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Wave Front Sets of Nilpotent Lie Group Representations

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PhD Dissertation

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Abstract

Let G be a nilpotent, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, and π a unitary representation of G. The main goal of this doctoral thesis is to prove that the wave front set of π coincides with the asymptotic cone of the orbital support of π, i.e. $\mathrm{WF}(\pi) = \mathrm{AC}(\bigcup_{\sigma \in \mathrm{supp}(\pi)} \mathcal{O}_\sigma)$, where $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$ is the coadjoint orbit associated to the irreducible unitary representation $\sigma \in \hat{G}$ by Kirillov. We use two different approaches: by induction over the dimension of $\mathfrak{g}$ as customary for nilpotent Lie groups and via integrated characters following the work by Harris, He and Ólafsson for real reductive, algebraic Lie groups.
Lastly, we apply our result to restrictions of unitary representations to nilpotent subgroups to obtain asymptotic information about their support.

Zusammenfassung

Sei G eine nilpotente, zusammenhängende, einfach zusammenhängende Lie-Gruppe mit Lie-Algebra $\mathfrak{g}$ und π eine unitäre Darstellung von G. Das Hauptziel dieser Doktorarbeit ist es, zu beweisen, dass die Wellenfrontmenge von π gleich dem asymptotischen Kegel des orbitalen Trägers von π ist, d.h. $\mathrm{WF}(\pi) = \mathrm{AC}(\bigcup_{\sigma \in \mathrm{supp}(\pi)} \mathcal{O}_\sigma)$, wobei $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$ der von Kirillov zur irreduziblen unitären Darstellung $\sigma \in \hat{G}$ assoziierte koadjungierte Orbit ist. Wir präsentieren zwei Herangehensweisen: erstens per Induktion über die Dimension von $\mathfrak{g}$ wie üblich für nilpotente Lie-Gruppen und zweitens über integrierte Charaktere, wie sie von Harris, He und Ólafsson für reell reduktive, algebraische Lie-Gruppen verwendet wurden.
Abschließend wenden wir unser Resultat auf die Einschränkung von unitären Darstellungen auf nilpotente Untergruppen an, um asymptotische Informationen über ihren Träger zu erhalten.

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Contents

1 Introduction

Let G be a Lie group, π a unitary representation of G, and $\hat{G}$ the unitary dual of G, that is the set of all equivalence classes of irreducible unitary representations. If G is type I, π decomposes uniquely into a direct integral over $\hat{G}$:

$$\pi \cong \int_{\hat{G}} \sigma^{\oplus m_\pi(\sigma)}\, d\mu_\pi(\sigma), \quad m_\pi(\sigma) \in \mathbb{N} \cup \{\infty\},$$

where $m_\pi(\sigma)$ is the multiplicity of σ and μ_π is a Borel measure on $\hat{G}$.
In the field of representation theory it is a major goal is to determine this decomposition of a unitary representation into irreducibles explicitly. That is why we are interested in the support

$$\operatorname{supp}(\pi) \coloneqq \operatorname{supp}(\mu_\pi) \subset \hat{G}.$$

Kirillov's idea was that $\hat{G}$ should be described by the space of coadjoint orbits $\mathfrak{g}^*/G$ and he proved that they are in fact isomorphic for nilpotent, connected, simply connected Lie groups (see Theorem 3.2). The irreducible unitary representation associated to a coadjoint orbit can be constructed explicitly as an induced representation from a one-dimensional representation. Also Kirillov's character formula connects the Fourier transforms of coadjoint orbits to the infinitesimal characters of the irreducible representations. Furthermore, one can show that each coadjoint orbit $\mathcal{O} \in \mathfrak{g}^*/G$ is a symplectic manifold and geometric quantization yields a Hilbert space with a corresponding operator. We give an overview over the structure theory of nilpotent Lie groups and Lie algebras in Section 3. The important definitions and concepts are illustrated in the examples of the Heisenberg group and the group K_3 in Sections 3.2 and 3.3, respectively.

Turning to a different concept for a moment: Studying partial differential equations Hörmander and Sato introduced the notion of wave front sets in order to obtain information about the regularity of the solutions. For a distribution u on a smooth manifold X the wave front set $\mathrm{WF}(u)$ is a closed subset of the cotangent bundle T^*X which measures its smoothness (see Section 2.1).
As outlined in Section 2.2 the wave front set of a unitary representation $(\pi, \mathcal{H})$ of a Lie group G with Lie algebra $\mathfrak{g}$ is defined as the closure of the unions of the wave front sets at the identity of the matrix coefficients of π:

$$\mathrm{WF}(\pi) = \overline{\bigcup_{v,w\in\mathcal{H}} \mathrm{WF}_e\left(\langle \pi(g)v, w\rangle_{\mathcal{H}}\right)} \subset iT_e^*G \cong i\mathfrak{g}^*.$$

This notion was first introduced in [How81].

Connecting these two concepts and using Kirillov's character formula [How81, Proposition 2.2] shows for the asymptotic support of a single irreducible unitary representation σ of the nilpotent Lie group G that

$$\mathrm{WF}(\sigma) = \mathrm{AC}\left(\mathcal{O}_\sigma\right), \tag{1.1}$$

where the asymptotic cone $\mathrm{AC}(S)$ for a subset S of a finite-dimensional vector space V is the closed cone defined by

$$\mathrm{AC}(S) = \{v \in V \mid \forall \text{ open cone } \mathcal{C} \ni v : \mathcal{C} \cap S \text{ unbounded}\} \cup \{0\}.$$

In order to formulate results analogous to (1.1) for non-irreducible representations we introduce the notion of the orbital support

$$\mathcal{O}-\mathrm{supp}(\pi) := \bigcup_{\sigma \in \mathrm{supp}\,\pi} \mathcal{O}_\sigma.$$

Thus, the resulting statements can be helpful in finding the decomposition of a unitary representation.
In fact, Kashiwara-Vergne [KV79] and Howe [How81] proved for compact, connected Lie groups

$$\mathrm{WF}(\pi) = \mathrm{AC}\left(\mathcal{O}-\mathrm{supp}(\pi)\right),$$

using that one can also associate a coadjoint orbit to each irreducible unitary representation for compact Lie groups.

More recently, [HHÓ16] gave the same result for real reductive, algebraic Lie groups under the assumption that π is weakly contained in the regular representation. In this case a finite union of coadjoint orbits can by associated to each irreducible tempered representation due to Duflo and Rossmann (see [Duf70], [Ros78] and [Ros80]).
For a more detailed presentation of the known results see Section 2.3.

Our main goal is to prove this statement for nilpotent Lie groups:

Theorem. *Let G be a nilpotent, connected, simply connected Lie group and π a unitary representation of G. Then*

$$\mathrm{WF}(\pi) = \mathrm{AC}(\mathcal{O}-\mathrm{supp}(\pi)).$$

Our proof in Section 4 uses explicit microlocal estimates of individual matrix coefficients. These microlocal estimates are obtained via induction over the dimension of G which is a common method for nilpotent Lie groups based on the extensive knowledge of the restriction of irreducible unitary representations to subgroups of co-dimension 1 and the corresponding orbits outlined in Section 3 (see Theorem 3.18 and Procedure 3.19). In the inductive step we make a case distinction based on the dimension of the center of the Lie algebra.

In Section 5 we give a second proof based on the approach of [HHÓ16, Theorem 1.2] for real reductive, algebraic Lie groups using the wave front set of integrated characters as an intermediate step. Since the tools given for real reductive groups do not work for nilpotent groups we had to find another description of the wave front set of a unitary representation which we prove in Proposition 2.19 for general Lie groups.

Unfortunately, in order to use integrated characters in the proof of the inclusion $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O}-\mathrm{supp}(\pi))$ we have to make assumptions on the unitary representation. We discuss these difficulties in the setting of nilpotent Lie groups at the end of Section 5.1.2 (see Remarks 5.9, 5.10 and 5.11). It was these difficulties that led to the approach of working with individual matrix coefficients instead of characters which was carried out in Section 4.
Subsequently, we try to finalize this proof for a general unitary representation in Section 5.2 by using our knowledge of the integral kernels of the integrated representations as studied in Section 3 (see Proposition 3.26). We have to make assumptions regarding the structure of the Campbell-Baker-Hausdorff formula in order to be able to compute and estimate the required integrals. However, we show that these are fulfilled in our examples of the Heisenberg group and K_3.

Lastly, recall that our result can provide asymptotic information about the decomposition of unitary representations into irreducibles since it is a statement about its (orbital) support. We apply it in Section 6 to restriction problems which are a fundamental problem in representation theory. Consider a Lie group G and a closed subgroup $H \subset G$ and let $\mathfrak{h}$, $\mathfrak{g}$ be the respective Lie algebras. Let $q : i\mathfrak{g}^* \to i\mathfrak{h}^*$ be the natural projection and π be a unitary representation of G. On the level of the wave front sets we know by [How81, Proposition 1.5] that $q(\mathrm{WF}(\pi)) \subset \mathrm{WF}(\pi|_H)$. Hence, we can expect representation theoretical results like

$$q(\mathrm{AC}(\mathcal{O}-\mathrm{supp}\,\pi)) \subset \mathrm{AC}\left(\mathcal{O}-\mathrm{supp}\left(\pi|_H\right)\right),$$

if one has statements like our main theorem for both G and H. Assuming we know the support of π (e.g. when it is irreducible) this tells us in which directions there have to be infinitely many points in the orbital support of $\pi|_H$, that is which irreducible unitary representations of H have to occur in the decomposition of $\pi|_H$.

2 Wave Front Sets

The aim of this section is to introduce the notion of wave front sets and to recall central results that will be used in the subsequent sections.
We first introduce the notion of the wave front set of a distribution on a manifold. After that we turn to the wave front set of a unitary representation of a Lie group.

2.1 Wave Front Sets of Distributions on Manifolds

Let W be a real, finite-dimensional vector space and fix a Lebesgue measure dx on W. We define the Fourier transform of a Schwartz function $\varphi \in \mathcal{S}(W)$ to be $\mathcal{F}(\varphi) \in \mathcal{S}(iW^*)$ with

$$\mathcal{F}(\varphi)(\zeta) := \int_W \varphi(x) e^{-2\pi\langle \xi, x\rangle}\, dx, \quad \xi \in iW^*,$$

and for a tempered distribution $u \in \mathcal{S}'(W)$ as $\mathcal{F}(u) \in \mathcal{S}'(iW^*)$ with

$$\mathcal{F}(u)(\psi) := u(\mathcal{F}(\psi)), \quad \psi \in \mathcal{S}(iW^*),$$

since $\mathcal{F}(\psi) \in \mathcal{S}(i(iW^*)^*) = \mathcal{S}(W)$.
The inversion formula for $\mathcal{F} : \mathcal{S}(W) \to \mathcal{S}(iW^*)$ gives us

$$\mathcal{F}^{-1} : \mathcal{S}(iW^*) \to \mathcal{S}(W), \quad \psi \mapsto \left(x \mapsto \int_{iW^*} \psi(\xi) e^{2\pi\langle \xi, x\rangle}\, d\xi \right)$$

for a suitable measure $d\xi$ on iW^*.

Furthermore we define the Fourier transform of a distribution $v \in \mathcal{E}'(W)$ with compact support to be

$$\mathcal{F}(v)(\xi) := v\left[e^{-2\pi\langle \xi, \bullet\rangle} \right], \quad \xi \in iW^*.$$

Definition 2.1. Let W be a real, finite-dimensional vector space and $u \in \mathcal{D}'(X)$ a distribution on an open subset $X \subset W$. Then we say $(x_0, \xi_0) \in X \times iW^* \setminus \{0\} \subset iT^*X$ is *not* in the *wave front set* $\mathrm{WF}(u) \subset iT^*X$ if there exist open neighborhoods U of x_0 and V of ξ_0 and a smooth compactly supported function $\phi \in C_c^\infty(U)$ with $\phi(x_0) \neq 0$ such that for all $N \in \mathbb{N}$ one of the following equivalent conditions hold:

(i) $\mathcal{F}(\phi u)(\tau\xi) = \mathcal{O}(\tau^{-N})$ for $\tau \to \infty$, uniformly in $\xi \in V$, i.e. the Fourier transform is rapidly decaying in V.

(ii) $\exists\, C_{N,\phi} > 0 : |\mathcal{F}(\phi u)(\tau\xi)| \leq C_{N,\phi} |\tau|^{-N} \quad \forall\, \tau \gg 0, \xi \in V$

(iii) $\exists\, C_{N,\phi} > 0 : |\mathcal{F}(\phi u)(\tau\xi)| \leq C_{N,\phi} \langle\tau\rangle^{-N} \quad \forall\, \tau \gg 0, \xi \in V$

where $\langle y\rangle = \sqrt{1 + y^2}$.

Note that $(x,0) \in iT^*X$ is never in the wave front set (contrary to Definition 2.11 for unitary representations) because in order to analyze the singularities of a function or distribution it only makes sense to look in the directions $\xi \neq 0$.

This connection of the wave front set to the singularities of distributions is specified with the following

Definition 2.2 (Singular support, see [Hör03, Definition 2.2.3])**.** For $u \in \mathcal{D}'(X)$, $X \subset W$ open, the *singular support*, denoted $\operatorname{singsupp}(u)$, is the set of points in X having no open neighborhood to which the restriction of u is a C^∞ function.

Then we have with the projection $pr_x : X \times iW^* \cong iT^*X \to X, (x,\xi) \mapsto x$

$$pr_x(\mathrm{WF}(u)) = \operatorname{singsupp}(u)$$

(see [Hör03, Definition 8.1.2]).

Lemma 2.3. *The wave front set* $\mathrm{WF}(u) \subset iT^*X$ *is a closed cone in the sense that*

$$\forall\, \tau > 0 : \; (x_0,\xi_0) \in \mathrm{WF}(u) \;\Rightarrow\; (x_0, \tau\xi_0) \in \mathrm{WF}(u).$$

Let us now consider two explicit examples:

Example 2.4. Let $\delta_0 \in \mathcal{D}'(\mathbb{R}^n)$ be the delta distribution (in 0), i.e. $\delta_0(\phi) = \phi(0)$, $\phi \in C_c^\infty(\mathbb{R}^n)$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\phi(0) = c \neq 0$. Then

$$\mathcal{F}(\phi\delta_0)(\tau\xi) = \delta_0(\phi e^{-2\pi\tau\langle x,\xi\rangle}) = \phi(0) = c \notin \mathcal{O}(|\tau|^{-N}) \quad \forall N \in \mathbb{N}.$$

This implies $\{(0,\xi) \in \mathbb{R}^n \times i\mathbb{R}^n \mid \xi \neq 0\} \subset \mathrm{WF}(\delta_0)$.
If now $x \neq 0$ then there exits $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi(x) \neq 0$ but $\varphi(0) = 0$. Hence analogously $\mathcal{F}(\varphi\delta_0)(\tau\xi) = 0 \in \mathcal{O}(|\tau|^{-N})$ for all $N \in \mathbb{N} \Rightarrow (x,\xi) \notin \mathrm{WF}(\delta_0)$. Summarizing we have

$$\mathrm{WF}(\delta_0) = \{(0,\xi) \in \mathbb{R}^n \times i\mathbb{R}^n \mid \xi \neq 0\}, \quad \operatorname{singsupp}(\delta_0) = \mathrm{pr}_x(\mathrm{WF}(\delta_0)) = \{0\}.$$

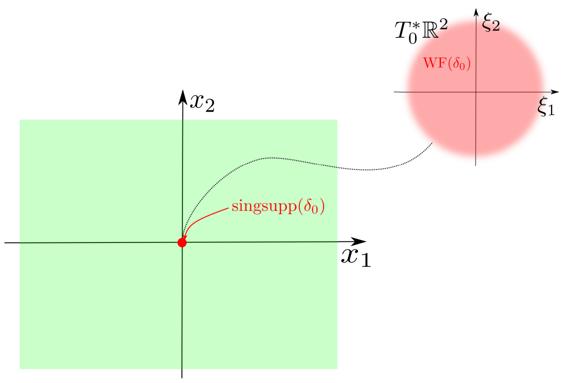


Figure 1: Wave front set of the delta distribution as in Example 2.4

Example 2.5. Consider the Heaviside-function

$$g(x_1, x_2) = \begin{cases} 1 & x_2 > 0 \\ 0 & x_2 \le 0 \end{cases}, \quad (x_1, x_2) \in \mathbb{R}^2$$

and denote by u_g the corresponding distribution.
Let $x \in \mathbb{R}^2$, $\xi \in i\mathbb{R}^2$ with $x_2 \neq 0$ and $\phi \in C_c^\infty(\mathbb{R}^2)$ with $\phi(x) \neq 0$ and $\text{supp}(\phi) \subset B_\epsilon(x)$ for some $\epsilon < \frac{x_2}{2}$. Then $\phi u_g = \phi$ or 0 and therefore $\phi u_g \in C_c^\infty(\mathbb{R}^2) \Rightarrow (x, \xi) \notin \text{WF}(u_g)$.

If now $x_2 = 0$ and $\xi_1 \neq 0$

$$\begin{aligned} \mathcal{F}(\phi u_g)(\tau\xi) &= \int_{\mathbb{R}^2} \phi(y) g(y) e^{-2\pi\tau\langle y, \xi\rangle}\, dy \\ &= \int_0^\infty \int_{-\infty}^\infty \phi(y_1, y_2) \left(\frac{\partial_{y_1}}{-2\pi\tau\xi_1}\right)^N e^{-2\pi\tau\langle y, \xi\rangle}\, dy_1\, dy_2 \\ &= (2\pi\tau\xi_1)^{-N} \int_0^\infty \int_{-\infty}^\infty \left[(\partial_{y_1})^N \phi(y_1, y_2)\right] e^{-2\pi\tau\langle y, \xi\rangle}\, dy_1\, dy_2. \end{aligned}$$

The last equality follows by integration by parts. Since $(\partial_{y_1})^N \phi \in C_c(\mathbb{R}^2)$ this now implies

$$|\mathcal{F}(\phi u_g)(\tau\xi)| \le C_N |\tau|^{-N} \Rightarrow (x, \xi) \notin \text{WF}(u_g).$$

At last, if $x_2 = 0, \xi_1 = 0$, and thus $\xi_2 \neq 0$, let $\phi \in C_c^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}} \phi(y_1, 0)\, dy_1 \neq 0$. Integrating by parts twice we obtain

$$\begin{aligned} \mathcal{F}(\phi u_g)(\tau\xi) &= \int_{\mathbb{R}^2} \phi(y) u_g(y) e^{-2\pi\tau\langle y, \xi\rangle}\, dy \\ &= -(2\pi\tau\xi_2)^{-1} \int_{-\infty}^\infty \phi(y_1, 0) e^{-2\pi\tau(y_1 \cdot 0 + 0 \cdot \xi_2)}\, dy_1 \\ &\quad + (2\pi\tau\xi_2)^{-2} \int_{-\infty}^\infty \partial_{y_2}\phi(y_1, 0) e^{-2\pi\tau(y_1 \cdot 0 + 0 \cdot \xi_2)}\, dy_1 \\ &\quad + (2\pi\tau\xi_2)^{-2} \int_0^\infty \int_{-\infty}^\infty \left[\partial_{y_2}^2 \phi(y_1, y_2)\right] e^{-2\pi\tau\langle y, \xi\rangle}\, dy_1\, dy_2 \\ &= -(2\pi\tau\xi_2)^{-1} \int_{-\infty}^\infty \phi(y_1, 0)\, dy_1 + (2\pi\tau\xi_2)^{-2} \int_{-\infty}^\infty \partial_{y_2}\phi(y_1, 0)\, dy_1 \\ &\quad + (2\pi\tau\xi_2)^{-2} \int_0^\infty \int_{-\infty}^\infty \left[\partial_{y_2}^2 \phi(y_1, y_2)\right] e^{-2\pi\tau\langle y, \xi\rangle}\, dy_1\, dy_2. \end{aligned}$$

Since the first summand has order 1 in τ and the other at least order 2, $\mathcal{F}(\phi u_g)$ cannot be rapidly decaying in τ. Collectively,

$$\text{WF}(u_g) = \{(x, \xi) \in \mathbb{R}^2 \times i\mathbb{R}^2 \mid x_2 = 0, \xi_1 = 0\}, \quad \text{singsupp}(u_g) = \mathbb{R} \times \{0\}$$

(see Figure 2).

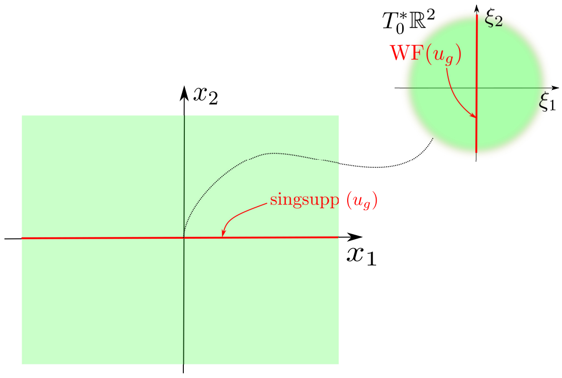


Figure 2: Wave front set of the Heaviside-distribution as in Example 2.5

In order to expand the notion of wave front sets to distributions on manifolds it is helpful to have the following generalizing proposition.

Proposition 2.6 (see [Dui96, Proposition 1.3.2])**.** *$(x_0, \xi_0) \notin \mathrm{WF}(u)$ if and only if for every function $\psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^p)$ with $d_x\psi(x_0, a_0) = \xi_0$ for $x_0 \in \mathbb{R}^n$, $a_0 \in \mathbb{R}^p$, there exist open neighborhoods U of x_0 and A of a_0 such that for every $\phi \in C_c^\infty(U)$ and $N \in \mathbb{N}$*

$$u\left(e^{-\tau\psi(\bullet, a)}\phi\right) = \mathcal{O}(\tau^{-N}) \text{ for } \tau \to \infty, \text{ uniformly in } a \in A.$$

Now, if $\psi : X \to Y$ is a diffeomorphism between two open sets and u is a distribution on Y, then

$$\psi^* \, \mathrm{WF}(u) = \mathrm{WF}(\psi^* u), \tag{2.2}$$

where the pullback on the cotangent bundle is defined by

$$\psi^*(y, \xi) = \left(\psi^{-1}(y), (D\psi(\psi^{-1}(y)))^T \xi\right), \quad (y, \xi) \in iT^*Y.$$

Thus the notion of the wave front set of a distribution on a smooth manifold is independent of the choice of local coordinates and is therefore well-defined.

Another characterization of wave front sets is given by [Fol89, Chapter 3, Section 2] in terms of the wave packet transform of a distribution:
Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function, $u \in \mathcal{S}'(\mathbb{R}^n)$ a tempered distribution and define

$$\phi^t(x) := t^{n/4}\phi(t^{1/2}), \ t > 0, \quad \text{and} \quad P_\phi^t u(\xi, x) := u\left(e^{2\pi i \xi \cdot \bullet - \pi i \xi \cdot x}\phi^t(\bullet - x)\right), \ \xi, x \in \mathbb{R}^n.$$

Then

$$P_\phi^t u(\xi, x) = e^{\pi i \xi x}\mathcal{F}\left(u\overline{\phi^t(\bullet - x)}\right)(\xi) = e^{\pi i \xi x}\left(\mathcal{F}(u) \star \mathcal{F}\left(\overline{\phi^t(\bullet - x)}\right)\right)(\xi),$$

where $\star$ denotes the convolution.

One can compute

$$\mathcal{F}\left(\overline{\phi^t(\bullet - x)}\right)(\xi) = e^{-2\pi i x\xi} t^{-n/4} \overline{\mathcal{F}(\phi)(-t^{-1/2}\xi)}.$$

Combined we have

$$P_\phi^t u(\xi, x) = e^{\pi i \xi x} t^{-n/4} \mathcal{F}(u)\left(e^{-2\pi i x(\xi - \bullet)} \mathcal{F}(\phi)(-t^{-1/2}(\xi - \bullet)\right). \tag{2.3}$$

Definition 2.7. We define the *wave front set of u with respect to ϕ*, denoted $\mathrm{WF}_\phi(u)$, by

$$(x_0, \xi_0) \notin \mathrm{WF}_\phi(u) \;:\Leftrightarrow\; \exists \text{ conic neighborhood } V \text{ of } (x_0, \xi_0) \text{ such that } \forall\, a, N \geq 1 : \\ |P_\phi^t u(t\xi, x)| \leq C_{a,N} t^{-N} \quad \forall t \geq 1, a^{-1} \leq |\xi| \leq a, (x, \xi) \in V.$$

It will turn out that $\mathrm{WF}_\phi(u) = \mathrm{WF}(u)$ for suitable functions ϕ. We have the following lemma which is also true for the standard wave front set.

Lemma 2.8 (see [Fol89, Lemma 3.21])**.** *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $u = 0$ on an open set $U \subset \mathbb{R}^n$, then $(x_0, \xi_0) \notin \mathrm{WF}_\phi(u)$ for all $x_0 \in U$ and all $\xi_0 \in \mathbb{R}^n \setminus \{0\}$.*

This means that $\mathrm{WF}_\phi(u)$ depends only on the local properties of u.

Remark 2.9. *Definition 2.7 is equivalent to*

$$(x_0, \xi_0) \in \mathrm{WF}_\phi(u) \;\Leftrightarrow\; \forall \text{ conic neighborhood } V \text{ of } (\xi_0, x_0)\; \exists\, a, N \geq 1 : \\ \exists t_m \to \infty, a^{-1} \leq |\xi_m| \leq a, (\xi_m, x) \in V : |P_\phi^{t_m} u(t_m \xi_m, x)| > C_{a,N} t_m^{-N}.$$

Thus, in order to show that $(\xi_0, x_0) \in \mathrm{WF}_\phi(u)$ it suffices to find for all $\varepsilon > 0$ an integer $N \in \mathbb{N}$, a constant $C > 0$ and sequences $(t_m)_{m\in\mathbb{N}} \subset \mathbb{R}_+$ with $t_m \to \infty$ and $(\xi_m)_{m\in\mathbb{N}} \subset B_1(0) \cap B_\varepsilon(\xi_0)$ such that

$$|P_\phi^{t_m} u(t_m \xi_m, x_0)| > C t_m^{-N}.$$

Theorem 2.10 (see [Fol89, Theorem 3.22])**.** *If $\phi \in \mathcal{S}(\mathbb{R}^n)$ is even and nonzero, then*

$$\mathrm{WF}_\phi(u) = \mathrm{WF}(u) \quad \forall\, u \in \mathcal{S}'(\mathbb{R}^n).$$

2.2 Wave Front Sets of Unitary Representations

In this section G is a n-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $(\pi, \mathcal{H})$ a unitary representation of G.
Denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on the Hilbert space $\mathcal{H}$ and by $J_1(\mathcal{H})$ the subspace of trace class operators with trace class norm $\|T\|_1$.

Definition 2.11. The *wave front set of a unitary representation* π is defined as the closure of the union of the wave front sets at the identity of the matrix coefficients of π:

$$\mathrm{WF}(\pi) := \overline{\bigcup_{v,w\in\mathcal{H}} \mathrm{WF}_e(\langle \pi(g)v, w\rangle_{\mathcal{H}})} \cup \{0\} \subset iT_e^*G \cong i\mathfrak{g}^*.$$

Here we use the convention that zero is always in the wave front set (contrary to Definition 2.1) because it makes the statements of the results for unitary representations cleaner.
Howe used in [How81] the equivalent definition

$$\mathrm{WF}(\pi) = \overline{\bigcup_{T\in J_1(\mathcal{H})} \mathrm{WF}_e(\mathrm{Tr}_\pi(T))} \cup \{0\}, \tag{2.4}$$

where $\mathrm{Tr}_\pi(T) := \mathrm{Tr}(\pi(\cdot)T)$, $T \in J_1(\mathcal{H})$, is a continuous bounded function on G regarded as a distribution on G by integration. The equivalence of these definitions was shown in [HHÓ16, Proposition 2.4].

Let us collect some well known basic properties of $\mathrm{WF}(\pi)$.

Lemma 2.12 (see [How81, Proposition 1.1 and Equations (1.7) and (1.8)])**.**
The wave front set $\mathrm{WF}(\pi) \subset i\mathfrak{g}^*$ *is a closed,* $\mathrm{Ad}^*(G)$*-invariant cone.*

Proof. We begin with proving that

$$\widetilde{\mathrm{WF}}(\pi) := \overline{\bigcup_{T\in J_1(\mathcal{H})} \mathrm{WF}(\mathrm{Tr}_\pi(T))} \subset iT^*G \cong G \times i\mathfrak{g}^*$$

is bi-invariant: For $g \in G$ using (2.2) we have

$$\begin{aligned} L_g(\mathrm{WF}(\mathrm{Tr}_\pi(T))) &= \mathrm{WF}(L_g \mathrm{Tr}_\pi(T)) = \mathrm{WF}(\mathrm{Tr}_\pi(T\pi(g)^{-1})), \\ R_g(\mathrm{WF}(\mathrm{Tr}_\pi(T))) &= \mathrm{WF}(L_g \mathrm{Tr}_\pi(T)) = \mathrm{WF}(\mathrm{Tr}_\pi(\pi(g)T)). \end{aligned}$$

The claim now follows from the definition of $\widetilde{\mathrm{WF}}(\pi)$ since $T\pi(g)^{-1}, \pi(g)T \in J_1(\mathcal{H})$ if $T \in J_1(\mathcal{H})$.
With Lemma 2.3 this also finishes the proof since every bi-invariant set in $T^*G \cong G \times \mathfrak{g}^*$ is of the form $G \times N$ with an $\mathrm{Ad}^*(G)$-invariant set $N \subset \mathfrak{g}^*$. □

The next result is rather technical and uses Howe's definition (2.4) but offers various equivalent descriptions of the wave front set.

Lemma 2.13 (see [HHÓ16, Lemma 2.5] and [How81, Theorem 1.4]).
The following assertions are all equivalent:

(i) $\xi_0 \notin \mathrm{WF}(\pi)$

(ii) *For every* $T \in J_1(\mathcal{H})$ *there is an open set* $e \in U \subset G$ *on which the logarithm is a well-defined diffeomorphism onto its image and an open set* $\xi_0 \in V \subset i\mathfrak{g}^*$ *such that for every* $\phi \in C_c^\infty(U)$ *the absolute value of the integral*

$$I(\phi,\xi,T)(\tau) := \int_G \mathrm{Tr}_\pi(T)(g) e^{\tau\xi(\log g)}\phi(g)\, dg$$

is rapidly decaying in τ *for* $\tau > 0$ *uniformly in* $\xi \in V$.

(iii) *There is an open set* $e \in U \subset G$ *on which the logarithm is a well-defined diffeomorphism onto its image and an open set* $\xi_0 \in V \subset i\mathfrak{g}^*$ *such that for every* $\phi \in C_c^\infty(U)$ *there exists a family of constants* $C_N(\phi) > 0$ *independent of both* $\xi \in V$ *and* $T \in J_1(H)$*, such that*

$$|I(\phi,\xi,T)(\tau)| \le C_N(\phi)\|T\|_1 \tau^{-N}$$

for $\tau \gg 0$*,* $\xi \in V$*,* $T \in J_1(H)$*.*

(iv) *There is an open set* $e \in U \subset G$ *on which the logarithm is a well-defined diffeomorphism onto its image and an open set* $\xi_0 \in V \subset i\mathfrak{g}^*$ *such that for every* $\phi \in C_c^\infty(U)$ *the quantity*

$$\|\pi(\phi(g)e^{\tau\xi(\log g)})\|_{op}$$

is rapidly decaying in τ *for* $\tau > 0$ *uniformly in* $\xi \in V$.

A first simple result which gives us an idea why wave front sets might be interesting for the decomposition of unitary representations is the following equality.

Proposition 2.14. *Let* $(\pi_1, \mathcal{H}_1)$*,* $(\pi_2, \mathcal{H}_2)$ *be two unitary representations of* G*, then*

$$\mathrm{WF}(\pi_1 \oplus \pi_2) = \mathrm{WF}(\pi_1) \cup \mathrm{WF}(\pi_2).$$

Proof. The matrix coefficients of $\pi_1 \oplus \pi_2$ have the form $\mathrm{Tr}_{\pi_1\oplus\pi_2}(T) = \mathrm{Tr}_{\pi_1}(T_1) + \mathrm{Tr}_{\pi_2}(T_2)$ with suitable $T_i \in J_1(\mathcal{H}_i)$. Now, the inclusions $\mathrm{WF}(\pi_i) \subset \mathrm{WF}(\pi_1 \oplus \pi_2)$ follow by setting one $T_j = 0$ and letting the other vary over all trace class operators. The other inclusion follows from the fact that $\mathrm{WF}(u+v) \subset \mathrm{WF}(u) \cup \mathrm{WF}(v)$ for two distributions $u, v \in \mathcal{D}'$ since the sum of two rapidly decaying integrals inherits this property. □

Example 2.15. Consider the action π of $\mathbb{R}^2$ on $L^2(\mathbb{R})$ via $(\pi(t_1,t_2)f)(x) = f(x-t_1)$ as a toy model for the right regular representation on a homogeneous space $L^2(G/H)$ (here $G = \mathbb{R}^2$, $H = \mathbb{R}$).

