Analysis of chemotactic migration in liquid environments

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Zusammenfassung

Diese Arbeit untersucht Anfangs-Randwertprobleme von Systemen partieller Differentialgleichungen, welche in der mathematischen Biologie zur theoretischen Beschreibung von Bakterienpopulationen auftreten. Die einzelnen Bakterien können dabei ihre Bewegung durch Reaktion auf einen Signalgradienten anpassen. Ein großer Anteil der häufig verwendeten Modelle vernachlässigt dabei den Einfluss der Umgebung auf die Bakterien. Um eine womöglich angemessenere Modellierung der Interaktion zwischen flüssiger Umgebung und Bakterien in die mathematische Beschreibung einfließen zu lassen, können verschiedene Keller–Segel Systeme mit den Stokes- oder Navier–Stokes-Gleichungen gekoppelt werden. Es ist das Ziel dieser Arbeit, die qualitativen Lösungseigenschaften in solchen ausgewählten Chemotaxis-Fluid Systemen zu untersuchen. Über Bedingungen für die globale Existenz von Lösungen in geeigneten Lösungskonzepten hinaus werden die Beschränktheit, eventuelle Regularität und die Konvergenz von Lösungen betrachtet.

Abstract

This work investigates initial-boundary value problems for systems of partial differential equations arising in mathematical biology to theoretically describe collective behavior in populations of bacteria which may adjust their motion in response to a signal gradient. Large quantities of the commonly used models neglect the influence of the environment on the bacteria. It is the aim of this work to study the qualitative solution behavior in selected chemotaxis-fluid systems, which by means of a coupling between various Keller–Segel systems and the Stokes or Navier–Stokes equations allow for a potentially more appropriate modeling of the interaction between liquid environments and bacterial populations. Beyond conditions for the global existence of solutions in suitable solvability concepts, we will inspect boundedness properties, eventual regularity and convergence of solutions.

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Contents

1 Introduction

The adaptation of migrational patterns according to an external stimulus is an effective strategy for survival in the form of an optimized nutrient acquisition and reduced energy depletion. The innate behavior of organisms to orient their movement in response to stimuli is known as taxis and famous examples include rheotaxis of fish ([3]), phototaxis of insects ([39]) and chemotaxis, which can even be observed on the smallest scales of life, with single-celled organisms and bacteria adjusting their locomotion according to the concentration gradient of a signal chemical ([1]). Though quite simple in their individual behavior, it has been observed in numerous experiments that, due to this chemotactic cell kinetics, larger populations of some bacteria can organize themselves in complex spatial patterns ([1, 11]). When Keller and Segel proposed a prototypical system of partial differential equations modeling the chemotactic migration of *Dictyostelium discoideum* ([40, 41]), they sparked the interest of many mathematical biologists and numerous studies trying to capture the mathematical mechanisms underlying the experimentally observed physical processes, such as pattern formation of cells, were initiated. Letting n and c denote the density of the cells and the concentration of the chemical, respectively, a very simple realization of the acclaimed system by Keller and Segel can be formulated as

$$\begin{cases} n_t = \Delta n - \nabla \cdot \big(n \nabla c\big), \\ c_t = \Delta c - c + n. \end{cases}$$

Remarkably, even in this simple form the Keller–Segel system has been proven to be able to describe self-organizing behavior of cells, like the spontaneous aggregation of *Dictyostelium discoideum*, without even including possible interaction between the cells and their surrounding environment ([35]). Experiments undertaken in [20, 85], however, highlighted that, in particular, with populations of aerobic bacteria suspended in drops of water, certain buoyancy and mixing effects should not be neglected. In order to capture plume-like convection patterns witnessed with colonies of *Bacillus subtilis* suspended in drops of water, it was suggested in [85] to consider the Keller–Segel system in combination with the Navier–Stokes equations, while incorporating a buoyancy effect as source term in the added fluid equation.

With the proposed coupling merging both fluid equations and chemotaxis equations, at first glance it appears to be quite hopeless to obtain a thorough understanding of the interplay between both of them, as each on their own still has a wide array of unanswered questions and difficult challenges to overcome. On the one hand there is the notable example of the open Millennium Prize Problem from the Clay Mathematics Institute for the celebrated Navier–Stokes equations, and on the other hand due to the cross-diffusive mechanism present in the chemotaxis equations, the possibility of solutions blowing up is a recurrent theme throughout the many variations of Keller–Segel type systems. In this respect, as the interplay between the equations moreover negates many strategies and methods applicable in the fluid-free Keller–Segel system, it is also not very surprising that, undeterred by the considerable activity in both fields, the knowledge on coupled chemotaxis-fluid equations is still only quite fragmentary. Especially, the workings in three-dimensional domains are mostly enigmatic, but the qualitative behavior in two-dimensional domains is also quite hard to access with the methods of mathematical analysis.

In light of this inherent difficulty and the close relation to real world applications, a carefully crafted solution theory is the very necessary foundation of the studies in this field. If we consider solution concepts too abstract, any meaningful insight obtained from the mathematical analysis may get lost upon interpretation of the results in the underlying natural model and hence results in studies of purely academic nature. Finding the optimal degree of generalization, however, is a task far from trivial and is, in particular in the context of Keller–Segel systems, deeply intertwined with the qualitative understanding of the model behind the equations. Depending on the precise variant of the Keller–Segel model, the existence of solutions blowing up in finite time, which in the physical interpretation is commonly linked to the aggregation of cells, may actually not be the desired outcome, as experiments may suggest that rather a steady state should be approached. Hence, with the importance of the solution theory in mind, a major question concerning the coupled chemotaxis-fluid systems is how much of the qualitative solution properties can be maintained from the fluid-free setting, despite the possibly deregularizing effect of the fluid. In particular, since the emergence of patterns and aggregation phenomena is closely related to the convergence towards non-constant steady states and blow-up of solutions, respectively, a great area of interest consists of the transference of global existence results in sufficiently well-behaved function spaces from the fluid-free setting to the chemotaxis-fluid framework.

Before we take a more in depth look at the main results of this thesis, let us first specify a quite general formulation of the chemotaxis-fluid system proposed in [85], where u, P and ϕ denote the fluid-velocity field, the associated pressure and a prescribed gravitational potential, respectively.

$$\begin{cases} n_t + \quad u\cdot\nabla n = \nabla\cdot\big(D(n)\nabla n - nS(n,c)\nabla c\big), \\ c_t + \quad u\cdot\nabla c = \Delta c + g(n,c), \\ u_t + (u\cdot\nabla)u = \Delta u - \nabla P + n\nabla\phi, \\ \qquad\quad \nabla\cdot u = 0. \end{cases}$$

In this setting $D(n)$ describes the diffusivity of cells, $S(n,c)$ models the chemotactic sensitivity of the organisms and the source term $g(n,c)$ is related to the interaction between cells and chemical substance. The simplest choices for these functions are $D(n)\equiv 1$, $S(n,c)\equiv 1$ and $g(n,c)=-c+n$ for a signal producing population of bacteria or $g(n,c)=-nc$ in the case of cells consuming the chemical. Briefly noting that, depending on the space dimension and the choices for the functions D, S and g, the solution of the system may, even without considering the fluid component, very well only exist on finite time intervals or globally, but only in generalized solution concepts requiring less regularity than the standard variational concept of weak solvability, we can now outline the core results of this work.

Main results of the thesis. The systems we are going to consider here can, with respect to the interaction between the bacteria and the signal chemical, be grouped in two major classes. In Chapters 2 and 3 we will investigate two slightly different chemotaxis-fluid settings where the signal is produced by the bacteria akin to the Keller–Segel model based on the aggregation of *Dictyostelium discoideum*, while Chapters 4 and 5 will be concerned with two different questions regarding signal consumption processes as witnessed with populations of *Bacillus subtilis* and *Escherichia coli*. Without going into too much detail, the corresponding results can be summarized in following ways.

In Chapter 2 we will discuss a case where the evolution of the fluid-velocity is described by the Stokes equations instead of the full Navier–Stokes equations. We will prove that in a two-dimensional domain a Keller–Segel–Stokes system with a sublinear production rate of the from $g(n,c) = -c + f(n)$ satisfying $0 \leq f(s) \leq K_0 s^\alpha$ for all $s \in [0,\infty)$ and some $0 < \alpha < 1$ always emits time-global and bounded classical solutions regardless of the size of the initial data. This result is maintained from the fluid-free setting ([54]) and the sublinear growth-rate is optimal in respect to α in the sense that for $g(n,c) = -c+n$ a critical mass phenomenon (i.e. blow-up occurs if the initial mass of n is above a critical number) is known from the fluid-free setting ([38, 64, 62]).

In Chapter 3 we are going to consider a nonlinear diffusion of porous-medium-type $D(n) = mn^{m-1}$, which, at least biologically, appears to be more appropriate, since densely packed cells suffer a larger portion of stress and try to move away from one another ([43]), and sensitivities of the type $S(n) = \frac{1}{(n+1)^\alpha}$, which can be motivated by the fact that whenever there are large amount of cells present in an area, the movement of the individuals is inhibited ([70]). While a sublinear production rate was the main ingredient for global existence in Chapter 2 and the Stokes equations even enabled us to discuss solutions in the classical concept of solvability, there have been studies for the fluid-free framework indicating that the growth rate of the ratio $\frac{S(n)}{D(n)}$ is also a crucial quantity distinguishing whether blow-up can occur or not ([80, 95]). Trying to capture the importance of this growth rate in the chemotaxis-fluid setting, in Chapter 3 we will consider a three-dimensional chemotaxis-Navier–Stokes system with cell diffusion of porous medium type, sensitivity functions satisfying a saturation effect essentially specified through $|S(n,c)| \lesssim \frac{1}{(1+n)^\alpha}$ and linear signal production rate. Depending on the parameters m and α, we will prove global existence of solutions in two different concepts of weak solvability, where we also note that the solution concepts exclude the formation of Dirac-type singularities and that the parameter range for the weakest concept coincides with the range excluding blow-up in the fluid-free system.

In Chapter 4 we will then consider a chemotaxis-Stokes setting with signal consumption, i.e. $g(n,c) = -nc$, combined with the singular sensitivity function $S(n,c) = \frac{1}{c}$, a setting much more unfavorable for global existence. Even in the fluid-free Keller–Segel system only so called global generalized solutions, which merely satisfy very mild regularity conditions, have been shown to exist ([102]). The concept of generalized solvability can be adjusted to the corresponding two-dimensional Keller–Segel–Stokes setting in a straightforward fashion ([87]) and despite the unobtainable global regularity result in good spaces, we may still hope for an eventual regularization process to occur. In the main results of this chapter we will prove that, under a smallness assumption on the initial mass, the global generalized solution of the two-dimensional chemotaxis-Stokes system will eventually become a classical solution and that the solution converges to a constant steady state, which in turn rules out the emergence of complex spatial patterns. As an interesting byproduct we also obtain a condition on global classical solvability under certain constraints on the initial data.

Finally, in Chapter 5 we make use of an eventual regularization process, as witnessed in the previous chapter, in order to get a grasp on some quantifiable difference between using the Stokes approximation and the full Navier–Stokes equations for the description of the fluid-velocity. The focus being on the differences between the evolution of the fluid-velocity, we consider a minimal chemotaxis-consumption system, in particular more well-behaved than the one present in Chapter 4. To be precise, we consider a chemotaxis-Navier–Stokes consumption system in a three-dimensional domain with linear diffusion, standard chemotactic sensitivity $S(n,c)=1$, and investigate the small-convection limit behavior of the corresponding solutions. While time-local convergence is quite easily obtainable, a result for time-global convergence is more intricate. To manage convergence on large time-scales we will first prove a result on eventual regularity, where the smoothing time is independent of the convection strength. Afterwards, we will finally conclude that in the Stokes limit, the weak solution of the chemotaxis-Navier–Stokes system converges towards a weak solution of the chemotaxis-Stokes system in the standard solution space for weak solutions of the chemotaxis-Navier–Stokes system.

For more details on the context and related works to each of these chapters as well as the precise statements of the individual theorems, we refer the interested reader to the introductions at the start of each chapter. We made sure that recurrences of arguments between chapters are kept to a minimum, but favored independent readability of the chapters when necessary, so that in fact each chapter may be read on its own.

1.1 Original publications

The chapters of this work have previously been published in peer reviewed journals. Except for minor changes these articles are almost identical to the content presented in each chapter and quotations from said articles will not be marked separately on each occasion.

Chapter 2:
[6]: T. Black. Sublinear signal production in a two-dimensional Keller–Segel–Stokes system. *Nonlinear Anal. Real World Appl.*, 31:593–609, 2016.

Chapter 3:
[9]: T. Black. Global solvability of chemotaxis-fluid systems with nonlinear diffusion and matrix-valued sensitivities in three dimensions. *Nonlinear Anal.*, 180:129–153, 2019.

and

[7]: T. Black. Global Very Weak Solutions to a Chemotaxis-Fluid System with Nonlinear Diffusion. *SIAM J. Math. Anal.*, 50(4):4087–4116, 2018.

Chapter 4:
[8]: T. Black. Eventual smoothness of generalized solutions to a singular chemotaxis-Stokes system in 2D. *J. Differ. Equ.*, 265(5):2296–2339, 2018.

Chapter 5:
[10]: T. Black. The Stokes limit in a three-dimensional chemotaxis-Navier-–Stokes system. *J. Math. Fluid Mech.*, 22(1):1, 2020.

2 Sublinear signal production in a two-dimensional Keller–Segel–Stokes system

2.1 Introduction

Keller–Segel models. The acclaimed chemotaxis system

$$\begin{cases} n_t = \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega,\, t>0,\\ c_t = \Delta c - c + n, & x\in\Omega,\, t>0, \end{cases} \tag{2.1.1}$$

by Keller and Segel ([40, 41]) alone has been studied intensively in the last decades and a wide array of interesting properties, such as finite time blow-up and spatial pattern formation, have been discovered (see also the surveys [4, 34, 35]). For instance, the initial-boundary value problem obtained from (2.1.1) with homogeneous Neumann boundary conditions where $\Omega\subset\mathbb{R}^N$ is a ball, emits blow-up solutions for $N\geq 2$, if the total initial mass of cells lies above a critical value ([33, 38, 62, 64, 97]), while all solutions remain bounded when either $N=1$, or $N=2$ and the initial total mass of cells is below the critical value ([68, 66]).
Through its application to various biological contexts, many variants of the Keller–Segel model have been proposed over the years. In particular, adaptations of (2.1.1) in the form of

$$n_t = \Delta n - \nabla\cdot(nS(x,n,c)\cdot\nabla c),\quad x\in\Omega,\ t>0, \tag{2.1.2}$$

with given chemotactic sensitivity function S, which can either be a scalar function, or more general a tensor valued function (see e.g. [108]), for the first equation or

$$c_t = \Delta c - ng(c),\quad x\in\Omega,\ t>0, \tag{2.1.3}$$

with given function g for the second equation, have been studied. Both of these adjustments are known to have an influence on the boundedness of solutions to their respective systems. For instance, if we replace the first equation of (2.1.1) with (2.1.2), where S is a scalar function of n satisfying $S(n)\leq C(1+n)^{-\gamma}$ for all $n>0$ and some $\gamma>1-\frac{2}{N}$, then all solutions to the corresponding Neumann problem are global and uniformly bounded. On the other hand if $N\geq 2$, $\Omega\subset\mathbb{R}^N$ is a ball and $S(n)\geq Cn^{-\gamma}$ for all $n\geq 1$ and some $\gamma<1-\frac{2}{N}$ then the solution may blow up ([36]).
Considering the variant of (2.1.1) with (2.1.3) as second equation, which basically corresponds to the model assumption that the cells consume some of the chemical instead

of producing it, it was shown in [81, Proposition 1.2] that for $N = 2$ the corresponding Neumann problem possesses a bounded classical solution for suitable regular initial data not depending on a smallness condition. For $N = 3$ it was proven that there exist global weak solutions which eventually become smooth and bounded after some waiting time. A combination of both adjustments, where S is matrix-valued with non-trivial nondiagonal parts, was studied in [101]. There it was shown that under fairly general assumptions on g and S at least one generalized solution exists which is global. This result does neither contain a restriction on the spatial dimension nor on the size of the initial data.
One last variant of (2.1.1) we would like to mention has only recently been studied thoroughly and concerns the system

$$\begin{cases} n_t = \Delta n - \nabla\cdot(n\nabla c), & x \in \Omega, t > 0, \\ c_t = \Delta c - c + f(n), & x \in \Omega, t > 0, \end{cases} \tag{2.1.4}$$

with $f \in C^1([0,\infty))$ satisfying $0 \le f(s) \le Ks^\alpha$ for any $s \ge 0$ with some $K > 0$ and $\alpha > 0$. In this setting, it is known that the system (2.1.4) does not emit any blow-up solution if $\alpha < \frac{2}{N}$ ([54]) but it remains an open question whether this exponent is indeed critical.
Similar forms of $f(n)$ have been treated before either in the linear case $f(n) = n$ ([59]) or (sub-)linear cases with an additional logistic growth term introduced to the first equation (e.g. [69, 96, 67]).
Chemotaxis-fluid systems. Nonetheless, one assumption is shared by all of these Keller–Segel-type models. That is, only the cell density n and the chemical concentration c are unknown and all other system parameters are fixed. In particular, the models assume that there is no interaction between the cells and their surroundings. However, experimental observations indicate that chemotactic motion inside a liquid can be substantially influenced by the mutual interaction between cells and fluid. For instance, in [85] the dynamical generation of patterns and emergence of turbulence in a population of aerobic bacteria suspended in sessile drops of water is reported, whereas examples involving instationary fluids are important in the context of broadcast spawning phenomena related to successful coral fertilization ([16, 58]).
A model considering the chemotaxis-fluid interaction building on experimental observations of Bacillus subtilis was given in [85]. In the system in question, the fluid-velocity $u = u(x,t)$ and the associated pressure $P = P(x,t)$ are introduced as additional unknown quantities utilizing the incompressible Navier–Stokes equations. One of the first theoretical results concerning the solvability in this context were shown in [56], where the local existence of weak solutions for $N \in \{2,3\}$ was shown. This setting, however, involved signal consumption in the form of per-capita oxygen consumption of the bacteria, which corresponds to an equation of the form (2.1.3). Since we want to focus on the case of signal production by the cells as realized in (2.1.1), a more suitable system in this context is the Keller–Segel–Navier–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c), & x \in \Omega, t > 0, \\ c_t + u\cdot\nabla c = \Delta c - c + n, & x \in \Omega, t > 0, \\ u_t + u\cdot\nabla u = \Delta u - \nabla P + n\nabla\phi, & x \in \Omega, t > 0, \\ \nabla\cdot u = 0, & x \in \Omega, t > 0, \end{cases} \tag{2.1.5}$$

where the fluid is supposed to be driven by forces induced by the fixed gravitational potential ϕ and transports both the cells and the chemical.
The mathematical analysis of (2.1.5) regarding global and bounded solutions is far from trivial, as on the one hand its Navier–Stokes subsystem lacks complete existence theory ([94]) and on the other hand the previously mentioned properties for the Keller–Segel system can still emerge. In order to weaken the necessary analytical effort, a commonly made simplification is to assume that the fluid flow is comparatively slow and thus the fluid-velocity evolution may be described by the Stokes equation rather than the full Navier–Stokes system.
Of course, all alterations to (2.1.1) described above can be included as adjustments to the systems in this Keller–Segel(–Navier)–Stokes setting as well. Their influences on global and bounded solutions are one focal point of recent studies. For instance, an adjustment making use of both sensitivity and chemical consumption has been applied to Keller–Segel–Stokes systems in [104], where for scalar valued sensitivity functions S the existence of global weak solutions for bounded three-dimensional domains has been established. Building on this existence result, it was shown in [105] that the solution approaches a spatially homogeneous steady state under fairly weak assumptions imposed on the parameter functions S and g. Under similar assumptions, the existence of global weak solutions for suitable non-linear diffusion types have been proven in [17] and the existence of bounded and global weak solutions even allowing matrix-valued S not requiring a decay assumption in [100].
A Keller–Segel–Stokes system corresponding to the adjustment made to (2.1.1) by only making use of rotational sensitivity was studied in [89], where it was shown that the Neumann problem for the 2D Keller–Segel–Stokes system possesses a unique global classical solution which remains bounded for all times, if we assume S to satisfy $|S(x,n,c)| \leq C_S(1+n)^{-a}$ with $C_S > 0$ for some $a > 0$.
Regarding the introduction of the additional logistic growth term $+rn - \mu n^2$ with $r \geq 0$ and $\mu > 0$ to the first equation, it was shown in [82, Theorem 1.1] for space dimension $N = 3$, that every solution remains bounded if $\mu \geq 23$ and thus any blow-up phenomena are excluded. Moreover, these solutions tend to zero ([82, Theorem 1.2]).
Some of these results have in part been transferred to the full chemotaxis Navier–Stokes system. This includes global existence of classical solutions for $N = 2$ with scalar valued sensitivity ([98]), large time behavior and eventual smoothness of such solutions ([105]) and even global existence of mild solution to double chemotaxis systems under the effect of incompressible viscous fluid ([44]). Boundedness results with matrix-valued sensitivity without decay requirements but for small initial data have been discussed in [13] and boundedness results under influence of a logistic growth term in [83].
Main results. The results above indicate that certain alterations to the systems are always favorable for the existence of global and bounded solutions and, if their respective influence is strong enough, they may even withstand the possibly deregularizing effect of the fluid interaction successfully. Motivated by this observation and the result of [54] for (2.1.4) mentioned above, we are now interested in whether the influence of a coupled slow moving fluid described by Stokes equation affects the possible choice for $\alpha \in (0,1)$, while still maintaining the exclusion of possible unbounded solutions. Henceforth, we

will consider that the evolution of (n, c, u, P) is governed by the Keller–Segel–Stokes System

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega, \quad t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + f(n), & x\in\Omega, \quad t>0,\\ u_t + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega, \quad t>0,\\ \nabla\cdot u = 0, & x\in\Omega, \quad t>0, \end{cases} \tag{2.1.6}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded and smooth domain and $f\in C^1([0,\infty))$ satisfies

$$0\le f(s)\le K_0 s^\alpha \quad \text{for all } s\in[0,\infty) \tag{2.1.7}$$

with some $\alpha\in(0,1]$ and $K_0>0$. We shall examine this system along with no-flux boundary conditions for n and c and a no-slip boundary condition for u,

$$\frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0 \quad \text{and} \quad u=0 \qquad \text{for } x\in\partial\Omega \text{ and } t>0, \tag{2.1.8}$$

and initial conditions

$$n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), \quad x\in\Omega. \tag{2.1.9}$$

For simplicity we will assume $\phi\in W^{2,\infty}(\Omega)$ and that for some $\vartheta>2$ and $\varrho\in(\frac12,1)$ the initial data satisfy the regularity and positivity conditions

$$\begin{cases} n_0\in C^0(\overline{\Omega}) \text{ with } n_0>0 \text{ in } \overline{\Omega},\\ c_0\in W^{1,\vartheta}(\Omega) \text{ with } c_0>0 \text{ in } \overline{\Omega},\\ u_0\in D(A^\varrho), \end{cases} \tag{2.1.10}$$

where here and below A^ϱ denotes the fractional power of the Stokes operator $A:=-\mathcal{P}\Delta$ regarding homogeneous Dirichlet boundary conditions, with the Helmholtz projection $\mathcal{P}$ from $L^2(\Omega;\mathbb{R}^2)$ to the solenoidal subspace $L^2_\sigma(\Omega):=\{\varphi\in L^2(\Omega;\mathbb{R}^2)\,|\,\nabla\cdot\varphi=0\}$ and domain $D(A):=W^{2,2}(\Omega;\mathbb{R}^2)\cap W^{1,2}_0(\Omega;\mathbb{R}^2)\cap L^2_\sigma(\Omega)$. In this framework we can state our main result in the following way:

Theorem 2.1.
Let $\vartheta>2$, $\varrho\in(\frac12,1)$ and $\Omega\subset\mathbb{R}^2$ be a bounded and convex domain with smooth boundary. Assume $\phi\in W^{2,\infty}(\Omega)$ and that n_0, c_0 and u_0 comply with (2.1.10)*. Then for any $\alpha\in(0,1)$, the PDE system* (2.1.6) *coupled with boundary conditions* (2.1.8) *and initial conditions* (2.1.9) *possesses a solution (n,c,u,P) satisfying*

$$\begin{cases} n\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\ c\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\ u\in C^0(\overline{\Omega}\times[0,\infty);\mathbb{R}^2)\cap C^{2,1}(\overline{\Omega}\times(0,\infty);\mathbb{R}^2),\\ P\in C^{1,0}(\overline{\Omega}\times[0,\infty)), \end{cases}$$

which solves (2.1.6) *in the classical sense and remains bounded for all times. This solution is unique within the class of functions which for all* $T \in (0,\infty)$ *satisfy the regularity properties*

$$\begin{cases} n \in C^0\big([0,T);L^2(\Omega)\big) \cap L^\infty\big((0,T);C^0(\overline{\Omega})\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T)\big)\,, \\ c \in C^0\big([0,T);L^2(\Omega)\big) \cap L^\infty\big((0,T);W^{1,\vartheta}(\Omega)\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T)\big)\,, \\ u \in C^0\big([0,T);L^2(\Omega;\mathbb{R}^2)\big) \cap L^\infty\left((0,T);D(A^\varrho)\right) \cap C^{2,1}\big(\overline{\Omega}\times(0,T);\mathbb{R}^2\big)\,, \\ P \in L^1\left((0,T);W^{1,2}(\Omega)\right)\,, \end{cases} \tag{2.1.11}$$

up to addition of functions $\hat{p}$ *to* P*, such that* $\hat{p}(\cdot,t)$ *is constant for any* $t \in (0,\infty)$*.*

In view of Theorem 2.1, there is no evident difference regarding α between the coupled system (2.1.6) and the chemotaxis system without fluid (2.1.4) for dimension $N=2$.
In Section 2.2 we will briefly discuss local existence of classical solutions and basic a priori estimates. Section 2.3 is dedicated to the connection between the regularity of n and the regularity of the spacial derivative of u, which plays a crucial part in obtaining additional information on the regularity of c. In Section 2.4 we will combine standard testing procedures with the results from the previous sections to prove the boundedness and globality of classical solutions to (2.1.6).

2.2 Local existence of classical solutions

The following lemma concerning the local existence of classical solutions and an extensibility criterion can be proven with exactly the same steps demonstrated in [98, Lemma 2.1] and [78, Lemma 2.1].

Lemma 2.2. - Local existence of classical solutions
Let $\vartheta > 2$, $\varrho \in (\frac{1}{2},1)$ *and* $\Omega \subset \mathbb{R}^2$ *be a bounded and convex domain with smooth boundary. Suppose* $\phi \in W^{2,\infty}(\Omega)$ *and that* n_0, c_0 *and* u_0 *satisfy* (2.1.10)*. Then there exist* $T_{max} \in (0,\infty]$ *and functions* (n,c,u,P) *satisfying*

$$\begin{cases} n \in C^0\big(\overline{\Omega}\times[0,T_{max})\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T_{max})\big)\,, \\ c \in C^0\big(\overline{\Omega}\times[0,T_{max})\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T_{max})\big)\,, \\ u \in C^0\big(\overline{\Omega}\times[0,T_{max});\mathbb{R}^2\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T_{max});\mathbb{R}^2\big)\,, \\ P \in C^{1,0}\big(\overline{\Omega}\times[0,T_{max})\big)\,, \end{cases}$$

which solve (2.1.6) *with* (2.1.8) *and* (2.1.9) *in the classical sense in* $\Omega\times(0,T_{max})$*. Moreover, we have* $n>0$ *and* $c>0$ *in* $\overline{\Omega}\times[0,T_{max})$ *and the alternative*

$$\begin{aligned} &\textit{either } T_{max} = \infty \textit{ or} \\ &\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} + \|A^\varrho u(\cdot,t)\|_{L^2(\Omega)} \to \infty \textit{ as } t \nearrow T_{max}. \end{aligned} \tag{2.2.1}$$

The solution is unique among all functions satisfying (2.1.11) *for all* $T \in (0,T_{max})$*, up to addition of functions* $\hat{p}$*, such that* $\hat{p}(\cdot,t)$ *is constant for any* $t \in (0,T)$*, to the pressure* P*.*

Local existence at hand, we can immediately prove two elementary properties, which will be the starting point for all of our regularity results to come.

Lemma 2.3.
Under the assumptions of Lemma 2.2, the solution of (2.1.6) *satisfies*

$$\int_\Omega n(x,t)\,\mathrm{d}x = \int_\Omega n_0 =: m \quad \textit{for all } t \in (0,T_{max}) \tag{2.2.2}$$

and there exists a constant $C > 0$ such that

$$\int_\Omega c(x,t)\,\mathrm{d}x \le C \quad \textit{for all } t \in (0,T_{max}). \tag{2.2.3}$$

Proof: The first property follows immediately from simple integration of the first equation in (2.1.6). For (2.2.3) we integrate the second equation of (2.1.6) and recall (2.1.7) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c + \int_\Omega c \le K_0 \int_\Omega n^\alpha \quad \text{for all } t \in (0,T_{max}).$$

Hence, making use of (2.2.2) and the fact $\alpha < 1$, $y(t) = \int_\Omega c(x,t)\,\mathrm{d}x$ satisfies the ODI

$$y'(t) + y(t) \le C_1 \|n_0\|^\alpha_{L^1(\Omega)} = C_2 \quad \text{for all } t \in (0,T_{max})$$

for some $C_1 > 0$ and $C_2 := C_1 m^\alpha > 0$ in view of (2.1.10). Upon integration we infer

$$y(t) \le y(0)e^{-t} + C_2\left(1 - e^{-t}\right) \quad \text{for all } t \in (0,T_{max}),$$

which, due to the assumed regularity of c_0 in (2.1.10), completes the proof. □

2.3 Regularity of u implied by regularity of n

Let us recall that $\mathcal{P}$ denotes the Helmholtz projection from $L^2\big(\Omega;\mathbb{R}^2\big)$ to the subspace $L^2_\sigma(\Omega) = \big\{\varphi \in L^2\big(\Omega;\mathbb{R}^2\big) \mid \nabla\cdot\varphi = 0\big\}$ and $A := -\mathcal{P}\Delta$ denotes the Stokes operator under homogeneous Dirichlet boundary conditions.
For now we limit our observations to a projected version of the Stokes subsystem $\frac{\mathrm{d}}{\mathrm{d}t}u + Au = \mathcal{P}(n\nabla\phi)$ in (2.1.6) without regard for the rest of the system. In contrast to the setting with the full Navier–Stokes equations we can make use of the absence of the convective term $(u\cdot\nabla)u$ in the Stokes equation to gain results concerning the regularity of the spatial derivative Du based on the regularity of the term $\mathcal{P}(n\nabla\phi)$, which in fact solely depends on the regularity of n, due to the assumed boundedness of $\nabla\phi$.
In [89, Lemma 2.4] this correlation between the regularity of u and n is proven in space dimension $N = 2$. The proof of [89, Lemma 2.4] is based on an approach employed in [100, Section 3.1], which makes use of general results for sectorial operators shown in [24], [32] and [28] and mainly relies on an embedding of the domains of fractional powers $D(A^\varrho)$ into $L^p(\Omega)$, see [32, Theorem 1.6.1] or [28, Theorem 3], for instance. Since we are only working in two-dimensional domains we will only state the result from [89, Lemma 2.4] here and refer the reader to [100, Corollary 3.4] and [89, Lemma 2.5] for the remaining details regarding the proof.

Lemma 2.4.
Let $p \in [1,\infty)$ and $r \in [1,\infty]$ be such that

$$\begin{cases} r < \frac{2p}{2-p} & \text{if } p \leq 2, \\ r \leq \infty & \text{if } p > 2. \end{cases}$$

Furthermore, assume $\Omega \subset \mathbb{R}^2$ to be a smoothly bounded domain and let $T > 0$ be such that $n : \Omega \times (0,T) \mapsto \mathbb{R}$ satisfies

$$\|n(\cdot,t)\|_{L^p(\Omega)} \leq L \quad \text{for all } t \in (0,T),$$

with some $L > 0$. Then for $u_0 \in D(A^\varrho)$ with $\varrho \in \left(\frac{1}{2},1\right)$ and $\phi \in W^{2,\infty}(\Omega)$ all solutions u of the third and fourth equations in (2.1.6) *fulfill*

$$\|Du(\cdot,t)\|_{L^r(\Omega)} \leq C \quad \text{for all } t \in (0,T),$$

with a constant $C = C(p,r,L,u_0,\phi) > 0$.

Evidently, a supposedly known bound for n at hand, we immediately obtain the desired boundedness of u in view of Sobolev embeddings. Nevertheless, since we only have the time independent L^1 bound of n from Lemma 2.3 as a starting point, obtaining a bound for n in $L^p(\Omega)$ with suitable large $p > 1$ will require additional work.

2.4 Global existence and boundedness in two-dimensional domains

For the rest of the chapter, unless stated otherwise, we fix $\vartheta > 2, \varrho \in (\frac{1}{2},1)$, initial data satisfying (2.1.10) and $\Omega \subset \mathbb{R}^2$ meeting all requirements of Lemma 2.2. We then let (n,c,u,P) denote the solution given by Lemma 2.2 and T_{max} its maximal time of existence. Making use of the connection between the regularity of u and n discussed in the previous section, we immediately obtain the following result.

Proposition 2.5.
For all $r < 2$ and all $q < \infty$ there exist constants $C_1 > 0$ and $C_2 > 0$ such that the solution to (2.1.6) *satisfies*

$$\|Du(\cdot,t)\|_{L^r(\Omega)} \leq C_1 \quad \text{for all } t \in (0,T_{max})$$

and

$$\|u(\cdot,t)\|_{L^q(\Omega)} \leq C_2 \quad \text{for all } t \in (0,T_{max}).$$

Proof: In light of (2.2.2) and (2.1.10) we can find $C_3 > 0$ satisfying $\|n(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)} \leq C_3$ for all $t \in (0,T_{max})$. Thus, we may apply Lemma 2.4 with $p = 1$ to obtain for any $r < 2$ that $\|Du(\cdot,t)\|_{L^r(\Omega)} \leq C_2$ for all $t \in (0,T_{max})$ with some $C_2 > 0$. The second claim then follows immediately from the Sobolev embedding theorem ([23, Theorem 5.6.6]). □

2.4.1 Obtaining a first information on the gradient of c

In order to derive the bounds necessary in our approach towards the boundedness result, we require an estimate on the gradient of c as a starting point. To obtain a first information in this matter, we apply standard testing procedures to derive an energy inequality involving integrals of $n \ln n$ and $|\nabla c|^2$. But first, let us briefly recall Young's inequality in order to fix notation.

Lemma 2.6.
Let $a, b, \varepsilon > 0$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \varepsilon a^p + C(\varepsilon, p, q) b^q,$$

where $C(\varepsilon, p, q) = (\varepsilon p)^{-\frac{q}{p}} q^{-1}$.

Before deriving an inequality for the time evolution of $\int_\Omega n \ln n$ we employ the Gagliardo–Nirenberg inequality to show one simple preparatory lemma on which we will rely multiple times later on.

Lemma 2.7.
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Let $r \geq 1$ and $s \geq 1$. Then for any $L > 0$ there exists $C > 0$ such that

$$\int_\Omega |\varphi|^{rs} \leq C \left(\int_\Omega |\nabla(|\varphi|^{r/2})|^2 \right)^{\frac{(rs-1)}{r}} + C$$

holds for all functions $\varphi \in L^1(\Omega)$ satisfying $\nabla(|\varphi|^{r/2}) \in L^2(\Omega; \mathbb{R}^2)$ and $\int_\Omega |\varphi| \leq L$.

Proof: By an application of the Gagliardo–Nirenberg inequality (see [52, Lemma 2.3] for a version including integrability exponents less than 1) we can pick $C_1 > 0$ such that

$$\int_\Omega |\varphi|^{rs} = \| |\varphi|^{r/2} \|_{L^{2s}(\Omega)}^{2s} \leq C_1 \| \nabla(|\varphi|^{r/2}) \|_{L^2(\Omega)}^{2sa} \| |\varphi|^{r/2} \|_{L^{\frac{2}{r}}(\Omega)}^{2s(1-a)} + C_1 \| |\varphi|^{r/2} \|_{L^{\frac{2}{r}}(\Omega)}^{2s}$$

holds for all $\varphi \in L^1(\Omega)$ with $\nabla(|\varphi|^{r/2}) \in L^2(\Omega; \mathbb{R}^2)$, with $a \in (0,1)$ provided by

$$a = \frac{\frac{r}{2} - \frac{1}{2s}}{\frac{r}{2} + \frac{1}{2} - \frac{1}{2}} = 1 - \frac{1}{rs}.$$

Since $\int_\Omega |\varphi| \leq L$ we have $\| |\varphi|^{r/2} \|_{L^{\frac{2}{r}}(\Omega)} = \left(\int_\Omega |\varphi| \right)^{\frac{r}{2}} \leq L^{\frac{r}{2}}$ and thus

$$\int_\Omega |\varphi|^{rs} \leq C_2 \left(\int_\Omega |\nabla(|\varphi|^{r/2})|^2 \right)^{\frac{(rs-1)}{r}} + C_2$$

for all $\varphi \in L^1(\Omega)$ satisfying $\nabla(|\varphi|^{r/2}) \in L^2(\Omega; \mathbb{R}^2)$, where $C_2 = C_1 \max\{L, L^{rs}\} > 0$. $\square$

The particular form in which we will need this inequality most often is the following:

Corollary 2.8.
There exists a constant $K_1 > 0$ such that the solution of (2.1.6) *fulfills*

$$\int_\Omega n^2 \le K_1 \int_\Omega |\nabla(n^{1/2})|^2 + K_1$$

for all $t \in (0, T_{max})$.

Testing the first equation of (2.1.6) with $1 + \ln n$ yields the following estimation.

Lemma 2.9.
There exists a constant $K_2 > 0$ such that the solution of (2.1.6) *fulfills*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n\ln n + \int_\Omega |\nabla(n^{1/2})|^2 \le K_2 \int_\Omega |\Delta c|^2 + K_2 \quad \textit{for all } t \in (0, T_{max}). \tag{2.4.1}$$

Proof: Making use of (2.2.2) and $\nabla\cdot u = 0$ in Ω, multiplication of the first equation in (2.1.6) with $1 + \ln n$ and integration by parts yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n\ln n + \int_\Omega \frac{|\nabla n|^2}{n} = \int_\Omega \nabla c\cdot\nabla n \quad \text{for all } t \in (0, T_{max}). \tag{2.4.2}$$

To further estimate the right hand side, we first let $K_1 > 0$ be as in Corollary 2.8. Then, integrating the right hand side of (2.4.2) once more by parts and applying Young's inequality with $p = q = 2$ and $\varepsilon = \frac{3}{K_1}$ (see Lemma 2.6) and Corollary 2.8, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n\ln n + 4\int_\Omega |\nabla(n^{1/2})|^2 &\le \frac{3}{K_1}\int_\Omega n^2 + C_1\int_\Omega |\Delta c|^2\\ &\le \frac{3}{K_1}\left(K_1\int_\Omega |\nabla(n^{1/2})|^2 + K_1\right) + C_1\int_\Omega |\Delta c|^2\end{aligned}$$

for all $t \in (0, T_{max})$ and some $C_1 > 0$. Reordering the terms appropriately completes the proof with $K_2 := \max\{3, C_1\}$. □

The second separate inequality treats the time evolution of $\int_\Omega |\nabla c|^2$.

Lemma 2.10.
Given any $\xi > 0$, there exists a constant $K_3 > 0$ such that

$$\frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^2 + \frac{\xi}{4}\int_\Omega |\Delta c|^2 + \xi\int_\Omega |\nabla c|^2 \le \frac{1}{2}\int_\Omega |\nabla(n^{1/2})|^2 + K_3 \tag{2.4.3}$$

holds for all $t \in (0, T_{max})$.

Proof: Testing the second equation of (2.1.6) with $-\xi\Delta c$ and integrating by parts we obtain

$$\frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^2 + \xi\int_\Omega |\Delta c|^2 + \xi\int_\Omega |\nabla c|^2 = -\xi\int_\Omega f(n)\Delta c + \xi\int_\Omega \Delta c\nabla c\cdot u$$

for all $t \in (0, T_{max})$. An application of Young's inequality to both integrals on the right side therefore implies that

$$\begin{aligned}\frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^2 + \xi\int_\Omega |\Delta c|^2 + \xi\int_\Omega |\nabla c|^2 \\ \leq \xi\int_\Omega f(n)^2 + \frac{\xi}{2}\int_\Omega |\Delta c|^2 + \xi\int_\Omega |\nabla c|^2|u|^2 \qquad (2.4.4)\end{aligned}$$

holds for all $t \in (0, T_{max})$. We fix $q > 2$ and make use of Hölder's inequality to see that

$$\xi\int_\Omega |\nabla c|^2|u|^2 \leq \xi\|\nabla c\|^2_{L^{\frac{2q}{q-2}}(\Omega)}\|u\|^2_{L^q(\Omega)} \qquad (2.4.5)$$

is valid for all $t \in (0, T_{max})$. An application of the Gagliardo–Nirenberg inequality combined with [73, Theorem 3.4] allows us to further estimate

$$\begin{aligned}\|\nabla c\|^2_{L^{\frac{2q}{q-2}}(\Omega)} &\leq C_1\|\Delta c\|^{\frac{4q+4}{3q}}_{L^2(\Omega)}\|c\|^{\frac{2q-4}{3q}}_{L^1(\Omega)} + C_1\|c\|^2_{L^1(\Omega)} \\ &\leq C_2\|\Delta c\|^{\frac{4}{3}+\frac{4}{3q}}_{L^2(\Omega)} + C_2 \quad \text{for all } t \in (0, T_{max})\end{aligned}$$

for some $C_1 > 0$ and $C_2 > 0$ in view of (2.2.3). Plugging this into (2.4.5) and recalling Proposition 2.5, we thus find $C_3 > 0$ such that

$$\xi\int_\Omega |\nabla c|^2|u|^2 \leq C_3\|\Delta c\|^{\frac{4}{3}+\frac{4}{3q}}_{L^2(\Omega)} + C_3 \quad \text{for all } t \in (0, T_{max}).$$

Since $q > 2$, we have $\frac{4}{3} + \frac{4}{3q} < 2$ and may apply Young's inequality to obtain

$$\xi\int_\Omega |\nabla c|^2|u|^2 \leq \frac{\xi}{4}\|\Delta c\|^2_{L^2(\Omega)} + C_4, \qquad (2.4.6)$$

for some $C_4 > 0$ and all $t \in (0, T_{max})$. To estimate the term containing $f(n)^2$ in (2.4.4) we let K_1 denote the positive constant from Corollary 2.8. Then, recalling (2.1.7) and making use of the fact $\alpha < 1$, an application of Young's inequality yields $C_5 > 0$ fulfilling $\xi f(n)^2 \leq \frac{1}{2K_1}n^2 + C_5$ for all $(x, t) \in \Omega \times (0, T_{max})$ and thus, by Corollary 2.8

$$\xi\int_\Omega f(n)^2 \leq \frac{1}{2K_1}\int_\Omega n^2 + C_5|\Omega| \leq \frac{1}{2}\int_\Omega |\nabla(n^{1/2})|^2 + C_6 \quad \text{for all } t \in (0, T_{max}) \qquad (2.4.7)$$

with $C_6 := \frac{1}{2} + C_5|\Omega|$. Combining (2.4.4), (2.4.6) and (2.4.7) completes the proof. □

Before we are able to combine the previous lemmata to derive an ODI appropriate for our purpose, we require one additional result which is a corollary from Lemma 2.7.

Corollary 2.11.
There exists a constant $K_4 > 0$ such that the solution to (2.1.6) *fulfills*

$$\frac{1}{2}\int_\Omega |\nabla(n^{1/2})|^2 \geq K_4\int_\Omega n\ln n - \frac{1}{2} \quad \textit{for all } t \in (0, T_{max}).$$

Proof: In view of the pointwise inequality $s \ln s \leq s^2$ for $s \in (0,\infty)$, the positivity of n ascertained in Lemma 2.2 therefore implies $n \ln n \leq n^2$ for all $t \in (0, T_{max})$ and thus an application of Corollary 2.8 immediately shows that there exists $C_1 > 0$ such that

$$\int_\Omega n \ln n \leq \int_\Omega n^2 \leq C_1 \|\nabla(n^{1/2})\|^2_{L^2(\Omega)} + C_1$$

holds for all $t \in (0, T_{max})$. Hence, multiplying by $K_4 := \frac{1}{2C_1}$ and reordering the terms appropriately proves the asserted inequality. □

Adding up suitable multiples of the differential inequalities in Lemma 2.9 and Lemma 2.10, we obtain a first bound on the gradient of c.

Proposition 2.12.
There exists a constant $C > 0$ such that the solution of (2.1.6) *fulfills*

$$\int_\Omega |\nabla c|^2 \leq C \quad \textit{for all } t \in (0, T_{max}). \tag{2.4.8}$$

Proof: Letting K_2 denote the positive constant from Lemma 2.9, we set $\xi = 4K_2+4$ and then $K_3 > 0$ as the corresponding constant given by Lemma 2.10. With the constants defined this way, we know that the inequality

$$(2K_2+2)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^2 + (K_2+1)\int_\Omega |\Delta c|^2 + (4K_2+4)\int_\Omega |\nabla c|^2 \leq \frac{1}{2}\int_\Omega |\nabla(n^{1/2})|^2 + K_3, \tag{2.4.9}$$

holds for all $t \in (0, T_{max})$ due to Lemma 2.10. Thus, adding up (2.4.1) and (2.4.9) entails

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega n \ln n + (2K_2+2)\int_\Omega |\nabla c|^2\Big) + \frac{1}{2}\int_\Omega |\nabla(n^{1/2})|^2 + \int_\Omega |\Delta c|^2 + (4K_2+4)\int_\Omega |\nabla c|^2 \leq C_1$$

for all $t \in (0, T_{max})$ with $C_1 = K_2 + K_3 > 0$. By Corollary 2.11 we can estimate $\frac{1}{2}\int_\Omega |\nabla(n^{1/2})|^2$ from below to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega n \ln n + (2K_2+2)\int_\Omega |\nabla c|^2\Big) &+ K_4\int_\Omega n \ln n \\ &+ \int_\Omega |\Delta c|^2 + 2(2K_2+2)\int_\Omega |\nabla c|^2 \leq C_2\end{aligned}$$

for all $t \in (0, T_{max})$, with $K_4 > 0$ provided by Corollary 2.11 and $C_2 = C_1 + \frac{1}{2} > 0$. Dropping the nonnegative term $\int_\Omega |\Delta c|^2$, we find that $y(t) := \int_\Omega n \ln n + (2K_2+2)\int_\Omega |\nabla c|^2$, $t \in [0, T_{max})$ satisfies

$$y'(t) + C_3 y(t) \leq C_2 \quad \text{for all } t \in (0, T_{max}),$$

where $C_3 := \min\{K_4, 2\} > 0$. Upon an ODE comparison ([86, Thm. IX]), this leads to the boundedness of y and hence (2.4.8) due to $n \ln n$ being bounded from below by the positivity of n. □

2.4.2 Further testing procedures

The L^2 bound of the gradient of c from the previous lemma will be our starting point in improving the regularity of both n and c. Preparation and combination of differential inequalities concerning n^p and $|\nabla c|^{2q}$, for appropriately chosen q and p, will be the main part of this section. The testing procedures employed in this approach are based on the application to a similar chemotaxis-Stokes system discussed in [100].
The following preparatory result, taken from [79, Lemma 2.6], will be a useful tool in estimations later on and is a simple derivation from Young's inequality.

Lemma 2.13.
Let $a > 0$ and $b > 0$ be such that $a + b < 1$. Then for all $\varepsilon > 0$ there exists $C > 0$ such that

$$x^a y^b \leq \varepsilon(x+y) + C \quad \textit{for all } x \geq 0 \textit{ and } y \geq 0.$$

The first step to improve the known regularities of n and c consists of an application of standard testing procedures to gain separate inequalities regarding the time evolution of $\int_\Omega n^p$ and $\int_\Omega |\nabla c|^{2q}$, respectively.

Lemma 2.14.
Let $p > 1$. Then the solution of (2.1.6) *satisfies*

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p + \frac{2(p-1)}{p^2}\int_\Omega |\nabla(n^{p/2})|^2 \leq \frac{p-1}{2}\int_\Omega n^p|\nabla c|^2 \tag{2.4.10}$$

for all $t \in (0, T_{max})$.

Proof: We multiply the first equation of (2.1.6) with n^{p-1} and integrate by parts to see that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p = -(p-1)\int_\Omega |\nabla n|^2 n^{p-2} + (p-1)\int_\Omega n^{p-1}\nabla c\cdot\nabla n - \frac{1}{p}\int_{\partial\Omega} n^p u\cdot\vec{\nu}$$

holds for all $t \in (0, T_{max})$, where we made use of the fact $\nabla\cdot u = 0$ and the divergence theorem to rewrite the last term accordingly. Due to the boundary condition imposed on u the last term disappears, and therefore an application of Young's inequality to the second to last term implies

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p + (p-1)\int_\Omega |\nabla n|^2 n^{p-2} \leq \frac{p-1}{2}\int_\Omega |\nabla n|^2 n^{p-2} + \frac{p-1}{2}\int_\Omega n^p|\nabla c|^2$$

for all $t \in (0, T_{max})$. Reordering the terms and rewriting $|\nabla n|^2 n^{p-2} = \frac{4}{p^2}|\nabla(n^{p/2})|^2$ completes the proof. □

Lemma 2.15.
Let $q > 1$. Then the solution of (2.1.6) *satisfies*

$$\frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^{2q} + \frac{2(q-1)}{q^2}\int_\Omega \Big|\nabla|\nabla c|^q\Big|^2 + \int_\Omega |\nabla c|^{2q}$$

$$\leq \left(K_0(q-1)+\frac{K_0}{\sqrt{2}}\right)^2 \int_\Omega n^{2\alpha}|\nabla c|^{2q-2} + \int_\Omega |\nabla c|^{2q}|Du| \tag{2.4.11}$$

for all $t \in (0, T_{max})$.

Proof: Differentiating the second equation of (2.1.6) and making use of the fact that $\Delta|\nabla c|^2 = 2\nabla c \cdot \nabla\Delta c + 2|D^2c|^2$, we obtain for all $(x,t) \in \Omega \times (0, T_{max})$ that

$$\begin{aligned}\frac{1}{2}\left(|\nabla c|^2\right)_t &= \nabla c \cdot \nabla\left(\Delta c - c + f(n) - u\cdot\nabla c\right)\\ &= \frac{1}{2}\Delta|\nabla c|^2 - |D^2c|^2 - |\nabla c|^2 + \nabla c\cdot\nabla f(n) - \nabla c \cdot \nabla\left(u\cdot\nabla c\right).\end{aligned}$$

Multiplying this identity by $\left(|\nabla c|^2\right)^{q-1}$ and integrating by parts, where, due to the Neumann boundary conditions imposed on n and c, every boundary integral except the one involving $\frac{\partial|\nabla c|^2}{\partial\nu}$ disappears, we find that

$$\begin{aligned}&\frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla c|^{2q} + \frac{q-1}{2}\int_\Omega|\nabla c|^{2q-4}\left|\nabla|\nabla c|^2\right|^2 + \int_\Omega|\nabla c|^{2q-2}|D^2c|^2 + \int_\Omega|\nabla c|^{2q}\\ &= \int_\Omega|\nabla c|^{2q-2}\nabla c\cdot\nabla f(n) - \int_\Omega|\nabla c|^{2q-2}\nabla c\cdot\nabla\left(u\cdot\nabla c\right) + \frac{1}{2}\int_{\partial\Omega}|\nabla c|^{2q-2}\frac{\partial|\nabla c|^2}{\partial\nu}\end{aligned} \tag{2.4.12}$$

holds for all $t \in (0, T_{max})$. Recalling (2.1.7), we integrate the first integral on the right by parts to see that

$$\int_\Omega|\nabla c|^{2q-2}\nabla c\cdot\nabla f\left(n\right) \leq K_0\int_\Omega\left|\nabla|\nabla c|^{2q-2}\right||\nabla c|\, n^\alpha + K_0\int_\Omega|\nabla c|^{2q-2}|\Delta c|\, n^\alpha$$

holds for all $t \in (0, T_{max})$. Since $\nabla|\nabla c|^{2q-2} = 2(q-1)|\nabla c|^{2q-4}D^2c\cdot\nabla c$ in $\Omega\times(0,T_{max})$, wherein the Cauchy-Schwarz inequality furthermore implies $|\Delta c| \leq \sqrt{2}|D^2c|$, we may apply Young's inequality to obtain

$$\begin{aligned}&\int_\Omega|\nabla c|^{2q-2}\nabla c\cdot\nabla f\left(n\right)\\ &\leq \int_\Omega|\nabla c|^{2q-2}|D^2c|^2 + \frac{\left(2K_0(q-1)+\sqrt{2}K_0\right)^2}{4}\int_\Omega|\nabla c|^{2q-2}n^{2\alpha}\\ &= \int_\Omega|\nabla c|^{2q-2}|D^2c|^2 + \left(K_0(q-1)+\frac{K_0}{\sqrt{2}}\right)^2\int_\Omega|\nabla c|^{2q-2}n^{2\alpha}\end{aligned} \tag{2.4.13}$$

for all $t \in (0, T_{max})$. To treat the second integral on the right hand side of (2.4.12), we first rewrite

$$\begin{aligned}&-\int_\Omega|\nabla c|^{2q-2}\nabla c\cdot\nabla\left(u\cdot\nabla c\right)\\ &= -\int_\Omega|\nabla c|^{2q-2}\nabla c\cdot\left(Du\cdot\nabla c\right) - \int_\Omega|\nabla c|^{2q-2}\nabla c\cdot\left(D^2c\cdot u\right)\end{aligned} \tag{2.4.14}$$

for all $t \in (0, T_{max})$, and then make use of the pointwise equality

$$|\nabla c|^{2q-2}\nabla c \cdot \left(D^2 c \cdot u\right) = \frac{1}{2q} u \cdot \nabla |\nabla c|^{2q} \text{ in } \Omega \times (0, T_{max}),$$

to see that, since u is divergence free,

$$-\int_\Omega |\nabla c|^{2q-2}\nabla c \cdot \left(D^2 c \cdot u\right) = \frac{1}{2q}\int_\Omega (\nabla \cdot u)|\nabla c|^{2q} = 0$$

holds for all $t \in (0, T_{max})$. Thus, (2.4.14) implies

$$-\int_\Omega |\nabla c|^{2q-2}\nabla c \cdot \nabla\left(u \cdot \nabla c\right) \le \int_\Omega |\nabla c|^{2q}|Du| \quad \text{for all } t \in (0, T_{max}). \tag{2.4.15}$$

For the remaining boundary integral in (2.4.12) we recall that the convexity of Ω ensures $\frac{\partial |\nabla c|^2}{\partial \nu} \le 0$ on $\partial\Omega$ (see [53, Lemme I.1, p.350]). Combining this with (2.4.12), (2.4.13) and (2.4.15) completes the proof due to the identity

$$\left|\nabla |\nabla c|^q\right|^2 = \frac{q^2}{4}|\nabla c|^{2q-4}\left|\nabla |\nabla c|^2\right|^2 \quad \text{in } \Omega \times (0, T_{max}). \qquad \Box$$

Before uniting the inequalities from (2.4.10) and (2.4.11) into a single energy-type inequality, we estimate the right hand sides therein separately.

Lemma 2.16.
Let $\infty > q > \max\{2, \frac{1}{\alpha}\}$, $p = \alpha q$. For any $\eta > 0$ there exist constants K_5, K_6 and $K_7 > 0$ such that

$$\frac{p-1}{2}\int_\Omega n^p |\nabla c|^2 \le \frac{\eta}{6}\left(\int_\Omega |\nabla (n^{p/2})|^2 + \int_\Omega \left|\nabla |\nabla c|^q\right|^2\right) + K_5, \tag{2.4.16}$$

$$\left(K_0(q-1) + \frac{K_0}{\sqrt{2}}\right)^2 \int_\Omega n^{2\alpha}|\nabla c|^{2q-2} \le \frac{\eta}{6}\left(\int_\Omega |\nabla (n^{p/2})|^2 + \int_\Omega \left|\nabla |\nabla c|^q\right|^2\right) + K_6 \tag{2.4.17}$$

and

$$\int_\Omega |\nabla c|^{2q}|Du| \le \frac{\eta}{6}\int_\Omega \left|\nabla |\nabla c|^q\right|^2 + K_7 \tag{2.4.18}$$

hold for all $t \in (0, T_{max})$.

Proof: To prove (2.4.16), we first fix some $\beta_1 > 1$ and apply Hölder's inequality to obtain

$$\frac{p-1}{2}\int_\Omega n^p |\nabla c|^2 \le \frac{p-1}{2}\left(\int_\Omega n^{p\beta_1}\right)^{\frac{1}{\beta_1}}\left(\int_\Omega |\nabla c|^{2\beta_1'}\right)^{\frac{1}{\beta_1'}} \tag{2.4.19}$$

for all $t \in (0, T_{max})$, where β_1' denotes the Hölder conjugate of β_1. By (2.2.2) and Lemma 2.7 applied to $\varphi = n$, $L = m$, $r = p$ and $s = \beta_1$, we can find $C_1 > 0$ such that

$$\left(\int_\Omega n^{p\beta_1}\right)^{\frac{1}{\beta_1}} \leq C_1 \left(\int_\Omega |\nabla(n^{p/2})|^2\right)^{1-\frac{1}{p\beta_1}} + C_1 \quad \text{for all } t \in (0, T_{max}). \tag{2.4.20}$$

An application of the Gagliardo–Nirenberg inequality ([52, Lemma 2.3]), similar to the one utilized in Lemma 2.7, shows that the second integral on the right in (2.4.19) satisfies

$$\left(\int_\Omega |\nabla c|^{2\beta_1'}\right)^{\frac{1}{\beta_1'}} \leq C_2 \left(\left\| \nabla |\nabla c|^q \right\|_{L^2(\Omega)}^{\frac{2b_1}{q}} \left\| |\nabla c|^q \right\|_{L^{\frac{2}{q}}(\Omega)}^{\frac{(2-2b_1)}{q}} + \left\| |\nabla c|^q \right\|_{L^{\frac{2}{q}}(\Omega)}^{\frac{2}{q}} \right) \tag{2.4.21}$$

for all $t \in (0, T_{max})$ with $C_2 > 0$ and $b_1 \in (0,1)$ provided by

$$b_1 = \frac{\frac{q}{2} - \frac{q}{2\beta_1'}}{\frac{q}{2} + \frac{1}{2} - \frac{1}{2}} = 1 - \frac{1}{\beta_1'} = \frac{1}{\beta_1}.$$

Since Proposition 2.12 implies the boundedness of $\||\nabla c|^q\|_{L^{\frac{2}{q}}(\Omega)}$, plugging (2.4.20) and (2.4.21) into (2.4.19) we obtain $C_3 > 0$ such that

$$\begin{aligned} \frac{p-1}{2} \int_\Omega n^p |\nabla c|^2 \leq{}& C_3 \left(\int_\Omega |\nabla(n^{p/2})|^2 \right)^{1-\frac{1}{p\beta_1}} \left(\int_\Omega \left|\nabla |\nabla c|^q\right|^2 \right)^{\frac{1}{q\beta_1}} \\ &+ C_3 \left(\int_\Omega |\nabla(n^{p/2})|^2 \right)^{1-\frac{1}{p\beta_1}} + C_3 \left(\int_\Omega \left|\nabla |\nabla c|^q\right|^2 \right)^{\frac{1}{q\beta_1}} + C_3 \end{aligned}$$

holds for all $t \in (0, T_{max})$. Due to $\alpha < 1$ the choice of $p = \alpha q$ implies $p < q$ and thus, $1 - \frac{1}{p\beta_1} + \frac{1}{q\beta_1} < 1$. Therefore, we may apply Lemma 2.13 with $\varepsilon = \frac{\eta}{12}$ to the three terms on the right hand side containing an integral and obtain for some $C_4 > 0$ that

$$\frac{p-1}{2} \int_\Omega n^p |\nabla c|^2 \leq \frac{\eta}{6} \left(\int_\Omega |\nabla(n^{p/2})|^2 + \int_\Omega \left|\nabla |\nabla c|^q\right|^2 \right) + C_4$$

holds for all $t \in (0, T_{max})$, which proves (2.4.16). The proof of (2.4.17) follows the same reasoning. First, we apply Hölder's inequality with $\beta_2 = \frac{q+1}{2}$ and β_2' as corresponding Hölder conjugate to obtain

$$\int_\Omega n^{2\alpha} |\nabla c|^{2q-2} \leq \left(\int_\Omega n^{2\alpha\beta_2} \right)^{\frac{1}{\beta_2}} \left(\int_\Omega |\nabla c|^{(2q-2)\beta_2'} \right)^{\frac{1}{\beta_2'}} \tag{2.4.22}$$

for all $t \in (0, T_{max})$. Since the choices of β_2 and p imply $\frac{2\alpha\beta_2}{p} = \frac{\alpha(q+1)}{\alpha q} > 1$, we can utilize Lemma 2.7 with $\varphi = n$, $r = p$ and $s = \frac{2\alpha\beta_2}{p}$ to estimate

$$\left(\int_\Omega n^{2\alpha\beta_2} \right)^{\frac{1}{\beta_2}} \leq C_5 \left(\int_\Omega |\nabla(n^{p/2})|^2 \right)^{\frac{2\alpha\beta_2 - 1}{p\beta_2}} + C_5 \quad \text{for all } t \in (0, T_{max}), \tag{2.4.23}$$

with some $C_5 > 0$. For the integral involving $|\nabla c|^{(2q-2)\beta_2'}$, we make use of the Gagliardo–Nirenberg inequality as shown before to obtain $C_6 > 0$ such that

$$\left(\int_\Omega |\nabla c|^{(2q-2)\beta_2'}\right)^{\frac{1}{\beta_2'}} \le C_6 \left(\int_\Omega \Big|\nabla |\nabla c|^q\Big|^2\right)^{\frac{(q-1)b_2}{q}} + C_6 \tag{2.4.24}$$

holds for all $t \in (0, T_{max})$, with $b_2 \in (0,1)$ determined by

$$b_2 = \frac{\frac{q}{2} - \frac{q}{2(q-1)\beta_2'}}{\frac{q}{2} + \frac{1}{2} - \frac{1}{2}} = 1 - \frac{1}{(q-1)\beta_2'} = 1 - \frac{1}{(q-1)} + \frac{1}{(q-1)\beta_2}.$$

Thus, a combination of (2.4.22), (2.4.23) and (2.4.24) leads to

$$\begin{aligned}
&\left(K_0(q-1) + \frac{K_0}{\sqrt{2}}\right)^2 \int_\Omega n^{2\alpha} |\nabla c|^{2q-2} \\
&\le C_7 \bigg(\int_\Omega |\nabla (n^{p/2})|^2\bigg)^{\frac{2\alpha\beta_2 - 1}{p\beta_2}} \left(\int_\Omega \Big|\nabla |\nabla c|^q\Big|^2\right)^{\frac{(q-1)b_2}{q}} \\
&\quad + C_7 \left(\int_\Omega |\nabla (n^{p/2})|^2\right)^{\frac{2\alpha\beta_2 - 1}{p\beta_2}} + C_7 \left(\int_\Omega \Big|\nabla |\nabla c|^q\Big|^2\right)^{\frac{(q-1)b_2}{q}} + C_7
\end{aligned}$$

for all $t \in (0, T_{max})$ with some $C_7 > 0$. Here the choice of p and the fact that $\alpha < 1$ imply

$$\begin{aligned}
\frac{2\alpha\beta_2 - 1}{p\beta_2} + \frac{(q-1)b_2}{q} &= \frac{2\alpha}{p} - \frac{1}{p\beta_2} + \frac{q-2}{q} + \frac{1}{q\beta_2} \\
&= \frac{2}{q} - \frac{1}{\alpha q \beta_2} + \frac{q-2}{q} + \frac{1}{q\beta_2} = 1 - \frac{1-\alpha}{\alpha q \beta_2} < 1.
\end{aligned}$$

Therefore, the requirements of Lemma 2.13 are satisfied again and an application thereof yields $C_8 > 0$ such that

$$\left(K_0(q-1) + \frac{K_0}{\sqrt{2}}\right)^2 \int_\Omega n^{2\alpha} |\nabla c|^{2q-2} \le \frac{\eta}{6} \left(\int_\Omega |\nabla (n^{p/2})|^2 + \int_\Omega \Big|\nabla |\nabla c|^q\Big|^2\right) + C_8$$

holds for all $t \in (0, T_{max})$ and thus proves (2.4.17). To verify (2.4.18) we fix $\beta_3 = \frac{3}{2}$ and $\beta_3' = 3$. Since $\beta_3 < 2$ Hölder's inequality yields

$$\int_\Omega |\nabla c|^{2q} |Du| \le \left(\int_\Omega |\nabla c|^{2q\beta_3'}\right)^{\frac{1}{\beta_3'}} \left(\int_\Omega |Du|^{\beta_3}\right)^{\frac{1}{\beta_3}} \le C_9 \Big\| |\nabla c|^q \Big\|_{L^6(\Omega)}^2$$

for some $C_9 > 0$, in view of the boundedness of $\|Du\|_{L^{\frac{3}{2}}(\Omega)}$ shown in Proposition 2.5. Similarly to the previous applications of the Gagliardo-Nirenberg and Young inequalities we can make use of the boundedness of $\||\nabla c|^q\|_{L^{\frac{2}{q}}(\Omega)}$ to obtain $C_{10} > 0$ such that

$$\int_\Omega |\nabla c|^{2q} |Du| \le \frac{\eta}{6} \int_\Omega \Big|\nabla |\nabla c|^q\Big|^2 + C_{10}$$

for all $t \in (0, T_{max})$, which completes the proof. □

Combining the three previous lemmata we are now in the position to control norms of n and ∇c in $L^p(\Omega)$ with arbitrarily high p. In fact, we have the following:

Proposition 2.17.
Let $\infty > q > \max\{2, \frac{1}{\alpha}\}$ and $p = \alpha q$. Then we can find $C > 0$ such that, the solution of (2.1.6) *satisfies*

$$\int_\Omega n^p \leq C \quad \textit{for all } t \in (0, T_{max}) \tag{2.4.25}$$

and

$$\int_\Omega |\nabla c|^{2q} \leq C \quad \textit{for all } t \in (0, T_{max}). \tag{2.4.26}$$

Proof: Given $q > \max\{2, \frac{1}{\alpha}\}$ and $p = \alpha q$ we fix $\eta = \min\left\{\frac{2(q-1)}{q^2}, \frac{2(p-1)}{p^2}\right\}$. By the Lemmata 2.14, 2.15 and 2.16, we can find $C_1 := K_5 + K_6 + K_7 > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\int_\Omega n^p + \frac{1}{2q}\int_\Omega |\nabla c|^{2q}\right) + \frac{2(p-1)}{p^2}\int_\Omega |\nabla(n^{p/2})|^2 + \frac{2(q-1)}{q^2}\int_\Omega \Big|\nabla|\nabla c|^q\Big|^2 \\ + \int_\Omega |\nabla c|^{2q} \leq \frac{\eta}{2}\left(\int_\Omega |\nabla(n^{p/2})|^2 + \int_\Omega \Big|\nabla|\nabla c|^q\Big|^2\right) + C_1\end{aligned}$$

holds for all $t \in (0, T_{max})$. Herein the choice of η implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\int_\Omega n^p + \frac{1}{2q}\int_\Omega |\nabla c|^{2q}\right) + \frac{p-1}{p^2}\int_\Omega |\nabla(n^{p/2})|^2 \\ + \frac{q-1}{q^2}\int_\Omega \Big|\nabla|\nabla c|^q\Big|^2 + \int_\Omega |\nabla c|^{2q} \leq C_1\end{aligned} \tag{2.4.27}$$

for all $t \in (0, T_{max})$. We drop the nonnegative term $\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla c|^q|^2$ and apply Lemma 2.7 to estimate $\int_\Omega |\nabla(n^{p/2})|^2$ from below in (2.4.27), to obtain $C_2, C_3 > 0$ such that $y(t) := \frac{1}{p}\int_\Omega n^p + \frac{1}{2q}\int_\Omega |\nabla c|^{2q}$, $t \in (0, T_{max})$ satisfies

$$y'(t) + C_2 y(t) \leq C_3 \quad \text{for all } t \in (0, T_{max}),$$

from which we infer the boundedness of y upon an ODE comparison and thus (2.4.25) and (2.4.26). □

2.4.3 Global existence and boundedness

We can now begin to verify the boundedness of the three quantities appearing in the extensibility criterion (2.2.1). The first of these quantities will be $\|A^\varrho u(\cdot, t)\|_{L^2(\Omega)}$.

