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Approximate Pure Nash Equilibria in Congestion, Opinion Formation and Facility Location Games

Dissertation

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Abstract

This thesis investigates approximate pure Nash equilibria in different game-theoretic models. In such an outcome, no player can improve her objective by more than a given factor through a deviation to another strategy.

In the first part, we investigate two variants of Congestion Games in which the existence of pure Nash equilibria is guaranteed through a potential function argument. However, the computation of such equilibria might be hard. We construct and analyze approximation algorithms that enable the computation of states with low approximation factors in polynomial time. To show their guarantees we use sub games among players, bound the potential function values of arbitrary states and exploit a connection between Shapley and proportional cost shares. Furthermore, we apply and analyze sampling techniques for the computation of approximate Shapley values in different settings.

In the second part, we concentrate on the existence of approximate pure Nash equilibria in games in which no pure Nash equilibria exist in general. In the model of Coevolving Opinion Formation Games, we bound the approximation guarantees for natural states nearly independent of the specific definition of the players' neighborhoods by applying a concept of virtual costs. For the special case of only one influential neighbor, we even show lower approximation factors for a natural strategy. Then, we investigate a two-sided Facility Location Game among facilities and clients on a line with an objective function consisting of distance and load. We show tight bounds on the approximation factor for settings with three facilities and infinitely many clients. For the general scenario with an arbitrary number of facilities, we bound the approximation factor for two promising candidates, namely facilities that are uniformly distributed and which are paired.

Zusammenfassung

Diese Dissertation untersucht approximative reine Nash-Gleichgewichte in verschiedenen spieltheoretischen Modellen. In einem solchen Zustand kann sich kein Spieler durch Veränderung seiner Strategie um einen gegebenen Faktor verbessern.

Im ersten Teil der Arbeit beschäftigen wir uns mit zwei Varianten von Congestion Games. In diesen Modellen ist die Existenz von reinen Nash-Gleichgewichten durch Potentialfunktionen garantiert, jedoch kann deren Berechnung schwierig sein. Wir analysieren Approximationsalgorithmen zur Berechnung von Zuständen mit kleinen Approximationsfaktoren. Für die Analyse benutzen wir Teilspiele, wir beweisen Schranken für die Potentialfunktion eines beliebigen Zustandes und zeigen eine Relation zwischen Shapley und proportionalen Kostenaufteilungen. Zusätzlich wenden wir Sampling-Methoden zur Approximation von Shapley-Werten in verschiedenen Szenarien an.

Im zweiten Teil konzentrieren wir uns auf die Existenz von approximativen reinen Nash-Gleichgewichten in Spielen, in denen im Allgemeinen keine reinen Gleichgewichte existieren. In einem Coevolving Opinion Formation Game können wir niedrige Approximationsgarantien für zwei natürliche Zustände zeigen, die unabhängig von der Definition der Nachbarschaft sind. Hierzu wenden wir ein Konzept von virtuellen Kosten an. Für den Spezialfall nur eines Nachbarn zeigen wir noch stärkere Approximationsfaktoren für einen Zustand, der durch eine natürliche Strategie entsteht. Des Weiteren untersuchen wir ein zweiseitiges Facility Location Game zwischen Facilities und Clients, die als Zielfunktion eine Kombination von Distanz und Auslastung haben. Hier zeigen wir scharfe Schranken für das Szenario mit drei Facilities und unendlich vielen Clients. Für das generelle Szenario mit einer beliebigen Anzahl an Facilities zeigen wir Approximationsfaktoren für einen Zustand mit gleichverteilten Facilities sowie mit Paaren von Facilities.

Contents

Preface

Writing this thesis was like a marathon that finally ended at the finish line by writing these sentences. Many ups and downs have accompanied me on this path, lots of new experiences I made on the way and many new people I met during this time. After an intensive final sprint, now everything comes to a great and satisfying end. The last five years have been a wonderful and valuable time. I've had the chance to meet a lot of fantastic people and places around the world and work on exciting and challenging topics, and I had a lot of fun.

Now it's time to say thank you: First and foremost, I would like to thank my adviser, Alexander Skopalik, who made all this possible for me. He introduced me to the topic of algorithmic game theory and I was allowed to work with him on many questions in this research field. I don't know how many hours we spent together with the research. He had time for even the smallest questions and he always gave me good and direct feedback. Right from the start he motivated me to take part in many workshops and research trips and introduced me to many people from the community so that I could quickly build up a good network. The European algorithmic game theory community consists of great people, many of whom I got to know over the last years, and it has many interesting connections to other research fields in computer science, operations research and economics.

All this would not be possible without my university here in Paderborn which provides a perfect research environment, especially in the Department of Computer Science, but also in the Heinz Nixdorf Institute. I would like to thank Friedhelm and the whole research group for the environment with all the people, who are not only colleagues but also friends. This also holds for all other research groups in the department, in particular, all the theory groups and our support staff.

Much of this work would not be possible without financing. Therefore, in the first place, I would like to thank the DFG with the Collaborative Research Centre 901, which made many things possible! But not only financially, also related to content – it offered a great platform for discussion and research on an interesting, interdisciplinary and highly relevant topic in an interdisciplinary environment. Here, our sub project A3, the K-Team, the board meetings, the doctoral seminars, etc. provided good opportunities for discussion and participation. Last but not least I really appreciate the support from the management in all kinds of issues.

But a good environment is not enough, you also need good collaborators to do good research. Besides Alex there are all my co-authors, the other scientific members of the research group, the student assistants, theses, fellows, etc. to

mention. Without all of them this thesis would not be possible. And in particular I would like to thank Martin Gairing who accompanied me from the very beginning through our joint papers and now also agreed to take part in this thesis as a reviewer.

After the intensive collaboration and research during the past years, there is one final document at the end, which I had to accomplish alone and which is now done with these sentences. Besides many great tools and services that made my life easier, I would like to thank everyone who gave me feedback and/or proofread parts of the thesis.

Finally I would like to thank Alex and Martin as well as all the other members of my doctoral panel who have agreed to participate in the completion of my doctorate: Friedhelm Meyer auf der Heide, Claus-Jochen Haake and Rainer Feldmann.

Last but not least, I thank my friends and my family. They had to give up my presence several times in the last years and especially the last weeks and still remained faithful to me. In particular I would like to thank my parents, who supported me in all my steps in life and with all my decisions!

This phase of my life is now coming to an end and I look back happily on the past years and all the experiences. Now it's time to look ahead, and I'm looking forward to the new challenges that come my way!

Thank you all!

Paderborn, November 2018 *Matthias Feldotto*

CHAPTER 1

Introduction

Game theory is the study of mathematical models of strategic interaction between rational decision makers [Mye97]. While it has been one of the main research areas in economics for decades, it recently became popular at the intersection of computer science and economic theory, largely motivated by the emergence of the Internet. Driven by computational questions that arise in traditional game-theoretic models as well as in new models describing applications in computer science, computer scientists have increasingly contributed to this research field.

The central element in this research field is a game that serves as a model for an interactive situation among at least two rational and intelligent players, where each player's payoff is determined by the strategies of all participating players. In a *non-cooperative game* players are competing against each other so that only self-enforcing coalitions are possible due to the absence of external means to enforce cooperative behavior. Rationality ensures that players only make choices that optimize their payoff. Intelligence means that the players have all the information that they need for this decision. In an equilibrium, we have some kind of stable state so that there is no possibility or incentive for the players to improve their current situation. Many different equilibrium solution concepts using different definitions and assumptions have been developed to capture this informal idea.

One notion that formally captures this concept is the *pure Nash equilibrium* that is a very popular solution concept due to its natural and simple definition. Given a set of players with different pure strategies, a pure Nash equilibrium is a state in the game in which no player can improve the value of her objective function by deviating to another pure strategy assuming the strategies of the other players are fixed. However, there are also several drawbacks in this concept: The assumptions are very strong and do not realistically model the behavior of participants in the real world. Additionally, there are many games in which the existence of a pure Nash equilibrium is not guaranteed or in which the computation of such an equilibrium is not efficiently possible. If the solution does not exist, or is not computable, how powerful is such a solution concept to make statements about the scenario it models? While they are one of the most popular solution concepts, pure Nash equilibria are not the only solution concept available by far. Over the years, lots of different variants or even completely new equilibrium concepts have been developed and investigated. It highly depends on the scenario and the research questions to determine which equilibrium concept is the most appropriate.

In this thesis, we focus on one alternative solution concept, the *approximate* *pure Nash equilibrium.* It extends the concept of the pure Nash equilibrium in the following way: A player deviates only to another strategy if she can improve her objective by more than a given factor. This solution concept is mainly investigated by the computer science community. Its definition and methods are motivated by the study of approximation algorithms which has been a well-studied field in theoretical computer science for decades.

Typical research questions in the study of approximate pure Nash equilibria are similar to the work with other solution concepts. First of all, we are interested in the existence or non-existence of equilibria in the considered model and game class. Additionally, motivated by computer science, the computational decision problem whether an equilibrium exists becomes more interesting if existence is not always guaranteed in the respective model. Furthermore, we study different natural dynamics and ask the question of convergence: Given a predefined dynamic and an arbitrary state of the game, can an equilibrium be reached after finite time or even in polynomial time? A slightly weaker question tries to analyze whether it is at least possible to compute such an equilibrium. Can we state a polynomial algorithm that computes an equilibrium? Or can we prove the computational hardness of this problem? Lastly, we are interested in the quality of our equilibria in terms of the costs of the absence of cooperation. Therefore, we compare the set of equilibria with a centrally computed optimal outcome for all players which does not necessary need to be an equilibrium.

The next section in this introduction elaborates on the concept of approximate equilibria and compares it to similar solution concepts. In the further course of this thesis we focus mainly on two of the mentioned research questions: In the first part we consider models in which the existence of pure Nash equilibria is given, but they can not be reached efficiently. Here, we focus on the *computation* problem for approximate equilibria: Can we construct algorithms that efficiently compute approximate pure Nash equilibria with low approximation factors? We show this by analyzing such algorithms. In the second part we consider two models for which the non-existence of pure equilibria is known. Here, we ask whether approximate equilibria exist, or more specifically for which factors we can prove the *existence* of an approximate equilibrium.

Both parts of this thesis deal with completely different models: In the first part, we work with two variants of *Congestion Games*, a very powerful and simple model which covers the topic of resource allocation. It has been extensively studied in recent years, especially in the algorithmic game theory community. In the second part, we work with two model classes that are especially popular in the economics literature: *Opinion Formation Games* and *Facility Location Games.* Compared to the more technically and economically motivated model of Congestion Games, these two are mainly motivated by the behavior and decisions of humans in the context of opinion formation and product differentiation.

1.1. Approximate Pure Nash Equilibrium

In non-cooperative game theory, and especially in algorithmic game theory, the pure Nash equilibrium and the mixed Nash equilibrium are often-used solution concepts. A pure Nash equilibrium provides a complete definition of how all players act in the game, so it assigns each player exactly one strategy out of her set of possible strategies. In contrast, a mixed Nash equilibrium only describes an assignment of probabilities to each strategy for each player. We know by Nash [Nas51] that every game with a finite number of players and a finite set of strategies has a mixed Nash equilibrium which is not true for the concept of the pure Nash equilibrium.

However, there is a heavy discussion in the literature about the usefulness of the concept. The main criticisms are based on the central technique of randomization which lacks behavioral support [Aum87]. Players do not make choices following a lottery, in general. And even further, people are not able to generate random outcomes without the help of random generators. But also the interpretation of such mixed strategies is difficult, since the actual outcome may be hard to imagine.

Other researchers justify the use of mixed strategies with the lack of knowledge [Har73] or the interpretation of the strategies as beliefs instead of actions [AB95]. A much simpler view on mixed strategies is based on a large population of players. Each individual agent only plays one pure strategy, but a player in the game-theoretic sense represents a mass of agents. Then the mixed strategy represents the distribution of pure strategies chosen by each population. While this interpretation is useful for several scenarios, it fails for games in which each player is an individual agent. We leave this discussion open in the thesis and refer to [Rub91] for a further overview. Instead we focus only on pure strategies.

However, as already stated, there are issues that question the powerfulness of the concept of pure Nash equilibria which we want to discuss in more detail now.

Non-Existence There exist many games in which we can not guarantee the existence of any pure Nash equilibrium, e.g., in the popular game Rock–paper-scissors. The solution concept becomes at least questionable if we can only state that there is no equilibrium since this gives no further insights in the considered scenario.

Non-Convergence Other games always have a pure Nash equilibrium, but no natural dynamics (e.g., best-response behavior) converge to such an equilibrium. So how should players act as it is given by the strategy profile?

Inefficient Convergence The convergence of dynamics needs exponentially many improvement steps, e.g., in Congestion Games [Ros73]. It is unlikely that strategic players then follow such dynamics and the question for alternatives arises.

Computational Hardness Furthermore we consider games in which pure Nash equilibria exist, but the computation of such takes exponentially long. If even a central algorithm is not able to compute an equilibrium efficiently, how should strategic players locally determine such an equilibrium?

Underlying Assumptions Lastly we have to discuss the main assumptions underlying the concept of pure Nash equilibria: All players have full information and the players are always rational and selfish. These assumptions are very strong and it is often more than questionable whether they are fulfilled.

This is only a brief outline about the drawbacks of the concept, but it should become clear that it is reasonable to also work with other solution concepts that make weaker assumptions. In this thesis, we focus on the approximate pure Nash equilibrium and discuss why it may offer a reasonable alternative solution concept. In this concept possible other strategies lead to deviations only if they cause an improvement by a given factor in the player's payoff. This definition is known as the multiplicative variant of approximate equilibria (e.g., [CS11; CKS11; OPS04]). For several types of games we can achieve new insights and use the concept to explain the given scenario if the concept of pure equilibria reaches its limits.

The concept of approximate pure Nash equilibria is motivated by the huge area of research in the field of approximation algorithms. Instead of computing optimal solutions for difficult problems, researchers work only with nearly optimal solutions. There is thus a trade-off between a minimal loss of quality and a gain in speed. In many situations, especially if the approximation factor is very low (the result is close to the optimum), the quality is often reasonable for the aim of the algorithms. And it can often be proven that an efficient algorithm for the optimum does not exist, or that there is a lower bound on the approximation factor. Therefore, approximative solutions are the best results reachable in the respective situation.

In game theory we work with the same idea: Instead of the inefficient computation of exact equilibria, we only compute nearly optimal solutions. As the alternative is often the non-existence of pure equilibria, approximate equilibria are a good "second best" option. Thus, by porting computer science methods to traditional economics models, we open new areas of research and show how an interdisciplinary approach enriches research in both fields.

We investigate three different specific topics here to show reasonable further applications for the approximate concept: If we concentrate on the modeling part of game theory, approximate moves and equilibria are one way to express the behavior of strategic agents better. Players often do not have full information or have other reasons to not select the best strategy. Approximate moves are one way to model such a not completely rational behavior.

Additionally, we can ask if a player would change her strategy for a minimal improvement in her objective. In some contexts we look at the game and the strategic behavior as a process and assume costs for the players for changing their strategies. If we now consider low changes in the payoffs on the one hand and additional changing costs on the other, players aiming for minimal improvements are more than doubtful.

Lastly, we can interpret approximation factors of approximate equilibria as a metric for the stability of states. This enables us to use approximate equilibria in new contexts, e.g., in the evaluation of simulations for game-theoretic scenarios. They provide a way to reduce the running time of the simulations and also directly serve as one metric for the evaluation.

To complete this introduction and discussion, we shortly present the additive variant of approximate equilibria. In this well-studied solution concept a player has no incentive to deviate if the absolute difference between the current payoff and the new one is bounded by a small constant. This concept exactly captures the idea of focusing on changing costs and was already introduced and investigated in the context of mixed Nash equilibria, starting with the work by Radner [Rad80] (see also [Mye13] and the references within). Nevertheless, the additive variant has some drawbacks for the models and research questions that we consider in our work. We focus here on two main aspects: The first point considers the scaling of payoffs – if we have given a game in which most of the outcomes have a very low payoff, while a few have a high payoff. The additive constant does not provide any new insights for the solutions. Either the constant is negligible, resulting in the same behavior as without any approximation, or it is too high, leading to trivial outcomes with many equilibria. Such a distribution of payoffs naturally occurs in the context of Congestion Games, for example. Secondly, we are interested in the quality of the equilibria which we measure through the comparison to a centrally computed optimal solution. Since this Price of Anarchy is a ratio, the natural fitting definition is the multiplicative version of approximation.

This was just a short overview of the concept of approximate pure Nash equilibria and some related concepts to help the reader to gain a better understanding of this thesis. We refer to the literature for details and a deeper comparison of solution concepts. In the further course of this thesis, we fully concentrate on the concept of the (approximate) pure Nash equilibrium in its multiplicative variant.

1.2. Contribution of the Thesis

The main contribution of this thesis is the application of the concept of approximate pure Nash equilibria in different game classes and with different central research questions. The range of considered research questions mainly covers the existence of approximate equilibria, as well as their computation and their quality.

In **Chapter 2** we give a general and formal introduction to the basic concepts used throughout the thesis. The remainder of the thesis is then divided into three parts that each focus on different research questions.

Part I: Computation of Approximate Pure Nash Equilibria

The first part of the thesis investigates different variants of the model of *Congestion Games*. In a Congestion Game, players compete for resources and their payoffs depend on the players using the same resources. In these models, the existence of pure Nash equilibria was shown by a potential function. Nevertheless, the computation of such an equilibrium can be computationally hard, which motivates an investigation of approximate equilibria.

Chapter 3 gives a detailed introduction to this game class together with a comprehensive examination of the relevant literature. Furthermore, we introduce a common model for this thesis. It generalizes several variants of Congestion Games and we later explicitly specify it for two considered variants. The chapter also informally describes the main underlying ideas for the algorithms. The content of the chapter is based on the following two publications, but extended with further literature and a new common model:

> Matthias Feldotto, Martin Gairing, and Alexander Skopalik. **Bounding the Potential Function in Congestion Games and Approximate Pure Nash Equilibria**. In: *Proceedings of the 10th International Conference on Web and Internet Economics (WINE)*. 2014. [FGS14]

> Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria in Shapley Value Weighted Congestion Games**. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE)*. 2017. [Fel+17a]

Chapter 4 investigates the model of unweighted Congestion Games in which all players have the same influence on the resources' costs. The main contribution of this chapter is twofold: Firstly, we investigate the underlying potential function of this game. We bound the value of an arbitrary outcome of a Congestion Game by a multiple of the minimum potential value. The factor depends only on the set of cost or utility functions in the game. This gives Price-of-Anarchy-like results regarding the potential function. Secondly, we use this bound for games with non-decreasing cost functions together with algorithmic ideas from [Car+11a] to obtain approximate equilibria with an approximation factor close to the bound given through the analysis of the potential function. Our technique significantly improves the approximation factor for polynomial cost functions. Moreover, our analysis identifies large and practically relevant classes of cost functions for which approximate equilibria with small factors can be computed in polynomial time. In particular, for Congestion Games with linear functions with strictly positive offset, the factor is smaller than 2. To the best of our knowledge, this is the first work to show that approximate equilibria with factors lower than 2 are polynomial-time-computable without restricting the strategy spaces. Chapter 4 is based mainly on the following publication:

> Matthias Feldotto, Martin Gairing, and Alexander Skopalik. **Bounding the Potential Function in Congestion Games and Approximate Pure Nash Equilibria**. In: *Proceedings of the 10th Inter-*

national Conference on Web and Internet Economics (WINE). 2014. [FGS14]

In **Chapter 5** we consider the weighted variant of Congestion Games with Shapley cost-sharing. Here, the players have different influence on the resources' costs and each player has to pay for her average marginal contribution to the costs. We present an algorithm to compute approximate pure Nash equilibria in this model. The algorithm is based on ideas by [Car+15], but directly works with the underlying potential function of the game. To the best of our knowledge this is the first algorithmic result on computation of approximate equilibria in models with non-proportional cost sharing. We show that our algorithm can also be applied to the weighted model with proportional shares and achieves nearly the same approximation guarantee. To achieve the results, we again investigate different properties of the potential function. Then, we show that Shapley cost shares and proportional cost shares can be approximated by each other within a constant factor. Furthermore, we derive bounds on the approximate Price of Anarchy in this model. The results of Chapter 5 are based mainly on the following publication:

Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria in Shapley Value Weighted Congestion Games**. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE)*. 2017. [Fel+17a]

Chapter 6 presents a method and its analysis to approximate Shapley values through sampling. We extend standard methods for two scenarios and construct a Fully Polynomial-Time Randomized Approximation Scheme (FPRAS). Firstly, we again investigate the model of weighted Congestion Games with Shapley cost-sharing and improve the analysis of the approximation algorithm. Instead of showing only a polynomial number of improvement steps, we show that we can also achieve polynomial running time for the computation of approximate equilibria. This is achieved by the consideration of approximations for the cost shares. In the second application, we consider a model to handle the aggregation of reputation values from different users and the disaggregation of information on composed bundles. Next to the application of our sampling method, this is the first work that handles both steps, aggregation and disaggregation, *together*. We discuss how to combine aggregation of evaluations across users and disaggregation of information on bundles to derive valuations for the single components. As a solution we propose to use the (weighted) average as an aggregation method in connection with the Shapley value as a disaggregation method, since this combination fulfills natural requirements in our context. Lastly, we show that a slightly modified Shapley value and the weighted average are still applicable if the evaluation profiles are incomplete. The results in the two scenarios are based on the following publications:

Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria**

in Shapley Value Weighted Congestion Games. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE)*. 2017. [Fel+17a]

Matthias Feldotto, Claus-Jochen Haake, Alexander Skopalik, and Nadja Stroh-Maraun. **Disaggregating User Evaluations Using the Shapley Value**. In: *Proceedings of the 13th Workshop on the Economics of Networks, Systems and Computation (NetEcon 2018)*. Irvine, California, USA, 2018. [Fel+18b]

Part II: Existence of Approximate Pure Nash Equilibria

In the second part of the thesis, we investigate two continuous game models – a model of coevolutionary opinion formation and a model for facility location with loads. Both models share the property that pure Nash equilibria do not exist in general. Therefore, our main focus is the existence of approximate equilibria in these models.

Chapter 7 studies a coevolutionary opinion formation model, called asymmetric K-NN games. It was shown in [BGM13] that pure Nash equilibria do not exist in general in these games. However, we can show that approximate equilibria with low approximation factors exist. To the best of our knowledge this is the first work to investigate an approximate concept of equilibria in the context of Opinion Formation Games. On the one hand, we analyze the approximation guarantee of natural states, namely if all players stay at their intrinsic opinions and if all players agree on an arbitrary, but common opinion. On the other hand, we propose and analyze a quite natural dynamic in which each player deviates only a certain fraction towards her best alternative. Chapter 7 is based on the following recent work:

Angelo Fanelli, Matthias Feldotto, and Alexander Skopalik. **Approximate Equilibria in Coevolutionary Opinion Formation Games**. Unpublished Manuscript. 2018 [FFS18]

In **Chapter 8** we investigate an extension of the Facility Location Game model of Hotelling [Hot90] by a load component [Koh83; PSV18]. It was shown that pure Nash equilibria exist only if the load component has a very high influence on the objectives of the players and if there is an even number of players. In contrast, we show the existence of approximate equilibria with low factors, also for objectives in which the load has less influence in comparison to the distance. Additionally, we consider the original Hotelling model with three players. So far, it was only known that no pure equilibrium exists. We extend this characterization by showing a tight approximation factor. Chapter 8 is based on the following recent work:

Matthias Feldotto, Pascal Lenzner, Louise Molitor, and Alexander Skopalik. **Two-Sided Facility Location: From Hotelling to Load Balancing**. Unpublished Manuscript. 2018 [Fel+18c]

Part III: Further Applications and Discussion

In the final part we extend the scope beyond the presented contributions and discuss further applications as well as next steps and challenges in the research of approximate equilibria.

In **Chapter 9** we look at approximate pure Nash equilibria from a different point of view. We show the application of this concept as a metric for simulations and present one concrete scenario and its evaluation. Our research is motivated by the analysis of electronic markets with thousands of participants and possibly complex strategic behavior. We model it with an infinitely repeated game and use techniques from the fields of game theory and simulations to analyze it. Our approach and the presented scenario and evaluations are based on the following publication:

> Matthias Feldotto and Alexander Skopalik. **A simulation framework for analyzing complex infinitely repeated games**. In: *Proceedings of the 4th International Conference on Simulation And Modeling Methodologies, Technologies And Applications (SIMULTECH)*. 2014. [FS14]

The final **Chapter 10** contains a short summary and an overview of the thesis as well as a more detailed discussion on open research questions in the different areas. For all considered problems in this thesis we show several open questions and possible next steps and discuss the main challenges.

1.3. Own Publications

Since the focus of this thesis is the concept of the approximate pure Nash equilibrium, I did not include results in other research areas that I also published during my research time. This section presents a list of the publications that I co-authored while studying the topics of this thesis. The publications are given in reverse chronological order. The considered research questions clearly go beyond the topics of the thesis. Therefore, the main purpose of the list is to put the topics of this thesis in context with my other research. Many contributions are based on interdisciplinary work, either in direct collaborations and publications or in much academic exchange resulting in new motivations and research questions from other areas.

The main part of my research focuses on resource allocation, mainly on variants of congestions games (see [Fel+17a; FLS18; FLS17; FLS16; FGS14]) and bandwidth allocation games (see [Dre+18; Dre+17; Dre+15]), but also in non-game-theoretic settings with Bin Packing (see [Fel+18a]). Recently, we investigated different other game-theoretic models (see [Fel+18c; FFS18]). Next to our work on the disaggregation of reputation values (see [Fel+18b]), we also worked (in a more practical setting) on gamification topics (see [Joh+17; Fel+17b]). Finally, I worked on a few more practical topics, namely peer-to-peer systems, parallel

XML compression and various simulation papers (see [FG16; FSG14; FS14; FG13; BFH13]).

2018

[Fel+18c] Matthias Feldotto, Pascal Lenzner, Louise Molitor, and Alexander Skopalik. **Two-Sided Facility Location: From Hotelling to Load Balancing**. Unpublished Manuscript. 2018

[FFS18] Angelo Fanelli, Matthias Feldotto, and Alexander Skopalik. **Approximate Equilibria in Coevolutionary Opinion Formation Games**. Unpublished Manuscript. 2018

[FLS18] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. **Congestion games with mixed objectives**. In: *J. Comb. Optim.* 36.4 (2018).

[Fel+18a] Björn Feldkord, Matthias Feldotto, Anupam Gupta, Guru Guruganesh, Amit Kumar, Sören Riechers, and David Wajc. **Fully-Dynamic Bin Packing with Little Repacking**. In: *Proceedings of the 45th International Colloquium on Automata, Languages, and Programming (ICALP)*. 2018.

[Fel+18b] Matthias Feldotto, Claus-Jochen Haake, Alexander Skopalik, and Nadja Stroh-Maraun. **Disaggregating User Evaluations Using the Shapley Value**. In: *Proceedings of the 13th Workshop on the Economics of Networks, Systems and Computation (NetEcon 2018)*. Irvine, California, USA, 2018.

[Dre+18] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. **Pure Nash equilibria in restricted budget games**. In: *J. Comb. Optim.* (2018)

2017

[Fel+17a] Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria in Shapley Value Weighted Congestion Games**. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE)*. 2017.

[Dre+17] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. **Pure Nash Equilibria in Restricted Budget Games**. In: *Proceedings of the 23rd International Conference on Computing and Combinatorics (COCOON)*. 2017.

[Joh+17] Thomas John, Matthias Feldotto, Paul Hemsen, Katrin Klingsieck, Dennis Kundisch, and Mike Langendorf. **Towards a Lean Approach to Gamifying Education**. In: *Proceedings of the 25th European Conference on Information Systems (ECIS)*. 2017.

[Fel+17b] Matthias Feldotto, Thomas John, Dennis Kundisch, Paul Hemsen, Katrin Klingsieck, and Alexander Skopalik. **Making Gamification Easy for the Professor: Decoupling Game and Content with the StudyNow Mobile App**. In: *Proceedings of the 12th International Conference on Designing the Digital Transformation (DESRIST)*. 2017.

[FLS17] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. **Congestion Games with Complementarities**. In: *Proceedings of the 10th International Conference on Algorithms and Complexity (CIAC)*. 2017.

2016

[FLS16] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. **Congestion Games with Mixed Objectives**. In: *Proceedings of the10th International Conference on Combinatorial Optimization and Applications (COCOA)*. 2016.

[FG16] Matthias Feldotto and Kalman Graffi. **Systematic evaluation of peer-to-peer systems using PeerfactSim.KOM**. in: *Concurrency and Computation: Practice and Experience* 28.5 (2016).

2015

[Dre+15] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. **On Existence and Properties of Approximate Pure Nash Equilibria in Bandwidth Allocation Games**. In: *Proceedings of the 8th International Symposium on Algorithmic Game Theory (SAGT)*. 2015.

2014

[FGS14] Matthias Feldotto, Martin Gairing, and Alexander Skopalik. **Bounding the Potential Function in Congestion Games and Approximate Pure Nash Equilibria**. In: *Proceedings of the 10th International Conference on Web and Internet Economics (WINE)*. 2014.

[FSG14] Matthias Feldotto, Christian Scheideler, and Kalman Graffi. **HSkip+: A self-stabilizing overlay network for nodes with heterogeneous bandwidths**. In: *Proceedings of the 14th IEEE International Conference on Peer-to-Peer Computing (P2P)*. 2014.

[FS14] Matthias Feldotto and Alexander Skopalik. **A simulation framework for analyzing complex infinitely repeated games**. In: *Proceedings of the 4th International Conference on Simulation And Modeling Methodologies, Technologies And Applications (SIMULTECH)*. 2014.

2013

[FG13] Matthias Feldotto and Kalman Graffi. **Comparative evaluation of peer-to-peer systems using PeerfactSim.KOM**. in: *Proceedings of the International Conference on High Performance Computing & Simulation, (HPCS)*. 2013.

[BFH13] Stefan Böttcher, Matthias Feldotto, and Rita Hartel. **Schema-based Parallel Compression and Decompression of XML Data**. In: *Proceedings of the 9th International Conference on Web Information Systems and Technologies (WEBIST)*. 2013

1.4. Outline of the Thesis

After this introductory chapter, the thesis continues in Chapter 2 with a more formal introduction to algorithmic game theory, important concepts and mathematical tools. The chapter serves to introduce several notations and concepts that are used throughout the entire thesis.

The remainder of the thesis is then divided into two major parts: Part I focuses on the computation of approximate pure Nash equilibria, mainly using the discrete model of Congestion Games, and Part II investigates the existence of approximate pure Nash equilibria in two continuous models.

The first part starts in Chapter 3 with an introduction to Congestion Games, the related work, the used model and the main algorithmic ideas. What follows, are applications of the presented ideas and their analysis for the standard model of unweighted Congestion Games (see Chapter 4). Afterwards, we turn our attention to the more powerful model of weighted Congestion Games using the Shapley cost-sharing method (see Chapter 5). The first part finishes in Chapter 6 with the approximation of the Shapley value through sampling, applied to the considered model of Shapley Congestion Games as well as the disaggregation of user evaluations.

While the first part considers the same basic model in all parts, Part II is structured a little bit differently. Here, we analyze two completely different continuous models and focus on the existence of approximate pure Nash equilibria and their quality. In Chapter 7, we consider a model for coevolutionary Opinion Formation Games and in Chapter 8 a model for Facility Location Games.

After the two main parts, the thesis continues in Chapter 9 with further applications of the concept in the context of simulations. The thesis finishes in Chapter 10 with a common conclusion and a discussion of open research questions in the various topics.

CHAPTER 2

Preliminaries

This chapter gives an introduction to the basic concepts in the area of algorithmic game theory. It also introduces the main notations that are used throughout the thesis. Furthermore, we introduce some techniques that are extensively used in the further course of the thesis, such as the potential function and the smoothness framework. More specialized concepts and notations are introduced in the respective chapters, e.g., the formal models of the game classes that we consider in this thesis.

Outline of This Chapter This chapter starts with an introduction of the basic concepts of non-cooperative game theory and especially algorithmic game theory (Section 2.1) that are used throughout the thesis. This section also serves as an introduction to our notation. At the end of this section, we introduce more complex games with a continuum of players and multiple rounds. In Section 2.2 we introduce important technical methods which we use several times during the analysis.

2.1. Non-Cooperative Game Theory

One of the main concepts of algorithmic game theory and also non-cooperative game theory is the strategic game and the definition and investigation of equilibria in this game. Here, we focus on pure Nash equilibria and approximate pure Nash equilibria, which are the main points of interest in this thesis. Next to the equilibrium as the main solution concept, algorithmic game theory deals with the Price of Anarchy as the most important metric to evaluate the quality of equilibria. First, we focus on one-shot games with a finite set of players, meaning that all players simultaneously select a strategy once and these strategy choices determine the outcome of the whole game. At the end of this section we turn our attention to the more complex models with a continuum of players as well as two-stage and infinitely repeated games.

Strategic Game A *strategic (cost-minimization) game* $\mathcal{G}$ consists of a set of n players $\mathcal{N} = \{1, 2, \ldots, n\}$. Each player $i \in \mathcal{N}$ has a set of pure strategies S_i which can be discrete or continuous and from which the player always chooses exactly one $s_i \in S_i$. Throughout the whole thesis we only work with pure strategies, so each player always chooses exact one strategy of this set and no distribution over strategies. A *strategy profile* or a *state/outcome* of the game is a choice of strategies

$\mathsf{s} = (s_1, s_2, \dots, s_n)$ by all players with $s_i \in S_i \, \forall \, i \in \mathcal{N}$. The set of all outcomes of this game is given by $\mathsf{S} = S_1 \times S_2 \times \cdots \times S_n$. For notational purposes, (s_{-i}, s'_i) denotes the outcome that results when player i changes her strategy in s from s_i to s'_i. Furthermore, for $A \subseteq \mathcal{N}$ let $(\mathsf{s}_A, \mathsf{s}'_{\mathcal{N} \setminus A})$ be the outcome that results when players $i \in A$ play their strategies in s and players $i \in \mathcal{N} \setminus A$ the strategies in s'.

Each player i has a cost function

$$c_i : \mathsf{S} \to \mathbb{R}$$

assigning a real value for each state. We call the value $c_i(\mathsf{s})$ the costs of player i in state s. In some parts of the thesis we also investigate *strategic utility-maximization games*. Then, each player i has assigned a utility function $u_i : \mathsf{S} \to \mathbb{R}$ which defines a utility value $u_i(\mathsf{s})$ for each strategy profile s.

The *social costs* of a game are defined as the sum over all players' costs with

$$SC(\mathsf{s}) = \sum_{i \in \mathcal{N}} c_i(\mathsf{s}).$$

Furthermore, we define the social costs of a subset of players $A \subseteq \mathcal{N}$ as $SC_A(\mathsf{s}) = \sum_{i \in A} c_i(\mathsf{s})$ (or $SC(\mathsf{s}) = \sum_{i \in \mathcal{N}} u_i(\mathsf{s})$ and $SC_A(\mathsf{s}) = \sum_{i \in A} u_i(\mathsf{s})$ the social utility for utility-maximization games).

Pure Nash Equilibrium A *pure Nash equilibrium* is an outcome s where no player has an incentive to deviate from her current pure strategy to another pure strategy assuming that all other players are fixed in their strategy choices. Formally, s is a pure Nash equilibrium if for each player $i \in \mathcal{N}$ and each alternative strategy $s'_i \in S_i$ for player i, we have

$$c_i(\mathsf{s}) \leq c_i(\mathsf{s}_{-i}, s'_i)$$

(or $u_i(\mathsf{s}) \geq u_i(\mathsf{s}_{-i}, s'_i)$). We refer to this inequality as the *Nash condition/inequality*. Furthermore, let PNE $\subseteq \mathsf{S}$ be the set of all pure Nash equilibria in the game.

Approximate Pure Nash Equilibrium An outcome s is a ρ-*approximate pure Nash equilibrium* if for each player $i \in \mathcal{N}$ and each alternative strategy $s'_i \in S_i$, for player i, we have

$$c_i(\mathsf{s}) \leq \rho \cdot c_i(\mathsf{s}_{-i}, s'_i)$$

(or $u_i(\mathsf{s}) \geq \rho \cdot u_i(\mathsf{s}_{-i}, s'_i)$). To improve readability, we also refer to these equilibria as *approximate equilibria*. We call ρ the approximation factor of the state. Notice that a pure Nash equilibrium is a 1-approximate pure Nash equilibrium. Let $\rho-$PNE $\subseteq \mathsf{S}$ be the set of all ρ-approximate pure Nash equilibria in the game.

Best-Response Moves We call a strategy change of player i from s_i to s_i', which results in a change from state s to (s_{-i}, s_i'), an *improvement move* if $c_i(\mathsf{s}_{-i}, s_i') < c_i(\mathsf{s})$. The move is called a *$\rho$-move* if $\rho \cdot c_i(\mathsf{s}_{-i}, s_i') < c_i(\mathsf{s})$. Furthermore, given a strategy profile s, we call strategy $\mathcal{BR}_i(\mathsf{s})$ the *best-response move* among all alternative strategies $s_i' \in S_i$ if

$$c_i(\mathsf{s}_{-i}, \mathcal{BR}_i(\mathsf{s})) \leq c_i(\mathsf{s}_{-i}, s_i') \, \forall \, s_i' \in S_i$$

Price of Anarchy The *Price of Anarchy (PoA)* is a measure to compare the strategic outcome of a game with a central optimum [KP99; Pap01; KP09] and hence, the quality of the equilibria. To a certain extent it is comparable to the concept of competitiveness in the analysis of online algorithms. While competitiveness measures the costs of unpredictability of the future, the Price of Anarchy measures the costs of the absence of cooperation. We define the optimum of the game by $\mathsf{s}^* = \min_{\mathsf{s} \in \mathsf{S}} SC(\mathsf{s})$. Then, the Price of Anarchy of the game is defined by

$$\mathrm{PoA} = \max_{\mathsf{s} \in \mathrm{PNE}} \frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)}.$$

Approximate Price of Anarchy Similarly, we define the *ρ-Approximate Price of Anarchy (ρ−PoA)* [CKS11; CKS09]. It is defined by

$$\rho\mathrm{-PoA} = \max_{\mathsf{s} \in \rho\mathrm{-PNE}} \frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)}.$$

Price of Stability In addition to the Price of Anarchy, the *Price of Stability (PoS)* is a further concept to measure the quality of equilibria [SM03; Ans+08]. Instead of using the worst equilibrium such as it it is the case for the PoA, we compare the social optimum with the best equilibrium. Using the same definitions for the optimum of the game $\mathsf{s}^* = \min_{\mathsf{s}' \in \mathsf{S}} SC(\mathsf{s}')$ and the set of all pure Nash equilibria $\mathrm{PNE} \subseteq \mathsf{S}$, the Price of Stability is defined by

$$\mathrm{PoS} = \min_{\mathsf{s}' \in \mathrm{PNE}} \frac{SC(\mathsf{s}')}{SC(\mathsf{s}^*)}.$$

2.1.1. Games with an Infinite Number of Players

In contrast to our original definition of players given by a finite set, we now consider players which are uniformly distributed on an interval, denoted by Z. Therefore, we have an infinite number of players. To prevent confusion, we call this type of players *agents*. Each agent $z \in Z$ again selects exactly one strategy $t_z \in T_z$. We assume that all agents have the same finite strategy space, therefore $T_z = T \, \forall \, z \in Z$. We describe the strategy profile by a measurable choice function $\mathsf{f} : Z \to T$. Thus the strategy of an agent $z \in Z$ is given by the function value $\mathsf{f}(z)$. The set of all possible strategy profiles is given by $\mathsf{F} = T^Z$.

Each agent $z \in Z$ has a cost function $C_z : \mathsf{F} \to \mathbb{R}$ assigning the agent a real value for each state. We call the value $C_z(\mathsf{f})$ the costs of agent z in state f.

To express strategy changes of single agents, we define by (f_{-z}, f'_z) the choice function which results, if only the mapping of the agent at z changes from the value $f(z)$ to the value $f'(z)$.

Pure Agent Nash Equilibrium A *Pure Agent Nash Equilibrium* is any measurable choice function f, such that no single agent $z \in Z$ can strictly decrease her cost by changing her strategy.

More formally, f is an agent equilibrium if and only if for any agent $z \in Z$, the inequality

$$C_z(\mathsf{f}) \leq C_z((\mathsf{f}_{-z}, f'_z))$$

for all alternative choice functions $\mathsf{f}' \in \mathsf{F}$ holds.

2.1.2. Two-Stage Games

In a two-stage game, we have two groups of players. The first group is defined similarly as in our one-shot game with a finite set of players $\mathcal{N}$. The second group is a mass of agents uniformly distributed on an interval, denoted by Z (as introduced in the previous section about games with an infinite number of players). They always make their decision given the strategies of the players of the first group. The costs/utilities of both groups of players depend on all strategy choices of all players and agents.

Formally, a stage game in this thesis always consists of two stages with the two different groups of active players. In the first stage, a finite set of players $\mathcal{N}$ is acting. Each player $i \in \mathcal{N}$ selects a strategy $s_i \in S_i$ (from a continuous or discrete strategy space). These choices of all players result in a player strategy profile $\mathsf{s} = (s_1, \dots, s_n)$. The set of all possible strategy profiles is given by $\mathsf{S} = S_1 \times S_2 \times \cdots \times S_n$.

In the second stage a second group of players Z, which we call agents, is acting given the decisions of the first group of players. Here, the agents are not given by a discrete set, but by a continuous space Z. Each agent $z \in Z$ selects a strategy $t_z \in T_z$. The strategy profile of the agents is given by the choice function $\mathsf{f} : \mathsf{S} \times Z \to T$. The set of all possible agent strategy profiles is given by $\mathsf{F} = T^{\mathsf{S} \times Z}$. To express strategy changes of single agents, we define by (f_{-z}, f'_z) the choice function which results, if only the mapping at agent z changes from the value $f(z)$ to the value $f'(z)$.

The complete strategy profile (of the players and the agents) is a pair $(\mathsf{s}, \mathsf{f}) \in \mathsf{S} \times \mathsf{F}$, where s is the vector of strategies of the players and f are the choice functions determining the strategies of the agents.

Each player $i \in \mathcal{N}$ has a utility function $u_i : \mathsf{S} \times \mathsf{F} \to \mathbb{R}$ assigning the player a real value for each state (s, f) (or a cost function $c_i : \mathsf{S} \times \mathsf{F} \to \mathbb{R}$). Similarly, each agent $z \in Z$ has a cost function $C_z : \mathsf{S} \times \mathsf{F} \to \mathbb{R}$. The *social costs* of the game are defined as the sum over all agents' costs with $SC(\mathsf{s}, \mathsf{f}) = \int_Z C_z(\mathsf{s}, \mathsf{f}) dz$.

Similar to the pure Nash equilibrium, we define a pure equilibrium in the two-stage game. To prevent the handling of two stages in the definition of the equilibrium, we use the concept of a subgame perfect equilibrium.

Pure Subgame Perfect Nash Equilibrium A strategy profile (s, f) is a *pure subgame perfect Nash equilibrium* (SPE) if and only if the following two conditions are satisfied:

1. for all $i \in \mathcal{N}$ and for all $s_i' \in S_i$,

$$u_i(\mathsf{s}, \mathsf{f}) \geq u_i((\mathsf{s}_{-i}, s_i'), \mathsf{f});$$

2. for all $\mathsf{s} \in \mathsf{S}$, for all $z \in Z$ and for any alternative choice function $\mathsf{f}' \in \mathsf{F}$,

$$C_z(\mathsf{s}, \mathsf{f}) \leq C_z(\mathsf{s}, (\mathsf{f}_{-z}, f_z')).$$

Let SPE $\subseteq \mathsf{S} \times \mathsf{F}$ the set of all subgame perfect Nash equilibria in the game. For the analysis we also investigate the two conditions separately and talk about an *agent equilibrium* if assuming a fixed player strategy profile s is given and the second condition holds. Equivalently we talk about a *player equilibrium* if the agent choice function f ensures that the second condition holds for all strategy deviation of players and then also the first condition holds. Thus, we analyze the two stages of the games separately.

Approximate Pure Subgame Perfect Nash Equilibrium We extend the definition to an approximate variant in which the players of the first stage are satisfied with approximate states while the client players in the second stage still play optimal strategies. A strategy profile (s, f) is a *ρ-approximate pure subgame perfect Nash equilibrium* (ρ-SPE) if and only if the following two conditions are satisfied:

1. for all $i \in \mathcal{N}$ and for all $s_i' \in S_i$,

$$u_i(\mathsf{s}, \mathsf{f}) \geq \rho \cdot u_i((\mathsf{s}_{-i}, s_i'), \mathsf{f});$$

2. for all $\mathsf{s} \in \mathsf{S}$, for all $z \in Z$ and for any alternative choice function $\mathsf{f}' \in \mathsf{F}$,

$$C_z(\mathsf{s}, \mathsf{f}) \leq C_z(\mathsf{s}, (\mathsf{f}_{-z}, f_z')).$$

Let $\rho-$SPE $\subseteq \mathsf{S} \times \mathsf{F}$ the set of all ρ-approximate subgame perfect Nash equilibria in the game.

2.1.3. Repeated Games

Another extension to the one-shot game is an *infinitely repeated game* which we will use in the context of simulations in Chapter 9. The main property fo this game is the infinitely number of rounds $R = (0, 1, 2, \ldots)$ [LR57; Aum59; Sha53b]. To capture the progress during the game we have a set of states Q. $q^j \in Q$ denotes the state of the game in round $j \in R$. We have one finite set of n players $\mathcal{N}$ and for each player $i \in \mathcal{N}$ a set of actions A_i. $a_i^j \in A_i$ denotes the action of player i in round j. The system has a transition probability function $\delta : Q \times A_1 \times A_2 \times \cdots \times A_n \times Q \to [0, 1]$ which defines the next state depending on all players' action.

The strategy of a player is a function or an algorithm that maps the current state of the game to the player's next action. Therefore, each player $i \in \mathcal{N}$ selects a strategy, which is a deterministic transition function $s_i : \tilde{Q} \to A_i$ that determines the player's next action, depending on the current observed state. An observed state $\tilde{q} \in \tilde{Q}$ contains a subset of the information in the current state of the game $q \in Q$. Here, the class of transition functions is restricted to functions that do not differentiate between information which is in the state of the game q, but not in the observed state $\tilde{q}$. Which information about the history each player observes, how much memory he possesses and which computations he is able to perform is left open here. Thus, this definition captures simple models such as positional games or stochastic games as well as more complex models such as machine games [Rub98; Rub86]. The strategy profile $\mathsf{s} = (s_1, s_2, \ldots, s_n)$ denotes all players' strategies. Note that a strategy profile and also the strategy of a player is fixed during all rounds of the game.

Each player i has a utility function $u_i^j : Q \times A_1 \times A_2 \times \cdots \times A_n \to \mathbb{R}$ assigning a payoff for a round of the game $j \in R$. The overall payoff of a player i is the discounted payoff over all rounds as $u_i = \sum_{j=0}^{\infty} \delta^j u_i^j(q^j, a_1^j, \ldots, a_n^j)$ with discount factor $\delta \in (0, 1)$.

Our original definition of the (approximate) pure Nash equilibrium still holds with the difference that a strategy profile is given by the different chosen algorithms fixed for the whole execution of the game.

2.1.4. Tools from Cooperative Game Theory

A cooperative or coalitional game is a game among groups of players instead single players. Cooperative game theory gives a framework to predict which coalitions will form in a given setting and which actions results in which payoffs. In contrast to the presented non-cooperative game theory, it uses other solution concepts and methods, e.g., stable sets [MV44], the core [Gil59], the kernel [DM65], the nucleolus [Sch69] or the Shapley value [Sha53a].

In this thesis we only work in the non-cooperative setting, but we utilize the concept of the *Shapley value (SV)* [Sha53a] for sharing costs or reputations.

Shapley Value In a coalitional game, there is a set N of $|N|$ players and a function v that maps subsets of players (coalitions) to real numbers: $v : 2^N \to \mathbb{R}$. Intuitively, the Shapley value distributes the value (or costs) of a game according to the average marginal contributions of a player to any coalition. Let $\Pi(\mathcal{N})$ be the set of all permutations of N. Given any $\pi \in \Pi(N)$, then $N^{i,\pi}$ is the set of all players that precede player i in the permutation π. Then the Shapley value for the uniform distribution over $\Pi(A)$ (called the standard Shapley value) is defined by

$$\begin{aligned} SV_i(N, v) &= E_{\pi \sim \Pi(N)} \left[v(N^{i,\pi} \cup \{i\}) - v(N^{i,\pi}) \right] \\ &= \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} (v(N^{i,\pi} \cup \{i\}) - v(N^{i,\pi})) \\ &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|\mathcal{N}|} (v(S \cup \{i\}) - v(S)). \end{aligned}$$

We use this concept as a method to share costs on a resource (see Chapter 5) or to disaggregate reputation values (see Section 6.4).

2.2. Analysis Methods and Tools

For the mathematical analysis of the concepts presented in this thesis, we make intensive use of some standard methods. A potential function is a powerful method to show the existence of equilibria, but it also serves as a good tool to analyze the computation of equilibria. Smoothness and similar concepts are used in the analysis of the Price of Anarchy.

Potential Function In several parts of this thesis we extensively work with a *potential function* [MS96]. Such a function $\Phi : \mathsf{S} \to \mathbb{R}$ maps each strategy profile of the game to a real value. It is called an exact potential function for a game if for any two outcomes s and (s_{-i}, s_i') that differ only in the strategy of player $i \in \mathcal{N}$, we have

$$\Phi(\mathsf{s}) - \Phi(\mathsf{s}_{-i}, s_i') = c_i(\mathsf{s}) - c_i(\mathsf{s}_{-i}, s_i')$$

(and analogously for utility maximizing games). Informally spoken, the change in the costs of the deviating player is always identical to the change in the potential function. Thus, the set of pure Nash equilibria corresponds to the set of local optima of the potential function.

Sub Games For several proofs we will use the concept of *sub games* among a subset of players $A \subseteq \mathcal{N}$. All other players of $\mathcal{N} \setminus A$ are frozen to their strategies. Note that the sub games defined in this context are different from the sub games in the definition of the subgame perfect equilibrium.

We restrict the potential function to this sub game and define the *A-limited potential* as $\Phi^A(\mathsf{s})$. Now consider sets A and B such that $B \subseteq A \subseteq \mathcal{N}$. Then the

B-partial potential of set A is defined by $\Phi_B^A(\mathsf{s}) = \Phi^A(\mathsf{s}) - \Phi^{A\setminus B}(\mathsf{s})$. If the set B contains only one player, i.e., $B = \{i\}$, then we write $\Phi_i^A(\mathsf{s}) = \Phi_B^A(\mathsf{s})$. In case of $A = \mathcal{N}$, $\Phi_B^{\mathcal{N}}(\mathsf{s}) = \Phi_B(\mathsf{s})$. Intuitively, $\Phi_B^A(\mathsf{s})$ is the value that the players in $B \subseteq A$ contribute to the A-limited potential.

Stretch of the Potential Function Similar to the approximate Price of Anarchy, we define a ratio with respect to the potential function, the *Stretch of the Potential Function*. Let $\hat{\mathsf{s}}$ be an outcome that minimizes the potential, i.e., $\hat{\mathsf{s}} = \min_{s' \in \mathsf{S}} \Phi(\mathsf{s}')$. Then the ρ-stretch is defined as

$$\rho\text{-}\Omega = \max_{\mathsf{s}' \in \rho-\mathrm{PNE}} \frac{\Phi(\mathsf{s}')}{\Phi(\hat{\mathsf{s}})}.$$

Additionally, we define a ρ-stretch restricted to players in a subset $A \subseteq \mathcal{N}$. Let $\rho-\mathrm{PNE}_A \subseteq \mathsf{S}$ be the set of ρ-approximate pure Nash equilibria where only players in A participate (the sub game among players in A). The other players $\mathcal{N} \setminus A$ have a fixed strategy $\bar{\mathsf{s}}_{\mathcal{N}\setminus A}$. Then we define the *A-limited ρ-stretch* as $\rho\text{-}\Omega_A = \max_{\mathsf{s}' \in \rho-\mathrm{PNE}_A} \frac{\Phi(\mathsf{s}')}{\Phi(\hat{\mathsf{s}})} = \max_{\mathsf{s}' \in \rho-\mathrm{PNE}_A} \frac{\Phi(\mathsf{s}'_A, \bar{\mathsf{s}}_{\mathcal{N}\setminus A})}{\Phi(\hat{\mathsf{s}}_A, \bar{\mathsf{s}}_{\mathcal{N}\setminus A})}$.

Smoothness [Rou09; Rou15] For showing bounds on the Price of Anarchy, a general technique was developed using smoothness properties of the underlying game. A cost-minimization game is (λ, μ)-smooth if for every two outcomes s and s' and for fixed values $\mu < 1$ and $\lambda > 0$,

$$\sum_{i \in \mathcal{N}} c_i(\mathsf{s}_{-i}, s'_i) \leq \lambda \cdot SC(\mathsf{s}') + \mu \cdot SC(\mathsf{s}).$$

If a game is (λ, μ)-smooth for $\mu < 1$ and $\lambda > 0$, then PoA $\leq \frac{\lambda}{1-\mu}$.

2.2.1. Further Mathematical Tools

In the remainder of this section we introduce further mathematical tools which are independent of game theory, but used in the proofs in this thesis.

Generalized Continued Fraction [Gau66] Let b_0 and $a_i, b_i \, \forall \, i \in 1, \ldots, n$ be real or complex values. Then we define the generalized continued fraction with the help of the Gauss notation as

$$x = b_0 + \mathop{\Large K}_{k=1}^{n} \frac{a_i}{b_i} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots \cfrac{a_n}{b_n}}}}$$

We call the a_i partial numerators, the b_i partial denominators and the b_0 the integer part of the continued fraction.

Chebyshev inequality [Che67] Let X be a random variable with expected value $E[X]$ and variance $Var[X]$. Then for any real number $k > 0$,

$$Pr[|X - E[X]| \geq k] \leq \frac{Var[X]}{k^2}.$$

Hoeffding's inequality [Hoe63a; Hoe63b] Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \in [a_i, b_i]$, let $X = \frac{1}{n}(X_1 + \cdots + X_n)$. Then for $\delta \geq 0$,

$$Pr[|X - E[X]| \geq \delta] \leq 2\exp\left(-\frac{2\delta^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right).$$

Chernoff inequality [Che52] Let $X_1, X_2, \ldots, X_n$ be independent random variables with $Pr[X_i = 1] = p$ and $Pr[X_i = 0] = 1 - p$. Let $X = \sum_{i=1}^{n} X_i$. Then for every $\delta > 0$,

$$Pr[X \geq (1 + \delta) \cdot pn] \leq \exp\left(-\frac{\min\{\delta, \delta^2\}pn}{3}\right)$$

and for every $\delta \in [0, 1]$,

$$Pr[X \leq (1 - \delta) \cdot pn] \leq \exp\left(-\frac{\delta^2 pn}{2}\right).$$

PART I

Computation of Approximate Pure Nash Equilibria

The first main part of this thesis concentrates on the efficient computation of approximate pure Nash equilibria. Using the examples of two variants of Congestion Games in which we know that there exist pure Nash equilibria, we construct and analyze approximation algorithms with extensive use of the potential function. Furthermore, we show how to efficiently approximate Shapley values by using sampling methods.

Outline of This Part We start this part with a detailed introduction to Congestion Games, the general model, an overview of related work and our algorithmic approach (see Chapter 3). Afterwards, we first consider the standard Congestion Game model with unweighted players and proportional cost-sharing (see Chapter 4) and we investigate weighted players and a Shapley value cost-sharing method (see Chapter 5). Part I ends with methods for the approximation of the Shapley value through sampling in two different scenarios (see Chapter 6).

CHAPTER 3

Congestion Games

Congestion Games are well-studied in algorithmic game theory and offer a very powerful framework to model resource allocation problems among strategic players. A large amount of research has been published around this model, dealing with a variety of different research questions and variants of the basic model. The focus of this first part of the thesis is the computation of approximate pure Nash equilibria in variants of this model.

Next to its powerfulness and extensive application examples (from traffic routing to computer networks), some variants of Congestion Games have nice game-theoretic properties; especially the existence of exact potential functions such that the existence of pure Nash equilibria is guaranteed.

