: \text{SDiff}(\mathcal{M}^{(n)})_{\bar{0},\Lambda}^{\geq 2} \rightarrow \text{SDiff}(\mathcal{M}^{(n)})_{\bar{0},\Lambda'}^{\geq 2}, \\ (X, f) &\mapsto (\overline{\mathcal{X}(\mathcal{M}^{(n)})_{\bar{0}}^{\geq 2}}_{\varrho}(X), f), \end{aligned}$$

then we also have $\varprojlim_n \text{SDiff}(\mathcal{M}^{(n)})_{\bar{0},\varrho}^{\geq 2} = \text{SDiff}(\mathcal{M})_{\bar{0},\varrho}^{\geq 2}$ under the above identification, for $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$. The naturality of $(\exp_{\mathcal{M},\Lambda}^{\geq 2})_{\Lambda \in \mathbf{Gr}}$ follows from Lemma C.3.2 because under the exponential map, the polynomial morphism of groups $\text{SDiff}(\mathcal{M}^{(n)})_{\bar{0},\varrho}^{\geq 2}$ corresponds to the linear map $\overline{\mathcal{X}(\mathcal{M}^{(n)})_{\bar{0}}^{\geq 2}}_{\varrho}$.

(b) This follows from the same arguments as in (a).

(c) Let \mathcal{M} be a σ -compact Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. We know that $\iota(\exp_{\mathcal{M}}^c): \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}} \rightarrow \iota(\text{Aut}_{\text{id}}^c(\mathcal{M}))$ is a global chart and in that chart we have

$$\ker(\iota(\text{Aut}_{\text{id}}^c(\mathcal{M}))_{\varepsilon_{\Lambda}}) \cong \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda+}$$

for all $\Lambda \in \mathbf{Gr}$. Here $\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda+}$ is considered as a polynomial group with the BCH multiplication with regard to the Lie bracket $\overline{[\cdot, \cdot]}_{\Lambda+}$. The induced action of $\text{Aut}_{\text{id}}^c(\mathcal{M})$ on $\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda+}$ is given by

$$\text{Aut}_{\text{id}}^c(\mathcal{M}) \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda+} \rightarrow \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda+}, \quad (f, (\lambda_I T f \circ_I X \circ f^{-1})_I).$$

Since $\text{SDiff}_{\text{id}}^c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$ induces the same group structure on $\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\Lambda+}$ and since the actions of $\text{Aut}_{\text{id}}^c(\mathcal{M})$ are the same by Lemma 4.4.12, we have $\iota(\text{Aut}_{\text{id}}^c(\mathcal{M})) \cong \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$. By naturality, $\iota(\text{Aut}_{\text{id}}^c(\mathcal{M}))_{\varrho}$ and $\text{SDiff}_c(\mathcal{M})_{\bar{0},\varrho}^{\geq 2}$ both correspond to $\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_{\varrho}$ for each $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$. Since $\exp_{\mathcal{M},\Lambda}^{c,\geq 2}$ restricts to the respective exponential maps, it is obvious that $\exp_{\mathcal{M},\Lambda}^{c,\geq 2}$ is a chart with respect to the smooth structure from Lemma 4.4.19. \square

Proposition 4.4.21. *Let \mathcal{M} be a supermanifold.*

(a) *Let $n \in \mathbb{N}$. There exists an isomorphism of groups*

$$\mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda_n} \cong \mathrm{Aut}_{T^n \mathbb{R}}(T^k \mathcal{M}^{(1)})|_{\mathcal{P}_{0,+}^n}^-$$

that is the identity on $\iota_\infty^1(\mathrm{Aut}(\mathcal{M}^{(1)})) \cong \mathrm{Aut}(\mathcal{M}^{(1)})$, such that $\mathrm{SDiff}_{\mathrm{id}}(\mathcal{M}^{(1)})_{\Lambda_n} \cong \mathcal{X}^n(\mathcal{M}^{(1)})_{\bar{0}}|_{\mathcal{P}_{0,+}^n}^-$ holds.

(b) *If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then an analogous statement to (a) holds for*

$$\mathrm{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0}, \Lambda_n} \cong \mathrm{Aut}_{T^n \mathbb{R}}(T^k \mathcal{M}^{(1)})_c|_{\mathcal{P}_{0,+}^n}^-.$$

(c) *If \mathcal{M} is a σ -compact Banach supermanifold and $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then $\mathrm{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0}}$ can be turned into a Lie supergroup such that*

$$\mathrm{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0}} \cong \iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))$$

holds.

Proof. (a) Let $[\cdot, \cdot]$ denote the Lie bracket of $\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}}$. From the definitions in Lemma 4.4.18 and Lemma 4.4.19, we know that $\mathrm{SDiff}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda_n}$ is isomorphic to $\overline{\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda_n^+}} \rtimes \mathrm{Aut}(\mathcal{M}^{(1)})$, where the group structure on $\overline{\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda_n^+}}$ is given by the BCH multiplication with respect to the Lie bracket $[\cdot, \cdot]_{\Lambda_n^+}$ and the action of $\mathrm{Aut}(\mathcal{M}^{(1)})$ is given by $\lambda_I X \mapsto \lambda_I \mathcal{T}f \circ X \circ f^{-1}$ for $I \in \mathcal{P}_{0,+}^n$, $X \in \mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}}$ and $f \in \mathrm{Aut}(\mathcal{M}^{(1)})$. But, by Lemma/Definition E.3.8, this is exactly the same group structure as the one induced by $\mathrm{Aut}_{T^n \mathbb{R}}(T^n \mathcal{M}^{(1)})|_{\mathcal{P}_{0,+}^n}^- = \mathcal{X}^n(\mathcal{M}^{(1)})_{\bar{0}}|_{\mathcal{P}_{0,+}^n}^- \rtimes \mathrm{Aut}(\mathcal{M}^{(1)})$ on $\overline{\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda_n^+}} \rtimes \mathrm{Aut}(\mathcal{M}^{(1)})$.

(b) This follows from the same arguments as in (a).

(c) It was shown in Lemma E.3.9 that $T^n \mathrm{Aut}_c(\mathcal{M}^{(1)})|_{\mathcal{P}_{0,+}^n}^- \cong \mathrm{Aut}_{T^n \mathbb{R}}(T^n \mathcal{M}^{(1)})|_{\mathcal{P}_{0,+}^n}^-$ holds. Moreover, we know from Proposition 2.3.16 that $\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))_{\Lambda_n} \cong T^n \mathrm{Aut}_c(\mathcal{M}^{(1)})|_{\mathcal{P}_{0,+}^n}^-$ holds as Lie groups such that $\ker(\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))_{\varepsilon_{\Lambda_n}}) \cong (T^n \mathrm{Aut}_c(\mathcal{M}^{(1)})|_{\mathcal{P}_{0,+}^n}^-)_{\mathrm{id}_{\mathcal{M}^{(1)}}}$. All that is left to see is that under these identifications $\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))_{\varrho}$ and $\mathrm{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0}, \varrho}$ are the same for all $\varrho \in \mathrm{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$. Since neither group morphism changes the contribution of $\mathrm{Aut}_c(\mathcal{M}^{(1)})$ in the respective semidirect product, it is enough to check this on $\ker(\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))_{\varepsilon_{\Lambda}})$ and $\mathrm{SDiff}_{\mathrm{id}}^c(\mathcal{M}^{(1)})_{\bar{0}, \Lambda}$. Let $\exp_{\Lambda}^{\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))} : \overline{\mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}, \Lambda^+}} \rightarrow \ker(\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))_{\varepsilon_{\Lambda}})$ denote the unique diffeomorphism from Proposition 3.2.6. Then the group structure induced on $\overline{\mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}, \Lambda^+}}$ by $\exp_{\Lambda}^{\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))}$, is given by the BCH multiplication with respect to $[\cdot, \cdot]_{\Lambda^+}$. If $\Psi_{\Lambda} : \ker(\mathrm{Aut}_c(\mathcal{M}^{(1)}))_{\varepsilon_{\Lambda}} \rightarrow \mathrm{SDiff}_{\mathrm{id}}^c(\mathcal{M}^{(1)})_{\bar{0}, \Lambda}$ is the isomorphism obtained above, uniqueness of the exponential maps implies $\Psi_{\Lambda} \circ \exp_{\Lambda}^{\iota(\mathrm{Aut}_c(\mathcal{M}^{(1)}))} = \exp_{\mathcal{M}, \Lambda}^c$ and since both exponential maps are natural transformations, so is $(\Psi_{\Lambda})_{\Lambda \in \mathbf{Gr}}$. \square

Corollary 4.4.22. *Let M be a finite-dimensional σ -compact manifold. Then we have*

$$\begin{aligned} \text{SDiff}_c(\iota(M)) &\cong \iota(\text{Diff}_c(M)) \quad \text{and} \\ \text{SDiff}_{\text{id}}^c(\iota(M)) \rtimes \text{Diff}(M) &\cong \iota(\text{Diff}(M)) \end{aligned}$$

as supergroups, turning $\text{SDiff}_c(\iota(M))$ and $\text{SDiff}_{\text{id}}^c(\iota(M)) \rtimes \text{Diff}(M)$ into Lie supergroups.

Proof. The first statement follows immediately by applying Proposition 4.4.21 to the trivial bundle $(\iota(M))^{(1)} \cong M \times \{0\}$. The second statement follows analogously to the proposition, using Lemma E.2.4. \square

Remarkably, even for the full diffeomorphism group $\text{Diff}(M)$, the nilpotent part $\ker(\iota(\text{Diff}(M))_{\varepsilon_\Lambda})$ is only identified with compactly supported superdiffeomorphisms. The reason behind this is that $\text{Diff}(M)$ is modelled on $\mathfrak{X}_c(M)$.

We now understand the respective structure of $\text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0}}$ and $\text{SDiff}(\mathcal{M}^{(1)})_{\bar{0}}$ very well. All that remains to do, is to calculate the semi-direct product for the case that \mathcal{M} is a supermanifold of Batchelor type.

Lemma 4.4.23. *Let \mathcal{M} be a supermanifold of Batchelor type modelled on $E \in \mathbf{SVec}_{lc}$. Then $\text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rtimes \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$ is a pro-polynomial group with the Lie algebra*

$$\overline{\mathcal{X}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}} \oplus \overline{\iota_\infty^1(\mathcal{X}(\mathcal{M}^{(1)})_{\bar{0}})_{\Lambda+}} \subseteq \overline{\mathcal{X}(\mathcal{M})_{\bar{0},\Lambda}}$$

for each $\Lambda \in \mathbf{Gr}$. If $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional, then the same holds for $\text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rtimes \text{SDiff}_{\text{id}}^c(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$ with the Lie algebra

$$\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}} \oplus \overline{\iota_\infty^1(\mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}})_{\Lambda+}}.$$

Proof. We already know from Lemma 4.4.18 that $\text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rtimes \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$ is a polynomial group with the correct Lie algebra. Likewise, we have seen in Lemma 4.4.20 that $\text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$ is a pro-polynomial group that also has the correct Lie algebra. It only remains to be seen that the action of $\text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$ on $\text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$ restricted to a map

$$\beta_\Lambda : \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \times \text{Aut}_{\text{id}}(\mathcal{M}) \rightarrow \text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$$

(where we consider $\text{Aut}_{\text{id}}(\mathcal{M}) \subseteq \text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$) is pro-polynomial in an appropriate way and leads to the correct Lie algebra. We have

$$g \circ_\Lambda f \circ_\Lambda g^{-1} \in f \circ \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$$

for $g \in \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$ and $f \in \text{Aut}_{\text{id}}(\mathcal{M})$, because the part not dependent on $\mathcal{P}(\Lambda)$ remains unchanged. It follows $\beta_\Lambda(g, f) = (f^{-1} \circ_\Lambda g \circ_\Lambda f \circ_\Lambda g^{-1}, f)$ if we consider $\text{SDiff}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} = \text{SDiff}_{\text{id}}(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rtimes \text{Aut}_{\text{id}}(\mathcal{M})$. Since \mathcal{M} is of Batchelor type, we may

consider $\text{Aut}_{\text{id}}(\mathcal{M}^{(n)}) \subseteq \text{Aut}_{\text{id}}(\mathcal{M})$ as a subset. Then

$$\begin{aligned} \beta^{(n)} : \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda} \times \text{Aut}_{\text{id}}(\mathcal{M}^{(n)}) &\rightarrow \overline{\mathcal{X}(\mathcal{M}^{(n)})}_{\bar{0}}^{\geq 2}_{\Lambda^+}, \\ (g, f) &\mapsto \overline{\pi_n^\infty}((\exp_{\mathcal{M}, \Lambda}^{\geq 2})^{-1}(f^{-1} \circ_{\Lambda} g \circ_{\Lambda} f \circ_{\Lambda} g^{-1})) \end{aligned}$$

is well-defined, because f_{Λ_k} for $n < k$ does not contribute to the right-hand side. Locally, it follows from Corollary 4.3.10, Lemma 4.4.1 and Lemma 4.2.2 that $\beta^{(n)}$ is polynomial and that we have $\exp_{\mathcal{M}, \Lambda}^{\geq 2} \circ \varprojlim_n \beta^{(n)} = \beta$.

For the iterated actions of $\text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda}$ on $\text{SDiff}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2}$, it follows again from Corollary 4.3.10 and Lemma 4.4.1 that the total multilinear degree in the elements of $\text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda}$ is bounded by the number of generators of Λ . Therefore, Lemma C.2.4 is satisfied and $\text{SDiff}(\mathcal{M})_{\bar{0}, \Lambda}^{\geq 2} \rtimes \text{SDiff}_{\text{id}}(\mathcal{M}^{(1)})_{\bar{0}, \Lambda}$ is a pro-polynomial group.

By the definition of the respective pro-polynomial group structures, it suffices to calculate the Lie bracket locally. Let $\mathcal{U} \subseteq \bar{E}$ be an open subfunctor and let $\text{pr}_{\mathcal{U}} : \mathcal{P}(\Lambda_n) \times \mathcal{U} \rightarrow \mathcal{U}$ be the projection. Further, let $X \in \mathcal{SC}^\infty(\mathcal{U}, \bar{E})_{\bar{0}}^{\geq 2}$ and $Y \in \widehat{\mathcal{SC}}^\infty(\mathcal{U}, \bar{E})_{\Lambda_n^+}$ with $Y = \sum_{I \in \mathcal{P}_{0,+}^n} \mathbf{v}_I \wedge_I Y$, where $_I Y \in \mathcal{SC}^\infty(\mathcal{U}^{(1)}, \bar{E}^{(1)}) \subseteq \mathcal{SC}^\infty(\mathcal{U}, \bar{E})$. Let m_Λ denote the multiplication in $\text{SDiff}(\mathcal{U})_{\bar{0}, \Lambda}^{\geq 2} \rtimes \text{SDiff}_{\text{id}}(\mathcal{U}^{(1)})_{\bar{0}, \Lambda_n}$. For $(m_\Lambda)_{1,1}$ as in C.2, we directly calculate

$$\begin{aligned} &((m_\Lambda)_{1,1}(\text{pr}_{\mathcal{U}} + X, \text{pr}_{\mathcal{U}} + Y))_k \\ &= \mathfrak{A}^k \left(\sum_{k \geq l \text{ even}} \frac{k!}{(k-l)!!l!} dX_{k-l}(\text{id}_{\mathcal{U}_{\mathbb{R}}})(Y_l, \text{pr}_{E_1}, \dots, \text{pr}_{E_1}) \right. \\ &\quad \left. + \sum_{k \geq l \text{ odd}} \frac{k!}{(k-l)!!l!} X_{k-l+1}(\text{id}_{\mathcal{U}_{\mathbb{R}}})(Y_l, \text{pr}_{E_1}, \dots, \text{pr}_{E_1}) \right) \\ &= (dX \circ (\text{id}_{\mathcal{U}}, Y))_k \end{aligned}$$

and with Corollary 4.3.10 we obtain

$$\begin{aligned} &((m_\Lambda)_{1,1}(\text{pr}_{\mathcal{U}} + Y, \text{pr}_{\mathcal{U}} + X))_k \\ &= \begin{cases} \sum_{\substack{k \geq l \text{ even,} \\ I \in \mathcal{P}_{0,+}^n, |I|=k-l}} \mathbf{v}_I \wedge (d_I Y_0(\text{id}_{\mathcal{U}_{\mathbb{R}}})(X_l)) & k \text{ even} \\ \sum_{\substack{k \geq l \text{ odd,} \\ I \in \mathcal{P}_{0,+}^n, |I|=k-l}} \mathbf{v}_I \wedge (_I Y_1(\text{id}_{\mathcal{U}_{\mathbb{R}}})(X_l)) & k \text{ odd} \end{cases} \\ &= \left(\sum_{I \in \mathcal{P}_{0,+}^n} \mathbf{v}_I \wedge d_I Y(\text{id}_{\mathcal{U}}, X) \right)_k, \end{aligned}$$

for $k > 0$ and the projection $\text{pr}_{E_1} : \mathbb{R}^n \times E_1 \rightarrow E_1$. It follows from Lemma A.2.18 and the definition of the Lie bracket $[\cdot, \cdot]$ of $\mathcal{X}(\mathcal{M})_{\bar{0}}$ that

$$(m_\Lambda)_{1,1}(\text{pr}_{\mathcal{U}} + Y, \text{pr}_{\mathcal{U}} + X) - (m_\Lambda)_{1,1}(\text{pr}_{\mathcal{U}} + X, \text{pr}_{\mathcal{U}} + Y) = \sum_{I \in \mathcal{P}_{0,+}^n} \mathbf{v}_I \wedge [_I Y, X],$$

as well as

$$(m_\Lambda)_{1,1}(\text{pr}_U + X, \text{pr}_U + Y) - (m_\Lambda)_{1,1}(\text{pr}_U + Y, \text{pr}_U + X) = \sum_{I \in \mathcal{P}_{0,+}^n} \mathbf{v}_I \wedge [X, {}_I Y],$$

hold. The same arguments apply in the case of compact support. \square

Proposition 4.4.24. *Let \mathcal{M} be a σ -compact Banach supermanifold of Batchelor type with finite-dimensional base $\mathcal{M}_\mathbb{R}$. Then $\text{SDiff}_c(\mathcal{M})_{\bar{0}}$ is a purely even Lie supergroup such that*

$$\text{SDiff}_c(\mathcal{M})_{\bar{0}} = \text{SDiff}_c(\mathcal{M})_{\bar{0}}^{\geq 2} \rtimes \text{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0}} \cong \iota(\text{Aut}_c(\mathcal{M})).$$

Proof. We already know from Lemma 4.4.19 that there exists a split short exact sequence of Lie groups

$$1 \rightarrow \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \hookrightarrow \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda} \rightarrow \text{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \rightarrow 1$$

that is natural in $\Lambda \in \mathbf{Gr}$. By Lemma/Definition 3.0.3, we only need to show that the action defined by

$$\text{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \times \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2} \rightarrow \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}, \quad (f, g) \mapsto f \circ_\Lambda g \circ_\Lambda f^{-1}$$

is supersmooth. By Corollary 2.2.9, it suffices to check the derivative at points from $\text{Aut}_c(\mathcal{M}^{(1)}) \times \text{Aut}_{\text{id}}^c(\mathcal{M})$ and by the definition of $\text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}^{\geq 2}$ in Lemma 4.4.20, we can consider the action

$$\beta_\Lambda: \text{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0},\Lambda} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_\Lambda \rightarrow \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}}^{\geq 2}}_\Lambda,$$

$$(f, X) \mapsto (\exp_{\mathcal{M},\Lambda}^{c,\geq 2})^{-1}(f \circ_\Lambda \exp_{\mathcal{M},\Lambda}^{c,\geq 2}(X) \circ_\Lambda f^{-1})$$

instead. Let $n \in \mathbb{N}$ and $\Lambda := \Lambda_n$. It follows from Lemma 4.4.12 that for $f \in \text{Aut}_c(\mathcal{M}^{(1)}) \subseteq \text{SDiff}_c(\mathcal{M}^{(1)})_{\bar{0},\Lambda}$, $X \in \mathcal{X}_c(\mathcal{M})_{\bar{0}}$ and $I \in \mathcal{P}_{0,+}^n$, we have

$$\beta_\Lambda(f, \lambda_I X) = \lambda_I T f \circ X \circ f^{-1}.$$

In particular $\beta_\Lambda(f, \cdot)$ is $\Lambda_{\bar{0}}$ -linear. To see supersmoothness, we therefore only need to check the derivative of $\alpha_\Lambda := \beta_\Lambda(\cdot, X)$. Let $\phi: U \rightarrow \text{Aut}_c(\mathcal{M}^{(1)})$ be the inverse of a chart with $U \subseteq \mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}}$, $0 \in U$ and $\phi(0) = \text{id}$, such that $T_0 \phi = \text{id}_{\mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}}}$ holds under the identification $T_{\text{id}} \text{Aut}_c(\mathcal{M}^{(1)}) \cong \mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}}$ from Lemma D.3.9. If $l_f: \text{Aut}_c(\mathcal{M}^{(1)}) \rightarrow \text{Aut}_c(\mathcal{M}^{(1)})$ denotes the left-multiplication by f , then $l_f \circ \phi$ is the inverse of a chart around f and $\iota(l_f \circ \phi) = \iota(l_f) \circ \iota(\phi)$ is a chart around the point in $\iota(\text{Aut}_c(\mathcal{M}^{(1)}))$ corresponding to f . We calculate

$$\beta_\Lambda(\iota(l_f \circ \phi)_\Lambda(Y), X) = \beta_\Lambda(\iota(l_f)_\Lambda, \beta_\Lambda(\iota(\phi)_\Lambda(Y), X))$$

for $Y \in \overline{(\mathcal{X}_c(\mathcal{M}^{(1)})_{\bar{0}}|_U)_\Lambda}$. Therefore, it suffices to calculate the derivative of α_Λ at the identity. Note that $T^n \phi$ is the identity on the axes, i.e., we have $T^n \phi(\varepsilon_I Y) =$

$\varepsilon_I Y$ in the notation of Section E.1. With the identifications from Proposition 4.4.21, it follows that

$$\iota(\phi)_\Lambda(\lambda_I Y) = T^n \phi|_{\mathcal{P}_{0,+}^n}(\lambda_I Y) = \exp_{\mathcal{M},\Lambda}^c(\lambda_I Y).$$

Using Lemma 4.4.23 and Lemma C.3.3, we obtain

$$\begin{aligned} \beta_\Lambda(\iota(\phi)(\lambda_I Y), X) &= (\exp_{\mathcal{M},\Lambda}^{c,\geq 2})^{-1}(\exp_{\mathcal{M},\Lambda}^c(\lambda_I Y) \cdot \exp_{\mathcal{M},\Lambda}^{c,\geq 2}(X) \cdot \exp_{\mathcal{M},\Lambda}^c(\lambda_I Y)^{-1}) \\ &= \lambda_I[Y, X] + X, \end{aligned} \quad (4.12)$$

where $[\cdot, \cdot]$ is the Lie bracket of $\mathcal{X}(\mathcal{M})_{\bar{0}}$. On the other hand, we have $d\alpha_\Lambda(\text{id}_{\mathcal{M}^{(1)}}, Y) = [Y, X]$ by Proposition 4.2.13. Differentiating (4.12) in the first component then shows that $(\beta_\Lambda)_{\Lambda \in \mathbf{Gr}}$ is supersmooth. Therefore, $\text{SDiff}_c(\mathcal{M})_{\bar{0}}$ is a purely even Lie supergroup and since $\text{SDiff}_c(\mathcal{M})_{\bar{0},\mathbb{R}} = \text{Aut}_c(\mathcal{M})$ holds, it follows that $\text{SDiff}_c(\mathcal{M})_{\bar{0}} \cong \iota(\text{Aut}_c(\mathcal{M}))$. \square

4.4.5. The Lie supergroup structure of the superdiffeomorphisms

The trivialization from Lemma 4.4.17, together with the Lie supergroup structure discussed in Proposition 4.4.24, now finally allows us to turn $\text{SDiff}_c(\mathcal{M})$ into a Lie supergroup for appropriate supermanifolds \mathcal{M} . The construction of this Lie supergroup is basically identical to the construction of a Lie supergroup from a super Harish-Chandra pair in Proposition 3.3.7.

Theorem 4.4.25. *Let \mathcal{M} be a σ -compact Banach supermanifold of Batchelor type such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional. Then the natural transformation from Lemma 4.4.17, defined by the bijections*

$$\text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}} \rightarrow \text{SDiff}_c(\mathcal{M})_\Lambda, \quad (f, X) \mapsto f \circ_\Lambda \exp_{\mathcal{M},\Lambda}^c(X)$$

for each $\Lambda \in \mathbf{Gr}$, turns $\text{SDiff}_c(\mathcal{M})$ into a Lie supergroup.