Since the action is independent of t_2 so is each matrix coefficient. If $\xi_2 \neq 0$ the insertion of the partial differential operator $\left(\frac{\partial_{t_2}}{-2\pi\tau\xi_2}\right)^N$ and integration by parts (compare the first part of Example 2.5) shows that

$$\mathrm{WF}_0(\langle \pi(\bullet) f_1, f_2 \rangle) \subset \left\{ \xi \in i\mathbb{R}^2 \mid \xi_2 = 0 \right\} \quad \forall\, f_1, f_2 \in L^2(\mathbb{R}),$$

and this implies

$$\mathrm{WF}(\pi) \subset \left\{ \xi \in i\mathbb{R}^2 \mid \xi_2 = 0 \right\}.$$

Let us show that we have not only an inclusion but in fact equality: We choose the indicator functions $f_1 = \chi_{[0,1]}$ and $f_2 = \chi_{[1,2]}$ and compute

$$\langle \pi(t_1, t_2) f_1, f_2 \rangle = \begin{cases} 0, & \text{if } t_1 < 0 \text{ or } t_1 \geq 2, \\ t_1, & \text{if } 0 \leq t_1 < 1, \\ 1 - t_1, & \text{if } 1 \leq t_1 < 2. \end{cases}$$

Consequently, for $\xi = (\xi_1, 0)$ we have

$$\begin{aligned} \mathcal{F}(\langle \pi(\bullet) f_1, f_2 \rangle)(\tau\xi) &= \int_0^1 t_1 e^{-2\pi\tau t_1 \xi_1}\, dt_1 + \int_1^2 (1 - t_1) e^{-2\pi\tau t_1 \xi_1}\, dt_1 \\ &= \int_0^1 t_1 \left(\frac{\partial_{t_1}}{-2\pi\tau\xi_1} \right) e^{-2\pi\tau t_1 \xi_1}\, dt_1 + \int_1^2 (1 - t_1) \left(\frac{\partial_{t_1}}{-2\pi\tau\xi_1} \right) e^{-2\pi\tau t_1 \xi_1}\, dt_1 \\ &= (-2\pi\tau\xi_1)^{-1} \left(e^{2\pi\tau\xi_1} - e^{-4\pi\tau\xi_1} \right) + (2\pi\tau\xi_1)^{-2} \left(-2e^{2\pi\tau\xi_1} + e^{-4\pi\tau\xi_1} + 1 \right), \end{aligned}$$

by integration by parts. Hence, $(\xi_1, 0) \in \mathrm{WF}_0(\langle \pi(\bullet) f_1, f_2 \rangle)$ and therefore

$$\mathrm{WF}(\pi) = \left\{ \xi \in i\mathbb{R}^2 \mid \xi_2 = 0 \right\}.$$

Now, we are interested in another description of the wave front set of π allowing for distributional limits of the functions $\mathrm{Tr}_\pi(T)$, at least for certain $T \in J_1(\mathcal{H})$. For this we will first show a general result for distributions and their convergence in a Sobolev space.

Lemma 2.16. *Let $(t_m)_{m \in \mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ with $\operatorname{supp}(t_m) \subset K$ for a compact $K \subset \mathbb{R}^n$ and $t_m \to 0 \in \mathcal{D}'(\mathbb{R}^n)$ as distributions. Then there exists $s \in \mathbb{R}$ such that $t_m \overset{H^s}{\to} 0$ in the Sobolev space $H^s(\mathbb{R}^n)$.*

Proof. By [DK10, Lemma 5.4] there exist $k \in \mathbb{N}$, $C > 0$ such that

$$|t_m(\phi)| \leq C \|\phi\|_{C^k} \quad \forall\, m \in \mathbb{N}, \phi \in C_c^\infty(K).$$

Since $\operatorname{supp}(t_m) \subset K$ for all $m \in \mathbb{N}$ the inequality holds for all $\phi \in C_c^\infty(\mathbb{R}^n)$.
Using [Hör03, Lemma 7.6.3] we can show that there exists a constant $C' > 0$ such that $\|\phi\|_{C^k} \leq C' \|\phi\|_{H^{r+k}}$ for all $\mathbb{N} \ni r > \frac{n}{2}$, $\phi \in C_c^\infty(\mathbb{R}^n)$.

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $H^{r+k}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$ this shows that for every $m \in \mathbb{N}$ the distribution t_m can be extended to a continuous functional on $H^{r+k}(\mathbb{R}^n)$ and we have $\|t_m\|_{H^{r+k}\to\mathbb{C}} \leq C'C =: \tilde{C}$ and by duality of the Sobolev spaces $\|t_m\|_{H^{-(r+k)}} \leq \tilde{C}$ for all $m \in \mathbb{N}$, i.e.

$$\int |\hat{t}_m(\xi)|^2 \langle\xi\rangle^{-2(r+k)}\, d\xi < \tilde{C}^2 \quad \forall m \in \mathbb{N}.$$

Here $\hat{t}_m(\xi)$ is the Fourier transform of t_m which is a smooth function in ξ.
Let $R > 0$. Since t_m converges to 0 we choose $m_R \in \mathbb{N}$ such that $\|\hat{t}_m\|_{\infty,\overline{B_R(0)}} < R^{-n}$ for all $m \geq m_R$.
Then we have for $m \geq m_R$:

$$\begin{aligned}
\|t_m\|^2_{H^{-(r+k+1)}} &= \int |\hat{t}_m(\xi)|^2 \langle\xi\rangle^{-2(r+k+1)}\, d\xi \\
&= \int_{B_R(0)} |\hat{t}_m(\xi)|^2 \langle\xi\rangle^{-2(r+k+1)}\, d\xi + \int_{|\xi|>R} |\hat{t}_m(\xi)|^2 \langle\xi\rangle^{-2(r+k+1)}\, d\xi \\
&\leq \operatorname{Vol}(B_R(0))R^{-2n} + \langle R\rangle^{-2} \int_{|\xi|>R} |\hat{t}_m(\xi)|^2 \langle\xi\rangle^{-2(r+k)}\, d\xi \\
&\leq MR^{-n} + \tilde{C}^2\langle R\rangle^{-2}.
\end{aligned}$$

Thus, by letting $R \to \infty$ we obtain $t_m \to 0$ for $m \to \infty$ in $H^{-(r+k+1)}(\mathbb{R}^n)$. □

For the alternative description of $\mathrm{WF}(\pi)$ we want to look at:

Definition 2.17.

$$\begin{aligned}
P(\pi) &:= \{\operatorname{Tr}(\pi(\bullet)T) \mid T \in \mathcal{B}(\mathcal{H}) \text{ non-negative and of finite rank } \} \subset C(G), \\
\overline{P(\pi)}^{\mathcal{D}'} &= \{\alpha \in \mathcal{D}'(G) \mid \exists (\alpha_m)_{m\in\mathbb{N}} \subset P(\pi) : \alpha_m \to \alpha \text{ in } \mathcal{D}'(G)\}.
\end{aligned}$$

Proposition 2.18. *Let $\alpha \in \mathcal{D}'(G)$. If there exists a neighborhood $e \in U \subset G$ such that $U \subset U_\kappa$ for a local coordinate system $\kappa : G \supset U_\kappa \to V_\kappa \subset \mathbb{R}^n$ of the smooth manifold G, $\kappa(e) = 0$, and $(\alpha_m)_{m\in\mathbb{N}} \subset P(\pi)$ with $\alpha_m \circ \kappa^{-1} \to \alpha \circ \kappa^{-1}$ in a Sobolev space $H^s(V_\kappa)$ with $s > \frac{n}{2}$, then*

$$\mathrm{WF}_e(\alpha) \subset \mathrm{WF}(\pi).$$

Proof. Let $\alpha_m = \operatorname{Tr}(\pi(g)A_m)$, $A_m \in \mathcal{B}(\mathcal{H})$ non-negative and of finite rank. Since the delta distribution δ_0 in 0 is an element of $H^{-s}(V_\kappa)$ for $s > \frac{n}{2}$ we have

$$\delta_0(\alpha \circ \kappa^{-1}) \leftarrow \delta_0(\alpha_m \circ \kappa^{-1}) = \operatorname{Tr}(\pi(\kappa^{-1}(0))A_m) \overset{\kappa^{-1}(0)=e}{=} \operatorname{Tr}(A_m) \overset{A_m \text{ pos.}}{=} \|A_m\|_1$$

where $\|.\|_1$ denotes the trace norm on $J_1(\mathcal{H})$. This shows that $(\|A_m\|_1)_{m\in\mathbb{N}} \subset \mathbb{R}$ is bounded by a constant $C' > 0$.

Now let $\xi \notin \mathrm{WF}(\pi)$. By Lemma 2.13 (iii) there exists a neighborhood $\Omega \subset i\mathfrak{g}^*$ of ξ such that for all $N \in \mathbb{N}$ und $J \in J_1(\mathcal{H})$ there exists a constant $C_N(\varphi) > 0$ independent of J such that

$$|\operatorname{Tr}(\pi(\bullet)J)[\varphi e^{-2\pi t\eta(\log)}]| \leq C_N(\varphi)\|J\|_1|t|^{-N} \quad \forall \eta \in \Omega,\ \varphi \in C_c^\infty(G).$$

In particular, $|\alpha_m[\varphi e^{-2\pi t\eta \log}]| \leq C' C_N(\varphi)|t|^{-N}$ independent of $m \in \mathbb{N}$. Therefore the limit distribution α also satisfies the inequality. This shows, again by Lemma 2.13 (iii), that $\xi \notin \mathrm{WF}_e(\alpha)$. □

With this we can find a lower bound for $\mathrm{WF}(\pi)$:

Proposition 2.19.

$$\overline{\bigcup_{\alpha \in \overline{P(\pi)}^{\mathcal{D}'}} \mathrm{WF}_e(\alpha)} \subset \mathrm{WF}(\pi)$$

Proof. Let $\alpha \in \overline{P(\pi)}^{\mathcal{D}'}$, i.e. there exist $\alpha_m = \operatorname{Tr}(\pi(g)A_m) \in P(\pi)$, $A_m \in \mathcal{B}(\mathcal{H})$ non-negative and of finite rank, such that $\alpha_m \to \alpha$ in $\mathcal{D}'$. Since we are only interested in the wave front set in $e \in G$ it suffices to look at a small neighborhood U of e such that $U \subset U_\kappa$ for a local coordinate system $\kappa : G \supset U_\kappa \to V_\kappa \subset \mathbb{R}^n$ of the manifold G and the image V_κ is (contained in) a relatively compact set $K \subset \mathbb{R}^n$. Now we replace α_m and α by $\alpha_m \cdot \chi_U$ and $\alpha \cdot \chi_U$, respectively, where $\chi_U \in C_c^\infty$ is a smooth cut-off function with $0 \leq \chi_U \leq 1$, $\chi_U(e) = 1$ and $\operatorname{supp}\chi_U \subset U$. This does not change the wave front set of α and the convergence in $\mathcal{D}'(G)$.

Choose a basis $\{X_i\}_{1\leq i\leq n}$ of $\mathfrak{g}$. Each element $X_i \in \mathfrak{g}$ induces differential operators L_{X_i}, R_{X_i} of order 1 on G. As the left and right regular representations of G are unitary we also have $L_{X_i}^* = L_{-X_i}$ and $R_{X_i}^* = R_{-X_i}$ as differential operators of order 1 on G. If we define $P := -\sum_{i=1}^n X_i^2 \in \mathfrak{U}(\mathfrak{g})$ and the associated differential operators L_P, R_P of order 2 on G, we see that these are non-negative operators on $L^2(G)$. With that one can show that $1 + L_P = L_{1+P}$ and $1 + R_P = R_{1+P}$ are invertible and that their inverse is continuous by the bounded inverse theorem. We also obtain by ellipticity of the associated differential operators on $V_\kappa \subset \mathbb{R}^n$ that $\tilde{L}_{(1+P)} := (\kappa^{-1})^* L_{1+P}\kappa^*$ and $\tilde{R}_{(1+P)} := (\kappa^{-1})^* R_{1+P}\kappa^*$ are bounded operators on $H^r(V_\kappa) \to H^{r-2}(V_\kappa)$ for all $r \in \mathbb{R}$.

As $\alpha_m \circ \kappa^{-1} \to \alpha \circ \kappa^{-1}$ as distributions on V_κ, Lemma 2.16 gives us the existence of $s \in \mathbb{R}$ such that the convergence is also given in the Sobolev space $H^s(V_\kappa)$. Now choose $N \in \mathbb{N}$ such that $s + 4N > \frac{n}{2}$ and define

$$\tilde{\alpha}_m := \tilde{R}_{(1+P)}^{-N}\tilde{L}_{(1+P)}^{-N}(\alpha_m \circ \kappa^{-1}), \quad \tilde{\alpha} := \tilde{R}_{(1+P)}^{-N}\tilde{L}_{(1+P)}^{-N}(\alpha \circ \kappa^{-1}) \in H^{s+4N}(V_\kappa).$$

Note that $\|\tilde{\alpha}_m - \tilde{\alpha}\|_{H^{s+4N}} \to 0$ by continuity of $\tilde{L}_{(1+P)}^{-N}$ and $\tilde{R}_{(1+P)}^{-N}$ and

$$\tilde{\alpha}_m \circ \kappa = R_{(1+P)}^{-N} L_{(1+P)}^{-N} \operatorname{Tr}(\pi(g)A_m) = \operatorname{Tr}(\pi(g)\pi(1+P)^{-N}A_m\,\pi(1+P)^{-N}) \in P(\pi),$$

since $\tilde{A}_m := \pi(1+P)^{-N} A_m \pi(1+P)^{-N}$ is non-negative, as a product ABA is non-negative if $B \geq 0$ and A is self-adjoint, and of finite rank since the set of finite rank operators is an ideal. By Proposition 2.18 we have $\mathrm{WF}_e(\tilde{\alpha} \circ \kappa) \subset \mathrm{WF}(\pi)$. Then by definition of wave front sets on manifolds and since the action of differential operators does not increase the wave front set, we obtain $\mathrm{WF}_e(\alpha) \subset \mathrm{WF}_e(\tilde{\alpha}) \subset \mathrm{WF}(\pi)$. Since the wave front set is closed the claim follows. □

2.3 Historic Overview of known Results for Wave Front Sets of Lie Groups

In this section we would like to illustrate the significance of the wave front sets of unitary representations.

In order to state the results consider the following setting: For a unitary representation π of a Lie group G which is type I we denote by $\mathrm{supp}(\pi) \subset \hat{G}$ the irreducible unitary representations that are weakly contained in π, i.e. occur in the direct integral decomposition of π.

Assume that there is a way to associate to any $\sigma \in \mathrm{supp}(\pi)$ a coadjoint orbit $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$. Then we define the orbital support of π to be

$$\mathcal{O} - \mathrm{supp}\,\pi := \bigcup_{\sigma \in \mathrm{supp}(\pi)} \mathcal{O}_\sigma \subset i\mathfrak{g}^*.$$

Since $\mathrm{WF}(\pi)$ is $\mathrm{Ad}^*(G)$-invariant one can ask for its connection to its orbital support. The right notion to do so is given in the following

Definition 2.20. If V is a finite-dimensional vector space and $S \subset V$, then we define the *asymptotic cone* of S to be

$$\mathrm{AC}(S) := \{v \in V \mid \forall \text{ open cone } \mathcal{C} \ni v \, : \, \mathcal{C} \cap S \text{ unbounded}\} \cup \{0\}.$$

One of the first works that provided statements like the ones we are interested in was "K-types and singular spectrum" by Kashiwara and Vergne in 1979 (see [KV79]). One of their results states that for compact Lie groups the singular spectrum provides asymptotic information about the decomposition of the representation into irreducibles. On the level of wave front sets their statement can be phrased with our notation as follows:

Theorem 2.21 (compare [KV79]). *Let G be a compact, connected Lie group and π a unitary representation. Then*

$$\mathrm{WF}(\pi) = \mathrm{AC}\left(\mathcal{O} - \mathrm{supp}\,\pi\right).$$

Here $\mathcal{O}_\sigma = \mathrm{Ad}^*(G)\lambda_\sigma$, $\sigma \in \hat{G}$, is the orbit of the associated highest weight $\lambda_\sigma \in \mathfrak{t}^* \subset \mathfrak{g}^*$.

Notice that in their paper the results are reduced to statements in a Weyl chamber in $\mathfrak{t}^*$. Since that is only possible for compact groups we work with the notation we introduced above.

A few years later, Howe gave a definition for the wave front set of unitary representations for arbitrary Lie groups. He gave his own proof for Theorem 2.21 in [How81, Proposition 2.3], but he also looked at nilpotent Lie groups and a single irreducible unitary representation.

Theorem 2.22 (see [How81, Proposition 2.2]). *Let G be a nilpotent, connected, simply connected Lie group and $\sigma \in \hat{G}$ irreducible. Then*

$$\mathrm{WF}(\sigma) = \mathrm{AC}(\mathcal{O}_\sigma).$$

Here $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$ is given by the isomorphism between the unitary dual $\hat{G}$ and the space of coadjoint orbits $i\mathfrak{g}^*/G$ (see Theorem 3.2).

In the proof he uses that if the representation admits a global character then its wave front set equals the wave front set of the representation:

Theorem 2.23. *Let G be a nilpotent, connected, simply connected Lie group. For $\sigma \in \hat{G}$ the trace linear functional $\theta_\sigma(\phi) = \mathrm{Tr}(\sigma(\phi))$ is a distribution on G, called the character of σ, and we have*

$$\mathrm{WF}_e(\theta_\sigma) = \mathrm{WF}(\sigma).$$

Remark 2.24. *By [How81, Theorem 1.8] the previous statement is actually true for general Lie groups G if $\sigma \in \hat{G}$ is so-called strong trace class, which implies that the trace linear functional defines a global character.*

Then from 1994 to 1998 Kobayashi studied in [Kob94], [Kob98a] and [Kob98b] the restriction of representations of real reductive Lie groups to subgroups for which he used the wave front set on the level of maximal compact subgroups (see also [Kob05]). In particular, Kobayashi further developed the idea to study the singularities of summed-up characters as an intermediate step in [Kob98b, Section 2].

Almost two decades later, this idea of studying the summed-up or integrated characters was then advanced in [HHÓ16] where Harris, He and Ólafsson use integrated characters when studying wave front sets for real reductive, algebraic Lie groups for which they proved the asymptotic result mentioned above:

Theorem 2.25 (see [HHÓ16, Theorem 21.2]). *Let G be a real reductive, algebraic Lie group and π be weakly contained in the regular representation of G, i.e. $\operatorname{supp}\pi \subset \hat{G}_{temp}$. Then*

$$\mathrm{WF}(\pi) = \mathrm{AC}\left(\mathcal{O} - \operatorname{supp}\pi\right).$$

Here $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$ is the finite union of coadjoint orbits Duflo and Rossmann associated to the irreducible representations which are weakly contained in $L^2(G)$ (see [Duf70], [Ros78] and [Ros80]).

Their proof uses, as highlighted above, the wave front set of integrated characters as an intermediate step (see [HHÓ16, Lemma 6.3 and the proof of Proposition 7.1]).

In [Har18] Harris then also studies representations of a real reductive Lie group that are a direct integral of singular, irreducible representation. They are the complement of the irreducible, tempered representations studied in [HHÓ16]. He shows that their wave front set is contained in the singular set. Thus, combining these two papers asymptotic results for arbitrary unitary representations of a real reductive group are obtained.

Since the association of orbital parameters to irreducible representations by Duflo and Rossmann only works on $\hat{G}_{temp}$, Harris and Oshima introduce in [HO17] a geometric character formula for the set of singular, irreducible representations. With this one can generalize the results from [HHÓ16] that are applications of Rossmann's classical character formula to harmonic analysis questions and branching problems.

We conclude that it would be worthwhile to have this kind of statement relating the wave front set to the asymptotic orbital support for as many types of Lie groups as possible. Potential applications are then restrictions and inductions of unitary representations between different groups.

Our main goal is to prove this statement for nilpotent Lie groups:

Theorem. *Let G be a nilpotent, connected, simply connected Lie group and π a unitary representation of G. Then*

$$\mathrm{WF}(\pi) = \mathrm{AC}\left(\mathcal{O} - \operatorname{supp} \pi\right).$$

We will give the proof in Section 4 using the structure theory of a nilpotent Lie group and its coadjoint orbits, presented in Section 3. In Section 5 we will provide alternative proofs for our main theorem. In particular, in Section 5.1 we try to follow the work of [HHÓ16] and their approach by integrated characters since traces are a well-understood tool for nilpotent Lie groups as Theorem 2.23 already suggests.
Lastly, we look at an application of our result to restrictions of unitary representations in Section 6.

Concerning the induction of unitary representations note that by [HW17] the wave front set of induced representations is explicitly known in quite large generality. For example one has:

Theorem 2.26 (see [HW17, Theorem 2.1]). *Suppose $X = G/H$ is a homogeneous space for a Lie group G equipped with a nonzero invariant density. Then*

$$\mathrm{WF}(L^2(X)) = \mathrm{WF}(\mathrm{Ind}_H^G 1) = \overline{\mathrm{Ad}^*(G) \cdot i(\mathfrak{g}/\mathfrak{h})^*}.$$

This gives a geometric interpretation and generalizes our elementary Example 2.15.

3 Nilpotent Lie Groups and Lie Algebras

We begin with the definition of nilpotent Lie algebras.

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra. The *descending central series of* $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(1)} := \mathfrak{g}, \quad \mathfrak{g}^{(N+1)} := [\mathfrak{g}, \mathfrak{g}^{(N)}], \; N \in \mathbb{N}.$$

We say that $\mathfrak{g}$ is a *nilpotent* Lie algebra if there is an integer N such that $\mathfrak{g}^{(N)} = \{0\}$. If this N is minimal we also call it the *degree of nilpotence* of $\mathfrak{g}$.
A Lie group G is called *nilpotent* if its Lie algebra is nilpotent.

The aim of this section is now to provide the structure theory of nilpotent Lie groups and Lie algebras and illustrate it in the examples of the Heisenberg group H_n and the group K_3. It is mostly based on the book by Corwin and Greenleaf [CG90].

3.1 General Theory

Let G be a nilpotent, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ of dimension n and $\mathfrak{g}^*$ its vector space dual. By $\hat{G}$ we denote the unitary dual of G, i.e. the set of unitary irreducible representations of G modulo unitary equivalence, which is equipped with the Fell topology.
In this section we would like to take a look at the nice structures a nilpotent Lie group and its Lie algebra provides. The three main results are the following Theorems 3.2, 3.4 and 3.5. All necessary definitions will be provided in the detailed discussions afterwards.

Theorem 3.2 (see [CG90, Theorems 2.2.1 - 2.2.4])**.** *Let $i\mathfrak{g}^*/G$ denote the space of coadjoint orbits. There exists a homeomorphism*

$$\begin{cases} \hat{G} & \to & i\mathfrak{g}^*/G, \\ \sigma & \mapsto & \mathcal{O}_\sigma, \\ \sigma_l & \leftarrow & \mathcal{O}_l = \mathrm{Ad}^*(G)l. \end{cases}$$

For the continuity of the map $i\mathfrak{g}^*/G \to \hat{G}$ see [Kir62, Theorem 8.2] and for the continuity of the map $\hat{G} \to i\mathfrak{g}^*/G$ see [Bro73].

This bijection then allows us to shift concepts from the side of irreducible unitary representations to the side of coadjoint orbits or vice versa. An important notion are the global characters.

Definition 3.3. For a unitary representation π on G and Schwartz function $\phi \in \mathcal{S}(G)$ the *integrated representation* is defined as

$$\pi(\phi) := \int_G \phi(g)\pi(g)\,dg.$$

Theorem 3.4 (see [CG90, Theorems 4.2.1 and 4.2.4]). *For $\sigma \in \hat{G}$ the integrated representation $\sigma(\phi)$, $\phi \in \mathcal{S}(G)$, is of trace class. The trace $\theta_\sigma(\phi) := \operatorname{Tr}\sigma(\phi)$ is a tempered distribution on G.*
Furthermore, the character formula (see also [Kir62, Theorem 7.4]) states that there exists a unique invariant measure ϑ_σ on the corresponding orbit $\mathcal{O}_\sigma$ such that

$$\theta_\sigma(\phi) = \int_{\mathcal{O}_\sigma} \mathcal{F}^+(\phi)(l)\, d\vartheta_\sigma(l), \tag{3.5}$$

where $\mathcal{F}^+(\phi)(l) = \int_{\mathfrak{g}} e^{2\pi l(X)}\phi(\exp X)\, dX$ denotes the Fourier transform of $\phi \in \mathcal{S}(G)$.

The third statement tells us more about the structure of the coadjoint orbits which ultimately gives us more information about the unitary dual and the global characters.

Theorem 3.5 (see [CG90, Theorem 3.1.14]). *Fix a (strong Malcev) basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$. Then there exits a finite set D of orbit types. Denote by $U_d \subset i\mathfrak{g}^*$ the union of all orbits of type $d \in D$. Moreover, all orbits in U_d have the same dimension.*
For each $d \in D$ there also exists a cross-section $\Sigma_d \subset i\mathfrak{g}^*$ of the orbits in U_d, i.e. each orbit $\mathcal{O} \subset U_d$ intersects Σ_d in a unique point. Then

$$\Sigma := \bigsqcup_{d \in D} \Sigma_d \cong i\mathfrak{g}^*/G$$

is a cross-section of all G-orbits.
Furthermore, for each $d \in D$ there exists a decomposition

$$i\mathfrak{g}^* = V_{S(d)} \oplus V_{T(d)}$$

as a direct sum of vector spaces and a birational, non-singular, surjective map

$$\psi_d \colon \Sigma_d \times V_{S(d)} \to U_d$$

such that for each $l \in \Sigma_d$ its orbit is given by $\mathcal{O}_l = \psi_d\left(l, V_{S(d)}\right)$.

Now we take a closer look at the ingredients and underlying concepts of these main statements starting on the level of nilpotent Lie algebras. These details will not only be presented as background material but their knowledge will be crucial for our own results. Firstly, they enable us to prove the last two Lemmas 3.28 and 3.29 of this section providing an estimate for the characters depending on $l \in \Sigma_d$ but we will also use them at various points throughout the next section.

Lemma 3.6 (see [CG90, Kirillov's Lemma 1.1.12]). *Let $\mathfrak{g}$ be a non-abelian nilpotent Lie algebra whose center $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}Z$ is one-dimensional. Then $\mathfrak{g}$ can be written as*

$$\mathfrak{g} = \mathbb{R}Z \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathfrak{w} = \mathbb{R}X \oplus \mathfrak{g}_0,$$

a vector space direct sum with a suitable subspace $\mathfrak{w}$. Furthermore, $[X, Y] = Z$ and $\mathfrak{g}_0 = \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$ is the centralizer of Y and an ideal.

Proof. We have $\dim \mathfrak{g} \geq 3$. We choose $Z \in \mathfrak{g}$ such that $\mathfrak{z}(\mathfrak{g}) = \mathbb{R} \cdot Z$.
Then $\overline{\mathfrak{g}} := \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent and $\dim \overline{\mathfrak{g}} \geq 2$. We choose $0 \neq \overline{Y} \in \mathfrak{z}(\overline{\mathfrak{g}})$ and $Y \in \mathfrak{g}$ such that $\overline{Y} = Y\mathfrak{z}(\mathfrak{g})$. By definition of $\overline{Y}$ we have $[W, Y] \in \mathbb{R}Z$ for all $W \in \mathfrak{g}$. Thus, we define a linear map

$$\alpha : \mathfrak{g} \to \mathbb{R}, \quad [W, Y] = \alpha(W)Z.$$

Since $Y \notin \mathfrak{z}(\mathfrak{g})$ we know $\alpha \neq 0$ and can choose $X \in \mathfrak{g}$ with $\alpha(X) = 1$, i.e. $[X, Y] = Z$.
Let $\mathfrak{g}_0 := \ker(\alpha)$. Then $Z, Y \in \mathfrak{g}_0$ are linear independent. Let $\mathfrak{w} \subset \mathfrak{g}_0$ be a subspace such that $\mathfrak{g}_0 = \mathbb{R}Z \oplus \mathbb{R}Y \oplus \mathfrak{w}$.
Now we check that that $\mathfrak{g}_0$ is a subalgebra: For $W_1, W_2 \in \mathfrak{g}_0$ we have

$$[[W_1, W_2], Y] = -[W_2, [W_1, Y]] + [W_1, [W_2, Y]] = -[W_2, 0] + [W_1, 0] = 0,$$

which implies $[W_1, W_2] \in \mathfrak{g}_0$. Since [CG90, Lemma 1.1.8] states that subalgebras of codimension 1 are ideals the proof is finished. □

Theorem 3.7 (see [CG90, Theorem 1.1.13]). *Let $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_k \subset \mathfrak{g}$ be subalgebras with $\dim \mathfrak{g}_j = m_j$. Then*

(i) $\mathfrak{g}$ has a basis $\{X_1, \ldots, X_n\}$ such that

a) for each m, $\mathfrak{h}_m = \operatorname{span}_{\mathbb{R}} \{X_1, \ldots, X_m\}$ is a subalgebra of $\mathfrak{g}$,

b) for $1 \leq j \leq k$, $\mathfrak{h}_{m_j} = \mathfrak{g}_j$.

(ii) If the $\mathfrak{g}_j$ are ideals of $\mathfrak{g}$, then one can pick the X_j so that a) is replaced by

c) for each m, $\mathfrak{h}_m = \operatorname{span}_{\mathbb{R}} \{X_1, \ldots, X_m\}$ is an ideal of $\mathfrak{g}$.

With this one can define a suitable basis for the Lie algebra which will be helpful on the level of the Lie group but also in order to find the parametrization of coadjoint orbits in Theorem 3.5:

Definition 3.8. We call a basis satisfying a) and b) of Theorem 3.7 a *weak Malcev basis* for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$; and one satisfying b) and c) a *strong Malcev basis* for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. In the case $k = 0$ we simply refer to a *weak/strong Malcev basis* for $\mathfrak{g}$.

From now on, let G be a nilpotent, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ which is by definition nilpotent. Firstly, we examine how G and $\mathfrak{g}$ are connected.

Theorem 3.9 (see [CG90, Theorem 1.2.1]). *The exponential map $\exp : \mathfrak{g} \to G$ is an analytic diffeomorphism.*

Lemma 3.10. *The* Campbell-Baker-Hausdorff formula

$$X * Y := \log(\exp X \exp Y) \in \mathfrak{g}$$

is defined for all elements $X, Y \in \mathfrak{g}$ and can be written as a finite sum of elements of $\mathfrak{g}$, *more precisely*

$$X * Y = \sum_{m=0}^{N} \frac{(-1)^{m+1}}{m} \sum_{p_i+q_i>0} \frac{X^{p_1}Y^{q_1}\cdots X^{p_m}Y^{q_m}}{p_1!q_1!\cdots p_m!q_m!}$$
$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \textit{(commutators in} \geq 4 \textit{ terms)},$$

where N is the degree of nilpotence of $\mathfrak{g}$.

Definition 3.11. We say that a map between two vector spaces is *polynomial* if it is described by polynomials in the coordinates for some (hence any) pair of bases.
We call a map f a *polynomial diffeomorphism* if f^{-1} exists and both f and f^{-1} are polynomial. Carrying this notion over to G via its identification with $\mathfrak{g}$ we call a map $\phi : G \to G$ a polynomial diffeomorphism if $\log\circ\phi\circ\exp : \mathfrak{g} \to \mathfrak{g}$ has this property.
We define a *polynomial coordinate map* for G as a map $\phi : \mathbb{R}^n \to G$, $n = \dim G$, for which $\log\circ\phi$ is a polynomial diffeomorphism.

For a (weak or strong) Malcev basis $\{X_1, \dots, X_n\}$ one can show that the exponential coordinates

$$g = \exp(s_1X_1)\cdots\exp(s_nX_n) = \exp(s_1X_1 * \cdots * s_nX_n) \in G$$

and the so-called Malcev coordinates $g = \exp(\sum \alpha_j X_j)$ are related by a polynomial diffeomorphism, i.e. the α_j are polynomial in the s_i and vice versa (see [CG90, Propositions 1.2.7 and 1.2.8]).

Since we want to integrate over the group but also use the structure of the Lie algebra we have the following:

Theorem 3.12 (see [CG90, Theorem 1.2.10])**.**

(i) The exponential map takes the Lebesgue measure on $\mathfrak{g}$ to a left- and right-invariant (Haar) measure on G. In particular, nilpotent LIe groups are unimodular so that the right-invariance implies left-invariance and vice versa.

(ii) Let $\phi : \mathbb{R}^n \to G$ be any polynomial coordinate map. Then ϕ takes the Lebesgue measure on $\mathbb{R}^n$ to a Haar measure on G. In particular, this is true if the Lebesgue measure is transferred to G via weak or strong Malcev coordinates.

We will also need to know how we can find a measure if we have given one on a subgroup and a quotient.

Lemma 3.13 (see [CG90, Lemma 1.2.13])**.** *Let H be a closed subgroup of the locally compact group G. Suppose that $H\backslash G$ has a right-invariant measure $d\dot{g}$ and that we have fixed a right Haar dh measure on H. Then a right Haar measure on G is given by*

$$\int_G \phi\, dg = \int_{H\backslash G} \left(\int_H \phi(hg)\, dh \right) d\dot{g} \quad \forall\, \phi \in C_c^\infty(G).$$

Now let us turn to the coadjoint orbits $\mathcal{O}_l = \mathrm{Ad}^*(G)l \subset i\mathfrak{g}^*$ for $l \in i\mathfrak{g}^*$ and examine their structure. We start off with their dimension:

Lemma 3.14 (see [CG90, Lemma 1.3.2]). *For $l \in i\mathfrak{g}^*$ we define the bilinear form $B_l(X,Y) = l([X,Y])$ on $\mathfrak{g}$. Then the radical*

$$\mathfrak{r}_l := \{X \in \mathfrak{g} \mid B_l(X,Y) = 0 \ \forall \ Y \in \mathfrak{g}\} = \{X \in \mathfrak{g} \mid \mathrm{ad}^*(X)l = 0\} \tag{3.6}$$

has even codimension in $\mathfrak{g}$. Hence coadjoint orbits are of even dimension.
They are actually symplectic manifolds with the non-degenerate skew symmetric 2-form $\omega(l') \in \Lambda^2(T_{l'}\mathcal{O}_l)$ such that $\omega(l')(-(\mathrm{ad}^* X)l', -(\mathrm{ad}^* Y)l') = l'([X,Y])$, $l' \in \mathcal{O}_l$. Note that ω is $\mathrm{Ad}^*(G)$ invariant.

