DISSERTATION

Essays on Cooperation in Differential Games

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Preface

This thesis is the result of an academic tour that started about 10 years ago in Hamburg. During my bachelor studies I was influenced by Thorsten Pampel who introduced me to the topic of dynamic optimization. I then did my master in Münster and participated in a seminar on bargaining theory held by Andrea Schneider. In the seminar I discussed a dynamic bargaining game which kicked off my everlasting interest in differential games. I then decided to pursue a doctorate and was lucky enough to find a position in Claus-Jochen's micro group at Paderborn University.

My immature plan was to somehow link classic cooperative games with differential games. Throughout I took several wrong turns and faced many dead ends. I would thus like to thank foremost Claus-Jochen for not giving up on my ideas and encouraging me on digging deeper. His thorough understanding of bargaining theory also gave me a hard time sometimes when thinking about challenging questions. Partial answers to these questions can be found in Chapters 3 and 4.

Next, I would like to thank Herbert Dawid for acting as the second reviewer and for letting me participate in the doctoral program of the Bielefeld Graduate School of Economics and Management. As the differential game community is

rather small, I am lucky that Bielefeld is close by and that I thus had the chance to meet him.

I am further thanking Burkhard and Wendeling for the completion of my doctoral panel.

I am also indebted to Georges Zaccour for hosting me at GERAD in Montréal. During my visit, I essentially wrote Chapter 2. I also benefited directly from his work, because Chapter 5 generalizes a paper he wrote back in 2003.

Furthermore, I am very grateful for the financial support I received from both the SFB 901 and the Faculty of Business Administration and Economics. They enabled me to present my work at various international conferences and hence get into touch with people I would otherwise never have met personally.

On a personal note, I would like to thank my colleagues from the Chair of Microeconomics. From Monday to Thursday the uni is basically my second home and I always enjoy going to office.

Also, all of this would not have been possible without the unconditional support of my family. I thank them for everything they have done for me and that they have encouraged and enabled me to go my own way.

Finally, I thank my beloved better half who was, is and hopefully will always be there for me.

Simon Hoof
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Chapter 1

Synopsis

1.1 Introduction

The whole is more than the sum of its parts.[1] This sentence describes the underlying synergy effects of teamwork in the sense that **T**ogether, **E**veryone **A**chieves **M**ore. This thesis is concerned with the question of how to make sure that a team does not fall apart over time. To illustrate the problem one may considers the following scenario: There is a group of people that is supposed to work on some joint project. Let us assume that all group members unanimously agree on an execution plan for the project such that the grand group forms. Even though it is in the best interest of the entire group that everybody sticks to the cooperative agreement over the entire planning horizon, there may be incentives for some individuals to defect on the agreement and act differently to what they agreed upon beforehand. The purpose of the thesis is to answer

[1] The sentence is a misquote of Aristotle. It rather says "[...] the totality is not, as it were, a mere heap, but the whole is something besides the parts, [...]" (Ross, 1908, Book VIII, Part 6).

the following question:

> **Q**: How can we sustain the cooperative agreement over time?

An, admittedly, almost trivial answer to the question is that for each group member sticking to the agreement dominates a given outside option. That is, every individual prefers to be part of the grand group to deviating from the grand group. This answer, however, gives rise to two follow-up questions in case an individual actually considers deviating:

> **Q**: How do the other individuals react if I deviate from the agreement?
>
> **Q**: What outcome do I receive if I deviate from the agreement?

The answers to these questions are up for debate. It might be the case that the other group members are very disappointed with someone who defects on the cooperative agreement and they thus try to retaliate, or they do not bother too much and still stick to the initial agreement.

To answer the questions I rely on game theory and introduce theoretical models of dynamic group behavior. Generally, game theory is a context-free mathematical toolbox that tries to predict the behavior of individuals in an interactive decision making environment. By using game theory as a modeling device one makes sure that the modeler is transparent on the underlying assumptions and that conclusions are derived from formal arguments. Each noncooperative game consists of at least three ingredients:

1. A set of agents.

2. For each agent a set of feasible actions.

3. For each agent a real valued function that takes each action profile – vector of all actions – to a payoff.

The first comprehensive study of games dates back to the seminal book *Theory of Games and Economic Behavior*, in which von Neumann and Morgenstern (1944) laid down the foundation of modern game theory. In this thesis I am concerned with dynamic games. That is, the agents play a game over a prescribed time interval and they thus need to take into account that current actions influence future payoffs. Here, time is continuous and the games under consideration belong to the class of *differential games* introduced by Isaacs (1965). In a differential game the payoff of an agent does not only depend on the actions of all agents, but also on some state variable. Further, the players control the evolution of the state over time via their actions, and the evolution is described by a differential equation (called: state equation). In contrast to static games, differential games are able to describe dynamic adjustment processes like the continuous evolution of prices or resource stocks. The number of real-world phenomena that can be formalizes by differential games is basically indefinite.[2]

For example, when two kids play tag in the yard and one of them is chasing the other, then this situation can be formally described by a two player zero sum differential game. The state of each kid is the position in the yard and the action is where to run. The goal of the chasing kid is to minimize the distance to the fleeing kid, while the fleeing kid tries to maximize the distance. This game belongs to the class of zero sum games, because the preferences are antagonistic. Another simple example is the so-called cake eating game when two or more

[2] A comprehensive differential game textbook with a focus on economic and management applications is Dockner et al. (2000).

players share a cake. The state variable is the size of the cake and the action variable is consumption. Each agent derives payoffs from eating the cake, but since current consumption decreases future consumption possibilities – the cake shrinks – we encounter an intertemporal decision making problem of selecting an optimal consumption plan.

We can also think of more complex situations such as environmental agreements. Environmental agreements usually call for the reduction of emissions to protect the environment. These agreements, however, are fragile in the sense that a nation may unilaterally deviates if it does not fear sanctions. If we cannot enforce the agreement, one may asks an economic theorist to design a game in which it is in the best interest of each nation to participate in the agreement and to stick to it over time.[3]

Another application from the area of industrial organization is provided by the scenario of our collaborative research center Sonderforschungsbereich (SFB) 901 – On-The-Fly-Computing. Within the scope of the SFB 901 we are concerned with designing and analyzing a market for software services. This market consists of a final good consumer who demands a composed service, a composed service producer and single service providers. An example of a composed service would be a picture classifier that checks whether the animal on a picture is a cat or not. The composed service, however, is a combination of different single services such as the classifier (a program) and training data (pictures of cats and dogs). We are thus concerned with three interdependent stages and on each stage the agents are faced with decisions of how much to buy, what prices to set or what quality to provide. Yet, we have analyzed the market from a non-

[3] This application will be considered in Chapter 5.

cooperative perspective such that each agent acts on her own. In Heinzel and Hoof (2020) we study price and quantity competition under the assumption that the single services can range from perfect complements to perfect substitutes. We find that the single service providers prefer price over quantity competition when the inputs are complements and vice versa when they are substitutes. The composed service producer and the final good consumer prefer price over quantity competition for all degrees of input differentiation. Building on the aforementioned paper I dynamize the model by allowing sticky prices and show that, depending on the degree of product differentiation, prices and quantities may periodically cycle around the steady state (Hoof, 2020). In the current phase of the project we want to study the endogenous emergence of vertical and/or horizontal mergers within the SFB scenario. The theoretical foundations for studying these coalition formation problems are partially laid down in this thesis.

If the agents were about to act fully noncooperatively and each agent thus maximizes her own payoffs, then the predominant solution concept is the Nash equilibrium (Nash, 1950*a*, 1951). A Nash equilibrium is an action profile such that no agent has an incentive to unilaterally change her action.[4] In its original form the Nash equilibrium is a solution concept for static one shot games. In dynamic games, however, one has to adjust this definition with respect to time. An action profile is now a time-dependent function and it is an equilibrium if no agent has an incentive to deviate from it at each time instant (Starr and Ho, 1969*a*,*b*).

[4]Actually, the idea can already be traced back to Cournot (1838) who considers a duopoly with quantity competition and who solves the game for the noncooperative equilibrium.

In this thesis the Nash equilibrium serves two purposes. In Chapters 2, 3 and 4 the noncooperative equilibrium payoffs are a benchmark a cooperative agreement is compared to and thus defines the given outside option. In Chapters 5 and 6, I consider games in which coalitions of agents play against each other. For a given coalition structure the noncooperative equilibrium payoffs define the worth of each coalition. By using this modeling device one can map a game in normal form to a cooperative game in partition function form and then study the endogenous formation of coalitions.