In this work, we only consider atomic variants of Congestion Games. In these variants, there is a finite set of players and resources and each player has different strategies, which all consist of subsets of these resources. In a state, each player chooses a strategy corresponding to a subset of resources that she allocates. All resources selected by a player will be completely used by her. Note that the strategy space for each player is always discrete.

Contribution and Underlying Work Since most contributions in this first main part of the thesis are overlapping, this chapter is devoted to presenting the common contributions together in one framework. The main content is the introduction and classification of Congestion Games and especially the existing research around approximate pure Nash equilibria in this model. Furthermore, this chapter gives a common model and the main ideas of our algorithmic approach for the results in the next chapters.

The model and the background are based mainly on the following two publications, extended with new material and a more detailed and uniform view on several aspects:

> Matthias Feldotto, Martin Gairing, and Alexander Skopalik. **Bounding the Potential Function in Congestion Games and Approximate Pure Nash Equilibria**. In: *Proceedings of the 10th International Conference on Web and Internet Economics (WINE)*. 2014. [FGS14]

> Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria in Shapley Value Weighted Congestion Games**. In: *Proceedings*

of the 13th International Conference on Web and Internet Economics (WINE). 2017. [Fel+17a]

Outline of This Chapter This chapter starts with a deeper introduction to and a motivation of Congestion Games as well as a detailed view on related work in this large field of research (see Section 3.1). Afterwards, we introduce a formal model of Congestion Games and several notations (see Section 3.2) which we use in the following chapters. The chapter ends with an intuitive description of our common algorithmic approach (see Section 3.3) which is used in Chapters 4 and 5.

3.1. Related Work

A central problem in large-scale networks such as the Internet is network congestion, or more generally contention for scarce resources. *Congestion Games* were introduced by Rosenthal [Ros73] and provide us with a general model for the non-cooperative sharing of resources. In a Congestion Game, we are given a set of resources and each player selects a subset of them (e.g., a path in a network). Each resource has a cost function that depends on the load induced by the players using it. Each player aims to minimize the sum of the resources' costs in her strategy given the strategies chosen by the other players.

Unweighted Congestion Games always admit a pure Nash equilibrium, where players pick a single strategy and do not randomize. Rosenthal [Ros73] showed this by means of a *potential function*. Such a function has the following property: If a single player deviates to a different strategy, then the value of the potential changes by the same amount as the cost of the deviating player. Pure Nash equilibria correspond to local optima of the potential function. Games admitting such a potential function are called potential games and each potential game is isomorphic to a Congestion Game [MS96].

Note that in unweighted Congestion Games each player using a resource has the same influence on the cost of this resource. To alleviate this limitation, [Mil96] and [FKS05] studied a natural generalization called *weighted Congestion Games* in which each player has a weight and the *joint cost* of the resource depends on the total weight of players using a resource. The joint cost of a resource has to be covered by the set of players using it. Each player's contribution is her cost share on this resource. The *cost sharing method* of the game defines how exactly the joint cost of a resource is divided into individual cost shares. For weighted Congestion Games, the most widely studied cost sharing method is *proportional sharing* (PS), where the cost share of a player is proportional to her weight. Unfortunately, weighted Congestion Games with proportional sharing in general do not admit a *pure* Nash equilibrium or even a potential function [LO01; GMV05; FKS05; HK12; HKM11].

On the basis of the Shapley value [HM89], Kollias and Roughgarden [KR15] proposed to use the *Shapley value* (SV) for sharing the cost of a resource in weighted Congestion Games. In the *Shapley cost-sharing* method, the cost share of a player on a resource is the average marginal cost increase caused by her over all permutations of the players. Using the Shapley value restores the existence of a potential function and therefore the existence of pure Nash equilibria to such games [KR15]. [GMW14] extends this result by proving that a weighted generalization of Shapley values is the only method that guarantees pure Nash equilibria. In contrast, proportional sharing does not guarantee the existence of equilibria in general [HK12].

Further variants of Congestion Games include player-specific games [Mil96; Mav+07; LHL11], in which each player has her own cost functions, and bottleneck Congestion Games [BO07; Har+13; HKM09], in which the maximum of the resources' costs is considered instead of their sum. Lots of extensions have been developed and generalize the standard models by using different objective functions (e.g., [FLS17; FLS18; Kuk07; Kuk15; Voi+09; BPJ09]). Additionally, restrictions on the combinatorial structure of the games were introduced, most famously singleton [Ieo+05] and matroid [ARV09; ARV08] Congestion Games. In these variants, the strategy sets are either singletons or limited to the bases of matroids. For results in models with non-atomic variants of Congestion Games we refer to the seminal work of Roughgarden and Tardos [RT02; RT04] and further related literature.

Besides establishing the existence of a pure Nash equilibrium, the potential function has played an important role for proving various results in the field of potential games. Its use ranges from bounding the *price of stability* [Car+11b; CG16; CK05a; KS17] to tracking the convergence rate of best response dynamics [Awe+08; CS11]. In addition to exact potential functions, also approximate variants of them have been used to establish the existence of approximate pure Nash equilibria [Chr+18; HKS14; CKS11; CR09].

Potential functions immediately arise a simple and natural search procedure for an equilibrium by performing iterative improvement steps starting from an arbitrary state. Unfortunately, this process may take exponentially many steps, even in the simple case of unweighted Congestion Games and linear cost functions [ARV08]. Note that in the unweighted case, proportional sharing and Shapley cost sharing coincide. Moreover, computing a pure Nash equilibrium in these games is intractable as the problem is PLS-complete. Fabrikant et al. [FPT04] showed that the problem of computing a pure Nash equilibrium in Congestion Games is PLS-complete, that is, computing a pure Nash equilibrium is as hard as the problem of finding a local optimum. PLS was defined as a class of local search problems where local optimality can be verified in polynomial time [JPY88]. Dunkel and Schulz [DS08] extended the result and showed that it is strongly NP-complete to determine whether an equilibrium exist in a given weighted Congestion Game. Ackermann and Skopalik [AS08] showed similar results for player-specific network Congestion Games. Computing a pure Nash equilibrium in Congestion Games stays a hard problem even for games with only three players [AS08] or if the cost functions are linear [ARV08]. This last result directly carries over to the game class with

Shapley cost-sharing. Efficient algorithms are known only for special cases, e. g. for symmetric network Congestion Games [FPT04] or when the strategies are restricted to be bases of matroids [ARV08].

The hardness of computing a pure Nash equilibrium in Congestion Games motivates relaxing the Nash equilibrium conditions and asking for approximations instead. One possible approximation is the ρ-approximate pure Nash equilibrium, a state in which no player can improve by a factor larger than ρ. Without restricting the cost (or utility) functions the problem of computing ρ-approximate pure Nash equilibria in Congestion Games remains PLS-complete for any fixed ρ [SV08]. However, the problem becomes tractable for certain subclasses of congestion and potential games with varying approximation guarantees. For *symmetric* Congestion Games certain best response dynamics converge in polynomial time to a $(1+\varepsilon)$-Nash equilibrium [CS11]. For *asymmetric* Congestion Games, it has been shown that for linear resource cost functions a $(2+\varepsilon)$-approximate Nash equilibrium can be computed in polynomial time [Car+11a]. More generally, for polynomial resource cost functions with maximum degree d, the best known approximation guarantee is $d^{O(d)}$ [Car+11a]. These results have recently been extended to Congestion Games with weighted players [Car+15]. For linear cost functions it is shown how to compute a $\frac{3+\sqrt{5}}{2}+\varepsilon$-approximate equilibrium. For polynomial cost functions it is proven that $d!$-approximate equilibria always exist and $d^{2d+o(d)}$-approximate equilibria can be computed in polynomial time. [GNS18] improved the algorithm and its analysis for weighted Congestion Games and showed the computation of $d^{d+o(d)}$-approximate equilibria. They utilize an approximate potential function directly on the original game instead of on a modified game. The bounds for the existence of approximate equilibria in weighted Congestion Games without any computation were improved by [HKS14] to $d+1$ for polynomial cost functions and $\frac{3}{2}$ for concave cost functions. Existing approaches [BCK10; Car+11a; CFG17] for computing ρ-approximate pure Nash equilibria (with small ρ) heavily build on the ability to compute intermediate states that approximate the optimum potential value and satisfy certain more local conditions. By focusing on approximating the potential function, our results significantly further this line of research.

Our bounds on the spread of the potential function for two outcomes are related to results on the *Price of Anarchy* (PoA). The PoA was introduced in [KP99] as the worst case ratio between the value of some global objective function in a Nash equilibrium and its optimum value. Most results on the PoA in Congestion Games use the *total latency* (i.e., the sum of the players' costs) as the global objective function. For this setting, Christodoulou and Koutsoupias [CK05b] showed that the PoA for non-decreasing affine cost functions is $\frac{5}{2}$. Gairing and Schoppmann [GS07] provided various bounds for singleton unweighted Congestion Games. Aland et al. [Ala+11; Ala+06] obtained the exact value on the PoA for polynomial cost functions. Roughgarden's [Rou15] smoothness framework determines the PoA with respect to any set of allowable cost functions. These results have been extended to the more general class of *weighted* Congestion Games [Ala+11; AAE13; BGR14;

CK05b; Bil16; BV17]. Additionally, the approximate Price of Anarchy in theses games was investigated [CKS11; Bil18; GNS18]. Gkatzelis et al. [GKR14] show that, among all cost-sharing methods that guarantee existence of pure Nash equilibria, Shapley values minimize the worst PoA. Furthermore, tight bounds on PoA for general cost-sharing methods were given [GKK15]. For the extended model with non-anonymous costs by using set functions it was also shown that Shapley cost-sharing is the best method and tight results are given [KS15; RS16]. For Congestion Games, the total latency and the value of the potential function are related. So, for Nash [Ala+11; CK05b; Rou15] and approximate Nash equilibria [CKS11], bounds on the potential function can be derived from the corresponding results on the Price of Anarchy. However, this approach yields much weaker guarantees than our approach.

In addition to the Price of Anarchy a more optimistic measure was introduced with the *Price of Stability* (PoS) using the best equilibria in the game for the comparison [SM03; Ans+08]. For unweighted Congestion Games tight bounds where shown for affine and polynomial cost functions [Car+11b; CG16; CK05a], bounds for weighted Congestion Games are given in [Chr+18].

3.2. Model

A Congestion Game is defined as $\mathcal{G} = (\mathcal{N}, E, (w_i)_{i\in\mathcal{N}}, (S_i)_{i\in\mathcal{N}}, (f_e)_{e\in E})$ where $\mathcal{N}$ is the set of players, E the set of resources, w_i the positive weight of player i (with $w_i = 1\,\forall\, i \in \mathcal{N}$ in the case of unweighted Congestion Games), $S_i \subseteq 2^E$ the strategy set of player i, and f_e the cost function of resource e (drawn from a set $\mathcal{F}$ of allowable cost functions). In this work, $\mathcal{F}$ is the set of polynomial functions with maximum degree d and non-negative coefficients.

In a given strategy profile $\mathsf{s} = (s_1, s_2, \ldots, s_n)$, each player i selects strategy $s_i \in S_i$, a subset of the resources. Then, the set of users of resource e is defined by $N_e(\mathsf{s}) = \{i : e \in s_i\}$, the number of users by $n_e(\mathsf{s}) = |N_e(\mathsf{s})|$ and the total weight on e by $w_e(\mathsf{s}) = \sum_{i\in N_e(\mathsf{s})} w_i$. Furthermore, let $N_e^A(\mathsf{s}) = \{i \in A : e \in s_i\}$, $n_e^A(\mathsf{s}) = |N_e^A(\mathsf{s})|$ and $w_e^A(\mathsf{s}) = \sum_{i\in N_e^A(\mathsf{s})} w_i$ be variants of these definitions with a restricted player set $A \subseteq \mathcal{N}$. Additionally, let $W_A = \sum_A w_j$ be the sum of the players' weights from a subset $A \subseteq \mathcal{N}$, independent of the current state.

The *cost share* of a player i on a resource e is given as a function of the player's identity, the resource's cost function and its users A, i.e., $\chi_e(i, A)$. If we consider all players of resource e in a specific state s for the computation of the costs share (which is the normal case), we shortly write $\chi_{ie}(\mathsf{s}) = \chi_e(i, N_e(\mathsf{s}))$.

The *joint cost* on a resource e is given by $c_e(w_e(\mathsf{s})) = w_e(\mathsf{s}) \cdot f_e(w_e(\mathsf{s}))$ and the costs of players are such that $c_e(w_e(\mathsf{s})) = \sum_{i\in N_e(\mathsf{s})} \chi_{ie}(\mathsf{s})$. The *total cost* of a player i equals the sum of her costs on the resources she uses, i.e., $c_i(\mathsf{s}) = \sum_{e\in s_i} \chi_{ie}(\mathsf{s})$.

The social costs are given by $SC(\mathsf{s}) = \sum_{e\in E} c_e(\mathsf{s}) = \sum_{e\in E} \sum_{i\in N_e(\mathsf{s})} \chi_{ie}(\mathsf{s}) = \sum_{i\in\mathcal{N}} c_i(\mathsf{s})$. Further define the social costs of a subset of players $A \subseteq \mathcal{N}$ with $SC_A(\mathsf{s}) = \sum_{i\in A} c_i(\mathsf{s})$.

The cost-sharing method is important for our analysis, as it defines how the joint cost on a resource e is distributed among its users. In this thesis, we focus on the proportional cost-sharing (cf. Chapter 4) and the Shapley value (cf. Chapter 5).

Proportional cost-sharing. The cost of a player i on a resource under proportional sharing is given by

$$\chi_e^{\text{Prop}}(i, A) = w_i \cdot f_e(W_A) \text{ or } \chi_{ie}^{\text{Prop}}(\mathsf{s}) = w_i \cdot f_e(w_e(\mathsf{s})).$$

For the unweighted case with $w_i = 1$ for all players $i \in \mathcal{N}$, the cost share is simply given by

$$\chi_{ie}^{\text{Prop}}(\mathsf{s}) = f_e(n_e(\mathsf{s})).$$

So the cost share of each player equals the cost function of the resource with the number of players as parameter.

Shapley cost-sharing. For a set of players A, let $\Pi(A)$ be the set of permutations $\pi : A \to \{1, \ldots, |A|\}$. For a $\pi \in \Pi(A)$, define as $A^{<i,\pi} = \{j \in A : \pi(j) < \pi(i)\}$ the set of players preceding player i in π and as $W_A^{<i,\pi} = \sum_{A^{<i,\pi}} w_j$ the sum of their weights.

For the uniform distribution over $\Pi(A)$, the Shapley value of a player i on resource e is given by

$$\chi_e^{\text{SV}}(i, A) = E_{\pi \sim \Pi(A)} \left[c_e \left(W_A^{<i,\pi} + w_i \right) - c_e \left(W_A^{<i,\pi} \right) \right]$$

and for a specific state s by

$$\chi_{ie}^{\text{SV}}(\mathsf{s}) = E_{\pi \sim \Pi(N_e(\mathsf{s}))} \left[c_e \left(W_{N_e(\mathsf{s})}^{<i,\pi} + w_i \right) - c_e \left(W_{N_e(\mathsf{s})}^{<i,\pi} \right) \right].$$

In this thesis we always consider the uniform distribution for the computation of the expected value, therefore we concentrate on the standard Shapley value. For affine cost functions $f_e \, \forall \, e \in E$, observe that $\chi_{ie}^{\text{SV}}(\mathsf{s}) = \chi_{ie}^{\text{Prop}}(\mathsf{s})$.

3.3. Algorithmic Approach for the Computation

Our algorithms in the following chapters are based on ideas by Caragiannis et al. [Car+11a; Car+12; Car+15]. Intuitively, we partition the players' costs into intervals $[b_1, b_2], [b_2, b_3], \ldots, [b_{m-1}, b_m]$ in decreasing order. The cost values in one interval are within a polynomial factor. Note that this ensures that every sequence of ρ-moves for $\rho > 1$ of players with costs in one or two intervals converges in polynomial time. Here, the main difference in the algorithms of both models (Chapters 4 and 5) comes into account. In the unweighted model in Chapter 4, the algorithm works with static groups: The partition of a player is fixed during the whole execution and is determined by the costs of this player in an empty game with no other players. In the weighted model in Chapter 5, we work with

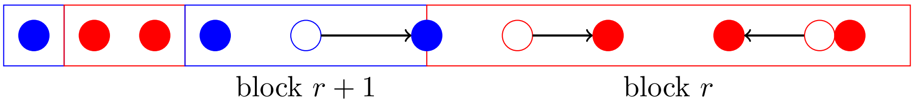

Figure 3.1.: The players are partitioned in blocks and two blocks are active.

dynamic groups, therefore the allocation of a player to a group may change during the execution and is always defined by the current costs.

After an initialization, the algorithm proceeds in phases r from 1 to $m-1$. In each phase r (see Fig. 3.1), players with costs in the interval $[b_r, b_{r-1}]$ do ρ-approximate moves where ρ is close to the desired approximation factor. In the second algorithm in Chapter 5 the interval is open at the upper border, $[b_r, \infty]$, to take into account that costs can increase and then a player could change her group. Additionally, players with costs in the interval $[b_{r+1}, b_r]$ make $1+\gamma$-moves for some small $\gamma > 0$. After a polynomial number of steps no such moves are possible and we freeze all players with costs in $[b_r, +\infty]$. These players will never be allowed to move again. We then proceed with the next phase. Note that at the time players are frozen, they are in an ρ-approximate equilibrium. The purpose of the $1+\gamma$-moves of players of the neighboring interval is to ensure that the costs of frozen players do not change significantly in later phases.

To that end we utilize a potential function argument. We argue about the potential of *sub games* among a subset of players. We can bound the potential value of an arbitrary ρ-approximate equilibrium with the minimal potential value (using the *stretch*). In comparison to the existing approaches in [Car+11a; Car+12; Car+15], we directly work with the exact potential functions of the games and with better bounds on the stretch which significantly improves the results, but also requires a more involved analysis. We show that the potential of the sub game in one phase is significantly smaller than b_r. Therefore, the costs experienced by players moving in phase r are considerably lower than the costs of any player in the interval $[b_1, b_{r-1}]$. The analysis heavily depends on the stretch of the potential function which we analyze in the respective chapters. For the first model in Chapter 4 we take use of an approach similar to the smoothness technique. In the second model in Chapter 5 we use an approximation between Shapley and proportional cost share and structural properties of costs-shares and the restricted potentials.

CHAPTER 4

Unweighted Congestion Games

Unweighted congestion games were first introduced by Rosenthal [Ros73]. In this model, all players have an equal weight of 1, formally $w_i = 1 \, \forall \, i \in \mathcal{N}$. Furthermore, the costs are distributed among the players on a resource by using the proportional sharing method, $\chi_{ie}(\mathsf{s}) = \chi_{ie}^{\text{Prop}}(\mathsf{s}) = f_e(n_e(\mathsf{s}))$. We first study the potential function in these Congestion Games with non-decreasing cost functions. We show that the value of the potential function $\Phi(\mathsf{s})$ of any outcome s of a Congestion Game approximates the optimum potential value $\Phi(\mathsf{s}^*)$ by a factor $\Psi_{\mathcal{F}}$ which depends only on the set of cost/ functions $\mathcal{F}$, and an additive term which is bounded by the sum of the total possible improvements of the players in the outcome s.

The significance of this result is twofold. On the one hand it provides *Price-of-Anarchy*-like results with respect to the potential function. On the other hand, we show that these approximations can be used to compute $(1 + \varepsilon) \cdot \Psi_{\mathcal{F}}$-approximate pure Nash equilibria for Congestion Games with non-decreasing cost functions. For the special case of polynomial cost functions, this significantly improves the guarantees from Caragiannis et al. [Car+11a]. Moreover, our machinery provides the first guarantees for general cost functions.

Contribution and Underlying Work In this chapter we show that for any outcome s of a Congestion Game the value of the potential function $\Phi(\mathsf{s})$ can be bounded by the optimum potential value $\Phi(\mathsf{s}^*)$, a factor $\Psi_{\mathcal{F}}$ which depends only on the set of cost functions $\mathcal{F}$, and an additive term $D(\mathsf{s}, \mathsf{s}^*)$ which is bounded by the sum of the total possible improvements of the players in the outcome s. As a direct corollary we get that any outcome s provides us with a bound on the optimum potential value. For non-decreasing cost functions, our lower bound is matching if some technical constraint on the resource functions $\mathcal{F}$ is fulfilled.

To achieve this result and bound the stretch of the potential function, we use a technique similar to the smoothness framework which is widely used for the analysis of the Price of Anarchy. We apply similar arguments to the potential function as objective.

Our result can be used to obtain ρ-approximate equilibria with small values of $\rho = \Psi_{\mathcal{F}} \cdot (1 + \varepsilon)$ with the method of [Car+11a]. Our technique significantly improves the approximation by [Car+11a] for polynomial cost functions. Moreover, our analysis suggests and identifies large and practically relevant classes of cost functions for which ρ-approximate equilibria with small ρ can be computed in polynomial time (see tables in Section 4.2).

For example, in games where resources have a certain cost offset, e.g., traffic networks, the approximation factor ρ drastically decreases with the increase of offsets or coefficients in delay functions. In particular for Congestion Games with linear functions with strictly positive offset, ρ is smaller than 2. To the best of our knowledge this is the first work to show that ρ-approximate equilibria with $\rho < 2$ are polynomial time computable without restricting the strategy spaces.

The model, analysis, and results presented in this chapter are based on the following publication, extended by a new and simpler analysis of the bounds for the potential function:

> Matthias Feldotto, Martin Gairing, and Alexander Skopalik. **Bounding the Potential Function in Congestion Games and Approximate Pure Nash Equilibria**. In: *Proceedings of the 10th International Conference on Web and Internet Economics (WINE)*. 2014. [FGS14]

Outline of This Chapter We start this chapter in Section 4.1 with a redefinition of our general model from Section 3.2 and the definition of the underlying potential function. Afterwards, the analysis of the potential function with regard to the stretch follows in Section 4.2. Finally, we present the computation algorithm which follows the described approach from Section 3.3 together with its analysis in Section 4.3.

4.1. Model and Further Notations

This chapter specifies a model for Congestion Games which is build on the general model of Section 3.2 with an equal weight, $w_i = 1 \, \forall \, i \in \mathcal{N}$, and the proportional cost-sharing method, $\chi_{ie}(\mathsf{s}) = \chi_{ie}^{\text{Prop}}(\mathsf{s}) = f_e(n_e(\mathsf{s}))$.

To simplify the further reading in this chapter, we restate the model in a simpler variant: An unweighted Congestion Game with proportional cost-sharing is a tuple $\mathcal{G} = (\mathcal{N}, E, (S_i)_{i \in \mathcal{N}}, (f_e)_{e \in E})$. The definitions of $\mathcal{N}$, E, $S_i \, \forall \, i \in \mathcal{N}$ and $f_e \, \forall \, e \in E$ stay the same. In cost minimizing games the *cost* for player i is defined by $c_i(\mathsf{s}) = \sum_{e \in s_i} f_e(n_e(\mathsf{s}))$. For two outcomes s and s' define $D(\mathsf{s}, \mathsf{s}') = \sum_{i \in \mathcal{N}} (c_i(\mathsf{s}_{-i}, s_i') - c_i(\mathsf{s}))$ as the possible improvement of each player separately.

Potential Function Unweighted Congestion Games admit the potential function $\Phi(\mathsf{s}) = \sum_{e \in E} \sum_{j=1}^{n_e(\mathsf{s})} f_e(j)$ introduced by Rosenthal [Ros73]. Thus, the set of pure Nash equilibria corresponds to the set of local optima of the potential function. For any resource $e \in E$ denote $\Phi_e(\mathsf{s}) = \sum_{j=1}^{n_e(\mathsf{s})} f_e(j)$ as the contribution of e to the potential.

For our analysis we will use sub games among a subset of players $A \subseteq \mathcal{N}$. All other players of $\mathcal{N} \setminus A$ are frozen to their strategies. In this sub game $n_e^A(\mathsf{s}) = |i \in A : e \in s_i|$ gives the number of participating players in A which uses resource e in strategy profile s and the latency function is then defined as $f_e^A(x) = f_e(x + n_e^{\mathcal{N}\setminus A}(\mathsf{s}))$ with $n_e^{\mathcal{N}\setminus A}(\mathsf{s}) = |i \in \mathcal{N} \setminus A : e \in s_i|$. Then, the partial potential is defined as $\Phi_A(\mathsf{s}) = \sum_{e\in E} \sum_{j=1}^{n_e^A(\mathsf{s})} f_e^A(j)$.

4.2. Approximating the Potential

Now we study the difference of potential function values of two strategy profiles s and s^*. We show that the ratio of the potential values of s and s^* can be bounded by $D(\mathsf{s}, \mathsf{s}^*)$ and a parameter $\Psi_{\mathcal{F}}$ of the class of cost functions which is defined as follows.

Definition 4.1. *For a class of functions $\mathcal{F}$ define*

$$\Psi_{\mathcal{F}} = \sup \left\{ \frac{(n-m)f(n+1) - nf(n) + \sum_{j=1}^{n} f(j)}{\sum_{j=1}^{n-m} f(j)} : f \in \mathcal{F}, n, m \in \mathbb{N}, n > m \right\}.$$

Now the following theorem holds for different states.

Theorem 4.2. *Let $\mathcal{F}$ be a set of non-decreasing cost functions. Consider a Congestion Game with cost functions in $\mathcal{F}$ and two arbitrary strategy profiles s and s^*. Then $\Phi(\mathsf{s}) \leq \Psi_{\mathcal{F}} \cdot \Phi(\mathsf{s}^*) - D(\mathsf{s}, \mathsf{s}^*)$.*

Proof. To prove the theorem, we utilize techniques similar to the smoothness framework, but applied to the potential function as objective instead of the social costs.

By definition,

$$\begin{aligned} D(\mathsf{s}, \mathsf{s}^*) &= \sum_{i\in\mathcal{N}} (c_i(\mathsf{s}_{-i}, s_i^*) - c_i(\mathsf{s})) \\ &= \sum_{i\in\mathcal{N}} \left(\sum_{e\in s_i^*} f_e(n_e(\mathsf{s}_{-i}, s_i^*)) - \sum_{e\in s_i} f_e(n_e(\mathsf{s})) \right) \end{aligned}$$

Since a player i is moving to a new strategy s_i^*, we have $f_e(n_e(\mathsf{s}_{-i}, s_i^*)) \leq f_e(n_e(\mathsf{s}) + 1)$. Thus,

$$\begin{aligned} D(\mathsf{s}, \mathsf{s}^*) &\leq \sum_{i\in\mathcal{N}} \left(\sum_{e\in s_i^*} f_e\left(n_e(\mathsf{s}) + 1\right) - \sum_{e\in s_i} f_e(n_e(\mathsf{s})) \right) \\ &= \sum_{e\in E} n_e(\mathsf{s}^*) \cdot f_e(n_e(\mathsf{s}) + 1) - \sum_{e\in E} n_e(\mathsf{s}) \cdot f_e(n_e(\mathsf{s})) \\ &= \sum_{e\in E} (n_e(\mathsf{s}^*) \cdot f_e(n_e(\mathsf{s}) + 1) - n_e(\mathsf{s}) \cdot f_e(n_e(\mathsf{s}))) \end{aligned}$$

We fix an arbitrary resource $e \in E$ and define $f := f_e$, $n := n_e(\mathsf{s})$ and $m := n_e(\mathsf{s}) - n_e(\mathsf{s}^*)$.

$$n_e(\mathsf{s}^*) \cdot f_e(n_e(\mathsf{s}) + 1) - n_e(\mathsf{s}) \cdot f_e(n_e(\mathsf{s})) = (n-m) \cdot f(n+1) - n \cdot f(n)$$

Now we start with the inequality of the theorem.

$$\Phi(\mathsf{s}) \leq \Psi_{\mathcal{F}} \cdot \Phi(\mathsf{s}^*) - D(\mathsf{s},\mathsf{s}^*) \Leftrightarrow \Psi_{\mathcal{F}} \geq \frac{\Phi(\mathsf{s}) + D(\mathsf{s},\mathsf{s}^*)}{\Phi(\mathsf{s}^*)}$$

Concentrating only on one resource and using the result from above yields

$$\begin{aligned}\Psi_{\mathcal{F}} &\geq \frac{\Phi_e(\mathsf{s}) + D_e(\mathsf{s},\mathsf{s}^*)}{\Phi_e(\mathsf{s}^*)} \\ &\geq \frac{\sum_{j=1}^{n} f(j) + (n-m)\cdot f(n+1) - n\cdot f(n)}{\sum_{j=1}^{n-m} f(j)}\end{aligned}$$

The theorem follows with the definition of $\Psi_{\mathcal{F}}$. □

Denote s_i' the best response of player $i \in \mathcal{N}$ to the outcome s. Then we can lower bound $D(\mathsf{s},\mathsf{s}^*) = \sum_{i\in\mathcal{N}}(c_i(\mathsf{s}_{-i}, s_i^*) - c_i(\mathsf{s})) \geq \sum_{i\in\mathcal{N}}(c_i(\mathsf{s}_{-i}, s_i') - c_i(\mathsf{s}))$ which yields the following corollary.

Corollary 4.3. *Given any outcome s denote s_i' the best response of player $i \in \mathcal{N}$ to s. Then, $\Phi(\mathsf{s}^*) \geq \frac{1}{\Psi_{\mathcal{F}}}(\Phi(\mathsf{s}) + \sum_{i\in\mathcal{N}}(c_i(\mathsf{s}_{-i}, s_i') - c_i(\mathsf{s})))$. So, any outcome s provides us with a lower bound on the minimum potential value.*

We have a matching lower bound in the case in which the set of functions $\mathcal{F}$ satisfies a technical condition. Let (f, n, m) be a tuple of parameters that determine $\Psi_{\mathcal{F}}$ in Definition 4.1 and define $D_{\mathcal{F}} = (n-m)f(n+1) - nf(n)$.

With these parameters at hand, we are ready to present the matching lower bound for families of functions $\mathcal{F}$ that satisfy $D_{\mathcal{F}} \geq 0$.

Theorem 4.4. *Given a class of non-decreasing functions $\mathcal{F}$ with $D_{\mathcal{F}} \geq 0$ there is a Congestion Game $\mathcal{G}_{\mathcal{F}}$ with cost functions in $\mathcal{F}$ and two strategy profiles s and s^* with $\Phi(\mathsf{s}) = \Psi_{\mathcal{F}} \cdot \Phi(\mathsf{s}^*)$ and $c_i(\mathsf{s}_{-i}, s_i^*) - c_i(\mathsf{s}) = 0$ for all players $i \in \mathcal{N}$.*

Proof. We construct a Congestion Game $\mathcal{G}_{\mathcal{F}}$ and two strategy profiles s and s^* that have the desired properties.

Let α and β be positive integers such that

$$\frac{\beta}{\alpha} = \frac{2n(n-m)((n-m)f(n+1) - nf(n))}{f(1)}.$$

The game $\mathcal{G}_{\mathcal{F}}$ consists of two sets of players, A and B, each of size $n \cdot (n-m)$. We denote the players by $A(i,j)$ and $B(i,j)$ with $1 \leq i \leq n$ and $1 \leq j \leq n-m$, respectively. There are three sets of resources: Q, R, and S. The former two have size $\alpha n \cdot (n-m)$, the later has size β. We denote the resources by $Q(i,j,k)$ and $R(i,j,k)$ with $1 \leq i \leq n$ and $1 \leq j \leq n-m$ and $1 \leq k \leq \alpha$ and $S(1), \ldots, S(\beta)$, respectively. Each resource has the cost function $f_{\mathcal{F}}$.

Each player has two strategies. The first strategy of every player $A(i,j)$ is $A_1(i,j) = \{Q(i^*, j, k^*) \mid \forall\, i^*, k^*)\}$. The second strategy of player $A(1,1)$ is

$A_2(1,1) = \{R(j,i^*,k^*) \mid \forall\, i^*k^*)\} \cup S$. The second strategies of the remaining players of set A are $A_2(i,j) = \{R(j,i^*,k^*) \mid \forall\, i^*k^*)\}$. The two strategies of the players of set B are $B_1(i,j) = \{R(i^*,j,k^*) \mid \forall\, i^*,k^*)\}$and $B_2(i,j) = \{Q(j,i^*,k^*) \mid \forall\, i^*k^*)\}$.

In strategy profile s every player plays her first strategy ($A_1(\cdot,\cdot)$ or $B_1(\cdot,\cdot)$). In strategy profile s^* every player plays her second strategy ($A_2(\cdot,\cdot)$ or $B_2(\cdot,\cdot)$).

For every player $i \neq A(1,1)$ it holds that

$$c_i(\mathsf{s}_{-i}, s_i^*) - c_i(\mathsf{s}) = \alpha(n-m)f(n+1) - \alpha n f(n).$$

For player $A(1,1)$ it is

$$c_1(\mathsf{s}_{-1}, s_1^*) - c_1(\mathsf{s}) = (n-m)f(n+1) - nf(n) - \beta f(1).$$

Therefore,

$$\begin{aligned} D = &\sum_{i \in A \cup B} c_i(\mathsf{s}_{-i}, s_i^*) \\ =&\beta f(1) + \alpha 2n(n-m)((n-m)f(n+1) - nf(n)) = 0. \end{aligned}$$

Furthermore,

$$\Phi(\mathsf{s}) = 2\alpha n(n-m)\sum_{i=1}^{n} f(i) + \beta f(1)$$

and

$$\Phi(\mathsf{s}^*) = 2\alpha n(n-m)\sum_{i=0}^{n-m} f(i).$$

Using the definition of β we get

$$\begin{aligned} \Phi(\mathsf{s}) = 2\alpha n(n-m)&\sum_{i=1}^{n} f(i) \\ &+ 2n(n-m)((n-m)f(n+1) - nf(n)) \end{aligned}$$

and obtain the desired result that

$$\frac{\Phi(\mathsf{s})}{\Phi(\mathsf{s}^*)} = \frac{\sum_{i=1}^{n} f(i) + ((n-m)f(n+1) - nf(n))}{\sum_{i=0}^{n-m} f(i)} = \Psi_{\mathcal{F}}$$

as needed. □

Let us remark that the above construction can be used to show lower bounds for other values of $D > D_\Psi$ by choosing other values for α and β and by choosing other cost functions for the resources in S. For example, if all resources in S were removed, we obtain a matching lower bound for the case of $D = D_\Psi$.

We do not know a way to construct a tight lower bound for the case $D < D_\Psi$. Unfortunately, there are some interesting classes of functions, e.g., some polynomials of certain degree, for which D_Ψ is negative. The best we can do here is to construct a lower bound by combining two games similar to the example above. In one of the games D is negative, in the other game the cost functions and the parameters n and m have to be chosen in such a way to compensate for this D.

Examples

We computed some values of $\Psi_\mathcal{F}$ for specific non-decreasing cost functions by using numerical analysis. For the exact values polynomials of degree can be found in Table 4.1a. Not surprisingly, the $\Psi_\mathcal{F}$ grows rapidly with the degree. However, if we consider polynomials of degree d with an additive offset of d^d (see Table 4.1b) or polynomials such as $(x+5)^d$ (see Table 4.1c), the approximation ratios are much smaller. The positive effect of the offset becomes even more evident in the case of linear or low-degree polynomial cost functions, which can be seen in Table 4.1d and Table 4.1e. Cost functions like log (see Table 4.1f) that grow slowly seem to allow for very small values for $\Psi_\mathcal{F}$.

d	$\Psi_\mathcal{F}$
1	2
2	6
3	20
4	111
5	571
6	4131
7	31289
8	200835
9	2547536
10	22512326

(a) Polynomials without offsets of the form $f(x) := x^d$.

d	$\Psi_\mathcal{F}$
1	1.500000
2	2.076923
3	3.004808
4	4.675938
5	7.376095
6	12.157520
7	20.600515
8	34.860689
9	58.564402
10	98.028377

(b) Polynomials of the form $f(x) := d^d + x^d$.

d	$\Psi_\mathcal{F}$
1	1.166667
2	1.416667
3	1.782407
4	2.375157
5	3.492036
6	5.401454
7	9.203115
8	15.912182
9	30.595039
10	60.114077

(c) Polynomials with offsets of the form $f(x) := (x+5)^d$.

e	$\Psi_\mathcal{F}$
0	2.000000
1	1.500000
2	1.333333
3	1.250000
4	1.200000
5	1.166667
6	1.142857
7	1.125000
8	1.111111
9	1.100000

(d) Linear functions with offsets of the form $f(x) := x + e$.

e	$\Psi_\mathcal{F}$
0	6.000000
1	2.750000
2	2.000000
3	1.687500
4	1.520000
5	1.416667
6	1.346939
7	1.296875
8	1.259259
9	1.230000

(e) Quadratic functions with offsets of the form $f(x) := (x+e)^2$.

e	$\Psi_\mathcal{F}$
0	1.830075
1	1.415037
2	1.203114
3	1.131517
4	1.095779
5	1.074525
6	1.060529
7	1.050668
8	1.043378
9	1.037789

(f) Logarithmic functions of the form $f(x) := \log(x+e)$.

Table 4.1.: Upper bounds on $\Psi_\mathcal{F}$ for different types of cost functions.

4.3. Computation of Approximate Pure Nash Equilibria

We use our results to compute approximate pure Nash equilibria with the general approach from Section 3.3. Our algorithm is based on the idea of [Car+11a] and we improve the analysis of their algorithm by using the results from the previous section.

$q = 1 + n^{-c}$, $\theta(q) = \frac{\Psi_{\mathcal{F}}}{1 - \frac{1-q}{q} n}$, $p = \left(\frac{1}{\theta(q)} - n^{-c}\right)^{-1}$
for all $i \in \mathcal{N}$ **do** $\ell_i = c_i\left(\mathcal{BR}_i(\mathbf{0})\right)$
$\ell_{\min} = \min_{i \in \mathcal{N}} \ell_i$, $\ell_{\max} = \max_{i \in \mathcal{N}} \ell_i$, $\Delta = \max_{e \in E} \frac{f_e(n)}{f_e(1)}$
$m = 1 + \lceil \log_{2\Delta n^{2c+2}} (\ell_{\max}/\ell_{\min}) \rceil$, $g = 2\Delta n^{2c+2}$, $b_r = \ell_{\max} \cdot g^{-r}\, \forall r \in [0, m]$
(Implicitly) partition players into *blocks* $B_1, B_2, \ldots, B_m$, such that $i \in B_r \Leftrightarrow \ell_i \in (b_{r+1}, b_r]$
for all $i \in \mathcal{N}$ **do** set i to play the strategy $s_i \leftarrow \mathcal{BR}_i(\mathbf{0})$
for all phases r from 1 to $m-1$ with $B_r \neq \emptyset$ **do**
    while there exists a player i that either belongs to B_r and has a p-move or belongs to B_{r+1} and has a q-move **do**
        i deviates to the best-response strategy $s_i \leftarrow \mathcal{BR}_i(\mathsf{s})$.

Algorithm 1: Computation of approximate equilibria in unweighted Congestion Games.

Algorithm 1 divides the set of players into polynomially many blocks depending on the costs of the players. For any strategy profile s we denote $\mathcal{BR}_i(\mathsf{s})$ the best response of player i to s. Let $\mathcal{BR}_i(\mathbf{0})$ be the best response of a player i if no other player participates in the game. We partition the players into blocks according to their costs in $\mathcal{BR}_i(\mathbf{0})$. The lower and upper bounds of theses blocks are polynomially related in n and Δ, where $\Delta = \max_{e \in E} \frac{f_e(n)}{f_e(1)}$. It is the maximal possible increase of a cost function. The algorithm proceeds in phases, starting with the blocks of players with high costs. In each phase the players of two consecutive blocks B_r and B_{r+1} are allowed to make approximate best response moves until they reach an approximate equilibrium. The players in B_r make p-approximate moves with p being slightly larger than $\Psi_{\mathcal{F}}$ and the players in Block B_{r+1} do q-approximate moves with q being slightly larger that 1. After polynomially many phases the algorithm terminates. It is crucial to note that the number of approximate best response moves in each round is bounded by a polynomial in n and Δ.

Theorem 4.5. *For every constant* $c > \max\{1, \log_n \Psi_{\mathcal{F}}\}$*, Algorithm 1 computes a* $(1 + O(n^{-c}))\Psi_{\mathcal{F}}$*-approximate pure Nash equilibrium for every Congestion Game with cost functions from* $\mathcal{F}$ *and* n *players. The algorithm terminates after at most* $O\left(\Delta^3 n^{5c+5}\right)$ *best-response moves.*

Proof. Let b_r be the boundaries of the different blocks, exactly $b_r = 2\Delta n^{2c+2} b_{r+1}$ with $b_1 = \ell_{\max}$. In addition to the set of players in a block B_r, we define R_r as the set of players who move during the phase r of the algorithm. With s^r we denote the strategy profile after phase r, where s^0 gives the strategy profile after the execution of step 6. For the following proof we will use sub games among a subset of players $F \subseteq \mathcal{N}$. All other players of $\mathcal{N} \setminus F$ are frozen to their strategies. In this sub game $n_e^F(\mathsf{s}) = |i \in F : e \in s_i|$ gives the number of players in F which uses resource e in strategy profile s and the latency function is defined as $f_e^F(x) = f_e(x + n_e^{\mathcal{N}\setminus F}(\mathsf{s}))$ with $n_e^{\mathcal{N}\setminus F}(\mathsf{s}) = |i \in \mathcal{N} \setminus F : e \in s_i|$. Then, the potential is defined as $\Phi_F = \sum_{e\in E} \sum_{j=1}^{n_e^F(\mathsf{s})} f_e^F(j)$.

We start with the three basic claims needed for the further lemmas in the proof:

Claim 4.6 (cf. [Car+11a])**.** *For any outcome* s *it holds that*

$$\sum_{e\in E} f_e(n_e(\mathsf{s})) \leq \Phi(\mathsf{s}) \leq \sum_{u\in\mathcal{N}} c_i(\mathsf{s}).$$

Proof (cf. [Car+11a]). The first inequality follows easily by the definition of function Φ. The second one can be obtained by the following derivation:

$$\begin{aligned}
\Phi(\mathsf{s}) &= \sum_{e\in E} \sum_{j=1}^{n_e(\mathsf{s})} f_e(j) \\
&\leq \sum_{e\in E} n_e(\mathsf{s}) \cdot f_e\left(n_e(\mathsf{s})\right) \\
&= \sum_{i\in\mathcal{N}} \sum_{e\in s_i} f_e\left(n_e(\mathsf{s})\right) \\
&= \sum_{i\in\mathcal{N}} c_i(\mathsf{s}).
\end{aligned}$$

□

Claim 4.7 (cf. [Car+11a])**.** *Let* s *be a state of the Congestion Game with a set of players* $\mathcal{N}$ *and let* $F \subseteq \mathcal{N}$*. Then,* $\Phi(\mathsf{s}) \leq \Phi_F(\mathsf{s}) + \Phi_{\mathcal{N}\setminus F}(\mathsf{s})$ *and* $\Phi(\mathsf{s}) \geq \Phi_F(\mathsf{s})$*.*

Proof (cf. [Car+11a]). We use the definition of the potential function for the original game and the sub games, the definitions of the modified latency functions $f_e^F(x) = f_e(x + n_e^{\mathcal{N}\setminus F}(\mathsf{s}))$ and $f_e^{\mathcal{N}\setminus F}(x) = f_e(x + n_e^F(\mathsf{s}))$, and the equality $n_e(\mathsf{s}) = n_e^F(\mathsf{s}) + n_e^{\mathcal{N}\setminus F}(\mathsf{s})$ to obtain

$$\begin{aligned}
\Phi(\mathsf{s}) &= \sum_{e\in E} \sum_{j=1}^{n_e(\mathsf{s})} f_e(j) \\
&= \sum_{e\in E} \sum_{j=1}^{n_e^F(\mathsf{s})} f_e(j) + \sum_{e\in E} \sum_{j=n_e^F(\mathsf{s})+1}^{n_e(\mathsf{s})} f_e(j) \\
&\leq \sum_{e\in E} \sum_{j=1}^{n_e^F(\mathsf{s})} f_e(j + n_e^{\mathcal{N}\setminus F}(\mathsf{s})) + \sum_{e\in E} \sum_{j=n_e^F(\mathsf{s})+1}^{n_e(\mathsf{s})} f_e(j)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{e\in E}\sum_{j=1}^{n_e^F(\mathsf{s})} f_e^f(j) + \sum_{e\in E}\sum_{j=1}^{n_e^{\mathcal{N}\setminus F}(\mathsf{s})} f_e(j+n_e^F(\mathsf{s}))\\
&= \sum_{e\in E}\sum_{j=1}^{n_e^F(\mathsf{s})} f_e^F(j) + \sum_{e\in E}\sum_{j=1}^{n_e^{\mathcal{N}\setminus F}(\mathsf{s})} f_e^{N\setminus F}(j)\\
&= \Phi_F(\mathsf{s}) + \Phi_{\mathcal{N}\setminus F}(\mathsf{s}),
\end{aligned}
$$

as desired for the first part of the claim. For the second part, we have

$$
\begin{aligned}
\Phi(\mathsf{s}) &= \sum_{e\in E}\sum_{j=1}^{n_e(\mathsf{s})} f_e(j)\\
&\geq \sum_{e\in E}\sum_{j=n_e^{\mathcal{N}\setminus F}(\mathsf{s})+1}^{n_e(\mathsf{s})} f_e(j)\\
&= \sum_{e\in E}\sum_{j=1}^{n_e^F(\mathsf{s})} f_e(j+n_e^{\mathcal{N}\setminus F}(\mathsf{s}))\\
&= \sum_{e\in E}\sum_{j=1}^{n_e^F(\mathsf{s})} f_e^F(j)\\
&= \Phi_F(\mathsf{s}).
\end{aligned}
$$

□

Claim 4.8 (cf. [Car+11a])**.** *Let c_i denote the costs of player $i \in R_r$ just after making her last move within phase r. Then,*

$$\Phi_{R_r}(\mathsf{s}^r) \leq \sum_{i\in R_r} c_i.$$

Proof (cf. [Car+11a]). We denote by s_i the strategy of player i at state s^r. We rank the players that use resource e in s^r according to the timing of their last moves (using consecutive integers $1, 2, \ldots$). We denote by $\mathrm{rank}_e(i)$ the number of players in R_r with the smaller ranking than i on resource e. Then, we get $c_i \geq \sum_{e\in s_i} f_e^{R_r}(\mathrm{rank}_e(i))$, since any resource e in s_i is occupied by at least $\mathrm{rank}_e(i)$ players from R_r at state s^r: i and the players with ranks $1, 2, \ldots, \mathrm{rank}_e(i) - 1$ that made their last move before i. Hence, by the definition of the potential function (expressed using the modified latency functions for the sub game among the players of R_r), we have

$$
\begin{aligned}
\Phi_{R_r}(\mathsf{s}^i) &= \sum_{e\in E}\sum_{j=1}^{n_e^{R_r}(\mathsf{s}^r)} f_e^{R_r}(j)\\
&= \sum_{e\in E}\sum_{i\in R_r: e\in s_i} f_e^{R_r}(\mathrm{rank}_e(i))
\end{aligned}
$$

$$= \sum_{i \in R_r} \sum_{e \in s_i} f_e^{R_r}(\mathrm{rank}_e(i))$$
$$\leq \sum_{i \in R_r} c_i,$$

and the claim follows. □

Now we bound the potential value of an arbitrary q-approximate equilibrium with the minimal potential value:

Lemma 4.9. *Let* s *be a* q*-approximate equilibrium and* s^* *be a strategy profile with minimal potential, then* $\Phi_F(\mathsf{s}) \leq \theta(q)\Phi_F(\mathsf{s}^*)$ *for every* $F \subseteq \mathcal{N}$*.*

Proof. Since s is a q-approximate equilibrium, we have $c_i(\mathsf{s}) \leq qc_i(\mathsf{s}_{-i}, s_i^*)$. Thus,

$$c_i(\mathsf{s}_{-i}, s_i^*) - c_i(\mathsf{s}) \geq \frac{1-q}{q} c_i(\mathsf{s}) \geq \frac{1-q}{q} \Phi_F(\mathsf{s}).$$

With the definition of $D(\mathsf{s}, \mathsf{s}^*)$, we get $-D(\mathsf{s}, \mathsf{s}^*) \geq n\frac{1-q}{q}\Phi_F(\mathsf{s})$ and using Theorem 4.2 gives us

$$\Phi_F(\mathsf{s}) \leq \Psi_{\mathcal{F}}\Phi_F(\mathsf{s}^*) + n\frac{1-q}{q}\Phi_F(\mathsf{s}),$$

or equivalently

$$\Phi_F(\mathsf{s}) \leq \frac{\Psi_{\mathcal{F}}}{1 - \frac{1-q}{q}n}\Phi_F(\mathsf{s}^*).$$

Together with our definition of $\theta(q)$ we have shown the lemma. □

Afterwards, we denote the key argument where we show with the help of Lemma 4.9 that the potential of the sub game is significantly smaller than b_r. Therefore, the costs experienced by players moving in phase r are considerably lower than the costs of any player in blocks $B_1, \ldots, B_{r-1}$.

Lemma 4.10. *For every phase* $r \geq 2$*, it holds that* $\Phi_{R_r}(\mathsf{s}^{r-1}) \leq \frac{b_r}{n^c}$*.*

Proof. We show this lemma by contradiction. We assume that $\Phi_{R_r}(\mathsf{s}^{r-1}) > \frac{b_r}{n^c}$ and show with help of Lemma 4.9 that state s^{r-1} cannot be a q-approximate equilibrium for the players in $R_r \cap B_r$. Therefore we will bound the potential of state s^{r-1}.

By Claim 4.7, we have $\Phi_{R_r}(\mathsf{s}^{r-1}) \leq \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) + \Phi_{R_r \cap B_{r+1}}(\mathsf{s}^{r-1})$. Now observe that a player i in B_{r+1} does not move in the first $r-1$ phases of the algorithm. As it is assigned to the strategy $\mathcal{BR}_i(\mathbf{0})$ in the initialization phase, it experiences costs bounded by Δb_{r+1} at state s^{r-1}. The potential is upper-bounded with the costs of all n players by Claim 4.6:

$$\Phi_{R_r \cap B_{r+1}}(\mathsf{s}^{r-1}) \leq n\Delta b_{r+1}. \tag{4.1}$$

Now we can use (4.1) together with the assumption that $\Phi_{R_r}(\mathsf{s}^{r-1}) > \frac{b_r}{n^c}$, Claim 4.7 and the definition of b_r:

$$\begin{aligned}
\Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) &\geq \Phi_{R_r}(\mathsf{s}^{r-1}) - \Phi_{R_r \cap B_{r+1}}(\mathsf{s}^{r-1}) \\
&> \frac{b_r}{n^c} - n\Delta b_{r+1} \\
&= \frac{2\Delta n^{2c+2} b_{r+1}}{n^c} - n\Delta b_{r+1} \\
&= (2\Delta n^{2c+2} - n\Delta) b_{r+1} \\
&\geq \Delta n^{c+1} b_{r+1}. \qquad (4.2)
\end{aligned}$$

Now we look at the sub game among the players in R_r at phase r in more detail. Every player in $B_r \cap R_r$ decreases the potential by at least $(p-1)c_i$. By using Claim 4.7 and inequalities (4.1) and (4.2), we get

$$\begin{aligned}
(p-1) \sum_{i \in R_r \cap B_r} c_i &\leq \Phi_{R_r}(\mathsf{s}^{r-1}) - \Phi_{R_r}(\mathsf{s}^r) \\
&\leq \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) + \Phi_{R_r \cap B_{r+1}}(\mathsf{s}^{r-1}) - \Phi_{R_r}(\mathsf{s}^r) \\
&\leq \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) + n\Delta b_{r+1} - \Phi_{R_r}(\mathsf{s}^r) \\
&< \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) + \frac{1}{n^c}\Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) - \Phi_{R_r}(\mathsf{s}^r) \\
&= \left(1 + \frac{1}{n^c}\right) \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) - \Phi_{R_r}(\mathsf{s}^r). \qquad (4.3)
\end{aligned}$$

Furthermore, all players in $R_r \cap B_{r+1}$ decreases their costs at most by deviating to strategy $\mathcal{BR}_i(\mathbf{0})$, which is $n\Delta b_{r+1}$ for all n players. Now by using Claim 4.8 and inequalities (4.2) and (4.3), we obtain

$$\begin{aligned}
\Phi_{R_r}(\mathsf{s}^r) &\leq \sum_{i \in R_r} c_i \\
&= \sum_{i \in R_r \cap B_{r+1}} c_i + \sum_{i \in R_r \cap B_r} c_i \\
&< n\Delta b_{r+1} + \frac{1}{p-1}\left(1 + \frac{1}{n^c}\right) \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) - \frac{1}{p-1}\Phi_{R_r}(\mathsf{s}^r) \\
&\leq \frac{1}{p-1}\left(1 + \frac{p}{n^c}\right) \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) - \frac{1}{p-1}\Phi_{R_r}(\mathsf{s}^r) \\
&< \left(\frac{1}{p} + \frac{1}{n^c}\right) \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}). \qquad (4.4)
\end{aligned}$$

We now define an intermediate state $\hat{\mathsf{s}}$ in which all players in $R_r \cap B_r$ play their strategies in s^r and all other players their strategies in s^{r-1}. The difference of the potential of this state $\hat{\mathsf{s}}$ to the next state s^r can be at most $n\Delta b_{r+1}$ as at most all players deviate from their strategy $\mathcal{BR}_i(\mathbf{0})$. Obviously, it yields that for a state s^* with minimal potential that $\Phi_{R_r \cap B_r}(\mathsf{s}^*) \leq \Phi_{R_r \cap B_r}(\hat{\mathsf{s}})$ and $\Phi_{R_r \cap B_r}(\hat{\mathsf{s}}) \leq \Phi_{R_r}(\hat{\mathsf{s}})$. Together with inequalities (4.2), (4.3) and (4.4), we have

$$\begin{aligned}
\Phi_{R_r \cap B_r}(\mathsf{s}^*) &\leq \Phi_{R_r \cap B_r}(\hat{\mathsf{s}}) \\
&\leq \Phi_{R_r}(\hat{\mathsf{s}}) \\
&\leq \Phi_{R_r}(\mathsf{s}^r) + n\Delta b_{r+1} \\
&< \left(\frac{1}{p} + \frac{2}{n^c}\right) \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}) \\
&\leq \frac{1}{\theta(q)} \Phi_{R_r \cap B_r}(\mathsf{s}^{r-1}).
\end{aligned}$$

With Lemma 4.9 the last inequality contradicts the fact that state s^{r-1} is a q-approximate equilibrium for the players in $R_r \cap B_r$. □

Further, we can show that for players in block B_t neither its costs increase considerably nor a deviation to another strategy with considerably lower costs is interesting, caused by the movements of the other players in all following rounds.

Lemma 4.11. *Let i be a player in the block B_t, where $t \leq m-2$. Let s_i' be a strategy different from the one assigned to i by the algorithm at the end of phase t. Then, for each phase $r \geq t$, it holds that*

$$c_i(\mathsf{s}^r) \leq p \cdot c_i(\mathsf{s}^r_{-i}, s_i') + \frac{p+1}{n^c} \sum_{k=t+1}^{r} b_k.$$

Proof. We will prove the lemma by using induction on r. For $r = t$, the claim follows by the definition of phase r of the algorithm. Assume that the claim is true for a phase r with $t \leq r \leq m-2$. In the following, we show that the claim is true for the phase $r+1$ as well.

First, we show that if

$$c_i(\mathsf{s}^{r+1}) \leq c_i(\mathsf{s}^r) + \frac{b_{r+1}}{n^c} \tag{4.5}$$

and

$$c_i(\mathsf{s}^r_{-i}, s_i') \leq c_i(\mathsf{s}^{r+1}_{-i}, s_i') + \frac{b_{r+1}}{n^c} \tag{4.6}$$

then the claim holds. By the hypothesis of induction, we have

$$c_i(\mathsf{s}^r) \leq p \cdot c_i(\mathsf{s}^r_{-i}, s_i') + \frac{p+1}{n^c} \sum_{k=t+1}^{r} b_k.$$

Combining the above three inequalities, we obtain that

$$\begin{aligned} c_i(\mathsf{s}^{r+1}) &\leq c_i(\mathsf{s}^r) + \frac{b_{r+1}}{n^c} \\ &\leq p \cdot c_i(\mathsf{s}^r_{-i}, s'_i) + \frac{p+1}{n^c} \sum_{k=t+1}^{r} b_k + \frac{b_{r+1}}{n^c} \\ &\leq p \cdot c_i(\mathsf{s}^{r+1}_{-i}, s'_i) + \frac{p+1}{n^c} \sum_{k=t+1}^{r+1} b_k, \end{aligned}$$

as desired.

