Aspects of global solvability in cross-diffusive parabolic systems

Am Institut für Mathematik der Universität Paderborn unter der Betreuung durch Herrn Univ.-Prof. Dr. Michael Winkler erarbeitete und der Fakultät für Elektrotechnik, Informatik und Mathematik der Universität Paderborn im November 2020 vorgelegte Dissertation

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Zusammenfassung Gegenstand dieser Arbeit sind kreuzdiffusive Systeme parabolischer partieller Differentialgleichungen. Neben grundlegenden Fragen zur Lösbarkeit wird unter anderem die Existenzzeit dieser entdeckten Lösungen diskutiert. Darüber hinaus werden einige spezifischere Situationen wie die Auswirkungen eines Richtungswechsels der Chemotaxis oder die Effekte beim Übergang von einer parabolischen Differentialgleichung zu einem elliptischen Pendant untersucht.

Abstract The subject of this work are cross-diffusive systems of parabolic partial differential equations. One main focus is the discussion of solvability, defining appropriate solution concepts and detecting functions satisfying these conditions. After establishing under which circumstances these solutions exist for all times, additional, more specific problems are tackled: these include changing the movement from chemoattraction to chemorepulsion and the transition from a parabolic equation to an elliptic one.

Danksagung Bis zum Abschluss dieser Dissertation hat es letztendlich wesentlich länger gedauert als ursprünglich gedacht oder geplant, häufig und lange Zeit schien er gar gänzlich außer Reichweite.

Ein ganz entscheidender Faktor dabei, dass diese Zeit zu einem versöhnlichen Ende gefunden hat, ist zweifelsohne die Mitwirkung durch eine Vielzahl wichtiger Menschen in meinem Umfeld.

Insbesondere in der kritischsten Phase kann die Bedeutung der Unterstützung durch meine Familie und guten Freunde nicht überbewertet werden.

Ausgesprochen lang ist die Liste derer, die einen Beitrag dazu leisteten, dass ich heute hier stehe. Beginnend mit den Lehrern zu Schulzeiten verdienen auch etliche Kommilitonen, Tutoren und Dozenten meinen Dank dafür, einerseits das Interesse an der Mathematik immer wieder und weiter entfacht und andererseits die Zuversicht auf ein Gelingen am Leben gehalten zu haben.

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Contents

Chapter I

Introduction and preamble; some results and an overview

In this first chapter, the foundation for the upcoming passages will be built; sometimes by merely citing well-known results, and sometimes by proving certain statements – for example when they take a different shape than elsewhere.

I.1 Previous publications and their relation to this dissertation

Chapters II through V correspond to the following publications in peer-reviewed journals; there will be no further references to these in the chapters themselves.

Chapter II: [21], Freitag, Blow-up profiles and refined extensibility criteria in quasilinear Keller–Segel systems, *J. Math. Anal. Appl. 463(2)*, 2018

Chapter III: [23], Freitag, Global existence and boundedness in a chemorepulsion system with superlinear diffusion, *Discrete Contin. Dyn. Sys. A 38(11)*, 2018

Chapter IV: [24], Freitag, The fast signal diffusion limit in nonlinear chemotaxis systems, *Discrete Contin. Dyn. Sys. B 25(3)*, 2020

Chapter V: [22], Freitag, Global solutions to a higher-dimensional system related to crime modeling, *Math. Methods Appl. Sci. 41(16)*, 2018

I.2 Overview

The dominating concept throughout the thesis is (weak) cross-diffusion: in addition to some variant of the heat equation, $v_t = \Delta v - v + u$ for example, we also have a partial differential equation for u which features not only Δu but also Δv. One context in which this phenomenon can be observed is chemotaxis, the movement of the molecules measured by u towards the gradient of v. For most of the upcoming chapters, the underlying idea is that u is some biological entity which follows some signal v; in chapter V however we will see applicability beyond this original purpose.

Let us begin by providing a brief summary of the upcoming chapters. In chapter II we consider the rather general equation

$$u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v)$$

where both D and S behave similarly to some power function. Depending on this asymptotic behaviour and the dimension n of the space we will detect some threshold $\mathfrak{p}$ and arrive at the following conclusion: whenever the solution to a system involving the equation above is such that

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} < \infty$$

for some $p > \mathfrak{p}$ and every $t > 0$, then the solution is global and both u and v are uniformly bounded.

While in this setting S is very unspecific, possibly even featuring a changing sign, the following chapter fixes this sensitivity function: we take $S = -id$ and exploit the effect this fixed sign has in the resulting equation

$$u_t = \nabla \cdot (D(u)\nabla u) + \nabla \cdot (u\nabla v)\,.$$

While the simple observation

$$\left\|u(\cdot,t)\right\|_{L^1(\Omega)} = \left\|u(\cdot,0)\right\|_{L^1(\Omega)}$$

for every $t > 0$, coupled with suitable initial data and inserted into our result from chapter II, gives us the global existence and boundedness of solutions if

$$D(u) \approx (u+1)^{m-1}$$

for some $m > 1 + \frac{n-2}{n}$, in chapter III we will prove that the specific choice of S allows for a better result: as it turns out, $m > 1 + \frac{(n-2)(n-1)}{n^2}$ is a sufficient condition.

In the same chapter we will begin to take an interest in weak solutions: previously, the diffusion function was strictly positive because we chose it roughly as some power of $u + 1 \geq 1$. As we have seen in lemma I.3.2, positive initial data for v in $\overline{\Omega}$ with $v_t = \Delta v - v + u$ allows us to find a uniform lower bound for v in $\Omega \times (0, T)$ for arbitrary but fixed $T \in (0, \infty)$. This is not possible for u: here the same argument can only prove nonnegativity which gives rise to the question: what happens if D is degenerate at $u = 0$? The second part of chapter III will detect weak solution to such problems using an approximation process.

In the fourth chapter we will discuss another modification of the system

$$\begin{cases} u_t = \nabla \cdot \left((u+1)^{m-1} \nabla u \right) u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u. \end{cases}$$

Later on we will give a few examples for this claim, but generally systems where the second equation is substituted by the elliptic equation

$$0 = \Delta v - v + u$$

are easier to handle and better understood. With this in mind, we will begin with the equations

$$\varepsilon v_t = \Delta v - v + u$$

for $\varepsilon \in (0, 1)$ instead and notice that they can be approached in a very similar way to what we have done so far. Seeing how this equation transitions from the initial parabolic version $v_t = \Delta v - v + u$ to the elliptic case $0 = \Delta v - v + u$ as ε approaches zero, one might ask: does this also mean that, at least for small ε, solutions to the parabolic-elliptic system roughly describe the behaviour of solutions to the fully parabolic system? We will find that the solutions actually converge, thereby creating a bridge between the two extreme cases of $\varepsilon = 0$ and $\varepsilon = 1$.

The second to last chapter then ventures beyond the – by then rather familiar – context. The system here will be given by

$$\begin{cases} u_t = \Delta u - \chi \cdot \left(\frac{u}{v} \nabla v \right) - uv + B_1, \\ v_t = \Delta v - v + uv + B_2 \end{cases}$$

for some $\chi > 0$ and two nonnegative sources B_1 and B_2. Retaining strong similarities to previously discussed equations and keeping some integral properties like cross-diffusion, the system considered in chapter V differs mathematically as well as with respect to the purpose for which it was modelled. Once more, proving the existence and boundedness of solutions drives us, and we will see some very different tools and approaches compared to previous chapters.

Our final chapter pushes our idea as to the question under which circumstances a function can be considered a solution to a system of partial differential equations even further: abandoning the positivity of $\liminf_{u \to \infty} D(u)$, we will encounter so-called very weak solutions as an additional generalisation of previously discussed weak solutions.

I.3 A closer look at the heat equation and some useful estimates

Since it is of great importance to the systems we are going to deal with, a more in-depth look at the heat equation is warranted. Let us begin with a very general form:

$$v_t(x, t) = \Delta v(x, t) + f\left(x, t, v(x, t)\right)$$

constitutes the inhomogeneous nonlinear heat equation and it describes the distribution of heat over time under the influence of some source f.

In order to produce a reasonable solution concept, we also introduce additional conditions, namely Neumann boundary conditions, that is $\frac{\partial v}{\partial \nu}\big|_{\partial\Omega} = 0$, wherein ν denotes the outward normal unit vector on the boundary $\partial\Omega$ of Ω, and we demand that for $t = 0$ any solution coincide with some given function v_0. In summary this leaves us with the system

$$\begin{cases} v_t = \Delta v + f & \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases}$$

A well-known and very helpful piece of information is provided by the following comparison principle (proposition 52.13 in [67]):

Lemma I.3.1. *Let $n \in \mathbb{N}$, some bounded domain $\Omega \subset \mathbb{R}^n$ and $T \in (0,\infty)$ be given. Furthermore, let f be some function in $C^0\left(\overline{\Omega} \times [0,T] \times \mathbb{R}\right)$ that is Lipschitz continuous with respect to its third component in the sense that for every compact $K \subset \mathbb{R}$ there is a positive constant C_K such that*

$$\left| f(x,t,v) - f(x,t,\tilde{v}) \right| \le C_K \left| v - \tilde{v} \right|$$

holds for every $x \in \overline{\Omega}$, $t \in [0,T]$, $v \in \mathbb{R}$ and $\tilde{v} \in \mathbb{R}$. Then for any combination of functions $\underline{v}$, $\overline{v}$ in $C^0\left(\overline{\Omega} \times [0,T]\right) \cap C^{2,1}\left(\Omega \times (0,T)\right)$ with

$$\underline{v}_t \le \Delta \underline{v} + f\left(\cdot,\cdot,\underline{v}\right) \quad \text{in } \Omega \times (0,T)$$

and

$$\overline{v}_t \ge \Delta \overline{v} + f\left(\cdot,\cdot,\overline{v}\right) \quad \text{in } \Omega \times (0,T),$$

as well as

$$\frac{\partial \underline{v}}{\partial \nu} \le \frac{\partial \overline{v}}{\partial \nu} \quad \text{on } \partial\Omega \times (0,T)$$

and

$$\underline{v}(\cdot,0) \le \overline{v}(\cdot,0) \quad \text{in } \Omega$$

we have the relation

$$\underline{v} \le \overline{v} \quad \text{in } \Omega \times (0,T).$$

This result not only plays an important role in ensuring the uniqueness of solutions to systems involving the heat equation, it also provides us with the following lower bound: Usually, v denotes some kind of concentration and accordingly, v_0 is assumed to be nonnegative or even strictly positive. We can already show that this property is carried over to solutions as long as f is nonnegative.

Lemma I.3.2. *For some bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, let $T > 0$ and some nonnegative function $f \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap C^1\left(\Omega \times (0,T)\right)$ be given. Then for any function*

$$v \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right)$$

solving

$$v_t = \Delta v - v + f$$

in $\Omega \times (0,T)$ and $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega \times (0,T)$ we already have $v \geq \delta e^{-t}$ in $\Omega \times (0,T)$ where $\delta := \inf_\Omega v(\cdot, 0)$.

Proof. We define

$$\underline{v}(x,t) := \delta e^{-t} \text{ for } (x,t) \in \Omega \times (0,T).$$

Then, using the nonnegativity of f, we see

$$\begin{aligned}\underline{v}_t(\cdot,t) &= -\underline{v}(\cdot,t)\\ &= \Delta\underline{v}(\cdot,t) - \underline{v}(\cdot,t)\\ &\leq \Delta\underline{v}(\cdot,t) - \underline{v}(\cdot,t) + f(\cdot,t)\end{aligned}$$

in Ω for every $t \in (0,T)$. Since obviously $\underline{v}(\cdot,0) = \delta \leq v(\cdot,0)$ in Ω, the parabolic comparison principle in lemma I.3.1 can be employed to finish the proof. **q.e.d.**

Accordingly, prescribing $v(\cdot,0) = v_0$ in Ω for some strictly positive $v_0 \in C^0\left(\Omega \times (0,T)\right)$ locally gives us a positive lower bound for v.

In this context there exists a notation similar to ordinary differential equations: the Neumann-semigroup for the homogeneous heat equation $v_t = \Delta v$, $\left(e^{t\Delta}\right)_{t\geq 0}$, which lets us represent and estimate in our setting via Duhamel formulae such as

$$v(\cdot,t) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} f(\cdot,s)\,\mathrm{d}s,$$

most importantly in this upcoming set of *L^p-L^q-estimates*. We also want to remark that in many cases we will be interested in the equation $v_t = \Delta v - v + u$ with some sufficiently smooth function u and that in such settings we use the modified semigroup $\left(e^{t(\Delta-1)}\right)_{t\geq 0}$ where $e^{t(\Delta-1)} = e^{-t}e^{t\Delta}$; accordingly most of the time this shift has little effect on the overall estimates.

For a reference to these results see [96] where several estimates of this name can be found. The estimates used in this thesis are collected in

Lemma I.3.3 (**L^p-L^q-estimates**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for some $n \in \mathbb{N}$. Then we can find some positive λ with the following properties: For $1 \leq p \leq q \leq \infty$ there is $C_1 > 0$ such that for every $\varphi \in C^0\left(\overline{\Omega}\right)$ with $\int_\Omega \varphi = 0$ we have*

$$\left\|e^{t\Delta}\varphi\right\|_{L^q(\Omega)} \leq C_1 \cdot \left(1 + t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right) e^{-\lambda t} \left\|\varphi\right\|_{L^p(\Omega)}$$

for every $t > 0$. *Furthermore, there is also some* $C_2 > 0$ *with*

$$\left\|\nabla e^{t\Delta}\varphi\right\|_{L^q(\Omega)} \leq C_2 \cdot \left(1 + t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right) e^{-\lambda t} \left\|\varphi\right\|_{L^p(\Omega)}$$

for every $\varphi \in C^0\left(\overline{\Omega}\right)$ *and every* $t > 0$. *Lastly, whenever* $q < \infty$ *or* $p < q$, *there is* $C_3 > 0$ *such that for every* $\Phi \in C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$ *with* $\Phi \cdot \nu|_{\partial\Omega} = 0$ *and every* $t > 0$

$$\left\|e^{t\Delta}\nabla \cdot \Phi\right\|_{L^q(\Omega)} \leq C_3 \cdot \left(1 + t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right) e^{-\lambda t} \left\|\Phi\right\|_{L^p(\Omega)}$$

holds.

With respect to the third inequality we remark that the proof in [96] originally only claimed to cover finite values for p and q; however, the density of $C_0^\infty(\Omega)$ in $L^1(\Omega)$ allows for the inclusion of $1 < p < q = \infty$.

Since from this time forward the partial differential equation for v remains mostly the same, let us fix the associated system: given $n \in \mathbb{N}$, some bounded domain $\Omega \subset \mathbb{R}^n$ as well as two functions u and v_0 (the details of which will be specified in the respective situations) it reads as follows:

$$\begin{cases} v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases} \tag{I.1}$$

An immediate consequence of these estimates is the upcoming well-established statement (see for example lemma 4.1 in [38]) linking the regularity of v to that of the right-hand side u:

Lemma I.3.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain for some* $n \in \mathbb{N}$. *Assume that for some time* $T > 0$ *as well as two given functions* $u \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right)$ *and* $v \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right)$ *the differential equation in* (I.1) *is solved classically in* $\Omega \times (0,T)$. *If* $\left\|v(\cdot,0)\right\|_{W^{1,\infty}(\Omega)} < \infty$, *then for every* $p \geq 1$ *we have the following result: assume that for some* $C_{u,p} > 0$

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \leq C_{u,p}$$

holds for every $t \in (0,T)$, *then for any* $q \in \left[1, \frac{np}{n-p}\right)$ *for* $p < n$, $q \in [1,\infty)$ *if* $p = n$ *and* $q \in [1,\infty]$ *for* $p > n$ *we can find* $C_{v,q} > 0$ *depending on the quantities* n, p, q *and* $C_{u,p}$ *as well as* $\left\|v(\cdot,0)\right\|_{W^{1,\infty}(\Omega)}$ *with*

$$\left\|v(\cdot,t)\right\|_{W^{1,q}(\Omega)} \leq C_{v,q}$$

for every $t \in (0,T)$.

Especially in conjunction with a *mass conservation* property of u, this is of fundamental importance in many of the following chapters. This phenomenon is described by the following statement:

Lemma I.3.5. *Assume that in some bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and for $T > 0$ some nonnegative $D \in C^2([0,\infty))$ and $S \in C^2([0,\infty))$ as well as two functions u and v belonging to $C^0\left(\overline{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right)$ be given.*
Then under the condition $D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega \times (0,T)$ from the differential equation

$$u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v)$$

in $\Omega \times (0,T)$ we find

$$\int_\Omega u(\cdot,t) = \int_\Omega u(\cdot,0)$$

for every $t \in (0,T)$.

Proof. From Gauß's theorem we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = \int_{\partial\Omega}\left(D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu}\right) = 0$$

and accordingly the mass remains on the initial level. **q.e.d.**

This result (or at least a comparably trivial statement) will help us in many situations by providing us with an upper bound for $\left\|u(\cdot,t)\right\|_{L^1(\Omega)}$.

Since the problems discussed in the upcoming chapters share some similarities, the same holds true for the results used to obtain what we seek. Therefore, it seems reasonable to note as much as possible as generally as possible beforehand.

Young's inequality is used as a foundation countless times in the upcoming proofs, therefore it seems warranted to display it properly:

Lemma I.3.6 (Young's inequality). *Let $p > 1$ and $p' := \frac{p}{p-1}$. Then*

$$AB \le \frac{A^p}{p} + \frac{B^{p'}}{p'}$$

holds for any combination of nonnegative numbers A and B.

The shape in which we want to use this result is quickly derived in

Corollary I.3.7. *Let $p > 1$ and $C_p := \frac{p-1}{p^{\frac{p}{p-1}}}$. Then for any $\varepsilon > 0$ and nonnegative a and b we have*

$$ab \le \varepsilon a^p + C_p \varepsilon^{-\frac{1}{p-1}} b^{\frac{p}{p-1}}.$$

Furthermore, given $\beta \in (0,1)$ and $\gamma \in (0,1)$ such that $\beta + \gamma < 1$, for every positive number ε there is some $C(\varepsilon) > 0$ with

$$\left(1 + a^\beta\right)(1 + b^\gamma) \le \varepsilon(a+b) + C(\varepsilon)$$

for every combination of $a \ge 0$ and $b \ge 0$.

Proof. With the notation from the previous lemma we see

$$\begin{aligned} ab &= \left(p^{\frac{1}{p}}\varepsilon^{\frac{1}{p}}a\right)\cdot\left(p^{-\frac{1}{p}}\varepsilon^{-\frac{1}{p}}b\right) \\ &\le \frac{p\varepsilon a}{p} + \frac{p^{-\frac{p'}{p}}\varepsilon^{-\frac{p'}{p}}b^{p'}}{p'}, \end{aligned}$$

for any $p > 1$, $p' := \frac{p}{p-1}$, $\varepsilon > 0$ as well as nonnegative a and b, proving the first portion of our claim. Accordingly, for fixed $\varepsilon > 0$ we find $C_1 > 0$ and $C_2 > 0$ such that

$$a^\beta \le \frac{\varepsilon}{2}a + C_1 \text{ and } b^\gamma \le \frac{\varepsilon}{2}b + C_2$$

hold for every $a \ge 0$ and $b \ge 0$. Since additionally $\frac{\gamma}{1-\beta} < 1$, there are also positive constants C_3 and C_4 such that

$$\begin{aligned} a^\beta b^\gamma &\le \frac{\varepsilon}{2}a + C_3 b^{\frac{\gamma}{1-\beta}} \\ &\le \frac{\varepsilon}{2}a + \frac{\varepsilon}{2}b + C_4 \end{aligned}$$

is true independently of the choices for $a \ge 0$ and $b \ge 0$. Therefore, after setting our final constant $C(\varepsilon) := 1 + C_1 + C_2 + C_4$, we have

$$\begin{aligned} \left(1 + a^\beta\right)(1 + b^\gamma) &= 1 + a^\beta + b^\gamma + a^\beta b^\gamma \\ &\le \varepsilon(a + b) + C(\varepsilon) \end{aligned}$$

for any combination of nonnegative numbers a and b. **q.e.d.**

Next we want to cite the celebrated Gagliardo-Nirenberg inequality, mainly so that we have a reference whenever the exact size of some constant is crucial. For a reference see Theorem 10.1 in [25] or the original works [63], [27] and [28] directly.

Lemma I.3.8 (Gagliardo-Nirenberg inequality). *For a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, assume that some $p, q \in [1, \infty]$ and $r \in (0, p)$ with $p < \infty$ for $q = n$ and $p \le \frac{nq}{n-q}$ for $q < n$ are given. Then for $a \in (0, 1]$ determined by*

$$-\frac{n}{p} = \left(1 - \frac{n}{q}\right)a - \frac{n}{r}(1 - a)$$

and some $C > 0$ we have

$$\|w\|_{L^p(\Omega)} \le C\,\|\nabla w\|^a_{L^q(\Omega)}\,\|w\|^{1-a}_{L^r(\Omega)} + C\,\|w\|_{L^r(\Omega)}$$

and

$$\|w\|_{L^p(\Omega)} \le C\,\|\nabla w\|^a_{W^{1,q}(\Omega)}\,\|w\|^{1-a}_{L^r(\Omega)}$$

for any $w \in C^1(\Omega)$.

Out of a class of generalisations, the following fractional variant is also needed later on. For this version we refer to Lemma 2.5 in [39].

Lemma I.3.9 (GNI for fractional Sobolev spaces). *For some bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and fixed $r \in \left(0, \frac{1}{2}\right)$ and $q \in (0, 2)$ there are two constants $C > 0$ and $a \in (0, 1)$ such that*

$$\|w\|_{W^{r+\frac{1}{2},2}(\Omega)} \le C\|\nabla w\|^a_{L^2(\Omega)}\|w\|^{1-a}_{L^q(\Omega)} + C\|w\|_{L^q(\Omega)}$$

holds for any $w \in W^{1,2}(\Omega)$.

Let us conclude this series of estimates with one that removes the need for Ω to be convex: in arbitrary domains we cannot deduce $\frac{\partial|\nabla w|^2}{\partial\nu} \le 0$ on $\partial\Omega$ from $\frac{\partial w}{\partial\nu} = 0$ on $\partial\Omega$, and so for terms involving $\Delta|\nabla w|^2$ integration by parts becomes less trivial.

Lemma I.3.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for some $n \in \mathbb{N}$ and let $s \ge 1$ and $q > 1$ with $\frac{s}{q} < 2$ be given. Then for every $K > 0$ there is $C > 0$ such that for any $w \in C^2(\Omega)$ with $\frac{\partial w}{\partial\nu}\big|_{\partial\Omega} = 0$ and $\|\nabla w\|_{L^s(\Omega)} \le K$ the estimate*

$$\int_\Omega |\nabla w|^{2q-2}\,\Delta|\nabla w|^2 \le -\frac{q-1}{q^2}\int_\Omega \left|\nabla|\nabla w|^q\right|^2 + C$$

holds.

Proof. Firstly, from [55] we receive some positive constant C_1 with

$$\frac{\partial|\nabla w|^2}{\partial\nu} \le C_1|\nabla w|^2$$

for every w meeting the conditions above. Additionally, using the variant of the Gagliardo Nirenberg inequality for fractional Sobolev spaces given in lemma I.3.9, for some arbitrary $r \in \left(0, \frac{1}{2}\right)$ the embedding $W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ leads to the detection of positive constants $a < 1$, C_2 and C_3 with

$$\begin{aligned}\int_{\partial\Omega}|\nabla w|^{2q} &= \left\||\nabla w|^q\right\|^2_{L^2(\partial\Omega)}\\ &\le C_2\left\||\nabla w|^q\right\|^2_{W^{r+\frac{1}{2},2}(\Omega)}\\ &\le C_3\left\|\nabla|\nabla w|^q\right\|^{2a}_{L^2(\Omega)}\left\||\nabla w|^q\right\|^{2(1-a)}_{L^{\frac{s}{q}}(\Omega)} + C_3\left\||\nabla w|^q\right\|^2_{L^{\frac{s}{q}}(\Omega)}\end{aligned}$$

for any such w. Together with Young's inequality this results in the existence of some $C_4 > 0$ such that

$$\int_{\partial\Omega}|\nabla w|^{2q-2}\frac{\partial|\nabla w|^2}{\partial\nu} \le 3\frac{q-1}{q^2}\int_\Omega|\nabla|\nabla w|^q|^2 + C_4$$

holds for every w enjoying the demanded properties. Therefore, integration by parts proves

$$\int_\Omega |\nabla w|^{2q-2} \Delta |\nabla w|^2 = -\int_\Omega \nabla |\nabla w|^{2q-2} \cdot \nabla |\nabla w|^2 + \int_{\partial\Omega} |\nabla w|^{2q-2} \frac{\partial |\nabla w|^2}{\partial \nu}$$
$$\leq -\frac{q-1}{q^2} \int_\Omega \left|\nabla |\nabla w|^q\right|^2 + C_4$$

for all of these w since

$$\begin{aligned}\nabla |\nabla w|^{2q-2} \cdot \nabla |\nabla w|^2 &= \nabla \left(\nabla |\nabla w|^q\right)^{\frac{2q-2}{q}} \cdot \nabla \left(\nabla |\nabla w|^q\right)^{\frac{2}{q}} \\ &= \frac{2q-2}{q} \cdot \frac{2}{q} |\nabla w|^{\frac{2q-2}{q}-1+\frac{2}{q}-1} \left|\nabla |\nabla w|^q\right|^2 \\ &= 4\frac{q-1}{q^2} \left|\nabla |\nabla w|^q\right|^2\end{aligned}$$

is true for any $w \in C^2(\Omega)$. **q.e.d.**

This result, applied to the setting in upcoming chapters, gives us the following estimate:

Lemma I.3.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for some $n \in \mathbb{N}$ and let $s \geq 1$ and $q > 1$ with $\frac{s}{q} < 2$ be given. Then for every $K > 0$ there is $C > 0$ such that, given any $u \in C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right)$ with $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega \times (0,\infty)$, the estimate*

$$\frac{1}{2q} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v|^{2q} + \frac{q-1}{4q^2} \int_\Omega \left|\nabla |\nabla v|^q\right|^2 + \int_\Omega |\nabla v|^{2q} \leq C + C \int_\Omega u^2 |\nabla v|^{2q-2}$$

holds in $(0,\infty)$ for every $v \in C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right)$ satisfying the differential equation and boundary condition in (I.1) *as well $\|\nabla v\|_{L^s(\Omega)} \leq K$.*

Proof Using the identities

$$\frac{\partial}{\partial t} |\nabla v|^2 = 2\nabla v \cdot \nabla v_t = 2\nabla v \cdot \nabla \Delta v - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v$$

and

$$\Delta |\nabla v|^2 = 2\nabla \cdot (D^2 v \nabla v) = 2|D^2 v|^2 + 2\nabla v \cdot \nabla \Delta v,$$

lemma I.3.10 yields some $C_1 > 0$ such that

$$\begin{aligned}\frac{1}{2q} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v|^{2q} &= \frac{1}{2} \int_\Omega |\nabla v|^{2q-2} \frac{\partial}{\partial t} |\nabla v|^2 \\ &= \frac{1}{2} \int_\Omega |\nabla v|^{2q-2} \left(\Delta |\nabla v|^2 - 2|D^2 v|^2 - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v\right) \\ &\leq -\frac{q-1}{2q^2} \int_\Omega |\nabla |\nabla v|^q|^2 - \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 \\ &\quad - \int_\Omega |\nabla v|^{2q} + \int_\Omega |\nabla v|^{2q-2} \nabla u \cdot \nabla v + C_1\end{aligned}$$

holds in $(0,T)$ for any function v meeting the demands of this lemma.

Continuing with the integral on the right-hand side, we use integration by parts and employ Young's inequality twice more to find

$$\begin{aligned}\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v =& -\int_\Omega u\nabla\cdot\left(|\nabla v|^{2q-2}\nabla v\right)\\ =&(q-1)\int_\Omega u|\nabla v|^{2q-4}\nabla v\cdot\nabla|\nabla v|^2 - \int_\Omega u|\nabla v|^{2q-2}\Delta v\\ \le& \frac{q-1}{16}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 + 4(q-1)\int_\Omega u^2|\nabla v|^{2q-2}\\ &+\frac{1}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2 + \frac{n}{4}\int_\Omega u^2|\nabla v|^{2q-2}\\ \le& \frac{q-1}{4q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + \left[4(q-1)+\frac{n}{4}\right]\int_\Omega (u+1)^2|\nabla v|^{2q-2}\\ &+\int_\Omega |\nabla v|^{2q-2}|D^2v|^2\end{aligned}$$

in $(0,T)$ for any such v, wherein for the last step we used the pointwise estimate

$$|\Delta v|^2 \le n|D^2 v|^2.$$

After cancellation, we see that in $(0,T)$ and for $C_2 := \max\left\{C_1, 4(q-1)+\frac{n}{4}\right\}$

$$\frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} + \frac{q-1}{4q^2}\int_\Omega \left|\nabla\,|\nabla v|^q\right|^2 + \int_\Omega |\nabla v|^{2q} \le C_2 + C_2\int_\Omega u^2\,|\nabla v|^{2q-2}$$

holds for every v enjoying the properties formulated in the lemma. **q.e.d.**

Finally, we want to cite the following result which can be found as a corollary in chapter 8 of [78] and with which we will be able to detect convergent subsequences in some of the upcoming chapters:

Lemma I.3.12 (Aubin-Lions lemma). *For $T > 0$, $p \in [1,\infty]$ and three Banach spaces*

$$X \overset{cpt.}{\hookrightarrow} B \hookrightarrow Y$$

let F be a bounded set of $L^p((0,T);X)$-functions. If furthermore the set of derivatives $\frac{\partial F}{\partial t} := \left\{\frac{\partial f}{\partial t} : f \in F\right\}$ is bounded in $L^1((0,T);Y)$ (for $1 \le p < \infty$) or in $L^r((0,T);Y)$ for some $r > 1$ (in the case $p = \infty$), then F is relatively compact in $L^p((0,T);B)$ or $C^0((0,T);B)$ respectively.

Chapter II

Blow-up profiles and refined extensibility criteria in some quasilinear Keller-Segel systems

II.1 Introduction and main result

The Keller-Segel systems considered in this chapter attempt to describe the behaviour of certain slime molds. In particular, given a position x and a time t, by $u(x,t)$ we denote the density of a cell population whose movement is motivated by the concentration $v(x,t)$ of a signal substance.

In these systems, which were proposed by Keller and Segel ([43]) in 1970 and of which there are several modifications (see for example [36]), the cross-diffusion makes solutions prone to blow-up and indeed blow-up detection is one of the most challenging tasks; to this day results remain fragmented. Even with the original system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0,\infty)\,, \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty)\,, \end{cases}$$

there is no trivial answer on occurrences of blow-up; and in cases where blow-up occurs the next task is to find out whether this happens in finite or infinite time. There are several results on corresponding parabolic-elliptic versions (motivating our fourth chapter) where the second equation has been replaced by either

$$0 = \Delta v - v + u$$

or

$$0 = \Delta v - \overline{\mathfrak{m}} + u$$

for $\overline{\mathfrak{m}} := \frac{1}{|\Omega|} \int_\Omega u_0$. In these versions there has been discovered ([40], [57] and [59]) a connection between blow-up and the mass $\mathfrak{m} := \int_\Omega u_0$, which is constant over time: In the case $n = 2$ and for large $\mathfrak{m}$, blow-up occurs in finite time if the mass is concentrated around a single point whereas solutions remain bounded if $\mathfrak{m}$ is small enough. In higher dimensions the situation changes drastically: if $n = 3$, then there is no such threshold and blow-up in finite time may occur for arbitrarily small masses $\mathfrak{m}$ as shown in [58] and [33]. Far less is known if we return to the original system. Beginning with a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary (and sufficiently regular initial data) we can state the following results: The case $n = 1$ has been studied (see [65]) with the conclusion that there is no blow-up at all. For the two-dimensional setting we know that if the initial mass $\int_\Omega u_0$ is smaller than 4π, then solutions are bounded (for this we refer to [29] and [61]) while for $n \geq 3$ a smallness condition on $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} + \|v_0\|_{W^{1,n}(\Omega)}$ can be used to infer the existence of such a solution (see [10]). For larger initial data on the other hand we generally only know that there are blow-up solutions for which unboundedness can happen either in finite or infinite time ([37]).

In some cases, the statements can be refined if we restrict ourselves to radially symmetric settings. For $\Omega = B_R(0) \subset \mathbb{R}^2$ and $\int_\Omega u_0 > 8\pi$, radially symmetric solutions that blow up in finite time have been found by [34] and [56]. On the other hand, in the case of $\Omega = B_R(0) \subset \mathbb{R}^n$ and $n \geq 3$ even for small initial masses some solutions blow up in finite time (see [98]).

In this chapter – as well as in parts of upcoming chapters – we modify the first equation and for some bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases} \tag{II.1}$$

with nonnegative functions D and S. For a helpful overview of many models arising out of this fundamental description we also refer to the survey [3].

Several choices for these functions have been proposed and studied in recent years. One suggestion is to couple them via some function $Q \in C^2([0,\infty))$ and the relations $D(u) = Q(u) - uQ'(u)$ and $S(u) = uQ'(u)$ for every $u \geq 0$. Here, Q is intended to describe the probability of a cell at (x,t) to find space nearby, [35] considers a decreasing function with decay at large densities as the best fit. In [101] an overview of hydrodynamic approaches or those involving cellular Potts models is given.

There are also authors who propose a signal dependence in D or S, that is to write $S(u,v)$ for example as done in [91], [36] and [79] to incorporate saturation effects or a threshold for the activation of cross-diffusion. Later on, when fixing the properties of

S, we will see that to some small degree this idea is also covered by our results. For similar changes to D we refer to the works [26], [51], [82] and [83].

One set of choices has been of particular interest, namely where D and S behave like powers of u, and the result heavily depends on the relation of these two quantities. Setting

$$D(s) = (s+1)^{m-1} \text{ for every } s \in [0,\infty)$$

and

$$S(s) = s(s+1)^{\kappa-1} \text{ for every } s \in [0,\infty)$$

for some $m \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ we find the following for $n \geq 2$: If $1+\kappa-m < \frac{2}{n}$ and if the initial data are reasonably smooth, then we can find global classical solutions that are bounded ([89]) and this even remains true for general nonnegative functions D and S with

$$\frac{S(s)}{D(s)} \leq Cs^{\alpha} \text{ for every } s \geq 1$$

for some $C > 0$ and $\alpha < \frac{2}{n}$. On the other hand, if $1+\kappa-m > \frac{2}{n}$ and if Ω is a ball, then for any $M > 0$ we can find some $T \in (0,\infty]$ and a radially symmetric solution (u,v) in $\Omega\times(0,T)$ with $\int_\Omega u(\cdot,t) = M$ for every $t \in (0,T)$ which is not bounded in $\Omega\times(0,T)$ ([94]). Once more there are also studies on more general choices of D and S (see [89], [94] as well as [46], [14], [74] and [38]) that find

$$\frac{S(s)}{D(s)} \geq Cs^{\alpha} \text{ for every } s \geq 1$$

for some $C > 0$ and $\alpha > \frac{2}{n}$ to be enough to obtain the same result. In [16] and [17], the specific choice of D and S as powers of $u+1$ has been examined in greater detail with respect to this blow-up phenomenon and the authors were able to prove that for $\kappa \geq 1$ or $m \geq 1$ a finite value of T is obtainable. They also showed that for $m < \frac{n-2}{n}$ and $\kappa < \frac{m}{2} - \frac{n-2}{2n}$ we have $T = \infty$ which means that the solution exists globally with $\limsup_{t\to\infty} \left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} = \infty$.

Here we want to refine the blow-up results in the case where $1+\kappa-m < \frac{2}{n}$ does not necessarily hold. We consider twice differentiable D and S allowing for the inequalities

$$C_D(s+1)^{m-1} \leq D(s) \leq \hat{C}_D\,(s+1)^{\hat{m}-1} \tag{II.2}$$

with some $m, \hat{m} \in \mathbb{R}$ and $C_D, \hat{C}_D > 0$ for any $s \in [0,\infty)$ and

$$\left|S(s)\right| \leq C_S\,(s+1)^{\kappa} \tag{II.3}$$

with $\kappa \in \mathbb{R}$ and some $C_S > 0$, again for every $s \in [0,\infty)$.
We note that the functions may even depend on the variables $(x,t) \in \Omega\times[0,\infty)$ and on the solution to the second equation in (II.1), $\sigma \in [0,\infty)$, as long as the overall boundedness remains unaltered, for example

$$\left|S(x,t,s,\sigma)\right| \leq C_S\,(s+1)^{\kappa}\ \forall\,(x,t,s,\sigma) \in \Omega\times[0,\infty)^3$$

is actually good enough. Since this compromises the legibility without substantially adding anything, this has not been incorporated into upcoming results.

It has already been shown that the trivial observation of the boundedness of

$$t \mapsto \int_\Omega u(\cdot,t) \equiv \int_\Omega u_0$$

suffices to prove that classical solutions are bounded and thereby global at least for $m > \kappa + \frac{n-2}{n}$ (see [89]). It is our endeavour to extend this result by considering any relation between κ and m and finding some $p_0 \geq 1$ such that boundedness of

$$t \mapsto \int_\Omega u^{p_0}(\cdot,t)$$

is sufficient for a solution to (II.1) to be global and bounded.

The authors of [30] (in the case $\kappa > 1$ and $(n-2)\kappa < n+2$) have examined the semilinear heat equation

$$u_t = \Delta u + u^\kappa$$

in a convex domain Ω and concluded that a positive blow-up solution u with maximum existence time $T \in (0,\infty)$ for some $C > 0$ satisfies

$$u(\cdot,t) \leq C(T-t)^{\frac{1}{1-\kappa}} \text{ for every } t \in (0,T),$$

giving us a more precise idea on the manner in which u blows up. From our main result we will deduce a statement of similar shape for blow-up solutions to (II.1).

The center of our computations and estimates is a threshold $\mathfrak{p}$ given by

$$\mathfrak{p} := \begin{cases} \frac{n}{2}(1+\kappa-m) & \text{if } \kappa < m+1, \\ n(\kappa-m) & \text{if } \kappa \geq m+1, \end{cases} \tag{II.4}$$

which gives us refined knowledge on the behaviour of blow-up solutions to (II.1). Note that this is the same as setting $\mathfrak{p} := \max\left\{\frac{n}{2}(1+\kappa-m), n(\kappa-m)\right\}$.

A standard argument guarantees the existence of local solutions and provides us with a criterion for the occurrence of blow-up.

Our main result asserts that such a solution can be extended to a solution in $\Omega \times [0,\infty)$ if we have more information on the $L^{p_0}(\Omega)$-norm of u for some $p_0 \geq 1$:

Theorem II.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for some $n \geq 2$. Let $D \in C^2([0,\infty))$ with* (II.2) *for some $m, \hat{m} \in \mathbb{R}$ and $C_D, \hat{C}_D > 0$ as well as $S \in C^2([0,\infty))$ with $S(0) = 0$ and* (II.3) *for some $\kappa \in \mathbb{R}$ and $C_S > 0$.*
Additionally let nonnegative initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$ be given. Then there is $T_{max} \in (0,\infty]$ alongside a solution

$$(u,v) \in \left(\in C^0\left(\overline{\Omega} \times [0,T_{max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T_{max})\right)\right)^2$$

to (II.1) *in* $\Omega \times (0, T_{max})$ *for which in the case of* $T_{max} < \infty$ *we automatically know*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^{p_0}(\Omega)} = \infty$$

for any $p_0 \geq 1$ *with* $p_0 > \mathfrak{p}$ *for* $\mathfrak{p}$ *as in* (II.4).

This indeed gives us an interpretation for the behaviour of u in comparison to some negative powers of the space variable x:

Corollary II.1.2. *In the setting of theorem II.1.1 assume that* $T_{max} < \infty$. *Then, upon picking a blow-up point* $x_0 \in \Omega$, *meaning there are sequences* $(x_k)_{k \in \mathbb{N}} \subset \Omega$ *and* $(t_k)_{k \in \mathbb{N}} \subset (0, T_{max})$ *with*

$$\begin{aligned} x_k &\to x_0, \\ t_k &\to T_{max} \text{ and} \\ u(x_k, t_k) &\to \infty \end{aligned}$$

as $k \to \infty$, *for any* $\alpha < \frac{n}{\mathfrak{p}}$ *with* $\mathfrak{p}$ *as in* (II.4) *we cannot find any* $C > 0$ *such that*

$$u(x, t) \leq C |x - x_0|^{-\alpha}$$

holds for every $(x, t) \in \Omega \times (0, T_{max})$.

Remark. *If* $n \geq 2$, $\kappa \geq 1$ *and* $m = 1$, *then from* $\mathfrak{p} = n(\kappa - 1)$ *we find that our condition for the exponent is* $\alpha < \frac{1}{\kappa - 1}$, *resembling the result in [30].*

We close this section with two comments concerning the compatibility of this result with previous works. Firstly, if $m \geq \kappa + \frac{n-2}{n}$, we can pick any $p_0 > 1$ and [89] shows that even $p_0 = 1$ is enough for $\|u\|_{L^\infty((0,T_{\max});L^{p_0}(\Omega))} < \infty$ to guarantee that u is global and bounded. On the other hand we do not require too much of p_0 comparing our result to one in [3]: lemma 3.2 in that work demands $\mathfrak{p} = \max\left\{\frac{n\kappa}{2}, n(\kappa - 1)\right\}$, which is exactly the same as our result for $m = 1$.