1.2 A primer on stationary differential games

In this section I briefly introduce the notation and the crucial ingredients of a differential game, because they will be recurring throughout the entire thesis. There is a group of n agents denoted by $N = \{1, 2, \ldots, n\}$. The game is played on some state space $X \subseteq \mathbb{R}$ and each agent $i \in N$ continuously executes an action so as to control the state from one position to another. The set of admissible actions for each agent $i \in N$ is denoted by A_i. In this thesis I solely deal with stationary differential games such that all functions are time invariant and when taking an action the agents are not concerned about the current time, but only about the current state. That is, each agent observes the current state $x \in X$ and then a feedback strategy $\sigma_i : X \to A_i$ fixes the action $a_i = \sigma_i(x)$. The set of admissible feedback strategies is denoted by Σ_i and the set of jointly admissible strategies by $\Sigma \subseteq \bigtimes_{i \in N} \Sigma_i$. To each strategy profile $\boldsymbol{\sigma} = (\sigma_i)_{i \in N} \in \Sigma$ the payoff function $u_i : X \times \Sigma \to \mathbb{R}$ assigns a value conditioned on the current state x. A stationary differential game in normal form $\Gamma(x)$ is then described by the triplet

$\langle N, (\Sigma_i)_{i \in N}, (u_i(x, \cdot))_{i \in N} \rangle$. The definition of a Nash equilibrium is straightforward. A strategy profile $\overline{\boldsymbol{\sigma}} \in \Sigma$ is a subgame perfect Nash equilibrium[5] of the game $\Gamma(x)$ if for all states $x \in X$ and for all agents $i \in N$ the following inequalities hold:

$$u_i(x, \overline{\boldsymbol{\sigma}}) \geq u_i(x, \sigma_i, \overline{\boldsymbol{\sigma}}_{-i}) \quad \forall \sigma_i \in \Sigma_i$$

where $\overline{\boldsymbol{\sigma}}_{-i} = (\overline{\sigma}_j)_{j \in N \setminus \{i\}}$ denotes the equilibrium strategies of the other agents. In Chapters 2, 3 and 4 I assume that the equilibrium payoff $u_i(x, \overline{\boldsymbol{\sigma}})$ serves as a noncooperative outside option, and that the agents have the possibility to jointly coordinate on some cooperative strategy profile $\boldsymbol{\sigma}^* \in \Sigma$. They will agree on playing $\boldsymbol{\sigma}^*$ if for each agent $i \in N$ the cooperative payoff $u_i(x, \boldsymbol{\sigma}^*)$ weakly exceeds the noncooperative equilibrium payoff over the entire state space X. In Chapter 2 the cooperative strategies are determined via a multi-objective dynamic optimization problem and in Chapters 3 and 4 they are determined via a cooperative bargaining solution.

In Chapters 5 and 6 the agents can form coalitions $S \subseteq N$. Let π denote a partition (coalition structure) of N and Π the set of all partitions. Then, the partition function $V : X \times 2^N \times \Pi \to \mathbb{R}$ assigns to each coalition $S \in \pi$ a worth conditioned on the state x. I am going to show how one can send a normal form differential game $\Gamma(x)$ to a cooperative partition function form game $\langle N, V \rangle$. The partition function is constructed by considering a noncooperative game played by the coalitions $S \in \pi$. I assume that the agents play cooperatively within, but noncooperatively across coalitions. That is, a coalition maximizes the sum

[5] In the literature one also finds the terms state-feedback or Markov perfect equilibrium.

of payoffs and a strategy profile $\overline{\sigma} \in \bigtimes_{S\in\pi} \bigtimes_{i\in S} \Sigma_i$ is an equilibrium if for all states $x \in X$ and coalitions $S \in \pi$ the following inequalities hold:

$$\sum_{i\in S} u_i(x, \overline{\sigma}) \geq \sum_{i\in S} u_i(x, \sigma_S, \overline{\sigma}_{-S}) \quad \forall \sigma_S \in \bigtimes_{i\in S} \Sigma_i$$

where $\overline{\sigma}_{-S} = (\overline{\sigma}_C)_{C\in\pi\setminus\{S\}}$ denotes the equilibrium strategies of the other coalitions. For a given coalition structure π, I can now define a state-dependent partition function $V(x, S, \pi) = \sum_{i\in S} u_i(x, \overline{\sigma})$ by assigning the noncooperative equilibrium payoff to each coalition $S \in \pi$. I then discuss the endogenous formation of coalitions.