It stays open to prove (4.5) and (4.6). We show this by contradiction with the help of Lemma 4.10.

Assume that (4.5) does not hold, i.e., $c_i(\mathsf{s}^{r+1}) > c_i(\mathsf{s}^r) + \frac{b_{r+1}}{n^c}$ for some player i of block B_t, where $t \leq r$. We will show that the potential $\Phi_{R_{r+1}}(\mathsf{s}^{r+1})$ at state s^{r+1} of the sub game among the players in R_{r+1} is larger than $\frac{b_{r+1}}{n^c}$. Since the potential decreases during phase $r+1$, $\Phi_{R_{r+1}}(\mathsf{s}^r)$ should also be larger than $\frac{b_{r+1}}{n^c}$, contradicting Lemma 4.10. Indeed, since player i does not move during phase $r+1$, the increase in her costs from state s^r to state s^{r+1} implies the existence of a set of resources $C \subseteq s_i$ in her strategy with the following properties: Each resource $e \in C$ is also used by at least one player of R_{r+1} in state s^{r+1} and, furthermore, $\sum_{e \in C} f_e(n_e(\mathsf{s}^{r+1})) > \frac{b_{r+1}}{n^c}$. By Claim 4.7, we obtain that $\Phi_{R_{r+1}}(\mathsf{s}^{r+1}) > \frac{b_{r+1}}{n^c}$.

Similarly, assume that (4.6) does not hold for a player i of block B_t and a strategy s'_i that is different from s_i, the strategy assigned to i in phase t, i.e., $c_i(\mathsf{s}^r_{-i}, s'_i) > c_i(\mathsf{s}^{r+1}_{-i}, s'_i) + \frac{b_{r+1}}{n^c}$ and $c_i(\mathsf{s}^r_{-i}, s'_i) - c_i(\mathsf{s}^{r+1}_{-i}, s'_i) > \frac{b_{r+1}}{n^c}$. Recall that player i does not move during phase $r+1$. This implies that there exists a set of resources $C \subseteq s'_i$ with $n_e(\mathsf{s}^r) > n_e(\mathsf{s}^{r+1}) \,\forall\, e \in C$. Therefore,

$$\begin{aligned} \Phi_{R_{r+1}}(\mathsf{s}^r) &\geq \sum_{e \in s'_i} f_e(n_e(\mathsf{s}^r)) \geq \sum_{e \in C} f_e(n_e(\mathsf{s}^r))) \\ &\geq \sum_{e \in C} f_e(n_e(\mathsf{s}^r)+1) - f_e(n_e(\mathsf{s}^{r+1})+1) \\ &\geq c_i(\mathsf{s}^r_{-i}, s'_i) - c_i(\mathsf{s}^{r+1}_{-i}, s'_i) \\ &> \frac{b_{r+1}}{n^c}. \end{aligned}$$

Again, this contradicts Lemma 4.10.

Hence, (4.5) and (4.6) hold and the proof of the inductive step is complete. □

Finally, we have to show that in the state s^{m-1}, computed by the algorithm after the last phase, no player has an incentive to deviate to another strategy in order to decrease her costs by a factor of at least $p\left(1 + \frac{4}{n^c}\right)$. The claim is obviously true for the players in the blocks B_{m-1} and B_m by the definition of the last phase of the algorithm. Let i be a player in block B_t with $t \leq m-2$ and let s_i' be any strategy different from the one assigned to i by the algorithm after phase t. We apply Lemma 4.11 to player i. By the definition of b_r, we have $\sum_{k=t+1}^{m} b_k \leq 2b_{t+1}$. Also, $c_i(\mathsf{s}_{-i}^{m-1}, s_i') \geq b_{t+1}$, since i belongs to block B_t. Hence, Lemma 4.11 implies that

$$c_i(\mathsf{s}^{m-1}) \leq p \cdot c_i(\mathsf{s}_{-i}^{m-1}, s_i') + \frac{2(p+1)}{n^c} c_i(\mathsf{s}_{-i}^{m-1}, s_i') \leq p\left(1+\frac{4}{n^c}\right) c_i(\mathsf{s}_{-i}^{m-1}, s_i'),$$

as desired. The last inequality follows since $p \geq 1$.

By the definition of the parameters q and p, we obtain that the computed state is a ρ-approximate equilibrium with $\rho \leq \left(\frac{1}{\theta(q)} - \frac{1}{n^c}\right)^{-1}\left(1+\frac{4}{n^c}\right)$, where $\theta(q) = \frac{\Psi_{\mathcal{F}}}{1-\frac{1-q}{q}n}$ and $q = 1+n^{-c}$. By making simple calculations, we obtain

$$\rho \leq \frac{1}{\frac{1+\frac{n}{n^c+1}}{\Psi_{\mathcal{F}}} - n^{-c}}\left(1+\frac{4}{n^c}\right) \leq \frac{\Psi_{\mathcal{F}}}{1+\frac{n}{n^c+1} - \frac{\Psi_{\mathcal{F}}}{n^c}}\left(1+\frac{4}{n^c}\right) \leq \left(1+O\left(\frac{1}{n^c}\right)\right)\Psi_{\mathcal{F}}$$

The last inequality holds as $c > \log_n \Psi_{\mathcal{F}}$.

We will consider the different phases of the algorithm to upper bound the total number of best-response moves. In line 6, n best-response moves are executed in which each player deviates to her best strategy if there would not be any other player. Afterwards, we have at most n remaining phases. We start by looking at the first phase: Due to the definition of block B_1 and the relation between the cost functions Δ, any player has a latency of at most Δb_1. We can bound the potential with $\Phi_{R_1}(\mathsf{s}^0) \leq \sum_{i\in R_1} c_i(\mathsf{s}^0) \leq n\Delta b_1$. On the other side no player which that has a latency smaller than b_3, otherwise she would not change in this phase. Hence, the decrease of the potential caused by each q-move is at least $(q-1)b_3$. Therefore, we can bound the maximal moves in the first phase with the definition of b_r by $\frac{n\Delta b_1}{(q-1)b_3} = 4\Delta^3 n^{5c+5}$.

For all other phases $r \geq 2$, Lemma 4.10 implies that $\Phi_{R_r}(\mathsf{s}^{r-1}) \leq \frac{b_r}{n^c}$. Equivalently to the first phase, each player in R_r experiences a latency of at least b_{r+2} and each move decreases the potential by at least $(q-1)b_{r+2}$. As a result, the maximal number of moves in each phase is bounded by $\frac{b_r}{n^c(q-1)b_{r+2}} = 4\Delta^2 n^{4c+4}$.

Altogether, we can upper bound the number of best-response moves during the execution of the algorithm by $O\left(\Delta^3 n^{5c+5}\right)$. □

Conclusion

The results of this chapter are twofold. Firstly, we bounded the potential function and showed results for the stretch of the potential function for very interesting subclasses of cost functions, especially polynomial functions with offsets. Secondly, we used these bounds together with the given algorithmic approach to efficiently compute approximate pure Nash equilibria with low approximation factors.

CHAPTER 5

Shapley Value Weighted Congestion Games

Shapley value (SV) weighted Congestion Games were introduced in [KR15]. This class of games considers weighted Congestion Games in which Shapley values are used as an alternative to proportional shares for distributing the total cost of each resource among its users. We focus on the interesting subclass of such games with polynomial resource cost functions and present an algorithm that computes approximate pure Nash equilibria with a polynomial number of strategy updates. The algorithm builds on the algorithmic ideas of [Car+15] which have been introduced in Section 3.3. However, to the best of our knowledge, this is the first algorithmic result on computation of approximate equilibria using a different sharing method than proportional shares in this setting. We present a novel relation that approximates the Shapley value of a player by her proportional share and vice versa. As side results, we upper bound the approximate Price of Anarchy of such games and significantly improve the best known factor for computing approximate pure Nash equilibria in weighted Congestion Games with proportional sharing of [Car+15].

Contribution and Underlying Work We present an algorithm to compute approximate pure Nash equilibria in weighted Congestion Games under Shapley cost sharing. In games with polynomial cost functions of degree at most d, our algorithm achieves an approximation factor asymptotically close to $\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$. Similar to [Car+15] our algorithm computes a sequence of improvement steps of polynomial length that yields a ρ-approximate Nash equilibrium. Hence, our algorithm performs only a polynomial number of strategy updates. We show that our algorithm can also be used to compute ρ-approximate pure Nash equilibria for weighted Congestion Games with proportional sharing which improves the approximation factor of $d^{2 \cdot d + o(d)}$ in [Car+15] to $\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$.

In the course of the analysis we exhibit an interesting relation between the Shapley cost -share of a player and her proportional share. In the case of polynomial cost functions with constant degree, each of them can be approximated by the other within a constant factor. This insight leads to an alternative proof to [HKS14] for the existence of approximate pure Nash equilibria in weighted Congestion Games with proportional cost sharing.

Finally, we derive bounds on the approximate Price of Anarchy which may be of independent interest as they allow to bound the quality of approximately stable states.

The model, analysis, and results presented in this chapter are based on the following publication:

> Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria in Shapley Value Weighted Congestion Games**. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE)*. 2017. [Fel+17a]

Outline of This Chapter The chapter starts in Section 5.1 with an extended version of our general model from Section 3.2. In the next section we analyze various properties of the Shapley value as well as of the potential function which are utilized later (see Section 5.2). Section 5.3 is devoted to the approximation of Shapley cost-sharing with proportional cost-sharing and vice versa and Section 5.4 analyzes the approximate Price of Anarchy and the stretch of the potential function as the last building block for our analysis of the algorithm. Finally, Section 5.5 states the formal algorithm and its analysis using the results from the previous sections and the approach from Section 3.3.

5.1. Model and Further Notations

In this chapter, we use the model of Section 3.2, mainly with the Shapley cost-sharing method. If not stated otherwise, we use $\chi_e = \chi_e^{\mathrm{SV}}$ for a better reading throughout this chapter. We also implicitly assume the Shapley cost-sharing as the underlying method for the cost calculation. If we refer to the proportional cost sharing method, we write $\chi_{ie}^{\mathrm{Prop}}$ or c_i^{Prop}. Otherwise, we completely stick to the introduced model of a weighted Congestion Game $\mathcal{G}$.

Potential Function Kollias and Roughgarden [KR15] prove that weighted Congestion Games under Shapley values are potential games by using the following potential. Given an outcome s and an arbitrary ordering τ of the players in $\mathcal{N}$, the potential is given by

$$\Phi(\mathsf{s}) = \sum_{e \in E} \Phi_e(\mathsf{s}) = \sum_{e \in E} \sum_{i \in N_e(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}). \tag{5.1}$$

We now restrict this potential function by allowing only a subset of players $A \subseteq \mathcal{N}$ to participate and define the A-limited potential as

$$\Phi^A(\mathsf{s}) = \sum_{e \in E} \Phi_e^A(\mathsf{s}) = \sum_{e \in E} \sum_{i \in N_e^A(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^A(\mathsf{s})\}). \tag{5.2}$$

Consider sets A and B such that $B \subseteq A \subseteq \mathcal{N}$. Then the B-partial potential of set A is defined by

$$\Phi_B^A(\mathsf{s}) = \Phi^A(\mathsf{s}) - \Phi^{A \setminus B}(\mathsf{s}) = \sum_{e \in E} \Phi_{e,B}^A(\mathsf{s}) = \sum_{e \in E} \Phi_e^A(\mathsf{s}) - \Phi_e^{A \setminus B}(\mathsf{s}). \tag{5.3}$$

If the set B contains only one player, i.e., $B = \{\{i\}\}$, then we write $\Phi_i^A(\mathsf{s}) = \Phi_B^A(\mathsf{s})$. In case of $A = \mathcal{N}$, $\Phi_B^{\mathcal{N}}(\mathsf{s}) = \Phi_B(\mathsf{s}) = \sum_{e \in E} \Phi_{e,B}(\mathsf{s})$. Intuitively, $\Phi_B^A(\mathsf{s})$ is the value that the players in $B \subseteq A$ contribute to the A-limited potential.

5.2. Shapley and Potential Properties

The following properties of Shapley values are extensively used in our proofs.

Proposition 5.1. *Fix a resource e. Then for any set of players S and $i \in S$, we have for $j, j_1, j_2, j', j_1', j_2', i_1, i_2 \notin S$:*

a. $\chi_e(i, S) \leq \chi_e(i, S \cup \{j\})$,

b. $\chi_e(i, S \cup \{j'\}) \geq \chi_e(i, S \cup \{j_1, j_2\})$, *with* $j' \neq i$ *and* $w_{j'} = w_{j_1} + w_{j_2}$,

c. $\chi_e(i, S \cup \{j_1, j_2\}) \geq \chi_e(i, S \cup \{j_1', j_2'\})$, *with* $w_{j_1'} = w_{j_2'} = \frac{w_{j_1} + w_{j_2}}{2}$,

d. $\chi_e(i, S) \geq \chi_e(i_1, S \backslash \{i\} \cup \{i_1\}) + \chi_e(i_2, S \backslash \{i\} \cup \{i_1, i_2\})$, *with* $w_{i_1} = w_{i_2} = \frac{w_i}{2}$.

Proof. Let $k := |S|$. By the definition of Shapley values

$$\begin{aligned}
\chi_e(i, S \cup \{j\}) &= \frac{1}{(k+1)!} \sum_{\pi \in \Pi(S \cup \{j\})} \left(c_e\left(W_{S \cup \{j\}}^{<i,\pi} + w_i\right) - c_e\left(W_{S \cup \{j\}}^{<i,\pi}\right)\right) \\
&\geq \frac{1}{(k+1)!} \sum_{\pi \in \Pi(S \cup \{j\})} \left(c_e\left(W_S^{<i,\pi} + w_i\right) - c_e\left(W_S^{<i,\pi}\right)\right) \\
&= \frac{1}{k!} \sum_{\pi \in \Pi(S)} \left(c_e\left(W_S^{<i,\pi} + w_i\right) - c_e\left(W_S^{<i,\pi}\right)\right) \\
&= \chi_e(i, S),
\end{aligned}$$

proving (a).

For (b) and (c), consider $\chi_e(i, S \cup \{j_1, j_2\})$. Observe that only for permutations $\pi \in \Pi(S \cup \{j_1, j_2\})$ where either $j_1 < i < j_2$ or $j_2 < i < j_1$, the corresponding contribution to $\chi_e(i, S \cup \{j_1, j_2\})$ changes if we change the weight of j_1, j_2 but keep their sum the same. Fix a permutation $\pi \in \Pi(S \cup \{j_1, j_2\})$ with $j_1 < i < j_2$ and pair it with the corresponding permutation $\hat{\pi}$ where only j_1 and j_2 are swapped. Then the contribution of π and $\hat{\pi}$ to $\chi_e(i, S \cup \{j_1, j_2\})$ is

$$\begin{aligned}
\frac{1}{(k+2)!} \cdot \Big(& c_e\left(W_S^{<i,\pi} + w_{j_1} + w_i\right) - c_e\left(W_S^{<i,\pi} + w_{j_1}\right) \\
& + c_e\left(W_S^{<i,\pi} + w_{j_2} + w_i\right) - c_e\left(W_S^{<i,\pi} + w_{j_2}\right)\Big). \qquad (5.4)
\end{aligned}$$

Since $c_e(x + w_i) - c_e(x)$ is convex in x, we get that

$$(5.4) \geq \frac{1}{(k+2)!} \cdot \Big(c_e\left(W_S^{<i,\pi} + w_{j_1'} + w_i\right) - c_e\left(W_S^{<i,\pi} + w_{j_1'}\right)$$

$$+c_e\left(W_S^{<i,\pi}+w_{j_2'}+w_i\right)-c_e\left(W_S^{<i,\pi}+w_{j_2'}\right)\Big),$$

and

$$(5.4)\leq\frac{1}{(k+2)!}\cdot\Big(c_e\left(W_S^{<i,\pi}+w_{j_1}+w_{j_2}+w_i\right)-c_e\left(W_S^{<i,\pi}+w_{j_1}+w_{j_2}\right)$$
$$+c_e\left(W_S^{<i,\pi}+0+w_i\right)-c_e\left(W_S^{<i,\pi}+0\right)\Big).$$

Part (c) and (b) follow, respectively. Part (d) of the proposition is shown in [GKK15]. □

We proceed to the properties of the restricted types of the potential function that we defined before.

Proposition 5.2. *Let A and B be sets of players such that $B\subseteq A\subseteq\mathcal{N}$, s and s' outcomes of the game such that the players in $A\subseteq\mathcal{N}$ use the same strategies in both s and s', and $z\in\mathcal{N}$ an arbitrary player. Then*

a. $\Phi_B^A(\mathsf{s})\leq\Phi_B(\mathsf{s})$, *b.* $\Phi_B^A(\mathsf{s})=\Phi_B^A(\mathsf{s}')$, *c.* $\Phi_z(\mathsf{s})=c_z(\mathsf{s})$.

Proof. We prove the different parts separately:

a. For each $e\in E$, let $I_e(\mathsf{s})=\Phi_e^A(\mathsf{s})-\Phi_e^{A\setminus B}(\mathsf{s})$. By definition of the B-partial potential (5.3), we have

$$\Phi_B^A(\mathsf{s})=\Phi^A(\mathsf{s})-\Phi^{A\setminus B}(\mathsf{s})=\sum_{e\in E}I_e(\mathsf{s}).\tag{5.5}$$

By the definition of limited potential (5.2), for an arbitrary τ, define $I_e(\mathsf{s})$, $\forall\, e\in E$, as

$$I_e(\mathsf{s})=\sum_{i\in N_e^A(\mathsf{s})}\chi_e(i,\{j:\tau(j)\leq\tau(i),j\in N_e^A(\mathsf{s})\})$$
$$-\sum_{i\in N_e^{A\setminus B}(\mathsf{s})}\chi_e(i,\{j:\tau(j)\leq\tau(i),j\in N_e^{A\setminus B}(\mathsf{s})\}).\tag{5.6}$$

Hart and Mas-Collel [HM89] proved that the potential is independent of the ordering τ that players are considered. As mentioned before, $\Phi^A(\mathsf{s})$ is a restriction of $\Phi(\mathsf{s})$ where only players in A participate. Thus, independence from τ also applies to the limited potential.

Firstly, we focus on the first term of (5.6) and choose an ordering where the players in set A are first. Then we observe that by substituting $N_e^A(\mathsf{s})$ with $N_e(\mathsf{s})$, the cost share remains the same. This is due to the fact that any player coming after the players in set A in the ordering has no impact in the cost computation. These are the players who belong in set $\mathcal{N} \setminus A$ (since we assume players in A are first). Therefore, the first term of (5.6) equals

$$\sum_{i \in N_e^A(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}).$$

Following the same technique for the second term of (5.6), we choose an ordering in which the players in $A \setminus B$ are first. Then we can substitute $N_e^{A \setminus B}(\mathsf{s})$ with $N_e^{\mathcal{N} \setminus B}(\mathsf{s})$ without affecting the term's value. Therefore, (5.6) is equivalent to

$$\sum_{i \in N_e^A(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}) - \sum_{i \in N_e^{A \setminus B}(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^{\mathcal{N} \setminus B}(\mathsf{s})\}). \tag{5.7}$$

For each $e \in E$, define $I'_e(\mathsf{s})$ to equal

$$\sum_{i \in N_e^{\mathcal{N} \setminus A}(\mathsf{s})} \Big(\chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}) - \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^{\mathcal{N} \setminus B}(\mathsf{s})\})\Big). \tag{5.8}$$

Note that $I'_e(\mathsf{s}) \geq 0, \forall\, e \in E$. Intuitively, the first term computes the cost with respect to all players using resource e, $N_e(\mathsf{s})$. Regarding the second term, if we take away some of these players, i.e., players in B, then due to convexity the costs of the remaining players either remain the same or are reduced. This depends on the position players in B had in the ordering. To simplify, for the rest of this proof, let

$$\chi_i^{\mathcal{N}}(\mathsf{s}) = \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}), \tag{5.9}$$

$$\chi_i^{\mathcal{N} \setminus B}(\mathsf{s}) = \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^{\mathcal{N} \setminus B}(\mathsf{s})\}). \tag{5.10}$$

Since $I'_e(\mathsf{s}) \geq 0$, we get that for each $e \in E$,

$$I_e(\mathsf{s}) \leq I_e(\mathsf{s}) + I'_e(\mathsf{s})$$

which, by (5.7), (5.8), (5.9) and (5.10), is equivalent to

$$\sum_{i \in N_e^A(\mathsf{s})} \chi_i^{\mathcal{N}}(\mathsf{s}) - \sum_{i \in N_e^{A \setminus B}(\mathsf{s})} \chi_i^{\mathcal{N} \setminus B}(\mathsf{s}) \leq \leq \sum_{i \in N_e^A(\mathsf{s})} \chi_i^{\mathcal{N}}(\mathsf{s}) - \sum_{i \in N_e^{A \setminus B}(\mathsf{s})} \chi_i^{\mathcal{N} \setminus B}(\mathsf{s}) + \sum_{i \in N_e^{\mathcal{N} \setminus A}(\mathsf{s})} \left(\chi_i^{\mathcal{N}}(\mathsf{s}) - \chi_i^{\mathcal{N} \setminus B}(\mathsf{s})\right). \tag{5.11}$$

By the assumption $B \subseteq A \subseteq \mathcal{N}$, we get that $(\mathcal{N} \setminus A) \cup (A \setminus B) = \mathcal{N} \setminus B$. Thus inequality (5.11) becomes

$$\sum_{i \in N_e^A(\mathsf{s})} \chi_i^{\mathcal{N}}(\mathsf{s}) - \sum_{i \in N_e^{A\setminus B}(\mathsf{s})} \chi_i^{\mathcal{N}\setminus B}(\mathsf{s}) \leq \sum_{i \in N_e(\mathsf{s})} \chi_i^{\mathcal{N}}(\mathsf{s}) - \sum_{i \in N_e^{\mathcal{N}\setminus B}(\mathsf{s})} \chi_i^{\mathcal{N}\setminus B}(\mathsf{s}).$$

Substituting $\chi_i^{\mathcal{N}}(\mathsf{s})$ and $\chi_i^{\mathcal{N}\setminus B}(\mathsf{s})$ from (5.9) and (5.10), we get by (5.7) that the previous is equivalent to

$$I_e(\mathsf{s}) \leq \Phi_e(\mathsf{s}) - \Phi_e^{\mathcal{N}\setminus B}(\mathsf{s}) \quad \Leftrightarrow \quad \sum_{e\in E} I_e(\mathsf{s}) \leq \sum_{e\in E} \Phi_e(\mathsf{s}) - \Phi_e^{\mathcal{N}\setminus B}(\mathsf{s}).$$

By (5.5), we conclude to the desirable $\Phi_B^A(\mathsf{s}) \leq \Phi_B(\mathsf{s})$.

b. By definition (5.3) of partial potential, we have

$$\Phi_B^A(\mathsf{s}) = \Phi^A(\mathsf{s}) - \Phi^{A\setminus B}(\mathsf{s}) = \sum_{e\in E} \left(\Phi_e^A(\mathsf{s}) - \Phi_e^{A\setminus B}(\mathsf{s})\right). \tag{5.12}$$

For each $e \in E$ and any $A' \subseteq A$, observe that $N_e^{A'}(\mathsf{s}) = N_e^{A'}(\mathsf{s}')$. Thus

$$\sum_{i \in N_e^A(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^A(\mathsf{s})\})$$
$$= \sum_{i \in N_e^A(\mathsf{s}')} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^A(\mathsf{s}')\}).$$

Similarly, we prove that $\Phi_e^{A\setminus B}(\mathsf{s}) = \Phi_e^{A\setminus B}(\mathsf{s}')$. Therefore, using (5.12), we have

$$\Phi_B^A(\mathsf{s}) = \sum_{e\in E} \left(\Phi_e^A(\mathsf{s}') - \Phi_e^{A\setminus B}(\mathsf{s}')\right) = \Phi_B^A(\mathsf{s}').$$

c. Let P be an outcome of the game. Her contribution in the potential value is given by

$$\Phi_z(\mathsf{s}) = \Phi(\mathsf{s}) - \Phi^{\mathcal{N}\setminus\{z\}}(\mathsf{s}) = \sum_{e\in E} \left(\Phi_e(\mathsf{s}) - \Phi_e^{\mathcal{N}\setminus\{z\}}(\mathsf{s})\right) = \sum_{e\in E} I_e(\mathsf{s}), \tag{5.13}$$

where $I_e(\mathsf{s})$ equals

$$\sum_{i \in N_e(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\})$$
$$- \sum_{i \in N_e^{\mathcal{N}\setminus\{z\}}(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^{\mathcal{N}\setminus\{z\}}\}).$$

Since the potential is independent of the players ordering, we choose the τ such that player z is last. Then (5.13) equals

$$\sum_{e\in E} \chi_e(z, \{j : \tau(j) \leq \tau(z), j \in N_e(\mathsf{s})\}) = \sum_{e\in E} \chi_e(z, j : j \in N_e(\mathsf{s}))$$

$$= \sum_{e \in E} \chi_{ze}(\mathsf{s}) = c_z(\mathsf{s})$$

which completes the proof. □

Next, we show that the potential property also holds for the partial potential.

Proposition 5.3. *Consider a subset $B \subseteq \mathcal{N}$ and a player $i \in B$. Given two states, s and s', which differ only in the strategy of player i, then $\Phi_B(\mathsf{s}) - \Phi_B(\mathsf{s}') = c_i(\mathsf{s}) - c_i(\mathsf{s}')$.*

Proof. By definition of the partial potential (5.3),

$$\Phi_B(\mathsf{s}) - \Phi_B(\mathsf{s}') = \Phi(\mathsf{s}) - \Phi^{\mathcal{N}\setminus B}(\mathsf{s}) - \left(\Phi(\mathsf{s}') - \Phi^{\mathcal{N}\setminus B}(\mathsf{s}')\right) = \Phi(\mathsf{s}) - \Phi(\mathsf{s}').$$

Since the underlying game (considering all players in $\mathcal{N}$) is a potential game [KR15], $\Phi(\mathsf{s}) - \Phi(\mathsf{s}') = c_i(\mathsf{s}) - c_i(\mathsf{s}')$. □

The next lemma establishes a relation between partial potential and Shapley values.

Lemma 5.4. *Given an outcome s of the game, a resource e and a subset $B \subseteq \mathcal{N}$, it holds that $\Phi_{e,B}(\mathsf{s}) \leq \sum_{i \in B} \chi_{ie}(\mathsf{s}) \leq \Phi_{e,B}(\mathsf{s}) \cdot (d+1)$.*

Proof. By definition (5.3), we have

$$\Phi_{e,B}(\mathsf{s}) = \Phi_e(\mathsf{s}) - \Phi_e^{\mathcal{N}\setminus B}(\mathsf{s}) = \sum_{e \in E} \left(\Phi_e(\mathsf{s}) - \Phi_e^{\mathcal{N}\setminus B}(\mathsf{s})\right) = I_e(\mathsf{s}), \tag{5.14}$$

where $I_e(\mathsf{s})$ equals

$$\sum_{i \in N_e(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}) - \sum_{i \in N_e^{\mathcal{N}\setminus B}(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e^{\mathcal{N}\setminus B}\}). \tag{5.15}$$

Then we break the first term of (5.15) to the sum of

$$\sum_{i \in N_e^{\mathcal{N}\setminus B}(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}) + \sum_{i \in N_e^{B}(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}).$$

We choose an ordering τ in which all players in $\mathcal{N} \setminus B$ come first. Then the previous sum is equivalent to

$$\sum_{i \in N_e^{\mathcal{N}\setminus B}(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^{\mathcal{N}\setminus B}(\mathsf{s})\})$$

$$+ \sum_{i \in N_e^B(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}).$$

Substituting the previous to the first term of (5.15) gives

$$\sum_{i \in N_e^B(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}).$$

Combining it with the definition of $I_e(\mathsf{s})$ yields

$$\begin{aligned} I_e(\mathsf{s}) &= \sum_{i \in N_e^A(\mathsf{s})} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in N_e(\mathsf{s})\}) \\ &\leq \sum_{i \in N_e^A(\mathsf{s})} \chi_e(i, j : j \in N_e(\mathsf{s})) = \sum_{i \in N_e^A(\mathsf{s})} \chi_{ie}(\mathsf{s}) = \sum_{i \in A} \chi_{ie}(\mathsf{s}). \end{aligned}$$

Equation (5.14) completes the proof of the lower bound.

For the upper bound consider a fixed ordering of the players in B. The partial potential can be written as

$$\begin{aligned} \Phi_{e,B}(\mathsf{s}) &= \left(\Phi_e(\mathsf{s}) - \Phi_e^{\mathcal{N} \setminus B}(\mathsf{s})\right) \\ &= \sum_{i \in N_e^B(\mathsf{s})} \chi_e \left(i, \left\{j : \tau(j) \leq \tau(i); j \in N_e^B(\mathsf{s})\right\} \cup N_e^{\mathcal{N} \setminus B}(\mathsf{s})\right) \\ &\geq \int_{w_e^{\mathcal{N} \setminus B}(\mathsf{s})}^{w_e^{\mathcal{N}}(\mathsf{s})} f_e(x) dx \\ &\geq \left[\frac{x \cdot f_e(x)}{d+1}\right]_{w_e^{\mathcal{N} \setminus B}(\mathsf{s})}^{w_e^{\mathcal{N}}(\mathsf{s})} \\ &= \frac{w_e^{\mathcal{N}}(\mathsf{s}) \cdot f_e(w_e^{\mathcal{N}}(\mathsf{s})) - w_e^{\mathcal{N} \setminus B}(\mathsf{s}) \cdot f_e(w_e^{\mathcal{N} \setminus B}(\mathsf{s}))}{d+1} \\ &= \frac{w_e(\mathsf{s}) \cdot f_e(w_e(\mathsf{s}))}{d+1} - \frac{w_e^{\mathcal{N} \setminus B}(\mathsf{s}) \cdot f_e(w_e^{\mathcal{N} \setminus B}(\mathsf{s}))}{d+1} \\ &= \frac{\sum_{i \in \mathcal{N}} \chi_{ie}(\mathsf{s})}{d+1} - \frac{w_e^{\mathcal{N} \setminus B}(\mathsf{s}) \cdot f_e(w_e^{\mathcal{N} \setminus B}(\mathsf{s}))}{d+1}, \end{aligned} \tag{5.16}$$

where the first inequality follows by repeatedly applying Proposition 5.1(c) and 5.1(d) and adding additional players of weight 0 (which do not change the cost shares). The second inequality holds, since f_e is a polynomial of maximum degree d with non-negative coefficients.

Observe that $w_e^{\mathcal{N} \setminus B}(\mathsf{s}) \cdot f_e(w_e^{\mathcal{N} \setminus B}(\mathsf{s}))$ is the social cost of s on resource e if only the players in $\mathcal{N} \setminus B$ are in the game. By Proposition 5.1(a), the cost shares of those players can increase only if the players in B are joining the game, i.e.:

$$w_e^{\mathcal{N} \setminus B}(\mathsf{s}) \cdot f_e(w_e^{\mathcal{N} \setminus B}(\mathsf{s})) \leq \sum_{i \in \mathcal{N} \setminus A} \chi_{ie}(\mathsf{s}).$$

Combining this with (5.16) completes the proof of the claim:

$$\Phi_{e,B}(\mathsf{s}) \geq \frac{\sum_{i\in\mathcal{N}} \chi_{ie}(\mathsf{s})}{d+1} - \frac{\sum_{i\in\mathcal{N}\setminus B} \chi_{ie}(\mathsf{s})}{d+1} = \frac{\sum_{i\in B} \chi_{ie}(\mathsf{s})}{d+1}.$$

□

Summing up over all resources $e \in E$ yields the next corollary.

Corollary 5.5. *Given an outcome* s *of the game and a subset* $B \subseteq \mathcal{N}$*, it holds that* $\Phi_B(\mathsf{s}) \leq \sum_{i\in B} c_i(\mathsf{s}) \leq \Phi_B(\mathsf{s}) \cdot (d+1)$.

Proof. By the definition of the partial potential (5.3) and by applying Lemma 5.4, we directly have

$$\Phi_B(\mathsf{s}) = \sum_{e\in E} \Phi_{e,B}(\mathsf{s}) \leq \sum_{e\in E}\sum_{i\in B} \chi_{ie}(\mathsf{s}) = \sum_{i\in B} c_i(\mathsf{s})$$

and

$$\sum_{i\in B} c_i(\mathsf{s}) = \sum_{i\in B}\sum_{e\in E} \chi_{ie}(\mathsf{s}) = \sum_{e\in E}\sum_{i\in B} \chi_{ie}(\mathsf{s}) \leq \sum_{e\in E} \Phi_{e,B}(\mathsf{s}) \cdot (d+1) = \Phi_B(\mathsf{s}) \cdot (d+1).$$

□

5.3. Approximating Shapley with Proportional Cost-Shares

In this section we approximate the Shapley value of a player with her proportional share. This approximation plays an important role in our proofs of the stretch and for the computation.

Lemma 5.6. *For a player* i*, a resource* e *and any state* s*, the following inequality holds between her Shapley and proportional cost:*

$$\frac{2}{d+1} \cdot \chi_{ie}(\mathsf{s}) \leq \chi_{ie}^{Prop}(\mathsf{s}) \leq \frac{d+3}{4} \cdot \chi_{ie}(\mathsf{s}).$$

Proof. Since f_e is a polynomial of maximum degree d with non-negative coefficients, it suffices to show the inequalities for all monomial cost functions $f_e(x) = x^r$, with $r = \{0, \dots, d\}$. Fix some resource e with monomial cost function and a player i assigned to e, i.e., $e \in s_i$. Denote $Y = \{j \neq i : e \in s_j\}$ and $w = w_i$. Define $y = \sum_{j\in Y} w_j$ and $z = \frac{w}{y}$. By Proposition 5.1 (b), we can upper bound $\chi_{ie}(\mathsf{s})$ by replacing Y with a single player of weight y, i.e.,

$$\begin{aligned}\chi_{ie}(\mathsf{s}) &\leq \frac{1}{2}\left((y+w)^{r+1} - y^{r+1}\right) + \frac{1}{2}\cdot w^{r+1} = y^{r+1}\cdot\frac{1}{2}\cdot\left((z+1)^{r+1} - 1 + z^{r+1}\right)\\ &= y^{r+1}\cdot\left(z^{r+1} + \frac{1}{2}\cdot\sum_{j=1}^{r}\binom{r+1}{j}\cdot z^j\right) =: A.\end{aligned}$$

Similarly, by repeatedly using Proposition 5.1 (c) and by adding additional players of weight 0, we can lower bound $\chi_{ie}(\mathsf{s})$ by

$$\frac{1}{y}\cdot\int_0^y\Big((x+w)^{r+1}-x^{r+1}\Big)\,dx=\frac{1}{y}\cdot\frac{1}{r+2}\cdot\Big((y+w)^{r+2}-y^{r+2}-w^{r+2}\Big)$$
$$=y^{r+1}\cdot\frac{1}{r+2}\cdot\Big((z+1)^{r+2}-1-z^{r+2}\Big)=y^{r+1}\cdot\frac{1}{r+2}\cdot\sum_{j=1}^{r+1}\binom{r+2}{j}\cdot z^j=:B.$$

The proportional cost of player i, $\chi_{ie}^{\mathrm{Prop}}(\mathsf{s})$, equals

$$w\cdot f_e(y+w)=w\cdot(y+w)^r=y^{r+1}\cdot z\cdot(z+1)^r=y^{r+1}\cdot\sum_{j=1}^{r+1}\binom{r}{j-1}\cdot z^j.$$

To complete the proof we give an upper bound on $\frac{A}{\chi_{ie}^{\mathrm{Prop}}(\mathsf{s})}$ and a lower bound on $\frac{B}{\chi_{ie}^{\mathrm{Prop}}(\mathsf{s})}$. We have,

$$\frac{A}{\chi_{ie}^{\mathrm{Prop}}(\mathsf{s})}=\frac{z^{r+1}+\frac{1}{2}\sum_{j=1}^{r}\binom{r+1}{j}\cdot z^j}{\sum_{j=1}^{r+1}\binom{r}{j-1}\cdot z^j}=\frac{z^{r+1}+\frac{1}{2}\sum_{j=1}^{r}\binom{r+1}{j}\cdot z^j}{z^{r+1}+\sum_{j=1}^{r}\binom{r}{j-1}\cdot z^j},$$

which is upper bounded by

$$\frac{A}{\chi_{ie}^{\mathrm{Prop}}(\mathsf{s})}\leq\max\left(1,\max_{1\leq j\leq r}\frac{\binom{r+1}{j}}{2\cdot\binom{r}{j-1}}\right)=\max\left(1,\max_{1\leq j\leq r}\frac{r+1}{2\cdot j}\right)\leq\frac{d+1}{2}. \quad (5.17)$$

This implies the lower bound on $\chi_{ie}^{\mathrm{Prop}}(\mathsf{s})$ in the statement of the lemma. On the other hand, by first order conditions,

$$\frac{B}{\chi_{ie}^{\mathrm{Prop}}(\mathsf{s})}=\frac{\frac{1}{r+2}\cdot\sum_{j=1}^{r+1}\binom{r+2}{j}\cdot z^j}{\sum_{j=1}^{r+1}\binom{r}{j-1}\cdot z^j},$$

which achieves its extreme values at the roots of

$$g(z):=\sum_{j=1}^{r+1}\sum_{k=1}^{r+1}(j-k)\binom{r+2}{j}\binom{r}{k-1}\cdot z^{k+j-1}.$$

Claim 5.7. *The function $g:z\to\sum_{j=1}^{r+1}\sum_{k=1}^{r+1}(j-k)\binom{r+2}{j}\binom{r}{k-1}\cdot z^{k+j-1}$ has a unique positive real root at $z=1$.*

Proof. We will show that g(z) has a unique positive real root at $z=1$, is negative for $z<1$ and positive for $z>1$. To this end, by combining coefficients of the same monomial, we get

$$g(z)=\sum_{\sigma=2}^{r+1}\sum_{j=1}^{\sigma-1}(2j-\sigma)\binom{r+2}{j}\binom{r}{\sigma-j-1}\cdot z^{\sigma-1}$$

$$+\sum_{\sigma=r+3}^{2r+2}\sum_{j=\sigma-r-1}^{r+1}(2j-\sigma)\binom{r+2}{j}\binom{r}{\sigma-j-1}\cdot z^{\sigma-1},$$

where by symmetry the coefficient for $\sigma = r+2$ is 0. Pairing summands j and $\sigma - j$, we get

$$g(z) = \sum_{\sigma=2}^{r+1}\sum_{j=1}^{\lfloor\frac{\sigma-1}{2}\rfloor}(2j-\sigma)\left(\binom{r+2}{j}\binom{r}{\sigma-j-1}-\binom{r+2}{\sigma-j}\binom{r}{j-1}\right)\cdot z^{\sigma-1}$$
$$+\sum_{\sigma=r+3}^{2r+2}\sum_{j=\lceil\frac{\sigma}{2}\rceil}^{r+1}(2j-\sigma)\left(\binom{r+2}{j}\binom{r}{\sigma-j-1}-\binom{r+2}{\sigma-j}\binom{r}{j-1}\right)\cdot z^{\sigma-1}.$$

Define $\beta(\sigma,j) := (2j-\sigma)\cdot\left(\binom{r+2}{j}\binom{r}{\sigma-j-1}-\binom{r+2}{\sigma-j}\binom{r}{j-1}\right)$. Now observe that

$$\binom{r+2}{j}\binom{r}{\sigma-j-1} = \frac{(\sigma-j)(r+2-(\sigma-j))}{j(r+2-j)}\cdot\binom{r+2}{\sigma-j}\binom{r}{j-1}.$$

Since $\frac{(\sigma-j)(r+2-(\sigma-j))}{j(r+2-j)} \geq 1$ for all (σ, j) where $2 \leq \sigma \leq r+1$ and $1 \leq j \leq \frac{\sigma-1}{2}$ and for all (σ, j) where $r+3 \leq \sigma \leq 2r+2$ and $\frac{\sigma}{2} \leq j \leq r+1$, we get that $\beta(\sigma, j) \leq 0$ when $\sigma \leq r+1$ and $\beta(\sigma, j) \geq 0$ when $\sigma \geq r+3$ for all j in the corresponding range. *Descartes' rule of signs* implies that $g(z)$ has at most one positive real root. Simple arithmetic shows that $z = 1$ is a root of $g(z)$. □

By the previous lemma, we conclude that $\frac{B}{\chi_{ie}^{\text{Prop}}(\mathsf{s})}$ is minimized for $z = 1$, i.e.,

$$\frac{B}{\chi_{ie}^{\text{Prop}}(\mathsf{s})} \geq \frac{\frac{1}{r+2}\cdot\sum_{j=1}^{r+1}\binom{r+2}{j}}{\sum_{j=1}^{r+1}\binom{r}{j-1}} = \frac{\frac{1}{r+2}\cdot(2^{r+2}-2)}{2^r} \geq \frac{4}{r+3} \geq \frac{4}{d+3},$$

which completes the proof of the upper bound in the lemma. □

Summing up over all $e \in E$ implies the following corollary.

Corollary 5.8. *For a player i and any state s, the following inequality holds between her Shapley and proportional cost:*

$$\frac{2}{d+1}\cdot c_i(\mathsf{s}) \leq c_i^{Prop}(\mathsf{s}) \leq \frac{d+3}{4}\cdot c_i(\mathsf{s}).$$

Lemma 5.9. *Any ρ-approximate pure Nash equilibrium for a SV weighted Congestion Game of degree d is a $\frac{(d+3)\cdot(d+1)}{8}\cdot\rho$-approximate pure Nash equilibrium for the weighted Congestion Game with proportional sharing.*

Proof. Let s be a ρ-approximate equilibrium in the SV weighted Congestion Game and s_i' an arbitrary other strategy of player i. Using the equilibrium condition and Corollary 5.8, we have

$$c_i^{\text{Prop}}(\mathsf{s}) \leq \frac{d+3}{4}\cdot c_i(\mathsf{s}) \leq \frac{d+3}{4}\cdot\rho\cdot c_i(\mathsf{s}_{-i}, s_i') \leq \frac{d+3}{4}\cdot\frac{d+1}{2}\cdot\rho\cdot c_i^{\text{Prop}}(\mathsf{s}_{-i}, s_i').$$

□

5.4. Approximate Price of Anarchy and Stretch

Firstly, we upper bound the approximate Price of Anarchy for our game class.

Lemma 5.10. *Let $\rho \geq 1$ and d the maximum degree of the polynomial cost functions. Then*

$$\rho\text{-}PoA \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

Proof. Let s be an ρ-approximate pure Nash equilibrium and s^* the optimal outcome:

$$SC(\mathsf{s}) = \sum_{i \in N} \sum_{e \in s_i} \chi_e(i, N_e(\mathsf{s})) \overset{\text{Def. } \rho\text{-PNE}}{\leq} \rho \cdot \sum_{i \in N} \sum_{e \in s_i^*} \chi_e(i, N_e(\mathsf{s}) \cup \{i\}).$$

Due to the convexity of the cost functions, note that the cost share of any player on any resource is always upperbounded by the marginal cost increase she causes to the resource cost when she is last in the ordering, $\chi_e(i, N_e(\mathsf{s}) \cup \{i\}) \leq c_e(w_e(\mathsf{s}) + w_i) - c_e(w_e(\mathsf{s}))$. Thus,

$$\begin{aligned} SC(\mathsf{s}) &\leq \rho \cdot \left(\sum_{i \in N} \sum_{e \in s_i^*} c_e(w_e(\mathsf{s}) + w_i) - c_e(w_e(\mathsf{s})) \right) \\ &\leq \rho \cdot \left(\sum_{e \in E} \sum_{i : e \in s_i^*} c_e(w_e(\mathsf{s}) + w_i) - c_e(w_e(\mathsf{s})) \right) \\ &\leq \rho \cdot \left(\sum_{e \in E} c_e(w_e(\mathsf{s}) + w_e(\mathsf{s}^*)) - c_e(w_e(\mathsf{s})) \right). \end{aligned} \tag{5.18}$$

The last inequality follows from assumption that c_e is a convex function in players' weights.

Claim 5.11. *Let $\lambda = 2^{\frac{d}{d+1}} \cdot \left(2^{\frac{1}{d+1}} - 1\right)^{-d}$ and $\mu = 2^{\frac{d}{d+1}} - 1$, then for $x, y > 0$ and $d \geq 1$, $(x+y)^{d+1} - x^{d+1} \leq \lambda \cdot y^{d+1} + \mu \cdot x^{d+1}$.*

Using this claim that was proven in [GKR14], (5.18) becomes

$$\begin{aligned} SC(\mathsf{s}) &\leq \rho \cdot \left(\sum_{e \in E} \lambda \cdot c_e(w_e(\mathsf{s}^*)) + \mu \cdot c_e(w_e(\mathsf{s})) \right) \\ &= \rho \cdot \lambda \cdot SC(\mathsf{s}^*) + \rho \cdot \mu \cdot SC(\mathsf{s}). \end{aligned}$$

Rearranging and substituting the values for λ and μ we get an upper bound on the ρ-PoA,

$$\rho\text{-PoA} \leq \frac{\rho \cdot \lambda}{1 - \rho \cdot \mu} = \frac{\rho \cdot 2^{\frac{d}{d+1}} \cdot \left(2^{\frac{1}{d+1}} - 1\right)^{-d}}{1 - \rho \cdot \left(2^{\frac{d}{d+1}} - 1\right)} = \rho \cdot \frac{2}{2^{\frac{1}{d+1}}} \cdot \frac{\left(2^{\frac{1}{d+1}} - 1\right)^{-d}}{1 - \rho \cdot \frac{2}{2^{\frac{1}{d+1}}} + \rho}$$

$$= \frac{2 \cdot \rho \left(2^{\frac{1}{d+1}} - 1\right)^{-d}}{2^{\frac{1}{d+1}} \cdot (1+\rho) - 2 \cdot \rho} = \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

□

Similar to the ρ-PoA, we also derive an upper bound on the ρ-stretch, which expresses the ratio between the local and global optimum of the potential function.

Lemma 5.12. *Let $\rho \geq 1$ and d be the maximum degree of the polynomial cost functions. Then an upper bound for the ρ-stretch of polynomial SV weighted Congestion Games is*

$$\rho\text{-}\Omega \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d} \cdot (d+1)}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

Proof. Let s be a ρ-approximate equilibrium, s^* the optimal outcome and $\hat{\mathsf{s}} = \min_{\mathsf{s}' \in \mathsf{S}} \Phi(\mathsf{s}')$ the minimizer of the potential, which is by definition a pure Nash equilibrium. Then the ρ-approximate Price of Anarchy equals

$$\rho\text{-PoA} = \max_{\mathsf{s} \in \rho\text{-PNE}} \frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)} \geq \max_{\mathsf{s} \in \rho\text{-PNE}} \frac{SC(\mathsf{s})}{SC(\hat{\mathsf{s}})} \overset{\text{Def. } \Phi}{\geq} \max_{\mathsf{s} \in \rho\text{-PNE}} \frac{\Phi(\mathsf{s})}{SC(\hat{\mathsf{s}})}.$$

By Lemma 5.10 and Corollary 5.5 for $A = N$, the ρ-PoA is bounded as follows

$$\max_{\mathsf{s} \in \rho\text{-PNE}} \frac{\Phi(\mathsf{s})}{(d+1) \cdot \Phi(\hat{\mathsf{s}})} \leq \rho\text{-PoA} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{\frac{-d}{d+1}} \cdot (1+\rho) - \rho}.$$

Rearranging the terms gives the desired upper bound of the ρ-stretch,

$$\rho\text{-}\Omega = \max_{\mathsf{s} \in \rho\text{-PNE}} \frac{\Phi(\mathsf{s})}{\Phi(\hat{\mathsf{s}})} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d} \cdot (d+1)}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$ □

We now proceed to the upper bound of the D-limited ρ-stretch. To do this, we use the ρ-PoA (Lemma 5.10) and Lemmas 5.13 and 5.14, which we prove next.

Lemma 5.13. *Let $\rho \geq 1$, d be the maximum degree of the polynomial cost functions and $\hat{\mathsf{s}} = \min_{s' \in S} \Phi(s')$. Then*

$$\frac{SC(s)}{SC(\hat{\mathsf{s}})} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

Proof. Let s be an ρ-approximate equilibrium and s^* the optimal outcome. Let $\hat{\mathsf{s}} = \min_{\mathsf{s}' \in \mathsf{S}} \Phi(\mathsf{s}')$ be the minimizer of the potential and by definition also a pure Nash equilibrium. Then we can lower bound the ρ-PoA as follows,

$$\rho\text{-PoA} = \max_{\mathsf{s} \in \rho\text{-PNE}} \frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)} \geq \max_{\mathsf{s} \in \rho\text{-PNE}} \frac{SC(\mathsf{s})}{SC(\hat{\mathsf{s}})}. \tag{5.19}$$

Lemma 5.10 and (5.19) give that $\max_{\mathsf{s} \in \rho\text{-PNE}} \frac{SC(\mathsf{s})}{SC(\hat{\mathsf{s}})} \leq \rho\text{-PoA} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{\frac{-d}{d+1}} \cdot (1+\rho) - \rho}$. □

Lemma 5.14. *Let $\rho \geq 1$, d be the maximum degree of the polynomial cost functions and $D \subseteq \mathcal{N}$ an arbitrary subset of players. Then*

$$\rho\text{-}\Omega_D \leq \frac{(d+1)^2 \cdot (d+3)}{8} \cdot \frac{SC(\mathsf{s})}{SC(\hat{\mathsf{s}})}.$$

Proof. To show the lemma we lower and upper bound the D-partial potential. Let e be an arbitrary resource. By using Lemma 5.4 and Lemma 5.6, we get

$$\Phi_{e,D}(\mathsf{s}) \leq \sum_{i \in D} \chi_{ie}(\mathsf{s}) \leq \frac{d+1}{2} \cdot \sum_{i \in D} \chi_{ie}^{\text{Prop}}(\mathsf{s}). \tag{5.20}$$

By definition of the proportional share χ_{ie}^{Prop}, (5.20) becomes

$$\begin{aligned} \Phi_{e,D}(\mathsf{s}) &\leq \frac{d+1}{2} \cdot \sum_{i \in D} w_i \cdot f_e(w_e(\mathsf{s})) = \frac{d+1}{2} \cdot w_e^D(\mathsf{s}) \cdot f_e(w_e(\mathsf{s})) \\ &= \frac{d+1}{2} \cdot \frac{w_e^D(\mathsf{s})}{w_e(\mathsf{s})} \cdot w_e(\mathsf{s}) \cdot f_e(w_e(\mathsf{s})) = \frac{d+1}{2} \cdot \frac{w_e^D(\mathsf{s})}{w_e(\mathsf{s})} \cdot \sum_{i \in N} \chi_{ie}(\mathsf{s}). \end{aligned} \tag{5.21}$$

Rearranging (5.21) gives a relation of the per unit contribution to Φ_D and Φ,

$$\frac{\Phi_{e,D}(\mathsf{s})}{w_e^D(\mathsf{s})} \leq \frac{d+1}{2} \cdot \frac{\sum_{i \in N} \chi_{ie}(\mathsf{s})}{w_e(\mathsf{s})},$$

and by summing up over all resources e, we get

$$\frac{\Phi_D(\mathsf{s})}{W_D} \leq \frac{d+1}{2} \cdot \frac{SC(\mathsf{s})}{W}, \tag{5.22}$$

where $W = \sum_{i \in N} w_i = \sum_{e \in E} w_e(\mathsf{s})$ and $W_D = \sum_{i \in D} w_i = \sum_{e \in E} w_e^D(\mathsf{s})$.

Similar to (5.21), we lower bound the D-partial potential with

$$\begin{aligned} \Phi_{e,D}(\mathsf{s}) \geq \frac{1}{d+1} \cdot \sum_{i \in D} \chi_{ie}(\mathsf{s}) &\geq \frac{4}{(d+1) \cdot (d+3)} \cdot \sum_{i \in D} w_i \cdot f_e(w_e(\mathsf{s})) \\ &= \frac{4}{(d+1) \cdot (d+3)} \cdot \frac{w_e^D(\mathsf{s})}{w_e(\mathsf{s})} \cdot \sum_{i \in N} \chi_{ie}(\mathsf{s}). \end{aligned}$$

The first inequality uses Lemma 5.4 and the second uses Lemma 5.6. Again we get a per unit contribution to Φ_D and Φ on one resource and in the whole game,

$$\begin{aligned} & \frac{\Phi_{e,D}(\mathsf{s})}{w_e^D(\mathsf{s})} \geq \frac{4}{(d+1) \cdot (d+3)} \cdot \frac{\sum_{i \in N} \chi_{ie}(\mathsf{s})}{w_e(\mathsf{s})} \\ \Leftrightarrow \quad & \frac{\Phi_D(\mathsf{s})}{W_D} \geq \frac{4}{(d+1) \cdot (d+3)} \cdot \frac{SC(\mathsf{s})}{W}. \end{aligned} \tag{5.23}$$

Combining (5.22) with (5.23) and rearranging the terms completes Lemma's 5.14 proof,

$$\frac{\Phi_D(\mathsf{s})}{\Phi_D(\hat{\mathsf{s}})} \leq \frac{d+1}{2} \cdot \frac{SC(\mathsf{s})}{W} \cdot \frac{W_D}{1} \cdot \frac{(d+1)\cdot(d+3)}{4} \cdot \frac{W}{SC(\hat{\mathsf{s}})} \cdot \frac{1}{W_D}$$
$$= \frac{(d+1)^2 \cdot (d+3)}{8} \cdot \frac{SC(\mathsf{s})}{SC(\hat{\mathsf{s}})}.$$

□

By Lemma 5.13 and Lemma 5.14, we get the following desirable corollary.

Corollary 5.15. *Let $\rho \geq 1$, d be the maximum degree of the polynomial cost functions and $D \subseteq \mathcal{N}$ an arbitrary subset of players. Then*

$$\rho\text{-}\Omega_D \leq \frac{(d+1)^2 \cdot (d+3)}{8} \cdot \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

5.5. Computation of Approximate Pure Nash Equilibria

To compute ρ-approximate pure Nash equilibria in SV congestion games, we construct an algorithm based on the idea by Caragiannis et al. [Car+15] which was already introduced in Section 3.3. The main idea is to separate the players in different blocks depending on their costs. The players who are processed first are the ones with the largest costs followed by the smaller ones. The size of the blocks and the distance between them is polynomially bounded by the number of players n and the maximum degree d of the polynomial cost functions f_e. Formally, we define $c_{\max} = \max_{i \in \mathcal{N}} c_i(\mathsf{s})$ as the maximum cost among all players before running the algorithm. Let $\mathcal{BR}_i(\mathbf{0})$ be a state of the game in which only player i participates and plays her best move. Then, define as $c_{\min} = \min_{i \in \mathcal{N}} c_i(\mathcal{BR}_i(\mathbf{0}))$ the minimum possible cost in the game. Let γ be an arbitrary constant such that $\gamma > 0$, $m = \log\left(\frac{c_{\max}}{c_{\min}}\right)$ is the number of different blocks and $b_r = c_{\max} \cdot g^{-r}$ the block size for any $r \in [0, m]$, where $g = 2 \cdot n \cdot (d+1) \cdot \gamma^{-3}$.

The algorithm is now executed in $m-1$ phases. Let s be the current state of the game and, for each phase $r \in [1, m-1]$, let s^r be the state before phase r. All players i with $c_i(\mathsf{s}) \in [b_r, +\infty]$ perform an p-move with $p = \left(\frac{1}{q\text{-}\Omega_D} - 2\gamma\right)^{-1}$ (almost q-Ω_D-approximate moves), while all players i with $c_i(\mathsf{s}) \in [b_{r+1}, b_r]$ perform a q-move with $q = 1 + \gamma$ (almost pure moves). Let $\mathcal{BR}_i(\mathsf{s})$ be the best response of player i in state s. The phase ends when the first and the second group of players are in an p- and q-approximate equilibrium, respectively. At the end of the phase, players with $c_i(\mathsf{s}) > b_r$ have irrevocably decided their strategy and have been added in the list of finished players. In addition, before the described phases are executed, there is an initial phase in which all players with $c_i(\mathsf{s}) \geq b_1$ can perform a q-move to prepare the first real phase.

