Regularity properties of infinite-dimensional Lie groups and exponential laws

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Abstract

In the first part of this thesis, we recall the concept of differentiability of vector-valued functions on topological groups along one-parameter subgroups and introduce a notion of $C^{k,l}$-functions on products of topological groups. We study the properties of C^k- and $C^{k,l}$-functions and of the locally convex spaces $C^k(G,E)$ and $C^{k,l}(G \times H, E)$. Further, we prove an exponential law of the form $C^{k,l}(G \times H, E) \cong C^k(G, C^l(H,E))$, which holds under suitable hypotheses on G and H.

In the second part of the thesis, we show that in cases where G is a locally exponential Lie group or a certain direct limit Lie group the above calculus of C^k-functions coincides with the differential calculus on G as a locally convex manifold.

In the third part, we discuss Lebesgue spaces $L^p([a,b],E)$ of Lusin-measurable vector-valued functions and the corresponding vector spaces $AC_{L^p}([a,b],E)$ of absolutely continuous functions. These are used to construct Lie groups $AC_{L^p}([a,b],G)$ of absolutely continuous functions with values in an infinite-dimensional Lie group G. We extend the notion of L^p-regularity of infinite-dimensional Lie groups introduced by Glöckner to this setting and adapt several results and tools.

Deutsche Zusammenfassung: Im ersten Teil der Dissertation wiederholen wir den Begriff der Differenzierbarkeit von vektorwertigen Funktionen auf topologischen Gruppen entlang der Einparametergruppen und wir führen den Begriff der $C^{k,l}$-Funktionen auf Produkten von topologischen Gruppen ein. Wir untersuchen die Eigenschaften der C^k- und $C^{k,l}$-Funktionen sowie der lokalkonvexen Räume $C^k(G, E)$ und $C^{k,l}(G \times H, E)$. Weiter beweisen wir ein Exponentialgesetz von der Form $C^{k,l}(G \times H, E) \cong C^k(G, C^l(H, E))$, welches unter bestimmten Voraussetzungen an G und H gilt.

Im zweiten Teil der Arbeit zeigen wir, dass falls G eine lokal exponentielle Liegruppe oder ein direkter Limes bestimmter Liegruppen ist, das obere Differentialkalkül mit dem Differentialkalkül auf G als lokalkonvexe Mannigfaltigkeit übereinstimmt.

Im dritten Teil untersuchen wir Lebesgue-Räume $L^p([a, b], E)$ der Lusin-messbaren vektorwertigen Funktionen und die Vektorräume $AC_{L^p}([a, b], E)$ der entsprechenden absolutstetigen Funktionen. Diese nutzen wir um Liegruppen $AC_{L^p}([a, b], G)$ der absolutstetigen Funktionen mit Werten in einer unendlich-dimensionalen Liegruppe G zu konstruieren. Wir erweitern den Begriff der L^p-Regularität von unendlich-dimensionalen Liegruppen, eingeführt von Glöckner, auf diesen Rahmen und passen einige Ergebnisse an.

1 Introduction

Exponential laws

Exponential laws of the form $C^\infty(M \times N, E) \cong C^\infty(M, C^\infty(N, E))$ for spaces of vector-valued smooth functions on manifolds are essential tools in infinite-dimensional calculus and infinite-dimensional Lie theory (cf. works by Kriegl and Michor [28], Kriegl, Michor and Rainer [29], Alzaareer and Schmeding [1], Glöckner [19], Glöckner and Neeb [21], Neeb and Wagemann [34], and others). Stimulated by the research by Beltiţă and Nicolae [4], we devote the first part of this work to providing exponential laws for function spaces on topological groups.

Let G be a topological group, $U \subseteq G$ be an open subset, $f\colon U \to E$ be a function to a locally convex space and $\mathfrak{L}(G) := \mathrm{Hom}_{cts}(\mathbb{R}, G)$ be the set of continuous one-parameter subgroups $\gamma\colon \mathbb{R} \to G$, endowed with the compact-open topology. For $x \in U$ and $\gamma \in \mathfrak{L}(G)$ let us write

$$D_\gamma f(x) := \lim_{t\to 0} \frac{1}{t}(f(x \cdot \gamma(t)) - f(x))$$

if the limit exists. Following Riss [40] and Boseck et al. [8], we say that f is C^k (where $k \in \mathbb{N}_0 \cup \{\infty\}$) if f is continuous, the iterated derivatives

$$d^{(i)} f(x, \gamma_1, \ldots, \gamma_i) := (D_{\gamma_i} \cdots D_{\gamma_1} f)(x)$$

exist for all $x \in U$, $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$, and the obtained maps $d^{(i)} f\colon U \times \mathfrak{L}(G)^i \to E$ are continuous. We endow the space $C^k(U, E)$ of all C^k-maps $f\colon U \to E$ with the compact-open C^k-topology (recalled in Definition 2.1.3). If G and H are topological groups and $f\colon G \times H \to E$ is C^∞, then $f^\vee(x) := f(x, \bullet) \in C^\infty(H, E)$ for all $x \in G$. With a view towards universal enveloping algebras, Beltiţă and Nicolae [4] verified that $f^\vee \in C^\infty(G, C^\infty(H, E))$ and showed that the linear map

$$\Phi\colon C^\infty(G \times H, E) \to C^\infty(G, C^\infty(H, E)), \quad f \mapsto f^\vee$$

is a topological embedding.

Recall that a Hausdorff space X is called a $k_\mathbb{R}$-space if functions $f\colon X \to \mathbb{R}$ are continuous if and only if $f\big|_K$ is continuous for each compact subset $K \subseteq X$. We obtain the following criterion for surjectivity of Φ (Theorem 2.5.5):

Theorem A. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, and E be a locally convex space. If $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_\mathbb{R}$-space for all $i, j \in \mathbb{N}_0$, then*

$$\Phi\colon C^\infty(U \times V, E) \to C^\infty(U, C^\infty(V, E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces.

The condition is satisfied, for example, if both G and H are locally compact or both G and H are metrizable (see Corollary 2.5.7).

Generalizing the case of open subsets U and V in locally convex spaces treated by Alzaareer and Schmeding [1] and Glöckner and Neeb [21], we introduce $C^{k,l}$-functions $f\colon U \times V \to E$ on open subsets $U \subseteq G$ and $V \subseteq H$ of topological groups with separate degrees $k, l \in \mathbb{N}_0 \cup \{\infty\}$ of differentiability in the two variables, and a natural topology on the space $C^{k,l}(U \times V, E)$ of such maps (see Definition 2.2.1 for details). Theorem A is a consequence of the following result (Theorem 2.5.4):

Theorem B. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. If $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$-space for all $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$, then*

$$\Phi\colon C^{k,l}(U \times V, E) \to C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces.

A special case of the above theorem (for subsets U, V of locally convex spaces) can also be found in [1].

Differentiability on Lie groups

Let $f\colon G \to E$ be a function on an infinite-dimensional Lie group G with values in a locally convex space E. Another possible concept of differentiability of such functions was popularized by Milnor [30], where G is considered as a differentiable infinite-dimensional manifold and the differential calculus arises from the calculus of functions between locally convex spaces, the so-called Keller-C^k_c-calculus [27], going back to A. Bastiani [2] (we will call such functions C^k_{mfd}*-functions*, see Definition 3.1.1). The second part of this work is devoted to the question, under which conditions do both the concepts of differentiability of vector-valued functions on infinite-dimensional Lie groups coincide. We obtain the following result (Theorem 3.2.10):

Theorem C. *Let E be a locally convex space and $k \in \mathbb{N}_0 \cup \{\infty\}$. If G is a locally exponential Lie group or a direct limit Lie group of an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite-dimensional Lie groups such that the inclusions $G_n \to G_{n+1}$ are continuous, then $C^k(G, E) = C^k_{\mathrm{mfd}}(G, E)$ as topological vector spaces.*

The interplay of differentiability along one-parameter subgroups and differentiability on a Lie group G as a manifold plays a role, for example, in the study of spaces of smooth vectors, cf [31], [33].

Measurable regularity of Lie groups

In [30], Milnor calls an infinite-dimensional Lie group G modelled on a sequentially complete locally convex space (with Lie algebra $\mathfrak{g}$ and identity element e) *regular* if for every smooth curve $\gamma\colon [0,1] \to \mathfrak{g}$ the initial value problem

$$\eta' = \eta.\gamma, \quad \eta(0) = e, \tag{1.1}$$

has a (necessarily unique) solution $\mathrm{Evol}(\gamma)\colon [0,1] \to G$ and the function

$$\mathrm{evol}\colon C^\infty([0,1],\mathfrak{g}) \to G, \quad \gamma \mapsto \mathrm{Evol}(\gamma)(1)$$

so obtained is smooth.

Further, Glöckner [20] and Neeb [32] deal with the concept of *C^k-regularity*, investigating whether the above initial value problem has a solution for every C^k-curve γ (the solution $\mathrm{Evol}(\gamma)$ being a C^{k+1}-curve then) and whether the function $\mathrm{evol}\colon C^k([0,1],\mathfrak{g}) \to G$ is smooth.

Generalizing this theory even more, in [17] Glöckner constructs Lebesgue spaces $L^p_B([a,b],E)$ of Borel measurable functions $\gamma\colon [a,b] \to E$ with values in Fréchet spaces E (for $p \in [1,\infty]$) and introduces spaces of certain absolutely continuous E-valued functions $\eta\colon [a,b] \to E$ (denoted by $AC_{L^p}([a,b],E)$) with derivatives in $L^p_B([a,b],E)$. Having a Lie group structure on the spaces $AC_{L^p}([0,1],G)$ available, in [17] a Fréchet-Lie group G is called *L^p-semiregular* if the initial value problem (1.1) has a solution $\mathrm{Evol}(\gamma) \in AC_{L^p}([0,1],G)$ for every $\gamma \in L^p_B([0,1],\mathfrak{g})$, and G is called *L^p-regular* if it is L^p-semiregular and the map $\mathrm{Evol}\colon L^p_B([0,1],\mathfrak{g}) \to AC_{L^p}([0,1],G), \gamma \mapsto \mathrm{Evol}(\gamma)$ is smooth.

Since the sum of two vector-valued Borel measurable functions may be not Borel measurable, certain assumptions need to be made to obtain a vector space structure on the space of the maps which can be considered. This implies that the concepts of *L^p-regularity* (mentioned above) only make sense for Fréchet-Lie groups (and some other classes of Lie groups described in [17]).

To loosen this limitation, in the third part of this work we recall the notion of *Lusin-measurable* functions in Definition 4.1.2, which have the advantage that vector-valued Lusin-measurable functions always form a vector space, and define the corresponding Lebesgue spaces $L^p([a,b],E)$ in Definition 4.1.10. Further, in Lemma 4.1.8, we recall that under certain conditions there is a close relation between Lusin and Borel measurable functions (known as Lusin's Theorem). This leads to the result that the Lebesgue spaces $L^p_B([a,b],E)$ constructed in [17] coincide with our Lebesgue spaces $L^p([a,b],E)$, due to the conditions needed for Borel measurable functions to form a vector space. (Note that Lebesgue spaces of Lusin-measurable functions are also considered by Florencio, Mayoral, Paúl [11], for example. Also Bourbaki [9], Thomas [43] work with Lusin measurability.)

We lean on the theory established in [17] and construct locally convex topological vector spaces $AC_{L^p}([a,b],E)$ of functions with values in sequentially complete locally convex spaces and Lie groups $AC_{L^p}([a,b],G)$. In Definition 4.3.7 we define the notion of L^p-regularity for infinite-dimensional Lie groups modelled on such spaces and adopt several useful results from [17]. In particular (Theorem 4.3.9):

Theorem D. *If G is an L^p-semiregular Lie group, then the function* $\mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to AC_{L^p}([0,1],G)$ *is smooth if and only if* Evol *is smooth as a function to $C([0,1],G)$.*

As a consequence, we get (Theorem 4.3.10):

Theorem E. *Let G be a Lie group modelled on a sequentially complete locally convex space and $p, q \in [1, \infty]$ with $q \geq p$. If G is L^p-regular, then G is L^q-regular. Furthermore, in this case G is C^0-regular.*

Moreover, we show (Theorem 4.3.13):

Theorem F. *Let G be a Lie group modelled on a sequentially complete locally convex space. Let $\Omega \subseteq L^p([0,1], \mathfrak{g})$ be an open 0-neighbourhood. If for every $\gamma \in \Omega$ there exists the corresponding $\mathrm{Evol}(\gamma) \in AC_{L^p}([0,1], G)$, then G is L^p-semiregular. If, in addition, the function $\mathrm{Evol}\colon \Omega \to AC_{L^p}([0,1], G)$ is smooth, then G is L^p-regular.*

Notation All topological spaces are assumed Hausdorff. We call a function $f\colon X \to Y$ between topological spaces X and Y a *topological embedding* if f is a homeomorphism onto its image (it is known that an injective map f is a topological embedding if and only if the topology on X is initial with respect to f, that is, X carries the coarsest topology making f continuous). All vector spaces are $\mathbb{R}$-vector spaces (and locally convex topological vector spaces are called "locally convex spaces" for short). Wherever we write $[a, b]$, we always mean an interval in $\mathbb{R}$ with $a < b$.

2 Exponential laws for spaces of differentiable functions on topological groups

In Section 2.1, we recall the notion of a C^k-function $f\colon U \to E$ on an open subset of a topological group with values in a locally convex space and the definition of the locally convex space $C^k(U,E)$. Further, in Section 2.2, we consider vector-valued functions on products of topological groups with different degrees of differentiability in the two factors (called $C^{k,l}$-functions) and the associated function spaces $C^{k,l}(U \times V, E)$. After studying some properties of differentiable functions and the function spaces (Sections 2.3 and 2.4), we prove the exponential law $C^{k,l}(U \times V, E) \cong C^k(U, C^l(V,E))$ in Section 2.5 (Theorem 2.5.4).

2.1 Differentiability on topological groups

Definition 2.1.1. Let G be a topological group. A *one-parameter subgroup* is a group homomorphism $\gamma\colon \mathbb{R} \to G$. We denote by $\mathfrak{L}(G) := \mathrm{Hom}_{cts}(\mathbb{R}, G)$ the set of all continuous one-parameter subgroups endowed with the compact-open topology.

Note that the space $\mathfrak{L}(G)$ does not have a topological vector space structure in general.

Remark 2.1.2. For a topological group G, the evaluation map $\mathfrak{L}(G) \times \mathbb{R} \to G, (\gamma, t) \mapsto \gamma(t)$ is continuous.

If G, H are topological groups, $\gamma \in \mathfrak{L}(G)$ and $\varphi\colon G \to H$ is a continuous group homomorphism, then $\varphi \circ \gamma \in \mathfrak{L}(H)$ and the function $\mathfrak{L}(\varphi)\colon \mathfrak{L}(G) \to \mathfrak{L}(H), \gamma \mapsto \varphi \circ \gamma$ is continuous (cf. [21, Appendix A.5], see also [14, Appendix B]).

For $\psi = (\gamma, \eta) \in C(\mathbb{R}, G \times H)$ it is easy to see that $\psi \in \mathfrak{L}(G \times H)$ if and only if $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$. Moreover, the natural map

$$(\mathfrak{L}(\mathrm{pr}_1), \mathfrak{L}(\mathrm{pr}_2))\colon \mathfrak{L}(G \times H) \to \mathfrak{L}(G) \times \mathfrak{L}(H)$$

(where $\mathrm{pr}_1\colon G \times H \to G$, $\mathrm{pr}_2\colon G \times H \to H$ are the coordinate projections) is a homeomorphism (cf. [21, Appendix A.5], [14, Appendix B]).

Now, we recall the notion of differentiability along one-parameter subgroups of vector-valued functions on topological groups:

Definition 2.1.3. Let $U \subseteq G$ be an open subset of a topological group G and E be a locally convex space. For a map $f\colon U \to E$, $x \in U$ and $\gamma \in \mathfrak{L}(G)$ we define

$$d^{(1)}f(x,\gamma) := df(x,\gamma) := D_\gamma f(x) := \lim_{t\to 0} \frac{1}{t}(f(x\cdot\gamma(t)) - f(x))$$

if the limit exists.

We call f a C^k*-map* for $k \in \mathbb{N}$ if f is continuous and for each $x \in U$, $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$ the iterative derivatives

$$d^{(i)}f(x,\gamma_1,\ldots,\gamma_i) := (D_{\gamma_i}\cdots D_{\gamma_1}f)(x)$$

exist and define continuous maps

$$d^{(i)}f\colon U \times \mathfrak{L}(G)^i \to E, \quad (x,\gamma_1,\ldots,\gamma_i) \mapsto (D_{\gamma_i}\cdots D_{\gamma_1}f)(x).$$

If f is C^k for each $k \in \mathbb{N}$, then we call f a C^∞*-map* or *smooth*. Further, we call continuous maps C^0 and write $d^{(0)}f := f$.

The set of all C^k-maps $f\colon U \to E$ will be denoted by $C^k(U,E)$ and we endow it with the initial topology with respect to the family $(d^{(i)})_{i\in\mathbb{N}_0, i\leq k}$ of maps

$$d^{(i)}\colon C^k(U,E) \to C(U \times \mathfrak{L}(G)^i, E), \quad f \mapsto d^{(i)}f$$

(where the right-hand side is equipped with the compact-open topology) turning $C^k(U,E)$ into a Hausdorff locally convex space. (This topology is known as the *compact-open* C^k*-topology.*)

Remark 2.1.4. Note that the compact-open topology on $C(U,E)$ coincides with the compact-open C^0-topology.

Remark 2.1.5. Let E, F be locally convex spaces and $f\colon U \to F$ be a continuous function on an open subset $U \subseteq E$. The directional derivative is defined as

$$df(x,y) := D_y f(x) := \lim_{t\to 0}\frac{1}{t}(f(x+ty) - f(x)),$$

for $x \in U$, $y \in E$. The function f is called C^k if for each $1 \leq i \leq k$ the differential

$$d^{(i)}f\colon U \times E^i \to F, \quad d^{(i)}f(x,y_1,\ldots,y_i) := (D_{y_i}\cdots D_{y_1}f)(x)$$

is defined and continuous. If f is C^k for each $k \in \mathbb{N}$, then f is called C^∞. This concept can be understood as a special case of the concept in Definition 2.1.3, as E is, in particular, a topological group and $E \cong \mathfrak{L}(E)$ via $y \mapsto \gamma_y$, where γ_y denotes the one-parameter subgroup of E of the form $t \mapsto ty$.

In the case $E = \mathbb{R}$, we write

$$f'\colon U \to F, \quad f'(s) := \lim_{t\to 0}\frac{1}{t}(f(s+t) - f(s)).$$

2.2 Differentiability on products of topological groups

Definition 2.2.1. Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space. For a map $f\colon U \times V \to E$, $x \in U$, $y \in V$, $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$ we define

$$d^{(1,0)}f(x,y,\gamma) := D_{(\gamma,0)}f(x,y) := \lim_{t\to 0}\frac{1}{t}(f(x\cdot\gamma(t),y) - f(x,y))$$

and

$$d^{(0,1)}f(x,y,\eta) := D_{(0,\eta)}f(x,y) := \lim_{t\to 0}\frac{1}{t}(f(x,y\cdot\eta(t)) - f(x,y))$$

whenever the limits exist.

We call a continuous map $f\colon U \times V \to E$ a $C^{k,l}$*-map* for $k,l \in \mathbb{N}_0 \cup \{\infty\}$ if the derivatives

$$d^{(i,j)}f(x,y,\gamma_1,\ldots,\gamma_i,\eta_1,\ldots,\eta_j) := (D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}f)(x,y)$$

exist for all $x \in U$, $y \in V$, $i,j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and $\gamma_1,\ldots,\gamma_i \in \mathfrak{L}(G)$, $\eta_1,\ldots,\eta_j \in \mathfrak{L}(H)$, and define continuous functions

$$\begin{aligned} d^{(i,j)}f\colon & U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \to E \\ & (x,y,\gamma_1,\ldots,\gamma_i,\eta_1,\ldots,\eta_j) \mapsto (D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}f)(x,y). \end{aligned}$$

We endow the space $C^{k,l}(U \times V, E)$ of all $C^{k,l}$-functions $f\colon U \times V \to E$ with the initial topology with respect to the family of maps

$$d^{(i,j)}\colon C^{k,l}(U\times V,E) \to C(U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, E), \quad f \mapsto d^{(i,j)}f,$$

with $i,j \in \mathbb{N}_0, i \leq k, j \leq l$, which is a Hausdorff locally convex vector topology (called the *compact-open* $C^{k,l}$*-topology*.)

Remark 2.2.2. If $k = 0$ or $l = 0$, then the definition of $C^{k,l}$-maps $f\colon U \times V \to E$ also makes sense if U or V, respectively, is any Hausdorff topological space. All further results for $C^{k,l}$-maps on topological groups carry over to this situation.

Remark 2.2.3. Simple computations show that for $k \geq 1$ a map $f\colon U \to E$ is C^k if and only if f is C^1 and $df\colon U \times \mathfrak{L}(G) \to E$ is $C^{k-1,0}$; in this case we have $d^{(i,0)}(df) = d^{(i+1)}f$ for all $i \in \mathbb{N}$ with $i \leq k-1$.

Similarly, we can show that a map $f\colon U \times V \to E$ is $C^{k,0}$ if and only if f is $C^{1,0}$ and $d^{(1,0)}f\colon U \times (V \times \mathfrak{L}(G)) \to E$ is $C^{k-1,0}$, then $d^{(i,0)}(d^{(1,0)}f) = d^{(i+1,0)}f$ for all i as above.

Further, if a map $f\colon U \times V \to E$ is $C^{k,l}$, then for each $i,j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and fixed $\gamma_1,\ldots,\gamma_i \in \mathfrak{L}(G)$, $\eta_1,\ldots,\eta_j \in \mathfrak{L}(H)$ the map

$$D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}f\colon U \times V \to E$$

is $C^{k,l-j}$ if $i = 0$, and $C^{k-i,0}$ otherwise.

We warn the reader that the full statement of Schwarz' Theorem does not carry over to non-abelian topological groups; for a C^2-function $f\colon G \to \mathbb{R}$ and $\gamma, \eta \in \mathfrak{L}(G)$ it can happen that $D_\gamma D_\eta f \neq D_\eta D_\gamma f$.

Example 2.2.4. Consider the following subgroup G of $GL_3(\mathbb{R})$:

$$G := \left\{ x = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

(known as the Heisenberg group) and $\gamma, \eta \in \mathfrak{L}(G)$ defined as

$$\gamma(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \qquad (\forall t \in \mathbb{R}).$$

Then $(G, \cdot) \cong (\mathbb{R}^3, *)$ as topological groups via

$$\varphi\colon G \to \mathbb{R}^3, \quad x := \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x_1, x_2, x_3),$$

where the group multiplication $*\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$\begin{aligned}(x_1, x_2, x_3) * (y_1, y_2, y_3) &:= \varphi(\varphi^{-1}(x_1, x_2, x_3) \cdot \varphi^{-1}(y_1, y_2, y_3)) \\ &= (x_1 + y_1, x_2 + x_1 y_3 + y_2, x_3 + y_3).\end{aligned}$$

Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^2-map in the usual sense and define

$$f := g \circ \varphi\colon G \to \mathbb{R},$$

which is a C^2-map by Lemma 2.3.2. Then for each $x \in G$, the derivatives $D_\gamma f(x)$, $D_\eta f(x)$, $(D_\eta D_\gamma f)(x)$ and $(D_\gamma D_\eta f)(x)$ can be expressed using the partial derivatives of g.

First, we have

$$\begin{aligned}D_\gamma f(x) &= \lim_{t \to 0} \frac{1}{t}(f(x \cdot \gamma(t)) - f(x)) = \lim_{t \to 0} \frac{1}{t}(g(\varphi(x \cdot \gamma(t))) - g(\varphi(x))) \\ &= \lim_{t \to 0} \frac{1}{t}(g(x_1 + t, x_2, x_3) - g(x_1, x_2, x_3)) \\ &= \lim_{t \to 0} \frac{1}{t}(g((x_1, x_2, x_3) + t(1, 0, 0)) - g(x_1, x_2, x_3)) = \frac{\partial}{\partial x_1} g(x_1, x_2, x_3).\end{aligned}$$

Further,

$$\begin{aligned}D_\eta f(x) &= \lim_{t \to 0} \frac{1}{t}(f(x \cdot \eta(t)) - f(x)) = \lim_{t \to 0} \frac{1}{t}(g(\varphi(x \cdot \eta(t))) - g(\varphi(x))) \\ &= \lim_{t \to 0} \frac{1}{t}(g(x_1, x_2 + t x_1, x_3 + t) - g(x_1, x_2, x_3)) \\ &= x_1 \cdot \frac{\partial}{\partial x_2} g(x_1, x_2, x_3) + \frac{\partial}{\partial x_3} g(x_1, x_2, x_3).\end{aligned}$$

Now,

$$\begin{aligned}(D_\eta D_\gamma f)(x) &= \lim_{t\to 0}\frac{1}{t}(D_\gamma f(x\cdot \eta(t)) - D_\gamma f(x))\\ &= \lim_{t\to 0}\frac{1}{t}\left(\frac{\partial}{\partial x_1}g(x_1, x_2 + tx_1, x_3 + t) - \frac{\partial}{\partial x_1}g(x_1,x_2,x_3)\right)\\ &= x_1\cdot\frac{\partial^2}{\partial x_1\partial x_2}g(x_1,x_2,x_3) + \frac{\partial^2}{\partial x_1\partial x_3}g(x_1,x_2,x_3).\end{aligned}$$

And, finally

$$\begin{aligned}&(D_\gamma D_\eta f)(x) = \lim_{t\to 0}\frac{1}{t}(D_\eta f(x\cdot\gamma(t)) - D_\eta f(x))\\ &= \lim_{t\to 0}\frac{1}{t}\left((x_1+t)\cdot\frac{\partial}{\partial x_2}g(x_1+t,x_2,x_3) + \frac{\partial}{\partial x_3}g(x_1+t,x_2,x_3)\right.\\ &\quad\left. - x_1\cdot\frac{\partial}{\partial x_2}g(x_1,x_2,x_3) - \frac{\partial}{\partial x_3}g(x_1,x_2,x_3)\right)\\ &= \lim_{t\to 0}\frac{x_1}{t}\left(\frac{\partial}{\partial x_2}g(x_1+t,x_2,x_3) - \frac{\partial}{\partial x_2}g(x_1,x_2,x_3)\right)\\ &\quad + \lim_{t\to 0}\frac{1}{t}\left(\frac{\partial}{\partial x_3}g(x_1+t,x_2,x_3) - \frac{\partial}{\partial x_3}g(x_1,x_2,x_3)\right) + \lim_{t\to 0}\frac{\partial}{\partial x_2}g(x_1+t,x_2,x_3)\\ &= x_1\cdot\frac{\partial^2}{\partial x_1\partial x_2}g(x_1,x_2,x_3) + \frac{\partial^2}{\partial x_1\partial x_3}g(x_1,x_2,x_3) + \frac{\partial}{\partial x_2}g(x_1,x_2,x_3)\\ &= (D_\eta D_\gamma f)(x) + \frac{\partial}{\partial x_2}g(x_1,x_2,x_3).\end{aligned}$$

Thus we see that if $\frac{\partial}{\partial x_2}g(x_1,x_2,x_3)\neq 0$, then $(D_\gamma D_\eta f)(x)\neq (D_\eta D_\gamma f)(x)$.