Proposition 2.18.
Let $\varrho \in (\frac{1}{2}, 1)$ be as in Lemma 2.2. There exists a constant $C > 0$ such that the solution of (2.1.6) *satisfies*

$$\|A^\varrho u(\cdot, t)\|_{L^2(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{max}).$$

Proof: The proof essentially follows the argumentation of [79, Lemma 2.4], whilst making use of the previously proven bound $\|n\|_{L^p(\Omega)} \leq C$ for all $t \in (0, T_{max})$ with some $p > 2$. Nonetheless, let us recount the main arguments.
It is well known, see [72, Theorem 38.6] and [75, p.204] for instance, that the Stokes operator A is a positive, sectorial operator and generates a contraction semigroup $\left(e^{-tA}\right)_{t\geq 0}$ in $L^2_\sigma(\Omega)$ with operator norm bounded by

$$\|e^{-tA}\| \leq e^{-\lambda_1 t} \quad \text{for all } t \geq 0,$$

with some $\lambda_1 > 0$. Furthermore, the operator norm of the fractional powers of the Stokes operator satisfy an exponential decay property ([72, Theorem 37.5]). That is, there exists $C_1 > 0$ such that

$$\left\|A^\varrho e^{-tA}\right\| \leq C_1 t^{-\varrho} e^{-\lambda_1 t} \quad \text{for all } t > 0. \tag{2.4.28}$$

Thus, representing u by its variation of constants formula

$$u(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}\left(n(\cdot,s)\nabla\phi\right)\mathrm{d}s, \quad t \in (0, T_{max}),$$

and applying the fractional power A^ϱ, we can make use of the fact that e^{-tA} commutes with A^ϱ ([75, IV.(1.5.16), p.206]), the contraction property and (2.4.28) to find $C_2 > 0$ such that

$$\begin{aligned}
&\|A^\varrho u(\cdot,t)\|_{L^2(\Omega)} \\
&\leq \|A^\varrho u_0\|_{L^2(\Omega)} + C_1 \int_0^t (t-s)^{-\varrho}\, e^{-\lambda_1(t-s)} \left\|\mathcal{P}\left(n(\cdot,s)\nabla\phi\right)\right\|_{L^2(\Omega)} \mathrm{d}s \\
&\leq \|A^\varrho u_0\|_{L^2(\Omega)} + C_2 \sup_{t\in(0,T_{max})} \|n(\cdot,t)\|_{L^2(\Omega)} \int_0^\infty \sigma^{-\varrho} e^{-\lambda_1\sigma}\,\mathrm{d}\sigma
\end{aligned} \tag{2.4.29}$$

holds for all $t \in (0, T_{max})$, by the boundedness of $\nabla\phi$. In light of (2.1.10) we have $\|A^\varrho u_0\|_{L^2(\Omega)} \leq C_3$ for some $C_3 > 0$. Furthermore, since $\varrho < 1$ the integral converges and by Proposition 2.17, applied with some $q > \frac{2}{\alpha}$, we can find $C_4 > 0$ such that $\|n(\cdot,t)\|_{L^2(\Omega)} \leq C_4$ for all $t \in (0, T_{max})$. Combining these facts with (2.4.29) yields

$$\|A^\varrho u(\cdot,t)\|_{L^2(\Omega)} \leq C_5 \quad \text{for all } t \in (0, T_{max})$$

with some $C_5 > 0$, which completes the proof. □

The next quantity of the extensibility criterion we treat is $\|c(\cdot,t)\|_{W^{1,\vartheta}(\Omega)}$. In view of Proposition 2.17, we can take some $q > \max\{\vartheta, \frac{1}{\alpha}\}$ to obtain the following corollary from a simple application of the Poincaré inequality.

Corollary 2.19.
There exists a constant $C > 0$ such that

$$\|c(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} \leq C$$

holds for all $t \in (0, T_{max})$.

Now, to prove the last remaining bound required for the extensibility criterion (2.2.1) as well as one of the estimates required for the boundedness result, we require some well known results concerning the Neumann heat semigroup $\left(e^{t\Delta}\right)_{t\geq 0}$. These semigroup estimates and Proposition 2.17 will be the main ingredients of our proof. For more details concerning the estimation process employed, we refer the reader to [12, Lemma 2.1], [97, Lemma 1.3] and [32].

Proposition 2.20.
There exists a constant $C > 0$ such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq C$$

holds for all $t \in (0, T_{max})$.

Proof: First, we fix $p > 2$ and represent n by its variation of constants formula

$$n(\cdot,t) = e^{t\Delta}n_0 - \int_0^t e^{(t-s)\Delta}\big(\nabla\cdot(n\nabla c) + u\cdot\nabla n\big)(\cdot,s)\ \mathrm{d}s, \quad t \in (0, T_{max}).$$

The fact $\nabla\cdot u = 0$ and the maximum principle then yield

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq \|n_0\|_{L^\infty(\Omega)} + \int_0^t \Big\|e^{(t-s)\Delta}\big(\nabla\cdot(n\nabla c + un)\big)(\cdot,s)\Big\|_{L^\infty(\Omega)}\ \mathrm{d}s$$

for all $t \in (0, T_{max})$. Now, we can make use of the well-known smoothing properties of the Neumann heat semigroup (see [12, Lemma 2.1 (iv)]), to estimate

$$\begin{aligned} \|n(\cdot,t)\|_{L^\infty(\Omega)} &\leq \|n_0\|_{L^\infty(\Omega)} \\ &\quad + C_1\int_0^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{1}{p}}\Big)\, e^{-\mu_1(t-s)}\, \|(n\,(\nabla c + u))\,(\cdot,s)\|_{L^p(\Omega)}\ \mathrm{d}s \end{aligned} \tag{2.4.30}$$

for all $t \in (0, T_{max})$ and some $C_1 > 0$, where μ_1 denotes the first nonzero eigenvalue of $-\Delta$ in Ω with regards to the homogeneous Neumann boundary conditions. To estimate $\|n(\nabla c + u)\|_{L^p(\Omega)}$ we apply Hölder's inequality to obtain some $C_2 > 0$ such that

$$\|n(\nabla c + u)(\cdot,t)\|_{L^p(\Omega)} \leq \|n(\cdot,t)\|_{L^{2p}(\Omega)}\big(\|\nabla c(\cdot,t)\|_{L^{2p}(\Omega)} + \|u(\cdot,t)\|_{L^{2p}(\Omega)}\big) \leq C_2$$

holds for all $t \in (0, T_{max})$, wherein the boundedness of all quantities on the right hand side followed in view of Propositions 2.5 and 2.17. Plugging this into (2.4.30) and recalling $n_0 \in C^0\big(\overline{\Omega}\big)$ yields $C_3 > 0$ such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq C_3 + C_3\int_0^\infty \Big(1 + \sigma^{-\frac{1}{2}-\frac{1}{p}}\Big)\, e^{-\mu_1\sigma}\ \mathrm{d}\sigma$$

is valid for all $t \in (0, T_{max})$. By the choice of p we have $-\frac{1}{2} - \frac{1}{p} > -1$ and thus there exists $C_4 > 0$ such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq C_4 \quad \text{for all } t \in (0, T_{max}),$$

which completes the proof. $\square$

Let us gather the previous results to prove our main theorem.

Proof of Theorem 2.1: As an immediate consequence of the bounds in Proposition 2.18, Corollary 2.19 and Proposition 2.20, we obtain $T_{max} = \infty$ in view of the extensibility criterion (2.2.1). Secondly, since $\vartheta > 2$ we have $W^{1,\vartheta}(\Omega) \hookrightarrow C^{\gamma_1}(\Omega)$ with $\gamma_1 = \frac{\vartheta-2}{\vartheta}$ ([23, Theorem 5.6.5]). Thus, Corollary 2.19 implies $\|c(\cdot,t)\|_{L^\infty(\Omega)} \leq C$ for all $t \in (0, T_{max})$. Additionally, since for $\varrho \in (\frac{1}{2}, 1)$ the fractional powers of the Stokes operator satisfy $D(A^\varrho) \hookrightarrow C^{\gamma_2}(\Omega)$ for any $\gamma_2 \in (0, 2\varrho - 1)$ (see [75, Lemma III.2.4.3] and [23, Theorem 5.6.5]), Proposition 2.18 shows that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C$ for all $t \in (0, T_{max})$ and the boundedness of $\|n(\cdot,t)\|_{L^\infty(\Omega)}$ for all $t \in (0, T_{max})$ follows directly from Proposition 2.20. $\square$

3 Global solvability of chemotaxis fluid systems with nonlinear diffusion and matrix-valued sensitivities in three dimensions

3.1 Introduction

A distinct feature of Keller–Segel-type models (even without fluid) is their possibility to capture the emergence of patterns arising from the aggregation of bacteria, which on the solution level of the corresponding PDE system

$$n_t = \nabla \cdot \big(D(n)\nabla n - nS(n,c)\nabla c\big) \qquad\qquad c_t = \Delta c - c + n, \tag{3.1.1}$$

can be observed as blow-up of solutions. Accordingly, obtaining results proving or excluding the possibility of blow-up has been a significant concern of the literature. For the Keller–Segel system of the form in (3.1.1), the quantity governing the behavior has been identified to be the growth ratio of $\frac{nS(n)}{D(n)}$, with its critical number given by $\frac{2}{N}$ and N being the space dimension (see [80] and references therein). Essentially, considering a corresponding Neumann-boundary value problem in a smooth domain $\Omega \subset \mathbb{R}^N$, the classical solutions emerging from suitably regular initial data remain bounded for all times, whenever $S = S(n)$ and

$$\frac{nS(n)}{D(n)} \leq C(1+n)^{\beta} \quad \text{for all } n \geq 0 \text{ with some } C > 0 \text{ and } \beta < \frac{2}{N}.$$

On the other hand, in [95] smooth solutions blowing-up in either infinite or finite time have been shown to exist under the assumption of

$$\frac{nS(n)}{D(n)} \geq Cn^{\gamma} \quad \text{for all } n > 1 \text{ with some } C > 0 \text{ and } \gamma > \frac{2}{N}.$$

(Finite time blow-up has also been witnessed in [15].) Especially, considering cell diffusion as covered by variants of the porous medium operator, but nondegenerate, i.e. $D(n) \equiv m(n+1)^{m-1}$, and a sensitivity function satisfying $S(n) \equiv (1+n)^{-\alpha}$, the condition for finite time blow-up to be excluded in (3.1.1) can be expressed as $m + \alpha > \frac{2N-2}{N}$. This number will act as our comparison point for conditions arising in the setting of a Keller–Segel system coupled to fluid equations, where the underlying model is motivated by the fact that studies on broadcast spawning ([16, 58]) indicate a great impact of the

the surrounding liquid on the migration process. The literature concerning global existence in systems incorporating both fluid interaction and signal production described by

$$\begin{cases} n_t + \quad u \cdot \nabla n \quad = \nabla \cdot \big(D(n)\nabla n - nS(x,n,c)\nabla c \big), \\ c_t + \quad u \cdot \nabla c \quad = \Delta c - c + n, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u - \nabla P + n\nabla\phi, \\ \qquad \nabla \cdot u \quad = 0, \end{cases} \tag{3.1.2}$$

where S may be a tensor-valued function, u denotes the fluid-velocity field, P the corresponding pressure and ϕ is a given gravitational potential, however, is not as rich and mostly focuses either on the case $D(n) \equiv 1$ or on $S(x,n,c) \equiv 1$. (A more common variant of (3.1.2) is concerned with signal consumption as proposed by [85]. For this setting the results are a bit more extensive and an overview of known results in three-dimensional domains can be found in the references of [7].) The tensor-valued sensitivity function present in (3.1.2) is motivated by the observation that movement of bacteria is biased, as witnessed in colonies of *Proteus mirabilis* ([109]), where spiraling streams always turn counterclockwise, or with *E. coli* cells always following clockwise, circular trajectories near solid boundaries ([19, 49]). Actually, a contribution to the chemotaxis term perpendicular to the signal gradient also appears when deriving macroscopic chemotaxis equations from a cell-based model incorporating swimming bias ([108]). Let us briefly recapitulate the recent developments for porous medium type diffusion $D(n) = mn^{m-1}$. In the case of $m = 1$ (i.e. linear diffusion) and tensor-valued $S(x,n,c)$ satisfying $|S(x,n,c)| \le (1+n)^{-\alpha}$ global weak solutions were shown to exist for $\alpha \ge \frac{3}{7}$ ([55]) and global very weak solutions were established whenever $\alpha > \frac{1}{3}$ ([88]). In space dimension $N = 2$ the optimal condition $\alpha > 0$ can even be reached with global bounded classical solutions ([90]). If we simplify to Stokes-fluid ($\kappa = 0$ in (3.1.2)) instead of full Navier–Stokes-fluid, more regular solutions can also be achieved in dimension $N = 3$, as indicated by the recent studies on bounded classical solutions in [106]. On the other hand, in the case of $S(x,n,c) \equiv 1$ (i.e. $\alpha = 0$) and $m > 1$ global weak solutions were obtained first for $m > 2$ in [111] and more recently for $m > \frac{5}{3}$ in [7], where also global very weak solutions were shown to exist whenever $m > \frac{4}{3}$. The results concerning $N = 3$ and Navier–Stokes-fluid can be illustrated by the picture on the left. Comparing with the either or cases from the fluid-free setting one would expect global very weak solutions to exist for all $m \ge 1$ and $\alpha \ge 0$ satisfying $m + \alpha > \frac{4}{3}$. However, connecting the currently known limit cases

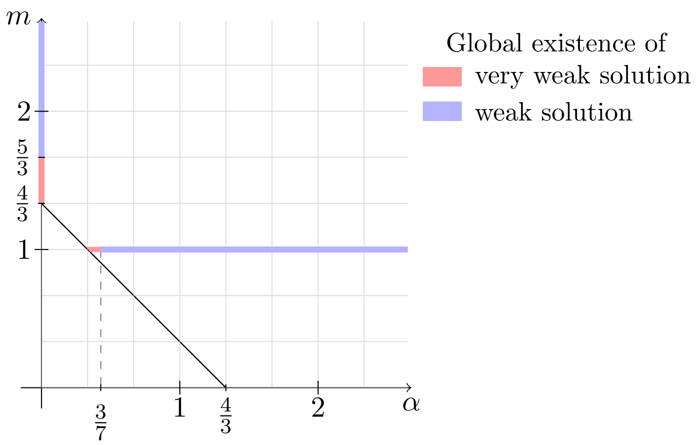

Fig. 3.1: Overview of global existence in (3.1.2) prior to this work

for weak solutions in the standard sense to exist leads to a line which appears to have a rather unnatural slope, posing the question whether the current condition $m = 1$ and $\alpha > \frac{3}{7}$ is critical in α for global weak solutions to exist. Our main interest thereby consists in extracting a priori estimates from the sparse information provided by the system, which, most importantly, capture optimal conditions on $m \geq 1$ and $\alpha \geq 0$.

Main results. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $m \geq 1$ and that for some $\alpha \geq 0$ and $S_0 > 0$ the tensor-valued sensitivity function $S \in C^2\big(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times 3}\big)$ satisfies

$$|S(x,n,c)| \leq \frac{S_0}{(1+n)^\alpha} \quad \text{for all } x \in \overline{\Omega}, n \geq 0 \text{ and } c \geq 0. \tag{3.1.3}$$

Under these assumptions we will consider

$$\begin{cases} n_t + \ u \cdot \nabla n \ = \Delta n^m - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, \quad t > 0, \\ c_t + \ u \cdot \nabla c \ = \Delta c - c + n, & x \in \Omega, \quad t > 0, \\ u_t + (u \cdot \nabla) u = \Delta u - \nabla P + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \qquad \nabla \cdot u \ = 0, & x \in \Omega, \quad t > 0, \end{cases} \tag{3.1.4}$$

complemented with boundary conditions

$$\Big(\nabla n^m(x,t) - n(x,t) S\big(x, n(x,t), c(x,t)\big) \nabla c(x,t)\Big) \cdot \nu = 0, \tag{3.1.5}$$
$$\nabla c(x,t) \cdot \nu = 0 \quad \text{and} \quad u(x,t) = 0 \qquad \text{for } x \in \partial\Omega \text{ and } t > 0,$$

and initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \quad x \in \Omega, \tag{3.1.6}$$

where the gravitational potential ϕ is assumed to satisfy

$$\phi \in W^{2,\infty}(\Omega). \tag{3.1.7}$$

Prescribing initial data which satisfy the conditions

$$\begin{cases} n_0 \in C^\gamma\big(\overline{\Omega}\big) \quad \text{for some } \gamma > 0 \quad \text{with } n_0 \geq 0 \text{ in } \overline{\Omega} \text{ and } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{with } c_0 \geq 0 \text{ in } \overline{\Omega}, \text{ and } c_0 \not\equiv 0, \\ u_0 \in W^{2,2}\big(\Omega; \mathbb{R}^3\big) \cap W_0^{1,2}\big(\Omega; \mathbb{R}^3\big) \quad \text{such that } \nabla \cdot u_0 = 0, \end{cases} \tag{3.1.8}$$

we obtain the following main results.

Theorem 3.1.
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Suppose that $m \geq 1$ and $\alpha \geq 0$ satisfy $m + 2\alpha > \frac{5}{3}$. Moreover, assume $S \in C^2\big(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times 3}\big)$ fulfills (3.1.3) *with some $S_0 > 0$ and that n_0, c_0 and u_0 comply with* (3.1.8). *Then* (3.1.4)–(3.1.6) *admits at least one global weak solution in the sense of Definition 3.6 below.*

If we merely prescribe $m + 2\alpha \leq \frac{5}{3}$, we have to weaken the solution concept in order to verify the existence of global solutions – which is due to the obtainable a priori information being so weak that we have to consider a sublinear functional of n for our testing methods.

Theorem 3.2.
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Suppose that $m \geq 1$ and $\alpha \geq 0$ satisfy $m+\alpha > \frac{4}{3}$. Moreover, assume $S \in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$ fulfills (3.1.3) and that n_0, c_0 and u_0 comply with (3.1.8). Then (3.1.4)–(3.1.6) admits at least one global very weak solution (n,c,u) in the sense of Definition 3.5 below. In particular, this global very weak solution satisfies

$$n \in L_{loc}^{2(m+\alpha)-\frac{4}{3}}\big(\overline{\Omega}\times[0,\infty)\big)\,,\ c \in L^2_{loc}\big([0,\infty);W^{1,2}(\Omega)\big)\,,\ u \in L^2_{loc}\big([0,\infty);W_0^{1,2}(\Omega;\mathbb{R}^3)\big),$$

and

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \quad \textit{for a.e. } t>0.$$

Remark 3.3.
For the linear diffusion case $m=1$ Theorem 3.1 provides the existence of a global weak solution for $\alpha > \frac{1}{3}$, extending the results of [88] and [55], which provided the existence of a global very weak solution for $\alpha > \frac{1}{3}$ and a global weak solution for $\alpha > \frac{3}{7}$, respectively.
Since we are considering Navier–Stokes-fluid, smooth global solutions can not be expected. However, it could be likely that the very weak solutions obtained for $m+\alpha > \frac{4}{3}$ may in fact become smooth solutions after some waiting time, i.e. there may exist some $T>0$ such that the global very weak solutions satisfies the additional regularity properties

$$n, c \in C^{2,1}\big(\overline{\Omega}\times[T,\infty)\big) \quad \text{and} \quad u \in C^{2,1}\big(\overline{\Omega}\times[T,\infty);\mathbb{R}^3\big)\,.$$

Effects of this kind have, in more generous settings featuring signal consumption instead of production, been observed in e.g. [105]. However, since the methods underlying the proof of those results rely heavily on the consumption of the chemical, a result on eventual smoothness in presence of signal production, to the best of our knowledge, is still open. Illustrating the previous diagram once more with the new results, we obtain the figure below, which neatly fits together with the expectations we obtained from Figure 3.1.

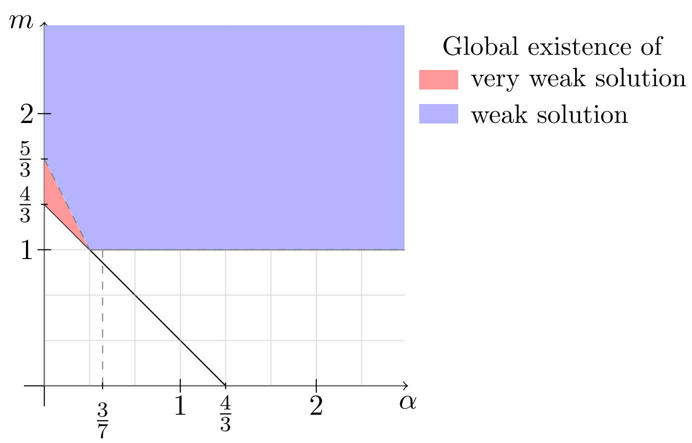

Fig. 3.2: Overview of global existence in (3.1.4) with Thm. 3.1 and Thm. 3.2

Mathematical difficulties. The absence of any energy-functional in this setting incorporating both fluid interaction and signal production is one of the main difficulties in obtaining estimates optimal with respect to m and α. Most of the problems resulting from this lack of an energy estimate can be combated by utilizing similar methods as displayed in [88] and our previous work [7], but even greater care has to be taken when trying to derive information on gradient terms and combined quantities without tightening the scope for m and α. After regularizing the problem in a suitable fashion, a functional of the form

$$\int_\Omega (n_\varepsilon+\varepsilon)^{m+2\alpha-1}(\cdot,t)+\int_\Omega c_\varepsilon^2(\cdot,t),\quad t>0,$$

(which, specifically, for small values of m and α is of sublinear growth with respect to n) will be the main cornerstone of our analysis and will also provide bounds on $\int_t^{t+1}\int_\Omega\big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big|^2$ as well as $\int_t^{t+1}\int_\Omega|\nabla c_\varepsilon|^2$ (Lemma 3.10), which by the Gagliardo–Nirenberg inequality can be refined into more spatio-temporal regularity information on n_ε (Lemma 3.11). The careful combination of these estimates with standard arguments for the Navier–Stokes-subsystem will enable us to conclude from compactness arguments the existence of the desired limit object (Proposition 3.19). Depending on the size of m and α, the convergence properties can be relied on to conclude Theorems 3.1 and 3.2.

3.2 The notions of weak and very weak solutions

Let us start by laying out the exact formulations of the different concepts of solvability we are going to discuss. The notion of very weak solvability present in Theorem 3.2 is adapted from the related works in [101, 88, 7] and the main distinguishing aspect when comparing to the standard notion of weak solvability is the fact that the first component of the system only has to satisfy a supersolution property.

Definition 3.4.
Let $\Phi\in C^2([0,\infty))$ be a nonnegative function satisfying $\Phi'>0$ on $(0,\infty)$. Assume that $n_0\in L^\infty(\Omega)$ is nonnegative with $\Phi(n_0)\in L^1(\Omega)$ and that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$ satisfies (3.1.3) for some $S_0>0$ and $\alpha\geq0$. Suppose that $c\in L^2_{loc}\big([0,\infty);W^{1,2}(\Omega)\big)$ and $u\in L^1_{loc}\big([0,\infty);W_0^{1,1}(\Omega;\mathbb{R}^3)\big)$ with $\nabla\cdot u\equiv0$ in $\mathcal{D}'\big(\Omega\times(0,\infty)\big)$. The nonnegative measurable function $n:\Omega\times(0,\infty)\to\mathbb{R}$ satisfying $n\in L^1_{loc}\big([0,\infty);W^{1,1}(\Omega)\big)$ will be named a global weak Φ–supersolution of the initial-boundary value problem

$$\begin{cases} n_t+u\cdot\nabla n=\Delta n^m-\nabla\cdot\big(nS(x,n,c)\nabla c\big), & x\in\Omega,\quad t>0,\\ \frac{\partial n}{\partial\nu}=0, & x\in\partial\Omega,\quad t>0,\\ n(x,0)=n_0(x), & x\in\Omega,\end{cases}\tag{3.2.1}$$

if

$$\begin{gathered}\Phi(n),\ \text{and}\ \ \Phi''(n)n^{m-1}|\nabla n|^2\ \text{belong to}\ L^1_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,,\\ \Phi'(n)n^{m-1}\nabla n,\ \text{and}\ \ \Phi(n)u\ \text{belong to}\ L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,,\\ \Phi'(n)n\ \text{belongs to}\ L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,,\ \text{and}\ \ \Phi''(n)n\nabla n\ \text{belongs to}\ L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,,\end{gathered}\tag{3.2.2}$$

and if for each nonnegative $\varphi \in C_0^\infty\left(\overline{\Omega}\times[0,\infty)\right)$ *the inequality*

$$\begin{aligned}
-\int_0^\infty\int_\Omega \Phi(n)\varphi_t - \int_\Omega \Phi(n_0)\varphi(\cdot,0)& \\
\geq -m\int_0^\infty\int_\Omega \Phi''(n)n^{m-1}|\nabla n|^2\varphi - m\int_0^\infty\int_\Omega \Phi'(n)n^{m-1}(\nabla n\cdot\nabla\varphi)& \qquad (3.2.3)\\
+\int_0^\infty\int_\Omega \Phi''(n)n\big(\nabla n\cdot S(x,n,c)\nabla c\big)\varphi + \int_0^\infty\int_\Omega \Phi'(n)n\big(S(x,n,c)\nabla c\cdot\nabla\varphi\big)&\\
+\int_0^\infty\int_\Omega \Phi(n)(u\cdot\nabla\varphi)&
\end{aligned}$$

is satisfied.

Let us briefly remark on the test function we will use later on. For $m \geq 1$ and $\alpha \geq 0$ satisfying the conditions $m+\alpha > \frac{4}{3}$ and $m+2\alpha < 2$ we will consider $\Phi(s) \equiv (s+1)^{m+2\alpha-1}$. Due to $m+2\alpha-1 < 1$ our main intention in the coming sections will be to obtain a priori bounds which allow for the conclusion that $n^{m+2\alpha-1} \in L^2_{loc}\big([0,\infty);W^{1,2}(\Omega)\big)$. Combining this with suitable regularity information on the other solution components is sufficient to determine that all of the integrals appearing in the supersolution property above are well defined (see also Corollary 3.12 and Lemma 3.24 below).

Complementing Definition 3.4 with the standard properties of weak solvability for the remaining subproblems of (3.1.4) will lead us to the following notion of global very weak solutions.

Definition 3.5.

A triple (n,c,u) *of functions*

$$\begin{aligned}
n &\in L^1_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,,\\
c &\in L^2_{loc}\big([0,\infty);W^{1,2}(\Omega)\big)\,,\\
u &\in L^1_{loc}\big([0,\infty);W_0^{1,1}(\Omega;\mathbb{R}^3)\big),
\end{aligned}$$

satisfying $n \geq 0$ *and* $c \geq 0$ *in* $\overline{\Omega}\times[0,\infty)$, $cu \in L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)$, *as well as* $u\otimes u \in L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^{3\times3}\big)$ *will be called a global very weak solution of* (3.1.4)*–*(3.1.6)*, if*

$$\int_\Omega n(\cdot,t) \leq \int_\Omega n_0 \quad \textit{for a.e. } t>0,$$

if $\nabla\cdot u = 0$ *in* $\mathcal{D}'\big(\Omega\times(0,\infty)\big)$, *if the equality*

$$-\int_0^\infty\int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi - \int_0^\infty\int_\Omega c\varphi + \int_0^\infty\int_\Omega n\varphi + \int_0^\infty\int_\Omega c(u\cdot\nabla\varphi) \qquad (3.2.4)$$

holds for all $\varphi \in L^\infty(\Omega\times(0,\infty))\cap L^2\left((0,\infty);W^{1,2}(\Omega)\right)$ *with* $\varphi_t \in L^2(\Omega\times(0,\infty))$, *which are compactly supported in* $\overline{\Omega}\times[0,\infty)$, *if*

$$-\int_0^\infty\int_\Omega u\cdot\psi_t - \int_\Omega u_0\cdot\psi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\psi + \int_0^\infty\int_\Omega (u\otimes u)\cdot\nabla\psi + \int_0^\infty\int_\Omega n\nabla\phi\cdot\psi \qquad (3.2.5)$$

is fulfilled for all $\psi \in C_0^\infty\left(\Omega \times [0,\infty);\mathbb{R}^3\right)$ *with* $\nabla \cdot \psi \equiv 0$ *in* $\Omega \times (0,\infty)$*, and if finally there exists some nonnegative* $\Phi \in C^2([0,\infty))$ *with* $\Phi' > 0$ *on* $[0,\infty)$ *such that* n *is a global weak* Φ*–supersolution of* (3.2.1) *in the sense of Definition 3.4.*

If, on the other hand, $m + 2\alpha > \frac{5}{3}$ we will obtain global weak solutions in the standard sense. Let us formulate this well-established concept for the sake of completeness in the following definition.

Definition 3.6.
Let $S \in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$ *satisfy* (3.1.3) *for some* $S_0 > 0$ *and* $\alpha \geq 0$*. A triple* (n,c,u) *of functions*

$$n \in L^1_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,,\quad c \in L^1_{loc}\big([0,\infty);W^{1,2}(\Omega)\big)\,,\quad u \in L^1_{loc}\big([0,\infty);W_0^{1,1}(\Omega;\mathbb{R}^3)\big),$$

satisfying $n \geq 0$ *and* $c \geq 0$ *in* $\overline{\Omega}\times[0,\infty)$ *will be called a global weak solution of* (3.1.4)*–*(3.1.6)*, if* $n \in L^1_{loc}\big([0,\infty);W^{1,1}(\Omega)\big)$ *and* $u\otimes u \in L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^{3\times3}\big)$*, if* $\nabla\cdot u = 0$ *in* $\mathcal{D}'\big((\Omega\times(0,\infty)\big)$*, if*

$$n^{m-1}\nabla n,\quad n\nabla c\quad \textit{and}\quad nu,\quad \textit{as well as}\quad cu\quad \textit{belong to } L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,,$$

if equality (3.2.4) *holds for all* $\varphi \in L^\infty(\Omega\times(0,\infty)) \cap L^2\left((0,\infty);W^{1,2}(\Omega)\right)$ *with* $\varphi_t \in L^2(\Omega\times(0,\infty))$*, which are compactly supported in* $\overline{\Omega}\times[0,\infty)$*, if* (3.2.5) *is fulfilled for all* $\psi \in C_0^\infty\left(\Omega\times[0,\infty);\mathbb{R}^3\right)$ *with* $\nabla\cdot\psi \equiv 0$ *in* $\Omega\times(0,\infty)$*, and if finally for each* $\varphi \in C_0^\infty\big(\overline{\Omega}\times[0,\infty)\big)$ *the equality*

$$\begin{aligned} -\int_0^\infty\!\!\int_\Omega n\varphi_t - \int_\Omega n_0\,\varphi(\cdot,0) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2.6)\\ = -m\int_0^\infty\!\!\int_\Omega n^{m-1}(\nabla n\cdot\nabla\varphi) + \int_0^\infty\!\!\int_\Omega n\big(S(x,n,c)\nabla c\cdot\nabla\varphi\big) + \int_0^\infty\!\!\int_\Omega n(u\cdot\nabla\varphi) \end{aligned}$$

is satisfied.

Remark 3.7.
i) If (3.2.3) *is satisfied for* $\Phi(s)\equiv s$ *with equality, then* (n,c,u) *is a global weak solution of* (3.1.4) *in the sense of Definition 3.6, which shows that every global weak solution is also a global very weak solution.*
ii) If the global very weak solution (n,c,u) *satisfies the regularity properties* $n,c \in C^0\big(\overline{\Omega}\times[0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty)\big)$ *and* $u \in C^0\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty);\mathbb{R}^3\big)$*, it can be checked that the solution is also a global classical solution, i.e. one can find* $P \in C^{1,0}\big(\overline{\Omega}\times(0,\infty)\big)$ *such that* (n,c,u,P) *solves* (3.1.4) *in the classical sense. See [101, Lemma 2.1] for the arguments involved.*

3.3 A family of regularized problems

As a first step in the construction of global solutions in either of the senses above, we will first adapt the approaches undertaken in [101, 88, 7] to our setting in order to approximate the system (3.1.4) by problems in which the no-flux boundary condition of the first component reduces to a homogeneous Neumann boundary condition and which are solvable globally in time. With a family $(\rho_\varepsilon)_{\varepsilon\in(0,1)} \subset C_0^\infty(\Omega)$ of cut-off functions in Ω satisfying

$$0 \le \rho_\varepsilon(x) \le 1 \text{ for all } x \in \Omega \quad \text{such that} \quad \rho_\varepsilon \nearrow 1 \text{ as } \varepsilon \searrow 0,$$

we define

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x) S(x,n,c), \quad (x,n,c) \in \Omega \times [0,\infty)^2 \tag{3.3.1}$$

and accordingly for $\varepsilon \in (0,1)$ consider regularized problems of the form

$$\begin{cases} n_{\varepsilon t} + \quad u_\varepsilon \cdot \nabla n_\varepsilon &= \nabla \cdot \big(m(n_\varepsilon+\varepsilon)^{m-1} \nabla n_\varepsilon - \frac{n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)}{(1+\varepsilon n_\varepsilon)^3} \nabla c_\varepsilon \big), & x \in \Omega, \ t>0, \\ c_{\varepsilon t} + \quad u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, & x \in \Omega, \ t>0, \\ u_{\varepsilon t} + \ (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon &= \Delta u_\varepsilon - \nabla P_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega, \ t>0, \\ \nabla \cdot u_\varepsilon &= 0, & x \in \Omega, \ t>0, \\ \partial_\nu n_\varepsilon = \partial_\nu c_\varepsilon = 0, & \qquad u_\varepsilon = 0, & x \in \partial\Omega, t>0, \\ n_\varepsilon(x,0) &= n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.3.2}$$

where the Yosida approximation Y_ε of the Stokes operator $A := -\mathcal{P}\Delta$ is given by

$$Y_\varepsilon \varphi := (1+\varepsilon A)^{-1}\varphi \quad \text{for } \varepsilon \in (0,1) \text{ and } \varphi \in L^2_\sigma(\Omega),$$

and $L^2_\sigma(\Omega) := \big\{ \varphi \in L^2\big(\Omega;\mathbb{R}^3\big) \mid \nabla\cdot\varphi = 0 \big\}$ denotes the solenoidal subspace of $L^2\big(\Omega;\mathbb{R}^3\big)$.

3.3.1 Global existence of approximating solutions and basic properties

By standard arguments involving well-established testing procedures and a Moser-type iteration one can readily verify that for all $m \ge 1$ and $\alpha \ge 0$ the classical solutions to the approximating system above are in fact global solutions, which in addition satisfy certain L^1 estimates.

Lemma 3.8.
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $\phi \in W^{2,\infty}(\Omega)$, $\vartheta > 3$, $m \ge 1$ and $\alpha \ge 0$. Suppose that $S \in C^2\big(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times3}\big)$ satisfies (3.1.3) for some $S_0 > 0$ and assume that n_0, c_0 and u_0 comply with (3.1.8). Then for any $\varepsilon \in (0,1)$, there exists a uniquely determined triple $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of functions satisfying

$$\begin{aligned} n_\varepsilon &\in C^0\big(\overline{\Omega}\times[0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty)\big), \\ c_\varepsilon &\in C^0\big(\overline{\Omega}\times[0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty)\big) \cap C^0\big([0,\infty); W^{1,\vartheta}(\Omega)\big), \\ u_\varepsilon &\in C^0\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty);\mathbb{R}^3\big), \end{aligned}$$

which, together with some $P_\varepsilon \in C^{1,0}\big(\overline{\Omega}\times(0,\infty)\big)$, solve (3.3.2) in the classical sense and fulfill $n_\varepsilon \ge 0$ and $c_\varepsilon \ge 0$ in $\overline{\Omega}\times[0,\infty)$, as well as

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \quad \textit{for all } t \in (0,\infty) \tag{3.3.3}$$

and

$$\int_\Omega c_\varepsilon(\cdot,t) \le \max\left\{\int_\Omega n_0, \int_\Omega c_0\right\} \quad \textit{for all } t \in (0,\infty). \tag{3.3.4}$$

Proof: Adapting well-established fixed point arguments as illustrated for similar chemotaxis frameworks in e.g. [78, Lemma 2.1], [48, Lemma 2.2] and [98, Lemma 2.1], we can readily achieve time-local existence of classical solutions. Moreover, denoting by $T_{max,\varepsilon} \in (0,\infty]$ the maximal existence time of the solution, the solution satisfies that if $T_{max,\varepsilon} < \infty$, then

$$\limsup_{t \nearrow T_{max,\varepsilon}} \left(\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon\cdot,t)\|_{W^{1,\vartheta}(\Omega)} + \|A^\varrho u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\right) = \infty \tag{3.3.5}$$

for all $\vartheta > 3$ and $\varrho \in (\frac{3}{4},1)$. The nonnegativity of n_ε and c_ε in $\overline{\Omega}\times[0,T_{max,\varepsilon})$ can then be established by relying on the maximum principle, whereas the L^1 estimates for n_ε and c_ε on $(0,T_{max,\varepsilon})$ follow immediately from integrating the corresponding equations and, for c_ε, an additional employment of an ODE comparison argument ([86, Thm. IX]). To verify that the solution is indeed global in time we pick some $T \in (0,T_{max,\varepsilon}]$ satisfying $T < \infty$, let $\gamma := \max\{m-1,6\}$ and fix $\varepsilon \in (0,1)$. Making use of the first equation in (3.3.2), integration by parts, the divergence-free property of u_ε and the fact that $|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)| \le S_0$ holds in $\Omega \times (0,T_{max,\varepsilon})$, we find that

$$\begin{aligned}\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^\gamma &= \int_\Omega n_\varepsilon^{\gamma-1}\nabla\cdot\Big(m(n_\varepsilon+\varepsilon)^{m-1}\nabla n_\varepsilon - \frac{n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)}{(1+\varepsilon n_\varepsilon)^3}\nabla c_\varepsilon\Big) - \frac{1}{\gamma}\int_\Omega \nabla\cdot(n_\varepsilon^\gamma u_\varepsilon)\\ &\le -(\gamma-1)m\int_\Omega (n_\varepsilon+\varepsilon)^{m-1}n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 + S_0(\gamma-1)\int_\Omega \frac{n_\varepsilon^{\gamma-1}}{(1+\varepsilon n_\varepsilon)^3}(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\end{aligned}$$

on $(0,T_{max,\varepsilon})$. Now, noticing that $m \ge 1$ implies $-(s+\varepsilon)^{m-1} \le -s^{m-1}$ for all $s \ge 0$, that $\frac{s}{1+\varepsilon s} \le \frac{1}{\varepsilon}$ for all $s \ge 0$ and that, by the choice of γ, the inequalities $\gamma - m + 1 \ge 0$ and $m - \gamma + 5 \ge 0$ are also satisfied, we draw on Young's inequality to obtain that

$$\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^\gamma \le -\frac{(\gamma-1)m}{2}\int_\Omega n_\varepsilon^{m+\gamma-3}|\nabla n_\varepsilon|^2 + \frac{S_0^2(\gamma-1)}{2m\varepsilon^{\gamma-m+1}}\int_\Omega |\nabla c_\varepsilon|^2 \quad \text{on } (0,T). \tag{3.3.6}$$

In a similar fashion, we multiply the second equation of (3.3.2) with $(c_\varepsilon+1)^{\gamma-1}$ and again using that u_ε is divergence-free, we integrate by parts and see that

$$\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (c_\varepsilon+1)^\gamma + (\gamma-1)\int_\Omega (c_\varepsilon+1)^{\gamma-2}|\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon(c_\varepsilon+1)^{\gamma-1} = \int_\Omega n_\varepsilon(c_\varepsilon+1)^{\gamma-1}$$

is valid on $(0,T)$, from which we infer by positivity of c_ε and an application of Young's inequality that

$$\begin{aligned}\frac{S_0^2}{\gamma m\varepsilon^{\gamma-m+1}}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (c_\varepsilon+1)^\gamma + \frac{S_0^2(\gamma-1)}{m\varepsilon^{\gamma-m+1}}\int_\Omega |\nabla c_\varepsilon|^2 \\ \le \frac{S_0^2}{\gamma m\varepsilon^{\gamma-m+1}}\int_\Omega n_\varepsilon^\gamma + \frac{S_0^2(\gamma-1)}{\gamma m\varepsilon^{\gamma-m+1}}\int_\Omega (c_\varepsilon+1)^\gamma \quad \text{on } (0,T).\end{aligned} \tag{3.3.7}$$

Thus, combining (3.3.6) and (3.3.7) and integrating the resulting inequality implies the existence of $C_1 := C_1(T, \varepsilon)$ satisfying

$$\int_\Omega n_\varepsilon^6(\cdot,t) + \int_\Omega (c_\varepsilon(\cdot,t)+1)^6 \le C_1 \quad \text{for all } t \in (0,T), \tag{3.3.8}$$

in light of the fact that $\gamma \ge 6$. In order to attain some information on the spatial gradient of c_ε we will first require information on u_ε. Due to the continuous embedding $D(A^\varrho) \hookrightarrow C^\theta(\overline{\Omega})$ for any $\theta \in (0, 2\varrho - \frac{3}{2})$ (see [75, Lemma III.2.4.3] and [23, Thm. 5.6.5]), we immediately obtain a bound for the norm of u_ε in $L^\infty(\Omega)$, if we find $C_2 > 0$ such that $\|A^\varrho u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_2$ holds for $t \in (0,T)$. For this, we first test the third equation of (3.3.2) by u_ε to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi \quad \text{for all } t \in (0,T),$$

where we used the facts that $\nabla \cdot u_\varepsilon \equiv 0$ and $\nabla \cdot (1+\varepsilon A)^{-1} u_\varepsilon \equiv 0$. In light of (3.1.7) and (3.3.8) this readily implies $\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_3$ on $(0,T)$ for some $C_3 > 0$. Relying on properties of the Yosida approximation Y_ε, we can also immediately find $C_4 > 0$ (cf. [60, p.462 (3.6)]) such that $v_\varepsilon := (1+\varepsilon A)^{-1} u_\varepsilon$ satisfies

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \|(1+\varepsilon A)^{-1} u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_4 \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_5 := C_3 C_4$$

for all $t \in (0,T)$. Finally, we can refine these bounds into the desired estimate for $\|A^\varrho u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ by a two-step procedure (see e.g. [104, Lemma 3.9]). Firstly, testing the equation $u_{\varepsilon t} + A u_\varepsilon = \mathcal{P}(-(v_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla\phi)$ against $A u_\varepsilon$ yields $C_6 > 0$ such that

$$\int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega |A^{\frac{1}{2}} u_\varepsilon|^2 \le C_6 \quad \text{for all } t \in (0,T),$$

and $C_7 > 0$ satisfying $\|\mathcal{P}((v_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla\phi)\|_{L^2(\Omega)} \le C_7$ for all $t \in (0,T)$. Secondly, we express $A^\varrho u_\varepsilon$ by its variation-of-constants representation and make use of well-known smoothing properties of the Stokes semigroup (e.g. [100, Lemma 3.1]) to obtain $C_8 > 0$ such that

$$\|A^\varrho u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_8 t^{-\varrho} \|u_0\|_{L^2(\Omega)} + \frac{C_8 T^{1-\varrho}}{1-\varrho} \quad \text{for all } t \in (0,T),$$

yielding a bound on both the quantity $\|A^\varrho u_\varepsilon\|_{L^2(\Omega)}$ appearing in the extensibility criterion and $\|u_\varepsilon\|_{L^\infty(\Omega)}$ by the previously mentioned embedding. These bounds at hand, we can now go to testing the second equation by $-\Delta c_\varepsilon$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 \le \int_\Omega n_\varepsilon^2 + \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla c_\varepsilon|^2$$

on $(0,T)$. Plugging in our previous bounds, this immediately entails the existence of $C_9 > 0$ such that $\|\nabla c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_9$ holds for all $t \in (0,T)$. The L^2 regularity of ∇c_ε at hand, we can now draw on well-known smoothing properties for the Neumann heat semigroup (e.g. [97, Lemma 1.3]) to find $C_{10} > 0$ such that $\|\nabla c_\varepsilon(\cdot,t)\|_{L^{\frac{11}{2}}(\Omega)} \le C_{10}$ for all $t \in (0,T)$, by expressing ∇c_ε through its variation-of-constants representation. In fact, this entails that

$$\frac{n_\varepsilon(\cdot,t)S_\varepsilon\big(x,n_\varepsilon(\cdot,t),c_\varepsilon(\cdot,t)\big)}{(1+\varepsilon n_\varepsilon(\cdot,t))^3}\nabla c_\varepsilon(\cdot,t) + n_\varepsilon(\cdot,t)u_\varepsilon(\cdot,t) \in L^q(\Omega)$$

for all $t \in (0,T)$ with some $q > 5$. Hence, we may employ a Moser-type iteration (see [80, Lemma A.1] for a version applicable to our system) to obtain $C_{11} > 0$ such that $\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{11}$ holds for all $t \in (0,T)$. In light the bounds attained above, we find that assuming $T_{max,\varepsilon} < \infty$ contradicts the extensibility criterion (3.3.5). Thus, we finally conclude that in fact $T_{max,\varepsilon} = \infty$. □

3.4 A priori estimates

Since our main focus will be on values $m \ge 1$ and $\alpha \ge 0$, which are both as small as possible, our main task will be to obtain regularity information independent on $\varepsilon \in (0,1)$, restricting m and α in the least possible way. As in particular no energy-structure is present in (3.3.2), we are thereby tasked with finding a testing procedure, capturing as optimal conditions on these parameters as possible. Even obtaining an estimate for the norm of n_ε in $L^2(\Omega)$ seems to be far out of reach without gravely restricting either m or α. Thus, similar to the approach in [7], we decide to investigate a functional, which for small values of m and α is of sublinear growth, hoping to obtain a spatio-temporal bound on the gradient of n_ε, which we can refine later to a regularity estimate beyond the L^1 estimate of Lemma 3.8.

3.4.1 Estimates capturing optimal conditions on m and α

Let us start with an elementary identity laying the groundwork to impending testing procedures.

Lemma 3.9.
Let $m \ge 1$, $\alpha \ge 0$, $\beta \ge 1$ be such that $m + \frac{\beta}{2}\alpha > 1$, assume that n_0, c_0 and u_0 comply with (3.1.8) and that $S \in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$. Then for any $\varepsilon \in (0,1)$ the classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.3.2) satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (n_\varepsilon+\varepsilon)^{m+\beta\alpha-1} \\
= &-\frac{m(m+\beta\alpha-1)(m+\beta\alpha-2)}{(m+\frac{\beta}{2}\alpha-1)^2}\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\beta}{2}\alpha-1}\big|^2 \\
&+\frac{(m+\beta\alpha-1)(m+\beta\alpha-2)}{m+\frac{\beta}{2}\alpha-1}\int_\Omega \frac{n_\varepsilon(n_\varepsilon+\varepsilon)^{\frac{\beta}{2}\alpha-1}}{(1+\varepsilon n_\varepsilon)^3}\big(\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\beta}{2}\alpha-1}\cdot S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\big)
\end{aligned} \tag{3.4.1}$$

on $(0,\infty)$.

Proof: Drawing on the first equation of (3.3.2), straightforward calculations show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (n_\varepsilon+\varepsilon)^{m+\beta\alpha-1}\\
&= (m+\beta\alpha-1)\int_\Omega (n_\varepsilon+\varepsilon)^{m+\beta\alpha-2}\nabla\cdot\Big(m(n_\varepsilon+\varepsilon)^{m-1}\nabla n_\varepsilon-\frac{n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\Big)\\
&\qquad-(m+\beta\alpha-1)\int_\Omega (n_\varepsilon+\varepsilon)^{m+\beta\alpha-2}(u_\varepsilon\cdot\nabla n_\varepsilon)
\end{aligned}$$

holds on $(0,\infty)$. Making use of the fact that $\nabla\cdot u_\varepsilon\equiv 0$ in $\Omega\times(0,\infty)$ as well as the imposed boundary conditions, we find that the asserted equality follows immediately upon integration by parts and appropriate reformulation of some terms. □

Depending on the sign of $m+2\alpha-2$, we will multiply the equality of Lemma 3.9 with either positive or negative constants and then estimate. Combining the resulting inequality with a standard testing procedure for the second equation, we will derive some information on $(n_\varepsilon+\varepsilon)^{m+2\alpha-1}$, $\nabla(n+\varepsilon)^{m+\alpha-1}$, c_ε^2 and ∇c_ε^2. This approach has been undertaken previously in e.g. [88, Lemma 4.1].

Lemma 3.10.
Let $m\geq 1$, $\alpha\geq 0$ be such that $m+\alpha>\frac{4}{3}$, suppose that n_0, c_0 and u_0 fulfill (3.1.8) *and assume that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$ satisfies* (3.1.3) *with some $S_0>0$. Then there exists some $C>0$ such that for all $\varepsilon\in(0,1)$ the global classical solution $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ of* (3.3.2) *satisfies*

$$\int_\Omega (n_\varepsilon+\varepsilon)^{m+2\alpha-1}(\cdot,t)+\int_\Omega c_\varepsilon^2(\cdot,t)+\int_t^{t+1}\int_\Omega\big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big|^2+\int_t^{t+1}\int_\Omega|\nabla c_\varepsilon|^2\leq C \tag{3.4.2}$$

for all $t\geq 0$.

Proof: We will first assume $m+2\alpha\neq 2$ and later give comments on the adjustments necessary for the case $m+2\alpha=2$. For $m+\alpha>\frac{4}{3}$ with $m+2\alpha\neq 2$ we employ Lemma 3.9 with $\beta=2$ and multiply the resulting identity by $\frac{\operatorname{sgn}(m+2\alpha-2)}{(m+2\alpha-1)}$. Relying on Young's inequality and the fact that $\left|\frac{n_\varepsilon(n_\varepsilon+\varepsilon)^{\alpha-1}S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)}{(1+\varepsilon n_\varepsilon)^3}\right|\leq S_0$ in $\Omega\times(0,\infty)$ we then obtain that for all $\varepsilon\in(0,1)$

$$\begin{aligned}
&\frac{\operatorname{sgn}(m+2\alpha-2)}{m+2\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (n_\varepsilon+\varepsilon)^{m+2\alpha-1} \\
&\qquad\leq -\frac{m|m+2\alpha-2|}{2(m+\alpha-1)^2}\int_\Omega\big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big|^2+\frac{S_0^2|m+2\alpha-2|}{2m}\int_\Omega|\nabla c_\varepsilon|^2
\end{aligned}\tag{3.4.3}$$

holds on $(0,\infty)$. Preparing a corresponding differential inequality for the signal chemical, we test the second equation of (3.3.2) by c_ε, and make use of the fact that u_ε is a solenoidal vector field, integration by parts, an application of Hölder's inequality and the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ to find that there exists $C_1 > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\varepsilon^2(\cdot,t) + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2 &\le \|c_\varepsilon(\cdot,t)\|_{L^6(\Omega)}\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)} \\ &\le C_1\big(\|\nabla c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\big)\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}\end{aligned}$$

is valid for all $t > 0$ and all $\varepsilon \in (0,1)$. Here, an additional application of Young's inequality entails that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\varepsilon^2(\cdot,t) + \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2 + \int_\Omega c_\varepsilon^2(\cdot,t) \le C_2\|n_\varepsilon(\cdot,t)\|^2_{L^{\frac{6}{5}}(\Omega)} \tag{3.4.4}$$

for all $t > 0$ and all $\varepsilon \in (0,1)$, with $C_2 := C_1^2$. Moreover, drawing on the Gagliardo–Nirenberg inequality (e.g. [52, Lemma 2.3]), the nonnegativity of n_ε, the mass conservation featured in Lemma 3.8 and the fact that $\varepsilon \in (0,1)$, we obtain $C_3 > 0$ such that

$$C_2\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \le C_2\big\|(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big\|^{\frac{2}{m+\alpha-1}}_{L^{\frac{6}{5(m+\alpha-1)}}(\Omega)} \le C_3\big\|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big\|^{\frac{2}{6(m+\alpha)-7}}_{L^2(\Omega)} + C_3$$

holds on $(0,\infty)$ for all $\varepsilon \in (0,1)$, and, since $m+\alpha > \frac{4}{3}$ implies $\frac{2}{6(m+\alpha)-7} < 2$, an application of Young's inequality thereby provides $C_4 > 0$ such that

$$C_2\|n_\varepsilon(\cdot,t)\|^2_{L^{\frac{6}{5}}(\Omega)} \le \frac{m^2}{4S_0^2(m+\alpha-1)^2}\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,t)\big|^2 + C_4 \tag{3.4.5}$$

for all $t > 0$ and all $\varepsilon \in (0,1)$. Thus, combining (3.4.3) with a suitable multiple of (3.4.4) and (3.4.5) consequently entails

$$y_\varepsilon'(t) + y_\varepsilon(t) + g_\varepsilon(t) \le C_4C_5 \quad \text{for all } t > 0 \text{ and all } \varepsilon \in (0,1), \tag{3.4.6}$$

where we have set $C_5 := \frac{S_0^2|m+2\alpha-2|}{m} > 0$,

$$y_\varepsilon(t) := \frac{\operatorname{sgn}(m+2\alpha-2)}{m+2\alpha-1}\int_\Omega (n_\varepsilon+\varepsilon)^{m+2\alpha-1}(\cdot,t) + C_5\int_\Omega c_\varepsilon^2(\cdot,t), \quad t > 0,$$

and

$$g_\varepsilon(t) := \frac{m|m+2\alpha-2|}{8(m+\alpha-1)^2}\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,t)\big|^2 + \frac{C_5}{2}\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2, \quad t > 0.$$

Because of $g_\varepsilon(t) \ge 0$ for all $t > 0$, an ODE comparison thereby implies

$$y_\varepsilon(t) \le C_6 := \max\left\{\frac{\operatorname{sgn}(m+2\alpha-2)}{m+2\alpha-1}\int_\Omega (n_0+1)^{m+2\alpha-1} + C_5\int_\Omega c_0^2, C_4C_5\right\} \tag{3.4.7}$$

for all $t > 0$ and all $\varepsilon \in (0,1)$. For $m+2\alpha > 2$ this indeed entails the boundedness of the first two terms appearing in (3.4.2), whereas in the case of $m+2\alpha < 2$ the asserted bound cannot be deduced yet, as $y_\varepsilon(t)$ may in fact be negative. Nevertheless, since in this case $m+2\alpha-1<1$, we conclude from Lemma 3.8 the existence of $C_7>0$ such that for all $\varepsilon\in(0,1)$ the estimate $\frac{1}{m+2\alpha-1}\int_\Omega(n_\varepsilon+\varepsilon)^{m+2\alpha-1}\le C_7$ is valid on $(0,\infty)$. Combining this with (3.4.7) shows that

$$C_5\int_\Omega c_\varepsilon^2(\cdot,t)=y_\varepsilon(t)+\frac{1}{m+2\alpha-1}\int_\Omega(n_\varepsilon+\varepsilon)^{m+2\alpha-1}(\cdot,t)\le C_6+C_7$$

for all $t>0$ and $\varepsilon\in(0,1)$, proving boundedness of the first two terms in (3.4.2) also in the case that $m+2\alpha<2$. In a second step, in order to obtain estimates on the spatio-temporal quantities featured in the formulation of the Lemma, we observe that integrating (3.4.6) with respect to time implies that

$$\int_t^{t+1}g_\varepsilon(s)\,\mathrm{d}s\le y_\varepsilon(t)-y_\varepsilon(t+1)-\int_t^{t+1}y_\varepsilon(s)\,\mathrm{d}s+C_4C_5\quad\text{for all } t>0 \text{ and all } \varepsilon\in(0,1).$$

By the definition of y_ε and nonnegativity of $\int_\Omega c_\varepsilon^2$, this leads to

$$\begin{aligned}\int_t^{t+1}g_\varepsilon(s)\,\mathrm{d}s\le y_\varepsilon(t)&+\frac{1}{m+2\alpha-1}\int_\Omega(n_\varepsilon+\varepsilon)^{m+2\alpha-1}(\cdot,t+1)\\&+\frac{1}{m+2\alpha-1}\int_t^{t+1}\int_\Omega(n_\varepsilon+\varepsilon)^{m+2\alpha-1}(\cdot,s)\,\mathrm{d}s+C_4C_5\end{aligned}$$

for all $t>0$ and all $\varepsilon\in(0,1)$, which, in light of (3.4.7) and the uniform bound on $\int_\Omega(n_\varepsilon+\varepsilon)^{m+2\alpha-1}$ obtained in the first part of this proof, completes the proof for the case $m+2\alpha\neq2$.

To verify the asserted bound in the case of $m+2\alpha=2$, we note that $m+\alpha-1=1-\alpha$ and that moreover $\alpha\le\frac12$ due to $m\ge1$. Thus, estimating

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(n_\varepsilon\ln n_\varepsilon)(\cdot,t)\le-\frac{m}{2(1-\alpha)^2}\int_\Omega\left|\nabla(n_\varepsilon+\varepsilon)^{1-\alpha}(\cdot,t)\right|^2+\frac{S_0^2}{2m}\int_\Omega|\nabla c_\varepsilon(\cdot,t)|^2,$$

for all $t>0$ and all $\varepsilon\in(0,1)$, and combining with (3.4.4) we obtain an inequality of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega(n_\varepsilon\ln n_\varepsilon)(\cdot,t)+\frac{S_0^2}{m}\int_\Omega c_\varepsilon^2(\cdot,t)\Big)&+C_7\int_\Omega\left|\nabla(n_\varepsilon+\varepsilon)^{1-\alpha}(\cdot,t)\right|^2\\&+C_7\int_\Omega|\nabla c_\varepsilon(\cdot,t)|^2+C_7\int_\Omega c_\varepsilon^2(\cdot,t)\le C_8,\end{aligned}$$

with some $C_7>0$ and $C_8>0$. By means of the Gagliardo–Nirenberg inequality and the evident estimate $s\ln s\le s^{5/3}$ for $s>0$ we have $C_9>0$ satisfying

$$\int_\Omega n_\varepsilon\ln n_\varepsilon\le\left\|(n_\varepsilon+\varepsilon)^{1-\alpha}\right\|_{L^{\frac{5}{3(1-\alpha)}}(\Omega)}^{\frac{5}{3(1-\alpha)}}\le C_9\left\|\nabla(n_\varepsilon+\varepsilon)^{(1-\alpha)}\right\|_{L^2(\Omega)}^{\frac{4}{5-6\alpha}}+C_9\quad\text{on}\quad(0,\infty).$$

Because of $\frac{4}{5-6\alpha}\le2$ for $\alpha\le\frac12$, this now allows to pursue a similar reasoning as before, while making use of the fact that $s\ln s\ge-\frac1e$ for all $s>0$. $\square$

While the main idea of utilizing the latter spatio-temporal bound for $\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}$ to establish time-space bounds for $n_\varepsilon+\varepsilon$ remains unchanged from the previous works [88, Lemma 4.2] and [7, Lemma 4.3], we have to treat the term more delicately in order to prepare sufficient information for the limiting procedure later on.