Proof. Let $[\cdot, \cdot]_\Lambda$ denote the Lie bracket of $\overline{\mathcal{X}_c(\mathcal{M})_\Lambda}$. Note that, by Proposition 4.4.24, $\text{SDiff}_c(\mathcal{M})_{\bar{0}} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b}}$ is indeed a supermanifold so that we just have to show that the group operations in $\text{SDiff}_c(\mathcal{M})$ are supersmooth. For this, we first show supersmoothness of the conjugation given by

$$\begin{aligned} \sigma_\Lambda: \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}} &\rightarrow \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}} \\ (f, X) &\mapsto (\exp_{\mathcal{M},\Lambda}^c)^{-1}(f \circ_\Lambda \exp_{\mathcal{M},\Lambda}^c(X) \circ_\Lambda f^{-1}). \end{aligned}$$

Note that σ_Λ is well-defined because by Lemma C.2.7, we have

$$\begin{aligned} &(\exp_{\mathcal{M},\Lambda}^c)^{-1}(\exp_{\mathcal{M},\Lambda}^c(Y) \circ_\Lambda \exp_{\mathcal{M},\Lambda}^c(X) \circ_\Lambda \exp_{\mathcal{M},\Lambda}^c(-Y)) \\ &= X + \sum_{k=1}^n \frac{1}{k!} \underbrace{[Y, \dots, [Y, X]_\Lambda \dots]_\Lambda}_{k \text{ times}} \end{aligned}$$

and since Lemma 4.4.12 shows that

$$(\exp_{\mathcal{M},\Lambda}^c)^{-1}(f \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(X) \circ_{\Lambda} f^{-1}) = \sum_{I \in \mathcal{P}_1^n} \lambda_I \bar{\Pi}(\mathcal{T}f) \circ_I X \circ f^{-1}$$

for $\Lambda = \Lambda_n$, $Y \in \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0}\Lambda^+}}$, $X = \sum_{I \in \mathcal{P}_1^n} \lambda_I X$ and $f \in \text{Aut}_c(\mathcal{M})$. By Corollary 2.2.9, we only need to calculate the derivative at points of the form $(f, 0) \in \text{Aut}_c(\mathcal{M}) \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}}$. But, $d_1\sigma_{\Lambda}(f, 0)(\bullet) = 0$ and $d_2\sigma_{\Lambda}(f, 0)(\bullet) = \sigma_{\Lambda}(f, \bullet)$ are both $\Lambda_{\bar{0}}$ -linear. It follows that σ is supersmooth. Next, we need to see that the purely odd multiplication, given by

$$\mu_{\Lambda}: \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}} \rightarrow \text{SDiff}_c(\mathcal{M})_{\Lambda}, \quad (X, Y) \mapsto \exp_{\mathcal{M},\Lambda}^c(X) \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(Y),$$

is supersmooth. Let Ψ_{Λ} be the map from Lemma 3.3.5. Then it follows from Lemma 3.3.4 and Lemma 3.3.6 that the morphism defined by

$$\begin{aligned} \Phi_{\Lambda}: \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}} &\rightarrow \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0},b\Lambda}} \times \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}}, \\ (X, Y) &\mapsto \Psi_{\Lambda}^{-1}((\exp_{\mathcal{M},\Lambda}^c)^{-1}(\mu_{\Lambda}(X, Y))) \end{aligned}$$

is supersmooth. Let $\text{pr}_0: \overline{\mathcal{X}_c(\mathcal{M})_b} \rightarrow \overline{\mathcal{X}_c(\mathcal{M})_{\bar{0},b}}$ and $\text{pr}_1: \overline{\mathcal{X}_c(\mathcal{M})_b} \rightarrow \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b}}$ denote the corresponding supersmooth projections. By definition, we have

$$\exp_{\mathcal{M},\Lambda}^c(\text{pr}_{0,\Lambda} \circ \Phi_{\Lambda}(X, Y)) \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(\text{pr}_{1,\Lambda} \circ \Phi_{\Lambda}(X, Y)) = \mu_{\Lambda}(X, Y)$$

for all $X, Y \in \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}}$. Because we have $\exp_{\mathcal{M}}^c|_{\overline{\mathcal{X}_c(\mathcal{M})_{\bar{0},b}}} = \exp^{\text{SDiff}_c(\mathcal{M})_{\bar{0}}}$, Proposition 3.2.6 and Lemma 3.3.6 now show that $(\mu_{\Lambda})_{\Lambda \in \mathbf{Gr}}$ is supersmooth. Next, we show the supersmoothness of the composition in $\text{SDiff}_c(\mathcal{M})$. Let $f, g \in \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}$ and $X, Y \in \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}}$. We abbreviate $\tilde{X} := (\exp_{\mathcal{M},\Lambda}^c)^{-1}(\sigma_{\Lambda}(g^{-1}, \exp_{\mathcal{M},\Lambda}^c(X)))$. Then, we have

$$\begin{aligned} f \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(X) \circ_{\Lambda} g \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(Y) &= f \circ_{\Lambda} g \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(\tilde{X}) \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(Y) \\ &= \underbrace{f \circ_{\Lambda} g \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(\text{pr}_{0,\Lambda} \circ \Phi_{\Lambda}(\tilde{X}, Y))}_{\in \text{SDiff}_c(\mathcal{M})_{\bar{0},\Lambda}} \circ_{\Lambda} \underbrace{\exp_{\mathcal{M},\Lambda}^c(\text{pr}_{1,\Lambda} \circ \Phi_{\Lambda}(\tilde{X}, Y))}_{\in \overline{\mathcal{X}_c(\mathcal{M})_{\bar{1},b\Lambda}}}. \end{aligned}$$

With the statements derived above, it follows that the composition is supersmooth, because inversion and composition in $\text{SDiff}_c(\mathcal{M})_{\bar{0}}$ are supersmooth. Likewise, the supersmoothness of the inversion follows from

$$(f \circ_{\Lambda} \exp_{\mathcal{M},\Lambda}^c(X))^{-1} = \exp_{\mathcal{M},\Lambda}^c(-X) \circ_{\Lambda} f^{-1} = f^{-1} \circ_{\Lambda} \sigma_{\Lambda}(f, \exp_{\mathcal{M},\Lambda}^c(-X)). \quad \square$$

From the construction of the Lie supergroup structure, it is obvious that the super Harish-Chandra pair associated to $\text{SDiff}_c(\mathcal{M})$ is given by $(\text{Aut}_c(\mathcal{M}), \mathcal{X}_c(\mathcal{M}))$ together with the action from Lemma 4.2.15.

Theorem 4.4.26. *Let \mathcal{M} be a σ -compact Banach supermanifold such that $\mathcal{M}_{\mathbb{R}}$ is finite-dimensional and let $g: \mathcal{M} \rightarrow \iota_{\infty}^1(\mathcal{M}^{(1)})$ be a Batchelor model. Then the*

isomorphisms of groups

$$\Theta_{g,\Lambda}: \text{SDiff}_c(\mathcal{M})_\Lambda \rightarrow \text{SDiff}_c(\iota_\infty^1(\mathcal{M}^{(1)}))_\Lambda, \quad f \mapsto g \circ f \circ (\text{id}_{\mathcal{P}(\Lambda)} \times g^{-1})$$

define a natural transformation Θ_g , i.e., an isomorphism of supergroups. The Lie supergroup structure induced by this isomorphism on $\text{SDiff}_c(\mathcal{M})$ does not depend on the Batchelor model.

Proof. Evidently, $\Theta_{g,\Lambda}$ is an isomorphism of groups for each $\Lambda \in \mathbf{Gr}$. For $\varrho \in \text{Hom}_{\mathbf{Gr}}(\Lambda, \Lambda')$ and $f \in \text{SDiff}_c(\mathcal{M})_\Lambda$, we calculate

$$\begin{aligned} \Theta_{g,\Lambda'}(\text{SDiff}_c(\mathcal{M})_{\varrho}(f)) &= g \circ f \circ (\mathcal{P}(\varrho) \times \text{id}_{\mathcal{M}}) \circ (\text{id}_{\mathcal{P}(\Lambda)} \times g^{-1}) = \\ &= g \circ f \circ (\text{id}_{\mathcal{P}(\Lambda)} \times g^{-1}) \circ (\mathcal{P}(\varrho) \times \text{id}_{\iota_\infty^1(\mathcal{M}^{(1)})}) = \text{SDiff}_c(\iota_\infty^1(\mathcal{M}^{(1)}))_{\varrho}(\Theta_{g,\Lambda}(f)), \end{aligned}$$

which shows naturality. Let g' also be a Batchelor model of \mathcal{M} and let $\text{SDiff}_c(\mathcal{M})'$ denote the Lie supergroup induced by g' . Then the identity $\text{SDiff}_c(\mathcal{M}) \rightarrow \text{SDiff}_c(\mathcal{M})'$ is supersmooth if and only if $\Theta_{g'} \circ \Theta_g^{-1}: \text{SDiff}_c(\iota_\infty^1(\mathcal{M}^{(1)})) \rightarrow \text{SDiff}_c(\iota_\infty^1(\mathcal{M}^{(1)}))$ is supersmooth. But, for each $\Lambda \in \mathbf{Gr}$, the map $\Theta_{g',\Lambda} \circ (\Theta_{g,\Lambda})^{-1}$ is just the conjugation by $g' \circ g^{-1}$ in $\text{SDiff}_c(\iota_\infty^1(\mathcal{M}^{(1)}))_\Lambda$, which defines a supersmooth morphism by Proposition 3.2.9. \square

Remark 4.4.27. If one wants to define supermorphisms or superdiffeomorphisms for k -supermanifolds \mathcal{M}, \mathcal{N} with $k \in \mathbb{N}_0$, one runs into the following problem. One cannot simply define $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N})_\Lambda$ as $\mathcal{SC}^\infty(\mathcal{P}(\Lambda)^{(k)} \times \mathcal{M}, \mathcal{N})$ because one would lose all $\lambda_I \in \Lambda$ with $|I| > k$. Locally, one can simply set

$$\widehat{\mathcal{SC}}^\infty(\mathcal{U}, \overline{F}^{(k)})_{\Lambda_n} := \bigoplus_{I \in \mathcal{P}_0^n} \mathfrak{v}_I \wedge \mathcal{SC}^\infty(\mathcal{U}, \overline{F}^{(k)}) \oplus \bigoplus_{I \in \mathcal{P}_1^n} \mathfrak{v}_I \wedge \mathcal{SC}^\infty(\mathcal{U}, \overline{\Pi(F)})^{(k)}$$

for $E, F \in \mathbf{SVec}_{lc}$ and an open subfunctor $\mathcal{U} \subseteq \overline{E}^{(k)}$. The composition of such supermorphisms is then just the usual composition, where one cuts off all maps whose multilinear degree in E_1 is greater than k . Next, one needs to check that this leads to a well-defined functor $\widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N})$. For superdiffeomorphisms, it appears to be easier to work directly with the pair $(\text{Aut}(\mathcal{M}), \mathcal{X}(\mathcal{M}))$ instead.

For any two supermanifolds \mathcal{M} and \mathcal{N} , we define an evaluation morphism

$$\text{ev}: \widehat{\mathcal{SC}}^\infty(\mathcal{M}, \mathcal{N}) \times \mathcal{M} \rightarrow \mathcal{N}$$

as follows. For morphisms $x: \mathcal{P}(\Lambda) \rightarrow \mathcal{M}$ and $f: \mathcal{P}(\Lambda) \times \mathcal{M} \rightarrow \mathcal{N}$, we let

$$\text{ev}_\Lambda(f, x) := f \circ (\text{id}_{\mathcal{P}(\Lambda)}, x): \mathcal{P}(\Lambda) \rightarrow \mathcal{N},$$

where we identify $\mathcal{M}_\Lambda \cong \mathcal{SC}^\infty(\mathcal{P}(\Lambda), \mathcal{M})$ and $\mathcal{N}_\Lambda \cong \mathcal{SC}^\infty(\mathcal{P}(\Lambda), \mathcal{N})$ via Corollary 4.3.3 (see [40, Section 8.3, p.416]). Molotov [40, Proposition 8.4.2, p.417] states that for any supersmooth action $\alpha: \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ of a Lie supergroup \mathcal{G} on \mathcal{M} ,

there exists a unique morphism $\widehat{\alpha}: \mathcal{G} \rightarrow \text{SDiff}(\mathcal{M})$ of supergroups such that

$$\text{ev} \circ (\widehat{\alpha} \times \text{id}_{\mathcal{M}}) = \alpha$$

holds. This approach for studying the properties of supersmooth actions of Lie supergroups on supermanifolds seems to be a promising area for future research.

A. Important Facts

A.1. Products, Inverse Limits and Direct Sums

Lemma A.1.1 ([21, Lemma 1.3, p.24]). *Let E, F be locally convex spaces, $U \subseteq E$ open and $f: U \rightarrow F$. If $f(U) \subseteq F'$ for a closed vector subspace $F' \subseteq F$, then f is smooth if and only if its co-restriction $f|^{F'}: U \rightarrow F'$ is smooth.*

Let J be a set and $(F_j)_{j \in J}$ be a family of locally convex spaces. Then the product $\prod_{j \in J} F_j$ equipped with the product topology is a Hausdorff locally convex space.

Lemma A.1.2 ([23]). *Let E be a locally convex space, $U \subseteq E$ open and $(F_j)_{j \in J}$ be a family of locally convex spaces. Let $F := \prod_{j \in J} F_j$ and let $\text{pr}_j: F \rightarrow F_j$ be the projection onto the j -th component. A map $f: U \rightarrow F$ is smooth if and only if $f_j := \text{pr}_j \circ f: U \rightarrow F_j$ is smooth for every $j \in J$. In this case, we have*

$$df(x, y) = (df_j(x, y))_{j \in J} \quad \text{for all } x \in U \text{ and } y \in E.$$

Let J be a directed index set. The inverse limit

$$\varprojlim_{j \in J} F_j := \left\{ (x_j)_{j \in J} \in \prod_{j \in J} F_j : q_i^j(x_j) = x_i \text{ for all } i \leq j \right\}$$

of an inverse system $((F_j)_{j \in J}, (q_i^j)_{i \leq j})$ of locally convex spaces, where $q_i^j: F_j \rightarrow F_i$ is continuous linear for all $i \leq j$, is a closed subset of $\prod_{j \in J} F_j$ and thus a Hausdorff locally convex space. A direct consequence of this is the following lemma.

Lemma A.1.3 ([23]). *Let E, F be locally convex spaces, $U \subseteq E$ be open and $f: U \rightarrow F$ be a map. Assume that $F = \varprojlim F_j$ for an inverse system $((F_i)_{i \in J}, (q_i^j)_{i \leq j})$ of locally convex spaces and continuous linear maps $q_i^j: F_j \rightarrow F_i$, with limit maps $q_i: F \rightarrow F_i$. Then f is smooth if and only if $q_i \circ f: U \rightarrow F_i$ is smooth for each $i \in J$. In this case, we have*

$$df(x, y) = (d(q_i \circ f)((x, y)))_{i \in J} \quad \text{for all } x \in U \text{ and } y \in E.$$

If I is a countable index set and $(E_i)_{i \in I}$ is a family of locally convex spaces, then the direct sum

$$E := \bigoplus_{i \in I} E_i := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} E_i : x_i = 0 \text{ for almost all } i \right\}$$

can be given a unique Hausdorff locally convex vector topology such that sets of the form $\bigoplus_{i \in I} U_i := \bigoplus_{i \in I} E_i \cap \prod_{i \in I} U_i$ for open zero-neighborhoods $U_i \subseteq E_i$ constitute a basis of zero-neighborhoods (see [29]).

Lemma A.1.4 ([50, Lemma 4.15, p.32 f.]). *Let I, J be countable index sets and $(E_i)_{i \in I}, (F_j)_{j \in J}$ be families of locally convex spaces. For every $i \in I$, let $U_i \subseteq E_i$ be an open zero neighborhood. Moreover, for every $j \in J$, let $I_j \subseteq I$ be a finite subset such that $\bigcup_{j \in J} I_j = I$ and such that every $i \in I$ is only contained in finitely many I_j . If for every $j \in J$, we have a smooth map $f_j: \bigoplus_{i \in I_j} U_i \rightarrow F_j$ with $f_j(0) = 0$, then the map*

$$f: \bigoplus_{i \in I} U_i \rightarrow \bigoplus_{j \in J} F_j, \quad (x_i)_{i \in I} \mapsto (f_j((x_i)_{i \in I_j}))_{j \in J}$$

is also smooth.

In particular with $I = J$ and $I_j := \{j\}$, we see that a map between direct sums defined by component-wise smooth maps is smooth (this is [17, Proposition 7.1, p.993]).

A.2. Mapping Spaces

Let E and F be locally convex spaces and $U \subseteq E$ open. We give $\mathcal{C}^\infty(U, F)$ the topology that turns

$$\mathcal{C}^\infty(U, F) \rightarrow \prod_{i \in \mathbb{N}_0} \mathcal{C}(U \times E^i, F), \quad \gamma \mapsto (d^i \gamma)_{i \in \mathbb{N}_0}$$

into an embedding. Here $\mathcal{C}(U \times E^i, F)$ denotes the space of continuous functions $U \times E^i \rightarrow F$ equipped with the *compact-open topology*, i.e., the vector space topology generated by the sets

$$[K, V] := \{\gamma \in \mathcal{C}(U \times E^i, F) : \gamma(K) \subseteq V\},$$

where K ranges through the compact subsets of $U \times E^i$ and V through the open subsets of F . This turns $\mathcal{C}^\infty(U, F)$ into a Hausdorff locally convex vector space. Note that for all $k \in \mathbb{N}_0$ the linear map

$$d^k: \mathcal{C}^\infty(U, F) \rightarrow \mathcal{C}^\infty(U \times E^k, F), \quad f \mapsto d^k f$$

is continuous. We refer to [23] and [18] for more details.

Proposition A.2.1 (c.f. [22, Proposition 2.5, p.7]). *Let E, F and H be locally convex spaces, $U \subseteq E$ open and $f: U \times F \rightarrow H$ smooth. Then the map*

$$f_*: \mathcal{C}^\infty(U, F) \rightarrow \mathcal{C}^\infty(U, H), \quad \gamma \mapsto f \circ (\text{id}_U, \gamma)$$

is smooth.

Lemma A.2.2 (c.f. [18, Lemma 4.4, p.23]). *Let E, E' and F be locally convex spaces and $U \subseteq E'$ and $V \subseteq F$ be open. If $f: U \rightarrow V$ is a smooth map, then so is*

the “pullback”

$$\mathcal{C}^\infty(f, E): \mathcal{C}^\infty(V, E) \rightarrow \mathcal{C}^\infty(U, E), \quad \gamma \mapsto \gamma \circ f.$$

Lemma A.2.3. *Let E, F be locally convex spaces and $U \subseteq E$ open. Then for every $p \in U$, the point evaluation*

$$\text{ev}_p: \mathcal{C}^\infty(U, F) \rightarrow F, \quad \gamma \mapsto \gamma(p),$$

is a continuous linear map.

Proof. Obviously ev_p is linear, and for any open subset $V \subseteq F$, we have

$$\text{ev}_p^{-1}(V) = \{\gamma \in \mathcal{C}^\infty(U, F) : \gamma(p) \in V\} = [\{p\}, V].$$

Since the inclusion $\mathcal{C}^\infty(U, F) \rightarrow \mathcal{C}(U, F)$ is continuous, the statement follows. \square

Lemma A.2.4 ([18, Lemma 4.6, p.24]). *Let E, F be locally convex spaces and $U \subseteq E$ open. For every open subset $V \subseteq U$, the restriction map*

$$\rho_V: \mathcal{C}^\infty(U, F) \rightarrow \mathcal{C}^\infty(V, F), \quad \gamma \mapsto \gamma|_V,$$

is continuous. Moreover, for every open cover \mathfrak{U} of U , the topology on $\mathcal{C}^\infty(U, F)$ is initial with respect to the family $(\rho_V)_{V \in \mathfrak{U}}$.

Lemma A.2.5 ([18, Lemma 4.5, p.23]). *Let E, F be locally convex spaces and $U \subseteq E$ open. If $f: U \rightarrow \mathbb{R}$ is a smooth map, then so is the pointwise product*

$$m_f: \mathcal{C}^\infty(U, F) \rightarrow \mathcal{C}^\infty(U, F), \quad \gamma \mapsto f \cdot \gamma.$$

Lemma A.2.6. *Let E, F be locally convex and $U, V \subseteq E$ open. If $h: U \rightarrow \mathbb{R}$ is a smooth map with support $\text{supp}(h) = K \subseteq U$, then the map*

$$m_h^\sim: \mathcal{C}^\infty(U, F) \rightarrow \mathcal{C}^\infty(V, F), \quad \gamma \mapsto (h \cdot \gamma)^\sim,$$

where

$$(h \cdot \gamma)^\sim(x) := \begin{cases} h(x) \cdot \gamma(x) & \text{if } x \in U \cap V, \\ 0 & \text{else,} \end{cases}$$

is well-defined, linear and continuous.

Proof. We may assume $U \cap V \neq \emptyset$ because the other case is trivial. We have $(h \cdot \gamma)^\sim|_{V \setminus K} = 0$ and $(h \cdot \gamma)^\sim|_{V \cap U} = h \cdot \gamma$. Because $(V \cap U) \cup (V \setminus K) = V$, it follows that $(h \cdot \gamma)^\sim$ is smooth. The claim now results from Lemma A.2.4 and Lemma A.2.5. \square

Proposition A.2.7 (cf. [18, Proposition 12.2, p.67] and [23]). *Let E, F be locally convex vector spaces, $V \subseteq F$ open, H be a finite-dimensional vector space and $U \subseteq H$ open. Then a mapping $g: V \rightarrow \mathcal{C}^\infty(U, E)$ is smooth if and only if*

$$g^\wedge: V \times U \rightarrow E, \quad g^\wedge(x, y) := g(x)(y)$$

is smooth. In this case, we have

$$(d^m g(x)(v))(u) = d^m g^\wedge(x, u)((v_1, 0), \dots, (v_m, 0))$$

for $m \in \mathbb{N}$, $x \in V$, $v = (v_1, \dots, v_m) \in F^m$ and $u \in U$.

Proposition A.2.8 (cf. [18, Proposition 11.1, p.59]). *Let M be a finite-dimensional manifold and E a locally convex space. Then the evaluation map*

$$\text{ev}: \mathcal{C}^\infty(M, E) \times M \rightarrow E, \quad \text{ev}(f, x) := f(x)$$

is smooth.

A.2.1. Spaces of partially multilinear smooth maps

Let $n \in \mathbb{N}$, let E_0, \dots, E_n and F be locally convex spaces and let $U \subseteq E_0$ be open. Recall Definition 2.2.11. It follows easily from Lemma A.2.3 that the image of the map

$$\mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F)) \rightarrow \mathcal{C}^\infty(U \times E_1 \times \dots \times E_n, F), \quad f \mapsto f^\wedge$$

is closed. We denote the space $\mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F))$ equipped with the induced topology by $\mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F))_c$ and define $\mathcal{C}^\infty(U, \text{Alt}^n(E_1; F))_c$ analogously.

We denote by $\mathcal{L}^n(E_1, \dots, E_n; F)_b$ the space $\mathcal{L}^n(E_1, \dots, E_n; F)$ equipped with the topology of uniform convergence on bounded sets (see for example [23]). We define $\text{Alt}^n(E_1; F)_b$ analogously. It is well-known that $\mathcal{L}^n(E_1, \dots, E_n; F)_b$ is a Banach space if all involved spaces are Banach spaces. In this case the evaluation map

$$\mathcal{L}^n(E_1, \dots, E_n; F)_b \times E_1 \times \dots \times E_n \rightarrow F, \quad (L, v) \mapsto L(v),$$

is continuous and so are compositions of the form

$$\begin{aligned} \mathcal{L}^n(E_1, \dots, E_n; F)_b \times \mathcal{L}^m(E'_1, \dots, E'_m; E_i)_b &\rightarrow \\ \mathcal{L}^{n+m}(E_1, \dots, E_{i-1}, E'_1, \dots, E'_m, E_{i+1}, \dots, E_n; F)_b, &\quad (L, L') \mapsto L(\cdot, L'(\cdot), \cdot), \end{aligned}$$

where $1 \leq i \leq n$, $m \in \mathbb{N}$, and E'_1, \dots, E'_m are also Banach spaces (see for example [33, Proposition 2.6, p.7]). One easily sees that in this case the projection

$$\mathfrak{A}^n: \mathcal{L}^n(E_1; F)_b \rightarrow \text{Alt}^n(E_1; F)_b$$

is continuous and that $\text{Alt}^n(E_1; F)_b$ is closed in $\mathcal{L}^n(E_1; F)_b$.

Lemma A.2.9. *Let $n \in \mathbb{N}$, E, F and H be locally convex spaces and let $U \subseteq H$ be open. Then the linear map*

$$\mathfrak{A}^n: \mathcal{C}^\infty(U, \mathcal{L}^n(E; F))_c \rightarrow \mathcal{C}^\infty(U, \text{Alt}^n(E; F))_c, \quad f \mapsto (x \mapsto \mathfrak{A}^n f(x))$$

is continuous. In particular, $\mathcal{C}^\infty(U, \mathcal{A}l^n(E; F))_c$ is a closed vector subspace of $\mathcal{C}^\infty(U, \mathcal{L}^n(E; F))_c$.

Proof. This follows immediately from Lemma A.2.2 and the definition of \mathfrak{A}^n . \square

Proposition A.2.10 ([20], cf. [23]). *Let $n \in \mathbb{N}$, E_0, \dots, E_n and F be locally convex spaces and $U \subseteq E_0$ open. If $f \in \mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F))$, then f is smooth as a map*

$$f: U \rightarrow \mathcal{L}^n(E_1, \dots, E_n; F)_c$$

and as a map

$$f: U \rightarrow \mathcal{L}^n(E_1, \dots, E_n; F)_b.$$

Furthermore, we have

$$d^m f(x)(w).v = d^m f^\wedge(x, v)((w_1, 0), \dots, (w_m, 0))$$

for all $m \in \mathbb{N}$, $x \in U$, $w = (w_1, \dots, w_m) \in E_0^m$ and $v \in E_1 \times \dots \times E_n$.

Corollary A.2.11. *Let $n \in \mathbb{N}$, let E_0, \dots, E_n and F be locally convex spaces and let $U \subseteq E_0$ open. If $f \in \mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F))$, then f is smooth as a map*

$$f: U \rightarrow \mathcal{L}^n(E_1, \dots, E_n; F)_b$$

and $d^m f(x)(w, v)$ as defined in Definition 2.2.11 coincides with $d^m f(x)(w).v$ for all $m \in \mathbb{N}$, $x \in E_0$, $v \in E_1 \times \dots \times E_n$ and $w \in E_0^m$.

Proof. This follows from Proposition A.2.10. \square

Lemma A.2.12. *Let $n \in \mathbb{N}$, E_1, \dots, E_n, F be Banach spaces, E_0 be finite-dimensional and $U \subseteq E_0$ open. Then*

$$\mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F)) \rightarrow \mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F)_b), \quad f \mapsto f,$$

is an isomorphism of vector spaces.

Proof. By Corollary A.2.11, the map is well-defined. Let $W := E_1 \times \dots \times E_n$, let $g \in \mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F)_b)$ and let $g^\wedge: U \times W \rightarrow F$, $(x, w) \mapsto g(x, w)$. The evaluation

$$\text{ev}_W: \mathcal{L}^n(E_1, \dots, E_n; F)_b \times W \rightarrow F, \quad (L, w) \mapsto L(w)$$

is smooth. Thus, the map $g^\wedge: U \times W \rightarrow F$, $(u, w) \mapsto \text{ev}_W(g(u), w)$ is smooth, which implies $g \in \mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F))$. \square

Corollary A.2.13. *Let $n \in \mathbb{N}_0$, E_1, \dots, E_n and F be Banach spaces, H a locally convex space, E_0 finite-dimensional and let $U \subseteq E_0$ as well as $V \subseteq H$ be open. Then a map*

$$f: V \rightarrow \mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F)_b)$$

is smooth if and only if

$$(f^\wedge)^\wedge: V \times U \times (E_1 \times \cdots \times E_n) \rightarrow F, \quad (x, u, v) \mapsto f(x)(u).v,$$

is smooth. We have

$$(d^m f(x)(v))(u).w = d^m (f^\wedge)^\wedge(x, u, w)((v_1, 0, 0), \dots, (v_m, 0, 0))$$

for $m \in \mathbb{N}$, $x \in V$, $v = (v_1, \dots, v_m) \in H^m$, $u \in U$ and $w \in E_1 \times \cdots \times E_n$.

Proof. Let $W := E_1 \times \cdots \times E_n$ and $\text{ev}_W: \mathcal{L}^n(E_1, \dots, E_n; F)_b \times W \rightarrow F$, $(L, v) \mapsto L(v)$. If f is smooth, then by Proposition A.2.7 $f^\wedge: V \times U \rightarrow \mathcal{L}^n(E_1, \dots, E_n; F)_b$ is smooth. But then, $(x, u, v) \mapsto f(x)(u).v = \text{ev}_W(f^\wedge(x, u), v)$ is smooth. Conversely, if $(f^\wedge)^\wedge$ is smooth, then so is f^\wedge by Corollary A.2.11, and it follows from Proposition A.2.7 that f is smooth. The formula for the derivative follows from the respective formulas for the derivative in the corollary and the proposition. \square

Lemma A.2.14. Let $n, l \in \mathbb{N}$, $E_1, \dots, E_n, E'_1, \dots, E'_l$ and F be Banach spaces, E_0 be finite-dimensional and $U \subseteq E_0$ be open. Abbreviate $W := \mathcal{C}^\infty(U, \mathcal{L}^n(E_1, \dots, E_n; F)_b)$, $W' := \mathcal{C}^\infty(U, \mathcal{L}^l(E'_1, \dots, E'_l; E_n)_b)$ and $W'' := \mathcal{C}^\infty(U, \mathcal{L}^l(E'_1, \dots, E'_l; E_0)_b)$.

(a) The bilinear map

$$\begin{aligned} W \times W' &\rightarrow \mathcal{C}^\infty(U, \mathcal{L}^{n+l-1}(E_1, \dots, E_{n-1}, E'_1, \dots, E'_l; F)_b), \\ (f, g) &\mapsto (x \mapsto f(x)(\bullet, g(x)(\bullet))) \end{aligned}$$

is continuous.

(b) For $m \in \mathbb{N}$, the bilinear map

$$\begin{aligned} W \times W'' &\rightarrow \mathcal{C}^\infty(U, \mathcal{L}^{n+m+l-1}(E_0, \dots, E_0, E'_1, \dots, E'_l, E_1, \dots, E_n; F)_b), \\ (f, g) &\mapsto (x \mapsto d^m f(x)(\bullet, g(x)(\bullet))(\bullet)) \end{aligned}$$

is continuous. This is also true if $n = 0$ or $l = 0$.