Nevertheless, we can prove the following restricted version of Schwarz' Theorem for $C^{k,l}$-maps:

Proposition 2.2.5. *Let $U\subseteq G$ and $V\subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $f\colon U\times V\to E$ be a $C^{k,l}$-map for some $k,l\in\mathbb{N}\cup\{\infty\}$. Then the derivatives*

$$(D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}f)(x,y)$$

exist for all $(x,y)\in U\times V$, $i,j\in\mathbb{N}$ with $i\leq k$, $j\leq l$ and $\gamma_1,\dots,\gamma_i\in\mathfrak{L}(G)$, $\eta_1,\dots,\eta_j\in\mathfrak{L}(H)$ and we have

$$\begin{aligned}&(D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}f)(x,y)\\ &\qquad\qquad = (D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}f)(x,y).\end{aligned}$$

Proof. First we prove the assertion for $j=1$ by induction on i.

Base case: Let $(x,y) \in U \times V$, $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$. For suitable $\varepsilon, \delta > 0$ we define the continuous map

$$h\colon]-\varepsilon,\varepsilon[\times]-\delta,\delta[\to E, \quad (s,t)\mapsto f(x\cdot\gamma(s), y\cdot\eta(t)),$$

and obtain the partial derivatives of h via

$$\begin{aligned}\frac{\partial h}{\partial s}(s,t) &= \lim_{r\to 0}\frac{1}{r}(h(s+r,t)-h(s,t))\\ &= \lim_{r\to 0}\frac{1}{r}(f(x\cdot\gamma(s)\cdot\gamma(r), y\cdot\eta(t)) - f(x\cdot\gamma(s), y\cdot\eta(t)))\\ &= D_{(\gamma,0)}f(x\cdot\gamma(s), y\cdot\eta(t)),\end{aligned}$$

and analogously,

$$\frac{\partial h}{\partial t}(s,t) = D_{(0,\eta)}f(x\cdot\gamma(s), y\cdot\eta(t))$$

and

$$\frac{\partial^2 h}{\partial s\partial t}(s,t) = (D_{(\gamma,0)}D_{(0,\eta)}f)(x\cdot\gamma(s), y\cdot\eta(t)).$$

The obtained maps $\frac{\partial h}{\partial s}$, $\frac{\partial h}{\partial t}$ and $\frac{\partial^2 h}{\partial s\partial t}$ are continuous, hence we apply [21, Lemma 1.3.18], which states that in this case also the partial derivative $\frac{\partial^2 h}{\partial t\partial s}$ exists and coincides with $\frac{\partial^2 h}{\partial s\partial t}$. Therefore, we have

$$\begin{aligned}(D_{(\gamma,0)}D_{(0,\eta)}f)(x,y) &= \frac{\partial^2 h}{\partial s\partial t}(0,0) = \frac{\partial^2 h}{\partial t\partial s}(0,0) = \lim_{r\to 0}\frac{1}{r}\left(\frac{\partial h}{\partial s}(0,r) - \frac{\partial h}{\partial s}(0,0)\right)\\ &= \lim_{r\to 0}\frac{1}{r}\left(D_{(\gamma,0)}f(x, y\cdot\eta(r)) - D_{(\gamma,0)}f(x,y)\right)\\ &= (D_{(0,\eta)}D_{(\gamma,0)}f)(x,y).\end{aligned}$$

Thus the assertion holds for $i = 1$.

Induction step: Now, let $2 \le i \le k$, $(x,y) \in U \times V$, $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$. Consider the map

$$g_1\colon U\times V\to E, \quad (x,y)\mapsto (D_{(\gamma_{i-1},0)}\cdots D_{(\gamma_1,0)}f)(x,y),$$

which is $C^{1,0}$ (see Remark 2.2.3). Further, g_1 is $C^{0,1}$, because

$$\begin{aligned}D_{(0,\eta)}g_1(x,y) &= (D_{(0,\eta)}D_{(\gamma_{i-1},0)}\cdots D_{(\gamma_1,0)}f)(x,y)\\ &= (D_{(\gamma_{i-1},0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta)}f)(x,y),\end{aligned}$$

by the induction hypothesis, and we see that

$$(D_{(\gamma_i,0)}D_{(0,\eta)}g_1)(x,y) = (D_{(\gamma_i,0)}D_{(\gamma_{i-1},0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta)}f)(x,y),$$

whence g_1 is $C^{1,1}$. By the base case, the derivative $(D_{(0,\eta)}D_{(\gamma_i,0)}g_1)(x,y)$ exists and equals $(D_{(\gamma_i,0)}D_{(0,\eta)}g_1)(x,y)$, thus we get

$$\begin{aligned}(D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta)}f)(x,y) &= (D_{(\gamma_i,0)}D_{(0,\eta)}g_1)(x,y)\\ &= (D_{(0,\eta)}D_{(\gamma_i,0)}g_1)(x,y)\\ &= (D_{(0,\eta)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}f)(x,y).\end{aligned}$$

Hence the assertion holds for $j=1$.

Now, let $2 \le j \le l$, $1 \le i \le k$, $\gamma_1,\ldots,\gamma_i \in \mathfrak{L}(G)$, $\eta_1,\ldots,\eta_j \in \mathfrak{L}(H)$ and $(x,y) \in U \times V$. By Remark 2.2.3, the map

$$g_2\colon U\times V \to E, \quad (x,y)\mapsto (D_{(0,\eta_{j-1})}\cdots D_{(0,\eta_1)}f)(x,y)$$

is $C^{k,1}$, whence we have

$$\begin{aligned}&(D_{(0,\eta_j)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_{j-1})}\cdots D_{(0,\eta_1)}f)(x,y)\\ &= (D_{(0,\eta_j)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}g_2)(x,y)\\ &= (D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_j)}g_2)(x,y)\\ &= (D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}f)(x,y), \end{aligned} \tag{2.1}$$

using the first part of the proof. But we also have

$$\begin{aligned}&(D_{(0,\eta_j)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_{j-1})}\cdots D_{(0,\eta_1)}f)(x,y)\\ &= (D_{(0,\eta_j)}D_{(0,\eta_{j-1})}\cdots D_{(0,\eta_1)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}f)(x,y),\end{aligned} \tag{2.2}$$

by induction, whence (2.2) equals (2.1), that is

$$\begin{aligned}&(D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}f)(x,y)\\ &= (D_{(\gamma_i,0)}\cdots D_{(\gamma_1,0)}D_{(0,\eta_j)}\cdots D_{(0,\eta_1)}f)(x,y),\end{aligned}$$

and the proof is finished. □

Corollary 2.2.6. *Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $k,l \in \mathbb{N}_0 \cup \{\infty\}$. A map $f\colon U\times V \to E$ is $C^{k,l}$ if and only if the map*

$$g\colon V\times U \to E, \quad (y,x)\mapsto f(x,y)$$

is $C^{l,k}$. Moreover, we have

$$d^{(j,i)}g(y,x,\eta_1,\ldots,\eta_j,\gamma_1,\ldots,\gamma_i) = d^{(i,j)}f(x,y,\gamma_1,\ldots,\gamma_i,\eta_1,\ldots,\eta_j)$$

for all $x\in U$, $y \in V$, $i,j\in\mathbb{N}_0$ with $i\le k$, $j \le l$ and $\gamma_1,\ldots,\gamma_i \in \mathfrak{L}(G)$, $\eta_1,\ldots,\eta_j \in \mathfrak{L}(H)$.

Proof. First, we assume that $l = 0$, that is, $f\colon U \times V \to E$ is $C^{k,0}$. Then for $x \in U$, $y \in V$ and $\gamma \in \mathfrak{L}(G)$ we have

$$\begin{aligned} d^{(1,0)} f(x, y, \gamma) &= \lim_{t \to 0} \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y)) \\ &= \lim_{t \to 0} \frac{1}{t}(g(y, x \cdot \gamma(t)) - g(y, x)) = d^{(0,1)} g(y, x, \gamma), \end{aligned}$$

and similarly we get $d^{(0,i)} g(y, x, \gamma_1, \ldots, \gamma_i) = d^{(i,0)} f(x, y, \gamma_1, \ldots, \gamma_i)$ for each $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$. The obtained differentials $d^{(0,i)} g\colon V \times U \times \mathfrak{L}(G)^i \to E$ are obviously continuous, hence g is $C^{0,k}$. The other implication, as well as the case $k = 0$, can be proven analogously.

If $k, l \geq 1$, then the assertion follows immediately from Proposition 2.2.5. □

Remark 2.2.7. Using Remark 2.2.3 and Corollary 2.2.6, we can easily show that if $f\colon U \times V \to E$ is $C^{k,l}$, then for all $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and fixed $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$, $\eta_1, \ldots, \eta_j \in \mathfrak{L}(H)$ the maps

$$D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} f\colon U \times V \to E$$

are $C^{k-i,l-j}$.

2.3 Properties of C^k- and $C^{k,l}$-functions

Lemma 2.3.1. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E, F be locally convex spaces, $\lambda\colon E \to F$ be a continuous linear map and $k, l \in \mathbb{N}_0 \cup \{\infty\}$.*

(a) If $f\colon U \to E$ is a C^k-map, then the map $\lambda \circ f\colon U \to F$ is C^k.

(b) If $f\colon U \times V \to E$ is a $C^{k,l}$-map, then the map $\lambda \circ f\colon U \times V \to F$ is $C^{k,l}$.

Proof. To prove *(a)*, let $x \in U$, $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough, then we have

$$\frac{\lambda(f(x \cdot \gamma(t))) - \lambda(f(x))}{t} = \lambda\left(\frac{f(x \cdot \gamma(t)) - f(x)}{t}\right) \to \lambda(df(x, \gamma)),$$

as $t \to 0$, because λ is assumed linear and continuous. Thus, the derivative $d(\lambda \circ f)(x, \gamma)$ exists and we have $d(\lambda \circ f)(x, \gamma) = (\lambda \circ df)(x, \gamma)$.

Proceeding similarly, for each $i \in \mathbb{N}$ with $i \leq k$, $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$ we obtain the derivatives $d^{(i)}(\lambda \circ f)(x, \gamma_1, \ldots, \gamma_i) = (\lambda \circ d^{(i)} f)(x, \gamma_1, \ldots, \gamma_i)$. Since each of the obtained maps $d^{(i)}(\lambda \circ f) = \lambda \circ d^{(i)} f\colon U \times \mathfrak{L}(G)^i \to F$ is continuous, we see that the map $\lambda \circ f$ is C^k.

Analogously, assertion *(b)* can be proved showing that for each $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ we have $d^{(i,j)}(\lambda \circ f) = \lambda \circ d^{(i,j)} f$. □

Lemma 2.3.2. *Let G and H be topological groups, E be a locally convex space. Let $\varphi\colon G \to H$ be a continuous group homomorphism and $f\colon V \to E$ be a C^k-map ($k \in \mathbb{N} \cup \{\infty\}$) on an open subset $V \subseteq H$. Then for $U := \varphi^{-1}(V)$ the map*

$$f \circ \varphi\big|_U \colon U \to E, \quad x \mapsto f(\varphi(x))$$

is C^k.

Proof. Obviously, the map $f \circ \varphi\big|_U$ is continuous. Now, let $x \in U$ and $\gamma \in \mathfrak{L}(G)$. For $t \neq 0$ small enough we have

$$\frac{f(\varphi(x \cdot \gamma(t))) - f(\varphi(x))}{t} = \frac{f(\varphi(x) \cdot \varphi(\gamma(t))) - f(\varphi(x))}{t} \to df(\varphi(x), \varphi \circ \gamma)$$

as $t \to 0$, since $\varphi \circ \gamma \in \mathfrak{L}(H)$, see Remark 2.1.2. Therefore $d(f \circ \varphi\big|_U)(x, \gamma)$ exists and is given by $df(\varphi(x), \varphi \circ \gamma)$.

Repeating the above steps, we obtain for $i \in \mathbb{N}$ with $i \leq k$, $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$ the derivatives $d^{(i)}(f \circ \varphi\big|_U)(x, \gamma_1, \ldots, \gamma_i) = d^{(i)} f(\varphi(x), \varphi \circ \gamma_1, \ldots, \varphi \circ \gamma_i)$.

Now, recall that the map $\mathfrak{L}(\varphi)\colon \mathfrak{L}(G) \to \mathfrak{L}(H), \eta \mapsto \varphi \circ \eta$ is continuous (Remark 2.1.2), whence also each of the maps

$$d^{(i)}(f \circ \varphi\big|_U) := (d^{(i)} f) \circ (\varphi\big|_U \times \underbrace{\mathfrak{L}(\varphi) \times \cdots \times \mathfrak{L}(\varphi)}_{i\text{-times}})\colon U \times \mathfrak{L}(G)^i \to E$$

is continuous. Hence $f \circ \varphi\big|_U$ is C^k. □

Lemma 2.3.3. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let $(E_\alpha)_{\alpha \in A}$ be a family of locally convex spaces with direct product $E := \prod_{\alpha \in A} E_\alpha$ and the coordinate projections $\mathrm{pr}_\alpha\colon E \to E_\alpha$. For $k, l \in \mathbb{N}_0 \cup \{\infty\}$ the following holds:*

(a) A map $f\colon U \to E$ is C^k if and only if all of its components $f_\alpha := \mathrm{pr}_\alpha \circ f$ are C^k.

(b) A map $f\colon U \times V \to E$ is $C^{k,l}$ if and only if all of its components $f_\alpha := \mathrm{pr}_\alpha \circ f$ are $C^{k,l}$.

Proof. To prove *(a)*, first recall that because each of the projections pr_α is continuous and linear, the compositions $\mathrm{pr}_\alpha \circ f$ are C^k if f is C^k, by Lemma 2.3.1 *(a)*.

Conversely, assume that each f_α is C^k and let $x \in U$, $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough. Then we have

$$\frac{1}{t}\left(f(x \cdot \gamma(t)) - f(x)\right) = \left(\frac{1}{t}\left(f_\alpha(x \cdot \gamma(t)) - f_\alpha(x)\right)\right)_{\alpha \in A}.$$

Since $\frac{1}{t}\left(f_\alpha(x \cdot \gamma(t)) - f_\alpha(x)\right)$ converges to $df_\alpha(x, \gamma)$ as $t \to 0$ for each $\alpha \in A$, the derivative $df(x, \gamma)$ exists and is given by $(df_\alpha(x, \gamma))_{\alpha \in A}$.

Repeating the above steps, we obtain for $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$ the derivatives $d^{(i)} f(x, \gamma_1, \ldots, \gamma_i) = \left(d^{(i)} f_\alpha(x, \gamma_1, \ldots, \gamma_i)\right)_{\alpha \in A}$, which define continuous maps

$$d^{(i)} f = \left(d^{(i)} f_\alpha\right)_{\alpha \in A} : U \times \mathfrak{L}(G)^i \to E.$$

Therefore, f is C^k.

The assertion *(b)* can be proven similarly, by using Lemma 2.3.1 *(b)* and showing that for all $i, j \in \mathbb{N}_0$, with $i \leq k$, $j \leq l$ we have $d^{(i,j)} f = \left(d^{(i,j)} f_\alpha\right)_{\alpha \in A}$. □

Lemma 2.3.4. *Let $U \subseteq G$ be an open subset of a topological group G, and E be a locally convex space. A continuous map $f \colon U \to E$ is C^1 if and only if there exists a continuous map*

$$f^{[1]} \colon U^{[1]} \to E$$

on the open set

$$U^{[1]} := \{(x, \gamma, t) \in U \times \mathfrak{L}(G) \times \mathbb{R} : x \cdot \gamma(t) \in U\}$$

such that

$$f^{[1]}(x, \gamma, t) = \frac{1}{t}(f(x \cdot \gamma(t)) - f(x))$$

for each $(x, \gamma, t) \in U^{[1]}$ with $t \neq 0$.

In this case we have $df(x, \gamma) = f^{[1]}(x, \gamma, 0)$ for all $x \in U$ and $\gamma \in \mathfrak{L}(G)$.

The above lemma is a special case of the following lemma:

Lemma 2.3.5. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space. A continuous map $f \colon U \times V \to E$ is $C^{1,0}$ if and only if there exists a continuous map*

$$f^{[1,0]} \colon U^{[1]} \times V \to E,$$

where

$$U^{[1]} := \{(x, \gamma, t) \in U \times \mathfrak{L}(G) \times \mathbb{R} : x \cdot \gamma(t) \in U\},$$

such that

$$f^{[1,0]}(x, \gamma, t, y) = \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y))$$

for each $(x, \gamma, t, y) \in U^{[1]} \times V$ with $t \neq 0$.

In this case we have $d^{(1,0)} f(x, y, \gamma) = f^{[1,0]}(x, \gamma, 0, y)$ for all $x \in U$, $y \in V$ and $\gamma \in \mathfrak{L}(G)$.

Proof. First, assume that the map $f^{[1,0]}$ exists and is continuous. Then for $x \in U$, $y \in V$, $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y)) = f^{[1,0]}(x, \gamma, t, y) \to f^{[1,0]}(x, \gamma, 0, y)$$

as $t \to 0$. Hence $d^{(1,0)}f(x, y, \gamma)$ exists and is given by $f^{[1,0]}(x, \gamma, 0, y)$, whence the map

$$d^{(1,0)}f\colon U \times V \times \mathfrak{L}(G) \to E, \quad (x, y, \gamma) \mapsto f^{[1,0]}(x, \gamma, 0, y)$$

is continuous. Thus f is $C^{1,0}$.

Conversely, let f be a $C^{1,0}$-map. Then we define

$$f^{[1,0]}\colon U^{[1]} \times V \to E, \quad f^{[1,0]}(x, \gamma, t, y)\colon = \begin{cases} \frac{f(x \cdot \gamma(t), y) - f(x,y)}{t} & \text{if } t \neq 0, \\ d^{(1,0)}f(x, y, \gamma) & \text{if } t = 0. \end{cases}$$

Since f is continuous, the map $f^{[1,0]}$ is continuous at each (x, γ, t, y) with $t \neq 0$ (by the continuity of the evaluation map $\mathfrak{L}(G) \times \mathbb{R} \to G$). Given $x_0 \in U$ and $\gamma_0 \in \mathfrak{L}(G)$, we have $(x_0, \gamma_0, 0) \in U^{[1]}$; the subset $W := U_{x_0} \times U_{\gamma_0} \times]-\varepsilon, \varepsilon[\subseteq U^{[1]}$ is an open neighborhood of $(x_0, \gamma_0, 0)$ in $U^{[1]}$, where $U_{x_0} \subseteq U$ and $U_{\gamma_0} \subseteq \mathfrak{L}(G)$ are open neighborhoods of x_0 and γ_0, respectively, and $\varepsilon > 0$. Now, for fixed $(x, \gamma, y) \in U_{x_0} \times U_{\gamma_0} \times V$ we define the continuous curve

$$h\colon]-\varepsilon, \varepsilon[\to E, \quad h(t) := f(x \cdot \gamma(t), y).$$

Then for $t \in]-\varepsilon, \varepsilon[$, $s \neq 0$ with $t + s \in]-\varepsilon, \varepsilon[$ we have

$$\begin{aligned} \frac{h(t+s) - h(t)}{s} &= \frac{f(x \cdot \gamma(t+s), y) - f(x \cdot \gamma(t), y)}{s} \\ &= \frac{f(x \cdot \gamma(t) \cdot \gamma(s), y) - f(x \cdot \gamma(t), y)}{s} \to d^{(1,0)}f(x \cdot \gamma(t), y, \gamma) \end{aligned}$$

as $s \to 0$. Thus, the derivative $h'(t)$ exists and is given by $d^{(1,0)}f(x \cdot \gamma(t), y, \gamma)$. The so obtained map $h'\colon]-\varepsilon, \varepsilon[\to E$ is continuous, hence h is a C^1-curve (see [21] for details on C^1-curves with values in locally convex spaces and also on weak integrals which we use in the next step). We use the Fundamental Theorem of Calculus ([21, Proposition 1.1.5]) and obtain for $t \neq 0$

$$\begin{aligned} f^{[1,0]}(x, \gamma, t, y) &= \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y)) = \frac{1}{t}(h(t) - h(0)) \\ &= \frac{1}{t}\int_0^t h'(\tau)d\tau = \frac{1}{t}\int_0^t d^{(1,0)}f(x \cdot \gamma(\tau), y, \gamma)d\tau \\ &= \frac{1}{t}\int_0^1 t d^{(1,0)}f(x \cdot \gamma(tu), y, \gamma)du = \int_0^1 d^{(1,0)}f(x \cdot \gamma(tu), y, \gamma)du. \end{aligned}$$

But if $t = 0$, then

$$\int_0^1 d^{(1,0)}f(x \cdot \gamma(0), y, \gamma)du = d^{(1,0)}f(x, y, \gamma) = f^{[1,0]}(x, \gamma, 0, y),$$

hence

$$f^{[1,0]}(x,\gamma,t,y) = \int_0^1 d^{(1,0)}f(x\cdot\gamma(tu),y,\gamma)du$$

for all $(x,\gamma,t,y) \in W \times V$. Since the map

$$W \times V \times [0,1] \to E, \quad (x,\gamma,t,y,u) \mapsto d^{(1,0)}f(x\cdot\gamma(tu),y,\gamma)$$

is continuous, also the parameter-dependent integral

$$W \times V \to E, \quad (x,\gamma,t,y) \mapsto \int_0^1 d^{(1,0)}f(x\cdot\gamma(tu),y,\gamma)du = f^{[1,0]}(x,\gamma,t,y)$$

is continuous (by [21, Lemma 1.1.11]), in particular in $(x_0,\gamma_0,0,y)$. Consequently, $f^{[1,0]}$ is continuous. □

We use Lemma 2.3.4, as well as the analogue for C^1-maps on locally convex spaces (which can be found in [21, Lemma 1.2.10]), for the proof of a chain rule for compositions of C^k-functions $f\colon G \to E$ and $g\colon E \to F$, which will be provided after the following version:

Lemma 2.3.6. *Let G be a topological group, P be a topological space and E, F be locally convex spaces. Let $U \subseteq G$, $V \subseteq E$ be open subsets, and $k \in \mathbb{N} \cup \{\infty\}$. If $f\colon U \times P \to E$ is a $C^{k,0}$-map such that $f(U \times P) \subseteq V$, and $g\colon V \to F$ is a C^k-map, then*

$$g \circ f\colon U \times P \to F$$

is a $C^{k,0}$-map.

Proof. We may assume that k is finite and prove the assertion by induction.

Base case: Assume that f is $C^{1,0}$, g is C^1 and let $x \in U$, $p \in P$ and $\gamma \in \mathfrak{L}(G)$. For $t \neq 0$ small enough we have

$$\begin{aligned}\frac{g(f(x\cdot\gamma(t),p)) - g(f(x,p))}{t} &= \frac{g\left(f(x,p) + t\frac{f(x\cdot\gamma(t),p)-f(x,p)}{t}\right) - g(f(x,p))}{t}\\ &= \frac{g(f(x,p) + t\cdot f^{[1,0]}(x,\gamma,t,p)) - g(f(x,p))}{t}\\ &= g^{[1]}(f(x,p), f^{[1,0]}(x,\gamma,t,p),t),\end{aligned}$$

where $g^{[1]}$, $f^{[1,0]}$ are the continuous maps from [21, Lemma 1.2.10] and Lemma 2.3.5. Consequently, we have

$$\begin{aligned}\frac{g(f(x\cdot\gamma(t),p)) - g(f(x,p))}{t} &\to g^{[1]}(f(x,p), f^{[1,0]}(x,\gamma,0,p),0)\\ &= dg(f(x,p), d^{(1,0)}f(x,p,\gamma))\end{aligned}$$

as $t \to 0$. Therefore, the derivative $d^{(1,0)}(g \circ f)(x,p,\gamma)$ exists and is given by the directional derivative $dg(f(x,p), d^{(1,0)}f(x,p,\gamma))$.

Consider the continuous map

$$h\colon U \times P \times \mathfrak{L}(G) \to E, \quad (x,p,\gamma) \mapsto f(x,p).$$

Since $d^{(1,0)}(g \circ f)(x,p,\gamma) = (dg \circ (h, d^{(1,0)}f))(x,p,\gamma)$, the map

$$d^{(1,0)}(g \circ f) = dg \circ (h, d^{(1,0)}f)\colon U \times P \times \mathfrak{L}(G) \to F$$

is continuous, whence $g \circ f$ is $C^{1,0}$.

Induction step: Now, assume that f is $C^{k,0}$ and g is C^k for some $k \geq 2$. By Remark 2.2.3, the map $d^{(1,0)}f\colon U \times (P \times \mathfrak{L}(G)) \to E$ is $C^{k-1,0}$, and it is easily seen that the map $h\colon U \times (P \times \mathfrak{L}(G)) \to E$ defined in the base case is $C^{k,0}$. Hence, using Lemma 2.3.3 *(b)*, we see that $(h, d^{(1,0)}f)\colon U \times (P \times \mathfrak{L}(G)) \to E \times E$ is a $C^{k-1,0}$-map. Since $dg\colon V \times E \to F$ is C^{k-1} (see [21, Definition 1.3.1]), the map

$$d^{(1,0)}(g \circ f) = dg \circ (h, d^{(1,0)}f)\colon U \times (P \times \mathfrak{L}(G)) \to F$$

is $C^{k-1,0}$, by the induction hypothesis, and from Remark 2.2.3, it follows that $g \circ f$ is $C^{k,0}$. □

Lemma 2.3.7. *Let G be a topological group, E, F be locally convex spaces and $k \in \mathbb{N} \cup \{\infty\}$. Let $U \subseteq G$, $V \subseteq E$ be open subsets. If $f\colon U \to E$ is a C^k-map with $f(U) \subseteq V$ and also $g\colon V \to F$ is a C^k-map, then the map*

$$g \circ f : U \to F$$

is C^k.

Proof. We may assume that k is finite and prove the assertion by induction.

Base case: Assume that f and g are C^1-maps. Analogously to the preceding lemma, for $x \in U$, $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(g(f(x \cdot \gamma(t))) - g(f(x))) = g^{[1]}(f(x), f^{[1]}(x,\gamma,t), t),$$

with continuous maps $f^{[1]}$ as in Lemma 2.3.4 and $g^{[1]}$ as in [21, Lemma 1.2.10]. Thus, the derivative $d(g \circ f)(x,\gamma)$ exists and we have

$$d(g \circ f)(x,\gamma) = g^{[1]}(f(x), f^{[1]}(x,\gamma,0), 0) = dg(f(x), df(x,\gamma)).$$

Using the continuous function

$$h\colon U \times \mathfrak{L}(G) \to E, \quad (x,\gamma) \mapsto f(x),$$

we see that

$$d(g \circ f) = dg \circ (h, df) : U \times \mathfrak{L}(G) \to F$$

is continuous, hence $g \circ f$ is a C^1-map.