Lemma 3.11.
Let $m\geq 1$, $\alpha\geq 0$ be such that $m+\alpha>\frac{4}{3}$ and assume that n_0,c_0 and u_0 comply with (3.1.8) and that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times 3}\big)$ fulfills (3.1.3) with some $S_0>0$. Then for all $p\in\big(1,6(m+\alpha-1)\big)$ there exists $C>0$ such that for all $\varepsilon\in(0,1)$ the solution $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ of (3.3.2) satisfies

$$\int_t^{t+1}\big\|n_\varepsilon(\cdot,s)+\varepsilon\big\|_{L^p(\Omega)}^{\frac{2p(m+\alpha-\frac{7}{6})}{p-1}}\,\mathrm{d}s\leq C\quad\text{for all } t\geq 0. \tag{3.4.8}$$

In particular, there exist $r\in(1,2)$ and $C>0$ such that

$$\int_t^{t+1}\big\|n_\varepsilon(\cdot,s)+\varepsilon\big\|_{L^{\frac{6r}{6-r}}(\Omega)}^{\frac{2r}{2-r}}\,\mathrm{d}s\leq C\quad\text{and}\quad\int_t^{t+1}\big\|n_\varepsilon(\cdot,s)+\varepsilon\big\|_{L^{2(m+\alpha)-\frac{4}{3}}(\Omega)}^{2(m+\alpha)-\frac{4}{3}}\,\mathrm{d}s\leq C \tag{3.4.9}$$

hold for each $\varepsilon\in(0,1)$ and all $t\geq 0$.

Proof: We employ reasoning similar to [88, Lemma 4.2]. Due to $p\in\big(1,6(m+\alpha-1)\big)$ and $m+\alpha>\frac{4}{3}>\frac{7}{6}$ we can utilize the Gagliardo–Nirenberg inequality to find $C_1>0$ such that with

$$a=\frac{m+\alpha-1-\frac{m+\alpha-1}{p}}{m+\alpha-1+\frac{1}{3}-\frac{1}{2}}=\frac{p-1}{p}\cdot\frac{6(m+\alpha-1)}{6m+6\alpha-7}\in(0,1)$$

the inequality

$$\begin{aligned}\int_t^{t+1}\big\|n_\varepsilon(\cdot,s)+\varepsilon\big\|_{L^p(\Omega)}^{\frac{2p(m+\alpha-\frac{7}{6})}{p-1}}\,\mathrm{d}s&=\int_t^{t+1}\big\|(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,s)\big\|_{L^{\frac{p}{m+\alpha-1}}(\Omega)}^{\frac{2p}{p-1}\cdot\frac{6m+6\alpha-7}{6(m+\alpha-1)}}\,\mathrm{d}s\\
\leq C_1\int_t^{t+1}\big\|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,s)\big\|_{L^2(\Omega)}^{\frac{2p}{p-1}\cdot\frac{6m+6\alpha-7}{6(m+\alpha-1)}\cdot a}&\big\|(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,s)\big\|_{L^{\frac{1}{m+\alpha-1}}(\Omega)}^{\frac{2p}{p-1}\cdot\frac{6m+6\alpha-7}{6(m+\alpha-1)}\cdot(1-a)}\,\mathrm{d}s\\
&+C_1\int_t^{t+1}\big\|(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,s)\big\|_{L^{\frac{1}{m+\alpha-1}}(\Omega)}^{\frac{2p}{p-1}\cdot\frac{6m+6\alpha-7}{6(m+\alpha-1)}}\,\mathrm{d}s\end{aligned}$$

holds for all $t\geq 0$ and all $\varepsilon\in(0,1)$. Combined with the mass conservation of n_ε, as established in Lemma 3.8, this implies the existence of $C_2>0$ such that

$$\int_t^{t+1}\big\|n_\varepsilon(\cdot,s)+\varepsilon\big\|_{L^p(\Omega)}^{\frac{2p(m+\alpha-\frac{7}{6})}{p-1}}\,\mathrm{d}s\leq C_2\int_t^{t+1}\big\|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,s)\big\|_{L^2(\Omega)}^2\,\mathrm{d}s+C_2$$

holds for all $t\geq 0$ and all $\varepsilon\in(0,1)$, which proves (3.4.8) under consideration of Lemma 3.10. For the first special case in (3.4.9) we note that due to $m+\alpha>\frac{4}{3}$ the interval

$I := \left(1, \min\left\{\frac{1+2(m+\alpha-\frac{7}{6})}{m+\alpha}, 2\right\}\right)$ is not empty and that, as $m+\alpha > \frac{7}{6}$, for arbitrary $r \in I$ we have $r < \frac{6(m+\alpha-1)}{m+\alpha}$ and $q := \frac{6r}{6-r} \in \big(1, 6(m+\alpha-1)\big)$. Hence,

$$\frac{2q(m+\alpha-\frac{7}{6})}{q-1} = \frac{12r(m+\alpha-\frac{7}{6})}{7r-6} > \frac{2r}{2-r}.$$

This entails the first part of (3.4.9), in light of Young's inequality and (3.4.8) employed to $p = \frac{6r}{6-r}$. For the second bound in (3.4.9) we work along similar lines noting that, again due to $m+\alpha > \frac{4}{3}$, $2(m+\alpha)-\frac{4}{3} \in \big(1, 6(m+\alpha-1)\big)$ and that $2(m+\alpha)-\frac{4}{3} = \frac{2(m+\alpha-\frac{7}{6})(2(m+\alpha)-\frac{4}{3})}{2(m+\alpha)-\frac{4}{3}-1}$, making the first part of the lemma applicable once more. $\square$

Let us also briefly establish some supplementary spatio-temporal estimates under the additional assumption that $m+2\alpha < 2$. These bounds follow in a straightforward fashion from Lemma 3.10 and Lemma 3.11, and will later form a cornerstone in obtaining the convergence properties necessary to pass to the limit in the integrals making up the global weak Φ-supersolution for $\Phi(s) = (s+1)^{m+2\alpha-1}$.

Corollary 3.12.
Let $m \geq 1$, $\alpha \geq 0$ be such that $\frac{4}{3} < m+\alpha$ and $m+2\alpha < 2$ hold. Suppose that n_0, c_0 and u_0 fulfill (3.1.8) *and assume that $S \in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$ satisfies* (3.1.3) *with some $S_0 > 0$. Then there exists some $C_1 > 0$ such that for all $\varepsilon \in (0,1)$ the global classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of* (3.3.2) *satisfies*

$$\int_t^{t+1}\int_\Omega \big|\nabla(n_\varepsilon+1)^{m+\alpha-1}\big|^2 + \int_t^{t+1}\int_\Omega \big|(n_\varepsilon+1)^{\frac{m+2\alpha-3}{2}}(n_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\nabla n_\varepsilon\big|^2 \leq C_1, \quad (3.4.10)$$

for all $t \geq 0$. Moreover, there exist $p > 2$, $r > 1$ and $C_2 > 0$ such that

$$\int_t^{t+1} \big\|(n_\varepsilon+1)^{m+\alpha-1}\big\|_{L^p(\Omega)}^p \leq C_2 \quad \text{and} \quad \int_t^{t+1} \big\|(n_\varepsilon+1)^{\alpha}(n_\varepsilon+\varepsilon)^{m-1}\big\|_{L^p(\Omega)}^p \leq C_2, \quad (3.4.11)$$

as well as

$$\int_t^{t+1} \big\|(n_\varepsilon+1)^{m+2\alpha-1}\big\|_{L^{\frac{6r}{6-r}}(\Omega)}^{\frac{2r}{2-r}} \leq C_2 \quad (3.4.12)$$

hold for each $\varepsilon \in (0,1)$ and all $t \geq 0$.

Proof: Due to $m+2\alpha < 2$ and $\alpha \geq 0$ we clearly also have $m+\alpha \in (\frac{4}{3}, 2)$. Hence, it is obvious that

$$\frac{1}{(m+\alpha-1)^2}\int_t^{t+1}\int_\Omega \big|\nabla(n_\varepsilon+1)^{m+\alpha-1}\big|^2 = \int_t^{t+1}\int_\Omega (n_\varepsilon+1)^{2(m+\alpha-2)}|\nabla n_\varepsilon|^2$$
$$\leq \int_t^{t+1}\int_\Omega (n_\varepsilon+\varepsilon)^{2(m+\alpha-2)}|\nabla n_\varepsilon|^2 = \frac{1}{(m+\alpha-1)^2}\int_t^{t+1}\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big|^2$$

holds for all $\varepsilon \in (0,1)$ and $t \geq 0$, whereupon the boundedness of the first term in (3.4.10) immediately follows from Lemma 3.10. The bound for the second term contained in (3.4.10) then is a direct consequence of the first bound in light of the fact that $m \geq 1$. Reiterating the proof of Lemma 3.11 for $(n_\varepsilon + 1)$ instead of $(n_\varepsilon + \varepsilon)$, while relying on (3.4.10), we find that for all $q \in \big(1, 6(m+\alpha-1)\big)$ there exists $C > 0$ such that

$$\int_t^{t+1} \big\|n_\varepsilon(\cdot,s)+1\big\|_{L^q(\Omega)}^{\frac{2q(m+\alpha-\frac{7}{6})}{q-1}} \,\mathrm{d}s \leq C \quad \text{for all } t \geq 0.$$

This spatio-temporal estimate at hand, straightforward calculations, similar to those undertaken to prove the special cases presented in Lemma 3.11, verify (3.4.11) and (3.4.12), due to the facts that $m \geq 1$, $\alpha \geq 0$, $m+\alpha > \frac{4}{3}$ and $m + 2\alpha < 2$ also entail that $\alpha < \frac{2}{3}$. □

3.4.2 Estimates involving the fluid component u_ε

We will briefly state [46, Lemma 3.4] without proof. This result will be applied to a differential inequality for $\int_\Omega |u_\varepsilon(\cdot,t)|^2$ in the lemma thereafter to obtain a first boundedness information on the fluid component, which can then be refined to additional spatio-temporal bounds.

Lemma 3.13.
For some $T \in (0,\infty]$ let $y \in C^1((0,T)) \cap C^0([0,T))$, $h \in C^0([0,T))$, $C > 0$, $a > 0$ satisfy

$$y'(t) + ay(t) \leq h(t), \qquad \int_{(t-1)_+}^t h(s)\,\mathrm{d}s \leq C$$

for all $t \in (0,T)$. Then $y \leq y(0) + \frac{C}{1-e^{-a}}$ throughout $(0,T)$.

Drawing on Lemmata 3.11 and 3.13 as well as Hölder's inequality, we are now in a position to utilize quite standard arguments, which have been successfully employed before in e.g. [104, Lemmata 3.5 and 3.6] and [88, Lemma 4.3].

Lemma 3.14.
Let $m \geq 1$, $\alpha \geq 0$ be such that $m + \alpha > \frac{4}{3}$ and assume that n_0, c_0 and u_0 comply with (3.1.8) *and that $S \in C^2\big(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times 3}\big)$ fulfills* (3.1.3) *with some $S_0 > 0$. Then there exists $C > 0$ such that for all $\varepsilon \in (0,1)$ the solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of* (3.3.2) *satisfies*

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_t^{t+1}\int_\Omega |\nabla u_\varepsilon|^2 + \int_t^{t+1} \|u_\varepsilon\|_{L^6(\Omega)}^2 \leq C$$

for all $t \geq 0$.

Proof: Multiplication of the third equation in (3.3.2) by u_ε, integration by parts and an application of the Hölder inequality shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\varepsilon|^2(\cdot,t) + \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \leq \|\nabla\phi\|_{L^\infty(\Omega)}\|u_\varepsilon(\cdot,t)\|_{L^6(\Omega)}\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)} \tag{3.4.13}$$

holds for all $t>0$ and all $\varepsilon\in(0,1)$. Recalling the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ and the Poincaré inequality we find $C_1>0$ satisfying

$$\|u_\varepsilon(\cdot,t)\|_{L^6(\Omega)}^2\le C_1\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2\quad\text{for all } t>0 \text{ and all } \varepsilon\in(0,1), \tag{3.4.14}$$

which upon combination with (3.4.13), (3.1.7) and Young's inequality entails the existence of $C_2>0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u_\varepsilon|^2(\cdot,t)+\frac{1}{2}\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2\le C_2\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}^2 \tag{3.4.15}$$

is valid for all $t>0$ and all $\varepsilon\in(0,1)$. Due to Lemma 3.11 implying the existence of $C_3>0$ satisfying $\int_t^{t+1}\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}^2\le C_3$ for all $t>0$, we find that by estimating the gradient term by means of the Poincaré inequality from below and then employing Lemma 3.13, there exists $C_4>0$ such that

$$\int_\Omega|u_\varepsilon|^2(\cdot,t)\le C_4\quad\text{for all } t>0 \text{ and all } \varepsilon\in(0,1).$$

The estimate for $\int_\Omega|u_\varepsilon|^2$ at hand, we can integrate (3.4.15) with respect to time to obtain that

$$\int_t^{t+1}\int_\Omega|\nabla u_\varepsilon|^2\le C_4+2C_2C_3\quad\text{for all } t>0 \text{ and all } \varepsilon\in(0,1),$$

which also immediately implies

$$\int_t^{t+1}\|u_\varepsilon\|_{L^6(\Omega)}^2\le C_1C_4+2C_1C_2C_3\quad\text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

in light of (3.4.14), and thus concludes the proof. □

With a first set of ε-independent estimates for the fluid component at hand, let us also briefly derive some spatio-temporal bounds for the combined quantities $n_\varepsilon u_\varepsilon$ and $(n_\varepsilon+1)^{m+2\alpha-1}u_\varepsilon$. These estimates will be a cornerstone in the treatment of the integrals appearing in the solution concepts of (3.3.2), which involve the fluid interaction.

Lemma 3.15.
Let $m\ge1$, $\alpha\ge0$ be such that $m+\alpha>\frac{4}{3}$ and assume that n_0, c_0 and u_0 comply with (3.1.8) and that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$ fulfills (3.1.3) with some $S_0>0$. Then there exist $r>1$ and $C_1>0$ such that for all $\varepsilon\in(0,1)$ the solution $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ of (3.3.2) satisfies

$$\int_t^{t+1}\int_\Omega|n_\varepsilon u_\varepsilon|^r\le C_1\quad\textit{for all } t\ge0.$$

If, additionally, $m+2\alpha<2$, then there are $s>1$ and $C_2>0$ such that

$$\int_t^{t+1}\int_\Omega\big|(n_\varepsilon+1)^{m+2\alpha-1}u_\varepsilon\big|^s\le C_2$$

hold for each $\varepsilon\in(0,1)$ and all $t\ge0$.

Proof: For any $r \in (1,2)$ an employment of the Hölder and Young inequalities shows that

$$\begin{aligned}\int_t^{t+1}\int_\Omega \big|n_\varepsilon u_\varepsilon\big|^r &\le \int_t^{t+1} \big\|(n_\varepsilon+\varepsilon)u_\varepsilon\big\|_{L^r(\Omega)}^r \le \int_t^{t+1} \|n_\varepsilon+\varepsilon\|^r_{L^{\frac{6r}{6-r}}(\Omega)}\|u_\varepsilon\|^r_{L^6(\Omega)}\\ &\le \int_t^{t+1} \|n_\varepsilon+\varepsilon\|^{\frac{2r}{2-r}}_{L^{\frac{6r}{6-r}}(\Omega)} + \int_t^{t+1}\|u_\varepsilon\|^2_{L^6(\Omega)} \quad \text{for all } t\ge 0.\end{aligned}$$

Thus, taking $r>1$ as provided by Lemma 3.11, the proof of the first assertion follows immediately from combining the estimate above with Lemmata 3.11 and 3.14. In a similar fashion we find that for $s\in(1,2)$ we have

$$\int_t^{t+1}\int_\Omega \big|(n_\varepsilon+1)^{m+2\alpha-1}u_\varepsilon\big|^s \le \int_t^{t+1}\big\|(n_\varepsilon+1)^{m+2\alpha-1}\big\|^{\frac{2s}{2-s}}_{L^{\frac{6s}{6-s}}(\Omega)} + \int_t^{t+1}\|u_\varepsilon\|^2_{L^6(\Omega)}$$

for all $t\ge 0$ and hence the second part of the lemma is implied by Corollary 3.12 and Lemma 3.14. $\square$

3.4.3 Time regularity

Having in mind an Aubin-Lions type argument to conclude the existence of limit objects of our approximate solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ when taking $\varepsilon \searrow 0$, we still require regularity estimates for the time derivatives. Relying on the bounds established in the previous sections alone does not yet yield sufficient information on terms appearing in our estimation process.

Lemma 3.16.
Let $m\ge 1$, $\alpha\ge 0$ be such that $m+\alpha>\frac{4}{3}$, suppose that n_0, c_0 and u_0 comply with (3.1.8) *and assume that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times 3}\big)$ satisfies* (3.1.3) *with some $S_0>0$. Then there exists $C>0$ such that for all $\varepsilon\in(0,1)$ the global classical solution $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ of* (3.3.2) *satisfies*

$$\int_t^{t+1}\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\alpha}{2}-1}\big|^2 \le C$$

for all $t\ge 0$.

Proof: Similar to the proof of Lemma 3.10 we first assume $m+\alpha\ne 2$, employ Lemma 3.9 for $\beta=1$ and multiply the equality by $\frac{\operatorname{sgn}(m+\alpha-2)}{m+\alpha-1}$ to obtain upon one application of Young's inequality that

$$\begin{aligned}\frac{\operatorname{sgn}(m+\alpha-2)}{m+\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(n_\varepsilon+\varepsilon)^{m+\alpha-1} + \frac{m|m+\alpha-2|}{2(m+\frac{\alpha}{2}-1)^2}\int_\Omega\big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\alpha}{2}-1}\big|^2 \qquad (3.4.16)\\ \le \frac{|m+\alpha-2|}{2m}\int_\Omega \frac{n_\varepsilon^2(n_\varepsilon+\varepsilon)^{\alpha-2}}{(1+\varepsilon n_\varepsilon)^6}|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)|^2|\nabla c_\varepsilon|^2\end{aligned}$$

holds on $(0,\infty)$. Noting that by $S_\varepsilon \le S$ on $\Omega \times [0,\infty)^2$ and (3.1.3) we have

$$\frac{n_\varepsilon^2(n_\varepsilon+\varepsilon)^{\alpha-2}}{(1+\varepsilon n_\varepsilon)^6}|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)|^2 \le \frac{S_0^2(n_\varepsilon+\varepsilon)^\alpha}{(1+n_\varepsilon)^{2\alpha}} \le S_0^2, \tag{3.4.17}$$

we integrate (3.4.16) to find that

$$\begin{aligned}\frac{\operatorname{sgn}(m+\alpha-2)}{m+\alpha-1}\int_\Omega(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,t+1)+\frac{m|m+\alpha-2|}{2(m+\frac{\alpha}{2}-1)^2}\int_t^{t+1}\int_\Omega\big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\alpha}{2}-1}\big|^2\\ \le \frac{\operatorname{sgn}(m+\alpha-2)}{m+\alpha-1}\int_\Omega(n_\varepsilon+\varepsilon)^{m+\alpha-1}(\cdot,t)+\frac{S_0^2|m+\alpha-2|}{2m}\int_t^{t+1}\int_\Omega|\nabla c_\varepsilon|^2\end{aligned}$$

for all $t \ge 0$, which proves the asserted bound for both $m+\alpha<2$ and $m+\alpha>2$, in light of Lemma 3.10. For $m+\alpha=2$, however, we consider the time-evolution of $\int_\Omega n_\varepsilon \ln n_\varepsilon$ to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon\ln n_\varepsilon+\frac{m}{2(1-\frac{\alpha}{2})^2}\int_\Omega\big|\nabla(n_\varepsilon+\varepsilon)^{1-\frac{\alpha}{2}}\big|^2 \le \frac{S_0^2}{2m}\int_\Omega|\nabla c_\varepsilon|^2 \tag{3.4.18}$$

on $(0,\infty)$, where we used estimations akin to those in (3.4.17) and the fact that in this case $m-1=1-\alpha$. Here, we rely on the elementary inequality $s\ln s \le s^{5/3}$ for $s>0$, the Gagliardo–Nirenberg inequality and the mass conservation (3.3.3) to estimate

$$\int_\Omega n_\varepsilon\ln n_\varepsilon \le \big\|(n_\varepsilon+\varepsilon)^{1-\frac{\alpha}{2}}\big\|_{L^{\frac{5}{3(1-\frac{\alpha}{2})}}(\Omega)}^{\frac{5}{3(1-\frac{\alpha}{2})}} \le C_1\big\|\nabla(n_\varepsilon+\varepsilon)^{1-\frac{\alpha}{2}}\big\|_{L^2(\Omega)}^{\frac{5a}{3(1-\frac{\alpha}{2})}}+C_1 \quad \text{on } (0,\infty),$$

with some $C_1>0$ and $a=\frac{12-6\alpha}{25-15\alpha}$. Since, in this case, $\alpha \le 1$ we have $\frac{5a}{3(1-\frac{\alpha}{2})} \le 2$ and hence (after an application of Young's inequality if necessary) there exists $C_2>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon\ln n_\varepsilon+C_2\int_\Omega n_\varepsilon\ln n_\varepsilon \le \frac{S_0^2}{2m}\int_\Omega|\nabla c_\varepsilon|^2+C_2 \quad \text{on } (0,\infty).$$

Due to Lemma 3.13 and Lemma 3.10 this implies on one hand that there exists $C_3>0$ satisfying $\int_\Omega n_\varepsilon\ln n_\varepsilon(\cdot,t) \le C_3$ for all $t\ge 0$ and on the other hand, upon returning to (3.4.18) and integrating with respect to time, that the asserted bound of the lemma holds in light of the fact that $s\ln s \ge -\frac{1}{e}$ for all $s>0$. $\square$

Now we can rely on standard reasoning to obtain the following:

Lemma 3.17.
Let $m\ge 1$, $\alpha \ge 0$ be such that $m+\alpha>\frac{4}{3}$ and assume that n_0, c_0 and u_0 comply with (3.1.8) and that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times 3}\big)$ fulfills (3.1.3) with some $S_0>0$. For every $T>0$ there exists $C(T)>0$ such that for any $\varepsilon\in(0,1)$ the solution $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ of (3.3.2) satisfies

$$\big\|\partial_t\big((n_\varepsilon+\varepsilon)^{m+\alpha-1}\big)\big\|_{L^1\left((0,T);(W_0^{3,2}(\Omega))^*\right)} \le C(T),$$

and

$$\|c_{\varepsilon t}\|_{L^1\left((0,T);(W_0^{3,2}(\Omega))^*\right)} \le C(T).$$

Proof: For fixed $T > 0$ we find $C_1 > 0$ such that

$$\|\varphi\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \le C_1 \|\varphi\|_{L^\infty\left((0,T);W_0^{3,2}(\Omega)\right)} \quad \text{for all } \varphi \in L^\infty\big((0,T);W_0^{3,2}(\Omega)\big),$$

in light of the continuous embedding of $W^{3,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. Here, noting that $L^\infty\big((0,T);W_0^{3,2}(\Omega)\big)$ is the dual space of $L^1\big((0,T);\big(W_0^{3,2}(\Omega)\big)^*\big)$, we fix an arbitrary $\varphi \in L^\infty\big((0,T);W_0^{3,2}(\Omega)\big)$ satisfying $\|\varphi\|_{L^\infty\left((0,T);W_0^{3,2}(\Omega)\right)} \le 1$ and make use of the first equation of (3.3.2), the Cauchy–Schwarz inequality and the bound (3.1.3) to obtain

$$\begin{aligned}
&\frac{1}{m+\alpha-1}\Big|\int_\Omega \partial_t\big((n_\varepsilon+\varepsilon)^{m+\alpha-1}\big)\varphi\Big| \\
&\le \frac{m|m+\alpha-2|C_1}{(m+\frac{\alpha}{2}-1)^2}\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\alpha}{2}-1}\big|^2 \\
&\quad + \frac{mC_1}{m+\frac{\alpha}{2}-1}\Big(\int_\Omega (n_\varepsilon+\varepsilon)^{2(m+\frac{\alpha}{2}-1)}\Big)^{\frac12}\Big(\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\alpha}{2}-1}\big|^2\Big)^{\frac12} \\
&\quad + \frac{|m+\alpha-2|S_0C_1}{m+\frac{\alpha}{2}-1}\Big(\int_\Omega \frac{n_\varepsilon^2(n_\varepsilon+\varepsilon)^{\alpha-2}}{(1+\varepsilon n_\varepsilon)^6(1+n_\varepsilon)^{2\alpha}}\big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\alpha}{2}-1}\big|^2\Big)^{\frac12}\Big(\int_\Omega |\nabla c_\varepsilon|^2\Big)^{\frac12} \\
&\quad + S_0C_1\Big(\int_\Omega \frac{n_\varepsilon^2(n_\varepsilon+\varepsilon)^{2(m+\alpha-2)}}{(1+\varepsilon n_\varepsilon)^6(1+n_\varepsilon)^{2\alpha}}\Big)^{\frac12}\Big(\int_\Omega |\nabla c_\varepsilon|^2\Big)^{\frac12} \\
&\quad + \frac{C_1}{m+\alpha-1}\Big(\int_\Omega |u_\varepsilon|^2\Big)^{\frac12}\Big(\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big|^2\Big)^{\frac12}
\end{aligned}$$

on $(0,T)$ for all $\varepsilon \in (0,1)$. Since $\frac{n_\varepsilon^2(n_\varepsilon+\varepsilon)^{\alpha-2}}{(1+\varepsilon n_\varepsilon)^6(1+n_\varepsilon)^{2\alpha}} \le \frac{(n_\varepsilon+\varepsilon)^\alpha}{(1+n_\varepsilon)^{2\alpha}} \le 1$, multiple applications of the Young inequality and integration over $(0,T)$ entails the existence of $C_2 > 0$ such that

$$\begin{aligned}
&\int_0^T \Big|\int_\Omega \partial_t\big((n_\varepsilon+\varepsilon)^{m+\alpha-1}\big)\varphi\Big| \\
&\qquad \le C_2\int_0^T\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\frac{\alpha}{2}-1}\big|^2 + C_2\int_0^T\int_\Omega \big|\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big|^2 \\
&\qquad\quad + C_2\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 + C_2\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{2m+\alpha-2} + C_2\int_0^T\int_\Omega |u_\varepsilon|^2 + C_2
\end{aligned}$$

holds for all $\varepsilon \in (0,1)$ and all $\varphi \in L^\infty\big((0,T);W_0^{3,2}(\Omega)\big)$ with $\|\varphi\|_{L^\infty\left((0,T);W_0^{3,2}(\Omega)\right)} \le 1$. Because of $2m+\alpha-2 < 2(m+\alpha)-\frac{4}{3}$, a combination of Lemmata 3.10, 3.11, 3.14 and 3.16 now leads to the existence of $C_3(T) > 0$ such that for all $\varphi \in L^\infty\big((0,T);W_0^{2,3}(\Omega)\big)$ with $\|\varphi\|_{L^\infty\left((0,T);W_0^{3,2}(\Omega)\right)} \le 1$

$$\int_0^T \Big|\int_\Omega \partial_t\big((n_\varepsilon+\varepsilon)^{m+\alpha-1}\big)\varphi\Big| \le C_3(T)$$

is satisfied. For the second part of the Lemma we follow a complementary reasoning for the second equation. For fixed φ as before we obtain $C_4 > 0$ such that

$$\Big|\int_\Omega c_{\varepsilon t}\varphi\Big| \le C_1\int_\Omega|\nabla c_\varepsilon|^2 + C_1\int_\Omega c_\varepsilon + C_1\int_\Omega n_\varepsilon + \frac{C_1}{2}\int_\Omega|u_\varepsilon|^2 + \frac{C_1}{2}\int_\Omega c_\varepsilon^2 + C_4$$

is valid on $(0,T)$ for all $\varepsilon\in(0,1)$. Hence, we can conclude the proof upon integration over $(0,T)$ in light of the bounds featured in Lemmata 3.8, 3.10 and 3.14. □

Enhancing similar arguments by known results for the Yosida approximation and the Stokes operator, a complementary result can be established for the third solution component.

Lemma 3.18.
Let $m\ge 1$, $\alpha\ge 0$ be such that $m+\alpha>\frac{4}{3}$ and suppose that n_0, c_0 and u_0 fulfill (3.1.8) and that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times3}\big)$ satisfies (3.1.3) with some $S_0>0$. For every $T>0$ there exists $C(T)>0$ such that for any $\varepsilon\in(0,1)$ the solution $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ of (3.3.2) satisfies

$$\int_0^T \|u_{\varepsilon t}\|^{\frac{4}{3}}_{(W_0^{1,2}(\Omega))^*} \le C(T). \tag{3.4.19}$$

Proof: In light of (3.4.9) from Lemma 3.11 there is $C_1>0$ such that $\int_t^{t+1}\|n_\varepsilon(\cdot,t)\|^{\frac{4}{3}}_{L^{\frac{6}{5}}(\Omega)}$ $\le C_1$ for all $t>0$ and hence we can follow the proof of [88, Lemma 5.5], where the related system with linear diffusion was discussed, to conclude the desired bound. Let us state a brief outline of the steps involved. We multiply the third equation of (3.3.2) with a fixed $\psi\in C_0^\infty(\Omega;\mathbb{R}^3)$ satisfying $\nabla\cdot\psi\equiv 0$ throughout Ω and employ Hölder's inequality to obtain

$$\begin{aligned}\Big|\int_\Omega u_{\varepsilon t}\cdot\psi\Big| \le \|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} + \|Y_\varepsilon u_\varepsilon\|_{L^6(\Omega)}\|u_\varepsilon\|_{L^3(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}&\\ + \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|\psi\|_{L^6(\Omega)}&\end{aligned}$$

on $(0,\infty)$ for all $\varepsilon\in(0,1)$. Next, we make use of known facts for the Yosida approximation and the Stokes operator, the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ and the Gagliardo–Nirenberg inequality to obtain $C_2>0$ such that

$$\begin{aligned}&\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^6(\Omega)} \le \|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)},\\ \text{and}\quad &\|u_\varepsilon(\cdot,t)\|^{\frac{4}{3}}_{L^3(\Omega)} \le C_2\|\nabla u_\varepsilon(\cdot,t)\|^{\frac{2}{3}}_{L^2(\Omega)}\|u_\varepsilon(\cdot,t)\|^{\frac{2}{3}}_{L^2(\Omega)}\end{aligned}$$

hold for all $t>0$ and all $\varepsilon\in(0,1)$. Combining the estimates above with Young's inequality shows that with some $C_3>0$ we have

$$\int_0^T\|u_{\varepsilon t}\|^{\frac{4}{3}}_{(W_0^{1,2}(\Omega))^*} \le C_3\int_0^T\|\nabla u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} + C_3\int_0^T\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}\|u_\varepsilon\|^{\frac{2}{3}}_{L^2(\Omega)} + C_3T$$

for all $T>0$ and all $\varepsilon\in(0,1)$, completing the proof in terms of Lemma 3.14. □

3.5 Limit functions and their regularity properties

The uniform bounds prepared in the previous section enable us to derive the existence of limit functions n, c, u, satisfying the regularity conditions imposed by Definition 3.5. With the precompactness properties contained in the previous lemmata, we also immediately obtain convergence properties favorable enough to pass to the limit in most of the integrals making up the solution concepts discussed in Section 3.2. In contrast to the scalar sensitivity case discussed in [7] and the linear diffusion case discussed in [88], the very weak solution concept features terms combining $n_\varepsilon + 1$ and $n_\varepsilon + \varepsilon$ in a slightly more varied way, which necessitates the preparation of additional convergence properties.

Proposition 3.19.
Let $m \geq 1$, $\alpha \geq 0$ be such that $m + \alpha > \frac{4}{3}$ and suppose that n_0, c_0, u_0 comply with (3.1.8) and assume that $S \in C^2(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times 3})$ fulfills (3.1.3) with some $S_0 > 0$. Then there exist a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ and functions

$$\begin{aligned} n &\in L^{2(m+\alpha)-\frac{4}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \quad \textit{with} \quad \nabla n^{m+\alpha-1} \in L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \\ c &\in L^2_{loc}\big([0,\infty); W^{1,2}(\Omega)\big)\,, \\ u &\in L^2_{loc}\big([0,\infty); W^{1,2}_0(\Omega;\mathbb{R}^3)\big), \end{aligned}$$

such that the solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.3.2) satisfy

$$(n_\varepsilon+\varepsilon)^{m+\alpha-1} \to n^{m+\alpha-1} \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \textit{ and a.e. in } \Omega\times(0,\infty), \tag{3.5.1}$$

$$\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1} \rightharpoonup \nabla n^{m+\alpha-1} \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \tag{3.5.2}$$

$$n_\varepsilon+\varepsilon \rightharpoonup n \quad \textit{in } L^{2(m+\alpha)-\frac{4}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,, \tag{3.5.3}$$

$$n_\varepsilon+\varepsilon \to n \quad \textit{and} \quad n_\varepsilon \to n \quad \textit{in } L^p_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \textit{ for any } p \in [1, 2(m+\alpha)-\tfrac{4}{3}), \tag{3.5.4}$$

$$c_\varepsilon \to c \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \textit{ and a.e. in } \Omega\times(0,\infty), \tag{3.5.5}$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \tag{3.5.6}$$

as well as

$$u_\varepsilon \to u \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big) \textit{ and a.e. in } \Omega\times(0,\infty), \tag{3.5.7}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^{3\times3}\big)\,, \tag{3.5.8}$$

$$Y_\varepsilon u_\varepsilon \to u \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \tag{3.5.9}$$

$$n_\varepsilon u_\varepsilon \to nu \quad \textit{in } L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \tag{3.5.10}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and such that $n \geq 0$, $c \geq 0$ a.e. in $\Omega\times(0,\infty)$. If, additionally, $m+2\alpha < 2$, then there exists a further subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}} \subset (0,1)$ such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ also satisfy

$$(n_\varepsilon+1)^{m+\alpha-1} \to (n+1)^{m+\alpha-1} \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,, \tag{3.5.11}$$

$$
\begin{aligned}
(n_\varepsilon+1)^{m+2\alpha-1} &\to (n+1)^{m+2\alpha-1} && \textit{in } L^1_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,, && (3.5.12)\\
\nabla(n_\varepsilon+1)^{m+\alpha-1} &\rightharpoonup \nabla(n+1)^{m+\alpha-1} && \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, && (3.5.13)\\
(n_\varepsilon+1)^{\alpha}(n_\varepsilon+\varepsilon)^{m-1} &\to (n+1)^{\alpha}n^{m-1} && \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,, && (3.5.14)\\
(n_\varepsilon+1)^{m+2\alpha-1}u_\varepsilon &\to (n+1)^{m+2\alpha-1}u && \textit{in } L^1_{loc}\big(\overline{\Omega}\times[0,\infty)\big) && (3.5.15)
\end{aligned}
$$

as well as

$$
(n_\varepsilon+1)^{\frac{m+2\alpha-3}{2}}(n_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\nabla n_\varepsilon \rightharpoonup (n+1)^{\frac{m+2\alpha-3}{2}}n^{\frac{m-1}{2}}\nabla n \quad \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \tag{3.5.16}
$$

as $\varepsilon=\varepsilon_{j_k}\searrow 0$.

Proof: Noticing that $2(m+\alpha-1)<2(m+\alpha)-\frac{4}{3}$, we find that by combining Lemmata 3.10, 3.11 and 3.17 with the Aubin-Lions lemma ([74, Corollary 8.4])

$$
\big\{(n_\varepsilon+\varepsilon)^{m+\alpha-1}\big\}_{\varepsilon\in(0,1)} \quad \text{is relatively compact in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big)
$$

and that hence there exists a sequence $\varepsilon_j\searrow 0$ such that (3.5.1) holds. Extracting an additional subsequence (still denoted by ε_j), we conclude from the spatio-temporal bounds featured in Lemma 3.10 and Lemma 3.11 that (3.5.2) and (3.5.3) hold as well. In light of Lemma 3.11 $\{(n_{\varepsilon_j}+\varepsilon_j)^p\}_{j\in\mathbb{N}}$ is equi-integrable for any $p<2(m+\alpha)-\frac{4}{3}$, and thus we can rely on the a.e. convergence of $n_\varepsilon+\varepsilon$ entailed by (3.5.1) and the Vitali convergence theorem to obtain the first part of (3.5.4), with the second part then being an immediate consequence of the uniform convergence of ε_j to zero. Along similar lines the Lemmata 3.10 and 3.17 together with the Aubin-Lions lemma imply that upon extraction of another subsequence also (3.5.5) and (3.5.6) hold. Moreover, applying these arguments once more for the third component of the approximate solutions while relying on Lemmata 3.14 and 3.18 proves (3.5.7) and (3.5.8), whereas (3.5.9) is a consequence of the dominated convergence theorem and the boundedness of $\|u_\varepsilon\|^2_{L^2(\Omega\times(0,T))}$ for any $T>0$ (see e.g. [104, Lemma 4.1]). The strong convergence property of the mixed term $n_\varepsilon u_\varepsilon$ in (3.5.10) can be concluded by combining the a.e. convergences contained in (3.5.1) and (3.5.7) with the equi-integrability of $\{|n_{\varepsilon_j}u_{\varepsilon_j}|^r\}_{j\in\mathbb{N}}$ for some $r>1$ implied by Lemma 3.15 and Vitali's convergence theorem. The assertions for the special case of $m+2\alpha<2$ follow from identical reasoning in light of Corollary 3.12 and Lemma 3.15. To be precise, we can conclude (upon extraction of another non-relabeled subsequence) (3.5.13) and (3.5.16) from (3.4.10). The properties (3.5.11), (3.5.12) and (3.5.14) are a consequence of (3.4.11), Vitali's convergence theorem and the fact that $m+2\alpha-1\le 2(m+\alpha-1)$, and finally, combining Lemma 3.15 with Vitali's theorem one last time shows (3.5.15). $\square$

3.6 Solution properties of the limit functions

3.6.1 Weak solution properties of c and u

Relying on the convergence properties prepared in Proposition 3.19, we can check in a straightforward manner that the limit objects c and u are weak solutions of their corresponding equations in (3.1.4).

Lemma 3.20.
Let $m \geq 1$, $\alpha \geq 0$ be such that $m + \alpha > \frac{4}{3}$, assume that n_0, c_0 and u_0 comply with (3.1.8) *and suppose that $S \in C^2\big(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times 3}\big)$ satisfies* (3.1.3) *with some $S_0 > 0$. Furthermore, let n, c, u denote the limit functions provided by Proposition 3.19. Then*

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \quad \textit{for a.e. } t > 0, \tag{3.6.1}$$

and c and u satisfy the weak solution properties (3.2.4) *and* (3.2.5)*, respectively, of Definition 3.5.*

Proof: The equality in (3.6.1) for almost every $t > 0$ is a direct result of the mass conservation (3.3.3) from Lemma 3.8 and (3.5.4). To verify that c solves its corresponding equation in the weak sense, we multiply the second equation of (3.3.2) by an arbitrary test function $\varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^2\big((0,\infty); W^{1,2}(\Omega)\big)$ with compact support in $\overline{\Omega} \times [0,\infty)$ and $\varphi_t \in L^2(\Omega \times (0,\infty))$ to find that

$$\begin{aligned} -\int_0^\infty\!\!\int_\Omega c_\varepsilon \varphi_t - \int_\Omega c_0 \varphi(\cdot,0) \\ = -\int_0^\infty\!\!\int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi - \int_0^\infty\!\!\int_\Omega c_\varepsilon \varphi + \int_0^\infty\!\!\int_\Omega n_\varepsilon \varphi + \int_0^\infty\!\!\int_\Omega c_\varepsilon (u_\varepsilon \cdot \nabla\varphi) \end{aligned}$$

holds for all $\varepsilon \in (0,1)$. In consideration of (3.5.5), (3.5.6), (3.5.4) and (3.5.7) we may pass to the limit in each of the integrals and conclude that (3.2.4) holds and that hence c solves its equation in the weak sense. In a similar fashion, we test the third equation of (3.3.2) by an arbitrary $\psi \in C_0^\infty\big(\Omega \times [0,\infty); \mathbb{R}^3\big)$ satisfying $\nabla \cdot \psi \equiv 0$ in $\Omega \times (0,\infty)$ to obtain

$$\begin{aligned} -\int_0^\infty\!\!\int_\Omega u_\varepsilon \psi_t - \int_\Omega u_0 \psi(\cdot,0) \\ = -\int_0^\infty\!\!\int_\Omega \nabla u_\varepsilon \cdot \nabla\psi + \int_0^\infty\!\!\int_\Omega (Y_\varepsilon u_\varepsilon \otimes u_\varepsilon) \cdot \nabla\psi + \int_0^\infty\!\!\int_\Omega n_\varepsilon (\nabla\phi \cdot \psi) \end{aligned}$$

for all $\varepsilon \in (0,1)$. Recalling (3.5.7), (3.5.8), (3.5.9), as well as (3.5.4) and (3.1.7) we can take $\varepsilon \searrow 0$ in all the integrals and find that u satisfies (3.2.5). □

3.6.2 Weak solution property of n for $m + 2\alpha > \frac{5}{3}$

The currently known compactness properties do not allow us to take $\varepsilon \searrow 0$ in some of the integrals appearing in the equation for n_ε corresponding to (3.2.6) of the weak solution concept in Definition 3.6. However, imposing the additional condition $m + 2\alpha > \frac{5}{3}$ we can obtain supplementary convergence properties to the ones in Proposition 3.19, which will allow us to pass to the limit in these crucial integrals.

Lemma 3.21.
Let $m \geq 1$, $\alpha \geq 0$ be such that $m + 2\alpha > \frac{5}{3}$, suppose that n_0, c_0 and u_0 comply with (3.1.8)*, and suppose that $S \in C^2\big(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times 3}\big)$ satisfies* (3.1.3) *with some $S_0 > 0$.*

Furthermore, let n, c, u denote the limit functions obtained in Proposition 3.19. Then $n \in L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big)$ and for any $\varphi \in C_0^\infty\big(\overline{\Omega}\times[0,\infty)\big)$ the weak solution property (3.2.6) *is satisfied.*

Proof: Multiplying the first equation of (3.3.2) by $\varphi \in C_0^\infty\big(\overline{\Omega}\times[0,\infty)\big)$ and integrating by parts, we find that

$$\begin{aligned}
-\int_0^\infty\int_\Omega n_\varepsilon\varphi_t - \int_\Omega n_0\varphi(\cdot,0)& \\
= -\frac{m}{m+\alpha-1}\int_0^\infty\int_\Omega (n_\varepsilon+\varepsilon)^{1-\alpha}\big(\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\cdot\nabla\varphi\big)& \qquad (3.6.2)\\
+\int_0^\infty\int_\Omega \frac{n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}\big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi\big) + \int_0^\infty\int_\Omega n_\varepsilon(u_\varepsilon\cdot\nabla\varphi)&
\end{aligned}$$

holds for all $\varepsilon\in(0,1)$, where we used $(n_\varepsilon+\varepsilon)^{m-1}\nabla n_\varepsilon = \frac{(n_\varepsilon+\varepsilon)^{1-\alpha}}{m+\alpha-1}\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}$. In light of (3.5.4) we see that

$$-\int_0^\infty\int_\Omega n_\varepsilon\varphi_t \to -\int_0^\infty\int_\Omega n\varphi_t \quad \text{as } \varepsilon=\varepsilon_j\searrow 0.$$

Moreover, since $m+2\alpha>\frac{5}{3}$, we have $2(1-\alpha)<2(m+\alpha)-\frac{4}{3}$, so that (3.5.4) implies that

$$(n_\varepsilon+\varepsilon)^{1-\alpha}\to n^{1-\alpha} \quad \text{in} \quad L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0,$$

which together with (3.5.2) shows

$$\begin{aligned}
-\frac{m}{m+\alpha-1}&\int_0^\infty\int_\Omega (n_\varepsilon+\varepsilon)^{1-\alpha}\big(\nabla(n_\varepsilon+\varepsilon)^{m+\alpha-1}\cdot\nabla\varphi\big)\\
&\to -\frac{m}{m+\alpha-1}\int_0^\infty\int_\Omega n^{1-\alpha}\big(\nabla n^{m+\alpha-1}\cdot\nabla\varphi\big) = -m\int_0^\infty\int_\Omega n^{m-1}(\nabla n\cdot\nabla\varphi)
\end{aligned}$$

as $\varepsilon=\varepsilon_j\searrow 0$. Additionally, since $2(1-\alpha)<2(m+\alpha)-\frac{4}{3}$, we can fix $2<s<\frac{2(m+\alpha)-\frac{4}{3}}{(1-\alpha)_+}$ and find that

$$\int_t^{t+1}\int_\Omega\Big|\frac{n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)}{(1+\varepsilon n_\varepsilon)^3}\Big|^s \le \int_t^{t+1}\int_\Omega \frac{S_0^s n_\varepsilon^s}{(1+n_\varepsilon)^{s\alpha}} \le \begin{cases} S_0^s|\Omega|, & \text{if } \alpha\ge 1\\ S_0^s\int_t^{t+1}\int_\Omega (n_\varepsilon+\varepsilon)^{s(1-\alpha)}, & \text{if } \alpha\in[0,1)\end{cases}$$

holds on $(0,\infty)$. Making use of the fact that $s(1-\alpha)<2(m+\alpha)-\frac{4}{3}$ and Lemma 3.11 we thus obtain that $\big\{n_{\varepsilon_j}^2 S_{\varepsilon_j}(x,n_{\varepsilon_j},c_{\varepsilon_j})^2(1+\varepsilon_j n_{\varepsilon_j})^{-6}\big\}_{j\in\mathbb{N}}$ is equi-integrable, which together with the a.e. convergences of $S_\varepsilon\to S$ and $\frac{n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}\to n$ in $\Omega\times(0,\infty)$ and Vitali's theorem shows that

$$\frac{n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)}{(1+\varepsilon n_\varepsilon)^3}\to nS(x,n,c) \quad \text{in} \quad L^2_{loc}\big(\overline{\Omega}\times[0,\infty)\big)$$

as $\varepsilon = \varepsilon_j \searrow 0$. Merging this convergence property with (3.5.6) we obtain that

$$\int_0^\infty\!\!\int_\Omega \frac{n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}\big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi\big) \to \int_0^\infty\!\!\int_\Omega n\big(S(x,n,c)\nabla c\cdot\nabla\varphi\big) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0.$$

Finally, relying on (3.5.10) we see that

$$\int_0^\infty\!\!\int_\Omega n_\varepsilon(u_\varepsilon\cdot\nabla\varphi) \to \int_0^\infty\!\!\int_\Omega n(u\cdot\nabla\varphi) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0.$$

In conclusion, we may pass to the limit in each of the integrals in (3.6.2) and find that (3.2.6) holds. □

Amalgamating the previous results finalizes the proof of Theorem 3.1.

Proof of Theorem 3.1: The proof is immediate after combination of Lemmata 3.20 and 3.21 with the regularity information on n, c and u presented in Proposition 3.19. □

3.6.3 Very weak solution property of n in the case of $m+\alpha > \frac{4}{3}$

Under the weaker assumption that only $m+\alpha > \frac{4}{3}$ is satisfied, the obtained limit function n does not appear to be regular enough to conclude that the integral $\int_0^\infty\!\int_\Omega n^{m-1}\nabla n\cdot\nabla\varphi$, appearing in (3.2.6), is finite. Weakening the solution concept appears to be the only way to compensate the missing regularity information, which is why we will only consider global very weak solutions as defined in Definition 3.5 for the parameter range of $m \geq 1$ and $\alpha \geq 0$ satisfying $m+\alpha > \frac{4}{3}$ and $m+2\alpha \leq \frac{5}{3}$. Working under these weaker hypothesis, however, the weak convergence statement for ∇c_ε is insufficient to pass to the limit in the integral containing both gradient terms. Therefore, we will have to attain a strong convergence result for ∇c_ε which we prepare with the following Lemma from [88].

Lemma 3.22.
Let $m \geq 1$, $\alpha \geq 0$ be such that $m+\alpha > \frac{4}{3}$ and assume that n_0, c_0 and u_0 comply with (3.1.8) and suppose that $S \in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times 3}\big)$ satisfies (3.1.3) with some $S_0 > 0$. Then there exists a null set $N \subset (0,\infty)$ such that the functions n, c and u obtained in Proposition 3.19 satisfy

$$\frac{1}{2}\int_\Omega c^2(\cdot,T) - \frac{1}{2}\int_\Omega c_0^2 + \int_0^T\!\!\int_\Omega |\nabla c|^2 \geq -\int_0^T\!\!\int_\Omega c^2 + \int_0^T\!\!\int_\Omega nc \quad \text{for all } T\in(0,\infty)\setminus N. \tag{3.6.3}$$

Proof: This is precisely [88, Lemma 7.1]. The same lemma has also been used in the setting with scalar sensitivity in [7, Lemma 6.3]. As the proof is quite technical, we will only provide a sketch of the main steps as featured in [7] and refer the reader to [88, 101] for an in-depth look at the details.
Based on (3.5.5), we know that $z(t) := \int_\Omega c^2(\cdot,t)$, $t>0$, satisfies $z \in L^1_{loc}([0,\infty))$. Hence, there exists a null set $N \subset (0,\infty)$ such that each $T \in (0,\infty)\setminus N$ is a Lebesgue point of z, so that

$$\frac{1}{\delta}\int_T^{T+\delta}\!\!\int_\Omega c^2(\cdot,t) \to \int_\Omega c^2(\cdot,T) \quad \text{for all } T\in(0,\infty)\setminus N \;\text{ as }\; \delta\searrow 0.$$

For given $T \in (0,\infty) \setminus N$, $\delta \in (0,1)$ and $r \in (0,1)$ we now consider

$$\zeta_\delta(t) := \begin{cases} 1, & t \in [0,T], \\ \frac{T+\delta-t}{\delta}, & t \in (T, T+\delta), \\ 0, & t \geq T+\delta, \end{cases} \quad \text{and} \quad \psi_r(s) := \frac{s}{1+rs}, \ s \geq 0$$

as well as

$$\tilde{c}_k(x,t) := \begin{cases} c(x,t), & (x,t) \in \Omega \times (0,\infty), \\ c_{0k}(x), & (x,t) \in \Omega \times (-1,0], \end{cases}$$

for $k \in \mathbb{N}$, where the nonnegative sequence $(c_{0k})_{k\in\mathbb{N}} \subset C^1(\overline{\Omega})$ is chosen such that $c_{0k} \to c_0$ in $L^2(\Omega)$ as $k \to \infty$. For $h \in (0,1)$ we then denote by

$$\big(A_h\psi_r(\tilde{c}_k)\big)(x,t) := \frac{1}{h}\int_{t-h}^{t} \psi_r(\tilde{c}_k)(x,s)\,\mathrm{d}s, \quad (x,t) \in \Omega \times [0,\infty),$$

the temporal Steklov average and let

$$\varphi(x,t) := \varphi_{\delta,k,h,r}(x,t) := \zeta_\delta(t) \cdot \big(A_h\psi_r(\tilde{c}_k)\big)(x,t), \quad (x,t) \in \Omega \times [0,\infty).$$

It can be checked that $\varphi \in L^\infty_{loc}\big(\overline{\Omega} \times [0,\infty)\big) \cap L^2\left((0,\infty); W^{1,2}(\Omega)\right)$, that φ has compact support in $\overline{\Omega} \times [0, T+1]$ and that $\varphi_t \in L^2(\Omega \times (0,\infty))$ and therefore we may use φ as a test function for (3.2.4). Inserting φ into (3.2.4) we obtain

$$\begin{aligned}
&-\int_0^\infty\int_\Omega c(x,t)\zeta_\delta'(t)\big(A_h\psi_r(\tilde{c}_k)\big)(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega c_0(x)\big(A_h\psi_r(\tilde{c}_k)\big)(x,0)\,\mathrm{d}x \\
&\quad -\int_0^\infty\int_\Omega c(x,t)\frac{\zeta_\delta(t)}{h}\big[\psi_r(\tilde{c}_k)(x,t) - \psi_r(\tilde{c}_k)(x,t-h)\big]\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^\infty\int_\Omega \nabla c(x,t)\cdot\zeta_\delta(t)\nabla\big(A_h\psi_r(\tilde{c}_k)\big)(x,t)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.6.4) \\
&\quad -\int_0^\infty\int_\Omega c(x,t)\zeta_\delta(t)\big(A_h\psi_r(\tilde{c}_k)\big)(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty\int_\Omega n(x,t)\zeta_\delta(t)\big(A_h\psi_r(\tilde{c}_k)\big)(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\int_0^\infty\int_\Omega c(x,t)\zeta_\delta(t)u(x,t)\cdot\nabla\big(A_h\psi_r(\tilde{c}_k)\big)(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Noting that $\psi_r(\tilde{c}_k) \in L^\infty(\Omega \times (0,T+1))$, that we have $\nabla\psi_r(\tilde{c}_k) \in L^2(\Omega \times (0,T+1);\mathbb{R}^3)$ in light of $c_{0k} \in C^1(\overline{\Omega})$ and that the primitive Ψ_r of ψ_r is given by $\Psi_r(s) : [0,\infty) \to \mathbb{R}$, $s \mapsto \frac{rs-\ln(1+rs)}{r^2}$, we rely on known results for Steklov averages (see e.g. [101, Lemma A.2]) to let $h \searrow 0$ in (3.6.4) and obtain

$$\begin{aligned}
&-\liminf_{h\to 0}\int_0^\infty\int_\Omega c(x,t)\frac{\zeta_\delta(t)}{h}\big[\psi_r(\tilde{c}_k)(x,t) - \psi_r(\tilde{c}_k)(x,t-h)\big]\,\mathrm{d}x\,\mathrm{d}t \\
&= \ -\int_0^\infty\int_\Omega \zeta_\delta(t)\frac{|\nabla c(x,t)|^2}{(1+rc(x,t))^2}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty\int_\Omega \zeta_\delta(t)\frac{c^2(x,t)}{1+rc(x,t)}\,\mathrm{d}x\,\mathrm{d}t \qquad (3.6.5)
\end{aligned}$$

$$+\int_0^\infty\int_\Omega \zeta_\delta(t)\frac{n(x,t)c(x,t)}{1+rc(x,t)}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\int_\Omega \zeta_\delta'(t)\frac{c^2(x,t)}{1+rc(x,t)}\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega \frac{c_0(x)c_{0k}(x)}{1+rc_{0k}(x)}\,\mathrm{d}x.$$

To estimate the remaining limit (compare (7.11)–(7.14) in [88, Lemma 7.1]), we make use of the convexity of Ψ_r, implying

$$\Psi_r(\tilde{c}_k(x,t+h))-\Psi_r(\tilde{c}_k(x,t))\geq \psi_r(\tilde{c}_k)(x,t)\big(c(x,t+h)-c(x,t)\big)$$

for a.e. $x\in\Omega$ and $t\in(0,T+1)$, the substitution $s=t+h$, Young's inequality and the definition of ζ_δ to find that

$$\begin{aligned}
&-\int_0^\infty\int_\Omega \frac{\zeta_\delta(t)}{h}\cdot\big[\psi_r(\tilde{c}_k(x,t)-\psi_r(\tilde{c}_k(x,t-h)\big]\cdot c(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
\leq\ &\int_0^\infty\int_\Omega \frac{\zeta_\delta(t+h)}{h}\big[\Psi_r(\tilde{c}_k(x,t+h))-\Psi_r(\tilde{c}_k(x,t))\big]\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_\Omega\frac{c_{0k}^2(x)}{(1+rc_{0k}(x))^2}\,\mathrm{d}x\\
&\quad+\frac{1}{2h}\int_0^h\int_\Omega c^2(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\int_\Omega\frac{\zeta_\delta(t+h)-\zeta_\delta(h)}{h}\psi_r(\tilde{c}_k(x,t))c(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Combining this with (3.6.5) shows that

$$\begin{aligned}
&\int_0^\infty\int_\Omega \zeta_\delta(t)|\nabla c(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_\Omega c_{0k}^2(x)\,\mathrm{d}x+\frac{1}{2}\int_\Omega c_0^2(x)\,\mathrm{d}x\\
\geq\ &-\int_0^\infty\int_\Omega\zeta_\delta(t)\frac{c^2(x,t)}{1+rc(x,t)}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\int_\Omega\zeta_\delta(t)\frac{n(x,t)c(x,t)}{1+rc(x,t)}\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\frac{c_0(x)c_{0k}(x)}{1+rc_{0k}(x)}\,\mathrm{d}x\\
&+\int_0^\infty\int_\Omega\zeta_\delta'(t)\Psi_r(\tilde{c}_k(x,t))\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\Psi_r(\tilde{c}_k(x,0))\,\mathrm{d}x
\end{aligned}$$

for all $k\in\mathbb{N}$ and $r\in(0,1)$. Drawing on the dominated convergence theorem, we may next let $r\searrow 0$ and then $k\to\infty$ to arrive at

$$\begin{aligned}
&\int_0^\infty\int_\Omega\zeta_\delta(t)|\nabla c(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x+\int_0^\infty\int_\Omega\zeta_\delta(t)c^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^\infty\int_\Omega\zeta_\delta(t)n(x,t)c(x,t)\,\mathrm{d}x\,\mathrm{d}t\geq\frac{1}{2}\int_\Omega c_0^2(x)\,\mathrm{d}x-\frac{1}{2\delta}\int_T^{T+\delta}\int_\Omega c^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Finally, recalling the Lebesgue point property of T we make use of the dominated convergence theorem once more to take $\delta\searrow 0$ and obtain (3.6.3). □

Relying on the spatio-temporal estimates of Section 3.4 and the inequality above, we can now pass to another subsequence along which $\nabla c_\varepsilon\to\nabla c$ in $L^2\big(\Omega\times(0,T);\mathbb{R}^3\big)$ holds as $\varepsilon\searrow 0$. Similar reasoning has been employed in e.g. [107, Lemma 4.4] and [88, Lemma 7.2].

Lemma 3.23.
Let $m\geq 1$, $\alpha\geq 0$ be such that $m+\alpha>\frac{4}{3}$ and assume that n_0, c_0 and u_0 comply with (3.1.8) *and suppose that $S\in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times 3}\big)$ satisfies* (3.1.3) *with some $S_0>0$. Furthermore, denote by $(\varepsilon_j)_{j\in\mathbb{N}}$ and n,c,u the sequence and limit functions provided by*

Proposition 3.19. Then there exist a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ and a null set $N \subset (0,\infty)$ such that for each $T \in (0,\infty)\setminus N$ the classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.3.2) satisfies

$$\nabla c_\varepsilon \to \nabla c \quad \text{in } L^2\big(\Omega\times(0,T);\mathbb{R}^3\big) \quad \text{as } \varepsilon = \varepsilon_{j_k} \searrow 0.$$

Proof: With $r \in (1,2)$ as given by Lemma 3.11 we note that, due to the bounds presented in Lemmata 3.10 and 3.11, the nonnegativity of n_ε and the Hölder and Young inequalities we have $C > 0$ satisfying

$$\int_t^{t+1}\int_\Omega |n_\varepsilon c_\varepsilon|^r \le \frac{2-r}{2}\int_t^{t+1}\|n_\varepsilon+\varepsilon\|_{L^{\frac{6r}{6-r}}(\Omega)}^{\frac{2r}{2-r}} + \frac{r}{2}\int_t^{t+1}\|c_\varepsilon\|_{L^6(\Omega)}^2 \le C$$

for all $t > 0$ and all $\varepsilon \in (0,1)$. Since $r > 1$, we can combine the a.e. convergence of $n_\varepsilon c_\varepsilon \to nc$ in $\Omega\times(0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$, as implied by Proposition 3.19, with Vitali's convergence theorem, to find that for all $T > 0$

$$\int_0^T\int_\Omega n_\varepsilon c_\varepsilon \to \int_0^T\int_\Omega nc \quad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Denoting by $N_1 \subset (0,\infty)$ the null set given by Lemma 3.22 we see that by Proposition 3.19 there exists another null set $N_2 \supset N_1$ and a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ such that

$$\int_\Omega c_\varepsilon^2(\cdot,T) \to \int_\Omega c^2(\cdot,T) \quad \text{for all } T \in (0,\infty)\setminus N_2 \quad \text{as } \varepsilon = \varepsilon_{j_k} \searrow 0.$$

Hence, for any such $T \in (0,\infty)\setminus N_2$, by testing the second equation of (3.3.2) by c_ε and making use of Lemma 3.22 and Proposition 3.19 we obtain

$$\begin{aligned}\int_0^T\int_\Omega |\nabla c|^2 &\ge -\frac{1}{2}\int_\Omega c^2(\cdot,T) + \frac{1}{2}\int_\Omega c_0^2 - \int_0^T\int_\Omega c^2 + \int_0^T\int_\Omega nc\\ &= \lim_{\varepsilon_{j_k}\searrow 0}\Big(-\frac{1}{2}\int_\Omega c_{\varepsilon_{j_k}}^2(\cdot,T) + \frac{1}{2}\int_\Omega c_0^2 - \int_0^T\int_\Omega c_{\varepsilon_{j_k}}^2 + \int_0^T\int_\Omega n_{\varepsilon_{j_k}}c_{\varepsilon_{j_k}}\Big)\\ &= \lim_{\varepsilon_{j_k}\searrow 0}\int_0^T\int_\Omega |\nabla c_{\varepsilon_{j_k}}|^2,\end{aligned}$$

which together with the fact that the norm in $L^2\big(\Omega\times(0,T);\mathbb{R}^3\big)$ is weakly lower semi-continuous and the weak convergence property in (3.5.6) implies that actually $\nabla c_\varepsilon \to \nabla c$ in $L^2\big(\Omega\times(0,T);\mathbb{R}^3\big)$ as $\varepsilon = \varepsilon_{j_k} \searrow 0$. $\square$

Finally, as a last step before proving Theorem 3.2, we can verify the Φ-supersolution property of Definition 3.4 for the choice of $\Phi(s) = (s+1)^{m+2\alpha-1}$ whenever $m \ge 1$ and $\alpha \ge 0$ satisfy $m+\alpha > \frac{4}{3}$ and $m+2\alpha < 2$. The restriction $m+2\alpha < 2$, however, is of no consequence for our Theorem, since for $m \ge 1$ and $\alpha \ge 0$ with $m+\alpha > \frac{4}{3}$ and $m+2\alpha \ge 2$, the existence of a global very weak solution is already established by Theorem 3.1 in light of the fact that every weak solution is also a very weak solution.