Proof. (a) As mentioned above, the composition

$$\begin{aligned} \Gamma: \mathcal{L}^n(E; F)_b \times \mathcal{L}^n(E'; E)_b &\rightarrow \mathcal{L}^{n+l-1}(E, \dots, E, E', \dots, E'; F)_b, \\ (L, L') &\mapsto L(\bullet, L'(\bullet)) \end{aligned}$$

is continuous. By Proposition A.2.8, it follows that

$$\begin{aligned} W \times W' \times U &\rightarrow \mathcal{L}^{n+l-1}(E_1, \dots, E_{n-1}, E'_1, \dots, E'_l; F)_b, \\ (f, g, u) &\mapsto \Gamma(\text{ev}_W(f, u), \text{ev}_{W'}(g, u)) \end{aligned}$$

is continuous, where $\text{ev}_W: W \times U \rightarrow \mathcal{L}^n(E_1, \dots, E_n; F)_b$ and $\text{ev}_{W'}: W' \times U \rightarrow \mathcal{L}^n(E'_1, \dots, E'_l; E_n)_b$ are the respective evaluations. The claim now follows from Proposition A.2.7.

(b) Using the continuity of the composition, we see with Lemma A.2.12 that

$$W \rightarrow \mathcal{C}^\infty(U, \mathcal{L}^{n+m}(E_0, \dots, E_0, E_1, \dots, E_n; F)_b), \quad f \mapsto (x \mapsto d^n f(x)(\cdot))$$

is continuous. Therefore, the claim follows from (a). \square

Lemma A.2.15. *Let E and F be locally convex spaces, let U be an open subset of a locally convex space and let $r, n \in \mathbb{N}$. Then*

$$\bigoplus_{I \in \mathcal{P}^n, |I| \leq r} \mathcal{C}^\infty(U, \mathcal{A}lt^{r-|I|}(E; F))_c \rightarrow \mathcal{C}^\infty(U, \mathcal{A}lt^r(\mathbb{R}^n \oplus E; F))_c,$$

$$(f_I)_{I \in \mathcal{P}^n, |I| \leq r} \mapsto \sum_{I \in \mathcal{P}^n, |I| \leq r} \mathbf{v}_I \wedge f_I$$

is an isomorphism of topological vector spaces. If E and F are Banach spaces and U is a subset of a finite-dimensional space, then

$$\bigoplus_{I \in \mathcal{P}^n, |I| \leq r} \mathcal{C}^\infty(U, \mathcal{A}lt^{r-|I|}(E; F)_b) \rightarrow \mathcal{C}^\infty(U, \mathcal{A}lt^r(\mathbb{R}^n \oplus E; F)_b),$$

$$(f_I)_{I \in \mathcal{P}^n, |I| \leq r} \mapsto \sum_{I \in \mathcal{P}^n, |I| \leq r} \mathbf{v}_I \wedge f_I$$

is an isomorphism, as well.

Proof. Both maps are obviously linear. That the first map is continuous follows because for $I \in \mathcal{P}^n$, $I = \{i_1, \dots, i_\ell\}$, $\ell \leq r$ and $f \in \mathcal{C}^\infty(U, \mathcal{A}lt^{r-\ell}(E; F))_c$, we have

$$(\mathbf{v}_I \wedge f)^\wedge = f^\wedge \circ (\text{id}_U, \gamma)$$

with $\gamma: (\mathbb{R}^n \oplus E)^r \rightarrow E^{r-\ell}$,

$$\gamma(v) := \sum_{\sigma \in \mathfrak{S}_r} \frac{\text{sgn}(\sigma)}{r!(r-\ell)!} (\mathbf{v}_I(\text{pr}_1(\cdot), \dots, \text{pr}_1(\cdot)) \cdot \text{pr}_2(\cdot), \dots, \text{pr}_2(\cdot))(v^\sigma)$$

and the projections $\text{pr}_1: \mathbb{R}^n \times E \rightarrow \mathbb{R}^n$ and $\text{pr}_2: \mathbb{R}^n \times E \rightarrow E$. We show that the inverse is given by the continuous map

$$\Psi: \mathcal{C}^\infty(U, \mathcal{A}lt^r(\mathbb{R}^n \oplus E; F))_c \rightarrow \bigoplus_{I \in \mathcal{P}^n, |I| \leq r} \mathcal{C}^\infty(U, \mathcal{A}lt^{r-|I|}(E; F))_c,$$

$$f \mapsto \left(x \mapsto f(x)((\mathbf{v}_{i_1}, 0), \dots, (\mathbf{v}_{i_{|I|}}, 0), j_2^{r-|I|}(\cdot)) \right)_{I \in \mathcal{P}^n, |I| \leq r},$$

where $I = \{i_1, \dots, i_{|I|}\}$ and where $j_2: E \rightarrow \mathbb{R}^n \oplus E$ is the inclusion. It suffices to show $f(x)(v, w) = \sum_I (\mathbf{v}_I \wedge \Psi(f))_I(x)(v, w)$ for all $x \in U$, $w \in (\{0\} \oplus E)^{r-\ell}$ and all $v = ((\mathbf{v}_{i_1}, 0), \dots, (\mathbf{v}_{i_\ell}, 0))$. Using the antisymmetry of $f(x)(v, \cdot)$ and the fact that the only contributing summand for fixed $J := \{i_1, \dots, i_\ell\}$ is from the set

$\mathfrak{v}_J \wedge \mathcal{A}^{r-\ell}(F; E)$, we calculate

$$f(x)(v, w) = \sum_{\sigma \in \mathfrak{S}_\ell, \tau \in \mathfrak{S}_{r-\ell}} \frac{\text{sgn}(\sigma\tau)}{\ell!(r-\ell)!} \mathfrak{v}_J(v^\sigma) \cdot \Psi(f)_J(x)(w^\tau) = (\mathfrak{v}_J \wedge \Psi(f)_J(x))(v, w).$$

Note that for every $\alpha \in \mathfrak{S}_r$ that is not of the form $(\sigma, \tau) \in \mathfrak{S}_\ell \times \mathfrak{S}_{r-\ell}$, we have $\mathfrak{v}_J \cdot \Psi(f)_J(x)(v, w)^\alpha = 0$. That the second map is an isomorphism follows easily with Corollary A.2.13 and the same inverse map. \square

Corollary A.2.16. *Let $n \in \mathbb{N}_0$, $E, F \in \mathbf{SVec}_{lc}$ and $U \subseteq E_0$ open. Identify $\bigwedge(\mathbb{R}^n)^* \cong \Lambda_n$ as in (4.8). Then $\prod_{l=0}^{\infty} \mathcal{C}^\infty(U, \mathcal{A}^l(\mathbb{R}^n \oplus E_1; F_l))_c$ is a topological $\Lambda_{n, \bar{0}}$ -module and we have*

$$\begin{aligned} \prod_{l=0}^{\infty} \mathcal{C}^\infty(U, \mathcal{A}^l(\mathbb{R}^n \oplus E_1; F_l))_c = \\ \prod_{l=0}^{\infty} \left(\bigoplus_{I \in \mathcal{P}_0^n} \mathfrak{v}_I \wedge \mathcal{C}^\infty(U, \mathcal{A}^l(E_1; F_l))_c \oplus \bigoplus_{I \in \mathcal{P}_1^n} \mathfrak{v}_I \wedge \mathcal{C}^\infty(U, \mathcal{A}^l(E_1; F_{l+1}))_c \right). \end{aligned}$$

If E and F are Banach spaces and E_0 is finite-dimensional, then $\prod_{l=0}^{\infty} \mathcal{C}^\infty(U, \mathcal{A}^l(\mathbb{R}^n \oplus E_1; F_l)_b)$ is a topological $\Lambda_{n, \bar{0}}$ -module in the same way.

Proof. The associativity of the module multiplication can be seen in the same way as the associativity of the usual wedge product (see for example [25, Proposition 5.30.1, p.141]). The rest follows from Lemma A.2.15. \square

Combinatorial formulas

Lemma A.2.17. *Let $l, m \in \mathbb{N}_0, m + l > 0$ and let E_0, E_1, E and F be \mathbb{R} -vector spaces. Further, let $g_i \in \mathcal{A}^i(F; E_0)$ for $i \in \mathbb{N}$ even, $g_j \in \mathcal{A}^j(F; E_1)$ for $j \in \mathbb{N}$ odd and $f \in L^{m+l}(E_0, \dots, E_0, E_1, \dots, E_1; E)$. Then for all $n \in \mathbb{N}$, $\tau \in \mathfrak{S}_l$ and $\tau' \in \mathfrak{S}_m$, we have*

$$\sum_{(\alpha, \beta) \in I_{m,l}^n} \mathfrak{A}^n f(g_\alpha \times g_\beta) = \sum_{(\alpha, \beta) \in I_{m,l}^n} \text{sgn}(\tau) \mathfrak{A}^n f(\cdot^{\tau'}, \cdot^\tau)(g_\alpha \times g_\beta).$$

Proof. Let $(\alpha, \beta) \in I_{m,l}^n$, $\tau \in \mathfrak{S}_l$, $\tau' \in \mathfrak{S}_m$ and $(v, w) \in F^m \times F^l$. We partition w into l blocks w^{β_i} of length β_i , define w^β as the tuple $(w_{\beta_1}, \dots, w_{\beta_l})$ and let $\beta_\tau = (\beta_{\tau(1)}, \dots, \beta_{\tau(l)})$. Analogously, we define v^α and $\alpha_{\tau'}$. With this notation, we easily see $f(\cdot^{\tau'}, \cdot^\tau)(g_\alpha \times g_\beta)(v, w) = f(g_{\alpha_{\tau'}}(v^{\alpha_{\tau'}}), g_{\beta_\tau}(w^{\beta_\tau}))$. Since the length of the blocks v^{α_i} is even and that of w^{β_i} is odd, there exists a $\sigma_{(\alpha, \beta)} \in \mathfrak{S}_n$ depending on τ' and τ such that $f(g_{\alpha_{\tau'}}(v^{\alpha_{\tau'}}), g_{\beta_\tau}(w^{\beta_\tau})) = f(g_{\alpha_{\tau'}} \times g_{\beta_\tau})(v, w)^{\sigma_{(\alpha, \beta)}}$ and $\text{sgn}(\tau) = \text{sgn}(\sigma_{(\alpha, \beta)})$. Now, we have

$$\sum_{(\alpha, \beta) \in I_{m,l}^n} \mathfrak{A}^n f(g_\alpha \times g_\beta) = \sum_{(\alpha, \beta) \in I_{m,l}^n} \mathfrak{A}^n f(g_{\alpha_{\tau'}} \times g_{\beta_\tau}) =$$

$$\begin{aligned} & \sum_{(\alpha, \beta) \in I_{m, l}^n} \operatorname{sgn}(\sigma_{(\alpha, \beta)}) \mathfrak{A}^n f(g_{\alpha_{\tau'}} \times g_{\beta_{\tau}})(\bullet^{\sigma_{(\alpha, \beta)}}) = \\ & \sum_{(\alpha, \beta) \in I_{m, l}^n} \operatorname{sgn}(\tau) \mathfrak{A}^n f(\bullet^{\tau'}, \bullet^{\tau})(g_{\alpha} \times g_{\beta}). \end{aligned} \quad \square$$

Lemma A.2.18. *Let $k \in \mathbb{N}$, E, E_1, \dots, E_k and F be \mathbb{R} -vector spaces, $f \in L^k(E_1, \dots, E_k; F)$, $\beta \in \mathbb{N}^k$ such that $|\beta| = n$ and $g_l \in \operatorname{Alt}^{\beta_l}(\mathbb{R}^n \oplus E; E_l)$ with $g_l = \mathfrak{v}_{I_l} \wedge_{I_l} g_l$, where $i_l := |I_l|$ and $_{I_l} g_l \in \operatorname{Alt}^{\beta_l - i_l}(\mathbb{R}^n \oplus E; E_l)$. Then, we have*

$$\begin{aligned} & \frac{n!}{\beta!} \mathfrak{A}^n f(g_1, \dots, g_k) = \\ & \frac{(-1)^N \cdot (n - i_1 - \dots - i_k - s)!}{(\beta_1 - i_1)! \dots (\beta_k - i_k)!} \mathfrak{v}_{I_1} \wedge \dots \wedge \mathfrak{v}_{I_k} \wedge \mathfrak{A}^{n - i_1 - \dots - i_k - s} f(_{I_1} g_1, \dots, _{I_k} g_k), \end{aligned}$$

where $N = \sum_{j=2}^k \sum_{l=1}^{j-1} i_j \cdot (\beta_l - i_l)$ and s is the number of indices such that $\beta_l - i_l = 0$, i.e., $_{I_l} g_l \in E_l$ constant.

Proof. For $1 \leq l \leq k$, $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_{\beta_l}$, there exists a $\tau' \in \mathfrak{S}_n$ with $\operatorname{sgn}(\tau') = \operatorname{sgn}(\tau)$ such that

$$\begin{aligned} & \operatorname{sgn}(\sigma) f(g_1, \dots, \operatorname{sgn}(\tau) \mathfrak{v}_{I_l} \cdot _{I_l} g_l(\bullet^{\tau}), \dots, g_k)(\bullet^{\sigma}) = \\ & \operatorname{sgn}(\sigma \tau') f(g_1, \dots, \mathfrak{v}_{I_l} \cdot _{I_l} g_l, \dots, g_k)(\bullet^{\tau' \sigma}). \end{aligned}$$

Furthermore, there exists a $\rho_l \in \mathfrak{S}_n$ with $\operatorname{sgn}(\rho_l) = (-1)^{(\beta_1 + \dots + \beta_{l-1}) \cdot i_l}$ such that

$$\mathfrak{v}_{I_l} \cdot f(g_1, \dots, _{I_l} g_l, \dots, g_k)(\bullet^{\rho_l}) = f(g_1, \dots, \mathfrak{v}_{I_l} \cdot _{I_l} g_l, \dots, g_k)(\bullet).$$

The sign arises from swapping a block of length i_l with the preceding block of length $\beta_1 + \dots + \beta_{l-1}$. Using this, we calculate

$$\begin{aligned} & \frac{n!}{\beta!} \mathfrak{A}^n f(g_1, \dots, g_l, \dots, g_k) \\ &= \frac{n!}{\beta!} \sum_{\sigma \in \mathfrak{S}_n} \frac{\operatorname{sgn}(\sigma)}{n!} f\left(g_1, \dots, \sum_{\tau \in \mathfrak{S}_{\beta_l}} \frac{\operatorname{sgn}(\tau)}{(\beta_l - i_l)! i_l!} (\mathfrak{v}_{I_l} \cdot _{I_l} g_l)(\bullet^{\tau}), \dots, g_k\right)(\bullet^{\sigma}) \\ &= \operatorname{sgn}(\rho_l) \frac{\beta_1! n!}{\beta! (\beta_1 - i_1)! i_1!} \mathfrak{A}^n \mathfrak{v}_{I_l} \cdot f(g_1, \dots, _{I_l} g_l, \dots, g_k) \\ &= \begin{cases} \operatorname{sgn}(\rho_l) \frac{\beta_1! (n - i_1)!}{\beta! (\beta_1 - i_1)!} \mathfrak{v}_{I_1} \wedge \mathfrak{A}^{n - i_1} f(g_1, \dots, _{I_l} g_l, \dots, g_k) & \text{if } \beta_l - i_l > 0, \\ \operatorname{sgn}(\rho_l) \frac{\beta_1! (n - i_1 - 1)!}{\beta! (\beta_1 - i_1)!} \mathfrak{v}_{I_1} \wedge \mathfrak{A}^{n - i_1 - 1} f(g_1, \dots, _{I_l} g_l, \dots, g_k) & \text{if } \beta_l - i_l = 0. \end{cases} \end{aligned}$$

Applying this equality inductively, we get

$$\begin{aligned} & \frac{n!}{\beta!} \mathfrak{A}^n f(\mathfrak{v}_{I_1} \wedge _{I_1} g_1, \dots, \mathfrak{v}_{I_k} \wedge _{I_k} g_k) = \\ & (-1)^N \frac{(n - i_1 - \dots - i_k - s)!}{(\beta_1 - i_1)! \dots (\beta_k - i_k)!} \mathfrak{v}_{I_1} \wedge \dots \wedge \mathfrak{v}_{I_k} \wedge \mathfrak{A}^{n - i_1 - \dots - i_k - s} f(_{I_1} g_1, \dots, _{I_k} g_k), \end{aligned}$$

where $N = \sum_{j=2}^k \sum_{l=1}^{j-1} i_j \cdot (\beta_l - i_l)$. \square

Lemma A.2.19. *Let $l, n \in \mathbb{N}_0$, E_1, E and F be \mathbb{R} -vector spaces, $f \in \text{Alt}^l(E_1; E)$ and $g_i \in \text{Alt}^i(F; E_1)$ for odd $i \in \mathbb{N}$. Define $g'_i := (0, g'_i): \mathbb{R}^n \times F^i \rightarrow \mathbb{R}^n \times E_1$ for $i > 1$ and $g'_1 := (\text{id}_{\mathbb{R}^n}, g_1)$. Then, we have*

$$\sum_{(\alpha, \beta) \in I_{0, i+l}^n} \frac{n!}{\beta!} \mathfrak{A}^n(\mathbf{v}_I \wedge f)(g'_\beta) = \mathbf{v}_I \wedge \sum_{(\alpha, \beta) \in I_{0, l}^{n-i}} \frac{(n-i)!(i+l)!}{\beta!l!} \mathfrak{A}^{n-i} f(g_\beta)$$

for $I \in \mathcal{P}^n$ with $i := |I|$.

Proof. We calculate

$$\begin{aligned} & \sum_{(\alpha, \beta') \in I_{0, i+l}^n} \frac{n!}{\beta'!} \mathfrak{A}^n(\mathbf{v}_I \wedge f)(g'_{\beta'}) \\ &= \sum_{(\alpha, \beta') \in I_{0, i+l}^n} \frac{n!}{\beta'!} \mathfrak{A}^n \left(\sum_{\tau \in \mathfrak{S}_{i+l}} \frac{\text{sgn}(\tau)}{i!l!} (\mathbf{v}_I \cdot f)(\cdot^\tau) \right) (g'_{\beta'}) \\ &= \sum_{(\alpha, \beta') \in I_{0, i+l}^n} \frac{n!}{\beta'!} \mathfrak{A}^n \frac{(i+l)!}{i!l!} (\mathbf{v}_I \cdot f)(g'_{\beta'}) \\ &= \sum_{(\alpha, \beta) \in I_{0, l}^{n-i}} \frac{n!(i+l)!}{\beta!i!l!} \mathfrak{A}^n \mathbf{v}_I \cdot (f(g_\beta)) \\ &= \sum_{(\alpha, \beta) \in I_{0, l}^{n-i}} \frac{(n-i)!(i+l)!}{\beta!l!} \mathbf{v}_I \wedge \mathfrak{A}^{n-i} f(g_\beta). \end{aligned}$$

To see the second equality note that by Lemma A.2.17, we have the same contribution to the outer sum for every $\tau \in \mathfrak{S}_{i+l}$. Therefore, we may simply add $(i+l)!$ -times the summand for $\tau = \text{id}$. For the third equality, we use that for $(\alpha, \beta') \in I_{0, i+l}^n$ the contribution to the sum is zero unless $\beta'_j = 1$ for $j \leq i$. Thus, we may assume $\beta' = (1, \dots, 1, \beta_1, \dots, \beta_l)$ for a unique $(\alpha, \beta) \in I_{0, l}^{n-i}$ with $\beta'! = \beta!$. \square

A.3. Lie Groups

A Lie group is a group object in the category **Man**. Just like for finite-dimensional Lie groups, we have an associated locally convex Lie algebra structure on $T_e G$ (where $e \in G$ is the unit element), denoted by $L(G)$. Setting $L(f) := T_e f$ for a morphism $f: G \rightarrow H$, one obtains a functor from the category of Lie groups to the category of locally convex topological Lie algebras. See [41] and [23] for more details.

Lemma A.3.1 ([55, Lemma A.3.3, p.133]). *Let G be a Lie group, M a manifold and $\alpha: G \times M \rightarrow M$ a group action such that for every $g \in G$ the map $\alpha(g, \cdot): M \rightarrow M$ is a diffeomorphism. Then α is smooth if and only if there exists an open unity neighborhood $U \subseteq G$ such that $\alpha|_{U \times M}$ is smooth.*

If G is a Lie group and $g \in G$, then $c_g: G \rightarrow G$, $h \mapsto ghg^{-1}$ is an isomorphism of Lie groups. We obtain an isomorphism of Lie algebras $\text{Ad}_g = \text{Ad}(g) := L(c_g): L(G) \rightarrow L(G)$.

Proposition A.3.2 ([23]). *Let G be a Lie group. Then*

$$\text{Ad}: G \times L(G) \rightarrow L(G), \quad (g, v) \mapsto \text{Ad}(g)(v),$$

is smooth. If $[\cdot, \cdot]$ is the Lie bracket of $L(G)$, we have

$$d\text{Ad}_\bullet(v)(w) = [w, v]$$

for $w, v \in L(G)$ and $\text{Ad}_\bullet(v): G \rightarrow L(G)$, $g \mapsto \text{Ad}_g(v)$.

Remark A.3.3. Let N and G be Lie groups and $\alpha: G \rightarrow \text{Aut}(N)$ be a morphism of groups that defines a smooth action $(g, n) \mapsto \alpha(g)(n)$ of G on N . Then the semidirect product $N \rtimes_\alpha G$ is a Lie group. Identify $N \cong N \times \{e_G\} \subseteq N \rtimes_\alpha G$ and $G \cong \{e_N\} \times G \subseteq N \rtimes_\alpha G$, where e_N and e_G are the respective neutral elements of N and G . With this, we have $gng^{-1} = \alpha(g)(n)$ for all $g \in G$ and $n \in N$ and therefore $\text{Ad}(g, w) = L(\alpha(g)).w$ for $w \in L(N)$. Proposition A.3.2 implies now that we have $d\text{Ad}_\bullet(v)(w) = [w, v]$ for $v \in L(G)$ and $w \in L(N)$.

Lemma A.3.4. *Let G, G', N, N' be groups and $\alpha: G \rightarrow \text{Aut}(N)$, $\alpha': G' \rightarrow \text{Aut}(N')$ morphisms of groups. If $f_0: G \rightarrow G'$ and $f_1: N \rightarrow N'$ are group morphisms, then*

$$f_1 \times f_0: N \rtimes_\alpha G \rightarrow N' \rtimes_{\alpha'} G'$$

is a group morphism if and only if $\alpha'(f_0(g)) \circ f_1 = f_1 \circ \alpha(g)$ holds for all $g \in G$.

Proof. This follows by comparing

$$(f_1 \times f_0)((n_1, g_1) \cdot (n_2, g_2)) = (f_1(n_1)f_1(\alpha(g_1)(n_2)), f_0(g_1)f_0(g_2))$$

and

$$(f_1(n_1), f_0(g_1)) \cdot (f_1(n_2), f_0(g_2)) = (f_1(n_1)\alpha'(f_0(g_1))(f_1(n_2)), f_0(g_1)f_0(g_2)). \quad \square$$

Proposition A.3.5 ([17, Proposition 7.3, p.995]). *Let $(G_i)_{i \in I}$ be a family of smooth Lie groups. Then there exists a uniquely determined Lie group structure on the weak direct product*

$$\prod_{i \in I}^* G_i := \left\{ (g_i)_{i \in I} \in \prod_{i \in I} G_i : g_i = e \text{ for almost all } i \in I \right\}$$

modelled on the locally convex direct sum $\bigoplus_{i \in I} L(G_i)$ such that, for some charts $\varphi_i: R_i \rightarrow S_i \subseteq L(G_i)$ of G_i around e taking e to 0, the mapping

$$\bigoplus_{i \in I} S_i \rightarrow \prod_{i \in I}^* G_i, \quad (x_i)_{i \in I} \mapsto (\varphi_i^{-1}(x_i))_{i \in I}$$

is a diffeomorphism of smooth manifolds onto an open subset of $\prod_{i \in I}^* G_i$.

Lemma A.3.6. *In the situation of Proposition A.3.5, the projection*

$$\text{pr}_j: \prod_{i \in I}^* G_i \rightarrow G_j, \quad (x_i)_{i \in I} \mapsto x_j$$

is smooth for every $j \in I$.

Proof. Since pr_j is a group morphism, it suffices to see this in a unity neighborhood. There it holds because the projection $\bigoplus_{i \in I} L(G_i) \rightarrow L(G_j)$ is smooth. \square

Lemma A.3.7. *Let G be a Lie group and M a compact manifold (possibly with boundary). Then the evaluation map*

$$\text{ev}: \mathcal{C}^\infty(M, G) \times M \rightarrow G, \quad (\gamma, x) \mapsto \gamma(x),$$

is smooth.

Proof. This follows from [2, Lemma 121, p.82] together with [3, Lemma 3.15, p.199]. \square

A.3.1. The Baker-Campbell-Hausdorff Series

If G is a Banach Lie group, then it has a smooth exponential map $\exp_G: L(G) \rightarrow G$ that is a diffeomorphism on a zero-neighborhood. For $X, Y \in L(G)$ close enough to zero, the multiplication in this diffeomorphism is given by the so called *BCH series*

$$\begin{aligned} & \exp_G^{-1}(\exp_G(X) \cdot \exp_G(Y)) = \\ & X + \sum_{\substack{k, m \geq 0, \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \dots + q_k + 1)} \cdot \frac{\text{ad}_X^{p_1} \text{ad}_Y^{q_1} \dots \text{ad}_X^{p_k} \text{ad}_Y^{q_k} \text{ad}_X^m(Y)}{p_1! q_1! \dots p_k! q_k! m!}, \end{aligned}$$

where $\text{ad}_X^n := \overbrace{[X, [X, \dots, [X, \cdot] \dots]]}^{n \text{ times}}: L(G) \rightarrow L(G)$ if $[\cdot, \cdot]$ is the Lie bracket of G ; see [41, Section IV.1, p.360ff.] for details and generalizations. If we have a nilpotent Lie algebra over a field of characteristic zero, this formula defines a polynomial group multiplication, which we call *BCH multiplication* (see Appendix C).

B. Multilinear Bundles

Multilinear bundles were introduced in [10] to describe higher order tangent bundles. As it turns out, the structure of supermanifolds is closely related to the structure of multilinear bundles. One important addition introduced below, is the inverse limit of multilinear bundles.

For this section, we fix the infinitesimal generators ε_k , $k \in \mathbb{N}$, with the relations $\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i$ and $\varepsilon_i \varepsilon_i = 0$. As usual, we set $\varepsilon_I := \varepsilon_{i_1} \cdots \varepsilon_{i_r}$ for $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, k\}$.

B.1. Multilinear Spaces

Definition B.1.1 ([10, MA.2, p.169]). Let $k \in \mathbb{N}$. A *(locally convex) k -dimensional cube* is a family $(E_I)_{I \in \mathcal{P}_+^k}$ of (locally convex) \mathbb{R} -vector spaces with the *total space*

$$E := \bigoplus_{I \in \mathcal{P}_+^k} E_I.$$

We denote the elements of E by $v = \sum_{I \in \mathcal{P}_+^k} v_I$ or by $v = (v_I)_{I \in \mathcal{P}_+^k}$ with $v_I \in E_I$. By abuse of notation, we will call E a k -dimensional cube as well. For convenience, we let a 0-dimensional cube be defined by the total space $\{0\}$. The spaces E_I are called the *axes* of E .