Remark. *For the entirety of this chapter, the following conditions are assumed: that firstly for some* $n \geq 2$ *we are given some bounded domain* $\Omega \subset \mathbb{R}^n$ *as well as functions* $D \in C^2([0, \infty))$ *with* (II.2) *for some* $m, \hat{m} \in \mathbb{R}$, $C_D, \hat{C}_D > 0$ *and* $S \in C^2([0, \infty))$ *with* $S(0) = 0$ *and* (II.3) *for some* $\kappa \in \mathbb{R}$ *and* $C_S > 0$. *Additionally, let* $u_0 \in C^0\left(\overline{\Omega}\right)$ *and* $v_0 \in C^1\left(\overline{\Omega}\right)$ *denote some nonnegative initial data.*

II.2 Setting parameters

As seen in the definition of the threshold in (II.4), there is a difference between a setting where $\kappa < m + 1$ and one where the inverse is true. Accordingly, some computations and estimates differ slightly. This section is not only dedicated to fixing the parameters used in upcoming lemmata, it also contains every step in which the relation between κ and m requires a different approach.

The following two subsections prove this central result:

Lemma II.2.1. *Let $p_0 \geq 1$ with*

$$p_0 > \max\left\{\frac{n}{2}(1+\kappa-m), n(\kappa-m)\right\}$$

and $p_0 < n$ whenever $\kappa < m+1$ be given. Then there is $p_1 \geq p_0$ such that for every $p > p_1$ we find $s > 1$, $q > n$, $\theta > 1$ and $\mu > \max\left\{\frac{p_0}{2}, 1\right\}$ such that (setting $\frac{1}{n-2} = \infty$ for $n = 2$) the following conditions are met:

(a) $\frac{nq}{nq-n+2} \leq \theta \leq \frac{n}{n-2} \cdot \frac{m+p-1}{-m+p-1+2\kappa}$,

(b) $\frac{1}{\theta} > 1 - \frac{2}{s}$,

(c) $\frac{nq}{2q+n-2} \leq \mu \leq \frac{n}{n-2} \cdot \frac{m+p-1}{2}$,

(d) $\frac{1}{\mu} > 1 - \frac{2(q-1)}{s}$,

(e) $p_0 \geq n$ *or* $s < \frac{np_0}{n-p_0}$,

(f) $\frac{s}{q} < 2$ *and*

(g) $\kappa < m+1$ *or* $s > \frac{np_0}{p_0-n(\kappa-m)}$.

Furthermore,

$$\beta_1 := \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_1 := \frac{\frac{n}{2}\left(\frac{2}{s} - \frac{1}{\theta'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

as well as

$$\beta_2 := \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_2 := \frac{\frac{n}{2}\left(\frac{2(q-1)}{s} - \frac{1}{\mu'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

are positive, additionally we have

$$\beta_1 + \gamma_1 < 1$$

and

$$\beta_2 + \gamma_2 < 1.$$

Note that the restriction $p_0 < n$ in the case of κ being less than $m+1$ firstly is not a contradiction to the lower bound for p_0. It also is of no consequence at this point since for any bounded domain Ω and any $1 \leq p \leq q$ we have $L^q(\Omega) \subset L^p(\Omega)$.

II.2.1 The case $\kappa \geq m + 1$

Let us begin with the slightly less involved route where $\kappa \geq m+1$ and accordingly $p_0 > n(\kappa - m) \geq n$. We begin with the parameters s and q, which here actually only means fixing one of them:

Lemma II.2.2. *If $\kappa \geq m+1$ and given some $p_0 > n(\kappa - m)$, we can find $p_1 \geq p_0$ with the following property: For any $p > p_1$ and any $s \geq 1$ with*

$$s > \frac{np_0}{p_0 - n(\kappa - m)} > 2$$

by defining

$$q := s \cdot \frac{m+p-1}{2p_0}$$

we achieve

$$q > \frac{m+p-1}{2p_0} > n$$

as well as

$$s < 2(q-1).$$

Furthermore, there are $\theta > 1$ and $\mu > \max\left\{1, \frac{p_0}{2}\right\}$ with

$$\frac{nq}{nq - n + 2} \leq \theta \leq \frac{n}{n-2} \cdot \frac{m+p-1}{-m+p-1+2\kappa}$$

and

$$\theta < \frac{s}{s-2}$$

as well as

$$\frac{nq}{2q+n-2} \leq \mu \leq \frac{n}{n-2} \cdot \frac{m+p-1}{2}.$$

Proof. As $s \geq 1$ and since p_1 can be assumed to be large enough that

$$\frac{m+p-1}{2p_0} > n$$

holds for every $p > p_1$ (an argument which will not be mentioned explicitly while moving along due to its equally trivial and tiresome nature), the first part of our lemma in need of proving is

$$s < 2(q-1).$$

Assume that conversely

$$s \geq 2(q-1) = s \cdot \frac{m+p-1}{p_0} - 2;$$

for $p > 4p_0 + 1 - m$ this results in

$$s > 4s - 2,$$

obviously contradicting $s > 2$. For the estimates belonging to θ and μ, let us take a closer look at the prescribed lower bounds: firstly we see

$$\frac{nq}{nq-n+2} = \frac{n}{n - \frac{n-2}{q}} = \frac{n}{n - \frac{2(n-2)p_0}{s(m+p-1)}} < \frac{n}{n - \frac{(n-2)p_0}{(m+p-1)}}$$

which (being identical to 1 for $n = 2$) can be assumed to be arbitrarily close to 1. Similarly,

$$\frac{nq}{2q+n-2} = \frac{n}{n + \frac{n-2}{q}} = \frac{n}{2 + \frac{2(n-2)p_0}{s(m+p-1)}} < \frac{n}{2 + \frac{(n-2)p_0}{(m+p-1)}} < \frac{n}{2}$$

is definitely less than $\frac{p_0}{2}$ and therefore a trivially guaranteed lower bound for any $\mu > \frac{p_0}{2}$. On the other hand,

$$\frac{n}{n-2} \cdot \frac{m+p-1}{-m+p-1+2\kappa}$$

tends towards $\frac{n}{n-2} > 1$ as p approaches ∞, $\frac{s}{s-2} > 1$ holds for any $s > 2$ and for μ we even have the arbitrarily large term

$$\frac{n}{n-2} \cdot \frac{m+p-1}{2}$$

as an upper bound. Hence, all of the quantities s, q, θ and μ can be chosen according to these restrictions as long as p_1 has been established to be sufficiently large. **q.e.d.**

With these parameters fixed, the next task is to ensure that the combination of them according to lemma II.2.1 behaves as claimed.

Lemma II.2.3. *If $\kappa \geq m+1$ and given some $p_0 > n(\kappa - m)$ and $p_1 \geq p_0$ from the previous lemma, we can find some $p_2 \geq p_1$ such that fixing $p > p_2$ results in the following: the quantities s, q, θ and μ found in that lemma allow for*

$$\beta_1 := \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_1 := \frac{\frac{n}{2}\left(\frac{2}{s} - \frac{1}{\theta'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

as well as

$$\beta_2 := \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_2 := \frac{\frac{n}{2}\left(\frac{2(q-1)}{s} - \frac{1}{\mu'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

to be positive and for

$$\beta_1 + \gamma_1 < 1$$

and

$$\beta_2 + \gamma_2 < 1$$

to hold.

Proof. Firstly, in this setting and with these choices

$$\frac{nq}{s} = \frac{n(m+p-1)}{2p_0}$$

holds, accordingly all of the denominators are the same and positive. Therefore, the positivity of β_1 is a trivial consequence of

$$\frac{-m+p-1+2\kappa}{p_0} > 1.$$

For the positivity of γ_1 we refer to condition (b) which has been ensured by the previous lemma and the only remaining ingredient necessary for the positivity of β_2 is the restriction $\mu > \frac{p_0}{2}$. Finally, since we demanded

$$s < 2(q-1),$$

γ_2 is positive as well.

Now we shall prove that the sums $\beta_j + \gamma_j$ for $j \in \{1, 2\}$ do not exceed 1.

We see

$$\beta_1 + \gamma_1 = \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta} + \frac{2}{s} - \frac{1}{\theta'}\right)}{1 - \frac{n}{2} - 1 + \frac{n(m+p-1}{2p_0}}$$
$$= \frac{\frac{-m+p-1+2\kappa}{p_0} + \frac{2}{s} - 1}{\frac{2}{n} - 1 + \frac{m+p-1}{p_0}},$$

accordingly we obtain the claimed upper bound if (and only if)

$$\frac{-m+p-1+2\kappa}{p_0} + \frac{2}{s} - 1 < \frac{2}{n} - 1 + \frac{m+p-1}{p_0}$$

holds. This in turn is equivalent to

$$\frac{1}{s} < \frac{1}{n} - \frac{\kappa - m}{p_0},$$

in other words: condition (g). For the final step we use

$$\frac{2(q-1)}{s} = \frac{m+p-1}{p_0} - \frac{2}{s}$$

which gives us

$$\beta_2 + \gamma_2 = \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu} + \frac{m+p-1}{p_0} - \frac{2}{s} - \frac{1}{\mu'}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$
$$= \frac{\frac{m+p+1}{p_0} - 1 + \frac{2}{s}}{\frac{2}{n} - 1 + \frac{m+p-1}{p_0}}.$$

This is less than 1 whenever

$$\frac{m+p+1}{p_0} - 1 + \frac{2}{s} < \frac{2}{n} - 1 + \frac{m+p-1}{p_0}$$

holds and this is equivalent to

$$\frac{1}{p_0} - \frac{1}{n} < \frac{1}{s},$$

a trivially correct statement since

$$p_0 > n(\kappa - m) \geq n.$$

q.e.d.

With these two lemmata we have proven lemma II.2.1 whenever $\kappa \geq m+1$.

II.2.2 The case $\kappa < m + 1$

The slightly more difficult task is to find the necessary parameters for the inverse case. In order to make sure that we do not fall into the trap of fixing first p, then q and then p afterwards without contemplating the consequence this last step has on the second one, we start slowly:

Lemma II.2.4. *If $\kappa < m + 1$ and given some $p_0 \geq 1$ with*

$$p_0 \in \left(\frac{n}{2}(1 + \kappa - m), n\right),$$

we can find $p_1 \geq p_0$ with the following property: For any $p > p_1$, with

$$s_- := \begin{cases} 1 & \text{if } \frac{np_0}{n-p_0} \leq 2, \\ 1 + \frac{np_0}{2(n-p_0)} & \text{if } \frac{np_0}{n-p_0} > 2 \end{cases}$$

and

$$s_+ := \frac{np_0}{n - p_0} > 1,$$

as well as

$$q_- := \max\left\{n, 1 + \frac{1}{2}\frac{np_0}{n - p_0}\right\}$$

and

$$q_+ := \frac{n}{2} \cdot \frac{m + p - 1}{n - p_0},$$

we can define two nonempty intervals (s_-, s_+) and (q_-, q_+). Furthermore, we can find $\theta > 1$ and $\mu > \max\left\{\frac{p_0}{2}, 1\right\}$ such that the following estimates hold:

$$\frac{nq}{nq - n + 2} \leq \theta \leq \frac{n}{n-2} \cdot \frac{m + p - 1}{-m + p - 1 + 2\kappa} \text{ for every } q \in (q_-, q_+),$$

$$\frac{1}{\theta} > 1 - \frac{2}{s} \text{ for every } s \in (s_-, s_+),$$

$$\frac{nq}{2q + n - 2} \leq \mu \leq \frac{n}{n-2} \cdot \frac{m + p - 1}{2} \text{ for every } q \in (q_-, q_+)$$

and

$$\frac{1}{\mu} > 1 - \frac{2(q-1)}{s} \text{ for every } q \in (q_-, q_+) \text{ and for every } s \in (s_-, s_+)$$

as well as

$$\frac{s}{q} < 2 \text{ for every } q \in (q_-, q_+) \text{ and for every } s \in (s_-, s_+).$$

Proof. Again the main argument is that fixing some p_1 large enough is the key to these estimates. While obviously $s_+ > 1$ for any $p_0 \in (1, n)$, the interval (q_-, q_+) contains at least one element if p is large enough. Since

$$q \mapsto \frac{nq}{nq - n + 2}$$

is increasing for $q > 1$, for the first condition on θ it suffices to prove

$$\frac{nq_+}{nq_+ - n + 2} \leq \frac{n}{n-2} \cdot \frac{m + p - 1}{-m + p - 1 + 2\kappa}.$$

Herein we see

$$\frac{nq_+}{nq_+ - n + 2} = \frac{n}{n - \frac{n-2}{q_+}} = \frac{n}{n - \frac{2(n-2)}{n} \cdot \frac{n-p_0}{m+p-1}}$$

and this means the term on the left approaches 1 from above while the term on the right tends to $\frac{n}{n-2} > 1$ as p approaches ∞. Similarly, to verify the feasibility of the second bound on θ, we take a look at the behaviour of $\frac{s}{s-2}$ for $s > 2$. First of all, this is only necessary if $s_+ > 2$ and s_- has been chosen in a way that it in this case is a constant larger than 2. Accordingly,

$$\frac{s}{s-2} > \frac{s_-}{s_- - 2} = \frac{1}{p_0} + \frac{n-1}{n} > 1$$

for every $s \in (s_-, s_+)$ in that case, meaning that this additional constraint can once more be resolved by enlarging p_1. The conditions for μ are even less difficult to manage: The only lower bound can be estimated via

$$\frac{nq}{2q + n - 2} \leq \frac{nq}{2q} = \frac{n}{2}$$

for $q \in (q_-, q_+)$ while the first upper bound can be pushed to any positive number. For the other constraint involving μ we see

$$\frac{2(q-1)}{s} > \frac{2(q_- - 1)}{s_+} \geq \frac{2 + \frac{np_0}{n-p_0} - 2}{\frac{np_0}{n-p_0}} = 1,$$

and this means that the term

$$1 - \frac{2(q-1)}{s}$$

is actually negative here for every $q \in (q_-, q_+)$ and every $s \in (s_-, s_+)$. We finish the proof with the – again very coarse – estimate

$$\frac{s}{q} < \frac{s_+}{q_-} \leq \frac{\frac{np_0}{n-p_0}}{1 + \frac{1}{2}\frac{np_0}{n-p_0}} < \frac{\frac{np_0}{n-p_0}}{\frac{1}{2}\frac{np_0}{n-p_0}} = 2$$

for every $q \in (q_-, q_+)$ and every $s \in (s_-, s_+)$.

q.e.d.

Having secured this result, we now proceed to fixing $q \in (q_-, q_+)$ and $s \in (s_-, s_+)$ such that the βs and γs in lemma II.2.1 behave as claimed:

Lemma II.2.5. *If $\kappa < m+1$ and given some $1 \le p_0$ with*

$$p_0 \in \left(\frac{n}{2}(1+\kappa-m), n\right),$$

we can find $p_1 \ge p_0$ such that for every $p > p_1$ we have the following: With s_-, s_+, q_- and q_+ as in the previous lemma, picking $\theta > 1$ and $\mu > \max\left\{\frac{p_0}{2}, 1\right\}$ in accordance with the conditions in that lemma and setting $\theta' := \frac{\theta}{\theta-1}$ as well as $\mu' := \frac{\mu}{\mu-1}$ we can find $q \in (q_-, q_+)$ and $s \in (s_-, s_+)$ such that for the positive quantities

$$\beta_1 := \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_1 := \frac{\frac{n}{2}\left(\frac{2}{s} - \frac{1}{\theta'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

as well as

$$\beta_2 := \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_2 := \frac{\frac{n}{2}\left(\frac{2(q-1)}{s} - \frac{1}{\mu'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

we have

$$\beta_1 + \gamma_1 < 1$$

and

$$\beta_2 + \gamma_2 < 1.$$

Proof. We consider some large $p > p_0$ and may assume that θ and μ are as we want them to be. Since $\frac{s}{q} < 2$ for every eligible s and q, all of the denominators are positive. The positivity of the numerators on the other hand is a direct consequence of most of the other constraints imposed upon our parameters in the previous lemma; the actual work lies in the upper bounds for $\beta_1 + \gamma_1$ and $\beta_2 + \gamma_2$. We want to consider them as evaluations of the functions

$$f(q,s) := \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}} + \frac{\frac{n}{2}\left(\frac{2}{s} - \frac{1}{\theta'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

and

$$g(q,s) := \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}} + \frac{\frac{n}{2}\left(\frac{2(q-1)}{s} - \frac{1}{\mu'}\right)}{1 - \frac{n}{2} + \frac{nq}{s}}$$

at certain points. We find that

$$1 > f(q_+, s_+) = \frac{1}{\frac{2}{n} - 1 + \frac{m+p-1}{p_0}} \left(\frac{-m + p - 1 + 2\kappa}{p_0} - 1 + 2\frac{n - p_0}{np_0} \right)$$

is equivalent to

$$\frac{2}{n} - 1 + \frac{m + p - 1}{p_0} > \frac{-m + p - 1 + 2\kappa}{p_0} - 1 + 2\frac{n - p_0}{np_0},$$

a statement which holds if and only if

$$\frac{2}{n} > 2\frac{\kappa - m}{p_0} + \frac{2}{p_0} - \frac{2}{n}$$

is true – and our condition $p_0 > \frac{n}{2}(1 + \kappa - m)$ ensures just that. Accordingly, utilising the continuity of f, some $\hat{s}_- \in (s_-, s_+)$ and $\hat{q}_- \in (q_-, q_+)$ can be found such that

$$f(q, s) < 1$$

holds for every $q \in (\hat{q}_-, q_+)$ and $s \in (\hat{s}_-, s_+)$. For g we use a similar strategy: with

$$\frac{2(q_+ - 1)}{s_+} = \frac{m + p - 1}{p_0} - \frac{2(n - p_0)}{np_0}$$

we see

$$\begin{aligned} g(q_+, s_+) &= \frac{1}{\frac{2}{n} - 1 + \frac{m+p-1}{p_0}} \left(\frac{2}{p_0} - 1 + \frac{m + p - 1}{p_0} - \frac{2(n - p_0)}{np_0} \right) \\ &= \frac{1}{\frac{2}{n} - 1 + \frac{m+p-1}{p_0}} \left(\frac{2}{n} - 1 + \frac{m + p - 1}{p_0} \right) \\ &= 1. \end{aligned}$$

This on its own does not yet help us; evaluating g at any other point, no matter how close, could still result in a value larger than 1. To show that this is not the case we consider a derivative: if $\frac{\partial g}{\partial q}(q, s_+)$ is positive for every $q \in (\hat{q}_-, q_+)$, then from the previous computation we get

$$g(q, s_+) < 1$$

for every $q \in (\hat{q}_-, q_+)$. Setting $L(q) := \frac{2s_+}{n}\left(1 - \frac{n}{2} + \frac{nq}{2}\right)^2$ yields

$$
\begin{aligned}
L(q)\frac{\partial g}{\partial q}(q, s_+) &= 2\left(1 - \frac{n}{2} + \frac{nq}{2}\right) - n\left(\frac{2(q-1)}{s_+} - \frac{1}{\mu'}\right) \\
&= 2 - n + \frac{2nq}{s_+} - \frac{2n(q-1)}{s_+} + \frac{n}{\mu'} \\
&= 2 - \frac{n}{\mu} + \frac{2n}{s_+} \\
&= 2 - \frac{n}{\mu} + \frac{2(n-p_0)}{np_0} \\
&= n\left(\frac{2}{p_0} - \frac{1}{\mu}\right)
\end{aligned}
$$

and therefore the desired behaviour for every $q \in (\hat{q}_-, q_+)$. Accordingly for some $s \in (\hat{s}_-, s_+)$ close to s_+ and any $q \in (\hat{q}_-, q_+)$ we have

$$g(q, s) < 1$$

which completes the proof. **q.e.d.**

With these tedious but ultimately rather basic computations out of the way, the proof of lemma II.2.1 is complete so that we can move on to more interesting steps.

II.3 Extending the maximal existence time

Locally, standard arguments involving the detection of a fixed point (for example in [38] or [14], see also the procedure in chapter V) give us classical solutions to (II.1) while simultaneously providing us with an extensibility criterion:

Lemma II.3.1. *There are $T_{max} \in (0, \infty]$ and a pair (u, v) of nonnegative functions in $C^0\left(\overline{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right)$ solving* (II.1) *classically in $\Omega \times (0, T_{max})$. Additionally we have*

$$T_{max} = \infty \quad \text{or} \quad \limsup_{t \nearrow T_{max}} \left(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\right) = \infty.$$

This tasks us with estimating

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} < \infty$$

for $t \in (0, T_{\max})$ and so we will proceed to find a less demanding condition from which this can be deduced. This is a process the first step of which – thanks to a pre-existing iteration procedure – is rather simple.

We start with a quick and coarse improvement for the boundedness condition in lemma II.3.1.

In a first step this condition can be relaxed quite easily using two general results:

Lemma II.3.2. *In the setting of theorem II.3.1 we can find $p \in (n+2, \infty)$ such that from*

$$\sup_{t \in (0,T_{max})} \left\| u(\cdot,t) \right\|_{L^p(\Omega)} < \infty$$

we automatically get

$$\sup_{t \in (0,T_{max})} \left(\left\| u(\cdot,t) \right\|_{L^\infty(\Omega)} + \left\| v(\cdot,t) \right\|_{W^{1,\infty}(\Omega)} \right) < \infty.$$

Proof. From lemma I.3.4 we immediately see that the assumed boundedness gives us some $C_1 > 0$ with

$$\left\| v(\cdot,t) \right\|_{W^{1,\infty}(\Omega)} \leq C_1$$

for every $t \in (0, T_{\max})$. For the other term we cite [89]: In the appendix of that paper bounds are derived for systems of the form

$$\begin{cases} w_t \leq \nabla \cdot (D(x,t,w)\nabla w) + \nabla \cdot f(x,t) + g(x,t), & \text{in } \Omega \times (0,T), \\ \frac{\partial w}{\partial \nu} \leq 0, & \text{on } \partial\Omega \times (0,T), \end{cases}$$

and its applicability to our system can easily be seen by setting

$$f(x,t) := S\left(u(x,t)\right) \cdot \nabla v(x,t)$$

for $(x,t) \in \Omega \times (0, T_{\max})$ and

$$g \equiv 0.$$

q.e.d.

This already gives us some $\hat{\mathfrak{p}} > n$ with the property that boundedness of

$$\sup_{t \in (0,T_{\max})} \left\| u(\cdot,t) \right\|_{L^p(\Omega)}$$

for some $p > \hat{\mathfrak{p}}$ suffices to deduce global existence and boundedness of a solution; however, this can be improved upon.

To this end we are going to use the parameters we found in the beginning of this chapter for several estimates eventually proving that in the previous lemma instead of 'some $\hat{\mathfrak{p}} > n$' we can actually achieve $\mathfrak{p}$ from (II.4) as a threshold.

As in many other cases, the starting point is quite straightforward, namely examining

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p$$

and we begin with

Lemma II.3.3. *Let, for some* $T \in (0, \infty]$, *the pair*

$$(u, v) \in \left(C^0\left(\overline{\Omega} \times [0, T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T)\right)\right)^2$$

denote a classical solution to the partial differential equations from (II.1) *in* $\Omega \times (0, T)$ *with* $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ *on* $\partial\Omega \times (0, T)$. *If* $p_0 \geq 1$ *with*

$$\left\|u(\cdot, t)\right\|_{L^{p_0}(\Omega)} < \infty$$

exists, then for every $p \geq 1$ *and every* $q > 1$ *we can find a positive constant* C *such that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right] + \frac{2(p-1)C_D}{(m+p-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2 \\ + \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \\ \leq C + C\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2 \\ + C\int_\Omega (u+1)^2|\nabla v|^{2q-2}\end{aligned}$$

holds for every $t \in (0, T)$.

Proof We begin by computing

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^p &= \int_\Omega (u+1)^{p-1}u_t \\ &= \int_\Omega (u+1)^{p-1}\nabla\cdot(D(u)\nabla u) - \int_\Omega (u+1)^{p-1}\nabla\cdot(S(u)\nabla v) \\ &= -(p-1)\int_\Omega D(u)(u+1)^{p-2}|\nabla u|^2 \\ &\quad + (p-1)\int_\Omega S(u)(u+1)^{p-2}\nabla u\cdot\nabla v \\ &\leq -(p-1)C_D\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\ &\quad + (p-1)C_S\int_\Omega (u+1)^{\kappa+p-2}|\nabla u||\nabla v|\end{aligned}$$

in $(0, T)$ using the assumed estimates for both D and S as well as integration by parts and continue by estimating the rightmost term. By Young's inequality, we see that

$$\begin{aligned}\int_\Omega (u+1)^{\kappa+p-2}|\nabla u||\nabla v| &= \int_\Omega (u+1)^{\frac{m+p-3}{2}}|\nabla u|\,(u+1)^{\frac{-m+p-1}{2}+\kappa}|\nabla v| \\ &\leq \frac{C_D}{2C_S}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\ &\quad + \frac{C_S}{2C_D}\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2\end{aligned}$$

holds in $(0,T)$ and so we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^p + \frac{(p-1)C_D}{2}\int_\Omega (u+1)^{m+p-3}|\nabla(u+1)|^2 \le \frac{(p-1)C_S^2}{2C_D}\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2$$

in $(0,T)$. Together with lemma I.3.11 this gives us some $C>0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right] + \frac{(p-1)C_D}{2}&\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\ &+ \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \\ &\le C\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2 \\ &+ C\int_\Omega (u+1)^2|\nabla v|^{2q-2} + C\end{aligned}$$

is true in $(0,T)$. **q.e.d.**

The terms on the right-hand side in this estimate, using the parameters θ and μ from lemma II.2.1, can be tackled further with Hölder's inequality.

Lemma II.3.4. *With $p_0 \ge 1$ and some $p_1 \ge p_0$ as in lemma II.2.1, for any $p > p_1$ and the associated parameters $q > n$, $s > 1$, $\theta > 1$ and $\mu > 1$ as well as β_1, β_2, γ_1 and γ_2 from that same lemma a positive constant C can be found such that for any $u \in C^1\left(\overline{\Omega}\right) \cap L^{p_0}(\Omega)$ and $v \in C^2\left(\overline{\Omega}\right)$ with $\nabla v \in L^s(\Omega)$ both of the estimates*

$$\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2 \le C\left[1+\left(\int_\Omega \left|\nabla(u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\beta_1}\right]\left[1+\left(\int_\Omega \left|\nabla|\nabla v|^q\right|^2\right)^{\gamma_1}\right]$$

and

$$\int_\Omega (u+1)^2|\nabla v|^{2q-2} \le C\left[1+\left(\int_\Omega \left|\nabla(u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\beta_2}\right]\left[1+\left(\int_\Omega \left|\nabla|\nabla v|^q\right|^2\right)^{\gamma_2}\right]$$

hold.

Proof Setting $\theta' := \frac{\theta}{\theta-1}$ and $\mu' := \frac{\mu}{\mu-1}$, the Hölder inequality allows for decomposing the integrals into their respective factors via

$$\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2 \le \left(\int_\Omega (u+1)^{(-m+p-1+2\kappa)\theta}\right)^{\frac{1}{\theta}}\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}}$$

and

$$\int_\Omega (u+1)^2|\nabla v|^{2q-2} \le \left(\int_\Omega (u+1)^{2\mu}\right)^{\frac{1}{\mu}}\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}}.$$

The second step is to employ the Gagliardo-Nirenberg inequality (lemma I.3.8) to these four integrals; apart from verifying the exponents this means ensuring that we are allowed to apply this result in the first place. For this we define

$$I_1 := \left(\int_\Omega (u+1)^{(-m+p-1+2\kappa)\theta} \right)^{\frac{1}{\theta}},$$

$$I_2 := \left(\int_\Omega |\nabla v|^{2\theta'} \right)^{\frac{1}{\theta'}}$$

and

$$I_3 := \left(\int_\Omega (u+1)^{2\mu} \right)^{\frac{1}{\mu}}$$

as well as

$$I_4 := \left(\int_\Omega |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}}$$

and discuss every integral separately.

In line with our assumptions we have $\|u\|_{L^{p_0}(\Omega)} \le C_0$ and $\|\nabla v\|_{L^s(\Omega)} \le C_0$ for some positive constant C_0, and we will now see that many of the constraints on our parameters have been put in place in order to ensure that this C_0 controls most of the arising terms.

Setting $k := \frac{2(-m+p-1+2\kappa)}{m+p-1}$, a first application of the Gagliardo-Nirenberg inequality gives us two positive constants C_1 and C_2 alongside some $a \in (0,1)$ with

$$\begin{aligned} I_1 &= \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{k\theta}(\Omega)}^{k} \\ &\le C_1 \left\| \nabla (u+1)^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{ka} \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{k(1-a)} \\ &\quad + C_1 \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{k} \\ &\le C_2 \left[1 + \left(\int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 \right)^{\beta_1} \right]. \end{aligned}$$

Herein condition (a) ensures that we may employ lemma I.3.8 and the identity

$$-\frac{n}{k\theta} = \left(1 - \frac{n}{2}\right) a - \frac{n(m+p-1)}{2p_0} (1-a)$$

shows that

$$\frac{ka}{2} = \beta_1$$

holds.

Next, with some positive constants C_3, C_4 and b such that

$$-\frac{nq}{2\theta'} = \left(1-\frac{n}{2}\right)b - \frac{nq}{s}(1-b)$$

we find

$$\begin{aligned}
I_2 &= \left\| |\nabla v|^q \right\|^{\frac{2}{q}}_{L^{\frac{2\theta'}{q}}(\Omega)} \\
&\le C_3 \left\| \nabla |\nabla v|^q \right\|^{\frac{2}{q}b}_{L^2(\Omega)} \left\| |\nabla v|^q \right\|^{\frac{2}{q}(1-b)}_{L^{\frac{s}{q}}(\Omega)} \\
&\quad + C_3 \left\| |\nabla v|^q \right\|^{\frac{2}{q}}_{L^{\frac{s}{q}}(\Omega)} \\
&\le C_4 \left[1 + \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\gamma_1} \right]
\end{aligned}$$

for which we use the lower bound in (a) as well as (b). Again, a simple computation proves $\frac{b}{q} = \gamma_1$.

For the third integral we refer to (c) and find positive constants C_5 and C_6 as well as some c uniquely defined by

$$-\frac{n(m+p-1)}{4\mu} = \left(1-\frac{n}{2}\right)c - \frac{n(m+p-1)}{2p_0}(1-c)$$

such that

$$\begin{aligned}
I_3 =& \left\| (u+1)^{\frac{m+p-1}{2}} \right\|^{\frac{4}{m+p-1}}_{L^{\frac{4\mu}{m+p-1}}(\Omega)} \\
\le& C_5 \left\| \nabla (u+1)^{\frac{m+p-1}{2}} \right\|^{\frac{4}{m+p-1}c}_{L^2(\Omega)} \left\| (u+1)^{\frac{m+p-1}{2}} \right\|^{\frac{4}{m+p-1}(1-c)}_{L^{\frac{2p_0}{m+p-1}}(\Omega)} \\
&+ C_5 \left\| (u+1)^{\frac{m+p-1}{2}} \right\|^{\frac{4}{m+p-1}}_{L^{\frac{2p_0}{m+p-1}}(\Omega)} \\
\le& C_6 \left[1 + \left(\int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 \right)^{\beta_2} \right]
\end{aligned}$$

holds. Here we used $\frac{2c}{m+p-1} = \beta_2$.

Finally, conditions (c) and (d) ensure that for certain $C_7 > 0$ and $C_8 > 0$ as well as $d \in (0,1)$ given by

$$-\frac{nq}{2(q-1)\mu'} = \left(1-\frac{n}{2}\right)d - \frac{nq}{s}(1-d)$$

we have

$$
\begin{aligned}
I_4 &= \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)}^{\frac{2(q-1)}{q}} \\
&\le C_7 \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{2(q-1)}{q}d} \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q}(1-d)} \\
&\quad + C_7 \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q}} \\
&\le C_8 \left[1 + \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\gamma_2} \right]
\end{aligned}
$$

wherein we have used $\frac{(q-1)d}{q} = \gamma_2$. **q.e.d.**

Applying this to the previous lemma II.3.3, we arrive at our penultimate estimate:

Lemma II.3.5. *Under the assumptions from lemma II.3.3 there is $p_1 \ge p_0$ such that for every $p > p_1$ we can find some $q \ge 1$ as well as two positive constants C_1 and C_2 such that*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p} \int_\Omega (u+1)^p + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \right] + C_1 \int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 + C_1 \int_\Omega \left| \nabla |\nabla v|^q \right|^2 \le C_2
$$

holds for every $t \in (0,T)$.

Proof. We use the parameters from lemma II.2.1 and lemma II.3.4 together with the fact that the conditions $\beta_1 + \gamma_1 < 1$ and $\beta_2 + \gamma_2 < 1$ allow us to employ the consequence of Young's inequality that we have derived in lemma I.3.7. This gives us two constants $C_1 > 0$ and $C_2 > 0$ with

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p} \int_\Omega (u+1)^p + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \right] \\
&+ \frac{2(p-1)C_D}{(m+p-1)^2} \int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 + \frac{q-1}{2q^2} \int_\Omega |\nabla |\nabla v|^q|^2 \\
&\quad \le C_1 + C_1 \left[1 + \left(\int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 \right)^{\beta_1} \right] \left[1 + \left(\int_\Omega |\nabla |\nabla v|^q|^2 \right)^{\gamma_1} \right] \\
&+ C_1 \left[1 + \left(\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \right)^{\beta_2} \right] \left[1 + \left(\int_\Omega |\nabla |\nabla v|^q|^2 \right)^{\gamma_2} \right] \\
&\quad \le C_2 + \frac{(p-1)C_D}{(m+p-1)^2} \int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 + \frac{q-1}{4q^2} \int_\Omega |\nabla |\nabla v|^q|^2
\end{aligned}
$$

in $(0,T)$. **q.e.d.**

Using this inequality, it is once more lemma I.3.8 that creates the crucial connection between the quantities in this estimate. With it, we proceed to derive the final ingredient needed for the proof of theorem II.1.1. Returning to the local solutions found in lemma II.3.1, we can find an ordinary differential equation involving the evolution of $\int_\Omega u^p$.

Lemma II.3.6. *In the setting of theorem II.1.1 let some $p_0 > \mathfrak{p}$ and $C_{p_0} > 0$ with $\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \le C_{p_0}$ for every $t \in (0, T_{max})$ be given. Then there is $p_1 \ge p_0$ such that for any $p > p_1$ we can find $q > 1$ and a positive constant C with*

$$y_t + y \le C$$

in $(0, T_{max})$ where $y(t) := \frac{1}{p}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}$.

Proof. From the previous lemma we have two constants K_1 and K_2 with

$$y_t + K_1 \int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2 + K_1 \int_\Omega \left|\nabla |\nabla v|^q\right|^2 \le K_2$$

in $(0,T)$. Accordingly, in this step we want to compare the two integrals on the left to y.

We begin by employing the Gagliardo-Nirenberg interpolation inequality in lemma I.3.8 to

$$\int_\Omega (u+1)^p = \left\|(u+1)^{\frac{m+p-1}{2}}\right\|^{\frac{2p}{m+p-1}}_{L^{\frac{2p}{m+p-1}}(\Omega)}$$

and since p is large enough this results in

$$\begin{aligned}\int_\Omega (u+1)^p \le & C_1 \left\|\nabla (u+1)^{\frac{m+p-1}{2}}\right\|^{\frac{2p}{m+p-1}a}_{L^2(\Omega)} \left\|(u+1)^{\frac{m+p-1}{2}}\right\|^{\frac{2p}{m+p-1}(1-a)}_{L^{\frac{2p_0}{m+p-1}}(\Omega)} \\ & + C_1 \left\|(u+1)^{\frac{m+p-1}{2}}\right\|^{\frac{2p}{m+p-1}}_{L^{\frac{2p_0}{m+p-1}}(\Omega)} \\ \le & C_2 \left[1 + \left(\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\frac{pa}{m+p-1}}\right]\end{aligned}$$

in $(0,T)$ for some positive constants C_1, C_2 and

$$a = \frac{\frac{n(m+p-1)}{2}\left(\frac{1}{p_0} - \frac{1}{p}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}.$$

This means that for

$$\lambda_1 := \frac{\frac{n}{2}\left(\frac{p}{p_0} - 1\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}} \in (0,1)$$

we have

$$\int_\Omega (u+1)^p \le C_2 + C_2 \left(\int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 \right)^{\lambda_1}$$

in $(0,T)$.

Since $\lambda_1 < 1$, Young's inequality gives us some $C_3 > 0$ with

$$\frac{1}{p} \int_\Omega (u+1)^p \le C_3 + K_1 \int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2$$

in $(0,T)$.

Using the boundedness of $\|u\|_{L^1(\Omega} \equiv \|u_0\|_{L^1(\Omega)}$ and lemma I.3.4, for

$$\lambda_2 := \frac{\frac{n}{2}(2q-1)}{1-\frac{n}{2}+nq}$$

and some positive constants C_4, C_5 the Gagliardo-Nirenberg inequality gives us

$$\begin{aligned}
\int_\Omega |\nabla v|^{2q} &= \left\| |\nabla v|^q \right\|^2_{L^2(\Omega)} \\
&\le C_4 \left\| \nabla |\nabla v|^q \right\|^{2\lambda_2}_{L^2(\Omega)} \left\| |\nabla v|^q \right\|^{2(1-\lambda_2)}_{L^{\frac{1}{q}}(\Omega)} \\
&\quad + C_4 \left\| |\nabla v|^q \right\|_{L^{\frac{1}{q}}(\Omega)} \\
&\le C_5 + C_5 \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\lambda_2}
\end{aligned}$$

in $(0,T)$.

As before, from Young's inequality and the trivial observation $\lambda_2 \in (0,1)$ we can then deduce the existence of some $C_6 > 0$ with

$$\frac{1}{q} \int_\Omega |\nabla v|^{2q} \le C_5 + K_1 \int_\Omega \left| \nabla |\nabla v|^q \right|^2$$

in $(0,T)$.

Adding these two inequalities to y_t and estimating according to the initially cited result from the lemma before yields

$$y_t + y \le K_2 + C_3 + C_5$$

in $(0,T)$. **q.e.d.**

Proof of theorem II.1.1. For any solution $y \in C^0([0, T_{\max})) \cap C^1((0, T_{\max}))$ to

$$\begin{cases} \dot{y} & \leq -y + C \text{ in } (0, T_{\max}), \\ y(0) & = y_0, \end{cases}$$

a comparison argument for ordinary differential equations allows us to see

$$y(t) \leq (y_0 - C)\, e^{-t} + C$$

for every $t \in (0, T_{\max})$ with the obvious upper bound $K := \max\{y_0, K\}$.

Thus, according to lemma II.3.6 for any $p > 1$ we find a constant $C_p > 0$ that admits the inequality

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_p$$

for every $t \in (0.T_{\max})$. Together with lemma II.3.2 this proves theorem II.1.1 in light of the extensibility criterion in lemma II.3.1. **q.e.d.**

Proof of corollary II.1.2. Assuming this to be wrong, for any $p_0 \in (\mathfrak{p}, \frac{n}{\alpha})$ and for some positive constants C_1 and C_2 we have

$$\frac{1}{C_1^{p_0}} \int_\Omega u^{p_0}(\cdot, t) \leq \int_\Omega |x - x_0|^{-\alpha p_0}\, \mathrm{d}x = C_2 + \int_{B_1(0)} |x|^{-\alpha p_0}\, \mathrm{d}x$$

and the right-hand side is bounded because of $\alpha p_0 < n$, leading to a contradiction in view of theorem II.1.1. **q.e.d.**

Chapter III

Global existence and boundedness in a chemorepulsion system with superlinear diffusion

III.1 Introduction and main result

This chapter is closely connected to the previous one and the system considered here can be expressed via (II.1) in the associated introduction by choosing $S(u) = -u$. We now want to consider this more specific system, namely

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) + \nabla \cdot (u\nabla v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \tag{III.1}$$

where $D(u) \geq C_D u^{m-1}$ with some $C_D > 0$ and $m \geq 1$ and we point out the crucial difference the changed sign in the first equation will make for $n \geq 3$. With D as here and $S(u) = -u$,

$$m > 1 + \frac{n-2}{n} \tag{III.2}$$

has been identified as a crucial relation for (II.1): if it holds, then [89] shows the global existence and boundedness of solutions (which is confirmed by our more general result in chapter I) while for m below this threshold blow-up may occur (see the introduction in chapter I for more detailed statements).