Lemma 3.24.
Let $m \geq 1$, $\alpha \geq 0$ satisfy $m+\alpha > \frac{4}{3}$ and $m+2\alpha < 2$. Assume that n_0, c_0, u_0 comply with (3.1.8) *and suppose that $S \in C^2\big(\overline{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times 3}\big)$ fulfills* (3.1.3) *with some $S_0 > 0$. Moreover, denote by n, c, u the limit functions provided by Proposition 3.19 and let $\Phi(s) := (s+1)^{m+2\alpha-1}$ for $s \geq 0$. Then n is a global Φ–supersolution of* (3.1.4) *in the sense of Definition 3.4.*

Proof: Because of $m+2\alpha < 2$ we may draw on the special case convergences discussed in Proposition 3.19, i.e. (3.5.11)–(3.5.16). With $\Phi(s) := (s+1)^{m+2\alpha-1}$ for $s \geq 0$, we find that the regularity requirements demanded by Definition 3.4 were already obtained in Proposition 3.19. In particular, we find that the conditions concerning n contained in (3.2.2) are implied by (3.5.12), (3.5.16), (3.5.13) together with (3.5.14), (3.5.15), (3.5.11) and (3.5.13), respectively, where we also used the fact that $\frac{n(n+1)^{\alpha-1}}{(1+n)^{\alpha}} \in L^{\infty}_{loc}\big(\overline{\Omega}\times[0,\infty)\big)$. What remains is the verification of (3.2.3). We pick an arbitrary nonnegative test function $\varphi \in C_0^{\infty}\big(\overline{\Omega}\times[0,\infty)\big)$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega\times(0,\infty)$ and then fix $T > 0$ such that $\varphi \equiv 0$ in $\Omega\times(T,\infty)$. Keeping in mind that $m+2\alpha < 2$, we multiply the first equation of (3.3.2) with $(m+2\alpha-1)(n_\varepsilon+1)^{m+2\alpha-2}\varphi$, integrate by parts and rewrite some terms to obtain that

$$
\begin{aligned}
&-\int_0^T\!\!\int_\Omega (n_\varepsilon+1)^{m+2\alpha-1}\varphi_t - \int_\Omega (n_0+1)^{m+2\alpha-1}\varphi(\cdot,0)\\
&= m(m+2\alpha-1)(2-(m+2\alpha))\int_0^T\!\!\int_\Omega \Big|(n_\varepsilon+1)^{\frac{m+2\alpha-3}{2}}(n_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\nabla n_\varepsilon\Big|^2\varphi\\
&\quad -\frac{m(m+2\alpha-1)}{m+\alpha-1}\int_0^T\!\!\int_\Omega (n_\varepsilon+1)^{\alpha}(n_\varepsilon+\varepsilon)^{m-1}\big(\nabla(n_\varepsilon+1)^{m+\alpha-1}\cdot\nabla\varphi\big) \qquad (3.6.6)\\
&\quad -\frac{(m+2\alpha-1)(2-(m+2\alpha))}{m+\alpha-1}\int_0^T\!\!\int_\Omega \frac{(n_\varepsilon+1)^{\alpha-1}n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}\big(\nabla(n_\varepsilon+1)^{m+\alpha-1}\cdot S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\big)\varphi\\
&\quad +(m+2\alpha-1)\int_0^T\!\!\int_\Omega (n_\varepsilon+1)^{m+\alpha-1}\frac{(n_\varepsilon+1)^{\alpha-1}n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}\big(S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi\big)\\
&\quad +\int_0^T\!\!\int_\Omega (n_\varepsilon+1)^{m+2\alpha-1}(u_\varepsilon\cdot\nabla\varphi)
\end{aligned}
$$

holds for all $\varepsilon \in (0,1)$. Making use of (3.1.3), we find that $\big|\frac{(n_\varepsilon+1)^{\alpha-1}n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}S_\varepsilon\big| \leq S_0$ for all $\varepsilon \in (0,1)$. Since moreover,

$$
\frac{(n_\varepsilon+1)^{\alpha-1}n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon) \to (n+1)^{\alpha-1}nS(\cdot,n,c) \quad \text{a.e. in } \Omega\times(0,\infty) \text{ as } \varepsilon\searrow 0
$$

we find that

$$
\frac{(n_\varepsilon+1)^{\alpha-1}n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \to (n+1)^{\alpha-1}nS(\cdot,n,c)\nabla c \quad \text{in } L^2\big(\Omega\times(0,T);\mathbb{R}^3\big)
$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, in light of Lemma 3.23 and [101, Lemma A.4]. Combining this strong convergence with (3.5.11) and (3.5.13) entails that

$$\int_0^T\int_\Omega (n_\varepsilon+1)^{m+\alpha-1}\frac{(n_\varepsilon+1)^{\alpha-1}n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}\big(S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi\big) \\ \to \int_0^T\int_\Omega (n+1)^{m+2\alpha-2}n\big(S(\cdot,n,c)\nabla c\cdot\nabla\varphi\big)$$

and

$$-\int_0^T\int_\Omega \frac{(n_\varepsilon+1)^{\alpha-1}n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3}\big(\nabla(n_\varepsilon+1)^{m+\alpha-1}\cdot S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\big)\varphi \\ \to -\int_0^T\int_\Omega (n+1)^{\alpha-1}n\big(\nabla(n+1)^{m+\alpha-1}\cdot S(\cdot,n,c)\nabla c\big)\varphi$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, respectively. Moreover, relying on (3.5.12), (3.5.13), (3.5.14) and (3.5.15) we obtain that

$$-\int_0^T\int_\Omega (n_\varepsilon+1)^{m+2\alpha-1}\varphi_t \to -\int_0^T\int_\Omega (n+1)^{m+2\alpha-1}\varphi_t,$$
$$-\int_0^T\int_\Omega (n_\varepsilon+1)^{\alpha}(n_\varepsilon+\varepsilon)^{m-1}\big(\nabla(n_\varepsilon+1)^{m+\alpha-1}\cdot\nabla\varphi\big) \\ \to -\int_0^T\int_\Omega (n+1)^{\alpha}n^{m-1}\big(\nabla(n+1)^{m+\alpha-1}\cdot\nabla\varphi\big)$$
$$\int_0^T\int_\Omega (n_\varepsilon+1)^{m+2\alpha-1}(u_\varepsilon\cdot\nabla\varphi) \to \int_0^T\int_\Omega n^{m+2\alpha-1}(u\cdot\nabla\varphi)$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$. Lastly, we depend on the lower semicontinuity of the norm in $L^2\big(\Omega\times(0,T);\mathbb{R}^3\big)$ with respect to weak convergence to conclude from (3.5.16) that

$$\liminf_{\varepsilon_{j_k}\searrow 0}\int_0^T\int_\Omega \big|(n_\varepsilon+1)^{\frac{m+2\alpha-3}{2}}(n_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\nabla n_\varepsilon\big|^2\varphi \geq \int_0^T\int_\Omega \big|(n+1)^{\frac{m+2\alpha-3}{2}}n^{\frac{m-1}{2}}\nabla n\big|^2\varphi.$$

Uniting the statements above with (3.6.6) and the fact that $m+2\alpha<2$ entails that

$$\begin{aligned}
&-\int_0^\infty\int_\Omega (n+1)^{m+2\alpha-1}\varphi_t - \int_\Omega (n_0+1)^{m+2\alpha-1}\varphi(\cdot,0)\\
&\geq m(m+2\alpha-1)(2-(m+2\alpha))\int_0^\infty\int_\Omega \big|(n+1)^{\frac{m+2\alpha-3}{2}}n^{\frac{m-1}{2}}\nabla n\big|^2\varphi\\
&-\frac{m(m+2\alpha-1)}{m+\alpha-1}\int_0^\infty\int_\Omega (n+1)^{\alpha}n^{m-1}\big(\nabla(n+1)^{m+\alpha-1}\cdot\nabla\varphi\big)\\
&-\frac{(m+2\alpha-1)(2-(m+2\alpha))}{m+\alpha-1}\int_0^\infty\int_\Omega (n+1)^{\alpha-1}n\big(\nabla(n+1)^{m+\alpha-1}\cdot S(\cdot,n,c)\nabla c\big)\varphi\\
&+(m+2\alpha-1)\int_0^\infty\int_\Omega (n+1)^{m+2\alpha-2}n\big(S(\cdot,n,c)\nabla c\cdot\nabla\varphi\big)+\int_0^\infty\int_\Omega (n+1)^{m+2\alpha-1}(u\cdot\nabla\varphi),
\end{aligned}$$

where we used that $\varphi \equiv 0$ in $\Omega \times (T, \infty)$. It can easily be checked that this is an equivalent formulation of (3.2.3) for our choice $\Phi(s) \equiv (s+1)^{m+2\alpha-1}$, which thereby completes the proof. $\square$

The previous lemma at hand, we can conclude Theorem 3.2 in a straightforward manner.

Proof of Theorem 3.2: The existence of a global very weak solution for $m \geq 1$ and $\alpha \geq 0$ satisfying $m + 2\alpha > \frac{5}{3}$ is already established in light of Theorem 3.1, since any global weak solution is also a global very weak solution for the choice $\Phi(s) \equiv s$. Evidently, we can restrict ourselves to verifying the Φ–supersolution property of Definition 3.4 for $m \geq 1$, $\alpha \geq 0$ satisfying $m + \alpha > \frac{4}{3}$ and $m + 2\alpha \leq \frac{5}{3}$. In this case Lemma 3.24 is applicable and therefore, an evident combination of Lemmata 3.20 and 3.24 with the regularity information presented in Proposition 3.19 completes the proof. $\square$

4 Eventual smoothness of generalized solutions to a singular chemotaxis-Stokes system in 2D

4.1 Introduction

Even among the smallest and most primitive organisms there are cases of complex and macroscopic collective behavior, for instance bacteria of species *E. coli* were confirmed to form migrating bands when subjected to a test environment featuring gradients of nutrient concentration ([1]). Following these experimental findings, chemotaxis systems of the form

$$\begin{cases} n_t = \Delta n - \nabla\cdot (nS(n,c)\nabla c), \\ c_t = \Delta c - nc, \end{cases} \tag{4.1.1}$$

were among the first phenomenological models proposed by Keller and Segel ([42]) to study these processes of chemotactic migration. Herein, in contrast to the models discussed in the previous chapters, the bacteria orient their movements towards a substance which serves as their food source and is thereby consumed in the process. In the mentioned reference the prototypical choice for $S(n,c)$ was the singular $S(n,c) = \frac{1}{c}$, modeling the assumption that the signal is perceived in accordance with the Weber-Fechner law ([42, 34]). An outstanding facet of this system, as already illustrated in [42], is the occurrence of wave-like solution behavior without any type of cell kinetics, which is known to be vital for such effects in standard reaction-diffusion equations. For studies on existence and stability properties of traveling wave solutions of (4.1.1), see [92, 51, 65] and references therein.

The results on global existence to systems of the form (4.1.1) are very sparse, with widely arbitrary initial data only being treated for the one-dimensional case ([84, 50]). In higher dimensions, the results were constrained to the Cauchy problem for (4.1.1) in $\mathbb{R}^n$ with $n \in \{2,3\}$, where smallness conditions on the initial data had to be imposed to show the existence of globally defined classical solutions ([93]). Only recently ([102]), so called global generalized solutions to (4.1.1) were constructed in the two-dimensional case. The solutions are obtained through the study of a suitably chosen regularization, guaranteeing that the regularized chemical concentration is strictly bounded away from zero for all times. These generalized solutions comply with the classical solution concept in the sense that generalized solutions which are sufficiently smooth also solve the system in the classical sense. In a sequel to the previously mentioned work, it was furthermore proved that if the initial mass is small, these generalized solutions eventually become classical solutions after some (possibly large) waiting time and that the solutions satisfy certain kind of asymptotic properties ([103]).

Eventual regularity and fluid interaction. Our interest slightly differing from the system proposed by Keller and Segel, we will consider the case that the bacteria may be affected by their liquid environment. Let us first recall the prototypical model developed in [85] to describe the experimental evidence of spontaneously emerging turbulence in populations of aerobic bacteria suspended in sessile drops of water. The proposed system, which not only incorporates the interaction by means of transport, but also in form of a feedback between cells and fluid-velocity stemming from a buoyancy effect, is of the form

$$\begin{cases} n_t + \quad u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c), \\ c_t + \quad u \cdot \nabla c &= \Delta c - nc, \\ u_t + \kappa (u \cdot \nabla) u &= \Delta u - \nabla P + n \nabla \phi, \\ \qquad \nabla \cdot u &= 0, \end{cases} \tag{4.1.2}$$

and has been the groundwork for many articles concerning the mathematical analysis of chemotaxis-fluid interaction since the first analytical results asserting local existence of weak solutions ([56]). Obtaining results concerning the global existence of solutions in this setting, however, is far from trivial. Even in the more favorable setting with $u \equiv 0$ the global existence of classical solutions is only known under a smallness condition on the initial data ([77]), or when $N = 2$ (e.g. [98]). The currently known results read similar in the case of $u \not\equiv 0$. In the two-dimensional setting, global classical solutions stemming from reasonably smooth initial data have also been shown to exist in [98], whereas many results treating variants of (4.1.2) in three-dimensional frameworks are again restricted to weak solutions emanating from small initial data (e.g. [44, 13]). Nevertheless, even in theses cases, where global regularity is hard to prove, some results concerning eventual regularity of solutions have been shown. In particular, for the fluid-free case eventual smoothness of solutions was shown in [81] for $N = 3$ and a result including fluid is contained in [105], where certain weak eventual energy solutions are considered.

Similar smoothing effects can also be observed in a setting where $N = 3$ and logistic growth terms of the form $+\rho n - \mu n^2$ ($\rho \geq 0, \mu > 0$) are included in the first equation. In this framework it is still unclear whether global classical solutions exist for small $\mu > 0$ and reasonably arbitrary initial data, but weak solutions which eventually become smooth are known to exist for any $\mu > 0$ and possibly large initial data, as indicated by the studies in e.g. [47].

Chemotaxis-fluid system with singular sensitivity. In light of the regularizing effects observed in the chemotaxis and chemotaxis-fluid problems mentioned above, it seems reasonable to assume that also in the case of singular sensitivity the smoothing effect of the second equation will eventually result in classical solutions, even if fluid interaction with the bacteria is present. As the construction of global weak solutions used in [87] does not work for the full Navier–Stokes subsystem (as included in (4.1.2)), we instead work with the simpler Stokes realization of the fluid, which was also employed in [87], instead. In fact we will study systems of the form

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot\left(\frac{n}{c}\nabla c\right), & x\in\Omega, \quad t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega, \quad t>0,\\ u_t + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega, \quad t>0,\\ \nabla\cdot u = 0, & x\in\Omega, \quad t>0, \end{cases} \tag{4.1.3}$$

with boundary conditions

$$\frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad \text{and} \quad u = 0 \quad \text{for } x\in\partial\Omega \text{ and } t>0, \tag{4.1.4}$$

and initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \quad x\in\Omega. \tag{4.1.5}$$

$\Omega\subset\mathbb{R}^2$ denotes a bounded domain with smooth boundary and the gravitational potential ϕ is assumed to satisfy

$$\phi\in C^2\big(\overline{\Omega}\big) \quad \text{with} \quad K_1 := \|\phi\|_{W^{1,\infty}(\Omega)}. \tag{4.1.6}$$

For the initial distributions we will prescribe the regularity assumptions

$$\begin{cases} n_0\in C^0\big(\overline{\Omega}\big) & \text{with } n_0\geq 0 \text{ in } \Omega \text{ and } n_0\not\equiv 0,\\ c_0\in W^{1,\infty}(\Omega) & \text{with } c_0>0 \text{ in } \overline{\Omega},\\ u_0\in D\big(A_r^{\varrho}\big) & \text{for all } r\in(1,\infty) \text{ and some } \varrho\in(\frac{1}{2},1), \end{cases} \tag{4.1.7}$$

with A_r denoting the Stokes operator $A_r := -\mathcal{P}_r\Delta$ in $L^r\big(\Omega;\mathbb{R}^2\big)$ with domain $D\left(A_r\right) = W^{2,r}\big(\Omega;\mathbb{R}^2\big)\cap W_0^{1,r}\big(\Omega;\mathbb{R}^2\big)\cap L_\sigma^r(\Omega)$, where $L_\sigma^r(\Omega) = \{\varphi\in L^r\big(\Omega;\mathbb{R}^2\big)\mid \nabla\cdot\varphi = 0\}$ stands for the solenoidal subspace of $L^r\big(\Omega,\mathbb{R}^2\big)$ obtained by the Helmholtz projection $\mathcal{P}_r$.
In this setting, building on the work [102], it was shown in [87] that for any (n_0,c_0,u_0) satisfying (4.1.7), the system (4.1.3) possesses at least on global generalized solution (in the sense of Definition 4.8 below). These solutions are constructed by a similar limiting procedure as in the fluid free setting, making sure that for each of the approximate solutions the quantity c remains strictly positive throughout Ω for all times. In a simplified version the result one global existence of generalized solutions and basic decay properties of c obtained in [87] can be summarized as follows:

Theorem A.
Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary. Then for all (n_0,c_0,u_0) satisfying (4.1.7), the problem (4.1.3)–(4.1.5) possesses at least one global generalized solution (n,c,u) in the sense of Definition 4.8 below. For each $p\in[1,\infty)$ the solution has the properties that $n(\cdot,t)\in L^p(\Omega)$ and $\frac{\nabla c(\cdot,t)}{c(\cdot,t)}\in L^2\big(\Omega;\mathbb{R}^2\big)$ for a.e. $t>0$. Moreover, c is continuous on $[0,\infty)$ as $L^\infty(\Omega)$–valued function with respect to the weak–$\star$ topology on $L^\infty(\Omega)$, and satisfies

$$c(\cdot,t)\overset{\star}{\rightharpoonup}0 \quad \text{in } L^\infty(\Omega) \qquad \text{and} \qquad c(\cdot,t)\to 0 \quad \text{in } L^p(\Omega) \qquad \text{as } t\to\infty.$$

Main results. The existence of global generalized solutions provided by Theorem A at hand, it is the purpose of the present chapter to study the question how far the eventual regularity and stabilization results for small data as obtained in [103] for (4.1.1), may be affected by the interaction of the bacteria with their liquid surroundings.

Theorem 4.1.
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then there exists some $m_\star > 0$ such that for any (n_0, c_0, u_0) satisfying (4.1.7) *as well as*

$$\int_\Omega n_0 \le m_\star, \tag{4.1.8}$$

the global generalized solution of (4.1.3)–(4.1.5) *from Theorem A has the property that there exists $T > 0$ such that*

$$n \in C^{2,1}\big(\overline{\Omega}\times[T,\infty)\big), \quad c \in C^{2,1}\big(\overline{\Omega}\times[T,\infty)\big) \quad \textit{and} \quad u \in C^{2,1}\big(\overline{\Omega}\times[T,\infty);\mathbb{R}^2\big), \tag{4.1.9}$$

that

$$c(x,t) > 0 \quad \textit{for all } x \in \overline{\Omega} \textit{ and any } t \ge T, \tag{4.1.10}$$

and such that (n, c, u) solve (4.1.3)–(4.1.5) *classically in $\Omega \times (T, \infty)$. Furthermore, this solution satisfies*

$$n(\cdot,t) \to \frac{1}{|\Omega|}\int_\Omega n_0 \quad \textit{in } L^\infty(\Omega), \qquad c(\cdot,t) \to 0 \quad \textit{in } L^\infty(\Omega), \qquad u(\cdot,t) \to 0 \quad \textit{in } L^\infty(\Omega), \tag{4.1.11}$$

and

$$\frac{\nabla c(\cdot,t)}{c(\cdot,t)} \to 0 \quad \textit{in } L^\infty\big(\Omega;\mathbb{R}^2\big) \tag{4.1.12}$$

as $t \to \infty$.

Our analysis will also in straightforward manner allow us to formulate a result for global classical solutions to (4.1.3)–(4.1.5) if certain smallness conditions are fulfilled by the initial distributions. Furthermore, these global classical solutions inherit the same asymptotic properties stated in Theorem 4.1. In order to completely formulate this outcome, we note that in two-dimensional domains by the Gagliardo–Nirenberg inequality and elliptic regularity theory one can find $K_2 > 0$ and $K_3 > 0$ such that

$$\|\varphi\|_{L^3(\Omega)}^3 \le K_2 \|\varphi\|_{W^{1,2}(\Omega)}^2 \|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in W^{1,2}(\Omega) \tag{4.1.13}$$

and

$$\|\nabla\varphi\|_{L^4(\Omega)}^4 \le K_3 \|\Delta\varphi\|_{L^2(\Omega)}^2 \|\nabla\varphi\|_{L^2(\Omega)}^2 \quad \text{for all } \varphi \in W^{2,2}(\Omega) \text{ with } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\,. \tag{4.1.14}$$

We obtain the following:

Theorem 4.2.
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then there exists $m_{\star\star} > 0$ such that for any (n_0, c_0, u_0) satisfying (4.1.7)*,*

$$\int_\Omega n_0 \le m_{\star\star}, \quad \text{and} \quad \int_\Omega |u_0|^4 \le m_{\star\star} \tag{4.1.15}$$

as well as

$$\int_\Omega n_0 \ln \frac{n_0}{\mu} + \frac{1}{2}\int_\Omega \frac{|\nabla c_0|^2}{c_0^2} < \min\left\{\frac{1}{4K_3}, \frac{1}{8K_2}\right\} - \frac{\mu|\Omega|}{e} \tag{4.1.16}$$

for some $\mu > 0$ and K_2, K_3 given by (4.1.13) *and* (4.1.14)*, respectively, there exists a triple (n, c, u) of functions, for each $\vartheta > 2$ uniquely determined by the inclusions*

$$\begin{cases} n \in C^0\big(\overline{\Omega}\times[0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty)\big), \\ c \in C^0\big(\overline{\Omega}\times[0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty)\big) \cap L^\infty_{loc}\big([0,\infty); W^{1,\vartheta}(\Omega)\big), \\ u \in C^0\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^2\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,\infty);\mathbb{R}^2\big), \end{cases}$$

such that $n > 0$ in $\overline{\Omega}\times(0,\infty)$ and $c > 0$ in $\overline{\Omega}\times[0,\infty)$, and such that (n, c, u) together with some $P \in C^{1,0}\big(\overline{\Omega}\times[0,\infty)\big)$ solve (4.1.3)*–*(4.1.5) *in the classical sense in $\Omega\times(0,\infty)$. Furthermore, this solution has the convergence properties stated in Theorem 4.1.*

In contrast to the known result for the system without fluid, obtained by taking $u \equiv 0$ in (4.1.3), where requiring only $\int_\Omega n_0 \ln \frac{n_0}{\mu} + \frac{1}{2}\int_\Omega \frac{|\nabla c_0|^2}{c_0^2}$ to be small was sufficient to obtain global classical solutions, in this case we require additional smallness conditions in the form of sufficiently small bounds for n_0 in $L^1(\Omega)$ and u_0 in $L^4(\Omega)$. Let us also briefly note that the approach utilized here can not be used to prove eventual smoothness of global generalized solutions in higher dimensions, mainly due to the Gagliardo-Nirenberg type inequalities (4.1.13) and (4.1.14). In particular, the functional $F_\mu(n, z) := \int_\Omega n \ln \frac{n}{\mu} + \frac{1}{2}\int_\Omega |\nabla z|^2$ (c.f. Sections 4.2.2 and 4.4.1) has to be nonincreasing for small mass (see Lemma 4.11 below), necessitating control on $\|\nabla z\|^4_{L^4(\Omega)}$ by $\|\Delta z\|^2_{L^2(\Omega)}\|\nabla z\|^2_{L^2(\Omega)}$ (c.f (4.4.6)), which is only possible in two dimensions. Similarly, problems stemming from dimension dependency of inequalities employed in the proofs also arise in Lemma 4.19. Moreover, one would also have to consider additional steps in order to control $\|u\|_{L^4(\Omega)}$ in Lemma 4.11 as Lemma 4.4 does not hold in higher dimensions.

Throughout the chapter, in addition to the previously mentioned assumptions in (4.1.6) and (4.1.7) for Ω, ϕ, the initial data, the Stokes operator and its semigroup, we will make use of the following notations. $\lambda_1 > 0$ will always denote the first positive eigenvalue of the Stokes operator in Ω with respect to homogeneous Dirichlet boundary data. Since $A_r^\varrho \varphi, e^{-tA_r}\varphi$ and $\mathcal{P}_r\psi$ are independent of $r \in (1,\infty)$ for $\varphi \in C_0^\infty(\Omega) \cap L^r_\sigma(\Omega)$ and $\psi \in C_0^\infty\big(\Omega;\mathbb{R}^2\big)$, we will drop the subscript whenever there is no danger of confusion. Similar to denoting by $L^r_\sigma(\Omega)$ all divergence free functions of $L^r(\Omega)$, the space of divergence free, smooth test functions with compact support in $\Omega \times (0,\infty)$ will be denoted by $C^\infty_{0,\sigma}(\Omega\times(0,\infty))$. Additionally, when talking about classical solutions to some of the featured systems in $\Omega\times(t_0,\infty)$ for some $t_0\geq 0$, we will often shorten the notation to $(n,c,u)\in C^0(\Omega\times[t_0,\infty))$, when we are actually considering $(n,c,u,P)\in C^0(\Omega\times[t_0,\infty))\times C^0(\Omega\times[t_0,\infty))\times C^0\big(\Omega\times[t_0,\infty);\mathbb{R}^2\big)\times C^{1,0}\big(\overline{\Omega}\times[t_0,\infty)\big)$. The notation $(n,c,u)\in C^{2,1}(\Omega\times(t_0,\infty))$ will be used in a similar fashion.

4.2 Basic properties of a family of generalized problems

The construction of the generalized solution mentioned above is based on a limit procedure of solutions to regularized problems and a transformation thereof. Since the original problem (4.1.3) and the family of approximate problems in question are quite similar, we will first consider the even more general family of problems

$$\begin{cases} n_t+u\cdot\nabla n=\Delta n-\nabla\cdot\left(\frac{nf'(n)}{c}\nabla c\right), & x\in\Omega,\quad t>0\\ c_t+u\cdot\nabla c=\Delta c-f(n)c, & x\in\Omega,\quad t>0,\\ u_t+\ \nabla P\ =\Delta u+n\nabla\phi, & x\in\Omega,\quad t>0,\\ \qquad\nabla\cdot u\ =0, & x\in\Omega,\quad t>0,\end{cases}\tag{4.2.1}$$

where we only require that the functions $f\in C^3([0,\infty))$ satisfy

$$f(0)=0\qquad\text{and}\qquad 0\leq f'\leq 1\text{ on }[0,\infty).\tag{4.2.2}$$

Upon proper choice of a subfamily of these functions (cf. (4.3.5) below), the system will be regularized in a way that ensures that c is bounded away from zero, from which one can easily obtain global and bounded solutions to the corresponding approximate problems. These global and bounded solutions are one of the main ingredients of the limit process involved in the construction of the generalized solution ([102, 87]).
The problems (4.2.1) will be regarded under the boundary conditions

$$\frac{\partial n}{\partial\nu}=\frac{\partial c}{\partial\nu}=0,\quad\text{and}\quad u=0\quad\text{for } x\in\partial\Omega\text{ and } t\in(0,T_{max}),\tag{4.2.3}$$

and the initial conditions

$$n(x,0)=n_0(x),\quad c(x,0)=c_0(x),\quad u(x,0)=u_0(x),\quad x\in\Omega.\tag{4.2.4}$$

For any $f\in C^3([0,\infty))$ satisfying the conditions above, local existence of classical solutions can be obtained by well-established fixed point methods. Since the necessary adaptions are quite straightforward, we will refer to local existence proofs in closely related situations for details.

Lemma 4.3.
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, $\vartheta > 2$ and suppose that $f \in C^3([0,\infty))$ satisfies (4.2.2). Then for all (n_0, c_0, u_0) satisfying (4.1.7) there exist $T_{max} \in (0,\infty]$ and uniquely determined functions

$$\begin{aligned} n &\in C^0\big(\overline{\Omega}\times[0,T_{max})\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T_{max})\big), \\ c &\in C^0\big(\overline{\Omega}\times[0,T_{max})\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T_{max})\big) \cap C^0\big([0,T_{max}); W^{1,\vartheta}(\Omega)\big), \\ u &\in C^0\big(\overline{\Omega}\times[0,T_{max}); \mathbb{R}^2\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,T_{max}); \mathbb{R}^2\big), \end{aligned}$$

which together with some $P \in C^{1,0}\big(\overline{\Omega}\times[0,T_{max})\big)$ solve (4.2.1)–(4.2.4) in the classical sense and satisfy $n > 0$ and $c > 0$ in $\overline{\Omega}\times(0,T_{max})$ as well as

$$\text{either } T_{max} = \infty, \text{ or } \liminf_{t\nearrow T_{max}} \inf_{x\in\Omega} c(x,t) = 0, \tag{4.2.5}$$
$$\text{or } \limsup_{t\nearrow T_{max}} \Big(\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} + \|A^\varrho u(\cdot,t)\|_{L^2(\Omega)} \Big) = \infty.$$

Furthermore, the solution has the properties that

$$\int_\Omega n(x,t)\,\mathrm{d}x = \int_\Omega n_0(x)\,\mathrm{d}x \quad \text{for all } t \in (0,T_{max}) \tag{4.2.6}$$

and

$$c(x,t) \le \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } (x,t) \in \overline{\Omega}\times[0,T_{max}). \tag{4.2.7}$$

Proof: Local existence, uniqueness and the blow-up criterion (4.2.5) can be obtained by straightforward adaptation of well-known arguments as detailed in [36, 26, 25] and [98] for related situations. Simple integration of the first equation in (4.2.1) proves (4.2.6), whereas by the nonnegativity of f, an application of the parabolic comparison principle to the second equation in (4.2.1), with $\bar{c} \equiv \|c_0\|_{L^\infty(\Omega)}$ taken as supersolution, immediately entails (4.2.7). $\square$

4.2.1 Regularity of the Stokes subsystem

It is known that the Stokes subsystem $\frac{\mathrm{d}}{\mathrm{d}t}u + Au = \mathcal{P}(n\nabla\phi)$ in (4.2.1) has the property that the regularity of the spatial derivative ∇u is solely reliant on the regularity of n (since $\nabla\phi$ is bounded). In fact, for Stokes systems of the form

$$\begin{cases} u_t = \Delta u - \nabla P + n\nabla\phi, & x \in \Omega, \quad t_0 > 0, \\ \nabla\cdot u = 0, & x \in \Omega, \quad t_0 > 0, \\ u = 0, & x \in \partial\Omega, \quad t_0 > 0, \end{cases} \tag{4.2.8}$$

we can obtain the following two results. The first of which is a refinement of a basic boundedness result e.g. featured in [89, Lemma 2.4].

Lemma 4.4.
Let $\phi \in C^2(\overline{\Omega})$. There exist constants $\lambda_1 > 0$ and $K_u > 0$ such that whenever $u \in C^0(\overline{\Omega} \times [t_0, T_0); \mathbb{R}^2) \cap C^{2,1}(\overline{\Omega} \times (t_0, T_0); \mathbb{R}^2)$ is a classical solution of (4.2.8) in $\Omega \times (t_0, T_0)$ for some $0 \le t_0 < T_0 \le \infty$ and $n \in C^0(\overline{\Omega} \times [t_0, T_0))$ satisfies

$$\int_\Omega |n(\cdot,t)| \le L \quad \text{for all } t \in (t_0, T_0),$$

with some $L > 0$, then

$$\|u(\cdot,t)\|_{L^4(\Omega)} \le K_u e^{-\lambda_1(t-t_0)} \|u(\cdot,t_0)\|_{L^4(\Omega)} + K_u L \quad \text{for all } t \in (t_0, T_0).$$

Proof: By the variation-of-constants representation for u we have

$$u(\cdot,t) = e^{-(t-t_0)A} u(\cdot,t_0) + \int_{t_0}^t e^{-(t-s)A} \mathcal{P}(n(\cdot,s)\nabla\phi)\,\mathrm{d}s \quad \text{for all } t \in (t_0, T_0).$$

Fixing any $\gamma \in (\frac{3}{4}, 1)$ we see that

$$\|u(\cdot,t)\|_{L^4(\Omega)} \le \|e^{-(t-t_0)A} u(\cdot,t_0)\|_{L^4(\Omega)} + \int_{t_0}^t \|A^\gamma e^{-(t-s)A} A^{-\gamma} \mathcal{P}(n(\cdot,s)\nabla\phi)\|_{L^4(\Omega)}\,\mathrm{d}s$$

holds for all $t \in (t_0, T_0)$. Now, in view of the well known regularity estimates for the Stokes semigroup (e.g. [100, Lemma 3.1]) we find constants $\lambda_1 > 0$ and $C_1 > 0$ such that

$$\|e^{-(t-t_0)A} u(\cdot,t_0)\|_{L^4(\Omega)} \le C_1 e^{-\lambda_1(t-t_0)} \|u(\cdot,t_0)\|_{L^4(\Omega)} \quad \text{for all } t > t_0,$$

and, since for $1 \le p < q < \infty$ and $\gamma \in (0,1)$ satisfying $\gamma > \frac{1}{p} - \frac{1}{q}$ it holds that $\|A^{-\gamma}\mathcal{P}\varphi\|_{L^q(\Omega)} \le C\|\varphi\|_{L^p(\Omega)}$ for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^2)$ ([89, Lemma 2.3]), there exists $C_2 > 0$ such that

$$\|A^\gamma e^{-(t-s)A} A^{-\gamma} \mathcal{P}(n(\cdot,s)\nabla\phi)\|_{L^4(\Omega)} \le C_2 (t-s)^{-\gamma} e^{-\lambda_1(t-s)} \|n(\cdot,s)\nabla\phi\|_{L^1(\Omega)}$$

for all $s \in (t_0, t)$, by choice of $\gamma \in (\frac{3}{4}, 1)$. Hence, relying on (4.1.6) and our assumption for $\int_\Omega |n(\cdot,t)|$, we may estimate

$$\|u(\cdot,t)\|_{L^4(\Omega)} \le C_1 e^{-\lambda_1(t-t_0)} \|u(\cdot,t_0)\|_{L^4(\Omega)} + C_2 K_1 L \int_0^\infty \sigma^{-\gamma} e^{-\lambda_1\sigma}\,\mathrm{d}\sigma$$

for all $t > 0$, which due to $\gamma < 1$ concludes the proof upon obvious choice for K_u. □

The second lemma regarding the Stokes subsystem concerns norms of the spatial gradient of u. These results are widely recognized, see e.g. [89, Lemma 2.5] and [100, Corollary 3.4] for details.

Lemma 4.5.
Assume $\varrho \in (\frac{1}{2}, 1)$, $t_0 \geq 0$ and $\phi \in C^2(\overline{\Omega})$ and let $p \in [1,\infty)$ and $r \in [1,\infty]$ be such that

$$\begin{cases} r < \frac{2p}{2-p} & \text{if } p \leq 2, \\ r \leq \infty & \text{if } p > 2. \end{cases}$$

Then for any $u(\cdot,t_0) \in D(A_r^\varrho)$ there exists a constant $C = C(u(\cdot,t_0),\phi,p,r,L) > 0$ such that whenever $u \in C^0(\overline{\Omega} \times [t_0,T_0);\mathbb{R}^2) \cap C^{2,1}(\overline{\Omega} \times (t_0,T_0);\mathbb{R}^2)$ is a classical solution of (4.2.8) in $\Omega \times (t_0,T_0)$ for some $0 \leq t_0 < T_0 \leq \infty$ and $n \in C^0(\overline{\Omega} \times [t_0,T_0))$ satisfies

$$\|n(\cdot,t)\|_{L^p(\Omega)} \leq L \quad \text{for all } t \in (t_0,T_0),$$

with some $L > 0$, then

$$\|\nabla u(\cdot,t)\|_{L^r(\Omega)} \leq Ce^{-\lambda_1(t-t_0)} + CL \quad \text{for all } t \in (t_0,T_0).$$

In particular, taking the mass conservation property of n and the Sobolev embedding theorem into consideration, we can easily obtain bounds independent of f for the quantity $\|u\|_{L^p(\Omega)}$ with $p < \infty$ from the previous lemma. For these potentially better bounds than the one provided by Lemma 4.4 however, we do not know the exact relation to $u(\cdot,t_0)$.

4.2.2 Logarithmic rescaling and basic a priori information on z

Now, a quite standard change in variables transformation obtained by taking n, c and u from Lemma 4.3 and setting

$$z := -\ln\left(\frac{c}{\|c_0\|_{L^\infty(\Omega)}}\right) \quad \text{and} \quad z_0 := -\ln\left(\frac{c_0}{\|c_0\|_{L^\infty(\Omega)}}\right),$$

will lead to the transformed systems

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n + \nabla\cdot(nf'(n)\nabla z), & x \in \Omega, \quad t > 0, \\ z_t + u\cdot\nabla z = \Delta z - |\nabla z|^2 + f(n), & x \in \Omega, \quad t > 0, \\ u_t + \nabla P = \Delta u + n\nabla\phi, & x \in \Omega, \quad t > 0, \\ \nabla\cdot u = 0, & x \in \Omega, \quad t > 0, \end{cases} \tag{4.2.9}$$

building the basis for our analysis of the energy-type inequalities featured in Section 4.4.1. This transformation has been thoroughly used in previous literature (see e.g. [93, 102, 103]) to analyze systems in similar settings. We will consider (4.2.9) along with the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \quad \text{and} \quad u = 0 \quad \text{for } x \in \partial\Omega \text{ and } t > 0, \tag{4.2.10}$$

and initial conditions

$$n(x,0) = n_0(x), \quad z(x,0) = z_0(x) := -\ln\left(\frac{c_0(x)}{\|c_0\|_{L^\infty(\Omega)}}\right), \quad u = u_0(x), \quad x \in \Omega.$$

Remark 4.6.
Let $f \in C^3([0,\infty))$ satisfy (4.2.2). Assume that $(n,z,u) \in C^{2,1}\big(\overline{\Omega} \times (T_1,T_2)\big)$ is a classical solution of the boundary value problem (4.2.9),(4.2.10) in $\Omega \times (T_1,T_2)$ with some $T_1 \geq 0$ and $T_2 \in (T_1,\infty]$. Then the solution satisfies the mass conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n(\cdot,t) = 0 \quad \textit{for all } t \in (T_1,T_2).$$

This reformulation of our previous systems at hand, we immediately obtain the following basic information – not depending on f – about the transformed chemical concentration z.

Lemma 4.7.
Let $m_0 > 0$. Suppose that for $f \in C^3([0,\infty))$ satisfying (4.2.2) and $t_0 \geq 0$ the triple $(n,z,u) \in C^{2,1}\big(\overline{\Omega} \times (t_0,\infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega \times (t_0,\infty)$ with the properties that $n \geq 0$ in $\Omega \times (t_0,\infty)$ and $\int_\Omega n(\cdot,t_0) \leq m_0$. Then

$$\int_\Omega z(\cdot,t) + \int_{t_0}^t\int_\Omega |\nabla z|^2 \leq \int_\Omega z(\cdot,t_0) + (t-t_0)m_0 \quad \textit{for all } t > t_0. \tag{4.2.11}$$

Proof: Integrating the second equation of (4.2.9) with respect to space shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z = \int_\Omega \Delta z - \int_\Omega |\nabla z|^2 + \int_\Omega f(n) - \int_\Omega u\cdot\nabla z$$

holds for all $t \in (t_0,\infty)$. Making use of $\nabla\cdot u = 0$, the Neumann boundary conditions for z, $n \geq 0$ and the fact that $f(s) \leq s$ for all $s \geq 0$ we obtain, upon integration by parts, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z + \int_\Omega |\nabla z|^2 \leq \int_\Omega n$$

is valid on $t \in (t_0,\infty)$. Due to the mass conservation we have $\int_\Omega n(\cdot,t) \leq m_0$ for all $t > t_0$ and therefore integrating this inequality immediately establishes (4.2.11). $\square$

4.3 Generalized solution concept and approximate solutions

Before going into more detail for our eventual smoothness result, let us briefly review the solution concept of generalized solutions and the exact form of the approximate problems. These were already used in [101, 102] for closely related settings without fluid and in [87] for the system with Stokes fluid.
A global generalized solution is defined as follows (see also [101, Definition 2.1–2.3],[87, Definition 2.1]).

Definition 4.8.
Assume that (n_0,c_0,u_0) satisfy (4.1.7). Suppose that a triple (n,c,u) of functions

$$\begin{cases} n \in L^1_{loc}\big(\overline{\Omega} \times [0,\infty)\big)\,, \\ c \in L^\infty_{loc}\big(\overline{\Omega} \times [0,\infty)\big) \cap L^2_{loc}\big([0,\infty); W^{1,2}(\Omega)\big)\,, \\ u \in L^1_{loc}\big([0,\infty); W^{1,1}_0(\Omega;\mathbb{R}^2)\big), \end{cases} \tag{4.3.1}$$

satisfies

$$n \geq 0, \quad \textit{and} \quad c > 0, \quad \textit{and} \quad \nabla \cdot u = 0 \quad \textit{a.e. in } \Omega \times (0,\infty) \tag{4.3.2}$$

as well as

$$\nabla \ln(n+1) \in L^2_{loc}\big(\overline{\Omega} \times [0,\infty); \mathbb{R}^2\big) \qquad \textit{and} \qquad \nabla \ln c \in L^2_{loc}\big(\overline{\Omega} \times [0,\infty); \mathbb{R}^2\big). \tag{4.3.3}$$

Then (n, c, u) *will be called a global generalized solution of* (4.1.3)–(4.1.5) *if* n *satisfies the mass conservation property*

$$\int_\Omega n(x,t)\,\mathrm{d}x = \int_\Omega n_0(x)\,\mathrm{d}x \quad \textit{for a.e. } t > 0,$$

if the inequality

$$\begin{aligned}
&-\int_0^\infty \int_\Omega \ln(n+1)\varphi_t - \int_\Omega \ln(n_0+1)\varphi(\cdot,0) \\
&\geq \int_0^\infty \int_\Omega |\nabla \ln(n+1)|^2 \varphi - \int_0^\infty \int_\Omega \nabla \ln(n+1) \cdot \nabla\varphi + \int_0^\infty \int_\Omega \frac{n}{n+1} \nabla \ln c \cdot \nabla \varphi \\
&\quad - \int_0^\infty \int_\Omega \frac{n}{n+1} \left(\nabla \ln(n+1) \cdot \nabla \ln c\right) \varphi + \int_0^\infty \int_\Omega \ln(n+1)(u \cdot \nabla \varphi)
\end{aligned} \tag{4.3.4}$$

holds for each nonnegative $\varphi \in C_0^\infty\left(\overline{\Omega} \times [0,\infty)\right)$, *if the identity*

$$\int_0^\infty \int_\Omega c\psi_t + \int_\Omega c_0 \psi(\cdot,0) = \int_0^\infty \int_\Omega \nabla c \cdot \nabla \psi + \int_0^\infty \int_\Omega nc\psi - \int_0^\infty \int_\Omega cu \cdot \nabla\psi$$

is valid for any $\psi \in L^\infty(\Omega \times (0,\infty)) \cap L^2\left((0,\infty); W^{1,2}(\Omega)\right)$ *compactly supported in* $\overline{\Omega} \times [0,\infty)$ *with* $\psi_t \in L^2(\Omega \times (0,\infty))$, *and if furthermore the equality*

$$\int_0^\infty \int_\Omega u \cdot \Psi_t + \int_\Omega u_0 \cdot \Psi(\cdot,0) = \int_0^\infty \int_\Omega \nabla u \cdot \nabla \Psi - \int_0^\infty \int_\Omega n \nabla\phi \cdot \Psi$$

holds for all $\Psi \in C_{0,\sigma}^\infty\left(\Omega \times [0,\infty); \mathbb{R}^2\right)$.

It can easily be verified that the supersolution property in (4.3.4) combined with the mass conservation (4.2.6) is sufficient to obtain that sufficiently regular global generalized solutions are also global solutions in the classical sense (see [102, Remark 2.1 ii)]), i.e. if (n, c, u) is a global generalized solution in the sense of Definition 4.8 and satisfies $n \geq 0$ and $c > 0$ in $\overline{\Omega} \times [0,\infty)$ as well as $(n,c,u) \in C^0\big(\overline{\Omega} \times [0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega} \times (0,\infty)\big)$ then (n, c, u) solves (4.2.1) in the classical sense.

Generalized solutions in the sense of Definition 4.8 are constructed by an approximation procedure relying on regularizations in the form of (4.2.9) with suitably chosen $f \equiv f_\varepsilon$ ([102, 103, 87]). For this we first fix a nonincreasing cut-off function $\rho \in C^\infty([0,\infty))$ fulfilling $\rho \equiv 1$ in $[0,1]$ and $\rho \equiv 0$ in $[2,\infty)$ and define the family of functions $\{f_\varepsilon\}_{\varepsilon \in (0,1)} \subseteq C^\infty([0,\infty))$ given by

$$f_\varepsilon(s) := \int_0^s \rho(\varepsilon\sigma)\,\mathrm{d}\sigma, \qquad s \geq 0. \tag{4.3.5}$$

Every function in this family evidently has the properties

$$f_\varepsilon(0) = 0 \quad \text{ and } \quad 0 \le f'_\varepsilon \le 1 \quad \text{ on } [0,\infty) \tag{4.3.6}$$

as well as

$$f_\varepsilon(s) = s \quad \text{for all } s \in [0, \tfrac{1}{\varepsilon}] \quad \text{ and } \quad f'_\varepsilon(s) = 0 \quad \text{for all } s \ge \tfrac{2}{\varepsilon}.$$

Furthermore, it holds that

$$f_\varepsilon(s) \nearrow s \quad \text{and} \quad f'_\varepsilon(s) \nearrow 1 \quad \text{as } \varepsilon \searrow 0$$

for each $s \ge 0$. According to this choice, we can ensure that for the local solutions to (4.2.1)–(4.2.4) n_ε is bounded throughout $\Omega \times (0, T_{max})$, and that c_ε is strictly positive on $\overline{\Omega} \times (0, T_{max})$, meaning that the most troublesome terms of the extensibility criterion in (4.2.5) remain bounded, whence the further estimation of remaining less troublesome terms in fact shows that the solution is actually global ([87]).
Relying on the logarithmic transformation again we obtain for this family of regularizing functions, (4.2.9)–(4.2.10) systems of the form

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon + \nabla \cdot (n_\varepsilon f'_\varepsilon(n_\varepsilon) \nabla z_\varepsilon), & x \in \Omega, \quad t > 0, \\ z_{\varepsilon t} - u_\varepsilon \cdot \nabla z_\varepsilon = \Delta z_\varepsilon - |\nabla z_\varepsilon|^2 + f_\varepsilon(n_\varepsilon), & x \in \Omega, \quad t > 0, \\ u_{\varepsilon t} + \quad \nabla P_\varepsilon \quad = \Delta u_\varepsilon + n_\varepsilon \nabla \phi, & x \in \Omega, \quad t > 0, \\ \qquad \nabla \cdot u_\varepsilon \quad = 0, & x \in \Omega, \quad t > 0, \end{cases} \tag{4.3.7}$$

with boundary conditions

$$\frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial z_\varepsilon}{\partial \nu} = 0, \quad \text{and} \quad u_\varepsilon = 0 \quad \text{for } x \in \partial\Omega \text{ and } t > 0, \tag{4.3.8}$$

and initial conditions

$$n_\varepsilon(x,0) = n_0(x), \quad z_\varepsilon(x,0) = z_0(x) = -\ln\left(\frac{c_0(x)}{\|c_0\|_{L^\infty(\Omega)}}\right), \quad u_\varepsilon(x,0) = u_0(x), \quad x \in \Omega. \tag{4.3.9}$$

As reported by [87] also these problems possess global classical solutions, with again n_ε and z_ε being nonnegative, n_ε still satisfying the mass conservation property as in Remark 4.6 and $(n_\varepsilon, z_\varepsilon, u_\varepsilon)$ correspond to solutions of systems of the form (4.2.1) by means of the substitution $z_\varepsilon = -\ln\left(\frac{c_\varepsilon}{\|c_0\|_{L^\infty(\Omega)}}\right)$.
The following result summarizes the outcome on the approximation of the generalized solutions established in [87, Lemma 2.5].

Proposition 4.9.
Let (4.1.7) *hold and denote by* (n, c, u) *the global generalized solution of* (4.1.3)–(4.1.5) *from Theorem A. Then there exists a sequence* $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1)$ *such that* $\varepsilon_j \searrow 0$ *as* $j \to \infty$ *and such that, for the choice of* $f \equiv f_\varepsilon$ *in* (4.2.1), *the corresponding solution* $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ *of* (4.2.1)–(4.2.4) *satisfies*

$$n_\varepsilon \to n \qquad \textit{and} \qquad c_\varepsilon \to c \qquad \textit{as well as} \qquad u_\varepsilon \to u \qquad \textit{a.e. in } \Omega \times (0,\infty).$$

as $\varepsilon = \varepsilon_j \searrow 0$.

4.4 Eventual smoothness of small-data generalized solutions

4.4.1 Nonincreasing energy for small mass

We will appropriately adjust the functional methods employed in [103] to our needs. In fact, we will study the behavior of functionals of the form

$$F_\mu(n,z) := \int_\Omega n \ln \frac{n}{\mu} + \frac{1}{2}\int_\Omega |\nabla z|^2 \tag{4.4.1}$$

for $\mu > 0$, $0 \le n \in C^0\big(\overline{\Omega}\big)$ and $z \in C^1\big(\overline{\Omega}\big)$. We will show that a suitably chosen condition on the size of $F_\mu\big(n(\cdot,t_0), z(\cdot,t_0)\big)$ for some $t_0 \ge 0$ implies that F_μ is non-increasing from that time onward along the trajectory of classical solutions to the system (4.2.9). Since we are working with the more generalized version of (4.3.7), almost all of the properties of F_μ also hold in the limit case $f(\xi) \equiv \xi$ obtained by taking $\varepsilon \searrow 0$ in (4.3.7). In particular, this will also hold true for the conditional regularity estimates discussed in Section 4.4.2.
We start with some basic relations between F_μ and the quantities appearing therein.

Lemma 4.10.
For $\mu > 0$ let F_μ be given by (4.4.1). *Then for all nonnegative $n \in C^0\big(\overline{\Omega}\big)$ and any $z \in C^1\big(\overline{\Omega}\big)$ we have*

$$\int_\Omega n|\ln n| \le F_\mu(n,z) + \ln\mu \int_\Omega n + \frac{2|\Omega|}{e} \tag{4.4.2}$$

and

$$\int_\Omega |\nabla z|^2 \le 2F_\mu(n,z) + \frac{2\mu|\Omega|}{e} \tag{4.4.3}$$

as well as

$$F_\mu(n,z) \ge -\frac{\mu|\Omega|}{e}. \tag{4.4.4}$$

Proof: Making use of the facts that n is nonnegative and that $-\xi\ln\xi \le \frac{1}{e}$ for all $\xi > 0$ we can see that

$$\int_\Omega n|\ln n| = F_\mu(n,z) - \frac{1}{2}\int_\Omega |\nabla z|^2 + \ln\mu\int_\Omega n - 2\int_{\{n<1\}} n\ln n \le F_\mu(n,z) + \ln\mu\int_\Omega n + \frac{2|\Omega|}{e},$$

proving (4.4.2). Similarly, we may compute

$$\frac{1}{2}\int_\Omega |\nabla z|^2 = F_\mu(n,z) - \mu\int_\Omega \frac{n}{\mu}\ln\frac{n}{\mu} \le F_\mu(n,z) + \frac{\mu|\Omega|}{e},$$

verifying (4.4.3) and, upon reordering and dropping one term, also (4.4.4). □

The main ingredient in showing that this generalized energy is non-increasing (after some waiting time) will be the following differential inequality.

Lemma 4.11.
Let $m > 0$ and $T \geq 0$ and assume that for $f \in C^3([0,\infty))$ fulfilling (4.2.2) the triple $(n,z,u) \in C^{2,1}\big(\overline{\Omega}\times(T,\infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega\times(T,\infty)$ satisfying $\int_\Omega|u(\cdot,T)|^4 \leq \ell$, and $\int_\Omega n(\cdot,t) \leq m$ for all $t > T$ as well as $n > 0$ in $\Omega\times(T,\infty)$. Then for all $\mu > 0$ and all $t > T$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\mu\big(n(\cdot,t),z(\cdot,t)\big) + \int_\Omega \frac{|\nabla n(\cdot,t)|^2}{n(\cdot,t)} + \left\{\frac{1}{2} - \frac{K_3}{2}\int_\Omega|\nabla z(\cdot,t)|^2 - K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}\big(\ell e^{-\lambda_1(t-T)}+m\big)\right\}\int_\Omega|\Delta z(\cdot,t)|^2 \leq 0,$$

with K_3 as in (4.1.14) and K_u, λ_1 provided by Lemma 4.4.

Proof: Since n is positive in $\overline{\Omega}\times(T,\infty)$ we see by utilizing integration by parts that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\mu(n,z) = -\int_\Omega\frac{|\nabla n|^2}{n} - \int_\Omega|\Delta z|^2 + \int_\Omega \Delta z|\nabla z|^2 + \int_\Omega \Delta z(u\cdot\nabla z) \tag{4.4.5}$$

holds for all $t > T$, where we used the first and second equations of (4.2.9) and $\nabla\cdot u = 0$. By Young's inequality and (4.1.14) we have

$$\int_\Omega \Delta z|\nabla z|^2 \leq \frac{1}{2}\int_\Omega|\Delta z|^2 + \frac{1}{2}\int_\Omega|\nabla z|^4 \leq \left\{\frac{1}{2}+\frac{K_3}{2}\int_\Omega|\nabla z|^2\right\}\int_\Omega|\Delta z|^2 \quad \text{for all } t > T. \tag{4.4.6}$$

To estimate the last term in (4.4.5), we note that by Hölder's inequality and (4.1.14) there holds $\|\nabla z\|_{L^4(\Omega)} \leq K_3^{\frac{1}{2}}|\Omega|^{\frac{1}{4}}\|\Delta z\|_{L^2(\Omega)}$ for all $t > T$, which together with Lemma 4.4 implies

$$\begin{aligned}\int_\Omega|\Delta z(u\cdot\nabla z)| &\leq \|\Delta z\|_{L^2(\Omega)}\|u\|_{L^4(\Omega)}\|\nabla z\|_{L^4(\Omega)}\\ &\leq K_3^{\frac{1}{2}}|\Omega|^{\frac{1}{4}}\|\Delta z\|_{L^2(\Omega)}^2\|u\|_{L^4(\Omega)}\\ &\leq K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}\big(\ell e^{-\lambda_1(t-T)}+m\big)\int_\Omega|\Delta z|^2 \quad \text{for all } t > T,\end{aligned} \tag{4.4.7}$$

since $\int_\Omega n \leq m$ in (T,∞). Combining (4.4.5)–(4.4.7) and reordering appropriately completes the proof. □

In view of the lemma above, we will need to depend on the nonnegativity of the term $\frac{1}{2} - \frac{K_3}{2}\int_\Omega|\nabla z(\cdot,t)|^2 - K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}(\ell e^{-\lambda_1(t-T)}+m)$ in order to obtain an inequality of the form $\frac{\mathrm{d}}{\mathrm{d}t}F_\mu\big(n(\cdot,t),z(\cdot,)\big) \leq 0$. Most of all, this will require some large waiting time t_0 and some small bound on $\int_\Omega n$ in order to treat the term $\ell e^{-\lambda_1(t-T)}+m$. Similarly to the fluid-free case, we further require that the energy at a certain time is already sufficiently small, which will provide control of the term containing $\int_\Omega|\nabla z|^2$.

Lemma 4.12.
Let $T \geq 0$ and $\left(4K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}\right)^{-1} > m_0 > 0$, with K_3 and K_u provided by (4.1.14) and Lemma 4.4, respectively. Suppose that for $f \in C^3([0,\infty))$ satisfying (4.2.2) the triple $(n,z,u) \in C^{2,1}\big(\overline{\Omega}\times(T,\infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega\times(T,\infty)$ satisfying $\int_\Omega |u(\cdot,T)|^4 \leq \ell$ and $m := \int_\Omega n(\cdot,T) \leq m_0$ as well as $n > 0$ in $\Omega\times(T,\infty)$ and $z \in C^0\big([T,\infty);W^{1,2}(\Omega)\big)$. Then if there exist $t_0 \geq T$ and $\mu > 0$ such that

$$\ell e^{-\lambda_1(t_0-T)} + m_0 \leq \frac{1}{4K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}} \tag{4.4.8}$$

and

$$F_\mu\big(n(\cdot,t_0),z(\cdot,t_0)\big) < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e}, \tag{4.4.9}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\mu\big(n(\cdot,t),z(\cdot,t)\big) \leq 0 \quad \text{for all } t > t_0. \tag{4.4.10}$$

Furthermore, one can find $\delta > 0$ such that

$$\int_{t_0}^t\int_\Omega \frac{|\nabla n|^2}{n} + \delta\int_{t_0}^t\int_\Omega |\Delta z|^2 < \frac{1}{4K_3} \quad \text{for all } t > t_0. \tag{4.4.11}$$

Proof: First, we note that in view of Remark 4.6, the inequality in (4.4.8) implies that

$$\ell e^{-\lambda_1(t-T)} + m < \ell e^{-\lambda_1(t_0-T)} + m_0 \leq \frac{1}{4K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}} \quad \text{for all } t > t_0. \tag{4.4.12}$$

Furthermore, recalling Lemma 4.10 we see that (4.4.9) implies

$$\frac{K_3}{2}\int_\Omega |\nabla z(\cdot,t_0)|^2 \leq K_3F_\mu\big(n(\cdot,t_0),z(\cdot,t_0)\big) + \frac{K_3\mu|\Omega|}{e} < \frac{1}{4}.$$

Therefore, the set

$$S := \left\{T' > t_0 \,\middle|\, \frac{K_3}{2}\int_\Omega |\nabla z(\cdot,t)|^2 < \frac{1}{4} \text{ for all } t \in [t_0,T')\right\}$$

is not empty and $T_S := \sup S$ is a well-defined element of $(t_0,\infty]$. In order to verify that actually $T_S = \infty$ we assume $T_S < \infty$ and derive a contradiction. To this end, we make use of Lemma 4.11 to obtain from the definition of T_S and (4.4.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\mu\big(n(\cdot,t),z(\cdot,t)\big) + \int_\Omega \frac{|\nabla n(\cdot,t)|^2}{n(\cdot,t)} + \delta\int_\Omega |\Delta z(\cdot,t)|^2 \leq 0 \quad \text{for all } t \in (t_0,T_S), \tag{4.4.13}$$

with some small $\delta > 0$. Due to the assumed $W^{1,2}(\Omega)$-valued continuity of z, the mapping $[t_0, \infty) \ni t \mapsto F_\mu\big(n(\cdot,t), z(\cdot,t)\big)$ is continuous as well and we infer from the definition of T_S that $\frac{K_3}{2} \int_\Omega |\nabla z|^2 < \frac{1}{4}$ for all $t \in (t_0, T_S)$, but

$$\frac{K_3}{2} \int_\Omega |\nabla z(\cdot, T_S)|^2 = \frac{1}{4}. \tag{4.4.14}$$

Integrating (4.4.13) we obtain

$$F_\mu\big(n(\cdot, T_S), z(\cdot, T_S)\big) \le F_\mu\big(n(\cdot, t_0), z(\cdot, t_0)\big),$$

which by Lemma 4.10 and (4.4.9) shows

$$\int_\Omega |\nabla z(\cdot, T_S)|^2 \le 2F_\mu\big(n(\cdot, T_S), z(\cdot, T_S)\big) + \frac{2\mu|\Omega|}{e} \le 2F_\mu\big(n(\cdot, t_0), z(\cdot, t_0)\big) + \frac{2\mu|\Omega|}{e} < \frac{1}{2K_3},$$

contradicting (4.4.14) and thus proving $T_S = \infty$. Therefore, the inequality (4.4.13) actually holds for all $t > t_0$, which firstly proves (4.4.10) and secondly, upon integration of (4.4.13) shows (4.4.11) due to (4.4.9). □

4.4.2 Conditional regularity estimates

In this section we will establish appropriate Hölder bounds for the components of our approximate solutions under the assumption that we already have control of $\int_\Omega |\nabla z|^p$ for some $p > 2$. In fact, as we will see in Section 4.4.3, obtaining the bound assumed throughout the section for the special value of $p = 4$ will only require bounds on $\int_\Omega n|\ln n|$ and $\int_\Omega |\nabla z|^2$. This, at least for possibly large times, can be obtained by relying on our analysis of F_μ (see Section 4.4.4). Our arguments here are inspired by an approach illustrated in [103, Section 4.2 and 4.3].