$\gamma > 0$, $q = 1 + \gamma$, $p = \left(\frac{1}{q\text{-}\Omega_D} - 2\gamma\right)^{-1}$
$c_{\max} = \max_{i \in \mathcal{N}} c_i(\mathsf{s})$, $c_{\min} = \min_{i \in \mathcal{N}} c_i(\mathcal{BR}_i(\mathbf{0}))$
$m = \log\left(\frac{c_{\max}}{c_{\min}}\right)$, $g = 2 \cdot n \cdot (d+1) \cdot \gamma^{-3}$, $b_r = c_{\max} \cdot g^{-r} \,\forall \in [0, m]$
while there is a player $i \in \mathcal{N}$ with $c_i(\mathsf{s}) \geq b_1$ and who can perform a q-move **do**
 $P \leftarrow (\mathsf{s}_{-i}, \mathcal{BR}_i(\mathsf{s}))$
for all phases r from 1 to $m-1$ **do**
 while there is a non-finished player $i \in \mathcal{N}$ either with $c_i(\mathsf{s}) \in [b_r, +\infty]$ and who can perform a p-move or with $c_i(\mathsf{s}) \in [b_{r+1}, b_r]$ and who can perform a q-move **do**
  $\mathsf{s} \leftarrow (\mathsf{s}_{-i}, \mathcal{BR}_i(\mathsf{s}))$
 Add all players $i \in \mathcal{N}$ with $c_i(\mathsf{s}) \geq b_r$ to the set of finished players.

Algorithm 2: Computation of approximate pure Nash equilibria

For the analysis, let D_r be the set of deviating players in phase r and $\mathsf{s}^{r,i}$ denote the state after player $i \in D_r$ has done her last move within phase r.

Theorem 5.16. *An ρ-approximate pure Nash equilibrium with $\rho \in \left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$ can be computed with a polynomial number of improvement steps.*

Proof. The main argument follows from bounding the D-partial potential of the moving players in each phase (see Lemma 5.18). To that end, we first prove that the partial potential is bounded by the sum of the costs of players when they did their last move (Lemma 5.17).

Lemma 5.17. *For every phase r, it holds that $\Phi_{D_r}(\mathsf{s}^r) \leq \sum_{i \in D_r} c_i(\mathsf{s}^{r,i})$.*

Proof. Let $D_r^i \subseteq D_r$ be the set of players who still have to perform their last move after player i in phase r. Then by definition of the partial potential 5.1, $\Phi_{D_r}(\mathsf{s}^r)$ equals

$$\Phi^{\mathcal{N}}(\mathsf{s}^r) - \Phi^{\mathcal{N} \setminus D_r}(\mathsf{s}^r) = \sum_{i=1}^{|D_r|} \left(\Phi^{N \setminus D_r^i}(\mathsf{s}^r) - \Phi^{N \setminus D_r^{i-1}}(\mathsf{s}^r)\right) = \sum_{i=1}^{|D_r|} \Phi_i^{N \setminus D_r^i}(\mathsf{s}^r). \quad (5.24)$$

For each player i, her strategy in state s^r is identical to her strategy in $\mathsf{s}^{r,i}$. By Proposition 5.2 (a), 5.2 (b) and 5.2 (c), we upperbound (5.24) by

$$\sum_{i=1}^{|D_r|} \Phi_i^{N \setminus D_r^i}(\mathsf{s}^r) = \sum_{i=1}^{|D_r|} \Phi_i^{N \setminus D_r^i}(\mathsf{s}^{r,i}) \leq \sum_{i=1}^{|D_r|} \Phi_i(\mathsf{s}^{r,i}) = \sum_{i=1}^{|D_r|} c_i(\mathsf{s}^{r,i}).$$

□

We now use the Lemma 5.17 and the stretch of the previous section to bound the potential of the moving players by the according block size.

Lemma 5.18. *For every phase r, it holds that $\Phi_{D_r}\left(\mathsf{s}^{r-1}\right) \leq \frac{n}{\gamma} \cdot b_r$.*

Proof. We show the lemma by contradiction. Thus, assume that $\Phi_{D_r}\left(\mathsf{s}^{r-1}\right) > \frac{n}{\gamma} \cdot b_r$. Let P_r, $Q_r \subseteq D_r$, be the set of players whose last move is an p-move and a q-move, accordingly, such that $P_r \cup Q_r = D_r$. First, we focus on the players in P_r. Let $i \in P_r$ be an arbitrary player. By definition of an p-move, player i decreases her costs in her last move during phase r by at least $(p-1) \cdot c_i(\mathsf{s}^{r,i})$. By Proposition 5.3, any such improvement step also decreases the i-partial potential by the same amount. Summing up over all players $i \in P_r$, we get a lower bound on the total decrease of the D_r-partial potential between states s^{r-1} and s^r: $\Phi_{D_r}(\mathsf{s}^{r-1}) - \Phi_{D_r}(\mathsf{s}^r) \geq (p-1) \cdot \sum_{i \in P_r} c_i(\mathsf{s}^{r,i})$. Rearranging, we upper bound the partial potential as follows,

$$\begin{aligned}
\Phi_{D_r}(\mathsf{s}^r) &\leq \Phi_{D_r}(\mathsf{s}^{r-1}) - (p-1) \cdot \sum_{i \in P_r} c_i(\mathsf{s}^{r,i}) \\
&\leq \Phi_{D_r}(\mathsf{s}^{r-1}) - (p-1) \cdot \left(\sum_{i \in D_r} c_i(\mathsf{s}^{r,i}) - \sum_{i \in Q_r} c_i(\mathsf{s}^{r,i}) \right) \\
&\leq \Phi_{D_r}(\mathsf{s}^{r-1}) - (p-1) \cdot \left(\sum_{i \in D_r} c_i(\mathsf{s}^{r,i}) - n \cdot b_r \right) \\
&\leq \Phi_{D_r}(\mathsf{s}^{r-1}) - (p-1) \cdot \left(\Phi_{D_r}\left(\mathsf{s}^r\right) - n \cdot b_r\right) \\
&\leq \Phi_{D_r}(\mathsf{s}^{r-1}) - (p-1) \cdot \left(\Phi_{D_r}\left(\mathsf{s}^r\right) - \gamma \cdot \Phi_{D_r}\left(\mathsf{s}^{r-1}\right)\right) \\
&\leq (1 + (p-1) \cdot \gamma) \cdot \Phi_{D_r}(\mathsf{s}^{r-1}) - (p-1) \cdot \Phi_{D_r}\left(\mathsf{s}^r\right),
\end{aligned}$$

where the third inequality follows from the fact that the cost of a player $i \in Q_r$ is upper bounded by the block border b_r, the fourth inequality by Lemma 5.17 and the fifth one by the assumption. Rearranging the terms gives

$$\Phi_{D_r}(\mathsf{s}^r) \leq \frac{1 + (p-1) \cdot \gamma}{p} \cdot \Phi_{D_r}(\mathsf{s}^{r-1}). \tag{5.25}$$

Let $\bar{\mathsf{s}}$ be an intermediate state between s^{r-1} and s^r such that all players in P_r have already finished their p-move and play their strategies in s^r, while the moving players in Q_r play their strategies in s^{r-1}. Consider a player $i \in Q_r$. The difference in her cost after her q-move is at most b_r. This is due to the fact that her initial cost is at most b_r (by the block construction) and the minimum cost she can improve to is zero. Then, by Proposition 5.3, the difference in the cost of player i equal the difference in the i-partial potential, that is, $\Phi_i(\bar{\mathsf{s}}) - \Phi_i(\mathsf{s}^r) = c_i(\mathsf{s}) - c_i(\mathsf{s}') \leq b_r$. Summing up over all players in Q_r, we get that the difference in the D_r-partial potential among states $\bar{\mathsf{s}}$ and s^r can be at most $n \cdot b_r$. Then, we get the following upper bound on the partial potential in state $\bar{\mathsf{s}}$,

$$\Phi_{D_r}(\bar{\mathsf{s}}) \leq \Phi_{D_r}(\mathsf{s}^r) + n \cdot b_r \leq \frac{1 + (p-1) \cdot \gamma}{p} \cdot \Phi_{D_r}(\mathsf{s}^{r-1}) + \gamma \cdot \Phi_{D_r}\left(\mathsf{s}^{r-1}\right)$$

$$= \left(\frac{1-\gamma}{p} + 2\cdot\gamma\right)\cdot\Phi_{D_r}(\mathsf{s}^{r-1}) < \left(\frac{1}{p} + 2\cdot\gamma\right)\cdot\Phi_{D_r}(\mathsf{s}^{r-1}),$$

where the second inequality holds by (5.25) and our assumption. Substituting p, we get

$$\Phi_{D_r}(\bar{\mathsf{s}}) < \frac{1}{q\text{-}\Omega_D}\cdot\Phi_{D_r}(\mathsf{s}^{r-1}),$$

which contradicts Corollary 5.15. □

It remains to show that the running time is bounded and that the approximation factor holds. For the first, since the partial potential is bounded and each deviation decreases the potential, we can limit the number of possible improvement steps (see Lemma 5.19).

Lemma 5.19. *The algorithm uses a polynomial number of improvement steps.*

Proof. At the beginning of the algorithm's execution, the sum of all players' costs is at most $n\cdot c_{\max}$. By Corollary 5.5, the potential is also upper bounded by the same amount. In the initial phase, each deviating player makes a q-move, therefore her cost improves by at least $(q-1)\cdot b_1$ (since her cost is at most b_1). The potential function also decreases by at least $(q-1)\cdot b_1$ in each step. Using the definition of b_1, we get that $(q-1)\cdot b_1 = \gamma\cdot g^{-1}\cdot c_{\max}$. Using both observations, we can compute the maximum number of improvement steps in the first phase:

$$\frac{n\cdot c_{\max}}{\gamma\cdot g^{-1}\cdot c_{\max}} = n\cdot\gamma^{-1}\cdot g = n\cdot\gamma^{-1}\cdot\frac{2\cdot n\cdot(d+1)}{\gamma^3} = 2\cdot n^2\cdot(d+1)\cdot\gamma^{-4}.$$

Consider an arbitrary phase $r \geq 1$. By Lemma 5.18, $\Phi_{D_r}(\mathsf{s}^{r-1}) \leq \frac{n}{\gamma}\cdot b_r$. Again, we look at the possible cost improvement in a deviation, which equals the potential decrease in this step. In this case, the cost improvement is at least $(q-1)\cdot b_{r+1}$. By definition of b_{r+1}, we have that $(q-1)\cdot b_{r+1} = b_r\cdot g^{-1}\cdot\gamma$. Similarly, the maximum number of improvement moves in this phase is

$$\frac{\frac{n}{\gamma}\cdot b_r}{b_r\cdot g^{-1}\cdot\gamma} = \frac{n\cdot g}{\gamma^2} = \frac{2\cdot n^2\cdot(d+1)\cdot\gamma^{-3}}{\gamma^2} = 2\cdot n^2\cdot(d+1)\cdot\gamma^{-5}.$$

In total, we have at most $2\cdot n^2\cdot(d+1)\cdot\gamma^{-4} + \log\left(\frac{c_{\max}}{c_{\min}}\right)\cdot 2\cdot n^2\cdot(d+1)\cdot\gamma^{-5} = \left(1+\log\left(\frac{c_{\max}}{c_{\min}}\right)\right)\cdot 2\cdot n^2\cdot(d+1)\cdot\gamma^{-9}$ improvement steps. □

We show next that every player who has already finished his movements will not get much worst costs at the end of the algorithm (see Lemma 5.20) and that there is no alternative strategy that is more attractive at the end (see Lemma 5.21).

Lemma 5.20. *Let i be a player who makes her last move in phase r of the algorithm. Then, $c_i(\mathsf{s}^{m-1}) \leq (1+\gamma^2)\cdot c_i(\mathsf{s}^r)$.*

Proof. We first show by contradiction the following. For $j \geq r$, the increase in the cost of player i from an arbitrary state s^j to state s^{j+1} is upper bounded by $\frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$. Thus, assume that $c_i(\mathsf{s}^{j+1}) - c_i(\mathsf{s}^j) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$. Since player i does not deviate during phase $j+1$, the increase in her cost is caused by other players deviating to the resources she uses. Thus, there exists a set of resources $E' \subseteq E$ such that each resource in E' is used by player i and by at least one player in D_{j+1} at state s^{j+1}. This yields

$$\begin{aligned}
&\sum_{e\in E'} \chi_{ie}(\mathsf{s}^{j+1}) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1} \\
\Rightarrow\ & \frac{\sum_{e\in E'} w_e(\mathsf{s}^{j+1}) \cdot f_e(w_e(\mathsf{s}^{j+1}))}{d+1} > \frac{n}{\gamma} \cdot b_{j+1} \\
\Leftrightarrow\ & \frac{SC_{D_{j+1}}(\mathsf{s}^{j+1})}{d+1} > \frac{n}{\gamma} \cdot b_{j+1} \\
\Rightarrow\ & \Phi_{D_{j+1}}(\mathsf{s}^{j+1}) > \frac{n}{\gamma} \cdot b_{j+1}.
\end{aligned}$$

The last step uses Corollary 5.5. Since the potential decreases during the execution of the algorithm, we get $\Phi_{D_{j+1}}(\mathsf{s}^j) \geq \Phi_{D_{j+1}}(\mathsf{s}^{j+1}) > \frac{n}{\gamma} \cdot b_{j+1}$, which contradicts Lemma 5.18. Therefore $c_i(\mathsf{s}^{j+1}) \leq c_i(\mathsf{s}^j) + \frac{n(d+1)}{\gamma} \cdot b_{j+1}$ and we use this to show the lemma as follows,

$$\begin{aligned}
c_i(\mathsf{s}^{m-1}) &\leq c_i(\mathsf{s}^{m-2}) + \frac{n\cdot(d+1)}{\gamma} \cdot b_{m-1} \\
&\leq c_i(\mathsf{s}^r) + \frac{n\cdot(d+1)}{\gamma} \sum_{j=r+1}^{m-1} b_j \\
&= c_i(\mathsf{s}^r) + \frac{n\cdot(d+1)}{\gamma} \sum_{j=r+1}^{m-1} c_{\max} \cdot g^{-j} \\
&= c_i(\mathsf{s}^r) + \frac{n\cdot(d+1)}{\gamma} \sum_{j=r+1}^{m-1} b_r \cdot g^{r-j} \\
&\leq c_i(\mathsf{s}^r) + \frac{n\cdot(d+1)}{\gamma} \cdot 2 \cdot b_r \cdot g^{-1} \\
&\leq c_i(\mathsf{s}^r) + \frac{2\cdot n\cdot(d+1)}{\gamma\cdot g} \cdot c_i(\mathsf{s}^r) \\
&= \left(1 + \frac{2\cdot n\cdot(d+1)}{\gamma\cdot g}\right) \cdot c_i(\mathsf{s}^r) = \left(1+\gamma^2\right) \cdot c_i(\mathsf{s}^r).
\end{aligned}$$

□

Lemma 5.21. *Let i be a player who makes her last move in phase r and let s_i' be an arbitrary strategy of i. Then, $c_i(\mathsf{s}_{-i}^{m-1}, \mathsf{s}_i') \geq (1-\gamma) \cdot c_i(\mathsf{s}_{-i}^r, \mathsf{s}_i')$.*

Proof. Similarly to previous lemma, we first show by contradiction the following. For two arbitrary successive phases j and $j+1$ and an arbitrary alternative strategy P_i' of player i, $c_i(\mathsf{s}_{-i}^{j+1}, \mathsf{s}_i') \geq c_i(\mathsf{s}_{-i}^{j}, \mathsf{s}_i') - \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$. Thus, assume that $c_i(\mathsf{s}_{-i}^{j}, \mathsf{s}_i') - c_i(\mathsf{s}_{-i}^{j+1}, \mathsf{s}_i') > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$. Since player i does not deviate during phase $j+1$, the increase in her costs is caused by other players deviating to the resources she uses. Thus, there exists a set of resources $E' \subseteq E$ such that each resource in E' is used by player i and by at least one player in D_{j+1} at state s^{j+1}. Therefore

$$\sum_{e\in E'} \chi_{ie}(\mathsf{s}_{-i}^{j}, \mathsf{s}_i') > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1} \Rightarrow \sum_{e\in E'} \chi_{ie}(\mathsf{s}_{-i}^{j}, \mathsf{s}_i) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}.$$

Following exactly the same steps as in proof of Lemma 5.20, the previous yields a contradiction of Lemma 5.18. Thus, $c_i(\mathsf{s}_{-i}^{j+1}, \mathsf{s}_i') \geq c_i(\mathsf{s}_{-i}^{j}, \mathsf{s}_i') - \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$, which we use to show the lemma's statement as follows,

$$\begin{aligned}
c_i(\mathsf{s}_{-i}^{m-1}, \mathsf{s}_i') &\geq c_i(\mathsf{s}_{-i}^{m-2}, \mathsf{s}_i') - \frac{n\cdot(d+1)}{\gamma} \cdot b_{m-1} \\
&\geq c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \frac{n\cdot(d+1)}{\gamma} \cdot \sum_{j=r+1}^{m-1} b_j \\
&= c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \frac{n\cdot(d+1)}{\gamma} \cdot \sum_{j=r+1}^{m-1} c_i(\cdot g^{-j}) \\
&= c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \frac{n\cdot(d+1)}{\gamma} \cdot \sum_{j=r+1}^{m-1} b_r \cdot g^{r-j} \\
&\geq c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \frac{n\cdot(d+1)}{\gamma} \cdot 2 \cdot b_r \cdot g^{-1} \\
&\stackrel{b_r}{=} c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \frac{2\cdot n\cdot(d+1)}{\gamma\cdot g} \cdot c_i(\mathsf{s}^r) \\
&\stackrel{g}{=} c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \gamma^2 \cdot c_i(\mathsf{s}^r) \\
&\stackrel{\gamma\leq\frac{1}{p}}{\geq} c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \frac{\gamma}{p} \cdot c_i(\mathsf{s}^r) \\
&\geq c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i') - \gamma \cdot c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}') = (1-\gamma) \cdot c_i(\mathsf{s}_{-i}^{r}, \mathsf{s}_i').
\end{aligned}$$

The second last inequality holds due to the p-approximate equilibrium for player i in s^r. ☐

Next, we bound the approximation factor of the whole algorithm.

Lemma 5.22. *After the last phase of the algorithm, every player i is in an ρ-approximate pure Nash equilibrium with $\rho = (1+O(\gamma)) \cdot q\text{-}\Omega_D$.*

Proof. Let i be an arbitrary player who took her last move in phase r and let s_i' be an arbitrary other strategy of player i. We use Lemma 5.20 and Lemma 5.21 and the fact that player i has no incentive to make a p-move in phase r (by definition of the algorithm):

$$\begin{aligned}\frac{c_i(\mathsf{s}^{m-1})}{c_i(\mathsf{s}^{m-1}_{-i}, \mathsf{s}'_i)} &\leq \frac{(1+\gamma^2)\cdot c_i(\mathsf{s}^r)}{(1-\gamma)\cdot c_i(\mathsf{s}^r_{-i}, \mathsf{s}'_i)} \\ &\leq \left(\frac{1+\gamma^2}{1-\gamma}\right)\cdot\left(\frac{1}{q\text{-}\Omega_D} - 2\gamma\right)^{-1} \\ &\leq \left(\frac{1+\gamma^2}{1-\gamma}\right)\cdot\left(\frac{1}{q\text{-}\Omega_D} - 2\gamma\right)^{-1}\end{aligned}$$

By minimizing the first part, we can get arbitrarily close to 1. For the second part, we need to fix a γ with $\gamma < \frac{1}{2q\text{-}\Omega_D}$. Therefore, the expression can be simplified to $\rho = (1+O(\gamma))\cdot q\text{-}\Omega_D$. □

The polynomial running time and the approximation factor of $\rho = (1+O(\gamma))\cdot q\text{-}\Omega_D$ follow directly from Lemma 5.19 and Lemma 5.22. Last, using Corollary 5.15, we show that $\rho \in \left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$.

Lemma 5.23. *The approximation factor ρ is in the order of $\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$.*

Proof. By Lemma 5.22 and Corollary 5.15, we get that our main factor ρ (from Lemma 5.22) equals

$$(1+O(\gamma))\cdot\frac{(d+1)^2\cdot(d+3)}{8}\cdot\frac{t\cdot(2^{\frac{1}{d+1}}-1)^{-d}}{2^{-\frac{d}{d+1}}\cdot(1+t)-t},$$

where γ is a small positive constant and $t = 1+\gamma$. Observe that factor ρ is essentially in the order of

$$\Theta(d^3)\cdot\left(\frac{1}{2^{\frac{1}{d+1}}-1}\right)^d.$$

We now claim that the order of the above is $\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$. To prove this, it is enough to show that $\frac{1}{2^{\frac{1}{d+1}}-1}$ is asymptotically similar to $\frac{d}{\ln(2)}$. Applying L'Hôspital's rule, this follows from the fact that

$$\lim_{d\to\infty}\frac{\frac{1}{d}}{2^{\frac{1}{d+1}}-1} = \lim_{d\to\infty}\frac{-\frac{1}{d^2}}{-\frac{2^{\frac{1}{d+1}}\cdot\ln(2)}{(d+1)^2}} = \frac{1}{\ln(2)},$$

which completes the proof. □

This Lemma also completes the proof of Theorem 5.16. □

We note that a significant improvement below $O\left(\left(\frac{d}{\ln 2}\right)^d\right)$ of the approximation factor would require new algorithmic ideas as the lower bound of the PoA in [GS07] immediately yields a corresponding lower bound on the stretch.

This algorithm can be used to compute also approximate pure Nash equilibria in weighted Congestion Games (with proportional sharing). Such a game can now be approximated by a Shapley game losing only a factor of $\frac{(d+3)(d+1)}{8}$ (by Lemma 5.9), which is included in $poly(d)$.

Corollary 5.24. *For any weighted Congestion Game with proportional sharing, an ρ-approximate pure Nash equilibrium with $\rho \in \left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$ can be computed with a polynomial number of improvement steps.*

Conclusion

In comparison to the previous Chapter 4 we used a similar, but more dynamic algorithm in this model. This required a more challenging analysis of the approximation guarantee. To handle this, we made intensive use of the properties of the potential function as well as the Shapley value. Our main technical method here was the approximation of Shapley cost shares with proportional cost shares and vice versa. For a side product we showed bounds on the approximate Price of Anarchy.

The polynomial running time is measured only in the number of improvement moves so far. In the next chapter we investigate this specific issue further and show how we can also achieve a truly polynomial running time.

CHAPTER 6

Sampling the Shapley Value

Exact computations of the Shapley value may be computationally hard in general since the number of coalitions exponentially grows with the number of players. Therefore, our computation of approximate equilibria in Chapter 5 does not immediately yield an algorithm with polynomial running time, since computing the Shapley cost share of a player, and hence an improvement step, is computationally hard in this model. In this chapter, we apply sampling techniques that allow us to achieve a polynomial running time by approximating the Shapley value.

Furthermore, we look at a different scenario which is independent of Congestion Games, and even non-cooperative game theory, in which we apply the Shapley value and also encounter the problem of efficiently computing the Shapley value. In this model, we consider a market where final products or services are compositions of a number of basic services. Users are asked to evaluate the quality of the composed product after purchase. The quality of the basic service influences the performance of the composed services but cannot be observed directly. The question we pose is whether it is possible to use user evaluations on composed services to assess the quality of basic services. We discuss how to combine the aggregation of evaluations across users and the disaggregation of information on composed services to derive valuations for the single components. For a solution we propose to use the (weighted) average as an aggregation method in connection with the Shapley value as a disaggregation method, since this combination fulfills natural requirements in our context. Here, we address multiple computational issues: We give an approximate solution concept by using only a limited number of evaluations which guarantees nearly optimal results with reduced running time. Lastly, we show that a slightly modified Shapley value and the weighted average are still applicable if the evaluation profiles are incomplete.

Contribution and Underlying Work We show that there is a fully polynomial-time randomized approximation scheme (FPRAS) that can be used to approximate the Shapley value. Applying it to the Shapley Congestion Games from the previous chapter, this results in a randomized polynomial time algorithm that computes a strategy profile that is an approximate pure Nash equilibrium with high probability.

We generalize the sampling method and the analysis and also apply it to the second scenario, the disaggregation of reputation values. With that, we can construct a fully polynomial-time randomized approximation scheme for our evaluation problem. The only assumption needed for this approach is a pair of bounds on the possible marginal contribution of a service to a composition. For the setting of reputation values, these bounds are naturally given by the minimal and maximal reputation values. Next to the computational challenges in this setting, we characterize the complete evaluation problem – aggregation and disaggregation – with natural axioms. To the best of our knowledge, this is the first work that handles both steps together. We show that a combination of the Shapley value as the disaggregator and the (weighted) average as the aggregator satisfies our normative requirements. Finally, we address the problem of missing valuations and apply the Data-Dependent Shapley Value [BSV17], which, in combination with the weighted average as aggregator, still yields a solution that satisfies our requirements.

The sampling approach, analysis, and results of both scenarios are based on the following publications:

> Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria in Shapley Value Weighted Congestion Games**. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE).* 2017. [Fel+17a]

> Matthias Feldotto, Claus-Jochen Haake, Alexander Skopalik, and Nadja Stroh-Maraun. **Disaggregating User Evaluations Using the Shapley Value**. In: *Proceedings of the 13th Workshop on the Economics of Networks, Systems and Computation (NetEcon 2018).* Irvine, California, USA, 2018. [Fel+18b]

Outline of This Chapter This chapter starts with a short overview of existing sampling methods for the Shapley value (see Section 6.1). Afterwards, in Section 6.2 we present the general approach, which we consider in both applications, and introduce and analyze our FPRAS. Then we go into detail for the two scenarios. First, we investigate the computation of the Shapley value in the context of the Congestion Games from Chapter 5 (see Section 6.3). Second, we introduce the disaggregation scenario in detail and also apply the sampling method there (see Section 6.4). At the end, we give an axiomatic characterization of the evaluation problem.

6.1. Related Work

Computing the Shapley value is known to be intractable for a number of cooperative games (see e.g., [Azi+09; Elk+09; MM01; DP94]. Only for very restricted classes polynomial-time algorithms for the exact computation haven been developed (e.g., [DP94; IS05]). Mann and Shapley [MS62] suggested a sampling algorithm, but without any theoretical guarantee, which was later analyzed by Bachrach et al. [Bac+10] for simple coalitional games and by Aziz and de Keijzer [AK14] for matching games. Finally, Liben-Nowell et al. [Lib+12] and Maleki [Mal15] consider cooperative games with supermodular functions that correspond to our class. If we restrict the games by the value function instead of any structural properties, [Lib+12] shows the approximation guarantee for supermodular games and [BSV17] for submodular games with bounded curvature. Additionally, [CGT09] and [Mal15] analyze the approximation in a more general setting, but require a given variance or range of the marginal values. Further approximation methods for the Shapley value have been discussed in [FWJ08; Owe72; Lee03; ZR94].

In the second application of our sampling method, we study the aggregation and disaggregation process. To the best of our knowledge, this is the first work which handles both steps together. However, there is a bunch of literature focusing on one of the two steps.

Aggregation functions are studied in both theoretical and empirical work. The most commonly used operators are the (weighted) mean, the median and the mode. An overview of their theoretical properties can be found in [Gra+11] and [Cal+02]. [GFJ09] compares the three aggregators empirically according to three different criteria, namely informativeness, robustness and strategyproofness. [Gar+09] extends this comparison to accuracy. They find evidence that although the mean is widely used it is not always the preferable aggregator regarding these five criteria. In an experiment the aggregators' influence on the users' rating behavior is studied in [GXF13]. It is shown that under certain circumstances the mean is preferable in terms of rating behavior. Different aggregators than the mean, median or mode are used by [DT10] who define an econometric framework to aggregate consumer preferences from online product reviews. They particularly take the heterogeneity of users' opinions, e.g., experts and non-professionals, into account when aggregating the evaluations. Another aspect of aggregation tackles the question of how to get the true quality of a (composed) product [MGR10].

In contrast to the widely studied aggregation process, the disaggregation process lacks this variety of studies. Although the problem of common evaluations for combined products or a similar problem with common evaluations of a group of firms selling regional products is known [WM05], few studies actually discuss the disaggregation of these evaluations. Nevertheless, there are few studies considering the Shapley value to disaggregate users' ratings of component services directly [Liu+12]. To decrease computational complexity a variant of the Shapley value might be used [Liu+14]. Additionally, researchers investigate the learning of classes of games by using the PAC model, i.e., [BPZ15].

To solve the disaggregation part of our problem we use the Shapley value. This concept is widely studied and characterized by some natural properties, namely symmetry, Pareto efficiency, additivity and the dummy player axiom [Sha53a]. Alternative characterizations use a fairness [Bri02] or a transfer property [Fel95] instead of additivity. [BSV17] and [AME10] deal with incomplete information in the context of the Shapley value. Similar concepts have been used by Smets in the context of decision making in the transferable belief model with pignistic transformations [Sme05].

6.2. FPRAS for the Shapley Value

This section gives a general approximation algorithm for the Shapley value. It is a fully polynomial-time randomized approximation scheme (FPRAS) for the Shapley value computation. The algorithm and analysis use sampling techniques that follow [Lib+12; MS62] and adjust them to our setting.

The sampling method of Algorithm 3 works in two nested loops. In the inner loop k random permutations are used to compute the marginal contributions for this setting. Then the average of all repetitions $\overline{MC_i}$ is used as intermediate result. This process is triggered by the outer loop which uses the median of $\log(\delta)$ averages $\widetilde{MC_i}$ as output. The number of repetitions of the inner and outer loops, given by the parameters k and δ will be defined by the analysis.

Given an arbitrary $\delta > 0$ and $k > 0$
for all r from 1 to $\log(\delta)$ **do**
  for all j from 1 to k **do**
    Pick uniformly at random a permutation π of the players N
    Compute marginal contribution $MC_i^j = v(N^{i,\pi} \cup \{i\}) - v(N^{i,\pi})$
  Let $\overline{MC_i} = \frac{1}{k}\sum_{j=1}^{k} MC_i^j$
Return the median $\widetilde{MC_i}$ of all $\overline{MC_i}$

Algorithm 3: Approximation of the Shapley value by sampling.

The following lemmas state the approximation guarantee and runtime for Algorithm 3. The first one assumes a monotone polynomial of degree d for the value function.

Lemma 6.1. *Given an arbitrary set of players N and an arbitrary value function v that is a monotone polynomial of degree d, Algorithm 3 with $k = \frac{4(|N|-1)}{\mu^2}$ computes a μ-approximation of $SV_i(N, v)$ for any player i in polynomial running time with a probability of at least $1 - (\delta)^{-1}$.*

Proof. The beginning of this proof follows the analysis in [Lib+12].

Let X be the marginal contribution of player i in a random permutation. By the definition of the Shapley value, $SV_i = E[X]$. Since v is a polynomial of degree d and monotone, we have $X \geq 0$. By the definition of the value function, the maximum possible value of X is achieved when i is the last player in the ordering. This happens in $1/|N|$ fraction of the permutations. X achieves the maximum value with a probability of at least $1/|N|$ and the maximum value is at most $|N| \cdot SV_i$ because of the expectation and the bounds of the values.

To upper bound the variance of X we define a second random variable Y, which is $|N| \cdot SV_i$ with probability $1/|N$ and 0 otherwise. Then,

$$Var(X) \leq Var(Y) = E[Y^2] - E[Y]^2 = (|N| - 1) \cdot SV_i^2$$

Since $\overline{MC_i} = \frac{1}{k}\sum_{j=1}^{k} MC_i^j$, $E[\overline{MC_i}] = E[X] = SV_i$ and the single permutations are independent of each other, we get $Var(\overline{MC_i}) = \frac{Var(X)}{k} \leq \frac{1}{k}(|N|-1) \cdot SV_i^2$. Using Chebyshev's inequality [Che67], we get

$$Pr[|\overline{MC_i} - SV_i| \geq \mu SV_i] \leq \frac{Var(\overline{MC_i})}{SV_i^2\mu^2} \leq \frac{(|N|-1)\cdot SV_i^2}{kSV_i^2\mu^2} = \frac{|N|-1}{k\cdot\mu^2}.$$

Let $k = \frac{4(|N|-1)}{\mu^2}$, then $\overline{MC_i}$ is a μ-approximation for SV_i with a probability of at least 3/4. If we repeat this procedure $l = \ln(1/\delta)$ times, using the median value of all runs and applying Chernoff bounds [Che52], we directly get a result with failure probability at most δ. □

The second use case assumes an arbitrary but bounded value function in $[-u, u]$.

Lemma 6.2. *Given an arbitrary set of players N and an arbitrary value function v whose values are in $[-u, u]$, Algorithm 3 with $k = 2\ln(8)u^2/\mu^2$ computes a μ-approximation of $SV_i(N, v)$ for any player i in polynomial running time with a probability of at least $1-(\delta)^{-1}$.*

Proof. Let X_r be a random variable for the marginal contribution of a service j in a random permutation r. By definition of the Shapley value $E[X_r] = SV_i$. Furthermore, the marginal contributions are bounded, i.e., $X_r \in [-u, u]$. Define the sum over k identical random variables as $X = \sum_{r=1}^{k} X_r$ with $E[X] = k \cdot SV_i$ and $\overline{MC_i} = \frac{X}{k}$. We have that $Pr\left[|X - E[X]| \geq t\right] = Pr\left[|X - k\cdot SV_i| \geq t\right] = Pr\left[|\overline{MC_i} - SV_i| \geq \frac{t}{k}\right]$.

Now using Hoeffding's inequality [Hoe63a; Hoe63b] and setting $t = \mu \cdot k$ yields

$$Pr\left[|\overline{MC_i} - SV_i| \geq \mu\right] \leq 2\exp\left(-\frac{2k^2\mu^2}{k(2u)^2}\right).$$

We bound the right-hand side with $2\exp\left(-\frac{2k^2\mu^2}{k(2u)^2}\right) \leq \frac{1}{4} \Leftrightarrow k \geq 2\ln(8)u^2\mu^{-2}$. Therefore, $\overline{MC_i}$ is a μ-approximation for SV_i with a probability of at least 3/4.

If we repeat this procedure $l = \ln(1/\delta)$ times, using the median value of all runs and applying Chernoff bounds [Che52], we directly get a result with a failure probability of at most δ. □

So far this approach and the algorithm is quite general for the computation of the Shapley value. The analysis requires specific bounds on the value function (here, either polynomial functions or bounded values), but it is easily extendable for further bounded value functions. In the following sections we apply and analyze this algorithm in two different settings.

6.3. Polynomial Runtime for Shapley Congestion Games

The previous chapter on the computation of approximate pure Nash equilibria in Shapley value Congestion Games gives an algorithm with polynomial running time with respect to the number of improvement steps (see Section 5.5). However, each improvement step requires multiple computations of Shapley values, which are hard to compute. For this reason, we aim at computing an approximated Shapley value with the presented sampling methods from Section 6.2. Since we are only interested in approximate equilibria, an execution of the algorithm with approximate values has a negligible impact on the final result. The technical properties of Shapley values stated in Section 5.2 also hold for sampled instead of exact Shapley values with high probability.

Theorem 6.3. *A ρ-approximate pure Nash equilibrium with $\rho \in \left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$ can be computed in polynomial time with high probability.*

Proof. We use the presented sampling technique from the previous Section 6.2. For using the sampling in the computation of an improvement step, a Shapley value has to be approximated for each alternative strategy of a player and for each resource in the strategy. In the worst case, each player has to be checked for an available improvement step.

Lemma 6.4. *Given an arbitrary state s and running Algorithm 3 with $k = \frac{4(|\mathcal{N}|-1)}{\mu^2}$ and $\delta = \left(n^c \cdot n \cdot \max_{i \in \mathcal{N}} |S_i| \cdot |E| \cdot \left(1 + \log\left(\frac{c_{max}}{c_{min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}\right)$ at most $n \cdot \max_{i \in N} |S_i| \cdot |E|$ times computes an improvement step for an arbitrary player with a probability of at least $1 - \left(n^c \cdot \left(1 + \log\left(\frac{c_{max}}{c_{min}}\right)\right) \cdot 2n^2 \cdot (d+1) \cdot \gamma^{-9}\right)^{-1}$.*

Proof. The result follows directly by applying the union bound:

$$
\begin{aligned}
&Pr[\exists\, i \in \mathcal{N} : \exists\, s_i' \in S_i : \exists\, e \in s_i' : |\overline{MC}_{ie}(\mathsf{s}_{-i}, s_i') - \chi_{ie}(\mathsf{s}_{-i}, s_i')| \geq \mu \cdot \chi_{ie}(\mathsf{s}_{-i}, s_i')] \\
&\leq n \cdot \max_{i \in \mathcal{N}} |S_i||E| \frac{1}{n^c \cdot n \cdot \max_{i \in N} |S_i||E| \left(1 + \log\left(\frac{c_{\max}}{c_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}} \\
&\leq \frac{1}{n^c \cdot \left(1 + \log\left(\frac{c_{\max}}{c_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}}.
\end{aligned}
$$

□

Lemma 5.19 gives a bound on the number of improvement steps. Using the sampling algorithm for $\mu = 1 + \gamma$, we can bound the total number of samplings:

Lemma 6.5. *During the whole execution of Algorithm 2 the sampling algorithm for $\mu = 1+\gamma$ is applied at most $n \cdot \max_{i \in \mathcal{N}} |S_i| \cdot |E| \cdot \left(1 + \log\left(\frac{c_{max}}{c_{min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}$ times and the computation of the approximate pure Nash equilibrium is correct with a probability of at least $1 - n^{-c}$ for an arbitrary constant c.*

Proof. The result follows directly by applying the union bound:

$$Pr[\exists \text{ an improvement step in which the sampling fails}] \leq$$
$$\leq \frac{\left(1+\log\left(\frac{c_{\max}}{c_{\min}}\right)\right)\cdot 2\cdot n^2\cdot(d+1)\cdot\gamma^{-9}}{n^c\cdot\left(1+\log\left(\frac{c_{\max}}{c_{\min}}\right)\right)\cdot 2\cdot n^2\cdot(d+1)\cdot\gamma^{-9}} \leq \frac{1}{n^c}. \qquad \square$$

Summing up, we show that a μ-approximation of one Shapley value can be computed in polynomial running time with high probability (Lemma 6.1) and the sampling algorithm is running at most a polynomial number of times (Lemma 6.5). Thus, Theorem 6.3 follows. □

6.4. Shapley Values for Disaggregating User Evaluations

In the second application of the sampling method, we propose a model and a solution to handle the aggregation of reputation values from different users and the disaggregation of information on composed services. As we work with evaluations on compositions from many users, we combine the weighted average as an *a*ggregation operation, which aggregates evaluations across users, and the Shapley value as a *d*isaggregation process, which evaluates each basic service on the basis of how the different compositions are rated.

6.4.1. Model for the Evaluation Problem

There are m basic services which can be combined to $2^m - 1$ different service compositions. Furthermore, n users can buy and evaluate these service compositions. We assume that each user i evaluates all compositions once, assigning a real number from a predefined scale $[0, u]$ to each service composition $S \subseteq \{1, \ldots, m\}, S \neq \emptyset$. This is quite a strong assumption. However, a user in our model can also represent a group of similar/identical users who evaluate all compositions together. The best evaluation is u, the worst is 0. The matrix E collects all available evaluations, hence $E \in \mathcal{M}\,(n \times 2^m - 1)$ and E_{iS} is user i's evaluation for service composition S $(i = 1, \ldots, n, S \subseteq \{1, \ldots, m\}, S \neq \emptyset)$. A solution (in the wider sense) to the problem of evaluating basic services assigns to each basic service a real number, which can be collected in a real valued matrix $e \in \mathcal{M}\,(1 \times m)$. Note that we do not require these values to lie in the interval $[0, u]$.

To find an appropriate single evaluation for each basic service, we need two different operators. A disaggregator is a mapping $D^{(q)} : \mathcal{M}\,(q \times 2^m - 1) \rightarrow \mathcal{M}\,(q \times m)$ that takes each of q (possibly different) evaluations of compositions and computes the corresponding q evaluations of the m basic services. An aggregator is a mapping $A^{(p)} : \mathcal{M}\,(n \times p) \rightarrow \mathcal{M}\,(1 \times p)$ that maps the evaluations of n users (either over $p = 2^m - 1$ service compositions or over $p = m$ basic services) to an aggregated evaluation. In the subsequent discussion, we use different specifications of $D^{(q)}$ and $A^{(p)}$. In what follows, we assume that aggregation and disaggregation

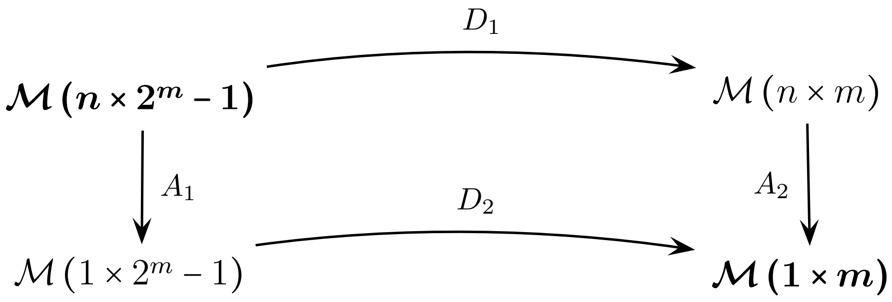

Figure 6.1.: Scheme of the evaluation problem

is anonymous in the sense that for an aggregator A (an instance of $A^{(p)}$) and evaluation matrix E the aggregated value in the i-th coordinate $(A(E))_i$ only depends on the i-th column of E and is calculated using the same function for each of the p coordinate. Similarly, a disaggregator D (an instance of $D^{(q)}$) is assumed to apply the same function for each of the q users that depends only on the corresponding row in the underlying evaluation matrix.

Two routes can be taken to transform an evaluation matrix E to a final evaluation of basic services (cf. Figure 6.1). On the one hand, it is possible to start with the aggregation of evaluations for each composed service. Applying the aggregator A_1 (a specification of $A^{(2^m-1)}$) on the input matrix results in the intermediate matrix $A_1(E) \in \mathcal{M}(1 \times 2^m - 1)$. A disaggregator D_2 (a specific $D^{(1)}$) then yields the final evaluation $D_2(A_1(E)) \in \mathcal{M}(1 \times m)$ of basic services.

On the other hand, one could also reverse the order of aggregation and disaggregation. Now, we first disaggregate each user's evaluation to an evaluation over basic services. Then, these evaluations are aggregated across users. Formally, we first apply D_1 (some $D^{(n)}$) to an evaluation matrix E resulting in $D_1(E) \in \mathcal{M}(n \times m)$ before aggregator A_2 (some $A^{(m)}$) yields the final evaluations $A_2(D_1(E)) \in \mathcal{M}(1 \times m)$. Both routes yield overall evaluations for the single services, which should not be restricted to the same range as the initial evaluations. We define $\underline{R}_{D_2,A_1}$ and $\overline{R}_{D_2,A_1}$ so that the range of $(D_2(A_1(\cdot)))_j$ is $[\underline{R}_{D_2,A_1}, \overline{R}_{D_2,A_1}]$. Analogously, $\underline{R}_{A_2,D_1}$ and $\overline{R}_{A_2,D_1}$ are defined such that the range of $(A_2(D_1(\cdot)))_j$ is $[\underline{R}_{A_2,D_1}, \overline{R}_{A_2,D_1}]$ ($j = 1, \dots, m$)). Note that with a continuous aggregator A and disaggregator D, the composed functions $(A(D(\cdot)))_j$ and $(D(A(\cdot)))_j$ are continuous and therefore the image of the compact set of evaluation matrices with entries in $[0, u]$ must be compact.

Applied in our scenario, the Shapley value for basic service j gives the average marginal contribution of j over any possible composition and is formally defined as follows:

$$\Phi_j(\hat{E}, i) = \sum_{S \subseteq \{1,\dots,m\} \setminus \{j\}} \frac{(m - |S| - 1)! \cdot |S|!}{m!} \left(\hat{E}_{iS \cup \{j\}} - \hat{E}_{iS} \right). \qquad (6.1)$$

The Shapley value as a disaggregating function when evaluations $\hat{E}$ are given is defined as follows:

$$D^{(q)}(\hat{E}) = \begin{pmatrix} \Phi(\hat{E},1) \\ \vdots \\ \Phi(\hat{E},q) \end{pmatrix} \tag{6.2}$$

with $\Phi(\hat{E},i) = (\Phi_1(\hat{E},i),\ldots,\Phi_m(\hat{E},i))$ where $\Phi_j(\hat{E},i)$ is user i's Shapley value for the basic service j with (or the aggregated Shapley value if $1 = i = q$).

As aggregator, we define the weighted average with user specific weights β_i, $\sum_{i=1}^{n} \beta_i = 1$ over evaluations $\bar{E}$:

$$A^{(p)}(\bar{E}) \text{ with } (A^{(p)}(\bar{E}))_T = \sum_{i=1}^{n} \beta_i \bar{E}_{iT} \quad \forall T \subseteq \{1,\ldots.m\}. \tag{6.3}$$

6.4.2. Approximation Through Sampling

The exact computation of the Shapley value is computationally hard in general since the number of coalitions exponentially grows with the number of basic services. Therefore, computing the solution of our evaluation problem could take take too long for practical scenarios. Nevertheless, by using sampling methods from Section 6.2 we can achieve at least a reasonable approximation for our reputation values with high probability. Indeed, we can construct a fully polynomial-time randomized approximation scheme (FPRAS) for our evaluation problem. The only assumption needed for this approach is a pair of bounds on the possible marginal contribution of a service to a composition. As explained in the previous section, for the setting of reputation values, these bounds are naturally given by $[-u, u]$ (arbitrary other bounds a and b are possible, too).

We apply Algorithm 3 several times in our matrix with $\delta = (n^{c+1} \cdot m^{c+1})^{-1}$ for any constant c and let $\tilde{\Phi}_j(\hat{E},i)$ bet the corresponding output of the algorithm.

Let $\tilde{\Phi}(\hat{E},i) = (\tilde{\Phi}_1(\hat{E},i),\ldots,\tilde{\Phi}_m(\hat{E},i))$ and

$$\tilde{D}^{(q)}(\hat{E}) = \begin{pmatrix} \tilde{\Phi}(\hat{E},1) \\ \vdots \\ \tilde{\Phi}(\hat{E},q) \end{pmatrix}. \tag{6.4}$$

Theorem 6.6. *Given any constant $\mu > 0$, $A(\tilde{D}(E))$ and $\tilde{D}(A(E))$ can be computed in polynomial time and $|A(\tilde{D}(E)) - e| \leq \mu$ and $|\tilde{D}(A(E)) - e| \leq \mu$ with high probability.*

Proof. Let $\delta = (n^{c+1} \cdot m^{c+1})^{-1}$ for any constant c. First aggregating and then disaggregating only needs m applications of our sampling algorithm as the last step (one for each service). With Lemma 6.2 we have for each service j that $Pr\left[|\tilde{\Phi}_j(\hat{E},1) - \Phi_j(\hat{E},1)| \geq \mu\right] \leq \left(n^{c+1} \cdot m^{c+1}\right)^{-1}$. We use the union bound since all approximations are independent results in $Pr\left[\exists\, j \in \{1,\ldots,m\} : |\tilde{\Phi}_j(\hat{E},1) - \Phi_j(\hat{E},1)| \geq \mu\right] \leq m\left(n^{c+1} \cdot m^{c+1}\right)^{-1} = \left(n^{c+1} \cdot m^{c}\right)^{-1}$.

For the other direction, we have to look at the two steps in more detail: In the disaggregation phase, we compute $n \cdot m$ independent approximations of the Shapley value. Again with Lemma 6.2 we can show that

$$Pr\left[\exists 1 \leq i \leq n : \exists 1 \leq j \leq m : |\tilde{\Phi}_j(\hat{E}, i) - \Phi_j(\hat{E}, i)| \geq \mu\right]$$
$$\leq n \cdot m \cdot n^{-c} \cdot m^{-c} = n^{-c} \cdot m^{-c}.$$

Taking the average over all player i affects neither the failure probability nor the range of values and the theorem is shown. □

6.4.3. Incomplete Evaluation Profiles

A natural question that arises in our work is the handling of incomplete information. Until now we assumed complete evaluations. But in reality, often each participating user rates only a subset of all possible service compositions. To tackle this problem we follow the line of research recently started by [BSV17] and apply the so-called *Data Dependent Shapley Value*.

Here we assume a distribution $\mathcal{D}$ which captures the frequency of the different compositions. The well-known Shapley axioms can be extended with regard to this distribution $\mathcal{D}$ and the value itself can be defined as

$$\Phi_j^{\mathcal{D}}(\hat{E}, i) = \sum_{S: j \in S} Pr[S \sim \mathcal{D}] \cdot \frac{E_{iS}}{|S|}. \tag{6.5}$$

Let $\Phi^{\mathcal{D}}(\hat{E}, i) = (\Phi_1^{\mathcal{D}}(\hat{E}, i), \ldots, \Phi_m^{\mathcal{D}}(\hat{E}, i))$ and

$$D^{\mathcal{D}(q)}(\hat{E}) = \begin{pmatrix} \Phi^{\mathcal{D}}(\hat{E}, 1) \\ \vdots \\ \Phi^{\mathcal{D}}(\hat{E}, q) \end{pmatrix}. \tag{6.6}$$

Since the distribution of the evaluations is known in this extension, the sampling algorithm from the previous section can be extended. The samples are now drawn from the underlying distribution $\mathcal{D}$ and again with help of the Hoeffding's inequality [Hoe63a; Hoe63b] the number of samples can be estimated.

Corollary 6.7. *Given evaluations E whose frequency follows distribution $\mathcal{D}$ and any constant $\mu > 0$, $A(D^{\mathcal{D}}(E))$ and $D^{\mathcal{D}}(A(E)))$ can be computed in polynomial time and $|A(D^{\mathcal{D}}(E)) - e| \leq \mu$ and $|D^{\mathcal{D}}(A(E)) - e| \leq \mu$ with high probability.*

6.4.4. Axiomatic Characterization

In the underlying publication of this chapter [Fel+18b], the axiomatic characterization is the main focus of our contribution. Here, we concentrate on the computation, and especially the sampling. Nevertheless we give the the axiomatic characterization of the described evaluation problem. In the pursuit of finding a plausible combination, we present a set of desirable properties for the interplay of aggregation and disaggregation. Two properties are central in the discussion. A *r*estricted influence requirement should forbid users to unilaterally change a service's overall evaluation to the best or worst possible valuation. *C*onsistency of our combined aggregation/disaggregation process requires that the order should not matter, i.e., the evaluation of an elementary service should not depend on whether we first aggregate user evaluations of compositions and then disaggregate or whether we first disaggregate each single user's valuation to her valuation of basic services and then aggregate these data. We state our natural axioms and show that weighted average and the Shapley value fulfill them in contrast to positional operators.

Axioms

A *s*olution to the problem of assigning valuations for basic services is a collection of aggregators A_1, A_2 and disaggregators D_1, D_2. In other words, a solution includes both types of operations, aggregation of values for compositions (A_1) and basic services (A_2) as well as disaggregation of valuations from single users (D_1) and from aggregated valuations (D_2).

We next formulate normative requirements (axioms) that a solution should satisfy, i.e., the combination of aggregators and disaggregators. The first key axiom is immediate and requires that the solution is independent of the order of aggregation and disaggregation.

Axiom 6.8 (CONSISTENCY). *A solution to the evaluation problem is* consistent *if it does not depend on the order of aggregation and disaggregation, i.e.,* $A_2(D_1(E)) = D_2(A_1(E))$ *for each evaluation matrix* E*.*

Apart from technical considerations (e.g., dynamic updates, etc.), without this requirement the ordering becomes a strategic question. The platform calculating single evaluations could influence the outcome by choosing the order of the aggregation and disaggregation in its favor.

The next two axioms capture extreme cases with one user or one service only, which makes either aggregation or disaggregation superfluous. Given a single user, $n = 1$, there is no difference between her evaluations and the aggregated ones. Hence, disaggregation should not give different results.

Axiom 6.9 (SINGLE USER). *A solution to the evaluation problem fulfills the* single user axiom *if in case* $n = 1$ *the disaggregation functions are equal,* $D_1 \hat{=} D_2$*,* D *in short, in order to yield the same result.*

Given only one single service, $m = 1$, the disaggregation step is vacuous.

Axiom 6.10 (SINGLE SERVICE). *A solution of the evaluation problem fulfills the* single service axiom *if in case* $m = 1$ *the aggregation functions are equal,* $A_1 \hat{=} A_2$*, shortly* A*, in order to yield the same aggregation result.*

Recall that our anonymity requirement from above together with the two previous axioms ensures that aggregation and disaggregation does not vary over coordinates.

We conclude our requirements section with one last aspect. A user might want to manipulate the final evaluation of a basic service with the help of her individual evaluation. Common aims of a manipulation are that a single service receives an extremely low or extremely high final evaluation. Therefore, the aggregation and disaggregation process should rule out these possibilities of manipulation. If a basic service is not already evaluated extremely high or extremely low, a newly evaluating user should not be able to choose her evaluations in such a way that the overall valuation of this basic service becomes extremely high or extremely low.

Axiom 6.11 (RESTRICTED INFLUENCE)**.** *Given evaluations E, a range $[0, u]$ and a service j with overall evaluations $(D_2(A_1(E)))_j \neq \overline{R}_{D_2,A_1}$, $(D_2(A_1(E)))_j \neq \underline{R}_{D_2,A_1}$, $(A_2(D_1(E)))_j \neq \overline{R}_{A_2,D_1}$ and $(A_2(D_1(E)))_j \neq \underline{R}_{A_2,D_1}$, a solution fulfills the restricted influence axiom, if there exists no $v \in \mathbb{R}^{2^m-1}$ and $\hat{E} = \begin{pmatrix} E \\ v \end{pmatrix}$ such that $(D_2(A_1(\hat{E})))_j = \overline{R}_{D_2,A_1}$, $(D_2(A_1(\hat{E})))_j = \underline{R}_{D_2,A_1}$, $(A_2(D_1(\hat{E})))_j = \overline{R}_{A_2,D_1}$ and $(A_2(D_1(\hat{E})))_j = \underline{R}_{A_2,D_1}$.*

(Dis-)Aggregations with Positional Operators

From a sequence of numbers, positional operators select the number that appears at a prespecified position after ordering the numbers. The most popular examples are the median, maximum and minimum operator, which are frequently used in social choice problems as an aggregation device. We investigate how far they are suitable for our goals.

First, we discuss aggregation and disaggregation with the median. The median is defined as the value dividing the evaluations in the lower and the higher half. It is often used to aggregate values because it is a robust statistical measure as it is not affected by outliers. But is it also a suitable disaggregation tool? It seems convincing that the valuation of a basic service j is described by the “middle value” between the evaluations of all composed services that include j. Taking the median as a tool to aggregate and disaggregate evaluations we can directly see, that the single user and the single service axioms are fulfilled as we use the same type of function for all operations. Unfortunately, we observe that using the median as an aggregation and disaggregation function will not always yield a consistent solution. If the number of evaluations is odd, the median is simply the value in the middle when the evaluations are arranged from the lowest to the highest value. If the number of evaluations is even, the median is defined by the mean of the two middle values.

Proposition 6.12. *The solution that uses the median as aggregator and as disaggregator is not consistent.*

Proof. Consider the following evaluations E with two basic services a and b which can be combined to a composed service ab (third column) and three users.

$$E = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \longrightarrow D(E) = \begin{pmatrix} 1.5 & 1.5 \\ 2 & 2 \\ 1.5 & 1.5 \end{pmatrix}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$A(E) = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{matrix} A(D(E)) = \begin{pmatrix} 1.5 & 1.5 \end{pmatrix} \\ D(A(E)) = \begin{pmatrix} 2 & 2 \end{pmatrix} \end{matrix}$$

One can easily verify that first disaggregating and then aggregating the evaluations with the help of the median each yields an evaluation of $(A(D(E)))_j = 1.5, j = a, b$, whereas first aggregating and then disaggregating yields $(D(A(E)))_j = 2, j = a, b$. Thus, the solution is not consistent. □

Next, we aggregate and disaggregate using the minimum or maximum operator. When choosing the minimum to disaggregate, the evaluation of a basic service j equals the evaluation of the worst composed service including j. When choosing the minimum to aggregate, the aggregate evaluation of a basic or composed service is the worst evaluation across users. Analogously, we can define (dis-)aggregation with the maximum operator. However, combining maximum and the minimum contradicts the axioms.