The positive sensitivity in (III.1), resulting in repulsion instead of attraction, promotes global existence and boundedness of solutions, especially for $m = 1$ we already have relevant results: If $n = 2$, global solutions and their boundedness have been established while for $n \in \{3, 4\}$ locally bounded global weak solutions have been found (both in [15]). For nonlinear sensitivity, [88] has found uniform-in-time bounds for classical solutions and convergence to the average of the initial mass. In this work we want to consider the case of superlinear diffusion and detect a threshold similar to (III.2). To this end we begin by proving

Theorem III.1.1. *For some $n \geq 3$ let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We assume that for a function $D \in C^1([0, \infty))$*

$$C_D(s+1)^{m-1} \leq D(s) \leq C'_D(s+1)^{M-1}$$

holds for every $s \geq 0$ with some positive constants C_D and C'_D as well as some $M \geq m > 1 + \frac{(n-2)(n-1)}{n^2}$. Then for any nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$ the system (III.1) *has a classical solution (u, v) consisting of two nonnegative functions $u \in C^0\left(\overline{\Omega} \times [0, \infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, \infty)\right)$ and $v \in C^0\left(\overline{\Omega} \times [0, \infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, \infty)\right)$. This solution is global and bounded in the sense that there is some $C > 0$ with*

$$\left\|u(\cdot, t)\right\|_{L^\infty(\Omega)} + \left\|v(\cdot, t)\right\|_{W^{1,\infty}(\Omega)} \leq C \text{ for every } t \in (0, \infty).$$

Moreover, for the case of degenerate diffusion (meaning $D(0) = 0$), this result can be used to detect global weak solutions that are locally bounded using the fact that D vanishing at zero influences the construction of solutions but not the size of the bounds derived in this chapter:

Theorem III.1.2. *For some $n \geq 3$ let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We assume that for a function $D \in C^1([0, \infty))$*

$$C_D s^{m-1} \leq D(s) \leq C'_D(s+1)^{M-1}$$

holds for every $s \geq 0$ with some positive constants C_D and C'_D as well as some exponents $M \geq m > 1 + \frac{(n-2)(n-1)}{n^2}$. Then for any $u_0 \in L^1(\Omega)$ and $v_0 \in W^{1,1}(\Omega)$ we find a locally bounded global weak solution to (III.1) *in the sense of definition III.3.1.*

Remark. *For the entirety of this chapter, we assume that for $n \geq 3$ some bounded domain $\Omega \subset \mathbb{R}^n$ be given. Furthermore, with some $C_D, C'_D > 0$ as well as $M \geq m > 1 + \frac{(n-2)(n-1)}{n^2}$, let $D \in C^1([0, \infty))$ be such that*

$$C_D s^{m-1} \leq D(s) \leq C'_D(s+1)^{M-1}$$

holds for every $s \geq 0$.

III.2 Uniform boundedness of classical solutions for nondegenerate diffusion and the proof of theorem III.1.1

In this part we will prove the existence of a uniformly bounded classical solution (u,v) to our system (III.1) in $\Omega\times(0,\infty)$ whenever D is positive in $[0,\infty)$. Furthermore, the bounds we find do not depend on $D(0)$ which enables us to utilise these results in an upcoming approximation process.

Under the overarching condition

$$m > 1 + \frac{(n-2)(n-1)}{n^2}, \tag{III.3}$$

which is less strict than (III.2) for every $n \geq 3$, we shall assume that we have been given some $T \in (0,\infty]$ and a pair of classical solutions to (III.1) in $\Omega\times(0,T)$. It is worth noting that in the upcoming results and especially their proofs none of the constants depend on the value of T or $D(0)$.

The main result of this section accordingly reads as follows:

Lemma III.2.1. *For every $K > 0$ there is a positive constant C such that whenever nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$ with*

$$\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{W^{1,\infty}(\Omega)} \leq K$$

are given, for any $T \in (0,\infty]$ and any classical solution (u,v) of (III.1) *in $\Omega\times(0,T)$ we have*

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|v(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} \leq C$$

for every $t \in (0,T)$.

When discussing chemoattraction systems, initial steps for the regularity of any solution often consist of proving boundedness of u in $L^1(\Omega)$ and then using L^p-L^q-estimates from lemma I.3.3 to show $v \in W^{1,q}(\Omega)$ for any $q \in [1,\frac{n}{n-1})$. Here however, the small difference in the first equation enables us to go even further as was first shown by [15].

Lemma III.2.2. *For any $K > 0$ we can find $C > 0$ such that, whenever we are given nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$ with*

$$\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{W^{1,\infty}(\Omega)} \leq K,$$

for any $T > 0$ and any classical solution (u,v) to (III.1) *in $\Omega\times(0,T)$ the estimates*

$$\left\|u(\cdot,t)\right\|_{L^1(\Omega)} \leq C$$

and

$$\left\|\nabla v(\cdot,t)\right\|_{L^2(\Omega)} \leq C$$

hold for every $t \in (0,T)$.

Proof From the simple observation $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = 0$ we see

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0$$

for every $t \in (0,T)$ and this proves the first claim.

Using integration by parts and the nonnegativity of D shows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u\ln u + \frac{1}{2}\int_\Omega |\nabla v|^2\right) &= \int_\Omega (1+\ln u)\, u_t + \int_\Omega \nabla v\cdot\nabla v_t \\ &= -\int_\Omega \frac{\nabla u}{u}\,(D(u)\nabla u + u\nabla v) - \int_\Omega |\Delta v|^2 - \int_\Omega |\nabla v|^2 + \int_\Omega \nabla u\cdot\nabla v \\ &= -\int_\Omega \frac{D(u)}{u}\,|\nabla u|^2 - \int_\Omega |\Delta v|^2 - \int_\Omega |\nabla v|^2 \\ &\le 0\end{aligned}$$

in $(0,T)$. Upon integration over $(0,t)$ for any $t \in (0,T)$, the easily verified estimate $x\ln x \ge -\frac{1}{e}$ for $x > 0$ results in

$$\begin{aligned}\int_\Omega |\nabla v(\cdot,t)|^2 &\le -2\int_\Omega u(\cdot,t)\ln u(\cdot,t) + 2\int_\Omega u_0\ln u_0 + \int_\Omega |\nabla v_0|^2 \\ &\le \frac{2|\Omega|}{e} + 2\int_\Omega u_0\ln u_0 + \int_\Omega |\nabla v_0|^2 \\ &\le \frac{2|\Omega|}{e} + 2|\Omega|K\cdot\ln(K+1) + |\Omega|K^2\end{aligned}$$

and this right-hand side is some constant depending only on Ω and K which completes the proof. **q.e.d.**

It is our goal to prove uniform boundedness of both components of any solution (u,v) to (III.1) and the next step on this way is concerned with a higher regularity for u. To prepare for this we prove the following estimate:

Lemma III.2.3. *Given $p > 1$ and $q > 1$, we can find two positive constants C_1 and C_2 such that for any $T > 0$ and any combination $u \in C^0\left(\overline{\Omega}\times[0,T)\right)\cap C^{2,1}\left(\overline{\Omega}\times(0,T)\right)$ and $v \in C^0\left(\overline{\Omega}\times[0,T)\right)\cap C^{2,1}\left(\overline{\Omega}\times(0,T)\right)$ satisfying the differential equations in* (III.1) *in $\Omega\times(0,T)$ as well as $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\Omega\times(0,T)$ the estimate*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\int_\Omega u^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right) + C_1\int_\Omega \left|\nabla u^{\frac{m+p-1}{2}}\right|^2 + C_1\int_\Omega |\nabla|\nabla v|^q|^2 \\ \le C_2\int_\Omega u^{-m+p+1}|\nabla v|^2 + C_2\int_\Omega u^2|\nabla v|^{2q-2}\end{aligned}$$

holds in $(0,T)$.

Proof Using integration by parts and the boundary conditions for u and v as well as our estimate for D and Young's inequality, we find positive constants C_1 and C_2 with

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p &= \int_\Omega u^{p-1}u_t\\ &= \int_\Omega u^{p-1}\nabla\cdot(D(u)\nabla u + u\nabla v)\\ &= -(p-1)\int_\Omega D(u)u^{p-2}\,|\nabla u|^2 - (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &\le (p-1)C_D\int_\Omega u^{m+p-3}\,|\nabla u|^2 - (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &\le -C_1\int_\Omega\left|\nabla u^{\frac{m+p-1}{2}}\right|^2 + C_2\int_\Omega u^{-m+p+1}|\nabla v|^2\end{aligned}$$

in $(0,T)$.

Due to

$$\Delta|\nabla v|^2 = 2\nabla v\cdot\nabla\Delta v + 2|D^2v|^2$$

in $\Omega\times(0,T)$ we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla v|^2 &= 2\nabla v\cdot\nabla\Delta v - 2|\nabla v|^2 + 2\nabla u\cdot\nabla v\\ &= \Delta|\nabla v|^2 - 2|D^2v|^2 - 2|\nabla v|^2 + 2\nabla u\cdot\nabla v\end{aligned}$$

in $\Omega\times(0,T)$ and therefore, together with some $C_3 > 0$ given by lemma I.3.10,

$$\begin{aligned}\frac{1}{q}\int_\Omega|\nabla v|^{2q} \le &-\frac{q-1}{q^2}\int_\Omega\left|\nabla\,|\nabla v|^q\right|^2 + C_3\\ &-2\int_\Omega|\nabla v|^{2q-2}|D^2v|^2 - 2\int_\Omega|\nabla v|^{2q} + 2\int_\Omega|\nabla v|^{2q-2}\nabla u\cdot\nabla v\end{aligned}$$

holds in $(0,T)$.

For the rightmost integral in this estimate we utilise integration by parts, Young's inequality and $|\Delta v|^2 \leq n|D^2 v|^2$ to see that for some $C_4 > 0$

$$\begin{aligned}
2\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v =& -2\int u\nabla\cdot\int_\Omega\Big(|\nabla v|^{2q-2}\,\nabla v\Big)\\
=& -2\int u\nabla\,|\nabla v|^{2q-2}\cdot\nabla v - 2\int u\,|\nabla v|^{2q-2}\cdot\Delta v\\
=& -2(q-1)\int u\,|\nabla v|^{2q-4}\,\nabla\,|\nabla v|^2\cdot\nabla v - 2\int u\,|\nabla v|^{2q-2}\cdot\Delta v\\
\leq& \frac{q-1}{2}\int_\Omega |\nabla v|^{2q-4}\left|\nabla\,|\nabla v|^2\right|^2 + C_4\int_\Omega u^2|\nabla v|^{2q-2}\\
&+\frac{2}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2\\
\leq& \frac{q-1}{2}\int_\Omega |\nabla v|^{2q-4}\left|\nabla\,|\nabla v|^2\right|^2 + C_4\int_\Omega u^2|\nabla v|^{2q-2}\\
&+2\int_\Omega |\nabla v|^{2q-2}|D^2 v|^2
\end{aligned}$$

holds in $(0,T)$. Adding this to the previous result completes the proof. **q.e.d.**

In the next lemma, which also is the source of our restriction for m, we will fix several parameters which combined with the previous result will go on to show that u belongs to $L^p(\Omega)$ for any finite p.

Lemma III.2.4. *There is $\hat{p} > 1$ with the following property: For any $p > \hat{p}$ we can find some $q > 2$, $\theta > 1$ and $\mu > \frac{n}{2}$ such that*

$$\theta \leq \frac{n}{n-2}\frac{m+p-1}{-m+p+1}, \tag{III.4}$$

$$\theta \geq \frac{1}{1-\frac{n-2}{nq}}, \tag{III.5}$$

$$\mu \leq \frac{n}{n-2}\frac{m+p-1}{2} \tag{III.6}$$

and

$$\mu \geq \frac{1}{1-\frac{n-2}{n}\cdot\frac{q-1}{q}}, \tag{III.7}$$

as well as

$$\frac{n}{2}\left(\frac{-m+p+1-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{1}{\theta}}{1-\frac{n}{2}+\frac{nq}{2}}\right) < 1 \tag{III.8}$$

and

$$\frac{n}{2}\left(\frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{q-2+\frac{1}{\mu}}{1-\frac{n}{2}+\frac{nq}{2}}\right) < 1 \tag{III.9}$$

hold.

Proof. Let us first verify that the lower bounds in (III.5) and (III.7) are in fact positive numbers: We need $1 - \frac{n-2}{nq} > 0$, which is equivalent to $q > \frac{n-2}{n}$ – an obviously true statement for every $q > 2$ – as well as

$$1 > \frac{n-2}{n} \cdot \frac{q-1}{q},$$

which again trivially holds.

Next, let us show that neither the combination of (III.4) and (III.5) nor that of (III.6) and (III.7) leads to a contradiction: We firstly need

$$\frac{1}{1 - \frac{n-2}{nq}} \leq \frac{n}{n-2} \cdot \frac{m+p-1}{-m+p+1}$$

for p and q above certain thresholds. Since for

$$\alpha : (m-1, \infty) \to (0, \infty), \; x \mapsto \frac{m+x-1}{-m+x+1}$$

the derivative

$$\alpha'(x) = -2\frac{m-1}{(-m+x+1)^2}$$

is strictly negative it suffices to show

$$\frac{1}{1 - \frac{n-2}{nq}} \leq \frac{n}{n-2}.$$

This is equivalent to

$$1 \leq \frac{n}{n-2} - \frac{1}{q}$$

which in turn is true for every $q \geq \frac{n-2}{n}$. For the remaining claim we want to have

$$\frac{1}{1 - \frac{n-2}{n} \cdot \frac{q-1}{q}} \leq \frac{n}{n-2} \cdot \frac{m+p-1}{2}$$

and this is true since for every $q > 2$ trivially

$$\frac{1}{1 - \frac{n-2}{n}\frac{q-1}{q}} < \frac{2n}{n+2}$$

while the right side can be assumed to be arbitrarily large. Only within this proof we also want to add the condition

$$\theta > \frac{n-2}{n} \cdot \frac{1}{2m + \frac{2}{n} - 3} \in (0, \infty). \tag{$*$}$$

For the positivity of this expression we use (III.3) and see

$$\begin{aligned}2m+\frac{2}{n}-3 >&2\left(1+\frac{(n-2)(n-1)}{n^2}\right)+\frac{2}{n}-3\\ =&\frac{2n^2+2(n^2-3n+2)+2n-3n^2}{n^2}\\ =&\frac{n^2-4n+4}{n^2}\\ =&\left(\frac{n-2}{n}^2\right).\end{aligned}$$

We also need to show

$$\frac{n-2}{n}\cdot\frac{1}{2m+\frac{2}{n}-3}<\frac{n}{n-2}\cdot\frac{m+p-1}{-m+p+1}$$

for large p so that again

$$\frac{n-2}{n}\cdot\frac{1}{2m+\frac{2}{n}-3}<\frac{n}{n-2}$$

is sufficient. With the estimate used to prove the positivity of the lower bound in $(*)$ this is evidently true.

Lastly, before tackling (III.8) and (III.9), we want to prove that for θ fulfilling $(*)$ and any $\mu>\frac{n}{2}$ we have

$$2m+\frac{2}{n}-3-\frac{1}{\theta}\cdot\frac{1-\frac{2}{n}}{\frac{2}{n}+1-\frac{1}{\mu}}>0. \tag{$**$}$$

This is equivalent to

$$2m+\frac{2}{n}-3>\frac{1}{\theta}\cdot\frac{1-\frac{2}{n}}{\frac{2}{n}+1-\frac{1}{\mu}}$$

and then

$$\theta>\frac{1-\frac{2}{n}}{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(2m+\frac{2}{n}-3\right)}.$$

This is true since for every $\mu>\frac{n}{2}$

$$\frac{1-\frac{2}{n}}{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(2m+\frac{2}{n}-3\right)}<\frac{\frac{n-2}{n}}{\left(\frac{2}{n}+1-\frac{2}{n}\right)\left(2m+\frac{2}{n}-3\right)}=\frac{n-2}{n}\cdot\frac{1}{2m+\frac{2}{n}-2},$$

at which point we refer to $(*)$. Accordingly, we can (and do) now pick $\theta>1$ and $\mu>\frac{n}{2}$ such that for every sufficiently large p and for every $q>\frac{n-2}{2}$ the conditions (III.4)-(III.7) and $(*)$ are met. We pick some such p and set

$$\tilde{q}(p) := \frac{n-2}{n} + \frac{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(\frac{2}{n}+m+p-2\right)}{2-\frac{1}{\mu}} > \frac{n-2}{n}$$

wherein silently another unproblematic constraint for p has been assumed to hold, using for example $\frac{\frac{2}{n}+1-\frac{1}{\mu}}{2-\frac{1}{\mu}} > \frac{1}{2}$ for any $\mu > 1$. The idea then is to show that for any $q \in \left(\frac{n-2}{2}, \tilde{q}\right)$ close enough to this p-dependent value the remaining claims (III.8) and (III.9) hold.

To this end we set

$$f(q) := \frac{-m+p+2\kappa-1-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{1}{\theta}}{1-\frac{n}{2}+\frac{nq}{2}}$$

and

$$g(q) := \frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{q-2+\frac{1}{\mu}}{1-\frac{n}{2}+\frac{nq}{2}}$$

for $q \in \left(\frac{n-2}{2}, \tilde{q}\right)$.

Straightforward computations show

$$\begin{aligned} f(\tilde{q}(p)) &= \frac{-m+p+1-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n-2}{2}+\frac{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)}{2-\frac{1}{\mu}}} \\ &= \frac{-m+p+1-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{1}{\theta}\left(2-\frac{1}{\mu}\right)}{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)} \\ &= \frac{-m+p+1+\frac{1}{\theta}\frac{1-\frac{2}{n}}{\frac{2}{n}+1-\frac{1}{\mu}}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \end{aligned}$$

and we immediately see that this tends to $\frac{2}{n}$ as p approaches ∞. Less obvious is the answer to the question as to whether this happens from above – the only case which would help us –, from below or without any monotonicity at all. However, considering this term as a function in and computing its derivative with respect to p,

$$\frac{1-\frac{n}{2}+\frac{n(m+p-1)}{2}-\frac{n}{2}\left(-m+p+1+\frac{1}{\theta}\frac{1-\frac{2}{n}}{\frac{2}{n}+1-\frac{1}{\mu}}\right)}{\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)^2} = \frac{2m+\frac{2}{n}-3-\frac{1}{\theta}\frac{1-\frac{2}{n}}{\frac{2}{n}+1-\frac{1}{\mu}}}{\frac{2}{n}\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)^2},$$

gives us (together with (∗∗)) the desired result: $f(\tilde{q}(p))$ is indeed less than $\frac{2}{n}$ and accordingly, using the obvious continuity, so is $f(q)$ for q close to $\tilde{q}(p)$.

Two more computations show

$$
\begin{aligned}
g(\tilde{q}(p)) &= \frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{n-2}{n}+\frac{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(\frac{2}{n}+m+p-2\right)}{2-\frac{1}{\mu}}-2+\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n-2}{2}+\frac{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)}{2-\frac{1}{\mu}}} \\
&= \frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{-\frac{2}{n}-1+\frac{1}{\mu}+\frac{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(\frac{2}{n}+m+p-2\right)}{2-\frac{1}{\mu}}}{\frac{\left(\frac{2}{n}+1-\frac{1}{\mu}\right)\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)}{2-\frac{1}{\mu}}} \\
&= \frac{2}{n}
\end{aligned}
$$

and

$$
\frac{\mathrm{d}g}{\mathrm{d}q}(q) = \frac{1-\frac{n}{2}+\frac{nq}{2}-\frac{n}{2}\left(q-2+\frac{1}{\mu}\right)}{\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)^2} = \frac{1+\frac{n}{2}\left(1-\frac{1}{\mu}\right)}{\left(1-\frac{n}{2}+\frac{n(m+p-1)}{2}\right)^2} > 0,
$$

resulting in the fact that $g(q) < \frac{2}{n}$ holds for $q \in (2, \tilde{q}(p))$. Therefore, for q close to $\tilde{q}(p)$ the remaining two conditions can be guaranteed as well.

q.e.d.

As announced, we now use these parameters together with lemma III.2.3 in order to obtain a useful regularity result for u.

Lemma III.2.5. *In the setting of lemma III.2.1, for any $p > 1$ there is $C > 0$ such that*

$$
\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \leq C
$$

holds for every $t \in (0, T)$.

Proof. Since Ω is bounded, it suffices to prove the statement for p above the thresholds in lemma III.2.4. We use q, θ and μ as provided by the same lemma, furthermore we fix $\theta' := \frac{1}{1-\frac{1}{\theta}}$ and $\mu' := \frac{1}{1-\frac{1}{\mu}}$. Applying Hölder's inequality to the right-hand side of the result of lemma III.2.3 motivates our using the Gagliardo-Nirenberg inequality to prove the existence of some $C > 0$ with

$$
\left(\int_\Omega u^{(-m+p+1)\theta}\right)^{\frac{1}{\theta}} \leq C + C\left(\int_\Omega \left|\nabla u^{\frac{m+p-1}{2}}\right|^2\right)^{\frac{n}{2}\frac{-m+p+1-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}}
$$

and

$$
\left(\int_\Omega u^{2\mu}\right)^{\frac{1}{\mu}} \leq C + C\left(\int_\Omega \left|\nabla u^{\frac{m+p-1}{2}}\right|^2\right)^{\frac{n}{2}\frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}}
$$

as well as

$$\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} \le C + C\left(\int_\Omega \left|\nabla |\nabla v|^q\right|^2\right)^{\frac{n}{2}\frac{1-\frac{1}{\theta'}}{1-\frac{n}{2}+\frac{nq}{2}}}$$

and

$$\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}} \le C + C\left(\int_\Omega \left|\nabla |\nabla v|^q\right|^2\right)^{\frac{n}{2}\frac{q-1-\frac{1}{\mu'}}{1-\frac{n}{2}+\frac{nq}{2}}}$$

in $(0,T)$.

Using (III.4) in lemma III.2.4, the mass conservation of u and the Gagliardo-Nirenberg inequality give us some $C_1 > 0$ and $C_2 > 0$ as well as a certain $a \in (0,1)$ with

$$\begin{aligned}
\left(\int_\Omega u^{(-m+p+1)\theta}\right)^{\frac{1}{\theta}} &= \left\| u^{\frac{m+p-1}{2}} \right\|^{2\frac{-m+p+1}{m+p-1}}_{L^{2\frac{(-m+p+1)\theta}{m+p-1}}(\Omega)} \\
&\le C_1 \left\|\nabla u^{\frac{m+p-1}{2}}\right\|^{2a\frac{-m+p+1}{m+p-1}}_{L^2(\Omega)} \left\| u^{\frac{m+p-1}{2}}\right\|^{2(1-a)\frac{-m+p+1}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\
&\quad + C_1 \left\| u^{\frac{m+p-1}{2}}\right\|^{2\frac{-m+p+1}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\
&\le C_2 + C_2\left(\int_\Omega \left|\nabla u^{\frac{m+p-1}{2}}\right|^2\right)^{\frac{n}{2}\cdot\frac{-m+p+1-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}}
\end{aligned}$$

in $(0,T)$. Herein the exponent a is determined by

$$-\frac{n(m+p-1)}{2(-m+p+1)\theta} = \left(1-\frac{n}{2}\right)a - \frac{n(m+p-1)}{2}(1-a)$$

which leads to

$$a = \frac{n(m+p-1)}{2(-m+p+1)} \frac{-m+p+1-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}.$$

Since $\mu > \frac{n}{2}$ and (III.6), we analogously have $C_3 > 0$ and $C_4 > 0$ such that

$$\begin{aligned}
\left(\int_\Omega u^{2\mu}\right)^{\frac{1}{\mu}} &= \left\| u^{\frac{m+p-1}{2}}\right\|^{\frac{4}{m+p-1}}_{L^{\frac{4\mu}{m+p-1}}(\Omega)} \\
&\le C_3 \left\|\nabla u^{\frac{m+p-1}{2}}\right\|^{\frac{4b}{m+p-1}}_{L^2(\Omega)} \left\| u^{\frac{m+p-1}{2}}\right\|^{\frac{4(1-b)}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} + C_3 \left\| u^{\frac{m+p-1}{2}}\right\|^{\frac{4}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\
&\le C_4 + C_4\left(\int_\Omega \left|\nabla u^{\frac{m+p-1}{2}}\right|^2\right)^{\frac{n}{2}\cdot\frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}}
\end{aligned}$$

holds in $(0,T)$ with $b = \frac{n(m+p-1)}{4} \frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}$.

With respect to the integrals containing v we use the boundedness of $\int_\Omega |\nabla v|^2$ in $(0,T)$ as well as (III.5) to find positive constants C_5 and C_6 with

$$\begin{aligned}
\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} &= \left\||\nabla v|^q\right\|^{\frac{2}{q}}_{L^{\frac{2\theta'}{q}}(\Omega)} \\
&\le C_5 \left\|\nabla |\nabla v|^q\right\|^{\frac{2c}{q}}_{L^2(\Omega)} \left\||\nabla v|^q\right\|^{\frac{2(1-c)}{q}}_{L^{\frac{2}{q}}(\Omega)} + C_5 \left\||\nabla v|^q\right\|^{\frac{2}{q}}_{L^{\frac{2}{q}}(\Omega)} \\
&\le C_6 + C_6 \left(\int_\Omega \left|\nabla |\nabla v|^q\right|^2\right)^{\frac{n}{2}\cdot\frac{1-\frac{1}{\theta'}}{1-\frac{n}{2}+\frac{nq}{2}}}
\end{aligned}$$

in $(0,T)$ and where $c = \frac{nq}{2} \frac{1-\frac{1}{\theta'}}{1-\frac{n}{2}+\frac{nq}{2}}$.

Condition (III.7) guarantees that $d = \frac{nq}{2(q-1)} \frac{q-1-\frac{1}{\mu'}}{1-\frac{n}{2}+\frac{nq}{2}}$ belongs to $(0,1)$ and that for certain $C_7 > 0$ and $C_8 > 0$

$$\begin{aligned}
\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}} &= \left\||\nabla v|^q\right\|^{\frac{2(q-1)}{q}}_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)} \\
&\le C_7 \left\|\nabla |\nabla v|^q\right\|^{\frac{2d(q-1)}{q}}_{L^2(\Omega)} \left\||\nabla v|^q\right\|^{\frac{2(1-d)(q-1)}{q}}_{L^{\frac{2}{q}}(\Omega)} + C_7 \left\||\nabla v|^q\right\|^{\frac{2(q-1)}{q}}_{L^{\frac{2}{q}}(\Omega)} \\
&\le C_8 + C_8 \left(\int_\Omega \left|\nabla |\nabla v|^q\right|^2\right)^{\frac{n}{2}\cdot\frac{q-1-\frac{1}{\mu'}}{1-\frac{n}{2}+\frac{nq}{2}}}
\end{aligned}$$

holds in $(0,T)$.

Now the estimate from the previous lemma III.2.3 combined with lemma I.3.7 gives us some positive constant C_9 with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\int_\Omega u^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right) + C_9 \int_\Omega \left|\nabla u^{\frac{m+p-1}{2}}\right|^2 + C_9 \int_\Omega \left|\nabla |\nabla v|^q\right|^2 \le \frac{1}{C_9}$$

in $(0,T)$.

Similarly to before, the Gagliardo-Nirenberg inequality allows for the comparison of the occurring terms: For sufficiently large p we have two constants $C_{10} > 0$ and $C_{11} > 0$

such that

$$\begin{aligned}\int_\Omega u^p &= \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2p}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1}} \\ &\le C_{10} \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{\frac{2ep}{m+p-1}} \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2(1-e)p}{m+p-1}} + C_{10} \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1}} \\ &\le C_{11} + C_{11} \left(\int_\Omega \left| \nabla u^{\frac{m+p-1}{2}} \right|^2 \right)^{\frac{n}{2} \cdot \frac{p-1}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}}\end{aligned}$$

holds in $(0,T)$ with $e = \frac{n(m+p-1)}{2p} \frac{p-1}{1-\frac{n}{2}+\frac{m+p-1}{2}}$.

Analogously we find that for some $C_{12} > 0$ and $C_{13} > 0$ we have

$$\begin{aligned}\int_\Omega |\nabla v|^{2q} &= \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^2 \\ &\le C_{12} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{2f} \left\| |\nabla v|^q \right\|_{L^{\frac{2}{q}}(\Omega)}^{2(1-f)} + C_{12} \left\| |\nabla v|^q \right\|_{L^{\frac{2}{q}}(\Omega)}^2 \\ &\le C_{13} + C_{13} \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\frac{n}{2} \cdot \frac{q-1}{1-\frac{n}{2}+\frac{nq}{2}}}\end{aligned}$$

in $(0,T)$ where $f = \frac{n}{2} \frac{q-1}{1-\frac{n}{2}+\frac{nq}{2}}$.

Therefore, from Young's inequality and the fact that the exponents in these two estimates belong to $(0,1)$, just like in the previous chapter, we now have a constant $C_{14} > 0$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{p} \int_\Omega u^p + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \right) + \frac{1}{p} \int_\Omega u^p + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \le C_{14}$$

in $(0,T)$ and this gives us the ordinary differential inequality

$$y_t + y \le C_{14}$$

in $(0,T)$ for $y(t) := \frac{1}{p} \int_\Omega u^p(\cdot,t) + \frac{1}{q} \int_\Omega \left| \nabla v(\cdot,t) \right|^{2q}$, $t \in (0,T)$.

The same comparison argument as in the previous chapter (see the proof of theorem II.1.1) shows

$$\frac{1}{p} \int_\Omega u^p(\cdot,t) \le y(t) \le \max\left\{ \frac{1}{p} \int_\Omega u_0^p + \frac{1}{q} \int_\Omega |\nabla v_0|^{2q}, C_{14} \right\}$$

for every $t \in (0,T)$ which completes the proof. **q.e.d.**

With this regularity result for u and the by now well-known estimate for v in lemma I.3.4, we are now able to prove boundedness of u and v, which directly results in the proof of the statement formulated at the beginning of this section:

Proof of lemma III.2.3. For any $p > n+2$, lemma III.2.5 gives us some $C_1 > 0$ with

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \le C_1 \in (0,T)$$

for every $t \in (0,T)$ which in turn, using lemma I.3.4, provides us with some positive C_2 such that

$$\left\|v(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} \le C_2$$

holds in $(0,T)$. Together with lemma A.1 from [89] we conclude as claimed. **q.e.d.**

Having secured this statement for arbitrary solutions to (III.1), for theorem III.1.1 we only need the existence of such a solution.

Proof of theorem III.1.1. Since $D(0) > 0$, using standard arguments (namely from [47]) we find a local solution to (III.1) in $\Omega \times (0, T_{\max})$ for some $T_{\max} \in (0,\infty]$. We also see that a finite value for $T_{\max}$ leads to

$$\limsup_{t \nearrow T_{\max}} \left(\left\|u(\cdot,t)\right\|_{L^p(\Omega)} + \left\|v(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} \right) = \infty.$$

However, the trivial observation $(s+1)^{m-1} \ge s^{m-1}$ for every $s \ge 0$ proves that the second alternative cannot occur as we have seen in lemma III.2.3.

q.e.d.

III.3 Introduction and existence of weak solutions

The crucial question concerning the existence of solutions is whether D vanishes at $u = 0$ or not. In the case of nondegenerate diffusion we have detected the global existence of classical solutions and proven their boundedness. Using an approximation process, this will also result in us finding weak solutions for systems where $D(0) = 0$ and for this it is crucial that the bounds from before do not depend on the precise value of D at $u = 0$.

Let us begin by defining an appropriate solution concept:

Definition III.3.1. *Setting* $\overline{D}(s) := \int_0^s D(\sigma)\, \mathrm{d}\sigma$ *for* $s \ge 0$ *and given nonnegative* $u_0 \in L^1(\Omega)$ *and* $v_0 \in L^1(\Omega)$*, by a locally bounded global weak solution to* (III.1) *we mean a pair of functions* $(u,v) : \Omega \times [0,\infty) \to \mathbb{R}^2$ *with the regularity*

$$u \in L^\infty_{loc}\left([0,\infty); L^\infty(\Omega)\right),$$
$$\overline{D}(u) \in L^2_{loc}\left([0,\infty); L^2(\Omega)\right) \text{ and}$$
$$v \in L^\infty_{loc}\left([0,\infty); W^{1,\infty}(\Omega)\right)$$

which solves

$$-\int_0^\infty\int_\Omega u\varphi_t-\int_\Omega u_0\varphi(\cdot,0)=-\int_0^\infty\int_\Omega\nabla\overline{D}(u)\nabla\varphi-\int_0^\infty\int_\Omega u\nabla v\nabla\varphi$$

and

$$-\int_0^\infty\int_\Omega v\varphi_t-\int_\Omega v_0\varphi(\cdot,0)=-\int_0^\infty\int_\Omega\nabla v\nabla\varphi-\int_0^\infty\int_\Omega v\varphi+\int_0^\infty\int_\Omega u\varphi$$

for any $\varphi\in C_0^\infty\left(\overline{\Omega}\times[0,\infty)\right)$.

To prove the existence of such a solution we will use $\varepsilon\in(0,1)$ and the function $D_\varepsilon:=D(\cdot+\varepsilon)$ to approximate D. Clearly, upon an appropriate discussion of the initial data, this choice allows for the employment of theorem III.1.1 since

$$C_D\varepsilon^{m-1}(s+1)^{m-1}\le D_\varepsilon(s)\le C_D'(s+1)^{M-1}$$

holds for every $s\ge 0$. While this may seem to couple the estimates to $\varepsilon\in(0,1)$, which is now a necessary part of such a lower bound for the diffusion, the proofs only rely on $D_\varepsilon(s)\ge C_D s^{m-1}$ which is valid independently of $\varepsilon\in(0,1)$.

As a basis for all following steps we want to fix the used approximations and their properties.

Lemma III.3.2. *Let nonnegative* $u_0\in L^1(\Omega)$ *as well as* $v_0\in W^{1,1}(\Omega)$ *be given and for* $\varepsilon\in(0,1)$ *define* $D_\varepsilon:=D(\cdot+\varepsilon)$. *Then we have*

$$C_D\varepsilon^{m-1}(s+1)^{m-1}\le D_\varepsilon(s)\le C_D'2^{M-1}(s+1)^{M-1}$$

and

$$D_\varepsilon(s)\ge C_D s^{m-1}$$

for every $s\ge 0$. *Additionally there are* $K>0$ *and two sequences of functions,* $(u_{0\varepsilon})_{\varepsilon\in(0,1)}\subset C^0(\overline{\Omega})$ *and* $(v_{0\varepsilon})_{\varepsilon\in(0,1)}\subset C^1(\overline{\Omega})$, *such that*

$$\|u_{0\varepsilon}\|_{L^\infty(\Omega)}+\|v_{0\varepsilon}\|_{W^{1,\infty}(\Omega)}\le K$$

for every $\varepsilon\in(0,1)$ *as well as*

$$\|u_0-u_{0\varepsilon}\|_{L^1(\Omega)}+\|v_0-v_{0\varepsilon}\|_{W^{1,1}(\Omega)}\to 0$$

as $\varepsilon\to 0$.

Proof The estimates for D_ε are an immediate consequence of the properties given to D and the rest is a matter of choosing a suitable approximation. **q.e.d.**

Having fixed this, we now consider a slightly different system than before. Aside from the adapted initial data ensuring sufficient regularity we also change the first equation in such a way that the diffusion is no longer degenerate. The resulting system is the following:

$$\begin{cases} u_{\varepsilon t} = \nabla \cdot (D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon + u_\varepsilon \nabla v_\varepsilon) & \text{in } \Omega \times (0,\infty), \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(0,\cdot) = u_{0\varepsilon}, \ v_\varepsilon(\cdot,0) = v_{0\varepsilon} & \text{in } \Omega. \end{cases} \tag{III.10}$$

This system meets all the requirements we have previously seen in deriving globally bounded classical solutions. Additionally, since all D_ε share the quality $D_\varepsilon(s) \geq C_D s^{m-1}$ for every $s \geq 0$, we find a common upper bound for the family of approximating solutions:

Corollary III.3.3. *For the quantities from lemma III.3.2 and every $\varepsilon \in (0,1)$ the system* (III.10) *has a classical solution $(u_\varepsilon, v_\varepsilon)$ that is global and there is $C > 0$ with*

$$\left\|u_\varepsilon(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} \leq C$$

for every $\varepsilon \in (0,1)$ and every $t \in (0,\infty)$.

Proof Firstly, theorem III.1.1 gives us classical solutions along with their global existence. The parameter-independent estimate $D_\varepsilon(s) \geq C_D s^{m-1}$ for every $s \geq 0$ and $\varepsilon \in (0,1)$ then guarantees the uniform boundedness together with lemma III.2.1 and thereby finishes the proof. **q.e.d.**

These solutions to the approximate problems (III.10) will be shown to converge to solutions of the actual system (III.1) in a suitable fashion. In preparation of the discussion of this convergence we will now find and fix several bounds that will enable us to start a process where we repeatedly choose subsequences along which u_ε and v_ε converge in a certain sense. Here and in the subsequent proof we follow ideas from [49].

Lemma III.3.4. *For any $\varepsilon \in (0,1)$, by $(u_\varepsilon, v_\varepsilon)$ we want to denote the pair of functions found in corollary III.3.3. Given $T \in (0,\infty)$ there is $C_T > 0$ with the following property: For any $\varepsilon \in (0,1)$ and the corresponding solution $(u_\varepsilon, v_\varepsilon)$ to the approximate problem we have*

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty((0,T);L^\infty(\Omega))} \leq C_T, \\ &\|v_\varepsilon\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \leq C_T \ \textit{and} \\ &\left\|D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\right\|_{L^2(\Omega\times(0,T))} \leq C_T \ \textit{as well as} \\ &\left\|\nabla u_\varepsilon^{m-1}\right\|_{L^2(\Omega\times(0,T))} \leq C_T, \\ &\int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-3} |\nabla u_\varepsilon|^2 \leq C_T, \\ &\|v_{\varepsilon t}\|_{L^2\left((0,T);(W_0^{1,1}(\Omega))^*\right)} \leq C_T \ \textit{and} \\ &\left\|(u_\varepsilon^{m-1})_t\right\|_{L^1\left((0,T);(W_0^{1,n+1}(\Omega))^*\right)} \leq C_T. \end{aligned}$$

Proof. The first two statements have already been proven by corollary III.3.3.

Let us therefore fix some $C_1 > 0$ such that

$$\|u_\varepsilon\|_{L^\infty((0,T);L^\infty(\Omega))} L^\infty((0,T);W^{1,\infty}(\Omega)) + \|v_\varepsilon\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \leq C_1$$

holds for every $\varepsilon \in (0,1)$.

For the next claim we define

$$\overline{\overline{D}}_\varepsilon(u) := \int_0^u \overline{D}_\varepsilon(\sigma)\, \mathrm{d}\sigma := \int_0^u \int_0^s D(\sigma)\, \mathrm{d}\sigma \mathrm{d}s$$

whenever some $\varepsilon \in (0,1)$ is given and rewrite the first differential equation in (III.10) as

$$u_{\varepsilon t} = \Delta \overline{D}_\varepsilon(u_\varepsilon) + \nabla \cdot (u_\varepsilon \nabla v_\varepsilon)$$

to which clearly $(u_\varepsilon, v_\varepsilon)$ remains a solution for every $\varepsilon \in (0,1)$. Testing this with $\overline{D}_\varepsilon(u_\varepsilon)$ and using Young's inequality gives us

$$\begin{aligned}\int_0^T \int_\Omega \Big(\overline{\overline{D}}_\varepsilon(u_\varepsilon)\Big)_t &= -\int_0^T \int_\Omega \big|\nabla \overline{D}_\varepsilon(u_\varepsilon)\big|^2 - \int_0^T \int_\Omega u_\varepsilon \nabla \overline{D}_\varepsilon(u_\varepsilon) \cdot \nabla v_\varepsilon \\ &\leq -\frac{1}{2}\int_0^T \int_\Omega \big|\nabla \overline{D}_\varepsilon(u_\varepsilon)\big|^2 + \frac{1}{2}\int_0^T \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^2\end{aligned}$$

for every $\varepsilon \in (0,1)$ which in turn shows

$$\frac{1}{2}\int_0^T \int_\Omega \big|\nabla \overline{D}_\varepsilon(u_\varepsilon)\big|^2 \leq \int_\Omega \overline{\overline{D}}_\varepsilon(u_{0\varepsilon}) - \int_\Omega \overline{\overline{D}}_\varepsilon(u_\varepsilon(\cdot, T)) + \frac{1}{2}\int_0^T \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^2$$

for every $\varepsilon \in (0,1)$ and the terms on the right are clearly bounded independently of $\varepsilon \in (0,1)$ due to the definition of $\overline{\overline{D}}_\varepsilon$ and the boundedness $u_\varepsilon^2 |\nabla v_\varepsilon|^2 \leq C_1^4$ for every $\varepsilon \in (0,1)$.

Next, from the first two claims we also get $\left|\int_\Omega v_{\varepsilon t} \varphi\right| \leq 3C_1 \|\varphi\|_{W^{1,1}(\Omega)}$ for any $\varphi \in C_0^\infty(\Omega)$ and every $\varepsilon \in (0,1)$; therefore, we have found a bound for $\|v_{\varepsilon t}\|_{L^2((0,T);(W_0^{1,1}(\Omega))^*)}$ which does not depend on the size of $\varepsilon \in (0,1)$.