Lemma 4.13.
Let $p > 2$, $m_0 > 0$, $M > 0$ and $\tau > 0$. Then there exists $C = C(p, m_0, M, \tau) > 0$ such that if for $f \in C^3([0,\infty))$ satisfying (4.2.2) and some $t_0 \ge 0$ the triple $(n, z, u) \in C^{2,1}\big(\overline{\Omega} \times (t_0, \infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega \times (t_0, \infty)$ with $n \ge 0$ in $\Omega \times (t_0, \infty)$ and

$$\int_\Omega n(\cdot, t_0) \le m_0 \tag{4.4.15}$$

as well as

$$\int_\Omega |\nabla z(\cdot, t)|^p \le M \quad \textit{for all } t > t_0,$$

then

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t \ge t_0 + \tau. \tag{4.4.16}$$

Proof: The proof is based on arguments employed in e.g. [103, Lemma 4.4]. We let $T > t_0 + 1$ and define

$$S(T) := \max\{S_1, S_2(T)\}$$

with

$$S_1 := \max_{t\in[t_0,t_0+1]} (t-t_0)\|n(\cdot,t)\|_{L^\infty(\Omega)} \quad \text{and} \quad S_2(T) := \max_{t\in[t_0+1,T]} \|n(\cdot,t)\|_{L^\infty(\Omega)}.$$

Now, in order to estimate $S(T)$ from above, we let $t_1(t) := \max\{t-1, t_0\}$ and for $t \in (t_0, T)$ represent $n(\cdot,t)$ according to

$$\begin{aligned} n(\cdot,t) &= e^{(t-t_1)\Delta} n(\cdot,t_1) + \int_{t_1}^t e^{(t-s)\Delta} \Big[\nabla\cdot\big(n(\cdot,s)f'(n(\cdot,s))\nabla z(\cdot,s)\big) - \big(u(\cdot,s)\cdot\nabla n(\cdot,s)\big)\Big]\,\mathrm{d}s \\ &=: e^{(t-t_1)\Delta} n(\cdot,t_1) + I(t_1,t), \end{aligned} \tag{4.4.17}$$

where $(e^{\sigma\Delta})_{\sigma\geq 0}$ denotes the heat semigroup with Neumann boundary data in Ω. Fixing some $q \in (2,p)$, we may rely on well known estimates for the heat semigroup (e.g. [97, Lemma 1.3] and [27, Lemma 3.3]) to find $C_1 > 0$ and $C_2 > 0$ such that for all $\sigma \in (0,1)$ there holds

$$\|e^{\sigma\Delta}\varphi\|_{L^\infty(\Omega)} \leq C_1\sigma^{-1}\|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in L^1(\Omega) \tag{4.4.18}$$

and

$$\|e^{\sigma\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \leq C_2\sigma^{-\gamma}\|\varphi\|_{L^q(\Omega)} \quad \text{for all } \varphi \in C^1\big(\overline{\Omega}\big) \text{ such that } \varphi\cdot\nu = 0 \text{ on } \partial\Omega, \tag{4.4.19}$$

with $\gamma := \frac{1}{2} + \frac{1}{q} < 1$. In the case $t \in (t_0, t_0+1]$, where $t_1(t) = t_0$, we thus have

$$\big\|e^{(t-t_0)\Delta} n(\cdot,t_0)\big\|_{L^\infty(\Omega)} \leq C_1 m_0 (t-t_0)^{-1}, \tag{4.4.20}$$

thanks to (4.4.15) and (4.4.18). Furthermore, making use of $\nabla\cdot u = 0$, the fact that $f' \leq 1$ on $[0,\infty)$, and (4.4.19) we see that

$$\|I(t_0,t)\|_{L^\infty(\Omega)} \leq C_2 \int_{t_0}^t (t-s)^{-\gamma}\Big(\big\|n(\cdot,s)\nabla z(\cdot,s)\big\|_{L^q(\Omega)} + \big\|n(\cdot,s)u(\cdot,s)\big\|_{L^q(\Omega)}\Big)\,\mathrm{d}s$$

holds for all $t \in (t_0, t_0+1]$. Herein, multiple applications of the Hölder inequality show that

$$\begin{aligned} \big\|n(\cdot,s)\nabla z(\cdot,s)\big\|_{L^q(\Omega)} &\leq \|n(\cdot,s)\|_{L^\infty(\Omega)}^a \|n(\cdot,s)\|_{L^1(\Omega)}^{1-a} \|\nabla z(\cdot,s)\|_{L^p(\Omega)} \\ &\leq m_0^{1-a} M^{\frac{1}{p}} \|n(\cdot,s)\|_{L^\infty(\Omega)}^a \quad \text{for all } s > t_0 \end{aligned} \tag{4.4.21}$$

with $a := 1 - \frac{p-q}{pq} \in (0,1)$ and

$$\big\|n(\cdot,s)u(\cdot,s)\big\|_{L^q(\Omega)} \leq C_3(1+m_0)m_0^{1-a}\|n(\cdot,s)\|_{L^\infty(\Omega)}^a \quad \text{for all } s > t_0, \tag{4.4.22}$$

for some $C_3 > 0$, where $\|u(\cdot,t)\|_{L^p(\Omega)} \le C_3(1+m_0)$ in view of Lemma 4.5 and Sobolev embeddings. In particular, recalling the definition of S_1 we have

$$\|I(t_0,t)\|_{L^\infty(\Omega)} \le C_4 S_1^a \int_{t_0}^t (t-s)^{-\gamma}(s-t_0)^{-a}\,\mathrm{d}s \quad \text{for all } t \in (t_0, t_0+1]. \tag{4.4.23}$$

with some $C_4 > 0$. Since for any $t \in (t_0, t_0+1]$

$$\int_{t_0}^t (t-s)^{-\gamma}(s-t_0)^{-a}\,\mathrm{d}s = (t-t_0)^{1-\gamma-a}\int_0^1 (1-\zeta)^{-\gamma}\zeta^{-a}\,\mathrm{d}\zeta$$

is finite according to the facts that $\gamma < 1$ and $a < 1$, we consequently see that collecting (4.4.17), (4.4.20), and (4.4.23) shows that there exists some $C_5 > 0$ such that

$$(t-t_0)\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_5 + C_5 S_1^a \quad \text{for all } t \in (t_0, t_0+1],$$

which, due to $a < 1$, implies that

$$S_1 \le C_6 := \max\left\{1, (2C_5)^{\frac{1}{1-a}}\right\}. \tag{4.4.24}$$

The estimation of $S_2(T)$ follows a similar path. We fix $t \in [t_0+1, T]$ and obtain from (4.4.17), (4.4.18), and (4.4.19) that

$$\begin{aligned}
&\|n(\cdot,t)\|_{L^\infty(\Omega)} \\
&\le \left\|e^{\Delta} n(\cdot,t-1)\right\|_{L^\infty(\Omega)} + \|I(t-1,t)\|_{L^\infty(\Omega)} \\
&\le C_1\|n(\cdot,t-1)\|_{L^1(\Omega)} + C_2\int_{t-1}^t (t-s)^{-\gamma}\Big(\left\|n(\cdot,s)\nabla z(\cdot,s)\right\|_{L^q(\Omega)} + \left\|n(\cdot,s)u(\cdot,s)\right\|_{L^q(\Omega)}\Big)\mathrm{d}s,
\end{aligned}$$

from which, again by relying on (4.4.15), (4.4.21) and (4.4.22), we infer that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 m_0 + C_2 m_0^{1-a}\big(M^{\frac{1}{p}} + C_3(1+m_0)\big)\int_{t-1}^t (t-s)^{-\gamma}\|n(\cdot,s)\|_{L^\infty(\Omega)}^a\,\mathrm{d}s$$

holds for all $t \in [t_0+1, T]$. By the definition of $S_2(T)$ we have $\|n(\cdot,s)\|^a_{L^\infty(\Omega)} \le S_2^a(T)$ for all $s \in [t_0+1, T]$, so that in both of the cases $t \in [t_0+1, t_0+2]$ and $t \in (t_0+2, T]$ we may estimate

$$\begin{aligned}
\int_{t-1}^t (t-s)^{-\gamma}\|n(\cdot,s)\|^a_{L^\infty(\Omega)}\,\mathrm{d}s &\le S_1^a\int_{t-1}^t (t-s)^{-\gamma}(s-t_0)^{-a}\,\mathrm{d}s + S_2^a(T)\int_{t-1}^t (t-s)^{-\gamma}\,\mathrm{d}s \\
&\le C_7 S_1^a + \frac{1}{1-\gamma}S_2^a(T).
\end{aligned}$$

with some $C_7 > 0$. Collecting these estimates and making use of (4.4.24) we find $C_8 > 0$ such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_8 + C_8 S_2^a(T) \quad \text{for all } t \in [t_0+1, T],$$

which implies $S_2(T) \le C_9 := \max\left\{1, (2C_8)^{\frac{1}{1-a}}\right\}$ for all $T > t_0+1$. Finally, combining both estimates for S_1 and $S_2(T)$ establishes (4.4.16) if we let $C := \max\{S_1, \frac{S_1}{\tau}, C_9\}$. $\square$

With the improved regularity for n at hand, we can easily derive the time local Hölder continuity of n and u under the same assumptions as above.

Lemma 4.14.
Let $p > 2$, $m_0 > 0$, $M > 0$ and $\tau > 0$. Then there exist some $\theta = \theta(p) \in (0,1)$ and $C = C(p, m_0, M, \tau) > 0$ such that if $f \in C^3([0,\infty))$ satisfies (4.2.2) and if for some $t_0 \geq 0$ the triple $(n,z,u) \in C^{2,1}\big(\overline{\Omega} \times (t_0,\infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega \times (t_0,\infty)$ with the properties that $n \geq 0$ in $\Omega \times (t_0,\infty)$ and

$$\int_\Omega n(\cdot,t_0) \leq m_0 \tag{4.4.25}$$

as well as

$$\int_\Omega |\nabla z(\cdot,t)|^p \leq M \quad \textit{for all } t > t_0, \tag{4.4.26}$$

then

$$\|n\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq C \quad \textit{and} \quad \|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq C \quad \textit{for all } t \geq t_0+\tau.$$

Proof: With ϱ given by (4.1.7) we fix $\beta \in \left(\frac{1}{2},\varrho\right)$. Then we apply the fractional power A^β of the L^2 realization of the Stokes operator to a variation-of-constants representation for u to achieve the identity

$$A^\beta u(\cdot,t) = A^\beta e^{-(t-t_1)A}u(\cdot,t_1) + \int_{t_1}^t A^\beta e^{-(t-s)A}\mathcal{P}\left(n(\cdot,s)\nabla\phi\right)\mathrm{d}s, \quad t \geq t_1,$$

where $t_1 := \max\{t-1,t_0\}$. Recalling that the positive sectorial Stokes operator A generates the contracting semigroup $\left(e^{-tA}\right)_{t\geq 0}$ in $L^2_\sigma(\Omega)$ and the fractional powers of the Stokes operator fulfill the decay property

$$\left\|A^\beta e^{-tA}\right\| \leq C_0 t^{-\beta}e^{-\lambda_1 t} \quad \text{for all } t > 0,$$

with some $C_0 > 0$ ([72, Theorem 37.5]), we can make use of the boundedness of $\mathcal{P}$ in $L^2(\Omega)$, (4.1.6), (4.4.25), and Lemma 4.5 to obtain $C_1 > 0$ such that

$$\begin{aligned}\left\|A^\beta u(\cdot,t)\right\|_{L^2(\Omega)} &\leq \left\|A^\beta e^{-(t-t_1)A}u(\cdot,t_1)\right\|_{L^2(\Omega)} + \int_{t_1}^t \left\|A^\beta e^{-(t-s)A}\mathcal{P}\left(n(\cdot,s)\nabla\phi\right)\right\|_{L^2(\Omega)}\mathrm{d}s \\ &\leq C_1(t-t_1)^{-\beta} + C_1K_1\int_{t_1}^t (t-s)^{-\beta}\|n(\cdot,s)\|_{L^2(\Omega)}\mathrm{d}s \end{aligned} \tag{4.4.27}$$

for all $t > t_1$. Since the assumptions (4.4.25) and (4.4.26) allow for an application of Lemma 4.13, we can find $C_2 > 0$ such that $\|n(\cdot,t)\|_{L^2(\Omega)} \leq C_2$ for all $t \geq t_0+\tau$. Combining $\beta < 1$ with the fact that in both cases $(t-t_1)^{1-\beta} \leq 1$ and $(t-t_1)^{-\beta} \leq 1+\tau^{-\beta}$ hold for $t \geq t_0+\tau$, we infer from (4.4.27) the existence of some $C_3 := C_3(p,m_0,M,\tau) > 0$ such that

$$\left\|A^\beta u(\cdot,t)\right\|_{L^2(\Omega)} \leq C_3 \quad \text{for all } t \geq t_0+\tau.$$

Considering that, since $\beta \in (\frac{1}{2}, \varrho)$ the domains of fractional powers of the Stokes semigroup satisfy $D(A^{\varrho}) \hookrightarrow D(A^{\beta}) \hookrightarrow C^{\theta_1}(\overline{\Omega})$ for any $\theta_1 \in (0, 2\beta - 1)$ ([75, Lemma III.2.4.3] and [23, Theorem 5.6.5]), the previous estimate entails the existence of some $C_4 > 0$ such that

$$\|u(\cdot,t)\|_{C^{\theta_1}(\overline{\Omega})} \le C_4 \quad \text{for all } t \ge t_0 + \tau.$$

Making use of similar arguments we can find $C_5 > 0$ such that

$$\big\|A^{\beta}u(\cdot,t) - A^{\beta}u(\cdot,t_2)\big\|_{L^2(\Omega)} \le C_5(t-t_2)^{1-\beta} \quad \text{for all } t_2 \ge t_0 + \tau \text{ and } t \in [t_2, t_2+1],$$

which together with (4.4.27) readily implies the Hölder regularity of u for some $\theta_2 := \min\{1-\beta, \theta_1\}$. For the regularity of n we first note that by Lemma 4.13 we obtain a constant $C_6 := C_6(p, m_0, M, \tau) > 0$ such that $n(x,t) \le C_6$ for all $x \in \Omega$ and $t \ge t_0 + \frac{\tau}{2}$. Hence, the function n is a bounded distributional solution to the parabolic equation

$$\tilde{n}_t - \nabla\cdot a(x,t,\tilde{n},\nabla\tilde{n}) = 0 \quad \text{in } \Omega \times (t_0, \infty),$$

with $a(x,t,\tilde{n},\nabla\tilde{n}) := \nabla\tilde{n} + n(x,t)f'\big(n(x,t)\big)\nabla z(x,t) - u(x,t)n(x,t)$ and $a(x,t,\tilde{n},\nabla\tilde{n})\cdot\nu = 0$ on the boundary of Ω. Considering that with the arguments illustrated in the first part of the proof, we can find $C_7 := C_7(p, m_0, M, \tau) > 0$ such that $|u(x,t)| \le C_7$ for all $x \in \Omega$ and $t \ge t_0 + \frac{\tau}{2}$, we let $\psi_0(x,t) := n(x,t)^2|\nabla z(x,t)|^2 + |u(x,t)n(x,t)|^2$ and $\psi_1(x,t) := C_6|\nabla z(x,t)| + C_6C_7$ and then see by means of Young's inequality and (4.2.2) that

$$a(x,t,\tilde{n},\nabla\tilde{n})\nabla\tilde{n} \ge \frac{1}{2}|\nabla\tilde{n}|^2 - \psi_0 \quad \text{and} \quad |a(x,t,\tilde{n},\nabla\tilde{n})| \le |\nabla\tilde{n}(x,t)| + \psi_1(x,t)$$

for all $(x,t) \in \Omega \times (t_0 + \frac{\tau}{2}, \infty)$. As moreover (4.4.26) provides a bound for $|\nabla z|^2$ in $L^{\infty}\big((t_0,\infty); L^{\frac{p}{2}}(\Omega)\big)$, we obtain from a well known result in [71, Theorem 1.3] that $\|n\|_{C^{\theta_3,\frac{\theta_3}{2}}(\overline{\Omega}\times[t,t+1])} \le C_8$ for all $t > t_0 + \tau$ with some $\theta_3(p) > 0$ and $C_8 > 0$. Picking $\theta \in (0, \min\{\theta_2, \theta_3\})$ the claim follows immediately. $\square$

In order to prepare a further improvement on the regularity we will show the following:

Lemma 4.15.
Let $p > 2$, $m_0 > 0$, $m_1 > 0$, $M > 0$ and $T > 0$. Then there is $C = C(p, m_0, m_1, M, T) > 0$ such that if for $f \in C^3([0,\infty))$ satisfying (4.2.2) *and $t_0 \ge 0$ the triple $(n,z,u) \in C^0\big(\overline{\Omega}\times[t_0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(t_0,\infty)\big)$ is a classical solution of* (4.2.9)–(4.2.10) *in $\Omega\times(t_0,\infty)$ with the properties that $n \ge 0$ in $\Omega \times (t_0,\infty)$ and*

$$\int_{\Omega} n(\cdot,t) \le m_0 \quad \textit{for all } t > t_0 \tag{4.4.28}$$

and

$$\int_{\Omega} z(\cdot,t_0) \le m_1 \tag{4.4.29}$$

as well as

$$\int_\Omega |\nabla z(\cdot,t)|^p \le M \quad \text{for all } t > t_0, \tag{4.4.30}$$

then

$$z(x,t) \le C \quad \text{for all } x \in \Omega \text{ and } t \in (t_0,T).$$

Proof: Because of the assumption $p > 2$, we have $W^{1,p}(\Omega) \hookrightarrow C^{1-\frac{2}{p}}(\Omega)$ and thus, there exists some constant $C_1 > 0$ such that for each $\varphi \in W^{1,p}(\Omega)$ it holds that

$$|\varphi(x) - \varphi(y)| \le C_1 |x-y|^{1-\frac{2}{p}} \|\nabla\varphi\|_{L^p(\Omega)} \quad \text{for all } x,y \in \Omega. \tag{4.4.31}$$

By Lemma 4.7, Remark 4.6 and the assumptions (4.4.28) and (4.4.29) we see that

$$\int_\Omega z(\cdot,t) \le \int_\Omega z(\cdot,t_0) + m_0(t-t_0) \le m_1 + m_0 T \quad \text{for all } t \in (t_0,T),$$

whence for any such $t \in (t_0,T)$ we can find $x_0(t) \in \Omega$ such that

$$z(x_0(t),t) \le \frac{m_1 + m_0 T}{|\Omega|}.$$

Therefore, (4.4.31) in conjunction with the assumption (4.4.30) shows that

$$\begin{aligned}
z(x,t) &\le z(x_0(t),t) + \big|z(x,t) - z(x_0(t),t)\big| \\
&\le \frac{m_1 + m_0 T}{|\Omega|} + C_1 |x - x_0(t)|^{1-\frac{2}{p}} \|\nabla z(\cdot,t)\|_{L^p(\Omega)} \\
&\le \frac{m_1 + m_0 T}{|\Omega|} + C_2 M^{\frac{1}{p}}
\end{aligned}$$

holds for all $x \in \Omega$, with C_2 only depending on p and the diameter of Ω. □

Drawing on the time-local bound for z, we can rely on the Hölder estimates for n and u and well-known parabolic regularity theory to show the following set of further bounds.

Lemma 4.16.
Let $p > 2, m_0 > 0, m_1 > 0, M > 0, T > 0$ and $\tau > 0$. Then there exist $\theta = \theta(p) \in (0,1)$ and $C = C(p,m_0,m_1,M,T,\tau) > 0$ such that if for $f \in C^3([0,\infty))$ satisfying (4.2.2) and $t_0 \ge 0$ the triple $(n,z,u) \in C^0\big(\overline{\Omega}\times[t_0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(t_0,\infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega \times (t_0,\infty)$ with the properties that $n \ge 0$ and $z \ge 0$ in $\Omega \times (t_0,\infty)$ and

$$\int_\Omega n(\cdot,t_0) \le m_0$$

and

$$\int_\Omega z(\cdot,t_0) \le m_1$$

as well as

$$\int_\Omega |\nabla z(\cdot,t)|^p \le M \quad \textit{for all } t > t_0,$$

then

$$\|n\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_0+\tau,T])} \le C,\ \|z\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_0+\tau,T])} \le C,\ \|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_0+\tau,T])} \le C. \tag{4.4.32}$$

Proof: By Lemma 4.15 and the fact that z is nonnegative we have

$$0 \le z \le C_1 \quad \text{in } \Omega\times(t_0,T)$$

with some $C_1 = C_1(p,m_0,m_1,M,T) > 0$. Thus, letting $\tilde{c} := e^{-z}$ we obtain

$$e^{-C_1} \le \tilde{c} \le 1 \quad \text{in } \Omega\times(t_0,T). \tag{4.4.33}$$

Since Lemma 4.14 entails the existence of $\theta_1 \in (0,1)$ and $C_2 = C_2(p,m_0,M,\tau,T) > 0$ such that

$$\|n\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[t_0+\frac{\tau}{4},T])} + \|u\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[t_0+\frac{\tau}{4},T])} \le C_2,$$

we find that $\tilde{c}$ solves the Neumann boundary value problem $\tilde{c}_t = \Delta\tilde{c} - u\nabla\tilde{c} - f(n)\tilde{c}$ in $\Omega\times(t_0,\infty)$ with Hölder continuous coefficients. Hence, according to standard parabolic Schauder theory ([45, III.5.1 and IV.5.3]), there exists some $\theta_2 \in (0,\theta_1)$ and $C_3 = C_3(p,m_0,m_1,M,T,\tau)$ such that

$$\|\tilde{c}\|_{C^{2+\theta_2,1+\frac{\theta_2}{2}}(\overline{\Omega}\times[t_0+\frac{\tau}{2},T])} \le C_3,$$

yielding the regularity assertion for z featured in (4.4.32) due to the lower bound for $\tilde{c}$ in (4.4.33). Relying on parabolic Schauder theory once more, we can conclude from the first equation that n satisfies (4.4.32). That, moreover, u satisfies (4.4.32) can be readily achieved by well known smoothing properties of the Stokes operator (see e.g. [30, Theorem 2.8], [2, Theorem 1.1]) and the boundedness of n established in Lemma 4.13. $\square$

4.4.3 Conditional estimates for $\int_\Omega |\nabla z|^4$ and $\int_\Omega n^2$

In this section, we will focus on attaining a bound on $\int_\Omega |\nabla z|^4$, which in view of Section 4.4.2 is the main requirement for the regularity estimates we will depend on later. As a preliminary step, we derive some basic differential inequalities through standard testing procedures.

Lemma 4.17.
Suppose that for $f \in C^3([0,\infty))$ satisfying (4.2.2) and $t_0 \ge 0$ the triple $(n,z,u) \in C^{2,1}(\overline{\Omega}\times(t_0,\infty))$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega\times(t_0,\infty)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^2 + \int_\Omega |\nabla n|^2 \le \int_\Omega n^2|\nabla z|^2 \quad \textit{for all } t > t_0. \tag{4.4.34}$$

Proof: By simply testing the first equation of (4.2.9) with n, we can rely on integration by parts, one application of Young's inequality, and the fact $|f'(n)| \le 1$ to easily arrive at (4.4.34). $\square$

Lemma 4.18.
For any $\eta \in (0, \frac{5}{4})$ there exists $C > 0$ such that if for $f \in C^3([0,\infty))$ satisfying (4.2.2) and $t_0 \ge 0$ the triple $(n, z, u) \in C^{2,1}\big(\overline{\Omega} \times (t_0,\infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega \times (t_0,\infty)$ with $n \ge 0$ in $\Omega \times (t_0,\infty)$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z|^4 + \Big(\frac{5}{2} - 2\eta\Big)\int_\Omega \Big|\nabla|\nabla z|^2\Big|^2 \\ \le 8\int_\Omega |\nabla z|^6 + \frac{12}{\eta}\int_\Omega n^2|\nabla z|^2 + 4\int_\Omega |\nabla z|^4|\nabla u| + C\Big(\int_\Omega |\nabla z|^2\Big)^2 \qquad (4.4.35)\end{aligned}$$

holds for all $t > t_0$.

Proof: We differentiate the second equation of (4.2.9) with regard to space and multiply by $|\nabla z|^2\nabla z$. In the resulting equality we can employ the identity $\nabla z \cdot \nabla\Delta z = \frac{1}{2}\Delta|\nabla z|^2 - |D^2 z|^2$ to obtain upon integration by parts that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z|^4 + 2\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + 4\int_\Omega |\nabla z|^2|D^2 z|^2 \\ = -4\int_\Omega |\nabla z|^2\nabla z \cdot \nabla|\nabla z|^2 - 4\int_\Omega |\nabla z|^2 f(n)\Delta z - 4\int_\Omega f(n)\nabla|\nabla z|^2 \cdot \nabla z \\ - 4\int_\Omega |\nabla z|^2\nabla z \cdot (\nabla u \cdot \nabla z) + 2\int_{\partial\Omega} |\nabla z|^2\frac{\partial|\nabla z|^2}{\partial\nu} \qquad (4.4.36)\end{aligned}$$

holds for all $t > t_0$, due to the fact that u is divergence free and the assumed boundary conditions. Drawing on the fact that $\frac{\partial|\nabla z|^2}{\partial\nu} \le C_1|\nabla z|^2$ on $\partial\Omega$ holds for some $C_1 > 0$ only depending on Ω ([61, Lemma 4.2]) and adapting arguments first employed in [37, Proposition 3.2] to find that for fixed $\eta \in (0, \frac{5}{4})$ there exists $C_2 > 0$ such that $2C_1\||\nabla z|^2\|^2_{L^2(\partial\Omega)} \le \eta\|\nabla|\nabla z|^2\|^2_{L^2(\Omega)} + C_2\|\nabla z\|^4_{L^2(\Omega)}$, we find that

$$2\int_{\partial\Omega} |\nabla z|^2\frac{\partial|\nabla z|^2}{\partial\nu} \le \eta\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + C_2\Big(\int_\Omega |\nabla z|^2\Big)^2 \quad \text{for all } t > t_0. \qquad (4.4.37)$$

For the remaining integrals, we note that since $f(n) \le n$ and $|\Delta z|^2 \le 2|D^2 z|^2$ by the Cauchy-Schwarz inequality, we can employ Young's inequality to see that

$$-4\int_\Omega |\nabla z|^2\nabla z \cdot \nabla|\nabla z|^2 \le \frac{1}{2}\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + 8\int_\Omega |\nabla z|^6 \quad \text{for all } t > t_0, \qquad (4.4.38)$$

$$\begin{aligned}-4\int_\Omega |\nabla z|^2 f(n)\Delta z &\le \eta\int_\Omega |\nabla z|^2|\Delta z|^2 + \frac{4}{\eta}\int_\Omega n^2|\nabla z|^2 \\ &\le 2\eta\int_\Omega |\nabla z|^2|D^2 z|^2 + \frac{4}{\eta}\int_\Omega n^2|\nabla z|^2 \quad \text{for all } t > t_0 \qquad (4.4.39)\end{aligned}$$

as well as

$$-4\int_\Omega f(n)\nabla|\nabla z|^2\cdot\nabla z \le \frac{\eta}{2}\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + \frac{8}{\eta}\int_\Omega n^2|\nabla z|^2 \quad \text{for all } t>t_0. \tag{4.4.40}$$

Collecting (4.4.36)–(4.4.40) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla z|^4 + \Big(\frac{3}{2}-\frac{3}{2}\eta\Big)\int_\Omega\big|\nabla|\nabla z|^2\big|^2 + (4-2\eta)\int_\Omega|\nabla z|^2|D^2z|^2\\ \le 8\int_\Omega|\nabla z|^6 + \frac{12}{\eta}\int_\Omega n^2|\nabla z|^2 + 4\int_\Omega|\nabla z|^4|\nabla u| + C_2\Big(\int_\Omega|\nabla z|^2\Big)^2\end{aligned}$$

for all $t>t_0$, which due to the pointwise inequality $\big|\nabla|\nabla z|^2\big|^2 \le 4|D^2z|^2|\nabla z|^2$ readily implies (4.4.35). □

The combination of the two prepared inequalities will now result in the desired bounds for $\int_\Omega|\nabla z|^4$ and $\int_\Omega n^2$, if we assume that we already have suitable bounds for the quantities $\int_\Omega n\ln n$ and $\int_\Omega|\nabla z|^2$. The bounds on these quantities will later on be acquired from the energy functional upon the requirement that $\int_\Omega n_0$ is small.

Lemma 4.19.
Let K_2 be as in (4.1.13)*. Then for all $m_0>0$, each $L>0$ and any $M\in\big(0,\frac{1}{4K_2}\big)$ and $\tau>0$ there exists $C>0$ such that if for $f\in C^3([0,\infty))$ satisfying* (4.2.2) *and some $t_0\ge0$ the triple $(n,z,u)\in C^{2,1}\big(\overline{\Omega}\times(t_0,\infty)\big)$ is a classical solution of* (4.2.9)–(4.2.10) *in $\Omega\times(t_0,\infty)$ satisfying $n\ge0$ in $\Omega\times(t_0,\infty)$ and*

$$\int_\Omega n(\cdot,t_0)\le m_0 \tag{4.4.41}$$

as well as

$$\int_\Omega n(\cdot,t)|\ln n(\cdot,t)|\le L \quad \textit{and} \quad \int_\Omega|\nabla z(\cdot,t)|^2\le M \quad \textit{for all } t>t_0, \tag{4.4.42}$$

then

$$\int_\Omega n^2(\cdot,t)\le C \quad \textit{and} \quad \int_\Omega|\nabla z(\cdot,t)|^4\le C \quad \textit{for all } t\ge t_0+\tau. \tag{4.4.43}$$

Proof: First, we note that due to $M<\frac{1}{4K_2}$, by continuity, one can find some small $\eta\in(0,1)$ such that

$$M<\frac{(2-2\eta)(1-\eta)}{8K_2(1+\eta)}. \tag{4.4.44}$$

Now, assuming (4.4.41) and (4.4.42) to hold, we combine the inequalities established in Lemma 4.17 and Lemma 4.18 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_\Omega n^2+\int_\Omega|\nabla z|^4\Big\}+\int_\Omega|\nabla n|^2+\Big(\frac{5}{2}-2\eta\Big)\int_\Omega\big|\nabla|\nabla z|^2\big|^2 \tag{4.4.45}$$

$$\leq \left(1+\frac{12}{\eta}\right)\int_\Omega n^2|\nabla z|^2 + 8\int_\Omega |\nabla z|^6 + 4\int_\Omega |\nabla z|^4|\nabla u| + C_1 M^2 \quad \text{for all } t>t_0,$$

with some $C_1>0$. Herein, Young's inequality provides $C_2>0$ such that

$$\left(1+\frac{12}{\eta}\right)\int_\Omega n^2|\nabla z|^2 \leq 8\eta\int_\Omega |\nabla z|^6 + C_2\int_\Omega n^3 \quad \text{for all } t>t_0. \tag{4.4.46}$$

To further control the term containing n^3, we recall that by a variant of the Gagliardo–Nirenberg inequality (cf. [5, (22)]) and Remark 4.6 we have

$$\begin{aligned} C_2\int_\Omega n^3 &\leq \frac{1}{2L}\left(\int_\Omega |\nabla n|^2\right)\left(\int_\Omega n|\ln n|\right) + C_3\left(\int_\Omega n\right)^3 + C_3 \\ &\leq \frac{1}{2}\int_\Omega |\nabla n|^2 + C_3 m_0^3 + C_3 \quad \text{for all } t>t_0, \end{aligned} \tag{4.4.47}$$

with some $C_3>0$. Returning to the analyzation of the remaining terms in (4.4.45), we observe that by Hölder's inequality, Lemma 4.5 combined with (4.4.41), the Gagliardo–Nirenberg inequality, and finally Young's inequality we can find $C_4, C_5, C_6>0$ such that

$$\begin{aligned} 4\int_\Omega |\nabla z|^4|\nabla u| &\leq 4\big\||\nabla z|^2\big\|^2_{L^6(\Omega)}\|\nabla u\|_{L^{\frac{3}{2}}(\Omega)} \leq C_4(1+m_0)\big\||\nabla z|^2\big\|^2_{L^6(\Omega)} \\ &\leq C_5\left(\int_\Omega \big|\nabla|\nabla z|^2\big|^2\right)^{\frac{5}{6}}\left(\int_\Omega |\nabla z|^2\right)^{\frac{1}{3}} + C_5\left(\int_\Omega |\nabla z|^2\right)^2 \\ &\leq \frac{1}{2}\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + C_6 M^2 \quad \text{for all } t>t_0. \end{aligned} \tag{4.4.48}$$

The estimation of the leftover term on the right in (4.4.45) is more involved. First, note that by (4.1.13) we have

$$\int_\Omega |\nabla z|^6 \leq K_2\left(\int_\Omega \big|\nabla|\nabla z|^2\big|^2\right)\left(\int_\Omega |\nabla z|^2\right) + K_2\left(\int_\Omega |\nabla z|^4\right)\left(\int_\Omega |\nabla z|^2\right) \quad \text{for all } t>t_0,$$

where additionally by the Cauchy-Schwarz inequality

$$\int_\Omega |\nabla z|^4 \leq \left(\int_\Omega |\nabla z|^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla z|^2\right)^{\frac{1}{2}} \quad \text{for all } t>t_0,$$

so that an application of Young's inequality combined with our assumption (4.4.42) implies that

$$\begin{aligned} \int_\Omega |\nabla z|^6 &\leq K_2\left(\int_\Omega \big|\nabla|\nabla z|^2\big|^2\right)\left(\int_\Omega |\nabla z|^2\right) + \eta\int_\Omega |\nabla z|^6 + \frac{K_2^2}{4\eta}\left(\int_\Omega |\nabla z|^2\right)^3 \\ &\leq K_2 M\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + \eta\int_\Omega |\nabla z|^6 + \frac{K_2^2 M^3}{4\eta} \quad \text{for all } t>t_0 \end{aligned}$$

and therefore

$$(8+8\eta)\int_\Omega |\nabla z|^6 \le \frac{8(1+\eta)K_2M}{1-\eta}\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + \frac{2(1+\eta)K_2^2M^3}{(1-\eta)\eta} \quad \text{for all } t>t_0. \tag{4.4.49}$$

Collecting (4.4.46)–(4.4.49), we infer from (4.4.45) that for some $C_8>0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega n^2 + \int_\Omega |\nabla z|^4\right\} + C_7\int_\Omega |\nabla n|^2 + C_7\int_\Omega \big|\nabla|\nabla z|^2\big|^2 \le C_8 \quad \text{for all } t>t_0, \tag{4.4.50}$$

where $C_7 := \min\left\{\frac{1}{2}, 2-2\eta-\frac{8(1+\eta)K_2M}{1-\eta}\right\}$ is positive due to (4.4.44). In order to conclude the desired bounds, we want to derive from the inequality above a differential inequality of the form $y'(t)+Cy^2(t)\le C$, where $y(t) := \int_\Omega n^2(\cdot,t)+\int_\Omega|\nabla z(\cdot,t)|^4$ and $C>0$. To this end, we still need to estimate the terms without time derivatives, arising in (4.4.50) on the left, from below. By making use of the Gagliardo–Nirenberg inequality, we firstly obtain upon use of the mass conservation and (4.4.41) that

$$\left(\int_\Omega n^2\right)^2 \le C_9\left(\int_\Omega |\nabla n|^2\right)\left(\int_\Omega n\right)^2 + C_9\left(\int_\Omega n\right)^4 \le C_9 m_0^2\int_\Omega |\nabla n|^2 + C_9 m_0^4$$

for all $t>t_0$ with some $C_9>0$, and secondly, relying on (4.4.42), we find $C_{10}>0$ such that

$$\begin{aligned}\left(\int_\Omega |\nabla z|^4\right)^2 &\le C_{10}\left(\int_\Omega \big|\nabla|\nabla z|^2\big|^2\right)\left(\int_\Omega |\nabla z|^2\right)^2 + C_{10}\left(\int_\Omega |\nabla z|^2\right)^4\\ &\le C_{10}M^2\int_\Omega \big|\nabla|\nabla z|^2\big|^2 + C_{10}M^4 \quad \text{for all } t>t_0.\end{aligned}$$

Thus, letting $C_{11} := \max\{2C_9m_0^2, 2C_{10}M^2\}$, we see that y satisfies

$$y'(t)+C_{12}y^2(t)\le C_{13} \quad \text{for all } t>t_0,$$

with $C_{12} := \frac{C_7}{C_{11}}$ and $C_{13} := C_8 + \frac{C_9m_0^4+C_{10}M^4}{C_{11}}$. By application of an ODE comparison argument, we observe that $\bar{y}(t) := \frac{2}{C_{12}(t-t_0)}+\sqrt{\frac{2C_{13}}{C_{12}}}$ satisfies $y(t)\le\bar{y}(t)$ for all $t>t_0$, implying that

$$y(t)\le \frac{2}{C_{12}\tau}+\sqrt{\frac{2C_{13}}{C_{12}}} \quad \text{for all } t\ge t_0+\tau$$

and thus proving (4.4.43). □

4.4.4 Eventual smoothness for generalized solutions with small mass

For our next proof we will require the following result demonstrated in [87, Lemma 2.6], which is based on an application of the Trudinger–Moser inequality combined with a spatio-temporal estimate on $\nabla\ln(n_\varepsilon+1)$ in L^2.

Lemma 4.20.
There exists $K_4 > 0$ such that for all $\varepsilon \in (0,1)$ the solution to (4.3.7)–(4.3.9) satisfies

$$\int_0^t \ln\left\{\frac{1}{|\Omega|}\int_\Omega (n_\varepsilon(x,s)+1)^2\,\mathrm{d}x\right\}\mathrm{d}s \le K_4\left(1+\int_\Omega n_0\right)t + K_4\left(\int_\Omega z_0 + \int_\Omega n_0\right) \quad \textit{for all } t>0.$$

Relying on the properties previously established for F_μ, we can now determine some possibly large time $t_\star$ depending on the initial data but not on $\varepsilon \in (0,1)$, for which $\int_\Omega n_\varepsilon |\ln n_\varepsilon|$, $\int_\Omega |\nabla z_\varepsilon|^2$ and $F_\mu(n_\varepsilon, z_\varepsilon)$ are sufficiently small for all times beyond $t_\star$. This in turn will then ensure that we can achieve the conditional estimates featured in Section 4.4.3 for times larger than $t_\star$.

Lemma 4.21.
Let K_2, K_3 be as in (4.1.13) and (4.1.14), respectively. Then, there exist constants $m_\star, \Gamma, M > 0$ and $\mu \in (0,1)$ such that

$$\Gamma < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e} \qquad \textit{and} \qquad M < \frac{1}{4K_2}, \tag{4.4.51}$$

and such that if the initial data (n_0, c_0, u_0) satisfy (4.1.7) as well as

$$m := \int_\Omega n_0 \le m_\star, \tag{4.4.52}$$

then one can find $t_\star > 0$ such that for each $\varepsilon \in (0,1)$ the solution $(n_\varepsilon, z_\varepsilon, u_\varepsilon)$ of (4.3.7)–(4.3.9) satisfies

$$F_\mu\big(n_\varepsilon(\cdot,t), z_\varepsilon(\cdot,t)\big) \le \Gamma \quad \textit{for all } t \ge t_\star \tag{4.4.53}$$

and

$$\int_\Omega n_\varepsilon(\cdot,t)\,|\ln n_\varepsilon(\cdot,t)| \le \frac{1}{4K_3} + \frac{2|\Omega|}{e} \quad \textit{for all } t \ge t_\star \tag{4.4.54}$$

as well as

$$\int_\Omega |\nabla z_\varepsilon(\cdot,t)|^2 \le M \quad \textit{for all } t \ge t_\star. \tag{4.4.55}$$

Proof: We fix $M \in \big(0, \frac{1}{4K_2}\big)$ and afterwards choose some small $\mu \in (0,1)$, such that

$$\frac{2\mu|\Omega|}{e} \le \frac{M}{2} \qquad \text{and} \qquad 0 < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e}. \tag{4.4.56}$$

Upon these choices, we can pick $\Gamma > 0$ fulfilling the first inequality in (4.4.51) as well as

$$\Gamma \le \frac{M}{4}. \tag{4.4.57}$$

Furthermore, letting K_4 be provided by Lemma 4.20 we can find $\eta \in (0,1)$ such that

$$\eta|\Omega|e^{16K_4} \le \frac{\Gamma}{4}. \tag{4.4.58}$$

Relying on the previous choices and with K_3, K_u given by (4.1.14) and Lemma 4.4, respectively, we introduce the positive number

$$m_\star := \min\left\{1, \frac{\Gamma}{4\ln\frac{1}{\eta\mu}}, \frac{\Gamma}{8}, \frac{1}{5K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}}\right\}, \tag{4.4.59}$$

where the positivity follows from the facts $\mu, \eta < 1$. Now given (n_0, c_0, u_0) such that (4.1.7) and (4.4.52) hold, we find $\ell > 0$ such that $\int_\Omega |u_0|^4 \le \ell$, due to $D(A^\varrho) \hookrightarrow L^4(\Omega)$ ([13, Lemma 2.3 iv)]). Moreover, since $\lambda_1 > 0$, we can easily find $t_0 \ge 0$ such that

$$\ell e^{-\lambda_1 t_0} + m_\star \le \frac{1}{4K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}} \tag{4.4.60}$$

holds. We next claim that the asserted inequalities are true if we fix some large $t_\star$ satisfying the conditions

$$(1+m)t_\star \ge \int_\Omega z_0 + m, \qquad mt_\star \ge \int_\Omega z_0, \qquad \text{and} \qquad t_\star > 2t_0, \tag{4.4.61}$$

with z_0 as defined in (4.3.9). To verify this claim we define the sets

$$S_1(\varepsilon) := \left\{t \in (0,t_\star) \;\middle|\; \ln\left\{\frac{1}{|\Omega|}\int_\Omega (n_\varepsilon(\cdot,t)+1)^2\right\} > 8K_4(1+m)\right\}$$

and

$$S_2(\varepsilon) := \left\{t \in (0,t_\star) \;\middle|\; \int_\Omega |\nabla z_\varepsilon(\cdot,t)|^2 > 8m\right\}$$

and estimate their respective sizes. By Lemma 4.20 we know that for all $\varepsilon \in (0,1)$ we have

$$I_1(\varepsilon) := \int_0^{t_\star} \ln\left\{\frac{1}{|\Omega|}\int_\Omega (n_\varepsilon(\cdot,t)+1)^2\right\} \mathrm{d}t \le K_4(1+m)t_\star + K_4\left(\int_\Omega z_0 + m\right),$$

so that the first condition in (4.4.61) combined with our definition of $S_1(\varepsilon)$ shows that

$$2K_4(1+m)t_\star \ge K_4(1+m)t_\star + K_4\left(\int_\Omega z_0 + m\right) \ge I_1(\varepsilon) \ge 8K_4(1+m)|S_1(\varepsilon)|$$

holds for all $\varepsilon \in (0,1)$, meaning that

$$|S_1(\varepsilon)| \le \frac{t_\star}{4} \quad \text{for all } \varepsilon \in (0,1). \tag{4.4.62}$$

In pursuance of a similar bound for the size of $|S_2(\varepsilon)|$, we recall that by Lemma 4.7 we have

$$I_2(\varepsilon) := \int_0^{t_\star}\int_\Omega |\nabla z_\varepsilon|^2 \le \int_\Omega z_0 + mt_\star \quad \text{for all } \varepsilon \in (0,1).$$

Relying on the second inequality in (4.4.61) and the definition of $S_2(\varepsilon)$ we infer that

$$2mt_\star \ge \int_\Omega z_0 + mt_\star \ge I_2(\varepsilon) \ge 8m|S_2(\varepsilon)|$$

holds for all $\varepsilon \in (0,1)$ and hence

$$|S_2(\varepsilon)| \le \frac{t_\star}{4} \quad \text{for all } \varepsilon \in (0,1). \tag{4.4.63}$$

Now, (4.4.62) and (4.4.63) guarantee that

$$\big|(0,t_\star)\setminus\big(S_1(\varepsilon)\cup S_2(\varepsilon)\big)\big| \ge \frac{t_\star}{2} \quad \text{for all } \varepsilon \in (0,1),$$

so that we conclude from the third inequality in (4.4.61) that for any $\varepsilon \in (0,1)$ we can pick some $t_\varepsilon \in (t_0, t_\star)$ such that

$$\ln\left\{\frac{1}{|\Omega|}\int_\Omega \big(n_\varepsilon(\cdot,t_\varepsilon)+1\big)^2\right\} \le 8K_4(1+m) \qquad \text{and} \qquad \int_\Omega |\nabla z_\varepsilon(\cdot,t_\varepsilon)|^2 \le 8m \tag{4.4.64}$$

hold. Relying on the elementary estimate $s\ln\frac{s}{\mu} \le \eta(s+1)^2 + s\ln\frac{1}{\eta\mu}$ for all $s>0$ (cf. [103, Lemma 5.5]), we can combine the mass conservation from Remark 4.6 with (4.4.52) and the first part of (4.4.64) to obtain that

$$\begin{aligned}\int_\Omega n_\varepsilon(\cdot,t_\varepsilon)\ln\frac{n_\varepsilon(\cdot,t_\varepsilon)}{\mu} &\le \eta\int_\Omega \big(n_\varepsilon(\cdot,t_\varepsilon)+1\big)^2 + \ln\frac{1}{\eta\mu}\int_\Omega n_\varepsilon(\cdot,t_\varepsilon)\\ &\le \eta|\Omega|e^{8K_4(1+m)} + m\ln\frac{1}{\eta\mu}.\end{aligned}$$

Now, recalling the first and second requirement for $m_\star$ from (4.4.59) as well as (4.4.58), we see that

$$\int_\Omega n_\varepsilon(\cdot,t_\varepsilon)\ln\frac{n_\varepsilon(\cdot,t_\varepsilon)}{\mu} \le \eta|\Omega|e^{16K_4} + m\ln\frac{1}{\eta\mu} \le \frac{\Gamma}{4}+\frac{\Gamma}{4} = \frac{\Gamma}{2}.$$

In a similar fashion, the second part of (4.4.64) in conjunction with the third inequality contained in (4.4.59) entails that

$$\frac{1}{2}\int_\Omega |\nabla z_\varepsilon(\cdot,t_\varepsilon)|^2 \le \frac{\Gamma}{2}$$

and thus we obtain that

$$F_\mu\big(n_\varepsilon(\cdot,t_\varepsilon), z_\varepsilon(\cdot,t_\varepsilon)\big) = \int_\Omega n_\varepsilon(\cdot,t_\varepsilon)\ln\frac{n_\varepsilon(\cdot,t_\varepsilon)}{\mu} + \frac{1}{2}\int_\Omega |\nabla z_\varepsilon(\cdot,t_\varepsilon)|^2 \le \Gamma.$$

In accordance with (4.4.51) and (4.4.60), this allows for the application of Lemma 4.12, implying that

$$F_\mu\big(n_\varepsilon(\cdot,t), z_\varepsilon(\cdot,t)\big) \le \Gamma \quad \text{for all } t \ge t_\varepsilon, \tag{4.4.65}$$

which, since $t_\varepsilon < t_\star$, immediately establishes (4.4.53) again due to (4.4.51). Now, to verify that also (4.4.54) and (4.4.55) hold, we recall that in view of Lemma 4.10 we have

$$\int_\Omega n_\varepsilon(\cdot,t)|\ln n_\varepsilon(\cdot,t)| \le F_\mu\big(n_\varepsilon(\cdot,t), z_\varepsilon(\cdot,t)\big) + \ln\mu \int_\Omega n_\varepsilon(\cdot,t) + \frac{2|\Omega|}{e}.$$

Therefore, (4.4.65), the fact $\mu < 1$ and once more (4.4.51) imply

$$\int_\Omega n_\varepsilon(\cdot,t)|\ln n_\varepsilon(\cdot,t)| \le \Gamma + \frac{2|\Omega|}{e} < \frac{1}{4K_3} + \frac{2|\Omega|}{e} \quad \text{for all } t \ge t_\varepsilon,$$

proving (4.4.54), because $t_\star > t_\varepsilon$. Similarly, again relying on Lemma 4.10 and (4.4.65), we conclude that due to (4.4.57) and the first restriction in (4.4.56), we have

$$\int_\Omega |\nabla z_\varepsilon(\cdot,t)|^2 \le 2F_\mu\big(n_\varepsilon(\cdot,t), z_\varepsilon(\cdot,t)\big) + \frac{2\mu|\Omega|}{e} \le 2\Gamma + \frac{2\mu|\Omega|}{e} \le \frac{M}{2} + \frac{M}{2} = M$$

for all $t \ge t_\varepsilon$, which proves (4.4.55). □

The bounds for $\int_\Omega n_\varepsilon \ln n_\varepsilon$ and $\int_\Omega |\nabla z_\varepsilon|^2$ at hand, we can first draw on the conditional estimates on $\int_\Omega |\nabla z_\varepsilon|^4$ from Section 4.4.3 and afterwards on the conditional regularity estimates from Section 4.4.2 to derive the following result.

Proposition 4.22.
Let $m_\star > 0$ be as provided by Lemma 4.21. Suppose that (n_0, c_0, u_0) satisfy (4.1.7) as well as

$$\int_\Omega n_0 \le m_\star,$$

and let (n, c, u) denote the global generalized solution of (4.1.3)–(4.1.5) from Theorem A. Then there exists $T > 0$ such that

$$n \in C^{2,1}\big(\overline{\Omega}\times[T,\infty)\big), \quad c \in C^{2,1}\big(\overline{\Omega}\times[T,\infty)\big) \quad \textit{and} \quad u \in C^{2,1}\big(\overline{\Omega}\times[T,\infty);\mathbb{R}^2\big), \tag{4.4.66}$$

that

$$c(x,t) > 0 \quad \textit{for all } x \in \overline{\Omega} \textit{ and any } t \ge T,$$

and such that (n, c, u) solves (4.1.3)–(4.1.5) classically in $\Omega \times (T,\infty)$. Moreover, one can find $\mu > 0$ such that

$$F_\mu\big(n(\cdot,t), z(\cdot,t)\big) < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e} \quad \textit{for all } t \ge T, \tag{4.4.67}$$

with $z := -\ln\big(\frac{c}{\|c_0\|_{L^\infty(\Omega)}}\big)$.

Proof: Let K_2, K_3 be provided by (4.1.13) and (4.1.14), respectively. In view of Lemma 4.21 we can find $\mu \in (0,1)$, $\Gamma \in \big(0, \frac{1}{4K_3} - \frac{\mu|\Omega|}{e}\big)$, $M \in (0, \frac{1}{4K_2})$, $L > 0$ and $t_\star > 0$ such that for any choice of $\varepsilon \in (0,1)$ we have

$$F_\mu\big(n_\varepsilon(\cdot,t), z_\varepsilon(\cdot,t)\big) \le \Gamma \quad \text{for all } t > t_\star \tag{4.4.68}$$

and

$$\int_\Omega n_\varepsilon(\cdot,t)|\ln n_\varepsilon(\cdot,t)| \le L \quad \text{as well as} \quad \int_\Omega |\nabla z_\varepsilon(\cdot,t)|^2 \le M \quad \text{for all } t > t_\star.$$

Since $M < \frac{1}{4K_2}$, we may employ Lemma 4.19 to obtain $C_1 > 0$ such that for any $\varepsilon \in (0,1)$ we have

$$\int_\Omega |\nabla z_\varepsilon(\cdot,t)|^4 \le C_1 \quad \text{for all } t > t_\star + 1.$$

This bound at hand, Lemma 4.16 yields $\theta \in (0,1)$ such that for each $T > t_\star + 2$ we can pick $C_2(T) > 0$ such that

$$\|n_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_\star+2,T])} + \|z_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_\star+2,T])} + \|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_\star+2,T])} \le C_2(T)$$

for all $\varepsilon \in (0,1)$. In view of the Arzelà-Ascoli theorem, we can find a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ of the sequence provided by Proposition 4.9, along which n_ε, z_ε and u_ε are convergent in $C^{2,1}_{loc}\big(\overline{\Omega}\times[t_\star+2,\infty)\big)$. The respective limits of n_ε, z_ε and u_ε must clearly coincide with n, z and u, which ensures that n, c and u have the desired regularity properties in (4.4.66). Additionally, the continuity of z implies $c > 0$ in $\overline{\Omega}\times[T,\infty)$ and passing to the limit for $\varepsilon = \varepsilon_{j_k} \searrow 0$ in (4.4.68), we easily obtain (4.4.67) due to $\Gamma < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e}$. Letting $\varepsilon = \varepsilon_{j_k} \searrow 0$ in (4.3.7), we first conclude that (n,z,u) solves (4.2.9)–(4.2.10) with $f(\xi) \equiv \xi$ classically in $\Omega \times (T,\infty)$, which then in combination with $c > 0$ in $\overline{\Omega}\times[T,\infty)$ entails that (n,c,u) solve (4.1.3)–(4.1.5) classically in $\Omega \times [T,\infty)$. □

4.4.5 Stabilization of solutions with small energy

This section discusses the last missing part for the proof of Theorem 4.1, which are the convergence properties featured therein. Since from the last section we already know that our generalized solutions will be classical solutions after some waiting time, we will concern our investigation only with convergence of classical solutions to (4.2.9). Before proving the desired large time behavior we require one additional preparation in form of a time-independent Hölder bound for ∇z.

Lemma 4.23.
For all $m_0 > 0$, $M > 0$, $\tau > 0$ there exist $\theta \in (0,1)$ and $C > 0$ such that if for $f \in C^3([0,\infty))$ satisfying (4.2.2) *and $t_0 \ge 0$ the triple $(n,z,u) \in C^0\big(\overline{\Omega}\times[t_0,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(t_0,\infty)\big)$ is a classical solution of* (4.2.9)–(4.2.10) *in $\Omega \times (t_0,\infty)$ satisfying*

$$\int_\Omega n(\cdot,t_0) \le m_0$$

and

$$\int_\Omega |\nabla z(\cdot,t)|^4 \le M \quad \textit{for all } t > t_0,$$

then

$$\|\nabla z(\cdot,t)\|_{C^\theta(\overline{\Omega})} \le C \quad \textit{for all } t \ge t_0 + \tau. \tag{4.4.69}$$

Proof: The arguments are quite similar to the ones employed in [103, Lemma 4.9] and we will not recount all details here. First, we note that by Lemma 4.13 we can find $C_1 > 0$ such that

$$\|n(\cdot,t)\|_{L^4(\Omega)} \le C_1 \quad \text{for all } t \ge \widetilde{t_0} := t_0 + \frac{\tau}{2}. \tag{4.4.70}$$

Now, we may choose some $\beta \in (0,1)$ close to 1 such that $\beta > \frac14$ and afterwards $q > 1$ satisfying $\frac14 < \frac1q < \frac54 - \beta$. With these values fixed we will make use of several well-known estimates for the Neumann heat semigroup $\big(e^{-sB}\big)_{s\ge0}$ in $L^4(\Omega)$, where $B := -\Delta + 1$ (e.g. [97]). In particular, for any fixed $\theta \in (0, 2\beta - \frac32)$ we have that $D\big(B^\beta\big) \hookrightarrow C^{1+\theta}\big(\overline{\Omega}\big)$ ([32, Theorem 1.6.1]) and hence

$$\|\nabla\varphi\|_{C^\theta(\overline{\Omega})} \le C_2 \|B^\beta\varphi\|_{L^4(\Omega)} \quad \text{for all } \varphi \in \mathrm{D}\big(B^\beta\big), \tag{4.4.71}$$

with some $C_2 > 0$. Letting

$$S_1 := \max_{t\in[\widetilde{t_0},\widetilde{t_0}+1]} (t-\widetilde{t_0})^\beta \|\nabla z(\cdot,t)\|_{C^\theta(\overline{\Omega})} \quad \text{and} \quad S_2(T) := \max_{t\in[\widetilde{t_0}+1,T]} \|\nabla z(\cdot,t)\|_{C^\theta(\overline{\Omega})}$$

for $T > \widetilde{t_0} + 1$ we continue by estimating $S(T) := \max\{S_1, S_2(T)\}$. Consequently, with $t_1(t) := \max\{t-1, \widetilde{t_0}\}$ we start by representing $z(\cdot,t)$ according to

$$\begin{aligned} z(\cdot,t) = \overline{z(\cdot,t_1)} + e^{t-t_1}e^{-(t-t_1)B}\Big(z(\cdot,t_1) - \overline{z(\cdot,t_1)}\Big) - \int_{t_1}^t e^{t-s}e^{-(t-s)B}|\nabla z(\cdot,s)|^2\,\mathrm{d}s \\ + \int_{t_1}^t e^{t-s}e^{-(t-s)B} f\big(n(\cdot,s)\big)\,\mathrm{d}s - \int_{t_1}^t e^{t-s}e^{-(t-s)B}u(\cdot,s)\nabla z(\cdot,s)\,\mathrm{d}s, \end{aligned} \tag{4.4.72}$$

where $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$ denotes the spatial average. In the case of $t - \widetilde{t_0} \le 1$ we make use of Young's inequality, (4.4.71), the semigroup estimates for the Neumann heat semigroup, and the fact that $f(s) \le s$ for all $s \ge 0$ to obtain $C_3 > 0$ such that

$$\begin{aligned} &\|\nabla z(\cdot,t)\|_{C^\theta(\overline{\Omega})} \\ &\le C_3 e(t-\widetilde{t_0})^{-\beta}\|z(\cdot,\widetilde{t_0}) - \overline{z(\cdot,\widetilde{t_0})}\|_{L^4(\Omega)} + C_3 e\int_{\widetilde{t_0}}^t (t-s)^{-\gamma}\big\||\nabla z(\cdot,s)|^2\big\|_{L^q(\Omega)}\,\mathrm{d}s \\ &\quad + C_3 e\int_{\widetilde{t_0}}^t (t-s)^{-\beta}\|n(\cdot,s)\|_{L^4(\Omega)}\,\mathrm{d}s + C_3 e\int_{\widetilde{t_0}}^t (t-s)^{-\beta}\|u(\cdot,s)\|_{L^8(\Omega)}^2\,\mathrm{d}s, \end{aligned} \tag{4.4.73}$$

holds for all $t \le \widetilde{t_0}+1$, where $\gamma := \beta + \frac{1}{q} - \frac{1}{4} < 1$. Herein, (4.4.70), Lemma 4.5 and the fact that $\beta < 1$ imply the existence of $C_4 > 0$ such that

$$C_3 e \int_{\widetilde{t_0}}^t (t-s)^{-\beta} \|n(\cdot,s)\|_{L^4(\Omega)}\,\mathrm{d}s + C_3 e \int_{\widetilde{t_0}}^t (t-s)^{-\beta} \|u(\cdot,s)\|_{L^8(\Omega)}^2\,\mathrm{d}s \\ \le C_4 \int_{\widetilde{t_0}}^t (t-s)^{-\beta}\,\mathrm{d}s \le \frac{C_4}{1-\beta}$$

for all $t \ge \widetilde{t_0}+1$, and the Poincaré inequality provides $C_5 > 0$ satisfying

$$\|z(\cdot,s) - \overline{z(\cdot,s)}\|_{L^4(\Omega)} \le C_5 \|\nabla z(\cdot,s)\|_{L^4(\Omega)} \le C_5 M^{\frac{1}{4}} \quad \text{for all } s \ge \widetilde{t_0}.$$

Furthermore, by means of the Hölder inequality we see that

$$\big\||\nabla z(\cdot,s)|^2\big\|_{L^q(\Omega)} \le \|\nabla z(\cdot,s)\|_{L^4(\Omega)}^{\frac{4}{q}} \|\nabla z(\cdot,s)\|_{L^\infty(\Omega)}^a \le M^{\frac{1}{q}} \|\nabla z(\cdot,s)\|_{C^\theta(\overline{\Omega})}^a$$

for all $s \ge \widetilde{t_0}$, with $a := \frac{2q-4}{q}$. Hence, for all $t \ge \widetilde{t_0}+1$ we have

$$\int_{\widetilde{t_0}}^t (t-s)^{-\gamma} \big\||\nabla z(\cdot,s)|^2\big\|_{L^q(\Omega)} \mathrm{d}s \\ \le M^{\frac{1}{q}} S_1^a (t-\widetilde{t_0})^{1-\gamma-\beta a} \int_0^1 (1-\sigma)^{-\gamma} \sigma^{-\beta a}\,\mathrm{d}\sigma \le C_6 M^{\frac{1}{q}} S_1^a (t-\widetilde{t_0})^{1-\gamma-\beta a},$$

where we used that $\int_0^1 (1-\sigma)^{-\gamma}\sigma^{-\beta a}\,\mathrm{d}\sigma =: C_6$ is finite due to the facts that $0 < a < 1$, $0 < \beta < 1$ and $\gamma < 1$. Accordingly, from (4.4.73) we infer that

$$(t-\widetilde{t_0})^\beta \|\nabla z(\cdot,t)\|_{C^\theta(\overline{\Omega})} \le C_3 C_5 e M^{\frac{1}{4}} + C_3 C_6 e M^{\frac{1}{q}} S_1^a (t-\widetilde{t_0})^{1-\gamma+(1-a)\beta} + \frac{C_4}{1-\beta} \\ \le C_7 + C_7 S_1^a$$

for all $t \in [\widetilde{t_0}, \widetilde{t_0}+1]$, with some $C_7 > 0$, which implies that $S_1 \le \max\{1, (2C_7)^{\frac{1}{1-a}}\}$. Similarly, in the case $t \in [\widetilde{t_0}+1, T]$ we conclude from (4.4.72) that

$$\|\nabla z(\cdot,t)\|_{C^\theta(\overline{\Omega})} \le C_8 M^{\frac{1}{4}} + C_8 M^{\frac{1}{q}} \int_{t-1}^t (t-s)^{-\gamma} \|\nabla z(\cdot,s)\|_{C^\theta(\overline{\Omega})}^a\,\mathrm{d}s + C_8 \int_{t-1}^t (t-s)^{-\beta}\,\mathrm{d}s,$$

for some $C_8 > 0$. In both of the cases $t \le \widetilde{t_0}+2$ and $t > \widetilde{t_0}+2$ we can estimate

$$\int_{t-1}^t (t-s)^{-\gamma} \|\nabla z(\cdot,s)\|_{C^\theta(\overline{\Omega})}^a\,\mathrm{d}s \le S_1^a \int_{t-1}^t (t-s)^{-\gamma}(s-\widetilde{t_0})^{-\beta a}\,\mathrm{d}s + S_2^a(T) \int_{t-1}^t (t-s)^{-\gamma}\,\mathrm{d}s \\ \le C_6 S_1^a + \frac{1}{1-\gamma} S_2^a(T)$$

with C_6 as defined above. Therefore, for suitable large $C_9 > 0$ we have

$$S_2(T) \le C_9 + C_9 S_2^a(T) \quad \text{for all } T > \widetilde{t_0}+1,$$

which implies that $S_2(T) \le \max\{1, (2C_9)^{\frac{1}{1-a}}\} =: S_2$ for all $T > \widetilde{t_0}+1$. Consequently, together with the previous estimate for S_1, this establishes (4.4.69) with $C := \max\{S_1, \frac{S_1}{\tau}, S_2\}$. □

Assuming that the energy $F_\mu(n,z)$ remains small for all times succeeding some waiting $T \geq 0$, which according to Proposition 4.22 is true for the generalized solutions with small mass, we will now show that any given solution to (4.2.9)–(4.2.10) in $\Omega \times (T,\infty)$ will satisfy the asymptotic properties described in Theorem 4.1. Here we explicitly allow $T = 0$ because, if the energy is already suitably small initially, we can transfer these asymptotic properties also to the global classical solutions discussed in Section 4.4.6.