Let (E_I) and (E'_I) be k -dimensional cubes. For each partition $\nu \in \mathcal{P}(I)$, $I \in \mathcal{P}_+^k$, let f^ν be an \mathbb{R} - $\ell(\nu)$ -multilinear map

$$f^\nu: E_\nu := E_{\nu_1} \times \dots \times E_{\nu_{\ell(\nu)}} \rightarrow E'_I,$$

$$v_\nu := (v_{\nu_1}, \dots, v_{\nu_{\ell(\nu)}}) \mapsto f^\nu(v_\nu).$$

A *morphism* of (locally convex) k -dimensional cubes E and E' is a (continuous) map of the form

$$f: E \rightarrow E', \quad \sum_{I \in \mathcal{P}_+^k} v_I \mapsto \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} f^\nu(v_\nu).$$

The composition of two morphisms is simply the composition of maps. We define the *product* $E \times E'$ by $(E \times E')_I := E_I \times E'_I$.

Clearly f is a polynomial map in the sense of Appendix C and thus a morphism f of k -multilinear bundles (that are also locally convex k -multilinear bundles) is continuous if and only if all f^ν are continuous.

Theorem B.1.2.

- (a) *The (locally convex) k -dimensional cubes and their (continuous) morphisms form a category, which we will call the category of (locally convex) k -multilinear spaces.*
- (b) *A morphism $f: E \rightarrow E'$ of k -dimensional cubes is invertible if and only if f^ν is a bijection for all partitions of the form $\nu = \{I\}$, $I \in \mathcal{P}_+^k$, i.e., for all partitions of length one. In this case f^{-1} is again a morphism of k -dimensional cubes.*
- (c) *If $f: E \rightarrow E'$ is a morphism of locally convex k -dimensional cubes such that $f^{\{I\}}$ is bijective with continuous inverse for all $I \in \mathcal{P}_+^k$, then f is invertible.*

Proof. Items (a) and (b) are just [10, MA.6, p.172] and (c) follows from the inductive construction in that proof. \square

We denote the category of k -multilinear spaces by $\mathbf{MSpace}^{(k)}$.

Remark B.1.3. It is calculated in the proof of [10, Theorem MA.6, p.172] that the composition of morphisms $f: E \rightarrow E'$, $g: E' \rightarrow E''$ of k -dimensional cubes E, E' and E'' is given by

$$(g \circ f)^\nu(v_\nu) = \sum_{\omega \preceq \nu} g^\omega \left(f^{\omega_1|\nu}(v_{\omega_1|\nu}), \dots, f^{\omega_{\ell(\omega)}|\nu}(v_{\omega_{\ell(\omega)}|\nu}) \right). \quad (\text{B.1})$$

Of course the sets $\omega_1|\nu, \dots, \omega_{\ell(\omega)}|\nu$ need not be in (graded) lexicographic order but by abuse of notation, we also write g^ω for the map that arises from permuting the factors.

Definition B.1.4. Let (E_I) be a k -dimensional cube. For $P \subseteq \mathcal{P}_+^k$, we define the *restriction* $((E|_P)_I)$ of (E_I) by

$$(E|_P)_I := \begin{cases} E_I & \text{if } I \in P \\ \{0\} & \text{if } I \in \mathcal{P}_+^k \setminus P. \end{cases}$$

It has the total space

$$E|_P = \bigoplus_{J \in P} E_J \oplus \bigoplus_{I \in \mathcal{P}_+^k \setminus P} \{0\}.$$

When convenient, we identify the restriction $E|_P$ with the respective n -dimensional cube in the obvious way if $P = \mathcal{P}_+^n \subseteq \mathcal{P}_+^k$ for $n \leq k$. If $E|_{\mathcal{P}_{0,+}^k} = E$ holds, i.e., if $E_I = \{0\}$ for $I \in \mathcal{P}_1^k$, we call E *purely even*.

Lemma B.1.5. *Let $P \subseteq \mathcal{P}_+^k$ be a subset such that $\{\sum_I \varepsilon_I a_I : I \in P, a_I \in \mathbb{R}\}$ is a subalgebra of $\mathbb{R}[\varepsilon_1, \dots, \varepsilon_k]$. Then every morphism of k -multilinear spaces $f: E \rightarrow E'$ can be restricted in a natural way to a morphism $f|_P: E|_P \rightarrow E'|_P$. This restriction defines a functor*

$$\mathbf{MSpace}^{(k)} \rightarrow \mathbf{MSpace}^{(k)}$$

that respects products.

Proof. Let E, E' be k -dimensional cubes and $f: E \rightarrow E'$ be a morphism given by

$$f^\nu: E_{\nu_1} \times \cdots \times E_{\nu_{\ell(\nu)}} \rightarrow E'_I,$$

for $\nu \in \mathcal{P}(I)$ and $I \in \mathcal{P}_+^k$. If $I \notin P$, then there exists $1 \leq i \leq \ell(\nu)$ such that $\nu_i \notin P$, which implies $f(E|_P) \subseteq E'|_P$. Therefore, we can define $f|_P$ by setting $(f|_P)^\nu := f^\nu$ if $\nu_1, \dots, \nu_{\ell(\nu)} \in P$ and $(f|_P)^\nu := 0$ else. Let E'' be another k -dimensional cube and $g: E' \rightarrow E''$ be a morphism. Since $g(E'|_P) = g|_P(E'|_P)$ holds, functoriality follows. That the restriction respects products is obvious. \square

The purely even k -dimensional cubes clearly form a full subcategory of $\mathbf{MSpace}^{(k)}$ which we denote by $\mathbf{MSpace}_0^{(k)}$. It follows from Lemma B.1.5 that we have an essentially surjective restriction functor

$$\mathbf{MSpace}^{(k)} \rightarrow \mathbf{MSpace}_0^{(k)}, \quad E \mapsto E|_{\mathcal{P}_{0,+}^k} \quad \text{and} \quad f \mapsto f|_{\mathcal{P}_{0,+}^k}$$

for $E, E' \in \mathbf{MSpace}^{(k)}$ and $f: E \rightarrow E'$ a morphism.

Definition B.1.6. Let $I \in \mathcal{P}_+^k$ and $\nu = \{\nu_1, \dots, \nu_\ell\} \in \mathcal{P}(I)$. We define a tuple $(\nu_1 | \cdots | \nu_\ell)$ by concatenating the elements of ν_1, \dots, ν_ℓ in ascending order and define $\text{sgn}(\nu)$, the *sign of ν* , as the sign of the permutation needed to bring this tuple into strictly ascending order.

This definition depends on the order one chooses on the partitions. However, the sign of a partition taken with regard to the lexicographic order is the same as when one takes it with regard to the graded lexicographic order, because changing the position of sets with even cardinality does not change the sign. We will only use these two orders in the following.

Example B.1.7. Let $\nu = \{\{2\}, \{1, 3\}\}$. Then we get the tuple $(\nu_1 | \nu_2) = (2, 1, 3)$ and to permute this tuple to $(1, 2, 3)$, the permutation $\sigma = (1, 2)$ is needed. Thus $\text{sgn}(\nu) = -1$.

Lemma B.1.8. Let $E, E' \in \mathbf{MSpace}_0^{(k)}$ and $f: E \rightarrow E'$ be a morphism defined by the family $(f^\nu)_{\nu \in \mathcal{P}(\{1, \dots, k\})}$. Setting $E^- := E$, we let the morphism $f^-: E^- \rightarrow E'^-$ be given by $(\text{sgn}(\nu)f^\nu)_{\nu \in \mathcal{P}(\{1, \dots, k\})}$. This defines a functor

$$- : \mathbf{MSpace}_0^{(k)} \rightarrow \mathbf{MSpace}_0^{(k)}.$$

The functor is inverse to itself and respects products.

Proof. Let $E, E', E'' \in \mathbf{MSpace}_0^{(k)}$ and let $f: E \rightarrow E', g: E' \rightarrow E''$ be morphisms. To check functoriality, it suffices to assume $I = \{i_1, \dots, i_s\} \in \mathcal{P}_{0,+}^k$ and $\nu \in \mathcal{P}_0(I)$ because if any $|\nu_j|$ is odd, then $f^\nu = 0$ holds. Recall formula (B.1) from Remark B.1.3. On the one hand, we have

$$((g \circ f)^-)^{\nu}(v_\nu) = \text{sgn}(\nu) \sum_{\omega \in \mathcal{P}_0(I), \omega \preceq \nu} g^\omega \left(f^{\omega_1 | \nu}(v_{\omega_1 | \nu}), \dots, f^{\omega_{\ell(\omega)} | \nu}(v_{\omega_{\ell(\omega)} | \nu}) \right).$$

On the other hand, we calculate

$$(g^- \circ f^-)^\nu(v_\nu) = \sum_{\substack{\omega \in \mathcal{P}_0(I), \\ \omega \preceq \nu}} \text{sgn}(\omega) \cdot \text{sgn}(\omega_1|\nu) \cdots \text{sgn}(\omega_{\ell(\omega)}|\nu) g^\omega \left(f^{\omega_1|\nu}(v_{\omega_1|\nu}), \dots, f^{\omega_{\ell(\omega)}|\nu}(v_{\omega_{\ell(\omega)}|\nu}) \right).$$

The sign $\text{sgn}(\nu)$ describes the reordering of the tuple $(\nu_1 | \cdots | \nu_{\ell(\nu)})$ to (i_1, \dots, i_s) . For $\omega \preceq \nu$ let $\omega_j|\nu = \{\nu_1^j, \dots, \nu_{\ell_j}^j\} \in \mathcal{P}(\omega_j)$. Then $\text{sgn}(\omega) \cdot \text{sgn}(\omega_1|\nu) \cdots \text{sgn}(\omega_{\ell(\omega)}|\nu)$ gives the sign of the reordering of the tuple $(\nu_1^1 | \cdots | \nu_{\ell_1}^1 | \cdots | \nu_1^\ell | \cdots | \nu_{\ell_\ell}^\ell)$ to (i_1, \dots, i_s) . Since we only need to consider ν_j with even cardinality, reordering $(\nu_1^1 | \cdots | \nu_{\ell_1}^1 | \cdots | \nu_1^\ell | \cdots | \nu_{\ell_\ell}^\ell)$ to $(\nu_1 | \cdots | \nu_{\ell(\nu)})$ does not change the sign and it follows $\text{sgn}(\nu) = \text{sgn}(\omega) \cdot \text{sgn}(\omega_1|\nu) \cdots \text{sgn}(\omega_{\ell(\omega)}|\nu)$. This implies $g^- \circ f^- = (g \circ f)^-$. That the functor respects products is obvious. \square

The motivation for the lemma is essentially to substitute the infinitesimal generators ε_i with λ_i . For more details see Remark B.2.10 below.

B.2. Multilinear Bundles

Definition B.2.1 (compare [10, 15.4, p.81]).

- (a) Let E be a locally convex k -dimensional cube. A *multilinear bundle* (with base M , of degree k) is a smooth fiber bundle F over a manifold M with typical fiber E together with an equivalence class of bundle atlases such that the change of charts leads to an isomorphism of locally convex k -dimensional cubes on the fibers.
- (b) Let F and F' be multilinear bundles of degree k with base M , resp. M' . A *morphism of multilinear bundles* is a smooth fiber bundle morphism $f: F \rightarrow F'$ that locally (i.e., in bundle charts) leads to a morphism of the respective k -dimensional cubes in each fiber.

We identify multilinear bundles of degree zero with their base manifold in the obvious way.

It follows from Theorem B.1.2 that the multilinear bundles form a category which we denote by **MBun**. Multilinear bundles of degree k form a full subcategory denoted by **MBun**^(k).

Remark B.2.2. The above definition means that in bundle charts a morphism $f: F \rightarrow F'$ of multilinear bundles of degree k with fiber E , resp. E' , has the form

$$U \times E \rightarrow U' \times E', \quad (x, (v_I)) \mapsto \left(\varphi(x), \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} f_x^\nu(v_\nu) \right),$$

where $\varphi: U \rightarrow U'$ is the local representation of the morphism induced by f on the base manifolds and

$$f_x^\nu: E_{\nu_1} \times \dots \times E_{\nu_{\ell(\nu)}} \rightarrow E'_I$$

is a multilinear map for each $x \in U$, $I \in \mathcal{P}_+^k$ and $\nu \in \mathcal{P}(I)$. A function of this form is smooth if and only if $\varphi: U \rightarrow U'$ is smooth and the maps $(x, v_\nu) \mapsto f_x^\nu(v_\nu)$ are all smooth. This can be easily checked by restricting to the closed subspaces of E defined by a given partition.

Definition B.2.3. Let F and F' be multilinear bundles of degree k over M and M' , with typical fiber E and E' . Further, let $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha \times E: \alpha \in A\}$ and $\{\psi_\beta: V'_\beta \rightarrow U'_\beta \times E': \beta \in B\}$ be bundle atlases of F and F' . We let the *product bundle* $F \times F'$ be the multilinear bundle of degree k over $M \times M'$ with typical fiber $E \times E'$ given by the bundle atlas

$$\{\varphi_\alpha \times \psi_\beta: V_\alpha \times V'_\beta \rightarrow (U_\alpha \times U'_\beta) \times (E \times E'): \alpha \in A, \beta \in B\}.$$

Lemma/Definition B.2.4. Let F be a multilinear bundle of degree k with typical fiber E and bundle atlas $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha \times E: \alpha \in A\}$. Let $P \subseteq \mathcal{P}_+^k$ be a subset such that $\{\sum_I \varepsilon_I a_I: I \in P, a_I \in \mathbb{R}\}$ is a subalgebra of $\mathbb{R}[\varepsilon_1, \dots, \varepsilon_k]$. Each change of bundle charts $\varphi_{\alpha\beta}: U_{\alpha\beta} \times E \rightarrow U_{\beta\alpha} \times E$ of F restricts to a map

$$\varphi_{\alpha\beta}|_P: U_{\alpha\beta} \times E|_P \rightarrow U_{\beta\alpha} \times E|_P.$$

We denote the multilinear bundle of degree k defined by the charts $\varphi_\alpha|_{(\varphi_\alpha)^{-1}(U_\alpha \times E|_P)}$ by $F|_P$ and call it a *subbundle* of F . If $P = \mathcal{P}_+^n \subseteq \mathcal{P}_+^k$ for $k \leq n$, we identify $F|_P$ with the respective multilinear bundle of degree n in the obvious way (compare to Definition B.1.4). Any morphism $f: F \rightarrow F'$ of multilinear bundles of degree k restricts to a morphism $f|_P: F|_P \rightarrow F'|_P$. This restriction defines a functor

$$\mathbf{MBun}^{(k)} \rightarrow \mathbf{MBun}^{(k)}$$

that respects products.

Proof. Applying Lemma B.1.5 pointwise to transition maps and morphisms of multilinear bundles in their chart representation shows that $F|_P$ and $f|_P$ are well-defined and that the restriction is functorial. That the restriction respects products is obvious. \square

Note that in the situation of the definition, the identification of $F|_{\mathcal{P}_+^n}$ with a bundle of degree n is not a morphism of multilinear bundles but only a diffeomorphism of manifolds. There are cases where a subbundle is a multilinear bundle of lesser degree in a natural way that are not contained in the above definition. One important example is the following.

Lemma/Definition B.2.5. Let $\pi: F \rightarrow M$ be a multilinear bundle of degree k with typical fiber E . For each $I \in \mathcal{P}_+^k$, we have a subbundle $F|_{\{I\}}$ which has the structure of a vector bundle with fiber E_I in a natural way. The $2^k - 1$ vector bundles obtained in this way are called the *axes* of F .

Proof. For any change of bundle charts $\varphi_{\alpha\beta}: U_{\alpha\beta} \times E \rightarrow U_{\beta\alpha} \times E$ of F , we have that the corresponding change of bundle charts

$$\varphi_{\alpha\beta}|_{\{I\}}: U_{\alpha\beta} \times E_I \rightarrow U_{\beta\alpha} \times E_I$$

of $F|_{\{I\}}$ is linear in the second component. Thus the restricted charts define a vector bundle with typical fiber E_I . \square

Bertram uses the above fact in [10, 15.4, p.81] to *define* multilinear bundles by letting these axes take an analogous role to the axes in cubes. It is easy to see that both definitions are equivalent but our definition via bundle charts makes the relation of multilinear bundles to supermanifolds more direct.

Definition B.2.6. A *purely even multilinear bundle* is a multilinear bundle F of degree k such that $F|_{\mathcal{P}_{0,+}^k} = F$. The purely even multilinear bundles form a full subcategory of \mathbf{MBun} (resp. $\mathbf{MBun}^{(k)}$), which we denote by $\mathbf{MBun}_{\bar{0}}$ (resp. $\mathbf{MBun}_{\bar{0}}^{(k)}$) and we have the essentially surjective restriction functor

$$\mathbf{MBun} \rightarrow \mathbf{MBun}_{\bar{0}}, \quad F \mapsto F|_{\mathcal{P}_{0,+}^k} \quad \text{and} \quad f \mapsto f|_{\mathcal{P}_{0,+}^k}$$

for $F, F' \in \mathbf{MBun}^{(k)}$ and $f \in \text{Hom}_{\mathbf{MBun}^{(k)}}(F, F')$ (resp. $\mathbf{MBun}^{(k)} \rightarrow \mathbf{MBun}_{\bar{0}}^{(k)}$).

Example B.2.7. Let $k \in \mathbb{N}$.

- (a) Let $U \subseteq E$ be an open subset of a locally convex vector space E . Define inductively $TU := U \times \varepsilon_1 E$, $T^2U = T(U \times \varepsilon_1 E) = U \times \varepsilon_1 E \times \varepsilon_2 E \times \varepsilon_1 \varepsilon_2 E$ and so on. Then $T^k U = U \times \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I E$ is a trivial multilinear bundle over U of degree k . The axes are the trivial vector bundles $U \times \varepsilon_I E \rightarrow U$.
- (b) Let M be a manifold with the atlas $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha: \alpha \in A\}$. Then $T^k M$ is a multilinear bundle over M of degree k with the bundle atlas $\{T^k \varphi_\alpha: T^k V_\alpha \rightarrow T^k U_\alpha: \alpha \in A\}$. Let $\varphi_{\alpha\beta}$ be a change of charts. Using (a), the corresponding change of bundle charts is given by

$$T^k \varphi_{\alpha\beta} \left(x, \sum_{I \in \mathcal{P}_+^k} \varepsilon_I v_I \right) = \left(\varphi_{\alpha\beta}(x), \sum_{m=1}^k \sum_{|I|=m} \varepsilon_I \sum_{\nu \in \mathcal{P}(I)} d^m \varphi_{\alpha\beta}(x)(v_\nu) \right)$$

(see [10, Theorem 7.5, p.47]). The axes of $T^k M$ are thus all isomorphic to TM and we write $\varepsilon_I TM$ to differentiate between them. It also follows from [10, Theorem 7.5, p.47] that for each smooth map $f: M \rightarrow N$ between manifolds, $T^k f$ is a morphism of multilinear bundles and we get a functor $T^k: \mathbf{Man} \rightarrow \mathbf{MBun}^{(k)}$ in this way.

Lemma B.2.8. Let $k \in \mathbb{N}$ and let $f: M \rightarrow N$ be a smooth map between manifolds. For each $I \in \mathcal{P}_1^k$, we have

$$T^k f(\varepsilon_I TM) \subseteq \varepsilon_I TN$$

with the notation of Example B.2.7(b).

Proof. This is obvious because $T^k f$ is a morphism of multilinear bundles. \square

Lemma B.2.9. *Applying the functor from Lemma B.1.8 pointwise to transition maps and local chart representations of morphisms, we get a functor*

$$^- : \mathbf{MBun}_{\bar{0}} \rightarrow \mathbf{MBun}_{\bar{0}}, \quad F \mapsto F^- \quad \text{and} \quad h \mapsto h^-,$$

where $F, F' \in \mathbf{MBun}_{\bar{0}}^{(k)}$ and $h \in \text{Hom}_{\mathbf{MBun}_{\bar{0}}^{(k)}}(F, F')$. This functor is an equivalence of categories and restricts to equivalences of categories $\mathbf{MBun}_{\bar{0}}^{(k)} \rightarrow \mathbf{MBun}_{\bar{0}}^{(k)}$. All these functors respect products.

Proof. Locally this is obvious in view of Remark B.2.2 and Lemma B.1.8. By functoriality, applying this to all the change of charts of F leads to new cocycles that define a bundle F^- . Likewise, applying it pointwise to the chart representation of a morphism $h: F \rightarrow F'$ of purely even multilinear bundles leads in a functorial way to a morphism $h^-: F^- \rightarrow F'^-$. Obviously $(F^-)^- \cong F$ and $(h^-)^- = h$ under this identification, which shows that the functor is an equivalence of categories. That these functors respect products also follows because it is true locally. \square

Remark B.2.10. The intuition behind the above equivalence of categories is as follows. One can take the case of higher tangent bundles as exemplary and define k -dimensional cubes as families $(\varepsilon_I E_I)$ of vector spaces. A morphism $f: E \rightarrow E'$ of k -multilinear spaces consists then as before of maps

$$f^\nu: E_{\nu_1} \times \cdots \times E_{\nu_{\ell(\nu)}} \rightarrow E'_I$$

for $I \in \mathcal{P}_+^k$, $\nu \in \mathcal{P}(I)$, where it is understood that

$$f^\nu(\varepsilon_{\nu_1} v_{\nu_1}, \dots, \varepsilon_{\nu_{\ell(\nu)}} v_{\nu_{\ell(\nu)}}) = \underbrace{\varepsilon_{\nu_1} \cdots \varepsilon_{\nu_{\ell(\nu)}}}_{=\varepsilon_I} f^\nu(v_{\nu_1}, \dots, v_{\nu_{\ell(\nu)}}).$$

Because of the relations of the infinitesimal generators, this point of view also explains why one only considers partitions for the morphisms and why the order of the partitions can usually be disregarded.

We would like to substitute the generators ε_i with the odd generators λ_i . One immediately sees that the order of the partition now plays a role, as a change of signs might occur. However, as we have shown in Lemma B.1.8, in the case of purely even multilinear bundles a consistent choice can be made such that this substitution leads to well-defined bundles and morphisms. In general this is not the case. With the notation of Lemma B.1.8 one could define a new composition law

$$(g^- \circ f^-)^\nu := \sum_{\omega \preceq \nu} \text{sgn}(\sigma_{\omega|\nu}) \text{sgn}(\omega) \text{sgn}(\omega_1|\nu) \cdots \text{sgn}(\omega_{\ell(\omega)}|\nu) g^\omega(f^{\omega_1|\nu}, \dots, f^{\omega_{\ell(\omega)}|\nu}),$$

where $\sigma_{\omega|\nu} \in \mathfrak{S}_{|I|}$ is the permutation that reorders $(\nu_1|\cdots|\nu_\ell)$ to $(\nu_1^1|\cdots|\nu_{\ell(\omega)}^{\ell(\omega)})$. If all ν_i have even cardinality then $\text{sgn}(\sigma_{\omega|\nu}) = 1$ and we get the same definition as above. In general the formula does not appear to lead to natural manifold structures though there is one interesting case where it does: If only those f^ν , where ν contains at most one set of odd cardinality, are not zero, the same argument as before applies. This means that for a supermanifold \mathcal{M} of Batchelor type, at least \mathcal{M}_Λ^- would be well-defined. However, morphisms remain problematic.

B.2.1. The tangent bundle of a multilinear bundle

Let F be a multilinear bundle of degree k over M with typical fiber E . Assume that M is modelled on E_0 and let $\varphi: U \times E \rightarrow V \times E$ be a change of bundle charts. Then by definition

$$\varphi(x, (v_I)_{I \in \mathcal{P}_+^k}) = \varphi_0(x) + \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} b^\nu(x, v_\nu),$$

where $\varphi_0: U \rightarrow V$ is a diffeomorphism and $b^\nu(x, \bullet): E_{\nu_1} \times \cdots \times E_{\nu_{\ell(\nu)}} \rightarrow E_I$ are multilinear maps for $x \in U$, $\nu \in \mathcal{P}(I)$. For $y \in E_0$ and $(w_I)_{I \in \mathcal{P}_+^k} \in E$, we calculate

$$\begin{aligned} d\varphi((x, (v_I)_{I \in \mathcal{P}_+^k}), (y, (w_I)_{I \in \mathcal{P}_+^k})) = \\ d\varphi_0(x, y) + \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} d_1 b^\nu(x, y, v_\nu) + \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} \sum_{i=1}^{\ell(\nu)} b^\nu(x, \widehat{v}_\nu^i), \end{aligned}$$

where $\widehat{v}_\nu^i := (v_{\nu_1}, \dots, v_{\nu_{i-1}}, w_{\nu_i}, v_{\nu_{i+1}}, \dots, v_{\nu_{\ell(\nu)}}) \in E_{\underline{\nu}}$. The corresponding change of charts for the tangent bundle TF is given by

$$(\varphi, d\varphi): (U \times E_0) \times E^2 \rightarrow (V \times E_0) \times E^2.$$

For $I \in \mathcal{P}_+^k$ let $\text{pr}_1^I: E_I \times E_I \rightarrow E_I$ be the projection to the first and $\text{pr}_2^I: E_I \times E_I \rightarrow E_I$ be the projection to the second component. Then

$$\begin{aligned} (\varphi, d\varphi)((x, (v_I)_{I \in \mathcal{P}_+^k}), (y, (w_I)_{I \in \mathcal{P}_+^k})) = (\varphi_0(x), d\varphi_0(x, y)) + \\ \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} \left(\text{pr}_1^I(b^\nu(x, v_\nu)) + \text{pr}_2^I(d_1 b^\nu(x, y, v_\nu) + \sum_{i=1}^{\ell(\nu)} b^\nu(x, \widehat{v}_\nu^i)) \right) \end{aligned}$$

holds. Thus, TF can be seen as a multilinear bundle of degree k over TM with typical fiber $E \times E$. The exact same calculation shows that for a morphism of multilinear bundles $f: F \rightarrow F'$, the tangent map $Tf: TF \rightarrow TF'$ is also a morphism of multilinear bundles. We have thus shown:

Lemma B.2.11. *For each $k \in \mathbb{N}_0$, the tangent functor $T: \mathbf{Man} \rightarrow \mathbf{Man}$ restricts to a functor*

$$T: \mathbf{MBun}^{(k)} \rightarrow \mathbf{MBun}^{(k)}.$$

The functor $T: \mathbf{MBun}^{(k)} \rightarrow \mathbf{MBun}^{(k)}$ commutes with restrictions of bundles:

Lemma B.2.12. *Let $k \in \mathbb{N}_0$, $F \in \mathbf{MBun}^{(k)}$ and $P \subseteq \mathcal{P}_+^k$ as in Lemma/Definition B.2.4. Then $(TF)|_P \cong T(F|_P)$ holds as multilinear bundles. If $f: F \rightarrow F'$ is a morphism of multilinear bundles, then $(Tf)|_P = T(f|_P): T(F|_P) \rightarrow T(F'|_P)$ holds under the above identification.*

Proof. Let F have typical fiber E and let the base M of F be modelled on E_0 . Since each change of charts $\varphi_{\alpha\beta}: U_{\alpha\beta} \times E \rightarrow U_{\beta\alpha} \times E$ of F restricts to a map

$$\varphi_{\alpha\beta}|_P: U_{\alpha\beta} \times E|_P \rightarrow U_{\beta\alpha} \times E|_P,$$

we have that $d\varphi_{\alpha\beta}$ restricts to

$$d\varphi_{\alpha\beta}|_P = d(\varphi_{\alpha\beta}|_P): (U_{\alpha\beta} \times E_0) \times (E|_P \times E|_P) \rightarrow U_{\beta\alpha} \times E|_P.$$

It follows that $(\varphi_{\alpha\beta}, d\varphi_{\alpha\beta})|_P = (\varphi_{\alpha\beta}|_P, d\varphi_{\alpha\beta}|_P)$ holds. We can repeat the same argument for morphisms. \square

By using this lemma, we shall simply write $TF|_P$, resp. $Tf|_P$, for the respective restrictions in the sequel.