Induction step: Now, let f and g be C^k-maps for some $k \geq 2$. Then the function $df\colon U \times \mathfrak{L}(G) \to E$ is $C^{k-1,0}$, by Remark 2.2.3, and the map $h\colon U \times \mathfrak{L}(G) \to E$ is obviously $C^{k,0}$. We use Lemma 2.3.3 *(b)* and see that $(h, df)\colon U \times \mathfrak{L}(G) \to E \times E$ is a $C^{k-1,0}$-map. By [21, Definition 1.3.1], the map $dg\colon V \times E \to F$ is C^{k-1}, hence by Lemma 2.3.6, the composition

$$d(g \circ f) = dg \circ (h, df)\colon U \times \mathfrak{L}(G) \to F$$

is $C^{k-1,0}$, whence $g \circ f$ is C^k, by Remark 2.2.3. □

2.4 Properties of spaces of C^k- and $C^{k,l}$-functions

The following two propositions provide a relation between C^k- and $C^{k,l}$-maps on products of topological groups (a version can also be found in [8]), in particular, we will conclude that $C^{\infty,\infty}(U \times V, E) = C^\infty(U \times V, E)$ (Corollary 2.4.3).

Proposition 2.4.1. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f\colon U \times V \to E$ is $C^{k,k}$, then f is C^k.*

Moreover, the inclusion map

$$\Psi\colon C^{k,k}(U \times V, E) \to C^k(U \times V, E), \quad f \mapsto f$$

is continuous and linear.

Proof. The case $k = 0$ is trivial. For $k \geq 1$, we show by induction on $i \in \mathbb{N}$ with $i \leq k$ that for all $(x, y) \in U \times V$, $(\gamma_1, \eta_1), \ldots, (\gamma_i, \eta_i) \in \mathfrak{L}(G \times H)$ the derivatives of f are given by

$$\begin{aligned} & d^{(i)} f((x,y),(\gamma_1,\eta_1),\ldots,(\gamma_i,\eta_i)) \\ & \qquad = \sum_{j=0}^{i} \sum_{I_{j,i}} d^{(j,i-j)} f(x, y, \gamma_{r_1}, \ldots, \gamma_{r_j}, \eta_{s_1}, \ldots, \eta_{s_{i-j}}) \end{aligned} \tag{2.3}$$

where $I_{j,i}$ ranges through the sets $\{r_1, \ldots, r_j\} \subseteq \{1, \ldots, i\}$ with $r_1 < \cdots < r_j$ and we write $\{1, \ldots, i\} \setminus I_{j,i} = \{s_1, \ldots, s_{i-j}\}$ with $s_1 < \cdots < s_{i-j}$.

Base case: Let $(x, y) \in U \times V$ and $(\gamma, \eta) \in \mathfrak{L}(G \times H)$, that is, $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$, see Remark 2.1.2. For $t \neq 0$ small enough we have

$$\begin{aligned} & \frac{f((x,y)\cdot(\gamma(t),\eta(t))) - f(x,y)}{t} \\ & = \frac{f(x\cdot\gamma(t), y\cdot\eta(t)) - f(x,y)}{t} \\ & = \frac{f(x\cdot\gamma(t), y\cdot\eta(t)) - f(x\cdot\gamma(t), y)}{t} + \frac{f(x\cdot\gamma(t), y) - f(x,y)}{t} \\ & = \frac{g(y\cdot\eta(t), x\cdot\gamma(t)) - g(y, x\cdot\gamma(t))}{t} + \frac{f(x\cdot\gamma(t), y) - f(x,y)}{t}, \end{aligned}$$

where g is the map $g\colon V \times U \to E, (y,x) \mapsto f(x,y)$. By Corollary 2.2.6, the map g is $C^{1,1}$, whence the map $g^{[1,0]}$ is defined and continuous, as well as $f^{[1,0]}$ (see Lemma 2.3.5). Thus we have

$$\begin{aligned}
&\frac{g(y\cdot\eta(t), x\cdot\gamma(t)) - g(y, x\cdot\gamma(t))}{t} + \frac{f(x\cdot\gamma(t), y) - f(x,y)}{t} \\
&= g^{[1,0]}(y,\eta,t,x\cdot\gamma(t)) + f^{[1,0]}(x,\gamma,t,y) \\
&\to g^{[1,0]}(y,\eta,0,x) + f^{[1,0]}(x,\gamma,0,y)
\end{aligned}$$

as $t \to 0$. Therefore, the derivative $df((x,y),(\gamma,\eta))$ exists and is given by

$$\begin{aligned}
df((x,y),(\gamma,\eta)) &= g^{[1,0]}(y,\eta,0,x) + f^{[1,0]}(x,\gamma,0,y) \\
&= d^{(1,0)}g(y,x,\eta) + d^{(1,0)}f(x,y,\gamma) \\
&= d^{(0,1)}f(x,y,\eta) + d^{(1,0)}f(x,y,\gamma).
\end{aligned}$$

Induction step: Assume that $2 \leq i \leq k$, $(x,y) \in U \times V$, $(\gamma_1,\eta_1),\dots,(\gamma_i,\eta_i) \in \mathfrak{L}(G \times H)$. Then for $t \neq 0$ small enough we have

$$\begin{aligned}
&\frac{1}{t}\Big(d^{(i-1)}f((x\cdot\gamma_i(t), y\cdot\eta_i(t)),(\gamma_1,\eta_1),\dots,(\gamma_{i-1},\eta_{i-1})) \\
&\quad - d^{(i-1)}f((x,y),(\gamma_1,\eta_1),\dots,(\gamma_{i-1},\eta_{i-1}))\Big) \\
&= \sum_{j=0}^{i-1}\sum_{I_{j,i-1}} \frac{1}{t}\Big(d^{(j,i-j-1)}f(x\cdot\gamma_i(t), y\cdot\eta_i(t), \gamma_{r_1},\dots,\gamma_{r_j},\eta_{s_1},\dots,\eta_{s_{i-j-1}}) \\
&\quad - d^{(j,i-j-1)}f(x,y,\gamma_{r_1},\dots,\gamma_{r_j},\eta_{s_1},\dots,\eta_{s_{i-j-1}})\Big) \\
&= \sum_{j=0}^{i-1}\sum_{I_{j,i-1}} \frac{1}{t}\Big((D_{(\gamma_{r_j},0)}\cdots D_{(\gamma_{r_1},0)}D_{(0,\eta_{s_{i-j-1}})}\cdots D_{(0,\eta_1)}f)(x\cdot\gamma_i(t), y\cdot\eta_i(t)) \\
&\quad - (D_{(\gamma_{r_j},0)}\cdots D_{(\gamma_{r_1},0)}D_{(0,\eta_{s_{i-j-1}})}\cdots D_{(0,\eta_1)}f)(x,y)\Big).
\end{aligned}$$

Each of the maps

$$D_{(\gamma_{r_j},0)}\cdots D_{(\gamma_{r_1},0)}D_{(0,\eta_{s_{i-j-1}})}\cdots D_{(0,\eta_1)}f\colon U \times V \to E$$

is $C^{1,1}$ (see Remark 2.2.7), hence C^1 (by the base case) and we have

$$\begin{aligned}
&\sum_{j=0}^{i-1}\sum_{I_{j,i-1}}\frac{1}{t}\Big((D_{(\gamma_{r_j},0)}\cdots D_{(\gamma_{r_1},0)}D_{(0,\eta_{s_{i-j-1}})}\cdots D_{(0,\eta_1)}f)(x\cdot\gamma_i(t),y\cdot\eta_i(t))\\
&\quad-(D_{(\gamma_{r_j},0)}\cdots D_{(\gamma_{r_1},0)}D_{(0,\eta_{s_{i-j-1}})}\cdots D_{(0,\eta_1)}f)(x,y)\Big)\\
&\to\sum_{j=0}^{i-1}\sum_{I_{j,i-1}}\Big((D_{(\gamma_{r_j},0)}\cdots D_{(\gamma_{r_1},0)}D_{(0,\eta_i)}D_{(0,\eta_{s_{i-j-1}})}\cdots D_{(0,\eta_1)}f)(x,y)\\
&\quad+(D_{(\gamma_i,0)}D_{(\gamma_{r_j},0)}\cdots D_{(\gamma_{r_1},0)}D_{(0,\eta_{s_{i-j-1}})}\cdots D_{(0,\eta_1)}f)(x,y)\Big)\\
&=\sum_{j=0}^{i}\sum_{I_{j,i}}d^{(j,i-j)}f(x,y,\gamma_{r_1},\ldots,\gamma_{r_j},\eta_{s_1},\ldots,\eta_{s_{i-j}})
\end{aligned}$$

as $t\to 0$ (using Proposition 2.2.5). Thus (2.3) holds, and we have

$$d^{(i)}f=\sum_{j=0}^{i}\sum_{I_{j,i}}d^{(j,i-j)}f\circ g_{I_{j,i}},$$

where

$$\begin{aligned}
&g_{I_{j,i}}\colon U\times V\times\mathfrak{L}(G\times H)^i\to U\times V\times\mathfrak{L}(G)^j\times\mathfrak{L}(H)^{i-j},\\
&(x,y,(\gamma_1,\eta_1),\ldots,(\gamma_i,\eta_i))\mapsto(x,y,\gamma_{r_1},\ldots,\gamma_{r_j},\eta_{s_1},\ldots,\eta_{s_{i-j}})
\end{aligned}$$

are continuous maps (see Remark 2.1.2). Hence f is C^k.

The linearity of the map Ψ is clear. Further, each of the maps

$$\begin{aligned}
g^*_{I_{j,i}}\colon C(U\times V\times\mathfrak{L}(G)^j\times\mathfrak{L}(H)^{i-j},E)&\to C(U\times V\times\mathfrak{L}(G\times H)^i,E),\\
h&\mapsto h\circ g_{I_{j,i}}
\end{aligned}$$

is continuous (see [21, Appendix A.5] or [14, Lemma B.9]), whence each of the maps

$$d^{(i)}\circ\Psi=\sum_{j=0}^{i}\sum_{I_{j,i}}g^*_{I_{j,i}}\circ d^{(j,i-j)}$$

is continuous. Since the topology on $C^k(U\times V,E)$ is initial with respect to the maps $d^{(i)}$, the continuity of Ψ follows. □

Proposition 2.4.2. *Let $U\subseteq G$, $V\subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $k,l\in\mathbb{N}_0$. If $f\colon U\times V\to E$ is a C^{k+l}-map, then f is $C^{k,l}$.*

Moreover, the inclusion map

$$\Psi\colon C^{k+l}(U\times V,E)\to C^{k,l}(U\times V,E),\quad f\mapsto f$$

is continuous and linear.

Proof. We denote by $\varepsilon_G \in \mathfrak{L}(G)$ the constant map $\varepsilon_G\colon \mathbb{R} \to G, t \mapsto e_G$, where e_G is the identity element of G, and $\varepsilon_H \in \mathfrak{L}(H)$ is defined analogously.

Let $x \in U$, $y \in V$, $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ and $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$ for some $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$. Then we obviously have

$$\begin{aligned} &d^{(i,j)} f(x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) \\ &\qquad = d^{(i+j)} f((x, y), (\gamma_1, \varepsilon_H), \dots, (\gamma_i, \varepsilon_H), (\varepsilon_G, \eta_1), \dots, (\varepsilon_G, \eta_j)). \end{aligned}$$

Each of the maps

$$\begin{aligned} \rho_{i,j}\colon\ & U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \to U \times V \times \mathfrak{L}(G \times H)^{i+j}, \\ & (x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) \mapsto (x, y, (\gamma_1, \varepsilon_H), \dots (\gamma_i, \varepsilon_H), (\varepsilon_G, \eta_1), \dots (\varepsilon_G, \eta_j)) \end{aligned}$$

is continuous (see Remark 2.1.2) and we have

$$d^{(i,j)} f = d^{(i+j)} f \circ \rho_{i,j}.$$

Therefore, f is $C^{k,l}$.

The linearity of the map Ψ is clear. Further, by [21, Appendix A.5] (see also [14, Lemma B.9]), each of the maps

$$\begin{aligned} \rho_{i,j}^*\colon C(U \times V \times \mathfrak{L}(G \times H)^{i+j}, E) &\to C(U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, E), \\ h &\mapsto h \circ \rho_{i,j} \end{aligned}$$

is continuous, whence each of the maps

$$d^{(i,j)} \circ \Psi = \rho_{i,j}^* \circ d^{(i+j)}$$

is continuous. Hence, the continuity of Ψ follows, since the topology on the space $C^{k,l}(U \times V, E)$ is initial with respect to the maps $d^{(i,j)}$. □

Corollary 2.4.3. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space. A map $f\colon U \times V \to E$ is C^∞ if and only if f is $C^{\infty,\infty}$.*

Moreover, the map

$$\Psi\colon C^\infty(U \times V, E) \to C^{\infty,\infty}(U \times V, E), \quad f \mapsto f$$

is an isomorphism of topological vector spaces.

Proof. The assertion is an immediate consequence of Propositions 2.4.1 and 2.4.2. □

2.5 The exponential law

We recall the classical Exponential Law for spaces of continuous functions which can be found, for example, in [21, Appendix A.5]:

Proposition 2.5.1. *Let X_1, X_2, Y be topological spaces. If $f\colon X_1 \times X_2 \to Y$ is a continuous map, then also the map*

$$f^\vee\colon X_1 \to C(X_2, Y), \quad x \mapsto f^\vee(x) := f(x, \bullet)$$

is continuous. Moreover, the map

$$\Phi\colon C(X_1 \times X_2, Y) \to C(X_1, C(X_2, Y)), \quad f \mapsto f^\vee$$

is a topological embedding.

If X_2 is locally compact or $X_1 \times X_2$ is a k-space, or $X_1 \times X_2$ is a $k_{\mathbb{R}}$-space and Y is completely regular, then Φ is a homeomorphism.

The following terminology is used here:

Remark 2.5.2. (a) A Hausdorff topological space X is called a *k-space* if functions $f\colon X \to Y$ to a topological space Y are continuous if and only if the restrictions $f\big|_K\colon K \to Y$ are continuous for all compact subsets $K \subseteq X$. All locally compact spaces and all metrizable spaces are k-spaces.

(b) A Hausdorff topological space X is called a *$k_{\mathbb{R}}$-space* if real-valued functions $f\colon X \to \mathbb{R}$ are continuous if and only if the restrictions $f\big|_K\colon K \to \mathbb{R}$ are continuous for all compact subsets $K \subseteq X$. Each k-space is a $k_{\mathbb{R}}$-space, hence also each locally compact and each metrizable space is a $k_{\mathbb{R}}$-space.

(c) A Hausdorff topological space X is called *completely regular* if its topology is initial with respect to the set $C(X, \mathbb{R})$. Each Hausdorff locally convex space (moreover, each Hausdorff topological group) is completely regular, see [22], as well as each Hausdorff locally compact space.

Theorem 2.5.3. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. Then the following holds:*

(a) If a map $f\colon U \times V \to E$ is $C^{k,l}$, then the map

$$f^\vee(x) := f(x, \bullet)\colon V \to E, \quad y \mapsto f^\vee(x)(y) := f(x, y)$$

is C^l for each $x \in U$ and the map

$$f^\vee\colon U \to C^l(V, E), \quad x \mapsto f^\vee(x)$$

is C^k.

(b) The map

$$\Phi\colon C^{k,l}(U \times V, E) \to C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is linear and a topological embedding.

Proof. *(a)* We will consider the following cases:

The case $k = l = 0$: This case is covered by the classical Exponential Law, Proposition 2.5.1.

The case $k = 0$, $l \geq 1$: Let $x \in U$; the map $f^\vee(x) = f(x, \bullet)$ is obviously continuous, and for $y \in V$, $\eta \in \mathfrak{L}(H)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(f^\vee(x)(y \cdot \eta(t)) - f^\vee(x)(y)) = \frac{1}{t}(f(x, y \cdot \eta(t)) - f(x, y)) \to D_{(0,\eta)} f(x, y)$$

as $t \to 0$. Thus $D_\eta(f^\vee(x))(y)$ exists and equals $D_{(0,\eta)} f(x, y) = (D_{(0,\eta)} f)^\vee(x)(y)$. Proceeding similarly, for each $j \in \mathbb{N}$ with $j \leq l$ and $\eta_1, \ldots \eta_j \in \mathfrak{L}(H)$, we obtain the derivatives

$$\left(D_{\eta_j} \cdots D_{\eta_1}(f^\vee(x))\right)(y) = (D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} f)^\vee(x)(y) \tag{2.4}$$

The obtained differentials $d^{(j)}(f^\vee(x)) = (d^{(0,j)} f)^\vee(x) \colon V \times \mathfrak{L}(H)^j \to E$ are continuous, therefore $f^\vee(x)$ is C^l.

Further, by the classical Exponential Law 2.5.1, each of the maps

$$f^\vee \colon U \to C(V, E), \quad x \mapsto f^\vee(x),$$
$$(d^{(0,j)} f)^\vee \colon U \to C(V \times \mathfrak{L}(H)^j, E), \quad x \mapsto (d^{(0,j)} f)^\vee(x)$$

is continuous, and we have $d^{(j)} \circ f^\vee = (d^{(0,j)} f)^\vee$ for all $j \in \mathbb{N}_0$ with $j \leq l$. Thus, the continuity of $f^\vee$ follows from the fact that the topology on $C^l(V, E)$ is initial with respect to the maps $d^{(j)}$.

The case $k \geq 1, l \geq 0$: By the preceding steps, the map $f^\vee(x)$ is C^l for each $x \in U$ (with derivatives given in (2.4)). Now we show by induction on $i \in \mathbb{N}$ with $i \leq k$ that

$$(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x) = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)^\vee(x) \tag{2.5}$$

for all $x \in U$ and $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$.

Base case: Since f is $C^{1,0}$, by Lemma 2.3.5 the map $f^{[1,0]} \colon U^{[1]} \times V \to E$ is continuous, hence so is the map $(f^{[1,0]})^\vee \colon U^{[1]} \to C(V, E)$ (see Proposition 2.5.1). Let $(x, \gamma, t) \in U^{[1]}$ such that $t \neq 0$ and let $y \in V$. Then we have

$$\begin{aligned}\frac{1}{t}(f^\vee(x \cdot \gamma(t))(y) - f^\vee(x)(y)) &= \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y)) \\ &= f^{[1,0]}(x, \gamma, t, y) = (f^{[1,0]})^\vee(x, \gamma, t)(y).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{t}(f^\vee(x \cdot \gamma(t)) - f^\vee(x)) &= (f^{[1,0]})^\vee(x, \gamma, t) \\ &\to (f^{[1,0]})^\vee(x, \gamma, 0) = (D_{(\gamma,0)} f)^\vee(x)\end{aligned}$$

as $t \to 0$. Thus, $D_\gamma(f^\vee)(x)$ exists and is given by $(D_{(\gamma,0)} f)^\vee(x)$.

Induction step: Now, let $2 \leq i \leq k$, $x \in U$ and $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$. For $t \neq 0$ small enough we have

$$\begin{aligned}&\frac{1}{t}\left((D_{\gamma_{i-1}} \cdots D_{\gamma_1}(f^\vee))(x \cdot \gamma_i(t)) - (D_{\gamma_{i-1}} \cdots D_{\gamma_1}(f^\vee))(x)\right)\\&= \frac{1}{t}\left((D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee(x \cdot \gamma_i(t)) - (D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)\right)\end{aligned}$$

by the induction hypothesis. But the map $D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f \colon U \times V \to E$ is $C^{1,0}$ (see Remark 2.2.3), hence by the base case we have

$$\begin{aligned}&\frac{1}{t}\left((D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee(x \cdot \gamma_i(t)) - (D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)})f^\vee(x)\right)\\&\to D_{\gamma_i}((D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee)(x) = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x),\end{aligned}$$

which shows that $(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x)$ exists and is given by $(D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)$, thus (2.5) holds.

From Remark 2.2.7, we know that each of the maps

$$D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f \colon U \times V \to E$$

is $C^{0,l}$, hence $(D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x) \in C^l(V,E)$ for each $x \in U$. Now, it remains to show that each of the maps

$$\begin{aligned}&d^{(i)}(f^\vee) \colon U \times \mathfrak{L}(G)^i \to C^l(V,E),\\&\qquad (x, \gamma_1, \ldots, \gamma_i) \mapsto (D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x) = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)\end{aligned}$$

is continuous. To this end, let $y \in V$, $j \in \mathbb{N}_0$ with $j \leq l$ and $\eta_1, \ldots, \eta_j \in \mathfrak{L}(H)$. Then we have

$$\begin{aligned}&(d^{(j)} \circ d^{(i)}(f^\vee))(x, \gamma_1, \ldots, \gamma_i)(y, \eta_1, \ldots, \eta_j)\\&= d^{(j)}(d^{(i)}(f^\vee)(x, \gamma_1, \ldots, \gamma_i))(y, \eta_1, \ldots, \eta_j)\\&= \left[D_{\eta_j} \cdots D_{\eta_1}[(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x)]\right](y)\end{aligned}$$

Using (2.5) and (2.4) in turn we obtain

$$\begin{aligned}&\left[D_{\eta_j} \cdots D_{\eta_1}[(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x)]\right](y)\\&= \left[D_{\eta_j} \cdots D_{\eta_1}[(D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)]\right](y)\\&= (D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)(y).\end{aligned}$$

Finally, from Proposition 2.2.5 we conclude

$$\begin{aligned}&(D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)(y)\\&= (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}f)^\vee(x)(y)\\&= d^{(i,j)}f(x, y, \gamma_1, \ldots, \gamma_i, \eta_1, \ldots, \eta_j)\\&= (d^{(i,j)}f \circ \rho_{i,j})(x, \gamma_1, \ldots, \gamma_i, y, \eta_1, \ldots, \eta_j)\\&= (d^{(i,j)}f \circ \rho_{i,j})^\vee(x, \gamma_1, \ldots, \gamma_i)(y, \eta_1, \ldots, \eta_j),\end{aligned}$$

where $\rho_{i,j}$ is the continuous map

$$\rho_{i,j}\colon U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j \to U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, \\ (x,\gamma,y,\eta) \mapsto (x,y,\gamma,\eta).$$

Now, from the classical Exponential Law 2.5.1, it follows that the maps

$$(d^{(i,j)}f \circ \rho_{i,j})^\vee \colon U \times \mathfrak{L}(G)^i \to C(V \times \mathfrak{L}(H)^j, E)$$

are continuous, and we have shown that

$$d^{(j)} \circ d^{(i)}(f^\vee) = (d^{(i,j)}f \circ \rho_{i,j})^\vee, \tag{2.6}$$

thus the continuity of $d^{(i)}(f^\vee)$ follows from the fact that the topology on $C^l(V,E)$ is initial with respect to the maps $d^{(j)}$, whence $f^\vee$ is C^k.

(b) The linearity and injectivity of Φ is clear. To show that Φ is a topological embedding we will prove that the given topology on $C^{k,l}(U \times V, E)$ is initial with respect to Φ. We define the functions

$$\rho^*_{i,j}\colon C(U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, E) \to C(U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j, E), \\ g \mapsto g \circ \rho_{i,j},$$

and

$$\Psi_{i,j}\colon C(U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j, E) \to C(U \times \mathfrak{L}(G)^i, C(V \times \mathfrak{L}(H)^j, E)), \\ g \mapsto g^\vee$$

for $i,j \in \mathbb{N}_0$ such that $i \le k$, $j \le l$. Then we have

$$(d^{(i,j)}f \circ \rho_{i,j})^\vee = (\Psi_{i,j} \circ \rho^*_{i,j} \circ d^{(i,j)})(f).$$

On the other hand, we have

$$d^{(j)} \circ d^{(i)}(f^\vee) = (C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi)(f),$$

where $C(U \times \mathfrak{L}(G)^i, d^{(j)})$ are the maps

$$C(U \times \mathfrak{L}(G)^i, C^l(V,E)) \to C(U \times \mathfrak{L}(G)^i, C(V \times \mathfrak{L}(H)^j, E)), \\ g \mapsto d^{(j)} \circ g.$$

Thus, from (2.6) follows the equality

$$C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi = \Psi_{i,j} \circ \rho^*_{i,j} \circ d^{(i,j)}.$$

The given topology on $C^{k,l}(U \times V, E)$ is initial with respect to the family $(d^{(i,j)})$ by definition, hence the topology is initial with respect to the family of maps $\Psi_{i,j} \circ \rho^*_{i,j} \circ d^{(i,j)}$ (see [21, Appendix A.5], Proposition 2.5.1 and [21, Appendix A.2] for transitivity of initial topologies). But by the above equality, this topology is also initial with respect to the maps $C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi$, hence it is is initial with respect to Φ. This completes the proof. □

Theorem 2.5.4. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. If $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$-space for all $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$, then*

$$\Phi \colon C^{k,l}(U \times V, E) \to C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces.

Proof. We need to show that if $g \in C^k(U, C^l(V, E))$, then the map

$$g^\wedge \colon U \times V \to E, \quad g^\wedge(x, y) := g(x)(y)$$

(which is continuous, since the locally convex space E is completely regular and we assumed that $U \times V$ is a $k_{\mathbb{R}}$-space, see Proposition 2.5.1) is $C^{k,l}$. As $\Phi(g^\wedge) = (g^\wedge)^\vee = g$, the map Φ will be surjective, hence an isomorphism of topological vector spaces (being a linear topological embedding by Theorem 2.5.3).