Now, we are interested in how we can define an irreducible unitary representation of G given an element $l \in i\mathfrak{g}^*$ (with Theorem 3.2 in mind).

Definition 3.15. A *polarizing subalgebra* for $l \in i\mathfrak{g}^*$ is a subalgebra $\mathfrak{m} \subset \mathfrak{g}$ that is a maximal isotropic subspace for the bilinear form $B_l : \mathfrak{g} \times \mathfrak{g} \to i\mathbb{R}$.
They are also called *maximal subordinate subalgebras* for l.

Theorem 3.16 (see [CG90, Proposition 1.3.4]). *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $l \in i\mathfrak{g}^*$. Then there exists a polarizing subalgebra for l.*

Now, for $l \in i\mathfrak{g}^*$ choose a polarizing $\mathfrak{m}$ and let $M = \exp \mathfrak{m}$. Then $\chi_l(\exp Y) = e^{2\pi l(Y)}$ is a one-dimensional representation of M since $l([\mathfrak{m},\mathfrak{m}]) = 0$. Hence, we can define

$$\sigma_l := \mathrm{Ind}_M^G(\chi_l).$$

More precisely,

$$\mathcal{H}_l = \left\{ f : G \to \mathbb{C} \text{ measurable} \mid f(mg) = \chi_l(m) f(g) \ \forall m \in M \text{ and } \int_{M\backslash G} \|f(g)\|^2 d\dot{g} < \infty \right\}$$

and

$$(\sigma_l(x)f)(g) = f(gx) \quad \forall \ x \in G, f \in \mathcal{H}_l.$$

With this construction one can prove the bijection $\hat{G} \cong i\mathfrak{g}^*/G$ which is our first main result Theorem 3.2. As [CG90, Theorems 2.2.1 and 2.2.2] show, σ_l is irreducible and is independent of the choice of $\mathfrak{m}$. [CG90, Theorems 2.2.3 and 2.2.4] continue to show that two representations σ_l and $\sigma_{l'}$ are unitary equivalent if and only if $l' \in \mathcal{O}_l$ and that every $\sigma \in \hat{G}$ is of the form $\sigma \cong \sigma_l$ for an $l \in i\mathfrak{g}^*$.

Below we want to provide the essential ingredients for these proof which will also be important for the proof of our main result in the next section.
The proof is by induction on the dimension of G. The inductive step is based an the following statement.

Proposition 3.17 (see [CG90, Proposition 1.3.4]). *Let* $\mathfrak{g}_0$ *be a subalgebra of codimension 1 in a nilpotent Lie algebra* $\mathfrak{g}$, *let* $l \in i\mathfrak{g}^*$, *and let* $l_0 = l\big|_{\mathfrak{g}_0}$. *Let* $\mathfrak{r}_l$ *be the radical defined in Equation (3.6). Then there are two mutually exclusive possibilities:*

- ***Case I*** *characterized by any of the following equivalent properties:*
  - *(i)* $\mathfrak{r}_l \notin \mathfrak{g}_0$;
  - *(ii)* $\mathfrak{r}_l \supset \mathfrak{r}_{l_0}$;
  - *(iii)* $\mathfrak{r}_{l_0}$ *of codimension 1 in* $\mathfrak{r}_l$.

  In this case, if $\mathfrak{m}$ *is a polarizing subalgebra for* l, *then* $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{g}_0$ *is a polarizing subalgebra for* l_0; $\mathfrak{m}_0$ *is of codimension 1 in* $\mathfrak{m}$ *and* $\mathfrak{m} = \mathfrak{r}_l + \mathfrak{m}_0$.

- ***Case II*** *characterized by any of the following equivalent properties:*
  - *(i)* $\mathfrak{r}_l \subset \mathfrak{g}_0$;
  - *(ii)* $\mathfrak{r}_l \subset \mathfrak{r}_{l_0}$;
  - *(iii)* $\mathfrak{r}_l$ *of codimension 1 in* $\mathfrak{r}_{l_0}$.

  In this case, any polarizing subalgebra for l_0 *is also polarizing for* l.

Even though this is a rather technical result its significance becomes clearer in the next statements since we also know how the irreducible representations and the orbits of G and G_0 are connected in these two cases.

Theorem 3.18 (see [CG90, Theorem 2.5.1]). *Let the notation be as above. Let* $p : i\mathfrak{g}^* \to i\mathfrak{g}_0^*$ *be the canonical projection and* $G_0 = \exp(\mathfrak{g}_0)$.

(i) In Case I, where $\mathfrak{r}_l \notin \mathfrak{g}_0$, *we have*

$$\sigma_{l_0} \cong \sigma_l\big|_{G_0} \quad \textit{and} \quad p : \mathcal{O}_l \to \mathcal{O}_{l_0} := \mathrm{Ad}^*(G_0)l_0 \ \textit{is a bijection}$$

(see Figure 3).

(ii) In Case II, where $\mathfrak{r}_l \subset \mathfrak{g}_0$, *we have*

$$\sigma_l \cong \mathrm{Ind}_{G_0}^{G}(\sigma_{l_0}), \quad p(\mathcal{O}_l) = \bigsqcup_{t \in \mathbb{R}} (\mathrm{Ad}^* \exp tX)\mathcal{O}_{l_0} \quad \textit{and} \quad \mathcal{O}_l = p^{-1}(p(\mathcal{O}_l)),$$

where X *is any element such that* $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_0$.

The common induction procedure in setting of nilpotent Lie groups is the following

Procedure 3.19 (Induction procedure on $\dim G$).

The base case consists of $\dim G = 1, 2$ which means that the group is abelian and therefore the orbits are zero-dimensional and all irreducible representations are one-dimensional (characters).

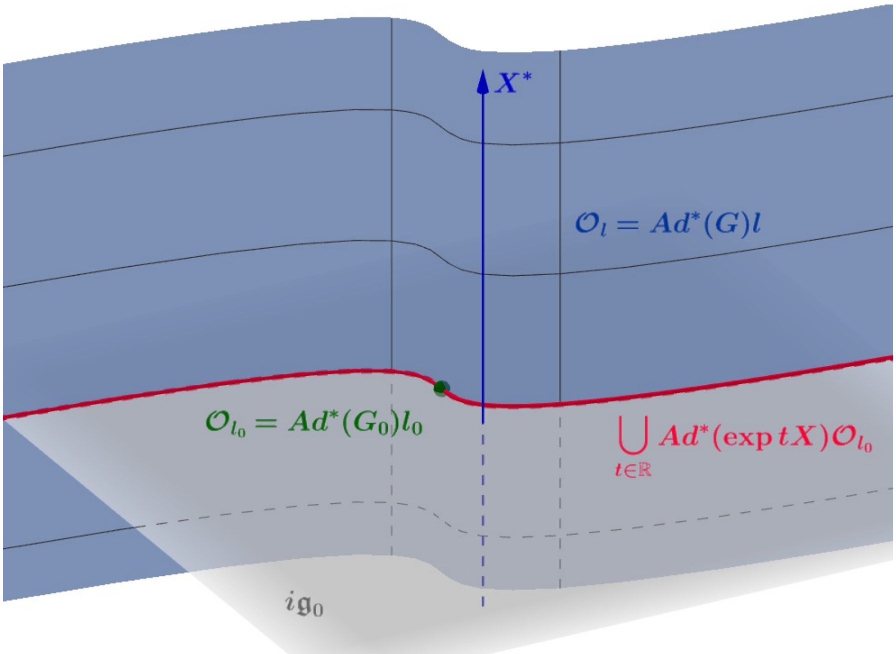


Figure 3: Orbits of G_0 and G in Case II

In the inductive step one distinguishes the two cases

(i) There exists $Z \in \mathfrak{z}(\mathfrak{g})$ with $l(Z) = 0$. i.e. either $\dim(\mathfrak{z}(\mathfrak{g})) > 1$ and $l \in i\mathfrak{g}^*$ arbitrary or $\dim(\mathfrak{z}(\mathfrak{g})) = 1$ and l is trivial on the center.
We define $\mathfrak{g}_0 = \mathfrak{g}/(\mathbb{R}\cdot Z)$ and $G_0 = G/\exp(\mathbb{R}\cdot Z)$ which unfortunately do not have to be a subalgebra and a subgroup, respectively. But every element of $\mathcal{O}_l \subset i\mathfrak{g}^*$ is trivial on $\mathbb{R}\cdot Z$ so that it can be regarded as a subset of $i\mathfrak{g}_0^*$ which coincides with $\mathcal{O}_{l_0} = \mathrm{Ad}^*(G_0)l_0$. Since l is trivial on Z every polarizing subalgebra $\mathfrak{m} \subset \mathfrak{g}$ contains $\mathbb{R}\cdot Z$ and $\mathfrak{m}/(\mathbb{R}\cdot Z)$ is a polarizing subalgebra for l_0. With the projection $P : G \to G_0$ we also know $\sigma_{l_0} \circ P \cong \sigma_l$. Thus, we are in a similar situation as in Case II of Theorem 3.18.

(ii) The center is $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\cdot Z$, i.e. one-dimensional, and $l(Z) \neq 0$.
By Kirillov's Lemma 3.6 there exist $X, Y \in \mathfrak{g}$ and an ideal $\mathfrak{g}_0 \subset \mathfrak{g}$ such that we have $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_0$ and $[X, Y] = Z$. Thus $X \notin \mathfrak{r}_l$ and for $tX + X_0 \in \mathfrak{r}_l$, $X_0 \in \mathfrak{g}_0$, we have $0 = l([tX + X_0, Y]) = l(tZ) = tl(Z)$ which implies $t = 0$, i.e. $\mathfrak{r}_l \subset \mathfrak{g}_0$ and we are in Case II of Proposition 3.17 and Theorem 3.18.

This induction procedure is slightly modified in the proof of Theorem 3.2 but still relies on the statements of the previous Theorem 3.18.

Consequently, we now turn to the study of the coadjoint orbits as Theorem 3.2 tells us that this suffices in order to obtain information about the irreducible unitary representations of G.

For the definition of orbit types, the construction of the cross-sections Σ_d and the parametrization of all orbits as stated in Theorem 3.5 we follow [CG90, Chapter 3.1]: Let $\{X_1, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ and $\{e_1, \ldots, e_n\}$ its dual basis in $\mathfrak{g}^*$. We define for $j = 1, \ldots, n$ the subspaces $V_j := i\,\mathrm{span}_{\mathbb{R}}(e_{j+1}, \ldots, e_n) \subset i\mathfrak{g}^*$. By the definition of the strong Malcev basis the V_j are $\mathrm{Ad}^*(G)$-invariant subspaces. Hence, G acts on each $i\mathfrak{g}^*/V_j$ and the canonical projection $p_j : i\mathfrak{g}^* \to i\mathfrak{g}^*/V_j$ is equivariant. Let

$$d_j(l) := \dim(p_j(\mathcal{O}_l)) = \dim(G \cdot p_j(l)), \quad l \in \mathfrak{g}^*,\ 1 \le j \le n.$$

Then $d(l) := (d_1(l), \ldots, d_n(l))$ is a non decreasing finite sequence of integers and the jumps, if any, are of size 1. For any $d \in \mathbb{N}^n$ define

$$U_d := \{l \in i\mathfrak{g}^* \mid d_j(l) = d_j,\ 1 \le j \le n\}, \quad D := \{d \in \mathbb{N}^n \mid U_d \neq \varnothing\}.$$

The set D is finite, each U_d is $\mathrm{Ad}^*(G)$-invariant by definition and for all $l \in U_d$ the orbits in $i\mathfrak{g}^*/V_j$, $1 \le j \le n$, have the same dimension. For $d \in D$ define

$$S(d) := \{1 \le j \le n \mid d_j = 1 + d_{j-1}\}, \quad T(d) := \{1 \le j \le n \mid d_j = d_{j-1}\}$$

(declaring $d_0 = 0$). Then $\{1, \ldots, n\} = S(d) \sqcup T(d)$ and we define further

$$V_{S(d)} := i\,\mathrm{span}_{\mathbb{R}}\{e_j : j \in S(d)\} \cong \mathbb{R}^k, \quad V_{T(d)} := i\,\mathrm{span}_{\mathbb{R}}\{e_j : j \in T(d)\} \cong \mathbb{R}^{n-k}.$$

With these definitions one can now prove the following theorem which is the detailed version of our third main result Theorem 3.5:

Theorem 3.20 (see [CG90, Theorem 3.1.14]). *There exists an ordering of the finitely many elements of D, i.e. $D = \{d^{(1)} > d^{(2)} > \ldots\}$, such that the following hold.*

(i) For each $d \in D$ the set $W_d := \bigcup_{d' \ge d} U_{d'}$ is G-invariant and Zariski-open in $i\mathfrak{g}^*$. In particular each U_d is the difference of two Zariski-open sets in $i\mathfrak{g}^*$.

(ii) The first element $d^{(1)}$ in the set is given by

$$d_j^{(1)} = \text{ maximal dimension of } G\text{-orbits in } i\mathfrak{g}^*/V_j$$

for $1 \le j \le n$ and $U_{d^{(1)}}$ is Zariski-open in $i\mathfrak{g}^*$.

(iii) Each G-orbit in U_d meets $V_{T(d)}$ in exactly one point. The cross-section

$$\Sigma_d := U_d \cap V_{T(d)}$$

is the difference of two Zariski-open sets. The disjoint union

$$\Sigma := \bigsqcup_{d \in D} \Sigma_d$$

is a cross-section for all G-orbits in $i\mathfrak{g}^*$.

(iv) *For each* $d \in D$ *there exists a birational, non-singular, surjective map*

$$\psi_d \colon \Sigma_d \times V_{S(d)} \to U_d$$

such that for each $l \in \Sigma_d$ *its orbit is* $\mathcal{O}_l = \psi_d\left(l, V_{S(d)}\right)$.

(v) *Let* $S(d) = \{j_1 < \ldots < j_k\}$. *If* $\psi_d(l,v) = \sum P_j^d(l,v)e_j$, *then for fixed* $l \in \Sigma_d$ *the function* P_j^d *is polynomial in* v *and only depends on the* v_i *with* $j_i \leq j$. *Moreover, we have*

a) $P_{j_i}^d(l,v) = v_i$ *for* $1 \leq i \leq k$,

b) $P_j^d(l,v) = l_j + R(l_1, \ldots, l_{j-1}, v_1, \ldots, v_i)$ *if* $j \in T(d)$ *and* i *is the largest integer such that* $j_i < j$. R *is rational. In particular,* $P_1^d(l,v) = l_1$.

Furthermore, $\mathcal{O}_l$ *is the graph of the polynomial map* $p_{T(d)} \circ \psi_d(l,\cdot)$ *where* $p_{T(d)}$ *denotes the projection of* $i\mathfrak{g}^*$ *onto* $V_{T(d)}$ *along* $V_{S(d)}$.

Remark 3.21. *Since* $\mathcal{O}_l \cong V_{S(d)}$ *is even dimensional (see Lemma 3.14) we know that the number of elements in* $S(d)$ *is even.*

Definition 3.22 (Basis realization of σ_l)**.** Let $\sigma_l \in \hat{G}$, $l \in i\mathfrak{g}^*$, $\mathfrak{m}$ a polarizing subalgebra for l and $\{X_1, \ldots, X_m, \ldots, X_n\}$ a weak Malcev basis through $\mathfrak{m}$, $m = \dim \mathfrak{m}$.
Set $k = \frac{1}{2} \dim \mathcal{O}_l = n - m$ and the map

$$\beta : \mathbb{R}^k \to G, t \mapsto \exp(t_1 X_{m+1}) \cdots \exp(t_k X_n).$$

Then $\beta(\mathbb{R}^k)$ is a cross-section for $M \backslash G$, $M = \exp \mathfrak{m}$, and the map carries the Lebesgue measure on $\mathbb{R}^k$ to a G-invariant measure on $M \backslash G$ (see [CG90, Theorem 1.2.12]).
The map $(m,t) \mapsto m \cdot \beta(t)$ is a homeomorphism $M \times \mathbb{R}^k \simeq G$ that allows us to define a natural isometry onto the Hilbert space $\mathcal{H}_l$ of σ_l:

$$J : L^2\left(\mathbb{R}^k\right) \to \mathcal{H}_l, \ (Jf)(m\beta(t)) = \chi_l(m) f(t) \quad \forall\, m \in M, t \in \mathbb{R}^k,$$

where $\chi_l(\exp Y) = e^{2\pi l(Y)}$, $Y \in \mathfrak{m}$, is the character from which σ_l is induced. Hence, we obtain a equivalent representation on $L^2(\mathbb{R}^k)$ which we call a *basis realization* of σ_l.

Remark 3.23. *For* $d \in D \subset \mathbb{N}^n$ *the basis realization from above gives us*

$$\mathcal{H}_l \cong L^2\left(\mathbb{R}^{d_n/2}\right) \quad \forall\, l \in \Sigma_d,$$

since $d_n = \dim \mathcal{O}_l$ *for all* $l \in \Sigma_d$.

Definition 3.24. If $S(d) = \{i_1, \ldots, i_{2k}\}$ denote by $\mathrm{Pf}_d(l)$ the *Pfaffian form* of the skew-symmetric matrix $\left(l([X_{i_j}, X_{i_m}])_{1 \leq j,m \leq 2k}\right)$, i.e. $\mathrm{Pf}_d(l)^2 = \det\left(l([X_{i_j}, X_{i_m}]\right)$.

One can show that it is an $\mathrm{Ad}^*(G)$-invariant polynomial on $i\mathfrak{g}^*$ which vanishes nowhere on U_d (see [CG90, Corollary 4.3.8] and preceding remarks).

At last we will study the trace of the integrated irreducible representations. We elaborate on our second main result Theorem 3.4 and give an estimate for the character's dependence on $l \in \Sigma_d$.

Recall that the integrated representation $\sigma(\phi)$ for $\phi \in \mathcal{S}(G)$ is of trace class. Here we use the following

Definition 3.25 (Schwartz functions)**.** We define $\mathcal{S}(G)$ to be the functions on G that correspond to $\mathcal{S}(\mathbb{R}^n)$ under a polynomial coordinate map $\phi : \mathbb{R}^n \to G$ (see Definition 3.11). Note that this does not depend on the particular choice of ϕ (see [CG90, Lemma A.2.1]).

Now we study the resulting tempered distribution (see Theorem 3.4)

$$\theta_\sigma(\phi) = \mathrm{Tr}(\sigma(\phi)) = \int_{\mathcal{O}_\sigma} \mathcal{F}^+(\phi)(y)\, d\vartheta_\sigma(y)$$

by using the parametrization of the coadjoint orbits. If $\sigma = \sigma_l$, $l \in i\mathfrak{g}^*$, by the identification $\hat{G} \cong i\mathfrak{g}^*/G$ we may also write $\theta_l, \vartheta_l, \mathcal{O}_l$ and so on.

Proposition 3.26 (see [CG90, Proposition 4.2.2])**.** *If we take the standard basis realizations of σ_l in $L^2\left(\mathbb{R}^k\right)$ relative to the weak Malcev basis $\{X_1, \dots, X_n\}$ through the polarization $\mathfrak{m}$, the kernel K_ϕ of the trace class operator $\sigma(\phi)$, $\phi \in \mathcal{S}(G)$, has the form*

$$K_\phi(s,t) = \int_M \chi_l(m)\phi\left(\beta(s)^{-1}m\beta(t)\right) dm \quad \textit{(absolutely convergent)},$$

where $\chi_l(\exp Y) = e^{2\pi l(Y)}$ for $Y \in \mathfrak{m}$ and $\beta\colon \mathbb{R}^k \to G, t \mapsto \exp(t_1 X_{p+1}) \cdots \exp(t_k X_n)$ and $p = \dim \mathfrak{m} = n - k$.

As each orbit $\mathcal{O}_\sigma$ is a $2k$-dimensional symplectic manifold, $k \in \mathbb{N}$, with the non-degenerate skew symmetric 2-form $\omega_\sigma(l) \in \Lambda^2(T\mathcal{O}_\sigma)_l$ such that

$$\omega_\sigma(l)(-(\mathrm{ad}^* X)l, -(\mathrm{ad}^* Y)l) = l([X,Y]), \quad l \in \mathcal{O}_\sigma,$$

it carries a canonical measure $\mu_\sigma = \omega_\sigma \wedge \ldots \wedge \omega_\sigma$ (k factors). With this we can specify the measure ϑ_σ from the character formula in Theorem 3.4:

$$\theta_\sigma(\phi) = \frac{1}{2^k k!} \int_{\mathcal{O}_\sigma} \mathcal{F}^+(\phi)(l)\, d\mu_\sigma(l) \quad \forall\, \phi \in \mathcal{S}(G).$$

Another way to obtain the measure ϑ_σ is by defining m_1' to be the Lebesgue measure on $V_{S(d)}$, normalized such that the cube determined by $\{ie_{i_1}, \dots, ie_{i_{2k}}\}$ has measure 1. Then $\nu_l = (\psi_d(l,\cdot))_* (m_1')$ is an $\mathrm{Ad}^*(G)$-invariant measure on the orbit $\mathcal{O}_l = \left\{\psi_d(l,v) \mid v \in V_{S(d)}\right\}$ by [CG90, Proposition 3.1.10] (where Theorem 3.20 (v) a) is used to show its invariance). Our goal is now to describe the measure ϑ_l in terms of the measure m_1'. We start by choosing a basis for $\mathfrak{g}$ through $\mathfrak{r}_l$:

Lemma 3.27 (see [CG90, Lemma 4.3.6]). *Let $\{X_1,\dots,X_n\}$ be the strong Malcev basis of $\mathfrak{g}$ as above and $S(d)=\{i_1,\dots,i_{2k}\}$. If $\{Y_1,\dots Y_r\}$ is a weak Malcev Basis for the radical $\mathfrak{r}_l$ of $l\in\Sigma_d$, then $\{Y_1,\dots,Y_r,X_{i_1},\dots,X_{i_{2k}}\}$ is a weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{r}_l$.*

The choice of this basis is convenient since it contains the same Lie algebra elements $\{X_1,\dots,X_{i_{2k}}\}\subset\mathfrak{g}$ independent of $l\in\Sigma_d$, even though the radical $\mathfrak{r}_l$ highly depends on l.

Lemma 3.28. *Let $l\in\Sigma_d$. Then*

$$\vartheta_l=|\operatorname{Pf}_d(l)|^{-1}\left(\psi_d(l,\cdot)\right)_*(m_1').$$

Proof. For the proof we put together different excerpts from [CG90, Sections 4.2, 4.3]. We start off with another way to obtain an $\operatorname{Ad}^*(G)$-invariant measure on $\mathcal{O}_l$ which turns out to be ϑ_l: Let m_1 be the Eucledian measure on $\mathbb{R}^{2k}$ such that the unit cube has mass 1. With the weak Malcev basis $\{Y_1,\dots,Y_r,X_{i_1},\dots,X_{i_{2k}}\}$ for $\mathfrak{g}$ through $\mathfrak{r}_l$ from the previous Lemma 3.27 we use the basis $\{\overline{X}_{i_j}:=X_{i_j}+\mathfrak{r}_l\}_{1\le j\le 2k}$ to identify $\mathfrak{g}/\mathfrak{r}_l$ with $\mathbb{R}^{2k}$. The bilinear form $\overline{B}_l(\overline{X},\overline{Y}):=l([X,Y])$ is well-defined on $\mathfrak{g}/\mathfrak{r}_l$ by definition of $\mathfrak{r}_l$. Denote by $\tilde{B}_l$ the corresponding bilinear form on $\mathbb{R}^{2k}$ and define for a Euclidean measure m on $\mathbb{R}^{2k}$ the Fourier transform

$$\mathcal{F}_{\tilde{B}_l}\phi(v)=\int_{\mathbb{R}^{2k}}\phi(v)e^{2\pi\tilde{B}_l(v,v')}\,dm(v'),\quad \phi\in\mathcal{S}(\mathbb{R}^{2k}).$$

Then there exists a so-called self-dual measure such that $\|\mathcal{F}_{\tilde{B}_l}\phi\|_{L^2(m)}=\|\phi\|_{L^2(m)}$.
By [CG90, Lemma 4.3.2] we know $m=|\operatorname{Pf}_d(l)|m_1$. Furthermore, the surjective map $f_l\colon\mathbb{R}^{2k}\to\mathcal{O}_l$ with $f_l(x)=\operatorname{Ad}^*(\exp x_1X_1\cdots\exp x_{2k}X_{2k})^{-1}l$ transforms m to ϑ_l, that is

$$\vartheta_l=(f_l)_*(m)=|\operatorname{Pf}_d(l)|(f_l)_*(m_1)$$

(see [CG90, Theorems 4.2.5 and 4.3.3]).
In addition to that the diffeomorphism $p_{S(d)}\circ f_l$ maps m_1 to a scalar multiple of m_1' after identifying $V_{S(d)}\cong\mathbb{R}^{2d}$ via the basis $\{ie_{r_j}\}_{1\le j\le 2k}$. In order to determine the scalar we differentiate

$$\begin{aligned}\langle d(p_{S(d)}\circ f_l)_0(X_{r_j}),X_{r_m}\rangle &= \lim_{t\to0}\left\langle\frac{(\operatorname{Ad}^*\exp(-tX_{r_j}))l-l}{t},X_{r_m}\right\rangle_{i\mathfrak{g}^*,\mathfrak{g}}\\ &= \langle(\operatorname{ad}^*(-X_{r_j}))l,X_{r_m}\rangle_{i\mathfrak{g}^*,\mathfrak{g}}=l([X_{r_j},X_{r_m}]),\end{aligned}$$

where $r_j,r_m\in S(d)$. This shows that $(p_{S(d)}\circ f_l)_*(m_1)=|\operatorname{Pf}_d(l)|^{-2}m_1'$ by definition of the Pfaffian and therefore

$$(f_l)_*(m_1)=|\operatorname{Pf}_d(l)|^{-2}\nu_l,$$

since the orbit $\mathcal{O}_l$ projects diffeomorphically onto $V_{S(d)}$ under $p_{S(d)}$ (see [CG90, Proposition 4.3.7]). □

With this description of the measure ϑ_l we obtain a useful upper bound for the trace which shows its dependence on the element $l\in i\mathfrak{g}^*$.

Lemma 3.29. *There exists a continuous map $\gamma_d : \mathcal{S}(G) \to \mathbb{R}$, $d \in D$, such that*

$$\forall\ l \in \Sigma_d,\ \phi \in \mathcal{S}(G)\ :\quad |\theta_l(\phi)| \le \gamma_d(\phi) \cdot |\operatorname{Pf}_d(l)|^{-1}.$$

Proof. We choose an inner product on $i\mathfrak{g}^*$ such that the decomposition $i\mathfrak{g}^* = V_{S(d)} \oplus V_{T(d)}$ is orthogonal. With the projection $p_{T(d)}$ we have due to Theorem 3.20 (v) a)

$$\|\psi_d(l,v)\|^2 = \|v + p_{T(d)}(\psi_d(l,v))\|^2 \ge \|v\|^2 \quad \forall\ v \in V_{S(d)}.$$

Since $\phi \in \mathcal{S}(G)$ we have $\mathcal{F}^+(\phi) \in \mathcal{S}(i\mathfrak{g}^*)$ and therefore

$$\|(1 + \|\cdot\|^2)^q \mathcal{F}^+(\phi)\|_\infty =: \alpha_q(\mathcal{F}^+(\phi)) < \infty \quad \forall\ q \in \mathbb{N}.$$

Now fix $q \in \mathbb{N}$ with $2q > \dim(V_{S(d)}) = 2k$. Then $\int_{V_{S(d)}} (1 + \|v\|^2)^{-q}\, dm_1'(v) =: I_q < \infty$ which will depend on d. We choose $\gamma_d(\phi) := \alpha_q(\mathcal{F}^+(\phi)) \cdot I_q$ and compute with the above

$$\begin{aligned}
|\theta_l(\phi)| &\le \int_{\mathcal{O}_l} |\mathcal{F}^+(\phi)(\tilde{l})|\, d\vartheta_l(\tilde{l}) = \int_{V_{S(d)}} |\mathcal{F}^+(\phi)(v + p_{T(d)}(\psi_d(l,v)))| \cdot |\operatorname{Pf}_d(l)|^{-1}\, dm_1'(v) \\
&\le \alpha_q(\mathcal{F}^+(\phi)) \int_{V_{S(d)}} (1 + \|v + p_{T(d)}(\psi_d(l,v))\|^2)^{-q} \cdot |\operatorname{Pf}_d(l)|^{-1}\, dm_1'(v) \\
&\le \alpha(\mathcal{F}^+(\phi)) \cdot \int_{V_{S(d)}} (1 + \|v\|^2)^{-q}\, dm_1'(v) \cdot |\operatorname{Pf}_d(l)|^{-1} = \gamma_d(\phi) \cdot |\operatorname{Pf}_d(l)|^{-1}.
\end{aligned}$$

The claim then follows since the Fourier transform $\mathcal{F}^+$ is continuous and $\alpha_q(\mathcal{F}^+(\phi)) \to 0$ if $\mathcal{F}^+(\phi) \to 0$ in $\mathcal{S}(i\mathfrak{g}^*)$. □

3.2 Heisenberg group

Let $G = H_n$ the Heisenberg group and $\mathfrak{g} = \mathfrak{h}_n$ the $(2n+1)$-dimensional Heisenberg algebra with basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ whose pairwise brackets are equal to zero except for $[X_j, Y_j] = Z$, $1 \le j \le n$ (see examples throughout [CG90]). One realization as a matrix algebra is given by

$$zZ + \sum_{j=1}^{n} (x_j X_j + y_j Y_j) = \begin{pmatrix} 0 & x & \ldots & x_n & z \\ & \cdot & & & y_1 \\ & & \cdot & & \vdots \\ & & & \cdot & y_n \\ 0 & & & & 0 \end{pmatrix} \in \mathfrak{h}_n,$$

$$(z,y,x) := \exp(zZ + \sum_{j=1}^{n} (x_j X_j + y_j Y_j)) = \begin{pmatrix} 1 & x & \ldots & x_n & z + \frac{1}{2} x \cdot y \\ 0 & 1 & & & y_1 \\ & & \ddots & & \vdots \\ & & & 1 & y_n \\ 0 & & & 0 & 1 \end{pmatrix} \in H_n.$$

A strong Malcev basis for $\mathfrak{h}_n$ is given by $\{Z, Y_1, \dots Y_n, X_1, \dots X_n\}$ and let $\{Z^*, Y_1^*, \dots Y_n^*, X_1^*, \dots X_n^*\}$ be the corresponding dual basis of $i\mathfrak{h}_n^*$.

For $l = \gamma Z^* + \sum_{j=1}^n (\beta_j Y_j^* + \alpha_j X_j^*) =: l_{\alpha,\beta,\gamma} \in i\mathfrak{g}^*$ one can compute

$$\mathrm{Ad}^*(z,y,x)l = \gamma Z^* + \sum_{j=1}^n ((\beta_j - \gamma x_j)Y_j^* + (\alpha_j + \gamma y_j)X_j^*) = l_{\alpha+\gamma y, \beta-\gamma x, \gamma}.$$

Thus, we have the following coadjoint orbits:

i) $\mathcal{O}_\gamma := \mathrm{Ad}^*(G) l_{0,0,\gamma} = \{ l_{\alpha',\beta',\gamma} \mid \alpha', \beta' \in \mathbb{R}^n \} \cong \mathbb{R}^{2n}$ for $\gamma \neq 0$,

ii) $\mathcal{O}_{\alpha,\beta} := \mathrm{Ad}^*(G) l_{\alpha,\beta,0} = \left\{ \sum_{j=1}^n (\beta_j Y_j^* + \alpha_j X_j^*) \right\} \cong \mathbb{R}^0$ for $\alpha, \beta \in \mathbb{R}^n$.

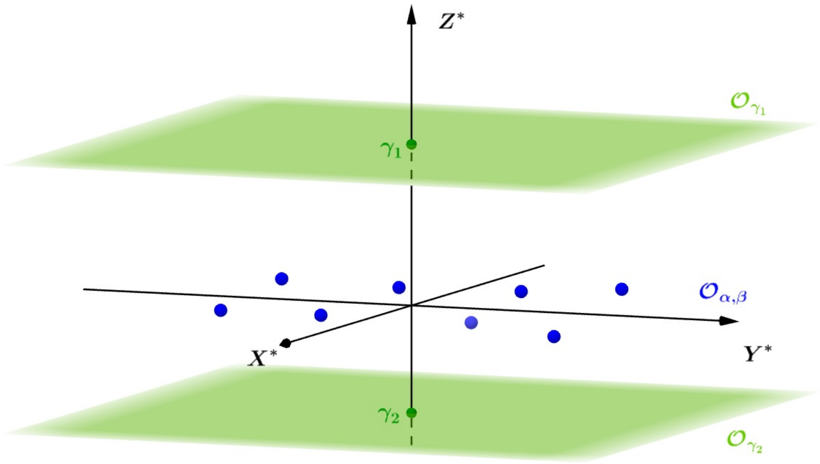


Figure 4: Coadjoint orbits of H_1 in $i\mathfrak{h}_1^* \cong \mathbb{R}^3$.