Proposition 6.13. *The solution that uses the minimum to aggregate and the maximum to disaggregate (the minimum to disaggregate and the maximum to aggregate), i.e.,* $A = \min\{\cdot\}$ *and* $D = \max\{\cdot\}$*, (*$D = \min\{\cdot\}$ *and* $A = \max\{\cdot\}$*), is not consistent.*

Proof. Assume E is the identity matrix. Minimizing first over rows or columns always leads to 0. Afterwards, the maximal evaluation within the resulting matrix is 0. Maximizing first over rows or columns always leads to 1. The minimal evaluation within the resulting matrix is 1. Thus, the results are not consistent. □

Let us now consider the cases where $A = D = \min\{\cdot\}$ and $A = D = \max\{\cdot\}$. Economically, the solutions can be interpreted as a lower and upper bound for the evaluations.

Proposition 6.14. *Using the minimum (maximum) to aggregate and to disaggregate,* $A = D = \min\{\cdot\}$ *(*$A = D = \max\{\cdot\}$*) yields a consistent solution.*

Proof. By using the minimum operator for first aggregation and then disaggregation to determine the overall valuation of basic service j, we take the lowest evaluation in each column of E and from those $2^m - 1$ many values the lowest one is the overall evaluation for j. In effect, this overall evaluation for j is the minimal entry in E among those that correspond to composed services, including j. It is easy to see, that precisely this value will also be attributed to j, when first disaggregating and then aggregating with the minimum operator. Analogous arguments show consistency of the solution that uses the maximum operator. □

Although we can achieve consistency using the minimum or maximum, the solutions are manipulable.

Proposition 6.15. *The solution using the minimum (maximum) to aggregate and to disaggregate, $A = D = \min\{\cdot\}$ ($A = D = \max\{\cdot\}$) violates the restricted influence axiom.*

Proof. As the users' evaluations lie within the range $[0, u]$ the solution of the aggregation and disaggregation process by using the minimum or maximum operator still lies within this range $[0, u]$. Consider the following evaluations with three users and two basic services a and b which can be combined to three composed service a, b, ab, $E = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & \alpha \end{pmatrix}$, and assume $u > 2$. All evaluations are equal except for E_{3ab}. It is easy to see that without user 3, all final evaluations are 2, which is neither the maximal nor the minimal possible valuation. If $\alpha = u$ and the maximum function is used to aggregate and disaggregate, the valuation for the basic services a and b is $\alpha = u$, $(D(A(E)))_a = (A(D(E)))_a = (D(A(E)))_b = (A(D(E)))_b = \alpha = u = E_{3ab}$. If $\alpha = 0$ and the minimum function is used to aggregate and disaggregate, the solution for the single services would be $E_{3ab} = 0$. Thus, the restricted influence axiom is violated. □

The maximum and minimum solutions can easily be manipulated: Each costumer who evaluates the (composed) services can determine the (dis-)aggregated evaluation of a single service j by simply assigning the best or worst possible evaluation to one of the composed services T with $j \in T$.

Disaggregation with the Shapley Value and Aggregation with the Weighted Average

Although minimum and maximum operators are easy to compute, they are proven to generate inappropriate solutions that do not meet all our axioms. Instead, using the weighted mean to aggregate and the Shapley value to disaggregate accomplish our axioms. While the single service and single axioms are obviously fulfilled, the others are proven in the next theorem.

Theorem 6.16. *The solution that uses the Shapley value as disaggregator and the weighted average as aggregator as defined in Equation 6.1 to 6.3 is consistent and fulfills the restricted influence axiom.*

Proof. To see consistency, we use the linearity of the Shapley value (in entries of E) and calculate

$$(D(A(E)))_j$$

$$= \sum_{S \subseteq \{1,\dots,m\} \setminus \{j\}} \frac{(m - |S| - 1)! \cdot |S|!}{m!} \Big([A(E)]_S - [A(E)]_{S \setminus \{j\}} \Big)$$

$$= \sum_{S \subseteq \{1,\dots,m\} \setminus \{j\}} \frac{(m - |S| - 1)! \cdot |S|!}{m!} \left(\sum_{i=1}^{n} \beta_i E_{iS \cup \{j\}} - \sum_{i=1}^{n} \beta_i E_{iS} \right)$$

$$= \sum_{i=1}^{n} \beta_i \sum_{S \subseteq \{1,\dots,m\} \setminus \{j\}} \frac{(m - |S| - 1)! \cdot |S|!}{m!} \Big(E_{iS \cup \{j\}} - E_{iS} \Big)$$

$$= \sum_{i=1}^{n} \beta_i [D(E)]_{ij} = (A(D(E)))_j$$

It is easy to see that this solution also fulfills the restricted influence axiom. If the users' evaluations lie within the range $[0, u]$, taking the mean does not change this range. By using the Shapley value the range changes to $[-u, u]$. Assume evaluations E such that after the first disaggregation step the evaluation of a basic service j is $(D(E))_{ij} \neq u$ and $(D(E))_{ij} \neq -u$ for all users i. A new user $n + 1$ gives evaluations v which results in $(D(v))_j$. The new evaluation matrix is given by $\hat{E} = \begin{pmatrix} E \\ v \end{pmatrix}$. By aggregating all $(D(E))_{ij}$ and $(D(v))_j$ to $(A(D(\hat{E})))_j$ with the weighted average, the aggregated value can never equal u or $-u$, $(A(D(\hat{E})))_j \neq u$ and $(A(D(\hat{E})))_j \neq -u$. As using the weighted mean and the Shapley value is consistent, we conclude that $(D(A(\hat{E})))_j \neq u$ and $(D(A(\hat{E})))_j \neq -u$ as well, showing that the restricted influence axiom is satisfied. □

Finally, we note that the range of the Shapley values of a basic service is $[-u, u]$. To see this, assume that the users' evaluations have already been aggregated. The highest marginal contribution of a basic service j is u, which is received when j is added to a composition that is evaluated with 0. Therefore, the Shapley value for j equals u, only when all compositions including j are evaluated with u and all compositions without j are valued 0. Similarly, basic service j receives a Shapley value of $-u$ when it is the other way round (compositions valued 0 when j is included and u, when j is not included). The change of scale from $[0, u]$ to $[-u, u]$ necessitates a different interpretation of overall evaluations. Interpreting evaluations as a proxy for quality, a positive Shapley value of service j suggests that it improves the quality of a composition, while a negative one is rather a deteriorating quality. In particular, two services that may be used as substitutes can be ranked in terms of quality enhancement by their Shapley values. The actual value in the overall solution has to be considered in relation to the values for other services. For instance, if all user evaluations for all compositions are u, then the final solution attributes u/m to each component service. If valuations for some service and compositions that include it fall, then the final values for all other services increase. Consequently, the interpretation of final values necessarily has to include the whole picture.

We can extend the result of Theorem 6.16 for the slightly different scenario with incomplete evaluation profiles that we introduced in Section 6.4.3.

Theorem 6.17. *Given evaluations E whose frequency follows distribution $\mathcal{D}$, using the Data-Dependent Shapley Value to disaggregate and the weighted average to aggregate as defined in Equation 6.6 and 6.3, yields a consistent solution that fulfills the restricted influence axiom.*

Proof. We follow the same argumentation as in the proof of Theorem 6.16:

$$(D^{\mathcal{D}}(A(E)))_j = \sum_{S:j\in S} Pr[S\sim\mathcal{D}]\cdot\frac{[A_1(E)]_S}{|S|}$$

$$= \sum_{S:j\in S} Pr[S\sim\mathcal{D}]\cdot\frac{1}{|S|}\sum_{i=1}^{n}\beta_i E_{iS} = \sum_{i=1}^{n}\beta_i\sum_{S:j\in S} Pr[S\sim\mathcal{D}]\cdot\frac{1}{|S|}E_{iS}$$

$$= \sum_{i=1}^{n}\beta_i[D_1^{\mathcal{D}}(E)]_{ij} = (A(D^{\mathcal{D}}(E)))_j$$

Since we still use the weighted average for aggregation, the restricted influence axiom holds with the same argumentation as in the proof of Theorem 6.16. □

Conclusion

This chapter introduced a general sampling approach for the Shapley value and we analyzed it for two specific settings. We could extend the results from the previous chapter to achieve a polynomial running time instead of a polynomial number of improvement moves.

Furthermore, we addressed the problem of eliciting information on the quality of components of a composed service (or product). As input we use evaluations of many users on compositions. Our focus was on the computational issues here and we showed that it is efficient and robust as the final valuations can be approximated in polynomial time, even if the evaluation matrix is incomplete. Additionally, we showed that a reasonable (in the sense of the stated axioms) approach is indeed to combine the Shapley value from cooperative game theory as disaggregator with the weighted average as aggregator.

An emerging demand for all-in-one services requires providers to compose their services out of a number of elementary services. For instance, cloud service providers such as Google, Amazon or Microsoft combine hardware resources with (several) software services and sell it as a composed service. Similarly, restaurants sell a composition of service and product (food). However, for the customer or user, it may be difficult to exactly identify the single parts in her good. As a consequence, she may not be able to observe their performances separately, when she experiences her good. For instance a "slow" computation service could result either from an inappropriate hardware or from inefficient algorithms.

Internet platforms offer possibilities to give evaluations for (composed) services or products, so that later users can base their purchase decision on it. However, as the components of a service cannot be identified, the user can only evaluate the composed service as a whole. Not only from the provider's perspective would it be worthwhile to know how we can infer from evaluation data on compositions on an evaluation of single services. The very fact that elementary services can be composed and sold in many different compositions allows us to assess how well a component service fits. Such a disaggregation of evaluations helps to sort out inefficient elementary services, for example, so that they should not be considered in the composition process.

We close by mentioning that our method can be applied in other scenarios as well. For instance, instead of using services and service compositions, we can think of workers and teams in a company. Our approach can then be used to measure the effectivity of a worker within a team on the basis of performance values for the different possible team compositions. The design of bonus systems or employment policies are possible fields of application. Another direction to use our framework is the understanding of learning algorithms. Imagine a complex learning environment, in which we compose the learning tools out of basic components such as pre- and postprocessors, different learning algorithms, computing environments and also training data. Here, we can often measure only the performance of the whole tool, but we need to make statements about the contribution of the various involved components to improve the learning tools.

PART II

Existence of Approximate Pure Nash Equilibria

The second part of this thesis investigates the existence of approximate pure Nash equilibria in models in which pure Nash equilibria do not exist or in which they only exist in restricted instances. Here, we work with extensions of popular models from the economics community. Both considered models share the property that their strategy spaces are continuous (compared to the discrete models of Congestion Games) which requires different methods and analysis. We construct candidates for good approximate equilibria and show their approximation factors. Additionally, we prove lower bounds on the approximation factor, the non-existence of pure Nash equilibria as well as bounds on the quality of our approximate equilibria.

Outline of This Part The first model in Chapter 7 deals with a variant of coevolutionary opinion formation [BGM13], based on the popular FJ model [FJ90] and DeGroot model [Deg74]. In the second model in Chapter 8, we investigate a generalization of the popular model of facility location by Hotelling [Hot90], extended with the load on the facilities as an additional objective to the distance [PSV18].

CHAPTER 7

Opinion Formation Games

Opinion Formation Games model scenarios in which each individual in a social network is endowed with private preferences on the strategy set and these preferences are counterbalanced by the influence of the neighbors in the social network. As its name suggests, this class of games has first been introduced to assess the sociological process of opinion formation by which an individual shapes her belief on a certain subject as a result of the interaction with her acquaintances, e.g., neighbors in a social network.

We study the coevolving process in which the costs of a player consist of two components, weighted by a parameter ω: The first term is the cost that the player incurs for disagreeing with the other players in her neighborhood; the second term represents the cost incurred by choosing a strategy far from her intrinsic opinion. A low stubbornness value ($\omega < 1$) results in a high weight for the first term, a high value ($\omega > 1$) for the second term. The crucial characteristic of this model lies in the definition of the neighborhood. It is not fixed through a given graph, but it coevolves through the changing opinions of the players. On special case of this model is the asymmetric K-Nearest-Neighbors (K-NN) games introduced in [BGM13]. Here, the neighborhood of a player is always given by the K neighbors with the closest expressed opinion to her own intrinsic opinion. Despite its appeal for modeling dynamically evolving social relationships and simultaneous changing opinions, this model lacks predictive power due to the non-existence of pure equilibria. In particular in the context of opinion formation, mixed and correlated equilibria might not to be a plausible solution concept. A way to alleviate this problem is the concept of approximate pure Nash equilibria, where small relative changes of the players' payoffs are deemed to be insufficient to trigger players to deviate from their opinions. This may even have a behavioral explanation as it might be due to uncertainty of the exact payoffs, the cost of switching strategies or the laziness of the players.

Contribution and Underlying Work It was shown by [BGM13] that pure Nash equilibria do not exist in general in asymmetric K-NN games. Since then, no further progress was made regarding the existence of equilibria or other solution concepts. We turn our attention to a more general model with a more general definition of the neighborhood and show the existence of approximate pure Nash equilibria with low approximation factors. To the best of our knowledge this is the first work to investigate an approximate concept of equilibria in the context of Opinion Formation Games.

Firstly, we analyze the approximation guarantees for natural outcomes, namely if all players stay at their opinions or if all players agree on an arbitrary common opinion. Even these simple strategies result in reasonable approximation factors for extreme values of the stubbornness of the players, helping to explain common behavior in society.

Secondly, we investigate the more restricted, but interesting case of the *Nearest-Neighbor Game*. Here, we propose and analyze a quite natural dynamic in which each player, starting from her intrinsic opinion, deviates only a certain fraction towards her best-response. This dynamic always converges to an approximate pure Nash equilibrium after one step, resulting in a factor depending on the stubbornness. For $\omega = 1$ (balanced weighting of the two objectives) a factor of 1.17 is achieved, and it decreases fast with rising ω. Next to the good approximation quality, the dynamic has two other main advantages because of its simplicity: First, it is an easily implementable protocol and secondly, through its natural approach (starting at the own opinion and changing it a little bit to the best response) it is a reasonable behavior for humans in multiple-round votes. The main challenge in the analysis is the changing neighborhood through the coevolutionary process. To overcome this issue, we introduce the concept of a virtual player with virtual costs. This enables us to separately analyze the opinion formation and the implications of changing neighborhoods by a very careful analysis of the scenario. Our developed technique and most results are independent of the concrete definition of the neighborhood. This concept should also be adaptable for the analysis of other coevolving opinion formation models.

The model, analysis, and results presented in this chapter are based on the following recent work:

> Angelo Fanelli, Matthias Feldotto, and Alexander Skopalik. **Approximate Equilibria in Coevolutionary Opinion Formation Games**. Unpublished Manuscript. 2018 [FFS18]

Outline of This Chapter We start this chapter with an overview of related work in Section 7.1. Afterwards, we formally introduce our concrete model in Section 7.2. Then, we first show some technical properties of the costs in Section 7.3 and go into the main contribution where we show different approximation factors in this model in Section 7.4.

7.1. Related Work

A long line of research in economic theory considers different models of opinion formation and their dynamics (for a good survey, we refer to [Jac08]). In the seminal FJ model by Friedkin and Johnsen [FJ90] each individual i has an intrinsic opinion o_i and an expressed opinion s_i which is updated by averaging between o_i and the social influence. This model departs from the long-standing line of work in economic theory, which builds on the basic model of *D*eGroot [Deg74], which instead considers the process by which a group of individuals in a social network can arrive at a shared opinion through a form of repeated averaging, without taking into account the counterbalancing effect of the intrinsic opinions. Another line of research focuses on models, in which the social influence of individuals is restricted to the confidence region of the others. Here, the *HK model* [HK02] and the *DW model* [Wei+02] are worth to mention as most prominent representatives.

To the best of our knowledge, Bindel et al. [BKO15; BKO11] have been the first to investigate opinion formation models following the algorithmic viewpoint. Each player i in the game has an intrinsic opinion o_i and an expressed opinion s_i which is the strategy of a player – both are real values. The social influence is defined by a graph G with fixed edge weights w_{ij}. Additionally, each player has a parameter w_i which models the player's stubbornness. The cost of each player in a state s is then defined as $c_i(\mathsf{s}) = \sum_{j \neq i} w_{ij} \cdot (s_i - s_j)^2 + w_i \cdot (o_i - s_i)^2$. Bindel et al. show the existence and convergence of a unique pure Nash equilibrium by coinciding Price of Anarchy and stability values in their analysis. For the case of undirected graphs, resulting in symmetric influences, the Price of Anarchy is tight with $9/8$; for the asymmetric setting it can be unbounded. Only in the case of G being a subclass of Eulerian graphs, they can derive constant upper bounds.

Chen et al. [CCL16] consider the same model and bound the Price of Anarchy for a more general class of directed graphs (asymmetric influences) in which a player's influence is bounded by the influence she experiences from others.

All the early works based on the FJ model share the common assumption that the social relationships remain fixed during the entire duration of the process, i.e., the social network is a static graph. This assumption has been relaxed in some recent works [BGM13; BFM16; Dur+12; HPZ11] which are based on the evidence that opinion formation and friend selection are often *c*oevolving processes in real life. In particular, Bhawalkar et al. [BGM13] consider the FJ model in which, for every individual, a specific function captures both the disagreement between the intrinsic opinion and a friend's opinion, and the strength of their relationship. The intrinsic opinion o_i and the expressed opinion s_i are still real values. In the symmetric variant of their model the cost of a player i is given by $c_i(\mathsf{s}) = \sum_{j \neq i} f_{ij}(s_i - s_j) + w_i \cdot g_i(o_i - s_i)$ where f_{ij} and g_i are *fixed real valued functions* and $f_{ij} = f_{ji}$. Observe that this is a direct generalization of the model of Bindel et al. [BKO15] which can be reobtained by setting $f_{ij}(x) = w_{ij} \cdot x^2$ and $g_i(x) = x^2$. As long as the functions f and g are either convex or differentiable, they show a tight bound on the Price of Anarchy of 2. Furthermore, for functions $f(x) = g(x) = |x|^\alpha$ they derive closed formula for the Price of Anarchy. Bhawalkar et al. [BGM13] bound the Price of Anarchy for this model utilizing local smoothness.

Additionally, they introduce an asymmetric variant with the cost function $c_i(\mathsf{s}) = \sum_{j \neq i} w_{ij}(\mathsf{s}) \cdot (s_i - s_j)^2 + \omega_i \cdot (o_i - s_i)^2$, where they show the existence of pure Nash equilibria if w_{ij} is a continuous function of the distance between o_i and all s_j.

For the asymmetric case, they also introduce the *K-NN (K-nearest-neighbors)* *game* which we investigate in this work. For a given integer $K > 0$, each player i is influenced by the K other players whose expressed opinions are closest to o_i. With defining this set of neighbors by $N_i(\mathsf{s})$ and the stubbornness of a player by ω, the costs of a player is given by $c_i(\mathsf{s}) = \sum_{j \in S(\mathsf{s},i)} (s_i - s_j)^2 + \omega \cdot K \cdot (o_i - s_i)^2$. Bhawalkar et al. prove that pure Nash equilibria do not necessarily exist in general, which raises the question for other solution concepts. Furthermore, they show that the robust Price of Anarchy of this game is at most $\frac{(7+\varepsilon)(2+\varepsilon)}{\varepsilon(1+\varepsilon)}$ for $\omega = 1 + \varepsilon > 1$ which is independent from K. The Price of Anarchy tends to 1 as ω increases and is unbounded as ω goes below 1, as they show a lower bound of at least $1/\omega^2$ for $\omega < 1$.

Epitropou et al. [Epi+17] investigate a variant of the Opinion Formation Game with aggregation aspects. They consider the average public opinion as a natural way to represent a global trend in the society. They show the existence of a unique equilibrium in their average-orient Opinion Formation Games. They also show the convergence of the best-response dynamics, even in a model with outdated information. For the Price of Anarchy they show a small bound of $9/8 + o(1)$, almost matching the tight bounds for games with aggregation. Finally, some results also hold for a class of games with negative influence.

Next to the continuous strategy sets as in the previous models, a further line of research investigates *discrete* variants of this model. The initial, intrinsic opinions are still real values for all players, but the strategy choice is now binary. A suitable setting, for example, is the decision to buy or not to buy a product given a more differentiated opinion. Ferraioli et al. [FGV16; FGV12] investigate the model of Bindel et al. [BKO15] under the assumptions that $s_i \in \{0, 1\}$. They show that these games are exact potential games [MS96], thus isomorphic to Congestion Games [Ros73]. Furthermore, they derive exact bounds of 2 in the case of integer edge weights and 1 in the case of rational edge weights for the Price of Stability. In contrast, the Price of Anarchy is always unbounded. Furthermore, they show several results bounding the rate of convergence of decentralized best-response dynamics and logit dynamics.

A special case with a discrete strategy choice of the asymmetric model in [BGM13] has been proposed and studied by Bilò et al. [BFM18]. In this work, the weight function $w_{ij}(\mathsf{s})$ has been specifically defined as $w_{ij}(\mathsf{s}) = (1 - |o_i - s_j|)^k$, for any fixed $k > 0$. In this work the strength by which an individual expresses an opinion s_j may influence another individual i is polynomially decreasing with the distance between s_j and the intrinsic opinion of i. This definition reflects the tendency of individuals to associate and bond with similar individuals. This definition of relationship is in sharp contrast with the model defined by Bhawalkar et al. [BGM13]. Here, for a given positive integer K, they investigate the particular case in which, for each individual i, the set of acquaintances is formed by the K individuals whose expressed opinion is at a minimum distance from o_i.

Chierichetti et al. [CKO18; CKO13] study the Price of Stability in a similar model with an unweighted social graph and show tight bounds. In their model, each player tries to minimize the distance from her internal opinion and the sum of distances from the expressed opinions of her adjacent players.

Further related works consider similar models as dynamic systems (e.g., [RRS17; Aul+17; Aul+16; FPS16; FV17]).

7.2. Model

We are given a set of n players $\mathcal{N} = \{1, 2, \ldots, n\}$. The strategy set of each player is the real interval $[0, 1]$, which we denote by S. We refer to the strategy of a player as her expressed opinion $s_i \in S$. Each player i has also a given intrinsic opinion $o_i \in S$. A state is a snapshot of the game in which every player has selected an expressed opinion. A state of the game is denoted by an n-vector $\mathsf{s} = (s_1, \ldots, s_n)$, where $s_i \in S$ is the expressed opinion of i.

Each player expresses cardinal preferences on the set of states through a *cost* function. The cost of player i in a state s is based on the contribution of two terms, corresponding to the two objectives of the player, together with the stubbornness $\omega \geq 0$, i.e.:

$$c_i(\mathsf{s}) = \frac{1}{|N_i(\mathsf{s})|} \sum_{j \in N_i(\mathsf{s})} (s_j - s_i)^2 + \omega(o_i - s_i)^2,$$

where $N_i(\mathsf{s})$ is a subset of $\mathcal{N} \setminus \{i\}$ that denotes the *neighborhood* of player i in state s. A minimal and fundamental property that we impose on $N_i(\mathsf{s})$ is that it does not depend on s_i. Thus the neighborhood is a function $N_i : \mathsf{S}_{-i} \to 2^{\mathcal{N}}$, mapping the strategy profile without the player i to a set of players. Formally, for every state s, for every player $i \in \mathcal{N}$ and for every strategy $s_i' \in S$, we have

$$N_i(\mathsf{s}) = N_i\big((\mathsf{s}_{-i}, s_i')\big). \tag{7.1}$$

The first term in the cost function is the cost that the player incurs for disagreeing with the other players; the second term represents the cost incurred by choosing a strategy far from the intrinsic opinion o_i. The trade-off between the two contributions to the cost is balanced by a positive parameter $\omega \geq 0$, which essentially controls the extent to which players are more concerned with agreeing with their neighbors (low ω) or with staying at their intrinsic opinion (high ω). We refer to ω as the *stubbornness* of the players. A value of $\omega = 1$ yields a balanced weighting of both objectives. Notice that this model is a natural extension of the FJ model [FJ90] where the network is not fixed but depends on the state of the game. The K-NN game [BGM13] is one variant of our model in which the neighborhood of player i is given by the K players $j \neq i$ with the smallest $|s_j - o_i|$.

7.3. Virtual Costs and Their Properties

In this section we prove that the individual cost of a player can be decomposed into two parts: the *v*irtual cost in which the influence of the neighboring players is replaced by their average position, and the *variance* which quantifies how the neighbors are distributed around the average. We show how to exploit the simple structure of the virtual costs to describe the approximate equilibria of the game.

We start with some technical definitions. For every subset of players $X \subseteq \mathcal{N}$ and state $\mathsf{s} = (s_j)_{j\in\mathcal{N}}$, we denote by $\text{avg}_{\mathsf{s}}(X)$ the *average* of their strategies, i.e., $\text{avg}_{\mathsf{s}}(X) = \frac{1}{|X|}\sum_{j\in X} s_j$, and by $\text{var}_{\mathsf{s}}(X)$ their *variance*, i.e., $\text{var}_{\mathsf{s}}(X) = \frac{1}{|X|}\sum_{j\in X}\left(s_j - \text{avg}_{\mathsf{s}}(X)\right)^2$. For every player i and state $\mathsf{s} = (s_j)_{j\in\mathcal{N}}$, let us define the *virtual cost* of i in s as

$$v_i(\mathsf{s}) = \left(\text{avg}_{\mathsf{s}}(N_i(\mathsf{s})) - s_i\right)^2 + \omega\left(o_i - s_i\right)^2. \tag{7.2}$$

The next theorem shows that the gap between the true cost and the virtual cost is given by $\text{var}_{\mathsf{s}}(N_i(\mathsf{s}))$.

Theorem 7.1. *For every player i and state $\mathsf{s} = (s_j)_{j\in\mathcal{N}}$, we have*

$$c_i(\mathsf{s}) = v_i(\mathsf{s}) + \text{var}_{\mathsf{s}}(N_i(\mathsf{s})).$$

Proof. The claim follows by proving that

$$\sum_{j\in N_i(\mathsf{s})} (s_j - s_i)^2 = |N_i(\mathsf{s})|\text{var}_{\mathsf{s}}(N_i(\mathsf{s})) + |N_i(\mathsf{s})|\left(\text{avg}_{\mathsf{s}}(N_i(\mathsf{s})) - s_i\right)^2.$$

Let us denote by x the average strategy of the players in $N_i(\mathsf{s})$, i.e., $x = \text{avg}_{\mathsf{s}}(N_i(\mathsf{s}))$. We have

$$\begin{aligned}
\sum_{j\in N_i(\mathsf{s})} (s_j - s_i)^2 &= \sum_{\substack{j\in N_i(\mathsf{s})\\ x\in[s_i,s_j]}} (|s_j - x| + |x - s_i|)^2 + \sum_{\substack{j\in N_i(\mathsf{s})\\ x\notin[s_i,s_j]}} (|s_j - x| - |x - s_i|)^2 \\
&= \sum_{\substack{j\in N_i(\mathsf{s})\\ x\in[s_i,s_j]}} \left((s_j - x)^2 + (x - s_i)^2 + 2|s_j - x||x - s_i|\right) \\
&\quad + \sum_{\substack{j\in N_i(\mathsf{s})\\ x\notin[s_i,s_j]}} \left((s_j - x)^2 + (x - s_i)^2 - 2|s_j - x||x - s_i|\right) \\
&= \sum_{j\in N_i(\mathsf{s})} \left((s_j - x)^2 + (x - s_i)^2\right) \\
&\quad + 2|x - s_i|\left(\sum_{\substack{j\in N_i(\mathsf{s})\\ x\in[s_i,s_j]}} |s_j - x| - \sum_{\substack{j\in N_i(\mathsf{s})\\ x\notin[s_i,s_j]}} |s_j - x|\right) \\
&= |N_i(\mathsf{s})|\text{var}_{\mathsf{s}}(N_i(\mathsf{s})) + |N_i(\mathsf{s})|(x - s_i)^2 \\
&\quad + 2|x - s_i|\left(\sum_{\substack{j\in N_i(\mathsf{s})\\ x\in[s_i,s_j]}} |s_j - x| - \sum_{\substack{j\in N_i(\mathsf{s})\\ x\notin[s_i,s_j]}} |s_j - x|\right).
\end{aligned} \tag{7.3}$$

We use A to denote the second term of the expression in (7.3), i.e.,

$$A = \left(\sum_{\substack{j \in N_i(\mathsf{s}) \\ x \in [s_i, s_j]}} |s_j - x| \; - \sum_{\substack{j \in N_i(\mathsf{s}) \\ x \notin [s_i, s_j]}} |s_j - x| \right).$$

In the remainder of the proof we show that $A = 0$, thus proving the claim.
Let us first assume $s_i \leq x$. In this case we have

$$\begin{aligned} A &= \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j \geq x}} |s_j - x| \; - \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j < x}} |s_j - x| \\ &= \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j \geq x}} (s_j - x) \; - \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j < x}} (x - s_j) \\ &= \sum_{j \in N_i(\mathsf{s})} s_j \; - Kx = 0. \end{aligned}$$

Otherwise, we obtain

$$\begin{aligned} A &= \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j \leq x}} |s_j - x| \; - \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j > x}} |s_j - x| \\ &= \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j \leq x}} (x - s_j) \; - \sum_{\substack{j \in N_i(\mathsf{s}) \\ s_j > x}} (s_j - x) \\ &= Kx - \sum_{j \in N_i(\mathsf{s})} s_j = 0. \end{aligned}$$

□

Next we show some properties regarding the average and variance:

Proposition 7.2. *For every player i, state s and any strategy s_i', we have*

(a) $\text{avg}_{\mathsf{s}}(N_i(\mathsf{s})) = \text{avg}_{\mathsf{s}}(N_i((\mathsf{s}_{-i}, s_i')))$,

(b) $\text{var}_{\mathsf{s}}(N_i(\mathsf{s})) = \text{var}_{\mathsf{s}}(N_i((\mathsf{s}_{-i}, s_i')))$.

Proof. (a) trivially follows by combining the neighborhood definition and the fact that every player in the neighborhood chooses the same strategy in the two different states. Therefore, since the average does not change, by applying the same argument we obtain that also (b) is true. □

It follows that the strategy that minimizes the virtual costs always also minimizes the real costs of a player.

Proposition 7.3. *For every player i and state s, we have*

$$\arg\min_{y \in S} c_i((\mathsf{s}_{-i}, y)) = \arg\min_{y \in S} v_i((\mathsf{s}_{-i}, y)).$$

Proof. By differentiating $c_i((\mathsf{s}_{-i}, y))$ with respect to y, we obtain

$$c'_i((\mathsf{s}_{-i}, y)) = v'_i((\mathsf{s}_{-i}, y)) + \mathrm{var}'_{\mathsf{s}}(N_i((\mathsf{s}_{-i}, y))) = v'_i((\mathsf{s}_{-i}, y)),$$

where the first equality follows from Theorem 7.1 and the second equality from point (b) of Proposition 7.2. The claim follows.

□

A similar, but weaker statement holds also for approximate equilibria, as stated by the next theorem. This theorem allows us to use the virtual cost instead of the real cost to describe approximate equilibria.

Theorem 7.4. *Given a number $\rho \geq 1$ and a state $\mathsf{s} = (s_j)_{j \in \mathcal{N}}$, if $v_i(\mathsf{s}) \leq \rho \min_{y \in S} v_i((\mathsf{s}_{-i}, y))$ for every player i, then s is a ρ-approximate equilibrium, i.e., $c_i(\mathsf{s}) \leq \rho \min_{y \in S} c_i((\mathsf{s}_{-i}, y))$ for every player i.*

Proof. Let us assume that $v_i(\mathsf{s}) \leq \rho \min_{y \in S} v_i((\mathsf{s}_{-i}, y))$ for every player i. Let $s'_i = \arg\min_{y \in S} v_i(\mathsf{s}_{-i}, y)$. Therefore we have

$$v_i(\mathsf{s}) \leq \rho v_i((\mathsf{s}_{-i}, s'_i)). \tag{7.4}$$

From Proposition 7.3, we know that s'_i is also minimizing c_i, i.e.,

$$s'_i = \arg\min_{y \in S} c_i((\mathsf{s}_{-i}, y)). \tag{7.5}$$

Additionally, by point (b) of Proposition 7.2, we have

$$\mathrm{var}_{\mathsf{s}}(N_i(\mathsf{s})) = \mathrm{var}_{\mathsf{s}}(N_i((\mathsf{s}_{-i}, s'_i))). \tag{7.6}$$

We can conclude that

$$\begin{aligned}
c_i(\mathsf{s}) &= v_i(\mathsf{s}) + \mathrm{var}_{\mathsf{s}}(N_i(\mathsf{s})) && (7.7)\\
&= v_i(\mathsf{s}) + \mathrm{var}_{\mathsf{s}}(N_i((\mathsf{s}_{-i}, s'_i))) && (7.8)\\
&\leq \rho v_i((\mathsf{s}_{-i}, s'_i)) + \mathrm{var}_{\mathsf{s}}(N_i((\mathsf{s}_{-i}, s'_i))) && (7.9)\\
&\leq \rho\Big[v_i((\mathsf{s}_{-i}, s'_i)) + \mathrm{var}_{\mathsf{s}}(N_i((\mathsf{s}_{-i}, s'_i)))\Big] && (7.10)\\
&= \rho c_i((\mathsf{s}_{-i}, s'_i))\\
&= \rho \min_{y \in S} c_i((\mathsf{s}_{-i}, y)), && (7.11)
\end{aligned}$$

where (7.7) follows from Theorem 7.1, (7.8) from (7.6), (7.9) from (7.4), (7.10) from the fact that $\rho \geq 1$ and (7.11) from (7.5).

□

The next lemma characterizes the strategy minimizing the virtual cost.

Lemma 7.5. *For every player i and state $\mathsf{s} = (s_j)_{j \in \mathcal{N}}$, we have* *(a)* $\arg\min_{y\in S} v_i((\mathsf{s}_{-i}, y)) = s_i + \frac{1}{(\omega+1)}\Big[(\mathrm{avg}_{\mathsf{s}}(N_i(\mathsf{s})) - s_i) - \omega(o_i - s_i)\Big]$,

(b) $\min_{y\in S} v_i((\mathsf{s}_{-i}, y)) = v_i(\mathsf{s}) - \frac{1}{(\omega+1)}\Big[(\mathrm{avg}_{\mathsf{s}}(N_i(\mathsf{s})) - s_i) - \omega(o_i - s_i)\Big]^2$.

Proof. Let us denote by x the average strategy of the players in $N_i(\mathsf{s})$, i.e., $x = \mathrm{avg}_{\mathsf{s}}(N_i(\mathsf{s}))$. We have $v_i(\mathsf{s}) = (x - s_i)^2 + \omega(o_i - s_i)^2$. Given any real ξ such that $s_i + \xi \in S$, by assuming that i changes her expressed opinion from s_i to $s_i + \xi$, the new value of v_i would be

$$\begin{aligned} v_i(\mathsf{s}_{-i}, s_i + \xi) &= (x - s_i - \xi)^2 + \omega(o_i - s_i - \xi)^2 && (7.12)\\ &= (x - s_i)^2 + \xi^2 - 2\xi(x - s_i) + \omega(o_i - s_i)^2 + \omega\xi^2 - 2\omega\xi(o_i - s_i)\\ &= v_i(\mathsf{s}) + \xi^2 - 2\xi(x - s_i) + \omega\xi^2 - 2\omega\xi(o_i - s_i)\\ &= v_i(\mathsf{s}) + \xi^2(\omega + 1) - 2\xi\Big[(x - s_i) - \omega(o_i - s_i)\Big], && (7.13)\end{aligned}$$

where (7.12) follows from the definition of virtual cost (7.2) and point (a) of Proposition 7.2. By differentiating (7.13) with respect to ξ, we obtain

$$v_i'(\mathsf{s}_{-i}, s_i + \xi) = 2\xi(\omega + 1) - 2\Big[(x - s_i) - \omega(o_i - s_i)\Big],$$

which implies that $v_i(\mathsf{s}_{-i}, s_i + \xi)$ is minimized when $\xi = \frac{1}{(\omega+1)}\Big[(x - s_i) - \omega(o_i - s_i)\Big]$, from which claim (a) follows. Moreover, by plugging the value of ξ into (7.13), we obtain

$$\begin{aligned} \min_{y\in S} v_i((\mathsf{s}_{-i}, y)) &= v_i(\mathsf{s}) + \frac{1}{(\omega+1)}\Big[(x - s_i) - \omega(o_i - s_i)\Big]^2\\ &\quad - \frac{2}{(\omega+1)}\Big[(x - s_i) - \omega(o_i - s_i)\Big]^2\\ &= v_i(\mathsf{s}) - \frac{1}{(\omega+1)}\Big[(x - s_i) - \omega(o_i - s_i)\Big]^2,\end{aligned}$$

from which claim (b) follows. □

7.4. Existence of Approximate Equilibria

We first show that even the two most natural outcomes yield reasonable approximation factors for different intervals of ω. In particular, we analyze the state in which every player expresses the same opinion, which yields low approximation factors for low values of ω.

Theorem 7.6. *There always exists a ρ-approximate equilibrium where*

$$\rho = \min\left\{(\omega + 1), \left(\frac{\omega + 1}{\omega}\right)\right\} \le 2.$$

Proof. We prove the theorem by considering two candidates: the state $\mathsf{s} = (s, s, \ldots, s)$ in which every player expresses the same arbitrary opinion $s \in S$ and the state $\mathsf{s} = (o_1, o_2, \ldots, o_n)$ in which every player chooses her intrinsic opinion. We show that the first state is a $(\omega + 1)$-approximate equilibrium providing a good approximation for small values of ω, while the second is a $\left(\frac{\omega+1}{\omega}\right)$-approximate equilibrium and hence a good candidate for large values of ω.

Let us first consider the state $\mathsf{s} = (s, s, \ldots, s)$ for any strategy s. For every player i we have

$$v_i(\mathsf{s}) = \left(\text{avg}_{\mathsf{s}}(N_i(\mathsf{s})) - s\right)^2 + \omega\left(o_i - s\right)^2 = \omega\left(o_i - s\right)^2, \tag{7.14}$$

where the second equality follows from the fact that $\text{avg}_{\mathsf{s}}(N_i(\mathsf{s})) = s$, irrespective of the definition of the neighborhood $N_i(\mathsf{s})$. By Lemma 7.5 we obtain

$$\begin{aligned}
\min_{y\in S} v_i((\mathsf{s}_{-i}, y)) &= v_i(\mathsf{s}) - \frac{1}{(\omega+1)}\Big[\left(\text{avg}_{\mathsf{s}}(N_i(\mathsf{s})) - s\right) - \omega\left(o_i - s\right)\Big]^2 \\
&= v_i(\mathsf{s}) - \frac{1}{(\omega+1)}\Big[\omega\left(o_i - s\right)\Big]^2 \\
&= v_i(\mathsf{s}) - \frac{\omega}{(\omega+1)} v_i(\mathsf{s}) = \frac{1}{(\omega+1)} v_i(\mathsf{s}),
\end{aligned} \tag{7.15}$$

where (7.15) follows from (7.14). By combining (7.15) with Theorem 7.4, we obtain that s is a $(\omega + 1)$-approximate equilibrium.

Let us now consider the state $\mathsf{o} = (o_1, o_2, \ldots, o_n)$. For every player i we have

$$v_i(\mathsf{o}) = \left(\text{avg}_{\mathsf{o}}(N_i(\mathsf{o})) - o_i\right)^2 + \omega\left(o_i - o_i\right)^2 = \left(\text{avg}_{\mathsf{o}}(N_i(\mathsf{o})) - o_i\right)^2. \tag{7.16}$$

By Lemma 7.5 we obtain

$$\begin{aligned}
\min_{y\in S} v_i((\mathsf{o}_{-i}, y)) &= v_i(\mathsf{o}) - \frac{1}{(\omega+1)}\Big[\left(\text{avg}_{\mathsf{o}}(N_i(\mathsf{o})) - o_i\right) - \omega\left(o_i - o_i\right)\Big]^2 \\
&= v_i(\mathsf{o}) - \frac{1}{(\omega+1)}\Big[\left(\text{avg}_{\mathsf{o}}(N_i(\mathsf{o})) - o_i\right)\Big]^2 \\
&= v_i(\mathsf{o}) - \frac{1}{(\omega+1)} v_i(\mathsf{o}) = \frac{\omega}{(\omega+1)} v_i(\mathsf{o}),
\end{aligned} \tag{7.17}$$

where (7.17) follows from (7.16). By combining (7.17) with Theorem 7.4, we obtain that o is a $\left(\frac{\omega+1}{\omega}\right)$-approximate equilibrium. □

7.4.1. Nearest-Neighbor Game

Until now, we allowed an arbitrary definition of the neighborhood $N_i(\mathsf{s})$ with the only restriction that it is independent of the current strategy of player i. From now on, the neighborhood corresponds to the the set of $K \geq 1$ players that minimize $|s_j - o_i|$. In [BGM13] the non-existence of pure Nash equilibria in the K-NN model was proven. We consider the interesting special case of $K = 1$, namely the *nearest-neighbor game* and improve the approximation factors of Theorem 7.6.

As a candidate, we consider the state $\mathsf{s} = (s_1, s_2, \ldots, s_n)$ where $s_i = o_i + \frac{\omega+1}{2}(\sqrt{\omega^2 + 2\omega + 5} - \omega - 1)\left(o_{\arg\min_{j \neq i} |o_j - o_i|} - o_i\right)$, that is, the state obtained from o after a simultaneous best-response, scaled by a factor α, of all the players. Informally spoken, each player moves an α-fraction towards her best alternative.

Theorem 7.7. *For $K = 1$, there always exists a ρ-approximate equilibrium where $\rho = \frac{\omega+1}{2\omega}(\omega - \sqrt{\omega^2 + 2\omega + 5} + 3)$.*

Proof. The idea behind state s is that every player moves simultaneously a fraction of α towards her best response, starting at the initial state o. Let $\alpha = \frac{\omega+1}{2}(\sqrt{\omega^2 + 2\omega + 5} - \omega - 1)$, which is chosen in the way to minimize the possible improvements in both of the following extreme cases.

Consider an arbitrary player i. W.l.o.g. we assume that $s_i \geq o_i$ and that $o_j \leq o_{j+1} \, \forall \, j < n$.

We investigate two different cases: First assume that $o_{i+1} - o_i \leq o_{i+2} - o_{i+1}$ or that i is the second-last player ($i + 1 = n$). In this case, player i and player $i + 1$ approach each other with their expressed opinion. Let d be the distance between their intrinsic opinions, formally $d = o_{i+1} - o_i$. For both players, their best response would be a move of $\frac{1}{1+\omega}d$ by Lemma 7.5. Accordingly, our approach now results in a move of $\frac{\alpha}{1+\omega}d$ each. Therefore, the distance between their expressed opinions becomes $(1 - \frac{2\alpha}{1+\omega})d$ and for each player the distance between her expressed and intrinsic opinion becomes $|s_i - o_i| = \frac{\alpha}{1+\omega}d$. Thus, the costs of player i are

$$c_i(\mathsf{s}) = \left(1 - \frac{2\alpha}{1+\omega})d\right)^2 + \omega \left(\frac{\alpha}{1+\omega}d\right)^2. \tag{7.18}$$

The best response move of player i in state s is a deviation of $\delta_i = \frac{1}{1+\omega}((1 - \frac{2\alpha}{1+\omega})d - \omega(\frac{\alpha}{1+\omega}d))$ (cf. Lemma 7.5). Assuming now that player i changes her expressed opinion to $s_i + \delta_i$, her new costs would be

$$\begin{aligned} c_i(\mathsf{s}_{-i}, s_i + \delta_i) &= ((1 - \frac{2\alpha}{1+\omega})d - \delta_i)^2 + \omega(\frac{\alpha}{1+\omega}d + \delta_i)^2 \\ &= ((1 - \frac{2\alpha}{1+\omega})d - \frac{1}{1+\omega}((1 - \frac{2\alpha}{1+\omega})d - \omega(\frac{\alpha}{1+\omega}d)))^2 \\ &\quad + \omega(\frac{\alpha}{1+\omega}d + \frac{1}{1+\omega}((1 - \frac{2\alpha}{1+\omega})d - \omega(\frac{\alpha}{1+\omega}d)))^2. \end{aligned} \tag{7.19}$$

Using (7.18) and (7.19) we can compute the approximation factor by

$$\frac{c_i(\mathsf{s})}{c_i(\mathsf{s}_{-i}, s_i + \delta_i)} = \frac{(\omega + 1)(\alpha^2\omega + 4\alpha^2 - 4\alpha\omega - 4\alpha + \omega^2 + 2\omega + 1)}{\omega(\alpha - \omega - 1)^2}.$$

Using the defined α we get the desired approximation bound

$$\frac{c_i(\mathsf{s})}{c_i(\mathsf{s}_{-i}, s_i + \delta_i)} = \frac{\omega + 1}{2\omega}(\omega - \sqrt{\omega^2 + 2\omega + 5} + 3).$$

Now consider the other case with $o_{i+1} - o_i > o_{i+2} - o_{i+1}$. In this case, the expressed opinions of player i and $i+1$ are right of their intrinsic opinions, so $s_i \geq o_i$ and $s_{i+1} > o_{i+1}$. Let again $d = o_{i+1} - o_i$ be the distance and additionally $e = o_{i+2} - o_{i+1}$ (with $e < d$). For player i, her best response is a move of $\frac{\alpha}{1+\omega}d$. For player $i+1$ it is $\frac{\alpha}{1+\omega}e$. The distance between both expressed opinions is therefore $(1 - \frac{\alpha}{1+\omega})d + \frac{\alpha}{1+\omega}e$. We know consider the worst case with $e = d$ and we have the following costs for player i:

$$\begin{aligned} c_i(\mathsf{s}) &= ((1 - \frac{\alpha}{1+\omega})d + \frac{\alpha}{1+\omega}d)^2 + \omega(\frac{\alpha}{1+\omega}d)^2 \\ &= d^2 + \omega(\frac{\alpha}{1+\omega}d)^2. \end{aligned} \tag{7.20}$$

Using Lemma 7.5 to define the best response gives us $\delta_i = \frac{1}{1+\omega}(d - \omega\frac{\alpha}{1+\omega}d)$. The costs in this new state for player i is given by

$$\begin{aligned} c_i(\mathsf{s}_{-i}, s_i + \delta_i) &= (d - \delta_i)^2 + \omega(\frac{\alpha}{1+\omega}d + \delta)^2 \\ &= (d - \frac{1}{1+\omega}(d - \omega\frac{\alpha}{1+\omega}d))^2 + \omega(\frac{\alpha}{1+\omega}d + \frac{1}{1+\omega}(d - \omega\frac{\alpha}{1+\omega}d))^2. \end{aligned} \tag{7.21}$$

Using (7.20) and (7.21) results in

$$\frac{c_i(\mathsf{s})}{c_i(\mathsf{s}_{-i}, s_i + \delta_i)} = \frac{(\omega+1)(\omega(\alpha^2 + \omega + 2) + 1)}{\omega(\alpha + \omega + 1)^2}$$

together with our α resulting in the same approximation factor. □

For the special case with $\omega_i = 1$ we get the following approximation guarantee.

Corollary 7.8. *For, $K = 1$ and $\omega = 1$, the state $\mathsf{s} = (s_1, s_2, \ldots, s_n)$, with $s_i = o_i + (\sqrt{2} - 1)(o_{\arg\min_{j \neq i} |o_j - o_i|} - o_i)$ for every $i \in \mathcal{N}$, is a $4 - 2\sqrt{2} \approx 1.17$-approximate equilibrium.*

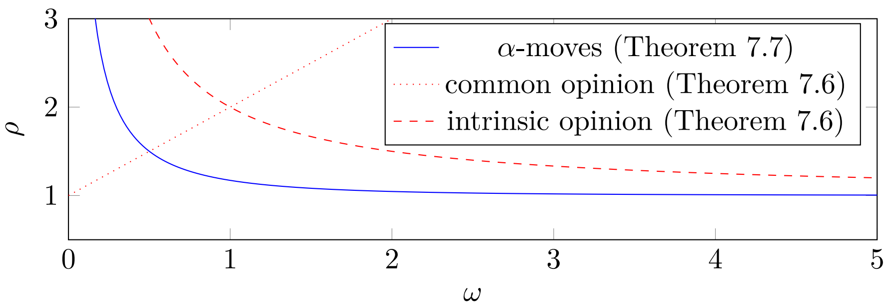

Figure 7.1.: The approximation factor ρ in dependency of ω for the different strategies.

With increasing parameter ω, the approximation factor ρ approaches 1, $\lim_{\omega\to\infty} \rho = 1$. For decreasing ω, the approximation factor approaches ∞, $\lim_{\omega\to 0} \rho = \infty$. Together with the states and approximation factors from Theorem 7.6 we can always guarantee an approximation factor of at most 1.5 for any ω (see Fig. 7.1).

Conclusion

To conclude this chapter we want to highlight two practical use cases of this model. Firstly, if we consider an autonomous system consisting of several independent and distributed agents. One typical task is the consensus problem in different variants in these systems. Our approach of α-moves can be easily implemented as a distributed protocol to solve this task for some settings.

If we now consider the opinion formation process among humans in groups, we see a similar behavior as in our strategy. People are slowly moving their opinions to each other to find a compromise. To fit the behavioral process even better and take different types of people into account, the model requires different ω_i for each player i.

CHAPTER 8

Facility Location Games

Facility Location Games are a classic object of study in economics, mathematics, operations research and game theory. In such games, rational players strategically select a location on some metric space. The utility of a player then depends on the chosen locations of all players. This class of models has a wide range of possible application scenarios which makes it quite powerful: In addition to the representation of physical locations, further typical applications for this basic model are the modeling of political orientation or product differentiations.

One of the landmark models in the area was proposed by Hotelling [Hot90], in which two players want to open competing facilities that offer the same service for the same price. They strategically select a position for their facility on a one-dimensional space, e.g., a line. Every point on the line represents a client who simply buys the service at the nearest facility. The utility of a facility is the induced total profit. In essence, the players compete for market shares in a linear market by strategically selecting locations. This model has been widely discussed and extended in the literature. A drawback in all of these models is the simplicity of the clients. They always use only the nearest facility to be served by and they do not act strategically in any further way.

In this chapter we depart from this classic setup by considering a two-sided Facility Location Game, in which both the facility players and client agents face non-trivial strategic choices that influence each other. We investigate a generalization of Hotelling's classical model, in which the utility of a client agent not only depends on her distance from her selected facility but also on its load, i.e., the number of other clients using the same facility [Koh83; PSV18]. Considering such strategic clients yields a more realistic model with enhanced predictive power for many potential applications.

Contribution and Underlying Work Previous research [Koh83; PSV18] investigated the existence of pure subgame perfect equilibria and shows the existence for arbitrary α in the case of two facilities and for an arbitrary influence of the load, but even number of players but with sufficiently high influence of the load. To the best of our knowledge, we are the first to consider the concept of approximate pure subgame perfect equilibria in this model. By using this concept we make two main contributions: First, we investigate the special case of three facilities which was open, even in the case of the standard Hotelling model without loads. Here, we show a tight approximation factor of 1.28 in the case without load and a general term decreasing with the fraction of the load's influence. Second, we consider an arbitrary number of facilities. In the model with loads, only results for very high load influence factors were shown so far, and nothing for the lower ranges. We give approximation guarantees for two classes of states, either pairs of facilities, or uniformly distributed facilities together with their qualities.

To analyze the subgame among the clients in this two-stage game, we construct a potential function to show the existence of equilibria. Since there is an infinite number of equilibria in this subgame, we show with a transformation lemma how to convert an arbitrary equilibrium to a proper equilibrium maintaining the same properties, especially the effects on the facilities. To show approximation factors for arbitrary numbers of facilities, we solve the linear system of equilibrium equalities with the concept of generalized continued fractions.

The model, analysis, and results presented in this chapter are based on the following work:

> Matthias Feldotto, Pascal Lenzner, Louise Molitor, and Alexander Skopalik. **Two-Sided Facility Location: From Hotelling to Load Balancing**. Unpublished Manuscript. 2018 [Fel+18c]

Outline of This Chapter The chapter starts with an overview of related work on Facility Location Games and different extended models in Section 8.1. Afterwards, we introduce our model of the two-sided Facility Location Game in Section 8.2. We investigate first the client behavior with a potential function in Section 8.3 and afterwards the approximate equilibria for the facilities in Section 8.4. Finally, in Section 8.5, we look at the quality of the considered states.

8.1. Related Work

The original model was proposed by Hotelling [Hot29; Hot90] as part of his research on product differentiation among two firms. Downs extended the research to also consider political competition in similar models [Dow57]. After these landmark papers, a huge number of papers appeared, working with different models and focusing on various research questions in this large field. Here, we focus only on research related to our model and analysis. For an overview of general location analysis we refer to the survey by Revelle and Eiselt [RE05] and for the competitive setting to Eiselt et al. [ELT93] and Brenner [Bre11].

Eaton and Lipsey [EL75] generalized the original model of Hotelling to more than two firms. They showed the existence of pure Nash equilibria for $n > 3$ firms. The principle of minimum differentiation was criticized by d'Aspremont et al. [dGT79], who showed that no pure price equilibrium exists if locations are sufficiently close. They show that under quadratic transportation costs, firms want to maximize differentiation. Osborne and Pitchik [OP86] investigate a different solution concept and characterize a mixed strategy pricing equilibrium for linear transportation costs in which firms have substantial differentiation.

The literature so far always assumes the line as the underlying structure. Other work in this area considers graphs [Pál11; FS16; Fou16], fixed locations on a circle [Sal79], finite sets of locations [NS16; NS17], and optimal interval division [Tia15].

The first model with negative network externalities and also the basis of our work was introduced by Kohlberg [Koh83]. He shows that there are no equilibria in which all firms locate at different locations. Peters et al. [PSV18] continued in this model and show the existence of pure equilibria for different configurations. Additionally, they investigate an asymmetric variant of this model.

Setting different prices is considered by Palma and Leruth [PL89], Navon et al. [NST+95], Grilo et al. [GST01] and Heikkinen [Hei14] under network externalities. Further special models investigate a duopoly [AA13], switching costs [LO13] and similarity of customers [FG05]. Ahlin and Ahlin [AA13] investigate a model with prices and load and show that the principle of minimum differentiation comes closer to holding in the presence of congestion costs.

Recent work in computer science investigates different attraction functions instead of the simple distance to facilities [BT17; FFO16]. [BT17] use a connection to the Shapley value to show the existence of pure equilibria through a potential function argument. Furthermore, it shows the convergence to approximate equilibria.

A related, but different line of research investigates variants of the popular optimization problem of (online) facility location [Mey01] from a game-theoretic perspective [Vet02; FT14; CH10; SM12; Ahn+04; DT07; Ban+15; Ahn+01; GS04]. In these models, the facility and client players coincide. Each of them is allowed to open a facility or use existing ones, depending on the concrete model and the underlying objective functions.

8.2. Model

We model the scenario with a two-stage game (see Section 2.1.2). The game contains two types of players, the set of *facilities* $\mathcal{N}$ and a continuous space of *clients* Z. Although an infinitely number of clients may not exist in reality, it is a good approximation for a very large number of clients compared to the number of facilities. We assume that all facilities offer the same service to the clients and that each client selects exactly one facility to get the service. We consider a two-sided scenario: The facility players will choose a location and their utility will depend on the distribution of the client agents with respect to the chosen locations of all facilities. Thus, the strategic location choices of the facility players will depend on the anticipated client distribution induced by any such choice.