1.3 Contribution

In what follows, I briefly summarize the chapters and highlight gaps in the literature as well as my own contribution. The overall idea is to define payoffs under cooperation and then check whether a group of agents has an invective to deviate from the agreement. In order to define the payoffs under cooperation I use three different approaches.

Cooperative dynamic advertising via state-dependent payoff weights

In Chapter 2, I reconsider Sorger's (1989) advertising game. There are two firms selling the same good, but they are located at different locations. The state space is $X = [0, 1]$ with $x \in X$ being the share of costumers of the first firm and $1 - x$ of the second firm. Each firm tries to attract costumers via advertising, but one should note that the preferences of the firms are perfectly diametrical in the sense that an increase of costumers for one firm results in a decrease for the other. The firms want to coordinate their strategies such that each firm is better off compared to the noncooperative equilibrium outcome. In order to derive efficient strategies I assume that the firms jointly maximize their weighted sum of payoffs. The solution of this maximization program is a pair of cooperative strategies

$$\sigma^*_\lambda \in \arg\max_{\sigma \in \Sigma} \{\lambda u_1(x, \sigma) + (1 - \lambda) u_2(x, \sigma)\}$$

that is parametrized in the weight $\lambda \in [0, 1]$. The cooperative payoff of each firm is then simply the payoff given that both firms play the cooperative strategies. I then check whether the cooperative payoff dominates the noncooperative equilibrium payoff $u_i(x, \sigma^*_\lambda) \geq u_i(x, \overline{\sigma})$ for both firms $i \in \{1, 2\}$ and for all feasible states $x \in X$. If this individually rationality criterion was not about to hold, the cooperative agreement is not installed in the first place. I show that no constant weight supports a payoff dominant cooperative solution. To bypass this obstacle I introduce a state-dependent weight in the spirit of Yeung and Petrosyan (2015).

<u>Contribution:</u> To the best of my knowledge this is the first paper that uses state-dependent payoff weights for infinite time horizon differential games. I consider an affine weighing function $\lambda(x) = 1 - x$ that adjusts the weight in the joint maximization problem with respect to the current state. I am then able to show that a state-dependent weight supports a payoff dominant cooperative solution over the entire state space.

<u>Outlook:</u> A state-dependent weighing function is generally useful if constant

weights fail to support a payoff dominant cooperative solution. Clearly, the functional form of $\lambda(x)$ depends on the application and cannot be stated generically. One could try, however, to approach the problem of identifying the functional form of $\lambda(x)$ on a restricted domain of games.

On a class of linear-state differential games with subgame individually rational and time consistent bargaining solutions

In Chapter 3, I consider a generic class of n-person linear-state games and the cooperative strategies are determined via bargaining solutions. A bargaining solution determines an element of a prescribed set of feasible alternatives. If the agents cannot reach an unanimous agreement, they receive a given disagreement outcome. Nash (1950*b*) laid the foundation of axiomatic bargaining theory in order to characterize a solution uniquely. Liu (1973) then extended the Nash bargaining solution to differential games in the sense that the agents bargain over strategies. The Nash bargaining solution, for instance, is then defined as

$$\arg\max_{\sigma \in \Sigma} \prod_{i \in N} [u_i(x, \sigma) - u_i(x, \overline{\sigma})].$$

In contrast to standard cooperative bargaining games, a solution is defined in the strategy space and not in the payoff space. I am concerned with two issues that were raised by Haurie (1976). An initial individual rational bargaining solution must not necessarily remain individually rational throughout the game. And an initial solution may not be robust to renegotiations at a later time instant. The first property is called subgame individual rationality (SIR) and the second one time consistency (TC).