Now we determine the orbit types and cross-sections from Theorem 3.20:
Following the definitions preceding this theorem we have with the choice of the Malcev basis $i\mathfrak{h}_n^*/V_1 \cong \mathbb{R} \cdot Z^*$ and

$$i\mathfrak{h}_n^*/V_j \cong \mathrm{span}\{Z^*, Y_1^*, \dots, Y_{j-1}^*\} \qquad \text{for } 2 \le j \le n+1,$$
$$i\mathfrak{h}_n^*/V_{n+m} \cong \mathrm{span}\{Z^*, Y_1^*, \dots, Y_n^*, X_1^*, \dots, X_{m-1}^*\} \qquad \text{for } 2 \le m \le n+1.$$

Thus, projecting onto these quotient we obtain the following orbit types $d^{(1)}, d^{(2)}$:

i) $\forall\ \gamma \neq 0:\quad \dim(p_1(\mathcal{O}_\gamma)) = 0,\quad \dim(p_j(\mathcal{O}_\gamma)) = j$ for $2 \leq j \leq n+1$,
$\dim(p_{n+m}(\mathcal{O}_\gamma)) = n+m$ for $2 \leq m \leq n+1$
$\Rightarrow\quad d^{(1)} = (0,1,2,\dots,2n)$ with jump indices $S(d^{(1)}) = \{2,3,\dots,2n+1\}$,

ii) $\forall\ \alpha,\beta \in \mathbb{R}^n:\quad \dim(p_j(\mathcal{O}_{\alpha,\beta})) = 0\quad \forall\ j = 1,\dots,2n$
$\Rightarrow\quad d^{(2)} = 0 \in \mathbb{N}^{2n+1}$ with jump indices $S(d^{(2)}) = \varnothing$.

The cross-sections are then defined as $\Sigma_d = U_d \cap V_{T(d)}$, $T(d) = \{1,\dots,2n+1\} \smallsetminus S(d)$:

$$\text{i)}\quad U_{d^{(1)}} = \bigcup_{\gamma\neq 0} \mathcal{O}_\gamma = \{l \in i\mathfrak{h}_n^* \mid l(Z) \neq 0\} \qquad \Rightarrow \qquad \Sigma_{d^{(1)}} = (\mathbb{R} \smallsetminus \{0\}) \cdot Z^*,$$

$$\text{ii)}\quad U_{d^{(2)}} = \bigcup_{\alpha,\beta} \mathcal{O}_{\alpha,\beta} = \{l \in i\mathfrak{h}_n^* \mid l(Z) = 0\} \qquad \Rightarrow \qquad \Sigma_{d^{(2)}} = \operatorname{span}\{Y_1^*,\dots,X_n^*\}.$$

With these we can find the parametrizations $\psi_d : \Sigma_d \times V_{S(d)} \to U_d$ of the orbits:

$$\text{i)}\quad \psi_1(\gamma Z^*, u) = \gamma Z^* + \sum_{j=1}^n u_j Y_j^* + u_{n+j} X_j^*,\quad u \in V_{S_1} = \operatorname{span}\{X_1^*,\dots,Y_n^*\} \cong \mathbb{R}^{2n},$$

$$\text{ii)}\quad \psi_2 = \mathrm{Id}_{\Sigma_2} \quad \text{since } V_{S_2} = \{0\}.$$

Finally, we can also find the Pfaffian for these two cases:

$$\text{i)}\ \mathrm{Pf}_1(\gamma Z^* + \sum_{j=1}^n (\beta_j Y_j^* + \alpha_j X_j^*)) = \gamma^n\ (\gamma \neq 0 \text{ on } U_{d^{(1)}}),\quad \text{ii)}\ \mathrm{Pf}_2 \equiv 1\ (\text{on } U_{d^{(2)}}).$$

Now, let us turn to the irreducible unitary representations of H_n and find their basis realization:

i) For $l = \gamma Z^* \in \mathcal{O}_\gamma$ we can choose $\mathfrak{m} = \operatorname{span}_{\mathbb{R}}\{Z, Y_1, \dots, Y_n\}$ as a polarizing subalgebra and compute

$$\mathcal{H}_\gamma \cong L^2(\mathbb{R}^n),\quad \sigma_\gamma(z,y,x)f(t) = e^{2\pi i\gamma(z+t\cdot y+\frac{1}{2}x\cdot y)} f(t+x) \text{ for } f \in L^2(\mathbb{R}^n).$$

ii) For $l = \sum_{j=1}^n (\beta_j Y_j^* + \alpha_j X_j^*) \in \mathcal{O}_{\alpha,\beta}$ we have $l([\mathfrak{h}_n, \mathfrak{h}_n]) = 0$ and hence $\mathfrak{r}_l = \mathfrak{h}_n = \mathfrak{m}$. Thus

$$\mathcal{H}_{\alpha,\beta} \cong \mathbb{C},\quad \sigma_{\alpha,\beta}(z,y,x) = \chi_{\alpha,\beta}(z,y,x) = e^{2\pi i(\beta\cdot y+\alpha\cdot x)}.$$

3.3 The Group K_3

Let $G = K_3$ with Lie algebra $\mathfrak{k}_3$ which is given by the basis $\{Z, Y, X, W\}$ whose pairwise brackets are equal to zero except for $[W, X] = Y$ and $[W, Y] = Z$ (see examples throughout [CG90] and [Kir04, Chapter 3 §3]). One realization as a matrix algebra is given by

$$zZ + yY + xX + wW = \begin{pmatrix} 0 & w & 0 & z \\ 0 & 0 & w & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k}_3,$$

$$(z, y, x, w) := \exp(zZ + yY + xX + wW) = \begin{pmatrix} 1 & w & \frac{w^2}{2} & z + \frac{wy}{2} + \frac{w^2x}{6} \\ 0 & 1 & w & y + \frac{wx}{6} \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K_3.$$

A strong Malcev basis for $\mathfrak{k}_3$ is given by $\{Z, Y, X, W\}$ and let $\{Z^*, Y^*, X^*, W^*\}$ be the corresponding dual basis of $i\mathfrak{k}_3^*$.

For $l = \delta Z^* + \gamma Y^* + \beta X^* + \alpha W^* \in i\mathfrak{k}_3$ one can compute

$$\mathrm{Ad}^*(z, y, x, w)l = \delta Z^* + (\gamma - w\delta)Y^* + \left(\beta - w\gamma + \frac{w^2\delta}{2}\right)X^* + \left(\alpha + x\gamma + (y - \frac{wx}{2})\delta\right)W^*.$$

Thus, we have the following coadjoint orbits

i) $\mathcal{O}_{\delta,\beta} = \mathrm{Ad}^*(G)(\delta Z^* + \beta X^*) = \delta Z^* + \left\{tY^* + \left(\beta + \frac{t^2}{2\delta}\right)X^* \mid t \in \mathbb{R}\right\} + \mathbb{R}\cdot W^*$ for $\delta \neq 0, \beta \in \mathbb{R}$ (see Figure 5),

ii) $\mathcal{O}_\gamma = \mathrm{Ad}^*(G)(\gamma Y^*) = \gamma Y^* + \mathbb{R}\cdot X^* + \mathbb{R}\cdot W^* \cong \mathbb{R}^2$ for $\gamma \neq 0$,

iii) $\mathcal{O}_{\alpha,\beta} = \mathrm{Ad}^*(G)(\beta X^* + \alpha W^*) = \{\beta X^* + \alpha W^*\} \cong \mathbb{R}^0$ for $\alpha, \beta \in \mathbb{R}$.

Now we determine the orbit types and cross-sections from Theorem 3.20:
Following the definitions preceding this theorem we have with the choice of the Malcev basis

$$i\mathfrak{h}_n^*/V_1 \cong \mathrm{span}\{Z^*\},\ i\mathfrak{h}_n^*/V_2 \cong \mathrm{span}\{Z^*, Y^*\},\ i\mathfrak{h}_n^*/V_3 \cong \mathrm{span}\{Z^*, Y^*, X^*\},\ i\mathfrak{h}_n^*/V_4 = i\mathfrak{h}_n^*.$$

Thus, projecting onto these quotient we obtain the following orbit types $d^{(1)}, d^{(2)}, d^{(3)}$:

i) $\forall\ \delta \neq 0, \beta \in \mathbb{R}:$ $\left(\dim(p_j(\mathcal{O}_{\delta,\beta}))\right)_{1\le j\le 4} = (0,1,1,2) = d^{(1)}$ with $S(d^{(1)}) = \{2, 4\}$,

ii) $\forall\ \gamma \neq 0:$ $\left(\dim(p_j(\mathcal{O}_\gamma))\right)_{1\le j\le 4} = (0,0,1,2) = d^{(2)}$ with $S(d^{(2)}) = \{3, 4\}$,

iii) $\forall\ \alpha, \beta \in \mathbb{R}^n:$ $\left(\dim(p_j(\mathcal{O}_{\alpha,\beta}))\right)_{1\le j\le 4} = (0,0,0,0) = d^{(3)}$ with $S(d^{(3)}) = \varnothing$.

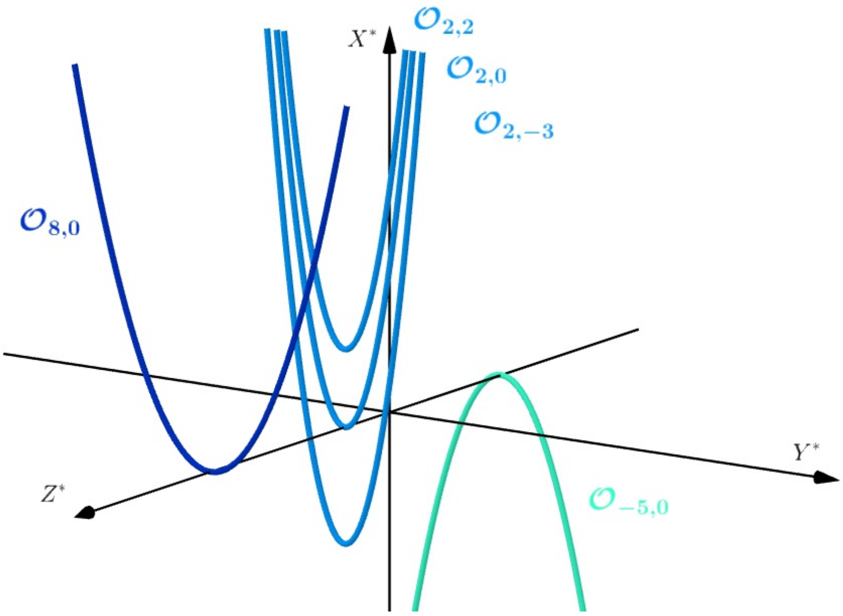


Figure 5: Projection of the coadjoint orbits of type d_1 onto $\mathrm{span}(Z^*, Y^*, X^*) \cong \mathbb{R}^3$.

The cross-sections are then defined as $\Sigma_d = U_d \cap V_{T(d)}$, $T(d) = \{1, \ldots, 2n+1\} \setminus S(d)$:

$$\text{i)} \quad U_{d^{(1)}} = \bigcup_{\delta \neq 0, \beta} \mathcal{O}_{\delta,\beta} = \{l \in i\mathfrak{k}_3^* \mid l(Z) \neq 0\} \quad \Rightarrow \quad \Sigma_{d^{(1)}} = (\mathbb{R} \setminus \{0\}) \cdot Z^* + \mathbb{R} \cdot X^*,$$

$$\text{ii)} \quad U_{d^{(2)}} = \bigcup_{\gamma \neq 0} \mathcal{O}_{\gamma} = \{l \in i\mathfrak{h}_n^* \mid l(Z) = 0, l(Y) \neq 0\} \quad \Rightarrow \quad \Sigma_{d^{(2)}} = (\mathbb{R} \setminus \{0\}) \cdot Y^*,$$

$$\text{iii)} \quad U_{d^{(3)}} = \bigcup_{\alpha, \beta} \mathcal{O}_{\alpha,\beta} = \{l \in i\mathfrak{k}_3^* \mid l(Z) = 0, l(Y) = 0\} \quad \Rightarrow \quad \Sigma_{d^{(3)}} = \mathbb{R} \cdot X^* + \mathbb{R} \cdot W^*.$$

With these we can find the parametrizations $\psi_d : \Sigma_d \times V_{S(d)} \to U_d$ of the orbits:

$$\text{i)} \quad \psi_1(\delta Z^* + \beta X^*, u) = \delta Z^* + u_1 Y^* + (\beta + \frac{u_1^2}{2\delta}) X^* + u_2 W^*,$$
$$u \in V_{S(d^{(1)})} = \mathbb{R}Y^* + \mathbb{R}W^* \cong \mathbb{R}^2 \text{ (see Figure 6)},$$
$$\text{ii)} \quad \psi_2(\gamma Y^*, u) = \gamma Y^* + u_1 X^* + u_2 W^*, \quad u \in V_{S(d^{(2)})} = \mathbb{R}X^* + \mathbb{R}W^* \cong \mathbb{R}^2,$$
$$\text{iii)} \quad \psi_3 = \mathrm{Id}_{\Sigma_{d^{(3)}}} \quad \text{since } V_{S(d^{(3)})} = \{0\}.$$

Finally, we can also find the Pfaffian for these three cases:

$$\text{i)} \ \mathrm{Pf}_1(\delta Z^* + \gamma Y^* + \beta X^* + \alpha W^*) = \delta \quad (\neq 0 \text{ on } U_{d^{(1)}}),$$
$$\text{ii)} \ \mathrm{Pf}_2(\gamma Y^* + \beta X^* + \alpha W^*) = \gamma \quad (\neq 0 \text{ on } U_{d^{(2)}}),$$
$$\text{iii)} \ \mathrm{Pf}_3 \equiv 1 \quad (\text{on } U_{d^{(3)}}).$$

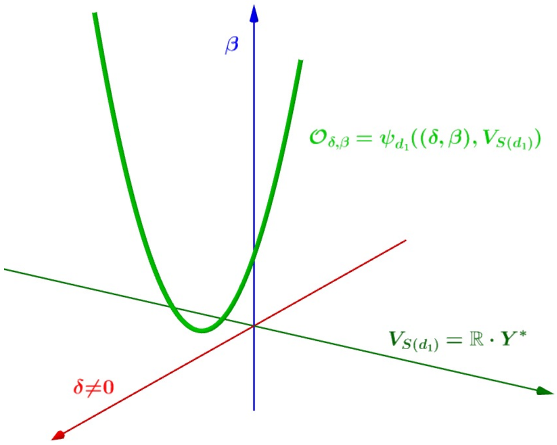


Figure 6: Orbit parametrization (type d_1) projected onto span$(Z^*, Y^*, X^*) \cong \mathbb{R}^3$.

Now, let us turn to the irreducible unitary representations of K_3 and find their basis realization:

i) For $l = \delta Z^* + \beta X^*$ we have $\mathfrak{r}_l = \mathbb{R}\cdot Z + \mathbb{R}\cdot X$ and may choose $\mathfrak{m} = \mathbb{R}\cdot Z + \mathbb{R}\cdot Y + \mathbb{R}\cdot X$ as a polarizing subalgebra since it is an abelian ideal of the correct dimension. One can compute with $\mathcal{H}_{\delta,\beta} \cong L^2(\mathbb{R})$

$$\sigma_{\delta,\beta}(z,y,x,w)f(t) = e^{2\pi i\beta x}e^{2\pi i\delta(z+ty+\frac{1}{2}t^2x+\frac{1}{2}twx+\frac{1}{2}wy+\frac{1}{6}w^2x)}f(t+w) \quad \text{for } f\in L^2(\mathbb{R}).$$

ii) For $l = \gamma Y^*$ we have $\mathfrak{r}_l = \mathbb{R}\cdot Z + \mathbb{R}\cdot Y$ and may choose the same polarizing subalgebra $\mathfrak{m} = \mathbb{R}\cdot Z + \mathbb{R}\cdot Y + \mathbb{R}\cdot X$ as above. One can compute with $\mathcal{H}_\gamma \cong L^2(\mathbb{R})$

$$\sigma_\gamma(z,y,x,w)f(t) = e^{2\pi i\gamma(y+tx+\frac{1}{2}wx)}f(t+w) \quad \text{for } f\in L^2(\mathbb{R}).$$

iii) For $l = \beta X^* + \alpha W^*$ we have $l([\mathfrak{k}_3,\mathfrak{k}_3]) = 0$ and hence $\mathfrak{r}_l = \mathfrak{k}_3 = \mathfrak{m}$. Thus

$$\mathcal{H}_{\alpha,\beta} = \mathbb{C}, \quad \sigma_{\alpha,\beta}(z,y,x,w) = \chi_l(z,y,x,w) = e^{2\pi i(\beta x+\alpha w)}.$$

4 Wave Front Sets of Nilpotent Lie Groups

Let G be a nilpotent, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ of dimension n and $\mathfrak{g}^*$ its vector space dual. By $\hat{G}$ we denote the unitary dual. It is isomorphic to the space of coadjoint orbits $i\mathfrak{g}^*/G$ (for details see Section 3).

Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then we can write

$$\pi \cong \int_{\hat{G}}^{\oplus} \sigma^{\oplus m(\pi,\sigma)} \, d\mu_\pi(\sigma), \quad \mathcal{H} \cong \int_{\hat{G}}^{\oplus} \mathcal{H}_\sigma^{\oplus m(\pi,\sigma)} \, d\mu_\pi(\sigma), \tag{4.7}$$

where $m(\pi, \sigma)$ keeps track of the multiplicity of σ in π. We recall that for such a representation the orbital support of π is given by

$$\mathcal{O} - \operatorname{supp} \pi = \bigcup_{\sigma \in \operatorname{supp}(\pi)} \mathcal{O}_\sigma \subset i\mathfrak{g}^*, \quad \operatorname{supp}(\pi) = \operatorname{supp}(\mu_\pi),$$

where $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$ is the orbit of the coadjoint action corresponding to $\sigma \in \hat{G}$ under the isomorphism $\hat{G} \cong i\mathfrak{g}^*/G$ (see Theorem 3.2).

Our main goal in this section is to prove the following

Theorem 4.1. *Let G be a nilpotent, connected, simply connected Lie group and π a unitary representation of G. Then*

$$\operatorname{WF}(\pi) = \operatorname{AC}(\mathcal{O} - \operatorname{supp} \pi).$$

We start by using the structure of nilpotent Lie groups and the unitary representations. As discussed in Section 3 (see Theorem 3.20) after fixing a strong Malcev basis of $\mathfrak{g}$ we have

$$\hat{G} \;\cong\; i\mathfrak{g}^*/G \;\cong\; \Sigma = \bigsqcup_{d \in D} \Sigma_d \subset i\mathfrak{g}^*,$$

where Σ is a cross-section of all G-orbits and Σ_d is a cross-section of all orbits of a certain type $d \in D$, which, in particular, all have the same dimension. Moreover, the set D is finite.
Thus, we can push μ_π forward to a positive measure on Σ and obtain

$$\begin{aligned} \pi &\cong \int_{\Sigma}^{\oplus} \sigma_l^{\oplus m(\pi,\sigma_l)} \, d\mu_\pi(l) \\ &= \bigoplus_{d \in D} \int_{\Sigma_d}^{\oplus} \sigma_l^{\oplus m(\pi,\sigma_l)} \, d\mu_\pi(l) =: \bigoplus_{d \in D} \pi_d. \end{aligned}$$

With this decomposition we have

$$\operatorname{WF}(\pi) = \bigcup_{d \in D} \operatorname{WF}(\pi_d), \qquad \operatorname{AC}(\mathcal{O} - \operatorname{supp} \pi) = \bigcup_{d \in D} \operatorname{AC}(\mathcal{O} - \operatorname{supp} \pi_d)$$

by Proposition 2.14 and the fact that $\mathrm{AC}\left(\bigcup_{i=1}^n S_i\right) = \bigcup_{i=1}^n \mathrm{AC}(S_i)$.
Therefore, it suffices to show that

$$\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_d) = \mathrm{WF}(\pi_d) \quad \forall\, d \in D. \tag{4.8}$$

From now on we fix $d \in D$ and may assume that all the irreducible representations in the support of π are of the form σ_l for an $l \in \Sigma_d \subset U_d$, where $U_d \subset i\mathfrak{g}^*$ is the set of all $l \in i\mathfrak{g}^*$ such that its orbit $\mathcal{O}_l = \mathrm{Ad}^*(G)l$ is of type d (see Theorem 3.20 (iii) and the preceding definitions).

4.1 Proof of the Inclusion $\mathrm{AC}(\mathcal{O} - \mathrm{supp}(\pi)) \subset \mathrm{WF}(\pi)$

For the first inclusion $\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi) \subset \mathrm{WF}(\pi)$ we use Lemma 2.13, in particular the equivalence of (i) and (iii) which states in our setting here:

$$\begin{aligned} \xi \notin \mathrm{WF}(\pi) \qquad &\Leftrightarrow \qquad \exists\, e \in U \subset G, \xi \in V \subset i\mathfrak{g}^* \ \forall\, \phi \in C_c^\infty(U)\ \exists\, C_N(\phi) > 0 : \\ &|\mathcal{F}(\langle \pi(\bullet)u, v\rangle \phi)(t\eta)| \le C_N(\phi) \|u\| \|v\| t^{-N} \quad \text{for } t \gg 0,\ \eta \in V,\ u, v \in \mathcal{H}, \end{aligned} \tag{4.9}$$

where the constants $C_N(\phi)$ may be chosen independent of both $\eta \in V$ and $u, v \in \mathcal{H}$.

We start by finding matrix coefficients whose Fourier transform is bounded from below.

Proposition 4.2. *Let $0 < \delta < 1$ such that $|\sin(2\pi x)| \le \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1}$ for all $|x| < \delta$. Fix an arbitrary inner product on $\mathfrak{g}$. Then there exists a chart $\kappa : G \to \mathfrak{g}$ with $D_e(\kappa^{-1} \circ \log) = \mathrm{Id}_G$ such that for $\zeta \in U_d \subset i\mathfrak{g}^*$ we can find vectors $u_\zeta \in \mathcal{H}_\zeta^\infty$, $v_\zeta \in \mathcal{H}_\zeta$ with $\|u_\zeta\| = \|v_\zeta\| = 1$ that depend measurably on ζ such that for all $\eta \in U_d$ with $|\eta - \zeta| < 4\delta$ we have the following estimate for all non-negative $\phi \in C_c^\infty(B_{1/4}(0))$:*

$$\mathrm{Re}\left(\int_{\mathfrak{g}} \langle \sigma_\zeta(\kappa^{-1}(X)) u_\zeta, v_\zeta \rangle \phi(X) e^{-2\pi\eta(X)}\, dX \right) \ge \left(\frac{1}{2}\right)^{2\dim \mathfrak{g}} \cdot \int_{\mathfrak{g}} \phi(X)\, dX \ge 0.$$

Proof. We prove this statement by induction on $n = \dim \mathfrak{g}$. If $n = 1, 2$, the group is abelian. In this case the irreducible unitary representations are one-dimensional, i.e. $\sigma_\zeta(g) = e^{2\pi\zeta(\log g)}$, $\mathcal{H}_\zeta = \mathbb{C}$. We choose $\kappa = \log$ and $u_\zeta = v_\zeta = 1$ and compute

$$\begin{aligned} \mathrm{Re}\left(\int_{\mathfrak{g}} \langle \sigma_\zeta(\kappa^{-1}(X)) u_\zeta, v_\zeta \rangle \phi(X) e^{-2\pi\eta(X)}\, dX \right) &= \mathrm{Re}\left(\int_{\mathfrak{g}} e^{2\pi(\zeta-\eta)(X)} \phi(X)\, dX \right) \\ = \int_{\mathfrak{g}} \mathrm{Re}\left(e^{2\pi(\zeta-\eta)(X)} \right) \phi(X)\, dX &= \int_{\mathfrak{g}} \cos(2\pi i(\eta - \zeta)(X)) \phi(X)\, dX \\ \ge \frac{1}{2} \int_{\mathfrak{g}} \phi(X)\, dX \ge \left(\frac{1}{2}\right)^{2\dim \mathfrak{g}} &\cdot \int_{\mathfrak{g}} \phi(X)\, dX, \end{aligned}$$

since $|i((\eta - \zeta)(X)| \le \|\eta - \zeta\| \cdot \|X\| \le 4\delta \cdot \frac{1}{4} = \delta$ on $\operatorname{supp} \phi$ and

$$\cos(2\pi x) = \sqrt{1-\sin(2\pi x)^2} \geq \sqrt{1-\left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1}} > \frac{1}{2} \quad \forall\, |x| < \delta. \tag{4.10}$$

Now we assume $n = \dim \mathfrak{g} \geq 3$. We will distinguish between the two cases following the ideas as described in Procedure 3.19.

Case I: $\dim \mathfrak{z}(\mathfrak{g}) > 1$. There exists $Z \in \mathfrak{z}(\mathfrak{g})$ with $\zeta(Z) = 0$ and $\|Z\| = 1$. We can choose an orthogonal complement $W < \mathfrak{g}$ such that $\mathfrak{g} = W \oplus \mathbb{R}Z$.
Then $\overline{\mathfrak{g}} = \mathfrak{g}/(\mathbb{R}\cdot Z)$ is isomorphic to W and has a well-defined Lie algebra structure given by $[v+\mathbb{R}Z, w+\mathbb{R}Z] = [v,w]_{\mathfrak{g}} + \mathbb{R}Z$ since $Z \in \mathfrak{z}(\mathfrak{g})$. The induction hypotheses gives us $\overline{G} = \kappa^{-1}(\overline{\mathfrak{g}})$ with a chart $\kappa : \overline{G} \to \overline{\mathfrak{g}}$. For G we choose the logarithm as a chart.

On $\overline{\mathfrak{g}}$ we use the inner product induced from the one we fixed on $\mathfrak{g}$. Using the corresponding inner products on $i\mathfrak{g}^*$ and $i\overline{\mathfrak{g}}^*$ we also obtain an orthogonal decomposition $i\mathfrak{g}^* = iW^* \oplus \mathbb{R}\eta_Z \cong i\overline{\mathfrak{g}}^* \oplus \mathbb{R}\eta_Z$ with $\|\eta_Z\| = 1$.

Note that $i\overline{\mathfrak{g}}^*$ is $\mathrm{Ad}^*(G)$-invariant (again due to $Z \in \mathfrak{z}(\mathfrak{g})$). We can identify ζ with an element $\overline{\zeta} \in i\overline{\mathfrak{g}}^*$. Let $\eta = \overline{\eta} + r\eta_Z \in i\mathfrak{g}^* = i\overline{\mathfrak{g}}^* \oplus \mathbb{R}\eta_Z$. By assumption $|r| = |(\eta-\zeta)_Z| \leq 4\delta$.

The induction hypothesis also gives us normalized vectors $u_{\overline{\zeta}} \in \mathcal{H}_{\overline{\zeta}}^\infty$, $v_{\overline{\zeta}} \in \mathcal{H}_{\overline{\zeta}}$. By Procedure 3.19 (i) $\mathcal{H}_{\overline{\zeta}} \cong \mathcal{H}_\zeta$ and $\sigma_{\overline{\zeta}} \circ P \cong \sigma_\zeta$ with the projection $P : G \to \overline{G}$ which is given by $P = \kappa^{-1} \circ \log$ here. Thus, we obtain corresponding vectors $u_\zeta = u_{\overline{\zeta}} \in \mathcal{H}_\zeta^\infty$, $v_\zeta = v_{\overline{\zeta}} \in \mathcal{H}_\zeta$ and compute

$$\begin{aligned}
R &:= \mathrm{Re}\left(\int_{\mathfrak{g}} \langle \sigma_\zeta(\exp(X))u_\zeta, v_\zeta\rangle \phi(X) e^{-2\pi\eta(X)}\, dX\right) \\
&= \mathrm{Re}\left(\int_{\overline{\mathfrak{g}}}\int_{\mathbb{R}} \langle \sigma_\zeta(\exp(\overline{X}+tZ))u_\zeta, v_\zeta\rangle \phi(\overline{X}+tZ) e^{-2\pi\eta(\overline{X}+tZ)}\, d\overline{X}\, dt\right) \\
&= \mathrm{Re}\left(\int_{\overline{\mathfrak{g}}}\int_{\mathbb{R}} \langle \sigma_\zeta(\exp(\overline{X})\exp(tZ))u_\zeta, v_\zeta\rangle \phi(\overline{X}+tZ) e^{-2\pi(\overline{\eta}(\overline{X})+r\eta_Z(tZ))}\, d\overline{X}\, dt\right) \\
&= \mathrm{Re}\left(\int_{\overline{\mathfrak{g}}}\int_{\mathbb{R}} \langle \sigma_{\overline{\zeta}}(\kappa^{-1}(\overline{X}))u_{\overline{\zeta}}, v_{\overline{\zeta}}\rangle \phi(\overline{X}+tZ) e^{-2\pi(\overline{\eta}(\overline{X})+r\eta_Z(tZ))}\, d\overline{X}\, dt\right) \\
&= \int_{\mathbb{R}} \cos(-2\pi rt)\, \mathrm{Re}\left(\int_{\overline{\mathfrak{g}}} \langle \sigma_{\overline{\zeta}}(\kappa^{-1}(\overline{X}))u_{\overline{\zeta}}, v_{\overline{\zeta}}\rangle \phi(\overline{X}+tZ) e^{-2\pi\overline{\eta}(\overline{X})}\, d\overline{X}\right) \\
&\qquad - \sin(-2\pi rt)\, \mathrm{Im}\left(\int_{\overline{\mathfrak{g}}} \langle \sigma_{\overline{\zeta}}(\kappa^{-1}(\overline{X}))u_{\overline{\zeta}}, v_{\overline{\zeta}}\rangle \phi(\overline{X}+tZ) e^{-2\pi\overline{\eta}(\overline{X})}\, d\overline{X}\right) dt.
\end{aligned}$$

Since $|rt| \leq 4\delta|t| \leq \delta$ for $\overline{X}+tZ \in \mathrm{supp}(\phi) \subset B_{1/4}(0)$ we have $\cos(-2\pi rt) > \frac{1}{2}$ as in (4.10) and $|\sin(-2\pi rt)| \leq \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1}$ by assumption. The induction hypothesis grants that the real part is non-negative and we can estimate

$$
\begin{aligned}
R &\geq \int_{\mathbb{R}} \frac{1}{2} \operatorname{Re}\left(\int_{\overline{\mathfrak{g}}} \langle \sigma_{\overline{\zeta}}(\kappa^{-1}(\overline{X})) u_{\overline{\zeta}}, v_{\overline{\zeta}}, \rangle \phi(\overline{X} + tZ) e^{-2\pi \overline{\eta}(\overline{X})} \, d\overline{X} \right) \\
&\quad - |\sin(-2\pi r t)| \left| \int_{\overline{\mathfrak{g}}} \langle \sigma_{\overline{\zeta}}(\kappa^{-1}(\overline{X})) u_{\overline{\zeta}}, v_{\overline{\zeta}}, \rangle \phi(\overline{X} + tZ) e^{-2\pi \overline{\eta}(\overline{X})} \, d\overline{X} \right| \, dt \\
&\geq \int_{\mathbb{R}} \frac{1}{2} \operatorname{Re}\left(\int_{\overline{\mathfrak{g}}} \langle \sigma_{\overline{\zeta}}(\kappa^{-1}(\overline{X})) u_{\overline{\zeta}}, v_{\overline{\zeta}}, \rangle \phi(\overline{X} + tZ) e^{-2\pi \overline{\eta}(\overline{X})} \, d\overline{X} \right) \\
&\quad - \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1} \int_{\overline{\mathfrak{g}}} \|u_{\overline{\zeta}}\| \|v_{\overline{\zeta}}\| \phi(\overline{X} + tZ) \, d\overline{X} \, dt.
\end{aligned}
$$

Now we can apply the induction hypothesis to the inner integral to finish the proof in this case: since $\|u_{\overline{\zeta}}\| = \|v_{\overline{\zeta}}\| = 1$ we obtain

$$
\begin{aligned}
R &\geq \left(\frac{1}{2} \left(\frac{1}{2}\right)^{2\dim \overline{\mathfrak{g}}} - \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1} \right) \cdot \int_{\mathbb{R}} \int_{\overline{\mathfrak{g}}} \phi(\overline{X} + tZ) \, d\overline{X} \, dt \\
&= \frac{3}{4} \left(\frac{1}{2}\right)^{2\dim \mathfrak{g} - 1} \cdot \int_{\mathfrak{g}} \phi(X) \, dX \ \geq \ \left(\frac{1}{2}\right)^{2\dim \mathfrak{g}} \cdot \int_{\mathfrak{g}} \phi(X) \, dX.
\end{aligned}
$$

Case II: $\mathfrak{z}(\mathfrak{g}) = \mathbb{R} \cdot Z$ is one-dimensional. Kirillov's Lemma 3.6 gives us $X, Y \in \mathfrak{g}$ and an ideal $\mathfrak{g}_0 \subset \mathfrak{g}$ with $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_0$ and $[X, Y] = Z$. We may choose X such that the decomposition is orthogonal. Since $\dim(\mathfrak{z}(\mathfrak{g}_0)) > 1$ as $Z, Y \in \mathfrak{z}(\mathfrak{g}_0)$ we are in Case I in the induction hypothesis for G_0 and therefore can use the chart $\log : G_0 \to \mathfrak{g}_0$. We define a chart for G via

$$
\kappa^{-1} : \mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}X \to G, \quad X_0 + tX \mapsto \exp(X_0)\exp(tX).
$$

If $\zeta(Z) = 0$ we can proceed analogously to Case I since for $w = Y_0 + sX \in W$ and $t \in \mathbb{R}$ we have

$$
\begin{aligned}
\kappa^{-1}(w + tZ) &= \kappa^{-1}(Y_0 + tZ + sX) = \exp(Y_0 + tZ)\exp(sX) = \exp(Y_0)\exp(tZ)\exp(sX) \\
&= \exp(Y_0)\exp(sX)\exp(tZ) = \kappa^{-1}(Y_0 + sX)\exp(tZ) = \kappa^{-1}(w)\exp(tZ).
\end{aligned}
$$

Thus, we now assume $\zeta(Z) \neq 0$. Then $X \notin \mathfrak{r}_l$ and we are in Case II of Proposition 3.17 and Theorem 3.18 with $G_0 = \exp(\mathfrak{g}_0) \subset G$ a normal subgroup and $p : i\mathfrak{g}^* \to i\mathfrak{g}_0^*$ the canonical projection. Let $\zeta = \zeta_0 + z\zeta_X, \eta = \eta_0 + r\eta_X \in \ker(p)^\perp \oplus \ker(p)$. Then by assumption $|z - r| = |(\zeta - \eta)_X| \leq 4\delta$.