There are n facility players $\mathcal{N} = \{1, 2, \ldots, n\}$, which strategically choose their location in the interval $S = [0, 1]$. We denote any strategy vector for the facility players as $\mathsf{s} = (s_1, s_2, \ldots, s_n)$, where $s_i \in S$ denotes the chosen location of facility player i. For the client agents we consider a continuum of infinitely many clients, which is represented by the interval $Z = [0, 1]$. Every point $z \in Z$ corresponds to a client agent who strategically selects some facility $i \in \mathcal{N}$ to get serviced. If multiple facilities are positioned at the same location, we always assume that the clients are equally distributed among these facilities. Hence, the strategy space of any client $z \in Z$ is the set of facilities $S_z = \mathcal{N} = \{1, 2, \ldots, n\}$, with $t_z \in \mathcal{N}$ being the current strategy selection. We define $\mathsf{f} : \mathsf{S} \times Z \to \mathcal{N}$ as the mapping induced by the clients' facility choices. Given a facility location vector s, a client $z \in Z$ selects the facility $\mathsf{f}(\mathsf{s}, z)$.

Since we eventually want to measure how many clients select a specific strategy, we consider only client choice functions f, where the interval Z is partitioned into $|\mathcal{N}|$ finite sets of intervals $\mathcal{J}_1(\mathsf{f}), \mathcal{J}_2(\mathsf{f}), \ldots, \mathcal{J}_{|\mathcal{N}|}(\mathsf{f})$, where $\mathcal{J}_i(\mathsf{f}) = \{\mathcal{I}_i^1(\mathsf{f}), \ldots, \mathcal{I}_i^{k_i}(\mathsf{f})\}$, for some k_i, with disjoint intervals $\mathcal{I}_i^j \subseteq [0, 1]$ and such that for all clients $z \in \mathcal{I}_i^j(\mathsf{f}) \, \forall j \in \{1, \ldots, k_i\}$ we have $\mathsf{f}(\mathsf{s}, z) = i$. We call such client mappings *measurable mappings*.

Given any measurable client mapping f and the corresponding induced partition into n finite sets of intervals $\mathcal{J}_1(\mathsf{s}, \mathsf{f}), \ldots, \mathcal{J}_n(\mathsf{s}, \mathsf{f})$, we define the *load of facility* i as

$$\ell_i(\mathsf{s}, \mathsf{f}) = \sum_{\mathcal{I}_i^j(\mathsf{s}, \mathsf{f}) \in \mathcal{J}_i(\mathsf{s}, \mathsf{f})} |\mathcal{I}_i^j(\mathsf{s}, \mathsf{f})|,$$

where $|\mathcal{I}_i^j(\mathsf{s}, \mathsf{f})|$ is the length of interval $\mathcal{I}_i^j(\mathsf{s}, \mathsf{f})$.

Given any facility location vector s and any measurable client mapping f, the *cost* C_z of a single client at some point $z \in Z$ is proportional to her distance from her chosen facility $\mathsf{f}(\mathsf{s}, z)$ and the current load $\ell_{\mathsf{f}(\mathsf{s}, z)}(\mathsf{s}, \mathsf{f})$ of that facility. The relative influence of these two objectives is adjusted via the parameter α. Thus, the cost of a client at point $z \in Z$ is

$$C_z(\mathsf{s}, \mathsf{f}) = (1 - \alpha) \cdot |s_{\mathsf{f}(\mathsf{s}, z)} - z| + \alpha \cdot \ell_{\mathsf{f}(\mathsf{s}, z)}(\mathsf{s}, \mathsf{f}).$$

For $\alpha = 0$, where clients simply ignore the facility loads, this corresponds to the client cost function from the original location game model by Hotelling [Hot90] in which clients simply select the closest facility. For $\alpha = 1$, where clients are oblivious to distances, this corresponds to the client cost function in simple load balancing games on identical machines [Nis+07, Chapter 20], in which clients select the least loaded facility.

The *utility* $u_i(\mathsf{s}, \mathsf{f})$ of a facility player i for facility location vector s and some client mapping f equals its induced load, that is,

$$u_i(\mathsf{s}, \mathsf{f}) = \ell_i(\mathsf{s}, \mathsf{f}).$$

8.3. Client Behavior in the Subgame

For any fixed facility location vector s, the clients will strategically select a facility that minimizes their cost. Hence, from all measurable client mappings we are interested in those mappings which are in a certain sense stable. For that, we use the notion of the client equilibrium from Section 2.1.1 and fix the facility strategy vector s.

A client equilibrium f is a measurable client mapping, i.e., for any facility i there exist finitely many intervals of clients that select facility i. We extend this definition to a much stronger notion of mappings, *proper client mappings*, in which all the clients that select some facility i form a single interval of $[0,1]$, formally $|\mathcal{J}_i(\mathsf{s},\mathsf{f})| = 1$ for every facility i. Thus, for any fixed facility location vector s we consider only client mappings f, where the interval $[0,1]$ is partitioned into n closed intervals $\mathcal{I}_1(\mathsf{s},\mathsf{f}),\ldots,\mathcal{I}_n(\mathsf{s},\mathsf{f})$ such that for all clients $z \in \mathcal{I}_i(\mathsf{s},\mathsf{f})$ we have $\mathsf{f}(\mathsf{s},z) = i$. We call such client mappings *proper mappings*. Moreover, by re-naming facilities we can always ensure that $s_1 \leq s_2 \leq \cdots \leq s_n$ which implies that the intervals $\mathcal{I}_1(\mathsf{s},\mathsf{f}),\ldots,\mathcal{I}_n(\mathsf{s},\mathsf{f})$ are consecutive in $[0,1]$ such that $\mathcal{I}_i(\mathsf{s},\mathsf{f}) = [\beta_{i-1},\beta_i]$ with $\beta_0 = 0$ and $\beta_n = 1$. A proper client mapping that is a client equilibrium is called *proper client equilibrium*. Later we show that we can transform any measurable client equilibrium into a proper client equilibrium without changing the utilities for the facilities. Peters et al. [PSV18] also show this transformation in a similar way. However, since our proofs take use of an exact potential function for the game among the clients in the second stage and show a constructive transformation, they provide a better understanding of the clients' behavior.

First, we establish the existence of a client equilibrium for any fixed facility location vector s and we prove that from a facility player's perspective all client equilibria are equivalent. For this, we use the following potential function $\Phi(\mathsf{s},\mathsf{f})$ which is defined for any measurable client mapping f:

$$\Phi(\mathsf{s},\mathsf{f}) = (1-\alpha)\int_0^1 \delta(x,\mathsf{f}(\mathsf{s},x))dx + \alpha\sum_{i=1}^{n}\frac{(\ell_i(\mathsf{s},\mathsf{f}))^2}{2},$$

where $\delta(x,\mathsf{f}(\mathsf{s},x))$ denotes the distance from x to her chosen facility $\mathsf{f}(\mathsf{s},x)$ at location $s_{\mathsf{f}(\mathsf{s},x)}$, i.e., $\delta(x,\mathsf{f}(\mathsf{s},x)) = |s_{\mathsf{f}(\mathsf{s},x)} - x|$.

Theorem 8.1. *For every given facility location vector s there exists a client equilibrium.*

Proof. The potential function is continuously differentiable and convex. The first-order optimality conditions for the function are precisely the equilibrium condition, therefore the local minima correspond to the equilibria. Since the function is continuous and has a compact space it has also a global minim that is an equilibrium.

We consider any measurable client mapping f that minimizes Φ. Clearly, f is also a local minimum of Φ and it follows that f^* is a client equilibrium. □

Next, we show some additional properties for any client equilibrium f. We will use these properties later to show that for any facility location vector s a client equilibrium f^* with $|\mathcal{J}_i(\mathsf{f}^*)| = 1$, exists for all $1 \leq i \leq n$.

Proposition 8.2. *Let* f *be a client equilibrium for some fixed facility location vector* s*. Furthermore, consider any two neighboring intervals* $\mathcal{I}_i^v(\mathsf{s},\mathsf{f}) = [a_i, b_i]$*,* $\mathcal{I}_j^w(\mathsf{s},\mathsf{f}) = [a_j, b_j]$ *with* $z = b_i = a_j$ *in which the clients have different strategies* s_i *and* s_j*. Then*

a. $(1-\alpha)\cdot|s_i - z| + \alpha\cdot\ell_i((\mathsf{s},\mathsf{f})) = (1-\alpha)\cdot|s_j - z| + \alpha\cdot\ell_j((\mathsf{s},\mathsf{f}))$ *and*

b. $(1-\alpha)\cdot|s_i - z'| + \alpha\cdot\ell_i((\mathsf{s},\mathsf{f})) \le (1-\alpha)\cdot|s_{j'} - z'| + \alpha\cdot\ell_{j'}((\mathsf{s},\mathsf{f}))\, \forall, z' \in \mathcal{I}_i^v(\mathsf{s},\mathsf{f}) j \in \mathcal{N}$*.*

c. Let $s_j < s_i$*. If* $s_i \ge b_i$*, then* $s_j \ge b_j$ *and if* $s_j \le a_j$*, then* $s_i \le a_i$*.*

Proof. Parts (a) and (b) follow directly from the equilibrium definition of the client at point z. We show part (c) by contradiction. W.l.o.g. we only consider the first statement. The second statement follows inversely.

Therefore, assume that $s_j < b_j$. We investigate three different cases and show with the help of the equilibrium condition of part (b) the contradiction for two carefully chosen points.

- Let $s_j \in [a_i, b_j)$ and $s_i \ge b_j$. Consider the two points $z_i \in (a_i, \min\{s_j, b_i\})$ and $z_j \in (\max\{s_j, a_j\}, b_j)$. Using the equilibrium conditions of (b) for both points results in $z_j \le s_j$ which is a contradiction to the choices of the points.
- Let $s_j \in [a_i, b_j)$ and $s_i < b_j$ and consider the points $z_i \in (a_i, \min\{s_j, b_i\})$ and $z_j \in (\max\{s_i, a_j\}, b_j)$. With the equilibrium inequality this yields $s_i \le s_j$ and therefore a contradiction to the statement of the proposition.
- Let $s_j < a_j$. Consider the points $z_j \in (\max\{s_i, a_j\}, b_j)$ and $z_i \in (a_i, b_i)$. This again results in a contradiction with the choices of points. □

Now we are ready to show the transformation lemma which allows us the conversion of an arbitrary client equilibrium into a proper client equilibrium with the same loads for all facilities.

Lemma 8.3. *Any client equilibrium* f *for some fixed facility location vector* s *with* $s_1 \le s_2 \le \cdots \le s_n$ *can be transformed into a proper client mapping* $\mathsf{f_s}$ *with the same potential function value and the same loads per facility as in* f*. That is,* $\Phi(\mathsf{s},\mathsf{f}) = \Phi(\mathsf{s},\mathsf{f_s})$ *and* $\ell_i(\mathsf{s},\mathsf{f}) = \ell_i(\mathsf{s},\mathsf{f_s})$ *for all facilities* $i \in \mathcal{N}$*.*

Proof. Since s is fixed during the entire proof, we will omit the reference to s.

We prove the lemma by first providing a single transformation step, which reduces the number of client intervals for some facility i by 1 if $\mathcal{J}_i(\mathsf{f})$ contains more than one client interval. Moreover, we ensure that this transformation step does not change the potential function value. Then we can iteratively apply single transformation steps until we obtain a proper client mapping $\mathsf{f_s}$ where all $\mathcal{J}_i(\mathsf{f_s})$ sets have cardinality one and $\Phi(\mathsf{f_s}) = \Phi(\mathsf{f})$. The latter together with Theorem 8.1 implies that $\mathsf{f_s}$ is a client equilibrium since it also locally minimizes Φ. During this process, we rearrange the client intervals such that $\mathcal{I}_i(\mathsf{f_s}) = [\beta_{i-1}, \beta_i]$ with $0 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_{n-1} \leq \beta_n = 1$.

We process the facilities and their intervals in ascending order, starting with facility 1 and $\mathcal{I}_1^1(\mathsf{f})$. A single transformation step works as follows: Let i be the current facility and $\mathcal{I}_i^v(\mathsf{f}) = [a_i, b_i]$ the current interval. As long as there exists an interval $\mathcal{I}_j^w(\mathsf{f}) = [a_j, b_j]$ with $b_j \leq a_i$ and $s_j > s_i$, we swap the interval $\mathcal{I}_i^v(\mathsf{f})$ with its direct predecessor $\mathcal{I}_k^u(\mathsf{f}) = [a_k, b_k]$ (k can be equal or unequal to j) meaning that the clients in those areas are reassigned. This results in the following borders:

$$\mathcal{I}_i^v(\mathsf{f}') = [a_k, a_k + (b_i - a_i)] \text{ and } \mathcal{I}_k^u(\mathsf{f}') = [a_k + (b_i - a_i), b_i].$$

Let f' be the obtained measurable mapping after the single transformation. It remains to show that $\Phi(\mathsf{f}') = \Phi(\mathsf{f})$. Since the transformation step only shifts client intervals, it follows that it is load neutral, i.e., the loads of all facilities remain unchanged by the transformation. Thus, we are left with proving that

$$\int_0^1 \delta(x, \mathsf{f}(x))dx = \int_0^1 \delta(x, \mathsf{f}'(x))dx$$

holds. Since only clients in the interval $[a_k, b_i]$ are reassigned, it suffices to show that

$$\begin{aligned} \int_{a_k}^{b_i} \delta(x, \mathsf{f}(x))dx &= \int_{a_k}^{b_i} \delta(x, \mathsf{f}'(x))dx \qquad (8.1)\\ \iff \int_{a_k}^{a_i} \delta(x, \mathsf{f}(x))dx + \int_{a_i}^{b_i} \delta(x, \mathsf{f}(x))dx &= \int_{a_k}^{a_k+(b_i-a_i)} \delta(x, \mathsf{f}'(x))dx \\ &\quad + \int_{a_k+(b_i-a_i)}^{b_i} \delta(x, \mathsf{f}'(x))dx \end{aligned}$$

is true. W.l.o.g. assume that $s_i \leq a_i$. From Proposition 8.2 it follows that $s_k \leq a_k$. By our transformation, we have that each client in an interval of length $b_i - a_i$ decreases her distance to her assigned facility i by $b_k - a_k$. Thus, we have

$$\int_{a_k}^{a_k+(b_i-a_i)} \delta(x, \mathsf{f}'(x))dx = \int_{a_i}^{b_i} \delta(x, \mathsf{f}(x))dx - (b_i - a_i)(b_k - a_k). \qquad (8.2)$$

For all clients in $\mathcal{I}_k^u(\mathsf{f})$ we know by Proposition 8.2 that each client is moved away from her assigned facility exactly by the distance $(b_i - a_i)$. Since the length of $\mathcal{I}_k^u(\mathsf{f})$ is $b_k - a_k$ we get

$$\int_{a_k+(b_i-a_i)}^{b_i} \delta(x, \mathsf{f}'(x))dx = \int_{a_k}^{b_k} \delta(x, \mathsf{f}(x))dx + (b_k - a_k)(b_i - a_i). \qquad (8.3)$$

Summing equations (8.2) and (8.3) yields equation (8.1).

We proceed with the next interval $\mathcal{I}_i^{v+1}(\mathsf{f})$ of facility i if it exists and then with the next facility $i+1$ in the same way. Thus, by applying the transformation step repeatedly we will eventually obtain a proper client mapping $\mathsf{f_s}$ with $\Phi(\mathsf{f}) = \Phi(\mathsf{f_s})$ and $\ell_i(\mathsf{f}) = \ell_i(\mathsf{f_s})$ for all facilities $i \in \mathcal{N}$. □

From Lemma 8.3 and its proof the next corollaries directly follow.

Corollary 8.4. *Given a client equilibrium with only one interval for each facility, formally* $|\mathcal{J}_i(s, f)| = 1$ *for every facility* $i \in \mathcal{N}$*. Then, for arbitrary two facilities* i *and* j *with* $s_i \leq s_j$*, also* $z_i \leq z_j \,\forall\, z_i \in I_i(s, f), z_j \in I_j(s, f)$*.*

Corollary 8.5. *All client equilibria with respect to* s *are unique in terms of the induced loads on the facilities.*

Furthermore, we can directly show that there is one unique proper equilibrium for any facility strategy profile s.

Lemma 8.6. *Given a fixed facility location vector* s*, then there always exists exactly one unique proper client equilibrium* f_{s}*.*

Proof. Because of the definition of a proper equilibrium, each facility has one corresponding interval with clients and all intervals are ordered in the same way as the facilities. Therefore, the $n-1$ borders between the intervals $0 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_{n-1} \leq \beta_n = 1$ determine the equilibrium.

Since 8.2 holds at each of these borders, it directly gives a system of linear equations with $n-1$ variables $\beta_1, \ldots, \beta_{n-1}$ and $n-1$ equations:

$$(1-\alpha)|s_i - \beta_i| + \alpha \cdot \ell_i(\mathsf{s}, \mathsf{f}) = (1-\alpha)|s_{i+1} - \beta_i| + \alpha \cdot \ell_{i+1}(\mathsf{s}, \mathsf{f}) \,\forall\, 1 \leq i \leq n-1.$$

This system has one unique solution.

Together with Theorem 8.1 and Lemma 8.3 and 8.6, we can conclude the existence of a unique proper client equilibrium. ☐

This is a somewhat surprising result, since even if the clients can be in many different equilibria, from the facilities' perspective all client equilibria look identical. This last Corollary 8.5 together with Lemma 8.6 allows us to consider only the unique proper equilibrium f_s in the following section as a proxy for all possible client equilibria.

8.4. Existence of Approximate Equilibria

Now we turn our attention to the game among the facility players. Because of Corollary 8.5 and Lemma 8.6, we always assume the clients to be in the unique proper client equilibrium for any facility location vector. This is possible since from a facility's perspective all client equilibria look identical because of identical loads. For any facility location vector s we call the corresponding unique proper client equilibrium f_s the *s-induced* client equilibrium.

Therefore, the client strategy mapping f_s is implicitly given and we omit it in the following definitions: For each facility i let $\mathcal{I}_i(\mathsf{s}) = \mathcal{I}_i(\mathsf{s}, \mathsf{f}_\mathsf{s}) = [\beta_{i-1}, \beta_i]$ be the interval of clients using facility i in this equilibrium with $\beta_0 = 0$ and $\beta_n = 1$. The load of facility $i \in \mathcal{N}$ is given by $\ell_i(\mathsf{s}) = \ell_i(\mathsf{s}, \mathsf{f}_\mathsf{s}) = |\mathcal{I}_i(\mathsf{s})|$, the utility of facility $i \in \mathcal{N}$ by $u_i(\mathsf{s}) = u_i(\mathsf{s}, \mathsf{f}_\mathsf{s}) = \ell_i(\mathsf{s})$. The costs of a client at position z are defined by $C_z(\mathsf{s}) = C_z(\mathsf{s}, \mathsf{f}_\mathsf{s})$.

Before starting with the approximate equilibria, we first focus on pure SPE. For the simple case of $n = 2$, Kohlberg [Koh83] has shown that there exists a SPE for all $\alpha > 0$ with $\mathsf{s} = (\frac{1}{2}, \frac{1}{2})$, extending the result of Hotelling [Hot90] to the model with loads. For $\alpha = 1$, there obviously always exist an infinite number of pure SPE, independent of the facilities' positions. For $\alpha = 0$, Eaton and Lipsey [EL75] have shown the existence of pure SPE for $n = 2, 4, 5$ (a unique equilibrium) and $n \geq 6$ (an infinite number of SPEs) as well as the non-existence of pure equilibria for $n = 3$. For sufficiently high values of α, Peters et al. [PSV18] have proposed a conjecture for the existence with arbitrary $n > 3$. Since our model is almost identical to theirs, we can adopt their Conjecture 1 also to our model. Simulations and computations for small values of n indicate that it is true in general.

Conjecture 8.7. *An SPE exists if and only if $n = 2k$ for some $k \in \mathbb{N}$ and α is sufficiently high. The SPE is unique and given by $s_j = 1 - s_{2k+1-j} = \frac{\frac{\alpha^j - 1}{\alpha - 1}}{2\frac{\alpha^k - 1}{\alpha - 1}} = \frac{\alpha^j - 1}{2(\alpha^k - 1)}$ and $u_j(s) = u_{2k+1-j}(s) = \frac{\alpha^{j-1}}{2\frac{\alpha^k - 1}{\alpha - 1}} = \frac{\alpha^{j-1}(\alpha - 1)}{2(\alpha^k - 1)}$ for all $i \in \{1, \ldots, k\}$.*

Since the restrictions for having pure SPE are very strict, we will investigate approximate pure subgame perfect Nash equilibria in the remainder of this chapter.

8.4.1. Hotelling with Three Facilities

In this section we consider the special but famous original Hotelling model, which corresponds to our model with $\alpha = 0$. It is known from literature [EL75] that for $n = 1$, $n = 2$ and $n > 3$ there always exist a pure subgame perfect Nash equilibrium in contrast to the case of $n = 3$. For $n = 3$ we show a tight approximation guarantee.

Theorem 8.8. *For $\alpha = 0$ and $n = 3$ the game always has a ρ-approximate pure subgame perfect Nash equilibrium with $\rho = \frac{1}{4}(1 + \sqrt{17}) \approx 1.28$.*

Proof. Consider the state $\mathsf{s} = (s_1, \frac{1}{2}, 1 - s_1)$. Because of the equilibrium among the clients, it holds that

$$\begin{aligned} \beta_1 - s_1 &= \frac{1}{2} - \beta_1 \\ \beta_2 - \frac{1}{2} &= s_2 - \beta_2. \end{aligned}$$

So the clients' intervals split at

$$\begin{aligned} \beta_1 &= \frac{1}{4}(1 + 2s_1) \\ \beta_2 &= \frac{1}{4}(3 - 2s_1). \end{aligned}$$

Since players 1 and 3 are equivalent we only consider player 1 w.l.o.g.. The best response of facility 1 is to locate next to facility 2 at $\frac{1}{2} - \varepsilon$ with $\varepsilon > 0$. Hence, facility 1 can improve by an approximation factor of $\rho_1 = \lim_{\varepsilon \to 0} \frac{\frac{1}{2} - \frac{\varepsilon}{2}}{\frac{1}{4}(1+2s_1)} = \frac{2}{1+2s_1}$.

We now consider facility 2 at $\frac{1}{2}$. W.l.o.g. , player 2 moves to the left. Her utility is $u_2(\mathsf{s}) = \beta_2 - \beta_1 = \frac{1}{2} - s_1$. Her best response is to jump to the left of facility 1 at position $s_1 - \varepsilon$. Therefore, facility 2 can improve by a factor of $\rho_2 = \lim_{\varepsilon \to 0} \frac{s_1 - \frac{\varepsilon}{2}}{\frac{1}{2} - s_1} = \frac{2s_1}{1-2s_1}$.

Choosing $s_1 = \frac{1}{4}(-3 + \sqrt{17})$ minimizes the maximum of ρ_1 and ρ_2 and both evaluate to $\rho = \frac{1}{4}(1 + \sqrt{17}) \approx 1.28$. □

We complete the tight characterization by a lower bound:

Theorem 8.9. *For $\alpha = 0$ and $n = 3$ the game does not have a ρ-approximate pure subgame perfect Nash equilibrium with $\rho < \frac{1}{4}(1 + \sqrt{17})$.*

Proof. Consider the state $\mathsf{s} = (s_1, s_2, s_3)$. Assume w.l.o.g. $s_1 \le s_2 \le s_3$. There are three different settings, apart from symmetry, where the facilities can be located. We show that all cases yield equal or worse approximation factors than the strategy profile of Theorem 8.8.

- $s_1 = s_2 = s_3$: According to our model, clients are equally distributed among facilities at the same locations. The clients' intervals split at $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{2}{3}$, so each facility has a utility of $u_i(\mathsf{s}) = \frac{1}{3}$. Each player is equivalent, so we only consider facility 1. W.l.o.g. assume $s_1 \le \frac{1}{2}$. The best response for facility 1 is to move ε closer to 1, resulting in a new $\mathsf{s}' = (\mathsf{s}_{-1}, s_1')$ with $s_1' = s_1 + \varepsilon$ and a utility of $u_i(\mathsf{s}') = 1 - s_1'$. So player 1 receives the highest utility if $\varepsilon \to 0$ and her approximation factor is $\rho_1 = \lim_{\varepsilon \to 0} \frac{1 - s_1 - \frac{\varepsilon}{2}}{\frac{1}{3}} = 3 - 3s_1$, which is minimized if $s_1 = \frac{1}{2}$. However $1.5 > \frac{1}{4}(1 + \sqrt{17})$.

- $s_1 < s_2 = s_3$: Since facilities 2 and 3 are at the same location, they have the same amount of load. So the clients' interval split at $\beta_1 = \frac{s_1 + s_2}{2}$, $\beta_2 = \frac{1}{4}(2 + s_1 + s_2)$. Assume that $0 \le s_1 \le \frac{1}{2}$ and $\frac{1}{2} \le s_2 \le 1$. Otherwise one of the facilities could at least double her utility, changing her strategy to the location $\frac{1}{2}$.

  Facility 1 has a utility of $u_1(\mathsf{s}) = \beta_1 = \frac{s_1 + s_2}{2}$. Her best response is to move ε close to s_2, resulting in $\mathsf{s}' = (\mathsf{s}_{-1}, s_1')$ with $s_1' = s_2 - \varepsilon$ and an utility of $u_1(\mathsf{s}') = s_2 - \varepsilon$. So player 1 receives the highest utility with $\varepsilon \to 0$ and her approximation factor is $\rho_1 = \lim_{\varepsilon \to 0} \frac{s_2 - \frac{\varepsilon}{2}}{\frac{s_1 + s_2}{2}} = \frac{2s_2}{s_1 + s_2}$. Since $\rho_1 < \frac{1}{4}(1 + \sqrt{17})$ it follows

$$\frac{7s_2 - \sqrt{17}s_2}{1 + \sqrt{17}} \le s_1 \le \frac{1}{2} \tag{8.4}$$

$$\frac{1}{2} \le s_2 \le \frac{-1 - \sqrt{17}}{-14 + 2\sqrt{17}}. \tag{8.5}$$

Facilities 2 and 3 are equivalent, so we only consider 2. Facility 2 has a utility of $u_2(\mathsf{s}) = \frac{1}{4}(2 - s_1 - s_2)$. A possible strategy change for player 2 could be to locate left, i.e., closer to 0, and she then has a utility of $u_2(\mathsf{s}') = s_1 - \varepsilon$ and an improvement of $\rho_2 = \lim_{\varepsilon \to 0} \frac{s_1 - \frac{\varepsilon}{2}}{\frac{1}{4}(2-s_1-s_2)} = -\frac{4s_1}{-2+s_1+s_2}$. It follows from $\rho_2 < \frac{1}{4}(1+\sqrt{17})$, (8.4) and (8.5)

$$\frac{7s_2 - \sqrt{17}s_2}{1+\sqrt{17}} \leq s_1 \leq \frac{2 + 2\sqrt{17} - s_2 - \sqrt{17}s_2}{17 + \sqrt{17}} \tag{8.6}$$

$$\frac{1}{2} \leq s_2 \leq \frac{-9-\sqrt{17}}{-30+2\sqrt{17}}. \tag{8.7}$$

The other opportunity for 2 is to locate right from player 3, i.e., closer to 1. This strategy change results in a utility of $u_2(\mathsf{s}'') = 1 - (s_2 + \varepsilon)$ and leads to an approximation factor of $\rho_3 = \lim_{\varepsilon \to 0} \frac{1-(s_2+\frac{\varepsilon}{2})}{\frac{1}{4}(2-s_1-s_2)} = -\frac{4(-1+s_2)}{-2+s_1+s_2}$. However, $\rho_3 < \frac{1}{4}(1+\sqrt{17})$, together with (8.6) and (8.7) does not have a valid solution.

- $s_1 < s_2 < s_3$: To avoid the obvious case that one of the facilities can double her current utility, we assume $0 < s_1 < \frac{1}{2}$ and $\frac{1}{2} \leq s_2 < s_3 < 1$. The clients' intervals split at $\beta_1 = \frac{s_1+s_2}{2}$ and $\beta_2 = \frac{s_2+s_3}{2}$.

  Facility 1 has a utility of $u_1(\mathsf{s}) = \beta_1 = \frac{s_1+s_2}{2}$. As in the previous case her best response is a move to $s_1' = s_2 - \varepsilon$. So (8.4) and (8.5) have to hold.

  Facility 2 has a utility of $u_2(\mathsf{s}) = \frac{1}{2}(s_3 - s_1)$. A possible strategy change for player 2 could be to locate left from 1, i.e., closer to 0, and receive a utility of $u_2(\mathsf{s}') = s_1 - \varepsilon$ and an improvement of $\rho_1 = \lim_{\varepsilon \to 0} \frac{s_1 - \frac{\varepsilon}{2}}{\frac{1}{2}(s_3-s_1)} = \frac{2s_1}{s_3-s_1}$. From $\rho_1 < \frac{1}{4}(1+\sqrt{17})$ it follows

$$s_1 \leq \frac{s_3 + \sqrt{17}s_3}{9+\sqrt{17}}. \tag{8.8}$$

  The other opportunity for 2 is to locate right from 3, i.e., closer to 1. This strategy change results in $u_2(\mathsf{s}'') = 1 - (s_3 + \frac{\varepsilon}{2})$ and leads to an approximation factor of $\rho_2 = \lim_{\varepsilon \to 0} \frac{1-(s_3+\frac{\varepsilon}{2})}{\frac{1}{2}(s_3-s_1)} = \frac{2s_1+s_3}{s_3-s_1}$. It follows from $\rho_2 < \frac{1}{4}(1+\sqrt{17})$ and (8.8)

$$s_1 \leq \frac{-8+9_f3+\sqrt{17}s_3}{1+\sqrt{17}} \quad \text{and} \quad \frac{8}{9+\sqrt{17}} < s_3 \leq \frac{9+\sqrt{17}}{10+2\sqrt{17}} \tag{8.9}$$

$$\text{or} \quad s_1 \leq \frac{s_3+\sqrt{17}s_3}{9+\sqrt{17}} \quad \text{and} \quad \frac{9+\sqrt{17}}{10+2\sqrt{17}} < s_3. \tag{8.10}$$

  Facility 3 has a utility of $u_3(\mathsf{s}) = \frac{1}{2}(2-s_2-s_3)$. A possible strategy change for player 3 could be to move closer to 2 and receive a utility of $u_3(\mathsf{s}') = 1-(s_2+\frac{\varepsilon}{2})$

and an improvement of $\rho_3 = \lim_{\varepsilon \to 0} \frac{1-(s_2+\frac{\varepsilon}{2})}{\frac{1}{2}(2-s_2-s_3)} = \frac{2(-1+s_2)}{-2+s_2+s_3}$. It follows from $\rho_3 < \frac{1}{4}(1+\sqrt{17})$

$$s_3 \leq \frac{-5+3\sqrt{17}}{2+2\sqrt{17}} \tag{8.11}$$

$$\text{or} \quad s_2 \geq \frac{-6+2\sqrt{17}-s_3-\sqrt{17}s_3}{-7+\sqrt{17}} \text{ and } s_3 > \frac{-5+3\sqrt{17}}{2+2\sqrt{17}}. \tag{8.12}$$

The other opportunity for player 3 is to locate left from facility 1, i.e., closer to 0. The strategy change results in $u_3(\mathsf{s}'') = s_1 - \frac{\varepsilon}{2}$ and leads to an approximation factor of $\rho_4 = \lim_{\varepsilon \to 0} \frac{s_1 - \frac{\varepsilon}{2}}{\frac{1}{2}(2-s_2-s_3)} = -\frac{2s_1}{-2+s_2+s_3}$. However, $\rho_4 < \frac{1}{4}(1+\sqrt{17})$, together with (8.4), (8.5), (8.9), (8.10), (8.11) and (8.12) does not have a valid solution. □

8.4.2. Three Facilities and an Arbitrary α

We generalize the Hotelling result from Theorem 8.8 to an arbitrary $0 \leq \alpha \leq 1$:

Theorem 8.10. *For $0 \leq \alpha \leq 1$ and $n = 3$ the game always has a ρ-approximate pure subgame perfect Nash equilibrium with $\rho = \frac{1-\alpha^2+\sqrt{17+\alpha(16+2\alpha+\alpha^3)}}{4-2(-2+\alpha)\alpha}$.*

Proof. Consider the state $\mathsf{s} = (s_1, \frac{1}{2}, 1-s_1)$. The clients' intervals split at β_1 and β_2 and because of the client equilibrium it holds that

$$\begin{aligned}(1-\alpha)(\beta_1 - s_1) + \alpha\beta_1 &= (1-\alpha)(\frac{1}{2} - \beta_1) + \alpha(\beta_2 - \beta_1)\\ \beta_2 &= 1 - \beta_1.\end{aligned}$$

Thus,

$$\beta_1 = \frac{1+\alpha+2s_1-2\alpha s_1}{4+2\alpha} \text{ and } \beta_2 = \frac{3+\alpha-2s_1+2\alpha s_1}{4+2\alpha}.$$

Since players 1 and 3 are equivalent we only consider player 1. The best response of player 1 is to locate at β_1'. So it follows from

$$\begin{aligned}\alpha\beta_1' &= (1-\alpha)(\frac{1}{2} - \beta_1') + \alpha(\beta_2' - \beta_1')\\ (1-\alpha)(\beta_2' - \frac{1}{2}) + \alpha(\beta_2' - \beta_1') &= (1-\alpha)|(1-s_1) - \beta_2'| + \alpha(1-\beta_2')\end{aligned}$$

that

$$\beta_1' = \frac{2+\alpha-2\alpha s_1+\alpha^2(-1+2s_1)}{4+4\alpha-2\alpha^2} \beta_2' = \frac{3+3\alpha+2\alpha^2(-1+s_1)-2s_1}{4+4\alpha-2\alpha^2}.$$

We can easily verify, after choosing s_1, that $s'_1 > \frac{1}{2}$ is not a best response by solving the system of equations. Thus, facility 1 (and 3, respectively) can improve by a factor of $\rho_1 = \frac{2(2+\alpha)(2+\alpha-2\alpha s_1+\alpha^2(-1+2s_1))}{(4+4\alpha-2\alpha^2)(1+\alpha+2s_1-2\alpha s_1)}$.

We now consider facility 2. W.l.o.g. player 2 moves ε to the left, so she gets closer to 0. Her utility is $u_2(\mathsf{s}) = \beta_2 - \beta_1 = \frac{1+2(-1+\alpha)s_1}{2+\alpha}$.

For her best response s'_2, we consider two cases:

- $s'_2 \leq s_1$: In this case, the utility of facility 2 equals the length of the first interval ending at point β_1. So, as discussed for player 1, the best response is $s'_2 = \beta'_1$. Hence, $s'_2 = \frac{\alpha+2s_1-2\alpha s_1}{2+2\alpha-\alpha^2}$ and facility 2 can improve by $\rho_2 = \frac{(2+\alpha)(\alpha+2s_1-2\alpha s_1)}{(2+2\alpha-\alpha^2)(1+2(-1+\alpha)s_1)}$.
- $s'_2 > s_1$: Note that $s'_2 < \frac{1}{2}$ is symmetric to $1 - s'_2$. So we have

$$(1-\alpha)|\beta'_1 - s_1| + \alpha\beta'_1 = (1-\alpha)(\frac{1}{2} - \varepsilon - \beta'_1) + \alpha(\beta'_2 - \beta'_1)$$
$$(1-\alpha)(\beta'_2 - (\frac{1}{2} - \varepsilon)) + \alpha(\beta'_2 - \beta'_1) = (1-\alpha)((1-s_1) - \beta'_2) + \alpha(1 - \beta'_2).$$

The utility $u_2(\mathsf{s}') = \beta'_2 - \beta'_1 = \frac{1-2\varepsilon+\alpha^2(1-2\varepsilon-2s_1)-2s_1+\alpha(-3+4\varepsilon+4s_1)}{(-4+\alpha)\alpha}$ becomes larger the greater $\varepsilon > 0$ gets. In particular, it is better for player 2 to be at the same location as player 1 than to be between players 1 and 3.

Choosing $s_1 = \frac{-3+(\alpha-4)\alpha+\sqrt{17+\alpha(16+2\alpha+\alpha^3)}}{4(a-1)^2}$ minimizes the maximum of ρ_1 and ρ_2 and both evaluate, for $0 < \alpha < 1$, to

$$\rho = \frac{1-\alpha^2+\sqrt{17+\alpha(16+2\alpha+\alpha^3)}}{4-2(-2+\alpha)\alpha}. \qquad \square$$

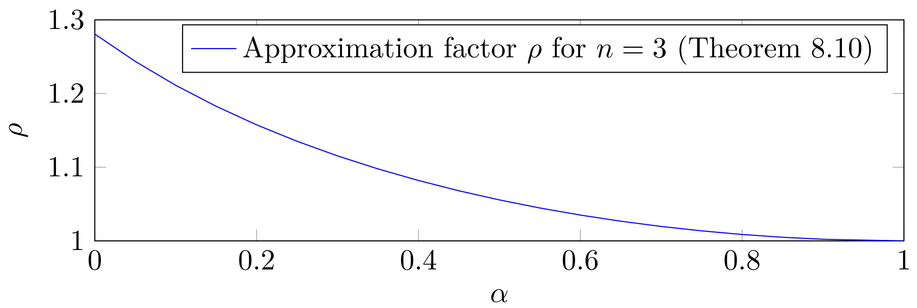

Figure 8.1.: Approximation factor ρ for $n = 3$ as a function of the parameter α.

The approximation factor corresponds with the result from Theorem 8.8 for $\alpha = 0$ and approaches 0 if α increases (see Fig. 8.1). The case of $\alpha = 1$ matches the simple load balancing games on identical machines for which obviously pure subgame perfect Nash equilibria exist.

8.4.3. Uniformly Distributed Facilities

In the remainder of this chapter we focus on games with an arbitrary number of facilities. We now consider the social optimum of the game, which is a uniform distribution of all facilities on the line, as a promising candidate. Thus, $\mathsf{s} = (s_1, \dots, s_n)$ with $s_i = \frac{2i-1}{2n} \,\forall\, i \in \{1, \dots, n\}$.

To bound the approximation factor of this state, we have to identify the player with the highest possible improvement factor. Intuitively, this is the case for the two border players 1 and n since they have the largest area around them with low influence of other players. The best new location for a border player is still in this area with an adjustment towards the next neighbor. A move to another position between the other players decreases her costs because a given area has to be shared among more players. To analytically show this observation, a new idea is needed to measure this influence. Otherwise, the number of required comparisons grows exponentially with the number of players in the game.

To overcome this issue and still have a strong argumentation we manually computed the possible improvement moves in this state for all participating players for games with $n \in \{4, \dots, 10\}$ players. By the additional execution of simulations for larger values of n we derive the following conjecture.

Conjecture 8.11. *Given a game with n facilities and the state $\mathsf{s} = (s_1, \dots, s_n)$ with $s_i = \frac{2i-1}{2n} \,\forall\, i \in \{1, \dots, n\}$., one of the border players (1 or n) has the highest possible improvement by moving towards the middle at the new interval border β'_1 or β'_{n-1}.*

With the help of generalized continued fractions, define

$$\tilde{K}^m := \operatorname*{K}_{j=1}^{m} \frac{-\alpha^2/4}{1}$$

and

$$\psi^{\text{opt}}_{n,\alpha} = \frac{n}{1 + \frac{2}{1+\alpha}\tilde{K}^{n-2}} \cdot \left(\frac{1-\alpha}{1+\alpha}\frac{3}{2n} + \sum_{k=2}^{n-1} \frac{1-\alpha}{1+\alpha}\frac{2k}{n} \prod_{j=n-k}^{n-2} \left(-\frac{2}{\alpha}\tilde{K}^j \right) + \frac{\alpha}{1+\alpha} \prod_{j=1}^{n-2} \left(-\frac{2}{\alpha}\tilde{K}^j \right) \right).$$

Using the conjecture and the definition of $\psi^{\text{opt}}_{n,\alpha}$ we can state the following approximation guarantee for arbitrary n.

Theorem 8.12. *Assuming that Conjecture 8.11 holds, for $0 < \alpha < 1$ and $n > 3$ facilities, the game always has a ρ-approximate pure subgame perfect Nash equilibrium with $\rho = \psi^{opt}_{n,\alpha}$.*

Proof. Consider the state $\mathsf{s} = (s_1, \dots, s_n)$ with $s_i = \frac{2i-1}{2n} \,\forall\, i \in \{1, \dots, n\}$. The clients' intervals split at $\beta_i = \frac{i}{n} \,\forall\, i \in \{1, \dots, n-1\}$, so each facility i has a utility of $u_i(\mathsf{s}) = \frac{1}{n}$.

Using Conjecture 8.11, we only consider player 1 with a move to the right at her new interval border β'_1. This enables us to omit the absolute values since we can directly sort borders and facility positions in the equilibrium conditions. The following system of linear equations characterizes the new state $\mathsf{s}' = (\mathsf{s}_{-1}, s'_1)$:

$$
\begin{aligned}
(1-\alpha)\,(\beta'_1 - s'_1) + \alpha(\beta'_1 - \beta'_0) &= (1-\alpha)\,(s_2 - \beta'_1) + \alpha(\beta'_2 - \beta'_1)\\
(1-\alpha)\,(\beta'_i - s_i) + \alpha(\beta'_i - \beta'_{1-1}) &= (1-\alpha)\,(s_{i+1} - \beta'_i) + \alpha(\beta'_{i+1} - \beta'_i)\\
&\quad \forall\, i \in \{2, \dots, n-1\}.
\end{aligned}
$$

Solving these equations to $\beta'_i \,\forall\, i \in \{1, \dots, n-1\}$ and setting $\beta'_0 = 0, \beta'_n = 1, s'_1 = \beta'_1$ results in:

$$
\begin{aligned}
\beta'_1 &= \frac{\alpha}{1+\alpha}\beta'_2 + \frac{1-\alpha}{1+\alpha}s_2\\
\beta'_i &= \frac{\alpha}{2}\beta'_{i-1} + \frac{\alpha}{2}\beta'_{i+1} + \frac{1-\alpha}{2}s_i + \frac{1-\alpha}{2}s_{i+1}\,\forall\, i \in \{2, \dots, n-2\}\\
\beta'_{n-1} &= \frac{\alpha}{2}\beta'_{n-2} + \frac{\alpha}{2} + \frac{1-\alpha}{2}s_{n-1} + \frac{1-\alpha}{2}s_n.
\end{aligned}
$$

We write the system of linear equations as an augmented matrix as follows

$$
\begin{pmatrix}
1 & -\frac{\alpha}{1+\alpha} & & & & & & \frac{1-\alpha}{1+\alpha}s_2\\
-\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & & \frac{1-\alpha}{2}s_2 + \frac{1-\alpha}{2}s_3\\
& -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & \frac{1-\alpha}{2}s_3 + \frac{1-\alpha}{2}s_4\\
& & & \ddots & & & & \vdots\\
& & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & \frac{1-\alpha}{2}s_{n-3} + \frac{1-\alpha}{2}s_{n-2}\\
& & & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & \frac{1-\alpha}{2}s_{n-2} + \frac{1-\alpha}{2}s_{n-1}\\
& & & & & -\frac{\alpha}{2} & 1 & \frac{1-\alpha}{2}s_{n-1} + \frac{1-\alpha}{2}s_n + \frac{\alpha}{2}
\end{pmatrix}
$$

We can now apply the Gaussian elimination (starting from the bottom line and going upwards). To improve readability we omit the right-hand side (so the last column in the matrix) for the moment. We start with the last three rows.

$$
\begin{pmatrix}
1 & -\frac{\alpha}{1+\alpha} & & & & & \\
-\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & \\
& -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & \\
& & & \ddots & & & \\
& & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & \\
& & & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2}\\
& & & & & -\frac{\alpha}{2} & 1
\end{pmatrix}
\begin{array}{l}
\\ \\ \\ \\ \\
\longleftarrow\!\!\rceil^{+}\\
\mid \cdot\,\gamma_{n-1}\ \rfloor
\end{array}
$$

$$
\text{with } \gamma_{n-1} = \frac{\alpha}{2} = \frac{-2}{\alpha}\tilde{K}^1
$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & -\frac{\alpha}{1+\alpha} & & & & & \\ -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & \\ & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & \\ & & & \ddots & & & \\ & & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & \\ & & & & -\frac{\alpha}{2} & 1-\frac{\alpha^2}{4} & 0 \\ & & & & & -\frac{\alpha}{2} & 1 \end{pmatrix} \begin{matrix} \longleftarrow\!\!\urcorner^{+} \\ \mid \cdot\,\gamma_{n-2} \lrcorner \end{matrix}$$

$$\text{with } \gamma_{n-2} = \frac{-2}{\alpha}\tilde{K}^2$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & -\frac{\alpha}{1+\alpha} & & & & & \\ -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & \\ & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & \\ & & & \ddots & & & \\ & & & -\frac{\alpha}{2} & 1-\frac{\alpha^2}{4}\frac{1}{1-\frac{\alpha^2}{4}} & 0 & \\ & & & & -\frac{\alpha}{2} & 1-\frac{\alpha^2}{4} & 0 \\ & & & & & -\frac{\alpha}{2} & 1 \end{pmatrix}$$

For two arbitrary rows i and $i+1$:

$$\begin{pmatrix} 1 & -\frac{\alpha}{1+\alpha} & & & & & \\ -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & \\ & & \ddots & & & & \\ & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & \\ & & & -\frac{\alpha}{2} & 1+\tilde{K}^{n-(i+1+1)} & 0 & \\ & & & & \ddots & & \\ & & & & -\frac{\alpha}{2} & 1-\frac{\alpha^2}{4} & 0 \\ & & & & & -\frac{\alpha}{2} & 1 \end{pmatrix} \begin{matrix} \longleftarrow\!\!\urcorner^{+} \\ \mid \cdot\,\gamma_{i+1} \lrcorner \end{matrix}$$

$$\text{with } \gamma_{i+1} = \frac{-2}{\alpha} \cdot \tilde{K}^{n-(i+1)}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & -\frac{\alpha}{1+\alpha} & & & & & & \\ -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & & \\ & & & \ddots & & & & \\ & & -\frac{\alpha}{2} & 1+\tilde{K}^{n-(i+1)} & 0 & & & \\ & & & -\frac{\alpha}{2} & 1+\tilde{K}^{n-(i+1+1)} & 0 & & \\ & & & & \ddots & & & \\ & & & & -\frac{\alpha}{2} & 1-\frac{\alpha^2}{4} & 0 \\ & & & & & -\frac{\alpha}{2} & 1 \end{pmatrix}$$

For the top three rows we have:

$$\begin{pmatrix} 1 & -\frac{\alpha}{1+\alpha} & & & & & & \\ -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & & \\ & -\frac{\alpha}{2} & 1+\tilde{K}^{n-4} & 0 & & & & \\ & & & \ddots & & & & \\ & & & -\frac{\alpha}{2} & 1+\tilde{K}^{2} & 0 & & \\ & & & & -\frac{\alpha}{2} & 1+\tilde{K}^{1} & 0 \\ & & & & & -\frac{\alpha}{2} & 1 \end{pmatrix} \begin{matrix} \\ \leftarrow\!\urcorner^{+} \\ \mid \cdot\gamma_3 \lrcorner \\ \\ \\ \\ \\ \end{matrix}$$

$$\text{with } \gamma_3 = \frac{-2}{\alpha} \cdot \tilde{K}^{n-3}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & -\frac{\alpha}{1+\alpha} & & & & & & \\ -\frac{\alpha}{2} & 1+\tilde{K}^{n-3} & 0 & & & & & \\ & -\frac{\alpha}{2} & 1+\tilde{K}^{n-4} & 0 & & & & \\ & & & \ddots & & & & \\ & & & -\frac{\alpha}{2} & 1+\tilde{K}^{2} & 0 & & \\ & & & & -\frac{\alpha}{2} & 1+\tilde{K}^{1} & 0 \\ & & & & & -\frac{\alpha}{2} & 1 \end{pmatrix} \begin{matrix} \leftarrow\!\urcorner^{+} \\ \mid \cdot\gamma_2 \lrcorner \\ \\ \\ \\ \\ \\ \end{matrix}$$

$$\text{with } \gamma_2 = \frac{-4}{\alpha(1+\alpha)} \cdot \tilde{K}^{n-2}$$

$$\Downarrow$$

$$\begin{pmatrix} 1+\frac{2}{\alpha+1}\tilde{K}^{n-2} & 0 & & & & & \\ -\frac{\alpha}{2} & 1+\tilde{K}^{n-3} & 0 & & & & \\ & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & \\ & & & \ddots & & & \\ & & & -\frac{\alpha}{2} & 1+\tilde{K}^{2} & 0 & \\ & & & & -\frac{\alpha}{2} & 1+\tilde{K}^{1} & 0 \\ & & & & & -\frac{\alpha}{2} & 1 \end{pmatrix}$$

Now we consider the right-hand side. Since we are only interested in the value of β_1', we focus on the first line. By applying all the operations again to the right-hand side, we end up in the following equation:

$$\begin{aligned} 1+\frac{2}{\alpha+1}\tilde{K}^{n-2}\beta_1' =& \frac{1-\alpha}{1+\alpha}s_2 + \sum_{i=2}^{n-2}\left(\frac{1-\alpha}{2}s_{i+1} + \frac{1-\alpha}{2}s_i\right)\prod_{j=2}^{i}(\gamma_j) \\ &+ \left(\frac{1-\alpha}{2}s_{n-1} + \frac{1-\alpha}{2}s_n + \frac{\alpha}{2}\right)\prod_{j=2}^{n-1}(\gamma_j) \end{aligned}$$

We now use the definition of $s_i = \frac{2i-1}{2n}\,\forall\, i \in \{1,\ldots,n\}$ and $\gamma_i = \frac{-2}{\alpha}\cdot\tilde{K}^{n-(i+1)}\,\forall\, i \in \{3,\ldots,n-1\}$ and $\gamma_2 = \frac{-4}{\alpha(1+\alpha)}\cdot\tilde{K}^{n-2}$.

$$\begin{aligned} 1+\frac{2}{\alpha+1}\tilde{K}^{n-2}\beta_1' =& \frac{1-\alpha}{1+\alpha}\frac{3}{2n} + \sum_{i=2}^{n-2}\left(\frac{1-\alpha}{2}\frac{2(i+1)-1}{2n} + \frac{1-\alpha}{2}\frac{2i-1}{2n}\right) \\ & \frac{-4}{\alpha(1+\alpha)}\cdot\tilde{K}^{n-2}\prod_{j=3}^{i}\left(\frac{-2}{\alpha}\cdot\tilde{K}^{n-j}\right) \\ &+ \left(\frac{1-\alpha}{2}\frac{2(n-1)-1}{2n} + \frac{1-\alpha}{2}\frac{2n-1}{2n} + \frac{\alpha}{2}\right) \\ & \frac{-4}{\alpha(1+\alpha)}\cdot\tilde{K}^{n-2}\prod_{j=3}^{n-1}\left(\frac{-2}{\alpha}\cdot\tilde{K}^{n-j}\right) \end{aligned}$$

This results in the interval border (written with generalized continued fractions)

$$\begin{aligned} \beta_1' =& \frac{1}{1+\frac{2}{1+\alpha}\tilde{K}^{n-2}}\cdot\left(\frac{1-\alpha}{1+\alpha}\frac{3}{2n} + \sum_{k=2}^{n-1}\frac{1-\alpha}{1+\alpha}\frac{2k}{n}\prod_{j=n-k}^{n-2}\left(-\frac{2}{\alpha}\tilde{K}^{j}\right)\right. \\ &\left. + \frac{\alpha}{1+\alpha}\prod_{j=1}^{n-2}\left(-\frac{2}{\alpha}j\right)\right). \end{aligned}$$

Since we consider the first facility, we have $u_1(\mathsf{s}') = \beta_1'$. Together with $u_1(\mathsf{s}) = \frac{1}{n}$ and Conjecture 8.11 we get

$$\rho = \frac{u_1(\mathsf{s}')}{u_1(\mathsf{s})} = \frac{\beta_1'}{\frac{1}{n}}\frac{n}{1+\frac{2}{1+\alpha}\tilde{K}^{n-2}}\cdot\left(\frac{1-\alpha}{1+\alpha}\frac{3}{2n} + \sum_{k=2}^{n-1}\frac{1-\alpha}{1+\alpha}\frac{2k}{n}\prod_{j=n-k}^{n-2}\left(-\frac{2}{\alpha}j\right)\right.$$

$$+\frac{\alpha}{1+\alpha}\prod_{j=1}^{n-2}\left(-\frac{2}{\alpha}\tilde{K}^j\right)\Bigg)$$

which equals $\psi_{n,\alpha}^{\text{opt}}$ by definition and finishes the proof. □

Using mathematical computing systems we derive a closed form of $\psi_{n,\alpha}^{\text{opt}}$ with

$$\psi_{n,\alpha}^{\text{opt}} = \frac{-\frac{2a^2}{\frac{2\sqrt{1-a^2}}{\left(\frac{-a^2+2\sqrt{1-a^2}+2}{a^2}\right)^n-1}+\sqrt{1-a^2}-1}+a-1}{2(a+1)\left(1-\frac{a^2}{(a+1)\left(\sqrt{1-a^2}\left(\frac{2}{\left(\frac{-a^2+2\sqrt{1-a^2}+2}{a^2}\right)^{n-2}-1}+1\right)+1\right)}\right)}.$$

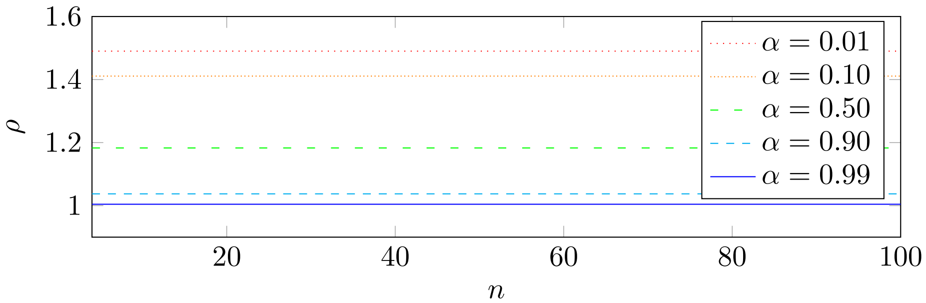

Figure 8.2.: Approximation factor ρ as a function of n.

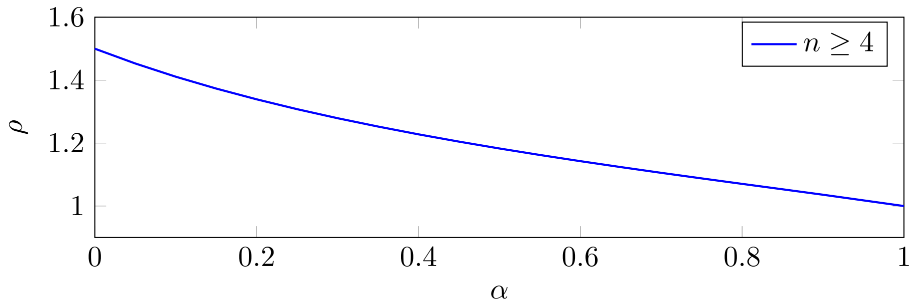

Figure 8.3.: Approximation factor ρ as a function of α.

Evaluating $\psi_{n,\alpha}^{\text{opt}}$ for different fixed values of α (see Fig. 8.2) we see that the factor is almost independent of the number of facilities.

Therefore we fix $n = 4$ with

$$\psi_{4,\alpha}^{\text{opt}} = \frac{a^3+7a^2-4a-12}{2\left(a^3+3a^2-4a-4\right)}$$

and $n = 6$ with

$$\psi^{\text{opt}}_{6,\alpha} = \frac{a^5 + 11a^4 - 12a^3 - 52a^2 + 16a + 48}{2a^5 + 10a^4 - 24a^3 - 40a^2 + 32a + 32}.$$

The approximation factor decreases monotonously in α and it is approaching 1 (see Fig. 8.3).

8.4.4. Pairing of Facilities

We now consider the state $s_{2i-1} = s_{2i} = \frac{2i-1}{2k} \, \forall\, i \in \{1, \ldots, k\}$, which is a pure subgame perfect Nash equilibrium for $\alpha = 0$ and $\alpha = 1$ (see [EL75]) and a second promising candidate for good approximation factors for arbitrary α.

The argumentation behind Conjecture 8.11 also holds for this state. Again by manually computing the possible improvement moves in this state for all participating players for games with $n \in \{4, \ldots, 10\}$ players and by the execution of simulations for larger values of n we derive the following conjecture.