To this end, we fix $x \in U$, then $g(x) \in C^l(U, E)$ and for $y \in V$, $\eta \in \mathfrak{L}(H)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(g^\wedge(x, y \cdot \eta(t)) - g^\wedge(x, y)) = \frac{1}{t}(g(x)(y \cdot \eta(t)) - g(x)(y)) \to d(g(x))(y, \eta)$$

as $t \to 0$. Thus, $d^{(0,1)}(g^\wedge)(x, y, \eta)$ exists and equals $d(g(x))(y, \eta) = (d^{(1)} \circ g)(x)(y, \eta) = (d^{(1)} \circ g)^\wedge(x, y, \eta)$. Analogously, for $j \in \mathbb{N}_0$ with $j \leq l$ and $\eta_1, \ldots, \eta_j \in \mathfrak{L}(H)$ we obtain the derivatives

$$d^{(0,j)}(g^\wedge)(x, y, \eta_1, \ldots, \eta_j) = (d^{(j)} \circ g)^\wedge(x, y, \eta_1, \ldots, \eta_j).$$

But for fixed $(y, \eta_1, \ldots, \eta_j)$ we have

$$\begin{aligned}(d^{(j)} \circ g)^\wedge(x, y, \eta_1, \ldots, \eta_j) &= (d^{(j)} \circ g)(x)(y, \eta_1, \ldots, \eta_j) \\ &= (\mathrm{ev}_{(y,\eta_1,\ldots,\eta_j)} \circ d^{(j)} \circ g)(x),\end{aligned}$$

where $\mathrm{ev}_{(y,\eta_1,\ldots,\eta_j)}$ is the continuous linear map

$$\mathrm{ev}_{(y,\eta_1,\ldots,\eta_j)} \colon C(V \times \mathfrak{L}(H)^j, E) \to E, \quad h \mapsto h(y, \eta_1, \ldots, \eta_j).$$

Since also $d^{(j)} \colon C^l(V, E) \to C(V \times \mathfrak{L}(H)^j, E)$ is continuous and linear, the composition $\mathrm{ev}_{(y,\eta_1,\ldots,\eta_j)} \circ d^{(j)} \circ g \colon U \to E$ is C^k, by Lemma 2.3.1. Thus for $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough we obtain

$$\begin{aligned}&\frac{1}{t}(d^{(0,j)}(g^\wedge)(x \cdot \gamma(t), y, \eta_1, \ldots, \eta_j) - d^{(0,j)}(g^\wedge)(x, y, \eta_1, \ldots, \eta_j)) \\ &= \frac{1}{t}((\mathrm{ev}_{(y,\eta_1,\ldots,\eta_j)} \circ d^{(j)} \circ g)(x \cdot \gamma(t)) - (\mathrm{ev}_{(y,\eta_1,\ldots,\eta_j)} \circ d^{(j)} \circ g)(x)) \\ &\to d(\mathrm{ev}_{(y,\eta_1,\ldots,\eta_j)} \circ d^{(j)} \circ g)(x, \gamma),\end{aligned}$$

as $t \to 0$. Thus $d^{(1,j)}(g^\wedge)(x,y,\gamma,\eta_1,\dots,\eta_j)$ is given by

$$\begin{aligned} d(\mathrm{ev}_{(y,\eta_1,\dots,\eta_j)} \circ d^{(j)} \circ g)(x,\gamma) &= (\mathrm{ev}_{(y,\eta_1,\dots,\eta_j)} \circ d^{(j)} \circ dg)(x,\gamma) \\ &= (d^{(j)} \circ dg)(x,\gamma)(y,\eta_1,\dots,\eta_j) \\ &= (d^{(j)} \circ dg)^\wedge(x,\gamma,y,\eta_1,\dots,\eta_j). \end{aligned}$$

Analogously, for each $i \in \mathbb{N}_0$ with $i \leq k$ and $\gamma_1,\dots,\gamma_i \in \mathfrak{L}(G)$ we obtain

$$d^{(i,j)}(g^\wedge)(x,y,\gamma_1,\dots,\gamma_i,\eta_1,\dots,\eta_j) = (d^{(j)} \circ d^{(i)} g)^\wedge(x,\gamma_1,\dots,\gamma_i,y,\eta_1,\dots,\eta_j).$$

To see that $g^\wedge$ is $C^{k,l}$ we need to show that the maps

$$\begin{aligned} d^{(i,j)}(g^\wedge) \colon U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \to E, \\ (x,y,\gamma_1,\dots,\gamma_i,\eta_1,\dots,\eta_j) \mapsto (d^{(j)} \circ d^{(i)} g)^\wedge(x,\gamma_1,\dots,\gamma_i,y,\eta_1,\dots,\eta_j) \end{aligned} \tag{2.7}$$

are continuous for all $i,j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$. To this end, consider the continuous maps

$$d^{(j)} \circ d^{(i)} g \colon U \times \mathfrak{L}(G)^i \to C(V \times \mathfrak{L}(H)^j, E).$$

By Proposition 2.5.1, the maps $(d^{(j)} \circ d^{(i)} g)^\wedge \colon U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j \to E$ are continuous, since E is completely regular and we assumed that $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_\mathbb{R}$-space, hence the maps $d^{(i,j)}(g^\wedge)$ are continuous and $g^\wedge$ is $C^{k,l}$. □

Theorem 2.5.5. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, and E be a locally convex space. If $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_\mathbb{R}$-space for all $i,j \in \mathbb{N}_0$, then*

$$\Phi \colon C^\infty(U \times V, E) \to C^\infty(U, C^\infty(V,E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces.

Proof. The assertion follows from Theorem 2.5.4, since $C^{\infty,\infty}(U \times V, E) = C^\infty(U \times V, E)$, by Corollary 2.4.3. □

Remark 2.5.6. From [24, Definition 3.25] we recall that a topological group G is called a *pro-Lie group* if G is complete and each identity neighborhood of G contains a normal subgroup N such that G/N is a Lie group. Theorem 3.39 in [24] states in particular that a topological group G is a pro-Lie group if and only if G is a projective limit of Lie groups. It is known that each almost connected locally compact topological group is a pro-Lie group (recall that a topological group G is called *almost connected* if G/G_0 is compact, where G_0 denotes the identity component of G).

Corollary 2.5.7. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $k,l \in \mathbb{N}_0 \cup \{\infty\}$. Assume that at least one of the following conditions is satisfied:*

(a) $l = 0$ and V is locally compact,

(b) $k, l < \infty$ *and* $U \times V \times \mathfrak{L}(G)^k \times \mathfrak{L}(H)^l$ *is a* $k_{\mathbb{R}}$*-space,*

(c) G *and* H *are metrizable,*

(d) G *and* H *are locally compact,*

(e) G *and* H *are almost connected pro-Lie groups.*

Then the map

$$\Phi\colon C^{k,l}(U \times V, E) \to C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces.

Proof. *(a)* As in the proof of Theorem 2.5.4, we need to show that for $g \in C^k(U, C(V, E))$ we have $g^\wedge \in C^{k,0}(U \times V, E)$. The computations of the derivatives of $g^\wedge$ carry over (with $j = 0$), hence it remains to show that the maps $d^{(i,0)}(g^\wedge)$ in (2.7) are continuous for all $i \in \mathbb{N}_0$ with $i \leq k$. But since V is assumed locally compact, each of the maps $(d^{(0)} \circ d^{(i)} g)^\wedge\colon U \times \mathfrak{L}(G)^i \times V \to E$ is continuous by Proposition 2.5.1, hence so is each of the maps $d^{(i,0)}(g^\wedge)$, as required.

(b) By [26, Proposition, p.62], if $U \times V \times \mathfrak{L}(G)^k \times \mathfrak{L}(H)^l$ is a $k_{\mathbb{R}}$-space, then so is $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ for all $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$. Hence, Theorem 2.5.4 applies and Φ is an isomorphism of topological vector spaces.

(c) Since G is metrizable, the space $C(\mathbb{R}, G)$ is metrizable (see [21, Appendix A.5] or [14, Lemma B.21]), whence so is $\mathfrak{L}(G) \subseteq C(\mathbb{R}, G)$ as well as $U \times \mathfrak{L}(G)^i$ for each $i \in \mathbb{N}_0$, $i \leq k$ as a finite product of metrizable spaces. With a similar argumentation we conclude that also $V \times \mathfrak{L}(H)^j$ is metrizable for each $j \in \mathbb{N}_0$ with $j \leq l$, whence so is $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$. But each metrizable space is a k-space, hence a $k_{\mathbb{R}}$-space. Therefore, Theorem 2.5.4 applies in this case and Φ is an isomorphism of topological vector spaces.

(d) As G is locally compact, it is known that the identity component G_0 of G (being a connected locally compact subgroup of G) is a pro-Lie group, see Remark 2.5.6. Hence, by [24, Theorem 3.12], $\mathfrak{L}(G)$ is a pro-Lie algebra, and from [24, Proposition 3.7], it follows that $\mathfrak{L}(G)$ is homeomorphic to $\mathbb{R}^I$ for some set I. Since also H is assumed locally compact, for all $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ we have $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \cong U \times V \times (\mathbb{R}^I)^i \times (\mathbb{R}^J)^j$ for some set J. Now, from [38, Theorem 5.6 (*ii*)], it follows that $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$-space, being homeomorphic to a product of locally compact spaces (hence completely regular locally pseudocompact $k_{\mathbb{R}}$-spaces), whence Theorem 2.5.4 applies and Φ is an isomorphism of topological vector spaces.

(e) Since G is a pro-Lie group, $\mathfrak{L}(G)$ is homeomorphic to $\mathbb{R}^I$ for some set I (see Remark 2.5.6 and [24, Theorem 3.12, Proposition 3.7]). Since, in addition, G is almost connected, there exist a compact subgroup C_G of G and a set K such that G is homeomorphic to $C_G \times \mathbb{R}^K$ (by Theorem 8.6 in [25]). Likewise, $\mathfrak{L}(H) \cong \mathbb{R}^J$ and $H \cong C_H \times \mathbb{R}^L$. Altogether, we have $G \times H \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \cong C_G \times \mathbb{R}^K \times C_H \times \mathbb{R}^L \times (\mathbb{R}^I)^i \times (\mathbb{R}^J)^j$, which is a $k_{\mathbb{R}}$-space, being homeomorphic to a product of locally compact spaces (hence completely regular locally pseudocompact $k_{\mathbb{R}}$-spaces) by [38, Theorem 5.6 (*ii*)]. From [6, Theorem,

p. 28], it follows that $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$-space, being an open subset of a completely regular $k_{\mathbb{R}}$-space. Hence Theorem 2.5.4 applies and Φ is an isomorphism of topological vector spaces. □

3 Differentiability along one-parameter subgroups compared to differentiability on Lie groups as manifolds

After recalling the concept of differentiability of functions on manifolds in Section 3.1, we prove that for functions defined on a locally exponential Lie group or a certain direct limit Lie group, the two concepts of differentiability coincide (Section 3.2, Theorem 3.2.10).

3.1 Differentiability on manifolds

Let M be a smooth manifold modeled on a locally convex space E, that is, M be a Hausdorff topological space together with an atlas of charts $\varphi\colon U_\varphi \to V_\varphi$ (homeomorphisms between open subsets of M and E) such that the transition maps $\psi \circ \varphi^{-1}$ are smooth functions (as in [16]). The definition of the tangent space T_xM, the tangent manifold TM and tangent maps between tangent spaces are defined in the usual way.

We recall the concept of differentiability for functions between manifolds.

Definition 3.1.1. Let M, N be manifolds modeled on locally convex spaces, $f\colon M \to N$ be a continuous function. For $k \in \mathbb{N} \cup \{\infty\}$, we call f a C^k_{mfd}-function if for each $x \in M$ there are charts φ for M around x and ψ for N around $f(x)$ such that the composition $\psi \circ f \circ \varphi^{-1}$ is C^k. In this case, if N is a locally convex space, then we write

$$d_{\mathrm{mfd}} f\colon TM \to N, \quad [\gamma] \mapsto (f \circ \gamma)'(0)$$

for the second component of the tangent map $Tf\colon TM \to N \times N$. Here $[\gamma]$ denotes a geometric tangent vector (equivalence class of the curves γ). Further, for a C^1_{mfd}-curve $f\colon \mathbb{R} \to N$ to a manifold N we define $f'\colon \mathbb{R} \to TN, f'(t) := Tf(t, 1)$.

We denote by $C^k_{\mathrm{mfd}}(M, E)$ the space of all C^k_{mfd}-functions $f\colon M \to E$ (where E is a locally convex space) and endow this space with the initial topology with respect to the family $(d^{(i)})_{i \in \mathbb{N}_0, i \leq k}$ of mappings

$$d^{(i)}\colon C^k_{\mathrm{mfd}}(M, E) \to C(V_\varphi \times F^i, E), \quad f \mapsto d^{(i)}(f \circ \varphi^{-1}),$$

for charts $\varphi\colon U_\varphi \to V_\varphi$ of the maximal atlas of M, where F is the modelling space of M. This topology turns $C^k_{\mathrm{mfd}}(M, E)$ into a Hausdorff locally convex vector space.

We will often use the following facts without further mention:

Remark 3.1.2. A function $f\colon E \to F$ is C^k if and only if f is C^1 and the differential $df\colon E \times E \to F$ is C^{k-1}. If $f\colon M \to E$ is C^k_{mfd}, then $d_{\mathrm{mfd}}f\colon TM \to E$ is C^{k-1}_{mfd}.

Further, compositions of composable C^k_{mfd}- (resp. C^k-) functions are C^k_{mfd} (resp. C^k). Each continuous linear function between locally convex spaces is C^∞.

A function between locally convex spaces is C^k_{mfd} if and only if it is C^k.

3.2 Differentiability on certain Lie groups

In this section, we will always assume that E is a locally convex topological vector space and that G is a smooth Lie group modeled on a locally convex space F, that is, G is a group endowed with a smooth manifold structure modeled on F such that the group multiplication $m_G\colon G \times G \to G$ and the inversion $j_G\colon G \to G$ are smooth functions.

We will prove that the concepts of differentiability on G as a topological group and as a manifold coincide if G is locally exponential or if G is a direct limit Lie group $G = \varinjlim G_n$ of certain Lie groups G_n. We denote by $\mathfrak{g} := T_eG$ the Lie algebra of G, where T_eG is the tangent space of G at the identity element e. We write $\sigma\colon G \times TG \to TG, (x, v) \mapsto x.v := T\lambda_x(v)$ for the smooth left action of G on the tangent group TG, where $T\lambda_x$ is the tangent map of the left translation $\lambda_x\colon y \mapsto x \cdot y$ on G.

Remark 3.2.1. (i) Recall that a Lie group G is called *locally exponential* if G has a C^∞_{mfd}-exponential function $\exp\colon \mathfrak{g} \to G$ and there exists an open 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\exp\big|_U$ is a diffeomorphism onto an open e-neighborhood $V \subseteq G$; we denote its inverse by $\log\colon V \to U$. In this case, if γ is a continuous one-parameter subgroup, then γ is C^∞_{mfd} and there exists a unique $v \in \mathfrak{g}$ such that $\gamma(t) = \exp(tv) =: \gamma_v(t)$. Moreover, the function $\Gamma\colon \mathfrak{g} \to \mathfrak{L}(G), v \mapsto \gamma_v$ is a homeomorphism with the inverse $\Gamma^{-1}\colon \mathfrak{L}(G) \to \mathfrak{g}, \gamma \mapsto [\gamma] = \gamma'(0)$. We equip $\mathfrak{L}(G)$ with the locally convex topological vector space structure making Γ an isomorphism of locally convex spaces (hence C^∞). (Details on locally exponential Lie groups can be found, for example, in [32] or [21].)

(ii) Consider an ascending sequence of finite-dimensional Lie groups $G_1 \subseteq G_2 \subseteq \cdots$ such that the inclusions $G_n \to G_{n+1}$ are continuous (hence C^∞_{mfd}, being homomorphisms between finite-dimensional Lie groups). Then $G := \bigcup_{n\in\mathbb{N}} G_n$ admits a Lie group structure such that $G = \varinjlim G_n$ in the category of Lie groups modeled on locally convex vector spaces (we call G a *direct limit Lie group*), for the Lie algebra $\mathfrak{g}$ of G we have $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ in the category of topological Lie algebras, where each $\mathfrak{g}_n$ denotes the Lie algebra of G_n (by [15, Theorem 4.3, (a)]). (Note that G always has a C^∞_{mfd}-exponential function, but is not necessarily locally exponential [13, Example 5.5].) Further, we have $\mathfrak{L}(G) = \bigcup_{n\in\mathbb{N}} \mathfrak{L}(G_n)$ ([13, 5.3]), that is, each $\gamma \in \mathfrak{L}(G)$ is a continuous one-parameter subgroup of some G_n, hence C^∞_{mfd}; moreover, each of the functions $\Gamma_n\colon \mathfrak{g}_n \to \mathfrak{L}(G_n)$ (as defined above) is a homeomorphism (each G_n being locally exponential). Hence so is the function $\varinjlim \Gamma_n\colon \mathfrak{g} \to \mathfrak{L}(G)$ (where $\mathfrak{L}(G) = \varinjlim \mathfrak{L}(G_n)$ is the direct limit in the category of topological spaces). But by [18, Theorem 4.4] the function $\Gamma\colon \mathfrak{g} \to \mathfrak{L}(G)$ is a homeomorphism (where $\mathfrak{L}(G)$ is equipped with the compact-open topology), thus the direct limit topology and the compact-open topology on $\mathfrak{L}(G)$ coincide. Since each finite-dimensional G_n and each $\mathfrak{L}(G_n) \cong \mathfrak{g}_n \cong \mathbb{R}^{dim(G_n)}$ is locally compact, we conclude that $G \times \mathfrak{L}(G)^k = \varinjlim (G_n \times \mathfrak{L}(G_n)^k)$, for each $k \in \mathbb{N}$ (see [12, Proposition 3.2], [23, Theorem 4.1]). As in (i), we can equip $\mathfrak{L}(G)$ with the locally convex topological vector space structure making Γ an isomorphism of locally convex spaces.

The following properties of differentiable functions on direct limit Lie groups will enable us to reduce the case of direct limit Lie groups to the case of locally exponential Lie groups.

Lemma 3.2.2. *Let G be a direct limit Lie group as in Remark 3.2.1 (ii) and $f\colon G \to E$ be a continuous function. For $k \in \mathbb{N} \cup \{\infty\}$ the following holds:*

(i) f is C^k if and only if $f\big|_{G_n}: G_n \to E$ is C^k for each $n \in \mathbb{N}$,

(ii) f is C^k_{mfd} if and only if $f\big|_{G_n}: G_n \to E$ is C^k_{mfd} for each $n \in \mathbb{N}$.

Proof. *(i)* First, assume that f is C^k. For $n \in \mathbb{N}$, the inclusion map $\mathrm{incl}_n\colon G_n \to G$ is a continuous homomorphism, whence $f \circ \mathrm{incl}_n$ is C^k by Lemma 2.3.2.

Conversely, let $x \in G$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ for some $i \leq k$. Since G and $\mathfrak{L}(G)$ are ascending unions of G_n and $\mathfrak{L}(G_n)$, respectively, there exists some $N \in \mathbb{N}$ such that $x, \gamma_1, \dots, \gamma_i \in G_N \times \mathfrak{L}(G_N)^i$. Hence $d^{(i)}f\colon G \times \mathfrak{L}(G)^i \to E$ is defined (with $d^{(i)}f(x, \gamma_1, \dots, \gamma_i) := d^{(i)}(f\big|_{G_N})(x, \gamma_1, \dots, \gamma_i)$). This differential is continuous if and only if $d^{(i)}f\big|_{G_n \times \mathfrak{L}(G_n)^i} = d^{(i)}(f\big|_{G_n})$ is continuous (see above) and this is satisfied by the assumption.

(ii) See [13, Proposition 4.2]. □

Proposition 3.2.3. *If G is a locally exponential Lie group and $f\colon G \to E$ is a C^k_{mfd}-map for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then f is C^k.*

Proof. For $k = 0$ the assertion is clear. For $k \geq 1$, we may assume that $k < \infty$ and proceed by induction.

Base case: For $x \in G$ we denote $\lambda_x\colon y \mapsto x \cdot y$, which is a C^∞_{mfd}-function. For $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ we have

$$\frac{f(x \cdot \gamma(t)) - f(x)}{t} = \frac{(f \circ \lambda_x \circ \gamma)(t) - (f \circ \lambda_x \circ \gamma)(0)}{t} \to (f \circ \lambda_x \circ \gamma)'(0),$$

as $t \to 0$, because the composition $f \circ \lambda_x \circ \gamma\colon \mathbb{R} \to E$ is a C^1-curve. We rewrite

$$df(x, \gamma) = (f \circ \lambda_x \circ \gamma)'(0) = d_{\mathrm{mfd}} f(\sigma(x, \Gamma^{-1}(\gamma))) \tag{3.1}$$

and see that the differential $df := d_{\mathrm{mfd}} f \circ \sigma \circ (\mathrm{id}_G \times \Gamma^{-1})\colon G \times \mathfrak{L}(G) \to E$ is continuous, hence the function f is C^1.

Induction step: Assume that f is C^k_{mfd} for $k \geq 2$. Then f is C^1, by the base case. Using (3.1), we see that the differential df can be written as a composition of C^{k-1}_{mfd}-functions, hence it is C^{k-1}_{mfd} on the locally exponential Lie group $G \times \mathfrak{L}(G)$. Therefore, the differential is C^{k-1}, by the induction hypothesis, whence f is C^k. □

Lemma 3.2.4. *Let G be a locally exponential Lie group, $f\colon G \to E$ be a C^1-map and $\gamma\colon \mathbb{R} \to G$ be a C^1_{mfd}-curve. Then $(f \circ \gamma)'(0) = df(\gamma(0), \kappa)$ for some one-parameter subgroup $\kappa \in \mathfrak{L}(G)$.*

Proof. First, we recall from Lemma 2.3.4 that the function $f^{[1]}\colon G \times \mathfrak{L}(G) \times \mathbb{R} \to E$ such that $f^{[1]}(x, \eta, t) = \frac{1}{t}(f(x \cdot \eta(t)) - f(x))$, for $t \neq 0$, is continuous on $G \times \mathfrak{L}(G) \times \mathbb{R}$, since f is assumed C^1, and we have $df(x, \eta) = f^{[1]}(x, \eta, 0)$. Now, for $\varepsilon > 0$ small enough consider the continuous curve

$$\eta\colon]-\varepsilon, \varepsilon[\to \mathfrak{g}, \quad \eta(t) := \begin{cases} \frac{1}{t} \log(\gamma(0)^{-1} \cdot \gamma(t)) & \text{if } t \neq 0, \\ \gamma(0)^{-1}.\gamma'(0) & \text{if } t = 0. \end{cases}$$

Note that the continuity of η in $t = 0$ follows from

$$\begin{aligned} \lim_{t \to 0} \frac{\log(\gamma(0)^{-1} \cdot \gamma(t))}{t} &= \lim_{t \to 0} \frac{\log(\gamma(0)^{-1} \cdot \gamma(t)) - \log(\gamma(0)^{-1} \cdot \gamma(0))}{t} \\ &= (\log \circ \lambda_{\gamma(0)^{-1}} \circ \gamma)'(0) \\ &= (d_{\mathrm{mfd}}(\log) \circ T\lambda_{\gamma(0)^{-1}})(\gamma'(0)) \\ &= \gamma(0)^{-1}.\gamma'(0), \end{aligned}$$

using that $d_{\mathrm{mfd}}(\log)$ is a restriction of $\mathrm{id}_{\mathfrak{g}}$ in this case. Now, for $0 \neq t \in]-\varepsilon, \varepsilon[$ we have

$$\frac{f(\gamma(t)) - f(\gamma(0))}{t} = \frac{f(\gamma(0) \cdot \kappa_t(t)) - f(\gamma(0))}{t} = f^{[1]}(\gamma(0), \kappa_t, t)$$

with the one-parameter subgroup $\kappa_t\colon \mathbb{R} \to G, \kappa_t(s) := \exp(s\eta(t))$. Then

$$(f \circ \gamma)'(0) = \lim_{t \to 0} f^{[1]}(\gamma(0), \kappa_t, t) = f^{[1]}(\gamma(0), \kappa_0, 0) = df(\gamma(0), \kappa_0),$$

and the proof is finished. □

Remark 3.2.5. Assume that $\mathfrak{L}(G)$ carries a topological vector space structure (for example, if G is a Lie group as in Remark 3.2.1). If a function $f\colon G \to E$ is C^1 and the differential df is C^{k-1} on the topological group $G \times \mathfrak{L}(G)$, then f is C^k with derivatives

$$d^{(i)} f(x, \gamma_1, \ldots, \gamma_i) = d^{(i-1)}(df)((x, \gamma_1), (\gamma_2, \bar{\gamma}_0), \ldots, (\gamma_i, \bar{\gamma}_0)), \tag{3.2}$$

where $\bar{\gamma}_0 \in \mathfrak{L}(\mathfrak{L}(G))$ denotes the one-parameter subgroup $t \mapsto \gamma_0$, where $\gamma_0 \in \mathfrak{L}(G)$ is the trivial one-parameter subgroup of G.

On the other hand, if f is C^k and df is linear in the second argument, then df is C^{k-1} with derivatives

$$\begin{aligned} &d^{(i)}(df)((x, \alpha), (\gamma_1, \eta_1), \ldots, (\gamma_i, \eta_i)) \\ &= d^{(i+1)} f(x, \alpha, \gamma_1, \ldots, \gamma_i) + \sum_{j=1}^{i} d^{(i)} f(x, \eta_j(1), \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_i). \end{aligned} \tag{3.3}$$

Proposition 3.2.6. *If G is a locally exponential Lie group and $f\colon G \to E$ is a C^k-map for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then f is C^k_{mfd}.*

Proof. For $k = 0$ the assertion is clearly true. Now, we assume that $1 \leq k < \infty$ and prove the assertion by induction.

Base case: Fix $g \in G$ and let $\varphi\colon U_\varphi \to V_\varphi \subseteq F$ be a chart for G around g, where F is the modelling space of G. To show that f is C^1_{mfd} we need to prove that $f \circ \varphi^{-1}\colon V_\varphi \to E$ is C^1. To this end, let $x \in V_\varphi$, $y \in F$ and define the C^1_{mfd}-curve $\gamma\colon]-\varepsilon, \varepsilon[\to G, t \mapsto \varphi^{-1}(x+ty)$ for suitable $\varepsilon > 0$. (Note that $T\varphi^{-1}(x, y) = [t \mapsto \varphi^{-1}(x+ty)] = [\gamma] = \gamma'(0)$.) Then we have

$$d(f \circ \varphi^{-1})(x, y) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} = (f \circ \gamma)'(0) = df(\gamma(0), \kappa)$$

with the one-parameter subgroup $\kappa := \kappa_0\colon \mathbb{R} \to G, t \mapsto \exp(t(\gamma(0)^{-1}.\gamma'(0)))$ from Lemma 3.2.4. Using the C^∞_{mfd}-function $\omega\colon TG \to \mathfrak{g}, v \mapsto \pi_{TG}(v)^{-1}.v$ (where π_{TG} denotes the bundle projection, which is C^∞_{mfd}) we rewrite

$$d(f \circ \varphi^{-1})(x, y) = df\left(\pi_{TG}([\gamma]), (\Gamma \circ \omega)([\gamma])\right),$$

hence the differential

$$d(f \circ \varphi^{-1}) = df \circ (\pi_{TG}, \Gamma \circ \omega) \circ T\varphi^{-1}\colon V_\varphi \times F \to E \tag{3.4}$$

is continuous and consequently $f \circ \varphi^{-1}$ is a C^1-function, as required. Note that we found

$$d_{\mathrm{mfd}} f([\gamma]) = df\left(\pi_{TG}([\gamma]), (\Gamma \circ \omega)([\gamma])\right), \tag{3.5}$$

for each $[\gamma] \in TG$.

Induction step: Now, let f be a C^k-map for $k \geq 2$ and φ be as above. By the base case, we know that f is C^1_{mfd} (that is, $f \circ \varphi^{-1}$ is C^1). To show that $d(f \circ \varphi^{-1})$ is C^{k-1}, consider the formula in (3.4). The composition $(\pi_{TG}, \Gamma \circ \omega) \circ T\varphi^{-1}\colon V_\varphi \times F \to G \times \mathfrak{L}(G)$ is C^∞_{mfd}. Further, using (3.1) we see that df is linear in the second argument, hence C^{k-1} (see Remark 3.2.5), Thus df is C^{k-1}_{mfd} on the locally exponential Lie group $G \times \mathfrak{L}(G)$, by the induction hypothesis. Therefore, the differential $d(f \circ \varphi^{-1})$ is C^{k-1}_{mfd}. Consequently, this differential is C^{k-1} as required. □

Proposition 3.2.7. *Let G be a direct limit Lie group as in Remark 3.2.1 (ii). A function $f\colon G \to E$ is C^k if and only if f is C^k_{mfd}, for each $k \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. Since each Lie group G_n is locally exponential (being finite dimensional), each of the restrictions $f\big|_{G_n}$ is C^k if and only if it is C^k_{mfd}, by Propositions 3.2.3 and 3.2.6. The remainder follows from Lemma 3.2.2. □

Using the fact that the differential df of each C^k-function f defined on a locally exponential Lie group G or on a direct limit Lie group G (as in Remark 3.2.1) is C^{k-1}, we show that in these cases the compact-open C^k-topology on $C^k(G, E)$ can be described in the following way (for finite k):

Lemma 3.2.8. *The compact-open C^k-topology $\mathcal{O}_1$ on the function space $C^k(G,E)$ coincides with the initial topology $\mathcal{O}_2$ with respect to the functions*

$$\begin{aligned} \mathrm{incl}\colon C^k(G,E) &\to C(G,E), \quad f \mapsto f,\\ D\colon C^k(G,E) &\to (C^{k-1}(G\times\mathfrak{L}(G),E),\mathcal{O}_1), \quad f\mapsto df \end{aligned}$$

for each $k\in\mathbb{N}$ if G is a Lie group as in Remark 3.2.1.