By Theorem 3.18, we know $\sigma_\zeta \cong \operatorname{Ind}_{G_0}^G(\sigma_{\zeta_0})$ with $\mathcal{H}_\zeta \cong L^2(A, \mathcal{H}_{\zeta_0})$, where $A = \exp(\mathbb{R} \cdot X)$. Thus, if we regard u and v as elements of $L^2(A, \mathcal{H}_{\zeta_0})$ and $\tilde{u}, \tilde{v} : G \to \mathcal{H}_{\zeta_0}$ the corresponding left-G_0-equivariant functions we have

$$\langle \sigma_\zeta(g_0 a)u, v\rangle_{\mathcal{H}_\zeta} = \int_A \langle [\sigma_\zeta(g_0 a)u](b), v(b)\rangle_{\mathcal{H}_{\zeta_0}}\, db \quad \text{and}$$
$$[\sigma_\zeta(g_0 a)\tilde{u}](b) = \tilde{u}(bg_0 a) = \tilde{u}(bg_0 b^{-1} ba) = \sigma_{\zeta_0}(bg_0 b^{-1})\tilde{u}(ba)$$

since $b^{-1}g_0 b \in G_0$ as $\mathfrak{g}_0$ is an ideal. This gives us $[\sigma_\zeta(g_0 a)u](b) = \sigma_{\zeta_0}(bg_0 b^{-1})u(ba)$.

Furthermore, the induction hypothesis gives us measurable, normalized vectors $u_{\zeta_0} \in \mathcal{H}^\infty_{\zeta_0}$, $v_{\zeta_0} \in \mathcal{H}_{\zeta_0}$. In order to find the suitable vectors $u_\zeta, v_\zeta \in \mathcal{H}_\zeta$ we begin with a cut-off function $\chi \in C^\infty_c(A)$ with $0 \le \chi \le 1$, $\chi = 1$ on $\exp\left(\left[-\frac{1}{4}, \frac{1}{4}\right] \cdot X\right)$ and $\|\chi\|_{L^2} = 1$. Define

$$u_\zeta := \chi e^{2\pi z \zeta_X \circ \log} \otimes u_{\zeta_0} \in C^\infty_c(A, \mathcal{H}^\infty_{\zeta_0}), \quad v_\zeta := \delta_e \otimes v_{\zeta_0} \in \mathcal{H}^{-\infty}_\zeta.$$

With these we can compute

$$\begin{aligned}
R &:= \operatorname{Re}\left(\int_{\mathfrak{g}} \langle \sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta\rangle \phi(X) e^{-2\pi\eta(X)}\, dX\right) \\
&= \operatorname{Re}\Bigg(\int_{\mathfrak{g}_0}\int_{\mathbb{R}} \left(\int_A \langle \sigma_{\zeta_0}(b\exp(X_0)b^{-1})u_\zeta(be^{tX}), v_\zeta(b)\rangle\, db\right) \cdot \\
&\qquad\qquad \phi(X_0 + tX)e^{-2\pi(\eta_0(X_0) + r\eta_X(tX))}\, dX_0\, dt\Bigg) \\
&= \operatorname{Re}\Bigg(\int_{\mathfrak{g}_0}\int_{\mathbb{R}} \left(\int_A \langle \sigma_{\zeta_0}(b\exp(X_0)b^{-1})u_{\zeta_0}, v_{\zeta_0}\rangle \chi(be^{tX}) e^{2\pi z\zeta_X(\log(be^{tX}))}\delta_e(b)\, db\right) \cdot \\
&\qquad\qquad \phi(X_0 + tX)e^{-2\pi(\eta_0(X_0) + r\eta_X(tX))}\, dX_0\, dt\Bigg) \\
&= \operatorname{Re}\left(\int_{\mathfrak{g}_0}\int_{\mathbb{R}} \langle \sigma_{\zeta_0}(\exp(X_0))u_{\zeta_0}, v_{\zeta_0}\rangle \chi(e^{tX}) e^{2\pi z\zeta_X(tX)} \phi(X_0 + tX) e^{-2\pi(\eta_0(X_0) + rt)}\, dX_0\, dt\right) \\
&= \int_{\mathbb{R}} \cos(2\pi(z-r)t)\chi(e^{tX}) \operatorname{Re}\left(\int_{\mathfrak{g}_0} \langle \sigma_{\zeta_0}(\exp(X_0))u_{\zeta_0}, v_{\zeta_0}\rangle \phi(X_0 + tX)e^{-2\pi\eta_0(X_0)}\, dX_0\right) \\
&\quad - \sin(2\pi(z-r)t)\chi(e^{tX}) \operatorname{Im}\left(\int_{\mathfrak{g}_0} \langle \sigma_{\zeta_0}(\exp(X_0))u_{\zeta_0}, v_{\zeta_0}\rangle \phi(X_0 + tX)e^{-2\pi\eta_0(X_0)}\, dX_0\right) dt.
\end{aligned}$$

Analogously to Case I we have $|(z-r)t| \le 4\delta|t| \le \delta$ for $\overline{X_0} + tX \in \operatorname{supp}(\phi) \subset B_{1/4}(0)$ and therefore $\cos(2\pi(z-r)t) > \frac{1}{2}$ as in (4.10) and $|\sin(2\pi(z-r)t)| \le \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1}$ by assumption.

Again, the induction hypothesis grants that the real part is non-negative and we can estimate

$$\begin{aligned}
R \ge \int_{\mathbb{R}} &\frac{1}{2}\chi(e^{tX}) \operatorname{Re}\left(\int_{\mathfrak{g}_0} \langle \sigma_{\zeta_0}(\exp(X_0))u_{\zeta_0}, v_{\zeta_0}\rangle \phi(X_0 + tX)e^{-2\pi\eta_0(X_0)}\, dX_0\right) \\
&- |\sin(2\pi(z-r)t)|\chi(e^{tX}) \left|\int_{\mathfrak{g}_0} \langle \sigma_{\zeta_0}(\exp(X_0))u_{\zeta_0}, v_{\zeta_0}\rangle \phi(X_0 + tX)e^{-2\pi\eta_0(X_0)}\, dX_0\right| dt,
\end{aligned}$$

and by unitarity of σ_{ζ_0}:

$$R \geq \int_{\mathbb{R}} \frac{1}{2}\chi(e^{tX})\,\mathrm{Re}\Big(\int_{\mathfrak{g}_0} \langle \sigma_{\zeta_0}(\exp(X_0))u_{\zeta_0}, v_{\zeta_0}\rangle \phi(X_0+tX)e^{-2\pi\eta_0(X_0)}\,dX_0\Big) - \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1}\chi(e^{tX})\int_{\mathfrak{g}_0}\|u_{\zeta_0}\|\|v_{\zeta_0}\|\phi(X_0+tX)\,dX_0\,dt.$$

Now we can apply the induction hypothesis to the inner integral to finish the estimation: since $\|u_{\zeta_0}\| = \|v_{\zeta_0}\| = 1$ we obtain

$$\begin{aligned} R &\geq \left(\frac{1}{2}\left(\frac{1}{2}\right)^{2\dim(\mathfrak{g}_0)} - \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1}\right)\int_{\mathbb{R}}\int_{\mathfrak{g}_0}\chi(e^{tX})\phi(X_0+tX)\,dX_0\,dt \\ &= \frac{3}{2}\left(\frac{1}{2}\right)^{2\dim\mathfrak{g}}\int_{\mathbb{R}}\int_{\mathfrak{g}_0}\phi(X_0+tX)\,dX_0\,dt = \frac{3}{2}\left(\frac{1}{2}\right)^{2\dim\mathfrak{g}}\int_{\mathfrak{g}}\phi(X)\,dX, \end{aligned}$$

where we used that $\chi\circ\exp = 1$ on $\mathrm{supp}\,\phi(X_0+\bullet)$ for all $X_0\in\mathfrak{g}_0$.

However, v_ζ is only a distributional vector. But we can approximate it by smooth vectors: there exists a sequence $(\varphi_n)_n \subset C_c^\infty(A)$ converging to the delta distribution δ_e in $\mathcal{D}'(A)$ with $\|\varphi_n\|_{L^1} = 1$ for all $n\in\mathbb{N}$. We define $v_\zeta^n := \varphi_n \otimes v_{\zeta_0}$ and study the functions

$$m_{u_\zeta, v_\zeta^n}(X) := \langle \sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta^n\rangle \in C^\infty(\mathfrak{g}). \tag{4.11}$$

We can show that on a compact set they have a uniformly convergent subsequence by the Arzela-Ascoli theorem (see [Rud76, Theorem 7.25]) - for details see the next Lemma 4.3. Since $m_{u_\zeta,v_\zeta^n} \to m_{u_\zeta,v_\zeta} := \langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta\rangle \in C^\infty(\mathfrak{g})$ point-wise we have on $\mathrm{supp}\,\phi$:

$$\exists\, N\in\mathbb{N}: \quad \|m_{u_\zeta,v_\zeta^N} - m_{u_\zeta,v_\zeta}\|_{L^\infty(\mathrm{supp}\,\phi)} \leq \left(\frac{1}{2}\right)^{2\dim\mathfrak{g}+1}.$$

We can now choose $v_\zeta^N \in \mathcal{H}_\zeta$ to finish the proof:

$$\begin{aligned} R_N &:= \mathrm{Re}\Big(\int_{\mathfrak{g}}\langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta^N\rangle\phi(X)e^{-2\pi\eta(X)}\,dX\Big) \\ &= \mathrm{Re}\Big(\int_{\mathfrak{g}}\langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta\rangle\phi(X)e^{-2\pi\eta(X)}\,dX\Big) \\ &\quad - \mathrm{Re}\Big(\int_{\mathfrak{g}}\big(\langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta\rangle - \langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta^N\rangle\big)\phi(X)e^{-2\pi\eta(X)}\,dX\Big) \\ &\geq \mathrm{Re}\Big(\int_{\mathfrak{g}}\langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta\rangle\phi(X)e^{-2\pi\eta(X)}\,dX\Big) \\ &\quad - \Big|\int_{\mathfrak{g}}\big(\langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta\rangle - \langle\sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta^N\rangle\big)\phi(X)e^{-2\pi\eta(X)}\,dX\Big|, \end{aligned}$$

and by induction hypothesis and the choice of v_ζ^N:

$$\begin{aligned} R_N \geq & \frac{3}{2}\left(\frac{1}{2}\right)^{2\dim\mathfrak{g}} \cdot \int_{\mathfrak{g}} \phi(X)\,dX - \|m_{u_\zeta,v_\zeta^N} - m_{u_\zeta,v_\zeta}\|_{L^\infty(\operatorname{supp}\phi)} \int_{\mathfrak{g}} \phi(X)\,dX \\ \geq & \left(\frac{1}{2}\right)^{2\dim\mathfrak{g}} \cdot \int_{\mathfrak{g}} \phi(X)\,dX. \end{aligned}$$

This is the desired estimate. □

A technical lemma used in the previous proof:

Lemma 4.3. *Let $K \subset \mathfrak{g}$ be a compact set. Then there exists a uniformly convergent subsequence of the matrix coefficients $m_{u_\zeta,v_\zeta^n}(X) := \langle \sigma_\zeta(\kappa^{-1}(X))u_\zeta, v_\zeta^n\rangle \in C^\infty(K)$, $n \in \mathbb{N}$, defined in the previous proof (see (4.11)).*

Proof. The matrix coefficients are uniformly bounded:

$$\begin{aligned} |m_{u_\zeta,v_\zeta^n}(W)| &= \left| \int_A \langle \sigma_{\zeta_0}(b\exp(W_0)b^{-1})u_{\zeta_0}, v_{\zeta_0}\rangle \chi(be^{W_X})e^{2\pi z\zeta_X(\log(be^{W_X}))}\varphi_n(b)\,db \right| \\ &\leq \|u_{\zeta_0}\|\|v_{\zeta_0}\|\|\chi\|_\infty \int_A |\varphi_n(b)|\,db \;=\; \|\chi\|_\infty \quad \forall\, W = W_0 + W_X \in \mathfrak{g}, n \in \mathbb{N}. \end{aligned}$$

Furthermore, their derivatives are bounded on K:

$$\begin{aligned} \partial_X m_{u_\zeta,v_\zeta^n}(W) &= \frac{d}{dt}\Big|_{t=0} \langle \sigma_\zeta(\kappa^{-1}(W+tX))u_\zeta, v_\zeta^n\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle \sigma_\zeta(\exp(W_0)\exp(W_X)\exp(tX))u_\zeta, v_\zeta^n\rangle \\ &= \langle \sigma_\zeta(\exp(W_0)\exp(W_X))d\sigma_\zeta(X)u_\zeta, v_\zeta^n\rangle. \end{aligned}$$

Here $d\sigma_\zeta(X)u_\zeta(b) = \big((T_b\chi)(X)e^{2\pi z\zeta_X(\log b)} + \chi(b)2\pi z e^{2\pi z\zeta_X(\log b)}\big) \otimes u_{\zeta_0}$ where $T_b\chi$ is the tangent mapping of χ at $b \in A$. With computations as above

$$|\partial_X m_{u_\zeta,v_\zeta^n}(W)| \leq \left\| T_\bullet\chi e^{2\pi z\zeta_X\circ\log} + \chi 2\pi z e^{2\pi z\zeta_X\circ\log}\right\|_{L^\infty} \leq \|T\chi\|_\infty \|X\| + 2\pi|z|\,\|\chi\|_\infty\,.$$

For the other directions $X_0 \in \mathfrak{g}_0$ we compute

$$\begin{aligned} \partial_{X_0} m_{u_\zeta,v_\zeta^n}(W) &= \frac{d}{dt}\Big|_{t=0} \langle \sigma_\zeta(\exp(W_0+tX_0)\exp(W_X))u_\zeta, v_\zeta^n\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle \sigma_\zeta(\exp(W_0)\exp(t\tilde{X}_0)\exp(W_X))u_\zeta, v_\zeta^n\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \int_A \langle \sigma_{\zeta_0}(b\exp(W_0)\exp(t\tilde{X}_0)b^{-1})u_{\zeta_0}, v_{\zeta_0}\rangle \chi(be^{W_X})e^{2\pi z\zeta_X(\log(be^{W_X}))}\varphi_n(b)\,db \\ &= \int_A \langle \sigma_{\zeta_0}(b\exp(W_0)b^{-1})d\sigma_{\zeta_0}(\mathrm{Ad}^*(b)\tilde{X}_0)u_{\zeta_0}, v_{\zeta_0}\rangle \chi(be^{W_X})e^{2\pi z\zeta_X(\log(be^{W_X}))}\varphi_n(b)\,db, \end{aligned}$$

where $\tilde{X}_0 = \int_0^1 e^{-s\,\mathrm{ad}\,W_0}X_0\,ds$ (see [DK01, Theorem 1.5.3]).

For $W \in K$ we can find constants $C_1, C_2 > 0$ such that

$$\|\tilde{X}_0\| \leq \int_0^1 \|e^{-s \operatorname{ad} W_0}\| \|X_0\| \, ds \leq \|X_0\| \int_0^1 e^{s\|-\operatorname{ad} W_0\|} \, ds \leq \|X_0\| \frac{e^{\|\operatorname{ad} W_0\|} - 1}{\|\operatorname{ad} W_0\|} \leq C_1 \|X_0\|,$$

$$\|\operatorname{Ad}^*(b)\tilde{X}_0\| \leq \|\operatorname{Ad}^*(b)\| \|\tilde{X}_0\| \leq C_2 C_1 \|X_0\|.$$

Let $\{X_i\}$ be a orthonormal basis for $\mathfrak{g}_0$. Then there exists a constant $C_3 > 0$ such that $\|d\sigma_{\zeta_0}(X_i)u_{\zeta_0}\| \leq C_3$ for all i. Now write $\operatorname{Ad}^*(b)\tilde{X}_0 = \sum \alpha_i X_i$ and we have

$$\begin{aligned}\|d\sigma_{\zeta_0}(\operatorname{Ad}^*(b)\tilde{X}_0)u_{\zeta_0}\| &\leq \sum |\alpha_i| \|d\sigma_{\zeta_0}(X_i)u_{\zeta_0}\| \\ &\leq C_3 \dim(\mathfrak{g}_0) \|\operatorname{Ad}^*(b)\tilde{X}_0\| \leq C_1 C_2 C_3 \dim \mathfrak{g}_0 \|X_0\|.\end{aligned}$$

With $C := C_1 C_2 C_3$ we can estimate as above

$$\begin{aligned}\left|\partial_{X_0} m_{u_\zeta, v_\zeta^n}(W)\right| &\leq \|\chi\|_{L^\infty} \|v_{\zeta_0}\| \int_A \|d\sigma_{\zeta_0}(\operatorname{Ad}^*(b)\tilde{X}_0)u_{\zeta_0}\| |\varphi_n(b)| \, db \\ &\leq C \dim(\mathfrak{g}_0) \|X_0\| \|\chi\|_\infty.\end{aligned}$$

This implies that the m_{u_ζ, v_ζ^n} are uniformly equicontinuous on K: Let $\varepsilon > 0$ and choose $\delta < \varepsilon(\dim(\mathfrak{g})M)^{-1}$ with $M = \max\{\|T\chi\|_\infty \|X\| + 2\pi|z| \|\chi\|_\infty, C \dim \mathfrak{g}_0 \|\chi\|_\infty\} < \infty$ on the compact set K. Then for $\|W - Y\| < \delta$ we have for some $0 \leq \theta \leq 1$

$$|m_{u_\zeta, v_\zeta^n}(W) - m_{u_\zeta, v_\zeta^n}(Y)| \leq \|\nabla m_{u_\zeta, v_\zeta^n}(W + \theta(Y - W))\| \|W - Y\| \leq \delta \dim(\mathfrak{g}) M < \varepsilon.$$

The Arzela-Ascoli theorem (see [Rud76, Theorem 7.25]) states that the uniform boundedness and the uniform equicontinuity imply the existence of a uniformly convergent subsequence. □

Now we can turn to the desired statement:

Theorem 4.4. *Let G be a nilpotent, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ and $(\pi, \mathcal{H}_\pi)$ a unitary representation of G. Then*

$$\operatorname{AC}(\mathcal{O} - \operatorname{supp} \pi) \subset \operatorname{WF}(\pi).$$

Proof. Let $\xi \in \operatorname{AC}(\mathcal{O} - \operatorname{supp} \pi)$. We may assume without loss of generality that $\|\xi\| = 1$. Defining the cones $\mathcal{C}_\varepsilon := \{\eta \in i\mathfrak{g}^* \mid \exists\, t > 0 : |\xi - t\eta| < \varepsilon\}$, then for all $\varepsilon > 0$ there exists a sequence $(t_m \eta_m)_m \subset \mathcal{C}_\varepsilon \cap \mathcal{O} - \operatorname{supp}(\pi)$ with $t_m \to \infty$ and $\eta_m \in B_\varepsilon(\xi)$, $\|\eta_m\| = 1$.
We now use Theorem 3.20: For all $m \in \mathbb{N}$ let $l_m \in \Sigma_d$ be the corresponding element in the cross-section of all orbits of type d, i.e. $\mathcal{O}_{l_m} = \mathcal{O}_{t_m \eta_m}$. Then there exists $v_m \in V_{S(d)}$ with $t_m \eta_m = \psi_d(l_m, v_m)$. For $l \in \Sigma_d$ near l_m we define $\zeta_l := \psi_d(l, v_m) \in \mathcal{O}_l$ which depends continuously on l (see Figure 7).
Now let $0 < \delta < 1$ as in Proposition 4.2, i.e. $|\sin(2\pi x)| \leq \left(\frac{1}{2}\right)^{2\dim(\mathfrak{g})+1}$ for all $|x| < \delta$. Then there exists a neighborhood $N_m \subset \Sigma_d$ of l_m such that $\psi_d(N_m, v_m) \subset B_\delta(t_m \eta_m)$ and $\mu_\pi(N_m) > 0$ since $l_m \in \mathcal{O} - \operatorname{supp}(\pi)$ (see also Figure 7).

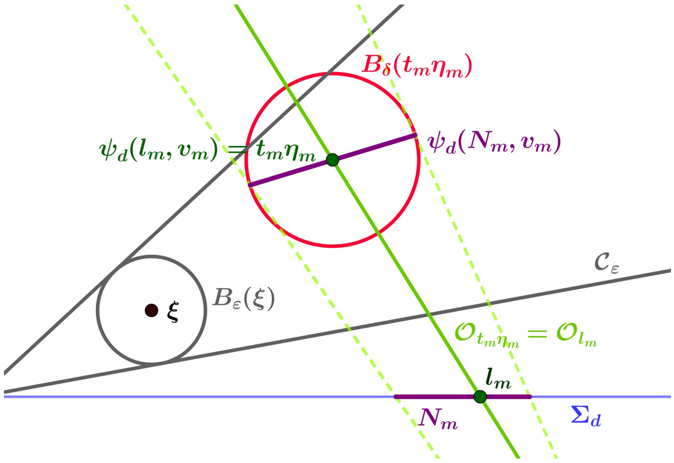


Figure 7: The choice of l_m and N_m.

Applying the above Proposition 4.2 to ζ_l, $l \in N_m$, we obtain measurable, normalized vectors $u_{\zeta_l}, v_{\zeta_l} \in \mathcal{H}_{\zeta_l}$. Since $\sigma_l \cong \sigma_{\zeta_l}$ and $\mathcal{H}_l \cong \mathcal{H}_{\zeta_l}$ we have corresponding measurable, normalized vectors $u_l, v_l \in \mathcal{H}_l$. With these we define

$$u^{(m)} := (\mu_\pi(N_m))^{-\frac{1}{2}} \int_{\Sigma_d} \chi_{N_m}(l) u_l \, d\mu_\pi(l) \ \in \mathcal{H}_\pi,$$

since the u_l are measurable in l and $\|u^{(m)}\|^2_{\mathcal{H}_\pi} = (\mu_\pi(N_m))^{-1} \int_{\Sigma_d} \chi_{N_m}(l) \|u_l\|^2 \, d\mu_\pi(l) = 1$. We define $v^{(m)} \in \mathcal{H}_\pi$ analogously.
Then we have for non-negative $\phi \in C_c^\infty(B_1(0))$, $\varphi = \phi \circ \log$ and the chart $\kappa : G \to \mathfrak{g}$ from Proposition 4.2 with the definition of N_m:

$$\begin{aligned}
&|\mathcal{F}(\langle(\pi \circ \kappa^{-1} \circ \log) u^{(m)}, v^{(m)}\rangle \varphi)(t\eta)| \\
&= \left| \int_G \int_{N_m} (\mu_\pi(N_m))^{-1} \langle \sigma_l(\kappa^{-1}(\log g)) u_l, v_l \rangle \varphi(g) e^{-2\pi t_m \eta_m (\log g)} \, dg \, d\mu_\pi(l) \right| \\
&\geq \left| \mathrm{Re}\left(\int_G \int_{N_m} (\mu_\pi(N_m))^{-1} \langle \sigma_l(\kappa^{-1}(\log g)) u_l, v_l \rangle \varphi(g) e^{-2\pi t_m \eta_m (\log g)} \, dg \, d\mu_\pi(l) \right) \right| \\
&= (\mu_\pi(N_m))^{-1} \left| \int_{N_m} \mathrm{Re}\left(\int_{\mathfrak{g}} \langle \sigma_l(\kappa^{-1}(X)) u_l, v_l \rangle \phi(X) e^{-2\pi t_m \eta_m (X)} \, dX \right) d\mu_\pi(l) \right| \\
&\overset{\text{Prop. 4.2}}{\geq} (\mu_\pi(N_m))^{-1} \int_{N_m} \left(\frac{1}{2}\right)^{2 \dim \mathfrak{g}} \|\phi\|_{L^1} \, d\mu_\pi(l) = \left(\frac{1}{2}\right)^{2 \dim \mathfrak{g}} \|\phi\|_{L^1} \|u^{(m)}\| \|v^{(m)}\|.
\end{aligned}$$

We can use this to show that $\xi \in \mathrm{WF}(\pi \circ \kappa^{-1} \circ \log)$: If we assume that $\xi \notin \mathrm{WF}(\pi \circ \kappa^{-1} \circ \log)$ we can employ the equivalence of Lemma 2.13 (i) and (iii) (see also (4.9)). It states that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that for all $\varphi \in C_c^\infty(\exp(B_{\varepsilon_2}(0)))$ and all $N \in \mathbb{N}$:

$$|\mathcal{F}(\langle(\pi \circ \kappa^{-1} \circ \log) u, v\rangle \varphi)(t\eta)| \leq C_N(\varphi) \|u\| \|v\| t^{-N} \quad \forall\, u, v \in \mathcal{H}_\pi,\ \eta \in B_{\varepsilon_1}(\xi),\ t > t_0.$$

Since the constant $C_N(\varphi)$ can be chosen independent of $\eta \in B_{\varepsilon_1}(\xi)$ and $u, v \in \mathcal{H}$, we found a sequence that violates this inequality for $N = 1$.

Now, we use [Hör03, Theorem 8.2.4] with $f = \kappa^{-1} \circ \log$, which is a homeomorphism with $Df(e) = \mathrm{Id}$, to see that

$$\mathrm{WF}_e(\langle(\pi \circ \kappa^{-1} \circ \log)u, v\rangle) \subset \mathrm{WF}_e(\langle \pi(\cdot)u, v\rangle) \quad \forall\, u, v \in \mathcal{H}_\pi,$$

and therefore

$$\xi \in \mathrm{WF}(\pi \circ \kappa^{-1} \circ \log) = \overline{\bigcup_{u,v\in\mathcal{H}} \mathrm{WF}_e(\langle(\pi \circ \kappa^{-1} \circ \log)u, v\rangle_{\mathcal{H}})} \subset \mathrm{WF}(\pi).$$

This finishes the proof. □

4.2 Proof of the Inclusion $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O} - \mathrm{supp}(\pi))$

Now let us turn to the other inclusion $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O} - \mathrm{supp}\,\pi)$. For its proof we will again estimate the Fourier transform of the matrix coefficients directly using the following

Proposition 4.5. *Let $n, N \in \mathbb{N}$. There exists a constant $C_{n,N} > 0$ such that for all nilpotent, connected, simply connected Lie groups G with Lie algebra $\mathfrak{g}$ and $\dim \mathfrak{g} = n$ there exists an inner product on $\mathfrak{g}$ and $i\mathfrak{g}^*$ and a chart $\kappa : G \to \mathfrak{g}$ with $D_e(\kappa^{-1} \circ \log) = \mathrm{Id}_G$ such that the following estimate holds for arbitrary Haar measure dX on $\mathfrak{g}$, all $\phi \in C_c^\infty(\mathfrak{g})$, $l, \eta \in i\mathfrak{g}^*$ and all $u, v \in \mathcal{H}_l$:*

$$\left| \int_{\mathfrak{g}} \langle \sigma_l(\kappa^{-1}(X))u, v\rangle_{\mathcal{H}_l} \phi(X) e^{-2\pi\eta(X)}\, dX \right| \le C_{n,N} \|u\|_{\mathcal{H}_l} \|v\|_{\mathcal{H}_l} \|\phi\|_{W^{N,1}(\mathfrak{g})} \langle d(\eta, \mathcal{O}_l)\rangle^{-N},$$

where $\|\phi\|_{W^{N,1}(\mathfrak{g})} := \sum_{|\alpha|\le N} \|D^\alpha \phi\|_{L^1(\mathfrak{g},dX)}$.

Proof. We prove this statement by induction on $\dim \mathfrak{g}$. If $n = \dim \mathfrak{g} = 1$ or 2, the group is abelian. In this case the irreducible unitary representations are one-dimensional, $\sigma_l(g) = e^{2\pi l(\log g)}$, and have a zero-dimensional orbit $\mathcal{O}_l = \{l\}$. We choose $\kappa = \log$ and compute

$$\begin{aligned} \left| \int_{\mathfrak{g}} \langle \sigma_l(\exp X)u, v\rangle_{\mathbb{C}} \phi(X) e^{-2\pi\eta(X)}\, dX \right| &= \left| \int_{\mathfrak{g}} \phi(g) e^{2\pi(l-\eta)(X)} u\bar{v}\, dX \right| \\ &= |\hat{\phi}(\eta - l)| \cdot |u| \cdot |v|. \end{aligned}$$

Fixing an inner product on $\mathfrak{g}$ we obtain a corresponding one on $i\mathfrak{g}^*$. Now let $\{X_i\}_{i=1}^n$ be an orthogonal basis for $\mathfrak{g}$ and pick $j \in \{1, n\}$ such that $|(l - \eta)(X_j)|$ is maximal.

With this choice we have for $N \in \mathbb{N}$

$$\begin{aligned}|\hat{\phi}(\eta - l)| &= \left|(2\pi(l-\eta)(X_j))^{-N} \int_{\mathfrak{g}} \phi(X) \partial_{X_j}^N e^{2\pi(l-\eta)(X)} dX\right| \\ &\leq (2\pi)^{-N} |(l-\eta)(X_j))|^{-N} \int_{\mathfrak{g}} |\partial_{X_j}^N \phi(X)| dX \\ &\leq (2\pi)^{-N} \sqrt{n}^N \|l-\eta\|^{-N} \|\phi\|_{W^{N,1}(\mathfrak{g})}.\end{aligned}$$

Thus we can choose $C_{n,N} = \left(\frac{\sqrt{n}}{2\pi}\right)^N$ for $n = 1, 2$.

Now we assume $n = \dim \mathfrak{g} \geq 3$. We will distinguish between the two cases following the ideas as described in Procedure 3.19:

Case I: $\dim \mathfrak{z}(\mathfrak{g}) > 1$. There exists $Z \in \mathfrak{z}(\mathfrak{g})$ with $l(Z) = 0$ and a subspace $W < \mathfrak{g}$ such that $\mathfrak{g} = W \oplus \mathbb{R}Z$. Then $\overline{\mathfrak{g}} = \mathfrak{g}/(\mathbb{R} \cdot Z)$ is isomorphic to W and has a well-defined Lie algebra structure $[v + \mathbb{R}Z, w + \mathbb{R}Z] = [v, w]_{\mathfrak{g}} + \mathbb{R}Z$ since $Z \in \mathfrak{z}(\mathfrak{g})$. The induction hypotheses gives us $\overline{G} = \kappa^{-1}(\overline{\mathfrak{g}})$ with a chart $\kappa : \overline{G} \to \overline{\mathfrak{g}}$. For G we choose the logarithm as a chart.

Given an inner product on $\overline{\mathfrak{g}}$ we choose one on $\mathfrak{g}$ such that the decomposition above is orthogonal. Furthermore, without loss of generality we may assume $\|Z\| = 1$. Using the corresponding inner product on $i\mathfrak{g}^*$ we also obtain an orthogonal decomposition $i\mathfrak{g}^* = iW^* \oplus \mathbb{R}\eta_Z \cong i\overline{\mathfrak{g}}^* \oplus \mathbb{R}\eta_Z$ with $\|\eta_Z\| = 1$.

Note that $i\overline{\mathfrak{g}}^*$ is $\mathrm{Ad}^*(G)$-invariant (again due to $Z \in \mathfrak{z}(\mathfrak{g})$). We can identify l and its orbit $\mathcal{O}_l^G \subset i\mathfrak{g}^*$ with an element $\overline{l} \in i\overline{\mathfrak{g}}^*$ and its orbit $\mathcal{O}_{\overline{l}}^{\overline{G}} \subset i\overline{\mathfrak{g}}^*$, respectively.
Let $\eta = \overline{\eta} + r\eta_Z \in i\mathfrak{g}^* = i\overline{\mathfrak{g}}^* \oplus \mathbb{R}\eta_Z$. Then by the choice of the inner product

$$d(\eta, \mathcal{O}_l^G)^2 = d(\overline{\eta}, \mathcal{O}_{\overline{l}}^{\overline{G}})^2 + r^2.$$

This implies that we are either in the case

$$\text{a) } r \geq \frac{1}{\sqrt{2}} d(\eta, \mathcal{O}_l^G) \quad \text{or} \quad \text{b) } d(\overline{\eta}, \mathcal{O}_{\overline{l}}^{\overline{G}}) \geq \frac{1}{\sqrt{2}} d(\eta, \mathcal{O}_l^G). \tag{4.12}$$

Turning to the integral we want to estimate:

$$\begin{aligned}J &:= \left|\int_{\mathfrak{g}} \langle \sigma_l(\exp(X))u, v\rangle_{\mathcal{H}_l} \phi(X) e^{-2\pi\eta(X)}\, dX\right| \\ &= \left|\int_{\overline{\mathfrak{g}}} \int_{\mathbb{R}} \langle \sigma_l(\exp(\overline{X} + tZ))u, v\rangle_{\mathcal{H}_l} \phi(\overline{X} + tZ) e^{-2\pi\eta(w+tZ)} d\overline{X}\, dt\right| \\ &= \left|\int_{\overline{\mathfrak{g}}} \int_{\mathbb{R}} \langle \sigma_l(\exp(\overline{X})\exp(tZ))u, v\rangle_{\mathcal{H}_l} \phi(\overline{X} + tZ) e^{-2\pi(\overline{\eta}(\overline{X}) + r\eta_Z(tZ))} d\overline{X}\, dt\right| \\ &= \left|\int_{\overline{\mathfrak{g}}} \int_{\mathbb{R}} \langle \sigma_l(\exp(\overline{X}))u, v\rangle_{\mathcal{H}_l} \phi(\overline{X} + tZ) e^{-2\pi(\overline{\eta}(\overline{X}) + r\eta_Z(tZ))} d\overline{X}\, dt\right|\end{aligned}$$

since every Haar measure dX on $\mathfrak{g}$ and the Lebesgue measure dt on $\mathbb{R}$ yields a Haar measure $d\overline{X}$ on $\overline{\mathfrak{g}}$ such that the first equality above holds. The last equality is due to

$l(Z)=0$ which implies $\sigma_l(g\exp(tZ))=\sigma_l(g)$ for all $g\in G$, $t\in\mathbb{R}$.