Proposition 4.24.
Assume $T \geq 0$, $\ell > 0$ and let $m_\star > 0$ be as in Lemma 4.21. Suppose that for $f \in C^3([0,\infty))$ satisfying (4.2.2) with $f > 0$ on $(0,\infty)$ the triple $(n,z,u) \in C^0\big(\overline{\Omega}\times[T,\infty)\big) \cap C^{2,1}\big(\overline{\Omega}\times(T,\infty)\big)$ is a classical solution of (4.2.9)–(4.2.10) in $\Omega \times (T,\infty)$ satisfying $z \in C^0\big([T,\infty);W^{1,2}(\Omega)\big)$, $m := \int_\Omega n(\cdot,T) < m_\star$, $0 \leq n \not\equiv 0$ and $\int_\Omega |u(\cdot,T)|^4 \leq \ell$ as well as

$$\inf_{t>T} F_\mu\big(n(\cdot,t),z(\cdot,t)\big) < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e} \tag{4.4.74}$$

for some $\mu > 0$. Then

$$n(\cdot,t) \to \overline{n_T} := \frac{1}{|\Omega|}\int_\Omega n(\cdot,T) \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty \tag{4.4.75}$$

and

$$\nabla z(\cdot,t) \to 0 \quad \text{in } L^\infty\big(\Omega;\mathbb{R}^2\big) \quad \text{as } t \to \infty \tag{4.4.76}$$

and

$$\inf_{x\in\Omega} z(x,t) \to \infty \qquad \text{as } t \to \infty \tag{4.4.77}$$

as well as

$$u(\cdot,t) \to 0 \quad \text{in } L^\infty\big(\Omega;\mathbb{R}^2\big) \quad \text{as } t \to \infty. \tag{4.4.78}$$

Proof: The convergence of n and z can be proven by relying on the methods shown in [103, Lemma 6.1], whereas the decay of u then follows by adapting the arguments illustrated in [100, Lemma 5.3]. For the sake of completeness we only recount the main steps and refer to the mentioned sources for more details. Recalling that $m_\star < \big(4K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}\big)^{-1}$, we can first find $t_0 > T$ such that $\ell e^{-\lambda_1(t_0-T)} + m_\star \leq \big(4K_3^{\frac{1}{2}}K_u|\Omega|^{\frac{1}{4}}\big)^{-1}$ and then rely on (4.4.74) and Lemma 4.12 to see that we can pick $t_\star > t_0 > T$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\mu\big(n(\cdot,t),z(\cdot,t)\big) \leq 0 \quad \text{for all } t > t_\star \tag{4.4.79}$$

and

$$F_\mu\big(n(\cdot,t),z(\cdot,t)\big) < C_1 := \frac{1}{4K_3} - \frac{\mu|\Omega|}{e} \quad \text{for all } t > t_\star \tag{4.4.80}$$

and that with some $\delta > 0$,

$$\int_{t_\star}^{\infty} \int_\Omega \frac{|\nabla n|^2}{n} + \delta \int_{t_\star}^{\infty} \int_\Omega |\Delta z|^2 \le C_2 := \frac{1}{4K_3}. \tag{4.4.81}$$

Since (n, z, u) solve (4.2.9) classically in $\Omega \times (T, \infty)$ by Remark 4.6 we have

$$\int_\Omega n(\cdot, t) = m \quad \text{for all } t > T, \tag{4.4.82}$$

and thus, making use of (4.4.2) and (4.4.80), we see that

$$\int_\Omega n(\cdot,t)|\ln n(\cdot,t)| \le F_\mu\big(n(\cdot,t), z(\cdot,t)\big) + \ln\mu \int_\Omega n(\cdot,t) + \frac{2|\Omega|}{e} \le C_1 + m\ln\mu + \frac{2|\Omega|}{e} \tag{4.4.83}$$

holds for all $t > t_\star$. Since $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$, a Poincaré–Sobolev inequality implies the existence of $C_3 > 0$ such that

$$\|\varphi - \overline{\varphi}\|_{L^2(\Omega)} \le C_3 \|\nabla\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in W^{1,1}(\Omega). \tag{4.4.84}$$

Similarly, by means of elliptic regularity theory we can find $C_4 > 0$ satisfying

$$\|\nabla\varphi\|_{L^2(\Omega)} \le C_4 \|\Delta\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in W^{2,2}(\Omega) \text{ such that } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\,. \tag{4.4.85}$$

According to (4.4.84) and the Cauchy-Schwarz inequality we thus have

$$\int_{t_\star}^{\infty} \|n(\cdot,t) - \overline{n_T}\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le C_3^2 \int_{t_\star}^{\infty} \|\nabla n\|_{L^1(\Omega)}^2 \,\mathrm{d}t \le mC_3^2 \int_{t_\star}^{\infty} \int_\Omega \frac{|\nabla n|^2}{n},$$

whereas (4.4.85) shows that

$$\int_{t_\star}^{T} \|\nabla z(\cdot,t)\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le C_4^2 \int_{t_\star}^{\infty} \int_\Omega |\Delta z|^2.$$

By combination of the two previous estimates with (4.4.81) we thereby see that

$$\int_{t_\star}^{\infty} \Big\{ \|n(\cdot,t) - \overline{n_T}\|_{L^2(\Omega)}^2 + \|\nabla z(\cdot,t)\|_{L^2(\Omega)}^2 \Big\} \,\mathrm{d}t \le mC_2C_3^2 + \frac{C_2C_4^2}{\delta} \tag{4.4.86}$$

which implies that there must exist $(t_k)_{k\in\mathbb{N}} \subset (t_\star, \infty)$ such that $t_k \to \infty$ and such that

$$n(\cdot, t_k) \to \overline{n_T} \quad \text{in } L^2(\Omega) \quad \text{and} \quad \nabla z(\cdot, t_k) \to 0 \quad \text{in } L^2\big(\Omega; \mathbb{R}^2\big) \tag{4.4.87}$$

as $k \to \infty$. Relying on the convexity of $0 < \xi \mapsto \xi\ln\xi$ and the Jensen inequality we see that

$$\int_\Omega \varphi\ln\varphi \,\mathrm{d}x \ge \int_\Omega \overline{\varphi}\ln\overline{\varphi} \quad \text{for all positive } \varphi \in C^0\big(\overline{\Omega}\big)\,,$$

and thus, we can make use of the mean value theorem, the Cauchy-Schwarz inequality, (4.4.82) and the first convergence in (4.4.87) to obtain

$$
\begin{aligned}
0 &\leq \int_\Omega n(\cdot,t_k)\ln n(\cdot,t_k) - \int_\Omega \overline{n_T}\ln\overline{n_T} = \int_\Omega n(\cdot,t_k)\big(\ln n(\cdot,t_k) - \ln\overline{n_T}\big)\\
&\leq \int_{\{n(\cdot,t_k)>\overline{n_T}\}} n(\cdot,t_k)\big(\ln n(\cdot,t_k) - \ln\overline{n_T}\big)\\
&\leq \frac{1}{\overline{n_T}}\|n(\cdot,t_k)\|_{L^2(\Omega)}\|n(\cdot,t_k)-\overline{n_T}\|_{L^2(\Omega)} \to 0 \quad \text{as } k\to\infty.
\end{aligned}
\tag{4.4.88}
$$

This, together with the definition of F_μ and the second convergence established in (4.4.87), shows that $F_\mu\big(n(\cdot,t_k),z(\cdot,t_k)\big) \to C_5 := \int_\Omega \overline{n_T}\ln\frac{\overline{n_T}}{\mu}$ as $k\to\infty$, which in turn by the monotonicity property (4.4.79) implies

$$
F_\mu\big(n(\cdot,t),z(\cdot,t)\big) \to C_5 \quad \text{as } t\to\infty.
$$

In view of (4.4.88) this convergence actually yields

$$
\begin{aligned}
\limsup_{t\to\infty}\int_\Omega|\nabla z(\cdot,t)|^2 &= 2\limsup_{t\to\infty}\left\{F_\mu\big(n(\cdot,t),z(\cdot,t)\big) - \int_\Omega n(\cdot,t)\ln\frac{n(\cdot,t)}{\mu}\right\}\\
&\leq 2C_5 - 2C_5 = 0.
\end{aligned}
\tag{4.4.89}
$$

Combining this with the bound provided by (4.4.83), we may first employ Lemma 4.19 and afterwards Lemma 4.14 and Lemma 4.23 to obtain $t_{\star\star} > t_\star$, $\theta\in(0,1)$ and $C_6>0$ such that

$$
\|n\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq C_6, \quad \|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq C_6, \quad \text{and} \quad \|\nabla z(\cdot,t)\|_{C^\theta(\overline{\Omega})} \leq C_6
\tag{4.4.90}
$$

for all $t\geq t_{\star\star}$. If the asserted convergence for n in (4.4.75) was false, we could find $(\tilde{t}_k)_{k\in\mathbb{N}}\subset(t_{\star\star},\infty)$ and $C_7>0$ such that $\tilde{t}_k\to\infty$ as $k\to\infty$ and

$$
\|n(\cdot,\tilde{t}_k)-\overline{n_T}\|_{L^\infty(\Omega)} \geq C_7 \quad \text{for all } k\in\mathbb{N},
$$

implying that, due to the uniform convergence of n in $\overline{\Omega}\times[t_{\star\star},\infty)$ asserted by (4.4.90), there exist $(x_k)_{k\in\mathbb{N}}\subset\Omega$, $r>0$, and $\tau>0$ such that $B_r(x_k)\subset\Omega$ for all $k\in\mathbb{N}$ and

$$
\big|n(x,t)-\overline{n_T}\big| \geq \frac{C_7}{2} \quad \text{for all } x\in B_r(x_k) \text{ and each } t\in(\tilde{t}_k,\tilde{t}_k+\tau).
$$

Consequently, this would show that

$$
\int_{\tilde{t}_k}^{\tilde{t}_k+\tau}\|n(\cdot,t)-\overline{n_T}\|_{L^2(\Omega)}^2\,\mathrm{d}t \geq \tau\frac{C_7^2}{4}\pi r^2 \quad \text{for all } k\in\mathbb{N},
$$

contradicting the spatial-temporal estimate (4.4.86) and thus proving (4.4.75). In a similar fashion, assuming that (4.4.76) is false, in view of the third portion of (4.4.90), we could find $(\hat{t}_k)_{k\in\mathbb{N}} \subset (t_{\star\star},\infty)$, $(\hat{x}_k)_{k\in\mathbb{N}} \subset \Omega$, $r>0$, and $C_8>0$ such that $\hat{t}_k \to \infty$ as $k\to\infty$ and $B_r(\hat{x}_k)\subset\Omega$ for all $k\in\mathbb{N}$ as well as

$$|\nabla z(x,\hat{t}_k)| \ge C_8 \quad \text{for all } x\in B_r(\hat{x}_k) \text{ and each } k\in\mathbb{N}.$$

This implies that

$$\int_\Omega |\nabla z(\cdot,\hat{t}_k)|^2 \ge C_8^2 \pi r^2 \quad \text{for all } k\in\mathbb{N},$$

which contradicts (4.4.89) and thereby proves (4.4.76). For (4.4.77) we make use of the fact that (4.4.75) together with the nontriviality of n establishes the existence of some $t_{\star\star\star}>T$ satisfying

$$n(x,t) > \frac{\overline{n_T}}{2} \quad \text{for all } x\in\Omega \text{ and } t>t_{\star\star\star},$$

whence, by relying on the nonnegativity of z, the fact that $f'\ge 0$ and parabolic comparison with the function $\overline{\Omega}\times[t_{\star\star\star},\infty)\ni(x,t)\mapsto f(\frac{\overline{n_T}}{2})(t-t_{\star\star\star})$, we see that

$$z(x,t) \ge f\big(\tfrac{\overline{n_T}}{2}\big)(t-t_{\star\star\star}) \quad \text{for all } x\in\Omega \text{ and } t>t_{\star\star\star},$$

ensuring (4.4.77) due to $f>0$ on $(0,\infty)$. In order to prove (4.4.78), we recall that the Stokes operator A in $L^2_\sigma(\Omega)$ is positive and self-adjoint with compact inverse and as such, there exists a complete orthonormal basis $(\psi_k)_{k\in\mathbb{N}}$ of eigenfunctions of A to positive eigenvalues λ_k, $k\in\mathbb{N}$. Since $\bigcup_{m\in\mathbb{N}} \operatorname{span}\{\psi_k \,|\, k\le m\}$ is dense in $L^2_\sigma(\Omega)$, in view of the uniform Hölder continuity of u in $\Omega\times(t_{\star\star},\infty)$ from (4.4.90), we only have to show that for each $k\in\mathbb{N}$ we have

$$\int_\Omega u(x,t)\cdot\psi_k(x)\,\mathrm{d}x \to 0 \quad \text{as } t\to\infty. \tag{4.4.91}$$

To this end we fix $k\in\mathbb{N}$ and let $y(t):=\int_\Omega u(x,t)\cdot\psi_k(x)\,\mathrm{d}x$, $t>T$. From the third equation in (4.2.9), the eigenfunction property of ψ_k as well as the fact that $\nabla\cdot\psi_k=0$ we obtain

$$y'(t) = -\lambda_k \int_\Omega u\cdot\psi_k + \int_\Omega \big(n-\overline{n_T}\big)\nabla\phi\cdot\psi_k \quad \text{for all } t>T. \tag{4.4.92}$$

Since $n\to\overline{n_T}$ in $L^\infty(\Omega)$ as $t\to\infty$ by (4.4.75), for any given real number $\chi>0$ we can find $t_\diamond>T$ such that

$$\left|\int_\Omega \big(n(x,t)-\overline{n_T}\big)\nabla\phi\cdot\psi_k(x)\,\mathrm{d}x\right| \le \frac{\chi\lambda_k}{2} \quad \text{for all } t>t_\diamond,$$

which shows upon integration of (4.4.92) that, due to the boundedness of u in $\Omega\times(T,\infty)$, we have

$$y(t) \le y(t_\diamond)e^{-\lambda_k(t-t_\diamond)} + \frac{\lambda_k\chi}{2}\int_{t_\diamond}^t e^{-\lambda_k(t-s)} < C_9 e^{-\lambda_k(t-t_\diamond)} + \frac{\chi}{2} \quad \text{for all } t>t_\diamond,$$

with some $C_9 > 0$. Now letting $t_{\diamond\diamond} := \max\left\{t_\diamond, t_\diamond + \frac{1}{\lambda_k}\ln\frac{2C_9}{\chi}\right\}$ we have

$$|y(t)| < \chi \quad \text{for all } t > t_{\diamond\diamond},$$

yielding (4.4.91) and thus completing the proof. □

All that is left is to gather the results of our previous two propositions to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1: With $m_\star > 0$ provided by Lemma 4.21, we obtain from Proposition 4.22 that for any initial data (n_0, c_0, u_0) satisfying (4.1.7) as well as (4.1.8), there exists $T > 0$ such that the solution (n, c, u) from Theorem A has the regularity properties featured in (4.1.9) and the positivity of c in $\overline{\Omega}\times(T,\infty)$ as claimed in (4.1.10) are valid. Since (4.4.67) from Proposition 4.22 furthermore guarantees that $\inf_{t>T} F_\mu\big(n(\cdot,t), z(\cdot,t)\big) < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e}$, we may employ Proposition 4.24 to obtain (4.1.11) and (4.1.12). □

4.4.6 Global classical solutions for small initial data

As mentioned in the Section 4.1, the result featured in Theorem 4.2 is a by-product of our previous analysis. Our main tools in the proof will on one hand be the fact that the assumed smallness conditions for the initial data, expressed in (4.1.15) and (4.1.16), allows for the choice of $t_0 = 0$ in Lemma 4.12, and on the other hand the uniqueness statement from Lemma 4.3. The uniqueness statement is essential, since we can only guarantee the global existence for our approximate solutions when $f(s) \equiv f_\varepsilon(s)$ with $f_\varepsilon(s)$ provided by (4.3.5).

Proof of Theorem 4.2: We denote by (n, c, u) the local classical solution from Lemma 4.3 for $f(s) \equiv s$, extended to its maximal existence time $T_{max} \in (0,\infty]$. Then, writing $z := -\ln\big(\frac{c}{\|c_0\|_{L^\infty(\Omega)}}\big)$ and $\tau := \min\{1, \frac{T_{max}}{2}\}$, we infer that $C_1 := \|n\|_{L^\infty(\Omega\times(0,\tau))}$ is finite by the continuity of n in $\overline{\Omega}\times[0,T_{max})$. On the other hand, let us also consider the approximate problems (4.3.7) and denote the corresponding solutions by $(n_\varepsilon, z_\varepsilon, u_\varepsilon)$ with $\varepsilon \in (0,1)$, which, according to [87, Section 2.1], are global for each $\varepsilon \in (0,1)$. For these solutions and μ as in (4.1.16) we have

$$F_\mu\big(n_\varepsilon(\cdot,0), z_\varepsilon(\cdot,0)\big) = C_2 := \int_\Omega n_0 \ln\frac{n_0}{\mu} + \frac{1}{2}\int_\Omega \frac{|\nabla c_0|^2}{c_0^2} \quad \text{for all } \varepsilon \in (0,1).$$

Furthermore, defining $m_{\star\star} := \frac{1}{8K_3^{\frac12}K_u|\Omega|^{\frac14}}$ we conclude that the inequalities contained in (4.1.15) imply

$$\int_\Omega |u_0|^4 e^{-\lambda_1 t} + \int_\Omega n_0 < \frac{1}{4K_3^{\frac12}K_u|\Omega|^{\frac14}} \quad \text{for all } t > 0.$$

In light of (4.2.6) and (4.1.16) we have $C_2 < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e}$ and Lemma 4.12 becomes applicable, asserting that

$$F_\mu\big(n_\varepsilon(\cdot,t), z_\varepsilon(\cdot,t)\big) \le C_2 \quad \text{for all } t > 0 \text{ and each } \varepsilon \in (0,1).$$

Thanks to Lemma 4.10, this implies that for any $\varepsilon \in (0,1)$ we have

$$\int_\Omega n_\varepsilon \,|\ln n_\varepsilon| \le C_2 + \ln \mu \int_\Omega n_0 + \frac{2|\Omega|}{e} \quad \text{and} \quad \int_\Omega |\nabla z_\varepsilon|^2 \le M := 2C_2 + \frac{2\mu|\Omega|}{e}$$

on $(0,\infty)$. Herein, the second restriction on C_2 from (4.1.16) shows that

$$M < \frac{2}{8K_2} - \frac{2\mu|\Omega|}{e} + \frac{2\mu|\Omega|}{e} = \frac{1}{4K_2}.$$

Hence, we may employ Lemma 4.19 to find $C_3 > 0$ such that

$$\int_\Omega |\nabla z_\varepsilon(\cdot,t)|^4 \le C_3 \quad \text{for all } t > \frac{\tau}{2} \text{ and each } \varepsilon \in (0,1).$$

In turn, Lemma 4.13 becomes applicable and provides $C_4 > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_4 \quad \text{for all } t > \tau \text{ and every } \varepsilon \in (0,1). \tag{4.4.93}$$

Now, fixing $\varepsilon \in (0,1)$ so small such that it satisfies $\varepsilon \le \min\left\{\frac{1}{C_1}, \frac{1}{C_4}\right\}$, we see that by the definition of f_ε in (4.3.5) we have

$$f_\varepsilon(n) = n \quad \text{in } \overline{\Omega} \times [0,\tau],$$

from which , in view of the uniqueness statement contained in Lemma 4.3 when applied to the system (4.2.1) with $f \equiv f_\varepsilon$, we infer that

$$(n,z,u) \equiv (n_\varepsilon, z_\varepsilon, u_\varepsilon) \quad \text{in } \overline{\Omega} \times [0,\tau]$$

for our fixed ε. On the other hand, relying on (4.4.93) and the second restriction on ε we also have $f_\varepsilon(n_\varepsilon) \equiv n_\varepsilon$ in $\overline{\Omega} \times (\tau,\infty)$, and $(n_\varepsilon, z_\varepsilon, u_\varepsilon)$ actually solves (4.2.9) in $\Omega \times (\tau,\infty)$ with $f(s) \equiv s$. Now, making use of the uniqueness result from Lemma 4.3 once more, when applied to (4.2.1) with $f(s) \equiv s$, guarantees that $T_{max} = \infty$ and that $(n,z,u) \equiv (n_\varepsilon, z_\varepsilon, u_\varepsilon)$ in $\Omega \times (0,\infty)$. The desired convergence properties easily follow from Proposition 4.24, since $C_2 < \frac{1}{4K_3} - \frac{\mu|\Omega|}{e}$. □

5 The Stokes limit in a three-dimensional chemotaxis-Navier–Stokes system

5.1 Introduction

The research of mathematical models which accurately describe natural phenomena often demands large analytical efforts, and even the most thorough studies encounter challenges for which the known mathematical tools reach their limit. In particular, models with inherent nonlinear structure may turn out to be very problematic. This is especially true for the models obtained by the interplay of Keller–Segel-type systems and Navier–Stokes equations, as their individual parts, chemotaxis equations on one hand and fluid equations on the other, each on their own feature significantly complex behavior. We witnessed one example in the previous chapter, but additional examples also reside in the apparently simpler consumption models of the form

$$\begin{cases} n_t + \quad u\cdot\nabla n &= \nabla\cdot\big(D(n)\nabla n - n\nabla c\big), \quad x\in\Omega, \quad t>0,\\ c_t + \quad u\cdot\nabla c &= \Delta c - cn, \quad x\in\Omega, \quad t>0,\\ u_t + \kappa(u\cdot\nabla)u &= \Delta u - \nabla P + n\nabla\phi, \quad x\in\Omega, \quad t>0,\\ \nabla\cdot u &= 0, \quad x\in\Omega, \quad t>0, \end{cases}$$

with $D\equiv const.$, where in $\Omega\subset\mathbb{R}^3$ neither of the subsystems is understood completely. For instance, working in the fluid-free three-dimensional setting, obtained upon letting $u\equiv 0$ in the system above, global bounded classical solutions were only obtained under the assumption that the initial chemical concentration $\|c(\cdot,0)\|_{L^\infty(\Omega)}$ is small ([77]). In contrast, for arbitrary initial data, global weak solutions have been shown to exist, which become smooth and classical after some waiting time ([81]). On the other hand, existence theory for the Navier–Stokes equations, which has already been garnering lots of interest for the better part of a century, beyond mere global weak solutions also remains dependent on various assumptions in the three-dimensional setting ([75]). Correspondingly, the known results for the given chemotaxis-Navier–Stokes system with arbitrary initial data also mainly cover global existence of weak solutions ([104]) and eventual smoothing properties ([105]). Even in more favorable scenarios, where the diffusion process is enhanced at large cell densities as e.g. incorporated by the choice $D(s)=s^{m-1}$, $s>0$, with $m>1$, only weak solutions could be established, as indicated by the results of [63, 110] $m>\frac{2}{3}$.

Neglecting the fluid convection term. In light of this difficulty, a substantial amount of the studies dedicated to the mathematical analysis of chemotaxis-fluid interaction mainly concentrates on systems where the fluid evolution is described by the Stokes equation obtained by letting $\kappa = 0$, i.e.

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega, \quad t > 0, \\ c_t + u \cdot \nabla c = \Delta c - cn, & x \in \Omega, \quad t > 0, \\ u_t = \Delta u - \nabla P + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, \quad t > 0. \end{cases} \tag{Λ_0}$$

In this setting, considerably stronger results besides mere global existence ([21, 98]) have been shown (see e.g [14, 22, 44] and [4, Section 4.1] for an additional non-exhaustive overview). The reasoning behind the neglection of the convection term, however, mostly originates from experimental observations indicating Reynolds numbers of order $\mathcal{R} \approx 10^{-4}$ ([57]) for the bacteria in question. Rigorous mathematical results appear to be mostly lacking. In fact, only recently it was shown in the two-dimensional setting that upon taking $\kappa \to 0$, the global classical solution $\big(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)}\big)$ of the chemotaxis-Navier–Stokes system convergences uniformly in time towards the global classical solution $\big(n^{(0)}, c^{(0)}, u^{(0)}\big)$ of (Λ_0) in the sense that there exist $C > 0$ and $\mu > 0$ such that whenever $\kappa \in (-1,1)$,

$$\begin{aligned} \big\| n^{(\kappa)}(\cdot,t) - n^{(0)}(\cdot,t) \big\|_{L^\infty(\Omega)} &+ \big\| c^{(\kappa)}(\cdot,t) - c^{(0)}(\cdot,t) \big\|_{L^\infty(\Omega)} \\ &+ \big\| u^{(\kappa)}(\cdot,t) - u^{(0)}(\cdot,t) \big\|_{L^\infty(\Omega)} \leq C|\kappa| e^{-\mu t} \end{aligned}$$

holds for all $t > 0$ ([91]).

Main results. Motivated by the temporally uniform convergence result for the limit $\kappa \to 0$ from [91], we aspire to quantify the effect of the Stokes approximation in the more intricate three dimensional setting beyond the expected mere time-local convergence. Before we take a brief look at the major challenges entailed by the increased space dimension, let us specify the framework and the main result obtained in this chapter. Under the assumptions that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary and that $\kappa \in [-1,1]$ we will consider

$$\begin{cases} n_t + \quad u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega, \quad t > 0, \\ c_t + \quad u \cdot \nabla c &= \Delta c - cn, & x \in \Omega, \quad t > 0, \\ u_t + \kappa (u \cdot \nabla) u &= \Delta u - \nabla P + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u &= 0, & x \in \Omega, \quad t > 0, \end{cases} \tag{Λ_κ}$$

with boundary conditions

$$\nabla n(x,t) \cdot \nu = 0, \quad \nabla c(x,t) \cdot \nu = 0 \quad \text{and} \quad u(x,t) = 0 \qquad \text{for } x \in \partial\Omega \text{ and } t > 0 \tag{5.1.1}$$

and initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \quad x \in \Omega, \tag{5.1.2}$$

where

$$\phi \in C^{1+\beta}(\overline{\Omega}) \quad \text{for some } \beta > 0. \tag{5.1.3}$$

Moreover, we assume the initial data to satisfy

$$\begin{cases} n_0 \in C^0(\overline{\Omega}) & \text{nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) & \text{with } c_0 > 0 \text{ in } \overline{\Omega}, \\ u_0 \in D(A^{\varrho}) & \text{for some } \varrho \in (\frac{3}{4}, 1), \end{cases} \tag{5.1.4}$$

where $A := -\mathcal{P}\Delta$ denotes the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^3)$ under homogeneous Dirichlet boundary conditions with its domain given by $D(A) := W^{2,2}(\Omega;\mathbb{R}^3) \cap W_0^{1,2}(\Omega;\mathbb{R}^3) \cap L^2_\sigma(\Omega)$. Herein, $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^3) \mid \nabla\cdot\varphi = 0\}$ stands for the Hilbert space of solenoidal vector fields in $L^2(\Omega;\mathbb{R}^3)$, and $\mathcal{P}$ represents the Helmholtz projection of $L^2(\Omega;\mathbb{R}^3)$ onto $L^2_\sigma(\Omega)$. Accordingly, we also abbreviate $W^{1,p}_{0,\sigma}(\Omega) := W_0^{1,p}(\Omega;\mathbb{R}^3) \cap L^2_\sigma(\Omega)$ and $C^\infty_{0,\sigma}(\Omega) := C_0^\infty(\Omega;\mathbb{R}^3) \cap L^2_\sigma(\Omega)$.
With the framework and notations clarified, we can now precisely state the main result.

Theorem 5.1.
Let $\Omega \subset \mathbb{R}^3$ be a bounded and smooth domain and suppose that ϕ and n_0, c_0, u_0 comply with (5.1.3) *and* (5.1.4)*, respectively. Let*

$$\begin{aligned} X := {} & L^\infty\big((0,\infty);L^1(\Omega)\big) \cap L^{\frac{5}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \cap L^{\frac{5}{4}}_{loc}\big([0,\infty);W^{1,\frac{5}{4}}(\Omega)\big) \\ & \times L^\infty\big(\Omega\times(0,\infty)\big) \cap L^4_{loc}\big([0,\infty);W^{1,4}(\Omega)\big) \\ & \times L^\infty_{loc}\big([0,\infty);L^2_\sigma(\Omega)\big) \cap L^{\frac{10}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big) \cap L^2_{loc}\big([0,\infty);W^{1,2}_{0,\sigma}(\Omega)\big). \end{aligned}$$

Then there exist a family $\{(n^{(\kappa)},c^{(\kappa)},u^{(\kappa)})\}_{\kappa\in[-1,1]} \subset X$ of global weak solutions, in the sense of Definition 5.2 below, to the corresponding family of chemotaxis-Navier–Stokes systems (Λ_κ),(5.1.1),(5.1.2) *and $T_\diamond > 0$ such that $(n^{(\kappa)},c^{(\kappa)},u^{(\kappa)})$ together with some $P^{(\kappa)} \in C^{1,0}\big(\overline{\Omega}\times(T_\diamond,\infty)\big)$ solve* (Λ_κ),(5.1.1),(5.1.2) *classically in $\Omega\times(T_\diamond,\infty)$. Moreover, for any null sequence $(\kappa_j)_{j\in\mathbb{N}} \subset [-1,1]$ there exist a subsequence $(\kappa_{j_k})_{k\in\mathbb{N}}$ and a global weak solution $(n,c,u) \in X$ of the chemotaxis-Stokes system* (Λ_0),(5.1.1),(5.1.2)*, such that*

$$\begin{aligned} \big(n^{(\kappa_{j_k})} - n\big) &\to 0 \quad \text{in } L^{p_1}\big(\Omega\times(0,\infty)\big) \quad \text{for any } p_1 \in [1,\tfrac{5}{3}), \\ \big(\nabla n^{(\kappa_{j_k})} - \nabla n\big) &\to 0 \quad \text{in } L^{p_2}\big(\Omega\times(0,\infty);\mathbb{R}^3\big) \quad \text{for any } p_2 \in [1,\tfrac{5}{4}), \\ \big(c^{(\kappa_{j_k})} - c\big) &\to 0 \quad \text{in } L^{q_1}\big(\Omega\times(0,\infty)\big) \quad \text{for any } q_1 \in [1,\infty), \\ \big(\nabla c^{(\kappa_{j_k})} - \nabla c\big) &\to 0 \quad \text{in } L^{q_2}\big(\Omega\times(0,\infty);\mathbb{R}^3\big) \quad \text{for any } q_2 \in [1,4), \\ \big(u^{(\kappa_{j_k})} - u\big) &\to 0 \quad \text{in } L^{r_1}\big(\Omega\times(0,\infty);\mathbb{R}^3\big) \quad \text{for any } r_1 \in [1,\tfrac{10}{3}), \\ \big(\nabla u^{(\kappa_{j_k})} - \nabla u\big) &\to 0 \quad \text{in } L^{r_2}\big(\Omega\times(0,\infty);\mathbb{R}^{3\times3}\big) \quad \text{for any } r_2 \in [1,2) \end{aligned} \tag{5.1.5}$$

as $\kappa_{j_k} \to 0$, and such that (n,c,u) together with some $P \in C^{1,0}\big(\overline{\Omega}\times(T_\diamond,\infty)\big)$ solve (Λ_0),(5.1.1),(5.1.2) *classically in $\Omega\times(T_\diamond,\infty)$.*

Mathematical challenges and the approach. In the two-dimensional setting investigated in [91], it is known that (Λ_κ) already emits a classical solution on $\Omega\times(0,\infty)$, which in turn allows for testing procedures immediately targeting the quasi-energy functional

$$\int_\Omega n^{(\kappa)} \ln n^{(\kappa)} + \frac{1}{2}\int_\Omega \frac{|\nabla c^{(\kappa)}|^2}{c^{(\kappa)}} + \eta\int_\Omega |u^{(\kappa)}|^2$$

for large $\eta > 0$ independent of $\kappa \in [-1,1]$ to derive, after some bootstrapping, κ-independent bounds in $C^1(\overline{\Omega}) \times C^2(\overline{\Omega}) \times D(A^{\varrho})$ uniform in time. These bounds, when combined with decay properties of (Λ_κ), then become the driving force of the exponential stabilization featured in [91]. In stark contrast, in the current three-dimensional framework we cannot utilize a corresponding quasi-energy functional immediately, as for (Λ_κ) only the global existence of a weak solution obtained by a limiting procedure from approximating systems is known ([104]). To transfer any reasonable information to this weak solution, however, we have to ensure that the precompactness properties used in the limit procedure are independent of κ. Even though the methods behind the derivation of the corresponding bounds are known (the same quasi-energy as above is exploited for the approximate system), their possible dependence on κ has not yet been ruled out and will be inspected in Sections 5.2 and 5.3. While the strong convergence properties entailed by these bounds (due to the independence of κ) would also entail a time-local convergence in certain L^p spaces in the limit $\kappa \to 0$, we strive for a stronger convergence result global in time. To expand the knowledge, however, we will need to meticulously adjust the analytic machinery behind the eventual smoothness results of [105, 47] in order to be able to carefully track the possible κ-dependence in the eventual smallness of oxygen, the eventual regularity estimates for $n_\varepsilon^{(\kappa)}$ and $c_\varepsilon^{(\kappa)}$ and their eventual stabilization properties presented in Sections 5.4 – 5.6. We can then utilize maximal Sobolev regularity estimates for the Stokes and Neumann heat semigroups to obtain an eventual smoothing time $T_\diamond > 0$, which does not depend on κ, ensuring that the triple $\big(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)}\big)$, obtained in the limit $\varepsilon \to 0$, solves (Λ_κ) classically on $\Omega \times (T_\diamond, \infty)$ (Section 5.7). Section 5.8 will then be devoted to gain insight in exponential decay estimates valid starting from the smoothing time $T_\diamond > 0$ and finally in Section 5.9, we will take $\kappa \to 0$ to obtain Theorem 5.1.

5.2 Preliminaries. Weak solutions and a priori information for a family of approximating systems

Before we start with our detailed analysis, let us also briefly specify what constitutes a weak solution as mentioned in Theorem 5.1. In the following definition, adapted from [104], we merely prescribe the weakest regularity necessary to ensure that all integrals in the equalities below are well defined. The solutions constructed later, however, will satisfy considerably stronger regularity assumptions.

Definition 5.2.
For $\kappa \in [-1,1]$ a triple $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ of functions

$$
\begin{aligned}
n^{(\kappa)} &\in L^1_{loc}\big([0,\infty); W^{1,1}(\Omega)\big)\,,\\
c^{(\kappa)} &\in L^1_{loc}\big([0,\infty); W^{1,1}(\Omega)\big)\,,\\
u^{(\kappa)} &\in L^1_{loc}\big([0,\infty); W^{1,1}_0(\Omega;\mathbb{R}^3)\big),
\end{aligned}
$$

satisfying $n^{(\kappa)} \geq 0$, $c^{(\kappa)} \geq 0$ and $\nabla \cdot u^{(\kappa)} = 0$ a.e. in $\overline{\Omega} \times [0,\infty)$ as well as the properties

$$n^{(\kappa)}c^{(\kappa)} \in L^1_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \quad \textit{and} \quad \kappa u^{(\kappa)} \otimes u^{(\kappa)} \in L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^{3\times 3}\big)$$

with

$$n^{(\kappa)}\nabla c^{(\kappa)}, \quad n^{(\kappa)}u^{(\kappa)} \quad \textit{and} \quad c^{(\kappa)}u^{(\kappa)} \quad \textit{belonging to} \quad L^1_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big),$$

will be called a weak solution of the system (Λ_κ), (5.1.1) *and* (5.1.2), *if the equality*

$$\begin{aligned}
&-\int_0^\infty\int_\Omega n^{(\kappa)}\varphi - \int_\Omega n_0\varphi(\cdot,0) \\
&\qquad = \int_0^\infty\int_\Omega n^{(\kappa)}u^{(\kappa)}\cdot\nabla\varphi - \int_0^\infty\int_\Omega \nabla n^{(\kappa)}\cdot\nabla\varphi + \int_0^\infty\int_\Omega n^{(\kappa)}\nabla c^{(\kappa)}\cdot\nabla\varphi
\end{aligned}$$

holds for each $\varphi \in C_0^\infty\big(\overline{\Omega}\times[0,\infty)\big)$, *if moreover*

$$\begin{aligned}
&-\int_0^\infty\int_\Omega c^{(\kappa)}\psi_t - \int_\Omega c_0\psi(\cdot,0) \\
&\qquad = \int_0^\infty\int_\Omega c^{(\kappa)}u^{(\kappa)}\cdot\nabla\psi - \int_0^\infty\int_\Omega \nabla c^{(\kappa)}\cdot\nabla\psi - \int_0^\infty\int_\Omega n^{(\kappa)}c^{(\kappa)}\psi
\end{aligned}$$

is fulfilled for every $\psi \in C_0^\infty\big(\overline{\Omega}\times[0,\infty)\big)$ *and if finally*

$$\begin{aligned}
&-\int_0^\infty\int_\Omega u^{(\kappa)}\cdot\Psi_t - \int_\Omega u_0\cdot\Psi(\cdot,0) \\
&\qquad = -\int_0^\infty\int_\Omega \nabla u^{(\kappa)}\cdot\nabla\Psi + \kappa\int_0^\infty\int_\Omega u^{(\kappa)}\otimes u^{(\kappa)}\cdot\nabla\Psi + \int_0^\infty\int_\Omega n^{(\kappa)}\Psi\cdot\nabla\phi
\end{aligned}$$

is valid for all $\Psi \in C_0^\infty\big(\Omega\times[0,\infty);\mathbb{R}^3\big)$ *satisfying* $\nabla\cdot\Psi \equiv 0$.

Weak solutions to (Λ_κ), in the sense above, will be constructed as limit objects from a family of appropriately regularized systems. The regularization we incorporate for our problem has previously (and in a more general fashion) been employed in [47, 104, 105]. To be precise, for $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ we will consider

$$\begin{cases}
n^{(\kappa)}_{\varepsilon t} + u^{(\kappa)}_\varepsilon\cdot\nabla n^{(\kappa)}_\varepsilon = \Delta n^{(\kappa)}_\varepsilon - \nabla\cdot\Big(\frac{n^{(\kappa)}_\varepsilon}{1+\varepsilon n^{(\kappa)}_\varepsilon}\nabla c^{(\kappa)}_\varepsilon\Big), & x\in\Omega, \quad t>0,\\
c^{(\kappa)}_{\varepsilon t} + u^{(\kappa)}_\varepsilon\cdot\nabla c^{(\kappa)}_\varepsilon = \Delta c^{(\kappa)}_\varepsilon - \frac{1}{\varepsilon}\ln\big(1+\varepsilon n^{(\kappa)}_\varepsilon\big)c^{(\kappa)}_\varepsilon, & x\in\Omega, \quad t>0,\\
u^{(\kappa)}_{\varepsilon t} + \kappa(Y_\varepsilon u^{(\kappa)}_\varepsilon\cdot\nabla)u^{(\kappa)}_\varepsilon = \Delta u^{(\kappa)}_\varepsilon - \nabla P^{(\kappa)}_\varepsilon + n^{(\kappa)}_\varepsilon\nabla\phi, & x\in\Omega, \quad t>0,\\
\nabla\cdot u^{(\kappa)}_\varepsilon = 0, & x\in\Omega, \quad t>0,\\
\partial_\nu n^{(\kappa)}_\varepsilon = 0, \qquad \partial_\nu c^{(\kappa)}_\varepsilon = 0, \qquad u^{(\kappa)}_\varepsilon = 0, & x\in\partial\Omega, \quad t>0,\\
n^{(\kappa)}_\varepsilon(x,0) = n_0(x), \quad c^{(\kappa)}_\varepsilon(x,0) = c_0(x), \quad u^{(\kappa)}_\varepsilon(x,0) = u_0(x), & x\in\Omega,
\end{cases} \tag{$\Lambda_{\varepsilon,\kappa}$}$$

where for $\varepsilon \in (0,1)$, Y_ε denotes the standard Yosida approximation ([60, 75]) given by

$$Y_\varepsilon\varphi := (1+\varepsilon A)^{-1}\varphi, \quad \text{for } \varphi \in L^2_\sigma(\Omega).$$

Let us also note that

$$\frac{1}{2}\min\{s,1\} \le \frac{1}{\varepsilon}\ln(1+\varepsilon s) \le s \quad \text{for all } s \ge 0 \text{ and all } \varepsilon \in (0,1), \tag{5.2.1}$$

which, due to the nonnegativity of $n_\varepsilon^{(\kappa)}$ we will establish later, are two useful estimates for one of the terms appearing in the second equation of $(\Lambda_{\varepsilon,\kappa})$ and will be used on multiple occasions throughout the chapter.
Now, let us start our analysis by gathering basic results for the family of approximating systems, most of which has already been discussed in works with fixed $\kappa = 1$ and can be obtained in well-known manner. Nevertheless, we have to ascertain that all of these familiar properties are κ-independent and therefore we will take a closer look at some (parts) of the proofs involved.

Lemma 5.3.
Let $q > 3$. For any $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ there exist $T_{max,\varepsilon}^{(\kappa)} \in (0,\infty]$ and a unique triplet $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of functions satisfying

$$\begin{aligned}
n_\varepsilon^{(\kappa)} &\in C^0(\overline{\Omega}\times[0,T_{max,\varepsilon}^{(\kappa)})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max,\varepsilon}^{(\kappa)})),\\
c_\varepsilon^{(\kappa)} &\in C^0(\overline{\Omega}\times[0,T_{max,\varepsilon}^{(\kappa)})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max,\varepsilon}^{(\kappa)})) \cap L^\infty\big((0,T_{max,\varepsilon}^{(\kappa)});W^{1,q}(\Omega)\big),\\
u_\varepsilon^{(\kappa)} &\in C^0(\overline{\Omega}\times[0,T_{max,\varepsilon}^{(\kappa)});\mathbb{R}^3) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max,\varepsilon}^{(\kappa)});\mathbb{R}^3),
\end{aligned}$$

which together with some $P_\varepsilon^{(\kappa)} \in C^{1,0}(\Omega\times(0,T_{max,\varepsilon}^{(\kappa)}))$ solves $(\Lambda_{\varepsilon,\kappa})$ classically in $\Omega \times (0,T_{max,\varepsilon}^{(\kappa)})$. In addition, if $T_{max,\varepsilon}^{(\kappa)} < \infty$, then

$$\|n_\varepsilon^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon^{(\kappa)}(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\varrho u_\varepsilon^{(\kappa)}(\cdot,t)\|_{L^2(\Omega)} \to \infty \ \text{ as } t \nearrow T_{max,\varepsilon}^{(\kappa)}$$

for all $\varrho \in (\frac{3}{4},1)$. The triplet $\big(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)}\big)$ moreover satisfies $n_\varepsilon^{(\kappa)} \ge 0$ and $c_\varepsilon^{(\kappa)} > 0$ in $\overline{\Omega}\times[0,T_{max,\varepsilon}^{(\kappa)})$ as well as

$$\int_\Omega n_\varepsilon^{(\kappa)}(\cdot,t) = \int_\Omega n_0 \ \text{ and } \ \|c_\varepsilon^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in [0,T_{max,\varepsilon}^{(\kappa)}) \tag{5.2.2}$$

and the mapping $t \mapsto \big\|c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)}$ is nonincreasing on $(0,\infty)$.

Proof: The proof of this local existence result draws on a standard reasoning involving semigroup estimates, Banach's fixed point theorem employed to a closed subset of $L^\infty\big((0,T);C^0\big(\overline{\Omega}\big)\times W^{1,q}(\Omega)\times D(A^\varrho)\big)$ and parabolic regularity theory. We refer the reader to [98, Lemma 2.1] for a detailed proof of the existence of a unique local solution, the extensibility criterion and the nonnegativity and positivity properties in a closely related setting. The conservation of mass $\int_\Omega n_\varepsilon^{(\kappa)} = \int_\Omega n_0$ on $(0,T_{max,\varepsilon}^{(\kappa)})$ then follows directly from integrating the first equation of $(\Lambda_{\varepsilon,\kappa})$, whereas the nonincreasing property of $t \mapsto \big\|c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)}$ on $(0,\infty)$ and bound for $\big\|c_\varepsilon^{(\kappa)}\big\|_{L^\infty(\Omega)}$ are an immediate consequence of the parabolic comparison principle employed to the second equation of $(\Lambda_{\varepsilon,\kappa})$. □

Since κ only impacts the third equation of $(\Lambda_{\varepsilon,\kappa})$ directly, we can, without any necessary change, adopt the results from [47, Lemmata 2.6 and 2.8] and [104, Lemma 3.4] to obtain the following:

Lemma 5.4.
There exists $K_0 > 0$ such that for all $\varepsilon \in (0,1)$ and all $\kappa \in [-1,1]$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\varepsilon^{(\kappa)} \ln n_\varepsilon^{(\kappa)} + \frac{1}{2}\int_\Omega \frac{\big|\nabla c_\varepsilon^{(\kappa)}\big|^2}{c_\varepsilon^{(\kappa)}}\right) + \frac{1}{K_0}\left(\int_\Omega \frac{\big|\nabla n_\varepsilon^{(\kappa)}\big|^2}{n_\varepsilon^{(\kappa)}} + \int_\Omega \frac{\big|D^2 c_\varepsilon^{(\kappa)}\big|^2}{c_\varepsilon^{(\kappa)}} + \int_\Omega \frac{\big|\nabla c_\varepsilon^{(\kappa)}\big|^4}{{c_\varepsilon^{(\kappa)}}^3}\right) \le K_0 \int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 + K_0$$

on $(0, T_{max,\varepsilon}^{(\kappa)})$.

Proof: Since the well-established testing procedures used to derive this inequality do not depend on κ in any way, we refer the reader to the detailed proofs in [47, Lemmata 2.6 and 2.8] (with $\kappa = 1$) and [104, Lemma 3.4] (in convex domains with $\kappa = 1$). □

Moreover, due to $u_\varepsilon^{(\kappa)}$ being divergence free, testing the third equation against $u_\varepsilon^{(\kappa)}$ itself also removes any dependence on κ and hence we readily transfer the result from [47, Lemma 2.9] to our setting.

Lemma 5.5.
For any $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 = \int_\Omega n_\varepsilon^{(\kappa)} \nabla\phi \cdot u_\varepsilon^{(\kappa)}$$

on $(0, T_{max,\varepsilon}^{(\kappa)})$.

Proof: Since $\nabla \cdot u_\varepsilon^{(\kappa)} = 0$ on $\Omega \times (0, T_{max,\varepsilon}^{(\kappa)})$ also implies that $\nabla \cdot Y_\varepsilon u_\varepsilon^{(\kappa)} = 0$ on $\Omega \times (0, T_{max,\varepsilon}^{(\kappa)})$, we have

$$\kappa \int_\Omega \big(Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla\big) u_\varepsilon^{(\kappa)} \cdot u_\varepsilon^{(\kappa)} = -\kappa \int_\Omega \nabla \cdot \big(Y_\varepsilon u_\varepsilon^{(\kappa)}\big) \big|u_\varepsilon^{(\kappa)}\big|^2 - \frac{\kappa}{2}\int_\Omega Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla \big|u_\varepsilon^{(\kappa)}\big|^2 = 0$$

on $\Omega \times (0, T_{max,\varepsilon}^{(\kappa)})$. Thus, we find that by multiplying the third equation of $(\Lambda_{\varepsilon,\kappa})$ by $u_\varepsilon^{(\kappa)}$ and integrating by parts

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 = \int_\Omega n_\varepsilon^{(\kappa)} \nabla\phi \cdot u_\varepsilon^{(\kappa)}$$

is valid on $(0, T_{max,\varepsilon}^{(\kappa)})$. □

A combination of the previous two lemmata now yields uniform a priori estimates which will be the basis for the remainder of our regularity analysis.

Lemma 5.6.
Let $K_0 > 0$ be provided by Lemma 5.4. There exists $K_1 > 0$ such that for all $\varepsilon \in (0,1)$ and each $\kappa \in [-1,1]$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_\Omega n_\varepsilon^{(\kappa)} \ln n_\varepsilon^{(\kappa)} + \frac{1}{2}\int_\Omega \frac{\big|\nabla c_\varepsilon^{(\kappa)}\big|^2}{c_\varepsilon^{(\kappa)}} + K_0 \int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2 \le K_1$$

on $(0, T_{max,\varepsilon}^{(\kappa)})$ and

$$\int_t^{t+\tau}\int_\Omega \frac{\big|\nabla n_\varepsilon^{(\kappa)}\big|^2}{n_\varepsilon^{(\kappa)}} + \int_t^{t+\tau}\int_\Omega \frac{\big|D^2 c_\varepsilon^{(\kappa)}\big|^2}{c_\varepsilon^{(\kappa)}} + \int_t^{t+\tau}\int_\Omega \frac{\big|\nabla c_\varepsilon^{(\kappa)}\big|^4}{{c_\varepsilon^{(\kappa)}}^3} + \int_t^{t+\tau}\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 + \int_t^{t+\tau}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 \le K_1$$

for all $t \in (0, T_{max,\varepsilon}^{(\kappa)} - \tau)$, where $\tau := \min\big\{1, \frac{1}{2}T_{max,\varepsilon}^{(\kappa)}\big\}$.

Proof: We replicate and adjust the steps of [47, Lemmata 2.10 and 2.11] and [104, Lemmata 3.6 and 3.8]. Adding up suitable multiples of the differential inequalities from Lemma 5.4 and Lemma 5.5 we find that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega n_\varepsilon^{(\kappa)} \ln n_\varepsilon^{(\kappa)} + \frac{1}{2}\int_\Omega \frac{\big|\nabla c_\varepsilon^{(\kappa)}\big|^2}{c_\varepsilon^{(\kappa)}} + K_0 \int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2\bigg) + K_0 \int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 \\ + \frac{1}{K_0}\bigg(\int_\Omega \frac{\big|\nabla n_\varepsilon^{(\kappa)}\big|^2}{n_\varepsilon^{(\kappa)}} + \int_\Omega \frac{\big|D^2 c_\varepsilon^{(\kappa)}\big|^2}{c_\varepsilon^{(\kappa)}} + \int_\Omega \frac{\big|\nabla c_\varepsilon^{(\kappa)}\big|^4}{{c_\varepsilon^{(\kappa)}}^3}\bigg) \le 2K_0 \int_\Omega n_\varepsilon^{(\kappa)} \nabla\phi \cdot u_\varepsilon^{(\kappa)} + K_0\end{aligned} \tag{5.2.3}$$

holds on $(0, T_{max,\varepsilon}^{(\kappa)})$. To estimate the right-hand side further, we make use of the boundedness of $\nabla\phi$ and Hölder's inequality, the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and the Poincaré inequality to obtain $C_1 > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$ we have

$$\begin{aligned}\int_\Omega n_\varepsilon^{(\kappa)} \nabla\phi \cdot u_\varepsilon^{(\kappa)} &\le \big\|\nabla\phi\big\|_{L^\infty(\Omega)} \big\|n_\varepsilon^{(\kappa)}\big\|_{L^{\frac{6}{5}}(\Omega)} \big\|u_\varepsilon^{(\kappa)}\big\|_{L^6(\Omega)} \\ &\le C_1 \big\|\nabla\phi\big\|_{L^\infty(\Omega)} \big\|n_\varepsilon^{(\kappa)}\big\|_{L^{\frac{6}{5}}(\Omega)} \big\|\nabla u_\varepsilon^{(\kappa)}\big\|_{L^2(\Omega)}.\end{aligned}$$

Here, we employ Young's inequality to find that

$$2K_0 \int_\Omega n_\varepsilon^{(\kappa)} \nabla\phi \cdot u_\varepsilon^{(\kappa)} \le \frac{K_0}{2}\big\|\nabla u_\varepsilon^{(\kappa)}\big\|_{L^2(\Omega)}^2 + 2K_0 C_1^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2 \big\|n_\varepsilon^{(\kappa)}\big\|_{L^{\frac{6}{5}}(\Omega)}^2 \tag{5.2.4}$$

on $(0, T_{max,\varepsilon}^{(\kappa)})$. According to the Gagliardo–Nirenberg inequality there is some $C_2 > 0$ such that

$$\|\varphi\|_{L^{\frac{12}{5}}(\Omega)}^4 \le C_2 \|\nabla\varphi\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}^3 + C_2 \|\varphi\|_{L^2(\Omega)}^4$$

holds for all $\varphi \in W^{1,2}(\Omega)$ and hence, in light of the mass conservation $\int_\Omega n_\varepsilon^{(\kappa)} = \int_\Omega n_0$ for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$ from Lemma 5.3, there exists some $C_3 > 0$ such that for each $\varepsilon(0,1)$ and $\kappa \in [-1,1]$

$$\begin{aligned}\big\|n_\varepsilon^{(\kappa)}\big\|_{L^{\frac{6}{5}}(\Omega)}^2 = \big\|n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^{\frac{12}{5}}(\Omega)}^4 &\le C_2\big\|\nabla n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^2(\Omega)}\big\|n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^2(\Omega)}^3 + C_2\big\|n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^2(\Omega)}^4 \\ &\le C_3\big\|\nabla n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^2(\Omega)} + C_3\end{aligned}$$

is valid on $(0, T_{max,\varepsilon}^{(\kappa)})$. Employing Young's inequality once more in (5.2.4), we thereby obtain $C_4 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$ the inequality

$$2K_0\int_\Omega n_\varepsilon^{(\kappa)}\nabla\phi\cdot u_\varepsilon^{(\kappa)} \le \frac{K_0}{2}\int_\Omega\big|\nabla u_\varepsilon^{(\kappa)}\big|^2 + \frac{1}{2K_0}\int_\Omega\frac{\big|\nabla n_\varepsilon^{(\kappa)}\big|^2}{n_\varepsilon^{(\kappa)}} + C_4$$

holds. Plugging this into (5.2.3) we find $C_5 := \max\{C_4 + K_0, \frac{2}{K_0}, 2K_0\} > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$ the functions

$$\begin{aligned} y_\varepsilon^{(\kappa)}(t) &:= \int_\Omega\big(n_\varepsilon^{(\kappa)}\ln n_\varepsilon^{(\kappa)}\big)(\cdot,t) + \frac{1}{2}\int_\Omega\frac{\big|\nabla c_\varepsilon^{(\kappa)}(\cdot,t)\big|^2}{c_\varepsilon^{(\kappa)}(\cdot,t)} + K_0\int_\Omega\big|u_\varepsilon^{(\kappa)}(\cdot,t)\big|^2 \\ \text{and } h_\varepsilon^{(\kappa)}(t) &:= \int_\Omega\frac{\big|\nabla n_\varepsilon^{(\kappa)}(\cdot,t)\big|^2}{n_\varepsilon^{(\kappa)}(\cdot,t)} + \int_\Omega\frac{\big|D^2c_\varepsilon^{(\kappa)}(\cdot,t)\big|^2}{c_\varepsilon^{(\kappa)}(\cdot,t)} + \int_\Omega\frac{\big|\nabla c_\varepsilon^{(\kappa)}(\cdot,t)\big|^4}{\big(c_\varepsilon^{(\kappa)}(\cdot,t)\big)^3} + \int_\Omega\big|\nabla u_\varepsilon^{(\kappa)}(\cdot,t)\big|^2\end{aligned}$$

satisfy the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon^{(\kappa)}(t) + \frac{1}{C_5}h_\varepsilon^{(\kappa)}(t) \le C_5. \tag{5.2.5}$$

Invoking the Poincaré inequality, Young's inequality, the boundedness of $c_\varepsilon^{(\kappa)}$, the inequality $z\ln z \le \frac{3}{2}z^{\frac{5}{3}}$ for $z \ge 0$ and the Gagliardo–Nirenberg inequality, it can be easily checked that there is some $C_6 > 0$ (independent of ε and κ) such that

$$y_\varepsilon^{(\kappa)}(t) \le C_6h_\varepsilon^{(\kappa)}(t) + C_6 \quad \text{for all } t \in (0, T_{max,\varepsilon}^{(\kappa)}).$$

And hence (5.2.5) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon^{(\kappa)}(t) + \frac{1}{2C_5}h_\varepsilon^{(\kappa)}(t) + \frac{1}{2C_5C_6}y_\varepsilon^{(\kappa)}(t) \le C_5 + \frac{1}{2C_5} \quad \text{for all } t \in (0, T_{max,\varepsilon}^{(\kappa)}),$$

which on the one hand implies for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$ that

$$y_\varepsilon^{(\kappa)}(t) \le C_7 := \max\Big\{\int_\Omega n_0\ln n_0 + \frac{1}{2}\int_\Omega\frac{|\nabla c_0|^2}{c_0} + K_0\int_\Omega|u_0|^2, 2C_5^2C_6 + C_6\Big\},$$

and, on the other hand, shows upon integration that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$

$$\frac{1}{2C_5}\int_t^{t+\tau} h_\varepsilon^{(\kappa)}(t)\,\mathrm{d}t \le y_\varepsilon^{(\kappa)}(0) + \big(C_5 + \frac{1}{2C_5}\big)\tau \le C_7 + C_5 + \frac{1}{2C_5} =: C_8$$

is valid for all $t \in (0, T_{max,\varepsilon}^{(\kappa)} - \tau)$ with $\tau := \min\{1, \frac{1}{2}T_{max,\varepsilon}^{(\kappa)}\}$. Moreover, drawing on the boundedness of $c_\varepsilon^{(\kappa)}$ obtained in Lemma 5.3, we find that

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 &\le \sup_{s\in[t,t+\tau]} \big\|c_\varepsilon^{(\kappa)}(\cdot,s)\big\|_{L^\infty(\Omega)}^3 \int_t^{t+\tau}\int_\Omega \frac{|\nabla c_\varepsilon^{(\kappa)}|^4}{{c_\varepsilon^{(\kappa)}}^3}\\ &\le \|c_0\|_{L^\infty(\Omega)}^3 \int_t^{t+\tau} h_\varepsilon^{(\kappa)}(t)\,\mathrm{d}t \le 2C_5C_8\|c_0\|_{L^\infty(\Omega)}^3\end{aligned}$$

is valid for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t \in (0, T_{max,\varepsilon}^{(\kappa)} - \tau)$, completing the proof upon obvious choice of $K_1 > 0$. □

Assuming a finite maximal existence time, we can now make use of the bounds from the previous lemma to derive a contradiction to the extensibility criterion featured in the local existence result.

Lemma 5.7.
For all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ the solution to $(\Lambda_{\varepsilon,\kappa})$ is global in time, i.e. $T_{max,\varepsilon}^{(\kappa)} = \infty$.

Proof: Assuming $T_{max,\varepsilon}^{(\kappa)}$ to be finite we will derive a contradiction to the extensibility criterion presented in Lemma 5.3. Reasoning along these lines is common in many related works and can e.g. be found in [104]. For the sake of completeness, we sketch the main parts of the proof. We first note that, due to $T_{max,\varepsilon}^{(\kappa)} < \infty$, Lemma 5.6 provides the existence of $C_1 > 0$ satisfying

$$\int_0^{T_{max,\varepsilon}^{(\kappa)}}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 \le C_1 \quad\text{and}\quad \int_\Omega \big|u_\varepsilon^{(\kappa)}(\cdot,t)\big|^2 \le C_1 \quad\text{for all } t \in (0, T_{max,\varepsilon}^{(\kappa)}). \tag{5.2.6}$$

Testing the first equation of $(\Lambda_{\varepsilon,\kappa})$ against $(n_\varepsilon^{(\kappa)})^3$, we find upon integrating by parts, utilizing the fact that $\frac{s}{1+\varepsilon s} \le \frac{1}{\varepsilon}$ for all $s \ge 0$ and invoking Young's inequality that

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {n_\varepsilon^{(\kappa)}}^4 + 3\int_\Omega {n_\varepsilon^{(\kappa)}}^2\big|\nabla n_\varepsilon^{(\kappa)}\big|^2 \le \int_\Omega {n_\varepsilon^{(\kappa)}}^2\big|\nabla n_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 + \frac{81}{64\varepsilon^4}\int_\Omega {n_\varepsilon^{(\kappa)}}^4$$

on $(0, T_{max,\varepsilon}^{(\kappa)})$, implying that there is some $C_2 > 0$ (possibly depending on ε) such that $\int_\Omega (n_\varepsilon^{(\kappa)})^4(\cdot,t) \le C_2$ holds for all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$, according to (5.2.6). Furthermore, in light of the embedding $D(1+\varepsilon A) = W^{2,2}(\Omega;\mathbb{R}^3) \cap W_{0,\sigma}^{1,2}(\Omega) \hookrightarrow L^\infty\big(\Omega;\mathbb{R}^3\big)$ and (5.2.6) we obtain $C_3, C_4 > 0$ satisfying

$$\begin{aligned}\big\|Y_\varepsilon u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)} &= \big\|(1+\varepsilon A)^{-1}u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)}\\ &\le C_3\big\|u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^2(\Omega)} \le C_4 \quad\text{for all } t \in (0, T_{max,\varepsilon}^{(\kappa)}).\end{aligned}$$

Hence, testing $u_{\varepsilon t}^{(\kappa)} + Au_\varepsilon^{(\kappa)} = \mathcal{P}\Big(-\kappa\big(Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla\big)u_\varepsilon^{(\kappa)} + n_\varepsilon^{(\kappa)}\nabla\phi\Big)$ against $Au_\varepsilon^{(\kappa)}$ we obtain some $C_5 > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|A^{\frac{1}{2}}u_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega \big|Au_\varepsilon^{(\kappa)}\big|^2 \leq \int_\Omega \big|Au_\varepsilon^{(\kappa)}\big|^2 + C_5\left(\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega n_\varepsilon^{(\kappa)2}\right)$$

on $(0, T_{max,\varepsilon}^{(\kappa)})$, in light of Young's inequality, (5.1.3) and the facts that $|\kappa| \leq 1$ and $\|\mathcal{P}\varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)}$ for all $\varphi \in L^2\big(\Omega;\mathbb{R}^3\big)$. Since $\int_\Omega |A^{\frac{1}{2}}\varphi| = \int_\Omega |\nabla\varphi|^2$ for $\varphi \in D(A)$, we thereby find $C_6 > 0$ fulfilling

$$\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}(\cdot,t)\big|^2 \leq C_6 \quad \text{for all } t \in (0, T_{max,\varepsilon}^{(\kappa)}).$$

Combining these bounds with well-known properties of the Stokes semigroup (see e.g. [29, p.201]) first provides a bound on $\big\|A^\varrho u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^2(\Omega)}$ for all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$, where ϱ is as in (5.1.4). By our choice of ϱ, the embedding $D(A^\varrho) \hookrightarrow L^\infty\big(\Omega;\mathbb{R}^3\big)$ also readily entails an L^∞ bound on the third component. Secondly, combining these bounds with semigroup estimates for the Neumann heat semigroup (e.g. [97, Lemma 1.3]), (5.2.1), (5.2.2) and (5.2.6), implies the boundedness of $\big\|\nabla c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^4(\Omega)}$ for all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$, which upon final combination with Neumann heat semigroup estimates with previous bounds also yields a bound on $\big\|n_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)}$ for all $t \in (0, T_{max,\varepsilon}^{(\kappa)})$, contradicting the extensibility criterion from Lemma 5.3, and hence we conclude $T_{max,\varepsilon}^{(\kappa)} = \infty$. □

In a straightforward manner, we can also draw on the Gagliardo–Nirenberg and Hölder inequalities to refine the spatio-temporal bounds on the gradient terms in Lemma 5.6 into slightly improved bounds for $n_\varepsilon^{(\kappa)}$, $\nabla n_\varepsilon^{(\kappa)}$ and $u_\varepsilon^{(\kappa)}$. The following lemma will play an important role in deriving the necessary precompactness properties to verify that the objects obtained from the limiting procedure actually constitute a weak solution of our system.

Lemma 5.8.
For every $T > 0$ there exists $C(T) > 0$ such that for any $\varepsilon \in (0,1)$ and all $\kappa \in [-1,1]$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_0^T\int_\Omega n_\varepsilon^{(\kappa)\frac{5}{3}} + \int_0^T\int_\Omega \big|\nabla n_\varepsilon^{(\kappa)}\big|^{\frac{5}{4}} + \int_0^T\int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^{\frac{10}{3}} \leq C(T).$$

Proof: These spatio-temporal bounds are an immediate consequence of the Gagliardo–Nirenberg and Hölder inequalities along with the bounds prepared in Lemma 5.6. Details on the steps involved are found in [104, Lemma 3.10]. □

5.3 Existence of a limit solution family when $\varepsilon \searrow 0$

In preparation of an Aubin–Lions type argument, which is the starting point for our convergence result, we will require information on the regularity of the time derivatives of our solution components. Again, taking care that our estimates do neither depend on ε nor on κ, these bounds on the time derivative will not only be useful for the ε–limit, but also for the κ–limit discussed in Section 5.9.