For the fourth and fifth claim some differences arise depending on the size of m. If $m = 2$, then we can use integration by parts and Young's inequality to see

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon \ln u_\varepsilon &= \int_\Omega (1 + \ln u_\varepsilon)\, \nabla \cdot (D_\varepsilon(u_\varepsilon) \nabla u_\varepsilon) + \int_\Omega (1 + \ln u_\varepsilon)\, \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) \\ &= -\int_\Omega \frac{D_\varepsilon(u_\varepsilon)}{u_\varepsilon} |\nabla u_\varepsilon|^2 + \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\ &\leq -\frac{1}{3}\int_\Omega \frac{D_\varepsilon(u_\varepsilon)}{u_\varepsilon} |\nabla u_\varepsilon|^2 - \frac{C_D}{3}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{3}{4C_D}\int_\Omega |\nabla v_\varepsilon|^2\end{aligned}$$

in $(0,T)$ and for every $\varepsilon \in (0,1)$. Rearranging and integration yield

$$\frac{1}{3}\int_0^T\int_\Omega \frac{D_\varepsilon(u_\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2 + \frac{C_D}{3}\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \le \frac{3|\Omega|C_1^2}{4C_D} + \int_\Omega u_{0\varepsilon}\ln u_{0\varepsilon} - \int_\Omega u_\varepsilon(\cdot,T)\ln u_\varepsilon(\cdot,T)$$

for every $\varepsilon \in (0,1)$ and, together with the observation $x\ln x > -\frac{1}{e}$ for $x > 0$, this shows that the for this case relevant quantities

$$\int_0^T\int_\Omega \frac{D_\varepsilon(u_\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2$$

and

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2$$

are bounded independently of $\varepsilon \in (0,1)$.

If conversely $m \ne 2$, then an almost identical computation for $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^{m-1}$ proves the equivalent of the two estimates above:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^{m-1} = -(m-2)\int_\Omega u_\varepsilon^{m-3}D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 - (m-2)\int_\Omega u_\varepsilon^{m-2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon$$

can be found in $(0,T)$ and for every $\varepsilon \in (0,1)$ rather quickly, but now we also have to pay attention to the sign of $m-2$ since m is just as likely to be smaller than 2 as it is to be larger.

For $m > 2$, the first integral on the right is negative and so we use the lower bound for D_ε, as well the decomposition used in the case $m = 2$ before, to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^{m-1} \le -\frac{m-2}{3}\int_\Omega u_\varepsilon^{m-3}D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 - \frac{C_D(m-2)}{3}\int_\Omega u_\varepsilon^{2m-4}\,|\nabla u_\varepsilon|^2 + \frac{3|\Omega|C_1^2}{4C_D}$$

in $(0,T)$ and for every $\varepsilon \in (0,1)$.

On the other hand, if $m-2 < 0$, then the estimate is reversed, giving us

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^{m-1} \ge \frac{2-m}{3}\int_\Omega u_\varepsilon^{m-3}D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \frac{C_D(2-m)}{3}\int_\Omega u_\varepsilon^{2m-4}\,|\nabla u_\varepsilon|^2 - \frac{3|\Omega|C_1^2}{4C_D}$$

in $(0,T)$ and for every $\varepsilon \in (0,1)$. In both cases we can now integrate over $(0,T)$ again and conclude as we did for $m = 2$:

$$\begin{aligned}\frac{|2-m|}{3}\int_0^T\int_\Omega u_\varepsilon^{m-3}D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \frac{C_D|m-2|}{3}\int_0^T\int_\Omega u_\varepsilon^{2m-4}\,|\nabla u_\varepsilon|^2 \\ \le T\cdot\frac{3|\Omega|C_1^2}{4C_D} + |\Omega|\cdot C_1^{m-1}\end{aligned}$$

for every $\varepsilon \in (0,1)$ proves the claim due to

$$u_\varepsilon^{2m-4} |\nabla u_\varepsilon|^2 = \frac{1}{(m-1)^2} \left|\nabla u_\varepsilon^{m-1}\right|^2.$$

Now let there be $C_2 > 0$ such that (using Sobolev's embedding theorem) for any $\Phi \in W_0^{1,n+1}(\Omega)$ with $\|\Phi\|_{W_0^{1,n+1}(\Omega)} \leq 1$ we have $\|\Phi\|_{L^\infty(\Omega)} \leq C_2$ and (using the boundedness of Ω) for any $\Phi \in L^\infty((0,T); W_0^{1,n+1}(\Omega))$ with $\|\Phi\|_{L^\infty((0,T);W_0^{1,n+1}(\Omega))} \leq 1$ we have $\|\Phi\|_{W^{1,2}(\Omega\times(0,T))} \leq C_2$ as well as $\|\Phi\|_{W^{1,1}(\Omega\times(0,T))} \leq C_2$.
In particular, for any $\Phi \in L^\infty((0,T); W_0^{1,n+1}(\Omega))$ with $\|\Phi\|_{L^\infty((0,T);W_0^{1,n+1}(\Omega))} \leq 1$ we also have $\|\Phi\|_{L^\infty((\Omega)\times(0,T))} \leq C_2$. Setting $X := L^1((0,T); (W_0^{1,n+1}(\Omega))^*)$ we see that $X^* = L^\infty((0,T); W_0^{1,n+1}(\Omega))$ and thus any $\varphi \in X^*$ with $\|\varphi\|_{X^*} \leq 1$ gives us

$$\begin{aligned}
\frac{1}{m-1}\left|\int_0^T \int_\Omega (u_\varepsilon^{m-1})_t \varphi\right| &= \left|\int_0^T \int_\Omega u_\varepsilon^{m-2} u_{\varepsilon t} \varphi\right| \\
&\leq \left|\int_0^T \int_\Omega u_\varepsilon^{m-2} \varphi \nabla \cdot (D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon)\right| \\
&\quad + \left|\int_0^T \int_\Omega u_\varepsilon^{m-2} \varphi \nabla \cdot (u_\varepsilon \nabla v_\varepsilon)\right| \\
&\leq |m-2| \left|\int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-3} \varphi |\nabla u_\varepsilon|^2\right| \\
&\quad + \left|\int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-2} \nabla u_\varepsilon \nabla \varphi\right| \\
&\quad + |m-2| \left|\int_0^T \int_\Omega u_\varepsilon^{m-2} \varphi \nabla u_\varepsilon \nabla v_\varepsilon\right| \\
&\quad + \left|\int_0^T \int_\Omega u_\varepsilon^{m-1} \nabla v_\varepsilon \cdot \nabla \varphi\right|
\end{aligned}$$

for every $\varepsilon \in (0,1)$.

Firstly we have

$$\begin{aligned}
I_1 :=& \left|\int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-3} \varphi |\nabla u_\varepsilon|^2\right| \\
&\leq C_2 \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-3} |\nabla u_\varepsilon|^2
\end{aligned}$$

which for every $\varepsilon \in (0,1)$ is bounded due to our previous estimate.

Next, Young's inequality and straightforward estimates show

$$
\begin{aligned}
I_2 &:= \left| \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-2} \nabla u_\varepsilon \nabla \varphi \right| \\
&\leq \frac{1}{2} \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-3} |\nabla u_\varepsilon|^2 + \frac{1}{2} \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-1} |\nabla \varphi|^2 \\
&\leq \frac{1}{2} \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-3} |\nabla u_\varepsilon|^2 + 2^{M-2} C_D' C_2 \left(1 + C_1\right)^{m+M-2}
\end{aligned}
$$

and the boundedness of the remaining integral has been proven for every $\varepsilon \in (0,1)$ in the previous step.

Moreover, with Young's inequality and for some $C_3 > 0$, we see

$$
\begin{aligned}
I_3 &:= \left| \int_0^T \int_\Omega u_\varepsilon^{m-2} \varphi \nabla u_\varepsilon \nabla v_\varepsilon \right| \\
&\leq C_1 \int_0^T \int_\Omega u_\varepsilon^{m-2} |\nabla u_\varepsilon| \varphi \\
&\leq C_3 + C_3 \left\| \nabla u_\varepsilon^{m-1} \right\|_{L^2(\Omega)}^2
\end{aligned}
$$

for every $\varepsilon \in (0,1)$ and for the final integral we have

$$
I_4 := \left| \int_0^T \int_\Omega u_\varepsilon^{m-1} \nabla v_\varepsilon \cdot \nabla \varphi \right| \leq C_1^m C_2
$$

for every $\varepsilon \in (0,1)$. **q.e.d.**

We can now prove the existence of a weak solution (u, v) to (III.1) by taking a zero sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ and solutions $(u_{\varepsilon_k}, v_{\varepsilon_k})$ to the approximating problems (III.10) for $\varepsilon = \varepsilon_k$ and letting $k \to \infty$.

Proof of theorem III.1.2. As in the previous lemma, the inspiration for the upcoming train of thought can be traced back to the proof of the corresponding theorem in [49]. For $\varepsilon \in (0,1)$ we again use $\overline{D}_\varepsilon(s) := \int_0^s D(\sigma) \, \mathrm{d}\sigma$ and by corollary III.3.3 we have a global classical solution to (III.10) that we denote by $(u_\varepsilon, v_\varepsilon)$.
We begin by proving the following claim: For every $k \in \mathbb{N}_0$ there is a zero sequence $(\varepsilon_{k,l})_{l\in\mathbb{N}} \subset (0,1)$ which for $k \in \mathbb{N}$ is a subsequence of $(\varepsilon_{k-1,l})_{l\in\mathbb{N}}$ and for which we have the following convergences as $l \to 0$:

$$
\begin{aligned}
u_{\varepsilon_{k,l}} &\text{ converges a.e. in } \Omega \times (0,k) \text{ and in } L^1(\Omega \times (0,k)), \\
\overline{D}_{\varepsilon_{k,l}}(u_{\varepsilon_{k,l}}) &\text{ converges weakly in } L^2\left((0,k); W_0^{1,2}(\Omega)\right), \\
v_{\varepsilon_{k,l}} &\text{ converges uniformly in } \Omega \times (0,k) \text{ and} \\
\nabla v_{\varepsilon_{k,l}} &\text{ converges weakly* in } L^\infty\left((0,k); L^\infty(\Omega)\right).
\end{aligned}
$$

We start with an arbitrary monotonous zero sequence $(\varepsilon_{0,l})_{l\in\mathbb{N}} \subset (0,1)$ and so we can assume that for some $k \in \mathbb{N}$ we have a sequence $(\varepsilon_{k-1,l})_{l\in\mathbb{N}}$ such that – after replacing k by $k-1$ – the properties above hold. Thanks to lemma III.3.4 we find $C_1(k) > 0$ with

$$\begin{aligned}
&\left\|u_{\varepsilon_{k-1,l}}\right\|_{L^\infty((0,k);L^\infty(\Omega))} \le C_1(k) && \forall l \in \mathbb{N},\\
&\left\|v_{\varepsilon_{k-1,l}}\right\|_{L^\infty((0,k);W^{1,\infty}(\Omega))} \le C_1(k) && \forall l \in \mathbb{N},\\
&\left\|D_{\varepsilon_{k-1,l}}(u_{\varepsilon_{k-1,l}})\nabla u_{\varepsilon_{k-1,l}}\right\|_{L^2(\Omega\times(0,k))} \le C_1(k) && \forall l \in \mathbb{N},\\
&\left\|\nabla u^{m-1}_{\varepsilon_{k-1,l}}\right\|_{L^2(\Omega\times(0,k))} \le C_1(k) && \forall l \in \mathbb{N},\\
&\left\|\left(v_{\varepsilon_{k-1,l}}\right)_t\right\|_{L^2((0,k);\left(W_0^{1,1}(\Omega)\right)^*)} \le C_1(k) && \forall l \in \mathbb{N} \text{ and}\\
&\left\|\left(u^{m-1}_{\varepsilon_{k-1,l}}\right)_t\right\|_{L^1((0,k);\left(W_0^{1,n+1}(\Omega)\right)^*)} \le C_1(k) && \forall l \in \mathbb{N}.
\end{aligned}$$

The abbreviation

$$d_k := \sup_{\varepsilon\in(0,1)} \sup_{0\le s\le C_1(k)} D_\varepsilon(s) \le C'_D\,(C_1(k)+2)^{M-1}$$

shows

$$\overline{D}_{\varepsilon_{k-1,l}}(u_{\varepsilon_{k-1,l}}(x,t)) \le \int_0^{C_1(k)} d_k = C_1(k)d_k$$

for every $l \in \mathbb{N}$ as well as $(x,t) \in \Omega\times(0,k)$, and so we find $C_2(k) > 0$ with

$$\left\|\overline{D}_{\varepsilon_{k-1,l}}(u_{\varepsilon_{k-1,l}})\right\|_{L^2((0,k);W^{1,2}(\Omega))} \le C_2(k)$$

for every $l \in \mathbb{N}$. Therefore, we may select a first subsequence $\left(\varepsilon^{(1)}_{k,l}\right)_{l\in\mathbb{N}} \subseteq (\varepsilon_{k-1,l})_{l\in\mathbb{N}}$ along which

$$\left(\overline{D}_\varepsilon(u_\varepsilon)\right)_{\varepsilon\in(0,1)}$$

converges weakly in $L^2((0,k);W^{1,2}(\Omega))$. Next we find $C_3(k) > 0$ as an upper bound in

$$\left\|u^{m-1}_{\varepsilon^{(1)}_{k,l}}\right\|_{L^2((0,k);W^{1,2}(\Omega))} \le C_3(k)$$

for every $l \in \mathbb{N}$ and note the boundedness of $\left(u^{m-1}_{\varepsilon^{(1)}_{k,l}}\right)_t$ in $L^1\left((0,k);\left(W_0^{1,n+1}(\Omega)\right)^*\right)$. Recalling $W^{1,2}(\Omega) \overset{\text{cpt.}}{\hookrightarrow} L^2(\Omega) \hookrightarrow \left(W_0^{1,n+1}(\Omega)\right)^*$, we invoke lemma I.3.12 to find a sequence $\left(\varepsilon^{(2)}_{k,l}\right)_{l\in\mathbb{N}} \subseteq \left(\varepsilon^{(1)}_{k,l}\right)_{l\in\mathbb{N}}$ such that $\left(u^{m-1}_{\varepsilon^{(2)}_{k,l}}\right)_{l\in\mathbb{N}}$ converges in $L^2(\Omega\times(0,k))$ for $l\to\infty$. Since $m > 1$, the mapping $[0,\infty) \ni y \mapsto y^{\frac{1}{m-1}}$ is continuous and so we have a sequence

$\left(\varepsilon_{k,l}^{(3)}\right)_{l\in\mathbb{N}} \subseteq \left(\varepsilon_{k,l}^{(2)}\right)_{l\in\mathbb{N}}$ giving us the convergence of

$$\left(u_{\varepsilon_{k,l}^{(3)}}\right)_{l\in\mathbb{N}}$$

almost everywhere in $\Omega\times(0,k)$ (instead of convergence only for its $(m-1)$-st power). Thanks to Lebesgue's dominated convergence theorem and the constant bound we also have convergence with respect to $\|\cdot\|_{L^1(\Omega\times(0,k))}$. Applying the same lemma I.3.12 to $W^{1,\infty}(\Omega) \overset{\text{cpt.}}{\hookrightarrow} C^0(\overline{\Omega}) \hookrightarrow \left(W_0^{1,1}(\Omega)\right)^*$ gives us another refinement $\left(\varepsilon_{k,l}^{(4)}\right)_{l\in\mathbb{N}} \subseteq \left(\varepsilon_{k,l}^{(3)}\right)_{l\in\mathbb{N}}$ with

$$\left(v_{\varepsilon_{k,l}^{(4)}}\right)_{l\in\mathbb{N}}$$

converging uniformly in $\Omega\times(0,k)$ while a final subsequence $\left(\varepsilon_{k,l}^{(5)}\right)_{l\in\mathbb{N}} \subseteq \left(\varepsilon_{k,l}^{(4)}\right)_{l\in\mathbb{N}}$ secures the weak*-convergence of

$$\left(\nabla v_{\varepsilon_{k,l}^{(5)}}\right)_{l\in\mathbb{N}}$$

in $L^\infty(\Omega\times(0,k))$ due to the boundedness

$$\left\|\nabla v_{\varepsilon_{k,l}^{(4)}}\right\|_{L^\infty((0,k);L^\infty(\Omega))} \le C_1(k)$$

for every $l\in\mathbb{N}$. This completes the induction and setting $(\varepsilon_k)_{k\in\mathbb{N}} := (\varepsilon_{k,k})_{k\in\mathbb{N}}$ we find three functions $u,\ v,\ z:\Omega\to\mathbb{R}$ and $\zeta:\Omega\to\mathbb{R}^n$ with

$$\begin{aligned}
u_{\varepsilon_k} &\to u && \text{in } L^1_{\text{loc}}([0,\infty);L^1(\Omega)) \text{ and a. e. in } \Omega\times(0,\infty),\\
v_{\varepsilon_k} &\to v && \text{in } L^\infty_{\text{loc}}([0,\infty);C^0(\overline{\Omega})),\\
\overline{D}_{\varepsilon_k}(u_{\varepsilon_k}) &\rightharpoonup z && \text{in } L^2_{\text{loc}}([0,\infty);W^{1,2}(\Omega)) \text{ and}\\
\nabla v_{\varepsilon_k} &\overset{*}{\rightharpoonup} \zeta && \text{in } L^\infty_{\text{loc}}([0,\infty);L^\infty(\Omega))
\end{aligned}$$

as $k\to\infty$. From $u_{\varepsilon_k}+\varepsilon_k \to u$ a.e. in $\Omega\times(0,\infty)$ as $k\to\infty$ and the continuity of $\overline{D}$ we see

$$\overline{D}_{\varepsilon_k}(u_{\varepsilon_k}) = \overline{D}_{\varepsilon_k}(u_{\varepsilon_k}+\varepsilon_k) - \overline{D}_{\varepsilon_k}(\varepsilon_k) \to \overline{D}(u)$$

as $k\to\infty$. Therefore we know $z=\overline{D}(u)$ while the second and fourth row combine to show $\zeta=\nabla v$. Additionally, the local boundedness of $\left\|u_{\varepsilon_k}\right\|_{L^\infty((0,k);L^\infty(\Omega))}$ and its convergence ensure $u\in L^\infty_{\text{loc}}([0,\infty);L^\infty(\Omega))$.
Thus we have proven that $(u_{\varepsilon_k},v_{\varepsilon_k})$ converges to a solution of (III.1) as $k\to\infty$: For any $\varphi\in C_0^\infty(\Omega\times[0,\infty))$ and any $k\in\mathbb{N}$ we have

$$-\int_0^\infty\int_\Omega u_{\varepsilon_k}\varphi_t - \int_\Omega u_{0\varepsilon_k}\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla\overline{D}_{\varepsilon_k}(u_{\varepsilon_k})\nabla\varphi - \int_0^\infty\int_\Omega u_{\varepsilon_k}\nabla v_{\varepsilon_k}\nabla\varphi$$

and

$$-\int_0^\infty \int_\Omega v_{\varepsilon_k}\varphi_t - \int_\Omega v_{0\varepsilon_k}\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v_{\varepsilon_k}\nabla\varphi - \int_0^\infty \int_\Omega v_{\varepsilon_k}\varphi + \int_0^\infty \int_\Omega u_{\varepsilon_k}\varphi$$

and together with the convergences established above and those of the initial data the proof is complete. **q.e.d.**

Chapter IV

The fast signal diffusion limit in nonlinear chemotaxis systems

IV.1 Introduction and main result

In [43], Keller and Segel initially examined the systems

$$\begin{cases} u_t = d_1 \Delta u - a_1 \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0,\infty)\,, \\ v_t = d_2 \Delta v - a_2 v + a_3 u & \text{in } \Omega \times (0,\infty) \end{cases}$$

with positive numbers d_1, d_2, a_1, a_2 and a_3 in order to describe the phenomenon that is known as chemotaxis. Here, u denotes the cell density of a slime mold and v is the concentration of a chemical substance produced by the cells themselves, both depending on a spatial parameter x and the time t.

With the substitutions

$$\frac{a_1}{d_1} = \chi, \quad \frac{d_1}{d_2} = \varepsilon, \quad \frac{a_2}{d_2} = \gamma \quad \text{and} \quad \frac{a_3}{d_2} = \alpha$$

and transforming the second variable from t to $\frac{t}{d_1}$, we arrive at

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \chi \nabla v) & \text{in } \Omega \times (0,\infty)\,, \\ \varepsilon v_t = \Delta v - \gamma v + \alpha u & \text{in } \Omega \times (0,\infty) \end{cases}$$

which brings us one step closer to our topic: If we take the limit $\varepsilon \to 0$, then the second equation in this system is formally turning into the inhomogeneous Helmholtz equation $-\Delta v + \gamma v = \alpha u$ and for the arising parabolic-elliptic system results seem to be more easily obtained.

Comparatively early, several works ([40], [57], [59], [7]) have detected solutions to blow-up in the parabolic-elliptic setting. On the other hand, results for the case of positive ε have dealt only with one example ([34]) or followed at a later time after a significantly higher amount of effort ([98], [56]).

The picture is similar when one wants to extract quantitative results from the systems. While there are numerous works for $\varepsilon = 0$ ([60], [62], [73], [87], [86], [81]), the findings for the fully-parabolic case are less abundant ([60], [55], [93]).

Accordingly, one might pose the following question: if we use the parabolic-elliptic system for the approximation of the non-simplified system, especially for small ε, that is in situations where the signal diffusion is much faster than that of the cells, how close are we? Until quite recently, a first hint was only given by numerical results in [52], but with [92] we now also have a theoretical work linking the two systems: in a suitable sense the solutions of the fully parabolic system for decreasing ε do in fact converge to a solution of the parabolic-elliptic simplification.

In this chapter, instead of a linear diffusion, for some $m > 1$ in the first equation we replace Δu by $\nabla \cdot \big((u+1)^{m-1}\nabla u\big)$, thereby operating near the system (II.1) in chapter II again.

In the fully parabolic system with $\varepsilon = 1$, the behaviour changes drastically when in the first equation the diffusion is no longer linear. While for $m = 1$ the importance of the initial data (or more specifically, the size thereof) cannot be stressed enough, superlinear diffusion removes the need for such conditions: In this case, demanding $m > 1 + \frac{n-2}{n}$ suffices to ensure global existence and boundedness of solutions as we have seen in chapter II.

As in the case of linear diffusion, we want to know in which sense (if any!) and under which conditions solutions to the parabolic-parabolic system

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1}\nabla u) - \nabla \cdot (u\nabla v) & \text{in } \Omega \times (0,\infty)\,, \\ \varepsilon v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty)\,, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \tag{IV.1}$$

with fast signal diffusion governed by $\varepsilon \in (0,1)$ converge to those of the parabolic-elliptic system where $\varepsilon = 0$, namely

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1}\nabla u) - \nabla \cdot (u\nabla v) & \text{in } \Omega \times (0,\infty)\,, \\ 0 = \Delta v - v + u & \text{in } \Omega \times (0,\infty)\,, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{cases} \tag{IV.2}$$

as $\varepsilon \to 0$. Here, we demand that $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with smooth boundary and that $m > 1 + \frac{n-2}{n}$. Furthermore, the nonnegative initial data fulfil $0 \not\equiv u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$.

Once more we mention the virtually pioneering work [92] which opened the door for the upcoming results.

We will first translate the existence and boundedness results from chapter II to versions of (IV.1) where $\varepsilon \in (0,1)$ instead of $\varepsilon = 1$ before eventually discussing the limit of the corresponding solutions. Our main result reads as follows:

Theorem IV.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for some $n \geq 2$. Additionally let $m > 1 + \frac{n-2}{n}$ and some nonnegative functions $0 \not\equiv u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ be given as well as some zero sequence $E \subset (0,1)$. Then for every $\varepsilon \in E$ the system* (IV.1) *has a global and uniformly bounded classical solution $(u_\varepsilon, v_\varepsilon)$. Furthermore, there is a classical solution (u,v) to* (IV.2) *such that for any $T > 0$ we can find a subsequence $E' \subset E$ with*

$$\begin{aligned}
&u_\varepsilon \to u \text{ in } C^0\left(\overline{\Omega} \times [0,T]\right),\\
&u_\varepsilon \rightharpoonup u \text{ in } L^2\left((0,T); W^{1,2}(\Omega)\right),\\
&v_\varepsilon \rightharpoonup v \text{ in } L^\infty_{loc}\left((0,T]; C^0\left(\overline{\Omega}\right)\right) \cap L^2_{loc}\left((0,T]; W^{1,2}(\Omega)\right),\\
&\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v \text{ in } L^\infty\left((0,T); W^{1,\infty}(\Omega)\right)
\end{aligned}$$

as $\varepsilon \to 0$ in E'.

Remark. *For the entirety of this chapter, for $n \geq 2$ let firstly $\Omega \subset \mathbb{R}^n$ be some bounded domain. Additionally, let $m > 1 + \frac{n-2}{n}$ as well as two nonnegative functions $0 \not\equiv u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ be given.*

IV.2 Existence of global classical solutions to the fully parabolic system and some bounds

Similarly to chapter II and [89], from the estimates in lemma I.3.3 we can show that the condition that

$$\left\|u_\varepsilon(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,\infty}(\Omega)}$$

has to have a bound that depends neither on $t \in (0,\infty)$ nor on $\varepsilon \in (0,1)$ can be weakened: as before, it is sufficient to prove boundedness of $\left\|u_\varepsilon(\cdot,t)\right\|_{L^p(\Omega)}$ for some large p. The important part is that we have to ensure that none of the arising constants are connected to ε.

Lemma IV.2.1. *There is a positive constant such that for every $\varepsilon \in (0,1)$ we can find a pair*

$$(u_\varepsilon, v_\varepsilon) \in \left(C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right)\right)^2$$

of classical solutions to (IV.1) *in* $\Omega \times (0,\infty)$ *with*

$$\left\|u_\varepsilon(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} \le C$$

for every $t \in (0,\infty)$.

Proof. Again, [47] gives us local solutions and our claim is that we can find bounds for these functions that do not depend on ε which simultaneously proves global existence and the central bound in this lemma.

Let us write $T_\varepsilon \in (0,\infty]$ as the maximum existence time corresponding to any given $\varepsilon \in (0,1)$.

Knowing that at least for $p = 1$ due to $\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0$ for every $\varepsilon \in (0,1)$ and $t \in (0,T_\varepsilon)$ we have some finite $C(p) > 0$ with

$$\left\|u_\varepsilon(\cdot,t)\right\|_{L^p(\Omega)} \le C(p) \tag{$*$}$$

for every $\varepsilon \in (0,1)$ and $t \in (0,T_\varepsilon)$, the following step will initially provide us with ε-independent bounds for v in $W^{1,q}(\Omega)$ whenever $q \in \left[1,\frac{n}{n-1}\right)$; returning with additional information in the form of $(*)$ for some $p > 1$ and $C(p) \in (0,\infty)$ will also allow us to pick larger q.

For $\varepsilon \in (0,1)$ and $t \in \left(0,\frac{T_\varepsilon}{\varepsilon}\right)$ we set

$$\hat{u}_\varepsilon(\cdot,t) := u_\varepsilon(\cdot,\varepsilon t)$$

and

$$\hat{v}_\varepsilon(\cdot,t) := v_\varepsilon(\cdot,\varepsilon t)$$

for which clearly the identity

$$\hat{v}_{\varepsilon t} = \Delta\hat{v}_\varepsilon - \hat{v}_\varepsilon + \hat{u}_\varepsilon$$

holds in $\left(0,\frac{T_\varepsilon}{\varepsilon}\right)$. The additional and equally trivial observations

$$\hat{v}_\varepsilon(\cdot,0) = v_0$$

and

$$\left.\frac{\partial\hat{v}_\varepsilon}{\partial\nu}\right|_{\partial\Omega} = 0$$

for every $\varepsilon \in (0,1)$ as well as the assumption that with some $p \ge 1$ and $C(p) \in (0,\infty)$ the estimate $(*)$ holds for every $\varepsilon \in (0,1)$ then enable us to employ lemma I.3.4 and find that for any $q \in \left[1,\frac{np}{n-p}\right)$ in the case $p < n$, any $q \in [1,\infty)$ for $p = n$ and any $q \in [1,\infty]$ if $p > n$ an ε-independent positive constant $C(q)$ with

$$\left\|\hat{v}_\varepsilon(\cdot,t)\right\|_{W^{1,q}(\Omega)} \le C(q)$$

for every $\varepsilon \in (0,1)$ and $t \in \left(0, \frac{T_\varepsilon}{\varepsilon}\right)$ exists. Returning to our original v_ε this translates to

$$\left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,q}(\Omega)} \leq C(q)$$

for every $\varepsilon \in (0,1)$ and $t \in (0,T_\varepsilon)$. Accordingly, lemma A.1 from [89] – if we choose the functions therein as $D(x,t,u_\varepsilon) = (u_\varepsilon(x,t)+1)^{m-1}$, $f(x,t) = u_\varepsilon(x,t)\nabla v_\varepsilon(x,t)$ and $g \equiv 0$ for $x \in \Omega$ and $t \in (0,T_\varepsilon)$, resulting in them being bounded in the necessary way and independently of ε – tells us that our proof is complete as soon as we find bounds in the form of $(*)$ for arbitrarily large p.

Fixing such a p along with some q and using the steps from chapter II (for $\kappa = 1$ and $p_0 = 1$) for the quantity

$$y_\varepsilon(t) := \frac{1}{p}\int_\Omega u_\varepsilon^p(\cdot,t) + \frac{\varepsilon}{q}\int_\Omega \left|\nabla v_\varepsilon(\cdot,t)\right|^{2q}$$

with $\varepsilon \in (0,1)$ and $t \in (0,T_\varepsilon)$ we find some constant $C > 0$ for which we have

$$y_{\varepsilon t}(t) + \frac{1}{p}\int_\Omega u_\varepsilon^p(\cdot,t) + \frac{1}{q}\int_\Omega \left|\nabla v_\varepsilon(\cdot,t)\right|^{2q} \leq C$$

for every $\varepsilon \in (0,1)$ and $t \in (0,T_\varepsilon)$. Here it is to be noted that the added ε in the second summand of y_ε disappears while computing the derivative so that the resulting term cannot immediately be written in terms of y_ε. However, since $\varepsilon \in (0,1)$, we can estimate from below to see

$$y_{\varepsilon t} + y_\varepsilon \leq C$$

for every $\varepsilon \in (0,1)$ in $(0,T_\varepsilon)$ which, as in lemma III.2.5, proves

$$\left\|u_\varepsilon(\cdot,t)\right\|_{L^p(\Omega)} \leq y_\varepsilon(t) \leq \max\{y_\varepsilon(0), C\}$$

for every $\varepsilon \in (0,1)$ and $t \in (0,T_\varepsilon)$. Trivially for every $\varepsilon \in (0,1)$ we can estimate $y_\varepsilon(0) \leq y_1(0)$ and this completes the proof. **q.e.d.**

Before discussing the convergence of the solutions to (IV.1), we collect a number of estimates. Here, $T \in (0,\infty)$ is at all times some arbitrarily large number and ε at least for now remains an arbitrary element of $(0,1)$.

We begin by showing the Hölder continuity of every u_ε in both arguments.

Lemma IV.2.2. *In the setting of lemma IV.2.1, fixing some $T \in (0,\infty)$ we find some $\theta \in (0,1)$ and $C > 0$ such that for any $\varepsilon \in (0,1)$ and the corresponding function u_ε found in lemma IV.2.1*

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}\left(\overline{\Omega}\times[0,T]\right)} \leq C$$

holds.

Proof. Taking v_ε from the previous lemma we can write $\partial_t u_\varepsilon = \nabla \cdot a\left(x, t, u_\varepsilon, \nabla u_\varepsilon\right)$ for

$$a(x,t,\alpha,\beta) := (\alpha+1)^{m-1}\beta - \alpha\nabla v_\varepsilon(x,t).$$

Fixing $C := \max\left\{1, \sup_{\varepsilon\in(0,1)} \|\nabla v_\varepsilon\|^2_{L^\infty(\Omega\times(0,T))}\right\}$ which is finite due to lemma IV.2.1, we estimate

$$a(x,t,\alpha,\beta)\cdot\beta \geq \frac{1}{2}\left(\alpha+1\right)^{m-1}|\beta|^2 - \frac{C}{2}\alpha^2$$

and

$$\left|a(x,t,\alpha,\beta)\right| \leq |\alpha+1|^{m-1}|\beta| + C|\alpha|.$$

These are the conditions needed to employ theorem 1.3 and remark 1.4 in [66] whence we deduce the claimed statement. **q.e.d.**

The boundedness of our solutions also gives us an estimate for the L^2-norm of ∇u_ε:

Lemma IV.2.3. *In the setting of lemma IV.2.1, fixing some $T \in (0,\infty)$ we find some $C > 0$ such that for any $\varepsilon \in (0,1)$ and the corresponding function u_ε found in lemma IV.2.1*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq C$$

holds.

Proof. Using lemma IV.2.1 we can fix the positive and finite quantity

$$C := \int_\Omega u_0^2 + T\cdot\sup_{\varepsilon\in E}\int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^2$$

while direct computation and Young's inequality yield

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^2 &= -2\int_\Omega (u_\varepsilon+1)^{m-1}|\nabla u_\varepsilon|^2 + 2\int_\Omega u_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\ &\leq -\int_\Omega (u_\varepsilon+1)^{m-1}|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^2\end{aligned}$$

in $(0,T)$ for every $\varepsilon\in(0,1)$. Integration over $(0,T)$ and a trivial estimate then show

$$\int_0^T\int_\Omega|\nabla u_\varepsilon|^2 \leq \int_\Omega u_0^2 - \int_\Omega u_\varepsilon^2(\cdot,T) + \int_0^T\int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^2 \leq C$$

for every $\varepsilon\in(0,1)$ as claimed. **q.e.d.**

In preparation for the limit process we define $W^{2,2}_N(\Omega) := \left\{\psi\in W^{2,2}(\Omega) : \frac{\partial\psi}{\partial\nu}\Big|_{\partial\Omega} = 0\right\}$ and prove a corresponding boundedness result for the derivative of u_ε with respect to time.

Lemma IV.2.4. *In the setting of lemma IV.2.1, fixing some $T \in (0, \infty)$ we find some $C > 0$ such that for any $\varepsilon \in (0,1)$ and the corresponding function u_ε found in lemma IV.2.1*

$$\int_0^T \left\| \partial_t u_\varepsilon(\cdot, t) \right\|^2_{\left(W_N^{2,2}(\Omega)\right)^*} \, \mathrm{d}t \le C$$

holds.

Proof. From lemma IV.2.1 we see that

$$C := \max \left\{ \frac{1}{m} \sup_{\varepsilon \in (0,1)} \left\| (u_\varepsilon + 1)^m \right\|_{L^2(\Omega)}, \sup_{\varepsilon \in (0,1)} \| u_\varepsilon \nabla v_\varepsilon \|_{L^2(\Omega)} \right\}$$

is finite and therefore we have

$$\left| \int_\Omega \partial_t u_\varepsilon \psi \right| = \left| \frac{1}{m} \int_\Omega (u_\varepsilon + 1)^m \, \Delta\psi + \int_\Omega u_\varepsilon \nabla v_\varepsilon \nabla \psi \right| \le C \left(\|\nabla \psi\|_{L^2(\Omega)} + \|\Delta \psi\|_{L^2(\Omega)} \right)$$

in $(0, T)$ for every $\varepsilon \in (0,1)$ and $\psi \in W_N^{2,2}(\Omega)$; accordingly, upon integration the proof is completed. **q.e.d.**

IV.3 Solutions of the parabolic-elliptic system

In the previous section we have found uniform local bounds for $\varepsilon \in (0,1)$ and the corresponding solutions to (IV.1).

To start this section off, we define a candidate (u, v) that is to become our solution to (IV.2). It is found by combining estimates from the previous section, fixing some zero sequence and consecutively picking subsequences.

Lemma IV.3.1. *Fixing some $T > 0$, $T' := T + 1$ and a zero sequence $E \subset (0,1)$, there exist a subsequence $E' \subset E$ and some $\theta \in (0,1)$ as well as functions $u \in C^{\theta, \frac{\theta}{2}}\left(\overline{\Omega} \times [0, T']\right)$ and $v \in L^2\left((0, T'); W^{1,2}(\Omega)\right)$ such that for corresponding functions u_ε and v_ε from lemma IV.2.1 we have*

$$\begin{aligned}
&u_\varepsilon \to u \text{ in } C^0\left(\overline{\Omega} \times [0, T']\right), \\
&u_\varepsilon \rightharpoonup u \text{ in } L^2\left((0, T'); W^{1,2}(\Omega)\right), \\
&v_\varepsilon \rightharpoonup v \text{ in } L^2\left((0, T'); W^{1,2}(\Omega)\right), \\
&\partial_t u_\varepsilon \rightharpoonup u_t \text{ in } L^2\left((0, T'); \left(W_N^{2,2}(\Omega)\right)^*\right)
\end{aligned}$$

as $\varepsilon \to 0$ along that subsequence.

Proof. This follows lemma IV.2.1, lemma IV.2.2, lemma IV.2.3 and lemma IV.2.4 together with the Arzelà-Ascoli theorem. **q.e.d.**

In [92], namely in lemma 5.1, we find an almost directly applicable result concerning a first property of u and v. However, unlike theirs, our system consists only of two components and therefore we cannot merely cite the lemma.

Lemma IV.3.2. *With T', u and v as in lemma IV.3.1, there is a null set $\mathcal{N} \subset (0, T')$ such that for every $t \in (0, T') \setminus \mathcal{N}$ we have $v(\cdot, t) \in W^{1,2}(\Omega)$ and*

$$\int_\Omega \nabla v \cdot \nabla \psi + \int_\Omega v\psi = \int_\Omega u\psi$$

for every $\psi \in W^{1,2}(\Omega)$.

Proof. Let E be a zero sequence provided by lemma IV.3.1. Then for every $\varepsilon \in E$ and every $\varphi \in C_0^\infty\left(\overline{\Omega} \times (0, T')\right)$ from the second equation in (IV.1) we have

$$-\varepsilon \int_0^{T'} \int_\Omega v_\varepsilon \varphi_t + \int_0^{T'} \int_\Omega \nabla v_\varepsilon \cdot \nabla \varphi + \int_0^{T'} \int_\Omega v_\varepsilon \varphi = \int_0^{T'} \int_\Omega u_\varepsilon \varphi$$

and the convergence in the previous lemma IV.3.1 shows

$$-\varepsilon \int_0^{T'} \int_\Omega v_\varepsilon \varphi_t \to 0,$$

$$\int_0^{T'} \int_\Omega \nabla v_\varepsilon \cdot \nabla \varphi \to \int_0^{T'} \int_\Omega \nabla v \cdot \nabla \varphi$$

and

$$\int_0^{T'} \int_\Omega v_\varepsilon \varphi \to \int_0^{T'} \int_\Omega v\varphi$$

as well as

$$\int_0^{T'} \int_\Omega u_\varepsilon \varphi \to \int_0^{T'} \int_\Omega u\varphi$$

as $\varepsilon \to 0$. Therefore we have

$$\int_0^{T'} \int_\Omega \nabla v \cdot \nabla \varphi + \int_0^{T'} \int_\Omega v\varphi = \int_0^{T'} \int_\Omega u\varphi$$

for every $\varphi \in C_0^\infty\left(\overline{\Omega} \times (0, T')\right)$.

As in [92], from the separability of $W^{1,2}(\Omega)$ and a mollification argument we get $(\psi_j)_{j\in\mathbb{N}} \subset C^\infty\left(\overline{\Omega}\right)$ such that $\Psi := \left\{\psi_j | j \in \mathbb{N}\right\}$ is dense in $W^{1,2}(\Omega)$. For $j \in \mathbb{N}$ and $t \in (0, T')$ we define

$$\zeta_j(t) := \int_\Omega \nabla v(\cdot,t) \cdot \nabla \psi_j \text{ and } \xi_j(t) := \int_\Omega v(\cdot,t)\psi_j$$

which clearly belong to $L^1((0,T'))$. Given any $j \in \mathbb{N}$ we therefore find such a null set $\mathcal{N}_j \subset (0,T')$ that every $t \in (0,T') \setminus \mathcal{N}_j$ is a Lebesgue point of ζ_j and ξ_j. Since obviously $v(\cdot,t) \in W^{1,2}(\Omega)$ for almost every $t \in (0,T')$, we can define the combined null set

$$\mathcal{N} := \left(\bigcup_{j\in\mathbb{N}} \mathcal{N}_j\right) \cup \left\{t \in (0,T') | v(\cdot,t) \notin W^{1,2}(\Omega)\right\},$$

so that $(0,T') \setminus \mathcal{N}$ only contains mutual Lebesgue points of every ζ_j and ξ_j and within which v belongs to $W^{1,2}(\Omega)$. Fixing $t_0 \in (0,T') \setminus \mathcal{N}$ as well as $h \in (0, T'-t_0)$ and a sequence $(\chi_k)_{k\in\mathbb{N}} \subset C_0^\infty((0,T'))$ with

$$\chi_k \overset{*}{\rightharpoonup} \chi_{(t_0,t_0+h)} \text{ in } L^\infty((0,T)) \text{ as } k \to \infty,$$

where $\chi_{(t_0,t_0+h)}$ is the characteristic function of (t_0, t_0+h), in $\Omega \times (0,T')$ we apply the identity from before to the function

$$\varphi(x,t) := \chi_k(t) \cdot \psi(x)$$

with fixed $k \in \mathbb{N}$ and $\psi \in \Psi$. Accordingly, for any $k \in \mathbb{N}$ we see

$$\int_0^{T'} \int_\Omega \chi_k \nabla v \cdot \nabla \psi + \int_0^{T'} \int_\Omega \chi_k v \psi = \int_0^{T'} \int_\Omega \chi_k u \psi$$

and that means for arbitrary $h \in (0, T'-t_0)$

$$\frac{1}{h}\int_{t_0}^{t_0+h} \int_\Omega \nabla v \cdot \nabla \psi + \frac{1}{h}\int_{t_0}^{t_0+h} \int_\Omega v\psi = \frac{1}{h}\int_{t_0}^{t_0+h} \int_\Omega u\psi.$$

Now, since t_0 is a Lebesgue point and due to the continuity of u, v and ∇v in $\overline{\Omega} \times (0,T')$ (see lemma IV.3.1) we can take the limit $h \to 0$ and see

$$\int_\Omega \nabla v(\cdot,t_0) \cdot \nabla \psi + \int_\Omega v(\cdot,t_0)\psi = \int_\Omega u(\cdot,t_0)\psi$$

for every $\psi \in \Psi$. Due to the density property of Ψ in $W^{1,2}(\Omega)$, the claim follows upon another approximation. **q.e.d.**

Outside this null set we can find two more results regarding boundedness and continuity properties of v:

Lemma IV.3.3. *With T', u and v as in lemma IV.3.1 and with $\mathcal{N}$ from lemma IV.3.2 there are $\theta \in (0,1)$ and $C > 0$ such that*

$$\left\| v(\cdot,t) \right\|_{W^{1,2}(\Omega)} \leq C \text{ for every } t \in (0,T') \setminus \mathcal{N}$$

and

$$\left\| v(\cdot,t) - v(\cdot,s) \right\|_{W^{1,2}(\Omega)} \leq C \, |t-s|^{\theta} \text{ for every } t,s \in (0,T') \setminus \mathcal{N}$$

hold. In particular, if necessary we can redefine $v(\cdot,t)$ for $t \in \mathcal{N} \cup \{0,T'\}$ in order to achieve

$$v \in C^{\theta}\left([0,T']; W^{1,2}(\Omega)\right).$$

Proof. Again as in [92], in $(0,T') \setminus \mathcal{N}$ we may pick $\psi = v$ in lemma IV.3.2 and together with Young's inequality this directly shows the first statement:

$$\int_{\Omega} |\nabla v|^2 + \int_{\Omega} v^2 = \int_{\Omega} uv \leq \frac{1}{2} \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} v^2$$

holds in $(0,T') \setminus \mathcal{N}$.