We start with case a) of (4.12) and define

$$\tilde{\phi}(t):=\int_{\overline{\mathfrak{g}}}\langle\sigma_l(\exp(\overline{X}))u,v\rangle_{\mathcal{H}_l}\phi(\overline{X}+tZ)e^{-2\pi\overline{\eta}(\overline{X})}\,d\overline{X}\ \in C_c^\infty(\mathbb{R}).$$

Then by integration by parts as in the abelian case with $l=0$ and $u=v=1$ we obtain

$$\begin{aligned}J=\left|\int_{\mathbb{R}}\tilde{\phi}(t)e^{-2\pi rt}\,dt\right|&\leq C_{1,N}\|\tilde{\phi}\|_{W^{N,1}(\mathbb{R})}\langle r\rangle^{-N}\\ &\overset{(4.12)a)}{\leq} C_{1,N}\sqrt{2}^N\|\tilde{\phi}\|_{W^{N,1}(\mathbb{R})}\langle d(\eta,\mathcal{O}_l^G)\rangle^{-N}.\end{aligned}$$

The claim now follows in this case with $C_{n,N}:=C_{1,N}\sqrt{2}^N$ and the following estimation:

$$\begin{aligned}\|\tilde{\phi}\|_{W^{N,1}(\mathbb{R})}&=\sum_{k=1}^N\|\partial_t^k\tilde{\phi}\|_{L^1(\mathbb{R},dt)}\\ &\leq\sum_{k=1}^N\int_{\mathbb{R}}\int_{\overline{\mathfrak{g}}}\left|\langle\sigma_l(\exp(\overline{X}))u,v\rangle_{\mathcal{H}_l}\partial_t^k\phi(\overline{X}+tZ)e^{-2\pi\overline{\eta}(\overline{X})}\right|\,d\overline{X}\,dt\\ &\leq\|u\|\|v\|\sum_{k=1}^N\int_{\mathbb{R}}\int_{\overline{\mathfrak{g}}}\left|\partial_t^k\phi(\overline{X}+tZ)\right|\,d\overline{X}\,dt\quad\leq\quad\|u\|\|v\|\|\phi\|_{W^{N,1}(\mathfrak{g})}.\end{aligned}$$

Now let's turn to case b) of (4.12). Note that by Procedure 3.19 (i) we know $\mathcal{H}_l\cong\mathcal{H}_{\overline{l}}$ and $\sigma_{\overline{l}}\circ P\cong\sigma_l$ with the projection $P:G\to\overline{G}$ which is given by $P=\kappa^{-1}\circ\log$ here. Thus, we have

$$J=\left|\int_{\overline{\mathfrak{g}}}\int_{\mathbb{R}}\langle\sigma_{\overline{l}}(\kappa^{-1}(\overline{X}))u,v\rangle_{\mathcal{H}_{\overline{l}}}\phi(\overline{X}+tZ)e^{-2\pi(\overline{\eta}(\overline{X})+r\eta_Z(tZ))}d\overline{X}\,dt\right|.$$

Now define

$$\check{\phi}(\overline{X}):=\int_{\mathbb{R}}\phi(\overline{X}+tZ)e^{-2\pi irt}\,dt\ \in C_c^\infty(\overline{\mathfrak{g}}).$$

Then

$$\begin{aligned}J&=\left|\int_{\overline{\mathfrak{g}}}\langle\sigma_{\overline{l}}(\kappa^{-1}(\overline{X}))u,v\rangle_{\mathcal{H}_l}\check{\phi}(\overline{X})e^{-2\pi\overline{\eta}(\overline{X})}d\overline{X}\right|\\ &\overset{\text{(IH)}}{\leq}C_{n-1,N}\|u\|\|v\|\|\check{\phi}\|_{W^{N,1}(\overline{\mathfrak{g}})}\langle d(\overline{\eta},\mathcal{O}_{\overline{l}}^{\overline{G}})\rangle^{-N}\\ &\overset{(4.12)a)}{\leq}C_{n-1,N}\sqrt{2}^N\|u\|\|v\|\|\check{\phi}\|_{W^{N,1}(\overline{\mathfrak{g}})}\langle d(\eta,\mathcal{O}_l^G)\rangle^{-N}.\end{aligned}$$

The claim now follows in this case with $C_{n,N} := C_{n-1,N}\sqrt{2}^N$ and the following estimation:

$$\begin{aligned}\|\check{\phi}\|_{W^{N,1}(\overline{\mathfrak{g}})} &= \sum_{|\alpha|<N} \|\partial^\alpha \check{\phi}\|_{L^1(\overline{\mathfrak{g}},dv)} \\ &= \sum_\alpha \int_{\overline{\mathfrak{g}}} \left| \int_{\mathbb{R}} \partial^\alpha_{\overline{X}} \phi(\overline{X}+tZ) e^{-2\pi i r t}\, dt \right| d\overline{X} \\ &\le \sum_\alpha \int_{\overline{\mathfrak{g}}} \int_{\mathbb{R}} \left| \partial^\alpha_{\overline{X}} \phi(\overline{X}+tZ) \right| dt\, d\overline{X} \quad \le \quad \|\phi\|_{W^{N,1}(\mathfrak{g})}.\end{aligned}$$

Case II: $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\cdot Z$ is one-dimensional. Kirillov's Lemma 3.6 gives us $X, Y \in \mathfrak{g}$ and an ideal $\mathfrak{g}_0 \subset \mathfrak{g}$ with $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_0$ and $[X,Y] = Z$. Given an inner product on $\mathfrak{g}_0$ we choose one on $\mathfrak{g}$ such that the decomposition is orthogonal. Since $\dim(\mathfrak{z}(\mathfrak{g}_0)) > 1$ as $Z, Y \in \mathfrak{z}(\mathfrak{g}_0)$ we are in Case I in the induction hypotheses for G_0 and therefore can use the chart $\log : G_0 \to \mathfrak{g}_0$. We define a chart for G via

$$\kappa^{-1} : \mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}X \to G, \quad X_0 + tX \mapsto \exp(X_0)\exp(tX).$$

If $l(Z) = 0$ we can proceed analogously to Case I since for $w = Y_0 + sX \in W$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}\kappa^{-1}(w+tZ) &= \kappa^{-1}(Y_0+tZ+sX) = \exp(Y_0+tZ)\exp(sX) = \exp(Y_0)\exp(tZ)\exp(sX) \\ &= \exp(Y_0)\exp(sX)\exp(tZ) = \kappa^{-1}(Y_0+sX)\exp(tZ) = \kappa^{-1}(w)\exp(tZ).\end{aligned}$$

Thus, we now assume $l(Z) \neq 0$. Then $X \notin \mathfrak{r}_l$ and we are in Case II of Proposition 3.17 and Theorem 3.18:

$$\begin{aligned}&p : i\mathfrak{g}^* \to i\mathfrak{g}_0^*, \quad l_0 := p(l), \eta_0 := p(\eta),\ \mathcal{O}_{l_0}^{G_0} := \mathrm{Ad}^*(G_0)l_0, \\ &p(\mathcal{O}_l^G) = \bigsqcup_{t\in\mathbb{R}} (\mathrm{Ad}^* \exp tX)\mathcal{O}_{l_0}^{G_0}, \quad \mathcal{O}_l^G = p^{-1}(p(\mathcal{O}_l^G)).\end{aligned}$$

where $G_0 = \exp(\mathfrak{g}_0) \subset G$ is a normal subgroup. Since $\ker(p) \subset i\mathfrak{g}^*$ is a one-dimensional subspace and $\ker(p)^\perp \cong i\mathfrak{g}_0^*$ as vector spaces we have a corresponding inner product on $i\mathfrak{g}_0^*$ which gives us for all $a \in A = \exp(\mathbb{R}X)$:

$$d(\eta_0, \mathcal{O}^{G_0}_{\mathrm{Ad}^*(a)l_0}) = d(\eta_0, \mathrm{Ad}^*(a)\mathcal{O}^{G_0}_{l_0}) \ge d(\eta_0, p(\mathcal{O}_l^G)) = d(\eta, \mathcal{O}_l^G). \tag{4.13}$$

In addition to that we have $\eta = \eta_0 + \eta_X$ with $\eta_X \in \ker(p)$.

Turning to the integral we want to estimate:

$$\begin{aligned}J &:= \left| \int_{\mathfrak{g}} \langle \sigma_l(\kappa^{-1}(X))u, v\rangle_{\mathcal{H}_l} \phi(X) e^{-2\pi\eta(X)}\, dX \right| \\ &= \left| \int_{\mathfrak{g}_0} \int_{\mathbb{R}} \langle \sigma_l(\exp(X_0)\exp(tX))u, v\rangle_{\mathcal{H}_l} \phi(X_0+tX) e^{-2\pi(\eta_0(X_0)+r\eta_X(tX))} dX_0\, dt \right|.\end{aligned}$$

By Theorem 3.18, we also know $\sigma_l \cong \mathrm{Ind}_{G_0}^{G}(\sigma_{l_0})$. Note that $\mathcal{H}_l \cong L^2(A, \mathcal{H}_{l_0})$. If we regard u and v as elements of $L^2(A, \mathcal{H}_{l_0})$ and $\tilde{u}, \tilde{v} : G \to \mathcal{H}_{l_0}$ the corresponding functions in the 'standard model' we have again

$$\langle \sigma_l(g_0 a)u, v\rangle_{\mathcal{H}_l} = \int_A \langle [\sigma_l(g_0 a)u](b), v(b)\rangle_{\mathcal{H}_{l_0}}\, db \quad \text{and}$$
$$[\sigma_l(g_0 a)\tilde{u}](b) = \tilde{u}(bg_0 a) = \tilde{u}(bg_0 b^{-1} ba) = \sigma_{l_0}(bg_0 b^{-1})\tilde{u}(ba)$$

since $b^{-1} g_0 b \in G_0$ as $\mathfrak{g}_0$ is an ideal. This gives us $[\sigma_l(g_0 a)u](b) = \sigma_{l_0}(bg_0 b^{-1})u(ba)$.
We deduce that

$$\begin{aligned} J &= \left| \int_{\mathfrak{g}_0} \int_{\mathbb{R}} \left(\int_A \langle \sigma_{l_0}(b \exp(X_0) b^{-1}) u(be^{tX}), v(b)\rangle_{\mathcal{H}_{l_0}} db \right) \cdot \right. \\ &\qquad\qquad \left. \phi(X_0 + tX) e^{-2\pi(\eta_0(X_0) + r\eta_X(tX))} dX_0\, dt \right| \\ &\leq \int_{\mathbb{R}} \int_A \left| \int_{\mathfrak{g}_0} \langle \sigma_{l_0}(b \exp(X_0) b^{-1}) u(be^{tX}), v(b)\rangle_{\mathcal{H}_{l_0}} \phi(X_0 + tX) e^{-2\pi\eta_0(X_0)} dX_0 \right| \cdot \\ &\qquad\qquad \left| e^{-2\pi r \eta_X(tX)} \right| db\, dt. \end{aligned}$$

The conjugation $C_b : G_0 \to G_0, g_0 \mapsto b^{-1} g_0 b$ is a group automorphism and we know that $\chi_{l_0} \circ C_b = \chi_{\mathrm{Ad}^*(b)l_0}$ for the character χ_{l_0} such that $\sigma_{l_0} = \mathrm{Ind}_M^{G_0}(\chi_{l_0})$, $M = \exp(\mathfrak{m})$ for a polarizing subalgebra $\mathfrak{m} \subset \mathfrak{g}_0$. Now, $\mathrm{Ad}(b)\mathfrak{m}$ is a polarizing subalgebra for $\mathrm{Ad}^*(b)l_0$ and $C_b^{-1} M = \exp(\mathrm{Ad}(b)\mathfrak{m})$. Thus, [CG90, Lemma 2.1.3] gives us

$$\sigma_{\mathrm{Ad}^*(b)l_0} = \mathrm{Ind}_{C_b^{-1}M}^{G_0}(\chi_{l_0} \circ C_b) \cong \mathrm{Ind}_M^{G_0}(\chi_{l_0}) \circ C_b = \sigma_{l_0} \circ C_b.$$

With this unitary equivalence of representations and the induction hypothesis in G_0 for $\mathrm{Ad}^*(b^{-1})l_0$ instead of l_0 we conclude

$$\begin{aligned} J &\leq \int_{\mathbb{R}} \int_A \left| \int_{\mathfrak{g}_0} \langle \sigma_{l_0}(b \exp(X_0) b^{-1}) u(be^{tX}), v(b)\rangle_{\mathcal{H}_{l_0}} \phi(X_0 + tX) e^{-2\pi\eta_0(X_0)} dX_0 \right| \cdot \\ &\qquad\qquad \left| e^{-2\pi r \eta_X(tX)} \right| db\, dt \\ &\overset{\text{(IH)}}{\leq} \int_{\mathbb{R}} \int_A C_{n-1,N} \|\phi(\bullet + tX)\|_{W^{N,1}(\mathfrak{g}_0)} \|u(be^{tX})\|_{\mathcal{H}_{l_0}} \|v(b)\|_{\mathcal{H}_{l_0}} \langle d(\eta_0, \mathcal{O}^{G_0}_{\mathrm{Ad}^*(b^{-1})l_0})\rangle^{-N} db\, dt \\ &\overset{(4.13)}{\leq} C_{n-1,N} \langle d(\eta, \mathcal{O}_l)\rangle^{-N} \int_{\mathbb{R}} \left(\int_A \|T_{\exp(tX)} u(b)\|_{\mathcal{H}_{l_0}} \|v(b)\|_{\mathcal{H}_{l_0}} db \right) \|\phi(\bullet + tX)\|_{W^{N,1}(\mathfrak{g}_0)} dt \\ &\leq C_{n-1,N} \langle d(\eta, \mathcal{O}_l)\rangle^{-N} \int_{\mathbb{R}} \|T_{\exp(tX)} u\|_{\mathcal{H}_l} \|v\|_{\mathcal{H}_l} \|\phi(\bullet + tX)\|_{W^{N,1}(\mathfrak{g}_0)} dt, \end{aligned}$$

where $T_{\exp(tX)}$ is the translation by $\exp(tX) \in A$ which is an isometry on $\mathcal{H}_l \cong L^2(A, \mathcal{H}_{l_0})$.

This gives us

$$\begin{aligned} J &\leq C_{n-1,N}\langle d(\eta,\mathcal{O}_l)\rangle^{-N}\|u\|_{\mathcal{H}_l}\|v\|_{\mathcal{H}_l}\int_{\mathbb{R}}\|\phi(\bullet+tX)\|_{W^{N,1}(\mathfrak{g}_0)}\,dt \\ &= C_{n-1,N}\langle d(\eta,\mathcal{O}_l)\rangle^{-N}\|u\|_{\mathcal{H}_l}\|v\|_{\mathcal{H}_l}\int_{\mathbb{R}}\sum_{|\alpha|\leq N}\int_{\mathfrak{g}_0}|\partial_{X_0}^N\phi(X_0+tX)|dX_0\,dt \\ &\leq C_{n-1,N}\langle d(\eta,\mathcal{O}_l)\rangle^{-N}\|u\|_{\mathcal{H}_l}\|v\|_{\mathcal{H}_l}\|\phi\|_{W^{N,1}(\mathfrak{g})}, \end{aligned}$$

which finishes the proof. □

Corollary 4.6. *The statement of the previous Proposition 4.5 also holds for* $u,v\in\mathcal{H}_l^{\oplus m_l}$ *with multiplicity* $m_l\in\mathbb{N}\cup\{\infty\}$.

Proof. For $u\in\mathcal{H}_l^{\oplus m_l}$ we have $u=(u_1,u_2,\dots)$ with (finitely or infinitely many) $0\neq u_i\in\mathcal{H}_l$ and $\sum_i\|u_i\|_{\mathcal{H}_l}^2<\infty$, $\|u\|=\left(\sum_i\|u_i\|^2\right)^{1/2}$. Thus

$$\begin{aligned} \Big|\int_{\mathfrak{g}}\langle\sigma_l(\kappa^{-1}(X))u,v\rangle_{\mathcal{H}_l}\phi(X)e^{-2\pi\eta(X)}\,dX\Big| &= \left|\int_{\mathfrak{g}}\sum_i\langle\sigma_l(\kappa^{-1}(X))u_i,v_i\rangle_{\mathcal{H}_l}\phi(X)e^{-2\pi\eta(X)}\,dX\right| \\ &= \left|\sum_i\int_{\mathfrak{g}}\langle\sigma_l(\kappa^{-1}(X))u_i,v_i\rangle_{\mathcal{H}_l}\phi(X)e^{-2\pi\eta(X)}\,dX\right| \\ &\overset{\text{Prop. 4.5}}{\leq} C_{n,N}\|\phi\|_{W^{N,1}(\mathfrak{g})}\langle d(\eta,\mathcal{O}_l)\rangle^{-N}\sum_i\|u_i\|\cdot\|v_i\| \\ &\leq C_{n,N}\|\phi\|_{W^{N,1}(\mathfrak{g})}\langle d(\eta,\mathcal{O}_l)\rangle^{-N}\left(\sum_i\|u_i\|^2\right)^{1/2}\cdot\left(\sum_i\|v_i\|^2\right)^{1/2} \\ &= C_{n,N}\|\phi\|_{W^{N,1}(\mathfrak{g})}\langle d(\eta,\mathcal{O}_l)\rangle^{-N}\|u\|\cdot\|v\|, \end{aligned}$$

where the interchanging of the order of integration and summation in the second equality is possible since $|\langle\sigma_l(\kappa^{-1}(X))u_i,v_i\rangle\phi(X)e^{-2\pi\eta(X)}|\leq\|u_i\|\cdot\|v_i\|\cdot|\phi(X)|\in L^1(\mathbb{N}\times\mathfrak{g})$. □

This inequality whose constant is in particular independent of $l\in i\mathfrak{g}^*$ now helps us to estimate the matrix coefficients of the big unitary representation π using its direct integral decomposition into the irreducibles σ_l.

Theorem 4.7. *Let* G *be a nilpotent, connected, simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and* $(\pi,\mathcal{H}_\pi)$ *a unitary representation of* G*. Then*

$$\mathrm{WF}(\pi)\subset\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi).$$

Proof. Let $\eta\notin\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi)$. Then there exists $\varepsilon>0$ and $t_0>0$ such that $d(t\eta,\mathcal{O}-\operatorname{supp}\pi)\geq\varepsilon t$ for all $t\geq t_0$. In particular, $d(t\eta,\mathcal{O}_l)\geq\varepsilon t$ for all $l\in\operatorname{supp}\pi$.
Again, we use $\mathcal{H}_\pi=\int_{\Sigma_d}\mathcal{H}_l^{\oplus m(\pi,\sigma_l)}\,d\mu_\pi(l)$ for the Hilbert space of the unitary representation π. If $u=(u_l),v=(v_l)\in\mathcal{H}$, $u_l,v_l\in\mathcal{H}_l^{\oplus m(\pi,\sigma_l)}$, in this direct integral decomposition

the matrix coefficient is

$$\langle \pi(g)u, v\rangle = \int_{\Sigma_d} \langle \sigma_l(g)u_l, v_l\rangle \, d\mu_\pi(l).$$

Let $\varphi \in C_c^\infty(G)$ with $\varphi(e) \neq 0$ and $\kappa : G \to \mathfrak{g}$ be the corresponding chart from Proposition 4.5 and Corollary 4.6. For $t \geq t_0$ and $\phi := \varphi \circ \exp \in C_c^\infty(\mathfrak{g})$ we conclude

$$\begin{aligned}
&|\mathcal{F}(\langle(\pi \circ \kappa^{-1} \circ \log)u, v\rangle\varphi)(t\eta)| \\
&\quad = \left| \int_G \langle \pi(\kappa^{-1}(\log g))u, v\rangle \varphi(g) e^{-2\pi t\eta(\log g)} \, dg \right| \\
&\quad = \left| \int_G \int_{\Sigma_d} \langle \sigma_l(\kappa^{-1}(\log g))u_l, v_l\rangle \varphi(g) e^{-2\pi t\eta(\log g)} \, d\mu_\pi(l) \, dg \right| \\
&\quad = \left| \int_{\Sigma_d} \left(\int_G \langle \sigma_l(\kappa^{-1}(\log g))u_l, v_l\rangle \phi(\log g) e^{-2\pi t\eta(\log g)} \, dg \right) d\mu_\pi(l) \right| \\
&\quad \leq \int_{\Sigma_d} \left| \int_{\mathfrak{g}} \langle \sigma_l(\kappa^{-1}(X))u_l, v_l\rangle \phi(X) e^{-2\pi t\eta(X)} \, dX \right| d\mu_\pi(l) \\
&\quad \overset{\text{Cor. 4.6}}{\leq} \int_{\Sigma_d} C_{n,N} \|u\|_{\mathcal{H}_l} \|v\|_{\mathcal{H}_l} \|\phi\|_{W^{N,1}(\mathfrak{g})} \langle d(t\eta, \mathcal{O}_l)\rangle^{-N} \, d\mu_\pi(l) \\
&\quad \leq C_{n,N} \|\phi\|_{W^{N,1}(\mathfrak{g})} \varepsilon^{-N} t^{-N} \int_{\Sigma_d} \|u_l\| \cdot \|v_l\| \, d\mu_\pi(l) \\
&\quad \leq C_{n,N} \|\phi\|_{W^{N,1}(\mathfrak{g})} \varepsilon^{-N} \|u\|_{\mathcal{H}_\pi} \cdot \|v\|_{\mathcal{H}_\pi} t^{-N} \ \in \mathcal{O}\left(t^{-N}\right).
\end{aligned}$$

This implies $\eta \notin \mathrm{WF}_e(\langle \pi(\kappa^{-1} \circ \log(\cdot))u, v\rangle)$.
Now, [Hör03, Theorem 8.2.4] with the map $f = \kappa^{-1} \circ \log$, which is a homeomorphism with $Df(e) = \mathrm{Id}$, implies $\eta \notin \mathrm{WF}_e(\langle \pi(\cdot)u, v\rangle)$. □

Theorems 4.4 and 4.7 prove our main result Theorem 4.1.

5 Alternative Proofs

In this section we present two alternative approaches to prove the two necessary inclusions $\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi) \subset \mathrm{WF}(\pi)$ and $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi)$. The first approach follows the strategy of [HHÓ16] using integrated characters. The second approach studies matrix coefficients via the formula for the kernels of integrated representations.

5.1 Integrated Characters

Here we try to follow the work of Harris, He and Ólafsson for real reductive, algebraic groups (see [HHÓ16, Chapter 6 and 7]) and use the integrated characters $\int_{\hat{G}} \theta_\sigma d\mu_\pi(\sigma)$. The inclusion $\mathrm{AC}(\mathcal{O}-\operatorname{supp}(\pi)) \subset \mathrm{WF}(\pi)$ can be proven with the wave front set of the integrated characters as an intermediate step even though in our setting there are more restrictions on the integrated characters as for the real reductive, algebraic groups. For the inclusion $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O}-\operatorname{supp}(\pi))$ this approach gives us, however, only a weaker statement (see Proposition 5.8, to be compared with Theorem 4.7 and the Remarks 5.9, 5.10 and 5.11).

Recall from (4.8), that we can assume without loss of generality that $\operatorname{supp}(\pi) \subset \Sigma_d$ for one $d \in D$.

5.1.1 The Inclusion $\mathbf{AC}(\mathcal{O}-\operatorname{\mathbf{supp}}(\pi)) \subset \mathbf{WF}(\pi)$

Since we want to use our knowledge of irreducible unitary representations of nilpotent Lie groups from Section 3, we take a closer look at the decomposition of the unitary representation $\pi = \int_{\hat{G}}^{\oplus} \sigma^{\oplus m(\pi,\sigma)}\, d\mu_\pi(\sigma)$ (see (4.7)) and in particular the corresponding measure μ_π on $\hat{G}$.

Remark 5.1. *Note that the positive measure μ_π is only well defined up to an equivalence relation. Here two measures μ and μ' are equivalent if and only if they are absolutely continuous with respect to each other. More precisely, Radon-Nikodym gives us a positive measurable function f such that $d\mu = f d\mu'$ and the unitary equivalence of the resulting integrated representations is given by the multiplication operator by $\sqrt{f}$. We will refer to f as the density function.*

This means that without loss of generality we can put certain conditions on the measure μ_π. We will find that the right choice for this subsection is the following:

Definition 5.2.

$$\mathcal{M}_d := \{\mu \text{ positive, finite measure on } \Sigma_d \text{ s.t. } \mathrm{Pf}_d^{-1} \in L^1(\mu)\},$$
$$\mathcal{M}_d(\pi) := \left\{\mu \in \mathcal{M}_d \mid \exists\, f \in L^1_{\mathrm{loc}}(\mu_\pi) \cap L^2_{\mathrm{loc}}(\mu_\pi) : \ \mu = f \cdot \mu_\pi\right\},$$

where the Pfaffian Pf_d is an $\mathrm{Ad}^*(G)$-invariant polynomial on $i\mathfrak{g}^*$ (see Definition 3.24).

Remark 5.3. *There exists $\mu_\pi \in \mathcal{M}_d$ such that $\pi \cong \int_{\Sigma_d}^{\oplus} \sigma_l^{\oplus m(\pi,\sigma_l)}\, d\mu_\pi(l)$. From now on we will only consider such a measure.*

Now let us introduce a distribution by integrating the characters $\theta_l = \mathrm{Tr}(\sigma_l)$ examined in Section 3. Its wave front set will provide an intermediate step in proving the first inclusion, i.e. $\mathrm{AC}(\mathcal{O}-\mathrm{supp}(\pi)) \subset \mathrm{WF}_e\left(\int_{\Sigma_d} \theta_l\, d\mu(l)\right) \subset \mathrm{WF}(\pi)$.

Lemma 5.4. *For every positive measure μ on Σ_d and every function f on Σ_d such that $f \cdot \mathrm{Pf}_d^{-1} \in L^1(\mu)$ the integral*

$$\int_{\Sigma_d} \theta_l f(l)\, d\mu(l)$$

defines a tempered distibution on G.
In particular, $\int_{\Sigma_d} \theta_l\, d\mu(l)$ is a tempered distribution for any $\mu \in \mathcal{M}_d$.

Proof. By Lemma 3.29

$$\begin{aligned} \left|\left[\int_{\Sigma_d} \theta_l f(l)\, d\mu(l)\right](\varphi)\right| &\leq \int_{\Sigma_d} |\theta_l(\varphi) f(l)|\, d\mu(l) \\ &\leq \int_{\Sigma_d} \gamma_d(\varphi) |\mathrm{Pf}_d(l)|^{-1} f(l)\, d\mu(l) \\ &= \gamma_d(\varphi) \cdot C < \infty. \end{aligned}$$

Hence the integral defines a distribution on G which is tempered since $\gamma_d(\varphi) \to 0$ if $\varphi \to 0$ in $\mathcal{S}(G)$. ☐

Proposition 5.5. *For every $\mu \in \mathcal{M}_d(\pi)$ the distribution $\int_{\Sigma_d} \theta_l\, d\mu(l)$ from Lemma 5.4 is an element of $\overline{P(\pi)}^{\mathcal{D}'}$.*

Proof. Let $\varphi \in C_c^\infty(G)$. Firstly, we choose a sequence $(\chi_{K_k})_{k\in\mathbb{N}} = (\chi_k)_{k\in\mathbb{N}}$ of compact characteristic functions that exhaust $\Sigma_d = \bigcup_{k\in\mathbb{N}} K_k$ such that Pf_d is bounded away from zero by a constant c_k on K_k for all $k \in \mathbb{N}$. Since $\theta_l(\varphi)$ is integrable by the previous lemma we can write

$$\int_{\Sigma_d} \theta_l(\varphi)\, d\mu(l) = \lim_{k\to\infty} \int_{\Sigma_d} \chi_k(l)\theta_l(\varphi)\, d\mu(l).$$

Now, let $(\eta_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^{d_n/2})$ with $d_n = \dim \mathcal{O}_l$ for all $l \in \Sigma_d$, which is well-defined due to Remark 3.23 and the definition of $d \in \mathbb{N}^n$. Since each $\sigma_l(\varphi)$ is trace class we have

$$\int_{\Sigma_d} \theta_l(\varphi)\, d\mu(l) = \lim_{k\to\infty} \int_{\Sigma_d} \lim_{N\to\infty} \chi_k(l) \left(\int_G \sum_{i=1}^{N} \varphi(g) \langle \sigma_l(g)\eta_i, \eta_i \rangle\, dg \right) d\mu(l).$$

Thus, we consider for $N \in \mathbb{N}$ the projections $P_N(\lambda) := \sum_{i=1}^{N} \langle \lambda, \eta_i \rangle \eta_i$, $\lambda \in L^2(\mathbb{R}^{d_n/2}) \cong \mathcal{H}_l$ for each $l \in \Sigma_d$, and the functions

$$\Psi_{N,k}(l,g) := \chi_k(l)\varphi(g) \sum_{i=1}^{N} \langle \sigma_l(g)\eta_i, \eta_i \rangle, \qquad l \in \Sigma_d,\ g \in G,$$
$$f_{N,k}(l) := \chi_k(l) \int_G \sum_{i=1}^{N} \varphi(g) \langle \sigma_l(g)\eta_i, \eta_i \rangle \, dg = \chi_k(l) \operatorname{Tr}(\sigma_l(\varphi) P_N).$$

We want to apply the dominated convergence theorem with regard to the limit in N and therefore estimate again with Lemma 3.29

$$\begin{aligned} |f_{N,k}(l)|^2 &\le \|P_N\|_{\text{op}}^2 \cdot \|\sigma_l(\varphi)\|_{\text{Tr}}^2 \cdot \chi_k(l)^2 \\ &= \operatorname{Tr}(\sigma_l(\varphi)^* \sigma_l(\varphi)) \cdot \chi_k(l) = \operatorname{Tr}(\sigma_l(\varphi^* \star \varphi)) \cdot \chi_k(l) = \theta_l(\varphi^* \star \varphi) \cdot \chi_k(l) \\ &\le \gamma_d(\varphi^* \star \varphi) \cdot |\operatorname{Pf}_d(l)|^{-1} \cdot \chi_k(l) = \gamma'_{d,k}(\varphi) \cdot \chi_k(l), \end{aligned}$$

since $\|P_N\|_{\text{op}} = 1$ and $|\operatorname{Pf}_d(l)|^{-1} \le c_k^{-1}$ on $\operatorname{supp} \chi_k = K_k$. By assumption $\mu = f \cdot \mu_\pi$ with $f \in L^1_{\text{loc}}(\mu_\pi)$ and we conclude that $f_{N,K}$ is integrable with respect to μ. Now we can interchange the limit in N and the integral over Σ_d and obtain for all $\varphi \in C_c^\infty(G)$:

$$\left[\int_{\Sigma_d} \theta_l \, d\mu(l) \right](\varphi) = \lim_{k \to \infty} \lim_{N \to \infty} \int_{\Sigma_d} \left(\int_G \sum_{i=1}^{N} \chi_k(l) \varphi(g) \langle \sigma_l(g)\eta_i, \eta_i \rangle \, dg \right) d\mu(l).$$

In order to find a sequence in $P(\pi)$ converging to the given distribution we first define the vectors

$$\eta_i^{k,\mu} := \int_{\Sigma_d} \eta_i \cdot \chi_k(l) \, d\mu(l) = \int_{\Sigma_d} \eta_i \cdot \chi_k(l) f(l) \, d\mu_\pi(l) \in \mathcal{H}_\pi \quad \forall\ i \in \mathbb{N},$$

as $f \in L^2_{\text{loc}}(\mu_\pi)$ and consider for $N \in \mathbb{N}$ the projections $P^\mu_{N,k}(\lambda) := \sum_{i=1}^{N} \langle \lambda, \eta_i^{k,\mu} \rangle \eta_i^{k,\mu}$, $\lambda \in \mathcal{H}_\pi$, which are non-negative operators of rank $N < \infty$. Consequently, the distributions $\alpha_{N,k} := \operatorname{Tr}(\pi(\varphi) P^\mu_{N,k})$ are elements of $P(\pi)$ and we obtain

$$\begin{aligned} \alpha_{N,k}(\varphi) &= \int_G \int_{\Sigma_d} \chi_k(l)^2 \varphi(g) \sum_{i=1}^{N} \langle \sigma_l(g)\eta_i, \eta_i \rangle \, d\mu(l) \, dg \\ &= \int_{\Sigma_d} \left(\int_G \chi_k(l) \varphi(g) \sum_{i=1}^{N} \langle \sigma_l(g)\eta_i, \eta_i \rangle \, dg \right) d\mu(l), \end{aligned}$$

since for fixed $N, k \in \mathbb{N}$ the absolute value of the function $\Psi_{N,k}$ is integrable on $G \times \Sigma_d$ with respect to $dg \, d\mu$:

$$\int_G \int_{\Sigma_d} |\Psi_N(l,g)| \, d\mu(l) \, dg \le N \|\varphi\|_\infty \mu(K_k) dg(\operatorname{supp} \varphi) < \infty.$$

This proves the claim. $\square$

As a result of this and Proposition 2.19 we have:

$$\mathrm{WF}_e\left(\int_{\Sigma_d} \theta_l \, d\mu(l)\right) \subset \mathrm{WF}(\pi) \quad \forall \, \mu \in \mathcal{M}_d(\pi). \tag{5.14}$$

Now we can prove the first inclusion:

Proposition 5.6. *Let G be a nilpotent, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ and $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Then*

$$\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi) \subset \mathrm{WF}(\pi).$$

Proof. Let $\xi \in \mathrm{AC}(\mathcal{O} - \operatorname{supp}\pi)$. Fix an inner product $\langle \cdot, \cdot \rangle$ on $i\mathfrak{g}^*$. Without loss of generality, we may assume $|\xi| = 1$. Define the cones

$$\mathcal{C}_\varepsilon := \{\eta \in i\mathfrak{g}^* \mid \exists\, t > 0 : |\xi - t\eta| < \varepsilon\}, \quad \mathcal{C}_{\varepsilon,R} := \{\eta \in \mathcal{C}_\varepsilon \mid |\eta| > R\} \quad \text{for } \varepsilon, R > 0.$$

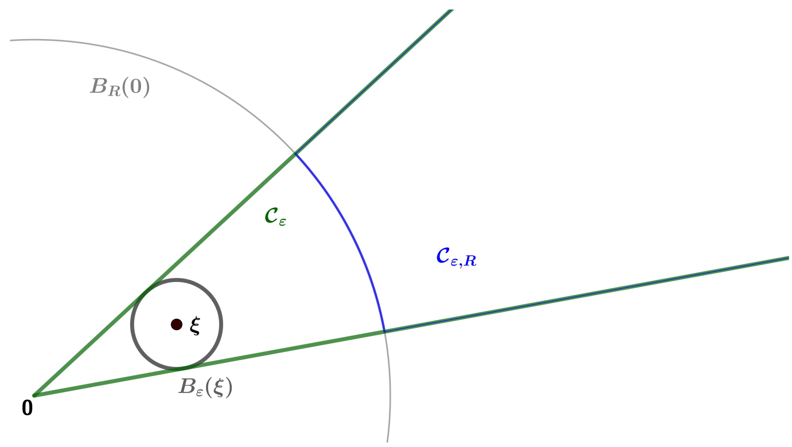


Figure 8: The cones $\mathcal{C}_\varepsilon$ and $\mathcal{C}_{\varepsilon,R}$

Let $\varepsilon > 0$. We recall from the discussions around Equation (4.8) that we may assume $\mathcal{O} - \operatorname{supp}(\pi) \subset U_d$. Thus, by the cross-section property of Σ_d we know that for any $l \in \mathcal{O} - \operatorname{supp}(\pi)$ the corresponding orbit meets Σ_d in exactly one point. Defining the set of all points of the orbital support in the cross-section Σ_d whose orbits intersect the cone $\mathcal{C}_\varepsilon$ outside of the ball of radius R as

$$A_{\varepsilon,R} := \{l \in \mathcal{O} - \operatorname{supp}(\pi) \cap \Sigma_d \mid \mathcal{O}_l \cap C_{\varepsilon,R} \neq \varnothing\} \subset \Sigma_d,$$

we have $A_{\varepsilon,R+\delta} \subset A_{\varepsilon,R}$ for all $\delta > 0$ and by assumption

$$A_{\varepsilon,R} = \bigsqcup_{k=0}^{\infty} \left(A_{\varepsilon,R+k} \setminus A_{\varepsilon,R+k+1}\right) \sqcup \left(\bigcap_{k=R}^{\infty} A_{\varepsilon,k}\right) \neq \varnothing \quad \forall \; R > 0.$$

We set $D_{\varepsilon,k} := A_{\varepsilon,k} \setminus A_{\varepsilon,k+1}$ and $L_\varepsilon = \bigcap_{k=R}^{\infty} A_{\varepsilon,k}$.