Conjecture 8.13. *Given a game with $n > 3$ facilities and $n = 2k$ with $k \in \mathbb{N}$ and the state $\mathbf{s} = (s_1, \ldots, s_n)$ with $s_{2i-1} = s_{2i} = \frac{2i-1}{2k} \, \forall\, i \in \{1, \ldots, k\}$, one of the border players (1, 2, $n-1$ or n) has the highest possible improvement by moving towards the middle to the new interval border β'_2 or β'_{n-2}.*

With the help of generalized continued fractions, define

$$\hat{K}^m := \mathop{\mathrm{K}}_{j=1}^{m} \frac{-\alpha/4}{1}$$

together with

$$\begin{aligned}
\beta'_1 = & \frac{1}{1 - \frac{\alpha}{2\left(\alpha+1+2\hat{K}^{n-3}\right)}} \left(\frac{1-\alpha}{2}\frac{1}{n} + \frac{1-\alpha}{2(\alpha+1)} \frac{1}{1+\frac{2}{\alpha+1}\hat{K}^{n-3}} \frac{3}{n} \right. \\
& + \left(-\frac{2}{\alpha+1} \frac{1}{1+\frac{2}{\alpha+1}\hat{K}^{n-3}} \right) \Bigg(\\
& \sum_{k=2}^{n/2-1} \left(\frac{(-2)^{2k-4}}{a^{\lfloor (2k-3)/2 \rfloor}} \prod_{j=n-3-(2k-4)}^{n-3} \hat{K}^j \left(\frac{1-\alpha}{2\alpha}\frac{2k-1}{n} + \frac{\alpha-1}{2\alpha}\frac{2k-1}{n} \right) \right. \\
& \left. + \frac{(-2)^{2k-3}}{a^{\lfloor (2k-2)/2 \rfloor}} \prod_{j=n-3-(2k-3)}^{n-3} \hat{K}^j \left(\frac{1-\alpha}{2}\frac{2k-1}{n} + \frac{1-\alpha}{2}\frac{2(k+1)-1}{n} \right) \right) \\
& \left. + \frac{(-2)^{n-4}}{a^{(n-4)/2}} \prod_{j=1}^{n-3} \hat{K}^j \left(\frac{1-\alpha}{2\alpha}\frac{n-1}{n} + \frac{\alpha-1}{2\alpha}\frac{n-1}{n} + \frac{1}{2} \right) \Bigg) \right),
\end{aligned}$$

$$\beta'_2 = \frac{1}{1 - \frac{1}{2}\frac{-4}{1+\alpha}\hat{K}^{n-3}} \left(\frac{\alpha}{1+\alpha}\beta'_1 + \frac{1-\alpha}{1+\alpha}\frac{3}{n} \right.$$

$$+\sum_{k=2}^{n/2-1}\left(\frac{-4}{1+\alpha}\frac{(-2)^{2k-4}}{a^{\lfloor(2k-3)/2\rfloor}}\prod_{j=n-3-(2k-4)}^{n-3}\hat{K}^j\left(\frac{1-\alpha}{2\alpha}\frac{2k-1}{n}+\frac{\alpha-1}{2\alpha}\frac{2k-1}{n}\right)\right.$$
$$\left.+\frac{-4}{1+\alpha}\frac{(-2)^{2k-3}}{a^{\lfloor(2k-2)/2\rfloor}}\prod_{j=n-3-(2k-3)}^{n-3}\hat{K}^j\left(\frac{1-\alpha}{2}\frac{2k-1}{n}+\frac{1-\alpha}{2}\frac{2k-1}{n}\right)\right)$$
$$\left.+\frac{-4}{1+\alpha}\frac{(-2)^{n-4}}{a^{(n-4)/2}}\prod_{j=1}^{n-3}\hat{K}^j\left(\frac{1-\alpha}{2\alpha}\frac{n-1}{n}+\frac{\alpha-1}{2\alpha}\frac{n-1}{n}+\frac{1}{2}\right)\right)$$

and

$$\psi_{n,\alpha}^{\text{pair}}=\frac{\beta_2'-\beta_1'}{1/n}$$

Using the conjecture and the definition of $\psi_{n,\alpha}^{\text{pair}}$ we can state the approximation guarantee for an arbitrary even number of facilities and an arbitrary α.

Theorem 8.14. *Assuming that Conjecture 8.13 holds, for $0<\alpha<1$ and $n>3$ facilities with $n=2k$ and $k\in\mathbb{N}$, the game always has a ρ-approximate pure subgame perfect Nash equilibrium with $\rho=\psi_{n,\alpha}^{pair}$.*

Proof. Consider the state $\mathsf{s}=(s_1,\ldots,s_n)$ with $s_{2i-1}=s_{2i}=\frac{2i-1}{2k}\,\forall\, i\in\{1,\ldots,k\}$. The clients' intervals split at $\beta_i=\frac{i}{2k}\,\forall\, i\in\{1,\ldots,2k-1\}$, so each facility i has a utility of $u_i(\mathsf{s})=\frac{1}{n}$.

Using Conjecture 8.13, we only consider player 2 with a move to the right at her new interval border β_2'. The following system of linear equations characterizes the new state $\mathsf{s}'=(\mathsf{s}_{-2},s_2')$:

$$\begin{aligned}(1-\alpha)(\beta_1'-s_1)+\alpha(\beta_1'-\beta_0')&=(1-\alpha)(s_2'-\beta_1')+\alpha(\beta_2'-\beta_1')\\(1-\alpha)(\beta_2'-s_2')+\alpha(\beta_2'-\beta_1')&=(1-\alpha)(s_3-\beta_2')+\alpha(\beta_3'-\beta_2')\\(1-\alpha)(\beta_{2i-1}'-s_{2i-1})+\alpha(\beta_{2i-1}'-\beta_{2i-2}')&=(1-\alpha)(\beta_{2i-1}'-s_{2i})+\alpha(\beta_{2i}'-\beta_{2i-1}')\\&\qquad\forall i\in\{2,\ldots,k-1\}\\(1-\alpha)(\beta_{2i}'-s_{2i})+\alpha(\beta_{2i}'-\beta_{2i-1}')&=(1-\alpha)(s_{2i+1}-\beta_{2i}')+\alpha(\beta_{2i+1}'-\beta_{2i}')\\&\qquad\forall i\in\{2,\ldots,k-1\}\\(1-\alpha)(\beta_{n-1}'-s_{n-1})+\alpha(\beta_{n-1}'-\beta_{n-2}')&=(1-\alpha)(\beta_{n-1}'-s_n)+\alpha(\beta_n'-\beta_{n-1}').\end{aligned}$$

Solving these equations to β_i' and setting $\beta_0'=0,\beta_n'=1,s_2'=\beta_2'$ results in:

$$\begin{aligned}\beta_1'&=\frac{1}{2}\beta_2'+\frac{1-\alpha}{2}s_1\\\beta_2'&=\frac{\alpha}{1+\alpha}\beta_1'+\frac{\alpha}{1+\alpha}\beta_3'+\frac{1-\alpha}{1+\alpha}s_3\\\beta_{2i-1}'&=\frac{1}{2}\beta_{2i-2}'+\frac{1}{2}\beta_{2i}'+\frac{1-\alpha}{2\alpha}s_{2i-1}+\frac{\alpha-1}{2\alpha}s_{2i}\,\forall\, i\in\{2,\ldots,k-1\}\end{aligned}$$

$$\begin{aligned}
\beta'_{2i} &= \frac{\alpha}{2}\beta'_{2i-1} + \frac{\alpha}{2}\beta'_{2i+1} + \frac{1-\alpha}{2}s_{2i} + \frac{1-\alpha}{2}s_{2i+1}\,\forall\, i \in \{2,\dots,k-1\} \\
\beta'_{n-1} &= \frac{1}{2}\beta'_{n-2} + \frac{1}{2}\beta'_{n} + \frac{1-\alpha}{2\alpha}s_{n-1} + \frac{\alpha-1}{2\alpha}s_{n}.
\end{aligned}$$

We write the system of linear equations as an augmented matrix as follows

$$\left(\begin{array}{cccccccc|l}
1 & -\frac{1}{2} & & & & & & & \frac{1-\alpha}{2}s_1 \\
-\frac{\alpha}{1+\alpha} & 1 & -\frac{\alpha}{1+\alpha} & & & & & & \frac{1-\alpha}{1+\alpha}s_3 \\
 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \frac{1-\alpha}{2\alpha}s_{2i-1} + \frac{\alpha-1}{2\alpha}s_{2i} \\
 & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & & \frac{1-\alpha}{2}s_{2i} + \frac{1-\alpha}{2}s_{2i+1} \\
 & & & \ddots & & & & & \vdots \\
 & & & -\frac{1}{2} & 1 & -\frac{1}{2} & & & \frac{1-\alpha}{2\alpha}s_{2i-1} + \frac{\alpha-1}{2\alpha}s_{2i} \\
 & & & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & \frac{1-\alpha}{2}s_{2i} + \frac{1-\alpha}{2}s_{2i+1} \\
 & & & & & -\frac{1}{2} & 1 & & \frac{1-\alpha}{2\alpha}s_{n-1} + \frac{\alpha-1}{2\alpha}s_n + \frac{1}{2}
\end{array}\right)$$

Again we omit the last column and determine the factors to solve this system. We start with the last three rows.

$$\begin{pmatrix}
1 & -\frac{1}{2} & & & & & & \\
-\frac{\alpha}{1+\alpha} & 1 & -\frac{\alpha}{1+\alpha} & & & & & \\
 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\
 & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{2} & 1 & -\frac{1}{2} & & \\
 & & & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} \\
 & & & & & -\frac{1}{2} & 1
\end{pmatrix}
\begin{array}{l} \\ \\ \\ \\ \\ \\ \longleftarrow\!\urcorner^{+} \\ \mid \cdot\, \gamma_{n-1} \lrcorner \end{array}$$

$$\text{with } \gamma_{n-1} = \frac{\alpha}{2} = -2\hat{K}^1$$

$$\Downarrow$$

$$\begin{pmatrix}
1 & -\frac{1}{2} & & & & & & \\
-\frac{\alpha}{1+\alpha} & 1 & -\frac{\alpha}{1+\alpha} & & & & & \\
 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\
 & & -\frac{\alpha}{2} & 1 & -\frac{\alpha}{2} & & & \\
 & & & \ddots & & & & \vdots \\
 & & & -\frac{1}{2} & 1 & -\frac{1}{2} & & \\
 & & & & -\frac{\alpha}{2} & 1-\frac{\alpha}{4} & 0 \\
 & & & & & -\frac{1}{2} & 1
\end{pmatrix}
\begin{array}{l} \\ \\ \\ \\ \\ \longleftarrow\!\urcorner^{+} \\ \mid \cdot\, \gamma_{n-2} \lrcorner \\ \\ \end{array}$$

$$\text{with } \gamma_{n-2} = \frac{1}{1-\frac{\alpha}{4}}\frac{1}{2} = \frac{-2}{\alpha}\hat{K}^2$$

$$\Downarrow$$

$$\begin{pmatrix}
1 & -\frac{1}{2} & & & & & & & \\
-\frac{\alpha}{1+\alpha} & 1 & -\frac{\alpha}{1+\alpha} & & & & & & \\
 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\
 & & -\frac{\alpha}{2} & 1 & & -\frac{\alpha}{2} & & & \\
 & & & \ddots & & & & & \vdots \\
 & & & -\frac{1}{2} & 1-\frac{\alpha}{4(1-\frac{\alpha}{4})} & & 0 & & \\
 & & & & -\frac{\alpha}{2} & & 1-\frac{\alpha}{4} & 0 & \\
 & & & & & & -\frac{1}{2} & 1 &
\end{pmatrix}$$

Processing further as in the proof of the distributed setting we can determine the following factors:

$$\gamma_{2k} = \frac{-2}{\alpha}\hat{K}^{n-2k}\,\forall\, k \in \{2,\ldots,n/2-1\}$$

$$\gamma_{2k-1} = -2\hat{K}^{n-(2k-1)}\,\forall\, k \in \{3,\ldots,n/2\}$$

$$\gamma_3 = \frac{-4}{1+\alpha}\hat{K}^{n-3}$$

$$\gamma_2 = \frac{1+\alpha}{2\left(\alpha+1+2\hat{K}^{n-3}\right)}$$

Now we consider the right-hand side. Since we are only interested in the value of β_1' and β_2', we focus on the first two row.

$$\begin{aligned}
1-\frac{\alpha}{2\left(\alpha+1+2\hat{K}^{n-3}\right)}\beta_1' =& \frac{1-\alpha}{2}s_1 \\
&+\gamma_2\frac{1-\alpha}{1+\alpha}s_3 \\
&+\gamma_2\gamma_3\frac{1-\alpha}{2\alpha}s_3+\frac{\alpha-1}{2\alpha}s_4 \\
&+\sum_{i=2}^{n-2}\left(\frac{1-\alpha}{2}s_{i+1}+\frac{1-\alpha}{2}s_i\right)\prod_{j=2}^{i}(\gamma_j) \\
&+\left(\frac{1-\alpha}{2}s_{n-1}+\frac{1-\alpha}{2}s_n+\frac{\alpha}{2}\right)\prod_{j=2}^{n-1}(\gamma_j)
\end{aligned}$$

We now use the definition of $s_i = \frac{2i-1}{2n}\,\forall\, i \in \{1,\ldots,n\}$ and $\gamma_i = \frac{-2}{\alpha}\cdot\hat{K}^{n-(i+1)}\,\forall\, i \in \{3,\ldots,n-1\}$ and $\gamma_2 = \frac{-4}{\alpha(1+\alpha)}\cdot\hat{K}^{n-2}$.

$$
\begin{aligned}
1+\frac{2}{\alpha+1}\hat{K}^{n-2}\beta_1' =& \frac{1-\alpha}{1+\alpha}\frac{3}{2n} + \sum_{i=2}^{n-2}\left(\frac{1-\alpha}{2}\frac{2(i+1)-1}{2n} + \frac{1-\alpha}{2}\frac{2i-1}{2n}\right) \\
& \frac{-4}{\alpha(1+\alpha)}\cdot\hat{K}^{n-2}\prod_{j=3}^{i}\left(\frac{-2}{\alpha}\cdot\hat{K}^{n-j}\right) \\
& + \left(\frac{1-\alpha}{2}\frac{2(n-1)-1}{2n} + \frac{1-\alpha}{2}\frac{2n-1}{2n} + \frac{\alpha}{2}\right) \\
& \frac{-4}{\alpha(1+\alpha)}\cdot\hat{K}^{n-2}\prod_{j=3}^{n-1}\left(\frac{-2}{\alpha}\cdot\hat{K}^{n-j}\right)
\end{aligned}
$$

This results in the interval borders (written with generalized continued fractions)

$$
\begin{aligned}
\beta_1' =& \frac{1}{1-\frac{\alpha}{2\left(\alpha+1+2\hat{K}^{n-3}\right)}}\left(\frac{1-\alpha}{2}\frac{1}{n} + \frac{1-\alpha}{2(\alpha+1)}\frac{1}{1+\frac{2}{\alpha+1}\hat{K}^{n-3}}\frac{3}{n}\right. \\
& + \left(-\frac{2}{\alpha+1}\frac{1}{1+\frac{2}{\alpha+1}\hat{K}^{n-3}}\right)\Bigg(\\
& \sum_{k=2}^{n/2-1}\left(\frac{(-2)^{2k-4}}{a^{\lfloor(2k-3)/2\rfloor}}\prod_{j=n-3-(2k-4)}^{n-3}\hat{K}^{j}\left(\frac{1-\alpha}{2\alpha}\frac{2k-1}{n} + \frac{\alpha-1}{2\alpha}\frac{2k-1}{n}\right)\right. \\
& \left. + \frac{(-2)^{2k-3}}{a^{\lfloor(2k-2)/2\rfloor}}\prod_{j=n-3-(2k-3)}^{n-3}\hat{K}^{j}\left(\frac{1-\alpha}{2}\frac{2k-1}{n} + \frac{1-\alpha}{2}\frac{2(k+1)-1}{n}\right)\right) \\
& \left. + \frac{(-2)^{n-4}}{a^{(n-4)/2}}\prod_{j=1}^{n-3}\hat{K}^{j}\left(\frac{1-\alpha}{2\alpha}\frac{n-1}{n} + \frac{\alpha-1}{2\alpha}\frac{n-1}{n} + \frac{1}{2}\right)\Bigg)\right)
\end{aligned}
$$

Combining the factors also for β_2' yields

$$
\begin{aligned}
\beta_2' =& \frac{1}{1-\frac{1}{2}\frac{-4}{1+\alpha}\hat{K}^{n-3}}\left(\frac{\alpha}{1+\alpha}\beta_1' + \frac{1-\alpha}{1+\alpha}\frac{3}{n}\right. \\
& + \sum_{k=2}^{n/2-1}\left(\frac{-4}{1+\alpha}\frac{(-2)^{2k-4}}{a^{\lfloor(2k-3)/2\rfloor}}\prod_{j=n-3-(2k-4)}^{n-3}\hat{K}^{j}\left(\frac{1-\alpha}{2\alpha}\frac{2k-1}{n} + \frac{\alpha-1}{2\alpha}\frac{2k-1}{n}\right)\right. \\
& \left. + \frac{-4}{1+\alpha}\frac{(-2)^{2k-3}}{a^{\lfloor(2k-2)/2\rfloor}}\prod_{j=n-3-(2k-3)}^{n-3}\hat{K}^{j}\left(\frac{1-\alpha}{2}\frac{2k-1}{n} + \frac{1-\alpha}{2}\frac{2k-1}{n}\right)\right) \\
& \left. + \frac{-4}{1+\alpha}\frac{(-2)^{n-4}}{a^{(n-4)/2}}\prod_{j=1}^{n-3}\mathop{\mathcal{K}}_{i=1}^{j}\left(\frac{1-\alpha}{2\alpha}\frac{n-1}{n} + \frac{\alpha-1}{2\alpha}\frac{n-1}{n} + \frac{1}{2}\right)\right)
\end{aligned}
$$

Since we consider the second facility, we have $u_2(\mathsf{s}') = \beta_2' - \beta_1'$. Together with $u_2(\mathsf{s}) = \frac{1}{n}$ and Conjecture 8.13 we get

$$\rho = \frac{u_2(\mathsf{s}')}{u_2(\mathsf{s})} = \psi_{n,\alpha}^{\text{pair}} = \frac{\beta_2' - \beta_1'}{1/n}$$

which proves the theorem. □

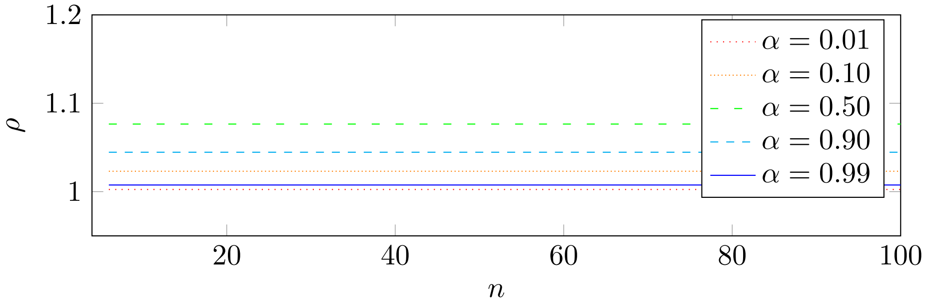

Figure 8.4.: Approximation factor ρ as a function of n.

Using mathematical computing systems we can derive also a closed form of $\psi_{n,\alpha}^{\text{pair}}$ but we omit it here because of its complexity. Again we have have an approximation factor that is nearly independent of n for $n \geq 4$ (see Fig. 8.4).

If we investigate the cases for $n = 4$ with

$$\psi_{4,\alpha}^{\text{pair}} = \frac{1}{4}\left(-a^2 + a + 4\right)$$

and $n = 6$ with

$$\psi_{6,\alpha}^{\text{pair}} = \frac{a^3 - 7a^2 - 4a + 16}{-2a^2 - 8a + 16}$$

we see low approximation factors for α near 0 and α near 1 (see Fig. 8.5).

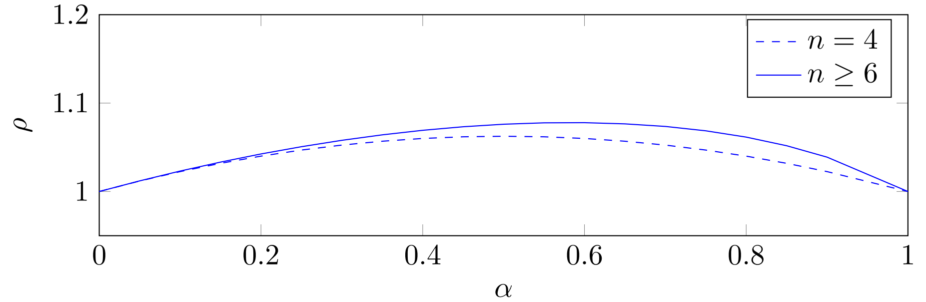

Figure 8.5.: Approximation factor ρ as a function of α.

8.5. Quality

Finally, we evaluate our approximate pure subgame perfect equilibria by comparing the social costs of the clients in the equilibrium with their social costs in a centrally computed optimum, a state with minimal costs. Similarly to the Price of Anarchy, we define the quality of an equilibrium as in [FS16]. We are interested in the costs of the client players, while the strategies of the players define the stable states. We define the optimum of the game by $(\mathsf{s},\mathsf{f})^* = \min_{(\mathsf{s},\mathsf{f})\in \mathsf{S}\times\mathsf{F}} SC(\mathsf{s},\mathsf{f})$. Then, the quality of an (approximate) pure subgame perfect Nash equilibrium (s,f) is defined by $\frac{SC(\mathsf{s},\mathsf{f})}{SC((\mathsf{s},\mathsf{f})^*)}$.

Since our setting with uniformly distributed facilities (see Theorem 8.12) is the social optimum by construction, its quality is also optimal regarding this measure. Therefore, we focus on the approximate equilibria with paired facilities (see Theorem 8.14).

Theorem 8.15. *Given a game with $n = 2k$ facilities for some $k \in \mathbb{N}$ and the state $\mathsf{s} = (s_1, \ldots, s_n)$ with $s_{2i-1} = s_{2_i} = \frac{2i-1}{2k}$ for $i \in \{1, \ldots, k\}$ and f_s, the quality of s is $\frac{2\alpha+2}{3\alpha+1}$.*

Proof. Consider the state $\mathsf{s} = (s_1, \ldots, s_n)$ with $s_{2i-1} = s_{2i} = \frac{2i-1}{2k}$ for $i \in \{1, \ldots, k\}$, which is a pure subgame perfect Nash equilibrium for $\alpha = 0$ and $\alpha = 1$ and for $0 < \alpha < 1$ a ρ-approximate pure subgame perfect Nash equilibrium with $\rho = \psi^{\text{pair}}_{n,\alpha}$.

Since for any two neighboring intervals $C_x = [\beta_{x-1}, \beta_x]$ and $C_{x+1} = [\beta_x, \beta_{x+1}]$, in which the clients have different strategies x and $x + 1$, the costs are equal for both strategies:

$$(1-\alpha)|s_x - \beta_x| + \alpha\ell_x = (1-\alpha)|s_{x+1} - \beta_x| + \alpha\ell_{x+1}.$$

The interval border $\beta_x = \frac{x}{2k}$ fulfills this linear equations for s. It holds that

- if $x = 2i-1$: $(1-\alpha)|\frac{2i-1}{2k} - \frac{2i-1}{2k}| + \alpha|\frac{2i-1}{2k} - \frac{2i-2}{2k}| = (1-\alpha)|\frac{2i-1}{2k} - \frac{2i-1}{2k}| + \alpha|\frac{2i}{2k} - \frac{2i-1}{2k}|$
- if $x = 2i$: $(1-\alpha)|\frac{2i-1}{2k} - \frac{2i}{2k}| + \alpha|\frac{2i}{2k} - \frac{2i-1}{2k}| = (1-\alpha)|\frac{2i+1}{2k} - \frac{2i}{2k}| + \alpha|\frac{2i+1}{2k} - \frac{2i}{2k}|$.

So each facility i has a utility of $u_i(\mathsf{s}) = \frac{1}{n}$ and is located at one of her interval borders. Hence, for the clients' cost it follows

$$SC(\mathsf{s},\mathsf{f}_\mathsf{s}) = n\left(\int_0^{\frac{1}{n}} (1-\alpha)\left(\frac{1}{n} - x\right) + \frac{\alpha}{n} dx\right) = \frac{1+\alpha}{2n}.$$

The state $\mathsf{s}^* = (s^*_1, \ldots, s^*_n)$ with $s^*_i = \frac{2i-1}{2n}$ for $i \in \{1, \ldots, n\}$ is the social optimum. In s^* each facility i has a utility of $u_i(\mathsf{s}^*) = \frac{1}{n}$ and is placed in the middle of her interval. Hence, for the clients' cost it follows

$$SC(\mathsf{s}^*,\mathsf{f}_{\mathsf{s}^*}) = n\left(\int_0^{\frac{1}{n}} (1-\alpha)\left|\frac{1}{2n} - x\right| + \frac{\alpha}{n} dx\right) = \frac{1+3\alpha}{4n}.$$

Thus, $\frac{SC(\mathsf{s},\mathsf{f}_\mathsf{s})}{SC(\mathsf{s}^*,\mathsf{f}_{\mathsf{s}^*})} = \frac{\frac{1+\alpha}{2n}}{\frac{1+3\alpha}{4n}} = \frac{2\alpha+2}{3\alpha+1} \in [1, 2]$. □

Conclusion

To close this chapter and the analysis of Facility Location Games, we compare the different computed approximation factors with the different methods (see Fig. 8.6).

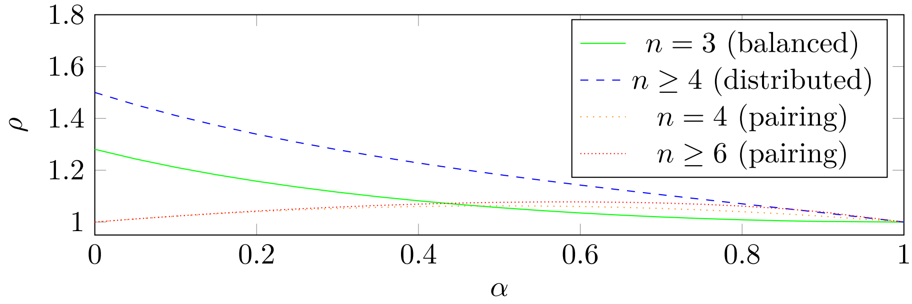

Figure 8.6.: Approximation factor ρ as a function of α.

As we can see, the paired states are much more stable and therefore also more attractive for the facilities if α is small. This supports the principle of minimum differentiation by Hotelling [Hot29]. Only for high values of α other states become more attractive and especially also the distributed placing of the facilities.

PART III

Further Applications and Discussion

The two main parts of this thesis investigated the concept of approximate pure Nash equilibria in different game models. We mainly focused on the three import research questions regarding existence, computation and quality of the equilibria. In this last part of the thesis we now open the discussion about the results for a better classification of the results. Here, we mainly focus on open research questions and discuss which challenges appear.
Additionally, we look at the concept of approximate equilibria from a completely different perspective. We use this concept as a metric for simulations in context of game theory.

Outline of This Part In Chapter 9, we look at approximate pure Nash equilibria as a metric for simulations. The thesis ends in Chapter 10 with a conclusion of the whole work as well as an detailed view on open questions in the different models.

CHAPTER 9

A Metric for Simulations

Research in the field of game theory has developed concepts and methods that describe and predict the outcome of strategic situations in which rationally acting individuals interact. However, when analyzing large-scale complex systems these methods quickly reach their limits. Even for games without repetition and only few players, the problem of determining the outcome is computationally difficult [DGP09; CDT09]. In games that are being played repeatedly the situation is even more unsatisfactory as the behavior of players is a lot more complex than the mere choice of an action. This makes the analysis of such games a daunting task even if one imposes severe restrictions on possible strategies for the players.

Therefore, we choose a simulation-based approach to analyze settings of repeated strategic interaction in complex scenarios. To that end, we model the strategic behavior of players as algorithms or automata which allow us to simulate the interaction of many players with possibly many different strategies over infinitely many periods. The results of the simulations provide us with information about the performances of strategies and which of them are plausibly chosen in real-world scenarios. To evaluate the results we make use of approximate pure Nash equilibria as a metric. Thus, our approach nicely complements experimental as well as theoretical methods.

Contribution and Underlying Work In this chapter we present a simulation-based approach to analyze complex infinitely repeated games. We have developed an abstract simulator for various scenarios. It is fully integrated in an automated workflow and simulation environment. This enables us to run and evaluate large sets of interesting configurations and different scenarios automatically and in parallel.

To show the powerfulness of the simulation approach we introduce a formal model for an electronic market with complex strategic behavior. Various participants strategically select actions over multiple rounds. All actions together as well as some random influence define the next state and the overall outcome. Four different specific scenarios in this model are evaluated with our self-developed simulation tool. As the main evaluation metric we propose the use of the approximate pure Nash equilibrium. Computing the approximation factor for the overall execution of a strategy allows us to judge the stability of this strategy and how robust it is against small perturbations.

We explore the capabilities and limitations of such simulations by performing an exhaustive search over the strategy space of repeated games with finite automata, also known as machine games [Rub98; Rub86]. Similar evaluations have already been investigated [Axe84; II06; PB12], but they all concentrated on standard game scenarios like the Repeated Prisoner's Dilemma.

The model, the simulator and the evaluations presented in this chapter are based on the following publication:

> Matthias Feldotto and Alexander Skopalik. **A simulation framework for analyzing complex infinitely repeated games**. In: *Proceedings of the 4th International Conference on Simulation And Modeling Methodologies, Technologies And Applications (SIMULTECH)*. 2014. [FS14]

Outline of This Chapter In Section 9.1 we introduce the electronic market as an evaluation example with complex strategic behavior and its formal model. Afterwards, we shortly present our simulation approach in Section 9.2 before we investigate different evaluations using the approximate equilibrium concept as metric in Section 9.3.

9.1. Scenario and Model

Our research is motivated by the analysis of electronic markets with thousands of participants and complex strategic behavior. We consider an example of a global market of composed IT services [Hap+13] which was investigated in an experimental setting [Mir+17] to demonstrate the use of our simulation technique. In this setting *s*ervice providers offer simple software services on an electronic market platform. These services can dynamically and flexibly be combined to more complex and individual service compositions by *s*ervice composers. An important characteristic of this market model is a certain degree of anonymity and information asymmetry. Service providers may have an incentive to exert low effort in providing their service, resulting in lower production cost and lower quality. The service composer would have to predict a service provider's choice of effort although she might not even be able to observe it in hindsight, refereed as credence goods in economics literature. We seek to understand the impact of this asymmetric information on the quality of the services and how it can be improved by market mechanisms like reputation systems [Bra+14]. The setting is similar to the disaggregation setting which we discussed in Chapter 6, but here we focus on a different property.

We model the market scenario as a repeated game from Section 2.1.3 that proceeds in rounds. The scenario is adapted from the experimental setup of [Mir+17]. Thus, all chosen parameters here are motivated by the evaluation of the experiments. In each round a service composer is asked to compose a certain service product. That is, she has to choose a set of services (out of a collection of possible choices) and composes the final product. The service providers which have been chosen by the composer can decide how much effort they exert in delivering their services. For simplicity we assume only two possible levels, high and low. The utility of a service provider is 20 if she only exerts low effort, 10 for high effort and 0 if she is not chosen by the composer. Each service composition consists of multiple services and, thus, its quality depends on the aggregated service qualities. Here we assume that the quality of a service composition is good if and only if each service was delivered with high quality. The realized quality of the composition can be correctly observed with a certain probability, which we chose to be 95% in our simulations. We implicitly assume that a player's $r$eputation depends on the observed quality of the services he participated in during the past rounds. We assign to each player a reputation vector of fixed length (in our example it will be two or three) storing the past (two or three) observed qualities. Note that the reputation depends on the choices of all participating service providers and the outcome of the random event of the observation. The payoff of the composer depends on her reputation and is 40 plus (minus) 10 for each positive (negative) entry in her reputation vector. The overall payoff of the repeated game is the discounted reward with discount factor of 10% for each player, formally $u_i = \sum_{j=1}^{\infty} \beta^j u_i^j$.

In general, the strategy of a player is modeled by a deterministic transition function $s_i : \tilde{q} \to A_i$ which determines the player's next action, depending on the current observed state. Here, we assume that a player's observable state is only her last two or there reputation values (her own reputation vector). Thus, the class of transition functions is restricted to functions that do not differentiate between states with the same reputation values. For this setting we use eight different deterministic strategies, which use the reputation values as only input. Since the focus of this work here is the tooling and methodology and not the interpretation of the scenario, we skip the description of the concrete strategies and refer to [Mir+17]. For the further understanding of this chapter, it is only relevant to remember that different action decisions can be modeled by different automata, i.e., based on their current reputation profiles the players deterministically chose the next action.

9.2. Simulator

In this section we present the technical components of our simulator and especially focus on the simulation workflow and environment.

9.2.1. Simulation Architecture

The simulator is a packaged Java application. This allows an execution on nearly every available execution environment nowadays without concerning any system-specific properties. In essence, the application consists of two modules (cf. Fig. 9.1).

The first one (solid rectangles) is responsible for the execution of the simulation. It consumes the input and starts the simulations with the correct settings in the

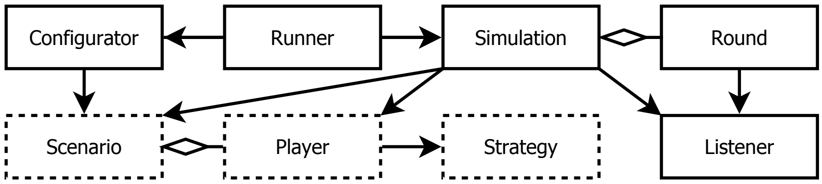

Figure 9.1.: The simulation architecture.

Runner and the *Configurator* components. An XML description of the scenario together with further individual properties yields as input. Furthermore, a *Simulation* and a *Round* component handle all simulation- and round-relevant aspects. A *Listener* component is responsible for collecting results and produces *csv* output files with relevant data like the players' utilities.

As the second module (dashed rectangles) we have the concrete application scenario, in our case the market for composed services. The most important components here are the *Scenario*, the *Player* and the *Strategy*. This last module can be replaced by any other scenario implementation, for example the well-known and extensively investigated Repeated Prisoner's Dilemma [Axe84] or a completely new designed scenario.

9.2.2. Simulation Workflow

The simulation is entirely integrated in an automated workflow. Beginning with an input all required processing steps are executed until the output is generated (cf. Fig. 9.2). The input consists of four parts:

- a general *configuration* which defines the scenario and all fixed parameters given as XML file,
- a list of *parameters* from which all combinations are evaluated in comparison,
- a list of possible *strategies* for each player in the scenario from which all strategy combinations are evaluated, and
- a list of additional *settings* for the simulation, e.g., the number of random seeds, the connection parameters for external clusters, etc.

The preprocessing step reads the inputs and generates all simulation configurations necessary for independent simulations (cf. Section 9.2.1). The execution step produces execution packages, distributes them on different available environments (see Section 9.2.3) and collects the results as soon as they are finished. Afterwards, the postprocessing step consumes the output of the different simulations and aggregates them according to different aspects. In the end, it produces tables with the measured values (**.csv*), ready-to-use graphics for presentation (**.png*), and papers (**.pdf*) and generation scripts (**.plt*) to manually adapt the graphics with gnuplot [WK12].

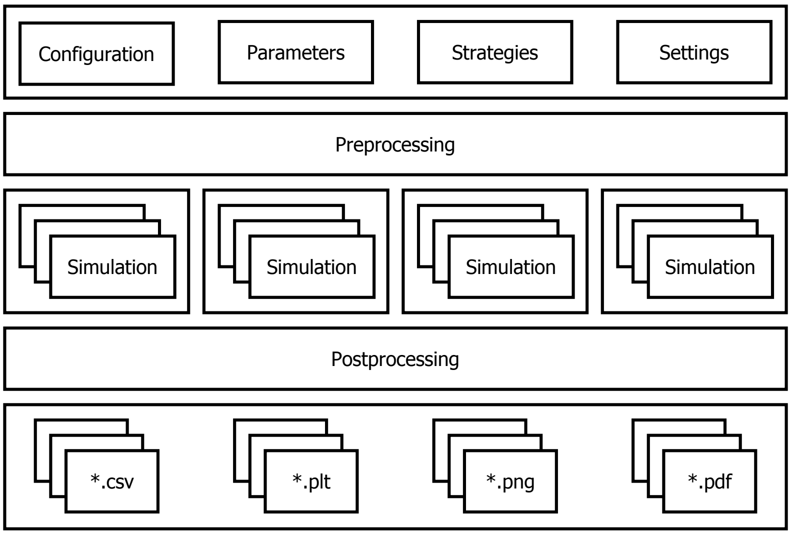

Figure 9.2.: The simulation workflow.

9.2.3. Simulation Environment

The simulation environment consists of two parts: On the one hand, the application running on a dedicated virtual server and, on the other hand, different external machines for running the simulations (cf. Fig. 9.3).

The application itself has two parts: a front-end managing the user input and output as well as a back-end responsible for the simulation workflow. The front-end is running as web application based on the Java *Spring Framework*. Therefore, the user can handle the simulations through a responsive HTML5 web interface from any device and has not to touch more technical parts. The back-end based on the *Spring Batch* manages the simulation workflow and is especially responsible for the pre- and postprocessing of the simulations as well as the distribution of the actual simulations. For this purpose four different approaches are implemented:

VM: The simulations run on the same virtual machine as the application itself in different threads, therefore a scheduler is integrated.

Pool: The simulations run on other desktop machines available in a pool. Also for this purpose, a scheduler exists in the application and the transfer of data is managed via the SSH protocol.

HTC: The simulations run on different virtual machines in a High Throughput Cluster. An existing front-end can be accessed via SSH and is responsible for the scheduling.

HPC: The simulations run on nodes in a High Performance Computing cluster. Also for this purpose a front-end exists to allocate the computation time.

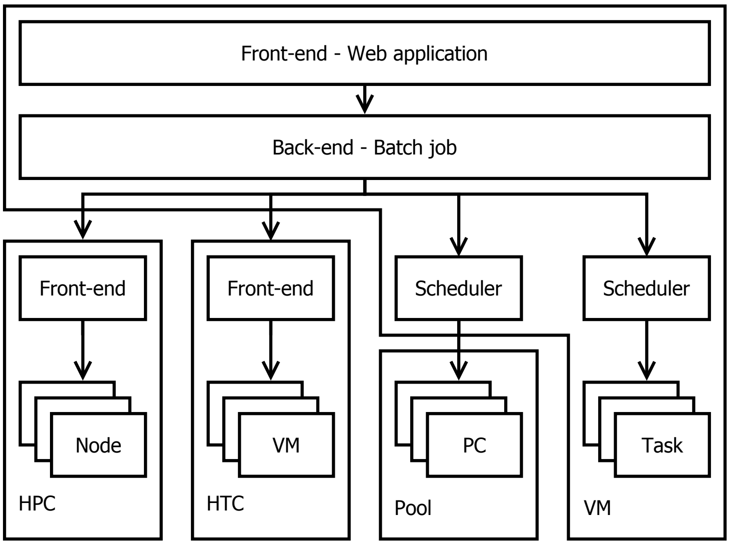

Figure 9.3.: The available simulation environment.

While the first two approaches are completely self-developed and independent of any environment, the latter two use existing managing approaches available in computing centers. They offer the advantage that the available computing time and memory is much higher and allow clearly more simulations in the same time.

9.3. Evaluations

In this section we investigate the capabilities and limitations of our tool by evaluating the market scenario. We start by investigating the possible scenarios for evaluation. Concentrating only on one strategy profile, i.e., one combination of strategies, the simulation framework allows very complex scenarios. We use the approximate equilibrium concept to evaluate the strategy profiles.

Players	**Combinations**	**Memory**	**Runtime**
2 reputation values			
2	256	2MB	30.17sec
3	4,096	21MB	232.18sec
4	65,536	375MB	344.19sec
3 reputation values			
2	65,536	0.32GB	0,09h
3	16,777,216	96.07GB	8,54h
4	*4,294,967,296*	*24,594.88GB*	*2186,37h*

Table 9.1.: Runtime and memory complexity of evaluations (The numbers written in italics are predicted).

However, for the analysis of strategic games and their outcomes we have to take into account all possible strategy profiles that may be chosen by the players. As the number of possible strategies grows exponentially with the size of the automata and, moreover, the number of strategy profiles grows exponentially with the number of players, evaluating all strategy profiles by simulations quickly becomes intractable (cf. Table 9.1). Already by using only 4 strategic players and a state space with 3 history values and running on a cluster with 100 available parallel computing nodes, the scenario can not be simulated with reasonable time and space input. Thus, we demonstrate our tool with a few small examples before we discuss further research beyond a mere exhaustive search.

#	Composition	Service Provider	Players	Reputation
1	composition of all	2 det. players	2	3 values
2	choose best one	2 det. players	2	3 values
3	choose best pair	3 det. players	3	3 values
4	choose with one monopolist	3 det. players	4	2 values

Table 9.2.: Four evaluated scenarios with 2, 3 and 4 deterministic strategic players.

Therefore, we consider 4 different scenarios with 2, 3 and 4 strategic players (see Table 9.2). Scenario 1 and 2 contain two service providers acting according to a deterministic transition function. In the first scenario, the composed service contains both separate service components, whereas in the second scenario only the best service (regarding the past reputation values) is included in the product. In the third scenario, three deterministic service providers are available and the composition consists of the two best services. The last scenario is an extension of scenario 3. There are 3 deterministic service providers, but the composition is also chosen by a player. Furthermore, one of the providers is fixed (it has a monopoly) and the choice is only between the other two.

All simulations are run for 100 rounds with 20 different initial seeds to get usable statistic results. In the following plots we compare the quality and the discounted utility of different providers together with the possible improvement factor of the first player visualized by the color. The incentive to change the strategy is higher in the brighter areas. This incentive factor corresponds to the approximation factor in our equilibrium definition.

In the first scenario in which both services are used for the combination we see two dark areas in the lower left and upper left corner of the plot (cf. Fig. 9.4). The service provider cannot improve its utility by serving higher quality. Furthermore, we can see in Fig. 9.5 that the player has all equilibria with high utility values.

In contrast, scenario 2 has another result. Here, always playing with a bad quality is not the best choice for the player (see Fig. 9.6). He has to cooperate and serve a better quality to be chosen. Furthermore, the equilibria are not with high utility values, states where no improvement can be reached are also found in

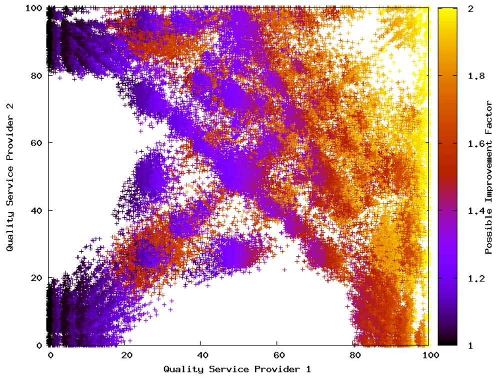

Figure 9.4.: Scenario 1: The quality of two service providers.

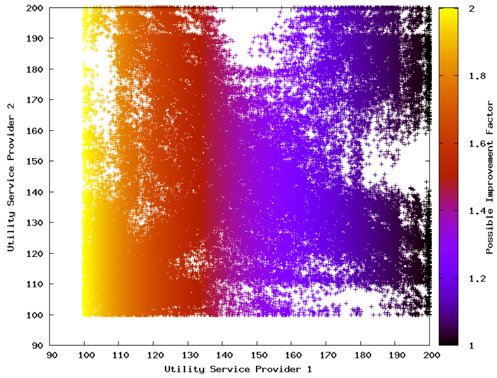

Figure 9.5.: Scenario 1: The payoff of two service providers.

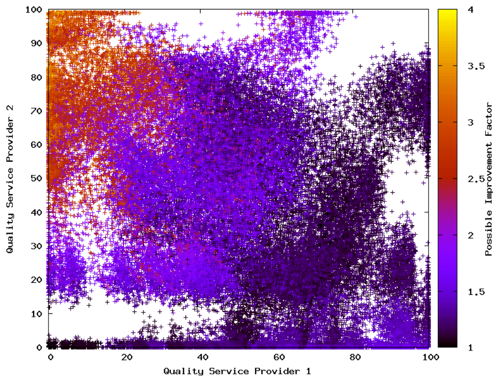

Figure 9.6.: Scenario 2: The quality of two service providers.

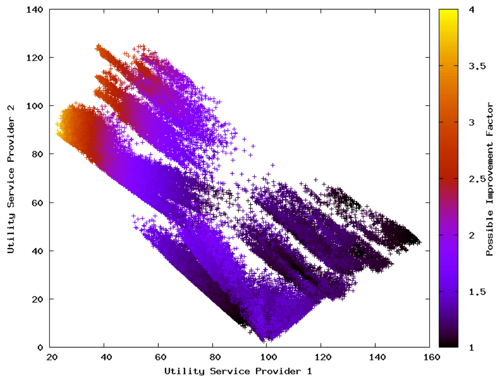

Figure 9.7.: Scenario 2: The payoff of two service providers.

the lower utility area (see Fig. 9.7). In the third scenario with three players we see similar results (cf. Fig. 9.8 and Fig. 9.9). Multiple areas with equilibria exist where no improvement for a single player is possible. A better analysis is possible by using 3D visualization tools with user interaction.

In the last scenario a monopolist and two other service providers are put together into compositions. We see a difference between the stable strategies of the monopolist (cf. Fig. 9.10) and the other providers (cf. Fig. 9.11). The monopolist is always in a nearly stable situation, not depending on its served quality. However, the other provider has only small stable areas.

The simulation framework helps us to study repeated games where the theoretic analysis is too complex and experimental data is not available. We can identify equilibria and stable strategies which are a plausible explanation of players' behavior. For the design of the electronic market and supporting market infrastructure (e.g., reputation systems), we can evaluate possible solutions in various different scenarios.

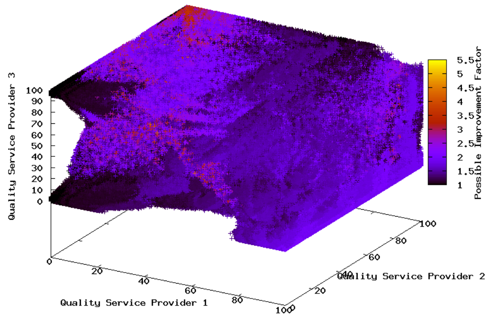

Figure 9.8.: Scenario 3: The quality of three service providers.

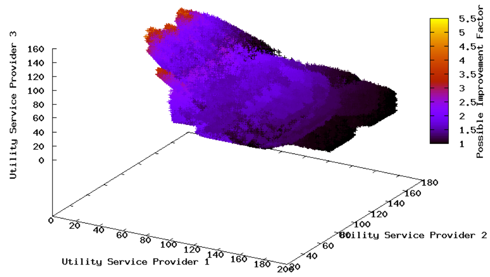

Figure 9.9.: Scenario 3: The payoff of three service providers.

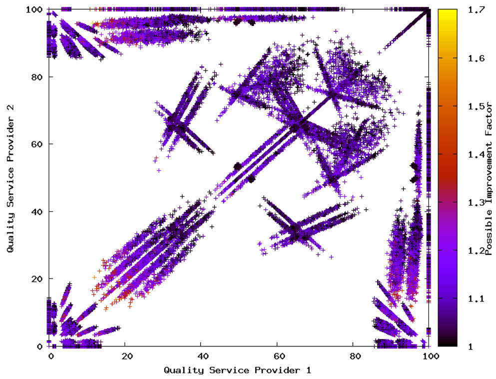

Figure 9.10.: Scenario 4: The quality of two service providers.

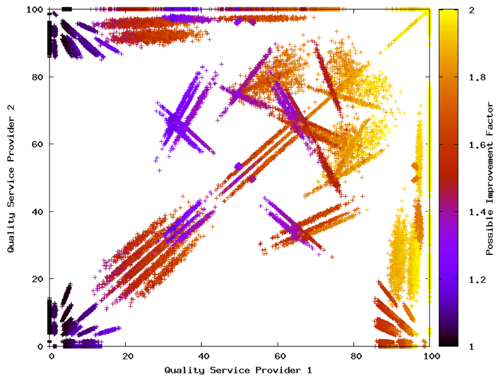

Figure 9.11.: Scenario 4: The quality of two service providers.

Conclusion

Not surprisingly, an exhaustive search over all strategy profiles quickly becomes intractable – even if one considers only strategies implemented by automata with very few states. Nevertheless, simulations give us interesting insights about the outcomes of strategic behavior in complex scenarios like the presented electronic market. By identifying equilibria and strategies with high utilities and low incentives to change, useful statements about the development of markets can be done and the impact of mechanisms like reputation systems can be studied before their implementation. The approximate equilibrium concept has been proven to be a reasonable metric in this context.

However, we are interested in even more realistic scenarios which allow players with more memory and more complex functions including randomization. A promising direction is the use of techniques from evolutionary game theory which uses a *population* of strategies, each additionally equipped with a fitness value. Utilizing replication and imitation dynamics [Wei97; TJ78; HS98], the population and the fitness values evolve over time. The concept of evolutionary equilibria [HR82] and evolutionary stable strategies (ESS) [SP73] describe states and strategies which are (approximately) stable with respect to the aforementioned dynamics. The simulation architecture presented here is easily adaptable towards such concepts and the computational results show that hundreds or even thousands of such experiments can be conducted in a very short time.

CHAPTER 10

Discussion and Open Research Questions

This thesis investigated the analysis of approximate pure Nash equilibria in various models. While the first part focused on the computation and the quality of approximate pure Nash equilibria in variants of Congestion Games, the second part considered two different models in which we showed the existence of approximate pure Nash equilibria with low approximation factors. We refrain from repeating the different results from the thesis here and refer to the contribution section in the introduction. To conclude this thesis, we would like to focus on various open research questions and further directions in the presented models. For each model we discuss the open questions and next steps. We especially consider the challenges that prevent us from already having solved them. In addition to investigating open questions regarding approximate equilibria in these models, we also look beyond them and discuss further applications of this concept.

Congestion Games The main algorithmic idea on which our algorithms in Chapters 3 to 5 are based is the division into polynomially related blocks of players and the iterative processing of them together with the preparation of the following block. This basis has meanwhile been implemented and extended in various publications in various models [Car+11a; Car+15; CFG17; FGS14; Fel+17a; GNS18]. Each of these contributions uses a slightly modified variant of the algorithm together with a new analysis. Of course, the question arises whether the algorithm can be prepared as a framework and applied directly instead of performing a complete analysis each time. So far, no one has succeeded in achieving this and the main reasons can be found in the way how it is analyzed: The key arguments of the proofs are directly based on the underlying potential functions. And since all mentioned works are based on different game models and therefore apply different potential functions, the unification is a tough challenge. Nevertheless, we see possibilities of constructing some kind of framework to simplify the analysis for further use cases.

So far all investigated models with this algorithmic idea are variants of Congestion Games or very related classes. A natural question is whether there exist other games, completely different from Congestion Games, that share similar properties. Two main aspects of the algorithm are to highlight here, which have to be solved for other problems as well: First, the approach only works with games that admit an (approximate) potential function. This is already a very tough restriction, since many games either do not have a potential function or it is not known. The second aspect is the need of a structure of the players in the game, that allows the classification of the players according to their influence in the game. Otherwise, the processing according to different blocks would not be efficiently possible. The presented Opinion Formation Games may be a promising candidate, since the structure is somehow comparable and different techniques from Congestion Games are already transferred to this domain (e.g., the smoothness technique).

To the best of our knowledge, this class of algorithms achieves the best results so far for computing approximate pure Nash equilibria in the context of Congestion Games. Nevertheless, there is still a gap for possible improvements. For example, in the case of weighted Congestion Games, we know about the existence of $d+1$-approximate equilibria [HKS14], but our computation achieves only an approximation factor of $({}^{d}\!/\!\ln 2)^{d} \cdot poly(d)$. Here, our technique reaches its limit since the factor heavily depends on the stretch of the potential function. The lower bound of the PoA in [GS07] immediately yields a corresponding lower bound on the stretch. Thus, a significant improvement below $O\left(({}^{d}\!/\!\ln 2)^{d}\right)$ of the approximation factor would require new algorithmic ideas.

Opinion Formation Games To the best of our knowledge, this is the first result of investigating approximate pure Nash equilibria in a coevolving Opinion Formation Game. Our work is therefore only the beginning of this line of research. In the nearest-neighbor game we only investigated a neighborhood of exactly one player so far. The straight-forward extension of our results is a neighborhood of $K > 1$ that is then indeed exactly the model of the K-nearest neighbors game by [BGM13]. The main challenge here is the changing neighborhood. Although we can still use our approach of virtual players and costs, we need to bound the influence of the changing neighborhood between the original strategy profile of intrinsic opinions compared to any kind of new state.

If we now open the discussion a little broader, our approach of virtual costs and our results for the natural states are valid for all possible neighborhood definitions that do not depend on the strategy of the corresponding player, but only on the others players. Here, different further concrete definitions of neighborhoods may be investigated.

The approximation factors that we can show by our analysis for the nearest-neighbor game are not tight. In contrast, it is only known that pure Nash equilibria do not exist. With the help of some simulations, we expect that the lower bound for the approximation factor is slightly below 1.05 for $\omega = 1$. Our results are quite good for a high weight of the own intrinsic opinion while they are improvable for low values. Here, one promising approach is the near grouping of expressed opinions at different points. However, at the moment we fail in the specification of these points, since any natural balanced strategy fails with high improvement factors for the center players.

Different extensions in this class of models are also possible. Starting from adding weights to the players or a higher dimension, up to the definition of generalized asymmetric games as defined by [BGM13] are reasonable and our techniques are easily extendable to these settings. But also the investigation of completely different objectives, or also other Opinion Formation models with the help of approximate equilibria is a good road map for this line of research.

Facility Location Games Two of our main results for the existence of approximative equilibria in Facility Location Games are based on conjectures, namely the identification of the player with the best possible improvement. So far we were not able to proof them analytically, due to the complexity of the game and the increasing possibilities for a player depending on the total number of players. For our conjectures, we computed the cases manually for values of $n \in [4, 10]$ to justify the statements. Additionally, we have started to work with simulations to get further insights and support our conjectures. An approach to prove the conjectures analytically applies insights about the available total space on the line: Moving a facility to an area with already existing facilities, there is less space available. This has to result in a displacement of other facilities which again require space at other areas. This process should lead to a contradiction since the space is limited on the line.

This work does not investigate scenarios with an odd number of $n > 4$ facilities. The simple paring is naturally not possible since one player always stays alone. For n players with $n \mod 4 = 1$ (e.g., $5, 9, \ldots$) we can apply the same strategy as in this work, resulting in a single player in the center. Our technique is still applicable and we achieve similar results and can create an expression with continued fractions. But here the computation of closed forms is no longer possible, since we receive an irregular continued fraction. For the other case with $n \mod 4 \neq 1$ (e.g., $7, 11, \ldots$), three facilities are left in the middle. Three single players, a group of three players or an asymmetric setting are possible, but none of them yield promising results so far.

The candidates in our work were given by existing results for similar models and they are not necessarily the best ones. Using an approach similar to the variant with three facilities would lead to tight results, but it fails on the complexity so far. To handle this and probably also further scenarios, our technique with the unified fractions as a result of a system of linear equations has to be formalized in a framework. Then we see opportunities to apply this further.

In the class of Facility Location Games, there are lots of different models and extensions to investigate further. A next natural step is the application of other distributions of clients by applying different density functions. Furthermore, the attraction to facilities is extensible using other functions than the linear combination of distance and load.

Further Applications and Outlook Apart from the discussed models in this thesis, the concept of approximate pure Nash equilibria may be further investigated in a variety of different games. Next to the discussed advantages in comparison to the pure variant, it enables a good bridge between theoretical analysis and simulations.

Bibliography

[AA13] Christian Ahlin and Peter D. Ahlin. **Product Differentiation Under Congestion: Hotelling was Right**. In: *Economic Inquiry* 51.3 (2013), pp. 1750–1763. DOI: 10.1111/j.1465-7295.2012.00489.x.