Fixing some $t \in (0,T') \setminus \mathcal{N}$ and $s \in (0,T') \setminus \mathcal{N}$, for $x \in \Omega$ we define

$$z(x) := v(x,t) - v(x,s)$$

which gives us a function belonging to $W^{1,2}(\Omega)$ so that it too may be inserted into the identity in lemma IV.3.2, yielding

$$\int_{\Omega} \nabla v(\cdot,\tau) \cdot (\nabla v(\cdot,t) - \nabla v(\cdot,s)) + \int_{\Omega} v(\cdot,\tau) \cdot (v(\cdot,t) - v(\cdot,s)) = \int_{\Omega} u(\cdot,\tau) \cdot (v(\cdot,t) - v(\cdot,s))$$

for every $\tau \in (0,T') \setminus \mathcal{N}$.

Evaluation in s and t and subtraction of the two identities give us

$$\int_{\Omega} |\nabla z|^2 + \int_{\Omega} z^2 = \int_{\Omega} (u(\cdot,t) - u(\cdot,s))\, z.$$

Accordingly, lemma IV.3.1 and Young's inequality allow for the estimate

$$\int_{\Omega} |\nabla z|^2 + \int_{\Omega} z^2 \leq \int_{\Omega} C \, |t-s|^{\frac{\theta}{2}} \, |z| \leq \frac{1}{2} \int_{\Omega} z^2 + \frac{C^2 \, |\Omega|}{2} \, |t-s|^{\theta}$$

with some $C > 0$ and $\theta \in (0,1)$. If necessary, we redefine $t \mapsto v(\cdot,t)$ in the null set $\mathcal{N} \cup \{0,T'\}$ which concludes the proof. **q.e.d.**

We will now see that v has further helpful properties and prove that the second equation in (IV.2) holds for the functions u and v provided by lemma IV.3.1.

Lemma IV.3.4. *With T', u and v as in lemma IV.3.1, there are $\theta \in (0,1)$ and $C > 0$ such that $\left\|v(\cdot,t)\right\|_{C^{2+\theta}(\overline{\Omega})} \leq C$ holds for every $t \in (0,T')$. Furthermore, we have*

$$\begin{cases} -\Delta v + v &= u \ \text{ in } \ \Omega \times (0,T')\,, \\ \quad \frac{\partial v}{\partial \nu} &= 0 \ \text{ on } \ \partial\Omega \times (0,T')\,. \end{cases}$$

Proof. Following the approach of [92], we start by proving the existence of some $q > n$ and some $C(q) > 0$ such that

$$\left\|v(\cdot,t)\right\|_{W^{2,q}(\Omega)} \leq C(q)$$

holds for every $t \in (0,T') \setminus \mathcal{N}$ where $\mathcal{N}$ is as in lemma IV.3.2. Since lemma IV.3.1 provides us with some $\theta_1 \in (0,1)$ and $C_1 > 0$ such that

$$\left\|u(\cdot,t)\right\|_{C^{\theta_1}(\overline{\Omega})} \leq C_1$$

holds for every $t \in (0,T')$, fixing some $q > n$ we find a positive constant C_2 with

$$\left\|u(\cdot,t)\right\|_{L^q(\Omega)} \leq C_2$$

for every $t \in (0,T')$. According to lemma IV.3.2, for every $t \in (0,T') \setminus \mathcal{N}$ the function v belongs to $W^{1,2}(\Omega)$ and it is a weak solution to the Neumann boundary value problem to $-\Delta v(\cdot,t) + v(\cdot,t) = u(\cdot,t)$ in Ω. Elliptic estimates ([31]) show

$$\left\|v(\cdot,t)\right\|_{W^{2,q}(\Omega)} \leq C_3 \left\|u(\cdot,t)\right\|_{L^q(\Omega)} \leq C_2 C_3$$

with some $C_3 > 0$ and for every $t \in (0,T') \setminus \mathcal{N}$. For any $\theta_2 \in \left(0, 1 - \frac{n}{q}\right)$ from the continuous embedding $W^{2,q}(\Omega) \hookrightarrow C^{1+\theta_2}\left(\overline{\Omega}\right)$ we can therefore conclude the boundedness of $(\nabla v(\cdot,t))_{t \in (0,T') \setminus \mathcal{N}} \subset C^{\theta_2}\left(\overline{\Omega}\right)$. With results from elliptic Schauder theory (see again [31]), the boundedness of $(u(\cdot,t))_{t \in (0,T') \setminus \mathcal{N}} \subset C^{\theta_1}\left(\overline{\Omega}\right)$, provides us with some $C_4 > 0$ such that

$$\left\|v(\cdot,t)\right\|_{C^{2+\theta_1}(\overline{\Omega})} \leq C_4$$

holds for every $t \in (0,T') \setminus \mathcal{N}$. Since $v \in C^{\theta}\left([0,T']; W^{1,2}(\Omega)\right)$ for some $\theta \in (0,1)$ by lemma IV.3.3 implies continuity of v with respect to time, the statement also holds for $t \in \mathcal{N}$. The rest follows from the identity in lemma IV.3.2. **q.e.d.**

We now combine these results in order to produce a Hölder estimate involving the second derivatives of v:

Lemma IV.3.5. *With T' and v as in lemma IV.3.1, there are $\theta \in (0,1)$ and $C > 0$ with*

$$\left\|v(\cdot,s) - v(\cdot,t)\right\|_{C^{2+\theta}(\overline{\Omega})} \leq C\,|t-s|^{\theta}$$

for every $s \in (0,T')$ and $t \in (0,T')$.

Proof. This is the same as in [92]. Lemma IV.3.3 gives us some $\theta_1 \in (0,1)$ such that $v \in C^{\theta_1}\left([0,T'];W^{1,2}(\Omega)\right)$. After fixing some $\theta_2 \in (0,\theta_1)$ via interpolation we find $a \in (0,1)$ and $C_1 > 0$ such that

$$\|\psi\|_{C^{2+\theta_2}(\overline{\Omega})} \le C_1 \|\psi\|^a_{C^{2+\theta_1}(\overline{\Omega})} \|\psi\|^{1-a}_{W^{1,2}(\Omega)}$$

holds for any $\psi \in C^{2+\theta_1}(\overline{\Omega})$. Accordingly, for any $s \in (0,T')$ and $t \in (0,T')$, upon inserting $v(\cdot,t) - v(\cdot,s)$ we see that $\left\|v(\cdot,t) - v(\cdot,s)\right\|_{C^{2+\theta_2}(\overline{\Omega})}$ is bounded from above by

$$C_1\left(\left\|v(\cdot,t)\right\|_{C^{2+\theta_1}(\overline{\Omega})} + \left\|v(\cdot,s)\right\|_{C^{2+\theta_1}(\overline{\Omega})}\right)^a \left\|v(\cdot,t) - v(\cdot,s)\right\|^{1-a}_{W^{1,2}(\Omega)}.$$

From lemma IV.3.3 and lemma IV.3.4 for some $C_2 > 0$ and $\theta_3 \in (0,1)$ we therefore get

$$\left\|v(\cdot,s) - v(\cdot,t)\right\|_{C^{2+\theta_2}(\overline{\Omega})} \le C_2 \,|t-s|^{\theta_3}$$

for every $s \in (0,T')$ and $t \in (0,T')$. To finish the proof we only need to identify some $\theta \in (0,1)$ (as well as a positive constant C_3) such that on both sides of the inequality the same θ can be put instead of θ_2 and θ_3. If $\theta_3 \le \theta_2$, then the inclusion

$$C^{2,\theta_2}\left(\overline{\Omega}\right) \subset C^{2,\theta_3}\left(\overline{\Omega}\right)$$

is the missing puzzle piece. If on the other hand $\theta_3 > \theta_2$, then we take a look at the term $|t-s|^{\theta_3}$. Since both times are contained in $(0,T')$, for $0 < \theta_3 - \theta_2 < 1$ and due to $T' > 1$ we have

$$|t-s|^{\theta_3} = |t-s|^{\theta_2} \cdot |t-s|^{\theta_3-\theta_2} \le T'|t-s|^{\theta_2}.$$

q.e.d.

As an additional ingredient for the proof of theorem IV.1.1, the following lemma, an analogon to lemma 5.5 in [92], proves that locally v_t belongs to L^2 with respect to both variables. We remark that in the following statements we will consider the original T instead of $T' = T+1$ and that the upcoming lemma is the reason for introducing T' in the first place.

Lemma IV.3.6. *With T and v as in lemma IV.3.1 we have $v_t \in L^2_{loc}\left(\overline{\Omega} \times (0,T]\right)$.*

Proof. Fixing some $\tau \in (0,T)$ as well as $h_0 := \min\{1, T-\tau\}$ and some $h \in (0,h_0)$, we define

$$z_h(x,t) := \frac{v(x,t+h) - v(x,t)}{h}$$

for $x \in \Omega$ and $t \in (\tau,T)$. From lemma IV.3.4, we know that for any $t \in (\tau,T)$ by $z_h(\cdot,t) \in C^2\left(\overline{\Omega}\right)$ we have found a classical solution for the Neumann boundary problem to

$$-\Delta z_h(\cdot,t) + z_h(\cdot,t) = \frac{u(\cdot,t+h) - u(\cdot,t)}{h}$$

in Ω. If B is the realisation of $-\Delta + 1$ in $W^{2,2}_N(\Omega)$, then we have

$$\left\|z_h(\cdot,t)\right\|_{L^2(\Omega)} = \left\|B^{-1}\frac{u(\cdot,t+h)-u(\cdot,t)}{h}\right\|_{L^2(\Omega)} = \left\|\frac{1}{h}\int_t^{t+h} B^{-1}u_t(\cdot,s)\,\mathrm{d}s\right\|_{L^2(\Omega)}$$

for every $t \in (\tau, T)$. From the Cauchy-Schwarz inequality we thus know that

$$\int_\tau^T \left\|z_h(\cdot,t)\right\|^2_{L^2(\Omega)}\,\mathrm{d}t \le \frac{1}{h}\int_\tau^T\int_t^{t+h}\left\|B^{-1}u_t(\cdot,s)\right\|^2_{L^2(\Omega)}\,\mathrm{d}s\,\mathrm{d}t$$

holds for every $h \in (0, h_0)$. Standard elliptic regularity theory in [100] shows that the mapping $B^{-1} : L^2(\Omega) \to W^{2,2}_N(\Omega)$ is continuous, providing us with some $C_1 > 0$ such that

$$\left\|B^{-1}\psi\right\|_{L^2(\Omega)} \le C_1 \|\psi\|_{\left(W^{2,2}_N(\Omega)\right)^*}$$

holds for every $\psi \in \left(W^{2,2}_N(\Omega)\right)^*$.

Accordingly, from the previous estimate we see

$$\int_\tau^T \left\|z_h(\cdot,t)\right\|^2_{L^2(\Omega)}\,\mathrm{d}t \le \frac{C_1}{h}\int_\tau^T\int_t^{t+h}\left\|u_t(\cdot,s)\right\|^2_{\left(W^{2,2}_N(\Omega)\right)^*}\,\mathrm{d}s\,\mathrm{d}t$$

for every $h \in (0, h_0)$ and with the abbreviation

$$w(s) := \left\|u_t(\cdot,s)\right\|^2_{\left(W^{2,2}_N(\Omega)\right)^*}$$

for $s \in (\tau, T)$, for such h we have

$$\int_\tau^T \left\|z_h(\cdot,t)\right\|^2_{L^2(\Omega)}\,\mathrm{d}t \le \frac{C_1}{h}\int_\tau^T\int_t^{t+h} w(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Using the Fubini theorem this can be rearranged, to this end we consider the sets

$$\{(s,t) \mid t \in (\tau,T),\ s \in (t, t+h)\} = X_1 \,\dot\cup\, X_2 \,\dot\cup\, X_3$$

where

$$\begin{aligned} X_1 &= \{(s,t) \mid s \in (\tau, \tau+h),\ t \in (\tau, s)\},\\ X_2 &= \{(s,t) \mid s \in (\tau+h, T),\ t \in (s-h, s)\} \text{ and}\\ X_3 &= \{(s,t) \mid s \in (T, T+h),\ t \in (s-h, T)\}. \end{aligned}$$

Accordingly we have

$$\begin{aligned} \frac{1}{h}\int_\tau^T\int_t^{t+h} w(s)\,\mathrm{d}s\,\mathrm{d}t =& \frac{1}{h}\int_\tau^{\tau+h}\int_\tau^s w(t)\,\mathrm{d}t\,\mathrm{d}s\\ &+ \frac{1}{h}\int_{\tau+h}^T\int_{s-h}^s w(t)\,\mathrm{d}t\,\mathrm{d}s\\ &+ \frac{1}{h}\int_T^{T+h}\int_{s-h}^T w(t)\,\mathrm{d}t\,\mathrm{d}s \end{aligned}$$

for every $h \in (0, h_0)$.

Let W be an antiderivative for w, then the three integrals on the right-hand side can be rewritten as

$$\frac{1}{h}\int_{\tau}^{\tau+h} \left(W(s) - W(\tau)\right) \,\mathrm{d}s + \frac{1}{h}\int_{\tau+h}^{T} \left(W(s) - W(s-h)\right) \,\mathrm{d}t \,\mathrm{d}s$$
$$+\frac{1}{h}\int_{T}^{T+h} \left(W(T) - W(s-h)\right) \,\mathrm{d}t \,\mathrm{d}s$$

and this can be rearranged to

$$\frac{1}{h}\int_{\tau}^{T} W(s) \,\mathrm{d}s - W(\tau) - \frac{1}{h}\int_{\tau+h}^{T} W(s-h)\mathrm{d}s + W(T) - \frac{1}{h}\int_{T}^{T+h} W(s-h) \,\mathrm{d}s.$$

Herein, most of the terms cancel each other out, leaving us with

$$\frac{1}{h}\int_{\tau}^{T}\int_{t}^{t+h} w(s) \,\mathrm{d}s \,\mathrm{d}t = W(T) - W(\tau) = \int_{\tau}^{T} w(t) \,\mathrm{d}t$$

for every $h \in (0, h_0)$.

From lemma IV.2.4 and the convergence result in lemma IV.3.1 we therefore get some $C_2 > 0$ such that

$$\int_{\tau}^{T} \left\| z_h(\cdot, t) \right\|_{L^2(\Omega)}^2 \le C_2$$

holds for every $h \in (0, h_0)$ and so there are some zero sequence $(h_k)_{k\in\mathbb{N}} \subset (0, h_0)$ and some function $z \in L^2\left(\Omega \times (\tau, T)\right)$ with $z_{h_k} \rightharpoonup z$ in $L^2\left(\Omega \times (\tau, T)\right)$ as $k \to \infty$. As per the definition of distributional derivatives, this z coincides with v_t almost everywhere in $\Omega \times (\tau, T)$ for any $\tau > 0$ which completes the proof. **q.e.d.**

We need one more convergence result for v_ε and v in order to prove our theorem, namely the following:

Lemma IV.3.7. *With T and v as well as a zero sequence E as in lemma IV.3.1 we have*

$$\begin{cases} v_\varepsilon \to v & \text{in } L^\infty_{loc}\left((0,T]; L^2(\Omega)\right) \\ \nabla v_\varepsilon \to \nabla v & \text{in } L^2_{loc}\left(\overline{\Omega} \times (0,T]\right) \end{cases}$$

as $\varepsilon \to 0$.

Proof. Adapting the argument in lemma 5.6 of [92], we begin with the definitions

$$z_\varepsilon(x,t) := v_\varepsilon(x,t) - v(x,t)$$

for $\varepsilon \in E$ and $(x,t) \in \Omega \times (0,T)$ as well as

$$y_\varepsilon(t) := \int_\Omega z_\varepsilon^2(\cdot, t)$$

for $\varepsilon \in E$ and $t \in (0,T)$ and from lemma IV.3.4 we see $z_\varepsilon \in L^\infty((0,T);L^\infty(\Omega))$ while $z_{\varepsilon t} = v_{\varepsilon t} - v_t \in L^2_{\text{loc}}\big((0,T];L^2(\Omega)\big)$ is guaranteed for every $\varepsilon \in E$ by lemma IV.3.6. Accordingly, we gather $y_\varepsilon \in W^{1,2}_{\text{loc}}((0,T])$, therefore y_ε is locally absolutely continuous in $(0,T]$ with

$$y_{\varepsilon t}(t) = 2\int_\Omega z_\varepsilon(\cdot,t) z_{\varepsilon t}(\cdot,t)$$

for every $\varepsilon \in E$ and almost every $t \in (0,T)$. From lemma IV.3.4 we know

$$\begin{aligned}\varepsilon z_{\varepsilon t} &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon - \varepsilon v_t\\ &= \Delta z_\varepsilon + \Delta v - z_\varepsilon - v + u_\varepsilon - \varepsilon v_t\\ &= \Delta z_\varepsilon - z_\varepsilon + u_\varepsilon - u - \varepsilon v_t\end{aligned}$$

for every $\varepsilon \in E$ and almost everywhere in $\Omega \times (0,T)$. We see

$$\frac{\varepsilon}{2} y_{\varepsilon t}(t) = \int_\Omega z_\varepsilon(\cdot,t)\Delta z_\varepsilon(\cdot,t) - \int_\Omega z_\varepsilon^2(\cdot,t) + \int_\Omega z_\varepsilon(\cdot,t)\left(u_\varepsilon(\cdot,t) - u(\cdot,t)\right) - \varepsilon\int_\Omega z_\varepsilon(\cdot,t) v_t(\cdot,t)$$

for every $\varepsilon \in E$ and almost every $t \in (0,T)$. According to lemma IV.2.1 and lemma IV.3.4, we have

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial v_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0$$

for every $\varepsilon \in E$ in $(0,T)$ and therefore integration by parts gives us

$$\int_\Omega z_\varepsilon \Delta z_\varepsilon = -\int_\Omega |\nabla z_\varepsilon|^2 + \int_{\partial\Omega} z_\varepsilon \frac{\partial z_\varepsilon}{\partial \nu} = -\int_\Omega |\nabla z_\varepsilon|^2$$

for every $\varepsilon \in E$ in $(0,T)$ while the last two terms are estimated via Young's inequality. Together this leads us to

$$\varepsilon y_{\varepsilon t}(t) + 2\int_\Omega \left|\nabla z_\varepsilon(\cdot,t)\right|^2 + \int_\Omega z_\varepsilon^2(\cdot,t) \le 2\int_\Omega \left|u_\varepsilon(\cdot,t) - u(\cdot,t)\right|^2 + 2\varepsilon^2 \int_\Omega v_t^2(\cdot,t) \qquad (*)$$

for every $\varepsilon \in E$ and almost every $t \in (0,T)$ and we already know from lemma IV.3.1 that the first term on the right vanishes as $\varepsilon \to 0$. We now fix some $\tau \in (0,T)$ and some $\eta > 0$.
Using $u_\varepsilon \to u$ in $L^\infty((0,T);L^\infty(\Omega))$ as $\varepsilon \to 0$ in E, lemma IV.3.6 and the boundedness of the sequence $(y_\varepsilon)_{\varepsilon\in E} \subset L^\infty((0,T))$ provided by lemma IV.3.4, we can fix $\varepsilon_0 > 0$ with the following property: Whenever $\varepsilon \in E$ is smaller than ε_0, we have

$$\begin{aligned}4\,|\Omega| \cdot \|u_\varepsilon - u\|^2_{L^\infty(\Omega\times(0,T))} &\le \frac{\eta}{3},\\ 2\varepsilon \int_{\frac{\tau}{2}}^{\tau}\int_\Omega v_t^2 &\le \frac{\eta}{3} \text{ and}\\ y_\varepsilon\left(\frac{\tau}{2}\right)\cdot e^{-\frac{\tau}{4\varepsilon}} &\le \frac{\eta}{3}.\end{aligned}$$

For the absolutely continuous function $[\frac{\tau}{2}, T] \ni t \mapsto e^{\frac{1}{2\varepsilon}\left(t-\frac{\tau}{2}\right)} y_\varepsilon(t)$, $\varepsilon \in E$, we see

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{1}{2\varepsilon}\left(t-\frac{\tau}{2}\right)} y_\varepsilon(t)\right) &= e^{\frac{1}{2\varepsilon}\left(t-\frac{\tau}{2}\right)}\left(y_{\varepsilon t}(t) + \frac{1}{2\varepsilon} y_\varepsilon(t)\right)\\ &\le \frac{e^{\frac{1}{2\varepsilon}\left(t-\frac{\tau}{2}\right)}}{\varepsilon}\left(-y_\varepsilon(t) + 2\,|\Omega| \cdot \|u_\varepsilon - u\|^2_{L^\infty(\Omega\times(0,T))} + 2\varepsilon^2 \int_\Omega v_t^2 + \frac{1}{2} y_\varepsilon(t)\right)\end{aligned}$$

for every $\varepsilon \in E$ and $t \in \left(\frac{\tau}{2}, T\right)$; upon integration this shows

$$\begin{aligned}y_\varepsilon(t) \le{}& y_\varepsilon\left(\frac{\tau}{2}\right) \cdot e^{-\frac{1}{2\varepsilon}\left(t-\frac{\tau}{2}\right)} - \frac{1}{2\varepsilon}\int_{\frac{\tau}{2}}^t e^{\frac{1}{2\varepsilon}(s-t)} y_\varepsilon(s)\,\mathrm{d}s\\ &+ \frac{2\,|\Omega| \cdot \|u_\varepsilon - u\|^2_{L^\infty(\Omega\times(0,T))}}{\varepsilon}\int_{\frac{\tau}{2}}^t e^{\frac{1}{2\varepsilon}(s-t)}\,\mathrm{d}s + 2\varepsilon \int_{\frac{\tau}{2}}^t e^{\frac{1}{2\varepsilon}(s-t)} \int_\Omega v_t^2(\cdot, s)\,\mathrm{d}s\end{aligned}$$

for every $\varepsilon \in E$ and $t \in (\tau, T)$. For any such t and for $\varepsilon \in E$ with $\varepsilon < \varepsilon_0$, the right-hand side is therefore bounded by η which proves

$$z_\varepsilon \to 0 \text{ in } L^\infty\left((\tau, T); L^2(\Omega)\right)$$

as $\varepsilon \to 0$. Integrating $(*)$, for any $\tau \in (0, T)$ we have

$$2\int_\tau^T \int_\Omega |\nabla z_\varepsilon|^2 \le \varepsilon y_\varepsilon(\tau) + 2\,|\Omega| \cdot (T-\tau) \cdot \|u_\varepsilon - u\|^2_{L^\infty(\Omega\times(0,T))} + 2\varepsilon^2 \int_\tau^T \int_\Omega v_t^2$$

for every $\varepsilon \in E$ and the terms on the right-hand side vanish as $\varepsilon \to 0$ due to the boundedness of y_ε in $L^\infty((0,T))$, the uniform convergence of u_ε as before and lemma IV.3.6. For any $\tau \in (0, T)$ we therefore have

$$z_\varepsilon \to 0 \text{ in } L^2\left((\tau, T); W^{1,2}(\Omega)\right)$$

as $\varepsilon \to 0$ which completes the proof. **q.e.d.**

As the final ingredient, we now want to show that also the first equation in (IV.2) holds together with the respective boundary condition:

Lemma IV.3.8. *With T and u as in lemma IV.3.1, in $\Omega \times (0, T)$ we have*

$$u_t = \nabla \cdot \left((u+1)^{m-1}\nabla u\right) - \nabla \cdot (u\nabla v)$$

and on $\partial\Omega \times (0, T)$ we have

$$\frac{\partial u}{\partial \nu} = 0.$$

Proof. With E as the zero sequence we have found in lemma IV.3.1, for any function $\varphi \in C_0^\infty\left(\overline{\Omega} \times [0,T)\right)$ and $\varepsilon \in E$ we have

$$-\int_0^T \int_\Omega u_\varepsilon \varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega (u_\varepsilon + 1)^{m-1} \nabla u_\varepsilon \cdot \nabla\varphi + \int_0^T \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi.$$

Lemma IV.3.1 and lemma IV.3.7 show the convergence of all occurring integrals, giving us

$$-\int_0^T \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega (u + 1)^{m-1} \nabla u \cdot \nabla\varphi + \int_0^T \int_\Omega u \nabla v \cdot \nabla\varphi$$

which means that $u \in L^2\left((0,T); W^{1,2}(\Omega)\right)$ (in the standard generalised sense found for example in [47]) defines a weak solution for the initial-boundary value problem given by the two statements in this lemma (and $u(\cdot,0) = u_0$). For the remainder of the proof we turn to the corresponding lemma 6.1 in [92]: classical results of parabolic regularity theory and Hölder continuity of u, v, ∇v and $D^2 v$ in $\overline{\Omega} \times [0,T]$ as given by lemma IV.3.1 and lemma IV.3.5 prove $u \in C^{1+\theta_1,\frac{1+\theta_1}{2}}\left(\overline{\Omega} \times (0,T]\right)$ for some $\theta_1 \in (0,1)$. This knowledge of Hölder regularity of ∇u lets us find some $\theta_2 \in (0,1)$ such that $u \in C^{2+\theta_2,1+\frac{\theta_2}{2}}\left(\overline{\Omega} \times (0,T]\right)$. Using the fundamental lemma of calculus of variations, this combined with the integral identity above yields the desired result. **q.e.d.**

We now collect the results from this section and return to the theorem we originally wanted to prove:

Proof of theorem IV.1.1. From lemma IV.3.1 we already have

$$u_\varepsilon \to u \text{ in } C^0\left(\overline{\Omega} \times [0,T]\right)$$

and

$$u_\varepsilon \to u \text{ in } L^2\left((0,T); W^{1,2}(\Omega)\right)$$

as $\varepsilon \to 0$.

Together with lemma IV.3.7 we also see

$$v_\varepsilon \to v \text{ in } L^\infty_{\text{loc}}\left((0,T]; L^2(\Omega)\right) \cap L^2_{\text{loc}}\left((0,T]; W^{1,2}(\Omega)\right) \tag{$*$}$$

as $\varepsilon \to 0$ along some suitable subsequence $E' \subset E$. On the other hand, lemma IV.2.1 provides us with some $C_1 > 0$ such that

$$\left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} \le C_1 \tag{$**$}$$

holds for every $t \in (0,T)$ and every $\varepsilon \in E$ which immediately gives us

$$\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v \text{ in } L^\infty\left((0,T); W^{1,\infty}(\Omega)\right)$$

as $\varepsilon \to 0$ upon another suitable restriction of E'. Using the Gagliardo-Nirenberg inequality we can use this a second time to see that for some $a \in (0,1)$ and some $C_2 > 0$ we have

$$\left\|v_\varepsilon(\cdot,t) - v(\cdot,t)\right\|_{C^0(\overline{\Omega})} \le C_2 \left\|v_\varepsilon(\cdot,t) - v(\cdot,t)\right\|^a_{W^{1,\infty}(\Omega)} \cdot \left\|v_\varepsilon(\cdot,t) - v(\cdot,t)\right\|^{1-a}_{L^2(\Omega)}$$

for every $t \in (0,T)$ and $\varepsilon \in E$ wherein the first term on the right-hand side is bounded via

$$\left\|v_\varepsilon(\cdot,t) - v(\cdot,t)\right\|^a_{W^{1,\infty}(\Omega)} \le \left(\left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} + \left\|v(\cdot,t)\right\|_{W^{1,\infty}(\Omega)}\right)^a.$$

Therefore, $(*)$ and $(**)$ combine to prove

$$v_\varepsilon \to v \text{ in } L^\infty_{\text{loc}}\left((0,T); C^0(\overline{\Omega})\right)$$

while lemma IV.3.8 and lemma IV.3.4 complete our proof by showing that (IV.2) is actually solved by u and v. **q.e.d.**

Chapter V

Global solutions to a higher-dimensional system related to crime modelling

V.1 Introduction and main result

The system discussed in this chapter, for specific choices for the parameters, has been used in [76] to describe and identify regions of disproportionally high crime levels. In particular Short, D'Orsogna, Pasour, Tita, Brantingham, Bertozzi and Chayes considered the reaction-advection-diffusion system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right) - uv + B_1, \\ v_t = \Delta v - v + uv + B_2. \end{cases}$$

Herein u denotes the density of criminal agents whereas v quantifies the attractiveness of regions at a given time. Several assumptions were used to build this model: In [18] and [20], opportunity is considered to be the primary factor leading to crime. The so-called 'repeat and near-repeat victimisation' as discussed in [41] and [77] observes an increased likelihood for burglarised houses and their vicinity to be the next burglar's target. Lastly, [44] suggested the 'broken-window theory': crime itself might lead to more crime.

Due to the biased movement toward high concentrations of the attractiveness value, which creates a connection to previous chapters, we see a conditional diffusion in the equation for u and the unconditional diffusion given by Δv mirrors the near-repeat victimisation effect. In the first equation, as uv roughly translates to the number of crimes, we see a decay $-uv$ as criminals are assumed to abstain from committing a second crime, whereas its positive counterpart in the second equation is intended to

incorporate the repeat victimisation effect. The external sources B_1 and B_2 introduce criminal agents into the system and portray the different attractiveness of certain regions at the beginning respectively.
In [75], the authors examined the emergence and suppression of the crime hotspots this system seeks to model and several works ([6], [9], [32], [45], [90]) investigated the existence and stability of localised patterns representing hotspots. Others considered a more generalised class of systems for the dynamics of criminal activity in [4] and in [5] an analysis of these models followed. The works [42], [68] and [104] discussed the effects of incorporation of law enforcement. [13] introduced 'commuter criminal agents' through Lévy flights while [54] and [64] chose an approach via dynamical systems. Other social phenomena were studied by [2], [72] and [80] while [19] gives a review of mathematical models and the theory for criminal activity.
From an analytical point of view there are similarities to the Keller-Segel model for chemotactical processes in biology, (II.1) in chapter II with $S = id$. Here the interplay of cross-diffusion and linear production in the second equation is known to have a strongly destabilising potential in higher dimensions: While global existence can be proven for $n = 1$ ([65]), blow-up in finite time is possible both for $n = 2$ ([34]) and $n \geq 3$ ([98], [3]).
Choosing $S(v) = \frac{a}{(1+bv)^\alpha}$ for some positive a and b as well as some $\alpha > 1$ on the other hand guarantees the existence of global and bounded solutions for every $n \in \mathbb{N}$ ([95]). For logarithmic sensitivities however, that is for $S(v) = \frac{\chi}{v}$ for some $\chi > 0$, global solutions are known to exist only under smallness conditions on χ as shown in [8], [97] and [48]. While the specific choice $\chi = 2$ seems important in the context which the system is used in, there are no general results for any $\chi \geq \sqrt{\frac{2}{n}}$. Apparently the only exception is [48] where for $n = 2$ some χ_0 slightly larger than 1 with the property that solutions exist globally for any $\chi \in (0, \chi_0)$ has been detected. Our result will contain such a restriction for χ as well. For $n \geq 2$ there are also global weak solutions in various generalised frameworks ([97], [85], [50], [103]).
In the model above, the production of the attractiveness value is even nonlinear so there is the possibility of still larger cross-diffusive gradients; even in the case $n = 1$ the corresponding Keller-Segel system proves difficult and there seem to be no results on global existence. Here however, the additional absorption $-uv$ might be the deciding advantage and it is exploited in order to establish one of the fundamental estimates in this chapter.
For the full system (V.1) below, [70] gave a first result on local existence and uniqueness, [53] and [69] dealt with modified versions containing additional regularising ingredients to infer global existence. Recently, [71] considered the one-dimensional version of this model and proved the existence and uniqueness of global solutions for arbitrary $\chi > 0$.

We want to conclude this introduction by citing [1] which firstly applies the results in this field to retroactively explain crime patterns. Even more interestingly, this model is also used for crime prediction, for example in Santiago de Chile.

Let us now formally present our result:

Theorem V.1.1 (Existence and uniqueness of global bounded solutions). *In some bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, let $q \in (n, \infty]$, $\chi > 0$ with*

$$\chi < \begin{cases} \sqrt{\frac{2}{n}} & \text{for } n \in \{2,3\}, \\ \frac{2}{n} & \text{for } n \geq 4, \end{cases}$$

nonnegative functions $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ with $\inf_{\overline{\Omega}} v_0 > 0$ as well as some $0 \leq B_1 \in C^1\left(\overline{\Omega} \times [0,\infty)\right)$ and $0 \leq B_2 \in C^1\left(\overline{\Omega} \times [0,\infty)\right)$ be given. Then there exists a unique pair of nonnegative functions (u, v) with

$$u \in C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right)$$
$$v \in C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right) \cap L^\infty_{loc}\left([0,\infty)\,; W^{1,q}(\Omega)\right)$$

that solves

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v\right) - uv + B_1 & \text{in } \Omega \times (0,\infty)\,, \\ v_t = \Delta v - v + uv + B_2 & \text{in } \Omega \times (0,\infty)\,, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \Omega \times (0,\infty) \end{cases} \tag{V.1}$$

classically in $\Omega \times (0,\infty)$.

We note that while this to the best of our knowledge is the first existence result in higher dimensions, the critical value $\chi = 2$ still eludes us as it does in similar problems (see [97]).

Remark. *For the entirety of this chapter, it shall be assumed that $n \geq 2$, some bounded domain $\Omega \subset \mathbb{R}^n$ and $q \in (n, \infty]$ are given as well as nonnegative functions $u_0 \in C^0\left(\overline{\Omega}\right)$, $v_0 \in W^{1,q}(\Omega)$ with $\inf_{\overline{\Omega}} v_0 > 0$, $B_1 \in C^1\left(\overline{\Omega} \times [0,\infty)\right)$ and $B_2 \in C^1\left(\overline{\Omega} \times [0,\infty)\right)$*

V.2 Local solutions and a criterion for global existence

The potential degeneracy in the cross-diffusion term makes it prudent to consider an altered version first: replacing $\frac{\chi}{v}$ in the first equation with some function f of v that does not vanish at $v = 0$, we will in a first step find solutions to a different system that only a posteriori reveals itself to be equivalent to the original.

Lemma V.2.1. *Given $\chi > 0$, let $C_f > 0$ and $\delta > 0$ and choose a $C^{1+\delta}$-function f with*

$$f(s) = \begin{cases} \chi \frac{2}{C_f}, & s \leq \frac{1}{2} C_f, \\ \chi \frac{1}{s}, & s \geq C_f \end{cases}$$

Then

$$\begin{cases} u_t = \Delta u - \nabla \cdot (uf(v)\nabla v) - uv + B_1 & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - v + uv + B_2 & \text{in } \Omega \times (0, T_{max}), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ u(\cdot, t) = u_0, \ v(\cdot, 0) = v_0 & \text{in } \Omega \times (0, T_{max}) \end{cases} \tag{V.2}$$

possesses a unique classical solution (u, v) *with*

$$\begin{aligned} &u \in C^0\left(\overline{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right) \\ &v \in C^0\left(\overline{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right) \cap L^\infty_{loc}\left([0, T_{max}) ; W^{1,q}(\Omega)\right) \end{aligned}$$

in $\Omega \times (0, T_{\max})$ *for some* $T_{\max} \in (0, \infty]$ *and in the case of finite* T_{max} *we have*

$$\lim_{t \nearrow T_{\max}} \left(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \right) = \infty.$$

Proof. Recalling and adjusting a well-established approach ([95]), we will first find such a solution using a fixed point method and then we will show that two arbitrary solutions to this problem must be identical. Pending the definition of the two parameters $R > 0$ and $T > 0$, let us consider the subset $B := \left\{(u, v) \in X | \left\|(u, v)\right\|_X \le R\right\}$ of the space $X := C^0\left([0, T]; C^0(\overline{\Omega})\right) \times L^\infty\left((0, T); W^{1,q}(\Omega)\right)$ with

$$\left\|(u, v)\right\|_X := \|u\|_{L^\infty(\Omega \times (0,T))} + \|v\|_{L^\infty((0,T); W^{1,q}(\Omega))}.$$

Furthermore, for $(u, v) \in X$ and $t \in [0, T]$ let

$$\begin{aligned} \psi_1(u, v)(\cdot, t) := &e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot \left(u(\cdot, s) f(v(\cdot, s)) \nabla v(\cdot, s)\right) \,\mathrm{d}s \\ &- \int_0^t e^{(t-s)\Delta} u(\cdot, s) v(\cdot, s) \,\mathrm{d}s + \int_0^t e^{(t-s)\Delta} B_1(\cdot, s) \,\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} \psi_2(u, v)(\cdot, t) := &e^{t(\Delta - 1)} v_0 + \int_0^t e^{(t-s)(\Delta-1)} u(\cdot, s) v(\cdot, s) \,\mathrm{d}s \\ &+ \int_0^t e^{(t-s)\Delta} B_2(\cdot, s) \,\mathrm{d}s \end{aligned}$$

which together give us a function $\psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : X \to X$. We now want to find values for R and T that result in ψ being a contraction that maps B into itself.

From the Sobolev inequality we know that for some constant $C_1 > 0$ and any function $w \in W^{1,q}(\Omega)$ we have

$$\|w\|_{L^\infty(\Omega)} \le C_1 \|w\|_{W^{1,q}(\Omega)} .$$

Given this first constant C_1 we now pick $L(R) > 0$ as a Lipschitz constant for f on $(-C_1R, C_1R)$. Then for any (u,v), $(\overline{u},\overline{v}) \in B$ and $t \in (0,T)$ we consider

$$I_1(\cdot,t) := \int_0^t e^{(t-s)\Delta}\nabla \cdot \left(u(\cdot,s)f(v(\cdot,s))\nabla v(\cdot,s) - \overline{u}(\cdot,s)f(\overline{v}(\cdot,s))\nabla\overline{v}(\cdot,s)\right) \,\mathrm{d}s$$

and

$$I_2(\cdot,t) := \int_0^t e^{(t-s)\Delta}\left(u(\cdot,s)v(\cdot,s) - \overline{u}(\cdot,s)\overline{v}(\cdot,s)\right) \,\mathrm{d}s$$

for every $t \in (0,T)$ respectively and we obviously have

$$-\psi_1(u,v) + \psi_1(\overline{u},\overline{v}) = I_1 + I_2$$

in $(0,T)$ for any $(u,v) \in B$ and $(\overline{u},\overline{v}) \in B$.