Lemma 5.9.
For any $T>0$ there exists $C>0$ such that for each $\varepsilon\in(0,1)$ and $\kappa\in[-1,1]$ the solution $(n_\varepsilon^{(\kappa)},c_\varepsilon^{(\kappa)},u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_0^T\big\|n_{\varepsilon t}^{(\kappa)}\big\|_{(W^{1,10}(\Omega))^*}^{\frac{10}{9}}+\int_0^T\Big\|\partial_t\sqrt{c_\varepsilon^{(\kappa)}}\Big\|_{\left(W^{1,\frac52}(\Omega)\right)^*}^{\frac53}+\int_0^T\big\|u_{\varepsilon t}^{(\kappa)}\big\|_{\left(W_{0,\sigma}^{1,5}(\Omega)\right)^*}^{\frac54}\le C.$$

Proof: The proof is basically contained in [104, Lemma 3.11] (where $\kappa=1$ was treated). To ensure that the constant does not depend on κ, we will illustrate the steps involved for the fluid component. For details regarding the other two estimation procedures (which work along similar lines), we refer the reader to the work mentioned above. Given any fixed $\varphi\in C_{0,\sigma}^\infty(\Omega)$, we test the third equation of $(\Lambda_{\varepsilon,\kappa})$ against φ and employ Hölder's inequality to obtain that, due to $|\kappa|\le1$,

$$\begin{aligned}
&\left|\int_\Omega u_{\varepsilon t}^{(\kappa)}(\cdot,t)\cdot\varphi\right|\\
=&\left|-\int_\Omega\nabla u_\varepsilon^{(\kappa)}(\cdot,t)\cdot\nabla\varphi-\kappa\int_\Omega(Y_\varepsilon u_\varepsilon^{(\kappa)}\otimes u_\varepsilon^{(\kappa)})(\cdot,t)\cdot\nabla\varphi+\int_\Omega n_\varepsilon^{(\kappa)}(\cdot,t)\nabla\phi\cdot\varphi\right|\\
\le&\left(\big\|\nabla u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^{\frac54}(\Omega)}+\big\|(Y_\varepsilon u_\varepsilon^{(\kappa)}\otimes u_\varepsilon^{(\kappa)})(\cdot,t)\big\|_{L^{\frac54}(\Omega)}+\big\|n_\varepsilon^{(\kappa)}(\cdot,t)\nabla\phi\big\|_{L^{\frac54}(\Omega)}\right)\|\nabla\varphi\|_{W^{1,5}(\Omega)}
\end{aligned}$$

is valid for all $t>0$. In light of (5.1.3) we can find $C_1>0$ such that $\|\nabla\phi\|_{L^\infty(\Omega)}\le C_1$ and hence Young's inequality entails that, with $C_2:=2^{\frac14}(1+C_1)>0$, we have

$$\begin{aligned}
&\int_0^T\big\|u_{\varepsilon t}^{(\kappa)}(\cdot,t)\big\|_{\left(W_{0,\sigma}^{1,5}(\Omega)\right)^*}^{\frac54}\,\mathrm{d}t\\
\le&\, C_2\int_0^T\int_\Omega\big|\nabla u_\varepsilon^{(\kappa)}\big|^{\frac54}+C_2\int_0^T\int_\Omega\big|Y_\varepsilon u_\varepsilon^{(\kappa)}\otimes u_\varepsilon^{(\kappa)}\big|^{\frac54}+C_2\int_0^T\int_\Omega n_\varepsilon^{(\kappa)\,\frac54}\qquad(5.3.1)\\
\le&\, C_2\int_0^T\int_\Omega\big|\nabla u_\varepsilon^{(\kappa)}\big|^2+C_2\int_0^T\int_\Omega\big|Y_\varepsilon u_\varepsilon^{(\kappa)}\big|^2+C_2\int_0^T\int_\Omega\big|u_\varepsilon^{(\kappa)}\big|^{\frac{10}{3}}+C_2\int_0^T\int_\Omega n_\varepsilon^{(\kappa)\,\frac53}+2C_2|\Omega|T
\end{aligned}$$

for all $T>0$. Drawing on the fact that $\|Y_\varepsilon v\|_{L^2(\Omega)}\le\|v\|_{L^2(\Omega)}$ holds for all $v\in L_\sigma^2(\Omega)$, we may employ Young's inequality once more to estimate $\int_0^T\int_\Omega\big|Y_\varepsilon u_\varepsilon^{(\kappa)}\big|^2\le\int_0^T\int_\Omega\big|u_\varepsilon^{(\kappa)}\big|^{\frac{10}{3}}+|\Omega|T$ and thus conclude the asserted bound from (5.3.1) in light of Lemmata 5.6 and 5.8. ☐

With the uniform bounds from Lemmata 5.3, 5.6, 5.8 and 5.9 we are now in the position to obtain limit functions $n^{(\kappa)}$, $c^{(\kappa)}$ and $u^{(\kappa)}$, which fulfill the regularity assumptions and integral equations required for the weak solution formulation of (Λ_κ).

Proposition 5.10.
There exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ with the property that for any $\kappa \in [-1,1]$ one can find functions

$$\begin{aligned} &n^{(\kappa)} \in L^{\frac{5}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \quad \textit{with} \quad \nabla n^{(\kappa)} \in L^{\frac{5}{4}}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \\ &c^{(\kappa)} \in L^{\infty}\left(\Omega\times(0,\infty)\right) \quad \textit{with} \quad \nabla c^{(\kappa)} \in L^{4}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, \\ &u^{(\kappa)} \in L^{2}_{loc}\big([0,\infty);W^{1,2}_{0,\sigma}(\Omega)\big), \end{aligned}$$

such that the solution $(n^{(\kappa)}_\varepsilon, c^{(\kappa)}_\varepsilon, u^{(\kappa)}_\varepsilon)$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\begin{aligned} n^{(\kappa)}_\varepsilon &\to n^{(\kappa)} && \textit{in } L^{p}_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \textit{ for any } p\in[1,\tfrac{5}{3}) \textit{ and a.e. in } \Omega\times(0,\infty), && (5.3.2)\\ \nabla n^{(\kappa)}_\varepsilon &\rightharpoonup \nabla n^{(\kappa)} && \textit{in } L^{\frac{5}{4}}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, && (5.3.3)\\ n^{(\kappa)}_\varepsilon &\rightharpoonup n^{(\kappa)} && \textit{in } L^{\frac{5}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big)\,, && (5.3.4)\\ c^{(\kappa)}_\varepsilon &\to c^{(\kappa)} && \textit{in } L^{p}_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \textit{ for any } p\in[1,\infty) \textit{ and a.e. in } \Omega\times(0,\infty), && (5.3.5)\\ c^{(\kappa)}_\varepsilon &\overset{\star}{\rightharpoonup} c^{(\kappa)} && \textit{in } L^{\infty}\left(\Omega\times(0,\infty)\right), && (5.3.6)\\ \nabla c^{(\kappa)}_\varepsilon &\rightharpoonup \nabla c^{(\kappa)} && \textit{in } L^{4}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, && (5.3.7)\\ u^{(\kappa)}_\varepsilon &\to u^{(\kappa)} && \textit{in } L^{2}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big) \textit{ and a.e. in } \Omega\times(0,\infty), && (5.3.8)\\ u^{(\kappa)}_\varepsilon &\rightharpoonup u^{(\kappa)} && \textit{in } L^{\frac{10}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)\,, && (5.3.9)\\ \nabla u^{(\kappa)}_\varepsilon &\rightharpoonup \nabla u^{(\kappa)} && \textit{in } L^{2}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^{3\times3}\big)\,, && (5.3.10) \end{aligned}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, the triple $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ is a global weak solution of (Λ_κ),(5.1.1),(5.1.2) in the sense of Definition 5.2.

Proof: Combining the bounds of Lemmata 5.8 and 5.9 with an Aubin–Lions type lemma ([74, Corollary 8.4]), we obtain that for any $\kappa \in [-1,1]$

$$\big\{n^{(\kappa)}_\varepsilon\big\}_{\varepsilon\in(0,1)} \quad \text{is relatively compact in } L^{\frac{5}{4}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big)$$

and that hence there is some sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j\to\infty$ such that $n^{(\kappa)}_\varepsilon \to n^{(\kappa)}$ in $L^{\frac{5}{4}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big)$ and a.e. in $\Omega\times(0,\infty)$. According to the spatio-temporal bounds in Lemma 5.8, we can furthermore conclude (5.3.3) and (5.3.4) along a subsequence (which we still denote by ε_j). Moreover, also by Lemma 5.8, $\{(n^{(\kappa)}_\varepsilon)^p\}_{\varepsilon\in(0,1)}$ is equi-integrable for any $p<\frac{5}{3}$ and therefore the a.e. convergence of $n^{(\kappa)}_\varepsilon$ together with Vitali's convergence theorem entails the strong convergence in (5.3.2). In a similar fashion we can make use of the bounds for $c^{(\kappa)}_\varepsilon$ in Lemmata 5.3, 5.6 and 5.9 to obtain (5.3.5)–(5.3.7) and the bounds for $u^{(\kappa)}_\varepsilon$ from Lemmata 5.6, 5.8 and 5.9 to verify (5.3.8)–(5.3.10) upon extraction of another subsequence. That $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ solves (Λ_κ) weakly in $\Omega\times(0,\infty)$ is then a straightforward consequence of the regularity and convergence properties we established just now, as these allow us to pass to the limit in all integrals making up the weak formulation of a solution, where we note that in particular (5.3.2) and (5.3.9) entail that for $\Psi \in C_0^\infty(\overline{\Omega}\times[0,\infty);\mathbb{R}^3)$ $\int_0^\infty\int_\Omega n_\varepsilon^{(\kappa)}u_\varepsilon^{(\kappa)}\cdot\Psi \to \int_0^\infty\int_\Omega n^{(\kappa)}u^{(\kappa)}\cdot\Psi$ and that (5.3.8) and the dominated convergence theorem imply that $Y_\varepsilon u_\varepsilon^{(\kappa)} \to u^{(\kappa)}$ in $L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big)$. □

5.4 Eventual smallness of oxygen concentration with waiting times independent of ε and κ

The main objective of this section will be to establish several eventual smallness results for the chemical concentration, where, most importantly, the necessary waiting time of each estimate is independent of $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$. While it is known that these stabilizations occur in the setting with fixed $\kappa = 1$ ([105]), the methods behind these results cannot be transferred directly if we want to maintain independence of the waiting time from the parameters ε and κ. We start with two rather mild eventual smallness properties akin to [105, Lemma 4.2].

Lemma 5.11.
For all $\delta > 0$ there exists $T > 0$ such that for each $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\inf_{t\in[0,T]}\int_t^{t+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} < \delta,$$

as well as

$$\inf_{t\in[0,T]}\int_t^{t+1}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 < \delta.$$

Proof: Given $\delta > 0$ we pick $T \in \mathbb{N}$ satisfying $\big(\|c_0\|_{L^\infty(\Omega)} + \|c_0\|^2_{L^\infty(\Omega)}\big)|\Omega|\delta^{-1} < T$. Then, utilizing the second and fourth equations of $(\Lambda_{\varepsilon,\kappa})$ and the prescribed boundary conditions, we find that for all $\varepsilon \in (0,1)$ and all $\kappa \in [-1,1]$ the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_{\varepsilon t}^{(\kappa)} = \int_\Omega \Delta c_\varepsilon^{(\kappa)} - \int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} - \int_\Omega u_\varepsilon^{(\kappa)}\cdot\nabla c_\varepsilon^{(\kappa)} = -\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)}$$

is valid on $(0,\infty)$. Integration over $(0,T)$ thus shows

$$\int_0^T\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} \le \int_\Omega c_0 \quad \text{for all } \varepsilon \in (0,1),\ \kappa \in [-1,1], \tag{5.4.1}$$

due to $c_\varepsilon^{(\kappa)}$ being nonnegative. Similarly, considering $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (c_\varepsilon^{(\kappa)})^2$ and making use of the fact that $\frac{1}{\varepsilon}\ln(1+\varepsilon s) \ge 0$ for all $\varepsilon \in (0,1)$ and $s \ge 0$, we find that

$$\int_0^T\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \le \int_\Omega c_0^2 \quad \text{for all } \varepsilon \in (0,1),\ \kappa \in [-1,1]. \tag{5.4.2}$$

From (5.4.1), (5.4.2) and Lemma 5.3, we first obtain that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ we have

$$\sum_{t=0}^{T-1}\Big(\int_t^{t+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} + \int_t^{t+1}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2\Big)$$
$$= \int_0^T\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} + \int_0^T\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \le \big(\|c_0\|_{L^\infty(\Omega)} + \|c_0\|_{L^\infty(\Omega)}^2\big)|\Omega| =: M$$

and infer from this that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ there exists some $t_0 \in [0,T]$ satisfying

$$\int_{t_0}^{t_0+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} + \int_{t_0}^{t_0+1}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \le \frac{M}{T} < \delta.$$

In conclusion, for all $\delta > 0$ one can find $T > 0$ such that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$

$$\inf_{t\in[0,T]}\Big(\int_t^{t+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} + \int_t^{t+1}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2\Big) < \delta,$$

which clearly implies the assertion of the lemma. □

Making use of the uniform bounds from the previous sections and the lemma above, we can also derive an additional eventual smallness property, which resembles the result of [105, Lemma 4.4].

Lemma 5.12.
For all $\delta > 0$ there exists $T > 0$ such that for any $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\inf_{t\in[0,T]}\int_t^{t+1}\int_\Omega c_\varepsilon^{(\kappa)} < \delta.$$

Proof: As previously employed in the proof of Lemma 5.6, we first note that the Gagliardo–Nirenberg inequality provides $C_1 > 0$ such that

$$\|\varphi\|_{L^{\frac{12}{5}}(\Omega)}^4 \le C_1\|\nabla\varphi\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}^3 + C_1\|\varphi\|_{L^2(\Omega)}^4 \tag{5.4.3}$$

holds for all $\varphi \in W^{1,2}(\Omega)$. Moreover, the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, together with the Poincaré inequality, entails the existence of $C_2 > 0$ satisfying

$$\|\varphi - \overline{\varphi}\|_{L^6(\Omega)} \le C_2\|\nabla\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{5.4.4}$$

where here and below we denote by $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega\varphi$ the spatial average. Preparing later estimates, we abbreviate $m := \int_\Omega n_0$ and set $C_3 := \frac{1}{2}\min\big\{|\Omega|, m\big\}$ and given any $\delta > 0$ we then fix

$$0 < \delta_0 < \min\left\{\frac{C_3\delta}{2|\Omega|}, \frac{C_3^2\delta^2}{4|\Omega|^2C_1C_2^2m^{\frac{3}{2}}\big(K_1^{\frac{1}{2}} + m^{\frac{1}{2}}\big)}\right\}, \tag{5.4.5}$$

where $K_1 > 0$ is the constant obtained in Lemma 5.6. According to Lemma 5.11, one can find $T > 0$ such that for any fixed $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ there is some $t_0 \in [0,T]$ satisfying

$$\int_{t_0}^{t_0+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} < \delta_0 \quad \text{and} \quad \int_{t_0}^{t_0+1}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 < \delta_0. \tag{5.4.6}$$

To show that in fact this $T > 0$ already fulfills the asserted property, we continue by recalling that $\frac{1}{\varepsilon}\ln\big(1+\varepsilon s\big) \ge \frac{1}{2}\min\{s,1\}$ for all $\varepsilon \in (0,1)$ and $s \ge 0$ and then estimate

$$\begin{aligned}
&\int_{t_0}^{t_0+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)} - \int_{t_0}^{t_0+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big(c_\varepsilon^{(\kappa)} - \overline{c}_\varepsilon^{(\kappa)}\big) \\
&\quad = \int_{t_0}^{t_0+1} \overline{c}_\varepsilon^{(\kappa)} \int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big) \ge \int_{t_0}^{t_0+1} \overline{c}_\varepsilon^{(\kappa)} \int_\Omega \frac{1}{2}\min\big\{n_\varepsilon^{(\kappa)},1\big\} = \frac{C_3}{|\Omega|}\int_{t_0}^{t_0+1}\int_\Omega c_\varepsilon^{(\kappa)}.
\end{aligned} \tag{5.4.7}$$

Making use of the Hölder inequality twice and drawing on (5.4.4) as well as the fact that $\frac{1}{\varepsilon}\ln(1+\varepsilon s) \le s$ for all $\varepsilon \in (0,1)$ and $s \ge 0$, we see that

$$\begin{aligned}
&-\int_{t_0}^{t_0+1}\int_\Omega \frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big(c_\varepsilon^{(\kappa)} - \overline{c}_\varepsilon^{(\kappa)}\big) \\
&\le \left(\int_{t_0}^{t_0+1}\big\|c_\varepsilon^{(\kappa)} - \overline{c}_\varepsilon^{(\kappa)}\big\|_{L^6(\Omega)}^2\right)^{\frac{1}{2}}\left(\int_{t_0}^{t_0+1}\big\|\tfrac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big\|_{L^{\frac{6}{5}}(\Omega)}^2\right)^{\frac{1}{2}} \\
&\le C_2\left(\int_{t_0}^{t_0+1}\big\|\nabla c_\varepsilon^{(\kappa)}\big\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\left(\int_{t_0}^{t_0+1}\big\|n_\varepsilon^{(\kappa)}\big\|_{L^{\frac{6}{5}}(\Omega)}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Plugging this into (5.4.7) and combining with (5.4.6) therefore implies that

$$\int_{t_0}^{t_0+1}\int_\Omega c_\varepsilon^{(\kappa)} \le \frac{|\Omega|\delta_0}{C_3} + \frac{|\Omega|C_2\delta_0^{\frac{1}{2}}}{C_3}\left(\int_{t_0}^{t_0+1}\big\|n_\varepsilon^{(\kappa)}\big\|_{L^{\frac{6}{5}}(\Omega)}^2\right)^{\frac{1}{2}}.$$

To further estimate the remaining term, we make use of (5.4.3), the Cauchy–Schwarz inequality and Lemma 5.6 to find that

$$\begin{aligned}
&\int_{t_0}^{t_0+1}\big\|n_\varepsilon^{(\kappa)}\big\|_{L^{\frac{6}{5}}(\Omega)}^2 = \int_{t_0}^{t_0+1}\big\|n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^{\frac{12}{5}}(\Omega)}^4 \le C_1 m^{\frac{3}{2}}\int_{t_0}^{t_0+1}\big\|\nabla n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^2(\Omega)} + C_1 m^2 \\
&\le C_1 m^{\frac{3}{2}}\left(\int_{t_0}^{t_0+1}\big\|\nabla n_\varepsilon^{(\kappa)\frac{1}{2}}\big\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} + C_1 m^2 \le C_1 m^{\frac{3}{2}}\big(K_1^{\frac{1}{2}} + m^{\frac{1}{2}}\big),
\end{aligned}$$

with $K_1 > 0$ provided by Lemma 5.6. This, in light of (5.4.5), establishes that

$$\int_{t_0}^{t_0+1}\int_\Omega c_\varepsilon^{(\kappa)} \le \frac{|\Omega|\delta_0}{C_3} + \frac{|\Omega|C_1^{\frac{1}{2}}C_2 m^{\frac{3}{4}}\big(K_1^{\frac{1}{2}} + m^{\frac{1}{2}}\big)^{\frac{1}{2}}}{C_3}\delta_0^{\frac{1}{2}} < \delta,$$

and thereby completes the proof. $\square$

Finally, augmenting the arguments of [47, Lemma 3.4] to cover our setting, we obtain the eventual smallness of the L^∞ norm of the oxygen concentration with waiting time uniform in ε and κ.

Lemma 5.13.
For all $\delta > 0$ there exists $T > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\big\|c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)} < \delta.$$

Proof: Similar to before, we first note that by the Gagliardo–Nirenberg inequality, we can find $C_1 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le C_1 \|\nabla\varphi\|_{L^4(\Omega)}^{\frac{12}{13}} \|\varphi\|_{L^1(\Omega)}^{\frac{1}{13}} + C_1\|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in W^{1,4}(\Omega). \tag{5.4.8}$$

Moreover, according to Lemma 5.6 there is $K_1 > 0$ such that

$$\int_t^{t+1}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 \le K_1 \tag{5.4.9}$$

is valid for all $t > 0$, $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$. Now, given $\delta > 0$ we fix $0 < \delta_0 < \min\big\{\frac{\delta}{2C_1}, \frac{\delta^{13}}{2^{13}C_1^{13}K_1^3}\big\}$ and note that in light of Lemma 5.12, we thus find $T_0 > 0$ such that for any fixed $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ there is $t_0 \in [0,T_0]$ satisfying

$$\int_{t_0}^{t_0+1}\int_\Omega c_\varepsilon^{(\kappa)} < \delta_0. \tag{5.4.10}$$

From a combination of (5.4.8) with two applications of Hölder's inequality, (5.4.9) and (5.4.10) we can directly conclude

$$\begin{aligned}
&\int_{t_0}^{t_0+1} \big\|c_\varepsilon^{(\kappa)}\big\|_{L^\infty(\Omega)} \\
&\le C_1\Big(\int_{t_0}^{t_0+1} \big\|\nabla c_\varepsilon^{(\kappa)}\big\|_{L^4(\Omega)}^4\Big)^{\frac{3}{13}} \Big(\int_{t_0}^{t_0+1} \big\|c_\varepsilon^{(\kappa)}\big\|_{L^1(\Omega)}\Big)^{\frac{1}{13}} + C_1\int_{t_0}^{t_0+1} \big\|c_\varepsilon^{(\kappa)}\big\|_{L^1(\Omega)} \\
&\le C_1 K_1^{\frac{3}{13}} \delta_0^{\frac{1}{13}} + C_1\delta_0,
\end{aligned}$$

which, by choice of δ_0 implies

$$\int_{t_0}^{t_0+1} \big\|c_\varepsilon^{(\kappa)}\big\|_{L^\infty(\Omega)} < \delta.$$

This entails that for all $\delta > 0$ there exists $T_0 > 0$ such that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ one can find $t_0 \in [0,T_0]$ such that

$$\inf_{t\in[t_0,t_0+1]} \big\|c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)} \le \int_{t_0}^{t_0+1} \big\|c_\varepsilon^{(\kappa)}\big\|_{L^\infty(\Omega)} < \delta,$$

which, by recalling that $t \mapsto \big\|c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)}$ is nonincreasing, immediately implies the assertion of the lemma with $T \ge T_0 + 1$. $\square$

5.5 Eventual L^p regularity estimates independent of ε and κ as consequence of small oxygen concentration

The uniform waiting time for smallness of $c_\varepsilon^{(\kappa)}$ in $L^\infty(\Omega)$ will be the key ingredient in obtaining additional regularity estimates for $n_\varepsilon^{(\kappa)}$ and $u_\varepsilon^{(\kappa)}$. We start by deriving a differential inequality for $n_\varepsilon^{(\kappa)}$ valid for all times after the chemical concentration has decayed below some threshold number η which, in a second step, together with Lemma 5.13 will then show that the norm of $n_\varepsilon^{(\kappa)}$ in $L^p(\Omega)$ is nonincreasing beyond some waiting time. A functional of similar form to the one we use in Lemma 5.14 to derive the differential inequality has previously been successfully employed in e.g. [99, Lemma 5.1] and [47, Lemma 3.5].

Lemma 5.14.
Let $T>0$, $p>1$, $\theta>0$ and $\eta>0$, $\varepsilon\in(0,1)$ and $\kappa\in[-1,1]$. If the solution $(n_\varepsilon^{(\kappa)},c_\varepsilon^{(\kappa)},u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\left\|c_\varepsilon^{(\kappa)}(\cdot,t)\right\|_{L^\infty(\Omega)}\le\eta\quad\textit{for all } t>T,$$

then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{n_\varepsilon^{(\kappa)p}}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\\
\le\ &-p(p-1)\int_\Omega\frac{n_\varepsilon^{(\kappa)p-2}}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2\\
&+\int_\Omega\left(\frac{p(p-1)}{\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}-\frac{2p\theta}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+1}}\right)n_\varepsilon^{(\kappa)p-1}\big(\nabla n_\varepsilon^{(\kappa)}\cdot\nabla c_\varepsilon^{(\kappa)}\big)\qquad(5.5.1)\\
&-\int_\Omega\left(\frac{\theta(\theta+1)}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+2}}-\frac{p\theta}{\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+1}}\right)n_\varepsilon^{(\kappa)p}\big|\nabla c_\varepsilon^{(\kappa)}\big|^2
\end{aligned}$$

on (T,∞).

Proof: First, we note that, due to $\left\|c_\varepsilon^{(\kappa)}(\cdot,t)\right\|_{L^\infty(\Omega)}\le\eta$ for all $t>T$, the mapping $t\mapsto\int_\Omega\frac{(n_\varepsilon^{(\kappa)}(\cdot,t))^p}{(2\eta-c_\varepsilon^{(\kappa)}(\cdot,t))^\theta}$ is well-defined on (T,∞) and then a straightforward computation, utilizing integration by parts, shows

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{n_\varepsilon^{(\kappa)p}}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\\
=\ &p\int_\Omega\frac{n_\varepsilon^{(\kappa)p-1}}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\left(\Delta n_\varepsilon^{(\kappa)}-\nabla\cdot\Big(\frac{n_\varepsilon^{(\kappa)}}{1+\varepsilon n_\varepsilon^{(\kappa)}}\nabla c_\varepsilon^{(\kappa)}\Big)-\nabla n_\varepsilon^{(\kappa)}\cdot u_\varepsilon^{(\kappa)}\right)\\
&+\theta\int_\Omega\frac{n_\varepsilon^{(\kappa)p}}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+1}}\left(\Delta c_\varepsilon^{(\kappa)}-\frac{1}{\varepsilon}\ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)c_\varepsilon^{(\kappa)}-\nabla c_\varepsilon^{(\kappa)}\cdot u_\varepsilon^{(\kappa)}\right)
\end{aligned}$$

$$\begin{aligned}\leq\ &-p(p-1)\int_\Omega\frac{n_\varepsilon^{(\kappa)p-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}-p\theta\int_\Omega\frac{n_\varepsilon^{(\kappa)p-1}\big(\nabla n_\varepsilon^{(\kappa)}\cdot\nabla c_\varepsilon^{(\kappa)}\big)}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+1}} \\ &+p(p-1)\int_\Omega\frac{n_\varepsilon^{(\kappa)p-1}\big(\nabla n_\varepsilon^{(\kappa)}\cdot\nabla c_\varepsilon^{(\kappa)}\big)}{\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}+p\theta\int_\Omega\frac{n_\varepsilon^{(\kappa)p}\big|\nabla c_\varepsilon^{(\kappa)}\big|^2}{\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+1}} \\ &-p\theta\int_\Omega\frac{n_\varepsilon^{(\kappa)p-1}\big(\nabla n_\varepsilon^{(\kappa)}\cdot\nabla c_\varepsilon^{(\kappa)}\big)}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+1}}-\theta(\theta+1)\int_\Omega\frac{n_\varepsilon^{(\kappa)p}\big|\nabla c_\varepsilon^{(\kappa)}\big|^2}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{\theta+2}} \\ &-\int_\Omega\frac{\nabla(n_\varepsilon^{(\kappa)})^p}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\cdot u_\varepsilon^{(\kappa)}-\int_\Omega n_\varepsilon^{(\kappa)p}\nabla\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{-\theta}\cdot u_\varepsilon^{(\kappa)}\end{aligned}\tag{5.5.2}$$

for all $t>T$, where we also made use of the fact that $\frac{1}{\varepsilon}\ln(1+\varepsilon s)\geq 0$ for $s\geq 0$. Herein, we have

$$-\int_\Omega\frac{\nabla(n_\varepsilon^{(\kappa)})^p}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\cdot u_\varepsilon^{(\kappa)}-\int_\Omega n_\varepsilon^{(\kappa)p}\nabla\big(2\eta-c_\varepsilon^{(\kappa)}\big)^{-\theta}\cdot u_\varepsilon^{(\kappa)}=-\int_\Omega\nabla\Big(\frac{n_\varepsilon^{(\kappa)p}}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\Big)\cdot u_\varepsilon^{(\kappa)}=0,$$

due to the imposed boundary conditions and $u_\varepsilon^{(\kappa)}$ being divergence-free. Therefore, rearranging the terms of (5.5.2) appropriately, we can immediately conclude (5.5.1). □

Waiting long enough for $c_\varepsilon^{(\kappa)}$ to decay past a certain threshold now entails the following:

Lemma 5.15.
For all $p>1$ there exist $K_2>0$ and $T>0$ such that for any $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and every $[t_1,t_2)\subseteq[T,\infty)$ the solution $(n_\varepsilon^{(\kappa)},c_\varepsilon^{(\kappa)},u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_\Omega n_\varepsilon^{(\kappa)p}(\cdot,t_2)+\int_{t_1}^{t_2}\int_\Omega n_\varepsilon^{(\kappa)p-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2\leq K_2\int_\Omega n_\varepsilon^{(\kappa)p}(\cdot,t_1).$$

Proof: Given $p>1$ we first fix $\theta\in(0,p-1)$ and then pick some $\eta>0$ satisfying

$$\eta<\min\left\{\frac{\theta+1}{2p},\sqrt{\frac{\theta(\theta+1-\frac{p}{p-1}\theta)}{p(p-1)}}\right\}.\tag{5.5.3}$$

For these choices of parameters, in light of Lemma 5.13, we find some $T=T(p)>0$ such that for all $\varepsilon\in(0,1)$ and $\kappa\in[-1,1]$ we have

$$\big\|c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)}\leq\eta\quad\text{for all } t\geq T.$$

Hence, the requirements of Lemma 5.14 are met and the inequality (5.5.1) is valid on (T,∞). Moreover, by choice of $\eta<\frac{\theta+1}{2p}$ and nonnegativity of $n_\varepsilon^{(\kappa)}$ and $c_\varepsilon^{(\kappa)}$ we have

$$\frac{\big(2\eta-c_\varepsilon^{(\kappa)}\big)p}{\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)(\theta+1)}\leq\frac{2\eta p}{\theta+1}<1\quad\text{on }[T,\infty)$$

and hence

$$\frac{p\theta}{\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big)\big(2\eta - c_\varepsilon^{(\kappa)}\big)^{\theta+1}} < \frac{\theta(\theta+1)}{\big(2\eta - c_\varepsilon^{(\kappa)}\big)^{\theta+2}} \quad \text{for all } t \geq T.$$

Therefore, we can cancel out the term containing $\big|\nabla c_\varepsilon^{(\kappa)}\big|^2$ in (5.5.1). In fact, an employment of Young's inequality in (5.5.1) shows that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{n_\varepsilon^{(\kappa)p}}{\big(2\eta - c_\varepsilon^{(\kappa)}\big)^\theta} \leq -\int_\Omega \left(\frac{p(p-1)}{\big(2\eta - c_\varepsilon^{(\kappa)}\big)^\theta} - \frac{1}{4}H\big(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}\big)\right) n_\varepsilon^{(\kappa)p-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2 \tag{5.5.4}$$

for all $t \geq T$, with

$$H(\sigma,\xi) := \frac{\left(\frac{p(p-1)}{(1+\varepsilon\sigma)(2\eta-\xi)^\theta} - \frac{2p\theta}{(2\eta-\xi)^{\theta+1}}\right)^2}{\frac{\theta(\theta+1)}{(2\eta-\xi)^{\theta+2}} - \frac{p\theta}{(1+\varepsilon\sigma)(2\eta-\xi)^{\theta+1}}}, \quad \text{for } \sigma \geq 0 \text{ and } \xi \in [0,2\eta).$$

To verify that $\frac{p(p-1)}{(2\eta-\xi)^\theta} - \frac{1}{4}H(\sigma,\xi) \geq 0$ for $\sigma \geq 0$ and $\xi \in [0,2\eta)$, we first write

$$\frac{H(\sigma,\xi)(2\eta-\xi)^\theta}{4p(p-1)} = \frac{\frac{p(p-1)(2\eta-\xi)^2}{(1+\varepsilon\sigma)^2} - \frac{4p\theta(2\eta-\xi)}{1+\varepsilon\sigma} + \frac{4p\theta^2}{p-1}}{4\theta(\theta+1) - \frac{4p\theta(2\eta-\xi)}{1+\varepsilon\sigma}} =: \frac{H_1(\sigma,\xi)}{H_2(\sigma,\xi)}$$

and note that by the nonnegativity of σ and ξ and latter part of (5.5.3) we have

$$H_1(\sigma,\xi) - H_2(\sigma,\xi) \leq p(p-1)4\eta^2 + \frac{4p}{p-1}\theta^2 - 4\theta(\theta+1) < 0.$$

Since, due to (5.5.3), we have $H_2(\sigma,\xi) \geq 4\theta(\theta+1) - 8p\theta\eta > 0$ for $\sigma \geq 0$ and $\xi \in [0,2\eta)$, this implies

$$\begin{aligned}\frac{H(\sigma,\xi)\big(2\eta-\xi\big)^\theta}{4p(p-1)} &= 1 + \frac{H_1(\sigma,\xi) - H_2(\sigma,\xi)}{H_2\big(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}\big)} \\ &\leq 1 + \frac{p(p-1)4\eta^2 + \frac{4p}{p-1}\theta^2 - 4\theta(\theta+1)}{4\theta(\theta+1) - 8p\theta\eta}\end{aligned}$$

for all $\sigma \geq 0$ and $\xi \in [0,2\eta)$, from which we infer that

$$\frac{p(p-1)}{(2\eta-\xi)^\theta} - \frac{1}{4}H(\sigma,\xi) \geq C_3\frac{p(p-1)}{(2\eta-\xi)^\theta} > 0 \quad \text{for all } \sigma \geq 0,\ \xi \in [0,2\eta),$$

with $C_3 := -\frac{p(p-1)\eta^2 + \frac{p}{p-1}\theta^2 - \theta(\theta+1)}{\theta(\theta+1) - 2p\eta\theta} > 0$. Hence, we conclude from (5.5.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{n_\varepsilon^{(\kappa)p}}{\big(2\eta - c_\varepsilon^{(\kappa)}\big)^\theta} + p(p-1)C_3\int_\Omega \frac{n_\varepsilon^{(\kappa)p-2}}{\big(2\eta - c_\varepsilon^{(\kappa)}\big)^\theta}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2 \leq 0 \quad \text{for all } t \geq T,$$

which for any $[t_1,t_2)\subseteq[T,\infty)$, upon integration with respect to time, shows that

$$\int_\Omega \frac{n_\varepsilon^{(\kappa)p}(\cdot,t_2)}{\big(2\eta-c_\varepsilon^{(\kappa)}(\cdot,t_2)\big)^\theta}+\int_{t_1}^{t_2}\int_\Omega \frac{n_\varepsilon^{(\kappa)p-2}}{\big(2\eta-c_\varepsilon^{(\kappa)}\big)^\theta}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2$$
$$\leq \frac{1}{\min\{1,p(p-1)C_3\}}\int_\Omega \frac{n_\varepsilon^{(\kappa)p}(\cdot,t_1)}{\big(2\eta-c_\varepsilon^{(\kappa)}(\cdot,t_1)\big)^\theta},$$

completing the proof, after taking into account that $\eta^\theta\leq(2\eta-c_\varepsilon^{(\kappa)})^\theta\leq(2\eta)^\theta$ on $\Omega\times[T,\infty)$. □

Making use of an inductive argument as exercised in [105, Lemma 6.3], we can get rid of the time dependence present in the right-hand side of the inequality provided by Lemma 5.15 entailing the eventual uniform L^p regularity of $n_\varepsilon^{(\kappa)}$ required for further analysis.

Lemma 5.16.
For all $p>1$ there exist $T>0$ and $K_3=K_3(p)>0$ such that for each $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and all $t>T$, the solution $(n_\varepsilon^{(\kappa)},c_\varepsilon^{(\kappa)},u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_\Omega n_\varepsilon^{(\kappa)p}(\cdot,t)\leq K_3 \quad \textit{and} \quad \int_T^\infty\int_\Omega n_\varepsilon^{(\kappa)p-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2\leq K_3.$$

Proof: Preparing an inductive argument, we first assume that there exist $p_0>1$, $C_0>0$ and $T_0\geq 0$ such that for all $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and $t>T_0$ we have

$$\int_t^{t+1}\|n_\varepsilon^{(\kappa)}\|_{L^{p_0}(\Omega)}\leq C_0. \tag{5.5.5}$$

In light of Lemma 5.15 we find for each $q\in(1,p_0]$ corresponding $T_1=T_1(q)>0$ and $K_2=K_2(q)>0$ with the property that for all $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and $[t_1,t)\subseteq[T_1,\infty)$ the inequality

$$\int_\Omega n_\varepsilon^{(\kappa)q}(\cdot,t)+\int_{t_1}^t\int_\Omega n_\varepsilon^{(\kappa)q-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2\leq K_2\int_\Omega n_\varepsilon^{(\kappa)q}(\cdot,t_1) \tag{5.5.6}$$

is valid. Letting $\bar T:=\max\{T_0,T_1\}$ we see that in view of (5.5.5) there exists $C_1>0$ such that for any $\varepsilon\in(0,1)$ and $\kappa\in[-1,1]$ we can find $t_*\in[\bar T,\bar T+1]$ such that $\|n_\varepsilon^{(\kappa)}(\cdot,t_*)\|_{L^q(\Omega)}\leq C_1$. Plugging this into (5.5.6) with $t_1=t_*$ we obtain for all $t>\bar T+1$ and any $\varepsilon\in(0,1)$ and $\kappa\in[-1,1]$ that

$$\int_\Omega n_\varepsilon^{(\kappa)q}(\cdot,t)+\int_{\bar T+1}^t\int_\Omega n_\varepsilon^{(\kappa)p_0-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2$$
$$\leq\int_\Omega n_\varepsilon^{(\kappa)q}(\cdot,t)+\int_{t_*}^t\int_\Omega n_\varepsilon^{(\kappa)q-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2\leq K_2\int_\Omega n_\varepsilon^{(\kappa)q}(\cdot,t_*)\leq K_2C_1, \tag{5.5.7}$$

proving that under the assumption (5.5.5) the asserted bounds are valid for $p\in(1,p_0]$. Moreover, due to the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ and Hölder's inequality, there is some $C_2=C_2(p_0)>0$ such that for all $t>\bar T+1$

$$\begin{aligned}\Big(\int_t^{t+1}\|n_\varepsilon^{(\kappa)}\|_{L^{3p_0}(\Omega)}\Big)^{p_0} &\leq \int_t^{t+1}\|n_\varepsilon^{(\kappa)\frac{p_0}{2}}\|_{L^6(\Omega)}^2\\ &\leq C_2\int_t^{t+1}\Big(\|\nabla n_\varepsilon^{(\kappa)\frac{p_0}{2}}\|_{L^2(\Omega)}^2+\|n_\varepsilon^{(\kappa)\frac{p_0}{2}}\|_{L^{\frac{2}{p_0}}(\Omega)}^2\Big)\\ &\leq \frac{C_2p_0^2}{4}\int_t^{t+1}\int_\Omega n_\varepsilon^{(\kappa)p_0-2}\big|\nabla n_\varepsilon^{(\kappa)}\big|^2+C_2m^{p_0}\\ &\leq \frac{C_1C_2K_2p_0^2}{4}+C_2m^{p_0},\end{aligned}$$

where we also made use of $\int_\Omega n_\varepsilon^{(\kappa)}(\cdot,t)=\int_\Omega n_0=:m$ for all $t>0$ and (5.5.7). Drawing on these calculations, the step from p_0 to $3p_0$ is possible and we only have to ensure that indeed the assumption (5.5.5) is fulfilled for some $p_0>1$. Now, in a similar fashion the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ and Lemma 5.6 provide $C_3>0$ and $K_1>0$ such that for all $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and $t>0$ we have

$$\begin{aligned}\int_t^{t+1}\|n_\varepsilon^{(\kappa)}\|_{L^3(\Omega)}&\leq C_3\int_t^{t+1}\Big(\|\nabla n_\varepsilon^{(\kappa)\frac12}\|_{L^2(\Omega)}^2+\|n_\varepsilon^{(\kappa)\frac12}\|_{L^2(\Omega)}^2\Big)\\ &\leq \frac{C_3}{4}\int_t^{t+1}\int_\Omega\frac{\big|\nabla n_\varepsilon^{(\kappa)}\big|^2}{n_\varepsilon^{(\kappa)}}+C_3\int_t^{t+1}\int_\Omega n_\varepsilon^{(\kappa)}\leq\frac{C_3K_1}{4}+C_3m,\end{aligned}$$

which shows that (5.5.7) is valid for $p_0=3$ and thereby concludes the proof. $\square$

An immediate consequence is the eventual boundedness of the forcing term in the third equation of $(\Lambda_{\varepsilon,\kappa})$, from which we extract new information on the gradient of $u_\varepsilon^{(\kappa)}$.

Lemma 5.17.
There exist $T>0$ and $C>0$ such that for each $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and all $t>T$ the solution $(n_\varepsilon^{(\kappa)},c_\varepsilon^{(\kappa)},u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_\Omega\big|\nabla u_\varepsilon^{(\kappa)}(\cdot,t)\big|^2\leq C.$$

Proof: Recalling that $\mathcal{P}$ denotes the Helmholtz projection from $L^2(\Omega;\mathbb{R}^3)$ to $L^2_\sigma(\Omega)$ and $A:=-\mathcal{P}\Delta$ the Stokes operator in $L^2(\Omega;\mathbb{R}^3)$ under homogeneous Dirichlet boundary conditions, we find that testing the projected third equation of $(\Lambda_{\varepsilon,\kappa})$ by $Au_\varepsilon^{(\kappa)}$ implies in view of Young's inequality that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\big|A^{\frac12}u_\varepsilon^{(\kappa)}\big|^2+\int_\Omega\big|Au_\varepsilon^{(\kappa)}\big|^2\\ &\qquad\leq\int_\Omega\big|Au_\varepsilon^{(\kappa)}\big|^2+\frac{|\kappa|}{2}\int_\Omega\Big|\big(Y_\varepsilon u_\varepsilon^{(\kappa)}\cdot\nabla\big)u_\varepsilon^{(\kappa)}\Big|^2+\frac12\int_\Omega\big|n_\varepsilon^{(\kappa)}\nabla\phi\big|^2\end{aligned}\tag{5.5.8}$$

is valid for all $t>0$, where we also made use of the facts that $\big\|A^{\frac12}u_\varepsilon^{(\kappa)}\big\|_{L^2(\Omega)}=\big\|\nabla u_\varepsilon^{(\kappa)}\big\|_{L^2(\Omega)}$ and $\|\mathcal{P}\varphi\|_{L^2(\Omega)}\leq\|\varphi\|_{L^2(\Omega)}$ for all $\varphi\in L^2(\Omega;\mathbb{R}^3)$. Moreover, since we have $D(1+\varepsilon A) = W^{2,2}(\Omega;\mathbb{R}^3) \cap W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow L^\infty\big(\Omega;\mathbb{R}^3\big)$, we see that there exists $C_1 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$ we have

$$\big\|Y_\varepsilon u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)} \leq C_1 \big\|u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^2(\Omega)} \leq C_1\sqrt{\frac{K_1}{K_0}} =: C_2,$$

where $K_0, K_1 > 0$ are the constants obtained in Lemma 5.4 and Lemma 5.6, respectively. In particular, we obtain from a combination with (5.5.8) and the fact that $|\kappa| \leq 1$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 \leq C_2 \int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)} \int_\Omega \big|n_\varepsilon^{(\kappa)}\big|^2$$

on $(0,\infty)$. Letting $y(t) := \int_\Omega |\nabla u_\varepsilon^{(\kappa)}(\cdot,t)|^2$ and $h(t) := \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega |n_\varepsilon^{(\kappa)}(\cdot,t)|^2$ we find that by Lemma 5.16 there exist $T > 0$ and $C_3 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t \geq T$ we have $h(t) \leq C_3$, and hence

$$y'(t) \leq C_2 y(t) + C_3 \quad \text{for all } t > T. \tag{5.5.9}$$

Recalling that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$ we moreover have

$$\int_t^{t+1}\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 \leq K_1,$$

with $K_1 > 0$ provided by Lemma 5.6, we infer that for any fixed $t > T+1$ and each $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ there exists some $t_* \in (t-1,t)$ such that

$$\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}(\cdot,t_*)\big|^2 \leq K_1,$$

which upon integrating the differential inequality (5.5.9) over (t_*,t) shows that

$$y(t) \leq y(t_*)e^{C_2(t-t_*)} + \int_{t_*}^t C_3 e^{C_2(t-t_*)} \leq K_1 e^{C_2} + C_3 e^{C_2},$$

completing the proof. □

As last step in this section, we also lift the regularity of the signal gradient for times beyond the waiting times from the previous lemmata.

Lemma 5.18.
There exist $T > 0$ and $C > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}(\cdot,t)\big|^4 \leq C.$$

Proof: We work along similar lines as in the proof of Lemma 5.17 by first establishing a differential inequality for the quantity $\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}(\cdot,t)\big|^4$. A standard testing procedure utilizing the pointwise identity $\nabla c_\varepsilon^{(\kappa)} \cdot \nabla \Delta c_\varepsilon^{(\kappa)} = \frac{1}{2}\Delta\big|\nabla c_\varepsilon^{(\kappa)}\big|^2 - \big|D^2 c_\varepsilon^{(\kappa)}\big|^2$, the fact that $\nabla \cdot u_\varepsilon^{(\kappa)} = 0$ on $\Omega \times (0,\infty)$ and the upper estimate of (5.2.1) shows that

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 \\
&\leq \frac{1}{2}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \Delta \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 - \int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \big|D^2 c_\varepsilon^{(\kappa)}\big|^2 - \int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \nabla c_\varepsilon^{(\kappa)} \cdot \big(\nabla u_\varepsilon^{(\kappa)} \cdot \nabla c_\varepsilon^{(\kappa)}\big) \\
&\quad + \int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \big|\Delta c_\varepsilon^{(\kappa)}\big| n_\varepsilon^{(\kappa)} c_\varepsilon^{(\kappa)} + 2\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)} \cdot \big(D^2 c_\varepsilon^{(\kappa)} \cdot \nabla c_\varepsilon^{(\kappa)}\big) n_\varepsilon^{(\kappa)} c_\varepsilon^{(\kappa)}\big| \qquad (5.5.10)
\end{aligned}$$

holds for all $t > 0$. To further estimate the first term on the right we draw on arguments employed in [37, Proposition 3.2]. Let us recall that there exists $C_1 > 0$ such that for any $\varphi \in C^2\big(\overline{\Omega}\big)$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ we have $\frac{\partial|\nabla\varphi|^2}{\partial\nu} \leq C_1|\nabla\varphi|^2$ on $\partial\Omega$ (cf. [61, Lemma 4.2]). Moreover, by utilizing the fact that for $r \in (0,\frac{1}{2})$ $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^1(\Omega)$ ([18]), Ehrling's lemma as well as trace embddings (e.g. [31, Thm. 4.24, Prop. 4.22]) we obtain for every fixed $\eta > 0$ some $C_2 > 0$ such that $\|\psi\|_{L^2(\partial\Omega)} \leq \eta\|\nabla\psi\|_{L^2(\Omega)} + C_2\|\psi\|_{L^1(\Omega)}$ holds for any $\psi \in W^{1,2}(\Omega)$. Hence, drawing on Lemmata 5.3 and 5.6 to estimate $\int_\Omega |\nabla c_\varepsilon^{(\kappa)}|^2 \leq 2K_1\|c_0\|_{L^\infty(\Omega)} =: C_3$, we find that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$

$$\begin{aligned}
\frac{1}{2}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \Delta\big|\nabla c_\varepsilon^{(\kappa)}\big|^2 &= -\frac{1}{2}\int_\Omega \Big|\nabla\big|\nabla c_\varepsilon^{(\kappa)}\big|^2\Big|^2 + \frac{1}{2}\int_{\partial\Omega} \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \frac{\partial\big|\nabla c_\varepsilon^{(\kappa)}\big|^2}{\partial\nu} \\
&\leq -\frac{1}{2}\int_\Omega \Big|\nabla\big|\nabla c_\varepsilon^{(\kappa)}\big|^2\Big|^2 + \frac{1}{2}\int_\Omega \Big|\nabla\big|\nabla c_\varepsilon^{(\kappa)}\big|^2\Big|^2 + C_2\Big(\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2\Big)^2 \\
&\leq C_2 C_3^2.
\end{aligned}$$

Combining this with (5.5.10), multiple employments of Young's inequality show that

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 \\
&\leq -\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \big|D^2 c_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 \big|\nabla u_\varepsilon^{(\kappa)}\big| + \frac{1}{12}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \big|\Delta c_\varepsilon^{(\kappa)}\big|^2 \\
&\quad + 3\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 n_\varepsilon^{(\kappa)2} c_\varepsilon^{(\kappa)2} + \frac{1}{4}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \big|D^2 c_\varepsilon^{(\kappa)}\big|^2 + 4\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 n_\varepsilon^{(\kappa)2} c_\varepsilon^{(\kappa)2} + C_2 C_3^2
\end{aligned}$$

is valid on $(0,\infty)$. In light of the pointwise estimate $\big|\Delta c_\varepsilon^{(\kappa)}\big|^2 \leq 3\big|D^2 c_\varepsilon^{(\kappa)}\big|^2$ and Hölder's inequality, this implies that

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4 + \frac{1}{2}\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^2 \big|D^2 c_\varepsilon^{(\kappa)}\big|^2 \qquad (5.5.11) \\
&\quad \leq \Big(\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^8\Big)^{\frac{1}{2}} \Big(\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2\Big)^{\frac{1}{2}} + 7\Big(\int_\Omega n_\varepsilon^{(\kappa)4} c_\varepsilon^{(\kappa)4}\Big)^{\frac{1}{2}} \Big(\int_\Omega \big|\nabla c_\varepsilon^{(\kappa)}\big|^4\Big)^{\frac{1}{2}} + C_2 C_3^2
\end{aligned}$$

for all $t > 0$. Making use of the Gagliardo–Nirenberg inequality we obtain $C_4 > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$

$$
\begin{aligned}
\Big(\int_\Omega |\nabla c_\varepsilon^{(\kappa)}|^8\Big)^{\frac12} &= \Big\| |\nabla c_\varepsilon^{(\kappa)}|^2 \Big\|_{L^4(\Omega)}^2 \\
&\le C_4 \Big\| \nabla |\nabla c_\varepsilon^{(\kappa)}|^2 \Big\|_{L^2(\Omega)}^{\frac95} \Big\| |\nabla c_\varepsilon^{(\kappa)}|^2 \Big\|_{L^1(\Omega)}^{\frac15} + C_4 \Big\| |\nabla c_\varepsilon^{(\kappa)}|^2 \Big\|_{L^1(\Omega)}^2 \\
&\le C_4 C_3^{\frac15} \Big\| \nabla |\nabla c_\varepsilon^{(\kappa)}|^2 \Big\|_{L^2(\Omega)}^{\frac95} + C_4 C_3^2 \\
&\le C_4 C_3^{\frac15} \Big(4 \int_\Omega |\nabla c_\varepsilon^{(\kappa)}|^2 |D^2 c_\varepsilon^{(\kappa)}|^2 \Big)^{\frac{9}{10}} + C_4 C_3^2.
\end{aligned}
$$

Therefore, again by using Young's inequality, we infer from (5.5.11) that there is $C_5 > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$

$$
\begin{aligned}
\frac14 \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla c_\varepsilon^{(\kappa)}|^4 \le C_5 \Big(\int_\Omega |\nabla u_\varepsilon^{(\kappa)}|^2 \Big)^5 &+ C_4 C_3^2 \Big(\int_\Omega |\nabla u_\varepsilon^{(\kappa)}|^2 \Big)^{\frac12} \\
&+ 7 \int_\Omega n_\varepsilon^{(\kappa)4} c_\varepsilon^{(\kappa)4} + \frac74 \int_\Omega |\nabla c_\varepsilon^{(\kappa)}|^4 + C_2 C_3^2,
\end{aligned}
$$

which in light of Lemmata 5.3, 5.16 and 5.17 entails that there are $T > 0$ and $C_6 > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ the function $y(t) := \int_\Omega |\nabla c_\varepsilon^{(\kappa)}(\cdot,t)|^4$ satisfies the differential inequality

$$
y'(t) \le 7y(t) + C_6. \tag{5.5.12}
$$

Now, according to Lemmata 5.3 and 5.6 for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$ we can estimate

$$
\int_t^{t+1} \int_\Omega |\nabla c_\varepsilon^{(\kappa)}|^4 \le \|c_0\|_{L^\infty(\Omega)}^3 \int_t^{t+1} \int_\Omega \frac{|\nabla c_\varepsilon^{(\kappa)}|^4}{c_\varepsilon^{(\kappa)3}} \le \|c_0\|_{L^\infty(\Omega)}^3 K_1,
$$

and hence for any fixed $t > T+1$ and each $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ we find $t_* \in (t-1,t)$ such that

$$
\int_\Omega |\nabla c_\varepsilon^{(\kappa)}(\cdot,t_*)|^4 \le \|c_0\|_{L^\infty(\Omega)}^3 K_1 =: C_7,
$$

which upon integrating (5.5.12) over (t_*,t) entails that

$$
y(t) \le C_7 e^7 + C_6 e^7
$$

as desired. □

5.6 Uniform eventual stabilization of $n_\varepsilon^{(\kappa)}$ and $u_\varepsilon^{(\kappa)}$ in some L^p spaces

Eventual decay of the signal component and uniform regularity estimates at hand, we can now turn towards obtaining eventual stabilization properties of the two remaining solution components. These will be an important cornerstone of the maximal Sobolev regularity type arguments we employ in Section 5.7 to obtain uniform bounds in Hölder spaces. We start with an eventual smallness result for a mixed quantity of $n_\varepsilon^{(\kappa)}$ and $\nabla c_\varepsilon^{(\kappa)}$.

Lemma 5.19.
For all $\delta > 0$ there exists $T > 0$ such that for any $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ and all $t > T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_t^{t+1} \left\| n_\varepsilon^{(\kappa)} \nabla c_\varepsilon^{(\kappa)} \right\|_{L^2(\Omega)}^2 < \delta.$$

Proof: According to Lemma 5.6, there is $K_1 > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$ we have

$$\int_t^{t+1} \int_\Omega \frac{\left|\nabla c_\varepsilon^{(\kappa)}\right|^4}{{c_\varepsilon^{(\kappa)}}^3} \leq K_1.$$

Similarly, drawing on Lemma 5.16, we find $K_3 > 0$ and $T_1 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T_1$ the estimate

$$\left\| n_\varepsilon^{(\kappa)}(\cdot,t) \right\|_{L^4(\Omega)}^4 \leq K_3$$

is valid. Now, given any $\delta > 0$ we fix

$$0 < \delta_0 < \min\left\{ \frac{\delta}{2K_1}, \sqrt{\frac{\delta}{2K_3}} \right\}$$

and, according to Lemma 5.13, obtain a corresponding $T_2 > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T_2$

$$\left\| c_\varepsilon^{(\kappa)}(\cdot,t) \right\|_{L^\infty(\Omega)} < \delta_0$$

is satisfied. Hence, by making use of the estimates above as well as Hölder's and Young's inequalities, we achieve for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > \max\{T_1, T_2\}$ that

$$\begin{aligned}
\int_t^{t+1} \left\| n_\varepsilon^{(\kappa)} \nabla c_\varepsilon^{(\kappa)} \right\|_{L^2(\Omega)}^2 &\leq \int_t^{t+1} \left\| n_\varepsilon^{(\kappa)} \right\|_{L^4(\Omega)}^2 \left\| \nabla c_\varepsilon^{(\kappa)} \right\|_{L^4(\Omega)}^2 \leq \int_t^{t+1} K_3^{\frac{1}{2}} \left\| \nabla c_\varepsilon^{(\kappa)} \right\|_{L^4(\Omega)}^2 \\
&\leq \int_t^{t+1} K_3^{\frac{1}{2}} \left\| c_\varepsilon^{(\kappa)} \right\|_{L^\infty(\Omega)}^{\frac{3}{2}} \left(\int_\Omega \frac{\left|\nabla c_\varepsilon^{(\kappa)}\right|^4}{{c_\varepsilon^{(\kappa)}}^3} \right)^{\frac{1}{2}} \\
&\leq \int_t^{t+1} K_3 \left\| c_\varepsilon^{(\kappa)} \right\|_{L^\infty(\Omega)}^2 + \int_t^{t+1} \left\| c_\varepsilon^{(\kappa)} \right\|_{L^\infty(\Omega)} \int_\Omega \frac{\left|\nabla c_\varepsilon^{(\kappa)}\right|^4}{{c_\varepsilon^{(\kappa)}}^3} \\
&\leq K_3 \delta_0^2 + K_1 \delta_0 < \delta,
\end{aligned}$$

completing the proof. $\square$

In order to successfully extract a uniform stabilization for $n_\varepsilon^{(\kappa)}$ and $u_\varepsilon^{(\kappa)}$ in certain L^p spaces we will require the following auxiliary lemma for ODEs, which we have taken from [91, Lemma 4.3].

Lemma 5.20.
Let I be any set and $\lambda > 0$, and for each $\iota \in I$ let $y_\iota \in C^0([0,\infty)) \cap C^1((0,\infty))$ and $f_\iota \in C^0((0,\infty))$ be nonnegative and such that

$$y_\iota'(t) + \lambda y_\iota(t) \le f_\iota(t) \quad \textit{for all } t > 0$$

and

$$\sup_{\iota\in I} y_\iota(0) < \infty \quad \textit{as well as} \quad \sup_{\iota\in I} \|f_\iota\|_{L^\infty((0,\infty))} < \infty$$

and

$$\sup_{\iota\in I} \int_t^{t+1} f_\iota(s)\,\mathrm{d}s \to 0 \quad \textit{as } t \to \infty$$

hold. Then

$$\sup_{\iota\in I} y_\iota(t) \to 0 \quad \textit{as } t \to \infty.$$

Tracking the time evolution of $y_\varepsilon^{(\kappa)}(t) := \int_\Omega \big(n_\varepsilon^{(\kappa)}(\cdot,t) - \overline{n_0}\big)^2$ and shifting the time appropriately, we can make use of the statement above to attain a uniform eventual smallness of $y_\varepsilon^{(\kappa)}$.

Lemma 5.21.
For all $\delta > 0$ there exists $T > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ he solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\big\|n_\varepsilon^{(\kappa)}(\cdot,t) - \overline{n_0}\big\|_{L^2(\Omega)}^2 < \delta.$$

Furthermore, for all $p \ge 2$ and $\delta' > 0$ there is $T' > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T'$ the solution satisfies

$$\int_t^{t+1} \big\|n_\varepsilon^{(\kappa)} - \overline{n_0}\big\|_{L^p(\Omega)}^p < \delta'.$$

Proof: We start with the case $p = 2$. Due to the Young and Poincaré inequalities we obtain $C_1 > 0$ fulfilling

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big(n_\varepsilon^{(\kappa)} - \overline{n_0}\big)^2 &\le -\int_\Omega \big|\nabla n_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega \big|n_\varepsilon^{(\kappa)}\nabla c_\varepsilon^{(\kappa)}\big|^2 \\ &\le -\frac{1}{C_1}\int_\Omega \big(n_\varepsilon^{(\kappa)} - \overline{n_0}\big)^2 + \big\|n_\varepsilon^{(\kappa)}\nabla c_\varepsilon^{(\kappa)}\big\|_{L^2(\Omega)}^2 \quad \text{for all } t > 0,\end{aligned}$$

where we also made use of the fact that $\nabla \cdot u_\varepsilon^{(\kappa)} = 0$ in $\Omega \times (0,\infty)$. Moreover, as

$$\big\|\big(n_\varepsilon^{(\kappa)}\nabla c_\varepsilon^{(\kappa)}\big)(\cdot,t)\big\|_{L^2(\Omega)}^2 \leq \big\|n_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^4(\Omega)}^2\big\|\nabla c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^4(\Omega)}^2 \quad \text{for all } t>0,$$

in light of Lemmata 5.16, 5.18 and 5.19, we find that there exists some $T_1>0$ such that

$$y_\varepsilon^{(\kappa)}(t) := \int_\Omega \big(n_\varepsilon^{(\kappa)}(t-T_1) - \overline{n_0}\big)^2$$

satisfies the conditions of Lemma 5.20. Hence, we conclude that for all $\delta>0$ there exists $\bar{T} \geq T_1$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > \bar{T}$ we have

$$\big\|n_\varepsilon^{(\kappa)}(\cdot,t) - \overline{n_0}\big\|_{L^2(\Omega)}^2 < \delta,$$

which, in particular, also immediately implies the second claim for $p=2$. For $p>2$ we let $K_3 := K_3(2p) > 0$ and $T_2>0$ be given by Lemma 5.16 and then, in consideration of (5.1.4), obtain $C_2>0$ such that for each $\varepsilon \in (0,1)$, $\kappa\in[-1,1]$ and all $t>T_2$ we have

$$\big\|n_\varepsilon^{(\kappa)}(\cdot,t) - \overline{n_0}\big\|_{L^{2p}(\Omega)}^{\frac{p(p-2)}{p-1}} \leq \big(K_3 + \|\overline{n_0}\|_{L^{2p}(\Omega)}\big)^{\frac{p(p-2)}{p-1}} \leq C_2.$$

Therefore, by means of Hölder interpolation and Hölder's inequality we find that for any $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and all $t>T_2$

$$\begin{aligned}\int_t^{t+1}\big\|n_\varepsilon^{(\kappa)} - \overline{n_0}\big\|_{L^p(\Omega)}^p &\leq \int_t^{t+1}\big\|n_\varepsilon^{(\kappa)} - \overline{n_0}\big\|_{L^{2p}(\Omega)}^{\frac{p(p-2)}{p-1}}\big\|n_\varepsilon^{(\kappa)} - \overline{n_0}\big\|_{L^2(\Omega)}^{\frac{p}{p-1}} \\ &\leq C_2\int_t^{t+1}\big\|n_\varepsilon^{(\kappa)} - \overline{n_0}\big\|_{L^2(\Omega)}^{\frac{p}{p-1}} \leq C_2\Big(\int_t^{t+1}\big\|n_\varepsilon^{(\kappa)} - \overline{n_0}\big\|_{L^2(\Omega)}^2\Big)^{\frac{p}{2p-2}}\end{aligned} \tag{5.6.1}$$

is valid, due to $\frac{p}{p-1}<2$. Finally, for given $\delta>0$ we let $0<\delta_0<\frac{\delta}{C_2}$ and then conclude the proof by making use of the first part of this Lemma to estimate the remaining term in (5.6.1) by δ_0 for $t>T_3$ large enough. □

The second conclusion we can draw from the Lemma 5.20 concerns the gradient of the fluid-velocity field and, by Sobolev embeddings, the fluid-velocity itself.

Lemma 5.22.
For all $\delta>0$ there exists $T>0$ such that for each $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and all $t>T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_t^{t+1}\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 < \delta. \tag{5.6.2}$$

Moreover, for all $p\in[1,6]$ and all $\delta'>0$ there exists $T'>0$ such that for each $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and all $t>T'$

$$\int_t^{t+1}\big\|u_\varepsilon^{(\kappa)}\big\|_{L^p(\Omega)}^2 < \delta' \tag{5.6.3}$$

holds.

Proof: Making use of Lemma 5.5 and the divergence-free property of $u_\varepsilon^{(\kappa)}$, we first find that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2 + \int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 = \int_\Omega \big(n_\varepsilon^{(\kappa)} - \overline{n_0}\big)\nabla\phi \cdot u_\varepsilon^{(\kappa)}.$$

Here, the Poincaré inequality provides $C_1 > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ we have $\|u_\varepsilon^{(\kappa)}\|^2_{L^2(\Omega)} \le C_1\|\nabla u_\varepsilon^{(\kappa)}\|^2_{L^2(\Omega)}$, which entails upon an application of Young's inequality that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2 + \frac{1}{2}\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 \le \frac{C_1\big\|\nabla\phi\big\|^2_{L^\infty(\Omega)}}{2}\int_\Omega \big(n_\varepsilon^{(\kappa)} - \overline{n_0}\big)^2 \tag{5.6.4}$$

is valid on $(0,\infty)$. Since the Poincaré inequality moreover implies that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2 + \frac{1}{C_1}\int_\Omega \big|u_\varepsilon^{(\kappa)}\big|^2 \le C_1\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega \big(n_\varepsilon^{(\kappa)} - \overline{n_0}\big)^2$$

holds, we find that in light of Lemmata 5.21, 5.6, 5.16 and (5.1.3), there exists some $T_1 > 0$ such that the function

$$y_\varepsilon^{(\kappa)}(t) := \int_\Omega \big|u_\varepsilon^{(\kappa)}(t - T_1)\big|^2$$

satisfies the conditions of Lemma 5.20. Hence, we find that for all $\delta_0 > 0$ there is some $\bar{T} \ge T_1$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > \bar{T}$

$$\big\|u_\varepsilon^{(\kappa)}(\cdot,t)\big\|^2_{L^2(\Omega)} < \delta_0$$

holds. Now, by making use of the first part of the proof and Lemma 5.21, given any $\delta > 0$, we find $T_2 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T_2$

$$\|u_\varepsilon^{(\kappa)}(\cdot,t)\|^2_{L^2(\Omega)} < \frac{\delta}{2} \quad \text{and} \quad \int_t^{t+1} \big\|n_\varepsilon^{(\kappa)} - \overline{n_0}\big\|^2_{L^2(\Omega)} < \frac{\delta}{2C_1\|\nabla\phi\|^2_{L^\infty(\Omega)}}.$$

Therefore, for $t > T_2$ integrating (5.6.4) with respect to time shows

$$\int_t^{t+1}\int_\Omega \big|\nabla u_\varepsilon^{(\kappa)}\big|^2 \le \int_\Omega \big|u_\varepsilon^{(\kappa)}(\cdot,t)\big|^2 + C_1\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_t^{t+1}\int_\Omega \big(n_\varepsilon^{(\kappa)} - \overline{n_0}\big)^2 < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

proving (5.6.2). Finally, (5.6.3) is an immediate consequence of (5.6.2) and $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$. □

Making use of semigroup estimates for the Stokes equation we can further refine the smallness results of the previous lemmata.