B.3. Inverse Limits of Multilinear Bundles

Lemma B.3.1. *Let $k \in \mathbb{N}_0$ and F be a multilinear bundle of degree k with typical fiber E and the bundle atlas $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha \times E: \alpha \in A\}$. For $n \leq k$, the projections*

$$(q_n^k)_\alpha: U_\alpha \times E \rightarrow U_\alpha \times E|_{\mathcal{P}_+^n}, \quad (x, (v_I)_{I \in \mathcal{P}_+^k}) \mapsto (x, (v_I)_{I \in \mathcal{P}_+^n})$$

define a smooth surjective morphism $q_n^k: F \rightarrow F|_{\mathcal{P}_+^n}$ with $\varphi_\alpha|_{\mathcal{P}_+^n} \circ q_n^k \circ \varphi_\alpha^{-1} = (q_n^k)_\alpha$.

Proof. We only need to show that q_n^k is well-defined, then smoothness and surjectivity follow immediately. Let $\alpha, \beta \in A$, $x \in U_{\alpha\beta}$ and $(v_I)_{I \in \mathcal{P}_+^k} \in E|_{\mathcal{P}_+^k}$. Then $\varphi_{\alpha\beta}(x, (v_I)_I) = \varphi_{\alpha\beta}|_{\mathcal{P}_+^n}(x, (v_I)_I)$ holds for the change of bundle charts $\varphi_{\alpha\beta}$. In particular, we have $\varphi_{\beta\alpha}|_{\mathcal{P}_+^n} \circ \varphi_{\alpha\beta}(x, (v_I)_I) = (x, (v_I)_I)$. It follows

$$\varphi_{\beta\alpha}|_{\mathcal{P}_+^n} \circ (q_n^k)_\beta \circ \varphi_{\alpha\beta} = (q_n^k)_\alpha$$

on $U_{\alpha\beta} \times E$. With this, the lemma follows from the local description of smooth maps between manifolds. \square

Definition B.3.2. Let $(F_k)_{k \in \mathbb{N}_0}$ be a family of multilinear bundles F_k of degree k with typical fiber $E^{(k)}$ and respective bundle atlas $\{\varphi_\alpha^{(k)}: V_\alpha^{(k)} \rightarrow U_\alpha \times E^{(k)}: \alpha \in A\}$ such that for all $n \leq k$, we have $E^{(k)}|_{\mathcal{P}_+^n} = E^{(n)}$ and $\varphi_\alpha^{(k)}|_{\mathcal{P}_+^n} = \varphi_\alpha^{(n)}$ with the identifications from Definition B.1.4 and Lemma/Definition B.2.4. In particular $F_k|_{\mathcal{P}_+^n} = F_n$ and all F_k are bundles over F_0 . Then the family

$$((F_k)_{k \in \mathbb{N}_0}, (q_n^k)_{n \leq k}),$$

where q_n^k is defined as in Lemma B.3.1, is called an *inverse system of multilinear bundles*. We shall simply write (F_k, q_n^k) in this situation. We call $\{\varphi_k^\alpha : k \in \mathbb{N}_0, \alpha \in A\}$ an *adapted atlas* of (F_k, q_n^k) . Two adapted atlases of (F_k, q_n^k) are *equivalent* if they lead to equivalent atlases for each F_k .

Let (F_k, q_n^k) and $(F'_k, q_n'^k)$ be inverse systems of multilinear bundles. A *morphism of inverse systems of multilinear bundles* is a family $(f_k)_{k \in \mathbb{N}_0}$ of morphisms $f_k : F_k \rightarrow F'_k$ of multilinear bundles such that $q_n'^k \circ f_k = f_n \circ q_n^k$ for all $n \leq k$. We write $(f_k)_{k \in \mathbb{N}_0} : (F_k, q_n^k) \rightarrow (F'_k, q_n'^k)$.

Proposition B.3.3. *The inverse system of multilinear bundles with their morphisms are a subcategory of the category of inverse systems of topological spaces and their morphisms. Let (F_k, q_n^k) be an inverse system of multilinear bundles and $\{\varphi_\alpha^{(k)} : k \in \mathbb{N}_0, \alpha \in A\}$ be an adapted atlas of (F_k, q_n^k) . Then $\{\varprojlim_k \varphi_\alpha^{(k)} : \alpha \in A\}$ is an atlas of $\varprojlim_k F_k$. Equivalent adapted atlases of (F_k, q_n^k) lead to equivalent atlases of $\varprojlim_k F_k$. With this manifold structure, $\varprojlim_k f_k : \varprojlim_k F_k \rightarrow \varprojlim_k F'_k$ is smooth for morphisms $(f_k)_{k \in \mathbb{N}_0} : (F_k, q_n^k) \rightarrow (F'_k, q_n'^k)$ of inverse systems of multilinear bundles.*

Proof. Let F_k have the typical fiber $E^{(k)}$ and let F_0 be modelled on E_\emptyset . It is clear from the local definition in Lemma B.3.1 that $q_m^n \circ q_n^k = q_m^k$ for all $m \leq n \leq k$. It then follows from the definition that inverse systems of multilinear bundles, resp. morphisms thereof, are inverse systems, resp. morphisms thereof, in the usual sense. Clearly, the composition of two morphisms of inverse systems of multilinear bundles is again a morphism of this type. Let $\{\varphi_\alpha^{(k)} : V_\alpha^{(k)} \rightarrow U_\alpha \times E^{(k)} : k \in \mathbb{N}_0, \alpha \in A\}$ be an adapted atlas of (F_k, q_n^k) . By definition $E_I^{(k)} = E_I^{(n)}$ holds for all $n \leq k$ and $I \in \mathcal{P}_+^n$. Thus, for each $\alpha \in A$, the local projection

$$(q_n^k)_\alpha : U^\alpha \times \prod_{I \in \mathcal{P}_+^k} E_I^{(k)} \rightarrow U^\alpha \times \prod_{I \in \mathcal{P}_+^n} E_I^{(n)}$$

is just the usual projection and $\varprojlim_k (U_\alpha \times E^{(k)}) = U_\alpha \times \prod_{I \subseteq \mathbb{N}, 0 < |I| < \infty} E_I^{(\max(I))}$, which is an open subset of the locally convex space $E_\emptyset \times \prod_{I \subseteq \mathbb{N}, 0 < |I| < \infty} E_I^{(\max(I))}$. Also by definition, $q_n^k \circ \varphi_\alpha^{(k)} = \varphi_\alpha^{(n)} \circ (q_n^k)_\alpha$ holds for all $\alpha \in A, n \leq k$ and therefore $\varprojlim_k \varphi_\alpha^{(k)} : \varprojlim_k (U_\alpha \times E^{(k)}) \rightarrow \varprojlim_k F_k$ is well-defined and a homeomorphism because each $\varphi_\alpha^{(k)}$ is so. We have already seen in Lemma B.3.1 that the changes of charts $\varphi_{\alpha\beta}^{(k)} : U_{\alpha\beta} \times E^{(k)} \rightarrow U_\beta \times E^{(k)}$ define a morphism of inverse systems of multilinear bundles and that we have

$$\varprojlim_k \varphi_\beta^{(k)} \circ \varprojlim_k (\varphi_\alpha^{(k)})^{-1}|_{U_{\alpha\beta} \times E^{(k)}} = \varprojlim_k \varphi_{\alpha\beta}^{(k)}.$$

Clearly, $\varprojlim_k (U_{\alpha\beta} \times E^{(k)})$ is an open subset of $\varprojlim_k (U_\alpha \times E^{(k)})$ and because $\varphi_{\alpha\beta}^{(k)}$ is smooth for each $k \in \mathbb{N}_0$, so is $\varprojlim_k \varphi_{\alpha\beta}^{(k)}$ by Lemma A.1.3.

It remains to be seen that $\varprojlim_k F_k$ is covered by $\{\varprojlim_k \varphi_\alpha^{(k)} : \alpha \in A\}$. Because the index set \mathbb{N}_0 is countable and the maps q_n^k are all surjective, the projections $q_n : \varprojlim_k F_k \rightarrow F_n$ are also surjective (see for example [16, Exercise 7.6.10, p. 269]).

For each $n \in \mathbb{N}_0$ and $\alpha \in A$, we have $(q_0^n)^{-1}((\varphi_\alpha^{(0)})^{-1}(U^\alpha)) = (\varphi_\alpha^{(n)})^{-1}(U_\alpha \times E^{(n)})$ which implies

$$q_0^{-1}((\varphi_\alpha^{(0)})^{-1}(U_\alpha)) = \varprojlim_n (\varphi_\alpha^{(n)})^{-1}(\varprojlim_n (U_\alpha \times E^{(n)})).$$

Since the sets $(\varphi_0^\alpha)^{-1}(U^\alpha)$ cover F_0 , the result follows. The change of charts with an adapted atlas leads to smooth maps in the same way. Because $q_0: \varprojlim_k F_k \rightarrow F_0$ is surjective and for each $x \in F_0$, we have that $q_0^{-1}(\{x\})$ is homeomorphic to the Hausdorff space $\varprojlim_k E^{(k)}$, it follows that $\varprojlim_k F_k$ is Hausdorff.

Now, let $(f_k)_{k \in \mathbb{N}_0}: (F_k, q_n^k) \rightarrow (F'_k, q_n'^k)$ be a morphism of inverse systems of multilinear bundles and $\{\psi_\beta^{(k)}: V'_\beta \rightarrow U'_\beta \times E'^{(k)}: k \in \mathbb{N}_0, \beta \in B\}$ be an adapted atlas of $(F'_k, q_n'^k)$. We define

$$f_k^{\alpha\beta} := \psi_\beta^{(k)} \circ f_k \circ (\varphi_\alpha^{(k)})^{-1}|_{\varphi_\alpha^{(k)} \circ f_k^{-1}(V_\beta'^{(k)})}$$

for $\beta \in B$ and $\alpha \in A$. Because f_k is a morphism of multilinear bundles, we have $\varphi_\alpha^{(k)} \circ f_k^{-1}(V_\beta'^{(k)}) = (\varphi_\alpha^{(0)} \circ f_0^{-1}(V_\beta'^{(0)})) \times E^{(k)}$ for all $k \in \mathbb{N}_0$. By definition,

$$(q_n^k)_\beta \circ f_k^{\alpha\beta} = f_n^{\alpha\beta} \circ (q_n^k)_\alpha$$

holds and thus

$$\varprojlim_k \psi_\beta^{(k)} \circ \varprojlim_k f_k \circ (\varprojlim_k \varphi_\alpha^{(k)})^{-1} = \varprojlim_k f_k^{\alpha\beta}$$

holds on $\varprojlim_k ((\varphi_\alpha^{(0)} \circ f_0^{-1}(V_\beta'^{(0)})) \times E^{(k)})$ for all $\alpha \in A, \beta \in B$. These maps are smooth by the same argument as above. \square

We denote by $\mathbf{MBun}^{(\infty)}$ the category of all manifolds arising as such a limit (together with an equivalence class of atlases that come from limits of equivalent adapted atlases) and morphisms that come from a respective limit of morphisms. Taking the inverse limit gives us a functor from the category of inverse systems of topological spaces to the category of topological spaces that respects products. By the above, if we restrict this functor to the subcategory of inverse systems of multilinear bundles (and the respective morphisms), we get a functor into the category $\mathbf{MBun}^{(\infty)}$. We also get a functor to \mathbf{Man} along the forgetful functor.

Example B.3.4. Let M be a manifold modelled on the locally convex space E with the atlas $\{\varphi_\alpha: V_\alpha \rightarrow U_\alpha: \alpha \in A\}$. For $n \in \mathbb{N}_0$ we set $\pi_n^n := \text{id}_{T^n M}$ and we have the natural projection $\pi_n^{n+1}: T^{n+1}M \rightarrow T^n M$. For $n < k$, we define inductively $\pi_n^k := \pi_{k-1}^k \circ \dots \circ \pi_n^{n+1}: T^k M \rightarrow T^n M$. Continuing from Example B.2.7(b), one easily sees that $(T^k M, \pi_n^k)$ is an inductive system of multilinear bundles with the adapted atlas $\{T^k \varphi_\alpha: T^k V_\alpha \rightarrow U_\alpha \times \prod_{I \in \mathcal{P}_+^k} \varepsilon_I E: \alpha \in A, k \in \mathbb{N}_0\}$. It follows from Proposition B.3.3 that $T^\infty M := \varprojlim_k T^k M$ is a manifold with the atlas $\{\varprojlim_k \varphi_\alpha: \alpha \in A\}$. For any smooth map $f: M \rightarrow N$ between manifolds, one obviously has $\pi_n^k \circ T^k f = T^n f \circ \pi_n^k$ if $\pi_n^k: T^k N \rightarrow T^n N$ denotes the projection. Thus $(T^k f)_{k \in \mathbb{N}_0}$ is a morphism of inductive systems of multilinear bundles

and $T^\infty f := \varprojlim_k T^k f: T^\infty M \rightarrow T^\infty N$ is smooth. Moreover, for any Lie group (G, μ, i, e) , we get a Lie group $(T^\infty G, T^\infty \mu, T^\infty i, e)$ because the inverse limit preserves products.

Lemma B.3.5. *If (F_k, q_n^k) is an inverse system of multilinear bundles, then so is (TF_k, Tq_n^k) and*

$$T \varprojlim_k ((F_k, q_n^k)) \cong \varprojlim_k (TF_k, Tq_n^k)$$

holds as manifolds. If $(f_k)_{k \in \mathbb{N}_0}: (F_k, q_n^k) \rightarrow (F'_k, q_n'^k)$ is a morphism of inverse systems of multilinear bundles, then so is $(Tf_k)_{k \in \mathbb{N}_0}$ and we have

$$\varprojlim_k Tf_k = T \varprojlim_k f_k: T \varprojlim_k ((F_k, q_n^k)) \rightarrow T \varprojlim_k ((F'_k, q_n'^k))$$

under the above identification. In particular, we may consider $T \varprojlim_k ((F_k, q_n^k))$ as an object in $\mathbf{MBun}^{(\infty)}$ in a natural way.

Proof. One easily sees from the local description of q_n^k in Lemma B.3.1 that Tq_n^k is the projection $TF \rightarrow TF|_{\mathcal{P}_+^n}$. If $\{\varphi_\alpha^{(k)}: k \in \mathbb{N}_0, \alpha \in A\}$ is an adapted atlas of F , it follows by functoriality of the tangent functor that $\{T\varphi_\alpha^{(k)}: k \in \mathbb{N}_0, \alpha \in A\}$ is an adapted atlas of (TF_k, Tq_n^k) . For the same reason $(Tf_k)_{k \in \mathbb{N}_0}: (TF_k, Tq_n^k) \rightarrow (TF'_k, Tq_n'^k)$ is again a morphism. By Lemma A.1.3 $d \varprojlim_k \varphi_{\alpha\beta}^{(k)} = \varprojlim_k d\varphi_{\alpha\beta}^{(k)}$ holds for any change of charts $\varphi_{\alpha\beta}^{(k)}$. Thus, the change of charts of $T \varprojlim_k ((F_k, q_n^k))$ and $\varprojlim_k (TF_k, Tq_n^k)$ is the same. The same argument works for morphisms. \square

In other words, taking the inverse limit commutes with the tangent functor.

C. Polynomial Groups

C.1. Basic Definitions

Polynomial groups were introduced in [10] to construct a suitable exponential map in the context of groups that have a multilinear structure. They are closely related to formal groups (see [13] or [51]). In characteristic zero, the categories of Lie algebras and formal groups are equivalent (see [51, p.112]) and as we will see, the category of polynomial groups corresponds to the category of nilpotent Lie algebras. In our discussion, we follow [10, PG, p.192ff.]. While generalizations to other base fields are possible, we only consider the real case to stay consistent with the rest of this work.

Definition C.1.1. Let E and F be (real) vector spaces. A *homogeneous polynomial map of degree $k \in \mathbb{N}$* is a map $p_k: E \rightarrow F$, such that for all $v \in E$, we have $p_k(v) = b_k(v, \dots, v)$ for an \mathbb{R} - k -multilinear map $b_k: E^k \rightarrow F$. A homogeneous polynomial map of degree 0 is a constant map $p_0: E \rightarrow F$. A *polynomial map* $p: E \rightarrow F$ is a finite sum $p = \sum p_k$ of homogeneous polynomial maps. The *degree of p* is the largest index such that p_k is non-zero.

Clearly, the composition of two polynomial maps is again a polynomial map. In characteristic zero one can recover the homogeneous parts of a polynomial by formal differential calculus (compare [10, PG, p.192]). In particular, if E and F are locally convex spaces then p is smooth if and only if all b_k are continuous (see [23]).

Let E, E' and F be \mathbb{R} -vector spaces and let $q: E \times E' \rightarrow F$ be a polynomial map of degree k . We write

$$q(x, y) = \sum_{r+s \leq k} q_{r,s}(x, y),$$

for the decomposition of $q(x, y)$ into parts $q_{r,s}(x, y)$ which are homogeneous of degree r in x and homogeneous of degree s in y .

Definition C.1.2. A *polynomial group of degree at most n* is a vector space E that has a group structure $(E, m, i, 0)$ such that $m: E \times E \rightarrow E$ and $i: E \rightarrow E$ are polynomial maps and all iterated products

$$m^{(k)}: E^k \rightarrow E, \quad (x_1, \dots, x_k) \mapsto x_1 \cdots x_k,$$

for $k \in \mathbb{N}$, are polynomial maps of degree at most n .

Note that the iterated products are automatically polynomial because so is $m = m^{(2)}$ and the composition of polynomial maps is again polynomial.

Example C.1.3.

- (a) Let E be a k -multilinear space. If E is a group such that the multiplication and inversion are morphisms of k -multilinear spaces, then E is a polynomial group of degree at most k . Indeed, that the maps are polynomials of degree at most k follows from the definition of morphisms of k -multilinear spaces and the iterated product maps are also morphisms of k -multilinear spaces.
- (b) Let (G, m, i, e) be a Lie group modelled on E_0 and $\varphi: U \rightarrow V$ be a chart of G such that $e \in U$ and $\varphi(e) = 0$. Then the vector space structure given by $T^k\varphi: (T^kG)_e \rightarrow (T^kE)_0$ turns $(T^kG)_e$ into a polynomial group of degree at most k for each $k \in \mathbb{N}$. This follows because by Example B.2.7(b) the multiplication (as well as the iterated product maps) and the inversion are morphisms of k -multilinear spaces in the chart $T^k\varphi$. This can also be done without a chart using the left (or right) trivialization from Section E.1 (compare [10, PG.2(2), p.193]).

C.2. The Lie Bracket and the Exponential Map

Now, let $(E, m, i, 0)$ be a polynomial group of degree n . By decomposing the iterated multiplication maps into multihomogeneous parts, we write

$$m^{(j)} = \sum_{\substack{p_1, \dots, p_j \geq 0 \\ p_1 + \dots + p_j \leq k}} m_{p_1, \dots, p_j}^{(j)}.$$

In other words

$$m_{p_1, \dots, p_j}^{(j)}(x_1, \dots, x_j) = b_l(\underbrace{x_1, \dots, x_1}_{p_1 \text{ times}}, \dots, \underbrace{x_j, \dots, x_j}_{p_j \text{ times}})$$

for a multilinear map $b_l: E^l \rightarrow E$ and $l = p_1 + \dots + p_j$. In this notation $m_{1,1}^{(2)} = m_{1,1}$ has a special meaning: One can show that

$$[x, y] := m_{1,1}(x, y) - m_{1,1}(y, x)$$

defines a nilpotent Lie bracket on E (see [10, PG.3, p.193f.] and [13, Ch. III, Par. 5, Prop. 1, p.299]). We denote E equipped with this Lie algebra structure by $L(E)$. To get some intuition, consider the following. It is quite obvious that $m_{0,1}(x, y) = y = m_{1,0}(y, x)$ and $i_1(x) = -x$ for all $x, y \in E$. Let now

$$\alpha: E \times E \rightarrow E, \quad (x, y) \mapsto m(x, m(y, i(x))) = xyx^{-1}$$

be the conjugation. The map α is clearly polynomial and we calculate

$$\alpha_{1,1}(x, y) = m_{1,1}(x, m_{1,0}(y, i_0(x))) + m_{0,1}(x, m_{1,1}(y, i_1(x)))$$

$$= m_{1,1}(x, y) + m_{1,1}(y, -x) = [x, y]$$

for $x, y \in E$. Furthermore, we define

$$\psi_j: E \rightarrow E, \quad x \mapsto \sum_{p_1, \dots, p_j > 0} m_{p_1, \dots, p_j}^{(j)}(x, \dots, x)$$

and denote the homogeneous part of degree j of ψ_j by $\psi_{j,j}$. Note that all the above maps are polynomials of degree at most n because $m^{(j)}$ is so. By using $\psi_{j,j}$, one obtains the following crucial theorem.

Theorem C.2.1 ([10, Theorem PG.6, p.196]). *Let E be a polynomial group of degree $n \in \mathbb{N}$. Then there exists a unique exponential map, given by*

$$\exp_E: L(E) \rightarrow E, \quad x \mapsto \sum_{j=1}^n \frac{1}{j!} \psi_{j,j}(x),$$

with the following properties:

- (a) the linear term of \exp_E is the identity,
- (b) for all $X \in E$ and $s, t \in \mathbb{R}$, one has $\exp_E((t+s)X) = \exp_E(tX) \cdot \exp_E(sX)$,
- (c) \exp_E is bijective and the inverse function is given by the polynomial map

$$\log_E: E \rightarrow L(E), \quad x \mapsto \sum_{j=1}^n \frac{(-1)^{j-1}}{j} \psi_j(x).$$

This exponential map has the same relation to one-parameter groups as the exponential map of a Lie group. For $X \in E$, there exists exactly one polynomial morphism of groups $\gamma_X: \mathbb{R} \rightarrow E$ such that $(\gamma_X)_1(1) = X$ and we have $\exp_E(X) = \gamma_X(1)$ (see the proof of Theorem C.2.1). From this, one easily deduces the functoriality of the exponential map like in the Lie group case.

Lemma C.2.2. *Let E and F be polynomial groups and $\varphi: E \rightarrow F$ be a polynomial morphism of polynomial groups. Then the linear part $L(\varphi)$ of φ is a Lie algebra morphism of $L(E)$ to $L(F)$ and the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \exp_E \uparrow & & \uparrow \exp_F \\ L(E) & \xrightarrow{L(\varphi)} & L(F) \end{array}$$

commutes.

Proof. Let m and m' be the multiplication of E and F . We directly calculate

$$\varphi_1 \circ m_{1,1} = (\varphi \circ m)_{1,1} = (m' \circ (\varphi \times \varphi))_{1,1} = m'_{1,1}(\varphi_1 \times \varphi_1),$$

which shows that $L(\varphi) = \varphi_1$ is a morphism of Lie algebras. Let $\gamma_X: \mathbb{R} \rightarrow E$ be the unique polynomial morphism of groups such that $(\gamma_X)_1(1) = X \in E$. Then $\varphi \circ \gamma_X$ is a polynomial morphism of groups and we have $(\varphi \circ \gamma_X)_1(1) = \varphi_1((\gamma_X)_1(1)) = L(\varphi)(X)$. Thus, $\varphi \circ \gamma_X = \gamma_{L(\varphi)(X)}$ holds and $\varphi \circ \exp_E(X) = \exp_F(L(\varphi)(X))$ follows. \square

Moreover, morphisms of Lie algebras uniquely determine morphisms of the respective polynomial groups.

Lemma C.2.3. *Let E and F be polynomial groups and $\phi: L(E) \rightarrow L(F)$ be a morphism of Lie algebras. Then there exists a unique polynomial morphism of groups $\Phi: E \rightarrow F$ such that $L(\Phi) = \phi$.*

Proof. Let $(L(E), *)$ be the nilpotent Lie algebra $L(E)$ equipped with the group structure given by the BCH multiplication. By [10, Theorem PG.8, p.198], $\exp_E: (L(E), *) \rightarrow E$ is an isomorphism of polynomial groups. Because ϕ is a morphism of Lie algebras, $\phi: (L(E), *) \rightarrow (L(F), *)$ is a polynomial morphism of groups. Then $\exp_F \circ \phi \circ \exp_E^{-1}: E \rightarrow F$ is a polynomial morphism of groups as well. We have $L(\exp_F \circ \phi \circ \exp_E^{-1}) = L(\phi) = \phi$ because ϕ is linear. Conversely, let $\Phi: E \rightarrow F$ be a polynomial morphism of groups with $L(\Phi) = \phi$. Then, we have

$$\exp_F^{-1} \circ \Phi \circ \exp_E(X) = \exp_F^{-1}(\gamma_{L(\Phi)(X)}(1)) = L(\Phi)(X) = \phi(X)$$

for $X \in E$. Uniqueness follows since \exp_E and \exp_F are isomorphisms. \square

Since every nilpotent Lie algebra equipped with the BCH multiplication is a polynomial group (see [10, PG.2(3), p.192]), the above lemmas together with [10, Theorem PG.8, p.198] implies that the categories of nilpotent Lie algebras and polynomial groups are equivalent (in characteristic zero).

Lemma C.2.4. *Let E and F be polynomial groups and $\alpha^\vee: E \rightarrow \text{Aut}(F)$ be a morphism of groups such that the action $\alpha: E \times F \rightarrow F$, $(x, y) \mapsto \alpha^\vee(x)(y)$ is a polynomial map and such that there exists $n \in \mathbb{N}$ so that all iterated actions*

$$\alpha^{(k)}: E^k \times F \rightarrow F, \quad (x_1, \dots, x_k, y) \mapsto \alpha(x_1, \alpha(x_2, \dots, \alpha(x_k, y) \dots))$$

are polynomials of degree at most n . Then the semidirect product $F \rtimes_\alpha E$ is a polynomial group. If $[\cdot, \cdot]_E$, $[\cdot, \cdot]_F$ and $[\cdot, \cdot]_{F \rtimes_\alpha E}$ are the Lie brackets of E , F and $F \rtimes_\alpha E$ respectively, we have

$$[(x, y), (x', y')]_{F \rtimes_\alpha E} = (\alpha_{1,1}(y, x') - \alpha_{1,1}(y', x) + [x, x']_F, [y, y']_E)$$

for $(x, y), (x', y') \in F \times E$.

Proof. That the semidirect product is a polynomial group is obvious. Let m^E be the multiplication of E , m^F the multiplication of F and m be the multiplication in $F \rtimes_\alpha E$, i.e., we have $m((x, y), (x', y')) = (x\alpha(y, x'), yy')$ for $(x, y), (x', y') \in F \times E$. With this, we calculate

$$m_{1,1}((x, y), (x', y')) = (m_{0,1}^F(x, \alpha_{1,1}(y, x')) + m_{1,1}^F(x, \alpha_{0,1}(y, x')), m_{1,1}^E(y, y'))$$

$$= (\alpha_{1,1}(y, x') + m_{1,1}^F(x, x'), m_{1,1}^E(y, y'))$$

and the assertion follows. \square

Next, we consider polynomial groups that, in addition, are Lie groups.