In order to find estimates for the $L^\infty(\Omega)$-norms of these integrals we use the abbreviation

$$F(u,v,\overline{u},\overline{v})(s) := u(\cdot,s)f(v(\cdot,s))\nabla v(\cdot,s) - \overline{u}(\cdot,s)f(\overline{v}(\cdot,s))\nabla\overline{v}(\cdot,s)$$

and lemma I.3.3 to find $C_2 > 0$ and $\lambda > 0$ with

$$\left\|I_1(\cdot,t)\right\|_{L^\infty(\Omega)} \le C_2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)e^{-\lambda(t-s)} \left\|F(u,v,\overline{u},\overline{v})(s)\right\|_{L^q(\Omega)} \mathrm{d}s$$

for every $t \in (0,T)$ and any $(u,v),(\overline{u},\overline{v}) \in B$. We use the conditions for elements taken from B and the fact that for such functions by $L(R)$ we have an applicable Lipschitz constant for f to estimate

$$\begin{aligned}
\left|F(u,v,\overline{u},\overline{v})(s)\right| =& \left|u(\cdot,s)f(v(\cdot,s))\nabla v(\cdot,s) - \overline{u}(\cdot,s)f(\overline{v}(\cdot,s))\nabla\overline{v}(\cdot,s)\right| \\
\le& \left|u(\cdot,s)\left[f(v(\cdot,s)) - f(\overline{v}(\cdot,s))\right]\nabla v(\cdot,s)\right| \\
&+ \left|\left[u(\cdot,s) - \overline{u}(\cdot,s)\right] f(\overline{v}(\cdot,s))\nabla v(\cdot,s)\right| \\
&+ \left|\overline{u}(\cdot,s)f(\overline{v}(\cdot,s))\left[\nabla v(\cdot,s) - \nabla\overline{v}(\cdot,s)\right]\right| \\
\le& R^2L(R)\left|v(\cdot,s) - \overline{v}(\cdot,s)\right| + \|f\|_{L^\infty(\mathbb{R})} R\left|u(\cdot,s) - \overline{u}(\cdot,s)\right| \\
&+ \|f\|_{L^\infty(\mathbb{R})} R\left|\nabla v(\cdot,s) - \nabla\overline{v}(\cdot,s)\right|
\end{aligned}$$

for every $s \in (0,T)$ and $(u,v),(\overline{u},\overline{v}) \in B$. Recalling the definition of $\|\cdot\|_X$, with the estimate above and the fact that $-\frac{1}{2} - \frac{n}{2q} > -1$ for some $C_3 > 0$ this results in

$$\|I_1\|_{L^\infty(\Omega\times(0,T))} \le C_3 \cdot \left(R^2L(R) + R\right)\left\|(u,v) - (\overline{u},\overline{v})\right\|_X$$

for any $(u,v) \in B$ and $(\overline{u},\overline{v}) \in B$.

Given any $(u,v) \in B$ and $(\overline{u},\overline{v}) \in B$, for $h(\cdot,s) := u(\cdot,s)v(\cdot,s) - \overline{u}(\cdot,s)\overline{v}(\cdot,s)$ we see

$$\begin{aligned}
\|h\|_{L^\infty((0,T);L^q(\Omega))} &\le \left\|(u-\overline{u})v\right\|_{L^\infty((0,T);L^q(\Omega))} + \left\|\overline{u}(v-\overline{v})\right\|_{L^\infty((0,T);L^q(\Omega))} \\
&\le R\left(\|u-\overline{u}\|_{L^\infty(\Omega\times(0,T))} + \|v-\overline{v}\|_{L^\infty((0,T);L^q(\Omega))}\right) \\
&\le R\left\|(u,v)-(\overline{u},\overline{v})\right\|_X
\end{aligned}$$

and we find some $C_4 > 0$ such that

$$\begin{aligned}
\|H\|_{L^\infty((0,T))} &\le \frac{1}{|\Omega|}\left\|\int_\Omega (u-\overline{u})v\right\|_{L^\infty((0,T))} + \frac{1}{|\Omega|}\left\|\int_\Omega \overline{u}(v-\overline{v})\right\|_{L^\infty((0,T))} \\
&\le \frac{1}{|\Omega|}\|v\|_{L^\infty((0,T);L^1(\Omega))}\cdot\|u-\overline{u}\|_{L^\infty(\Omega\times(0,T))} \\
&\quad + \frac{1}{|\Omega|}\|\overline{u}\|_{L^\infty(\Omega\times(0,T))}\cdot\|v-\overline{v}\|_{L^\infty((0,T);L^1(\Omega))} \\
&\le C_4\cdot R\left\|(u,v)-(\overline{u},\overline{v})\right\|_X .
\end{aligned}$$

holds for any $(u,v) \in B$ and $(\overline{u},\overline{v}) \in B$, h as above and $H(s) := \frac{1}{|\Omega|}\int_\Omega h(\cdot,s)$. Similarly to before, for additional constants $C_5 > 0$ and $C_6 > 0$, we then find that

$$\begin{aligned}
\left\|I_2(\cdot,t)\right\|_{L^\infty(\Omega)} &\le \int_0^t \left\|e^{(t-s)\Delta}\left(h(\cdot,s)-H(s)\right)\right\|_{L^\infty(\Omega)}\,\mathrm{d}s + \int_0^t \left\|e^{(t-s)\Delta}H(s)\right\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\le C_5\int_0^t\left(1+(t-s)^{-\frac{n}{2q}}\right)e^{-\lambda(t-s)}\left\|h(\cdot,s)-H(s)\right\|_{L^q(\Omega)}\,\mathrm{d}s + T\|H\|_{L^\infty(0,T)} \\
&\le C_5\int_0^t\left(1+(t-s)^{-\frac{n}{2q}}\right)e^{-\lambda(t-s)}\left\|h(\cdot,s)\right\|_{L^q(\Omega)}\,\mathrm{d}s + (C_5+T)\|H\|_{L^\infty(0,T)} \\
&\le (C_6+T)\cdot R\cdot\left\|(u,v)-(\overline{u},\overline{v})\right\|_X
\end{aligned}$$

holds for every $t \in (0,T)$ and any combination of $(u,v) \in B$ and $(\overline{u},\overline{v}) \in B$. Analogously we treat the second component: for $t \in (0,T)$ we want to find suitable estimates for

$$\begin{aligned}
I_3(\cdot,t) &:= \psi_2(u,v)(\cdot,t) - \psi_2(\overline{u},\overline{v})(\cdot,t) \\
&= \int_0^t e^{(t-s)(\Delta-1)}\left(u(\cdot,s)v(\cdot,s) - \overline{u}(\cdot,s)\overline{v}(\cdot,s)\right)\,\mathrm{d}s \\
&= \int_0^t e^{(t-s)(\Delta-1)}h(\cdot,s)\,\mathrm{d}s
\end{aligned}$$

in $W^{1,q}(\Omega)$. Similarly to the previous step there are positive constants C_7, C_8 and C_9 such that

$$\begin{aligned}
\left\|I_3(\cdot,t)\right\|_{L^q(\Omega)} &\le \int_0^t \left\|e^{(t-s)\Delta}\left(h(\cdot,s)-H(s)\right)\right\|_{L^\infty(\Omega)}\,\mathrm{d}s + \int_0^t \left\|e^{(t-s)\Delta}H(s)\right\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\le C_7\cdot\left(R^2L(R)+R+RT\right)\cdot\left\|(u,v)-(\overline{u},\overline{v})\right\|_X
\end{aligned}$$

and

$$\left\|\nabla I_3(\cdot,t)\right\|_{L^q(\Omega)} \le C_8 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} e^{-\lambda(t-s)}\right) \left\|h(\cdot,s)\right\|_{L^q(\Omega)} \,\mathrm{d}s$$
$$\le C_9 \cdot R \cdot \left\|(u,v) - (\overline{u},\overline{v})\right\|_X$$

hold for every $t \in (0,T)$ any $(u,v), (\overline{u},\overline{v}) \in B$. Accordingly there is $C_{10} > 0$ with

$$\left\|\psi(u,v) - \psi(\overline{u},\overline{v})\right\|_X \le C_{10} \cdot \left(R^2 \cdot L(R) + R + RT\right) \cdot \left\|(u,v) - (\overline{u},\overline{v})\right\|_X$$

for any choices $(u,v) \in B$ and $(\overline{u},\overline{v}) \in B$. From this, by setting $(\overline{u},\overline{v}) = (0,0)$, we see that ψ actually defines a mapping $B \to B$ for sufficiently small R and T. Moreover, if R and T are even beneath a second set of thresholds, then we also have that ψ is a contraction.

This allows us to identify a fixed point $(u,v) \in B$ of ψ which at first is a weak solution of our system. However, standard parabolic regularity arguments provided by [47] prove that in fact these functions also enjoy all the regularity necessary to be considered classical solutions. Moreover, from the maximum principle we also see that any solution of (V.2) conserves nonnegativity of the initial data. The next question concerns the existence time. The crucial information on which T depends are $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{W^{1,q}(\Omega)}$, therefore we can extend our solution with new starting points $u(\cdot,T)$ and $v(\cdot,T)$ until either quantity becomes unbounded - giving the alternative in our statement.

We are left with one open task, that of proving uniqueness of solutions to (V.2). Assume that for some $T > 0$ we are given two solutions (u,v) and $(\overline{u},\overline{v})$ to (V.2) in $\Omega \times (0,T]$. We set

$$R := \max\left\{ \max_{g\in\{u,v,\overline{u},\overline{v}\}} \|g\|_{L^\infty(\Omega\times(0,T))}\, ,\, \max_{g\in\{v,\overline{v}\}} \|g\|_{L^\infty((0,T);W^{1,q}(\Omega))} \right\} < \infty$$

and choose some $L(R) \ge \|f\|_{L^\infty(\mathbb{R})}$ such that

$$\left|f(x) - f(y)\right| \le L(R)\, |x-y|$$

holds for any real numbers x and y. Together with h from before we are also going to use $w := u - \overline{u}$ and $z := v - \overline{v}$ for which we immediately see both

$$w_t = \Delta w - \nabla \cdot (uf(v)\nabla v - \overline{u}f(\overline{v})\nabla\overline{v}) - h$$

and

$$z_t = \Delta z - z + h$$

in $(0,T)$. Computing

$$\int_\Omega z_t^2 = \int_\Omega |\Delta z|^2 + 2\int_\Omega |\nabla z|^2 + \int_\Omega z^2 + \int_\Omega h^2 - 2\int_\Omega hz + 2\int_\Omega h\Delta z$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z|^2 = -\int_\Omega |\Delta z|^2 - \int_\Omega |\nabla z|^2 - \int_\Omega h\Delta z$$

as well as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z^2 = -\int_\Omega |\nabla z|^2 - \int_\Omega z^2 + \int_\Omega zh$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^2 = -\int_\Omega |\nabla w|^2 + \int_\Omega (uf(v)\nabla v - \overline{u}f(\overline{v})\nabla\overline{v}) \cdot \nabla w - \int_\Omega wh$$

which are valid in the entirety of $(0,T)$, we set

$$y := \int_\Omega |\nabla z|^2 + \int_\Omega z^2 + \int_\Omega w^2$$

in $(0,T)$. This in a first step gives us the identity

$$\int_\Omega z_t^2 + \frac{1}{2}y_t = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^2 + \int_\Omega hz_t$$

in $(0,T)$ where

$$\int_\Omega hz_t \le \frac{1}{2}\int_\Omega h^2 + \frac{1}{2}\int_\Omega z_t^2$$

in $(0,T)$, and further estimates will lead to an ordinary differential inequality: Lipschitz continuity of f, coupled with Young's inequality, shows that for some positive constant C_{11} we have

$$\begin{aligned}
\left|\int_\Omega (uf(v)\nabla v - \overline{u}f(\overline{v})\nabla\overline{v}) \cdot \nabla w\right| &\le \int_\Omega \left|u\,(f(v) - f(\overline{v}))\,\nabla v + uf(\overline{v})\nabla v - \overline{u}f(\overline{v})\nabla\overline{v}\right| \cdot |\nabla w| \\
&\le L(R)\int_\Omega |u\nabla v||z||\nabla w| + L(R)\int_\Omega |u\nabla v - \overline{u}\nabla\overline{v}||\nabla w| \\
&\le L(R)R\int_\Omega |z|\,|\nabla v|\,|\nabla w| + L(R)\int_\Omega |w|\,|\nabla v|\,|\nabla w| \\
&\quad + L(R)R\int_\Omega |\nabla z|\,|\nabla w| \\
&\le \frac{1}{2}\int_\Omega |\nabla w|^2 + C_{11}\int_\Omega z^2\,|\nabla v|^2 + C_{11}\int_\Omega w^2\,|\nabla v|^2 \\
&\quad + C_{11}\int_\Omega |\nabla z|^2
\end{aligned}$$

in $(0,T)$.

We want this right-hand side to consist only of two components: one that can be controlled by $\int_\Omega |\nabla w|^2$ and one that is a multiple of y.

Since $q > 2$, Hölder's inequality provides us with the estimate

$$\int_\Omega z^2 |\nabla v|^2 \leq \left(\int_\Omega |\nabla v|^q\right)^{\frac{2}{q}} \cdot \left(\int_\Omega z^{\frac{2q}{q-2}}\right)^{\frac{q-2}{q}} \leq (1+R) \cdot \|z\|^2_{L^{\frac{2q}{q-2}}(\Omega)}$$

in $(0,T)$ wherein the Gagliardo-Nirenberg interpolation inequality and Young's inequality let us find some $C_{12} > 0$ with

$$\|z\|^2_{L^{\frac{2q}{q-2}}(\Omega)} \leq C_{12} \int_\Omega z^2 + C_{12} \int_\Omega |\nabla z|^2 \leq C_{12} y$$

in $(0,T)$.

Again from Hölder's inequality we get

$$\int_\Omega w^2 |\nabla v|^2 \leq (1+R) \cdot \|w\|^2_{L^{\frac{2q}{q-2}}(\Omega)}$$

in $(0,T)$.

Using the Gagliardo-Nirenberg interpolation inequality once more, we firstly find some $a \in (0,1)$ and $C_{13} > 0$ such that

$$C_{11} \int_\Omega w^2 |\nabla v|^2 \leq C_{13} \|\nabla w\|^{2a}_{L^2(\Omega)} \cdot \|w\|^{2(1-a)}_{L^2(\Omega)} + C_{13}$$

holds in $(0,T)$ and Young's inequality even lets us find some positive constant C_{14} with

$$\begin{aligned} C_{11} \int_\Omega w^2 |\nabla v|^2 &\leq \frac{1}{4} \int_\Omega |\nabla w|^2 + C_{14} \int_\Omega w^2 \\ &\leq \frac{1}{4} \int_\Omega |\nabla w|^2 + C_{14} y \end{aligned}$$

in $(0,T)$.

For the other term we see

$$h = uv - \overline{uv} = (u - \overline{u})v - \overline{u}(v - \overline{v}) = vw + \overline{u}z$$

in $(0,T)$ and therefore

$$\int_\Omega h^2 \leq R^2 \int_\Omega (|w| + |z|)^2 \leq 2R^2 y$$

in $(0,T)$ so that all terms that could be or are positive can be estimated from above by $C_{15}y$ for some $C_{15} > 0$ – in conclusion this means

$$\begin{cases} y_t & \leq C_{15}y, \quad \text{in } (0,T), \\ y(0) & = 0 \end{cases}$$

and Grönwall's lemma then completes the proof by showing that for any such function $y \equiv 0$ holds in $(0,T)$. **q.e.d.**

We immediately see the relevance of this lemma to our problem in

Lemma V.2.2. *Given $\chi > 0$, assume that for some time $T > 0$ in lemma V.2.1 we have found a pair of functions $(u,v) \in \left(C^0\left(\overline{\Omega}\times[0,T)\right) \cap C^{2,1}\left(\overline{\Omega}\times(0,T)\right)\right)^2$ solving (V.2) in $\Omega\times(0,T)$ for the specific choice $f(s) = \frac{\chi}{s}$ for $s \geq \mathfrak{v}_0 e^{-T}$. Then this solution also solves the original system (V.1) in $\Omega\times(0,T)$.*

Proof. As we have seen in the similar setting of lemma I.3.2, due to the maximum principle the solution (u,v) to (V.2) satisfies

$$v(\cdot,t) \geq \mathfrak{v}_0 e^{-t} \text{ in } \Omega$$

for any $t \in (0,T)$. Therefore we immediately have $f(v) \equiv \frac{\chi}{v}$ in $\Omega\times(0,T)$. **q.e.d.**

To prove theorem V.1.1 we therefore assume that a local solution is given and show its boundedness and thereby extensibility to a global solution.

V.3 Initial estimates for u and v

As before, we will find some bound for $\left\|u(\cdot,t)\right\|_{L^1(\Omega)}$ as a starting point. However, in contrast to previous chapters the current setting (more precisely: the second equation in (V.1)) does not let us deduce boundedness of $\left\|v(\cdot,t)\right\|_{W^{1,\hat{q}}(\Omega)}$ for every $\hat{q} \in \left[1, \frac{n}{n-1}\right)$. Instead, a smaller step is needed:

Lemma V.3.1. *Given $\chi > 0$ and $T > 0$, there is $C > 0$ such that for any solution (u,v) to (V.1) in $\Omega\times(0,T)$*

$$\int_\Omega u(\cdot,t) \leq C \text{ and } \int_\Omega v(\cdot,t) \leq C$$

hold for every $t \in (0,T)$. Additionally these functions inherit the properties $u \geq 0$ and $v \geq 0$ in $\Omega\times(0,T)$.

Proof. We compute the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u + \int_\Omega v\right) = \int_\Omega \nabla\cdot\left(\nabla u - \chi\frac{u}{v}\nabla v + \nabla v\right) - \int_\Omega v + \int_\Omega B_1 + B_2$$

in $(0,T)$ and see that $\int_\Omega \nabla\cdot\left(\nabla u - \chi\frac{u}{v}\nabla v + \nabla v\right)$ vanishes due to the boundary conditions imposed on u and v. Since for some $C_1 > 0$

$$\int_\Omega B_1 + B_2 \le C_1$$

holds in $(0,T)$, the nonnegativity of v allows us to see upon integration

$$\begin{aligned}\int_\Omega u(\cdot,t) + \int_\Omega v(\cdot,t) &\le -\int_0^t\int_\Omega v + \int_\Omega u_0 + \int_\Omega v_0 + C_1\cdot t\\ &\le -\int_0^t\int_\Omega v + \int_\Omega u_0 + \int_\Omega v_0 + C_1\cdot T\end{aligned}$$

for every $t \in (0,T)$. The nonnegativity of u and v is a trivial consequence of the parabolic comparison principle and therefore, using the finiteness of T as well as the nonnegativity of all functions involved in this estimate, this results in our claim. **q.e.d.**

Lemma V.3.2. *Let $\chi > 0$ and $T > 0$ as well as a solution (u,v) to* (V.1) *in $\Omega\times(0,T)$ be given. Then for any real numbers p and r we can compute the identity*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p v^{-r} = &-p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 - r(\chi p + r + 1)\int_\Omega u^p v^{-r-2}|\nabla v|^2\\ &+ p(\chi(p-1)+2r)\int_\Omega u^{p-1}v^{-r-1}\nabla u\cdot\nabla v\\ &- p\int_\Omega u^p v^{-r+1} + r\int_\Omega u^p v^{-r} - r\int_\Omega u^{p+1}v^{-r}\\ &+ p\int_\Omega u^{p-1}v^{-r}B_1 - r\int_\Omega u^p v^{-r-1}B_2\end{aligned}$$

in $(0,T)$.

Proof. To achieve this, we consider the two natural halves of the integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p v^{-r} = p\int_\Omega u^{p-1}v^{-r}u_t - r\int_\Omega u^p v^{-r-1}v_t$$

in $(0,T)$. Using integration by parts, we can transform these summands into

$$\begin{aligned}
I_1 :=& \int_\Omega u^{p-1} v^{-r} u_t \\
=& \int_\Omega u^{p-1} v^{-r} \left(\Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right) - uv + B_1 \right) \\
=& -(p-1) \int_\Omega u^{p-2} v^{-r} |\nabla u|^2 + r \int_\Omega u^{p-1} v^{-r-1} \nabla u \cdot \nabla v + \chi (p-1) \int_\Omega u^{p-1} v^{-r-1} \nabla u \cdot \nabla v \\
& - \chi r \int_\Omega u^p v^{-r-2} |\nabla v|^2 - \int_\Omega u^p v^{-r+1} + \int_\Omega u^{p-1} v^{-r} B_1
\end{aligned}$$

and

$$\begin{aligned}
I_2 :=& \int_\Omega u^p v^{-r-1} v_t \\
=& \int_\Omega u^p v^{-r-1} \left(\Delta v - v + uv + B_2 \right) \\
=& -p \int_\Omega u^{p-1} v^{-r-1} \nabla u \cdot \nabla v + (r+1) \int_\Omega u^p v^{-r-2} |\nabla v|^2 - \int_\Omega u^p v^{-r} \\
& + \int_\Omega u^{p+1} v^{-r} + \int_\Omega u^p v^{-r-1} B_2
\end{aligned}$$

in $(0,T)$ respectively. Addition shows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p v^{-r} =& p I_1 - r I_2 \\
=& -p(p-1) \int_\Omega u^{p-2} v^{-r} |\nabla u|^2 - r(\chi p + r + 1) \int_\Omega u^p v^{-r-2} |\nabla v|^2 \\
& + p(\chi(p-1) + 2r) \int_\Omega u^{p-1} v^{-r-1} \nabla u \cdot \nabla v \\
& - p \int_\Omega u^p v^{-r+1} + r \int_\Omega u^p v^{-r} - r \int_\Omega u^{p+1} v^{-r} \\
& + p \int_\Omega u^{p-1} v^{-r} B_1 - r \int_\Omega u^p v^{-r-1} B_2
\end{aligned}$$

in $(0,T)$ as claimed. **q.e.d.**

Although this at first does not look significantly more fruitful than the previously discarded idea, from this result we actually find bounds for $\int_\Omega u^p v^{-r}$ relatively quickly whenever p and r are carefully chosen.

Lemma V.3.3. *Given* $\chi \in (0,1)$ *and* $T > 0$, *and with the definition*

$$r_\pm(\pi) := \frac{\pi - 1}{2} \left(1 \pm \sqrt{1 - \chi^2 \pi} \right) \tag{V.3}$$

for any $\pi \in \left(1, \frac{1}{\chi^2}\right)$, *choosing any* $p \in \left(1, \frac{1}{\chi^2}\right)$ *and* $r \in (r_-(p), r_+(p))$ *there is some* $C > 0$ *such that*

$$\int_\Omega u^p(\cdot,t)v^{-r}(\cdot,t) \ \leq C$$

holds in $(0,T)$ *whenever* (u,v) *is a classical solution to* (V.1) *in* $\Omega \times (0,T)$.

Proof Since clearly $r > 0$, the signs of most of the terms on the right-hand side in the estimate in the previous lemma are known. This is especially helpful for the terms containing $|\nabla u|^2$ and $|\nabla v|^2$. There is another integral featuring both of the gradients and in order to see that it is controlled by those two terms we estimate

$$\begin{aligned}&\left|p(\chi(p-1)+2r)\int_\Omega u^{p-1}v^{-r-1}\nabla u\cdot\nabla v\right|\\ &\quad\leq p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 + \frac{p\left(\chi(p-1)+2r\right)^2}{4(p-1)}\int_\Omega u^p v^{-r-2}|\nabla v|^2\end{aligned}$$

in $(0,T)$ via Young's inequality, using the largest possible coefficient for the integral featuring ∇u.

Young's inequality also allows us to find a positive constant C_1 that depends on $\left\|\frac{1}{v}\right\|_{L^\infty(\Omega\times(0,T))}$ as well as $\|B_1\|_{L^\infty(\Omega\times(0,T))}$ and for which we have

$$p\int_\Omega u^{p-1}v^{-r}B_1 \leq \int_\Omega u^p v^{-r} + C_1$$

in $(0,T)$.

Utilising the knowledge regarding the signs of some terms, we therefore find a constant $C_2 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p v^{-r} \leq C_2 + C_2\int_\Omega u^p v^{-r} + A(p,r)\int_\Omega u^p v^{-r-2}|\nabla v|^2$$

holds in $(0,T)$ and for

$$A(p,r) := -r\left(\chi p + r + 1\right) + \frac{p\left(\chi(p-1)+2r\right)^2}{4(p-1)}.$$

To complete the proof, we now need to show $A(p,r) \leq 0$ and this is where our first restriction for χ stems from. We see

$$\begin{aligned}4(p-1)A(p,r) &= -4r(p-1)\left(\chi p + r + 1\right) + p\left(\chi(p-1)+2r\right)^2\\ &= -4r(p-1)\chi p - 4(p-1)r(r+1) + \chi^2 p(p-1)^2 + 4\chi p(p-1)r + 4pr^2\\ &= 4pr^2 + \chi^2 p(p-1)^2 - 4(p-1)r(r+1)\\ &= 4\left(r^2 - (p-1)r + \frac{\chi^2}{4}p(p-1)^2\right)\end{aligned}$$

and this quadratic form in r is negative between the designated constraints $r_\pm(p)$. For some $C_3 > 0$ we therefore conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p v^{-r} \le C_3 + C_3 \int_\Omega u^p v^{-r}$$

in $(0,T)$, and together with the boundedness of $\int_\Omega u_0^p v_0^{-r}$ and Grönwall's lemma this concludes our proof. **q.e.d.**

This result clearly couples boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ to suitable bounds for v; the question then is whether the previously attained boundedness of $\|v(\cdot,t)\|_{L^1(\Omega)}$ suffices. We begin to explore this path with

Lemma V.3.4. *Let $\chi \in (0,1)$, $T > 0$ and $p \in \left(1, \frac{1}{\chi^2}\right)$ be given. Then for any combination of $p_0 \in \left(p, \frac{1}{\chi^2}\right)$ and $r \in (r_-(p_0), r_+(p_0))$ (where the interval is again given by the definition* (V.3) *in the previous lemma) we can find some $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C \|v(\cdot,t)\|^{\frac{r}{p_0}}_{L^{\frac{rp}{p_0-p}}(\Omega)}$$

holds for every $t \in (0,T)$ and any classical solution (u,v) to (V.1) *in $\Omega \times (0,T)$.*

Proof. By Hölder's inequality we see that

$$\begin{aligned}\int_\Omega u^p(\cdot,t) &= \int_\Omega u^p(\cdot,t) v^{-\frac{rp}{p_0}}(\cdot,t) v^{\frac{rp}{p_0}}(\cdot,t) \\ &\le \left(\int_\Omega u^{p_0}(\cdot,t) v^{-r}(\cdot,t)\right)^{\frac{p}{p_0}} \left(\int_\Omega v^{\frac{qp_0}{p_0-p}}(\cdot,t)\right)^{1-\frac{p}{p_0}}\end{aligned}$$

holds for every $t \in (0,T)$ and by lemma V.3.3 we therefore find $C_1 > 0$ such that

$$\begin{aligned}\int_\Omega u^p(\cdot,t) &\le C_1 \left(\int_\Omega v^{\frac{rp}{p_0-p}}(\cdot,t)\right)^{\frac{p_0-p}{p_0}} \\ &= C_1 \|v(\cdot,t)\|^{\frac{rp}{p_0}}_{L^{\frac{rp}{p_0-p}}(\Omega)}\end{aligned}$$

holds for every $t \in (0,T)$. **q.e.d.**

As indicated above, we would like to combine this result with the boundedness of $v(\cdot,t)$ in $L^1(\Omega)$. For $n \ge 3$ this means that we have to restrict χ further:

Lemma V.3.5. *Let $\chi \in (0,1)$ with*

$$\chi < \begin{cases} \sqrt{\frac{2}{n}} & \textit{for } n \in \{2,3\}, \\ \frac{2}{n} & \textit{for } n \ge 4 \end{cases}$$

be given. Then (using once more the definition (V.3)*) there are some* $p_0 \in \left(\frac{n}{2}, \frac{1}{\chi^2}\right)$ *as well as* $r \in (r_-(p_0), r_+(p_0))$ *and* $p \in \left(\frac{n}{2}, p_0\right)$ *with*

$$\frac{pr}{p_0 - p} \le 1.$$

Proof. We immediately see that our claim is equivalent to $p \le \frac{p_0}{1+r}$, meaning that we have to prove the existence of eligible p_0 and r such that $\frac{n}{2} < \frac{p_0}{1+r}$ holds. In a first step we instead consider $\frac{n}{2} < \frac{p_0}{1+r_-(p_0)}$ before fixing r close enough to $r_-(p_0)$ for the relation to remain intact. Rearranging yields the equivalent condition

$$\sqrt{1 - p_0\chi^2} > \frac{(n-4)p_0 + n}{n(p_0 - 1)}$$

which we deal with differently based on the dimension n.
In the case $n = 2$ the term on the right is negative for any $p_0 > 1$ which remains true for $n = 3$ and any $p_0 \in \left(\frac{n}{2}, \frac{1}{\chi^2}\right)$ with $p_0 < n$. Therefore, in those cases, nothing more needs to be done. In higher dimensions, namely $n \ge 4$, we choose $p_0 := n - 1 > \frac{n}{2}$ and see

$$\begin{aligned}\frac{(n-4)p_0 + n}{n(p_0 - 1)} &= \frac{(n-4)(n-1) + n}{n(n-2)} \\ &= \frac{n^2 - 4n + 4}{n(n-2)} \\ &= 1 - \frac{2}{n}.\end{aligned}$$

Taking the square then results in the condition

$$1 - (n-1)\chi^2 > 1 - \frac{4}{n} + \frac{4}{n^2}$$

and this is equivalent to

$$\chi^2 < \frac{1}{n-1}\left(\frac{4}{n} - \frac{4}{n^2}\right) = \frac{4}{n^2},$$

ergo our constraint for χ.

q.e.d.

Remark. *Already for* $n = 4$ *we see that the condition* $\chi < \sqrt{\frac{2}{n}}$ *is not sufficient in higher dimensions: With the right-hand side reduced to* $\frac{1}{p_0 - 1}$*, the necessary condition in the proof becomes*

$$\chi^2 < \frac{p_0 - 2}{(p_0 - 1)^2}.$$

Since clearly $p_0 - 2 < \frac{1}{2}(p_0 - 1)^2$ *for every* $p_0 > 2$*, a further restriction for* χ *is inevitable.*

Corollary V.3.6. *Let* $\chi \in (0,1)$ *with*

$$\chi < \begin{cases} \sqrt{\frac{2}{n}} & \text{for } n \in \{2,3\}, \\ \frac{2}{n} & \text{for } n \geq 4 \end{cases}$$

and some $T > 0$ *be given. Then for* p_0*,* p *and* r *from lemma V.3.5 we can find some* $C > 0$ *such that*

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \leq C$$

holds for every $t \in (0,T)$ *and any classical solution* (u,v) *to* (V.1) *in* $\Omega \times (0,T)$*.*

Proof. For the quantities from lemma V.3.5 we see in lemma V.3.4 that there are constants $C_1 > 0$ and $\alpha > 0$ with

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \leq C_1 \sup_{t\in(0,T)} \left\|v(\cdot,t)\right\|^{\alpha}_{L^1(\Omega)}$$

for every $t \in (0,T)$. Combining this with lemma V.3.1 completes the proof. **q.e.d.**

As in previous chapters, this information concerning the boundedness of $u(\cdot,t)$ in $L^p(\Omega)$ for some $p > \frac{n}{2}$ will be used to deduce boundedness even in $L^\infty(\Omega)$. However, before we can achieve this, we also need estimates for the other component of our solution (in addition to such information being needed to prove $T_{\max} = \infty$ in view of our extensibility criterion).

Lemma V.3.7. *Let* $\chi \in (0,1)$ *with*

$$\chi < \begin{cases} \sqrt{\frac{2}{n}} & \text{for } n \in \{2,3\}, \\ \frac{2}{n} & \text{for } n \geq 4 \end{cases}$$

and some $T > 0$ *be given. Then for any choices* $p \in \left(\frac{n}{2}, n\right)$ *and* $\kappa \in (n,\infty)$ *with* $\kappa \leq q$ *and* $\kappa < \frac{np}{n-p}$ *we can find some constant* $C > 0$ *such that*

$$\left\|v(\cdot,t)\right\|_{W^{1,\kappa}(\Omega)} \leq C + C\left\|u\right\|^{1+\frac{n\kappa}{\kappa-n}}_{L^\infty((0,T));L^p(\Omega))}$$

holds for every $t \in (0,T)$ *and any classical solution* (u,v) *to* (V.1) *in* $\Omega \times (0,T)$*. Additionally, if we already have* $\sup_{t\in(0,T)} \left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} < \infty$*, then we also find some* $C' > 0$ *with*

$$\left\|v(\cdot,t)\right\|_{W^{1,\infty}(\Omega)} \leq C'$$

for every $t \in (0,T)$*.*

Proof. We begin by employing the Gagliardo-Nirenberg interpolation inequality, more precisely the second estimate in lemma I.3.8, to obtain

$$\left\|v(\cdot,t)\right\|_{L^\infty(\Omega)} \leq C_1 \left\|v(\cdot,t)\right\|_{W^{1,\kappa}(\Omega)}^a \cdot \left\|v(\cdot,t)\right\|_{L^1(\Omega)}^{1-a}$$

for every $t \in (0,T)$ with $a := \frac{n\kappa}{n\kappa+\kappa-n}$ and some constant $C_1 > 0$. Lemma V.3.1 then provides us with some $C_2 > 0$ such that

$$\left\|v(\cdot,t)\right\|_{L^\infty(\Omega)} \leq C_2 \left\|v(\cdot,t)\right\|_{W^{1,\kappa}(\Omega)}^a$$

for all times $t \in (0,T)$. Representing v via

$$v(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)} \left(u(\cdot,s)v(\cdot,s) + B_2(\cdot,s)\right) \, \mathrm{d}s$$

for $t \in (0,T)$, we once more use the estimates in lemma I.3.3. With $\lambda > 0$ taken from that lemma and utilising $-\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p} - \frac{1}{\kappa}\right) > -1$ as well as the subsequent observation

$$\int_0^\infty \left(1 + \tau^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{\kappa}\right)}\right) e^{-\lambda\tau} \, \mathrm{d}\tau < \infty,$$

they provide us with some positive constants C_3 and C_4 such that

$$\begin{aligned}\int_0^t \left\|e^{(t-s)(\Delta-1)}u(\cdot,s)v(\cdot,s)\right\|_{W^{1,\kappa}(\Omega)} \, \mathrm{d}s \leq & C_3 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{\kappa}\right)}\right) e^{-\lambda(t-s)} \cdot \\ & \left\|u(\cdot,s)v(\cdot,s)\right\|_{L^p(\Omega)} \, \mathrm{d}s \\ \leq & C_4 \sup_{s\in(0,t)} \left\|u(\cdot,s)v(\cdot,s)\right\|_{L^p(\Omega)}\end{aligned}$$

holds for every $t \in (0,T)$. Together with the previous estimate this results in

$$\int_0^t \left\|e^{(t-s)(\Delta-1)}u(\cdot,s)v(\cdot,s)\right\|_{W^{1,\kappa}(\Omega)} \, \mathrm{d}s \leq C_5 \sup_{s\in(0,t)} \left(\left\|u(\cdot,s)\right\|_{L^p(\Omega)} \cdot \left\|v(\cdot,s)\right\|_{W^{1,\kappa}(\Omega)}^a\right)$$

for $C_5 := C_2 \cdot C_4$ and every $t \in (0,T)$. Since $v_0 \in W^{1,q}(\Omega)$ and $\kappa \leq q$, there is some $C_6 > 0$ with

$$\left\|e^{t(\Delta-1)}v_0\right\|_{W^{1,\kappa}(\Omega)} \leq C_6$$

for every $t \in (0,T)$ and the estimates in lemma I.3.3 give us some $C_7 > 0$ such that

$$\int_0^t \left\|e^{(t-s)(\Delta-1)}B_2(\cdot,s)\right\|_{W^{1,\kappa}(\Omega)} \, \mathrm{d}s \leq \int_0^T \left\|e^{(t-s)(\Delta-1)}B_2(\cdot,s)\right\|_{W^{1,\kappa}(\Omega)} \, \mathrm{d}s \leq C_7$$

holds for every $t \in (0,T)$. From the representation formula for v we therefore find that with some constant $C_8 > 0$

$$\left\|v(\cdot,t)\right\|_{W^{1,\kappa}(\Omega)} \leq C_8 + C_8 \sup_{s\in(0,t)} \left(\left\|u(\cdot,s)\right\|_{L^p(\Omega)} \cdot \left\|v(\cdot,s)\right\|_{W^{1,\kappa}(\Omega)}^a\right)$$

holds for every $t \in (0,T)$. This means that for

$$M_t := \sup_{s\in(0,t)} \left\|v(\cdot,s)\right\|_{W^{1,\kappa}(\Omega)}$$

with $t \in (0,T]$ we have

$$\begin{aligned} M_t \leq& C_8 + C_8 \cdot \|u\|_{L^\infty((0,t);L^p(\Omega))} \cdot M_T^a \\ \leq& C_8 + C_8 \cdot \|u\|_{L^\infty((0,T);L^p(\Omega))} \cdot M_T^a. \end{aligned}$$

Now, we either have $\|u\|_{L^\infty((0,T);L^p(\Omega))} \cdot M_T^a < 1$ which immediately give us $M_T < 2C_8$; or, conversely, from

$$M_T \leq 2C_8 \|u\|_{L^\infty((0,T);L^p(\Omega))} \cdot M_T^a$$

we can deduce

$$M_T \leq (2C_8)^{\frac{1}{1-a}} \cdot \|u\|_{L^\infty((0,T);L^p(\Omega))}^{\frac{1}{1-a}}.$$

Together these two alternatives prove our first claim.
For the second claim we swiftly find $C_9 > 0$ and $C_{10} > 0$ with

$$\left\|e^{t(\Delta-1)}v_0\right\|_{W^{1,\infty}(\Omega)} \leq C_9$$

and

$$\left\|\int_0^t e^{(t-s)(\Delta-1)}B_2(\cdot,s)\,\mathrm{d}s\right\|_{W^{1,\infty}(\Omega)} \leq C_{10}$$

for every $t \in (0,T)$ and from the estimates in lemma I.3.3 and the Gagliardo-Nirenberg inequality in lemma I.3.8 as well as lemma V.3.1 we get a series C_{11}, C_{12}, C_{13}, C_{14} of positive constants with

$$\begin{aligned} &\int_0^t \left\|e^{(t-s)(\Delta-1)}u(\cdot,s)v(\cdot,s)\right\|_{W^{1,\infty}(\Omega)}\,\mathrm{d}s \\ &\leq C_{11}\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-\lambda(t-s)} \cdot \left\|u(\cdot,s)v(\cdot,s)\right\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\leq C_{12}\sup_{s\in(0,T)} \left\|v(\cdot,s)\right\|_{L^\infty(\Omega)} \\ &\leq C_{13}\sup_{s\in(0,T)} \left(\left\|v(\cdot,s)\right\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n-1}} \cdot \left\|v(\cdot,s)\right\|_{L^1(\Omega)}^{\frac{1}{n+1}}\right) \\ &\leq C_{14}\sup_{s\in(0,T)} \left\|v(\cdot,s)\right\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n-1}} \end{aligned}$$

for every $t \in (0,T)$. This in turn leads to the detection of some $C_{15} > 0$ with

$$\sup_{s\in(0,T)} \left\|v(\cdot,s)\right\|_{W^{1,\infty}(\Omega)} \leq C_{15} + C_{15}\sup_{s\in(0,T)} \left\|v(\cdot,s)\right\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+1}},$$

ensuring

$$\sup_{s\in(0,T)} \left\|v(\cdot,s)\right\|_{W^{1,\infty}(\Omega)} \leq \max\left\{1,(2C_{15})^{n+1}\right\}$$

which completes our proof. **q.e.d.**

V.4 Higher regularity for u and the proof of theorem V.1.1

Once more we see that the global existence of our solution is tied to $\left\|u(\cdot,t)\right\|_{L^p(\Omega)}$ for some finite p. Unlike before, the structure of the first differential equation in this setting does not allow us to simply use lemma A.1 from [89]. Therefore, we derive such bounds manually.

Lemma V.4.1. *Let* $\chi \in (0,1)$ *with*

$$\chi < \begin{cases} \sqrt{\frac{2}{n}} & \text{for } n \in \{2,3\}, \\ \frac{2}{n} & \text{for } n \geq 4 \end{cases}$$

and $T > 0$ *be given. Then for every* $K > 0$ *there is some* $C > 0$ *with the following property: if there is* $p > \frac{n}{2}$ *such that for some solution* (u,v) *to* (V.1) *in* $\Omega \times (0,\infty)$

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \leq K$$

holds for every $t \in (0,T)$*, then we even have*

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \leq C$$

for every $t \in (0,T)$*.*

Proof. Without loss of generality we can assume $p < n$. Due to $p > \frac{n}{2}$ we have $n < \frac{np}{n-p}$ and accordingly we can fix $\theta \in \left(n, \frac{np}{n-p}\right)$ with $\theta < q$ and $\varphi \in \left(\theta, \frac{np}{n-p}\right)$ with $\varphi \leq q$. As seen in the proof of lemma V.2.2, due to $u \geq 0$ and $B_2 \geq 0$, the comparison principle gives us

$$v(\cdot,t) \geq \left(\inf_{\overline{\Omega}} v_0\right) e^{-t} \geq \left(\inf_{\overline{\Omega}} v_0\right) e^{-T_{\max}}$$

for every $t \in (0,T)$.