In case that for all $\varepsilon > 0$ we have $L_\varepsilon \neq \varnothing$, there exists $l_0 \in \mathcal{O} - \operatorname{supp}\pi$ such that $\mathcal{O}_{l_0} \cap \mathcal{C}_\varepsilon$ is unbounded. Thus, $\xi \in \mathrm{AC}(\mathcal{O}_{l_0}) = \mathrm{WF}(\theta_{l_0})$ and it suffices to show $\theta_{l_0} \in \overline{P(\pi)}^{\mathcal{D}'}$ due to Proposition 2.19:

Consider a family of positive functions $\phi_\delta \in C_c^\infty(\Sigma_d)$ such that $\int_{\Sigma_d} \phi_\delta \, d\mu_\pi = 1$ for all $\delta > 0$ whereas the support of ϕ_δ converges to the point $l_0 \in \Sigma_d$ as δ tends to zero. Since $\phi_\delta \in L^1_{\mathrm{loc}}(\mu_\pi) \cap L^2_{\mathrm{loc}}(\mu_\pi)$, Proposition 5.5 gives us $\int_{\Sigma_d} \theta_l \phi_\delta(l)\, d\mu_\pi(l) \in \overline{P(\pi)}^{\mathcal{D}'}$ for all $\delta > 0$. Recall that for $\varphi \in C_c^\infty(G)$ with (3.5) and Lemma 3.28

$$\theta_l(\varphi) = \int_{\mathcal{O}_l} \mathcal{F}^+(\varphi)\, d\vartheta_l = |\mathrm{Pf}_d(l)|^{-1} \int_{V_{S(d)}} \mathcal{F}^+(\varphi)(\Psi_d(l,v))\, dm_1'(v).$$

Hence, $\theta_l(\varphi)$ is continuous in l and we deduce that

$$\begin{aligned} \theta_{l_0}(\varphi) &= \lim_{\delta \to 0} \int_{\Sigma_d} \theta_l(\varphi)\phi_\delta(l)\, d\mu_\pi(l) \quad \forall\, \varphi \in C_c^\infty(G) \\ \Rightarrow \quad \theta_{l_0} &= \lim_{\delta \to 0} \int_{\Sigma_d} \theta_l \phi_\delta(l)\, d\mu_\pi(l) \in \overline{P(\pi)}^{\mathcal{D}'}. \end{aligned}$$

Now consider the case where $L_{\varepsilon_0} = \varnothing$ for some $\varepsilon_0 > 0$ and hence for all $0 < \varepsilon \le \varepsilon_0$. Then the disjoint union $\bigsqcup_{k=R}^\infty D_{\varepsilon,k}$ is non-empty for all $R > 0$. Since $D_{\varepsilon,k} \subset \mathcal{O} - \operatorname{supp}\pi$ for all k and any $0 < \varepsilon \le \varepsilon_0$, there exists a sequence $k_m \to \infty$ such that $\mu_\pi(D_{\varepsilon,k_m}) > 0$ for all $m \in \mathbb{N}$. Now choose for every $m \in \mathbb{N}$ an element $l_m \in D_{\varepsilon,k_m}$ and a corresponding $t_m\eta_m \in \mathcal{O}_{l_m} \cap \mathcal{C}_\varepsilon$ with $\eta_m \in B_\varepsilon(\xi) \cap B_1(0)$ and $t_m \to \infty$.

To show that $\xi \in \mathrm{WF}_e\left(\int_{\Sigma_d} \theta_l\, d\mu(l)\right)$ for a suitable $\mu \in \mathcal{M}_d(\pi)$, we now want to use Folland's characterization of a wave front set in terms of the wave packet transform of the distribution (see Theorem 2.10) analogously to the beginning of [HHÓ16, Proof of Proposition 6.1].
Fix an even Schwartz function $\mathcal{F}(\varphi) \in \mathcal{S}(i\mathfrak{g}^*)$ such that $\mathcal{F}(\varphi)(x) \ge 0$ for all x and $\mathcal{F}(\varphi)(x) = 1$ if $|x| \le 1$. Then $\mathcal{F}(\varphi)$ is the Fourier transform of an even Schwartz function $\varphi \in \mathcal{S}(\mathfrak{g})$. By Remark 2.9 and (2.3) in order to show that

$$(0,\xi) \in \mathrm{WF}_\varphi\left(\int_{\Sigma_d} \theta_l\, d\mu(l)\right) \overset{\text{Thm. 2.10}}{=} \mathrm{WF}\left(\int_{\Sigma_d} \theta_l\, d\mu(l)\right),$$

we must find for $\varepsilon > 0$ a constant $C > 0$, an integer $N \in \mathbb{N}$, and a sequence $(t_m\eta_m)_{m\in\mathbb{N}} \subset i\mathfrak{g}^*$ with $\eta_m \in B_\varepsilon(\xi) \cap B_1(0)$ and $t_m \to \infty$ such that

$$\begin{aligned} Ct_m^{-N} &\le \left| e^{\pi t_m \eta_m(0)} t_m^{-n/4} \mathcal{F}\left(\int_{\Sigma_d} \theta_l\, d\mu(l)\right)\left[e^{-2\pi\langle t_m\eta_m - \bullet, 0\rangle}\mathcal{F}(\varphi)(t_m^{-1/2}(t_m\eta_m - \bullet))\right]\right| \\ &= t_m^{-n/4}\left|\left(\int_{\Sigma_d} \vartheta_l\, d\mu(l)\right)\left[\mathcal{F}(\varphi)(t_m^{-1/2}(t_m\eta_m - \bullet))\right]\right| \\ &= t_m^{-n/4}\left|\int_{\Sigma_d}\left(\int_{\mathcal{O}_l} \mathcal{F}(\varphi)\left(\frac{t_m\eta_m - \zeta}{\sqrt{t_m}}\right) d\vartheta_l(\zeta)\right) d\mu(l)\right|. \end{aligned} \tag{5.15}$$

Our first goal is to obtain polynomial estimates for the inner integral. We start by defining for $l'_m \in \Sigma_d$ near l_m the sets

$$B_{m,l'_m} := \left\{ \zeta \in \mathcal{O}_{l'_m} \cap \mathcal{C}_\varepsilon \mid |\zeta - t_m\eta_m| < 1 \right\} \subset i\mathfrak{g}^*.$$

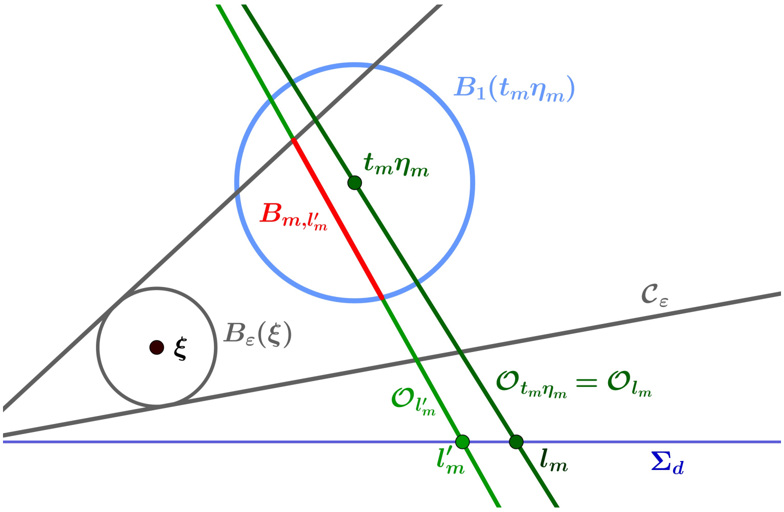


Figure 9: Definition of the sets B_{m,l'_m}

Again, by Lemma 3.28 we know that $\vartheta_l = |\operatorname{Pf}_d(l)|^{-1} \left(\operatorname{Id}_{V_{S(d)}} + p_{T(d)}(\psi_d(l,\cdot))\right)_* (m'_1)$ where m'_1 is the Eucledian measure on the vector space $V_{S(d)} \subset \mathfrak{g}^*$, that the Pfaffian $\operatorname{Pf}_d(l)$ is a polynomial in l and that the parametrization $p_{T(d)} \circ \psi_d(l,v)$ is a polynomial in $v \in V_{S(d)}$ as Theorem 3.20 states.
Consider the tangent spaces to the points $\eta \in S^{n-1} \cap \mathcal{C}_\varepsilon$. Given $\varepsilon' > 0$, after possibly shrinking $\mathcal{C}_\varepsilon$, i.e. $\varepsilon > 0$, the tangent spaces $T_\eta \mathcal{O}_\eta$ only vary by an angle less than ε'. If we choose ε' small enough, we can approximate $B_{m,l_m} \subset \mathcal{O}_{t_m\eta_m}$ by its tangent space. With all of the above, the choice of $t_m \to \infty$ (in particular, $t_m\varepsilon \geq 2$ for sufficiently large m) and since the coadjoint action is linear, we can approximate the orbital measure and deduce that for sufficiently large m and some $k \in \mathbb{N}$:

$$\nu_{l_m}(B_{m,l_m}) := \left(\operatorname{Id}_{V_{S(d)}} + p_{T(d)}(\psi_d(l_m,\cdot))\right)_* (m'_1)(B_{m,l_m}) \geq t_m^{-k}.$$

With this definition and the choice of $\mathcal{F}(\varphi)$ we can estimate

$$\int_{B_{m,l_m}} \mathcal{F}(\varphi)\left(\frac{t_m\eta_m - \zeta}{\sqrt{t_m}}\right) d\nu_{l_m}(\zeta) \geq \int_{B_{m,l_m}} 1 \, d\nu_{l_m}(\zeta) \geq t_m^{-k}.$$

Since $\int_{B_{m,l}} \mathcal{F}(\varphi)\left(\frac{t_m\eta_m - \zeta}{\sqrt{t_m}}\right) d\nu_l(\zeta)$ is a continuous function of $l \in \Sigma_d$ we can find for each index m neighborhoods $N_m \subset \Sigma_d$ of l_m such that

$$\int_{B_{m,l}} \mathcal{F}(\varphi)\left(\frac{t_m\eta_m-\zeta}{\sqrt{t_m}}\right) d\nu_l(\zeta) \geq \frac{1}{2}t_m^{-k} \quad \forall\, l \in N_m. \tag{5.16}$$

Now, we can find a suitable measure $\mu \in \mathcal{M}_d(\pi)$ such that the estimate (5.15) holds: First of all, we may also assume $\mu_\pi(N_m) > 0$ and

$$N_m \cap \left(\bigcup_{k\neq k_m} D_k\right) = \varnothing \quad \Rightarrow \quad N_m \cap N_k \neq \varnothing \quad \forall\, k \neq m.$$

After possibly shrinking the sets $N_m \subset \Sigma_d$ there exist constants $C_m > 0$ such that

$$\frac{1}{2}C_m \leq \inf_{l\in N_m} |\mathrm{Pf}_d(l)| \leq \sup_{l\in N_m} |\mathrm{Pf}_d(l)| \leq C_m,$$

since the Pfaffian Pf_d is continuous.
Let $\chi_m := \chi_{N_m}$ be the characteristic functions of the sets $N_m \subset \Sigma_d$. Now we choose

$$f := \sum_m C_m t_m^{-2}\mu_\pi(N_m)^{-1}\cdot\chi_m \quad \text{and} \quad \mu := f\cdot\mu_\pi.$$

Notice that $f \in L^1_{\mathrm{loc}}(\mu_\pi) \cap L^2_{\mathrm{loc}}(\mu_\pi)$ and

$$\begin{aligned}\int_{\Sigma_d} |\mathrm{Pf}_d(l)|^{-1}\,d\mu(l) &= \sum_m C_m t_m^{-2}\mu_\pi(N_m)^{-1}\int_{N_m} |\mathrm{Pf}_d(l)|^{-1}\,d\mu_\pi(l)\\ &\leq \sum_m C_m t_m^{-2}\mu_\pi(N_m)^{-1}\mu_\pi(N_m)\cdot 2C_m^{-1} = 2\sum_m t_m^{-2} < \infty,\end{aligned}$$

since $t_m \geq k_m \geq m$. This shows $\mu \in \mathcal{M}_d(\pi)$.
Finally, we obtain by definition of $B_{m,l}$ and N_m and the choice of μ the desired estimation:

$$\begin{aligned}&t_m^{-n/4}\left|\int_{\Sigma_d}\left(\int_{\mathcal{O}_l}\mathcal{F}(\varphi)\left(\frac{t_m\eta_m-\zeta}{\sqrt{t_m}}\right)d\vartheta_l(\zeta)\right)d\mu(l)\right|\\
&\geq t_m^{-n/4}\left|\int_{N_m}\left(\int_{B_{m,l}}\mathcal{F}(\varphi)\left(\frac{t_m\eta_m-\zeta}{\sqrt{t_m}}\right)d\vartheta_l(\zeta)\right)d\mu(l)\right|\\
&= t_m^{-n/4}\left|\int_{N_m}\left(\int_{B_{m,l}}\mathcal{F}(\varphi)\left(\frac{t_m\eta_m-\zeta}{\sqrt{t_m}}\right)|\mathrm{Pf}_d(l)|^{-1}\,d\nu_l(\zeta)\right)d\mu(l)\right|\\
&\overset{(5.16)}{\geq} \frac{1}{2}t_m^{-n/4-k}\cdot\int_{N_m}|\mathrm{Pf}_d(l)|^{-1}\,d\mu(l)\\
&= \frac{1}{2}t_m^{-n/4-k-2}C_m\mu_\pi(N_m)^{-1}\int_{N_m}|\mathrm{Pf}_d(l)|^{-1}\,d\mu_\pi(l) \;\geq\; \frac{1}{2}t_m^{-n/4-k-2}.\end{aligned}$$

The claim now follows with (5.14). □

5.1.2 The Inclusion $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O} - \mathrm{supp}(\pi))$

Now we turn to the second inclusion $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi)$. Here we would like to use the wave front set of the integrated characters again as an intermediate step due to the following

Lemma 5.7. *For every positive measure μ on Σ_d and every function f on Σ_d such that $f \cdot \mathrm{Pf}_d^{-1} \in L^1(\mu)$ (that is as in Lemma 5.4), we have*

$$\mathrm{WF}_e\left(\int_{\Sigma_d} \theta_l f(l)\, d\mu(l)\right) \subset \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi).$$

Proof. This follows by [Hör03, Lemma 8.4.17] from the fact that $\int_{\Sigma_d} \theta_l f(l)\, d\mu(l)$ is the Fourier transform of the tempered distribution $\int_{\Sigma_d} \vartheta_l f(l)\, d\mu(l)$ which is supported in $\mathcal{O} - \operatorname{supp} \pi = \bigcup_{\sigma \in \operatorname{supp} \pi} \mathcal{O}_\sigma$. □

Following the proof of [HHÓ16, Proposition 7.1] we obtain a weaker statement in our setting:

Proposition 5.8. *If there exists $\varepsilon > 0$ such that $|\mathrm{Pf}_d(l)| > \varepsilon$ for all $l \in \operatorname{supp} \mu_\pi \subset \Sigma_d$ then*

$$\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi).$$

Proof. Recall again with (4.8) that $\mathcal{H}_\pi = \int_{\Sigma_d} \mathcal{H}_l^{\oplus m(\pi,\sigma_l)}\, d\mu_\pi(l)$ for the Hilbert space of the unitary representation π. By Remark 3.23 we know that for all $l \in \Sigma_d$ we have $\mathcal{H}_l \cong L^2(\mathbb{R}^{d_n/2})$, $d_n = \dim \mathcal{O}_l$ by definition of $d \in \mathbb{N}^n$. If $u = (u_l), v = (v_l) \in \mathcal{H}_\pi$ in this direct integral decomposition the matrix coefficient is

$$\langle \pi(g)u, v\rangle = \int_{\Sigma_d} \langle \sigma_l(g)u_l, v_l\rangle\, d\mu_\pi(l).$$

Thus for $\eta \in i\mathfrak{g}^*$, $\phi \in C_c^\infty(G)$

$$\begin{aligned}
|\mathcal{F}(\langle \pi(\cdot)u,v\rangle\phi)(t\eta)| &= \left|\int_G \langle \pi(g)u, v\rangle \phi(g) e^{-2\pi t\eta(\log g)}\, dg\right| \\
&= \left|\int_G \int_{\Sigma_d} \langle \sigma_l(g)u_l, v_l\rangle \phi(g) e^{-2\pi t\eta(\log g)}\, d\mu_\pi(l)\, dg\right| \\
&= \left|\int_{\Sigma_d} \left(\int_G \langle \sigma_l(g)u_l, v_l\rangle \phi(g) e^{-2\pi t\eta(\log g)}\, dg\right) d\mu_\pi(l)\right| \\
&= \left|\int_{\Sigma_d} \langle \sigma_l(\phi e^{-2\pi t\eta(\log)})u_l, v_l\rangle\, d\mu_\pi(l)\right| \\
&\le \int_{\Sigma_d} \|\sigma_l(\phi e^{-2\pi t\eta(\log)})u_l\| \cdot \|v_l\|\, d\mu_\pi(l) \\
&\le \left(\int_{\Sigma_d} \|\sigma_l(\phi e^{-2\pi t\eta(\log)})\|_{\mathrm{HS}}^2 \cdot \|u_l\|^2\, d\mu_\pi(l)\right)^{1/2} \cdot \left(\int_{\Sigma_d} \|v_l\|^2\, d\mu_\pi(l)\right)^{1/2}. \qquad (5.17)
\end{aligned}$$

Here the third equality holds as $|\langle\sigma_l(g)u_l, v_l\rangle\phi(g)e^{2\pi t\eta(\log g)}| \leq |\phi(g)| \cdot \|u_l\|\|v_l\|$ with compactly supported $\phi \in C_c^\infty(G)$ and $u, v \in L^2(\Sigma_d, \mu_\pi)$ so that we may interchange the order of integration.
Following the calculation in [How81, p. 128] we obtain

$$\begin{aligned}
\|\sigma(\varphi)\|_{\mathrm{HS}}^2 &= \mathrm{Tr}(\sigma(\varphi)^*\sigma(\varphi)) = \mathrm{Tr}(\sigma(\varphi^* \star \varphi)) = \int_G \mathrm{Tr}(\sigma(h))(\varphi^* \star \varphi)(h)\, dh \\
&= \int_G \mathrm{Tr}(\sigma(h)) \int_G \varphi^*(g)\varphi(g^{-1}h)\, dg\, dh \\
&= \int_G \varphi^*(g) \Big(\int_G \mathrm{Tr}(\sigma(h)) L_g(\varphi)(h)\, dh \Big)\, dg \\
&= \int_G \varphi^*(g)\, \mathrm{Tr}(\sigma(L_g\varphi))\, dg \qquad \forall\ \sigma \in \hat{G},\ \varphi \in C_c^\infty(G),
\end{aligned}$$

where the order of integration can be interchanged since both φ and φ^* have compact support. Applying this equality here we have

$$\|\sigma_l(\phi e^{-2\pi t\eta(\log)})\|_{\mathrm{HS}}^2 = \int_G \overline{\phi}\left(g^{-1}\right) e^{-2\pi t\eta(\log(g))} \cdot \theta_l \left[L_g(\phi(\bullet) e^{-2\pi t\eta(\log(\bullet))}) \right] dg,$$

and integrating both sides over Σ_d with respect to $|u_l|^2\, d\mu_\pi(l)$ yields

$$\begin{aligned}
&\int_{\Sigma_d} \|\sigma_l(\phi e^{-2\pi t\eta(\log)})\|_{\mathrm{HS}}^2 \cdot \|u_l\|^2\, d\mu_\pi(l) \\
&\quad = \int_{\Sigma_d} \int_G \overline{\phi}\left(g^{-1}\right) e^{-2\pi t\eta(\log(g^{-1}))} \cdot \theta_l \left[L_g(\phi(\bullet) e^{-2\pi t\eta(\log(\bullet))}) \right] \|u_l\|^2\, dg\, d\mu_\pi(l) \\
&\quad = \int_G \overline{\phi}(g^{-1}) e^{-2\pi t\eta(\log(g^{-1}))} \Big(\int_{\Sigma_d} \theta_l \|u_l\|^2\, d\mu_\pi(l) \Big) \left[L_g(\phi(\bullet) e^{-2\pi t\eta(\log(\bullet))}) \right] dg.
\end{aligned}$$

The interchanging of the order of integration is possible since we have the estimate

$$\begin{aligned}
&\left| \overline{\phi}\left(g^{-1}\right) e^{-2\pi t\eta(\log(g^{-1}))} \cdot \theta_l \left[L_g(\phi e^{-2\pi t\eta(\log)}) \right] \|u_l\|^2 \right| \\
&\qquad\qquad \leq \left| \overline{\phi}\left(g^{-1}\right) \right| \gamma_d \Big(L_g(\phi e^{-2\pi t\eta(\log)}) \Big) |\mathrm{Pf}_d(l)|^{-1} \|u_l\|^2.
\end{aligned}$$

Here $\overline{\phi}$ has compact support and

$$G \to C_c^\infty \to \mathbb{R}_{>0},\ g \mapsto L_g(\phi e^{-2\pi t\eta(\log)}) \mapsto \gamma_d \Big(L_g(\phi e^{-2\pi t\eta(\log)}) \Big)$$

is continuous (the second map by Lemma 3.29) so that it is integrable over the compact set $\mathrm{supp}(\overline{\phi})^{-1}$.
In the integral over Σ_d we have $\|u_l\|^2 \in L^1(\mu_\pi)$ by definition and $\|u_l\|^2 |\mathrm{Pf}_d|^{-1} \leq \frac{1}{\varepsilon}\|u_l\|^2$ almost surely with respect to μ_π since we assumed $|\mathrm{Pf}_d| > \varepsilon$ on $\mathrm{supp}\,\mu_\pi$. This gives us $\|u_l\|^2\, \mathrm{Pf}_d^{-1} \in L^1(\mu_\pi)$.

Now let $\xi \notin \mathrm{AC}(\mathcal{O} - \operatorname{supp}\pi)$. With Lemma 5.7 we know $\xi \notin \mathrm{WF}_e\left(\int_{\Sigma_d} \theta_l \|u_l\|^2 \, d\mu_\pi(l)\right)$ and by the same argument $\xi \notin \mathrm{SS}_e\left(\int_{\Sigma_d} \theta_l \|u_l\|^2 \, d\mu_\pi(l)\right)$, the singular spectrum which is a similar concept to the wave front set (see [HHÓ16, Definitions 2.2 and 2.3]). Furthermore, $u = \int_{\Sigma_d} \exp^* \theta_l \|u_l\|^2 \, d\mu_\pi(l)$ is a tempered distribution on $\mathfrak{g}$ with $\xi \notin \mathrm{SS}_0(u)$.

When we take the analytic map $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (Y, X) \mapsto X * Y$, [HHÓ16, Proposition 7.2] gives us of the existence of open sets $\log U_1 \subset \log U \subset \log \tilde{U}$ containing 0 such that the closure of $\log U_1$ is contained in the interior of $\log U$ together with an open set $\Omega \subset i\mathfrak{g}^*$ containing ξ such that for every $N \in \mathbb{N}$ there exists a constant $C = C_{N,U_1,U} > 0$ with

$$\left| \int_{\Sigma_d} \left(\int_{\mathfrak{g}} \theta_l(\exp Y \exp X)(\exp^* \phi)(X) e^{-2\pi t \eta(X)} \, dX \right) \|u_l\|^2 \, d\mu_\pi(l) \right| \le C^N t^{-N},$$

whenever $\eta \in \Omega, Y \in \log \tilde{U}$ and $t > 0$. Going back the group we obtain

$$\left| \int_{\Sigma_d} \left(\int_G \theta_l(gh)\phi(h) e^{-2\pi t \eta(\log h)} \, dh \right) \|u_l\|^2 \, d\mu_\pi(l) \right| \le C^N t^{-N},$$

and the assertion of Lemma 5.4 yields

$$\left| \left(\int_{\Sigma_d} \theta_l \|u_l\|^2 \, d\mu_\pi(l) \right) \left[L_g(\phi e^{-2\pi t \eta(\log)}) \right] \right| \le C^N t^{-N},$$

whenever $g \in \tilde{U}$, $\eta \in \Omega$ and $t > 0$. Now, if we integrate over g in a precompact set in G with respect to a smooth density multiplied by a bounded function, then this will simply multiply the bound by a constant, which we may absorb in C. Thus we obtain

$$\left| \int_G \overline{\phi}(g^{-1}) e^{-2\pi t \eta(\log(g^{-1}))} \left(\int_{\Sigma_d} \theta_l \|u_l\|^2 \, d\mu_\pi(l) \right) \left[L_g(\phi e^{-2\pi t \eta(\log)}) \right] dg \right| \le C^N t^{-N}$$

for $\eta \in \Omega$ and $t > 0$. Tracing back our calculations this gives us

$$|\mathcal{F}(\langle \pi(\cdot)u, v\rangle \phi)(t\eta)| \le C^{N/2} t^{-N/2}$$

for $\eta \in \Omega$ and $t > 0$. By replacing N by $2N$ (and C_{2N} by C_N) we see $\xi \notin \mathrm{WF}(\pi)$ which proves the claim. □

Remark 5.9. *The assumption on the support* $\operatorname{supp}\mu_\pi$ *in the previous Proposition 5.8 is in fact a restriction: it can already be seen in the example of the Heisenberg group* H_1 *where* $\Sigma_1 = (\mathbb{R} \setminus \{0\}) \cdot Z^*$ *and* $\mathrm{Pf}_1(l) = l$, $l \in \mathbb{R} \setminus \{0\}$.

Remark 5.10. *Comparing Proposition 5.8 with Theorem 4.7 we only get a weaker statement with the above proof. Analyzing this proof in order to determine where one essentially looses something, we see that the crucial estimation is in (5.17) where we basically estimate* one *matrix coefficient* $|\langle \sigma(\varphi)^* \sigma(\varphi) u_l, u_l\rangle| = \|\sigma(\varphi) u_l\|$ *by its trace* $\mathrm{Tr}(\sigma(\varphi)^* \sigma(\varphi))$, *thus summing up* all *matrix coefficients (of an orthogonal basis).*

Remark 5.11. *The above proof does work for real reductive, algebraic groups in [HHÓ16] since there the integrated characters $\int \theta_\sigma f(\sigma)\,d\mu(\sigma)$ define a tempered distribution for all $f \in L^1(\mu)$ (see [HHÓ16, Lemma 6.2] which relies on a result of Harish-Chandra for the so-called invariant integral) whereas here in the nilpotent case we need $f \cdot \mathrm{Pf}_d^{-1} \in L^1(\mu)$ (see Lemma 5.4 and Lemma 5.7).*

5.2 Integral Kernels of Integrated Representations

In order to show the second inclusion $\mathrm{WF}(\pi) \subset \mathrm{AC}(\mathcal{O}\!-\!\operatorname{supp}\pi)$ for general π, i.e. unitary representations that contain irreducible representation arbitrary close to the zeros of the Pfaffian, we take another look at the Fourier transform of the matrix coefficient with regard to the integral kernels of the trace class operators $\sigma_l(\phi e^{R\eta(\log)})$ as given by Proposition 3.26:

$$\begin{aligned}\mathcal{F}(\langle\pi(\cdot)u,v\rangle\phi)(R\eta) &= \int_G \langle\pi(g)u,v\rangle\phi(g)e^{-2\pi R\eta(\log g)}\,dg \\ &= \int_{\Sigma_d} \left\langle \sigma_l\left(\phi(g)e^{-2\pi R\eta(\log)}\right)u_l, v_l\right\rangle d\mu_\pi(l) \\ &= \int_{\Sigma_d}\int_{\mathbb{R}^{2k}}\int_{\mathfrak{m}} e^{2\pi(l(Y)-R\eta\log(\beta(s)^{-1}\exp(Y)\beta(t)))}\,\cdot \\ &\qquad \phi(\beta(s)^{-1}\exp(Y)\beta(t))u_l(s)\overline{v_l(t)}\,dY\,d(s,t)\,d\mu_\pi(l),\end{aligned}$$

where $\{X_1,\dots,X_n\}$ is a weak Malcev basis through the polarizing subalgebra $\mathfrak{m}$ for $l \in i\mathfrak{g}^*$ with $p = \dim\mathfrak{m} = n-k$, $k = \frac{1}{2}\dim\mathcal{O}_l$ and $\beta\colon \mathbb{R}^k \to G, t \mapsto \exp(t_1X_{p+1})\cdots\exp(t_kX_n)$.

Let $\mathfrak{h} := \operatorname{span}\{X_{p+1},\dots,X_n\}$ be a compliment of $\mathfrak{m}$ in $\mathfrak{g}$. Assuming there exists a coordinate change

$$(X, P(s,t)) := \log(\beta(s)^{-1}\exp(Y)\beta(t)) \in \mathfrak{g} = \mathfrak{m}\oplus\mathfrak{h},$$

$dX = dY$, with $Y = Q(X,s,t)$ and polynomials $P : \mathbb{R}^{2k} \to \mathfrak{h}$ and $Q : \mathbb{R}^p \times \mathbb{R}^{2k} \to \mathfrak{m}$, we have

$$\mathcal{F}(\langle\pi(\cdot)u,v\rangle\phi)(R\eta) = \int\limits_{\Sigma_d}\int\limits_{\mathbb{R}^{2k}}\int\limits_{\mathfrak{m}} e^{2\pi(l(Q(X,s,t))-R\eta(X,P(s,t)))}\phi(\exp(X,P(s,t)))u_l(s)\overline{v_l(t)}\,dX\,d(s,t)\,d\mu_\pi(l).$$

If in addition to that the first part of the exponent is linear in X, i.e.

$$l(Q(\bullet,s,t)) = \sum_{j=1}^n q_j(s,t,l)X_j^* =: f(s,t,l) \in \mathfrak{m}^*,$$

and we choose a cut-off function ϕ such that $\phi(\exp(X,Z)) = \phi_X(X)\phi_r(Z)$, $X \in \mathfrak{m}$, $Z \in \mathfrak{h}$, we obtain

$$\begin{aligned}
&\mathcal{F}(\langle\pi(\cdot)u,v\rangle\phi)(R\eta) = \\
&\quad\int\limits_{\Sigma_d}\int\limits_{\mathbb{R}^{2k}}\int\limits_{\mathfrak{m}} e^{2\pi(f(s,t,l)(X)-R\eta(X,P(s,t)))}\phi(\exp(X,P(s,t)))u_l(s)\overline{v_l(t)}\,dX\,d(s,t)\,d\mu_\pi(l) \\
&\quad=\int\limits_{\Sigma_d}\int\limits_{\mathbb{R}^{2k}} \hat{\phi}_X\left((R\eta_j - q_j(s,t,l))_{j\le p}\right)\phi_r(P(s,t))e^{-2\pi R\eta_r(P(s,t))}u_l(s)\overline{v_l(t)}\,d(s,t)\,d\mu_\pi(l),
\end{aligned}$$

where $\eta = (\sum_{j=1}^p \eta_j X_j^*) + \eta_r \in i\mathfrak{m}^* \oplus i\mathfrak{h}^*$.