Lemma C.2.5. *A locally convex space E that is a polynomial group with the multiplication m is a Lie group with respect to the global chart \exp_E if and only if the Lie bracket is continuous, or equivalently, if $m_{1,1}: E \times E \rightarrow E$ is continuous.*

Proof. The Lie bracket is bilinear and thus smooth if it is continuous. That this is equivalent to the continuity of $m_{1,1}$ follows directly from the definition. The multiplication in the chart \exp_E is the BCH multiplication, which consists of a finite sum of iterated Lie brackets and thus is smooth if and only if the Lie bracket is continuous. The inversion is just the multiplication by -1 and thus smooth. \square

Lemma C.2.6. *Let E be a locally convex vector space that is a polynomial Lie group (with the same vector space structure). Then \exp_E is a diffeomorphism and is the exponential map of E . Moreover, $(L(E), *)$ is a Lie group and $\exp_E: (L(E), *) \rightarrow E$ is a polynomial isomorphism of polynomial Lie groups.*

Proof. By Theorem C.2.1, smoothness of \exp_E and \exp_E^{-1} follows from the smoothness of the iterated multiplications. Likewise, it follows that the Lie bracket is smooth and thus $(L(E), *)$ is a Lie group. We know from [10, Theorem PG.8, p.198] that \exp_E is a polynomial isomorphism of groups. Let $\gamma: \mathbb{R} \rightarrow E$ be a one-parameter group. Then γ is completely determined by $\gamma'(0) = \gamma_1(1) = X \in E$ and it follows $\gamma(t) = \exp_E(tX)$. Hence, \exp_E is the exponential map of the Lie group E . \square

Lemma C.2.7. *Let E be a polynomial group with the Lie bracket $[\cdot, \cdot]$ and let $X, Y \in E$. Then, we have*

$$\exp_E^{-1}(\exp_E(X) \cdot \exp_E(Y) \cdot \exp_E(Y)^{-1}) = Y + \sum_{n \in \mathbb{N}} \frac{1}{n!} \text{ad}_X^n(Y),$$

where $\text{ad}_X^n(Y) := \underbrace{[X, [X, \dots [X, Y] \dots]]}_{n \text{ times}}.$

Proof. First, note that the series terminates because the Lie bracket of E is nilpotent. Let $\langle X, Y \rangle$ denote the Lie algebra generated by X and Y . By nilpotency of the Lie bracket, $\langle X, Y \rangle$ is a finite-dimensional nilpotent Lie subalgebra of $L(E)$. Thus, it carries a unique topology turning it into a topological Lie algebra and equipped with the BCH multiplication, it becomes a finite-dimensional polynomial Lie group. Then $\exp_E(\langle X, Y \rangle)$ is a subgroup of E that has the structure of a finite-dimensional Lie group whose exponential map is simply the restricted exponential map of E . For finite-dimensional Lie groups, the formula is well-known (see for example [28, Formula (9.11), p. 307]). \square

C.3. Pro-Polynomial Groups

For $k \in \mathbb{N}$, let $(E_k, {}_k m, {}_k i, e)$ be a polynomial group of degree at most k and for each $l \leq k$, let $\pi_l^k: E_k \rightarrow E_l$ be polynomial morphisms of groups such that $((E_k)_{k \in \mathbb{N}}, (\pi_l^k)_{l \leq k})$ is an inverse system of groups. The category of *pro-polynomial groups* consists of objects of the form $E := \varprojlim_k E_k$ (in the category of groups) and group morphisms $\varprojlim_k \varphi_k: \varprojlim_k E_k \rightarrow \varprojlim_k (F_k)_{k \in \mathbb{N}}, (\pi_l^k)_{l \leq k}$ for polynomial morphisms of groups $\varphi_k: E_k \rightarrow F_k$ such that $\varphi_l \circ \pi_l^k = \pi_l^k \circ \varphi_k$ for $l \leq k$. Moreover, we define $L(\varprojlim_k E_k) := \varprojlim_k L(E_k)$ and $L(\varphi) := \varprojlim_k L(\varphi_k)$, using that every $L(\varphi_k)$ is a polynomial morphism of polynomial groups. Finally, we let $\exp_E := \varprojlim_k \exp_{E_k}$, which is well-defined by Lemma C.2.2. Note that \exp_E is bijective with the inverse $\exp_E^{-1} := \varprojlim_k \exp_{E_k}^{-1}$ because taking the inverse limit is functorial. We define the Lie bracket of E as $[\cdot, \cdot] := \varprojlim_k [\cdot, \cdot]_k$, where $[\cdot, \cdot]_k$ denotes the Lie bracket of E_k and the limit is taken in the category of Lie algebras. By definition, we have

$$\begin{aligned} [(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}] &= \varprojlim_k ({}_k m_{1,1}(x_k, y_k) - {}_k m_{1,1}(y_k, x_k)) \\ &= \varprojlim_k {}_k m_{1,1}((x_k, y_k)_{k \in \mathbb{N}}) - \varprojlim_k {}_k m_{1,1}((y_k, x_k)_{k \in \mathbb{N}}). \end{aligned}$$

The idea for considering pro-polynomial groups comes from [10, PG.9, p.198].

Lemma C.3.1. *Let $E := \varprojlim_k E_k$ be a pro-polynomial group such that all E_k are locally convex spaces and the projections π_l^k are continuous. Then E is a Lie group and the diffeomorphism $\varprojlim_k \exp_{E_k}$ is the exponential map of E .*

Proof. Because all E_k are nilpotent Lie groups isomorphic to $L(E_k)$ equipped with the BCH multiplication, E is a Lie group by Lemma A.1.3 (compare [41, Example IV.1.13, p.364]). Let $\pi_k: E \rightarrow E_k$ be the canonical projection. If $\gamma: \mathbb{R} \rightarrow E$ is a one-parameter group, then $\gamma_k := \pi_k \circ \gamma$ is a one-parameter group of E_k and $\varprojlim_k \gamma_k = \gamma$ holds. By Lemma A.1.3, we have $\gamma'(0) = \varprojlim_k \gamma'_k(0)$ and therefore Lemma C.2.6 implies that \exp_E is the exponential map of E . \square

Lemma C.3.2. *Let E and F be pro-polynomial groups and let $\varphi: E \rightarrow F$ be a morphism of pro-polynomial groups. Then, we have $\varphi \circ \exp_E = \exp_F \circ L(\varphi)$.*

Proof. This follows immediately from Lemma C.2.2 because taking the inverse limit is functorial. \square

Lemma C.3.3. *Let $E = \varprojlim_k E_k$ be a pro-polynomial group with the Lie bracket $[\cdot, \cdot] = \varprojlim_k [\cdot, \cdot]_k$ and $X = (X_k)_{k \in \mathbb{N}}, Y = (Y_k)_{k \in \mathbb{N}} \in E$. Then, we have*

$$\begin{aligned} \exp_E^{-1}(\exp_E(X) \cdot \exp_E(Y) \cdot \exp_E(X)^{-1}) &= \\ &= \left(Y_k + \sum_{n=1}^k \frac{1}{n!} \underbrace{[X_k [X_k, \dots [X_k, Y_k]_k \dots]_k]_k}_{n \text{ times}} \right)_{k \in \mathbb{N}}. \end{aligned}$$

Proof. This follows immediately from Lemma C.2.7. \square

D. The Lie Group of Vector Bundle Automorphisms

To turn the automorphisms $\text{Aut}(\mathcal{M})$ of a supermanifold \mathcal{M} into a Lie group, it is crucial to have a Lie group structure on the vector bundle automorphisms $\text{Aut}(\mathcal{M}^{(1)})$. We achieve this for $\text{Aut}_c(\mathcal{M}^{(1)})$ if \mathcal{M} is a Banach supermanifold with σ -compact finite-dimensional base. The general idea for this is as follows.

Based on [56] it was shown in [50] that the group of compactly supported automorphisms of a principal bundle (with a Banach Lie group as structure group) can be turned into a Lie group. It is well-known that finite-dimensional vector bundles are associated to their frame bundle. In our situation, where the typical fiber is a Banach space, the same construction works. We then transfer the Lie group structure of the group of compactly supported frame bundle automorphisms to the group of compactly supported vector bundle automorphisms. We need additional calculations that do not follow from [50], such as the smoothness of the evaluation map and an exact description of the Lie algebra, and therefore, we cannot avoid to recall some details from [50].

For this section, we fix a vector bundle $\pi: F \rightarrow M$ with the Banach space E_1 as typical fiber, where M is a finite-dimensional σ -compact manifold modelled on E_0 . We denote by $\text{Aut}(F)$ the group of vector bundle automorphisms of F , i.e., diffeomorphisms $f: F \rightarrow F$ such that there exists a diffeomorphism $\varphi: M \rightarrow M$ with

$$\begin{array}{ccc} F & \xrightarrow{f} & F \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\varphi} & M. \end{array}$$

Mapping an automorphism f to its diffeomorphism φ on the base defines a projection $q: \text{Aut}(F) \rightarrow \text{Diff}(M)$ to the diffeomorphism group $\text{Diff}(M)$ of M . This projection is obviously a morphism of groups. We define the *gauge group of F* as $\text{Gau}(F) := \ker(q)$. Furthermore, we define the group of *compactly supported automorphisms of F* by

$$\text{Aut}_c(F) := \{f \in \text{Aut}(F) : \exists K \subseteq M \text{ compact with } f|_{\pi^{-1}(M \setminus K)} = \text{id}_{F|_{\pi^{-1}(M \setminus K)}}\}.$$

Obviously, $\text{Aut}_c(F)$ is a subgroup of $\text{Aut}(F)$ and we have the exact sequence of groups

$$\text{Gau}_c(F) \hookrightarrow \text{Aut}_c(F) \xrightarrow{q} \text{Diff}_c(M),$$

where $\text{Gau}_c(F) := \text{Gau}(F) \cap \text{Aut}_c(F)$ and

$$\text{Diff}_c(M) := \{f \in \text{Diff}(M) : \exists K \subseteq M \text{ compact with } f|_{M \setminus K} = \text{id}_{M \setminus K}\}$$

is the group of diffeomorphisms with compact support .

For the remainder of this section, we fix locally finite open covers $\mathfrak{U} := \{U_i \subseteq M : i \in \mathbb{N}\}$ and $\mathfrak{V} := \{V_i \subseteq M : i \in \mathbb{N}\}$ of M such that for all $i \in \mathbb{N}$, we have that

- U_i is relatively compact,
- \mathfrak{V} is a refinement of \mathfrak{U} such that $\overline{V_i} \subseteq U_i$,
- $\overline{U_i}$ and $\overline{V_i}$ are submanifolds with boundary of M and
- there exists a smooth trivialization $\tau_i : \overline{U_i} \times E_1 \rightarrow F$.

Additionally, one can assume that around each $\overline{U_i}$ a chart can be defined. For the existence of such covers, we refer to [50]. By abuse of notation, we will also write τ_i for the trivializations restricted to U_i , $\overline{V_i}$ and V_i . For $i, j \in \mathbb{N}$, we have smooth transition functions $k_{ij} : U_i \cap U_j \rightarrow \text{Gl}_{E_1}$ defined by $\tau_i^{-1}(\tau_j(x, v)) = (x, k_{ij}(x).v)$ for $x \in U_i \cap U_j$, $v \in E_1$. Here $\text{Gl}_{E_1} \subseteq \mathcal{L}(E_1; E_1)_b$ denotes the group of continuous automorphisms of E_1 . It is well-known that Gl_{E_1} is a Banach Lie group modelled on $\mathfrak{gl}_{E_1} := \mathcal{L}(E_1; E_1)_b$, whose exponential map $\exp_{\text{Gl}_{E_1}} : \mathfrak{gl}_{E_1} \rightarrow \text{Gl}_{E_1}$ is a diffeomorphism around a zero-neighborhood (see [13, III, §1.1, p.213]).

The frame bundle

We define the frame bundle

$$\text{Fr}(F) := \bigcup_{x \in M} \text{Iso}(E_1; F_x),$$

where $\text{Iso}(E_1; F_x) \subseteq \mathcal{L}(E_1; F_x)_b$ is the Banach space of continuous vector space isomorphisms between E_1 and $F_x = \pi^{-1}(\{x\})$. Together with the obvious projection $\text{Fr}(F) \rightarrow M$, this defines a principal Gl_{E_1} -bundle with the action

$$\text{Fr}(F) \times \text{Gl}_{E_1} \rightarrow \text{Fr}(F), \quad (\varphi, A) \mapsto \varphi \circ A.$$

Every trivialization $\tau : U \times E_1 \rightarrow F$ of F corresponds to a local section $\sigma : U \rightarrow \text{Fr}(F)$, $x \mapsto \tau(x, \bullet) \in \text{Iso}(E_1, F_x)$ and a trivialization $\tau' : U \times \text{Gl}_{E_1} \rightarrow \text{Fr}(F)$, $(x, A) \mapsto \tau(x, \bullet) \circ A$ of the frame bundle. The transition functions $k : V \rightarrow \text{Gl}_{E_1}$ of the corresponding trivializations are the same. Given a vector bundle automorphism $f : F \rightarrow F$, one obtains an automorphism of principal bundles $f' : \text{Fr}(F) \rightarrow \text{Fr}(F)$ by setting

$$f'(\tau'(x, A)) := f_x \circ \tau(x, \bullet) \circ A \text{ for } (x, A) \in U \times \text{Gl}_{E_1}.$$

Conversely, for $f' : \text{Fr}(F) \rightarrow \text{Fr}(F)$, we let

$$f(\tau(x, v)) := f'(\tau(x, \bullet)).v \text{ for } (x, v) \in U \times E_1.$$

It is easy to see that these constructions are inverse to each other. Furthermore, both automorphisms induce the same diffeomorphism on the base and compactly supported automorphisms correspond to compactly supported automorphisms.

We denote by $\text{Aut}_c(\text{Fr}(F))$ the group of compactly supported automorphisms of $\text{Fr}(F)$ and let $\text{Gau}_c(\text{Fr}(F))$ be the subgroup of these automorphisms over the identity. The Lie group structure on $\text{Aut}_c(\text{Fr}(F))$ is realized as an extension of Lie groups

$$\text{Gau}_c(\text{Fr}(F)) \rightarrow \text{Aut}_c(\text{Fr}(F)) \rightarrow \text{Diff}_c(M)_{[\text{Fr}(F)]},$$

where $\text{Diff}_c(M)_{[\text{Fr}(F)]}$ is the open subgroup of $\text{Diff}_c(M)$ such that $f^*F \cong F$ for all $f \in \text{Diff}_c(M)_{[\text{Fr}(F)]}$, i.e., the subgroup fixing the equivalence class $[\text{Fr}(F)]$ under pullbacks (compare [41, Example V.1.6(c), p.392]). Analogously, we want to turn

$$\text{Gau}_c(F) \rightarrow \text{Aut}_c(F) \rightarrow \text{Diff}_c(M)_{[F]} \quad (\text{D.1})$$

into an extension of Lie groups, where again $\text{Diff}_c(M)_{[F]}$ denotes the open subgroup of $\text{Diff}_c(M)$ that fixes the equivalence class $[F]$ under pullbacks (compare [41, Example V.1.7(d), p.392]).

D.1. The Lie Group Structure of the Gauge Group

Recall Proposition A.3.5. If τ'_i are the trivializations of $\text{Fr}(F)$ corresponding to the trivializations τ_i of F and if we let $\sigma_i(x) := \tau_i(x, \bullet)$, then the Lie group structure on $\text{Gau}_c(\text{Fr}(F))$ is given by the embedding

$$\text{Gau}_c(\text{Fr}(F)) \hookrightarrow \prod_{i \in \mathbb{N}}^* C^\infty(\overline{U}_i, \text{Gl}_{E_1}), \quad \psi' \mapsto (\text{pr}_2 \circ \tau_i'^{-1} \circ \psi' \circ \sigma_i)_{i \in \mathbb{N}}$$

(see [50, Theorem 4.18, p.36]). Note that the Lie group structure induced on $\text{Gau}_c(\text{Fr}(F))$ does not depend on the trivialization by [50, Proposition 4.16, p.33]. Defining the embedding

$$\Phi: \text{Gau}_c(F) \hookrightarrow \prod_{i \in \mathbb{N}}^* C^\infty(\overline{U}_i, \text{Gl}_{E_1}), \quad \psi \mapsto (\text{pr}_2 \circ \tau_i^{-1} \circ \psi \circ \sigma_i)_{i \in \mathbb{N}}, \quad (\text{D.2})$$

we see that we obtain the same closed subgroup of the weak direct product in both cases because the transition maps are the same. Therefore, we get a Lie group structure on $\text{Gau}_c(F)$ that is independent of the trivialization.

Remark D.1.1. A concrete chart can be described in the following way. Let

$$\mathfrak{g}_{\overline{U}} := \left\{ (\eta_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} C^\infty(\overline{U}_i, \mathfrak{gl}_{E_1}) : \eta_i(x) = \text{Ad}(k_{ij}(x)) \cdot \eta_j(x) \text{ for } x \in \overline{U}_i \cap \overline{U}_j \right\}$$

and let $G_{\overline{U}} := \text{im}(\Phi)$. If $P \subseteq \mathfrak{gl}_{E_1}$ is open such that $\exp_{\text{Gl}_{E_1}}|_P$ is a diffeomorphism to $\exp_{\text{Gl}_{E_1}}(P) =: P'$ let

$$(\exp_{\text{Gl}_{E_1}})_*^i: \mathcal{C}^\infty(\overline{U}_i, P) \rightarrow \mathcal{C}^\infty(\overline{U}_i, P'), \quad \eta_i \mapsto \exp_{\text{Gl}_{E_1}} \circ \eta_i.$$

Let $C := \mathfrak{g}_{\overline{U}} \cap \bigoplus_{i \in \mathbb{N}} \mathcal{C}^\infty(\overline{U}_i, \widetilde{W})$ and $C' := G_{\overline{U}} \cap \prod_{i \in \mathbb{N}}^* \mathcal{C}^\infty(\overline{U}_i, P')$. Then

$$(\exp_{\text{Gl}_{E_1}})_*: C \rightarrow C', \quad (\eta_i)_{i \in \mathbb{N}} \mapsto ((\exp_{\text{Gl}_{E_1}})_*^i(\eta_i))_{i \in \mathbb{N}}$$

is a diffeomorphism and the inverse of a chart of $\text{Gau}_c(F)$ around the identity.

D.2. The Lie Group Structure of the Diffeomorphism Group

Recall the vector fields $\mathfrak{X}_c(M)$ of M with compact support from 1.3.2. We turn $\mathfrak{X}_c(M)$ into a locally convex space such that

$$\rho: \mathfrak{X}_c(M) \hookrightarrow \bigoplus_{i \in \mathbb{N}} C^\infty(U_i, E_0), \quad Y \mapsto (\text{pr}_2 \circ \tau_i^{-1} \circ Y|_{U_i})_{i \in \mathbb{N}} \quad (\text{D.3})$$

becomes an embedding. Fix a Riemannian metric g on M and let \exp_g be the corresponding Riemannian exponential map. It was shown in [19] that there exists an open zero-neighborhood $\Omega'' \subseteq \mathfrak{X}_c(M)$ such that the map

$$\varphi_g^{-1}: \Omega'' \rightarrow \text{Diff}(M), \quad Y \mapsto \exp_g \circ Y$$

is well-defined and bijective onto its image \mathcal{O}'' (see also [23], cf. [36]). The Lie group structure on $\text{Diff}(M)$ arises then from \mathcal{O}'' with the chart $\varphi_g: \mathcal{O}'' \rightarrow \Omega''$ and $\text{Diff}_c(M)$ is an open subgroup. One has that

$$T_{\text{id}} \text{Diff}(M) \rightarrow \mathfrak{X}_c(M), \quad [t \mapsto v_t] \mapsto (p \mapsto [t \mapsto v_t(p)])$$

is an isomorphism of locally convex Lie algebras. Following [19], we choose Ω'' small enough such that $f(\overline{V}_i) \subseteq U_i$ for all $i \in \mathbb{N}$ and $f \in \mathcal{O}''$. For technical reasons, we need to consider even smaller open unity neighborhoods than \mathcal{O}'' . For this, we fix a smooth partition of unity $(h_i)_{i \in \mathbb{N}}$ subordinate to $(V_i)_{i \in \mathbb{N}}$. It was shown in [50, Remark 5.10, p.44 f.] that there exists an open unity neighborhood $\mathcal{O}' \subseteq \mathcal{O}''$ such that \mathcal{O}' is symmetrical, $\mathcal{O}' \circ \mathcal{O}' \circ \mathcal{O}' \subseteq \mathcal{O}''$ and such that $(h_1 + \dots + h_n) \cdot \varphi_g(f) \in \mathcal{O}''$ for all $f \in \mathcal{O}'$, $n \in \mathbb{N}$. With the same argument, we also find an open symmetrical unity neighborhood $\mathcal{O} \subseteq \mathcal{O}'$ such that $(h_1 + \dots + h_n) \cdot \varphi_g(f) \in \mathcal{O}'$ holds for all $f \in \mathcal{O}$, $n \in \mathbb{N}$. We fix the neighborhoods $\mathcal{O}'', \mathcal{O}', \mathcal{O}$ and set $\Omega := \varphi_g^{-1}(\mathcal{O})$.

Remark D.2.1. We define the sets $W'_i := \bigcup_{f \in \mathcal{O}} f(V_i)$ and $W_i := \bigcup_{f \in \mathcal{O}} f(W'_i)$. The properties $\mathcal{O}' \circ \mathcal{O}' \circ \mathcal{O}' \subseteq \mathcal{O}''$ and $f(\overline{V}_i) \subseteq U_i$ for $f \in \mathcal{O}''$ imply $\overline{V}_i \subseteq \overline{W'_i} \subseteq \overline{W_i} \subseteq U_i$ and obviously $(W'_i)_{i \in \mathbb{N}}$ and $(W_i)_{i \in \mathbb{N}}$ are open locally finite covers of M .

Remark D.2.2. Let $\varphi_i: U_i \rightarrow V_i$ be charts of M and let $\rho_i: \mathfrak{X}_c(M) \rightarrow C^\infty(U_i, E_0)$ be the components of the embedding (D.3). Recall the covering $(W_i)_{i \in \mathbb{N}}$ from Remark D.2.1. We fix smooth maps $h'_i: U_i \rightarrow \mathbb{R}$ that are constantly 1 on W_i such that $\text{supp}(h'_i) \subseteq U_i$. The maps $\xi_i: C^\infty(U_i, E_0) \rightarrow \mathfrak{X}_c(M)$, given by

$$C^\infty(U_i, E_0) \rightarrow C^\infty_{\text{supp}(h'_i)}(U_i, E_0) \rightarrow \mathfrak{X}_{\text{supp}(h'_i)}(U_i) \rightarrow \mathfrak{X}_c(M),$$

$$f \mapsto h'_i \cdot f \mapsto T\varphi_i^{-1}(\varphi_i(\cdot), (h'_i \cdot f_i)(\cdot)) =: Y_i \mapsto \tilde{Y}_i$$

where \tilde{Y}_i is the continuation of Y_i by 0, are smooth because of [18, Proposition 8.13, p.50] and [18, Lemma 4.24, p.34]. Let $R := \bigoplus_{i \in \mathbb{N}} R_i \subseteq \bigoplus C^\infty(U_i, E_0)$ be an open zero-neighborhood such that $\rho^{-1}(R) \subseteq \Omega$. We define the open zero-neighborhoods $\tilde{\Omega}_i := \xi_i^{-1}(\Omega) \cap R_i$ and $\tilde{\Omega} := \rho^{-1}(\bigoplus_{i \in \mathbb{N}} \tilde{\Omega}_i) \subseteq \Omega$. This enables us to define smooth maps

$$\zeta_i: \tilde{\Omega}_i \rightarrow \mathcal{O}, \quad Y_i \mapsto \varphi_g^{-1}(\xi_i(Y_i))$$

with the property $\varphi_g^{-1}(Y)|_{W_i} = \zeta_i(\rho_i(Y))|_{W_i}$ for all $Y \in \tilde{\Omega}$, $i \in \mathbb{N}$.

D.3. The Automorphism Group

Next, we construct a local section $S: \mathcal{O} \rightarrow \text{Aut}_c(F)$ such that $\text{Aut}_c(F)$ becomes an extension of Lie groups. For this, we define smooth maps $s_i: \mathcal{O} \rightarrow \mathcal{O}' \circ \mathcal{O}'$ by

$$\begin{aligned} s_1(f) &:= \varphi_g^{-1}(h_1 \cdot \varphi_g(f)) \quad \text{and} \\ s_i(f) &:= \left(\varphi_g^{-1}((h_1 + \dots + h_{i-1}) \cdot \varphi_g(f)) \right)^{-1} \circ \varphi_g^{-1}((h_1 + \dots + h_i) \cdot \varphi_g(f)) \end{aligned}$$

for $i > 1$. In the remainder, we abbreviate $s_i(f) = f_i$ for $f \in \mathcal{O}$. From the definition, it follows $\text{supp}(f_i) \subseteq \bar{V}_i$ for all $i \in \mathbb{N}$ and all $f \in \mathcal{O}$ and $\lim_{i \rightarrow \infty} f_1 \circ \dots \circ f_i = f$ as a pointwise limit. The limit becomes stationary because we have $f_j = \text{id}_M$ if $V_j \cap \text{supp}(f) = \emptyset$ and this condition holds for almost all $j \in \mathbb{N}$.

Remark D.3.1. Let $f \in \mathcal{O}$. By definition, we have

$$f_i \circ f_{i+1} \circ \dots \circ f_{i+p} = \underbrace{\left(\varphi_g^{-1}((h_1 + \dots + h_{i-1}) \cdot \varphi_g(f)) \right)^{-1}}_{\in \mathcal{O}'} \circ \underbrace{\varphi_g^{-1}((h_1 + \dots + h_{i+p}) \cdot \varphi_g(f))}_{\in \mathcal{O}'}$$

and

$$f_1 \circ f_2 \circ \dots \circ f_p = \varphi_g^{-1}((h_1 + \dots + h_p) \cdot \varphi_g(f)) \in \mathcal{O}',$$

for all $p, i > 1$.

We lift a diffeomorphism $f \in \text{Diff}_c(M)$ with $\text{supp}(f) \subseteq \bar{V}_i$, to $\tilde{f} \in \text{Aut}_c(F)$ via

$$\begin{aligned} \tilde{f}(\tau_i(x, v)) &:= \tau_i(f(x), v) \text{ for } (x, v) \in V_i \times E_1 \text{ and} \\ \tilde{f}(p) &:= p \text{ for } p \notin \pi^{-1}(V_i). \end{aligned} \tag{D.4}$$

An easy direct calculation shows that this corresponds to the construction in [50] for the frame bundle. Note that \tilde{f} is an automorphism over f and that $\widetilde{f^{-1}} = \tilde{f}^{-1}$. For $f \in \text{Diff}_c(M)$, we define the local section (cf. [50, Definition 5.12, p.46])

$$S: \mathcal{O} \rightarrow \text{Aut}_c(M), \quad f \mapsto \lim_{i \rightarrow \infty} \tilde{f}_1 \circ \cdots \circ \tilde{f}_i.$$

The limit is well-defined because $s_j(f) = \text{id}_F$ if $V_j \cap \text{supp}(f) = \emptyset$. Also S is a section because $S(f)$ is an automorphism over $\lim_{i \rightarrow \infty} f_1 \circ \cdots \circ f_i = f$. It follows in particular that $\text{Diff}_c(M)_{[F]}$ is indeed an open subgroup of $\text{Diff}_c(M)$.

Remark D.3.2. To see the smoothness of several natural actions of $\text{Aut}_c(F)$, it is crucial to have a local formula in terms of transition functions for $S(f)(\tau_i(x, v))$ that is valid in neighborhoods of f and x . Such a formula was derived in [50, Remark 5.13, p.46] for the frame bundle and since the transition maps are identical, our formula will be essentially the same. However, since there was a small mistake in the construction of the open neighborhoods of f and x , we will give a corrected version here. Note that the correction carries over to the more general case of principal fiber bundles in [50] and does not impact any other arguments made there.