Contribution: To the best of my knowledge no one has studied bargaining solutions of stationary differential games. It seemed therefore worthwhile to further investigate these kind of games. As it turns out, the SIR and TC property can be used to characterize families of bargaining solutions which satisfy SIR and TC. In Chapter 3, I consider a class of linear-state games which are analytical tractable in sense that one can obtain closed form solutions for the equilibrium strategies as well as value functions. I show that all bargaining solutions which are individually rational at the beginning of the game (Overall Individually Rational [OIR]) are also individually rational throughout the game (SIR) and robust to renegotiations over the entire state space (TC) if the cooperative strategies are restricted to constants. The restriction to constants is not arbitrary, because there also exists a subgame perfect equilibrium in constant strategies. I thus restrict the set of admissible cooperative strategies to functions that are equivalent to the noncooperative equilibrium strategies.

A pure bargaining game of dynamic cake eating

The frameworks of Chapters 3 and 4 are identical. In Chapter 4, however, I do not consider linear-state games, but a specific nonlinear cake eating game (Clemhout and Wan, 1989). The motivating question was to check whether the SIR and TC property can be used to single out bargaining solution that satisfy those properties.

Contribution: Again, I first solve for the noncooperative equilibrium and show that there exists an equilibrium in linear strategies. I then follow the same approach as in Chapter 3 by allowing cooperative strategies that are functionally equivalent to the equilibrium strategies. That is, the set of admissible cooperative strategies is restricted to linear functions. In contrast to Chapter 3, however, not all OIR solutions satisfy SIR and TC. In fact, only those OIR solutions that maximize a *linear homogenous* function also satisfy SIR and TC. As a byproduct, I thus identified an inverse relationship between the complexity of the differential game and bargaining solutions that satisfy SIR and TC. The more complex the model, the less solutions satisfy SIR and TC.

Outlook Chapters 3 and 4: Cooperative bargaining theory characterizes solutions by a number of axioms. I am positive that the SIR and TC property can be used as axioms in order to discriminate between different bargaining solutions that satisfy those properties. It would be worthwhile to check whether there exists a class of games such that those properties single out, for instance, the Nash bargaining solution.

Linear-state differential games in partition function form

In Chapter 5, I introduce differential games in partition function form and provide a method of how to compute a partition function. A partition function $V : X \times 2^N \times \Pi \to \mathbb{R}$ assigns to each partition of agents $\pi \in \Pi$ a characteristic function $v : X \times 2^N \to \mathbb{R}$.

Contribution: To the best of my knowledge this is the first paper that studies differential games in partition function form. I consider, again, linear-state differential games, but now the agents are allowed to form coalitions. For a given partition of agents the coalitions play a noncooperative game (Zhao, 1992). The worth of a coalition is then its noncooperative equilibrium payoff. I also consider dynamic core concepts in the sense that an allocation is in the core if no coalition of agents has an incentive to deviate from the grand coalition over the entire time horizon. If the cooperative game is strongly convex, i.e., convex over the entire state space, then the results from static game theory (Hafalir, 2007) directly carry over and the core with singleton expectations (left out players become singletons) as well as the core with cautions expectations (left out players harm deviating coalition) are nonempty.

Equilibrium coalition structures of differential games in partition function form

In Chapter 6, I consider the cake eating game in partition function form. Instead of relying on standard ad-hoc core concepts, I study the endogenous formation of coalitions by means of a noncooperative extensive form game (Bloch, 1996). The equilibrium of the game yields an equilibrium coalition structure (ECS). As our partition function is derived from a differential game, the ECS generally depends on the current size of the cake (state variable). Therefore, the ECS could be time inconsistent in the sense that it changes with respect to the state.
Contribution: I first explicitly compute the partition function. Then I show that the ECS is time consistent and the initial ECS is thus the ECS over the entire time horizon. The ECS is generally the solution of a finite dynamic programming problem with n stages. I provide a generic algorithm of how to compute the ECS and also solve for the ECS for up to 800,000,000 agents.
Outlook Chapters 5 and 6: Partition function form (PFF) games in general, and differential games in partition function form (DGPFF) in particular are rather unexplored. As I already mentioned Chapter 5 basically introduced DGPFF. As the approach allows to construct a PFF game from any normal form game, the range of applications is broad. Within the SFB 901 scenario I am currently

trying to solve a model of cartel formation in dynamic oligopolies with differentiated products. It is safe to assume that the results depend on the competition mode (price vs. quantity), degree of product differentiation (complements vs. substitutes) and the current price level. This line of research might help us to understand why some cartels are stable and others are not.

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