Using this and Hölder's inequality, for $\varphi' := \frac{\theta\varphi}{\varphi-\theta}$ we see

$$\left\|\frac{u(\cdot,t)}{v(\cdot,t)}\nabla v(\cdot,t)\right\|_{L^\theta(\Omega)} \leq C_1 \left\|u(\cdot,t)\right\|_{L^{\varphi'}(\Omega)} \cdot \left\|\nabla v(\cdot,t)\right\|_{L^\varphi(\Omega)}$$

for some $C_1 > 0$ and every $t \in (0,T)$. With lemma V.3.7 applied to $q > n$ we also find $C_2 > 0$, $C_3 > 0$ and some $\alpha > 1$ such that

$$\left\|\nabla v(\cdot,t)\right\|_{L^\varphi(\Omega)} \leq C_2 + C_2 \sup_{s\in(0,t)} \left\|u(\cdot,s)\right\|^\alpha_{L^p(\Omega)} \leq C_3$$

holds for every $t \in (0,T)$. Observing $\varphi' > p$, for some $C_4 > 0$ we estimate

$$\left\|u(\cdot,t)\right\|_{L^{\varphi'}(\Omega)} \le \left(\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)}^{\varphi'-p} \int_\Omega u(\cdot,t)^p\right)^{\frac{1}{\varphi'}} \le C_4 \left\|u(\cdot,t)\right\|_{L^\infty(\Omega)}^{1-\frac{p}{\varphi'}}$$

for every $t \in (0,T)$.
Together with lemma I.3.3 this gives us positive constants C_5, C_6 and λ with

$$\begin{aligned}
&\int_0^t \left\| e^{(t-s)\Delta}\nabla \cdot \left(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\right)\right\|_{L^\infty(\Omega)} \,\mathrm{d}s \\
&\quad \le C_5 \int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2\theta}}\right) e^{-\lambda(t-s)} \left\|\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\right\|_{L^\theta(\Omega)} \,\mathrm{d}s \\
&\quad \le C_6 \sup_{s\in(0,t)} \left\|u(\cdot,s)\right\|_{L^\infty(\Omega)}^{1-\frac{p}{\varphi'}}
\end{aligned}$$

for every $t \in (0,T)$ wherein the condition $-\frac{1}{2}-\frac{n}{2\theta} > -1$ is crucial. Furthermore, there are $C_7 > 0$ and $C_8 > 0$ such that

$$\begin{aligned}
\int_0^t \left\|e^{(t-s)\Delta} B_1(\cdot,s)\right\|_{L^\infty(\Omega)} \,\mathrm{d}s &\le \int_0^t \left\|B_1(\cdot,s)\right\|_{L^\infty(\Omega)} \,\mathrm{d}s \\
&\le \int_0^{T_{\max}} \left\|B_1(\cdot,s)\right\|_{L^\infty(\Omega)} \,\mathrm{d}s \\
&\le C_7
\end{aligned}$$

and

$$\left\|e^{t\Delta}u_0\right\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \le C_8$$

hold for every $t \in (0,T)$. Utilising the representation and obvious estimate

$$\begin{aligned}
0 \le u(\cdot,t) &= e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla \cdot \left(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\right) \,\mathrm{d}s \\
&\quad - \int_0^t e^{(t-s)\Delta} u(\cdot,s)v(\cdot,s) \,\mathrm{d}s + \int_0^t e^{(t-s)\Delta} B_1(\cdot,s) \,\mathrm{d}s \\
&\le e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla \cdot \left(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\right) \,\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)\Delta} B_1(\cdot,s) \,\mathrm{d}s
\end{aligned}$$

with $t \in (0,T)$, we set $M := \sup_{t\in(0,T)} \left\|u(\cdot,t)\right\|_{L^\infty(\Omega)}$ and see that there is some $C_9 > 0$ such that

$$M \le C_9 + C_9 M^{1-\frac{p}{\varphi'}}$$

holds. Using the same argument as in the previous proof this results in

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \leq \max\left\{1, (2C_9)^{\frac{\varphi'}{p}}\right\}$$

for every $t \in (0,T)$. **q.e.d.**

We now have all the tools necessary to verify our central statement:

Proof of theorem V.1.1. Let us first fix some $\delta > 0$ and $\eta := \inf_{\overline{\Omega}} v_0 > 0$. Choosing the function f_1 in lemma V.2.1 such that $f_1 = \frac{\chi}{\cdot}$ in $\left(\eta e^{-1}, \infty\right)$, we find some $T_1 \in (0,1)$ and a classical solution (u_1, v_1) to (V.2) in $\Omega \times (0, T_1)$ which – according to lemma V.2.2 – simultaneously solves (V.1). The uniqueness of solutions to (V.2) allows us to state the following: if for some $0 < T' < T''$ solutions (u', v') and (u'', v'') to (V.1) in $\Omega \times (0, T')$ and $\Omega \times (0, T'')$ respectively are given, then we have $u''|_{\Omega\times(0,T')} = u'$ and $v''|_{\Omega\times(0,T')} = v'$. We now define two sequences $(T_k)_{k\in\mathbb{N}} \subset (0,\infty)$ and $(f_k)_{k\in\mathbb{N}} \subset C^{1+\delta}(\mathbb{R})$ such that firstly $T_k \in [k-1, k)$ for every $k \in \mathbb{N}$, which immediately gives us $T_{k+1} > T_k$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty} T_k = \infty$. For the other sequence we pick any suitable functions such that

$$f_k(s) = \begin{cases} \chi \frac{\eta e^{-k}}{2}, & s \leq \frac{1}{2}\eta e^{-k}, \\ \chi \frac{1}{s}, & s \geq \eta e^{-k} \end{cases}$$

holds. Now we either find that for every $k \in \mathbb{N}$ our solution to (V.1) can be extended to a solution in $\Omega \times \left(0, T'_k\right)$ for some $T'_k \in (T_k, T_{k+1})$ ensuring that it actually exists globally. Alternatively, if for some $k \in \mathbb{N}$ we were to find that the corresponding maximum existence time in lemma V.2 is finite and less than T_k, then this would lead to a contradiction to lemmata V.4.1 and V.3.7.

q.e.d.

Chapter VI

Energy solutions for eventually vanishing diffusion in a subcritical setting

VI.1 Introduction and main result

In chapter III we have seen that $D(0) = 0$ can lead to difficulties in the detection of solutions. On the other hand, $D(u) \approx u^{m-1}$ for some $m > 1$ and every $u \geq 0$ ensured that positive lower bounds exist in $[\delta, \infty)$ for every $\delta > 0$. Here we want to discuss the effects of $\lim_{u\to\infty} D(u) = 0$, which could stem from $D(u) = e^{-u}$ for example, on the solvability of our systems. In chapters II and III we already encountered the criticality of $\frac{2}{n}$ regarding the exponent describing the relation between the diffusion and sensitivity functions and this chapter is no exception: again we consider the subcritical case, meaning

$$\frac{S(u)}{D(u)} \leq C_{SD} u^{\alpha}$$

for some $C_{SD} > 0$, $\alpha \in \left(0, \frac{2}{n}\right)$ and every $u \geq 0$. Under the additional assumption that D be bounded from above by some constant value, [84] has found global weak solutions with several properties concerning their regularity and some nonincreasing energy. Without such a requirement for D and for the critical case that $\frac{S(u)}{D(u)} \approx u^{\frac{2}{n}}$ (we refer to chapter II), but demanding instead that the initial mass $\int_\Omega u_0$ be small, [93] detected global very weak energy solutions; further generalising previously examined concepts, the most striking feature of such solutions (properly introduced in our definition VI.2.1) is that instead of the 'difficult' function u the integrals central to this concept feature $\chi(u)$ where χ enjoys some favourable properties. It is the purpose of this chapter to examine a more general case allowing for D to be unbounded from above. Our assumptions are as follows: let $n \geq 3$, $C_{SD} > 0$ and $\alpha \in \left(0, \frac{2}{n}\right)$ as well as two functions $D \in C^2([0,\infty))$ and $S \in C^2([0,\infty))$ be given such that the following list of conditions holds:

For the diffusion we demand

$$D(u) > 0 \tag{VI.1}$$

for every $u \geq 0$, the sensitivity function S fulfil

$$S(0) = 0 < S(u) \tag{VI.2}$$

for every $u > 0$ and together we want

$$\frac{S(u)}{D(u)} \leq C_{SD} u^{\alpha} \tag{VI.3}$$

to hold for every $u \geq 0$.

With these parameters and functions we next fix some $\Omega \subset \mathbb{R}^n$ as well as two more functions $u_0 \in W^{1,\infty}(\Omega)$ with $u_0 > 0$ in $\overline{\Omega}$ and nonnegative $v_0 \in W^{2,\infty}(\Omega)$. In order to derive results for the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty), \\ D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, \ v(\cdot,0) = v_0 & \text{in } \Omega \times (0,\infty) \end{cases} \tag{VI.4}$$

we start, as in chapter III, by discussing a similar system in which the potential degeneracy of D as $u \to \infty$ has been eliminated. To this end we define $D_\varepsilon(s) := \frac{D(s)+\varepsilon}{1+\varepsilon D(s)}$ and $S_\varepsilon := \frac{S(s)}{1+\varepsilon D(s)}$ for $\varepsilon \in (0,1)$ and $s \geq 0$, enabling us to detect global solutions $(u_\varepsilon, v_\varepsilon)$ to the approximating systems

$$\begin{cases} u_t = \nabla \cdot (D_\varepsilon(u)\nabla u - S_\varepsilon(u)\nabla v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty), \\ D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, \ v(\cdot,0) = v_0 & \text{in } \Omega \times (0,\infty) \end{cases} \tag{VI.5}$$

rather quickly. These functions do in fact converge to some kind of solution to the original system. The introduction of a suitable solution concept and the detection of such a solution are the purpose of this chapter. Our result reads as follows:

Theorem VI.1.1. *Let $n \geq 3$, some constants $C_{SD} > 0$ and $\alpha \in \left(0, \frac{2}{n}\right)$ as well as two functions $D \in C^2([0,\infty))$ and $S \in C^2([0,\infty))$ with* (VI.1)*,* (VI.2) *and* (VI.3) *for every $u \geq 0$ be given. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let two more functions $u_0 \in W^{1,\infty}(\Omega)$ with $u_0 > 0$ in $\overline{\Omega}$ and nonnegative $v_0 \in W^{2,\infty}(\Omega)$ be given. Then the system* (VI.4) *possesses a global generalised solution in the sense of the upcoming definition VI.2.1 which can be found via a limit process.*

Remark. *For the entirety of this chapter, let us assume that we have some* $n \geq 3$*, some constants* $C_{SD} > 0$ *and* $\alpha \in \left(0, \frac{2}{n}\right)$ *as well as two functions* $D \in C^2([0,\infty))$ *and* $S \in C^2([0,\infty))$ *with* (VI.1)*,* (VI.2) *and* (VI.3) *for every* $u \geq 0$*. Note that the restriction* $\alpha > 0$ *actually does not reduce the set of eligible functions. Furthermore, let some bounded domain* $\Omega \subset \mathbb{R}^n$*,* $u_0 \in W^{1,\infty}(\Omega)$ *with* $u_0 > 0$ *in* $\overline{\Omega}$ *and some nonnegative* $v_0 \in W^{2,\infty}(\Omega)$ *be given.*

VI.2 Introduction of very weak energy solutions

In previous chapters we have already seen an example for a different concept of solutions: roughly speaking, by using integration by parts, for suitable test functions a differential equation can be transformed into an integral equation. The set in which we look for solutions to this new problem is larger which increases the chance of finding some. On the other hand we always want to ensure the compatibility with the classical solution concept: if such a weak solution is sufficiently smooth, then it should also solve the differential equation in the original, classical sense.

Definition VI.2.1. *A pair*

$$(u,v) \in L^\infty_{loc}\left([0,\infty); L^1(\Omega)\right) \times L^2_{loc}\left([0,\infty); W^{1,2}(\Omega)\right)$$

with $u > 0$ *and* $v \geq 0$ *almost everywhere in* $\Omega \times (0,\infty)$ *is called a global very weak energy solution of* (VI.4) *if all of the following conditions are met: defining*

$$h(s) := \frac{S(s)}{D(s)}$$

for $s \geq 0$*,*

$$G(s) := \int_1^s \int_1^\sigma \frac{\mathrm{d}\sigma \mathrm{d}\tau}{h(\tau)}$$

for $s > 0$ *as well as*

$$K_\Sigma := \frac{h(1)}{h(1)+1} \in (0,1)$$

and

$$\Sigma(s) := \begin{cases} K_\Sigma \cdot \sqrt{\frac{S(s)}{S(s)+1}} & \text{for } s \in [0,1], \\ \frac{h(s)}{h(s)+1} \cdot \sqrt{\frac{S(s)}{S(s)+1}} & \text{for } s > 1, \end{cases}$$

we demand

$$u(\cdot,t)v(\cdot,t) \in L^1(\Omega) \quad \text{for almost every } t > 0$$

and

$$G\left(u(\cdot,t)\right) \in L^1(\Omega) \quad \text{for almost every } t > 0$$

as well as

$$v_t \in L^2_{loc}\left(\overline{\Omega} \times [0,\infty)\right)$$

and

$$\frac{\Sigma(u)}{h(u)}\nabla u \in L^2_{loc}\left(\overline{\Omega} \times [0,\infty); \mathbb{R}^n\right).$$

With

$$\mathcal{F}(\varphi,\psi) := \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{2}\int_\Omega \psi^2 - \int_\Omega \varphi\psi + \int_\Omega G(\varphi)$$

for any two functions $\varphi \in L^1(\Omega)$ *with* $\varphi > 0$ *almost everywhere in* Ω *and* $\psi \in W^{1,2}(\Omega)$ *as well as*

$$\mathcal{D}(t) := \int_\Omega v_t^2(\cdot,t) + \int_\Omega \left| \Sigma(u(\cdot,t))\frac{D(u(\cdot,t))}{\sqrt{h(u(\cdot,t))}}\nabla u(\cdot,t) - \sqrt{\Sigma(u(\cdot,t))}\nabla v(\cdot,t)\right|^2$$

for $t > 0$ *such functions additionally need to fulfil*

$$\mathcal{F}\left(u(\cdot,t), v(\cdot,t)\right) + \int_0^t \mathcal{D}(s)\, \mathrm{d}s \le \mathcal{F}\left(u_0, v_0\right)$$

for almost every $t > 0$. *Alongside*

$$v(\cdot,0) = v_0 \quad \textit{in } \Omega,$$

for any $\varphi \in C_0^\infty\left(\overline{\Omega} \times [0,\infty)\right)$ *the second component has to solve*

$$\int_0^\infty\int_\Omega v_t\varphi = -\int_0^\infty\int_\Omega \nabla v\cdot\nabla\varphi - \int_0^\infty\int_\Omega v\varphi + \int_0^\infty\int_\Omega u\varphi.$$

Furthermore, for almost every $t > 0$ *we demand*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0$$

to be true. Lastly, we need that for every $\chi \in C^\infty([0,\infty))$ *with* $\chi' \ge 0$ *in* $[0,\infty)$, $\operatorname{supp}\chi' \subset\subset [0,\infty)$ *and* $\chi'' \le 0$ *as well as every nonnegative* $\varphi \in C_0^\infty\left(\overline{\Omega}\times[0,\infty)\right)$ *the inequality*

$$\begin{aligned}
-\int_0^\infty\int_\Omega \chi(u)\varphi_t - \int_\Omega \chi(u_0)\varphi(\cdot,0)& \\
\ge -\int_0^\infty\int_\Omega \chi''(u)D(u)\,|\nabla u|^2\,\varphi &+ \int_0^\infty\int_\Omega \chi''(u)S(u)\,(\nabla u\cdot\nabla v)\,\varphi \\
-\int_0^\infty\int_\Omega \chi'(u)D(u)\nabla u\cdot\nabla\varphi &+ \int_0^\infty\int_\Omega \chi'(u)S(u)\nabla v\cdot\nabla\varphi
\end{aligned}$$

holds.

Remark. *Our regularity assumptions for v ensure that the pointwise identity*

$$v(\cdot, 0) = v_0 \quad \text{in } \Omega$$

actually is meaningful; from the integrability of v_t we find that $v \in C^0\left([0,\infty); L^2(\Omega)\right)$.

It is not immediately apparent that such functions in a sense really generalise the classical concept of solutions to (VI.4), let us therefore justify this approach:

Lemma VI.2.2. *Let (u, v) be a solution to* (VI.4) *in the sense of definition VI.2.1 such that both functions belong to*

$$C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right).$$

Then these functions solve the system in the classical sense as well.

Proof We adapt the proof of lemma 2.1 in [99] where a sufficiently similar situation has been discussed. For the second equation we refer to the standard procedure; alternatively it can be viewed as a simpler analogon to the upcoming argument.

Essentially, choosing a sequence of φ concentrating around suitable regions in $\overline{\Omega}$ will allow us to transform the integral inequality into a pointwise statement. Picking a sequence $(\zeta_j)_{j\in\mathbb{N}} \subset C_0^\infty([0,\infty))$ with the properties

- $0 \le \zeta_j \le 1$ for every $j \in \mathbb{N}$,
- $\zeta_j(0) = 1$ for every $j \in \mathbb{N}$,
- $\zeta_j' \le 0$ for every $j \in \mathbb{N}$ and
- $\operatorname{supp}\zeta_j \subset \left[0, \frac{1}{j}\right]$ for every $j \in \mathbb{N}$,

for arbitrary nonnegative $\psi \in C_0^\infty(\Omega)$ and every $j \in \mathbb{N}$ we set

$$\varphi_j(x,t) := \psi(x) \cdot \zeta_j(t)$$

for every $(x,t) \in \Omega \times (0,\infty)$ and see that for any eligible χ the central inequality results in

$$\begin{aligned}
&-\int_0^\infty \int_\Omega \chi(u)\psi\zeta_j' - \int_\Omega \chi(u_0)\psi \\
&\quad \ge -\int_0^\infty \int_\Omega \chi''(u)D(u)\,|\nabla u|^2\,\psi\zeta_j + \int_0^\infty \int_\Omega \chi''(u)S(u)\,(\nabla u \cdot \nabla v)\,\psi\zeta_j \\
&\qquad - \int_0^\infty \int_\Omega \chi'(u)D(u)\zeta_j \nabla u \cdot \nabla\psi + \int_0^\infty \int_\Omega \chi'(u)S(u)\zeta_j \nabla v \cdot \nabla\psi
\end{aligned}$$

for every $j \in \mathbb{N}$. Due to our integrability demands and according to the dominated convergence theorem, the right-hand side in this tends to 0 as $j \to \infty$. On the left we have $\zeta_j'(t) \to -\delta_0(t)$ as $j \to \infty$ which results in

$$\int_\Omega \psi \cdot (\chi(u(\cdot,0)) - \chi(u_0)) \geq 0$$

and after another process of choosing suitable test functions this gives us

$$\chi(u(\cdot,0)) \geq \chi(u_0)$$

in Ω; since χ is nondecreasing, this even means

$$u(\cdot,0) \geq u_0$$

in Ω. Assuming this inequality to be strict at some point $x \in \Omega$, due to the continuity of the functions involved this would leave us with some set $\omega \subset \Omega$ such that $|\omega| > 0$ and $u(\cdot,0) > u_0$ in ω. This and the requirement on the behaviour of the mass of u yield the contradiction

$$\int_\Omega u_0 < \int_\Omega u(\cdot,0) = \int_\Omega u_0.$$

Choosing instead any $\varphi \in C_0^2\left(\overline{\Omega} \times [0,\infty)\right)$, the integral on $\partial\Omega$ does not vanish and this means that from

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \chi(u)\varphi = \int_\Omega \chi'(u)\varphi u_t + \int_\Omega \chi(u)\varphi_t$$

we get

$$\begin{aligned}
\int_0^\infty \int_\Omega \chi'(u)\varphi u_t &= -\int_0^\infty \int_\Omega \chi(u)\varphi_t - \int_\Omega \chi(u_0)\varphi(\cdot,0) \\
&\geq \int_0^\infty \int_\Omega \chi'(u)\,(D(u)\nabla u - S(u)\nabla v)\cdot\nabla\varphi \\
&\quad - \int_0^\infty \int_\Omega \varphi\nabla\chi'(u)\cdot(D(u)\nabla u - S(u)\nabla v) \\
&= -\int_0^\infty \int_\Omega \nabla\left(\chi'(u)\varphi\right)\cdot(D(u)\nabla u - S(u)\nabla v) \\
&= \int_0^\infty \int_\Omega \chi'(u)\varphi\nabla\cdot(D(u)\nabla u - S(u)\nabla v) \\
&\quad - \int_0^\infty \int_{\partial\Omega} \chi'(u)\varphi\left(D(u)\frac{\partial u}{\partial\nu} - S(u)\frac{\partial v}{\partial\nu}\right).
\end{aligned}$$

Upon rearranging, this gives us

$$\int_0^\infty \int_\Omega \chi'(u)\varphi\left[u_t - \nabla\cdot(D(u)\nabla u - S(u)\nabla v)\right] + \int_0^\infty \int_{\partial\Omega} \chi'(u)\varphi\left(D(u)\frac{\partial u}{\partial\nu} - S(u)\frac{\partial v}{\partial\nu}\right) \geq 0$$

whence suitable choices for φ and the previous argument show

$$u_t \geq \nabla\cdot(D(u)\nabla u - S(u)\nabla v)$$

in $\Omega\times(0,\infty)$ and

$$D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu} \geq 0$$

on $\partial\Omega\times(0,\infty)$. Similarly to before, assuming that there are some open and nontrivial $\omega\subset\Omega$ as well as $0<t_1<t_2<\infty$ with

$$u_t > \nabla\cdot(D(u)\nabla u - S(u)\nabla v)$$

in $\omega\times(t_1,t_2)$, for any $t>t_1$ we see

$$\begin{aligned}\int_\Omega u(\cdot,t) - \int_\Omega u_0 &= \int_0^t\int_\Omega u_t\\ &> \int_0^t\int_\Omega \nabla\cdot(D(u)\nabla u - S(u)\nabla v)\\ &= \int_0^t\int_{\partial\Omega} D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu}\\ &\geq 0\end{aligned}$$

once more contradicting $\int_\Omega u(\cdot,t) = \int_\Omega u_0$. The same estimate can be achieved assuming that somewhere $D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu}$ is strictly larger than 0. **q.e.d.**

VI.3 Global solutions to the approximating systems

Let us begin by collecting a series of trivial to simple properties of the functions defining our system (VI.5) as well as some aspects of the solution concept introduced in definition VI.2.1.

Lemma VI.3.1. *Under the assumptions from theorem VI.1.1, for $s\geq 0$ and $\varepsilon\in(0,1)$ we define*

$$D_\varepsilon(s) := \frac{D(s)+\varepsilon}{1+\varepsilon D(s)},$$

and

$$S_\varepsilon(s) := \frac{S(s)}{1+\varepsilon D(s)},$$

such that for every $\varepsilon\in(0,1)$ the functions D_ε and S_ε belong to $C^2([0,\infty))$. Additionally, again for every $\varepsilon\in(0,1)$ and $s\geq 0$ we define

$$h_\varepsilon(s) := \frac{S_\varepsilon(s)}{D_\varepsilon(s)}$$

and, whenever $s \neq 0$, also

$$G_\varepsilon(s) := \int_1^s \int_1^\sigma \frac{\mathrm{d}\sigma \mathrm{d}\tau}{h_\varepsilon(\tau)}.$$

For $\varepsilon \in (0,1)$ and $K_{\Sigma_\varepsilon} := \frac{h_\varepsilon(1)}{h_\varepsilon(1)+1} \in (0,1)$, we want to set

$$\Sigma_\varepsilon(s) := \begin{cases} K_{\Sigma_\varepsilon} \cdot \sqrt{\frac{S_\varepsilon(s)}{S_\varepsilon(s)+1}} & \text{for } s \in [0,1], \\ \frac{h_\varepsilon(s)}{h_\varepsilon(s)+1} \cdot \sqrt{\frac{S_\varepsilon(s)}{S_\varepsilon(s)+1}} & \text{for } s > 1. \end{cases}$$

Then a number of helpful estimates hold: we have

$$\varepsilon \leq D_\varepsilon(s) \leq \frac{1}{\varepsilon}$$

as well as

$$D_\varepsilon(s) \geq \frac{D(s)}{D(s)+1}$$

for every $s \geq 0$ and $\varepsilon \in (0,1)$. Additionally, there is some $C_S > 0$ such that for every $s \geq 0$ and $\varepsilon \in (0,1)$

$$\frac{S_\varepsilon(s)}{D_\varepsilon(s)} \leq C_{SD} s^\alpha$$

and such that for every $\varepsilon \in (0,1)$ and $s \in (0,1)$

$$\frac{S_\varepsilon(s)}{s} \leq C_S$$

hold.

Lastly, with h and Σ from definition VI.2.1, for every $s \geq 0$ and every $\varepsilon \in (0,1)$ we have $h_\varepsilon(s) \leq h(s)$ and the nonnegative functions Σ and Σ_ε are continuous on $[0,\infty)$ with

$$\Sigma_\varepsilon \leq \sqrt{S_\varepsilon}$$

in $[0,\infty)$ and

$$\Sigma_\varepsilon \to \Sigma$$

in $L^\infty_{loc}([0,\infty))$ as $\varepsilon \to 0$.

Proof The regularity of D_ε and S_ε is obvious as are the statements regarding h_ε and Σ_ε. For the diffusion we immediately see $D_\varepsilon(s) \geq \frac{D(s)}{D(s)+1}$ for every $\varepsilon \in (0,1)$ and $s \in (0,1)$, the other bounds follow from the positivity of the derivative of $\frac{x+\varepsilon}{1+x\varepsilon}$ with respect to $x \geq 0$ for every $\varepsilon \in (0,1)$. On the other hand, by the mean value theorem we see

$$\frac{S(s)}{s(1+\varepsilon D(s))} \leq \frac{S(s)}{s} \leq \|S'\|_{L^\infty((0,1))}$$

for every $\varepsilon \in (0,1)$ and $s \in (0,1)$. With respect to the quotient $\frac{S_\varepsilon}{D_\varepsilon}$, for every $\varepsilon \in (0,1)$ and $s \in (0,1)$ we see

$$\frac{S(s)}{1+\varepsilon D_\varepsilon(s)} \cdot \frac{1+\varepsilon D_\varepsilon(s)}{D_\varepsilon(s)+\varepsilon} \leq \frac{S(s)}{D(s)} \leq C_{SD} s^\alpha$$

and thus the proof has been completed. **q.e.d.**

These choices allow us to find global solutions to the approximating problems with an additional energy identity and further helpful properties.

Lemma VI.3.2. *In the setting of lemma VI.3.1, for every $\varepsilon \in (0,1)$ we find a pair of global classical solutions to* (VI.5) *with the following additional properties: u_ε and v_ε remain nonnegative in $\overline{\Omega} \times [0,\infty)$,*

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0$$

holds for every $t > 0$ and $\varepsilon \in (0,1)$ and we have

$$v_\varepsilon \in \bigcap_{q \geq 1} C^0\left([0,\infty); W^{1,q}(\Omega)\right)$$

for every $\varepsilon \in (0,1)$. For any $\varepsilon \in (0,1)$ as well as two functions $\varphi \in L^1(\Omega)$ with $\varphi > 0$ almost everywhere in Ω and $\psi \in W^{1,2}(\Omega)$ we write

$$\mathcal{F}_\varepsilon(\varphi,\psi) := \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{2}\int_\Omega \psi^2 - \int_\Omega \varphi\psi + \int_\Omega G_\varepsilon(\varphi)$$

with G_ε as in the previous lemma. Furthermore, for $t > 0$ and $\varepsilon \in (0,1)$ we set

$$\overline{\mathcal{D}}_\varepsilon(t) := \int_\Omega v_{\varepsilon t}^2(\cdot,t) + \int_\Omega \left| \frac{D_\varepsilon(u_\varepsilon(\cdot,t))}{\sqrt{S(u_\varepsilon(\cdot,t))}} \nabla u_\varepsilon(\cdot,t) - \sqrt{S(u_\varepsilon(\cdot,t))} \nabla v_\varepsilon(\cdot,t) \right|^2 .$$

Then for every $t > 0$ and $\varepsilon \in (0,1)$ we have

$$\mathcal{F}_\varepsilon(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)) + \int_0^t \overline{\mathcal{D}}(s)\, \mathrm{d}s = \mathcal{F}_\varepsilon(u_0, v_0).$$

Proof Since the previous lemma shows that D_ε is bounded from above and below by constants, the statement regarding the relation between sensitivity and diffusion results in the first part of our claim according to chapter II. While the mass conservation property is once more as apparent as the nonnegativity of both functions in view of the comparison principle and the additionally claimed regularity for v_ε another byproduct of the cited result, the energy result requires additional straightforward computation: fixing some $\varepsilon \in (0,1)$ and starting with

$$\mathcal{F}_\varepsilon\left(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)\right) = \mathcal{F}_\varepsilon\left(u_0, v_0\right) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\varepsilon\left(u_\varepsilon(\cdot,s), v_\varepsilon(\cdot,s)\right)\ \mathrm{d}s$$

for every $t > 0$ and $\varepsilon \in (0,1)$, computing the derivative gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\varepsilon\left(u_\varepsilon, v_\varepsilon\right) &= \int_\Omega \nabla v_\varepsilon \cdot \nabla v_{\varepsilon t} + \int_\Omega v_\varepsilon v_{\varepsilon t} - \int_\Omega u_\varepsilon v_{\varepsilon t} - \int_\Omega u_{\varepsilon t} v_\varepsilon + \int_\Omega G_\varepsilon'(u_\varepsilon) u_{\varepsilon t} \\ &= -\int_\Omega v_{\varepsilon t}^2 + \int_\Omega \left(G_\varepsilon'(u_\varepsilon) - v_\varepsilon\right) u_{\varepsilon t}\end{aligned}$$

in $(0,\infty)$ and for every $\varepsilon \in (0,1)$. With the first term already making up the first half of $\overline{\mathcal{D}}$, we examine the integral involving G_ε' further: using integration by parts we see

$$\begin{aligned}\int_\Omega \left(G_\varepsilon'(u_\varepsilon) - v_\varepsilon\right) u_{\varepsilon t} &= -\int_\Omega \left(G_\varepsilon''(u_\varepsilon)\nabla u_\varepsilon - \nabla v_\varepsilon\right) \cdot \left(D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon - S_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\right) \\ &= -\int_\Omega G_\varepsilon''(u_\varepsilon) D_\varepsilon(u_\varepsilon) \left|\nabla u_\varepsilon\right|^2 - \int_\Omega \left(G_\varepsilon''(u_\varepsilon) S_\varepsilon(u_\varepsilon) + D_\varepsilon(u_\varepsilon)\right) \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\ &\quad + \int_\Omega S_\varepsilon(u_\varepsilon) \left|\nabla v_\varepsilon\right|^2 \\ &= -\int_\Omega \left[\frac{D_\varepsilon^2(u_\varepsilon)}{S_\varepsilon(u_\varepsilon)} \left|\nabla u_\varepsilon\right|^2 - 2 D_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon + S_\varepsilon(u_\varepsilon) \left|\nabla v_\varepsilon\right|^2\right]\end{aligned}$$

in $(0,\infty)$ and for every $\varepsilon \in (0,1)$. Rearranging the integrand then gives us the claimed result. **q.e.d.**

In lemma I.3.4, we used the the mass conservation property of the first component of our solutions to show that for every $q \in \left[1, \frac{n}{n-1}\right)$ the functions $v_\varepsilon(\cdot,t)$ belong to $W^{1,q}(\Omega)$ for every $t > 0$ and $\varepsilon \in (0,1)$. If we are interested in integrals containing only v_ε as opposed to ones also featuring its gradient, even larger q become available:

Lemma VI.3.3. *For every $q \in \left[1, \frac{n}{n-2}\right)$ there is some $C(q) > 0$ with*

$$\left\|v_\varepsilon(\cdot,t)\right\|_{L^q(\Omega)} \leq C(q)$$

for every $t > 0$ and $\varepsilon \in (0,1)$ and where v_ε denotes the second component of a solution found in lemma VI.3.2.

Proof Fixing $C_1 := \int_\Omega u_0 \equiv \int_\Omega u_\varepsilon$ from the L^p-L^q-estimates in lemma I.3.3 we get some additional constants $C_2 > 0$ and $\lambda > 0$ such that for every $\varepsilon \in (0,1)$ and every $t > 0$

$$\begin{aligned}\left\|v_\varepsilon(\cdot,t)\right\|_{L^q(\Omega)} &\le \left\|e^{t(\Delta-1)}v_0\right\|_{L^q(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)\right\|_{L^q(\Omega)} \,\mathrm{d}s \\ &\le C_2 + C_2\int_0^t \left(1 + (t-s)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}\right)e^{-\lambda(t-s)}\left\|u_\varepsilon(\cdot,s)\right\|_{L^1(\Omega)} \,\mathrm{d}s \\ &\le C_2 + C_1\cdot C_2\int_0^\infty \left(1+\tau^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}\right)e^{-\lambda\tau}\,\mathrm{d}\tau\end{aligned}$$

holds and the final integral herein is bounded due to the constraint on q. **q.e.d.**

In view of the nonnegativity of $\overline{\mathcal{D}}_\varepsilon$, initial boundedness of $\mathcal{F}_\varepsilon$ implies boundedness at all times. Accordingly, establishing this quantity as an upper bound for some terms will be a fruitful endeavour. We begin with the following preparation:

Lemma VI.3.4. *In the setting of lemma VI.3.1, there is some $C > 0$ such that for every $s > 0$ and $\varepsilon \in (0,1)$ we have*

$$G_\varepsilon(s) \ge \frac{1}{C}s^{2-\alpha} - C\cdot(s+1).$$

Proof Fixing $C_1 := \frac{1}{(2-\alpha)(1-\alpha)C_{SD}}$ and $C_2 := \frac{1}{(2-\alpha)C_{SD}}$, for $s \ge 1$ we see

$$\begin{aligned}G_\varepsilon(s) &\ge \frac{1}{C_{SD}}\int_1^s\int_1^\sigma \tau^{-\alpha}\,\mathrm{d}\tau\,\mathrm{d}\sigma \\ &= \frac{1}{C_{SD}}\int_1^s \frac{\sigma^{1-\alpha}-1}{\alpha-1}\,\mathrm{d}\sigma \\ &= \frac{1}{(1-\alpha)C_{SD}}\left[\frac{s^{2-\alpha}-1}{2-\alpha} - (s-1)\right] \\ &= C_1 s^{2-\alpha} - C_2(s+1).\end{aligned}$$

If on the other hand $s \in (0,1)$, then the elementary observation

$$G_\varepsilon(s) - C_1 s^{2-\alpha} \ge -C_1 s^{2-\alpha} \ge -C_1$$

completes the proof for $C := \max\left\{\frac{1}{C_1}, C_1, C_2\right\}$. **q.e.d.**

As a direct result of this, the following lower bound for $\mathcal{F}_\varepsilon$ can be established.

Lemma VI.3.5. *In the setting of lemma VI.3.1, we can find some constant $C > 0$ such that*

$$\mathcal{F}_\varepsilon\left(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)\right) \ge \frac{1}{2}\int_\Omega G_\varepsilon(u_\varepsilon(\cdot,t)) - C$$

holds for every $t > 0$ and $\varepsilon \in (0,1)$.

Proof According to lemma VI.3.4 we can fix some $C_1 > 0$ with

$$G_\varepsilon(s) \geq C_1 s^{2-\alpha} - C_2 \cdot (s+1).$$

for every $s > 0$ and $\varepsilon \in (0,1)$. Additionally, by Young's inequality, there is some $C_3 > 0$ such that

$$xy \leq \frac{C_1}{2} x^{2-\alpha} + C_3 y^{\frac{2-\alpha}{1-\alpha}}$$

holds for any nonnegative x and y. Since $\alpha \in \left[0, \frac{2}{n}\right)$, we immediately see

$$\frac{\alpha n}{2(1-\alpha)} < \frac{n}{n-2}$$

and due to $\frac{2}{n} < \frac{4}{n+2}$ we also have

$$\frac{\alpha n}{2(1-\alpha)} < \frac{2-\alpha}{1-\alpha} < \frac{2n}{n-2}.$$

Accordingly, we can fix some

$$\max\left\{1, \frac{\alpha n}{2(1-\alpha)}\right\} < q < \min\left\{\frac{n}{n-2}, \frac{2-\alpha}{1-\alpha}\right\}$$

and for this parameter we have

$$\frac{(2-\alpha)n}{(1-\alpha)q} - n = \frac{2n}{q} + \frac{\alpha n}{(1-\alpha)q} - n < 2 - n + \frac{2n}{q}.$$

This estimate legitimises the upcoming employment of the Gagliardo-Nirenberg inequality wherein for the exponent

$$a := \frac{\frac{n}{q} - \frac{n(1-\alpha)}{2-\alpha}}{1 - \frac{n}{2} + \frac{n}{q}}$$

we have

$$\frac{2-\alpha}{1-\alpha} \cdot \frac{a}{2} = \frac{\frac{1}{2}\left(\frac{(2-\alpha)n}{(1-\alpha)q} - n\right)}{1 - \frac{n}{2} + \frac{n}{q}} < 1.$$

Together with Young's inequality for some $C_4 > 0$ and $C_5 > 0$, upon fixing

$$\beta := \max\left\{\frac{2(2-\alpha)(1-a)}{2(1-\alpha) - (2-\alpha)a}, \frac{2-\alpha}{1-\alpha}\right\},$$

this results in

$$\begin{aligned} C_3 \|\psi\|^{\frac{2-\alpha}{1-\alpha}}_{L^{\frac{2-\alpha}{1-\alpha}}(\Omega)} &\leq C_4 \|\nabla\psi\|^{\frac{2-\alpha}{1-\alpha}a}_{L^2(\Omega)} \cdot \|\psi\|^{\frac{2-\alpha}{1-\alpha}(1-a)}_{L^q(\Omega)} + C_4 \|\psi\|^{\frac{2-\alpha}{1-\alpha}}_{L^q(\Omega)} \\ &\leq \frac{1}{2}\int_\Omega |\nabla\psi|^2 + C_5 \cdot \left(1 + \|\psi\|^\beta_{L^q(\Omega)}\right) \end{aligned}$$

for every $\psi \in W^{1,2}(\Omega)$. Lemma VI.3.3 provides us with some $C_6 > 0$ such that

$$C_5 \cdot \left(1 + \left\|v_\varepsilon(\cdot,t)\right\|_{L^q(\Omega)}^\beta\right) \le C_6$$

holds for every $t > 0$ and $\varepsilon \in (0,1)$. Setting $C_7 := C_6 + \frac{C_2}{2}\int_\Omega (u_0+1) < \infty$, the combination of the previous steps results in

$$\begin{aligned}
\int_\Omega u_\varepsilon v_\varepsilon &\le \frac{C_1}{2}\int_\Omega u_\varepsilon^{2-\alpha} + C_3\int_\Omega v_\varepsilon^{\frac{2-\alpha}{1-\alpha}} \\
&\le \frac{C_1}{2}\int_\Omega u_\varepsilon^{2-\alpha} + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + C_6 \\
&\le \frac{1}{2}\int_\Omega G_\varepsilon(u_\varepsilon) + \frac{C_2}{2}\int_\Omega (u_\varepsilon + 1) + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + C_6 \\
&\le \frac{1}{2}\int_\Omega G_\varepsilon(u_\varepsilon) + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + C_7
\end{aligned}$$

in $(0,\infty)$ and for every $\varepsilon \in (0,1)$. According to the definition of $\mathcal{F}_\varepsilon$, this completes the proof by showing

$$\begin{aligned}
\mathcal{F}_\varepsilon\left(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)\right) &= \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_\Omega v_\varepsilon^2 - \int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega G_\varepsilon(u_\varepsilon) \\
&\ge \frac{1}{2}\int_\Omega G_\varepsilon(u_\varepsilon(\cdot,t)) - C_7
\end{aligned}$$

for every $t > 0$ and $\varepsilon \in (0,1)$. **q.e.d.**

The next lemma now reaps the rewards of these estimates and provides us with several helpful results, among others we see that for u_ε we have information beyond the usual mass conservation.

Lemma VI.3.6. *In the setting of lemma VI.3.1, there is some constant $C > 0$ such that*

$$\int_\Omega u_\varepsilon^{2-\alpha}(\cdot,t) \le C$$

and

$$\int_\Omega \left|\nabla v_\varepsilon(\cdot,t)\right|^2 \le C$$

as well as

$$\int_0^t \int_\Omega v_{\varepsilon t}^2 \le C$$

and

$$\int_0^t \int_\Omega \left| \Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(u_\varepsilon)\nabla v_\varepsilon \right|^2 \le C$$

hold for every $t > 0$ and $\varepsilon \in (0,1)$. Furthermore, with

$$\mathcal{D}_\varepsilon(t) := \int_\Omega v_{\varepsilon t}^2(\cdot,t) + \int_\Omega \left| \Sigma(u_\varepsilon(\cdot,t)) \frac{\nabla u_\varepsilon(\cdot,t)}{h_\varepsilon(u_\varepsilon(\cdot,t))} - \Sigma(u_\varepsilon(\cdot,t)) \nabla v_\varepsilon(\cdot,t) \right|^2$$

whenever $t > 0$ and $\varepsilon \in (0,1)$, the estimate

$$\mathcal{F}_\varepsilon\left(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)\right) + \int_0^t \mathcal{D}_\varepsilon(s)\,\mathrm{d}s \le \mathcal{F}_\varepsilon\left(u_0, v_0\right)$$

holds for every $t > 0$ and $\varepsilon \in (0,1)$.