Fix $f \in \mathcal{O}$ and $x \in \overline{V}_i$. Let j_1 be the largest index such that $x \in \overline{V}_{j_1}$. For $v \in E_1$, we calculate

$$\tilde{f}_{j_1}(\tau_i(x, v)) = \tilde{f}_{j_1}(\tau_{j_1}(x, k_{j_1 i}(x).v)) = \tau_{j_1}(f_{j_1}(x), k_{j_1 i}(x).v).$$

Note that for $x \in U_{j_1} \setminus V_{j_1}$, we have $f_{j_1}(x) = x$ and the formula is still valid. Next, let j_2 be the largest index smaller than j_1 with $f_{j_1}(x) \in \overline{V}_{j_2}$. Then we get

$$\begin{aligned} \tilde{f}_{j_2}(\tau_{j_1}(f_{j_1}(x), k_{j_1 i}(x).v)) &= \tilde{f}_{j_2}(\tau_{j_2}(f_{j_1}(x), k_{j_2 j_1}(f_{j_1}(x)) \cdot k_{j_1 i}(x).v)) \\ &= \tau_{j_2}(f_{j_2} \circ f_{j_1}(x), k_{j_2 j_1}(f_{j_1}(x)) \cdot k_{j_1 i}(x).v). \end{aligned}$$

Eventually, this leads to $j_\ell < \cdots < j_1$ such that there is no index $j < j_\ell$ with $f_{j_\ell} \circ f_{j_\ell-1} \circ \cdots \circ f_{j_1}(x) \in \overline{V}_j$ and we have the formula

$$\begin{aligned} S(f)(\tau_i(x, v)) &= \tau_{j_\ell}(f(x), k_{j_\ell j_{\ell-1}}(f_{j_{\ell-1}} \circ \cdots \circ f_{j_1}(x)) \cdots k_{j_1 i}(x).v) \\ &= \tau_{j_\ell}(f(x), k_{i j_\ell}(f(x)) \cdot k_{j_\ell j_{\ell-1}}(f_{j_{\ell-1}} \circ \cdots \circ f_{j_1}(x)) \cdots k_{j_1 i}(x).v), \end{aligned} \quad (\text{D.5})$$

where the last equality holds because $f(\overline{V}_i) \subseteq U_i$. Note that by construction $f(x) = f_1 \circ \cdots \circ f_{j_1}(x) = f_{j_\ell} \circ f_{j_\ell-1} \circ \cdots \circ f_{j_1}(x)$ holds because the omitted factors do not change the respective function value. There is an open neighborhood $U \subseteq U_i$ of x such that $U \subseteq U_{j_1}$ and such that there is no index $j > j_1$ with $x' \in \overline{V}_j$ for all $x' \in U$. The conditions for formula (D.5) to hold for some other diffeomorphism $f' \in \mathcal{O}$ and $x' \in U$ are as follows:

$$\begin{aligned} f'_{j_\ell} \circ \cdots \circ f'_{j_1}(x') &\notin \overline{V}_j \text{ for } j < \ell, \\ f'_{j_p} \circ \cdots \circ f'_{j_1}(x') &\notin \overline{V}_j \text{ for } j_{p+1} < j < j_p, \quad 1 \leq p \leq \ell - 1 \text{ and} \end{aligned}$$

$$f'_{j_p} \circ \cdots \circ f'_{j_1}(x') \in U_{j_p} \cap U_{j_{p+1}} \text{ for } 1 \leq p \leq \ell - 1.$$

The first condition means that the smallest relevant index is j_ℓ . The second condition implies that $\tilde{f}_{j_{p+1}}$ is indeed the next relevant map after $\tilde{f}_{j_p} \circ \cdots \circ \tilde{f}_{j_1}$ and the last condition guarantees that the transition maps $k_{j_{p+1}j_p}$ are defined for all $\ell - 1 \leq p \leq 1$. Since the evaluation map $\text{Diff}_c(M) \times M \rightarrow M$, $(f', x') \mapsto f'(x')$ is smooth and in particular continuous by [19, Proposition 6.2, p.28], the finitely many conditions above yield, after intersection, open neighborhoods \mathcal{O}_f of f and $U_x \subseteq U_i$ of x such that formula (D.5) holds for all $(f', x') \in \mathcal{O}_f \times U_x$.

Remark D.3.3. Note that we may substitute terms of the form $f_{j_p} \circ \cdots \circ f_{j_1}$, $p > \ell$ in formula (D.5) with $f_{j_p} \circ f_{j_{p+1}} \circ \cdots \circ f_{j_1} \circ f_{j_1}$ because, by definition, the additional maps do not change the respective function value. Now, Remark D.3.1 implies that $f_{j_p} \circ f_{j_{p+1}} \circ \cdots \circ f_{j_1} \circ f_{j_1}$ equals

$$\left(\varphi_g^{-1}((h_1 + \cdots + h_{j_{p-1}}) \cdot \varphi_g(f)) \right)^{-1} \circ \varphi_g^{-1}((h_1 + \cdots + h_{j_1}) \cdot \varphi_g(f))$$

for $j_p > 1$ and

$$\varphi_g^{-1}((h_1 + \cdots + h_{j_1}) \cdot \varphi_g(f))$$

for $j_p = 1$. In particular, the value of $S(f)(\tau_i(x, v))$ only depends on $f|_{W_i}$ for $(x, v) \in V_i \times E_1$, with $V_i \subseteq W'_i \subseteq W_i \subseteq U_i$ as in Remark D.2.1.

To see that this turns $\text{Aut}_c(F)$ into a Lie group, one has to check that the map

$$\omega: \mathcal{O} \times \mathcal{O} \rightarrow \text{Gau}_c(F), \quad (f, f') \mapsto S(f) \circ S(f') \circ S(f \circ f')^{-1}$$

is smooth and that for each $f \in \text{Diff}_c(M)_{[F]}$, there exists an open identity neighborhood $\mathcal{O}_f \subseteq \mathcal{O}$ such that

$$\omega_f: \mathcal{O}_f \rightarrow \text{Gau}_c(F), \quad f' \mapsto \psi \circ S(f') \circ \psi^{-1} \circ S(f \circ f' \circ f^{-1})^{-1}$$

is smooth, where $\psi \in \text{Aut}_c(F)$ is an automorphism over f . With the correspondence established above, this yields exactly the same maps as in [50]. Thus, $\text{Aut}_c(F)$ becomes a Lie group such that (D.1) is an extension of Lie groups. This Lie group does not depend on the various choices made in the construction of S (compare [50, Remark 5.23, p.59]). Let C and C' be as in Remark D.1.1. Then

$$\Upsilon^{-1}: C \times \Omega \rightarrow C' \circ S(\mathcal{O}), \quad (G, X) \mapsto (\exp_{\text{Gl}_{E_1}})_*(G) \circ S(\varphi_g(X)) \quad (\text{D.6})$$

is the inverse of a chart Υ of $\text{Aut}_c(F)$ around the identity.

Lemma D.3.4. *The evaluation map*

$$\text{ev}_{\text{Gau}}: \text{Gau}_c(F) \times F \rightarrow F, \quad (\psi, p) \mapsto \psi(p)$$

is smooth.

Proof. Recall the embedding $\Phi: \text{Gau}_c(F) \hookrightarrow \prod_{i \in \mathbb{N}}^* C^\infty(\overline{U_i}, \text{Gl}_{E_1})$. For $i \in \mathbb{N}$, the evaluation $\text{ev}_i: C^\infty(\overline{U_i}, \text{Gl}_{E_1}) \times \overline{U_i} \rightarrow \text{Gl}_{E_1}$ is smooth by Lemma A.3.7 and the projection $\text{pr}_i: \prod_{i \in \mathbb{N}}^* C^\infty(\overline{U_i}, \text{Gl}_{E_1}) \rightarrow C^\infty(\overline{U_i}, \text{Gl}_{E_1})$ is smooth by Lemma A.3.6. Because we have $\text{ev}_{\text{Gau}}(\psi, \tau_i(x, v)) = \text{ev}_i(\text{pr}_i(\psi), x).v$, the lemma follows since the action $\text{Gl}_{E_1} \times E_1 \rightarrow E_1$ is smooth. \square

Lemma D.3.5 (cf. [56, Proposition 2.15, p.20]). *The evaluation map*

$$\text{ev}: \text{Aut}_c(F) \times F \rightarrow F, \quad (\psi, p) \mapsto \text{ev}_p(\psi) := \psi(p)$$

is smooth.

Proof. By [55, Lemma A.3.3, p.133] it suffices to check smoothness on $U \times F$ for some open unity neighborhood $U \subseteq \text{Aut}_c(F)$. We choose $U = \text{Gau}_c(F) \circ S(\mathcal{O})$. Let $(x, v) \in V_i \times E_1$. By Lemma D.3.4 the evaluation $\text{Gau}_c(F) \times F \rightarrow F$ is smooth. Therefore, we only need to show that the evaluation map depends smoothly on the elements of \mathcal{O} . For $f \in \mathcal{O}$, we choose open neighborhoods \mathcal{O}_f of f and U_x of x as in Remark D.3.2 such that for all $(f', x') \in \mathcal{O}_f \times U_x$, we have

$$S(f')(\tau_i(x', v)) = \tau_i\left(f'(x'), k_{ij_\ell}(f'(x')) \cdot k_{j_\ell j_{\ell-1}}(f'_{j_{\ell-1}} \circ \cdots \circ f'_{j_1}(x')) \cdots k_{j_1 i}(x').v\right)$$

for fixed indices $j_\ell < \cdots < j_1$. Since the evaluation map $\text{Diff}_c(M) \times M \rightarrow M$ is smooth (see [19, Proposition 6.2, p.28]), it follows that this is a composition of smooth maps. \square

Remark D.3.6. It follows from Lemma D.3.5 that the tangent map

$$T\text{ev}: T\text{Aut}_c(F) \times TF \rightarrow TF$$

is smooth. Let $\Psi := [t \mapsto \psi_t] \in T\text{Aut}_c(F)$ and $v := [t \mapsto v_t] \in TF$. Then we calculate

$$T\text{ev}(\Psi, v) = [t \mapsto \psi_t(v_0)] + [t \mapsto \psi_0(v_t)] = [t \mapsto \psi_t(v_0)] + T\psi_0(v).$$

In particular, the map

$$\text{ev}_T: \text{Aut}_c(F) \times TF \rightarrow TF, \quad (\psi, v) \mapsto T\psi(v)$$

is smooth as well.

D.3.1. The Lie algebra of $\text{Aut}_c(F)$

Lemma D.3.7. *Recall Remark E.3.6. Any bundle isomorphism over the identity $\Theta: F \oplus_M TM \oplus_M F \rightarrow TF$ induces an isomorphism*

$$\mathfrak{X}_c(M) \times \mathfrak{gau}_c(F) \rightarrow \mathfrak{X}_c(F)_b, \quad (X, G) \mapsto (p \mapsto \Theta(p, X(\pi(p)), G(p)))$$

of locally convex spaces.

Proof. Let $(X, G) \in \mathfrak{X}_c(M) \times \mathfrak{gau}_c(F)$ and

$$Y: F \rightarrow TF, \quad Y(p) := \Theta(p, X(\pi(p)), G(p)).$$

Clearly, Y is smooth and by definition, we have $\pi_{TF} \circ Y = \text{id}_F$ for the projection $\pi_{TF}: TF \rightarrow F$. Let $\varphi^i = (\varphi_0^i, \varphi_1^i): \pi^{-1}(V_i) \rightarrow \tilde{V}_i \times E_1$ be a bundle chart of F and $G^{\varphi^i} := \varphi^i \circ G \circ (\varphi^i)^{-1}$. By [10, Theorem 10.5, p.62], we have

$$T\varphi^i \circ Y \circ (\varphi^i)^{-1}(x, v) = (x, v, X^{\varphi_0^i}(x), G^{\varphi^i}(x, v) + b_x^i(v, X^{\varphi_0^i}(x))), \quad (\text{D.7})$$

where b^i is the Christoffel symbol corresponding to Φ in the chart φ^i (see [10, 10.4, p.61f]). Note that $b^i: \tilde{V}_i \times E_1 \times E_0 \rightarrow E_1$, $(x, v, w) \mapsto b_x^i(v, w)$ is a smooth map such that b_x^i is bilinear for all $x \in \tilde{V}_i$. In particular, the map $\tilde{V}_i \times E_0 \rightarrow \mathcal{L}(E_1; E_1)_b$, $(x, w) \mapsto b_x^i(\bullet, w)$ is smooth by Proposition A.2.10. We see that the support of Y is the same as the union of the supports of X and G and that Y is indeed an element of $\mathfrak{X}_c(F)$. Since the map

$$C^\infty(\tilde{V}_i, E_0) \rightarrow C^\infty(\tilde{V}_i, \text{Gl}_{E_1}), \quad f \mapsto (x \mapsto b_x^i(\bullet, f(x)))$$

is smooth by Proposition A.2.1, smoothness follows from formula (D.7) together with Lemma A.1.4. Conversely, for $Y \in \mathfrak{X}_c(F)$, we define $X := \text{pr}_{TM} \circ \Theta^{-1} \circ Y \circ z_M$ and $G := \text{pr}_{\varepsilon_F} \circ \Theta^{-1} \circ Y$, where $z_M: M \rightarrow TF$ is the zero section and $\text{pr}_{\varepsilon_F}: TF \rightarrow F$, $T\pi: TF \rightarrow TM$ are the projectors given in [10, 10.7, p.63]. It is easy to see that this construction is inverse to the above and smoothness follows essentially in the same way as before because of the local description of Θ^{-1} given in [10, Theorem 10.5, p.62]. \square

Remark D.3.8. Since M is σ -compact and finite-dimensional, one can always construct a linear connection on TF . By [10, Theorem 10.5, p.62] one then obtains an associated isomorphism Θ as in Lemma D.3.7. It will be useful in the following to make this construction explicit in terms of the partition of unity $(h_i)_i$ and the trivializations τ_i . By abuse of notation, we will write b^i for the Christoffel symbol expressed with the trivialization τ_i (instead of a bundle chart). The condition [10, (10.2), p.62] for a family (b^i) to define a connection translates then to

$$b_x^i(k_{ij}(x).u, v) = k_{ij}(x).b^j(u, v) + dk_{ij}(v).u,$$

for $x \in V_i \cap V_j$, $v \in T_x M$ and $u \in E_1$. For $x \in M$ let I_x be the finite set $\{i \in \mathbb{N}: x \in V_i\}$. We define

$$b_x^i(u, v) := \sum_{l \in I_x} h_l(x) dk_{il}(v).(k_{li}(x).u).$$

Note that $k_{ij}(x).k_{jl}(x) = k_{il}(x)$ implies

$$dk_{ij}(v).(k_{jl}(x)) + k_{ij}(x).(dk_{jl}(v)) = dk_{il}(v).$$

With this, the following direct calculation shows that these b^i define a connection.

$$\begin{aligned}
 & k_{ij}(x).b_x^j(u, v) + dk_{ij}(v).u \\
 &= \sum_{l \in I_x} h_l(x) \left(k_{ij}(x).(dk_{jl}(v).(k_{lj}(x).u)) \right) + dk_{ij}(v).u \\
 &= \sum_{l \in I_x} h_l(x) \left(dk_{il}(v).k_{lj}(x).u - dk_{ij}(v).u \right) + dk_{ij}(v).u \\
 &= \sum_{l \in I_x} h_l(x) dk_{il}(v).k_{lj}(x).u = b_x^i(k_{ij}(x).u, v).
 \end{aligned}$$

Lemma D.3.9. *Recall the evaluation map $\text{ev}: \text{Aut}_c(F) \times F \rightarrow F$ from Lemma D.3.5. The map*

$$\Psi: T_{\text{id}}\text{Aut}_c(F) \rightarrow \mathfrak{X}_c(F), \quad v \mapsto (p \mapsto T\text{ev}_p(v))$$

is an isomorphism of locally convex Lie algebras.

Proof. By the definition of the respective topologies and Lemma A.1.4, it suffices to see the smoothness of

$$\mathbf{gau}_c(F) \times \mathfrak{X}_c(M) \cong T_{\text{id}}\text{Aut}_K(F) \rightarrow \mathfrak{X}_c(F)$$

in a trivialization. The local calculation will also show that Ψ is well-defined. Let $(G, X) \in C \times \Omega$ and Υ be as in (D.6). It follows

$$T_{\text{id}}\text{ev}_p T_0 \Upsilon^{-1}(G, X) = T_{\text{id}}\text{ev}_p T_0 \Upsilon^{-1}(G, 0) + T_{\text{id}}\text{ev}_p T_0 \Upsilon^{-1}(0, X).$$

For $p = \tau_i(x, v)$, we calculate $T_{\text{id}}\text{ev}_p T_0 \Upsilon^{-1}(G, 0) = T\tau_i(x, v, 0, G^{\tau_i}(x, v))$ and

$$\begin{aligned}
 & T_{\text{id}}\text{ev}_p T_0 \Upsilon^{-1}(0, X) = \\
 & [t \mapsto s_1(\exp_g \circ (tX))^{\sim}(p)] + \cdots + [t \mapsto s_N(\exp_g \circ (tX))^{\sim}(p)],
 \end{aligned}$$

where \sim denotes the lift defined in (D.4). Using the identification $\mathfrak{X}_c(M) \cong T_{\text{id}}\text{Diff}(M)$ via $X \mapsto [t \mapsto \exp_g \circ (tX)]$, we see that

$$\begin{aligned}
 & T_{\text{id}}s_1[t \mapsto \exp_g \circ (tX)] \cong h_1 \cdot X \quad \text{and} \\
 & T_{\text{id}}s_i[t \mapsto \exp_g \circ (tX)] \cong -(h_1 + \cdots + h_{i-1})X + (h_1 + \cdots + h_i) \cdot X = h_i \cdot X.
 \end{aligned}$$

Let $I_x := \{j \in \mathbb{N}: x \in V_j\}$ and for $j \in I_x$ let $v_j \in E_1$ such that $\tau_j(x, v_j) = p$. It follows $[t \mapsto s_j(\exp_g \circ (tX))^{\sim}(p)] = [t \mapsto p] = T\tau_i(x, v, 0, 0)$ for $\pi(p) \notin V_j$ and

$$\begin{aligned}
 & [t \mapsto s_j(\exp_g \circ (tX))^{\sim}(p)] \\
 &= [t \mapsto \tau_j(s_j(\exp_g \circ (tX))(x), v_j)] = T_{(x, v_j)}\tau_j[t \mapsto (t(h_j(x) \cdot X(x)), 0)] \\
 &= T\tau_i(x, v, (h_j \cdot X)(x), dk_{ij}((h_j \cdot X)(x)).(k_{ji}(x).v))
 \end{aligned}$$

if $\pi(p) \in V_j$. Summing up, we arrive at

$$\begin{aligned} T_{\text{id}} \text{ev}_p T_0 \Upsilon^{-1}(G, X) = \\ T\tau_i \left(x, v, X(x), G^{\tau_i}(x, v) + \sum_{j \in I_x} h_j(x) dk_{ij}(X(x)) \cdot (k_{ji}(x) \cdot v) \right). \end{aligned}$$

With the connection from Remark D.3.8, we see that this gives us exactly the isomorphism from Lemma D.3.7. Next, we show that we have a morphism of Lie algebras. Let $\mathfrak{X}^r(\text{Aut}_c(F))$ be the Lie algebra of the right-invariant vector fields of $\text{Aut}_c(F)$. Because $\mathfrak{X}^r(\text{Aut}_c(F))$ is a Lie subalgebra of $\mathfrak{X}(\text{Aut}_c(F))$, we have an embedding $\mathfrak{X}^r(\text{Aut}_c(F)) \hookrightarrow \mathfrak{X}(\text{Aut}_c(F) \times F)$ of Lie algebras via $X \mapsto X \times z_F$. Let $\Psi_1: T_{\text{id}}\text{Aut}_c(F) \rightarrow \mathfrak{X}^r(\text{Aut}_c(F))$, $\tilde{X}_0 \mapsto X$ be defined by $X(f) := T\rho_f(\tilde{X}_0)$, where ρ_f is the multiplication from the right with f in $\text{Aut}_c(F)$. It is well-known that Ψ_1 is an isomorphism, which we use to give $T_{\text{id}}\text{Aut}_c(F)$ a Lie algebra structure. Let $\tilde{X}_0 = [t \mapsto v_t] \in T_{\text{id}}\text{Aut}_c(F)$. Then

$$T \text{ev}(\Psi_1(\tilde{X}_0)(f), z_F(p)) = [t \mapsto v_t \circ f(p)] = \Psi(\tilde{X}_0)(f(p))$$

for all $p \in F$ and $f \in \text{Aut}_c(F)$. In other words $X_0 := \Psi(\tilde{X}_0)$ is ev-related to $X := \Psi_1(\tilde{X}_0)$ under the above embedding, which implies that Ψ is a morphism of Lie algebras. The idea for the last part of this proof is due to Milnor [37, p. 1041]. \square

Lemma D.3.10. *The action*

$$\text{Aut}_c(F) \times \mathcal{X}_c(F) \rightarrow \mathcal{X}_c(F), \quad (f, X) \mapsto Tf \circ X \circ f^{-1}$$

induced by the action of $\text{Diff}(F)$ on $\mathfrak{X}(F)$ is the same as the one induced by the action of $\text{Aut}_c(F)$ on $T_{\text{id}}\text{Aut}_c(F)$.

Proof. Let $\Psi: T_{\text{id}}\text{Aut}_c(F) \rightarrow \mathcal{X}_c(F)$ be the isomorphism from Lemma D.3.9. Let $f \in \text{Aut}_c(F)$ and $\tilde{X}_0 = [t \mapsto v_t] \in T_{\text{id}}\text{Aut}_c(F)$. If c_f is the conjugation with f in $\text{Aut}_c(F)$, we calculate

$$\begin{aligned} \Psi(Tc_f(\tilde{X}_0)) &= \Psi([t \mapsto f \circ v_t \circ f^{-1}]) = (p \mapsto [t \mapsto f \circ v_t \circ f^{-1}(p)]) \\ &= \left(p \mapsto Tf \circ \Psi(\tilde{X}_0)(f^{-1}(p)) \right) \end{aligned}$$

as needed. \square

Lemma D.3.11. *Let $v := [t \mapsto v_t] \in T^2F$ and $Y \in \mathcal{X}_c(F)$. Then*

$$T \text{ev}_T(Y, v) = v + TY(v_0)$$

holds for the evaluation from Remark D.3.6, where Y is identified with an element of $T_{\text{id}}\text{Aut}_c(F)$ via Lemma D.3.9 and the sum is taken in $T_{v_0}TF$.

Proof. Let $[t \mapsto f_t] \in T_{\text{id}}\text{Aut}_c(F)$ correspond to Y . We calculate

$$\begin{aligned} T \text{ev}_T([t \mapsto (f_t, v_t)]) &= T \text{ev}_T([t \mapsto (f_0, v_t)]) + T \text{ev}_T([t \mapsto (f_t, v_0)]) \\ &= [t \mapsto v_t] + [t \mapsto T f_t(v_0)]. \end{aligned}$$

For $v_0 = [s \mapsto x_s] \in TF$, we have

$$\begin{aligned} [t \mapsto T f_t([s \mapsto x_s])] &= [t \mapsto [s \mapsto f_t(x_s)]] = [s \mapsto [t \mapsto f_t(x_s)]] \\ &= [s \mapsto Y(x_s)] = TY(v_0), \end{aligned}$$

where the second equality follows from Schwarz's theorem. □

E. Higher Order Tangent Groups

E.1. Higher Order Tangent Lie Groups

This section is mainly taken from [10, Chapter 24, p.116 ff.]. We use the same notation for the infinitesimal generators ε_i as in Chapter B. Let G be a Lie group with multiplication $m: G \times G \rightarrow G$ and Lie algebra \mathfrak{g} . Then $T^k G$ becomes a Lie group with the multiplication $T^k m$. The inclusions of the axes induce inclusions $\mathfrak{g} = (TG)_e \rightarrow \varepsilon_I(TG)_e \subseteq (T^k G)_e$. Obviously, $(T^k G)_e$ is a closed Lie subgroup of $T^k G$ and we have a short exact sequence of Lie groups

$$1 \rightarrow (T^k G)_e \rightarrow T^k G \rightarrow G \rightarrow 1,$$

which splits along the zero section $G \rightarrow T^k G$ such that $T^k G = (T^k G)_e \rtimes G$ as Lie groups. It is well known that $TG \cong \mathfrak{g} \times G$ and in the same way, we get an iterated (left) trivialization

$$\Psi_k: \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g} \times G \rightarrow T^k G, \quad ((\varepsilon_I v_I)_I, g) \rightarrow g \cdot \prod_{I \in \mathcal{P}_+^k}^{\uparrow} \varepsilon_I v_I,$$

where all the products are taken in $T^k G$ and \prod^{\uparrow} indicates that the product is taken in ascending lexicographic order of the index sets (see [10, 24.3, p. 117]). Formulas for the multiplication and inversion of the induced Lie group structure on $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g}$ consist of iterated Lie brackets and are given in [10, Theorem 24.7, p. 119]. The map Ψ_k is an isomorphism of multilinear bundles over G that is the identity map on the axes. It follows that $(T^k G)_e$ is a polynomial group and from this, we get the following theorem.

Theorem E.1.1. *Let G be a Lie group and $k \in \mathbb{N}$. There exists a unique diffeomorphism $\exp_{(T^k G)_e}: (T^k \mathfrak{g})_0 \rightarrow (T^k G)_e$ such that*

- (a) *the representation of $\exp_{(T^k G)_e}$ with respect to the left trivialization is polynomial,*
- (b) $T_0 \exp_{(T^k G)_e} = \text{id}_{(T^k \mathfrak{g})_0},$
- (c) *for all $n \in \mathbb{Z}$ and $v \in (T^k \mathfrak{g})_0$, we have $\exp_{(T^k G)_e}(nv) = \exp_{(T^k G)_e}(v)^n.$*

The inverse of $\exp_{(T^k G)_e}$ is polynomial. For every Lie group automorphism $\varphi: (T^k G)_e \rightarrow (T^k G)_e$, we have $\varphi \circ \exp_{(T^k G)_e} = \exp_{(T^k G)_e} \circ T_0 \varphi$. Identifying $(T^k \mathfrak{g})_0 \cong \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g}$ in the usual way, the exponential map can be extended in

a G -invariant way to an isomorphism of multilinear bundles

$$\exp_{T^k G}: G \times \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g} \rightarrow T^k G.$$

With respect to $\exp_{(T^k G)_e}$, the group multiplication of $(T^k G)_e$ is given by the BCH multiplication with the nilpotent Lie algebra $(T^k \mathfrak{g})_0$ and the inversion is given by multiplication with -1 . On $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g}$, we get the Lie bracket

$$\begin{aligned} \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g} \times \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g} &\rightarrow \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g}, \\ (\varepsilon_I(v_I, w_I))_{I \in \mathcal{P}_+^k} &\mapsto \sum_{I \in \mathcal{P}_+^k, \nu \in \mathcal{P}_2(I)} \varepsilon_I([v_{\nu_1}, w_{\nu_2}] + [v_{\nu_2}, w_{\nu_1}]), \end{aligned}$$

where $[\cdot, \cdot]$ denotes the Lie bracket of \mathfrak{g} .

Proof. This is essentially a combination of [10, Theorem 25.2, Theorem 25.4 and Theorem 25.5, p. 124f.]. That $\exp_{(T^k G)_e}$ is a diffeomorphism follows from Lemma C.2.6. With [10, Theorem 7.5, p. 47f.], we get the formula for the Lie bracket on $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g}$ by differentiating the bilinear Lie bracket $[\cdot, \cdot]$. \square

Corollary E.1.2. *Let $k \in \mathbb{N}$, G be a Lie group with Lie algebra \mathfrak{g} , $g \in G$ and $\text{Ad}_g: \mathfrak{g} \rightarrow \mathfrak{g}$ be the adjoint action. Then, with respect to $\exp_{(T^k G)_e}$ from Theorem E.1.1, the action of G on $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g}$ induced by the action of G on $(T^k G)_e$ is given by*

$$G \times \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g} \rightarrow \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{g}, \quad (g, (\varepsilon_I v_I)_I) \mapsto (\varepsilon_I \text{Ad}_g(v_I))_I.$$

Proof. This follows by applying Theorem E.1.1 to the automorphism $T^k c_g: T^k G \rightarrow T^k G$, where $c_g: G \rightarrow G$, $h \mapsto ghg^{-1}$ is the conjugation. \square

As described in Remark B.2.10, applying the functor $^-$ to purely even multilinear spaces can be understood as a substitution of the generators ε_i with the generators λ_i . We use this point of view to make the following statement more readable.