Lemma 5.23.
For all $\delta > 0$ and any $p > 3$ there exists $T > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\left\|u_\varepsilon^{(\kappa)}(\cdot,s)\right\|_{L^p(\Omega)} < \delta \quad \textit{for any } s \in [t, t+1].$$

Proof: This is a consequence of Lemmata 5.21, 5.22 and a fixed point argument relying on the regularizing effects of the Stokes semigroup. The proof we give here is based on [105, Lemma 7.5] and [47, Lemma 3.8]. We fix some $p_0 \in (3,p)$ satisfying $p_0 \le 6$ and then let $\gamma := \frac{3}{2}(\frac{1}{p_0} - \frac{1}{p})$. We note that by these choices γ fulfills $\gamma \in (0, \frac{1}{2} - \frac{3}{2p})$ and hence the constant

$$C_1 := \int_0^1 (1-\sigma)^{-\frac{1}{2}-\frac{3}{2p}} \sigma^{-2\gamma}\,\mathrm{d}\sigma$$

is finite. Moreover, according to the well known smoothing properties of the Stokes operator ([29]), there exist $C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}
\|e^{-tA}\mathcal{P}\varphi\|_{L^p(\Omega)} &\le C_2 t^{-\gamma}\|\varphi\|_{L^{p_0}(\Omega)} && \text{for all } \varphi \in L^{p_0}(\Omega;\mathbb{R}^3) \text{ and all } t > 0,\\
\|e^{-tA}\mathcal{P}\varphi\|_{L^p(\Omega)} &\le C_3 \|\varphi\|_{L^p(\Omega)} && \text{for all } \varphi \in L^{p}(\Omega;\mathbb{R}^3) \text{ and all } t > 0, \quad (5.6.5)\\
\|e^{-tA}\mathcal{P}\nabla\cdot\varphi\|_{L^p(\Omega)} &\le C_4 t^{-\frac{1}{2}-\frac{3}{2p}}\|\varphi\|_{L^{\frac{p}{2}}(\Omega)} && \text{for all } \varphi \in L^{\frac{p}{2}}(\Omega;\mathbb{R}^3) \text{ and all } t > 0.
\end{aligned}$$

Now, given $\delta > 0$ we next fix $\delta_0 \in (0,\delta)$ such that

$$\delta_0 3^{\frac{1}{2}-\frac{3}{2p}-\gamma} C_1 C_4 < \frac{1}{3}$$

and then, in light of Lemma 5.22, Lemma 5.21 and (5.1.3), pick $T_0 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T_0$

$$\int_t^{t+1} \left\|u_\varepsilon^{(\kappa)}\right\|_{L^{p_0}(\Omega)} < \frac{\delta_0}{3C_2} \quad \text{and} \quad \int_t^{t+3} \left\|n_\varepsilon^{(\kappa)} - \overline{n_0}\right\|_{L^p(\Omega)} < \frac{\delta_0}{3^{1+\gamma} C_3 \|\nabla\phi\|_{L^\infty(\Omega)}},$$

which in particular also entails that for any fixed $t_1 > T_0$, each $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ there exists $t_\star \in (t_1, t_1+1)$ such that $\left\|u_\varepsilon^{(\kappa)}(\cdot,t_\star)\right\|_{L^{p_0}(\Omega)} < \frac{\delta_0}{3C_2}$ holds. Letting

$$X := \Big\{\varphi : \Omega \times (t_\star, t_\star+3) \to \mathbb{R}^3 \,\big|\, \|\varphi\|_X := \sup_{s\in(t_\star,t_\star+3)} (t-t_\star)^\gamma \|\varphi(\cdot,s)\|_{L^p(\Omega)} < \infty\Big\},$$

we now consider the map Ψ acting on the closed subset $S := \{\varphi \in X \,|\, \|\varphi\|_X \le \delta_0\}$ defined by

$$\Psi(\varphi)(\cdot,t) := e^{-(t-t_\star)A} u_\varepsilon^{(\kappa)}(\cdot,t_\star) + \int_{t_\star}^t e^{(t-s)A}\mathcal{P}\Big(-\kappa\nabla\cdot(Y_\varepsilon\varphi\otimes\varphi)(\cdot,s) + n_\varepsilon^{(\kappa)}(\cdot,s)\nabla\phi\Big)\,\mathrm{d}s.$$

Drawing on (5.6.5), the contraction property of Y_ε and the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}\|\Psi(\varphi)(\cdot,t)\|_{L^p(\Omega)} \le C_2(t-t_\star)^{-\gamma}\big\|u_\varepsilon^{(\kappa)}(\cdot,t_\star)\big\|_{L^{p_0}(\Omega)} + C_4\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\|\varphi\|^2_{L^p(\Omega)}\,\mathrm{d}s\\ + C_3\|\nabla\phi\|_{L^\infty(\Omega)}\int_{t_\star}^t \big\|n_\varepsilon^{(\kappa)}(\cdot,t)-\overline{n_0}\big\|_{L^p(\Omega)}\,\mathrm{d}s \qquad (5.6.6)\end{aligned}$$

for all $t\in(t_\star,t_\star+3)$. In light of our choice for δ_0, the definition of S and the fact that $|t-t_\star|\le 3$, (5.6.6) implies that

$$\begin{aligned}&(t-t_\star)^\gamma\|\Psi(\varphi)(\cdot,t)\|_{L^p(\Omega)}\\ &\le C_2\big\|u_\varepsilon^{(\kappa)}(\cdot,t_\star)\big\|_{L^{p_0}(\Omega)} + \delta_0^2(t-t_\star)^\gamma C_4\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}(s-t_\star)^{-2\gamma}\,\mathrm{d}s\\ &\qquad + 3^\gamma C_3\|\nabla\phi\|_{L^\infty(\Omega)}\int_{t_\star}^{t_\star+3}\big\|n_\varepsilon^{(\kappa)}(\cdot,t)-\overline{n_0}\big\|_{L^p(\Omega)}\,\mathrm{d}s\\ &< \frac{\delta_0}{3} + \delta_0^2(t-t_\star)^{\gamma-\frac{1}{2}-\frac{3}{2p}-2\gamma+1}C_4\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{3}{2p}}\sigma^{-2\gamma}\,\mathrm{d}\sigma + \frac{\delta_0}{3}\\ &\le \frac{\delta_0}{3} + \delta_0\left(\delta_0 3^{\frac{1}{2}-\frac{3}{2p}-\gamma}C_1C_4\right) + \frac{\delta_0}{3} < \delta_0, \qquad (5.6.7)\end{aligned}$$

and hence Ψ maps S onto itself. Now, taking into account that for any $\varphi,\psi\in L^p(\Omega;\mathbb{R}^3)$

$$\|Y_\varepsilon\varphi\otimes\varphi - Y_\varepsilon\psi\otimes\psi\|_{L^{\frac{p}{2}}(\Omega)} \le (\|\varphi\|_{L^p(\Omega)}+\|\psi\|_{L^p(\Omega)})\|\varphi-\psi\|_{L^p(\Omega)}$$

we find that for any $\varphi,\psi\in S$

$$\begin{aligned}\|\Psi(\varphi)-\Psi(\psi)\|_{L^p(\Omega)} &\le C_4\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\|\varphi\otimes\varphi-\psi\otimes\psi\|_{L^{\frac{p}{2}}(\Omega)}\,\mathrm{d}s\\ &\le 2\delta_0C_4\|\varphi-\psi\|_X\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}(s-t_\star)^{-2\gamma}\,\mathrm{d}s \quad \text{on } (t_\star,t_\star+3),\end{aligned}$$

so that

$$(t-t_\star)^\gamma\|(\Psi(\varphi)-\Psi(\psi))(\cdot,t)\|_{L^p(\Omega)} \le 2\delta_0 3^{\frac{1}{2}-\frac{3}{2p}-\gamma}C_1C_4\|\varphi-\psi\|_X \;\text{ for all } t\in(t_\star,t_\star+3),$$

with $2\delta_0 3^{\frac{1}{2}-\frac{3}{2p}-\gamma}C_1C_4 < \frac{2}{3}$. Thus, $\Psi: S\to S$ is a contracting map and therefore, there exists a unique fixed point of Ψ on S, which has to coincide with $u_\varepsilon^{(\kappa)}$ on $(t_\star,t_\star+3)$ ([75, Theorem V.2.5.1]) and we conclude from (5.6.7) and the fact that $(t_1+2,t_1+3)\subset(t_\star+1,t_\star+3)$ that for any $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$

$$\big\|u_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^p(\Omega)} < \delta \quad \text{for all } t\in(t_1+2,t_1+3),$$

completing the proof. □

5.7 Uniform eventual smoothness estimates

In order to obtain an improvement on the regularity of our solution components, we will incorporate arguments shown in [47, Lemmata 3.9, 3.10 and 3.11]. For this to work, however, we will require the following cut-off functions (cf. [105] and [47]).

Definition 5.24.
Given any monotonically increasing function $\xi_0 \in C^\infty(\mathbb{R})$ satisfying

$$0 \le \xi_0 \le 1 \text{ on } \mathbb{R}, \quad \xi_0 \equiv 0 \text{ on } (-\infty, 0] \quad \text{and} \quad \xi_0 \equiv 1 \text{ on } (1, \infty)$$

and some $t_0 > 0$ we set

$$\xi_{t_0}(t) := \xi_0(t - t_0), \quad t \in \mathbb{R}.$$

Relying on well known maximal Sobolev estimates for the Stokes equation we can establish a uniform bound for $u_\varepsilon^{(\kappa)}$ in certain Hölder spaces.

Lemma 5.25.
There exist $\gamma \in (0,1)$, $T > 0$ and $C > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\left\| u_\varepsilon^{(\kappa)} \right\|_{C^{1+\gamma, \frac{\gamma}{2}}(\overline{\Omega} \times [t, t+1])} \le C. \tag{5.7.1}$$

Proof: The proof follows the approach undertaken in [47, Lemma 3.9], which relies on maximal Sobolev regularity properties of the Stokes equation and the uniform bounds already prepared.
Let us first fix the following parameters. Let $s > 3$, $r > 1$ and then we pick $s_1 > 2s$ and s_1' such that $\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{s}$. Then according to Lemma 5.16 we can find $T' > 0$ and $C_1 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T'$

$$\int_t^{t+1} \left\| n_\varepsilon^{(\kappa)} \right\|_{L^s(\Omega)}^s \le C_1 \tag{5.7.2}$$

holds. Moreover, drawing on Lemmata 5.22 and 5.23 we can fix $T > T'$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ we also have

$$\left\| u_\varepsilon^{(\kappa)} \right\|_{L^\infty((t,t+2);L^r(\Omega))} \le C_1, \ \left\| u_\varepsilon^{(\kappa)} \right\|_{L^\infty((t,t+2);L^s(\Omega))} \le C_1, \ \left\| u_\varepsilon^{(\kappa)} \right\|_{L^\infty((t,t+2);L^{s_1'}(\Omega))} \le C_1. \tag{5.7.3}$$

Now, for $t_0 > T$ we let $\xi := \xi_{t_0}$ denote the cut-off function given by Definition 5.24 and find that $\xi u_\varepsilon^{(\kappa)}$ fulfills

$$\begin{aligned}
&\big(\xi u_\varepsilon^{(\kappa)}\big)_t = \Delta\big(\xi u_\varepsilon^{(\kappa)}\big) - \kappa\big(Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla\big)\xi u_\varepsilon^{(\kappa)} - \nabla\big(\xi P_\varepsilon^{(\kappa)}\big) + \xi n_\varepsilon^{(\kappa)} \nabla\phi + \xi' u_\varepsilon^{(\kappa)} \text{ in } \Omega \times (t_0, \infty), \\
&\qquad \nabla \cdot \big(\xi u_\varepsilon^{(\kappa)}\big) = 0 \text{ in } \Omega \times (t_0, \infty), \\
&\text{with } \big(\xi u_\varepsilon^{(\kappa)}\big)(\cdot, t_0) = 0 \text{ in } \Omega \qquad \text{and} \qquad \big(\xi u_\varepsilon^{(\kappa)}\big) = 0 \text{ on } \partial\Omega \times (t_0, \infty).
\end{aligned}$$

Thus, the maximal Sobolev regularity estimate for the Stokes semigroup ([30]) provides $C_2 > 0$ such that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$

$$\begin{aligned}
&\int_{t_0}^{t_0+2} \big\|\big(\xi u_\varepsilon^{(\kappa)}\big)_t\big\|_{L^s(\Omega)}^s + \int_{t_0}^{t_0+2} \big\|D^2\big(\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s \\
&\quad\le C_2 \cdot 0 + C_2 \int_{t_0}^{t_0+2} \big\|\mathcal{P}\big(\big(Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla\big)\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s + C_2 \int_{t_0}^{t_0+2} \big\|\mathcal{P}\big(\xi n_\varepsilon^{(\kappa)} \nabla\phi\big)\big\|_{L^s(\Omega)}^s \\
&\qquad + C_2 \int_{t_0}^{t_0+2} \big\|\mathcal{P}\big(\xi' u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s. \qquad (5.7.4)
\end{aligned}$$

According to (5.7.2) and (5.7.3) we obtain $C_3 > 0$ such that for any $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ we may estimate

$$\int_{t_0}^{t_0+2} \big\|\mathcal{P}\big(\xi n_\varepsilon^{(\kappa)} \nabla\phi\big)\big\|_{L^s(\Omega)}^s + \int_{t_0}^{t_0+2} \big\|\mathcal{P}\big(\xi' u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s \le C_3. \qquad (5.7.5)$$

Moreover, we can find $C_4 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t_0 > T$

$$\begin{aligned}
\big\|\mathcal{P}\big(\big(Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla\big)\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s &\le C_4 \big\|Y_\varepsilon u_\varepsilon^{(\kappa)}\big\|_{L^{s_1'}(\Omega)}^s \big\|\nabla\big(\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^{s_1}(\Omega)}^s \\
&\le C_4 C_1^s \big\|\nabla\big(\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^{s_1}(\Omega)}^s
\end{aligned}$$

on (t_0, t_0+2), due to Hölder's inequality and the fact that $\|Y_\varepsilon\varphi\|_{L^{s_1'}(\Omega)} \le \|\varphi\|_{L^{s_1'}(\Omega)}$ holds for all $\varphi \in L^{s_1'}\big(\Omega;\mathbb{R}^3\big)$. Employing the Gagliardo–Nirenberg inequality we then obtain $C_5 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t_0 > T$

$$\begin{aligned}
\big\|\mathcal{P}\big(\big(Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla\big)\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s &\le C_5 C_4 C_1^s \big\|D^2\big(\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^{as} \big\|\xi u_\varepsilon^{(\kappa)}\big\|_{L^r(\Omega)}^{(1-a)s} \\
&\le C_5 C_4 C_1^{(2-a)s} \big\|D^2\big(\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^{as}
\end{aligned}$$

holds on (t_0, t_0+2), where $a = \frac{\frac{1}{3} - \frac{1}{s_1} + \frac{1}{r}}{\frac{2}{3} - \frac{1}{s} + \frac{1}{r}} \in (\frac{1}{2}, 1)$. Hence, upon integration with respect to time, an application of Young's inequality provides $C_6 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t_0 > T$

$$C_2 \int_{t_0}^{t_0+2} \big\|\mathcal{P}\big(\big(Y_\varepsilon u_\varepsilon^{(\kappa)} \cdot \nabla\big)\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s \le \frac{1}{2} \int_{t_0}^{t_0+2} \big\|D^2\big(\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s + 2C_6 C_2,$$

which combined with (5.7.4) and (5.7.5) shows that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t_0 > T$ we have

$$\int_{t_0}^{t_0+2} \big\|\big(\xi u_\varepsilon^{(\kappa)}\big)_t\big\|_{L^s(\Omega)}^s + \frac{1}{2} \int_{t_0}^{t_0+2} \big\|D^2\big(\xi u_\varepsilon^{(\kappa)}\big)\big\|_{L^s(\Omega)}^s \le 2C_6 C_2 + C_3 C_2.$$

Due to $\xi \equiv 1$ on (t_0+1, t_0+2), this readily implies that for any $s > 1$ there exist $C_7 > 0$ and $T > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$

$$\int_t^{t+1} \big\|u_{\varepsilon t}^{(\kappa)}\big\|_{L^s(\Omega)}^s + \int_t^{t+1} \big\|u_\varepsilon^{(\kappa)}\big\|_{W^{2,s}(\Omega)}^s \le C_7,$$

and in light of known embedding results (e.g. [2, Theorem 1.1]) entails (5.7.1). $\square$

Arguments along the same lines of the previous lemma (and previously also employed in [47, Lemmata 3.10 and 3.11]), this time drawing on maximal Sobolev estimates for the Neumann heat semigroup also help us derive Hölder bounds for the remaining components. We proceed with proving a corresponding bound for the signal chemical.

Lemma 5.26.
For any $p \in (1,\infty)$ there exist $T > 0$ and $C > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\int_t^{t+1} \big\| c_{\varepsilon t}^{(\kappa)} \big\|_{L^p(\Omega)} + \int_t^{t+1} \big\| c_\varepsilon^{(\kappa)} \big\|_{W^{2,p}(\Omega)} \leq C.$$

Furthermore, there exist $\gamma \in (0,1)$, $T > 0$ and $C' > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$

$$\big\| c_\varepsilon^{(\kappa)} \big\|_{C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \leq C'. \tag{5.7.6}$$

Proof: Given an arbitrary $p \in (1,\infty)$ we first fix $q \in (1,p)$. Now, according to Lemmata 5.25 and 5.16 we can pick $T' > 0$ and $C_1 > 0$ such that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$ we have

$$\big\| u_\varepsilon^{(\kappa)} \big\|_{L^\infty(\Omega\times(T',\infty))} + \big\| n_\varepsilon^{(\kappa)} \big\|_{L^\infty((T',\infty);L^p(\Omega))} \leq C_1.$$

Then, for any $t_0 > T'$ we denote by $\xi := \xi_{t_0}$ a temporal cutoff function as given by Defintion 5.24 and observe that $\xi c_\varepsilon^{(\kappa)}$ then satisfies

$$\big(\xi c_\varepsilon^{(\kappa)}\big)_t + \xi u_\varepsilon^{(\kappa)} \cdot \nabla c_\varepsilon^{(\kappa)} = \Delta\big(\xi c_\varepsilon^{(\kappa)}\big) + \frac{1}{\varepsilon}\xi c_\varepsilon^{(\kappa)} \ln\big(1+\varepsilon n_\varepsilon^{(\kappa)}\big) + \xi' c_\varepsilon^{(\kappa)} \quad \text{on } \Omega\times(t_0,\infty)$$

with $\frac{\partial(\xi c_\varepsilon^{(\kappa)})}{\partial\nu} = 0$ on $\partial\Omega\times(t_0,\infty)$ and $\xi c_\varepsilon^{(\kappa)}(\cdot,t_0) = 0$ in Ω. In light of the maximal Sobolev regularity estimates for the Neumann heat semigroup ([30]), (5.2.1) and (5.2.2), this implies the existence of $C_2 > 0$ such that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$

$$\begin{aligned}
&\int_{t_0}^{t_0+2} \big\| \big(\xi c_\varepsilon^{(\kappa)}\big)_t \big\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2} \big\| \Delta\big(\xi c_\varepsilon^{(\kappa)}\big) \big\|_{L^p(\Omega)}^p \\
&\leq C_2\Big(0 + \int_{t_0}^{t_0+2} \big\| \xi u_\varepsilon^{(\kappa)} \cdot \nabla c_\varepsilon^{(\kappa)} \big\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2} \big\| \xi c_\varepsilon^{(\kappa)} n_\varepsilon^{(\kappa)} \big\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2} \big\| \xi' c_\varepsilon^{(\kappa)} \big\|_{L^p(\Omega)}^p \Big) \\
&\leq C_2 C_1^p \int_{t_0}^{t_0+2} \big\| \nabla\big(\xi c_\varepsilon^{(\kappa)}\big) \big\|_{L^p(\Omega)}^p + 2C_2C_1^p \|c_0\|_{L^\infty(\Omega)}^p + 2C_2\|c_0\|_{L^\infty(\Omega)}^p \|\xi'\|_{L^\infty(\mathbb{R})}^p
\end{aligned} \tag{5.7.7}$$

holds. Since the Gagliardo–Nirenberg inequality entails the existence of $C_3 > 0$ such that for all $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and $t > t_0$ we have

$$\big\|\nabla\big(\xi c_\varepsilon^{(\kappa)}\big)(\cdot,t)\big\|_{L^p(\Omega)}^p \leq C_3 \big\|\Delta\big(\xi c_\varepsilon^{(\kappa)}\big)(\cdot,t)\big\|_{L^p(\Omega)}^{ap} \big\|\xi c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^q(\Omega)}^{(1-a)p} + C_3 \big\|\xi c_\varepsilon^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)}^p,$$

with $a = \frac{\frac{1}{3} - \frac{1}{p} + \frac{1}{q}}{\frac{2}{3} - \frac{1}{p} + \frac{1}{q}}$ satisfying $a \in (\frac{1}{2}, 1)$, an employment of Young's inequality together with Hölder's inequality and (5.2.2) provides $C_4 > 0$ such that for all $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and any $t_0 > T'$ the inequality from (5.7.7) reads like

$$\int_{t_0}^{t_0+2} \big\| \big(\xi c_\varepsilon^{(\kappa)}\big) \big\|_{L^p(\Omega)}^p + \frac{1}{2} \int_{t_0}^{t_0+2} \big\| \Delta\big(\xi c_\varepsilon^{(\kappa)}\big) \big\|_{L^p(\Omega)}^p \\ \leq \|c_0\|_{L^\infty(\Omega)}^p \big(2C_4 + 2C_3 C_2 C_1^p + 2C_2 C_1^p + 2C_2 \|\xi'\|_{L^\infty(\mathbb{R})}^p\big).$$

Since $\xi \equiv 1$ on (t_0+1, t_0+2), this shows that for any $p > 1$ one can find $C_5 > 0$ and $T := T' + 1 > 0$ such that for any $t > T$ and all $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ we have

$$\int_t^{t+1} \|c_{\varepsilon t}^{(\kappa)}\|_{L^p(\Omega)}^p + \int_t^{t+1} \|c_\varepsilon^{(\kappa)}\|_{W^{2,p}(\Omega)} \leq C_5.$$

The asserted Hölder regularity finally results from an application of an embedding result e.g. presented in [2, Theorem 1.1] by taking p large enough. □

A final iteration of similar arguments entails a uniform Hölder bound for the first solution component.

Lemma 5.27.
There exist $\gamma \in (0,1)$, $T > 0$ and $C > 0$ such that for each $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ and all $t > T$ the solution $(n_\varepsilon^{(\kappa)}, c_\varepsilon^{(\kappa)}, u_\varepsilon^{(\kappa)})$ of $(\Lambda_{\varepsilon,\kappa})$ satisfies

$$\|n_\varepsilon^{(\kappa)}\|_{C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \leq C. \tag{5.7.8}$$

Proof: We work along similar lines as in the previous lemma. First, given any $p > 1$ we pick $q \in (1,p)$ and, in light of Lemmata 5.16, 5.25 and 5.26, can then find $T' > 0$ and $C_1 > 0$ such that for any $\varepsilon \in (0,1)$, $\kappa \in [-1,1]$ we have

$$\|n_\varepsilon^{(\kappa)}\|_{L^\infty((T',\infty);L^p(\Omega))} + \|n_\varepsilon^{(\kappa)}\|_{L^\infty((T',\infty);L^q(\Omega))} + \|n_\varepsilon^{(\kappa)}\|_{L^\infty((T',\infty);L^{2p}(\Omega))} \leq C_1,$$

and $C_2, C_3 > 0$ such that for any $\varepsilon \in (0,1)$, each $\kappa \in [-1,1]$ and all $t > T'$ the estimates

$$\|\nabla c_\varepsilon^{(\kappa)}\|_{L^\infty(\Omega\times(t,t+2))} \leq C_2, \quad \int_t^{t+2} \|\Delta c_\varepsilon^{(\kappa)}\|_{L^{2p}(\Omega)}^p \leq C_2, \quad \text{and} \quad \|u_\varepsilon^{(\kappa)}\|_{L^\infty(\Omega\times(t,t+2))} \leq C_3$$

hold. Now, for $t_0 > T'$ we once more denote by $\xi := \xi_{t_0}$ the cutoff function from Definition 5.24 and the maximal Sobolev regularity estimates ([30]) then again entail the existence of $C_4 > 0$ such that for all $\varepsilon \in (0,1)$ and $\kappa \in [-1,1]$

$$\int_{t_0}^{t_0+2} \big\| \big(\xi n_\varepsilon^{(\kappa)}\big)_t \big\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2} \big\| \Delta\big(\xi n_\varepsilon^{(\kappa)}\big) \big\|_{L^p(\Omega)}^p \tag{5.7.9}$$
$$\leq C_4 \int_{t_0}^{t_0+2} \left(\|\xi u_\varepsilon^{(\kappa)} \cdot \nabla n_\varepsilon^{(\kappa)}\|_{L^p(\Omega)}^p + \left\| \xi \nabla \cdot \left(\frac{n_\varepsilon^{(\kappa)}}{1+\varepsilon n_\varepsilon^{(\kappa)}} \nabla c_\varepsilon^{(\kappa)} \right) \right\|_{L^p(\Omega)}^p + \|\xi' n_\varepsilon^{(\kappa)}\|_{L^p(\Omega)}^p \right)$$

$$\le C_4C_3^p\int_{t_0}^{t_0+2}\|\nabla n_\varepsilon^{(\kappa)}\|_{L^p(\Omega)}^p + C_4\int_{t_0}^{t_0+2}\left\|\xi\nabla\cdot\left(\frac{n_\varepsilon^{(\kappa)}}{1+\varepsilon n_\varepsilon^{(\kappa)}}\nabla c_\varepsilon^{(\kappa)}\right)\right\|_{L^p(\Omega)}^p + 2C_1^pC_4\|\xi'\|_{L^\infty(\mathbb{R})}.$$

Next, to estimate mixed derivative term, we note that by the bounds prepared at the start of the lemma

$$\begin{aligned}\left\|\xi\nabla\cdot\left(\frac{n_\varepsilon^{(\kappa)}}{1+\varepsilon n_\varepsilon^{(\kappa)}}\nabla c_\varepsilon^{(\kappa)}\right)\right\|_{L^p(\Omega)}^p &= \left\|\xi\tfrac{\nabla n_\varepsilon^{(\kappa)}\cdot\nabla c_\varepsilon^{(\kappa)}}{(1+\varepsilon n_\varepsilon^{(\kappa)})^2}+\xi\frac{n_\varepsilon^{(\kappa)}}{1+\varepsilon n_\varepsilon^{(\kappa)}}\Delta c_\varepsilon^{(\kappa)}\right\|_{L^p(\Omega)}^p\\ &\le 2^pC_2^p\big\|\nabla\big(\xi n_\varepsilon^{(\kappa)}\big)\big\|_{L^p(\Omega)}^p + 2^pC_1^p\|\Delta c_\varepsilon^{(\kappa)}\|_{L^{2p}(\Omega)}^p \qquad (5.7.10)\end{aligned}$$

is valid on $(t_0, t_0{+}2)$. Moreover, the Gagliardo–Nirenberg inequality implies the existence of $C_5 > 0$ such that

$$\|\nabla\varphi\|_{L^p(\Omega)}^p \le C_5\|\Delta\varphi\|_{L^p(\Omega)}^{ap}\|\varphi\|_{L^q(\Omega)}^{(1-a)p} + \|\varphi\|_{L^q(\Omega)}^p \quad \text{for all } \varphi\in W^{2,p}(\Omega),$$

where again $a = \frac{\frac13-\frac1p+\frac1q}{\frac23-\frac1p+\frac1q}\in(\frac12,1)$, and hence we infer from Young's inequality that there is $C_6>0$ such that

$$C_4(C_3^p+2^pC_2^p)\|\nabla\varphi\|_{L^p(\Omega)}^p \le \frac12\|\Delta\varphi\|_{L^p(\Omega)}^p + C_6\|\varphi\|_{L^q(\Omega)}^p \quad \text{for all } \varphi\in W^{2,p}(\Omega). \quad (5.7.11)$$

Thus, collecting (5.7.9)–(5.7.11), we conclude for all $\varepsilon\in(0,1)$ and $\kappa\in[-1,1]$

$$\begin{aligned}&\int_{t_0}^{t_0+2}\big\|\big(\xi n_\varepsilon^{(\kappa)}\big)_t\big\|_{L^p(\Omega)}^p + \int_{t_0}^{t_0+2}\big\|\Delta\big(\xi n_\varepsilon^{(\kappa)}\big)\big\|_{L^p(\Omega)}^p\\ &\le C_4(C_3^p+2^pC_2^p)\int_{t_0}^{t_0+2}\big\|\nabla\big(\xi n_\varepsilon^{(\kappa)}\big)\big\|_{L^p(\Omega)}^p + 2^pC_1^p\int_{t_0}^{t_0+2}\|\Delta c_\varepsilon^{(\kappa)}\|_{L^{2p}(\Omega)}^p + 2C_1^pC_4\|\xi'\|_{L^\infty(\mathbb{R})}\\ &\le \frac12\int_{t_0}^{t_0+2}\big\|\Delta\big(\xi n_\varepsilon^{(\kappa)}\big)\big\|_{L^p(\Omega)}^p + 2C_6C_1^p + 2^pC_1^pC_2 + 2C_1^pC_4\|\xi'\|_{L^\infty(\mathbb{R})},\end{aligned}$$

which, due to $\xi\equiv1$ on (t_0+1, t_0+2) implies the existence of $C_7>0$ such that for any $\varepsilon\in(0,1)$, $\kappa\in[-1,1]$ and all $t>T:=T'+1$ we have

$$\int_t^{t+1}\|n_{\varepsilon t}^{(\kappa)}\|_{L^p(\Omega)}^p + \frac12\int_t^{t+1}\|\Delta n_\varepsilon^{(\kappa)}\|_{L^p(\Omega)}^p \le C_7.$$

Taking p large enough, the desired Hölder regularity is again an immediate consequence of the embedding result in e.g. [2, Theorem 1.1]. □

In light of standard parabolic theory and the Arzelà–Ascoli theorem, we can make use of the uniform estimates from the previous three lemmata to conclude that, after an eventual smoothing time $T_\diamond>0$, the solution obtained in Proposition 5.10 is actually a classical solution.

Lemma 5.28.
There exist $\gamma \in (0,1)$ and $T_\diamond > 0$ such that for each $\kappa \in [-1,1]$ the weak solution $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ of (Λ_κ) obtained in Proposition 5.10 satisfies

$$n^{(\kappa)}, c^{(\kappa)} \in C^{2+\gamma,1+\frac{\gamma}{2}}\big(\overline{\Omega}\times[T_\diamond,\infty)\big) \quad \text{and} \quad u^{(\kappa)} \in C^{2+\gamma,1+\frac{\gamma}{2}}\big(\overline{\Omega}\times[T_\diamond,\infty);\mathbb{R}^3\big).$$

In particular, $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ together with some $P^{(\kappa)} \in C^{1,0}\big(\overline{\Omega}\times(T_\diamond,\infty)\big)$ solves (Λ_κ) classically in $\Omega\times(T_\diamond,\infty)$. Moreover, there exists $C>0$ such that for all $\kappa \in [-1,1]$ and all $t \geq T_\diamond$

$$\begin{aligned}\big\|n^{(\kappa)}(\cdot,t)\big\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} + \big\|c^{(\kappa)}(\cdot,t)\big\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])}&\\ + \big\|u^{(\kappa)}(\cdot,t)\big\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \leq C.& \qquad (5.7.12)\end{aligned}$$

Proof: We employ standard parabolic regularity theory in a similar fashion as e.g. displayed in [47, Lemma 3.12]. Drawing on Lemmata 5.25, 5.26 and 5.27, we can pick some $\widehat{\gamma} \in (0,1)$, $T>0$ and $C>0$ such that (5.7.1), (5.7.6) and (5.7.8) hold for any $t>T$, each $\varepsilon \in (0,1)$ and every $\kappa \in [-1,1]$. Accordingly, by making use of the Arzelà–Ascoli theorem we find that for some $\gamma' \in (0,\widehat{\gamma})$ we have

$$n_\varepsilon^{(\kappa)} \to n^{(\kappa)},\ c_\varepsilon^{(\kappa)} \to c^{(\kappa)} \quad \text{in } C^{1+\gamma',\frac{\gamma'}{2}}\big(\overline{\Omega}\times[t,t+1]\big)$$

and

$$u_\varepsilon^{(\kappa)} \to u^{(\kappa)} \quad \text{in } C^{1+\gamma',\frac{\gamma'}{2}}\big(\overline{\Omega}\times[t,t+1];\mathbb{R}^3\big)$$

along a subsequence of the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ obtained in Proposition 5.10, the members of which, for convenience, we still label ε_j. Now, letting $\xi := \xi_T$ be given by Definition 5.24 we note that $\xi c^{(\kappa)}$ solves

$$c_t = \Delta c + g, \quad c(T) = 0, \quad \frac{\partial c}{\partial \nu}\big|_{\partial\Omega} = 0,$$

in the weak sense with $g = -\xi n^{(\kappa)} c^{(\kappa)} - \xi u^{(\kappa)} \nabla c^{(\kappa)} + c^{(\kappa)}\xi' \in C^{\gamma',\frac{\gamma'}{2}}\big(\overline{\Omega}\times(T,\infty)\big)$. In light of standard parabolic theory (e.g. [45, Theorems IV.5.3 and III.5.1]), we can hence conclude that for some $\gamma_1 \in (0,\gamma')$ $c^{(\kappa)} \in C^{2+\gamma_1,1+\frac{\gamma_1}{2}}\big(\overline{\Omega}\times[T+1,\infty)\big)$ and that for $\gamma \leq \gamma_1$ there is $C_1>0$ such that for any $\kappa \in [-1,1]$ (5.7.12) is true for $c^{(\kappa)}$. In a similar fashion, we observe that $\xi n^{(\kappa)}$ is a solution of

$$n_t = \Delta n - a\cdot\nabla n + b, \quad n(T) = 0, \quad \frac{\partial n}{\partial \nu}\big|_{\partial\Omega} = 0,$$

with $a = \nabla c^{(\kappa)} + u^{(\kappa)}$ and $b = -\xi n^{(\kappa)}\Delta c + n^{(\kappa)}\xi'$ both being of class $C^{\gamma_1,\frac{\gamma_1}{2}}\big(\overline{\Omega}\times(T,\infty)\big)$ and employing parabolic regularity theory (e.g. [45, Theorems IV.5.3 and III.5.1]) once more, we find $\gamma_2 \in (0,\gamma_1)$ such that $n^{(\kappa)} \in C^{2+\gamma_2,1+\frac{\gamma_2}{2}}\big(\overline{\Omega}\times[T+1,\infty)\big)$ and that for

$\gamma \le \gamma_2$ there is $C_2 > 0$ such that (5.7.12) is valid for $n^{(\kappa)}$. Lastly, since $\xi u^{(\kappa)}$ solves

$$u_t = \Delta u + h := \mathcal{P}\big(\xi' u^{(\kappa)} - \kappa\xi(u^{(\kappa)} \cdot \nabla)u^{(\kappa)}) + \xi n^{(\kappa)}\nabla\phi\big),$$
$$\nabla \cdot u = 0, \quad u(T-1) = 0, \quad u\big|_{\partial\Omega} = 0,$$

where h is again Hölder continuous due to the bounds from Lemmata 5.25, 5.26, 5.27 and (5.1.3). Hence, Schauder theory for Stokes equation (e.g. [76, Theorem 1.1]) combined with the uniqueness property ([75, V.1.5.1]) entails that for some $\gamma_3 \in (0, \gamma_2)$ we have $u^{(\kappa)} \in C^{2+\gamma_3, 1+\frac{\gamma_3}{2}}\big(\overline{\Omega} \times [T+1, \infty); \mathbb{R}^3\big)$ and that for $\gamma \le \gamma_3$ (5.7.12) is also valid for $u^{(\kappa)}$. Letting $\gamma := \gamma_3$ we obtain the inclusion in the asserted function spaces, whereas the existence of a corresponding $P^{(\kappa)} \in C^{1,0}\big(\overline{\Omega} \times (T_\diamond, \infty)\big)$ such that $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)}, P^{(\kappa)})$ solves (Λ_κ) classically in $\Omega \times (T_\diamond, \infty)$ is an immediate consequence of these regularity properties ([75]). $\square$

5.8 Uniform exponential decay after the smoothing time

For the remainder of the chapter we will denote by $T_\diamond > 0$ the smoothing time obtained in Lemma 5.28. Employing a second Aubin–Lions type argument for taking $\kappa \to 0$, we are still left with the obstacle that this limit procedure will only yield convergence on compact subsets of $\overline{\Omega} \times [0, \infty)$. In order to extend the convergence beyond compact subsets, our next objective will be to improve the previously obtained stabilization properties to a more detailed decay including an exponential rate of convergence, which, on the one hand, will still be independent of $\kappa \in [-1, 1]$ and, on the other, will be valid for all $t > T_\diamond$. We start with supplementing our decay results by the following lemma.

Lemma 5.29.
For all $\delta > 0$ one can find $T \ge T_\diamond$ such that for each $\kappa \in [-1, 1]$ and all $t > T$ the solution $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ of (Λ_κ) satisfies

$$\big\|n^{(\kappa)}(\cdot, t) - \overline{n_0}\big\|_{L^\infty(\Omega)} < \delta.$$

Proof: According to the Gagliardo–Nirenberg inequality there is $C_1 > 0$ such that for any $\kappa \in [-1, 1]$ and all $t > T_\diamond$

$$\big\|n^{(\kappa)}(\cdot, t) - \overline{n_0}\big\|_{L^\infty(\Omega)}^5 \le C_1 \big\|n^{(\kappa)}(\cdot, t) - \overline{n_0}\big\|_{W^{1,\infty}(\Omega)}^3 \big\|n^{(\kappa)}(\cdot, t) - \overline{n_0}\big\|_{L^2(\Omega)}^2. \qquad (5.8.1)$$

Moreover, by (5.7.12) and (5.1.4) we can find $C_2 > 0$ such that for any $\kappa \in [-1, 1]$ and $t > T_\diamond$

$$C_1 \big\|n^{(\kappa)}(\cdot, t) - \overline{n_0}\big\|_{W^{1,\infty}(\Omega)}^3 \le C_1 \big(\big\|n^{(\kappa)}(\cdot, t)\big\|_{C^1(\overline{\Omega})} + \|\overline{n_0}\|_{L^\infty(\Omega)}\big)^3 \le C_2, \qquad (5.8.2)$$

due to $\overline{n_0}$ being spatially homogeneous. Then, given $\delta > 0$ we set $\delta_0 := \frac{\delta^5}{C_2}$ and rely on Lemma 5.21 to find $T > T_\diamond$ such that for any $\kappa \in [-1, 1]$ and $t > T$

$$\big\|n^{(\kappa)}(\cdot, t) - \overline{n_0}\big\|_{L^2(\Omega)}^2 < \delta_0.$$

A combination of this with (5.8.1) and (5.8.2) yields that for any $\kappa \in [-1, 1]$ and all $t > T$

$$\left\|n^{(\kappa)}(\cdot, t) - \overline{n_0}\right\|_{L^\infty(\Omega)} < (C_2\delta_0)^{\frac{1}{5}} = \delta$$

holds, finalizing the proof. $\square$

Combining the previous lemma with the fact that $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ solves (Λ_κ) classically on $\Omega \times (T_\diamond, \infty)$, we can improve the eventual decay of the oxygen, which in Lemma 5.13 was still of a quite general nature, to a decay with exponential rate.

Lemma 5.30.
There exist $\mu > 0$ and $C > 0$ such that for each $\kappa \in [-1, 1]$ and all $t > 0$ the solution $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ of (Λ_κ) satisfies

$$\left\|c^{(\kappa)}(\cdot, t)\right\|_{L^\infty(\Omega)} \leq Ce^{-\mu t}. \tag{5.8.3}$$

Moreover, for any $p \geq 1$ there exist $\mu' > 0$ and $C' > 0$ such that for each $\kappa \in [-1, 1]$ and all $t > T_\diamond$

$$\left\|c^{(\kappa)}(\cdot, t)\right\|_{W^{1,p}(\Omega)} \leq C'e^{-\mu' t}. \tag{5.8.4}$$

Proof: We follow the reasoning of [91, Lemmata 4.5 and 4.6]. Drawing on the κ–independent stabilization property obtained in Lemma 5.29, we can fix $T > T_\diamond$ such that

$$n^{(\kappa)} \geq C_1 := \frac{\overline{n_0}}{2} \quad \text{in } \Omega \times (T, \infty),$$

where C_1 is positive due to (5.1.4). Noting that $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ is a classical solution of (Λ_κ) on $\Omega \times (T_\diamond, \infty)$, we can make use of the second equation of (Λ_κ) to find that

$$c_t^{(\kappa)} \leq \Delta c^{(\kappa)} - u^{(\kappa)} \cdot \nabla c^{(\kappa)} - C_1 c^{(\kappa)} \quad \text{in } \Omega \times (T, \infty),$$

and therefore, the comparison principle combined with (5.2.2) implies that

$$c^{(\kappa)}(\cdot, t) \leq \left\|c^{(\kappa)}(\cdot, T)\right\|_{L^\infty(\Omega)} e^{-C_1(t-T)} \leq \|c_0\|_{L^\infty(\Omega)} e^{-C_1(t-T)} \quad \text{for all } t > T.$$

Relying once more on (5.2.2), we find that (5.8.3) also holds for $0 < t \leq T$ by letting $C := \|c_0\|_{L^\infty(\Omega)} e^{C_1 T}$ and $\mu := \frac{\overline{n_0}}{2}$. As for the decay involving the gradient, we note that, assuming $p > 3$, the Gagliardo–Nirenberg inequality provides $C_2 > 0$ such that

$$\left\|c^{(\kappa)}(\cdot, t)\right\|_{W^{1,p}(\Omega)} \leq C_2 \left\|c^{(\kappa)}(\cdot, t)\right\|_{C^2(\overline{\Omega})}^{\frac{p-3}{2p}} \left\|c^{(\kappa)}(\cdot, t)\right\|_{L^\infty(\Omega)}^{\frac{p+3}{2p}}$$

is valid for all $t > T_\diamond$, which according to (5.8.3) and (5.7.12) implies (5.8.4). $\square$

With the previous result at hand, we cannot only transfer the exponential rate of convergence to the first solution component, but also establish this decay starting from the smoothing time $T_\diamond$, clarifying the convergence statement from Lemma 5.29.

Lemma 5.31.
There exist $\mu > 0$ and $C > 0$ such that for each $\kappa \in [-1,1]$ and all $t > T_\diamond$ the solution $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ of (Λ_κ) satisfies

$$\left\|n^{(\kappa)}(\cdot,t) - \overline{n_0}\right\|_{L^\infty(\Omega)} < Ce^{-\mu t}. \tag{5.8.5}$$

Moreover, for any $p \geq 1$ there exist $\mu' > 0$ and $C' > 0$ such that for each $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$\left\|n^{(\kappa)}(\cdot,t) - \overline{n_0}\right\|_{W^{1,p}(\Omega)} \leq C'e^{-\mu' t}.$$

Proof: We adjust the arguments of [91, Lemma 4.7] to our setting and start by working along similar lines as in Lemma 5.21, while this time making sure we keep the $L^2(\Omega)$ norm of $\nabla c_\varepsilon^{(\kappa)}$ to make full use of the exponential decay established in Lemma 5.30. In fact, drawing on the first equation in (Λ_κ) as well as integration by parts, Young's inequality and the Poincaré inequality we obtain $C_1 > 0$ such that for any $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big(n^{(\kappa)} - \overline{n_0}\big)^2 \leq -\frac{1}{C_1}\int_\Omega \big(n^{(\kappa)} - \overline{n_0}\big)^2 + \sup_{\kappa' \in [-1,1]} \left\|n^{(\kappa')}\right\|_{L^\infty(\Omega\times(T_\diamond,\infty))}\int_\Omega \left|\nabla c^{(\kappa)}\right|^2$$

holds. Hence, according to (5.7.12) and Lemma 5.30, we can fix $\mu_1 > 0$ with $\frac{1}{C_1} > \mu_1$ such that for any $\kappa \in [-1,1]$ the function $y^{(\kappa)}(t) := \int_\Omega (n^{(\kappa)}(\cdot,t) - \overline{n_0})^2$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}y^{(\kappa)}(t) + \frac{1}{C_1}y^{(\kappa)}(t) \leq C_2 e^{-\mu_1 t}$ for all $t > T_\diamond$, which implies

$$y^{(\kappa)}(t) \leq \left(C_3 + \frac{C_1 C_2}{1 - C_1\mu_1}\right) e^{\mu_1 T_\diamond} e^{-\mu_1 t} \quad \text{for all } t > T_\diamond,$$

with $C_3 := \sup_{\kappa \in [-1,1]} \int_\Omega \big(n^{(\kappa)}(\cdot,T_\diamond) - \overline{n_0}\big)^2$ being finite, again due to (5.7.12) and (5.1.4). Interpolation using the Gagliardo–Nirenberg inequality and, once more, (5.7.12) finally extends to (5.8.5) upon appropriately adjusting the constants. For the decay of the Sobolev norm, we assume, again without loss of generality, that $p > 3$ and draw on the Gagliardo–Nirenberg inequality to find $C_4 > 0$ such that

$$\begin{aligned}\left\|n^{(\kappa)}(\cdot,t) - \overline{n_0}\right\|_{W^{1,p}(\Omega)} &\leq C_4\left\|n^{(\kappa)}(\cdot,t) - \overline{n_0}\right\|_{W^{2,\infty}(\Omega)}^{\frac{p-3}{2p}}\left\|n^{(\kappa)}(\cdot,t) - \overline{n_0}\right\|_{L^\infty(\Omega)}^{\frac{p+3}{2p}}\\ &\leq C_4\Big(\left\|n^{(\kappa)}(\cdot,t)\right\|_{C^2(\overline{\Omega})} + \left\|\overline{n_0}\right\|_{L^\infty(\Omega)}\Big)^{\frac{p-3}{2p}}\left\|n^{(\kappa)}(\cdot,t) - \overline{n_0}\right\|_{L^\infty(\Omega)}^{\frac{p+3}{2p}}\end{aligned}$$

for all $t > T_\diamond$, because $\overline{n_0}$ is constant in space. Hence, the claimed exponential decay is a consequence of (5.8.5), (5.1.4) and Lemma 5.28. □

In the final part of this section, we extend the exponential stabilization of the first component to the fluid-velocity field.

Lemma 5.32.
There exist $\mu > 0$ and $C > 0$ such that for each $\kappa \in [-1,1]$ and all $t > T_\diamond$ the solution $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ of (Λ_κ) satisfies

$$\left\|u^{(\kappa)}(\cdot,t)\right\|_{L^\infty(\Omega)} < Ce^{-\mu t}. \tag{5.8.6}$$

Moreover, for any $p \geq 1$ there exists $C' > 0$ such that for each $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$\left\|u^{(\kappa)}(\cdot,t)\right\|_{W^{1,p}(\Omega)} \leq C' e^{-\mu t}.$$

Proof: Similar to the previous two lemmata and inspired by [91, Lemma 4.8], we proceed to derive an exponential decay estimate for the fluid-velocity. Due to $\nabla \cdot u^{(\kappa)} = 0$ in $\Omega \times (0,\infty)$ and $u^{(\kappa)} = 0$ on $\partial\Omega \times (0,\infty)$, we obtain upon testing the third equation of (Λ_κ) against $u^{(\kappa)}$ and employing Hölder's inequality that for any $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|u^{(\kappa)}\right|^2 + \int_\Omega \left|\nabla u^{(\kappa)}\right|^2 &= \int_\Omega \left(n^{(\kappa)} - \overline{n_0}\right)\nabla\phi \cdot u^{(\kappa)} \\ &\leq \sqrt{|\Omega|}\|\nabla\phi\|_{L^\infty(\Omega)}\left\|n^{(\kappa)} - \overline{n_0}\right\|_{L^\infty(\Omega)}\left\|u^{(\kappa)}\right\|_{L^2(\Omega)}.\end{aligned} \tag{5.8.7}$$

The Poincaré inequality provides $C_1 > 0$ such that for any $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$C_1 \int_\Omega \left|u^{(\kappa)}\right|^2 \leq \int_\Omega \left|\nabla u^{(\kappa)}\right|^2$$

holds, and hence we can rely on Young's inequality to conclude from (5.8.7) that for any $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|u^{(\kappa)}\right|^2 &+ \frac{C_1}{2}\int_\Omega \left|u^{(\kappa)}\right|^2 + \frac{1}{2}\int_\Omega \left|\nabla u^{(\kappa)}\right|^2 \\ &\leq \frac{\sqrt{|\Omega|}}{\sqrt{C_1}}\|\nabla\phi\|_{L^\infty(\Omega)}\left\|n^{(\kappa)} - \overline{n_0}\right\|_{L^\infty(\Omega)}\left\|\nabla u^{(\kappa)}\right\|_{L^2(\Omega)} \\ &\leq C_2\left\|n^{(\kappa)} - \overline{n_0}\right\|_{L^\infty(\Omega)}^2 + \frac{1}{2}\int_\Omega \left|\nabla u^{(\kappa)}\right|^2,\end{aligned}$$

where $C_2 := \frac{|\Omega|\|\nabla\phi\|_{L^\infty(\Omega)}^2}{2C_1}$. Thus, making use of the decay estimate from Lemma 5.31, we can find $\mu_1 \in (0, C_1)$ and $C_3 > 0$ such that $y^{(\kappa)}(t) := \int_\Omega \left|u^{(\kappa)}(\cdot,t)\right|^2$, $t > T_\diamond$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}y^{(\kappa)}(t) + C_1 y^{(\kappa)}(t) \leq C_3 e^{-\mu_1 t} \quad \text{for all } t > T_\diamond,$$

implying that for any $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$y^{(\kappa)}(t) \leq \left(y^{(\kappa)}(T_\diamond) + \frac{C_3}{C_1 - \mu_1}\right) e^{C_1 T_\diamond} e^{-\mu_1 t} =: C_4 e^{-\mu_1 t}, \tag{5.8.8}$$

where $C_4 = (y^{(\kappa)}(T_\diamond) + \frac{C_3}{C_1-\mu_1})e^{C_1 T_\diamond}$ does not depend on κ and is finite due to (5.7.12). Now, with $\varrho \in (\frac{3}{4}, 1)$ given by (5.1.4) and according to [32, Theorem 1.4.4], there are $C_5, C_6 > 0$ such that for any $\kappa \in [-1,1]$ and each $t > T_\diamond$

$$\begin{aligned}\big\|A^\varrho u^{(\kappa)}(\cdot,t)\big\|_{L^2(\Omega)} &\le C_5 \big\|Au^{(\kappa)}(\cdot,t)\big\|_{L^2(\Omega)}^{\varrho}\big\|u^{(\kappa)}(\cdot,t)\big\|_{L^2(\Omega)}^{1-\varrho}\\ &\le C_6 \big\|u^{(\kappa)}(\cdot,t)\big\|_{C^2(\overline{\Omega})}^{\varrho}\big\|u^{(\kappa)}(\cdot,t)\big\|_{L^2(\Omega)}^{1-\varrho},\end{aligned}$$

and drawing once more on (5.7.12) and (5.8.8) together with the embedding $D(A^\varrho) \hookrightarrow L^\infty\left(\Omega;\mathbb{R}^3\right)$ ([32, Theorem 1.6.1]) provides $C_7 > 0$ such that for any $\kappa \in [-1,1]$ and all $t > T_\diamond$

$$\big\|u^{(\kappa)}(\cdot,t)\big\|_{L^\infty(\Omega)} \le C_7 e^{-\frac{\mu_1(1-\varrho)t}{2}}$$

holds and hence proves (5.8.6). Employing the Gagliardo–Nirenberg inequality in a similar fashion as in the proofs of the previous two lemmata finally entails the exponential decay of the desired Sobolev norms. □

5.9 The second limit. Taking $\kappa \to 0$

The uniform exponential decay starting from the smoothing time $T_\diamond$ was the last missing ingredient for proving our theorem. Before we give the proof of the theorem however, we first collect many of the prepared estimates for the following second limit procedure.

Proposition 5.33.
Given any null sequence $(\kappa_j)_{j\in\mathbb{N}} \subset [-1,1]$ one can find a subsequence $(\kappa_{j_k})_{k\in\mathbb{N}}$ and functions

$$\begin{aligned} n &\in L^{\frac{5}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \quad \textit{with} \quad \nabla n \in L^{\frac{5}{4}}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big),\\ c &\in L^\infty(\Omega\times(0,\infty)) \quad \textit{with} \quad \nabla c \in L^4_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big),\\ u &\in L^2_{loc}\big([0,\infty);W^{1,2}_{0,\sigma}(\Omega)\big),\end{aligned}$$

such that the global weak solution $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ of (Λ_κ),(5.1.1),(5.1.2) satisfies

$$\begin{aligned} n^{(\kappa)} &\to n && \textit{in } L^p_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \textit{ for any } p\in[1,\tfrac{5}{3}) \textit{ and a.e. in } \Omega\times(0,\infty),\\ \nabla n^{(\kappa)} &\rightharpoonup \nabla n && \textit{in } L^{\frac{5}{4}}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big),\\ n^{(\kappa)} &\rightharpoonup n && \textit{in } L^{\frac{5}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty)\big),\\ c^{(\kappa)} &\to c && \textit{in } L^p_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \textit{ for any } p\in[1,\infty) \textit{ and a.e. in } \Omega\times(0,\infty),\\ c^{(\kappa)} &\overset{\star}{\rightharpoonup} c && \textit{in } L^\infty(\Omega\times(0,\infty)),\\ \nabla c^{(\kappa)} &\rightharpoonup \nabla c && \textit{in } L^4_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big),\\ u^{(\kappa)} &\to u && \textit{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big) \textit{ and a.e. in } \Omega\times(0,\infty),\\ u^{(\kappa)} &\rightharpoonup u && \textit{in } L^{\frac{10}{3}}_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^3\big),\end{aligned}$$

$$\nabla u^{(\kappa)} \rightharpoonup \nabla u \quad \text{in } L^2_{loc}\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^{3\times3}\big)\,,$$

as $\kappa = \kappa_{j_k} \to 0$. *The triple* (n,c,u) *is a global weak solution of the chemotaxis-Stokes system* (Λ_0),(5.1.1),(5.1.2) *in the sense of Definition 5.2, and one can find* $P \in C^{1,0}\big(\overline{\Omega}\times(T_\diamond,\infty)\big)$ *such that* (n,c,u,P) *are a classical solution of* (Λ_0),(5.1.1),(5.1.2) *in* $\Omega\times(T_\diamond,\infty)$. *Moreover, there exist* $\mu>0$ *and* $C>0$ *such that for all* $t>T_\diamond$,

$$\|n(\cdot,t)-\overline{n_0}\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{L^\infty(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} < Ce^{-\mu t} \tag{5.9.1}$$

and for any $p\geq 1$ *there are* $\mu'>0$ *and* $C'>0$ *such that*

$$\|n(\cdot,t)-\overline{n_0}\|_{W^{1,p}(\Omega)} + \|c(\cdot,t)\|_{W^{1,p}(\Omega)} + \|u(\cdot,t)\|_{W^{1,p}(\Omega)} < C'e^{-\mu' t} \tag{5.9.2}$$

is valid for all $t>T_\diamond$.

Proof: As the bounds in Lemmata 5.6, 5.8 and 5.9 are independent of $\varepsilon\in(0,1)$ and $\kappa\in[-1,1]$, they are inherited by the limit functions $n^{(\kappa)}, c^{(\kappa)}$ and $u^{(\kappa)}$ obtained in Proposition 5.10. Hence, an identical reasoning, drawing on the Aubin–Lions Lemma [74, Corollary 8.4] and Vitali's theorem, as previously done in Proposition 5.10, establishes the asserted convergence properties and weak solution properties of the limit functions n, c and u. That there exists some $P\in C^{1,0}\big(\overline{\Omega}\times(T_\diamond,\infty)\big)$, which together with (n,c,u) solves (Λ_0) classically in $\Omega\times(T_\diamond,\infty)$ is then a consequence of Lemma 5.28 and the Arzelà–Ascoli theorem. The exponential decay estimates for times larger than the smoothing time $T_\diamond$, as stated in (5.9.1) and (5.9.2), are a consequence of Lemmata 5.30, 5.31 and 5.32. □

With the limit objects and local convergence properties prepared by the previous lemma, we can finally draw on the uniform exponential decay for large times established in Section 5.8 to extend the local convergence to convergence beyond compact subsets of $\overline{\Omega}\times[0,\infty)$, as claimed in the main theorem.

Proof of Theorem 5.1: The existence, the regularity and the solution properties of the claimed functions were already established in Propositions 5.10 and 5.33. We are left with verifying the convergence with respect to the desired norms as in (5.1.5). According to Lemma 5.31 and (5.9.1), given any $p_1\in[1,\frac{5}{3})$ we can fix $\mu>0$ and $C_1>0$ such that for any $\kappa\in[-1,1]$

$$\big\|n^{(\kappa)}(\cdot,t)-\overline{n_0}\big\|_{L^{p_1}(\Omega)}^{p_1} + \|n(\cdot,t)-\overline{n_0}\|_{L^{p_1}(\Omega)}^{p_1} < C_1e^{-\mu t} \tag{5.9.3}$$

holds for all $t>T_\diamond$. Now, given $\delta>0$ we pick $T_\star \geq \max\big\{T_\diamond, \frac{1}{p_1\mu}\ln\big(\frac{2^{p_1+1}C_1}{p_1\mu\delta}\big)\big\}$ and obtain from (5.9.3) that

$$\begin{aligned}&\big\|n^{(\kappa)}-n\big\|_{L^{p_1}((T_\star,\infty),L^{p_1}(\Omega))}^{p_1}\\ &\leq 2^{p_1-1}\int_{T_\star}^\infty \big\|n^{(\kappa)}(\cdot,t)-\overline{n_0}\big\|_{L^{p_1}(\Omega)}^{p_1}\,\mathrm{d}t + 2^{p_1-1}\int_{T_\star}^\infty \big\|n(\cdot,t)-\overline{n_0}\big\|_{L^{p_1}(\Omega)}^{p_1}\,\mathrm{d}t\end{aligned}$$

$$\leq 2^{p_1} C_1 \int_{T_\star}^{\infty} e^{-p_1 \mu t} \, \mathrm{d}t = \frac{2^{p_1} C_1}{p_1 \mu} e^{-p_1 \mu T_\star} \leq \frac{\delta}{2} \tag{5.9.4}$$

for any $\kappa \in [-1,1]$. Next, given a null sequence $(\kappa_j)_{j\in\mathbb{N}} \subset [-1,1]$ and denoting by $(\kappa_{j_k})_{k\in\mathbb{N}}$ the subsequence from Proposition 5.33, we can conclude from the convergence statements in Proposition 5.33, the uniform bound in Lemma 5.8 and the Vitali theorem that due to $p_1 < \frac{5}{3}$ actually

$$n^{(\kappa_{j_k})} \to n \quad \text{in } L^{p_1}_{loc}([0,\infty), L^{p_1}(\Omega)) \quad \text{as } \kappa_{j_k} \to 0.$$

Hence, for the given $\delta > 0$ there is some $k_0 \in \mathbb{N}$ such that

$$\left\| n^{(\kappa_{j_k})} - n \right\|^{p_1}_{L^{p_1}([0,T_\star];L^{p_1}(\Omega))} \leq \frac{\delta}{2} \quad \text{is valid for all } k \geq k_0. \tag{5.9.5}$$

Combination of (5.9.4) and (5.9.5) shows that for all $k \geq k_0$ we have

$$\begin{aligned} &\left\| n^{(\kappa_{j_k})} - n \right\|^{p_1}_{L^{p_1}([0,\infty);L^{p_1}(\Omega))} \\ &= \left\| n^{(\kappa_{j_k})} - n \right\|^{p_1}_{L^{p_1}([0,T_\star];L^{p_1}(\Omega))} + \left\| n^{(\kappa_{j_k})} - n \right\|^{p_1}_{L^{p_1}((T_\star,\infty),L^{p_1}(\Omega))} \leq \delta, \end{aligned}$$

from which we conclude the first part of (5.1.5). Similarly, drawing on Lemma 5.31 and (5.9.2), for given $p_2 \in [1, \frac{5}{4})$ we can fix $\mu' > 0$ and $C_2 > 0$ such that for any $\kappa \in [-1,1]$

$$\left\| n^{(\kappa)}(\cdot,t) - \overline{n_0} \right\|^{p_2}_{W^{1,p_2}(\Omega)} + \left\| n(\cdot,t) - \overline{n_0} \right\|^{p_2}_{W^{1,p_2}(\Omega)} \leq C_2 e^{-\mu' t}$$

holds for all $t > T_\diamond$, from which we once again conclude that for the given $\delta > 0$ we can pick $T'_\star \geq \max\{T_\diamond, \frac{1}{p_2\mu'} \ln(\frac{2^{p_2+1}C_2}{p_2\mu'\delta})\}$ such that

$$\left\| \nabla n^{(\kappa)} - \nabla n \right\|^{p_2}_{L^{p_2}((T'_\star,\infty);L^{p_2}(\Omega))} \leq \frac{\delta}{2} \quad \text{for any } \kappa \in [-1,1]. \tag{5.9.6}$$

Since we know from Lemma 5.33 that

$$\nabla n^{(\kappa_{j_k})} \rightharpoonup \nabla n \quad \text{in } L^{p_2}_{loc}([0,\infty); L^{p_2}(\Omega)) \quad \text{as } \kappa_{j_k} \to 0,$$

we can make use of the fact that $p_2 < \frac{5}{4}$ to employ Vitali's theorem in combination with the uniform bounds presented in Lemma 5.8 to find that actually

$$\nabla n^{(\kappa_{j_k})} \to \nabla n \quad \text{in } L^{p_2}_{loc}([0,\infty); L^{p_2}(\Omega)) \quad \text{as } \kappa_{j_k} \to 0.$$

From this we conclude that there is some $k'_0 \in \mathbb{N}$ such that

$$\left\| \nabla n^{(\kappa_{j_k})} - \nabla n \right\|^{p_2}_{L^{p_2}([0,T'_\star];L^{p_2}(\Omega))} \leq \frac{\delta}{2} \quad \text{holds for all } k \geq k'_0, \tag{5.9.7}$$

so that a combination of (5.9.6) and (5.9.7) again entails the convergence in the desired topology. Analogous arguments drawing on Lemma 5.30, Lemma 5.32, (5.9.1), (5.9.2) and the uniform bounds in the Lemmata 5.6 and 5.8 finally entail the remaining properties listed in (5.1.5), completing the proof. $\square$

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