Proof. First, we show that $\mathcal{O}_2 \subseteq \mathcal{O}_1$. This will hold, if both functions

$$\mathrm{incl}\colon (C^k(G,E),\mathcal{O}_1) \to C(G,E), \quad f\mapsto f, \tag{3.6}$$

$$D\colon (C^k(G,E),\mathcal{O}_1) \to (C^{k-1}(G\times\mathfrak{L}(G),E),\mathcal{O}_1), \quad f\mapsto df \tag{3.7}$$

are continuous. The first function (3.6) is continuous by the definition of the topology $\mathcal{O}_1$, since we have $\mathrm{incl} = d^{(0)}$. The continuity of the second function (3.7) will follow from the continuity of the compositions

$$\begin{aligned} d^{(i)}\circ D\colon (C^k(G,E),\mathcal{O}_1) &\to C(G\times\mathfrak{L}(G)\times\mathfrak{L}(G\times\mathfrak{L}(G))^i,E),\\ f &\mapsto d^{(i)}(df), \end{aligned}\tag{3.8}$$

where $d^{(i)}\colon (C^{k-1}(G\times\mathfrak{L}(G),E),\mathcal{O}_1) \to C(G\times\mathfrak{L}(G)\times\mathfrak{L}(G\times\mathfrak{L}(G))^i,E)$ for all $0\le i\le k-1$. Using the two continuous functions

$$\begin{aligned} \rho_i\colon G\times\mathfrak{L}(G)\times\mathfrak{L}(G\times\mathfrak{L}(G))^i &\to G\times\mathfrak{L}(G)^{i+1},\\ (x,\alpha,(\gamma_1,\eta_1),\ldots,(\gamma_i,\eta_i)) &\mapsto (x,\alpha,\gamma_1,\ldots,\gamma_i),\\ \rho_{j,i}\colon G\times\mathfrak{L}(G)\times\mathfrak{L}(G\times\mathfrak{L}(G))^i &\to G\times\mathfrak{L}(G)^i,\\ (x,\alpha,(\gamma_1,\eta_1),\ldots,(\gamma_i,\eta_i)) &\mapsto (x,\eta_j(1),\gamma_1,\ldots,\gamma_{j-1},\gamma_{j+1},\ldots,\gamma_i), \end{aligned}$$

for $1\le j\le i$, and Equation (3.3) from Remark 3.2.5, we obtain

$$d^{(i)}(df) = (d^{(i+1)}f\circ\rho_i) + \sum_{j=1}^{i}(d^{(i)}f\circ\rho_{j,i})$$

for each $f\in C^k(G,E)$. Hence, using the continuous pullbacks $\rho_i^*\colon g\mapsto g\circ\rho_i$ and $\rho_{j,i}^*\colon g\mapsto g\circ\rho_{j,i}$ we can write each of the maps from (3.8) as

$$d^{(i)}\circ D = (\rho_i^*\circ d^{(i+1)}) + \sum_{j=1}^{i}(\rho_{j,i}^*\circ d^{(i)}).$$

(Note that the functions $d^{(i+1)}$, $d^{(i)}$ on the right-hand side are the differential operators on $(C^k(G,E),\mathcal{O}_1)$.) From the definition of the compact-open C^k-topology $\mathcal{O}_1$ we conclude that the composition is continuous, as required.

Now, we show that $\mathcal{O}_1\subseteq\mathcal{O}_2$, which will be the case if for all $0\le i\le k$ the functions

$$d^{(i)}\colon (C^k(G,E),\mathcal{O}_2)\to C(G\times\mathfrak{L}(G)^i,E), \quad f\mapsto d^{(i)}f \tag{3.9}$$

are continuous. For $i = 0$ we have $d^{(0)} = \mathrm{incl}$, hence the continuity follows from the definition of the topology $\mathcal{O}_2$. Now, using the continuous functions

$$\xi_i \colon G \times \mathfrak{L}(G)^i \to G \times \mathfrak{L}(G) \times \mathfrak{L}(G \times \mathfrak{L}(G))^{i-1},$$
$$(x, \gamma_1, \dots, \gamma_i) \mapsto ((x, \gamma_1), (\gamma_2, \bar{\gamma}_0), \dots, (\gamma_i, \bar{\gamma}_0)),$$

where $\bar{\gamma}_0 \in \mathfrak{L}(\mathfrak{L}(G))$ denotes the one-parameter subgroup $t \mapsto \gamma_0$, where $\gamma_0 \in \mathfrak{L}(G)$ is the trivial one-parameter subgroup of G, and Equation (3.2) from Remark 3.2.5 we can express the functions in (3.9) as

$$d^{(i)} = \xi_i^* \circ d^{(i-1)} \circ D,$$

with the continuous differential operators $d^{(i-1)}$ on $(C^{k-1}(G \times \mathfrak{L}(G), E), \mathcal{O}_1)$ on the right-hand side and the continuous pullbacks $\xi_i^* \colon g \mapsto g \circ \xi_i$. From the definition of the topology $\mathcal{O}_2$ on $C^k(G, E)$ we conclude that the composition above is continuous for each i, as required. □

Analogously, one can prove the following known fact:

Lemma 3.2.9. *The compact-open C^k_{mfd}-topology $\mathcal{O}_1$ on the space $C^k_{\mathrm{mfd}}(G, E)$ coincides with the initial topology $\mathcal{O}_2$ with respect to the functions*

$$\mathrm{incl} \colon C^k_{\mathrm{mfd}}(G, E) \to C(G, E), \quad f \mapsto f,$$
$$d_{\mathrm{mfd}} \colon C^k_{\mathrm{mfd}}(G, E) \to (C^{k-1}_{\mathrm{mfd}}(TG, E), \mathcal{O}_1), \quad f \mapsto d_{\mathrm{mfd}} f$$

for each $k \in \mathbb{N}$ if G is a Lie group as in Remark 3.2.1.

Using these descriptions of the topologies on the function spaces, we finally get the main result:

Theorem 3.2.10. *If G is a locally exponential Lie group or a direct limit Lie group (as in Remark 3.2.1), E is a locally convex space and $k \in \mathbb{N}_0 \cup \{\infty\}$, then $C^k(G, E) = C^k_{\mathrm{mfd}}(G, E)$ as topological vector spaces.*

Proof. From Propositions 3.2.3, 3.2.6 and 3.2.7, it follows that the function spaces coincide as sets, it remains to show that also the topologies coincide.

The topologies on $C^\infty(G, E)$ and $C^\infty_{\mathrm{mfd}}(G, E)$ are initial with respect to the inclusion maps $C^\infty(G, E) \to C^k(G, E)$ and $C^\infty_{\mathrm{mfd}}(G, E) \to C^k_{\mathrm{mfd}}(G, E)$ for $k \in \mathbb{N}_0$, respectively (this is easy to verify using the definitions). Hence it suffices to prove the continuity of both inclusion maps $\mathrm{incl}_k \colon C^k(G, E) \to C^k_{\mathrm{mfd}}(G, E)$ and $\mathrm{incl}^k \colon C^k_{\mathrm{mfd}}(G, E) \to C^k(G, E)$ by induction on k.

Base case: The inclusion maps incl_0 and incl^0 coincide with the functions incl from Lemma 3.2.8 and Lemma 3.2.9, respectively, hence they are continuous.

Induction step: By Lemma 3.2.9, the continuity of the inclusion map incl_k will follow from the continuity of the compositions

$$\mathrm{incl} \circ \mathrm{incl}_k \colon C^k(G, E) \to C(G, E), \quad f \mapsto f, \tag{3.10}$$
$$d_{\mathrm{mfd}} \circ \mathrm{incl}_k \colon C^k(G, E) \to C^{k-1}_{\mathrm{mfd}}(TG, E), \quad f \mapsto d_{\mathrm{mfd}} f. \tag{3.11}$$

The first composition (3.10) is continuous, by Lemma 3.2.8. Now, for $f \in C^k(G,E)$ and $v \in TG$ we have

$$d_{\mathrm{mfd}} f(v) = df(\pi_{TG}(v), (\Gamma \circ \omega)(v)),$$

using (3.5). Recall from Lemma 3.2.8 that

$$D\colon C^k(G,E) \to C^{k-1}(G \times \mathfrak{L}(G), E) \quad f \mapsto df$$

is continuous. Using the induction hypothesis, we conclude that D is continuous as a function to $C^{k-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E)$. Further, the operator

$$(\pi_{TG}, \Gamma \circ \omega)^*\colon C^{k-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E) \to C^{k-1}_{\mathrm{mfd}}(TG, E), \quad f \mapsto f \circ (\pi_{TG}, \Gamma \circ \omega)$$

is continuous (see [21]) and we have

$$d_{\mathrm{mfd}} = (\pi_{TG}, \Gamma \circ \omega)^* \circ D,$$

by the above. Therefore, also the composition in (3.11) is continuous and the first assertion is proved.

Now, by Lemma 3.2.8 the continuity of the inclusion map incl^k will follow from the continuity of the functions

$$\mathrm{incl} \circ \mathrm{incl}^k\colon C^k_{\mathrm{mfd}}(G,E) \to C(G,E), \quad f \mapsto f, \tag{3.12}$$

$$D \circ \mathrm{incl}^k\colon C^k_{\mathrm{mfd}}(G,E) \to C^{k-1}(G \times \mathfrak{L}(G), E), \quad f \mapsto df. \tag{3.13}$$

The first composition (3.12) is continuous by Lemma 3.2.9. Further, for $f \in C^k_{\mathrm{mfd}}(G,E)$ we have

$$df = d_{\mathrm{mfd}} f \circ \sigma \circ (\mathrm{id}_G \times \Gamma^{-1}),$$

by (3.1). The function $d_{\mathrm{mfd}}\colon C^k_{\mathrm{mfd}}(G,E) \to C^{k-1}_{\mathrm{mfd}}(TG,E)$ is continuous by Lemma 3.2.9, and also the operator

$$(\sigma \circ (\mathrm{id}_G \times \Gamma^{-1}))^*\colon C^{k-1}_{\mathrm{mfd}}(TG,E) \to C^{k-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E),$$
$$f \mapsto f \circ \sigma \circ (\mathrm{id}_G \times \Gamma^{-1})$$

is continuous (see [21]), hence so is the composition

$$D = (\sigma \circ (\mathrm{id}_G \times \Gamma^{-1}))^* \circ d_{\mathrm{mfd}}\colon C^k_{\mathrm{mfd}}(G,E) \to C^{k-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E).$$

But by the induction hypothesis, it is continuous as a function to $C^{k-1}(G \times \mathfrak{L}(G), E)$, hence the composition in (3.13) is continuous and the proof is finished. □

4 Measurable regularity of infinite-dimensional Lie groups based on Lusin measurability

In Section 4.1, we recall the definition of Lusin-measurable functions between topological spaces and construct Lebesgue spaces $L^p([a,b],E)$ of vector-valued Lusin-measurable functions $\gamma\colon [a,b]\to E$. Further, Section 4.2 describes the construction of vector spaces $AC_{L^p}([a,b],E)$ and Lie groups $AC_{L^p}([a,b],G)$. In Section 4.3, we introduce the definition of L^p-regular Lie groups and prove some important results (Theorems 4.3.9, 4.3.10 and 4.3.13).

Note that whenever we deal with C^k-functions on Lie groups in this chapter, we always mean the differentiability in the sense of Definition 3.1.1 (that is, on the Lie group as a manifold).

4.1 Measurable functions and Lebesgue spaces

This section starts with the description of measurability of functions between topological spaces, which is also known as *Lusin measurability.* This concept of measurability was used, for example, in [9], [11], [42] and others. Further, we present the definition of $\mathcal{L}^p$- and L^p-spaces of such functions (as in [11]) and show that these spaces coincide with the L^p-spaces considered in [17] whenever the latter spaces are defined. Finally, we discuss some basic properties of L^p-spaces and of functions between them.

Recall that the Borel σ-algebra $\mathcal{B}(X)$ on a topological space X is the σ-algebra generated by the open subsets of X. A function $\gamma\colon X\to Y$ between topological spaces is called *Borel measurable* if the preimage $\gamma^{-1}(A)$ of every open (resp. closed) subset $A\subseteq Y$ is in $\mathcal{B}(X)$. A measure $\mu\colon \mathcal{B}(X)\to[0,\infty]$ is called *inner regular* if $\mu(B)=\sup\{\mu(K) : K \text{ is compact}, K\subseteq B\}$ for every Borel set $B\subseteq X$. Further, μ is called *locally finite* if for every $x\in X$ there exists a neighborhood $U\subseteq X$ such that $\mu(U)<\infty$. A locally finite inner regular measure is called a *Radon measure.* Whenever there is a subset $N\in\mathcal{B}(X)$ such that $\mu(N)=0$ and an assertion holds for all $x\in X\setminus N$ we will say that the assertion holds *μ-almost everywhere* or *μ-a.e.*

Remark 4.1.1. Note that if μ is a Radon measure and $K\subseteq X$ is a compact subset, then $\mu(K)<\infty$, as μ is locally finite. Further, from the inner regularity of μ, it follows that for every Borel set B with $\mu(B)<\infty$ and every $\varepsilon>0$ there exists a compact subset $K\subseteq B$ such that $\mu(B\setminus K)\leq\varepsilon$.

Definition 4.1.2. Let μ be a Radon measure on a topological space X. A function $\gamma\colon X \to Y$ to a topological space Y is called *Lusin μ-measurable* if for every compact subset $K \subseteq X$ and every $\varepsilon > 0$ there exists a compact subset $K' \subseteq K$ with $\mu(K \setminus K') \leq \varepsilon$ such that the restriction $\gamma|_{K'}$ is continuous.

Lemma 4.1.3. *If $\gamma\colon X \to Y$ is Lusin μ-measurable, then for every Borel set B with $\mu(B) < \infty$ and every $\varepsilon > 0$ exists a compact subset $K \subseteq B$ such that $\mu(B \setminus K) \leq \varepsilon$ and $\gamma|_K$ is continuous.*

Proof. As μ is inner regular, there exists a compact set $L \subseteq A$ with $\mu(A \setminus L) \leq \varepsilon/2$, further there exists a compact subset $K \subseteq L$ such that $\mu(L \setminus K) \leq \varepsilon/2$ and $\gamma|_K$ is continuous. Since $\mu(A \setminus K) = \mu(A \setminus L) + \mu(L \setminus K) \leq \varepsilon$, the assertion is proved. □

Remark 4.1.4. Obviously, every continuous function is Lusin μ-measurable.

If X is a compact space, then it is easy to see that a function $\gamma\colon X \to Y$ is Lusin μ-measurable if and only if for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subseteq X$ such that $\mu(X \setminus K_\varepsilon) \leq \varepsilon$ and $\gamma|_{K_\varepsilon}$ is continuous.

Further, consider two functions $\gamma, \eta\colon X \to Y$ such that $\gamma(x) = \eta(x)$ μ-a.e. If γ is Lusin μ-measurable, then so is η.

In fact, let $K \subseteq X$ be a compact set and let $\varepsilon > 0$. We denote by N a Borel set such that $\mu(N) = 0$ and $\gamma(x) = \eta(x)$ for $x \in X \setminus N$. For $B := K \setminus N$, Lemma 4.1.3 provides a compact subset $K' \subseteq B$ such that $\mu(B \setminus K') \leq \varepsilon$ and $\eta|_{K'} = \gamma|_{K'}$ is continuous. Since $\mu(K \setminus K') = \mu(N) + \mu(B \setminus K') \leq \varepsilon$, we see that η is Lusin μ-measurable.

The following essential criterion for Lusin μ-measurability can be found in [9], or also in [3] as a part of Lusin's Theorem.

Lemma 4.1.5. *Let X, Y be topological spaces, $\mu\colon \mathcal{B}(X) \to [0, \infty]$ be a Radon measure on X. A function $\gamma\colon X \to Y$ is Lusin μ-measurable if and only if for each compact subset $K \subseteq X$ there exists a (pairwise disjoint) sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets $K_n \subseteq K$ such that $\mu(K \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0$ and every restriction $\gamma|_{K_n}$ is continuous.*

Proof. Assume that γ is Lusin μ-measurable and let $K \subseteq X$ be a compact subset. There exists a compact set $K_1 \subseteq K$ such that $\mu(K \setminus K_1) \leq 1$ and $\gamma|_{K_1}$ is continuous. Define $L_1 := K \setminus K_1 \in \mathcal{B}(X)$. As $\mu(L_1) < \infty$, by Lemma 4.1.3 there exists a compact set $K_2 \subseteq L_1$ such that $\mu(L_1 \setminus K_2) \leq 1/2$ and $\gamma|_{K_2}$ is continuous. Let $L_2 := L_1 \setminus K_2 \in \mathcal{B}(X)$. Then $\mu(L_2) < \infty$, therefore there exists a compact set $K_3 \subseteq L_2$ such that $\mu(L_2 \setminus K_3) \leq 1/3$ and $\gamma|_{K_3}$ is continuous. Proceeding this way, we obtain a sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ in K such that $\mu(K \setminus \bigcup_{i=1}^n K_i) = \mu(L_{n-1} \setminus K_n) \leq 1/n$ for every n. Then

$$\mu(K \setminus \bigcup_{n\in\mathbb{N}} K_n) = \mu(\bigcap_{n\in\mathbb{N}} (K \setminus \bigcup_{i=1}^n K_i)) = \lim_{n\to\infty} \mu(K \setminus \bigcup_{i=1}^n K_i) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

As every $\gamma|_{K_n}$ is continuous, the first part of the proof is finished.

Conversely, given $\varepsilon > 0$, a compact set $K \subseteq X$ and a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets $K_n \subseteq K$, we have

$$\lim_{n\to\infty} \mu(K \setminus \bigcup_{m=1}^{n} K_m) = \mu(\bigcap_{n\in\mathbb{N}} (K \setminus \bigcup_{m=1}^{n} K_m)) = \mu(K \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0.$$

Consequently, there is some $N \in \mathbb{N}$ such that $\mu(K \setminus \bigcup_{m=1}^{N} K_m) \leq \varepsilon$. The finite union $K' := \bigcup_{m=1}^{N} K_m$ is a compact subset of K and $\gamma|_{K'}$ is continuous, thus γ is Lusin μ-measurable. □

Lemma 4.1.6. *Let X be a topological space and μ be a Radon measure on X.*

(i) If Y, Z are topological spaces, $\gamma\colon X \to Y$ is Lusin μ-measurable and $f\colon Y \to Z$ is continuous, then the composition $f \circ \gamma$ is Lusin μ-measurable.

(ii) If $(Y_n)_{n\in\mathbb{N}}$ are topological spaces, then $\gamma := (\gamma_n)_{n\in\mathbb{N}}\colon X \to \prod_{n\in\mathbb{N}} Y_n$ is Lusin μ-measurable if and only if every component $\gamma_n\colon X \to Y_n$ is Lusin μ-measurable.

Proof. (*i*) Let $K \subseteq X$ be a compact subset and $\varepsilon > 0$. Then there exists a compact subset $K' \subseteq K$ such that $\mu(K \setminus K') \leq \varepsilon$ and $\gamma|_{K'}$ is continuous. Then obviously $(f \circ \gamma)|_{K'}$ is continuous, whence $f \circ \gamma$ is Lusin μ-measurable.

(*ii*) If $\gamma\colon X \to \prod_{n\in\mathbb{N}} Y_n$ is Lusin μ-measurable, then from (*i*), it follows that every $\gamma_m = \mathrm{pr}_m \circ \gamma$ is Lusin μ-measurable, where $\mathrm{pr}_m\colon \prod_{n\in\mathbb{N}} Y_n \to Y_m$ are the coordinate projections.

On the other hand, fix $\varepsilon > 0$ and a compact subset $K \subseteq X$. Then for each $n \in \mathbb{N}$ there is a compact subset $K_n \subseteq K$ such that $\gamma_n|_{K_n}$ is continuous and $\mu(K \setminus K_n) \leq \varepsilon/2^n$, as each γ_n is Lusin μ-measurable. Then the intersection $K_\varepsilon := \bigcap_{n\in\mathbb{N}} K_n$ is a compact subset of K with

$$\mu(K \setminus K_\varepsilon) = \mu(\bigcup_{n\in\mathbb{N}} (K \setminus K_n)) \leq \sum_{n=1}^{\infty} \mu(K \setminus K_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Since $K_\varepsilon \subseteq K_n$ for each $n \in \mathbb{N}$, the restriction $\gamma|_{K_\varepsilon}$ is continuous, thus γ is Lusin μ-measurable. □

Remark 4.1.7. From Lemma 4.1.6 easily, it follows that vector-valued Lusin μ-measurable functions form a vector space.

From now on we will work with functions defined on an interval $[a, b] \subseteq \mathbb{R}$, the considered measure will always be the Lebesgue-Borel measure λ and we will call Lusin λ-measurable functions $\gamma\colon [a, b] \to X$ just *measurable* for short.

The relation between Lusin μ-measurable functions and Borel measurable functions is known as Lusin's Theorem and can be found in several versions in [3], [10], and others. We prove a special case which will suffice for our purposes.

Lemma 4.1.8. *Let X be a topological space. If $\gamma\colon [a,b] \to X$ is measurable, then there exists a Borel measurable function $\bar{\gamma}\colon [a,b] \to X$ such that $\bar{\gamma}(t) = \gamma(t)$ a.e.*

On the other hand, if X has a countable base, then every Borel measurable function $\gamma\colon [a,b] \to X$ is measurable.

Proof. If $\gamma\colon [a,b] \to X$ is measurable, then by Lemma 4.1.5 there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of $[a,b]$ such that $\gamma|_{K_n}$ is continuous for every $n \in \mathbb{N}$ and $\lambda([a,b] \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0$. We define

$$\bar{\gamma}(t) := \gamma(t), \text{ if } t \in K_n \text{ for some } n \in \mathbb{N},$$

and

$$\bar{\gamma}(t) := x_0 \in X, \text{ otherwise},$$

and show that the obtained function is Borel measurable. Let $U \subseteq X$ be an open subset and consider the preimage

$$\bar{\gamma}^{-1}(U) = (\bar{\gamma}^{-1}(U) \cap N) \cup (\bar{\gamma}^{-1}(U) \cap \bigcup_{n\in\mathbb{N}} K_n),$$

where $N := [a,b] \setminus \bigcup_{n\in\mathbb{N}} K_n$. The subset

$$\bar{\gamma}^{-1}(U) \cap \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} (\bar{\gamma}^{-1}(U) \cap K_n) = \bigcup_{n\in\mathbb{N}} (\bar{\gamma}|_{K_n})^{-1}(U) = \bigcup_{n\in\mathbb{N}} (\gamma|_{K_n})^{-1}(U)$$

is a Borel subset of $[a,b]$. Further, $\bar{\gamma}^{-1}(U) \cap N = N$ if $x_0 \in U$, and $\bar{\gamma}^{-1}(U) \cap N = \emptyset$ otherwise, hence $\bar{\gamma}^{-1}(U)$ is a Borel set, as required.

Now, assume that $\gamma\colon [a,b] \to X$ is Borel measurable and X has a countable base denoted by $(V_n)_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$, the preimage $\gamma^{-1}(V_n)$ is a Borel subset of $[a,b]$, hence using the regularity of the Lebesgue measure λ, for a fixed $\varepsilon > 0$ we obtain open subsets $U_n \subseteq [a,b]$ and compact subsets $K_n \subseteq [a,b]$ such that

$$K_n \subseteq \gamma^{-1}(V_n) \subseteq U_n \text{ and } \lambda(U_n \setminus K_n) \leq \varepsilon/2^{n+1}$$

for every $n \in \mathbb{N}$. We define the Borel set $B := \bigcup_{n\in\mathbb{N}}(U_n \setminus K_n)$ and show that for $A := [a,b] \setminus B$ the restriction $\gamma|_A$ is continuous.

In fact, for every $n \in \mathbb{N}$ we have

$$\gamma^{-1}(V_n) \cap A \subseteq U_n \cap A = ((U_n \setminus K_n) \cup K_n) \cap A = K_n \cap A \subseteq \gamma^{-1}(V_n) \cap A,$$

in other words,

$$\gamma^{-1}(V_n) \cap A = U_n \cap A$$

is an open set in A.

Finally, by Remark 4.1.1 there exists a compact subset $K \subseteq A$ such that $\lambda(A\setminus K) \leq \varepsilon/2$. Since

$$\lambda([a,b] \setminus K) = \lambda(B) + \lambda(A \setminus K) \leq \sum_{n=1}^{\infty} \varepsilon/2^{n+1} + \varepsilon/2 = \varepsilon$$

and $\gamma|_K$ is continuous, we conclude that γ is measurable. $\square$

The next lemma can be found in [11].

Lemma 4.1.9. *Let E be a locally convex space, $\gamma\colon [a,b] \to E$ be a measurable function. Then the following assertions are equivalent:*

(i) $\gamma(t) = 0$ *a.e.,*

(ii) $\alpha(\gamma(t)) = 0$ *a.e., for each continuous linear functional* α *on* E*,*

(iii) $q(\gamma(t)) = 0$ *a.e., for each continuous seminorm* q *on* E*.*

Definition 4.1.10. Let $\gamma\colon [a,b] \to \mathbb{R}$ be a measurable function and $\bar{\gamma}\colon [a,b] \to \mathbb{R}$ be a Borel measurable function such that $\bar{\gamma}(t) = \gamma(t)$ a.e. (as in Lemma 4.1.8). We define

$$\int_a^b \gamma(t)\,dt := \int_a^b \bar{\gamma}(t)\,dt.$$

For a locally convex space E and $p \in [1,\infty[$, we denote by $\mathcal{L}^p([a,b],E)$ the vector space of measurable functions $\gamma\colon [a,b] \to E$ such that for every continuous seminorm q on E we have

$$\int_a^b q(\gamma(t))^p\,dt < \infty.$$

We endow $\mathcal{L}^p([a,b],E)$ with the locally convex topology defined by the family of seminorms

$$\|\gamma\|_{\mathcal{L}^p,q} := \left(\int_a^b q(\gamma(t))^p\,dt\right)^{\frac{1}{p}}.$$

Further, we denote by $\mathcal{L}^\infty([a,b],E)$ the vector space of measurable maps $\gamma\colon [a,b] \to E$ such that there exists some Borel set $N \subseteq [a,b]$ such that $\lambda(N) = 0$ and $\gamma([a,b] \setminus N)$ is bounded. We endow $\mathcal{L}^\infty([a,b],E)$ with the locally convex topology defined by the family of seminorms

$$\|\gamma\|_{\mathcal{L}^\infty,q} := \operatorname{ess\,sup}_{t\in[a,b]} q(\gamma(t))$$

for continuous seminorms q on E.