[AAE13] Baruch Awerbuch, Yossi Azar, and Amir Epstein. **The Price of Routing Unsplittable Flow**. In: *SIAM J. Comput.* 42.1 (2013), pp. 160–177. DOI: 10.1137/070702370.

[AB95] Robert Aumann and Adam Brandenburger. **Epistemic Conditions for Nash Equilibrium**. In: *Econometrica* 63.5 (1995), pp. 1161–1180. DOI: 10.2307/2171725.

[Ahn+01] Hee-Kap Ahn, Siu-Wing Cheng, Otfried Cheong, Mordecai J. Golin, and René van Oostrum. **Competitive Facility Location along a Highway**. In: *Proceedings of the 7th Annual International Conference on Computing and Combinatorics (COCOON).* 2001, pp. 237–246. DOI: 10.1007/3-540-44679-6_26.

[Ahn+04] Hee-Kap Ahn, Siu-Wing Cheng, Otfried Cheong, Mordecai J. Golin, and René van Oostrum. **Competitive facility location: the Voronoi game**. In: *Theor. Comput. Sci.* 310.1-3 (2004), pp. 457–467. DOI: 10.1016/j.tcs.2003.09.004.

[AK14] Haris Aziz and Bart de Keijzer. **Shapley meets Shapley**. In: *Proceedings of the 31st International Symposium on Theoretical Aspects of Computer Science (STACS).* 2014, pp. 99–111. DOI: 10.4230/LIPIcs.STACS.2014.99.

[Ala+06] Sebastian Aland, Dominic Dumrauf, Martin Gairing, Burkhard Monien, and Florian Schoppmann. **Exact Price of Anarchy for Polynomial Congestion Games**. In: *Proceedings of the 23rd Annual Symposium on Theoretical Aspects of Computer Science (STACS).* 2006, pp. 218–229. DOI: 10.1007/11672142_17.

[Ala+11] Sebastian Aland, Dominic Dumrauf, Martin Gairing, Burkhard Monien, and Florian Schoppmann. **Exact Price of Anarchy for Polynomial Congestion Games**. In: *SIAM J. Comput.* 40.5 (2011), pp. 1211–1233. DOI: 10.1137/090748986.

[AME10] Néstor E. Aguilera, Silvia C. Di Marco, and Mariana S. Escalante. **The Shapley value for arbitrary families of coalitions**. In: *European Journal of Operational Research* 204.1 (2010), pp. 125–138. DOI: 10.1016/j.ejor.2009.09.031.

[Ans+08] Elliot Anshelevich, Anirban Dasgupta, Éva Tardos, and Tom Wexler. **Near-Optimal Network Design with Selfish Agents**. In: *Theory of Computing* 4.1 (2008), pp. 77–109. DOI: 10.4086/toc.2008.v004a004.

[ARV08] Heiner Ackermann, Heiko Röglin, and Berthold Vöcking. **On the impact of combinatorial structure on congestion games**. In: *J. ACM* 55.6 (2008), 25:1–25:22. DOI: 10.1145/1455248.1455249.

[ARV09] Heiner Ackermann, Heiko Röglin, and Berthold Vöcking. **Pure Nash equilibria in player-specific and weighted congestion games**. In: *Theor. Comput. Sci.* 410.17 (2009), pp. 1552–1563. DOI: 10.1016/j.tcs.2008.12.035.

[AS08] Heiner Ackermann and Alexander Skopalik. **Complexity of Pure Nash Equilibria in Player-Specific Network Congestion Games**. In: *Internet Mathematics* 5.4 (2008), pp. 323–342. DOI: 10.1080/15427951.2008.10129170.

[Aul+16] Vincenzo Auletta, Ioannis Caragiannis, Diodato Ferraioli, Clemente Galdi, and Giuseppe Persiano. **Generalized Discrete Preference Games**. In: *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence (IJCAI)*. 2016, pp. 53–59.

[Aul+17] Vincenzo Auletta, Ioannis Caragiannis, Diodato Ferraioli, Clemente Galdi, and Giuseppe Persiano. **Robustness in Discrete Preference Games**. In: *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems (AAMAS)*. 2017, pp. 1314–1322.

[Aum59] Robert J Aumann. **Acceptable Points in General Cooperative n-Person Games**. In: *Contributions to the Theory of Games* 4 (1959), pp. 287–324.

[Aum87] Robert J Aumman. **What is Game Theory Trying to Accomplish**. In: *Frontiers of Economics, edited by Kenneth J. Arrow and S. Honkapohja* (1987), pp. 28–100.

[Awe+08] Baruch Awerbuch, Yossi Azar, Amir Epstein, Vahab S. Mirrokni, and Alexander Skopalik. **Fast convergence to nearly optimal solutions in potential games**. In: *Proceedings of the 9th ACM Conference on Electronic Commerce (EC)*. 2008, pp. 264–273. DOI: 10.1145/1386790.1386832.

[Axe84] Robert Axelrod. **The Evolution of Cooperation**. In: *Basic Books* (1984).

[Azi+09] Haris Aziz, Oded Lachish, Mike Paterson, and Rahul Savani. **Power Indices in Spanning Connectivity Games**. In: *Proceedings of the 5th International Conference on Algorithmic Aspects in Information and Management (AAIM)*. 2009, pp. 55–67. DOI: 10.1007/978-3-642-02158-9_7.

[Bac+10] Yoram Bachrach, Evangelos Markakis, Ezra Resnick, Ariel D. Procaccia, Jeffrey S. Rosenschein, and Amin Saberi. **Approximating power indices: theoretical and empirical analysis**. In: *Autonomous Agents and Multi-Agent Systems* 20.2 (2010), pp. 105–122. DOI: 10.1007/s10458-009-9078-9.

[Ban+15] Sayan Bandyapadhyay, Aritra Banik, Sandip Das, and Hirak Sarkar. **Voronoi game on graphs**. In: *Theor. Comput. Sci.* 562 (2015), pp. 270–282. DOI: 10.1016/j.tcs.2014.10.003.

[BCK10] Anand Bhalgat, Tanmoy Chakraborty, and Sanjeev Khanna. **Approximating pure nash equilibrium in cut, party affiliation, and satisfiability games**. In: *Proceedings of the 11th ACM Conference on Electronic Commerce (EC)*. 2010, pp. 73–82. DOI: 10.1145/1807342.1807353.

[BFH13] Stefan Böttcher, Matthias Feldotto, and Rita Hartel. **Schema-based Parallel Compression and Decompression of XML Data**. In: *Proceedings of the 9th International Conference on Web Information Systems and Technologies (WEBIST)*. 2013, pp. 77–86. DOI: 10.5220/0004366300770086.

[BFM16] Vittorio Bilò, Angelo Fanelli, and Luca Moscardelli. **Opinion Formation Games with Dynamic Social Influences**. In: *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*. 2016, pp. 444–458. DOI: 10.1007/978-3-662-54110-4_31.

[BFM18] Vittorio Bilò, Angelo Fanelli, and Luca Moscardelli. **Opinion formation games with dynamic social influences**. In: *Theor. Comput. Sci.* 746 (2018), pp. 73–87. DOI: 10.1016/j.tcs.2018.06.025.

[BGM13] Kshipra Bhawalkar, Sreenivas Gollapudi, and Kamesh Munagala. **Coevolutionary opinion formation games**. In: *Proceedings of the 45th ACM Symposium on Theory of Computing (STOC)*. 2013, pp. 41–50. DOI: 10.1145/2488608.2488615.

[BGR14] Kshipra Bhawalkar, Martin Gairing, and Tim Roughgarden. **Weighted Congestion Games: The Price of Anarchy, Universal Worst-Case Examples, and Tightness**. In: *ACM Trans. Economics and Comput.* 2.4 (2014), 14:1–14:23. DOI: 10.1145/2629666.

[Bil16] Vittorio Bilò. **On the Robustness of the Approximate Price of Anarchy in Generalized Congestion Games**. In: *Proceedings of the 9th International Symposium on Algorithmic Game Theory (SAGT)*. 2016, pp. 93–104. DOI: 10.1007/978-3-662-53354-3_8.

[Bil18] Vittorio Bilò. **A Unifying Tool for Bounding the Quality of Non-Cooperative Solutions in Weighted Congestion Games**. In: *Theory Comput. Syst.* 62.5 (2018), pp. 1288–1317. DOI: 10.1007/s00224-017-9826-1.

[BKO11] David Bindel, Jon M. Kleinberg, and Sigal Oren. **How Bad is Forming Your Own Opinion?** In: *Proceedings of the IEEE 52nd Annual Symposium on Foundations of Computer Science (FOCS)*. 2011, pp. 57–66. DOI: 10.1109/FOCS.2011.43.

[BKO15] David Bindel, Jon M. Kleinberg, and Sigal Oren. **How bad is forming your own opinion?** In: *Games and Economic Behavior* 92 (2015), pp. 248–265. DOI: 10.1016/j.geb.2014.06.004.

[BO07] Ron Banner and Ariel Orda. **Bottleneck Routing Games in Communication Networks**. In: *IEEE Journal on Selected Areas in Communications* 25.6 (2007), pp. 1173–1179. DOI: 10.1109/JSAC.2007.070811.

[BPJ09] Andrew Byde, Maria Polukarov, and Nicholas R. Jennings. **Games with Congestion-Averse Utilities**. In: *Proceedings of the Second International Symposium on Algorithmic Game Theory (SAGT)*. 2009, pp. 220–232. DOI: 10.1007/978-3-642-04645-2_20.

[BPZ15] Maria-Florina Balcan, Ariel D. Procaccia, and Yair Zick. **Learning Cooperative Games**. In: *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence (IJCAI)*. 2015, pp. 475–481.

[Bra+14] Sonja Brangewitz, Alexander Jungmann, Marie Christin Platenius, and Ronald Petrlic. **Towards a Flexible and Privacy-Preserving Reputation System for Markets of Composed Services**. In: *Proceedings of the 6th International Conferences on Advanced Service Computing (SERVICE COMPUTATION)*. 2014.

[Bre11] Steffen Brenner. **Location (Hotelling) Games and Applications**. In: *Wiley Encyclopedia of Operations Research and Management Science*. 2011. Chap. 1, pp. 1–11. DOI: 10.1002/9780470400531.eorms0477.

[Bri02] René van den Brink. **An axiomatization of the Shapley value using a fairness property**. In: *Int. J. Game Theory* 30.3 (2002), pp. 309–319. DOI: 10.1007/s001820100079.

[BSV17] Eric Balkanski, Umar Syed, and Sergei Vassilvitskii. **Statistical Cost Sharing**. In: *Advances in Neural Information Processing Systems 30: Annual Conference on Neural Information Processing Systems 2017*. 2017, pp. 6222–6231.

[BT17] Omer Ben-Porat and Moshe Tennenholtz. **Shapley Facility Location Games**. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE)*. 2017, pp. 58–73. DOI: 10.1007/978-3-319-71924-5_5.

[BV17] Vittorio Bilò and Cosimo Vinci. **On the Impact of Singleton Strategies in Congestion Games**. In: *Proceedings of the 25th Annual European Symposium on Algorithms (ESA)*. 2017, 17:1–17:14. DOI: 10.4230/LIPIcs.ESA.2017.17.

[Cal+02] Tomasa Calvo, Anna Kolesárová, Magda Komorníková, and Radko Mesiar. **Aggregation Operators: Properties, Classes and Construction Methods**. In: *Aggregation Operators: New Trends and Applications*. Heidelberg, 2002, pp. 3–104. DOI: 10.1007/978-3-7908-1787-4_1.

[Car+11a] Ioannis Caragiannis, Angelo Fanelli, Nick Gravin, and Alexander Skopalik. **Efficient Computation of Approximate Pure Nash Equilibria in Congestion Games**. In: *Proceedings of the IEEE 52nd Annual Symposium on Foundations of Computer Science (FOCS)*. 2011, pp. 532–541. DOI: 10.1109/FOCS.2011.50.

[Car+11b] Ioannis Caragiannis, Michele Flammini, Christos Kaklamanis, Panagiotis Kanellopoulos, and Luca Moscardelli. **Tight Bounds for Selfish and Greedy Load Balancing**. In: *Algorithmica* 61.3 (2011), pp. 606–637. DOI: 10.1007/s00453-010-9427-8.

[Car+12] Ioannis Caragiannis, Angelo Fanelli, Nick Gravin, and Alexander Skopalik. **Approximate pure nash equilibria in weighted congestion games: existence, efficient computation, and structure**. In: *Proceedings of the 13th ACM Conference on Electronic Commerce (EC)*. 2012, pp. 284–301. DOI: 10.1145/2229012.2229036.

[Car+15] Ioannis Caragiannis, Angelo Fanelli, Nick Gravin, and Alexander Skopalik. **Approximate Pure Nash Equilibria in Weighted Congestion Games: Existence, Efficient Computation, and Structure**. In: *ACM Trans. Economics and Comput.* 3.1 (2015), 2:1–2:32. DOI: 10.1145/2614687.

[CCL16] Po-An Chen, Yi-Le Chen, and Chi-Jen Lu. **Bounds on the price of anarchy for a more general class of directed graphs in opinion formation games**. In: *Oper. Res. Lett.* 44.6 (2016), pp. 808–811. DOI: 10.1016/j.orl.2016.10.001.

[CDT09] Xi Chen, Xiaotie Deng, and Shang-Hua Teng. **Settling the complexity of computing two-player Nash equilibria**. In: *J. ACM* 56.3 (2009), 14:1–14:57. DOI: 10.1145/1516512.1516516.

[CFG17] Ioannis Caragiannis, Angelo Fanelli, and Nick Gravin. **Short Sequences of Improvement Moves Lead to Approximate Equilibria in Constraint Satisfaction Games**. In: *Algorithmica* 77.4 (2017), pp. 1143–1158. DOI: 10.1007/s00453-016-0143-x.

[CG16] George Christodoulou and Martin Gairing. **Price of Stability in Polynomial Congestion Games**. In: *ACM Trans. Economics and Comput.* 4.2 (2016), 10:1–10:17. DOI: 10.1145/2841229.

[CGT09] Javier Castro, Daniel Gómez, and Juan Tejada. **Polynomial calculation of the Shapley value based on sampling**. In: *Computers & OR* 36.5 (2009), pp. 1726–1730. DOI: 10.1016/j.cor.2008.04.004.

[CH10] Jean Cardinal and Martin Hoefer. **Non-cooperative facility location and covering games**. In: *Theor. Comput. Sci.* 411.16-18 (2010), pp. 1855–1876. DOI: 10.1016/j.tcs.2010.02.005.

[Che52] Herman Chernoff. **A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on the sum of Observations**. In: *Ann. Math. Statist.* 23.4 (Dec. 1952), pp. 493–507. DOI: 10.1214/aoms/1177729330.

[Che67] Pafnutii Lvovich Chebyshev. **Des valeurs moyennes, Liouville's**. In: *J. Math. Pures Appl.* 12 (1867), pp. 177–184.

[Chr+18] George Christodoulou, Martin Gairing, Yiannis Giannakopoulos, and Paul G. Spirakis. **The Price of Stability of Weighted Congestion Games**. In: *Proceedings of the 45th International Colloquium on Automata, Languages, and Programming (ICALP)*. 2018, 150:1–150:16. DOI: 10.4230/LIPIcs.ICALP.2018.150.

[CK05a] George Christodoulou and Elias Koutsoupias. **On the Price of Anarchy and Stability of Correlated Equilibria of Linear Congestion Games**. In: *Proceedings of the 13th Annual European Symposium on Algorithms (ESA)*. 2005, pp. 59–70. DOI: 10.1007/11561071_8.

[CK05b] George Christodoulou and Elias Koutsoupias. **The price of anarchy of finite congestion games**. In: *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*. 2005, pp. 67–73. DOI: 10.1145/1060590.1060600.

[CKO13] Flavio Chierichetti, Jon M. Kleinberg, and Sigal Oren. **On discrete preferences and coordination**. In: *Proceedings of the 14th ACM Conference on Electronic Commerce (EC)*. 2013, pp. 233–250. DOI: 10.1145/2482540.2482573.

[CKO18] Flavio Chierichetti, Jon M. Kleinberg, and Sigal Oren. **On discrete preferences and coordination**. In: *J. Comput. Syst. Sci.* 93 (2018), pp. 11–29. DOI: 10.1016/j.jcss.2017.11.002.

[CKS09] George Christodoulou, Elias Koutsoupias, and Paul G. Spirakis. **On the Performance of Approximate Equilibria in Congestion Games**. In: *Proceedings of the 17th Annual European Symposium on Algorithms (ESA)*. 2009, pp. 251–262. DOI: 10.1007/978-3-642-04128-0_22.

[CKS11] George Christodoulou, Elias Koutsoupias, and Paul G. Spirakis. **On the Performance of Approximate Equilibria in Congestion Games**. In: *Algorithmica* 61.1 (2011), pp. 116–140. DOI: 10.1007/s00453-010-9449-2.

[CR09] Ho-Lin Chen and Tim Roughgarden. **Network Design with Weighted Players**. In: *Theory Comput. Syst.* 45.2 (2009), pp. 302–324. DOI: 10.1007/s00224-008-9128-8.

[CS11] Steve Chien and Alistair Sinclair. **Convergence to approximate Nash equilibria in congestion games**. In: *Games and Economic Behavior* 71.2 (2011), pp. 315–327. DOI: 10.1016/j.geb.2009.05.004.

[Deg74] Morris H. Degroot. **Reaching a Consensus**. In: *Journal of the American Statistical Association* 69.345 (1974), pp. 118–121. DOI: 10.1080/01621459.1974.10480137.

[DGP09] Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitriou. **The Complexity of Computing a Nash Equilibrium**. In: *SIAM J. Comput.* 39.1 (2009), pp. 195–259. DOI: 10.1137/070699652.

[dGT79] C. d'Aspremont, J. Jaskold Gabszewicz, and J.-F. Thisse. **On Hotelling's "Stability in Competition"**. In: *Econometrica* 47.5 (1979), pp. 1145–1150. DOI: 10.2307/1911955.

[DM65] Morton Davis and Michael Maschler. **The kernel of a cooperative game**. In: *Naval Research Logistics Quarterly* 12.3 (1965), pp. 223–259. DOI: 10.1002/nav.3800120303.

[Dow57] Anthony Downs. **An Economic Theory of Political Action in a Democracy**. In: *Journal of Political Economy* 65.2 (1957), pp. 135–150. DOI: 10.1086/257897.

[DP94] Xiaotie Deng and Christos H. Papadimitriou. **On the Complexity of Cooperative Solution Concepts**. In: *Math. Oper. Res.* 19.2 (1994), pp. 257–266. DOI: 10.1287/moor.19.2.257.

[Dre+15] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. **On Existence and Properties of Approximate Pure Nash Equilibria in Bandwidth Allocation Games**. In: *Proceedings of the 8th International Symposium on Algorithmic Game Theory (SAGT)*. 2015, pp. 178–189. DOI: 10.1007/978-3-662-48433-3_14.

[Dre+17] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. **Pure Nash Equilibria in Restricted Budget Games**. In: *Proceedings of the 23rd International Conference on Computing and Combinatorics (COCOON)*. 2017, pp. 175–187. DOI: 10.1007/978-3-319-62389-4_15.

[Dre+18] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. **Pure Nash equilibria in restricted budget games**. In: *J. Comb. Optim.* (2018). DOI: 10.1007/s10878-018-0269-7.

[DS08] Juliane Dunkel and Andreas S. Schulz. **On the Complexity of Pure-Strategy Nash Equilibria in Congestion and Local-Effect Games**. In: *Math. Oper. Res.* 33.4 (2008), pp. 851–868. DOI: 10.1287/moor.1080.0322.

[DT07] Christoph Dürr and Nguyen Kim Thang. **Nash Equilibria in Voronoi Games on Graphs**. In: *Proceedings of the 15th Annual European Symposium on Algorithms (ESA)*. 2007, pp. 17–28. DOI: 10.1007/978-3-540-75520-3_4.

[DT10] Reinhold Decker and Michael Trusov. **Estimating aggregate consumer preferences from online product reviews**. In: *International Journal of Research in Marketing* 27.4 (2010), pp. 293–307. DOI: 10.1016/j.ijresmar.2010.09.001.

[Dur+12] Richard Durrett, James P. Gleeson, Alun L. Lloyd, Peter J. Mucha, Feng Shi, David Sivakoff, Joshua E. S. Socolar, and Chris Varghese. **Graph fission in an evolving voter model**. In: *Proceedings of the National Academy of Sciences* (2012). DOI: 10.1073/pnas.1200709109.

[EL75] B Eaton and Richard Lipsey. **The Principle of Minimum Differentiation Reconsidered: Some New Developments in the Theory of Spatial Competition**. In: *The Review of Economic Studies* 42.1 (Jan. 1975), pp. 27–49. DOI: 10.2307/2296817.

[Elk+09] Edith Elkind, Leslie Ann Goldberg, Paul W. Goldberg, and Michael Wooldridge. **On the computational complexity of weighted voting games**. In: *Ann. Math. Artif. Intell.* 56.2 (2009), pp. 109–131. DOI: 10.1007/s10472-009-9162-5.

[ELT93] Horst A. Eiselt, Gilbert Laporte, and Jacques-François Thisse. **Competitive Location Models: A Framework and Bibliography**. In: *Transportation Science* 27.1 (1993), pp. 44–54. DOI: 10.1287/trsc.27.1.44.

[Epi+17] Markos Epitropou, Dimitris Fotakis, Martin Hoefer, and Stratis Skoulakis. **Opinion Formation Games with Aggregation and Negative Influence**. In: *Proceedings of the 10th International Symposium on Algorithmic Game Theory (SAGT)*. 2017, pp. 173–185. DOI: 10.1007/978-3-319-66700-3_14.

[Fel+17a] Matthias Feldotto, Martin Gairing, Grammateia Kotsialou, and Alexander Skopalik. **Computing Approximate Pure Nash Equilibria in Shapley Value Weighted Congestion Games**. In: *Proceedings of the 13th International Conference on Web and Internet Economics (WINE)*. 2017, pp. 191–204. DOI: 10.1007/978-3-319-71924-5_14.

[Fel+17b] Matthias Feldotto, Thomas John, Dennis Kundisch, Paul Hemsen, Katrin Klingsieck, and Alexander Skopalik. **Making Gamification Easy for the Professor: Decoupling Game and Content with the StudyNow Mobile App**. In: *Proceedings of the 12th International Conference on Designing the Digital Transformation (DESRIST)*. 2017, pp. 462–467. DOI: 10.1007/978-3-319-59144-5_32.

[Fel+18a] Björn Feldkord, Matthias Feldotto, Anupam Gupta, Guru Guruganesh, Amit Kumar, Sören Riechers, and David Wajc. **Fully-Dynamic Bin Packing with Little Repacking**. In: *Proceedings of the 45th International Colloquium on Automata, Languages, and Programming (ICALP)*. 2018, 51:1–51:24. DOI: 10.4230/LIPIcs.ICALP.2018.51.

[Fel+18b] Matthias Feldotto, Claus-Jochen Haake, Alexander Skopalik, and Nadja Stroh-Maraun. **Disaggregating User Evaluations Using the Shapley Value**. In: *Proceedings of the 13th Workshop on the Economics of Networks, Systems and Computation (NetEcon 2018)*. Irvine, California, USA, 2018. DOI: 10.1145/3230654.3230659.

[Fel+18c] Matthias Feldotto, Pascal Lenzner, Louise Molitor, and Alexander Skopalik. **Two-Sided Facility Location: From Hotelling to Load Balancing**. Unpublished Manuscript. 2018.

[Fel95] Vincent Feltkamp. **Alternative axiomatic characterizations of the Shapley and Banzhaf values**. In: *International Journal of Game Theory* 24.2 (June 1995), pp. 179–186. DOI: 10.1007/BF01240041.

[FFO16] Michal Feldman, Amos Fiat, and Svetlana Obraztsova. **Variations on the Hotelling-Downs Model**. In: *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence (AAAI)*. 2016, pp. 496–501.

[FFS18] Angelo Fanelli, Matthias Feldotto, and Alexander Skopalik. **Approximate Equilibria in Coevolutionary Opinion Formation Games**. Unpublished Manuscript. 2018.

[FG05] James W. Firedman and Isabel Grilo. **A Market with Social Consumption Externality**. In: *The Japanese Economic Review* 56.3 (2005), pp. 251–272. DOI: 10.1111/j.1468-5876.2005.00331.x.

[FG13] Matthias Feldotto and Kalman Graffi. **Comparative evaluation of peer-to-peer systems using PeerfactSim.KOM**. In: *Proceedings of the International Conference on High Performance Computing & Simulation, (HPCS)*. 2013, pp. 99–106. DOI: 10.1109/HPCSim.2013.6641399.

[FG16] Matthias Feldotto and Kalman Graffi. **Systematic evaluation of peer-to-peer systems using PeerfactSim.KOM**. In: *Concurrency and Computation: Practice and Experience* 28.5 (2016), pp. 1655–1677. DOI: 10.1002/cpe.3716.

[FGS14] Matthias Feldotto, Martin Gairing, and Alexander Skopalik. **Bounding the Potential Function in Congestion Games and Approximate Pure Nash Equilibria**. In: *Proceedings of the 10th International Conference on Web and Internet Economics (WINE)*. 2014, pp. 30–43. DOI: 10.1007/978-3-319-13129-0_3.

[FGV12] Diodato Ferraioli, Paul W. Goldberg, and Carmine Ventre. **Decentralized Dynamics for Finite Opinion Games**. In: *Proceedings of the 5th International Symposium on Algorithmic Game Theory (SAGT)*. 2012, pp. 144–155. DOI: 10.1007/978-3-642-33996-7_13.

[FGV16] Diodato Ferraioli, Paul W. Goldberg, and Carmine Ventre. **Decentralized dynamics for finite opinion games**. In: *Theor. Comput. Sci.* 648 (2016), pp. 96–115. DOI: 10.1016/j.tcs.2016.08.011.

[FJ90] Noah E. Friedkin and Eugene C. Johnsen. **Social influence and opinions**. In: *The Journal of Mathematical Sociology* 15.3-4 (1990), pp. 193–206. DOI: 10.1080/0022250X.1990.9990069.

[FKS05] Dimitris Fotakis, Spyros C. Kontogiannis, and Paul G. Spirakis. **Selfish unsplittable flows**. In: *Theor. Comput. Sci.* 348.2-3 (2005), pp. 226–239. DOI: 10.1016/j.tcs.2005.09.024.

[FLS16] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. **Congestion Games with Mixed Objectives**. In: *Proceedings of the10th International Conference on Combinatorial Optimization and Applications (COCOA)*. 2016, pp. 655–669. DOI: 10.1007/978-3-319-48749-6_47.

[FLS17] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. **Congestion Games with Complementarities**. In: *Proceedings of the 10th International Conference on Algorithms and Complexity (CIAC)*. 2017, pp. 222–233. DOI: 10.1007/978-3-319-57586-5_19.

[FLS18] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. **Congestion games with mixed objectives**. In: *J. Comb. Optim.* 36.4 (2018), pp. 1145–1167. DOI: 10.1007/s10878-017-0189-y.

[Fou16] Gaëtan Fournier. **General distribution of consumers in pure Hotelling games**. In: *CoRR* abs/1602.04851 (2016). arXiv: `1602.04851`.

[FPS16] Dimitris Fotakis, Dimitris Palyvos-Giannas, and Stratis Skoulakis. **Opinion Dynamics with Local Interactions**. In: *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence (IJCAI)*. 2016, pp. 279–285.

[FPT04] Alex Fabrikant, Christos H. Papadimitriou, and Kunal Talwar. **The complexity of pure Nash equilibria**. In: *Proceedings of the 36th Annual ACM Symposium on Theory of Computing (STOC)*. 2004, pp. 604–612. DOI: `10.1145/1007352.1007445`.

[FS14] Matthias Feldotto and Alexander Skopalik. **A simulation framework for analyzing complex infinitely repeated games**. In: *Proceedings of the 4th International Conference on Simulation And Modeling Methodologies, Technologies And Applications (SIMULTECH)*. 2014, pp. 625–630. DOI: `10.5220/0005110406250630`.

[FS16] Gaëtan Fournier and Marco Scarsini. **Location Games on Networks: Existence and Efficiency of Equilibria**. In: *CoRR* abs/1601.07414 (2016). arXiv: `1601.07414`.

[FSG14] Matthias Feldotto, Christian Scheideler, and Kalman Graffi. **HSkip+: A self-stabilizing overlay network for nodes with heterogeneous bandwidths**. In: *Proceedings of the 14th IEEE International Conference on Peer-to-Peer Computing (P2P)*. 2014, pp. 1–10. DOI: `10.1109/P2P.2014.6934300`.

[FT14] Dimitris Fotakis and Christos Tzamos. **On the Power of Deterministic Mechanisms for Facility Location Games**. In: *ACM Trans. Economics and Comput.* 2.4 (2014), 15:1–15:37. DOI: `10.1145/2665005`.

[FV17] Diodato Ferraioli and Carmine Ventre. **Social Pressure in Opinion Games**. In: *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence (IJCAI)*. 2017, pp. 3661–3667. DOI: `10.24963/ijcai.2017/512`.

[FWJ08] S. Shaheen Fatima, Michael Wooldridge, and Nicholas R. Jennings. **A linear approximation method for the Shapley value**. In: *Artif. Intell.* 172.14 (2008), pp. 1673–1699. DOI: `10.1016/j.artint.2008.05.003`.

[Gar+09] Florent Garcin, Boi Faltings, Radu Jurca, and Nadine Joswig. **Rating aggregation in collaborative filtering systems**. In: *Proceedings of the 2009 ACM Conference on Recommender Systems (RecSys)*. 2009, pp. 349–352. DOI: `10.1145/1639714.1639785`.

[Gau66] C.F. Gauss. **Carl Friedrich Gauss Werke: Bd. Analysis**. Carl Friedrich Gauss Werke. Dieterichsche Universitäts-Druckerei W. Fr. Kaestner, 1866.

[GFJ09] Florent Garcin, Boi Faltings, and Radu Jurca. **Aggregating reputation feedback**. In: *Proceedings of the First International Conference on Reputation: Theory and Technology*. 2009, pp. 62–74.

[Gil59] Donald B Gillies. **Solutions to general non-zero-sum games**. In: *Contributions to the Theory of Games* 4.40 (1959), pp. 47–85.

[GKK15] Martin Gairing, Konstantinos Kollias, and Grammateia Kotsialou. **Tight Bounds for Cost-Sharing in Weighted Congestion Games**. In: *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP)*. 2015, pp. 626–637. DOI: 10.1007/978-3-662-47666-6_50.

[GKR14] Vasilis Gkatzelis, Konstantinos Kollias, and Tim Roughgarden. **Optimal Cost-Sharing in Weighted Congestion Games**. In: *Proceedings of the 10th International Conference on Web and Internet Economics (WINE)*. 2014, pp. 72–88. DOI: 10.1007/978-3-319-13129-0_6.

[GMV05] Michel X. Goemans, Vahab S. Mirrokni, and Adrian Vetta. **Sink Equilibria and Convergence**. In: *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. 2005, pp. 142–154. DOI: 10.1109/SFCS.2005.68.

[GMW14] Ragavendran Gopalakrishnan, Jason R. Marden, and Adam Wierman. **Potential Games Are *Necessary* to Ensure Pure Nash Equilibria in Cost Sharing Games**. In: *Math. Oper. Res.* 39.4 (2014), pp. 1252–1296. DOI: 10.1287/moor.2014.0651.

[GNS18] Yiannis Giannakopoulos, Georgy Noarov, and Andreas S. Schulz. **An Improved Algorithm for Computing Approximate Equilibria in Weighted Congestion Games**. In: *CoRR* abs/1810.12806 (2018). arXiv: 1810.12806.

[Gra+11] Michel Grabisch, Jean-Luc Marichal, Radko Mesiar, and Endre Pap. **Aggregation functions: Means**. In: *Inf. Sci.* 181.1 (2011), pp. 1–22. DOI: 10.1016/j.ins.2010.08.043.

[GS04] Michel X. Goemans and Martin Skutella. **Cooperative facility location games**. In: *J. Algorithms* 50.2 (2004), pp. 194–214. DOI: 10.1016/S0196-6774(03)00098-1.

[GS07] Martin Gairing and Florian Schoppmann. **Total Latency in Singleton Congestion Games**. In: *Proceedings of the Third International Workshop on Internet and Network Economics (WINE)*. 2007, pp. 381–387. DOI: 10.1007/978-3-540-77105-0_42.

[GST01] Isabel Grilo, Oz Shy, and Jacques-François Thisse. **Price competition when consumer behavior is characterized by conformity or vanity**. In: *Journal of Public Economics* 80.3 (2001), pp. 385–408. DOI: 10.1016/S0047-2727(00)00115-8.

[GXF13] Florent Garcin, Lirong Xia, and Boi Faltings. **How aggregators influence human rater behavior**. In: *Proceedings of the Workshop at 14th ACM Conference on Electronic Commerce (EC)*. 2013.

[Hap+13] Markus Happe, Friedhelm Meyer auf der Heide, Peter Kling, Marco Platzner, and Christian Plessl. **On-The-Fly Computing: A novel paradigm for individualized IT services**. In: *Proccedings of the 16th IEEE International Symposium on Object/Component/Service-Oriented Real-Time Distributed Computing (ISORC)*. 2013, pp. 1–10. DOI: 10.1109/ISORC.2013.6913232.

[Har+13] Tobias Harks, Martin Hoefer, Max Klimm, and Alexander Skopalik. **Computing pure Nash and strong equilibria in bottleneck congestion games**. In: *Math. Program.* 141.1-2 (2013), pp. 193–215. DOI: 10.1007/s10107-012-0521-3.

[Har73] John C. Harsanyi. **Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points**. In: *International Journal of Game Theory* 2.1 (Dec. 1973), pp. 1–23. DOI: 10.1007/BF01737554.

[Hei14] T. Heikkinen. **A spatial economic model under network externalities: symmetric equilibrium and efficiency**. In: *Operational Research* 14.1 (2014), pp. 89–111. DOI: 10.1007/s12351-013-0136-3.

[HK02] Rainer Hegselmann and Ulrich Krause. **Opinion dynamics and bounded confidence: models, analysis and simulation**. In: *J. Artificial Societies and Social Simulation* 5.3 (2002).

[HK12] Tobias Harks and Max Klimm. **On the Existence of Pure Nash Equilibria in Weighted Congestion Games**. In: *Math. Oper. Res.* 37.3 (2012), pp. 419–436. DOI: 10.1287/moor.1120.0543.

[HKM09] Tobias Harks, Max Klimm, and Rolf H. Möhring. **Strong Nash Equilibria in Games with the Lexicographical Improvement Property**. In: *Proceedings of the 5th International Workshop on Internet and Network Economics (WINE)*. 2009, pp. 463–470. DOI: 10.1007/978-3-642-10841-9_43.

[HKM11] Tobias Harks, Max Klimm, and Rolf H. Möhring. **Characterizing the Existence of Potential Functions in Weighted Congestion Games**. In: *Theory Comput. Syst.* 49.1 (2011), pp. 46–70. DOI: 10.1007/s00224-011-9315-x.

[HKS14] Christoph Hansknecht, Max Klimm, and Alexander Skopalik. **Approximate Pure Nash Equilibria in Weighted Congestion Games**. In: *Proceedings of the 17th International Workshop on Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM)*. 2014, pp. 242–257. DOI: 10.4230/LIPIcs.APPROX-RANDOM.2014.242.

[HM89] Sergiu Hart and Andreu Mas-Colell. **Potential, Value, and Consistency**. In: *Econometrica* 57.3 (1989), pp. 589–614. DOI: 10.2307/1911054.

[Hoe63a] Wassily Hoeffding. **Probability Inequalities for Sums of Bounded Random Variables**. In: *Journal of the American Statistical Association* 58.301 (1963), pp. 13–30. DOI: 10.2307/2282952.

[Hoe63b] Wassily Hoeffding. **Probability Inequalities for Sums of Bounded Random Variables**. In: *Journal of the American Statistical Association* 58.301 (1963), pp. 13–30. DOI: 10.1080/01621459.1963.10500830.

[Hot29] Harold Hotelling. **Stability in Competition**. In: *The Economic Journal* 39.153 (1929), pp. 41–57. DOI: 10.2307/2224214.

[Hot90] Harold Hotelling. **Stability in Competition**. In: *The Collected Economics Articles of Harold Hotelling*. New York, NY, 1990, pp. 50–63. DOI: 10.1007/978-1-4613-8905-7_4.

[HPZ11] Adam Douglas Henry, Paweł Prałat, and Cun-Quan Zhang. **Emergence of segregation in evolving social networks**. In: *Proceedings of the National Academy of Sciences* 108.21 (2011), pp. 8605–8610. DOI: 10.1073/pnas.1014486108.

[HR82] Jack Hirshleifer and Paul H Rubin. **Evolutionary Models in Economics and Law**. Jai Press, 1982.

[HS98] Josef Hofbauer and Karl Sigmund. **Evolutionary Games and Population Dynamics**. Cambridge University Press, 1998.

[Ieo+05] Samuel Ieong, Robert McGrew, Eugene Nudelman, Yoav Shoham, and Qixiang Sun. **Fast and Compact: A Simple Class of Congestion Games**. In: *Proceedings of the Twentieth National Conference on Artificial Intelligence and the Seventeenth Innovative Applications of Artificial Intelligence Conference*. 2005, pp. 489–494.

[II06] Segismundo S. Izquierdo and Luis R. Izquierdo. **On the Structural Robustness of Evolutionary Models of Cooperation**. In: *Proceedings of the 7th International Conference on Intelligent Data Engineering and Automated Learning (IDEAL)*. 2006, pp. 172–182. DOI: 10.1007/11875581_21.

[IS05] Samuel Ieong and Yoav Shoham. **Marginal contribution nets: a compact representation scheme for coalitional games**. In: *Proceedings of the 6th ACM Conference on Electronic Commerce (EC)*. 2005, pp. 193–202. DOI: 10.1145/1064009.1064030.

[Jac08] Matthew O. Jackson. **Social and Economic Networks**. Princeton, NJ, USA: Princeton University Press, 2008.

[Joh+17] Thomas John, Matthias Feldotto, Paul Hemsen, Katrin Klingsieck, Dennis Kundisch, and Mike Langendorf. **Towards a Lean Approach to Gamifying Education**. In: *Proceedings of the 25th European Conference on Information Systems (ECIS)*. 2017.

[JPY88] David S. Johnson, Christos H. Papadimitriou, and Mihalis Yannakakis. **How Easy is Local Search?** In: *J. Comput. Syst. Sci.* 37.1 (1988), pp. 79–100. DOI: 10.1016/0022-0000(88)90046-3.

[Koh83] Elon Kohlberg. **Equilibrium store locations when consumers minimize travel time plus waiting time**. In: *Economics Letters* 11.3 (1983), pp. 211–216. DOI: 10.1016/0165-1765(83)90137-4.

[KP09] Elias Koutsoupias and Christos H. Papadimitriou. **Worst-case equilibria**. In: *Computer Science Review* 3.2 (2009), pp. 65–69. DOI: 10.1016/j.cosrev.2009.04.003.

[KP99] Elias Koutsoupias and Christos H. Papadimitriou. **Worst-case Equilibria**. In: *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*. 1999, pp. 404–413. DOI: 10.1007/3-540-49116-3_38.

[KR15] Konstantinos Kollias and Tim Roughgarden. **Restoring Pure Equilibria to Weighted Congestion Games**. In: *ACM Trans. Economics and Comput.* 3.4 (2015), 21:1–21:24. DOI: 10.1145/2781678.

[KS15] Max Klimm and Daniel Schmand. **Sharing Non-anonymous Costs of Multiple Resources Optimally**. In: *Proceedings of the 9th International Conference on Algorithms and Complexity (CIAC)*. 2015, pp. 274–287. DOI: 10.1007/978-3-319-18173-8_20.

[KS17] Pieter Kleer and Guido Schäfer. **Potential Function Minimizers of Combinatorial Congestion Games: Efficiency and Computation**. In: *Proceedings of the 2017 ACM Conference on Economics and Computation (EC)*. 2017, pp. 223–240. DOI: 10.1145/3033274.3085149.

[Kuk07] Nikolai S. Kukushkin. **Congestion games revisited**. In: *Int. J. Game Theory* 36.1 (2007), pp. 57–83. DOI: 10.1007/s00182-007-0090-5.

[Kuk15] Nikolai S. Kukushkin. **Rosenthal's potential and a discrete version of the Debreu-Gorman Theorem**. In: *Automation and Remote Control* 76.6 (2015), pp. 1101–1110. DOI: 10.1134/S0005117915060144.

[Lee03] Dennis Leech. **Computing Power Indices for Large Voting Games**. In: *Management Science* 49.6 (2003), pp. 831–837. DOI: 10.1287/mnsc.49.6.831.16024.

[LHL11] Lok Man Law, Jianwei Huang, and Mingyan Liu. **Price of anarchy of congestion games with player-specific constants**. In: *Proccedings of the International Conference on Wireless Communications & Signal Processing (WCSP)*. 2011, pp. 1–5. DOI: 10.1109/WCSP.2011.6096882.

[Lib+12] David Liben-Nowell, Alexa Sharp, Tom Wexler, and Kevin Woods. **Computing Shapley Value in Supermodular Coalitional Games**. In: *Proceedings of the 18th Annual International Conference on Computing and Combinatorics (COCOON)*. 2012, pp. 568–579. DOI: 10.1007/978-3-642-32241-9_48.

[Liu+12] An Liu, Qing Li, Liusheng Huang, and Shiting Wen. **Shapley Value Based Impression Propagation for Reputation Management in Web Service Composition**. In: *Proceedings of the IEEE 19th International Conference on Web Services*. 2012, pp. 58–65. DOI: 10.1109/ICWS.2012.85.

[Liu+14] An Liu, Qing Li, Xiaofang Zhou, Lu Li, Guanfeng Liu, and Yunjun Gao. **Rating Propagation in Web Services Reputation Systems: A Fast Shapley Value Approach**. In: *Proceedings of the 19th International Conference on Database Systems for Advanced Applications (DASFAA)*. 2014, pp. 466–480. DOI: 10.1007/978-3-319-05810-8_31.

[LO01] Lavy Libman and Ariel Orda. **Atomic Resource Sharing in Noncooperative Networks**. In: *Telecommunication Systems* 17.4 (2001), pp. 385–409. DOI: 10.1023/A:1016770831869.

[LO13] Luca Lambertini and Raimondello Orsini. **On Hotelling's 'stability in competition' with network externalities and switching costs**. In: *Papers in Regional Science* 92.4 (2013), pp. 873–883. DOI: 10.1111/j.1435-5957.2012.00469.x.

[LR57] Robert Duncan Luce and Howard Raiffa. **Games and Decisions: Introduction and Critical Surveys**. Wiley New York, 1957.

[Mal15] Sasan Maleki. "Addressing the computational issues of the Shapley value with applications in the smart grid". PhD thesis. University of Southampton, UK, 2015.

[Mav+07] Marios Mavronicolas, Igal Milchtaich, Burkhard Monien, and Karsten Tiemann. **Congestion Games with Player-Specific Constants**. In: *Proceedings of the 32nd International Symposium on Mathematical Foundations of Computer Science (MFCS)*. 2007, pp. 633–644. DOI: 10.1007/978-3-540-74456-6_56.

[Mey01] Adam Meyerson. **Online Facility Location**. In: *Proceedings of the 42nd Annual Symposium on Foundations of Computer Science (FOCS)*. 2001, pp. 426–431. DOI: 10.1109/SFCS.2001.959917.

[MGR10] Mary McGlohon, Natalie S. Glance, and Zach Reiter. **Star Quality: Aggregating Reviews to Rank Products and Merchants**. In: *Proceedings of the Fourth International Conference on Weblogs and Social Media (ICWSM)*. 2010.

[Mil96] Igal Milchtaich. **Congestion Games with Player-Specific Payoff Functions**. In: *Games and Economic Behavior* 13.1 (1996), pp. 111–124. DOI: 10.1006/game.1996.0027.

[Mir+17] Behnud Mir Djawadi, Rene Fahr, Claus-Jochen Haake, and Sonja Recker. **Maintaining vs. Milking Good Reputation When Customer Feedback is Inaccurate**. In: *SSRN* (2017). DOI: 10.2139/ssrn.3024561.

[MM01] Yasuko Matsui and Tomomi Matsui. **NP-completeness for calculating power indices of weighted majority games**. In: *Theor. Comput. Sci.* 263.1-2 (2001), pp. 305–310. DOI: 10.1016/S0304-3975(00)00251-6.

[MS62] Irwin Mann and Lloyd S Shapley. *Values of Large Games. 6: Evaluating the Electoral College Exactly.* Tech. rep. DTIC Document, 1962.

[MS96] Dov Monderer and Lloyd S. Shapley. **Potential Games**. In: *Games and Economic Behavior* 14.1 (1996), pp. 124–143. DOI: 10.1006/game.1996.0044.

[MV44] Oskar Morgenstern and John Von Neumann. **Theory of Games and Economic Behavior**. Princeton University Press, 1944.

[Mye13] Roger B Myerson. **Game theory**. Harvard university press, 2013.

[Mye97] Roger B. Myerson. **Game theory - Analysis of Conflict**. Harvard University Press, 1997.

[Nas51] John Nash. **Non-Cooperative Games**. In: *Annals of Mathematics* 54.2 (1951), pp. 286–295. DOI: 10.2307/1969529.

[Nis+07] Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V Vazirani. **Algorithmic game theory**. Vol. 1. Cambridge University Press Cambridge, 2007.

[NS16] Matías Núñez and Marco Scarsini. **Competing over a finite number of locations**. In: *Economic Theory Bulletin* 4.2 (Oct. 2016), pp. 125–136. DOI: 10.1007/s40505-015-0068-6.

[NS17] Matías Núñez and Marco Scarsini. **Large Spatial Competition**. In: *Spatial Interaction Models : Facility Location Using Game Theory*. Cham, 2017, pp. 225–246. DOI: 10.1007/978-3-319-52654-6_10.

[NST+95] Ami Navon, Oz Shy, Jacques-François Thisse, et al. **Product differentiation in the presence of positive and negative network effects**. Centre for Economic Policy Research, 1995.

[OP86] Martin J Osborne and Carolyn Pitchik. **The nature of equilibrium in a location model**. In: *International Economic Review* (1986), pp. 223–237. DOI: 10.2307/2526617.

[OPS04] James B. Orlin, Abraham P. Punnen, and Andreas S. Schulz. **Approximate Local Search in Combinatorial Optimization**. In: *SIAM J. Comput.* 33.5 (2004), pp. 1201–1214. DOI: 10.1137/S0097539703431007.

[Owe72] Guillermo Owen. **Multilinear Extensions of Games**. In: *Management Science* 18.5-part-2 (1972), pp. 64–79. DOI: 10.1287/mnsc.18.5.64.

[Pál11] Dénes Pálvölgyi. **Hotelling on graphs**. In: *Mimeo.* 2011.

[Pap01] Christos H. Papadimitriou. **Algorithms, games, and the internet**. In: *Proceedings on 33rd Annual ACM Symposium on Theory of Computing (STOC)*. 2001, pp. 749–753. DOI: 10.1145/380752.380883.

[PB12] Charles H Pence and Lara Buchak. **Oyun: A New, Free Program for Iterated Prisoner's Dilemma Tournaments in the Classroom**. In: *Evolution: Education and Outreach* 5.3 (2012), pp. 467–476. DOI: 10.1007/s12052-012-0434-x.

[PL89] André de Palma and Luc Leruth. **Congestion and Game in Capacity: A Duopoly Analysis in the Presence of Network Externalities**. In: *Annales d'Économie et de Statistique* 15/16 (1989), pp. 389–407. DOI: 10.2307/20075765.

[PSV18] Hans Peters, Marc Schröder, and Dries Vermeulen. **Hotelling's location model with negative network externalities**. In: *International Journal of Game Theory* (Feb. 2018). DOI: 10.1007/s00182-018-0615-0.

[Rad80] Roy Radner. **Collusive behavior in noncooperative epsilon-equilibria of oligopolies with long but finite lives**. In: *Journal of Economic Theory* 22.2 (1980), pp. 136–154. DOI: 10.1016/0022-0531(80)90037-X.

[RE05] Charles S. Revelle and Horst A. Eiselt. **Location analysis: A synthesis and survey**. In: *European Journal of Operational Research* 165.1 (2005), pp. 1–19. DOI: 10.1016/j.ejor.2003.11.032.

[Ros73] Robert W. Rosenthal. **A class of games possessing pure-strategy Nash equilibria**. In: *International Journal of Game Theory* 2.1 (Dec. 1973), pp. 65–67. DOI: 10.1007/BF01737559.

[Rou09] Tim Roughgarden. **Intrinsic robustness of the price of anarchy**. In: *Proceedings of the 41st Annual ACM Symposium on Theory of Computing (STOC)*. 2009, pp. 513–522. DOI: 10.1145/1536414.1536485.

[Rou15] Tim Roughgarden. **Intrinsic Robustness of the Price of Anarchy**. In: *J. ACM* 62.5 (2015), 32:1–32:42. DOI: 10.1145/2806883.

[RRS17] Susana Iglesias Rey, Patricio Reyes, and Alonso Silva. **Evolution of social power for opinion dynamics networks**. In: *Proceedings of the 55th Annual Allerton Conference on Communication, Control, and Computing*. 2017, pp. 716–723. DOI: 10.1109/ALLERTON.2017.8262808.

[RS16] Tim Roughgarden and Okke Schrijvers. **Network Cost-Sharing without Anonymity**. In: *ACM Trans. Economics and Comput.* 4.2 (2016), 8:1–8:24. DOI: 10.1145/2841228.

[RT02] Tim Roughgarden and Éva Tardos. **How bad is selfish routing?** In: *J. ACM* 49.2 (2002), pp. 236–259. DOI: 10.1145/506147.506153.

[RT04] Tim Roughgarden and Éva Tardos. **Bounding the inefficiency of equilibria in nonatomic congestion games**. In: *Games and Economic Behavior* 47.2 (2004), pp. 389–403. DOI: 10.1016/j.geb.2003.06.004.

[Rub86] Ariel Rubinstein. **Finite Automata Play the Repeated Prisoner's Dilemma**. In: *Journal of Economic Theory* 39.1 (1986), pp. 83–96. DOI: 10.1016/0022-0531(86)90021-9.

[Rub91] Ariel Rubinstein. **Comments on the Interpretation of Game Theory**. In: *Econometrica* 59.4 (1991), pp. 909–924. DOI: 10.2307/2938166.

[Rub98] Ariel Rubinstein. **Modeling bounded rationality**. Vol. 1. MIT press, 1998.

[Sal79] Steven C. Salop. **Monopolistic Competition with Outside Goods**. In: *The Bell Journal of Economics* 10.1 (1979), pp. 141–156. DOI: 10.2307/3003323.

[Sch69] D. Schmeidler. **The Nucleolus of a Characteristic Function Game**. In: *SIAM Journal on Applied Mathematics* 17.6 (1969), pp. 1163–1170. DOI: 10.1137/0117107.

[Sha53a] Lloyd S Shapley. **A value for n-person games**. In: *Ann. Math. Studies*. Princeton, 1953, pp. 307–317.

[Sha53b] Lloyd S Shapley. **Stochastic Games**. In: *Proceedings of the National Academy of Sciences of the United States of America* 39.10 (1953), p. 1095.

[SM03] Andreas S. Schulz and Nicolás E. Stier Moses. **On the performance of user equilibria in traffic networks**. In: *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2003, pp. 86–87.

[SM12] Daniela Sabán and Nicolás Stier Moses. **The Competitive Facility Location Problem in a Duopoly: Connections to the 1-Median Problem**. In: *Proceeding of the 8th International Workshop on Internet and Network Economics (WINE)*. 2012, pp. 539–545. DOI: 10.1007/978-3-642-35311-6_44.

[Sme05] Philippe Smets. **Decision making in the TBM: the necessity of the pignistic transformation**. In: *Int. J. Approx. Reasoning* 38.2 (2005), pp. 133–147. DOI: 10.1016/j.ijar.2004.05.003.

[SP73] J Maynard Smith and GR Price. **The Logic of Animal Conflict**. In: *Nature* 246 (1973), p. 15.

[SV08] Alexander Skopalik and Berthold Vöcking. **Inapproximability of pure nash equilibria**. In: *Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC)*. 2008, pp. 355–364. DOI: 10.1145/1374376.1374428.

[Tia15] Jianrong Tian. **Optimal Interval Division**. In: *SSRN* (2015). DOI: 10.2139/ssrn.2697430.

[TJ78] Peter D Taylor and Leo B Jonker. **Evolutionary Stable Strategies and Game Dynamics**. In: *Mathematical Biosciences* 40.1 (1978), pp. 145–156. DOI: 10.1016/0025-5564(78)90077-9.

[Vet02] Adrian Vetta. **Nash Equilibria in Competitive Societies, with Applications to Facility Location, Traffic Routing and Auctions**. In: *Proceedings of the 43rd Symposium on Foundations of Computer Science (FOCS)*. 2002, p. 416. DOI: 10.1109/SFCS.2002.1181966.

[Voi+09] Thomas Voice, Maria Polukarov, Andrew Byde, and Nicholas R. Jennings. **On the Impact of Strategy and Utility Structures on Congestion-Averse Games**. In: *Proceedings of the 5th International Workshop on Internet and Network Economics (WINE)*. 2009, pp. 600–607. DOI: 10.1007/978-3-642-10841-9_61.

[Wei+02] Gérard Weisbuch, Guillaume Deffuant, Frédéric Amblard, and Jean-Pierre Nadal. **Meet, discuss, and segregate!** In: *Complexity* 7.3 (2002), pp. 55–63. DOI: 10.1002/cplx.10031.

[Wei97] Jörgen W Weibull. **Evolutionary Game Theory**. MIT press, 1997.

[WK12] Thomas Williams and Colin Kelley. **gnuplot 4.6: An Interactive Plotting Program**. http://www.gnuplot.info. 2012.

[WM05] Jason A. Winfree and Jill J. McCluskey. **Collective Reputation and Quality**. In: *American Journal of Agricultural Economics* 87.1 (2005), pp. 206–213. DOI: 10.1111/j.0002-9092.2005.00712.x.

[ZR94] Gilad Zlotkin and Jeffrey S. Rosenschein. **Coalition, Cryptography, and Stability: Mechanisms for Coalition Formation in Task Oriented Domains**. In: *Proceedings of the 12th National Conference on Artificial Intelligence*. 1994, pp. 432–437.

Nomenclature

Preliminaries

SPE	Set of all subgame perfect Nash equilibria	17
$\rho-$SPE	Set of all ρ-approximate subgame perfect Nash equilibria	17
v	Value function of a coalitional game	19
SV_i	Shapley value of player i	19
Φ	Potential function	19
Φ^A	A-limited potential	19
ρ-Ω	ρ-stretch of the potential function	20

Congestion Games

E	Set of resources	29
w_i	Positive weight of player i	29
f_e	Cost function of resource e	29
$\mathcal{F}$	Set of allowable cost function	29
$N_e(\mathsf{s})$	Set of users of resource e	29
$n_e(\mathsf{s})$	Number of users of resource e	29
$w_e(\mathsf{s})$	Total weight on resource e	29
W_A	Sum of all players $i \in A \subseteq \mathcal{N}$	29
$\chi_e(i, A)$	Costs share of player i on resource e with players A	29
$\chi_{ie}(\mathsf{s})$	Cost share of player i on a resource e	29
$c_e(w_e(\mathsf{s}))$	Joint cost on a resource e	29
$SC(\mathsf{s})$	Social costs of the game	29
χ_e^{Prop}(i,A)	Proportional cost share of player i on resource e with players A	30
$\chi_{ie}^{\text{Prop}}(\mathsf{s})$	Proportional cost share of player i on resource e	30
χ_e^{SV}(i, A)	Shapley cost share of player i on resource e with players A	30
$\chi_{ie}^{\text{SV}}(\mathsf{s})$	Shapley cost share of player i on resource e in state s	30
$\Phi_e(\mathsf{s})$	Contribution to the potential by resource e	34

Opinion Formation Games

o_i	Intrinsic opinion of player i	93
ω	Stubbornness of the players	93
$N_i(\mathsf{s})$	Neighborhood of player i in state s	93

Facility Location Games

$\ell_i(\mathsf{s}, \mathsf{f})$	Load of facility i in state (s, f)	106
α	Influence of the load in the client's cost function	106