Lemma 5.12. *We make the following assumptions:*

(i) There exist $\delta,\varepsilon>0$ such that $\|P(s,t)\|<\delta$ implies $\|s-t\|<\varepsilon$.

(ii) $\forall\,\eta\notin \mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi)\quad \exists\, j\in\{1,\dots,p\}, R_0,c_1,c_2>0,\, b\in\mathbb{N}\quad \forall\, R>R_0:$

$$|R\eta_j - q_j(s,t,l)| \ge c_1 R^{\frac{1}{b}} - c_2 \quad \forall\,\|s-t\|<\varepsilon,\ l\in\operatorname{supp}(\mu_\pi).$$

Then

$$\mathrm{WF}(\pi)\subset \mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi).$$

Proof. Let $\eta\notin\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi)$ and ϕ be a cut-off function ϕ with $\phi=\phi_X\cdot\phi_r$ as before and $\operatorname{supp}\phi_r\subset B_\delta(0)$. Then we have with the computations above:

$$\begin{aligned}
F_R &:= |\mathcal{F}(\langle\pi(\cdot)u,v\rangle\phi)(R\eta)| \\
&= \left|\int\limits_{\Sigma_d}\int\limits_{\mathbb{R}^{2k}} \hat{\phi}_X\left((R\eta_j - q_j(s,t,l))_{j\le p}\right)\phi_r(P(s,t))e^{-2\pi R\eta_r(P(s,t))}u_l(s)\overline{v_l(t)}\,d(s,t)\,d\mu_\pi(l)\right| \\
&\le \int\limits_{\Sigma_d}\int\limits_{\mathbb{R}^{2k}} C_N\,|R\eta_j - q_j(s,t,l)|^{-N}\,\phi_r(P(s,t))|u_l(s)|\cdot|v_l(t)|\,d(s,t)\,d\mu_\pi(l) \\
&\le C_N\int\limits_{\Sigma_d}\int\limits_{B_\varepsilon(0)}\int\limits_{\mathbb{R}^k} |R\eta_j - q_j(s,t,l)|^{-N}\,|u_l(s)|\cdot|v_l(s+a)|\,d(s,a)\,d\mu_\pi(l)
\end{aligned}$$

by assumption (i). With assumption (ii) we can further estimate:

$$\begin{aligned}
F_R &\le C_N|c_1R^{\frac{1}{b}}-c_2|^{-N}\int_{\Sigma_d}\int_{B_\varepsilon(0)}\int_{\mathbb{R}^k}|u_l(s)|\cdot|v_l(s+a)|\,d(s,a)\,d\mu_\pi(l) \\
&= C_N|c_1R^{\frac{1}{b}}-c_2|^{-N}\int_{\Sigma_d}\int_{B_\varepsilon(0)}\int_{\mathbb{R}^k}|u_l(s)|\cdot|T_a v_l(s)|\,d(s,a)\,d\mu_\pi(l) \\
&\le C_N|c_1R^{\frac{1}{b}}-c_2|^{-N}\int_{\Sigma_d}\int_{B_\varepsilon(0)}\langle u_l, T_a v_l\rangle\,da\,d\mu_\pi(l),
\end{aligned}$$

where T_a is the translation by $a \in B_\varepsilon(0)$ which is an isometry on L^2. This gives us

$$\begin{aligned}
F_R &\leq C_N |c_1 R^{\frac{1}{b}} - c_2|^{-N} \int_{\Sigma_d} \int_{B_\varepsilon(0)} \|u_l\|_{L^2} \|T_a v_l\|_{L^2} \, da \, d\mu_\pi(l) \\
&= C_N |c_1 R^{\frac{1}{b}} - c_2|^{-N} \int_{\Sigma_d} \int_{B_\varepsilon(0)} \|u_l\|_{L^2} \|v_l\|_{L^2} \, da \, d\mu_\pi(l) \\
&\leq C_{N,\varepsilon} |c_1 R^{\frac{1}{b}} - c_2|^{-N} \int_{\Sigma_d} \|u_l\|_{L^2} \|v_l\|_{L^2} \, d\mu_\pi(l) = C_{N,\varepsilon} |c_1 R^{\frac{1}{b}} - c_2|^{-N} \left\langle \|u_l\|_{L^2}, \|v_l\|_{L^2} \right\rangle_{\mathcal{H}_\pi} \\
&\leq C_{N,\varepsilon} |c_1 R^{\frac{1}{b}} - c_2|^{-N} \|u\|_{\mathcal{H}_\pi} \|v\|_{\mathcal{H}_\pi} \in \mathcal{O}\left(R^{-\frac{N}{b}}\right).
\end{aligned}$$

This finishes the proof. □

Remark 5.13. *In particular, if we have $\eta \notin \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi)$ and want to check assumption (ii) of the previous Lemma it suffices to find $j \in \{1, \ldots, p\}$ such that*

$$\eta_j \neq 0 \quad \wedge \quad \operatorname{sign}(\eta_j) \cdot q_j(s,t,l) < Const. \quad \forall\, s,t \in \mathbb{R}^k, l \in \operatorname{supp} \mu_\pi.$$

Even though the assumptions in the previous lemma seem to be quite explicit it is actually all we needed in the following examples in order to show the desired inclusion without any assumptions regarding the unitary representation π.

Example 5.14. For the Heisenberg group H_n (see Section 3.2) we know in the case of the maximal dimensional orbits of type $d^{(1)}$, i.e. $2n$-dimensional affine planes, that $\mathrm{Pf}(l) = l(Z)^n$ and $\Sigma = \mathbb{R}^\times \cdot Z^*$. Now define for $\varepsilon > 0$ the set

$$N := \{l \in \Sigma \mid l(Z) > \sqrt[n]{\varepsilon}\} \cong \mathbb{R} \setminus B_{\sqrt[n]{\varepsilon}}(0)$$

and the characteristic function $\chi = \chi_N$. Let π be a unitary representation of H_n and define

$$\pi_1 := \int_{\mathbb{R}^\times} \sigma_l \cdot \chi \, d\mu_\pi(l), \quad \pi_2 := \int_{\mathbb{R}^\times} \sigma_l \cdot (1 - \chi) \, d\mu_\pi(l) \quad \Rightarrow \quad \pi = \pi_1 + \pi_2. \tag{5.18}$$

Then Proposition 5.8 gives us directly $\mathrm{WF}(\pi_1) \subset \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_1)$ and for the second part we can use Lemma 5.12 and Remark 5.13: Since

$$\mathrm{AC}(\mathcal{O}_l) = \operatorname{span}(X_1^*, \ldots, X_n^*, Y_1^*, \ldots Y_n^*) \quad \forall\, l \in \Sigma$$

we only have to regard $\eta \in \mathbb{R} \cdot Z^*$.
With the weak Malcev basis $\{Z, Y_1, \ldots Y_n, X_1, \ldots X_n\}$ through $\mathfrak{m} = \operatorname{span}(Z, Y_1, \ldots Y_n)$ and $\mathfrak{h} = \operatorname{span}\{X_1, \ldots X_n\}$ we compute

$$\beta(s)^{-1}\exp(zZ+\sum y_iY_i)\beta(t)=\exp\Big((z-\frac{1}{2}(t+s)\cdot y)Z+\sum y_iY_i+\sum(t_j-s_j)X_j\Big)$$
$$\Rightarrow\quad P(s,t)=\sum(t_j-s_j)X_j,\quad Q(z,y,t,s)=(z+\frac{1}{2}(t+s)\cdot y)Z+\sum y_iY_i$$
$$\Rightarrow\quad q_1(s,t,l)=l,\quad q_j(s,t,l)=\frac{1}{2}(t_j+s_j)l\ \text{ for } 2\le j\le n.$$

It is clear that q_1 is bounded on $\operatorname{supp}(1-\chi)=B_{\sqrt[n]{\varepsilon}}(0)$. Thus, we have $\eta\notin\mathrm{WF}(\pi_2)$ by Lemma 5.12 and Remark 5.13. This proves $\mathrm{WF}(\pi_2)\subset\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi_2)$. Combined:

$$\mathrm{WF}(\pi)\subset\mathrm{WF}(\pi_1)\cup\mathrm{WF}(\pi_2)\subset\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi_1)\cup\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi_2)=\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi).$$

The only other case consists of 0-dimensional orbits. But then $\mathrm{Pf}=1$ and the assumption of Proposition 5.8 is fulfilled for any unitary representation π.

Example 5.15. We consider the group K_3. From Section 3.3 we know that we have the three orbit types $D=\{d^{(1)}=(0,1,1,2),d^{(2)}=(0,0,1,2),d^{(3)}=(0,0,0,0)\}$.

$d^{(1)}$: In the case of maximal dimension we know that $\Sigma_1=\mathbb{R}^\times\cdot Z^*+\mathbb{R}\cdot X^*\ni(\delta,\beta)=l$ and $\mathrm{Pf}_1(l)=l(Z)=\delta$. Now, for $\varepsilon>0$ define the set

$$N:=\{l\in\Sigma\mid|l(Z)|>\varepsilon\}\cong(\mathbb{R}\setminus B_\varepsilon(0))\times\mathbb{R}$$

and the characteristic function $\chi=\chi_N$. Let π be a unitary representation of K_3 (supported in Σ_1). Defining π_1 and π_2 as in (5.18) we again have $\pi=\pi_1+\pi_2$ and $\mathrm{WF}(\pi_1)\subset\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi_1)$ by Proposition 5.8.

We know $\mathcal{O}_l=\delta Z^*+\{tY^*+(\beta+\frac{t^2}{2\delta})X^*+sW^*\mid s,t\in\mathbb{R}\}$ and its asymptotic cone is

$$\mathrm{AC}(\mathcal{O}_l)=\operatorname{sign}(\delta)\mathbb{R}_+\cdot X^*+\mathbb{R}\cdot W^*\quad\forall\ l=(\delta,\beta)\in\Sigma_1.$$

In particular, this gives us $\mathbb{R}\cdot W^*\subset\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi)$ and we only have to check the remaining three directions. To do this we further decompose $\pi_2:=\pi_2^++\pi_2^-$ with regard to the sign of δ:

$$\pi_2^+:=\int_{\mathbb{R}_+}\int_{\mathbb{R}}\sigma_{\delta,\beta}\cdot(1-\chi)d\mu_\pi(\delta,\beta),\quad \pi_2^-:=\int_{\mathbb{R}_-}\int_{\mathbb{R}}\sigma_{\delta,\beta}\cdot(1-\chi)d\mu_\pi(\delta,\beta).$$

We start with π_2^+: As $0<\delta<\varepsilon$ for all $(\delta,\beta)\in\mathcal{O}-\operatorname{supp}\pi_2$ we know

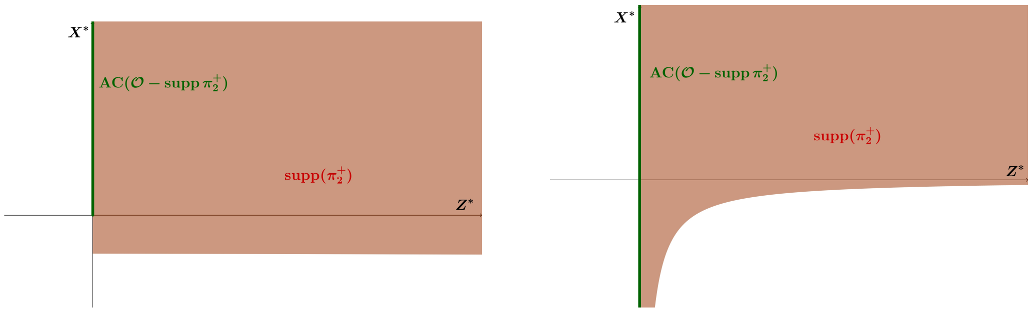
$$\mathbb{R}_+\cdot X^*+\mathbb{R}\cdot W^*\subset\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi_2^+)\subset\mathbb{R}\cdot Y^*+\mathbb{R}\cdot X^*+\mathbb{R}\cdot W^*.$$

In addition to that, we only have three different possibilities:

1) $\mathrm{AC}(\mathcal{O}-\operatorname{supp}\pi_2^+)=\mathbb{R}_+\cdot X^*+\mathbb{R}\cdot W^*\ \Leftrightarrow\ \exists\ C\in\mathbb{R}\ \forall\ (\delta,\beta)\in\operatorname{supp}\mu_{\pi_2^+}:\ \beta>C$ (see Figure 10a)

2) $\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^+) = \mathbb{R} \cdot X^* + \mathbb{R} \cdot W^* \;\Leftrightarrow\; \{\beta \mid (\delta, \beta) \in \operatorname{supp} \mu_{\pi_2^+}\}$ is not bounded from below but $\{-\delta\beta \mid (\delta, \beta) \in \operatorname{supp} \mu_{\pi_2^+}\} \subset \mathbb{R}$ is bounded from above (see Figure 10b)

3) $\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^+) = \mathbb{R} \cdot Y^* + \mathbb{R} \cdot X^* + \mathbb{R} \cdot W^* \;\Leftrightarrow\; \{-\delta\beta \mid (\delta, \beta) \in \operatorname{supp} \mu_{\pi_2^+}\}$ is not bounded from above.



(a) Case 1) **(b)** Case 2)

Figure 10: Relation between possible support of π_2^+ and the asymptotic orbital support projected onto the X^*, Z^*- plane

With the weak Malcev basis $\{Z, Y, X, W\}$ through $\mathfrak{m} = \operatorname{span}(Z, Y, X)$ and $\mathfrak{h} = \mathbb{R} \cdot W$, we compute

$$\beta(s)^{-1} \exp(zZ + yY + xX)\beta(t) = \exp\Big((z - \frac{1}{2}(t+s)y + \frac{1}{12}(t^2 + s^2 + 4st)x)Z + (y - \frac{1}{2}(t+s)x)Y + xX + (t-s)W\Big)$$

$$\Rightarrow Q(z, y, t, s) = (z - \frac{1}{2}(t+s)y + \frac{1}{12}(t^2 + s^2 + 4st)x)Z + (y - \frac{1}{2}(t+s)x)Y + xX$$

$$\Rightarrow q_1(s, t, l) = \delta, \quad q_2(s, t, l) = \frac{1}{2}(t+s)\delta, \quad q_3(s, t, l) = \Big(\frac{1}{6}(t^2 + ts + s^2)\delta + \beta\Big).$$

Now we check the assumption (ii) of Lemma 5.12 in the three cases from above in reverse order since it is a condition for all points that are *not* in the asymptotic cone:

3) It is clear that q_1 is bounded on $\operatorname{supp}(1 - \chi)$. Thus we have $\pm Z^* \notin \mathrm{WF}(\pi_2^+)$ (for all cases) by Lemma 5.12 and Remark 5.13.

2) If $\pm Y^* \notin \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^+)$ the set $\{-\delta\beta | (\delta, \beta) \in \operatorname{supp} \mu_\pi\} \subset \mathbb{R}$ is bounded from above.
Assuming that for all $c > 0$ there exists an sufficiently large R and $s \in \mathbb{R}$, $a \in [-\varepsilon, \varepsilon]$ such that $|q_2(s, s+a, \delta, \beta) \pm R| \leq \frac{1}{2}R$ (otherwise we would be already

done with $j = 2$), then $|s + \frac{1}{2}a| \geq \frac{R}{2\delta}$ and we estimate for $j = 3$:

$$\begin{aligned}
&|\,(RY^* - q(s, s+a, \delta, \beta))_3\,| = |q_3(s, s+a, \delta, \beta)| \\
&\quad = \left|\frac{1}{6}\left((s+a)^2 + (s+a)s + s^2\right)\delta + \beta\right| = \left|\frac{1}{6}(3s(s+a) + a^2)\delta + \beta\right| \\
&\quad = \left|\frac{1}{6}\left(3\left(s + \frac{1}{2}a\right)^2 - \frac{1}{2}a^2\right)\delta + \beta\right| \geq \frac{\delta}{6}\left(3\left(s + \frac{1}{2}a\right)^2 - \frac{1}{2}a^2\right) - |\beta| \\
&\quad \geq \frac{\delta}{6}\left(\frac{3R^2}{4\delta^2} - \frac{1}{2}a^2\right) - |\beta| \geq \frac{3R^2}{8\delta} - \frac{1}{12}a^2\delta - |\beta|.
\end{aligned}$$

If now $|\beta|$ is bounded, we immediately see the desired estimation as δ and a are bounded as well. If that is not the case we write further for $|\beta| > 1$ (w.l.o.g.):

$$|q_3(s, s+a, \delta, \beta)| \geq |\beta|\left(\frac{3R^2}{8\delta|\beta|} - \frac{a^2\delta}{12|\beta|} - 1\right) \geq \frac{3R^2}{8C_1} - \frac{\varepsilon^3}{12} - 1,$$

with the bound for $-\delta\beta$, which proves assumption (ii) in this case.

1) If $-X^* \notin \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^+)$ then there exists a constant $C \in \mathbb{R}$ such that $\beta > C$. Since $t^2 + ts + s^2 \geq 0$ for all $s, t \in \mathbb{R}$ we have

$$-q_3(s, t, l) = -\left(\frac{1}{6}(t^2 + ts + s^2)\delta + \beta\right) < -2C,$$

and Lemma 5.12 and Remark 5.13 give us $-X^* \notin \mathrm{WF}(\pi_2^+)$.

Now turning to π_2^- we have $-\varepsilon < \delta < 0$ and therefore

$$\mathbb{R}_- \cdot X^* + \mathbb{R} \cdot W^* \subset \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^-) \subset \mathbb{R} \cdot Y^* + \mathbb{R} \cdot X^* + \mathbb{R} \cdot W^*.$$

This means we only have the three following possibilities:

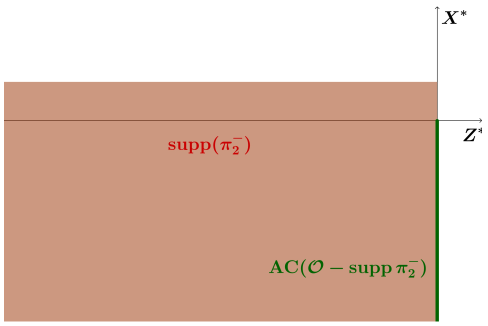
1') $\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^-) = \mathbb{R}_- \cdot X^* + \mathbb{R} \cdot W^* \Leftrightarrow \exists\, C \in \mathbb{R}\ \forall\ (\delta, \beta) \in \operatorname{supp} \mu_{\pi_2^-} :\ \beta < C$ (see Figure 11a).

2') $\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^-) = \mathbb{R} \cdot X^* + \mathbb{R} \cdot W^* \Leftrightarrow \left\{\beta \mid (\delta, \beta) \in \operatorname{supp} \mu_{\pi_2^-}\right\}$ is not bounded from above but $\left\{-\delta\beta \mid (\delta, \beta) \in \operatorname{supp} \mu_{\pi_2^-}\right\} \subset \mathbb{R}$ is bounded from above (see Figure 11b).

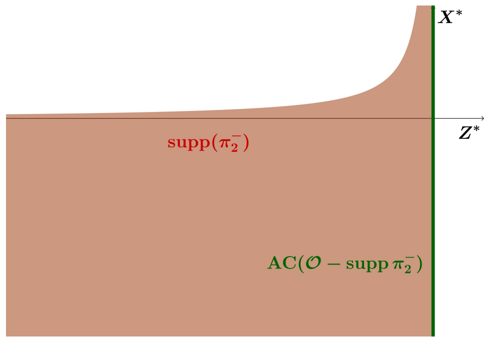
3') $\mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^-) = \mathbb{R} \cdot Y^* + \mathbb{R} \cdot X^* + \mathbb{R} \cdot W^* \Leftrightarrow \left\{-\delta\beta \mid (\delta, \beta) \in \operatorname{supp} \mu_{\pi_2^-}\right\}$ is not bounded from above.

Thus the computations are analogously to the ones for π_2^+. The only difference is: If $X^* \notin \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi_2^-)$ we have

$$q_3(s, t, l) = \frac{1}{6}(t^2 + ts + s^2)\delta + \beta < \frac{1}{3}(t^2 + ts + s^2)\varepsilon + 2C.$$



(a) Case 1')



(b) Case 2')

Figure 11: Relation between possible support of π_2^- and the asymptotic orbital support projected onto the X^*,Z^*– plane

Again, Lemma 5.12 and Remark 5.13 give us $X^* \notin \mathrm{WF}(\pi_2^-)$ in this case as well.

$d^{(2)}$: Here, we know $\Sigma_2 = \mathbb{R}^\times \cdot Y^* \ni \gamma$, $\mathcal{O}_\gamma = \gamma Y^* + \mathbb{R} \cdot X^* + \mathbb{R} \cdot W^*$, $\chi_\gamma(z, y, x, 0) = e^{2\pi i \gamma y}$ and $\mathrm{Pf}_2(\gamma) = \gamma$. In addition to that we can use the same $\mathfrak{m}$ as for $d^{(1)}$, thus have the same P and Q as in the case above and can compute

$$q_2(s,t,\gamma) = \gamma, \quad q_3(s,t,\gamma) = \frac{1}{2}(s+t)\gamma, \quad q_1 = 0 = q_4.$$

We have $\pm Z^* \notin \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi)$ for all π supported in Σ_2 and q_1 is obviously bounded, thus we know $\pm Z^* \notin \mathrm{WF}(\pi)$.
Since $\pm X^*, \pm W^* \in \mathrm{AC}(\mathcal{O}_\gamma)$ for all γ, we know $\pm X^*, \pm W^* \in \mathrm{AC}(\mathcal{O} - \operatorname{supp} \pi)$.
For the direction Y^* we use the same approach as above for Z^* in the Heisenberg group example (see (5.18)) and use Proposition 5.8 if γ is bounded away from 0, and Lemma 5.12 if $|\gamma| < \varepsilon$.

$d^{(3)}$: Again as in the Heisenberg group example, the trivial case consists of 0-dimensional orbits where $\mathrm{Pf} = 1$ and the assumption of Proposition 5.8 is fulfilled for any unitary representation π.

6 Applications to Restrictions of Representations

A fundamental problem in representation theory is the branching problem, i.e. restricting a unitary representation of a Lie group to a closed subgroup. More precisely, it is known (see for example [Kob05, Theorem 3.1.2]) that for $H \subset G$ real connected Lie groups of type I the restriction $\pi|_H$ of a unitary representation π of G decomposes uniquely into a direct integral:

$$\pi|_H = \int_{\hat{H}} \tau^{\oplus m_\pi(\tau)}\, d\mu_\pi(\tau), \quad m_\pi(\tau) \in \mathbb{N} \cup \{\infty\},$$

where $m_\pi(\tau)$ is the multiplicity of τ and μ_π is a Borel measure on $\hat{H}$, the unitary dual of H. Then the branching problem consists of determining the measure μ_π and the multiplicities $m_\pi(\tau)$. It is very hard to find explicit branching laws for general G, π and H. Let us first take a look which results are already known:
If G and H are both nilpotent and π is unitary irreducible Kirillov shows that the support of the measure μ_π is the projection of the corresponding orbit $\mathcal{O}_\pi \subset i\mathfrak{g}^*$ onto $i\mathfrak{h}^*$ (see [Kir76, §15.5 Theorem 1]).
Ten years later Corwin, Greenleaf and Grelaud were able to compute the measure and the multiplicities in [CGG87] and gave a geometric description of the multiplicities in [CG88]: Let $q : i\mathfrak{g}^* \to i\mathfrak{h}^*$ be the natural projection and π be irreducible. Then by [CG88, Theorem 1.1 b)]

$$\pi|_H \cong \int_{q(\mathcal{O}_\pi^G)/\,\mathrm{Ad}^*(H)} \tau_l^{\oplus m_\pi(l)}\, d\dot{\mu}(l),$$

where $\mathcal{O}_\pi^G \subset i\mathfrak{g}^*$ is the coadjoint orbit associated to $\pi \in \hat{G}$, $\tau_l \in \hat{H}$ the irreducible unitary representation associated to $l \in i\mathfrak{h}^*$, and $[\dot{\mu}]$ the measure class on the quotient space (a push forward of the invariant measure μ on $\mathcal{O}_\pi^G$).
Furthermore, they show in [CG88, Theorem 1.3] that the multiplicity of each $\tau_l \in \hat{H}$, $l \in i\mathfrak{h}^*$, in the direct integral above is given by

$$m_\pi(l) = \text{ number of } \mathrm{Ad}^*(H)\text{-orbits in } \mathcal{O}_\pi^G \cap q^{-1}\left(\mathcal{O}_l^H\right) \subset i\mathfrak{g}^*.$$

In particular,

$$\tau \in \mathrm{supp}\,(\pi|_H) \quad \Leftrightarrow \quad \mathcal{O}_\tau^H \subset q\left(\mathcal{O}_\pi^G\right). \tag{6.19}$$

At the same time Lipsman obtained the same results in [Lip89] but used different methods in his proofs which he then used to generalize the results for completely solvable groups in [Lip90]. Shortly after that, Fujiwara gave the branching law for exponential solvable Lie groups in [Fuj91].
However, already for compact G and H it becomes rather complicated. For example, Heckmann works in [Hec82] with so called asymptotic multiplicity functions and therefore obtains only asymptotic analogues to the results mentioned above. Another example

is [Hec82, Theorem 7.5] where more assumptions are necessary only to have the one inclusion $\mathrm{supp}(\mu_\pi) \subset q(\mathcal{O}_\pi)$. For other types of Lie groups progress was made by putting further conditions on the representation: For example, Kobayashi started studying real reductive groups in the case that the restriction is discretely decomposable (and has finite multiplicities) in [Kob94],[Kob98a] and [Kob98b], and explicit branching laws were obtained (see e.g. [Osh15] and references therein). But the case where the restriction contains both continuous and discrete spectrum has not been treated systematically, a general strategy was introduced by Frahm and Weiske this year in [FW20].
Other examples are highest weight modules (see e.g. [Kob08]) and principle series representations (see e.g. [Vog81], [HT93]).

However, very little seems to be know when the two groups are not of the same type. To obtain at least asymptotic information about the support of the restriction we can look at the wave front sets and their connection to the asymptotic orbital support of a unitary representation as already discussed in Section 2.3. We use the two following results by Howe regarding the wave front sets of restrictions:

Theorem 6.1 (see [How81, Proposition 1.5]). *Let G be a Lie group with Lie algebra $\mathfrak{g}$, π a unitary representation of G and $H \subset G$ a Lie subgroup with Lie algebra $\mathfrak{h}$. With the natural projection $q : i\mathfrak{g}^* \to i\mathfrak{h}^*$ we have*

$$q(\mathrm{WF}(\pi)) \subset \mathrm{WF}(\pi|_H).$$

Theorem 6.2 (see [How81, Proposition 1.6]). *If we have $\mathrm{WF}(\pi) \cap \ker(q) = \{0\}$ in addition to the setting of the previous theorem, then*

$$q(\mathrm{WF}(\pi)) = \mathrm{WF}(\pi|_H).$$

Combining Theorem 6.1 with Howe's result about the wave front set for compact Lie groups (see [How81, Proposition 2.3], compare Theorem 2.21) we obtain

Theorem 6.3 (see [How81, Proposition 1.5 and Proposition 2.3]). *Let G be a Lie group, $K \subset G$ a compact Lie subgroup and π a unitary representation of G. Then*

$$q(\mathrm{WF}(\pi)) \subset \mathrm{AC}\left(\mathcal{O} - \mathrm{supp}\left(\pi|_K\right)\right).$$

We recall that the asymptotic cone of $S \subset i\mathfrak{g}^*$ is defined as

$$\mathrm{AC}(S) = \{\eta \in i\mathfrak{g}^* \mid \forall \text{ open cone } \mathcal{C} \ni \eta : \mathcal{C} \cap S \text{ unbounded}\} \cup \{0\}$$

(see Definition 2.20) and the orbital support of π as $\mathcal{O} - \mathrm{supp}\,\pi = \bigcup_{\sigma \in \mathrm{supp}(\pi)} \mathcal{O}_\sigma \subset i\mathfrak{g}^*$.

Then these kinds of statements tell us in which directions there have to be infinitely many points in the orbital support of the restriction $\pi|_K$, i.e. which irreducible unitary representations of K have to occur in the decomposition of $\pi|_K$.

Now, let G be a Lie group with Lie algebra $\mathfrak{g}$, π a unitary representation of G and $N \subset G$ a nilpotent, connected, simply connected Lie subgroup with Lie algebra $\mathfrak{n}$ with the natural projection $q : i\mathfrak{g}^* \to i\mathfrak{n}^*$. Then our main result Theorem 4.1 combined with Howe's Theorem 6.1 gives us

$$q(\mathrm{WF}(\pi)) \subset \mathrm{AC}\left(\mathcal{O} - \mathrm{supp}\left(\pi|_N\right)\right).$$

With Theorem 6.2 we also have

$$\mathrm{WF}(\pi) \cap \ker(q) = \{0\} \quad \Rightarrow \quad q(\mathrm{WF}(\pi)) = \mathrm{AC}\left(\mathcal{O} - \mathrm{supp}\left(\pi\big|_N\right)\right).$$

If G itself is also nilpotent, connected, simply connected, Theorem 4.1 applied to the left hand side as well gives us

$$q(\mathrm{AC}(\mathcal{O} - \mathrm{supp}\,\pi)) \subset \mathrm{AC}\left(\mathcal{O} - \mathrm{supp}\left(\pi\big|_N\right)\right).$$

This can be viewed as half of an asymptotic version of (6.19).

In order to generalize this statement to other types of Lie groups G we have to assume that there is a way to associate to any $\sigma \in \mathrm{supp}(\pi) \subset \hat{G}$ a coadjoint orbit $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$. This is for example the case if G is a real reductive, algebraic Lie group and π is weakly contained in the regular representation of G: Duflo and Rossmann associated to each $\sigma \in \hat{G}_{temp}$ a finite union $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$ of coadjoint orbits (see [Duf70], [Ros78] and [Ros80]). Then with [HHÓ16, Theorem 1.2] and the above we have the following

Theorem 6.4. *Let G be a real reductive, algebraic Lie group and π be weakly contained in the regular representation of G. If $N \subset G$ is a nilpotent, connected, simply connected Lie subgroup, then*

$$q(\mathrm{AC}(\mathcal{O} - \mathrm{supp}\,\pi)) \subset \mathrm{AC}\left(\mathcal{O} - \mathrm{supp}\left(\pi\big|_N\right)\right).$$

This again tells us which irreducible unitary representations of N have to occur in the decomposition of $\pi|_N$. While this statement can now be entirely stated in representation theoretic terms its proof is based on the notion of wave front sets and microlocal analysis.

7 Outlook

To finish this thesis we want to give a short outlook. As mentioned before it would be worthwhile to know that the wave front set equals the asymptotic orbital support of a unitary representation for as many Lie groups as possible, i.e.

$$\mathrm{WF}(\pi) = \mathrm{AC}\left(\bigcup_{\sigma \in \mathrm{supp}(\pi)} \mathcal{O}_\sigma\right). \tag{7.20}$$

Then one could obtain asymptotic information about the decomposition of unitary representations, for example restrictions or induced representations between different type of Lie groups.
Admittedly, there has to exist an orbit method for the Lie group, i.e. a relation between a set of coadjoint orbits and the unitary dual, in order to have a well-defined right hand side of statement (7.20).

Let us take a closer look at the classes of Lie groups for which the statement is proven and what the methods of proof is.
Firstly, for compact, connected Lie groups one uses the coadjoint orbits of the highest weights. In this setting Howe reduces the statement to a maximal torus, i.e. the abelian case, where it can be computed explicitly (see [How81, Propositions 2.1 and 2.3]). However, this only works for compact Lie groups.
Secondly, for real reductive Lie groups Duflo and Rossmann associated a finite union of coadjoint orbits to each irreducible, tempered representation and provided a character formula. Using this relation Harris, He and Ólafsson obtained the statement in [HHÓ16, Theorem 1.2]. In their proof they use integrated characters whose wave front set provides an intermediate step in the desired equality (7.20). That these integrated characters are well-defined in general in this setting is based on a result by Harish-Chandra.
Lastly, we studied nilpotent, connected, simply connected Lie groups in this thesis. Their comprehensive orbit method was the first of its kind and is due to Kirillov who also provided a character formula. For this reason we started following the approach by [HHÓ16] but found that the integrated characters need more assumptions. This means that the proof does not work in all generality (see Proposition 5.8 and Remarks 5.9, 5.10 and 5.11). The solution was to turn to the methods of proof used in the context of nilpotent Lie groups: by induction over the dimension of the Lie algebra we could prove microlocal estimates for the Fourier transform of matrix coefficients directly (see Propositions 4.2 and 4.5).
In conclusion it looks like one has to employ the methods of proof that are conventional in the given setting instead of a general line of argumentation in order to give an elegant proof of the desired statement.

Another case in which an orbit methods exists are solvable Lie groups. Thus, a next step could be to prove the statement in this case. As Kirillov summarizes in [Kir04, Chapter 4 §1] the orbit method for exponential Lie groups is very similar to the one for nilpotent Lie groups, only the construction of the irreducible, unitary representations

and the character formula have to be modified slightly. The proofs are due to Pukánszky, [Puk67] (see also [BCD+67] and [Bus73]).
Solvable, but non-exponential Lie groups do not have to be of type I. But if they are the orbit method still works after appropriate amendments: one hast to restrict to a subset of coadjoint orbits, the so-called rigged coadjoint orbits, and use holomorphic induction in the construction of the irreducible unitary representations. These results are due to Auslander and Kostant, [AK71].
In both cases the proofs are again by induction over the dimension. This suggests that our approach can be used (and modified) to obtain the statement (7.20) in these settings.

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