For $p \in [1,\infty]$, define

$$N_p := \{\gamma \in \mathcal{L}^p([a,b],E) : \gamma(t) = 0 \text{ a.e.}\}.$$

From Lemma 4.1.9, it follows that $N_p = \overline{\{0\}}$ in $\mathcal{L}^p([a,b],E)$, thus we obtain Hausdorff locally convex spaces

$$L^p([a,b],E) := \mathcal{L}^p([a,b],E)/N_p$$

consisting of equivalence classes

$$[\gamma] := \{\eta \in \mathcal{L}^p([a,b],E) : \eta(t) = \gamma(t) \text{ a.e.}\},$$

with seminorms

$$\|[\gamma]\|_{L^p,q} := \|\gamma\|_{\mathcal{L}^p,q}.$$

Remark 4.1.11. By [11], $\gamma \in \mathcal{L}^\infty([a,b],E)$ if and only if for every continuous seminorm q on E the composition $q \circ \gamma$ is essentially bounded. By definition, $\gamma \in \mathcal{L}^p([a,b],E)$ if and only if $q \circ \gamma \in \mathcal{L}^p([a,b])$ (as in [11]).

In [17], the author defines Lebesgue spaces of Borel measurable functions with values in Fréchet spaces as follows.

Definition 4.1.12. Let E be a Fréchet space. For $p \in [1,\infty[$, the space $\mathcal{L}^p_B([a,b],E)$ is the vector space of Borel measurable functions $\gamma\colon [a,b] \to E$ such that $\gamma([a,b])$ is separable and $q \circ \gamma \in \mathcal{L}^p([a,b])$ for each continuous seminorm q on E. The locally convex topology on $\mathcal{L}^p_B([a,b],E)$ is defined by the (countable) family of seminorms

$$\|\gamma\|_{\mathcal{L}^p,q} := \|q \circ \gamma\|_{\mathcal{L}^p} = \left(\int_a^b q(\gamma(t))^p\,dt\right)^{\frac{1}{p}}.$$

Further, the vector space $\mathcal{L}^\infty_B([a,b],E)$ consists of Borel measurable maps $\gamma\colon [a,b] \to E$ such that $\gamma([a,b])$ is separable and bounded. The locally convex topology on the space $\mathcal{L}^\infty_B([a,b],E)$ is defined by the family of seminorms

$$\|\gamma\|_{\mathcal{L}^\infty,q} := \|q \circ \gamma\|_{\mathcal{L}^\infty} = \operatorname{ess\,sup}_{t\in[a,b]} q(\gamma(t)).$$

For $N_p := \{\gamma \in \mathcal{L}^p_B([a,b],E) : \gamma(t) = 0 \text{ a.e.}\}$, the Hausdorff locally convex spaces

$$L^p_B([a,b],E) := \mathcal{L}^p_B([a,b],E)/N_p$$

consist of equivalence classes

$$[\gamma] := \{\eta \in \mathcal{L}^p_B([a,b],E) : \eta(t) = \gamma(t) \text{ a.e.}\},$$

and are endowed with the topologies defined by seminorms

$$\|[\gamma]\|_{L^p,q} := \|\gamma\|_{\mathcal{L}^p,q}.$$

Remark 4.1.13. For locally convex spaces E having the property that every separable closed vector subspace $S \subseteq E$ can be written as a union $S = \bigcup_{n\in\mathbb{N}} F_n$ of vector subspaces $F_1 \subseteq F_2 \subseteq \cdots$ which are Fréchet spaces in the induced topology (such spaces are called (FEP)-*spaces* in [17]), the spaces $\mathcal{L}^p_B([a,b],E)$ and $L^p_B([a,b],E)$ are constructed in [17] in the same way.

Definition 4.1.14. If E is an arbitrary locally convex space, then the vector space $\mathcal{L}^\infty_{rc}([a,b],E)$ consists of Borel measurable functions $\gamma\colon [a,b] \to E$ such that $\overline{\gamma([a,b])}$ is compact and metrizable. The seminorms $\|\gamma\|_{\mathcal{L}^\infty,q} := \operatorname{ess\,sup}_{t\in[a,b]} q(\gamma(t))$ define the locally convex topology on $\mathcal{L}^\infty_{rc}([a,b],E)$.

For $N_{rc} := \{\gamma \in \mathcal{L}^\infty_{rc}([a,b],E) : \gamma(t) = 0 \text{ a.e.}\}$ the Hausdorff locally convex space

$$L^\infty_{rc}([a,b],E) := \mathcal{L}^\infty_{rc}([a,b],E)/N_{rc}$$

consists of equivalence classes

$$[\gamma] := \{\eta \in \mathcal{L}^\infty_{rc}([a,b],E) : \eta(t) = \gamma(t) \text{ a.e.}\},$$

with seminorms

$$\|[\gamma]\|_{L^\infty,q} := \|\gamma\|_{\mathcal{L}^\infty,q}.$$

Note that in [17], the author constructs all of the above Lebesgue spaces even in a more general form, consisting of Borel measurable functions $\gamma\colon X \to E$ defined on arbitrary measure spaces (X,Σ,μ).

Using the close relation between the two concepts of measurability, we prove that the spaces $L^p_B([a,b],E)$ coincide with our Lebesgue spaces.

Proposition 4.1.15. *If E is a Fréchet space, then $L^p_B([a,b],E) \cong L^p([a,b],E)$ as topological vector spaces, for each $p \in [1,\infty]$.*

Proof. Clearly, using Lemma 4.1.8 we see that $L^p_B([a,b],E) \subseteq L^p([a,b],E)$ because for $\gamma \in L^p_B([a,b],E)$ the image $\gamma([a,b])$ is separable and metrizable, hence has a countable base.

On the other hand, for every $\gamma \in \mathcal{L}^p([a,b],E)$ we can construct some Borel measurable $\bar\gamma$ with $[\gamma] = [\bar\gamma]$ such that the image $\bar\gamma([a,b]) = \{x_0\} \cup \bigcup_{n\in\mathbb{N}} \gamma(K_n)$ is separable (and bounded if $p = \infty$), using Lemma 4.1.8 again. Therefore, $L^p_B([a,b],E) \cong L^p([a,b],E)$ as topological vector spaces (as the equality of the topologies is obvious). □

Remark 4.1.16. If E is an (FEP)-space, then also $L^p_B([a,b],E) \cong L^p([a,b],E)$ as topological vector spaces. To see this, we only need to show that every $\gamma \in \mathcal{L}^p_B([a,b],E)$ is measurable, the rest of the proof is identical to the above.

Since $\operatorname{im}(\gamma)$ is separable, the vector subspace $\overline{\operatorname{span}(\operatorname{im}(\gamma))}$ is separable and closed, hence there is an ascending sequence $F_1 \subseteq F_2 \subseteq \cdots$ of vector subspaces such that

$$\overline{\operatorname{span}(\operatorname{im}(\gamma))} = \bigcup_{n\in\mathbb{N}} F_n$$

and each F_n is a separable Fréchet space (see [17, Lemma 1.39]). Consider the sets $B_1 := \gamma^{-1}(F_1)$, $B_n := \gamma^{-1}(F_n \setminus F_{n-1})$ for $n \geq 2$. Then $[a,b]$ is a disjoint union of the Borel sets $(B_n)_{n\in\mathbb{N}}$, and $\gamma|_{B_n}\colon B_n \to F_n$ is Borel measurable, hence measurable by Lemma 4.1.8. Therefore, $\gamma\colon [a,b] \to E$ is measurable.

Remark 4.1.17. For an arbitrary locally convex space E we have $L^\infty_{rc}([a,b],E) \subseteq L^\infty([a,b],E)$. Again, it suffices to prove that each $\gamma \in \mathcal{L}^\infty_{rc}([a,b],E)$ is measurable. This is true (by Lemma 4.1.8), since the closure of the image of γ is compact and metrizable, hence has a countable base.

We discuss some properties of Lebesgue spaces and functions between them.

Remark 4.1.18. For $1 \leq p \leq r \leq \infty$ we have

$$C([a,b],E) \subseteq \mathcal{L}^\infty([a,b],E) \subseteq \mathcal{L}^r([a,b],E) \subseteq \mathcal{L}^p([a,b],E) \subseteq \mathcal{L}^1([a,b],E)$$

with continuous inclusion maps, as for a continuous seminorm q on E we have

$$\|\gamma\|_{\mathcal{L}^p,q} \leq (b-a)^{\frac{1}{p}-\frac{1}{r}} \|\gamma\|_{\mathcal{L}^r,q},$$

resp.

$$\|\gamma\|_{\mathcal{L}^r,q} \leq (b-a)^{\frac{1}{p}} \|\gamma\|_{\mathcal{L}^\infty,q}.$$

(Here, $C([a,b],E)$ is endowed with the topology of uniform convergence, with continuous seminorms $\|\gamma\|_{\infty,q} = \|\gamma\|_{\mathcal{L}^\infty,q}$.)

Lemma 4.1.19. *Let E, F be locally convex spaces and $f\colon E \to F$ be continuous and linear. If $\gamma \in \mathcal{L}^p([a,b],E)$ for $p \in [1,\infty]$, then $f \circ \gamma \in \mathcal{L}^p([a,b],F)$ and the map*

$$\mathcal{L}^p([a,b],f)\colon \mathcal{L}^p([a,b],E) \to \mathcal{L}^p([a,b],F), \quad \gamma \mapsto f \circ \gamma$$

is continuous and linear.

Proof. From Lemma 4.1.6, it follows that $f \circ \gamma$ is measurable. Further, for every continuous seminorm q on F, the composition $q \circ f$ is a continuous seminorm on E, whence $q \circ (f \circ \gamma) \in \mathcal{L}^p([a,b])$. Therefore $f \circ \gamma \in \mathcal{L}^p([a,b],F)$.

Since

$$\|f \circ \gamma\|_{\mathcal{L}^p,q} = \|\gamma\|_{\mathcal{L}^p,q\circ f},$$

the linear function $\mathcal{L}^p([a,b],f)$ is continuous. □

Remark 4.1.20. From Lemma 4.1.19 we easily conclude that for locally convex spaces E and F we have

$$\mathcal{L}^p([a,b],E \times F) \cong \mathcal{L}^p([a,b],E) \times \mathcal{L}^p([a,b],F)$$

as locally convex spaces. In fact, the function

$$\mathcal{L}^p([a,b],E \times F) \to \mathcal{L}^p([a,b],E) \times \mathcal{L}^p([a,b],F), \quad \gamma \mapsto (\mathrm{pr}_1 \circ \gamma, \mathrm{pr}_2 \circ \gamma)$$

is continuous linear (where $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections on the first, resp., second component of $E \times F$) and is a linear bijection with the continuous inverse

$$\mathcal{L}^p([a,b],E) \times \mathcal{L}^p([a,b],F) \to \mathcal{L}^p([a,b],E \times F), \quad (\gamma_1,\gamma_2) \mapsto \lambda_1 \circ \gamma_1 + \lambda_2 \circ \gamma_2,$$

where $\lambda_1\colon E \to E \times F, x \mapsto (x,0)$ and $\lambda_2\colon F \to E \times F, y \mapsto (0,y)$ are continuous and linear.

Note that in Lemma 4.1.19 and Remark 4.1.20, one can replace $\mathcal{L}^p$ with L^p.

The following property of L^p-spaces is called *locality axiom* in [17].

Lemma 4.1.21. *For any $a = t_0 < t_1 < \ldots < t_n = b$, the function*

$$\Gamma_E\colon L^p([a,b],E) \to \prod_{j=1}^{n} L^p([t_{j-1},t_j],E), \quad [\gamma] \mapsto \Big([\gamma|_{[t_{j-1},t_j]}]\Big)_{j=1\ldots,n}$$

is an isomorphism of topological vector spaces.

Proof. If $[\gamma] \in L^p([a,b],E)$, then obviously $[\gamma|_{[t_{j-1},t_j]}] \in L^p([t_{j-1},t_j],E)$ for every $j \in \{1,\ldots,n\}$, and the continuity of the linear injective function Γ_E follows from the fact that $\|\gamma|_{[t_{j-1},t_j]}\|_{\mathcal{L}^p,q} \leq \|\gamma\|_{\mathcal{L}^p,q}$ for every continuous seminorm q on E.

On the other hand, given $([\gamma_1],\ldots,[\gamma_n]) \in \prod_{j=1}^{n} L^p([t_{j-1},t_j],E)$, the function

$$\gamma(t) := \gamma_j(t), \text{ if } t \in [t_{j-1},t_j[, \quad \gamma(t) := \gamma_n(t), \text{ if } t \in [t_{n-1},t_n]$$

is measurable and $q \circ \gamma \in \mathcal{L}^p([a,b])$ for every continuous seminorm q on E. Further, if $p < \infty$, then we have

$$\|\gamma\|_{\mathcal{L}^p,q} \leq \sum_{j=1}^{n} \|\gamma_j\|_{\mathcal{L}^p,q},$$

otherwise, we have

$$\|\gamma\|_{\mathcal{L}^\infty,q} \leq \max_{j=1,\ldots,n} \|\gamma_j\|_{\mathcal{L}^\infty,q}.$$

Therefore, the function Γ_E is surjective and the inverse Γ_E^{-1} is continuous, hence the proof is finished. □

Remark 4.1.22. From the above lemma, it readily follows that a function $\gamma\colon [a,b] \to E$ is in $\mathcal{L}^p([a,b],E)$ if and only if $\gamma|_{[t_{j-1},t_j]} \in \mathcal{L}^p([t_{j-1},t_j],E)$ for some partition $a = t_0 < t_1 < \ldots < t_n = b$.

As in [17, Lemma 2.1], we obtain the following result.

Lemma 4.1.23. *Let X be a topological space, $U \subseteq X$ be an open subset and E, F be locally convex spaces. Let $f\colon U \times E \to F$ be a continuous function which is linear in the second argument. If $\eta \in C([a,b],U)$ and $\gamma \in \mathcal{L}^p([a,b],E)$ for $p \in [1,\infty]$, then $f \circ (\eta,\gamma) \in \mathcal{L}^p([a,b],F)$.*

Proof. By Lemma 4.1.6, the composition $f \circ (\eta,\gamma)$ is a measurable function.

Now, consider the continuous function

$$h_\eta\colon [a,b] \times E \to F, \quad h_\eta(t,v) := f(\eta(t),v).$$

Let q be a continuous seminorm on F. Then $h_\eta([a,b] \times \{0\}) = \{0\} \subseteq B_1^q(0)$, thus $[a,b] \times \{0\} \subseteq V$, where $V := h_\eta^{-1}(B_1^q(0))$ is an open subset of $[a,b] \times E$. Using the Wallace Lemma, we find an open subset $W \subseteq E$ such that $[a,b] \times \{0\} \subseteq [a,b] \times W \subseteq V$. Then there is a continuous seminorm π on E such that

$$[a,b] \times \{0\} \subseteq [a,b] \times \overline{B_1^\pi(0)} \subseteq [a,b] \times W \subseteq V.$$

We show that for each $(t,v) \in [a,b]\times E$ we have $q(h_\eta(t,v)) \le \pi(v)$. In fact, if $\pi(v) > 0$, then (using the linearity of f in v) we have $(1/\pi(v))q(h_\eta(t,v)) = q(h_\eta(t,(1/\pi(v))v)) \le 1$. If $\pi(v) = 0$, then for each $r > 0$ we have $rv \in \overline{B_1^\pi(0)}$, whence $rq(h_\eta(t,v)) = q(h_\eta(t,rv)) \le 1$, that is $q(h_\eta(t,v)) \le 1/r$, consequently $q(h_\eta(t,v)) = 0 = \pi(v)$.

Now, if $p < \infty$, then

$$\int_a^b q(f(\eta(t),\gamma(t)))^p\,dt = \int_a^b q(h_\eta(t,\gamma(t)))^p\,dt \le \int_a^b \pi(\gamma(t))^p\,dt < \infty,$$

thus $q\circ(f\circ(\eta,\gamma)) \in \mathcal{L}^p([a,b])$.

If $p = \infty$, then $q(f(\eta(t),\gamma(t))) \le \pi(\gamma(t))$, whence

$$\operatorname{ess\,sup}_{t\in[a,b]}(q(f(\eta(t),\gamma(t)))) \le \operatorname{ess\,sup}_{t\in[a,b]}(\pi(\gamma(t))) < \infty,$$

thus $q\circ(f\circ(\eta,\gamma)) \in \mathcal{L}^\infty([a,b])$. □

Lemma 4.1.24. *Let E_1, E_2, E_3 and F be locally convex spaces, $U \subseteq E_1$, $V \subseteq E_2$ be open subsets and the function $f\colon U\times V\times E_3 \to F$ be a C^1-function and linear in the third argument. Then the function*

$$\tilde f\colon U\times C([a,b],V)\times L^p([a,b],E_3) \to L^p([a,b],F),$$
$$(u,\eta,[\gamma]) \mapsto [f(u,\bullet)\circ(\eta,\gamma)]$$

is continuous.(Here $C([a,b],V)$ is endowed with the topology of uniform convergence.)

Proof. Fix some $(\bar u,\bar\eta,[\bar\gamma]) \in U\times C([a,b],V)\times L^p([a,b],E_3)$ and let q be a continuous seminorm on F. The subset $K := \{\bar u\}\times\bar\eta([a,b]) \subseteq U\times V$ is compact, hence from Lemma [17, Lemma 1.61], it follows that there are seminorms π on $E_1\times E_2$ and π_3 on E_3 such that $K + B_1^\pi(0) \subseteq U\times V$ and

$$q(f(u,v,w) - f(u',v',w')) \le \pi_3(w-w') + \pi(u-u',v-v')\pi_3(w')$$

for all $(u,v),(u',v') \in K+B_1^\pi(0)$, $w,w' \in E_3$. We may assume $\pi(x,y) = \max\{\pi_1(x),\pi_2(y)\}$ for some continuous seminorms π_1 on E_1, π_2 on E_2. Then, setting

$$U_0 := B_1^{\pi_1}(\bar u), \quad V_0 := \bar\eta([a,b]) + B_1^{\pi_2}(0),$$

we define an open neighborhood

$$\Omega := U_0\times C([a,b],V_0)\times L^p([a,b],E_3)$$

of $(\bar u,\bar\eta,[\bar\gamma])$ and see that if $(u,\eta,[\gamma]) \to (\bar u,\bar\eta,[\bar\gamma])$ in Ω, then $\tilde f(u,\eta,[\gamma]) \to \tilde f(\bar u,\bar\eta,[\bar\gamma])$ in $L^p([a,b],E_3)$, because

$$\|\tilde f(u,\eta,[\gamma]) - \tilde f(\bar u,\bar\eta,[\bar\gamma])\|_{L^p,q}$$
$$\le \|[\gamma-\bar\gamma]\|_{L^p,\pi_3} + \max\{\pi_1(u-\bar u),\|\eta-\bar\eta\|_{L^\infty,\pi_2}\}\|[\bar\gamma]\|_{L^p,\pi_3} \to 0.$$

In other words, $\tilde f$ is continuous in $(\bar u,\bar\eta,[\bar\gamma])$. □

Proposition 4.1.25. *Let E_1, E_2, F be locally convex spaces, let $V \subseteq E_1$ be open and the function $f\colon V \times E_2 \to F$ be C^{k+1} for $k \in \mathbb{N} \cup \{0, \infty\}$ and linear in the second argument. Then for $p \in [1, \infty]$ the function*

$$\Theta_f\colon C([a,b],V) \times L^p([a,b],E_2) \to L^p([a,b],F), \quad (\eta,[\gamma]) \mapsto [f \circ (\eta,\gamma)]$$

is C^k.

Proof. For $k = 0$, the assertion holds by Lemma 4.1.24. Further, we may assume $k < \infty$ and proceed by induction.

Base case: $k = 1$. The map Θ_f is continuous by the previous step; we show that for all $(\eta,[\gamma]) \in C([a,b],V) \times L^p([a,b],E_2)$ and $(\bar{\eta},[\bar{\gamma}]) \in C([a,b],E_1) \times L^p([a,b],E_2)$ the directional derivative

$$d(\Theta_f)(\eta,[\gamma],\bar{\eta},[\bar{\gamma}]) := \lim_{h\to 0} \frac{\Theta_f(\eta + h\bar{\eta}, [\gamma + h\bar{\gamma}]) - \Theta_f(\eta,[\gamma])}{h}$$

exists in $L^p([a,b],F)$ and equals $[df \circ (\eta,\gamma,\bar{\eta},\bar{\gamma})]$.

Given $\eta, [\gamma], \bar{\eta}, [\bar{\gamma}]$ as above, we note that $\eta([a,b])$ is a compact subset of the open subset $V \subseteq E_1$, thus there exists an open 0-neighborhood $U \subseteq E_1$ such that $\eta([a,b]) + U \subseteq V$. Further, there is some balanced 0-neighborhood $W \subseteq U$ such that $W + W \subseteq U$. As $\bar{\eta}([a,b])$ is bounded in E_1 (being compact), for some $\varepsilon > 0$ we have $\bar{\eta}([a,b]) \subseteq {}^1\!/\!_\varepsilon W$. In this manner we obtain an open subset

$$\Omega :=]-\varepsilon,\varepsilon[\times (\eta([a,b]) + W) \times \frac{1}{\varepsilon} W \times E_2 \times E_2 \subseteq \mathbb{R} \times V \times E_1 \times E_2 \times E_2$$

such that $]-\varepsilon,\varepsilon[\times \eta([a,b]) \times \bar{\eta}([a,b]) \times \gamma([a,b]) \times \bar{\gamma}([a,b]) \subseteq \Omega$ and for all $(t,w,\bar{w},x,\bar{x}) \in \Omega$ we have $(w + t\bar{w}, x + t\bar{x}) \in V \times E_2$ (that is, Ω corresponds to a subset of $(V \times E_2)^{[1]}$ constructed as in Lemma 2.3.4).

Now $f^{[1]}\colon (V \times E_2)^{[1]} \to F$ is C^k and thus C^1 (see [5]). Hence the function

$$\Omega \to F, \quad (t,w,\bar{w},x,\bar{x}) \mapsto f^{[1]}(w,x,\bar{w},\bar{x},t)$$

is C^1 and linear in $(x,\bar{x})$, thus from Lemma 4.1.24, it follows (identifying the vector space $L^p([a,b],E_2 \times E_2)$ with $L^p([a,b],E_2) \times L^p([a,b],E_2)$, see Remark 4.1.20) that

$$(t,\varphi,\bar{\varphi},[\psi],[\bar{\psi}]) \mapsto [f^{[1]}(\bullet,t) \circ (\varphi,\psi,\bar{\varphi},\bar{\psi})] \in L^p([a,b],F)$$

is continuous on

$$]-\varepsilon,\varepsilon[\times C([a,b],\eta([a,b]) + W) \times C([a,b],{}^1\!/\!_\varepsilon W) \times L^p([a,b],E_2) \times L^p([a,b],E_2).$$

Hence

$$]-\varepsilon,\varepsilon[\to L^p([a,b],F), \quad t \mapsto [f^{[1]}(\bullet,t) \circ (\eta,\gamma,\bar{\eta},\bar{\gamma})]$$

is continuous. Therefore, we have

$$\begin{aligned} d(\Theta_f)(\eta,[\gamma],\bar\eta,[\bar\gamma]) &= \lim_{h\to 0}\frac{1}{h}\left(\Theta_f(\eta+t\bar\eta,[\gamma+t\bar\gamma])-\Theta_f(\eta,[\gamma])\right)\\ &= \lim_{h\to 0}\frac{1}{t}\left([f\circ(\eta+t\bar\eta,\gamma+t\bar\gamma)]-[f\circ(\eta,\gamma)]\right)\\ &= \lim_{h\to 0}[f^{[1]}(\bullet,h)\circ(\eta,\gamma,\bar\eta,\bar\gamma)]\\ &= [f^{[1]}(\bullet,0)\circ(\eta,\gamma,\bar\eta,\bar\gamma)] = [df\circ(\eta,\gamma,\bar\eta,\bar\gamma)] \end{aligned}$$

in $L^p([a,b],F)$.

It remains to show that

$$d(\Theta_f)\colon C([a,b],V)\times L^p([a,b],E_2)\times C([a,b],E_1)\times L^p([a,b],E_2)\to L^p([a,b],F)$$

is continuous. But as the function

$$V\times E_1\times E_2\times E_2\to F,\quad (w,\bar w,x,\bar x)\mapsto df(w,x,\bar w,\bar x) \tag{4.1}$$

is C^1 and linear in $(x,\bar x)$, by base case

$$\begin{aligned} C([a,b],V)\times C([a,b],E_1)\times L^p([a,b],E_2)\times L^p([a,b],E_2)\to L^p([a,b],F),\\ (\varphi,\bar\varphi,[\psi],[\bar\psi])\mapsto [df\circ(\varphi,\psi,\bar\varphi,\bar\psi)] \end{aligned}$$

is continuous (we identify the L^p-spaces again, as above), hence $d(\Theta_f)$ is continuous. Therefore, Θ_f is C^1.

Induction step: Now, assume that f is C^{k+2}. Then Θ_f is C^1 by base case and df is C^{k+1}. Then the map in (4.1) is C^{k+1} and linear in $(x,\bar x)$, hence by induction hypothesis, the map $(\varphi,\bar\varphi,[\psi],[\bar\psi])\mapsto [df\circ(\varphi,\psi,\bar\varphi,\bar\psi)] = d(\Theta_f)(\varphi,[\psi],\bar\varphi,[\bar\psi])$ is C^k. Hence Θ_f is C^{k+1}. □

Remark 4.1.26. The properties proved in Lemma 4.1.23 and Proposition 4.1.25 are called *pushforward axioms* in [17] (see [17, Lemma 2.4] and [17, Propositions 2.2, 2.3], respectively).

4.2 Spaces $AC_{L^p}([a,b],E)$ and $AC_{L^p}([a,b],G)$

Similarly to [17], we construct locally convex spaces $AC_{L^p}([a,b],E)$ and Lie groups $AC_{L^p}([a,b],G)$.

A function $\gamma\colon [a,b]\to\mathbb{R}$ is called *absolutely continuous* if for every $\varepsilon>0$ there exists some $\delta>0$ such that $\sum_{k=1}^n|\gamma(\beta_k)-\gamma(\alpha_k)|<\varepsilon$ whenever $a\le\alpha_1<\beta_1\le\alpha_2<\beta_2\le\cdots\le\alpha_n<\beta_n\le b$ with $\sum_{k=1}^n(\beta_k-\alpha_k)<\delta$. Every absolutely continuous function γ is continuous and the derivative γ' exists a.e.

Further, we recall the Fundamental Theorem of Calculus for Lebesgue Integrals from [10]:

Lemma 4.2.1. *If $\gamma \in \mathcal{L}^1([a,b])$, then*

$$\eta(t) := \int_a^t \gamma(s)\,ds$$

is an absolutely continuous function on $[a,b]$ and $\eta'(t) = \gamma(t)$ a.e.

On the other hand, assume that $\eta\colon [a,b] \to \mathbb{R}$ is absolutely continuous and define $\gamma(t) := \eta'(t)$ whenever the derivative exists, otherwise $\gamma(t) := 0$. Then $\gamma \in \mathcal{L}^1([a,b])$ and

$$\eta(t) = \eta(a) + \int_a^t \gamma(s)\,ds$$

for every $t \in [a,b]$.

For vector-valued functions, the following concept of integrability is well known.

Definition 4.2.2. Let E be a locally convex space and let $\gamma\colon [a,b] \to E$ be such that $\alpha \circ \gamma \in \mathcal{L}^1([a,b])$ for every continuous linear form $\alpha \in E'$. If there exists some $w \in E$ such that

$$\alpha(w) = \int_a^b \alpha(\gamma(t))\,dt$$

for every α, then w is called the *weak integral of γ from a to b*, and we write $\int_a^b \gamma(t)\,dt := w$.