Corollary E.1.3. *Let $k \in \mathbb{N}$ and $G = (G, m, i, e)$ be a Lie group with the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. With $\exp_{T^k G}$ as in Theorem E.1.1, we have an isomorphism of multilinear bundles*

$$\exp_{T^k G}|_{\mathcal{P}_{0,+}^k}^- : G \times \bigoplus_{I \in \mathcal{P}_{0,+}^k} \lambda_I \mathfrak{g} \rightarrow T^k G|_{\mathcal{P}_{0,+}^k}^-.$$

This isomorphism restricts to a polynomial map

$$\exp_{(T^k G)_e}|_{\mathcal{P}_{0,+}^k}^- : \bigoplus_{I \in \mathcal{P}_{0,+}^k} \lambda_I \mathfrak{g} \rightarrow (T^k G)_e|_{\mathcal{P}_{0,+}^k}^-.$$

and turns $((T^k G)_e|_{\mathcal{P}_{0,+}^k}^-, m|_{\mathcal{P}_{0,+}^k}^-, i|_{\mathcal{P}_{0,+}^k}^-, e)$ into a polynomial group. The induced Lie bracket $[\cdot, \cdot]_k$ on $\bigoplus_{I \in \mathcal{P}_{0,+}^k} \lambda_I \mathfrak{g}$ is given by

$$[\lambda_I v, \lambda_J w]_k = \lambda_I \lambda_J [v, w]$$

for $I, J \in \mathcal{P}_{0,+}^k$ and $v, w \in \mathfrak{g}$.

Proof. With the functoriality of the restriction to $\mathcal{P}_{0,+}^k$ and the definition of $-$ in Lemma B.2.9, this follows immediately from Theorem E.1.1. \square

E.2. Higher Order Diffeomorphism Groups

This section is mainly taken from [10, Section 28, p.139ff.]. We subsequently generalize these results to the group of vector bundle automorphisms.

Let M be a manifold modelled on E_0 . We denote the space of sections of the fiber bundle $T^k M \rightarrow M$ by $\mathfrak{X}^k(M)$. A chart $\varphi: U \rightarrow V$ of M gives rise to a bundle chart $T^k \varphi: T^k U \rightarrow T^k V$ and in this chart a section $X: M \rightarrow T^k M$ is given by

$$X^\varphi(x) := T^k \varphi \circ X \circ \varphi^{-1}(x) = x + \sum_{I \in \mathcal{P}_+^k} \varepsilon_I X_I^\varphi(x), \quad x \in V,$$

with smooth maps $X_I^\varphi: V \rightarrow E_0$. There exists a natural group structure on $\mathfrak{X}^k(M)$ that, in a chart representation, is given by

$$(X \cdot Y)^\varphi(x) = x + \sum_{I \in \mathcal{P}_+^k} \varepsilon_I \left(X_I^\varphi(x) + Y_I^\varphi(x) + \sum_{\ell=2}^{|I|} \sum_{\nu \in \mathcal{P}_\ell(I)} \sum_{j=1}^{\ell} d^{\ell-1} X_{\nu_j}^\varphi(x) (Y_{\nu_1}^\varphi(x), \dots, \widehat{Y_{\nu_j}^\varphi(x)}, \dots, Y_{\nu_\ell}^\varphi(x)) \right), \quad (\text{E.1})$$

where X_I^φ and Y_I^φ are the respective summands in the chart representation of $X^\varphi, Y^\varphi \in \mathfrak{X}^k(M)$ and $\widehat{Y_{\nu_j}^\varphi(x)}$ means that this term is omitted. The inclusions of the axes $TM \rightarrow \varepsilon_I TM \subseteq T^k M$ induce inclusions

$$\mathfrak{X}(M) \rightarrow \mathfrak{X}^k(M), \quad X \mapsto \varepsilon_I X.$$

Sections of this type are called *vectorial*. Using vectorial sections and the group structure of $\mathfrak{X}^k(M)$, one gets a canonical bijection

$$\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}(M) \rightarrow \mathfrak{X}^k(M), \quad (\varepsilon_I X_I)_I \mapsto \prod_{I \in \mathcal{P}_+^k}^{\uparrow} \varepsilon_I X_I, \quad (\text{E.2})$$

where \prod^{\uparrow} is the product taken in $\mathfrak{X}^k(M)$ in ascending lexicographic order of the indices I (see [10, Theorem 29.2, p. 144]).

Remark E.2.1. Since we will use similar left trivializations at various other points, let us explain in some detail why (E.2) is bijective. We use induction over n in the following way. Let $X \in \mathfrak{X}^k(M)$, $n < k$ and let $X^\varphi(x) = x + \sum_{I \in \mathcal{P}_+^{n+1}} \varepsilon_I X_I^\varphi(x)$ for all charts φ . Note that the property $X_I^\varphi = 0$ for $I \notin \mathcal{P}_+^n$ is invariant under change of charts. Thus, $X^{(n)}$ defined by $(X^{(n)})^\varphi(x) = x + \sum_{I \in \mathcal{P}_+^n} \varepsilon_I X_I^\varphi(x)$ is also an element of $\mathfrak{X}^k(M)$. By induction hypothesis, we can write $X^{(n)} = \prod_{I \in \mathcal{P}_+^n}^\uparrow \varepsilon_I X_I$. It follows from (E.1) that $\tilde{X} := X \cdot (X^{(n)})^{-1}$ has the local form $\tilde{X}^\varphi(x) = x + \sum_{I \in \mathcal{P}_+^{n+1} \setminus \mathcal{P}_+^n} \varepsilon_I \tilde{X}_I^\varphi(x)$. The $\tilde{X}_{\{n+1\}}^\varphi$ are the local description of an element $\varepsilon_{\{n+1\}} \tilde{X}_{\{n+1\}} \in \mathfrak{X}^k(M)$ because other components do not contribute under change of charts. Then, in the local description of $\tilde{X} \cdot \varepsilon_{\{n+1\}}(-\tilde{X}_{\{n+1\}})$, only components with $I \in \mathcal{P}_+^{n+1}$, $n+1 \in I$ and $I \neq \{n+1\}$ contribute. Continuing this process inductively with all remaining sets $I \in \mathcal{P}_+^{n+1} \setminus \mathcal{P}_+^n$ in lexicographic order finishes the proof.

This turns $\mathfrak{X}^k(M)$ into a k -multilinear space that is a polynomial group of degree at most k with the induced vector space structure. As a consequence, one has the following theorem.

Theorem E.2.2 ([10, Theorem 29.3, p.145]). *Let $k \in \mathbb{N}$ and let M be a manifold. There exists a unique bijective exponential map*

$$\exp_k: \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}(M) \rightarrow \mathfrak{X}^k(M)$$

such that:

- (a) In every chart representation \exp_k is a polynomial map,
- (b) for all $n \in \mathbb{Z}$ and $X \in \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}(M)$ we have $\exp_k(nX) = (\exp_k(X))^n$ and
- (c) we have $\exp_k(\varepsilon_I X) = \varepsilon_I X$ for all $I \in \mathcal{P}_+^k$, $X \in \mathfrak{X}(M)$.

Moreover, the multiplication on $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}(M)$ with respect to \exp_k is given by the BCH multiplication with respect to the Lie bracket

$$\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}(M) \times \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}(M) \rightarrow \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}(M),$$

$$(\varepsilon_I(X_I, Y_I))_{I \in \mathcal{P}_+^k} \mapsto \sum_{I \in \mathcal{P}_+^k, \nu \in \mathcal{P}_2(I)} (\varepsilon_{\nu_1} \varepsilon_{\nu_2} [X_{\nu_1}, Y_{\nu_2}] + \varepsilon_{\nu_2} \varepsilon_{\nu_1} [X_{\nu_2}, Y_{\nu_1}]),$$

where $[\cdot, \cdot]$ denotes the Lie bracket of $\mathfrak{X}(M)$. The inversion with respect to \exp_k is simply the multiplication with -1 and both multiplication and inversion are morphisms of k -multilinear spaces.

Proof. For the most part, this is just [10, Theorem 29.3, p.145]. That we have the desired Lie bracket follows readily from the multiplication formula in [10, Theorem 29.2, p.144]. With this Lie bracket, one sees from the formula of the BCH multiplication that \exp_k is a morphism of k -multilinear spaces. For the inversion this is obvious. \square

Moreover, the pullback by diffeomorphisms defines a group action by automorphisms:

$$\text{Ad}: \text{Diff}(M) \times \mathfrak{X}^k(M) \rightarrow \mathfrak{X}^k(M), \quad (f, X) \mapsto \text{Ad}_f(X) := T^k f \circ X \circ f^{-1}.$$

The resulting semidirect product $\mathfrak{X}^k(M) \rtimes \text{Diff}(M)$ is isomorphic to the group $\text{Diff}_{T^k\mathbb{R}}(T^k M)$ of automorphisms of $T^k M$ that are smooth over the ring $T^k\mathbb{R}$. This is primarily seen in [10, Theorem 28.3(3), p.140], but since in the theorem the action of the diffeomorphisms is not given on the level of $\mathfrak{X}^k(M)$, we will briefly explain why our description is the same. It is shown in [10, Theorem 28.3(1), p.139] that any $X \in \mathfrak{X}^k(M)$ can be uniquely extended to a diffeomorphism $\tilde{X} \in \text{Diff}_{T^k\mathbb{R}}(T^k M)$ such that $\tilde{X} \circ z = X$ for the zero section $z: M \rightarrow T^k M$. Furthermore, it is shown that the action of $f \in \text{Diff}(M)$ on \tilde{X} is given by $T^k f \circ \tilde{X} \circ T^k f^{-1}$. Clearly, we have $T^k f^{-1} \circ z = z \circ f^{-1}$. Therefore, it follows

$$T^k f \circ \tilde{X} \circ T^k f^{-1} \circ z = T^k f \circ \tilde{X} \circ z \circ f^{-1} = T^k f \circ X \circ f^{-1}.$$

In other words, $T^k f \circ \tilde{X} \circ T^k f^{-1}$ is an extension of $T^k f \circ X \circ f^{-1}$ and by uniqueness this shows that both descriptions are the same.

Remark E.2.3. Let $\mathfrak{X}_c^k(M) \subseteq \mathfrak{X}^k(M)$ be the sections of $T^k M$ with compact support, i.e., if $X \in \mathfrak{X}_c^k(M)$, then there exists a smallest compact set $\text{supp}(X) \subseteq M$ such that $X|_{M \setminus \text{supp}(X)}$ equals the zero section. With the local product formula, we easily see that for $X, Y \in \mathfrak{X}_c^k(M)$, we have $\text{supp}(X \cdot Y) \subseteq \text{supp}(X) \cap \text{supp}(Y)$. On the other hand, if X has compact support but Y does not, or vice versa, then $(X \cdot Y)$ clearly does not have compact support. Thus, $\mathfrak{X}_c^k(M)$ is closed under multiplication and inversion and therefore a subgroup of $\mathfrak{X}^k(M)$. Of course, the inclusions of the axes are now given by $\mathfrak{X}_c(M) \rightarrow \varepsilon_I \mathfrak{X}_c(M) \rightarrow \mathfrak{X}_c^k(M)$ and by the same argument as for the inversion, the bijection (E.2) restricts to a bijection

$$\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}_c(M) \rightarrow \mathfrak{X}_c^k(M).$$

Furthermore, one also sees that for $f \in \text{Diff}(M)$ and $X \in \mathfrak{X}_c^k(M)$, we have $\text{supp}(T^k f \circ X \circ f^{-1}) = f(\text{supp}(X))$ and therefore the semidirect product $\mathfrak{X}_c^k(M) \rtimes \text{Diff}(M)$ is defined and we get a respective subgroup $\text{Diff}_{T^k\mathbb{R}}(T^k M)_c \subseteq \text{Diff}_{T^k\mathbb{R}}(T^k M)$. Finally, we denote by $\text{Diff}_{T^k\mathbb{R}}(T^k M)_c^c$ the subgroup of $\text{Diff}_{T^k\mathbb{R}}(T^k M)$ that corresponds to $\mathfrak{X}_c^k(M) \rtimes \text{Diff}_c(M)$.

Lemma E.2.4. *If M is a σ -compact finite-dimensional manifold, then there exist natural isomorphisms of groups*

$$(T^k \text{Diff}(M))_{\text{id}} \cong \mathfrak{X}_c^k(M)$$

and

$$T^k \text{Diff}(M) \cong \text{Diff}_{T^k\mathbb{R}}(T^k M)_c.$$

The latter restricts to an isomorphism

$$T^k \text{Diff}_c(M) \cong \text{Diff}_{T^k \mathbb{R}}(T^k M)_c^c.$$

Proof. We identify $(T^k \text{Diff}(M))_{\text{id}}$ and $\mathfrak{X}_c^k(M)$ with $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathfrak{X}_c(M)$ via the respective exponential maps. It follows from Theorem E.1.1 and Theorem E.2.2 that both bijections induce exactly the same group structure on the direct sum. Since the groups $T^k \text{Diff}(M)$ and $\text{Diff}_{T^k \mathbb{R}}(T^k M)_c$ both arise as semidirect products with the above direct sum, all that is left to show is that the action of $\text{Diff}(M)$ on the direct sum is the same for both groups. For this, it suffices to compare the actions on the axes. Let $f \in \text{Diff}(M)$ and $X \in \mathfrak{X}_c(M)$. In the case of $\varepsilon_I X \in T^k \text{Diff}(M)$, we calculated in Corollary E.1.2 that f acts by $\text{Ad}_f(\varepsilon_I X) = \varepsilon_I T f \circ X \circ f^{-1}$. If we interpret $\varepsilon_I X \in \mathfrak{X}_c^k(M)$, then we have seen above that the action of f is given by $T^k f \circ \varepsilon_I X \circ f^{-1}$ but by Lemma B.2.8, we have

$$T^k f \circ \varepsilon_I X \circ f^{-1} = \varepsilon_I T f \circ X \circ f^{-1}. \quad \square$$

It is apparent that Bertram defined $\text{Diff}_{T^k \mathbb{R}}(T^k M)$ with the above connection to $T^k \text{Diff}(M)$ in mind.

E.3. Higher Order Bundle Automorphism Groups

Definition E.3.1. Let $\pi: F \rightarrow M$ be a vector bundle with typical fiber E_1 , where M is modelled on E_0 . We define the subspace of bundle sections $\mathcal{X}(F) \subseteq \mathfrak{X}(F)$ as the sections $X: F \rightarrow TF$ such that for every bundle chart $\varphi: V \rightarrow U \times E_1$, we have the local description $X^\varphi = (X_0^\varphi, X_1^\varphi)$ with smooth maps $X_0^\varphi: U \rightarrow E_0$ and $X_1^\varphi: U \times E_1 \rightarrow E_1$, where the latter is linear in the second component.

Lemma E.3.2. *In the situation of Definition E.3.1, $\mathcal{X}(F)$ is a Lie subalgebra of $\mathfrak{X}(F)$. If $\pi': F' \rightarrow M'$ is another vector bundle and $X \in \mathcal{X}(F)$ and $X' \in \mathcal{X}(F')$ are Ψ -related for a vector bundle isomorphism $\Psi: F \rightarrow F'$, then we have $X' \in \mathcal{X}(F')$.*

Proof. Let $\varphi: V \rightarrow U \times E_1$ be a bundle chart of F and $X, Y \in \mathcal{X}(F)$. For $(x, v) \in U \times E_1$, we calculate

$$\begin{aligned} dX^\varphi((x, v), Y_0^\varphi(x, v)) \\ = (dX_0^\varphi(x, Y_0^\varphi(x)), d_1 X_1^\varphi((x, v), Y_0^\varphi(x)) + X_1^\varphi(x, Y_1^\varphi(x, v))). \end{aligned}$$

The second component of this expression is linear in v and thus so is the second component of

$$[X, Y]^\varphi(x, v) = dX^\varphi((x, v), Y^\varphi(x, v)) - dY^\varphi((x, v), X^\varphi(x, v)).$$

For the second claim, let F' have typical fiber E'_1 , let M' be modelled on E'_0 and let $\varphi': V' \rightarrow U' \times E'_1$ be a bundle chart of F' . Let $\psi := (\psi_0, \psi_1)$ be the local

description of Ψ in φ and φ' . After restricting, we may assume $\psi_0(U) = U'$. We have

$$(T\Psi \circ X \circ \Psi^{-1})^{\varphi'} = \text{pr}_2 \circ d\psi \circ (\text{id}_{U \times E_1}, X^\varphi) \circ \psi^{-1}.$$

Let $\psi^{-1} = (\psi_0^{-1}, \psi_1^{-1})$. For $(x, v) = (\psi^{-1}(x'), \psi_1^{-1}(x', v'))$, where $(x', v') \in U' \times E_1$, we calculate

$$\begin{aligned} & \text{pr}_2 \circ d\psi \circ (\text{id}_{U \times E_1}, X^\varphi)(x, v) \\ &= \left(d\psi_0(x, X_0^\varphi(x)), d_1\psi_1((x, v), X_0^\varphi(x)) + \psi_1(x, X_1^\varphi(x, v)) \right). \end{aligned}$$

Because the second component of this expression is linear in v , we have shown $T\Psi \circ X \circ \Psi^{-1} \in \mathcal{X}(F')$. \square

Lemma/Definition E.3.3. Let $\pi: F \rightarrow M$ be a vector bundle with typical fiber E_1 , where M is modelled on E_0 . We define $\mathcal{X}^k(F) \subseteq \mathfrak{X}^k(F)$ as those sections $X: F \rightarrow T^k F$ that with respect to a bundle chart $\varphi: V \rightarrow U \times E_1$ (and the respective chart $T^k\varphi$) are represented by

$$X^\varphi(x, v) = (x, v) + \sum_{I \in \mathcal{P}_+^k} \varepsilon_I X_I^\varphi(x, v),$$

with $X_I^\varphi = (X_{I,0}^\varphi, X_{I,1}^\varphi)$. Here $X_{I,0}^\varphi: U \rightarrow E_0$ and $X_{I,1}^\varphi: U \times E_1 \rightarrow E_1$ are smooth maps and the latter is linear in the second component. Then $\mathcal{X}^k(F)$ is a polynomial subgroup of $\mathfrak{X}^k(F)$ and the trivialization (E.2) restricts to the trivialization

$$\bigoplus_{I \in \mathcal{P}_+^k} \mathcal{X}(F) \rightarrow \mathcal{X}^k(F), \quad (X_I)_I \mapsto \prod_{I \in \mathcal{P}_+^k}^{\uparrow} \varepsilon_I X_I,$$

where the product is taken in $\mathcal{X}^k(F)$ in ascending lexicographic order. Likewise, the exponential map from Theorem E.2.2 restricts to a bijective map

$$\exp_k: \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathcal{X}(F) \rightarrow \mathcal{X}^k(F)$$

with analogous properties. Finally, we define

$$\text{Aut}_{T^k\mathbb{R}}(T^k F) := \mathcal{X}^k(F) \rtimes \text{Aut}(F)$$

which is a subgroup of $\text{Diff}_{T^k\mathbb{R}}(T^k F)$.

Proof. Since $\mathfrak{X}^k(F)$ is a polynomial group, every Lie subalgebra is also a polynomial subgroup. Thus, $\mathcal{X}^k(F)$ is a polynomial subgroup by Lemma E.3.2. We have again inclusions $\mathcal{X}(F) \hookrightarrow \mathcal{X}^k(F)$, $X \mapsto \varepsilon_I X$ of the axes and we use the same arguments as in Remark E.2.1 to see that the trivialization is bijective. From this, we deduce that the exponential map restricts as needed. Lemma E.3.2 shows that the action of $\text{Aut}(F) \subseteq \text{Diff}(F)$ on $\mathcal{X}^k(F)$ is well-defined. \square

Remark E.3.4. Bertram introduces an additional infinitesimal generator to describe the structure of $T^k F$ in [10, Theorem 15.5, p.81], which one could also use to define $\text{Aut}_{T^k \mathbb{R}}(T^k F)$. However, this does not fit our needs because it would not correspond to the way the generators λ_I relate to superdiffeomorphisms.

Definition E.3.5. Let $F \rightarrow M$ be a vector bundle such that M is finite-dimensional and let $\pi_F: TF \rightarrow F$, as well as $\pi_M: F \rightarrow M$, be the projections. For $k \in \mathbb{N}$, we define $\text{supp}(X)$, the *support* of $X \in \mathcal{X}^k(F)$, as the smallest closed subset $K \subseteq M$ such that $X|_{\pi_F^{-1}(\pi_M^{-1}(M \setminus K))} = 0$. We then let

$$\mathcal{X}_c^k(F) := \{X \in \mathcal{X}^k(F) : \text{supp}(X) \text{ is compact}\}.$$

Remark E.3.6. Let $\pi: F \rightarrow M$ be a vector bundle with typical fiber E_1 . Comparing Definition E.3.1 and Remark 4.1.15, we see that $\mathcal{X}(F)$ and $\mathcal{X}(\iota_1^1(F))_{\bar{0}}$ can be identified. Taking into account Definition 4.1.8 and Definition E.3.5, the same holds for $\mathcal{X}_c(F)$ and $\mathcal{X}_c(\iota_1^1(F))_{\bar{0}}$ if M is finite-dimensional. If, in addition, E_1 is a Banach space, we denote by $\mathcal{X}_c(F)_b$ the space of sections $\mathcal{X}_c(F)$ equipped with the topology induced by $\mathcal{X}_c(\iota_1^1(F))_{\bar{0},b}$. In other words, we consider $X \in \mathcal{X}_c(F)_b$ to have the local form

$$X^\varphi = (X_0^\varphi, X_1^\varphi) \in \mathcal{C}^\infty(U_\varphi, E_0) \times \mathcal{C}^\infty(U_\varphi, \text{Gl}_{E_1}),$$

where E_0 is the model space of M .

Lemma E.3.7. *Let F be a vector bundle with finite-dimensional base manifold M . Then $\mathcal{X}_c(F)$ is a Lie subalgebra of $\mathcal{X}(F)$ and $\mathcal{X}_c^k(F)$ is a polynomial subgroup of $\mathcal{X}^k(F)$ for every $k \in \mathbb{N}$. Moreover, the left trivialization and the exponential map restrict to bijections*

$$\bigoplus_{I \in \mathcal{P}_+^k} \mathcal{X}_c(F) \rightarrow \mathcal{X}_c^k(F), \quad (X_I)_I \mapsto \prod_{I \in \mathcal{P}_+^k}^{\uparrow} \varepsilon_I X_I$$

and

$$\exp_k^c: \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathcal{X}_c(F) \rightarrow \mathcal{X}_c^k(F).$$

Finally, the restricted action of $\text{Aut}_c(F)$ on $\mathcal{X}_c^k(F)$ is well-defined and we let

$$\text{Aut}_{T^k \mathbb{R}}(T^k F)_c := \mathcal{X}_c^k(F) \rtimes \text{Aut}_c(F).$$

Proof. It follows immediately from the local description in Lemma E.3.2 that $\text{supp}(X \cdot Y) \subseteq \text{supp}(X) \cap \text{supp}(Y)$. Thus, we can use the same arguments as in Remark E.2.3 to see that $\mathcal{X}_c^k(F)$ is a subgroup of $\mathcal{X}^k(F)$ and that the left trivialization restricts as claimed. This implies that the exponential map restricts as well. It also follows from the calculations in Lemma E.3.2 that the action of $\text{Aut}_c(F)$ on $\mathcal{X}_c^k(F)$ is well-defined. \square

Lemma/Definition E.3.8. Let $F \rightarrow M$ be a vector bundle and $k \in \mathbb{N}$. We consider the polynomial group $\mathcal{X}^k(F) = (\mathcal{X}^k(F), m, i, 0)$ as a k -multilinear space via $\exp_k: \bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathcal{X}(F) \rightarrow \mathcal{X}^k(F)$. Then $\mathcal{X}^k(F)|_{\mathcal{P}_{0,+}^k}^- = (\mathcal{X}^k(F)|_{\mathcal{P}_{0,+}^k}^-, m|_{\mathcal{P}_{0,+}^k}^-, i|_{\mathcal{P}_{0,+}^k}^-, 0)$ is also a polynomial group and the group structure induced on $\bigoplus_{I \in \mathcal{P}_{0,+}^k} \lambda_I \mathcal{X}(F)$ via $\exp_k|_{\mathcal{P}_{0,+}^k}^-$ is given by the BCH multiplication with regard to the Lie bracket defined by

$$(\lambda_I X, \lambda_J Y) \mapsto \lambda_I \lambda_J [X, Y] \quad \text{for } X, Y \in \mathcal{X}(F),$$

where $[\cdot, \cdot]$ denotes the Lie bracket of $\mathcal{X}(F)$. We have a group action by automorphisms

$$\text{Ad}|_{\mathcal{P}_{0,+}^k}^- : \text{Aut}(F) \times \mathcal{X}^k(F) \rightarrow \mathcal{X}^k(F), \quad (f, X) \mapsto \text{Ad}_f|_{\mathcal{P}_{0,+}^k}^-(X)$$

and the induced action on $\bigoplus_{I \in \mathcal{P}_{0,+}^k} \lambda_I \mathcal{X}(F)$ is given by

$$(f, \lambda_I X) \mapsto \lambda_I \text{Ad}_f(X) = \lambda_I T f \circ X \circ f^{-1}$$

for $f \in \text{Aut}(F)$, $X \in \mathcal{X}(F)$ and $I \in \mathcal{P}_{0,+}^k$. We define the group

$$\text{Aut}_{T^k \mathbb{R}}(T^k F)|_{\mathcal{P}_{0,+}^k}^- := \mathcal{X}^k(F)|_{\mathcal{P}_{0,+}^k}^- \rtimes \text{Aut}(F).$$

If M is finite dimensional, we analogously define

$$\text{Aut}_{T^k \mathbb{R}}(T^k F)_c|_{\mathcal{P}_{0,+}^k}^- := \mathcal{X}_c^k(F)|_{\mathcal{P}_{0,+}^k}^- \rtimes \text{Aut}_c(F)$$

via \exp_k^c from Lemma E.3.7.

Proof. That $\mathcal{X}^k(F)|_{\mathcal{P}_{0,+}^k}^-$ is a group is obvious since m and i are morphisms of k -multilinear spaces and the involved functors preserve products. The description of the Lie bracket of $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathcal{X}(F)$ from Theorem E.2.2 shows that the Lie bracket is as claimed. Since Ad_f is an automorphism of the k -multilinear space $\mathcal{X}^k(F)$, $\text{Ad}_f|_{\mathcal{P}_{0,+}^k}^-$ acts well-defined as needed. The same arguments carry over to the case of compact support. \square

Lemma E.3.9. Let $k \in \mathbb{N}$ and F be a Banach vector bundle over a σ -compact finite-dimensional base M . Then we have isomorphisms of groups

$$T^k \text{Aut}_c(F) \cong \text{Aut}_{T^k \mathbb{R}}(T^k F)_c$$

and

$$T^k \text{Aut}_c(F)|_{\mathcal{P}_{0,+}^k}^- \cong \text{Aut}_{T^k \mathbb{R}}(T^k F)_c|_{\mathcal{P}_{0,+}^k}^-.$$

Proof. By Theorem E.1.1 and Lemma E.3.7, the Lie group $(T^k \text{Aut}_c(F))_{\text{id}}$ and the polynomial group $\mathcal{X}_c^k(F)$ induce the same group structure on $\bigoplus_{I \in \mathcal{P}_+^k} \varepsilon_I \mathcal{X}_c(F)$ because the induced Lie algebra on $\mathcal{X}_c(F)$ is the same by Lemma D.3.9. That $\text{Aut}_c(F)$ acts the same way in both cases follows like in Lemma E.2.4 using Lemma D.3.10 and Corollary E.1.2. From this, we also obtain the second isomorphism. \square

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