Proof Setting $0 < C_1 := \min_{x\in\overline{\Omega}} u_0(x)$ and $C_2 := \max_{x\in\overline{\Omega}} u_0(x)$, we can estimate

$$\mathcal{F}_\varepsilon\left(u_0, v_0\right) \le C_3$$

where

$$C_3 := \frac{1}{2}\|v_0\|_{W^{1,2}(\Omega)}^2 + |\Omega| \max\left\{ \int_{C_1}^1 \int_\sigma^1 \frac{D(\tau)+1}{S(\tau)}\,\mathrm{d}\tau\,\mathrm{d}\sigma, \int_1^{C_2} \int_1^\sigma \frac{D(\tau)+1}{S(\tau)}\,\mathrm{d}\tau\,\mathrm{d}\sigma \right\}.$$

Accordingly, our previous results in lemmata VI.3.4, VI.3.5 and VI.3.2 provide us with positive constants C_4, C_5 and C_6 such that

$$\begin{aligned} C_4 \int_\Omega u_\varepsilon^{2-\alpha}(\cdot,t) + \int_0^t \overline{\mathcal{D}}_\varepsilon(s)\,\mathrm{d}s &\le \frac{1}{2}\int_\Omega G_\varepsilon(u_\varepsilon(\cdot,t)) + C_5 + \int_0^t \overline{\mathcal{D}}_\varepsilon(s)\,\mathrm{d}s \\ &\le \mathcal{F}_\varepsilon\left(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)\right) + C_6 + \int_0^t \overline{\mathcal{D}}_\varepsilon(s)\,\mathrm{d}s \\ &= \mathcal{F}_\varepsilon\left(u_0, v_0\right) + C_6 \\ &\le C_3 + C_6 \end{aligned}$$

holds for every $t > 0$ and $\varepsilon \in (0,1)$. Since for every $\varepsilon \in (0,1)$ and $s \ge 0$ by lemma VI.3.1 we have $0 \le \Sigma_\varepsilon(s) \le \sqrt{S_\varepsilon(s)}$, most of the remaining claims follow in view of $\overline{\mathcal{D}}_\varepsilon \ge 0$; for the estimate involving the $L^2(\Omega)$-norm of ∇v_ε we once more refer to lemma I.3.4 in chapter I alongside the observation

$$2 < \frac{n(2-\alpha)}{n-2+\alpha}$$

and the bound we derived for u_ε two steps before. **q.e.d.**

Apart from this estimate for $\int_\Omega u_\varepsilon^{2-\alpha}$, we can also establish bounds for u_ε in the space $L^p\left((0,\infty); L^p(\Omega)\right)$ for every $p > 1$ while simultaneously gaining some knowledge regarding ∇u_ε.

Lemma VI.3.7. *In the setting of lemma VI.3.1, for every $p \geq 2$ we can find some $C(p)$ such that*

$$\int_0^T \int_\Omega u_\varepsilon^p \leq C(p) + C(p) \cdot T$$

and

$$\int_0^T \int_\Omega \left| \nabla \left(u_\varepsilon^{\frac{p}{2}} \right) \right|^2 \leq C(p) + C(p) \cdot T$$

hold for every $T > 0$ and $\varepsilon \in (0,1)$.

Proof Defining

$$\Phi_{p,\varepsilon}(s) := \int_0^s \int_0^\sigma \frac{\tau^{p-2}}{D_\varepsilon(\tau)} \,\mathrm{d}\tau \,\mathrm{d}\sigma$$

for $s \geq 0$, which gives us a nonnegative function in $C^2([0,\infty))$, through integration by parts and Young's inequality we derive the estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Phi_{p,\varepsilon}(u_\varepsilon) &= -\int_\Omega \Phi''_{p,\varepsilon}(u_\varepsilon) D_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \int_\Omega \Phi''_{p,\varepsilon}(u_\varepsilon) S_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot v_\varepsilon \\
&= -\int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^{p-2} \frac{S_\varepsilon(u_\varepsilon)}{D_\varepsilon(u_\varepsilon)} \nabla u_\varepsilon \cdot v_\varepsilon \\
&\leq -\frac{1}{2} \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \frac{1}{2} \int_\Omega u_\varepsilon^{p-2} \frac{S_\varepsilon^2(u_\varepsilon)}{D_\varepsilon^2(u_\varepsilon)} |\nabla v_\varepsilon|^2 \\
&\leq -\frac{1}{2} \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \frac{C_{SD}^2}{2} \int_\Omega u_\varepsilon^{p+2\alpha-2} |\nabla v_\varepsilon|^2
\end{aligned}$$

in $(0,\infty)$ for every $\varepsilon \in (0,1)$. By Young's inequality we see herein that again in $(0,\infty)$ and for every $\varepsilon \in (0,1)$

$$\int_\Omega u_\varepsilon^{p+2\alpha-2} |\nabla v_\varepsilon|^2 \leq \int_\Omega u_\varepsilon^{p+\alpha} + \int_\Omega |\nabla v_\varepsilon|^{\frac{2(p+\alpha)}{2-\alpha}}$$

holds. Similarly we derive a bound using the second differential equation in (VI.5) and the boundedness of $(\nabla v_\varepsilon)_{\varepsilon \in (0,1)} \subset L^\infty\left((0,\infty); L^2(\Omega)\right)$ proven in lemma VI.3.6. With $q := \frac{p}{2-\alpha} > 1$ due to lemma I.3.11 and Young's inequality, for some $C_1 > 0$ and every $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
\frac{1}{2q} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v_\varepsilon|^{2q} + \frac{q-1}{4q^2} \int_\Omega \left| \nabla |\nabla v_\varepsilon|^q \right|^2 + \int_\Omega |\nabla v_\varepsilon|^{2q} &\leq C_1 + C_1 \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2q-2} \\
&\leq C_1 + C_1 \int_\Omega u_\varepsilon^{p+\alpha} + C_1 \int_\Omega |\nabla v_\varepsilon|^{\frac{2(p+\alpha)}{2-\alpha}}
\end{aligned}$$

in $(0,\infty)$. With our previous line of estimates as well as $C_2 := C_1 + \frac{C_{SD}^2}{2}$ we see

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega \Phi_{p,\varepsilon}(u_\varepsilon) + \frac{1}{2q}\int_\Omega |\nabla v_\varepsilon|^{2q}\right] + \frac{2}{p^2}\int_\Omega \left|\nabla\left(u_\varepsilon^{\frac{p}{2}}\right)\right|^2 + \frac{q-1}{2q^2}\int_\Omega \left|\nabla\,|\nabla v_\varepsilon|^q\right|^2$$
$$\leq C_2 + C_2\int_\Omega u_\varepsilon^{p+\alpha} + C_2\int_\Omega |\nabla v_\varepsilon|^{\frac{2(p+\alpha)}{2-\alpha}}$$

in $(0,\infty)$ for every $\varepsilon \in (0,1)$. Using our knowledge regarding $\int_\Omega u_\varepsilon$ and $\int_\Omega |\nabla v_\varepsilon|^2$, for

$$a := \frac{\frac{np}{2} - \frac{n}{2}}{1 - \frac{np}{2(p+\alpha)} + \frac{np}{2}} = \frac{p-1}{\frac{2}{n} - \frac{p}{p+\alpha} + p}$$

and

$$b := \frac{\frac{np}{2(2-\alpha)} - \frac{n}{2}}{1 - \frac{np}{2(p+\alpha)} + \frac{np}{2(2-\alpha)}} = \frac{\frac{p}{2-\alpha} - 1}{\frac{2}{n} - \frac{p}{p+\alpha} + \frac{p}{2-\alpha}}$$

from the Gagliardo-Nirenberg inequality we get a set of positive constants C_i with $i \in \{3,4,5,6\}$ such that

$$C_2\int_\Omega u_\varepsilon^{p+\alpha} = C_2 \left\|u_\varepsilon^{\frac{p}{2}}\right\|_{L^{\frac{2(p+\alpha)}{p}}(\Omega)}^{\frac{2(p+\alpha)}{p}}$$
$$\leq C_3 \left\|\nabla\left(u_\varepsilon^{\frac{p}{2}}\right)\right\|_{L^2(\Omega)}^{\frac{2(p+\alpha)}{p}a} \cdot \left\|u_\varepsilon^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+\alpha)}{p}(1-a)} + C_3\left\|u_\varepsilon^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+\alpha)}{p}}$$
$$\leq C_4 + C_4\left(\int_\Omega \left|\nabla\left(u_\varepsilon^{\frac{p}{2}}\right)\right|^2\right)^{\frac{2(p+\alpha)}{p}a}$$

and

$$C_2\int_\Omega |\nabla v_\varepsilon|^{\frac{2(p+\alpha)}{2-\alpha}} = C_2\left\||\nabla v_\varepsilon|^q\right\|_{L^{\frac{2(p+\alpha)}{q(2-\alpha)}}(\Omega)}^{\frac{2(p+\alpha)}{q(2-\alpha)}}$$
$$\leq C_5\left\|\nabla\,|\nabla v_\varepsilon|^q\right\|_{L^2(\Omega)}^{\frac{2(p+\alpha)}{q(2-\alpha)}b} \cdot \left\||\nabla v_\varepsilon|^q\right\|_{L^{\frac{2}{q}}(\Omega)}^{\frac{2(p+\alpha)}{q(2-\alpha)(1-b)}} + C_5\left\||\nabla v_\varepsilon|^q\right\|_{L^{\frac{2}{q}}(\Omega)}^{\frac{2(p+\alpha)}{q(2-\alpha)}}$$
$$\leq C_6 + C_6\left(\int_\Omega \left|\nabla\,|\nabla v_\varepsilon|^q\right|^2\right)^{\frac{p+\alpha}{p}b}$$

hold in $(0,\infty)$ for every $\varepsilon \in (0,1)$. Since straightforward estimates show that both of the exponents $\frac{2(p+\alpha)}{p}a$ and $\frac{p+\alpha}{p}b$ are less than 1, from Young's inequality we get some $C_7 > 0$ with

$$C_2 + C_2\int_\Omega u_\varepsilon^{p+\alpha} + C_2\int_\Omega |\nabla v_\varepsilon|^{\frac{2(p+\alpha)}{2-\alpha}} \leq C_7 + \frac{1}{p^2}\int_\Omega \left|\nabla\left(u_\varepsilon^{\frac{p}{2}}\right)\right|^2 + \frac{q-1}{4q^2}\int_\Omega \left|\nabla\,|\nabla v_\varepsilon|^q\right|^2$$

in $(0,\infty)$ for every $\varepsilon \in (0,1)$. From Hölder's and Young's inequalities we see that for some $C_8 > 0$ we have

$$\begin{aligned}\int_\Omega u_\varepsilon^p &\le \left(\int_\Omega u_\varepsilon^{p+\alpha}\right)^{\frac{p}{p+\alpha}} \cdot \left(\int_\Omega 1\right)^{\frac{\alpha}{p+\alpha}} \\ &\le C_2 \int_\Omega u_\varepsilon^{p+\alpha} + C_8\end{aligned}$$

in $(0,\infty)$ for every $\varepsilon \in (0,1)$ which means that both our claims follow upon integration of our inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega \Phi_{p,\varepsilon}(u_\varepsilon) + \frac{1}{2q}\int_\Omega |\nabla v_\varepsilon|^{2q}\right] + \frac{1}{p^2}\int_\Omega \left|\nabla\left(u_\varepsilon^{\frac{p}{2}}\right)\right|^2 + \frac{q-1}{4q^2}\int_\Omega \left|\nabla |\nabla v_\varepsilon|^q\right|^2 \le C_7$$

which is valid in $(0,\infty)$ for every $\varepsilon \in (0,1)$: in doing so, since

$$\begin{aligned}\Phi_{p,\varepsilon}(u_0) &= \int_0^{u_0}\int_0^\sigma \frac{\tau^{p-2}}{D_\varepsilon(\tau)}\,\mathrm{d}\tau\mathrm{d}\sigma \\ &\le \int_0^{u_0}\int_0^\sigma \frac{\tau^{p-2}\left(D(\tau)+1\right)}{D(\tau)}\,\mathrm{d}\tau\mathrm{d}\sigma\end{aligned}$$

is bounded independently of $\varepsilon \in (0,1)$ and due to the nonnegativity of the terms involved, we arrive at

$$\int_0^T \int_\Omega \left|\nabla\left(u_\varepsilon^{\frac{p}{2}}\right)\right|^2 \le C_9 + C_9 \cdot T$$

for some $C_9 > 0$ as well as every $T > 0$ and $\varepsilon \in (0,1)$. **q.e.d.**

Several of the results so far have established bounds for different norms of u_ε and v_ε, often by alternating between the two: knowledge regarding the one leads to more information on the other. Following this tradition, the previous lemma helps us find two more estimates for v_ε, starting with

Lemma VI.3.8. *In the setting of lemma VI.3.1, for every $p \ge 2$ and $0 < \tau < T < \infty$ there is a positive constant C such that*

$$\int_\tau^T \left\|v_\varepsilon(\cdot,t)\right\|_{W^{2,p}(\Omega)}^p\,\mathrm{d}t \le C$$

holds for every $\varepsilon \in (0,1)$.

Proof. For the first part of this proof we once more use the L^p-L^q-estimates from lemma I.3.3 for p and $r := np$. Setting

$$\begin{aligned}\theta &:= \left[-\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right)\right] \cdot \frac{r}{r-1} \\ &= \left[-\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p} - \frac{1}{np}\right)\right] \cdot \frac{np}{np-1} \\ &= \left(-\frac{1}{2} - \frac{n-1}{2p}\right) \cdot \frac{np}{np-1} \\ &= -\frac{np + n^2 - n}{2(np-1)} \\ &\geq -\frac{n^2+n}{2(n^2-1)} \\ &\geq -\frac{3}{4}\end{aligned}$$

for any admissible n and p, for some $C_1 > 0$ from the aforementioned estimate and $C_2 > 0$ from Young's inequality we have

$$\begin{aligned}\left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,p}(\Omega)} &\leq C_1 + C_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\right) \left\|u_\varepsilon(\cdot,s)\right\|_{L^r(\Omega)} \,\mathrm{d}r \\ &\leq C_1 + C_2 \int_0^t \left(1+\sigma^\theta\right) \mathrm{d}\sigma + C_2 \int_0^t \left\|u_\varepsilon(\cdot,s)\right\|_{L^r(\Omega)}^r \,\mathrm{d}s\end{aligned}$$

for every $t > 0$ and $\varepsilon \in (0,1)$. Accordingly, for fixed $T > 0$ from lemma VI.3.7 we get some $C_3 > 0$ with

$$\int_0^T \left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,p}(\Omega)}^p \,\mathrm{d}t \leq C_3$$

for every $\varepsilon \in (0,1)$. For the remaining portion of the norm we use results from maximal Sobolev regularity for which we refer to [102] and [11] or to lemma 2.2 in [12] for a directly applicable formulation. For fixed p, τ and T this in tandem with lemma VI.3.7 gives us some positive constant C_4 such that

$$\int_\tau^T \int_\Omega |\Delta v_\varepsilon|^p \leq C_4$$

holds for every $\varepsilon \in (0,1)$ and this completes the proof. **q.e.d.**

A second estimate derived from lemma VI.3.7 allows us to detect the following pointwise bound for ∇v_ε:

Lemma VI.3.9. *In the setting of lemma VI.3.1, there is some $C > 0$ such that for every $(x,t) \in \Omega \times (0,\infty)$ and every $\varepsilon \in (0,1)$ the estimate*

$$\left|\nabla v_\varepsilon(x,t)\right| \leq C$$

holds.

Proof Fixing some arbitrary $p > n+2$, once more the L^p-L^q-estimates from lemma I.3.3 provide us with some $C_1 > 0$ such that

$$\begin{aligned}\left\|\nabla v_\varepsilon(\cdot,t)\right\|_{L^\infty(\Omega)} &= \left\|\nabla e^{t(\Delta-1)}v_0 + \int_0^t \nabla e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)\,\mathrm{d}s\right\|_{L^\infty(\Omega)}\\ &\le \|v_0\|_{W^{1,\infty}(\Omega)} + C_1\int_0^t\left(1+(t-s)^{-\frac12-\frac{n}{2p}}\right)\left\|u_\varepsilon(\cdot,s)\right\|_{L^p(\Omega)}\,\mathrm{d}s\end{aligned}$$

holds for every $t > 0$ and every $\varepsilon \in (0,1)$. By Young's inequality, herein we can estimate

$$\begin{aligned}\int_0^t\left(1+(t-s)^{-\frac12-\frac{n}{2p}}\right)\left\|u_\varepsilon(\cdot,s)\right\|_{L^p(\Omega)}\,\mathrm{d}s &\le \int_0^t\left(1+(t-s)^{-\frac12-\frac{n}{2p}}\right)^{\frac{p}{p-1}}\,\mathrm{d}s\\ &\quad+\int_0^t\left\|u_\varepsilon(\cdot,s)\right\|^p_{L^p(\Omega)}\,\mathrm{d}s\end{aligned}$$

for every $\varepsilon \in (0,1)$. Since $p > n+2$ allows for the estimate

$$\begin{aligned}-\frac12\left(1+\frac1p\right)\cdot\frac{p}{p-1} &= -\frac12\cdot\frac{n+p}{p-1}\\ &> -\frac12\cdot\frac{n+(n+2)}{(n+2)-1}\\ &= -\frac12\frac{2n+2}{n+1}\\ &= -1,\end{aligned}$$

lemma VI.3.7 ensures the claimed boundedness. **q.e.d.**

We want to close this section by establishing a uniform result regarding the positivity of the u_ε, given by the following

Lemma VI.3.10. *In the setting of lemma VI.3.1, for every $T > 0$ there is some $C(T) > 0$ such that*

$$\int_\Omega \ln u_\varepsilon(\cdot,t) \ge -C(T)$$

holds for every $t \in (0,T)$ and every $\varepsilon \in (0,1)$.

Proof Given any $t \in (0,\infty)$, it is sufficient to consider the integral over the set $\Omega \cap \{u_\varepsilon(\cdot,t) < 1\}$, therefore we define $\Gamma \in C^2((0,\infty))$ via

$$\Gamma(s) := \begin{cases}\ln s & \text{for } s < 1\\ \frac{s^3}{6} - s^2 + \frac{5s}{2} - \frac53 & \text{for } 1 \le s < 2\\ \frac{s}{2} - \frac13 & \text{for } 2 \le s.\end{cases}$$

We have $\Gamma'' \le 0$ in $(0,\infty)$ and more specifically $\Gamma''(s) = 0$ whenever $s \in [2,\infty)$. The mass conservation property of u_ε, coupled with a straightforward examination of Γ in $[1,2]$, yields $C_1 > 0$ with

$$\int_\Omega \Gamma(u_\varepsilon) \le \int_{\Omega\cap\{u_\varepsilon<1\}} \ln u_\varepsilon + C_1$$

in $(0,\infty)$ for every $\varepsilon \in (0,1)$. From Young's inequality and integration by parts we learn

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Gamma(u_\varepsilon) &= -\int_\Omega \Gamma''(u_\varepsilon)D_\varepsilon(u_\varepsilon)\,|\nabla u|^2 + \int_\Omega \Gamma''(u_\varepsilon)S_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\ &\ge \frac{1}{4}\int_\Omega \frac{\Gamma''(u_\varepsilon)S_\varepsilon^2(u_\varepsilon)}{D_\varepsilon(u_\varepsilon)}\,|\nabla v_\varepsilon|^2\end{aligned}$$

in $(0,\infty)$ for every $\varepsilon \in (0,1)$ which results in

$$\int_{\Omega\cap\{u_\varepsilon(\cdot,t)<1\}} \ln u_\varepsilon(\cdot,t) \ge -C_1 + \frac{1}{4}\int_0^t\int_\Omega \frac{\Gamma''(u_\varepsilon)S_\varepsilon^2(u_\varepsilon)}{D_\varepsilon(u_\varepsilon)}\,|\nabla v_\varepsilon|^2$$

for every $t \in (0,\infty)$ and $\varepsilon \in (0,1)$. To estimate further, we fix the two positive and finite constants $C_2 := \max_{\tau\in[0,2]} \frac{(S(\tau)+1)^2}{D(\tau)}$ and, with lemma VI.3.1, $C_3 := \frac{\left(\sup_{\tau\in(0,1)} \frac{S_\varepsilon(\tau)}{\tau}\right)^2}{\min_{\sigma\in[0,1]} D(\sigma)}$ so that firstly for every $\varepsilon \in (0,1)$ and $s \in [1,2]$ we have

$$\left|\frac{\Gamma''(s)S_\varepsilon^2(s)}{D_\varepsilon(s)}\right| = \frac{(2-s)\left(\frac{S(s)}{1+\varepsilon D(s)}\right)^2\cdot(1+\varepsilon D(s))}{D(s)+\varepsilon} \le C_2.$$

On the other hand, for $\varepsilon \in (0,1)$ and $s \in (0,1)$ we see

$$\left|\frac{\Gamma''(s)S_\varepsilon^2(s)}{D_\varepsilon(s)}\right| \le \frac{1}{D_\varepsilon(s)}\cdot\left(\frac{S_\varepsilon(s)}{s}\right) \le C_3.$$

Setting $C_4 := \max\left\{C_1, \frac{C_2}{4}, \frac{C_3}{4}\right\}$ we arrive at

$$\int_\Omega \ln u_\varepsilon(\cdot,t) \ge -C_4 - C_4\int_0^t\int_\Omega |\nabla v_\varepsilon|^2$$

for every $t \in (0,\infty)$ and $\varepsilon \in (0,1)$ whereupon an application of lemma VI.3.9 then completes the proof. **q.e.d.**

It is now time to introduce a crucial component of our solution concept: instead of a direct discussion of u_ε, our focus at many points is on $\chi(u_\varepsilon)$ for χ taken from a class of functions with helpful properties.

Lemma VI.3.11. *In the setting of lemma VI.3.1, let some* $\frac{n}{2} < m \in \mathbb{N}$ *be given. Then for every* $T > 0$ *and* $\chi \in C^\infty([0,\infty))$ *with* $\operatorname{supp}\chi' \subset\subset [0,\infty)$ *we can find some* $C > 0$ *such that*

$$\int_0^T \left\|\partial_t \chi(u_\varepsilon(\cdot,t))\right\|_{(W^{m,2}(\Omega))^*} \,\mathrm{d}t \le C$$

holds for every $\varepsilon \in (0,1)$.

Proof This is virtually the same as lemma 5.7 in [93]. In order to estimate this integrand we need to find some constant C such that for any $\psi \in C^\infty\left(\overline{\Omega}\right)$ the integral $I_\psi := \left|\int_\Omega \partial_t \chi(u_\varepsilon(\cdot,t))\psi\right|$ is bounded by $C \cdot \|\psi\|_{W^{m,2}(\Omega)}$. Given any such ψ, from Young's inequality and the Cauchy-Schwarz inequality we see

$$\begin{aligned}
I_\psi &= \left|\int_\Omega \chi'(u_\varepsilon)u_{\varepsilon t}\psi\right| \\
&= \left|-\int_\Omega (\chi''(u_\varepsilon)\psi\nabla u_\varepsilon + \chi'(u_\varepsilon)\nabla\psi)\cdot D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon - S_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\right| \\
&\le \int_\Omega \left|\chi''(u_\varepsilon)\right| D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2\,\psi + \int_\Omega \left|\chi''(u_\varepsilon)\right| S_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|\cdot|\nabla v_\varepsilon|\cdot|\psi| \\
&\quad + \int_\Omega \left|\chi'(u_\varepsilon)\right| D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|\cdot|\nabla\psi| + \int_\Omega \left|\chi'(u_\varepsilon)\right| S_\varepsilon(u_\varepsilon)\,|\nabla v_\varepsilon|\cdot|\nabla\psi| \\
&\le \|\psi\|_{L^\infty(\Omega)}\cdot\left[\int_\Omega \left|\chi''(u_\varepsilon)\right| D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \int_\Omega \left|\chi''(u_\varepsilon)\right| S_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|\cdot|\nabla v_\varepsilon|\right] \\
&\quad + \|\psi\|_{L^2(\Omega)}\cdot\left[\left(\int_\Omega \chi'(u_\varepsilon)^2 D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2\right)^{\frac{1}{2}} + \left(\int_\Omega \chi'(u_\varepsilon)^2 S_\varepsilon(u_\varepsilon)\,|\nabla v_\varepsilon|^2\right)^{\frac{1}{2}}\right] \\
&\le \|\psi\|_{L^\infty(\Omega)}\cdot\left[\int_\Omega \left|\chi''(u_\varepsilon)\right| D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \int_\Omega \chi''(u_\varepsilon)^2 S_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2\right] \\
&\quad + \|\psi\|_{L^2(\Omega)}\cdot\left[1 + \int_\Omega \chi'(u_\varepsilon)^2 D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \int_\Omega \chi'(u_\varepsilon)^2 S_\varepsilon(u_\varepsilon)\,|\nabla v_\varepsilon|^2\right]
\end{aligned}$$

in $(0,\infty)$ and for every $\varepsilon \in (0,1)$. Since $W^{m,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, this means that there is some $C_1 > 0$ with

$$\begin{aligned}
\frac{1}{C_1}\left\|\partial_t\chi(u_\varepsilon)\right\|_{(W^{m,2}(\Omega))^*} \le{}& \int_\Omega \left|\chi''(u_\varepsilon)\right| D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \int_\Omega \chi''(u_\varepsilon)^2 S_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 \\
&+ 1 + \int_\Omega \chi'(u_\varepsilon)^2 D_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \int_\Omega \chi'(u_\varepsilon)^2 S_\varepsilon(u_\varepsilon)\,|\nabla v_\varepsilon|^2
\end{aligned}$$

in $(0,\infty)$ for every $\varepsilon \in (0,1)$ which in light of lemma VI.3.6 leaves us with the task of estimating four integrands: obviously for any $\varepsilon \in (0,1)$ we have $D_\varepsilon \le D + \varepsilon$ and

$S_\varepsilon \leq S$, and choosing some $C_2 > 0$ with $\operatorname{supp}\chi' \subset [0, C_2]$ we see

$$\begin{aligned}
\left|\chi''(u_\varepsilon)\right| \cdot D_\varepsilon(u_\varepsilon) &\leq C_3 := \left\|\chi''\right\|_{L^\infty((0,\infty))} \cdot \|D+1\|_{L^\infty((0,C_2))}, \\
\chi''(u_\varepsilon)^2 S_\varepsilon^2(u_\varepsilon) &\leq C_4 := \left\|\chi''\right\|^2_{L^\infty((0,\infty))} \cdot \|S\|^2_{L^\infty((0,C_2))}, \\
\chi'(u_\varepsilon)^2 \cdot D_\varepsilon^2(u_\varepsilon) &\leq C_5 := \left\|\chi'\right\|^2_{L^\infty((0,\infty))} \cdot \|D+1\|^2_{L^\infty((0,C_2))} \quad \text{and} \\
\chi'(u_\varepsilon)^2 S_\varepsilon^2(u_\varepsilon) &\leq C_6 := \left\|\chi'\right\|^2_{L^\infty((0,\infty))} \cdot \|S\|^2_{L^\infty((0,C_2))}
\end{aligned}$$

in $\Omega \times (0, \infty)$ for every $\varepsilon \in (0, 1)$. Collecting all these results shows

$$\left\|\partial_t \chi(u_\varepsilon)\right\|_{(W^{m,2}(\Omega))^*} \leq C_1 \cdot \left[1 + (C_3 + C_4 + C_5) \int_\Omega |\nabla u_\varepsilon|^2 + (1 + C_6) \int_\Omega |\nabla v_\varepsilon|^2\right]$$

in $(0, \infty)$ for every $\varepsilon \in (0, 1)$ upon which the two previous lemmata VI.3.7 and VI.3.6 complete the proof. **q.e.d.**

The corresponding result for the second solution component is more readily obtained, especially after having established the general strategy in the previous proof.

Lemma VI.3.12. *In the setting of lemma VI.3.1, let some* $\frac{n}{2} < m \in \mathbb{N}$ *be given. Then for every* $T > 0$ *we can find some* $C > 0$ *such that*

$$\int_0^T \left\|v_{\varepsilon t}(\cdot, t)\right\|_{\left(W_0^{m,2}(\Omega)\right)^*} \,\mathrm{d}t \leq C$$

holds for every $\varepsilon \in (0, 1)$.

Proof Just like before, we estimate for arbitrary $\psi \in W_0^{m,2}(\Omega)$ and see that

$$\left|\int_\Omega v_{\varepsilon t} \psi\right| \leq \int_\Omega |\nabla v_{\varepsilon t}| \cdot |\psi| + \int_\Omega v_\varepsilon |\psi| + \int_\Omega u_\varepsilon |\psi|$$

holds in $(0, \infty)$ for every $\varepsilon \in (0, 1)$. The mass conservation property enjoyed by every u_ε and the same arguments used in the proof of lemma VI.3.11 then establish the claimed bound. **q.e.d.**

VI.4 Passing to the limit and proving the main result

The approximate solutions and their bounds discussed in the previous section allow us to find a zero sequence $E \subset (0, 1)$ along which we can detect convergence like in lemma 5.10 of [93].

Lemma VI.4.1. *In the setting of theorem VI.1.1, we can find some zero sequence* $E \subset (0,1)$ *as well as two functions*

$$u \in L^\infty\left((0,\infty); L^{2-\alpha}(\Omega)\right) \cap \bigcap_{p>1} L^p_{loc}\left(\overline{\Omega} \times [0,\infty)\right) \cap L^2_{loc}\left([0,\infty); W^{1,2}(\Omega)\right)$$

and

$$\begin{aligned} v \in & L^\infty\left((0,\infty); W^{1,2}(\Omega)\right) \cap L^\infty_{loc}\left([0,\infty); W^{1,\infty}(\Omega)\right) \\ & \cap L^2_{loc}\left([0,\infty); W^{2,2}(\Omega)\right) \cap \bigcap_{p>1} L^p_{loc}\left((0,\infty); W^{2,p}(\Omega)\right) \end{aligned}$$

with the property that $u > 0$ *and* $v \geq 0$ *almost everywhere in* $\Omega \times (0,\infty)$ *and such that*

$$\begin{aligned} & u_\varepsilon \to u && \text{in } \bigcap_{p>1} L^p_{loc}\left(\overline{\Omega} \times [0,\infty)\right) \text{ and almost everywhere in } \Omega \times (0,\infty), \\ & u_\varepsilon(\cdot,t) \to u(\cdot,t) && \text{in } \bigcap_{p>1} L^p(\Omega) \text{ and almost everywhere in } \Omega \text{ for almost every } t > 0, \\ & \nabla u_\varepsilon \rightharpoonup \nabla u && \text{in } L^2_{loc}\left(\overline{\Omega} \times [0,\infty)\right), \\ & v_\varepsilon(\cdot,t) \to v(\cdot,t) && \text{in } W^{1,\infty}(\Omega) \text{ for almost every } t > 0, \\ & v_\varepsilon \rightharpoonup v && \text{in } \bigcap_{p>1} L^p_{loc}\left((0,\infty); W^{2,p}(\Omega)\right) \text{ and} \\ & v_{\varepsilon t} \rightharpoonup v_t && \text{in } L^2\left(\Omega \times (0,\infty)\right) \end{aligned}$$

as $\varepsilon \to 0$ *along* E*. Furthermore, these functions solve* (VI.4) *in the sense of definition VI.2.1.*

Proof We fix some sequence $(\chi_k)_{k\in\mathbb{N}} \subset C^\infty([0,\infty))$ with $\chi_k(s) = s$ for every $k \in \mathbb{N}$ as well as $s \in [0,k]$ and such that $\operatorname{supp} \chi_k'$ is bounded for every $k \in \mathbb{N}$. Due to these properties, for fixed $k \in \mathbb{N}$ and $T > 0$ we find some constant $C_1 > 0$ such that

$$\begin{aligned} \|\chi_k(u_\varepsilon)\|^2_{L^2((0,T);W^{1,2}(\Omega))} &= \int_0^T \int_\Omega \chi_k^2(u_\varepsilon) + \int_0^T \int_\Omega \chi_k'(u_\varepsilon)^2 \cdot |\nabla u_\varepsilon|^2 \\ &\leq C_1 |\Omega| T + C_1 \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \end{aligned}$$

which by lemma VI.3.7 is bounded independently of $\varepsilon \in (0,1)$. Accordingly,

$$(\chi_k(u_\varepsilon))_{\varepsilon\in(0,1)} \subset L^2_{\mathrm{loc}}\left([0,\infty); W^{1,2}(\Omega)\right)$$

is bounded for every $k \in \mathbb{N}$. Fixing some $m > \frac{n}{2}$, from lemma VI.3.11 we find that also

$$(\partial_t \chi_k(u_\varepsilon))_{\varepsilon\in(0,1)} \subset L^1_{\mathrm{loc}}\left([0,\infty); \left(W^{1,2}(\Omega)\right)^*\right)$$

is bounded for every $k \in \mathbb{N}$. As in chapter III, we now use the Aubin-Lions lemma I.3.12 to detect some measurable $u : \Omega \times (0,\infty) \to [0,\infty)$ and some zero sequence $E \subset (0,1)$ with

$$u_\varepsilon \to u \qquad \text{almost everywhere in } \Omega \times (0,\infty)$$

as $\varepsilon \to 0$ in E using a diagonal argument involving the parameter $k \in \mathbb{N}$ as well.

Furthermore, $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty\left((0,\infty); L^{2-\alpha}(\Omega)\right)$ according to lemma VI.3.6 as well as in $L^p(\Omega \times (0,T))$ for any $p > 1$ and $T > 0$ by lemma VI.3.7. The pointwise convergence of u_ε enables us to employ Vitali's theorem whereupon – after an adjustment to E – we find that the remaining regularity and convergence results for u hold.

Turning our attention to $(v_\varepsilon)_{\varepsilon\in(0,1)}$, we firstly see from lemma VI.3.6 its boundedness in $L^\infty\left((0,\infty); W^{1,2}(\Omega)\right)$. On the other hand, lemma VI.3.8 ensures the corresponding boundedness in $\bigcap_{p>1} L^p_{\mathrm{loc}}\left((0,\infty); W^{2,p}(\Omega)\right)$, while the boundedness of the sequence $(v_{\varepsilon t})_{\varepsilon\in(0,1)} \subset L^2(\Omega \times (0,\infty))$ as guaranteed by lemma VI.3.6 together with the observation $|\Delta v_\varepsilon| \le |v_{\varepsilon t}| + v + u$ proves that $(v_\varepsilon)_{\varepsilon\in(0,1)} \subset L^2_{\mathrm{loc}}\left([0,\infty); W^{2,2}(\Omega)\right)$ is bounded. Another employment of the Aubin-Lions lemma, by passing to another subsequence, completes this first portion of the proof, leaving us with discussing the solution properties of u and v.

The regularity requirements for u and v result from the inclusions proven before and as a byproduct we find that

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0$$

is true for almost every $t > 0$.

For the positivity of u we employ lemma VI.3.10 and Fatou's lemma so that for every $T > 0$ the convergence proven before gives us some positive $C(T)$ with

$$\int_\Omega \ln(u(\cdot,t)) \ge -C(T)$$

for almost every $t \in (0,T)$. Assuming therefore that $u = 0$ in a subset $M \subset \Omega \times (0,\infty)$ leads to a contradiction in light of the fact that – due to the elemental estimate $\ln(x) < x$ for every $x > 0$ – the potentially positive portion of this integral is clearly bounded from above by $\int_\Omega u_0$.

The remaining regularity requirements

$$u(\cdot,t)v(\cdot,t) \in L^1(\Omega) \quad \text{and} \quad G\left(u(\cdot,t)\right) \in L^1(\Omega)$$

for almost every $t > 0$ and

$$v_t \in L^2_{\mathrm{loc}}\left(\overline{\Omega} \times [0,\infty)\right)$$

as well as

$$\frac{\Sigma(u)}{h(u)} \nabla u \in L^2_{\mathrm{loc}}\left(\overline{\Omega} \times [0,\infty); \mathbb{R}^n\right)$$

and the validity of the energy estimate

$$\mathcal{F}\left(u(\cdot,t),v(\cdot,t)\right)+\int_0^t\mathcal{D}(s)\,\mathrm{d}s\le\mathcal{F}\left(u_0,v_0\right)$$

for almost every $t>0$ are a consequence of lemma VI.3.6, the previously secured convergences as well as Fatou's lemma and the lower semicontinuity of L^2-norms with respect to weak convergence.

One result requiring rather little work concerns the second equation in (VI.4): given the convergences above and the fact that $v_{\varepsilon t}=\Delta v_\varepsilon-v_\varepsilon+u_\varepsilon$ holds in $\Omega\times(0,\infty)$ for every $\varepsilon\in(0,1)$, it is easily verified that indeed for every $\varphi\in C_0^\infty\left(\overline{\Omega}\times(0,\infty)\right)$ the identity

$$\int_0^\infty\int_\Omega v_{\varepsilon t}\varphi=-\int_0^\infty\int_\Omega\nabla v_\varepsilon\cdot\nabla\varphi+\int_0^\infty\int_\Omega v_\varepsilon\varphi-\int_0^\infty\int_\Omega u_\varepsilon\varphi$$

holds.

In order to complete this proof, we now need to verify that the first equation in (VI.4) is solved in the sense of definition VI.2.1.

To this end we fix some $\chi\in C^\infty\left([0,\infty)\right)$ with $\operatorname{supp}\chi'\subset\subset[0,\infty)$ and $\chi''\le0$ in $[0,\infty)$ as well as a nonnegative test function $\varphi\in C_0^\infty\left(\overline{\Omega}\times[0,\infty)\right)$. Then for every $\varepsilon\in(0,1)$

$$\begin{aligned}-\int_0^\infty\int_\Omega\chi''(u_\varepsilon)D_\varepsilon(u_\varepsilon)\left|\nabla u_\varepsilon\right|^2\varphi=&-\int_0^\infty\int_\Omega\chi(u_\varepsilon)\varphi_t-\int_\Omega\chi(u_0)\varphi(\cdot,0)\\&-\int_0^\infty\int_\Omega\chi''(u_\varepsilon)S_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon\varphi\\&+\int_0^\infty\int_\Omega\chi'(u_\varepsilon)D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\varphi\\&-\int_0^\infty\int_\Omega\chi'(u_\varepsilon)S_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla\varphi\end{aligned}\qquad(*)$$

holds. The pointwise convergence $u_\varepsilon\to u$ almost everywhere in $\Omega\times(0,\infty)$ coupled with the obvious boundedness of χ and Vitali's theorem gives us

$$-\int_0^\infty\int_\Omega\chi(u_\varepsilon)\varphi_t\to-\int_0^\infty\int_\Omega\chi(u)\varphi_t$$

as $\varepsilon\to0$ in E. Analogously we can see the additional convergences

$$\begin{aligned}&\chi''(u_\varepsilon)S_\varepsilon(u_\varepsilon)\to\chi''(u)S(u)\quad\text{in }L^4_{\mathrm{loc}}\left(\overline{\Omega}\times[0,\infty)\right),\\&\chi'(u_\varepsilon)D_\varepsilon(u_\varepsilon)\to\chi'(u)D(u)\quad\text{in }L^2_{\mathrm{loc}}\left(\overline{\Omega}\times[0,\infty)\right)\text{ and}\\&\chi'(u_\varepsilon)S_\varepsilon(u_\varepsilon)\to\chi'(u)S(u)\quad\text{in }L^{\frac{4}{3}}_{\mathrm{loc}}\left(\overline{\Omega}\times[0,\infty)\right)\end{aligned}$$

as $\varepsilon \to 0$ in E so that the convergences $\nabla u_\varepsilon \rightharpoonup \nabla u$ in $L^2_{\text{loc}}\left(\overline{\Omega} \times [0,\infty)\right)$ and $\nabla v_\varepsilon \rightharpoonup \nabla v$ in $L^1_{\text{loc}}\left(\overline{\Omega} \times [0,\infty)\right)$ ensure

$$-\int_0^\infty \int_\Omega \chi''(u_\varepsilon) S_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon \varphi \to -\int_0^\infty \int_\Omega \chi''(u) S(u) \nabla u \cdot \nabla v \varphi$$

as well as

$$\int_0^\infty \int_\Omega \chi'(u_\varepsilon) D_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi \to \int_0^\infty \int_\Omega \chi'(u) D(u) \nabla u \cdot \nabla \varphi$$

and

$$-\int_0^\infty \int_\Omega \chi'(u_\varepsilon) S_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi \to -\int_0^\infty \int_\Omega \chi'(u) S(u) \nabla v \cdot \nabla \varphi$$

as $\varepsilon \to 0$ in E upon checking that the exponents complement each other. For the remaining term we once more use the convergence of $(u_\varepsilon)_{\varepsilon \in (0,1)}$ (as well as the fact that $\chi''(s) D_\varepsilon(s) \leq 0$ for every $s \geq 0$ and $\varepsilon \in (0,1)$) to see

$$\sqrt{-\chi''(u_\varepsilon) D_\varepsilon(u_\varepsilon)} \nabla u_\varepsilon \rightharpoonup \sqrt{-\chi''(u) D(u)} \nabla u \quad \text{in } L^2_{\text{loc}}\left(\overline{\Omega} \times [0,\infty)\right)$$

as $\varepsilon \to 0$ in E and again from lower semicontinuity we then find

$$-\int_0^\infty \int_\Omega \chi''(u) D(u) |\nabla u|^2 \varphi \leq -\liminf_{E \ni \varepsilon \to 0} \int_0^\infty \int_\Omega \chi''(u_\varepsilon) D_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 \varphi$$

which upon rearrangement of the identity $(*)$ completes the proof. **q.e.d.**

Proof of theorem VI.1.1 Obviously everything has been achieved with the previous lemma.

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