As the continuous linear forms separate the points on E, the weak integral of a function γ is unique if it exists.

Further, in this case we have

$$q\left(\int_a^b \gamma(t)\,dt\right) \leq \int_a^b q(\gamma(t))\,dt$$

for every continuous seminorm q on E.

Remark 4.2.3. In [41], the weak integral $\int_X \gamma\,d\mu$ is defined for any suitable function $\gamma\colon X \to E$ on a measure space X. It is known, that if the locally convex space E has the *metric convex compactness property* (that is, the closed convex hull of every metrizable compact subset of E is compact), then every continuous function $\gamma\colon X \to E$ on a compact space X has a weak integral $\int_X \gamma\,d\mu$.

Remark 4.2.4. Since $|\alpha|$ is a continuous seminorm on E, for every $\alpha \in E'$, every $\gamma \in \mathcal{L}^1([a,b],E)$ satisfies the condition $\alpha \circ \gamma \in \mathcal{L}^1([a,b])$.

We use [7, §5] and get a result for functions in $\mathcal{L}^1([a,b],E)$, which is similar to the first part of the Fundamental Theorem of Calculus.

Proposition 4.2.5. *Let E be a locally convex space, let $\gamma \in \mathcal{L}^1([a,b],E)$. If the function*

$$\eta\colon [a,b] \to E, \quad \eta(t) := \int_a^t \gamma(s)\,ds \tag{4.2}$$

is defined, then η is continuous.

Moreover, if E is metrizable, then for almost every $t \in [a,b]$ the derivative $\eta'(t)$ exists and equals $\gamma(t)$.

Proof. To prove the continuity of η in every $t \in [a,b]$, let q be a continuous seminorm on E and let $\varepsilon > 0$. Then there exists some $\delta > 0$ such that whenever $|t - r| < \delta$, we have $\left|\int_r^t q(\gamma(s))\,ds\right| < \varepsilon$ (follows from the Fundamental Theorem of Calculus, Lemma 4.2.1). Therefore

$$\begin{aligned} q(\eta(t) - \eta(r)) &= q\left(\int_a^t \gamma(s)\,ds - \int_a^r \gamma(s)\,ds\right) \\ &= q\left(\int_r^t \gamma(s)\,ds\right) \leq \left|\int_r^t q(\gamma(s))\,ds\right| < \varepsilon, \end{aligned}$$

whence η is continuous in t.

Now, assume that E is metrizable and recall that by Lemma 4.1.5 there exists a sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ of $[a,b]$ such that $\lambda([a,b] \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0$ and $\gamma|_{K_n}$ is continuous. We may assume that $\gamma(t) = 0$ for each $t \notin \bigcup_{n\in\mathbb{N}} K_n$. Our aim is to show that for almost every $t \in [a,b]$ the difference quotient

$$\frac{1}{r}(\eta(t+r) - \eta(t)) = \frac{1}{r}\left(\int_a^{t+r} \gamma(s)\,ds - \int_a^t \gamma(s)\,ds\right) = \frac{1}{r}\int_t^{t+r} \gamma(s)\,ds$$

tends to $\gamma(t)$ as $r \to 0$. That is, for every $\varepsilon > 0$ and continuous seminorm q_m on E we have

$$q_m\left(\frac{1}{r}\int_t^{t+r} \gamma(s)\,ds - \gamma(t)\right) = q_m\left(\frac{1}{r}\int_t^{t+r} \gamma(s) - \gamma(t)\,ds\right) < \varepsilon \tag{4.3}$$

for $r \neq 0$ small enough.

We fix some $\varepsilon > 0$ and some continuous seminorm q_m. The set $\gamma([a,b]) \subseteq \{0\} \cup \bigcup_{n\in\mathbb{N}} \gamma(K_n) \subseteq E$ is separable, say $\gamma([a,b]) = \overline{\{a_k : k \in \mathbb{N}\}}$. Thus for every $t \in [a,b]$ we find some $a_m(t)$ such that

$$q_m(\gamma(t) - a_m(t)) < \frac{1}{3}\varepsilon,$$

hence for every $r \neq 0$ small enough we have

$$\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(t) - a_m(t))\,ds\right| < \frac{1}{3}\varepsilon. \tag{4.4}$$

Furthermore, each of the functions

$$h_{k,m}\colon [a,b]\to\mathbb{R},\quad h_{k,m}(t):=q_m(\gamma(t)-a_k)$$

is in $\mathcal{L}^1([a,b])$, hence by the Fundamental Theorem of Calculus (see Lemma 4.2.1) there exist some sets $N_{k,m}\subseteq[a,b]$ such that $\lambda(N_{k,m})=0$ and for every $t\notin N_{k,m}$ we have

$$\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(s)-a_k)\,ds - q_m(\gamma(t)-a_k)\right| < \frac{1}{3}\varepsilon \tag{4.5}$$

for $r\neq 0$ small enough.

Consequently, for $t\notin\bigcup_{k,m\in\mathbb{N}} N_{k,m}$ we have

$$\begin{aligned}\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(s)-\gamma(t))\,ds\right| &\le \left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(s)-\gamma(t)) - q_m(\gamma(s)-a_m(t))\,ds\right|\\ &\quad+\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(s)-a_m(t)) - q_m(\gamma(t)-a_m(t))\,ds\right|\\ &\quad+\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(t)-a_m(t))\,ds\right|\\ &<\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(s)-\gamma(t)) - q_m(\gamma(s)-a_m(t))\,ds\right|\\ &\quad+\frac{1}{3}\varepsilon+\frac{1}{3}\varepsilon,\end{aligned}$$

using the estimates in (4.5) and (4.4). Finally,

$$\begin{aligned}&\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(s)-\gamma(t)) - q_m(\gamma(s)-a_m(t))\,ds\right|\\ &\le\frac{1}{|r|}\left|\int_t^{t+r} |q_m(\gamma(s)-\gamma(t)) - q_m(\gamma(s)-a_m(t))|\,ds\right|\\ &\le\frac{1}{|r|}\left|\int_t^{t+r} q_m(\gamma(s)-\gamma(t)-\gamma(s)+a_m(t))\,ds\right|\\ &=\frac{1}{|r|}\left|\int_t^{t+r} q_m(\gamma(t)-a_m(t))\,ds\right| < \frac{1}{3}\varepsilon,\end{aligned}$$

using (4.4) again.

Altogether, we have

$$\begin{aligned}q_m\left(\frac{1}{r}\int_t^{t+r}\gamma(s)-\gamma(t)\,ds\right) &= \frac{1}{|r|}q_m\left(\int_t^{t+r}\gamma(s)-\gamma(t)\,ds\right)\\ &\le\frac{1}{|r|}\left|\int_t^{t+r} q_m(\gamma(s)-\gamma(t))\,ds\right|\\ &\le\left|\frac{1}{r}\int_t^{t+r} q_m(\gamma(s)-\gamma(t))\,ds\right| < \varepsilon\end{aligned}$$

by the above. Thus the desired estimate (4.3) holds for each $t\notin\bigcup_{k,m\in\mathbb{N}} N_{k,m}$ and $\lambda(\bigcup_{k,m\in\mathbb{N}} N_{k,m})=0$, whence the proof is finished. □

Even if the range E is not metrizable, the next lemma can be used to show that η as in (4.2) uniquely determines the corresponding $[\gamma] \in L^1([a,b],E)$.

Lemma 4.2.6. *Let E be a locally convex space and let $\gamma \in \mathcal{L}^1([a,b],E)$ such that $\int_a^t \gamma(s)\,ds = 0$ for all $t \in [a,b]$. Then $\gamma(s) = 0$ a.e.*

Proof. Let α be a continuous linear functional on E. Then we have

$$\int_a^t (\alpha \circ \gamma)(s)\,ds = \alpha\left(\int_a^t \gamma(s)\,ds\right) = 0$$

for every $t \in [a,b]$. From the Fundamental Theorem of Calculus (see Lemma 4.2.1), it follows that $(\alpha \circ \gamma)(t) = 0$ a.e. As $\alpha \in E'$ was arbitrary, from Lemma 4.1.9, it follows that $\gamma(t) = 0$ a.e. □

Now, we discuss the existence of weak integrals of $\mathcal{L}^p$-functions.

Proposition 4.2.7. *Let E be a sequentially complete locally convex space. Then each $\gamma \in \mathcal{L}^1([a,b],E)$ has a weak integral $\int_a^b \gamma(t)\,dt \in E$.*

Proof. As γ is measurable, pick a disjoint sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets $K_n \subseteq [a,b]$ such that $\gamma|_{K_n}$ is continuous and $\lambda([a,b] \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0$ (see Lemma 4.1.5). Then for each $n \in \mathbb{N}$ the weak integral $\int_{K_n} \gamma(t)\,dt$ exists in E (by [41, 3.27 Theorem]). We define a sequence $w_m := \sum_{n=1}^m \int_{K_n} \gamma(t)\,dt$ and show that $(w_m)_{m\in\mathbb{N}}$ is convergent in E and that $w := \lim_{m\to\infty} w_m$ is the weak integral of γ from a to b.

Fix $\varepsilon > 0$ and a seminorm q on E. We have

$$\sum_{n=1}^{\infty} \int_{K_n} q(\gamma(t))\,dt = \int_a^b q(\gamma(t))\,dt < \infty,$$

hence $\left(\sum_{n=1}^m \int_{K_n} q(\gamma(t))\,dt\right)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, that means that there exists some $N \in \mathbb{N}$ such that

$$\sum_{n=k+1}^{m} \int_{K_n} q(\gamma(t))\,dt < \varepsilon,$$

for all $m > k \geq N$. Then

$$q(w_m - w_k) = q\left(\sum_{n=k+1}^{m} \int_{K_n} \gamma(t)\,dt\right) \leq \sum_{n=k+1}^{m} \int_{K_n} q(\gamma(t))\,dt < \varepsilon,$$

therefore $(w_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in E. As E was assumed sequentially complete, the Cauchy sequence $(w_m)_{m\in\mathbb{N}}$ converges to some $w \in E$.

Finally, for any continuous linear form $\alpha \in E'$ we have

$$\alpha(w) = \lim_{m\to\infty} \alpha(w_m) = \lim_{m\to\infty} \sum_{n=1}^{m} \int_{K_n} \alpha(\gamma(t))\,dt = \int_a^b \alpha(\gamma(t))\,dt,$$

as required. □

Definition 4.2.8. Let E be a sequentially complete locally convex space. For $p \in [1,\infty]$ we denote by $AC_{L^p}([a,b],E)$ the vector space of continuous functions $\eta\colon [a,b] \to E$ such that for some $\gamma \in \mathcal{L}^p([a,b],E)$ we have

$$\eta(t) = \eta(a) + \int_a^t \gamma(s)\,ds \qquad \text{for all } t \in [a,b].$$

As η uniquely determines $[\gamma]$ (see Lemma 4.2.6), we write $\eta' := [\gamma]$.

The function

$$\Phi\colon AC_{L^p}([a,b],E) \to E \times L^p([a,b],E), \quad \eta \mapsto (\eta(a),\eta') \tag{4.6}$$

is an isomorphism of vector spaces and we endow $AC_{L^p}([a,b],E)$ with the Hausdorff locally convex topology making Φ a homeomorphism.

We consider some properties of AC-spaces and functions between them.

Lemma 4.2.9. *Let E is a sequentially complete locally convex space and endow the vector space $C([a,b],E)$ with the topology of uniform convergence (which is defined by the family of seminorms $\|\eta\|_{\infty,q} := \sup_{t\in[a,b]} q(\eta(t))$ with continuous seminorms q on E). Then for $1 \le p \le r \le \infty$, we have*

$$\begin{aligned} AC_{L^\infty}([a,b],E) \subseteq AC_{L^r}([a,b],E) &\subseteq AC_{L^p}([a,b],E) \\ &\subseteq AC_{L^1}([a,b],E) \subseteq C([a,b],E) \end{aligned}$$

with continuous inclusion maps.

Proof. We only show the continuity of the inclusion map $AC_{L^1}([a,b],E) \to C([a,b],E)$, as the remainder follows from Remark 4.1.18. Let $\eta \in AC_{L^1}([a,b],E)$ and denote $\eta' = [\gamma]$. For a continuous seminorm q on E and $t \in [a,b]$ we have

$$q(\eta(t)) = q\left(\eta(a) + \int_a^t \gamma(s)\,ds\right) \le q(\eta(a)) + \int_a^t q(\gamma(s))\,ds \le q(\eta(a)) + \|\gamma\|_{\mathcal{L}^1,q}$$

Thus

$$\|\eta\|_{\infty,q} \le q(\eta(a)) + \|\gamma\|_{\mathcal{L}^1,q},$$

whence the (linear) inclusion map is continuous. □

Remark 4.2.10. From the previous lemma, it follows that for any open set $U \subseteq V$, the subset $AC_{L^p}([a,b],U) = \mathrm{incl}^{-1}(C([a,b],U))$ is open in $AC_{L^p}([a,b],E)$.

Remark 4.2.11. It is well known that the evaluation map $C([a,b],E) \to E, \eta \mapsto \eta(\alpha)$ is continuous linear for $\alpha \in [a,b]$. By Lemma 4.2.9, so is the inclusion map $\mathrm{incl}\colon AC_{L^p}([a,b],E) \to C([a,b],E)$, hence the evaluation map

$$\mathrm{ev}_\alpha\colon AC_{L^p}([a,b],E) \to E, \quad \eta \mapsto \eta(\alpha)$$

is continuous, linear.

Remark 4.2.12. For any $t_0 \in [a,b]$, consider the vector space $AC^{t_0}_{L^p}([a,b],E)$ consisting of continuous functions $\eta\colon [a,b] \to E$ such that for some $\gamma \in \mathcal{L}^p([a,b],E)$ we have

$$\eta(t) = \eta(t_0) + \int_{t_0}^{t} \gamma(s)\,ds \quad \text{for all } t \in [a,b],$$

endowed with the locally convex topology making

$$\Phi^{t_0}\colon AC^{t_0}_{L^p}([a,b],E) \to E \times L^p([a,b],E), \quad \eta \mapsto (\eta(t_0),[\gamma])$$

an isomorphism of topological vector spaces. Using Remark 4.2.11, we can easily see that $AC^{t_0}_{L^p}([a,b],E) = AC_{L^p}([a,b],E)$ as topological vector spaces.

Lemma 4.2.13. *Let E be a sequentially complete locally convex space, $p \in [1,\infty]$ and $a = t_0 < t_1 < \ldots < t_n = b$. Then the function*

$$\Psi\colon AC_{L^p}([a,b],E) \to \prod_{j=1}^{n} AC_{L^p}([t_{j-1},t_j],E), \quad \eta \mapsto (\eta|_{[t_{j-1},t_j]})_{j=1,\ldots,n} \tag{4.7}$$

is a linear topological embedding with closed image.

Proof. Clearly, for $\eta \in AC_{L^p}([a,b],E)$ with $\eta' = [\gamma]$ and every $j \in \{1,\ldots,n\}$ we have $\eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],E)$ with $(\eta|_{[t_{j-1},t_j]})' = \left[\gamma|_{[t_{j-1},t_j]}\right]$, by Lemma 4.1.21, that is, the function Ψ is defined. Also the linearity and injectivity are clear.

We show that each of the components

$$AC_{L^p}([a,b],E) \to AC_{L^p}([t_{j-1},t_j],E), \quad \eta \mapsto \eta|_{[t_{j-1},t_j]}$$

is continuous, which will be the case if each

$$AC_{L^p}([a,b],E) \to E \times L^p([t_{j-1},t_j],E), \quad \eta \mapsto \left(\eta(t_{j-1}), \left[\gamma|_{[t_{j-1},t_j]}\right]\right)$$

is continuous (using the isomorphism as in Definition 4.2.8). But the first component is a continuous evaluation map on $AC_{L^p}([a,b],E)$, see Remark 4.2.11, and the second component is a composition of the continuous maps $AC_{L^p}([a,b],E) \to L^p([a,b],E), \eta \mapsto [\gamma]$ and $L^p([a,b],E) \to L^p([t_{j-1},t_j],E), [\gamma] \mapsto \left[\gamma|_{[t_{j-1},t_j]}\right]$, see Definition 4.2.8 and Lemma 4.1.21. Therefore, Ψ is continuous.

Note that $(\eta_1,\ldots,\eta_n) \in \operatorname{im}(\Psi) \subseteq \prod_{j=1}^{n} AC_{L^p}([t_{j-1},t_j],E)$ if $\eta_j(t_j) = \eta_{j+1}(t_j)$ for all $j \in \{1,\ldots,n-1\}$, thus the map

$$\Gamma(\eta_1,\ldots,\eta_n)\colon [a,b] \to E, \quad t \mapsto \eta_j(t) \text{ for } t \in [t_{j-1},t_j]$$

is continuous and it is easy to show that $\Gamma(\eta_1,\ldots,\eta_n) \in AC_{L^p}([a,b],E)$ and that

$$\Gamma\colon \operatorname{im}(\Psi) \to AC_{L^p}([a,b],E)$$

is the inverse of $\Psi|^{\mathrm{im}(\Psi)}$. The continuity of Γ follows from the continuity of

$$\begin{aligned}\mathrm{im}(\Psi) &\to E \times L^p([a,b],E),\\ (\eta_1,\dots,\eta_n) &\mapsto (\eta_1(a), \Gamma(\eta_1,\dots,\eta_n)') = (\eta_1(a), \Gamma_E^{-1}(\eta_1',\dots,\eta_n')),\end{aligned}$$

where Γ_E is the isomorphism from Lemma 4.1.21. Hence Ψ is a topological embedding.

Finally, let $(\eta_{1,\alpha},\dots,\eta_{n,\alpha})_{\alpha\in A}$ be a net in $\mathrm{im}(\Psi)$ which converges to $(\eta_1,\dots,\eta_n) \in \prod_{j=1}^n AC_{L^p}([t_{j-1},t_j],E)$. Then for every $j \in \{1,\dots,n-1\}$ we have

$$\eta_j(t_j) = \lim_{\alpha\in A} \eta_{j,\alpha}(t_j) = \lim_{\alpha\in A} \eta_{j+1,\alpha}(t_j) = \eta_{j+1}(t_j),$$

therefore $(\eta_1,\dots,\eta_n) \in \mathrm{im}(\Psi)$. □

Remark 4.2.14. By the above lemma, a continuous function $\eta\colon [a,b] \to E$ is in $AC_{L^p}([a,b],E)$ if and only if $\eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],E)$ for any $a = t_0 < t_1 < \dots < t_n = b$.

Lemma 4.2.15. *Let E be a locally convex space, $U \subseteq E$ be an open subset and $f\colon U \to \mathbb{R}$ be continuous. Then for every compact subset $K \subseteq U$ and every $\varepsilon > 0$ there exists a continuous seminorm q on E such that $K + B_1^q(0) \subseteq U$ and*

$$|f(x) - f(y)| < \varepsilon \quad \text{for } x \in K, y \in B_1^q(x).$$

We show that *C^1-functions act on AC_{L^p}*. The following lemma is a variant of [17, Lemma 3.18 (a)].

Lemma 4.2.16. *Let E, F be sequentially complete locally convex spaces, $V \subseteq E$ be an open subset and $p \in [1,\infty]$. If $f\colon V \to F$ is a C^1-function then*

$$f \circ \eta \in AC_{L^p}([a,b],F)$$

for every $\eta \in AC_{L^p}([a,b],V)$ and

$$(f\circ\eta)' = [t \mapsto df(\eta(t),\gamma(t))] \tag{4.8}$$

if $\eta' = [\gamma]$.

Proof. The composition $f \circ \eta$ is continuous and the differential $df\colon V \times E \to F$ is continuous and linear in the second argument, thus $df \circ (\eta,\gamma) \in \mathcal{L}^p([a,b],F)$ for $[\gamma] = \eta'$, by Lemma 4.1.23. In other words, the function

$$\zeta\colon [a,b] \to F, \quad \zeta(t) := f(\eta(a)) + \int_a^t df(\eta(s),\gamma(s))\,ds \tag{4.9}$$

is in $AC_{L^p}([a,b],F)$.

We claim that for each continuous linear form $\alpha \in E'$, the composition $\alpha \circ f \circ \eta$ is in $AC_{L^p}([a,b],\mathbb{R})$ (hence almost everywhere differentiable) and that $(\alpha \circ f \circ \eta)'(t) = \alpha(df(\eta(t),\gamma(t)))$ for almost every $t \in [a,b]$. From this, it will follow that

$$\begin{aligned}\alpha(f(\eta(t))) &= (\alpha \circ f \circ \eta)(a) + \int_a^t \alpha(df(\eta(s),\gamma(s)))\,ds \\ &= \alpha\left(f(\eta(a)) + \int_a^t df(\eta(s),\gamma(s))\,ds\right) = \alpha(\zeta(t)),\end{aligned}$$

for each $\alpha \in E'$ and $t \in [a,b]$, therefore $f \circ \eta = \zeta \in AC_{L^p}([a,b],F)$, as E' separates points on E. Notably, (4.8) holds.

To prove the claim, we may assume that $F = \mathbb{R}$ and we show that the composition $f \circ \eta\colon [a,b] \to \mathbb{R}$ is absolutely continuous.

As $\eta([a,b])$ is a compact subset of the open subset V, there is some open neighborhood $U \subseteq V$ of $\eta([a,b])$ and some continuous seminorm q on E such that

$$|f(u) - f(\bar{u})| \leq q(u - \bar{u}) \tag{4.10}$$

for all $u, \bar{u} \in U$, by [17, Lemma 1.60]. Given $\varepsilon > 0$ there exists some $\delta > 0$ such that $\sum_{j=1}^n |\sigma(b_j) - \sigma(a_j)| < \varepsilon$ whenever $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$ with $\sum_{j=1}^n |b_j - a_j| < \delta$, because the function

$$\sigma\colon [a,b] \to \mathbb{R}, \quad \sigma(t) := \int_a^t q(\gamma(s))\,ds$$

is absolutely continuous (see the Fundamental Theorem of Calculus, Lemma 4.2.1). Therefore, we have

$$\begin{aligned}\sum_{j=1}^n |f(\eta(b_j)) - f(\eta(a_j))| &\leq \sum_{j=1}^n q(\eta(b_j) - \eta(a_j)) \\ &= \sum_{j=1}^n q\left(\int_{a_j}^{b_j} \gamma(s)\,ds\right) \leq \sum_{j=1}^n \int_{a_j}^{b_j} q(\gamma(s))\,ds < \varepsilon,\end{aligned}$$

where we used (4.10) in the first step. Hence $f \circ \eta$ is absolutely continuous, thus, by Lemma 4.2.1 there is some $\varphi \in \mathcal{L}^1([a,b])$ such that

$$f(\eta(t)) = f(\eta(a)) + \int_a^t \varphi(s)\,ds,$$

in other words

$$f \circ \eta \in AC_{L^1}([a,b],\mathbb{R}) \quad \text{and} \quad (f \circ \eta)'(t) = \varphi(t) \text{ for a.e. } t \in [a,b].$$

Now, we want to show that $\varphi(t) = df(\eta(t),\gamma(t))$ for almost every $t \in [a,b]$, that is, $\varphi \in \mathcal{L}^p([a,b])$. To this end, we may assume that there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of $[a,b]$ (as in Lemma 4.1.5) such that for every $n \in \mathbb{N}$ the restriction $\gamma|_{K_n}$ is continuous , $\lambda([a,b] \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0$ and $\gamma(t) = 0$ for every $t \notin \bigcup_{n\in\mathbb{N}} K_n$.

Each of the sets

$$L_n := \eta([a,b]) \cup \bigcup_{m=1}^{n} \gamma(K_m)$$

is compact and metrizable, hence by [17, Lemma 1.11], there exists a locally convex topology $\mathcal{T}_{X_n}$ on each vector subspace

$$X_n := \operatorname{span}(L_n),$$

which is metrizable, separable and coarser than the induced topology $\mathcal{O}_{X_n}$. Then on each X_n, there is a countable family Λ_n of continuous (with respect to $\mathcal{T}_{X_n}$) linear functionals separating the points (see [42, Chapter II, Prop. 4]). Consequently, the countable family $\Lambda := \bigcup_{n\in\mathbb{N}} \Lambda_n$ separates the points on the vector space $X := \bigcup_{n\in\mathbb{N}} X_n$, which enables to define a metrizable locally convex topology $\mathcal{T}_X$ coarser than the induced topology $\mathcal{O}_X$. On the other hand, each of the m-fold sums

$$L_{m,n} := [-m,m]L_n + \cdots + [-m,m]L_n$$

is compact (with respect to $\mathcal{T}_{X_n}$ and $\mathcal{O}_{X_n}$), and $X_n = \bigcup_{m\in\mathbb{N}} L_{m,n}$, thus

$$X = \bigcup_{m,n\in\mathbb{N}} L_{m,n}$$

is σ-compact.

The space $X \times X \times \mathbb{R}$ has a locally convex metrizable σ-compact topology $\mathcal{T}$, say, $X \times X \times \mathbb{R} = \bigcup_{n\in\mathbb{N}} C_n$. Then $\mathcal{O}_{C_n} = \mathcal{T}_{C_n}$, where $\mathcal{O}_{C_n}$, $\mathcal{T}_{C_n}$ are the topologies on C_n induced by $E \times E \times \mathbb{R}$ and $X \times X \times \mathbb{R}$, respectively. Hence $V^{[1]} \cap C_n \in \mathcal{T}_{C_n}$ and $\mathcal{T}_{C_n}$ is compact and metrizable, hence second countable. Therefore, $V^{[1]} \cap C_n$ is σ-compact (being locally compact with countable base), that is, $V^{[1]} \cap C_n$ is a countable union of compact subsets, hence so is $(V \cap X)^{[1]} = V^{[1]} \cap (X \times X \times \mathbb{R}) = \bigcup_{n\in\mathbb{N}}(V^{[1]} \cap C_n)$, so that we may write $(V \cap X)^{[1]} = \bigcup_{n\in\mathbb{N}} A_n$ with compact subsets A_n.

Next, we will construct a metrizable locally convex topology on X such that $\eta \in AC_{L^p}([a,b], V \cap X)$ with $\eta' = [\gamma] \in L^p([a,b],X)$ and such that $f^{[1]}|_{(V\cap X)^{[1]}}$ remains continuous. From Lemma 4.2.15, it follows that for every $k,n \in \mathbb{N}$ there exists a continuous seminorm $q_{n,k}$ on $E \times E \times \mathbb{R}$ such that for all $(x,y,t) \in A_n$ we have

$$|f^{[1]}(x,y,t) - f^{[1]}(v,w,s)| < \frac{1}{k} \qquad \forall (v,w,s) \in B_1^{q_{n,k}}(x,y,t).$$

Consequently, there is a continuous seminorm $\pi_{n,k}$ on E and $\delta > 0$ such that

$$|f^{[1]}(x,y,t) - f^{[1]}(v,w,s)| < \frac{1}{k}$$
$$\forall (v,w,s) \in B_1^{\pi_{n,k}}(x) \times B_1^{\pi_{n,k}}(y) \times]t-\delta, t+\delta[\subseteq B_1^{q_{n,k}}(x,y,t).$$

We endow X with the metrizable locally convex topology $\mathcal{T}$ defined by the countable family $\{\pi_{n,k}|_X : n,k \in \mathbb{N}\}$. This topology is coarser than the induced topology, hence $\eta\colon [a,b] \to V \cap X$ remains continuous and $\eta(t) - \eta(a) = \int_a^t \gamma(s)\, ds$ is the weak integral of γ in X for every $t \in [a,b]$. To see this, let α be a continuous linear functional on $(X, \mathcal{T})$. Then α is continuous with respect to the induced topology on X (which is finer than $\mathcal{T}$) hence there is some continuous linear extension $\mathcal{A} \in E'$ of α. Thus

$$\alpha(\eta(t) - \eta(a)) = \mathcal{A}(\eta(t) - \eta(a)) = \int_a^t \mathcal{A}(\gamma(s))\, ds = \int_a^t \alpha(\gamma(s))\, ds.$$

Therefore, $\eta \in AC_{L^p}([a,b], V \cap X)$ with $\eta' = [\gamma]$ and, by the construction of the topology, the map $f^{[1]}$ is continuous in every $(x,y,t) \in (V \cap X)^{[1]}$ with respect to the obtained topology on $X \times X \times \mathbb{R}$. As $\mathcal{T}$ is metrizable, the map $\eta\colon [a,b] \to V \cap X$ is differentiable in almost every $t \in [a,b]$ with $\eta'(t) = \gamma(t)$ (see Proposition 4.2.5), so that in every such t we have

$$\begin{aligned}\frac{1}{h}(f(\eta(t+h) - f(\eta(t))) &= \frac{1}{h}(f(\eta(t) + \frac{\eta(t+h) - \eta(t)}{h}) - f(\eta(t)))\\ &= f^{[1]}(\eta(t), \frac{\eta(t+h) - \eta(t)}{h}, h) \to df(\eta(t), \gamma(t))\end{aligned}$$

as $h \to 0$. That means, for almost every $t \in [a,b]$ we have

$$\varphi(t) = (f \circ \eta)'(t) = df(\eta(t), \gamma(t)),$$

whence $\varphi \in \mathcal{L}^p([a,b])$ and $f \circ \eta \in AC_{L^p}([a,b], \mathbb{R})$. □

Proposition 4.2.17. *Let E, F be sequentially complete locally convex spaces, let $V \subseteq E$ be an open subset and $p \in [1, \infty]$. If $f\colon V \to F$ is a C^{k+2}-function (for $k \in \mathbb{N} \cup \{0, \infty\}$), then the map*

$$AC_{L^p}([a,b], f)\colon AC_{L^p}([a,b], V) \to AC_{L^p}([a,b], F), \quad \eta \mapsto f \circ \eta$$

is C^k.

Proof. The map $AC_{L^p}([a,b], f)$ is defined by Lemma 4.2.16, by definition of the topology on $AC_{L^p}([a,b], F)$ (see Definition 4.2.8), $AC_{L^p}([a,b], f)$ will be C^k if each of the components of

$$AC_{L^p}([a,b], V) \to F \times L^p([a,b], F), \quad \eta \mapsto (f(\eta(a)), (f \circ \eta)') \tag{4.11}$$

is C^k. The first component

$$AC_{L^p}([a,b], V) \to F, \quad \eta \mapsto (f \circ \mathrm{pr}_1 \circ \Phi)(\eta) = f(\eta(a))$$

is indeed C^k, where Φ is as in Definition 4.2.8 and pr_1 is the projection on the first component. Further, for $\eta' = [\gamma] \in L^p([a,b], E)$ we have $(f \circ \eta)' = [df \circ (\eta, \gamma)]$ by (4.8) and

$$C([a,b], V) \times L^p([a,b], E) \to L^p([a,b], F), \quad (\eta, [\gamma]) \mapsto [df \circ (\eta, \gamma)]$$

is C^k, the derivative $df\colon V \times E \to F$ being C^{k+1} and linear in the second argument (see Proposition 4.1.25). Hence, the second component of (4.11) is C^k, as required. □

Remark 4.2.18. In particular, the above proposition states that *smooth functions act smoothly on* AC_{L^p} ([17]), that is, for any smooth function $f\colon V \to F$, the function $AC_{L^p}([a,b],f)$ is smooth.

Remark 4.2.19. Since any continuous linear function $f\colon E \to F$ is smooth, we conclude from Proposition 4.2.17 that

$$AC_{L^p}([a,b],E\times F) \cong AC_{L^p}([a,b],E) \times AC_{L^p}([a,b],F)$$

as locally convex spaces (proceeding as in Remark 4.1.20).

The properties of the spaces $AC_{L^p}([a,b],E)$, proved in the preceding, enable us to define spaces of AC-functions with values in manifolds M modeled on sequentially complete locally convex spaces.

Definition 4.2.20. Let M be a manifold modeled on a sequentially complete locally convex space E. For $p \in [1,\infty]$, denote by $AC_{L^p}([a,b],M)$ the set of continuous functions $\eta\colon [a,b] \to M$ such that there exists some partition $a = t_0 < t_1 < \ldots < t_n = b$ with

$$\varphi_j \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],E)$$

for some charts $\varphi_j\colon U_j \to V_j$ such that $\eta([t_{j-1},t_j]) \subseteq U_j$ for $j = 1,\ldots,n$.

The following lemma shows, in particular, that given an AC-function with values in a manifold, the defining property holds for any suitable partition and charts.

Lemma 4.2.21. *Let $\eta \in AC_{L^p}([a,b],M)$, let $[\alpha,\beta] \subseteq [a,b]$ and $\varphi\colon U \to V$ be any chart for M such that $\eta([\alpha,\beta]) \subseteq U$. Then*

$$\varphi \circ \eta|_{[\alpha,\beta]} \in AC_{L^p}([\alpha,\beta],E).$$

Proof. We have $\alpha \in [t_k,t_{k+1}]$ and $\beta \in [t_{l-1},t_l]$ for some k,l, for the sake of simplicity we may assume $\alpha = t_k$, $\beta = t_l$. For $j \in \{k+1,\ldots,l\}$ we have

$$\varphi \circ \eta|_{[t_{j-1},t_j]} = (\varphi \circ \varphi_j^{-1}) \circ (\varphi_j \circ \eta|_{[t_{j-1},t_j]}).$$

Since $\varphi \circ \varphi_j^{-1}$ is a smooth function and $\varphi_j \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],E)$, the above composition is in $AC_{L^p}([t_{j-1},t_j],E)$ by Lemma 4.2.16. From Remark 4.2.14, it follows $\varphi \circ \eta|_{[\alpha,\beta]} \in AC_{L^p}([\alpha,\beta],E)$. □

Remark 4.2.22. If M is a sequentially complete locally convex space, then $AC_{L^p}([a,b],M)$ coincides with the set defined in Definition 4.2.8, by the previous lemma.

Lemma 4.2.23. *Let M, N be manifolds modeled on sequentially complete locally convex spaces E and F, respectively. If $f\colon M \to N$ is a C^1-map, then $f \circ \eta \in AC_{L^p}([a,b],N)$ for each $\eta \in AC_{L^p}([a,b],M)$ and $p \in [1,\infty]$.*

Proof. Consider a partition $a = t_0 < t_1 < \ldots < t_n = b$ and charts $\varphi_j \colon U_j \to V_j$ for M such that $\eta([t_{j-1}, t_j]) \subseteq U_j$ and $\varphi_j \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1}, t_j], E)$ for each $j \in \{1, \ldots, n\}$. Since $f \circ \eta|_{[t_{j-1},t_j]}$ is continuous, we find a partition $t_{j-1} = s_0 < s_1 < \cdots < s_m = t_j$ and charts $\psi_i \colon P_i \to Q_i$ for N such that $f(\eta([s_{i-1}, s_i])) \subseteq P_i$ for each $i \in \{1, \ldots, m\}$. Then

$$\psi_i \circ f \circ \eta|_{[s_{i-1},s_i]} = (\psi_i \circ f \circ \varphi_j^{-1}) \circ (\varphi_j \circ \eta|_{[s_{i-1},s_i]}) \in AC_{L^p}([s_{i-1}, s_i], F),$$

by Remark 4.2.14 and Lemma 4.2.16. Hence $f \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1}, t_j], N)$ for each $j \in \{1, \ldots, n\}$, whence $f \circ \eta \in AC_{L^p}([a, b], N)$. □

Remark 4.2.24. For manifolds M and N, from the above lemma, it follows that the sets $AC_{L^p}([a, b], M \times N)$ and $AC_{L^p}([a, b], M) \times AC_{L^p}([a, b], N)$ coincide.

Remark 4.2.25. Let G be a group, $U \subseteq G$ be a symmetric subset containing the identity element of G. Assume that U is endowed with a smooth manifold structure modeled on a locally convex space E such that the inversion $U \to U, x \to x$ on U is smooth, the subset $U_m := \{(x, y) \in U \times U : xy \in U\}$ is open in $U \times U$ and the multiplication $U_m \to U, (x, y) \mapsto xy$ is smooth on U_m. Further, assume that for each $g \in G$, there exists an open identity neighborhood $W \subseteq U$ such that $gWg^{-1} \subseteq U$ and $W \to U, x \mapsto gxg^{-1}$ is smooth. Then G can be endowed with a unique smooth manifold structure modeled on E such that G becomes a smooth Lie group and U with the given manifold structure becomes an open smooth submanifold.

Lemma 4.2.26. *Let E_1, E_2 and F be sequentially complete locally convex spaces. Let M be a smooth manifold modeled on E_1 and $V \subseteq E_2$ be an open subset. If $f \colon M \times V \to F$ is a C^{k+2}-map for some $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\zeta \in AC_{L^p}([a, b], M)$ for $p \in [1, \infty]$, then*

$$AC_{L^p}([a, b], V) \to AC_{L^p}([a, b], F), \quad \eta \mapsto f \circ (\zeta, \eta) \tag{4.12}$$

is a C^k-map.

Proof. Since $(\zeta, \eta) \in AC_{L^p}([a, b], M \times V)$, the above map is defined by Lemma 4.2.23; it will be C^k if for a partition $a = t_0 < t_1 < \ldots < t_n = b$ for ζ (as in Definition 4.2.20) the function

$$AC_{L^p}([a, b], V) \to \prod_{j=1}^{n} AC_{L^p}([t_{j-1}, t_j], F),$$
$$\eta \mapsto \Big(f \circ (\zeta, \eta)|_{[t_{j-1},t_j]}\Big)_{j=1,\ldots,n}$$

is C^k (where we use the topological embedding with closed image on $AC_{L^p}([a, b], F)$ as in Lemma 4.2.13). This will hold if every component

$$AC_{L^p}([a, b], V) \to AC_{L^p}([t_{j-1}, t_j], F), \quad \eta \mapsto f \circ (\zeta, \eta)|_{[t_{j-1},t_j]} \tag{4.13}$$

is C^k.

Now, given charts $\varphi\colon U_j \to V_j$ for M with $\zeta([t_{j-1},t_j]) \subseteq U_j$, for every $j \in \{1,\dots,n\}$, the function

$$AC_{L^p}([a,b],V) \to AC_{L^p}([t_{j-1},t_j],V_j \times V), \quad \eta \mapsto \Big(\varphi_j \circ \zeta|_{[t_{j-1},t_j]}, \eta|_{[t_{j-1},t_j]}\Big)$$

is smooth by Lemma 4.2.13 (identifying $AC_{L^p}([t_{j-1},t_j],V_j \times V)$ with $AC_{L^p}([t_{j-1},t_j],V_j)\times AC_{L^p}([t_{j-1},t_j],V)$, see Remark 4.2.19). As the composition $f\circ(\varphi_j^{-1}\times \mathrm{id}_V)\colon V_j \times V \to F$ is C^{k+2}, by Proposition 4.2.17 the function

$$\begin{aligned} &AC_{L^p}([a,b],V) \to AC_{L^p}([t_{j-1},t_j],F),\\ &\qquad \eta \mapsto \Big(f\circ(\varphi_j^{-1}\times \mathrm{id}_V)\circ(\varphi_j \circ \zeta|_{[t_{j-1},t_j]}, \eta|_{[t_{j-1},t_j]}\Big)\\ &\qquad\quad = f\circ(\zeta,\eta)|_{[t_{j-1},t_j]} \end{aligned}$$

is C^k. Therefore, the function in (4.13) is C^k and the proof is finished. □

Now, we are able to endow the sets $AC_{L^p}([a,b],G)$ with unique Lie group structures.

Proposition 4.2.27. *Let G be a smooth Lie group modeled on a sequentially complete locally convex space E, let $p \in [1,\infty]$. Then there exists a unique Lie group structure on $AC_{L^p}([a,b],G)$ such that for each open symmetric e_G-neighborhood $U \subseteq G$ the subset $AC_{L^p}([a,b],U)$ is open in $AC_{L^p}([a,b],G)$ and such that*

$$AC_{L^p}([a,b],\varphi)\colon AC_{L^p}([a,b],U) \to AC_{L^p}([a,b],V), \quad \eta \mapsto \varphi\circ\eta$$

is a smooth diffeomorphism for every chart $\varphi\colon U \to V$ for G.

Proof. Step 1: $AC_{L^p}([a,b],G)$ is a group.

As m_G and j_G are smooth, we have $m_G\circ(\eta,\xi), j_G\circ\eta \in AC_{L^p}([a,b],G)$ for all η, $\xi \in AC_{L^p}([a,b],G)$, by Lemma 4.2.23 (identifying $AC_{L^p}([a,b],G\times G)$ with $AC_{L^p}([a,b],G)\times AC_{L^p}([a,b],G)$). Then $\tilde G := AC_{L^p}([a,b],G)$ is a group with multiplication

$$m_{\tilde G} := AC_{L^p}([a,b],m_G)\colon \tilde G \times \tilde G \to \tilde G, \quad (\eta,\xi)\mapsto m_G\circ(\eta,\xi) =: \eta\cdot\xi,$$

inversion

$$j_{\tilde G} := AC_{L^p}([a,b],j_G)\colon \tilde G \to \tilde G, \quad \eta \mapsto j_G\circ\eta =: \eta^{-1}$$

and identity element $e_{\tilde G}\colon t \mapsto e_G$.

Step 2: Existence of a Lie group structure on $AC_{L^p}([a,b],G)$.

Consider an open symmetric e_G-neighborhood $U\subseteq G$ and a chart $\varphi\colon U\to V$. As $\tilde V := AC_{L^p}([a,b],V)$ is open in $AC_{L^p}([a,b],E)$ (see Remark 4.2.10), we endow the symmetric subset $\tilde U := AC_{L^p}([a,b],U) := \{\eta \in AC_{L^p}([a,b],G) : \eta([a,b]) \subseteq U\}$ with the C^∞-manifold structure turning the bijection

$$\tilde\varphi := AC_{L^p}([a,b],\varphi)\colon \tilde U \to \tilde V, \quad \eta\mapsto\varphi\circ\eta$$

into a global chart (the map is defined by Lemma 4.2.23). Obviously, $e_{\tilde{G}} \in \tilde{U}$.

Further, by Lemma 4.2.16, the function

$$AC_{L^p}([a,b], \varphi \circ j_G|_U \circ \varphi^{-1})\colon \tilde{V} \to \tilde{V}, \quad \eta \mapsto (\varphi \circ j_G|_U \circ \varphi^{-1}) \circ \eta$$

is smooth. Thus, writing

$$\begin{aligned}\tilde{U} \to \tilde{U}, \quad \eta \mapsto & (\tilde{\varphi}^{-1} \circ AC_{L^p}([a,b], \varphi \circ j_G|_U \circ \varphi^{-1}) \circ \tilde{\varphi})(\eta) \\ &= \varphi^{-1} \circ \varphi \circ j_G|_U \circ \varphi^{-1} \circ \varphi \circ \eta \\ &= j_G \circ \eta,\end{aligned}$$

we see that the inversion on $\tilde{U}$ is smooth.

Now, consider the open subset $U_m := \{(x,y) \in U \times U : xy \in U\}$ of $U \times U$. As $V_m := (\varphi \times \varphi)(U_m)$ is open in $E \times E$, the set $\tilde{V}_m := AC_{L^p}([a,b], V_m)$ is open in $AC_{L^p}([a,b], E) \times AC_{L^p}([a,b], E)$, whence $\tilde{U}_m := (\tilde{\varphi}^{-1} \times \tilde{\varphi}^{-1})(\tilde{V}_m)$ is open in $\tilde{U} \times \tilde{U}$. Again, by Lemma 4.2.16, the function

$$\begin{aligned}AC_{L^p}([a,b], \varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|_{V_m})\colon \tilde{V}_m \to \tilde{V}, \\ \eta \mapsto (\varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|_{V_m}) \circ \eta\end{aligned}$$

is smooth. Therefore

$$\begin{aligned}&\tilde{U}_m \to \tilde{U}, \\ &(\eta, \xi) \mapsto (\tilde{\varphi}^{-1} \circ AC_{L^p}([a,b], \varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|_{V_m}) \circ (\tilde{\varphi} \times \tilde{\varphi}))(\eta, \xi) \\ &\quad = \varphi^{-1} \circ \varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|_{V_m} \circ (\varphi \times \varphi) \circ (\eta, \xi) \\ &\quad = m_G \circ (\eta, \xi),\end{aligned}$$

which is the multiplication on $\tilde{U}_m$, is smooth.

Finally, fix some $\eta \in \tilde{G}$ and write $K := \operatorname{im}(\eta) \subseteq G$. As the function

$$h\colon G \times G \to G, (x,y) \mapsto xyx^{-1}$$

is smooth and $h(K \times \{e_G\}) = \{e_G\} \subseteq U$, the compact set $K \times \{e_G\}$ is a subset of the open set $h^{-1}(U) \subseteq G \times G$. By the Wallace Lemma, there are open subsets W_K, W of G such that $K \times \{e_G\} \subseteq W_K \times W \subseteq h^{-1}(U)$. We may assume $W \subseteq U$, then we see that $\tilde{W} := AC_{L^p}([a,b], W)$ is open in $\tilde{U}$ and for each $\xi \in \tilde{W}$ we have

$$\eta \cdot \xi \cdot \eta^{-1} = h \circ (\eta, \xi) \in \tilde{U}$$

by Lemma 4.2.23. Using Lemma 4.2.26, we see that the function

$$\begin{aligned}AC_{L^p}([a,b], \varphi(W)) \to \tilde{V}, \\ \xi \mapsto (\varphi \circ h \circ (\mathrm{id}_{W_K} \times \varphi^{-1}|_{\varphi(W)})) \circ (\eta, \xi)\end{aligned}$$

is smooth, whence

$$\begin{aligned}\tilde{W} \to \tilde{U}, \quad \xi \mapsto \varphi^{-1} &\circ (\varphi \circ h \circ (\mathrm{id}_{W_K} \times \varphi^{-1}|_{\varphi(W)})) \circ (\eta, \varphi \circ \xi) \\ &= h \circ (\eta, \xi) \\ &= \eta \cdot \xi \cdot \eta^{-1}\end{aligned}$$

is smooth.

Consequently, by Remark 4.2.25, there exists a unique Lie group structure on $\tilde{G}$ turning $\tilde{U}$ into a smooth open submanifold and $\tilde{\varphi}$ into a $\tilde{G}$-chart around $e_{\tilde{G}}$.

Step 3: Uniqueness of the Lie group structure on $AC_{L^p}([a,b],G)$.

Let $U' \subseteq G$ be an open symmetric e_G-neighborhood and $\varphi'\colon U' \to V'$ be a G-chart around e_G. Denote by $\tilde{G}'$ the group $AC_{L^p}([a,b],G)$ endowed with the Lie group structure turning $\tilde{U}' := AC_{L^p}([a,b],U')$ into an open submanifold and $\tilde{\varphi}'\colon \tilde{U}' \to AC_{L^p}([a,b],V')$ into a chart (constructed as in Step 2). We show that both identity maps $\mathrm{id}\colon \tilde{G}' \to \tilde{G}$ and $\mathrm{id}\colon \tilde{G} \to \tilde{G}'$ are continuous, that is, both Lie group structures coincide.

The set $U' \cap U$ is open in U', hence $\varphi'(U' \cap U)$ is open in V', thus $AC_{L^p}([a,b],\varphi'(U' \cap U))$ is open in $AC_{L^p}([a,b],V')$, and consequently $\tilde{U}' \cap \tilde{U} = \tilde{\varphi}'^{-1}(AC_{L^p}([a,b],\varphi'(U' \cap U)))$ is open in $\tilde{G}'$. Writing

$$\mathrm{id}_{\tilde{U}' \cap \tilde{U}} = \tilde{\varphi}^{-1} \circ AC_{L^p}([a,b],\varphi \circ \varphi'^{-1}|_{U' \cap U}) \circ \tilde{\varphi}'|_{\tilde{U}' \cap \tilde{U}} \colon \tilde{U}' \cap \tilde{U} \to \tilde{G}$$

and using Proposition 4.2.17, we see that $\mathrm{id}\colon \tilde{G}' \to \tilde{G}$ is smooth on the open identity neighborhood $\tilde{U}' \cap \tilde{U}$, hence smooth. In the same way, we show that also $\mathrm{id}\colon \tilde{G} \to \tilde{G}'$ is smooth, as required. □

Remark 4.2.28. One can easily show that

$$AC_{L^p}([a,b],G \times H) \cong AC_{L^p}([a,b],G) \times AC_{L^p}([a,b],H)$$

as Lie groups.

Lemma 4.2.29. *The inclusion map*

$$\mathrm{incl}\colon AC_{L^p}([a,b],G) \to C([a,b],G), \quad \eta \mapsto \eta$$

is a smooth homomorphism.

Proof. Let $U \subseteq G$ be an open identity neighborhood, $\varphi\colon U \to V$ be a chart for G. Then $C([a,b],\varphi)\colon C([a,b],U) \to C([a,b],V), \eta \mapsto \varphi \circ \eta$ is a chart for $C([a,b],G)$ and $AC_{L^p}([a,b],\varphi)\colon AC_{L^p}([a,b],U) \to AC_{L^p}([a,b],V), \eta \mapsto \varphi \circ \eta$ is a chart for $AC_{L^p}([a,b],G)$. The function

$$\begin{aligned}&AC_{L^p}([a,b],V) \to C([a,b],V), \\ &\qquad \eta \mapsto \left(C([a,b],\varphi) \circ \mathrm{incl} \circ AC_{L^p}([a,b],\varphi)^{-1}\right)(\eta) = \eta\end{aligned}$$

is smooth, being a restriction of the smooth inclusion map from Lemma 4.2.9. Hence the group homomorphism incl is smooth. □

Lemma 4.2.30. *For any $\alpha \in [a,b]$, the evaluation map*

$$\mathrm{ev}_\alpha \colon AC_{L^p}([a,b],G) \to G, \quad \eta \mapsto \eta(\alpha)$$

is a smooth homomorphism.

Proof. The function is a composition of the smooth inclusion map from Lemma 4.2.29 and the smooth evaluation map on $C([a,b],G)$, hence smooth. □

Lemma 4.2.31. *Let G be a Lie group modeled on a sequentially complete locally convex space E, let $p \in [1,\infty]$. Then the function*

$$\Gamma_G \colon AC_{L^p}([a,b],G) \to \prod_{j=1}^n AC_{L^p}([t_{j-1},t_j],G), \quad \eta \mapsto \left(\eta|_{[t_{j-1},t_j]}\right)_{j=1,\ldots,n}$$

is a smooth homomorphism and a smooth diffeomorphism onto a Lie subgroup of the product $\prod_{j=1}^n AC_{L^p}([t_{j-1},t_j],G)$.

Proof. First of all we introduce some notations. For $j = 1,\ldots,n$ we denote $G_j := AC_{L^p}([t_{j-1},t_j],G)$, and for an open identity neighborhood $U \subseteq G$ and a chart $\varphi\colon U \to V$ we write $U_j := AC_{L^p}([t_{j-1},t_j],U)$, $V_j := AC_{L^p}([t_{j-1},t_j],V)$ and $\varphi_j \colon U_j \to V_j, \zeta \mapsto \varphi \circ \zeta$.

Clearly, the map Γ_G is a group homomorphism and

$$\mathrm{im}(\Gamma_G) = \{(\eta_j)_{j=1,\ldots,n} \in \prod_{j=1}^n G_j : \eta_{j-1}(t_j) = \eta_j(t_j) \text{ for all } j \in \{2,\ldots,n\}\}$$

is a subgroup of $\prod_{j=1}^n G_j$. Moreover, the function

$$\psi := \prod_{j=1}^n \varphi_j \colon \prod_{j=1}^n U_j \to \prod_{j=1}^n V_j, \quad (\zeta_j,\ldots,\zeta_n) \mapsto (\varphi\circ\zeta_1,\ldots,\varphi\circ\zeta_n)$$

is a chart for $\prod_{j=1}^n G_j$ and $\psi(\mathrm{im}(\Gamma_G)\cap\prod_{j=1}^n U_j) = \mathrm{im}(\Gamma_E)\cap\prod_{j=1}^n V_j$, where Γ_E is the linear topological embedding with closed image from Lemma 4.2.13. Therefore, $\mathrm{im}(\Gamma_G)$ is a Lie subgroup modeled on the closed vector subspace $\mathrm{im}(\Gamma_E)$ of $\prod_{j=1}^n AC_{L^p}([t_{j-1},t_j],E)$.

Finally, both compositions

$$\psi \circ \Gamma_G \circ AC_{L^p}([a,b],\varphi)^{-1} \colon AC_{L^p}([a,b],V) \to \prod_{j=1}^n V_j, \quad \eta \mapsto \Gamma_E(\eta)$$

and

$$AC_{L^p}([a,b],\varphi) \circ \Gamma_G^{-1} \circ \left(\psi|_{\mathrm{im}(\Gamma_G)}\right)^{-1} : \mathrm{im}(\Gamma_E) \cap \prod_{j=1}^n V_j \to AC_{L^p}([a,b],V), \quad \eta \mapsto \Gamma_E^{-1}(\eta)$$

are smooth maps, thus we conclude that Γ_G is a smooth diffeomorphism onto its image. □

4.3 Measurable regularity of Lie groups

Definition 4.3.1. Let G be a Lie group modeled on a sequentially complete locally convex space E, let $p \in [1,\infty]$. Consider $\eta \in AC_{L^p}([a,b],G)$, a partition $a = t_0 < t_1 < \ldots < t_n = b$ and charts $\varphi_j \colon U_j \to V_j$ for G such that $\eta([t_{j-1},t_j]) \subseteq U_j$ for all j and

$$\eta_j := \varphi_j \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],E).$$

Denote $\eta_j' := [\gamma_j] \in L^p([t_{j-1},t_j],E)$ and set

$$\gamma(t) := T\varphi_j^{-1}(\eta_j(t),\gamma_j(t))$$

for $t \in [t_{j-1},t_j[$, and

$$\gamma(b) := T\varphi_n^{-1}(\eta_n(b),\gamma_n(b)).$$

The constructed function $\gamma\colon [a,b] \to TG$ is measurable and we write $\dot\eta := [\gamma]$.

Further, define the *left logarithmic derivative of* η via

$$\delta(\eta) := [\omega_l \circ \gamma],$$

where $[\gamma] = \dot\eta$ and $\omega_l \colon TG \to \mathfrak{g}, v \mapsto \pi_{TG}(v)^{-1}.v$ with the bundle projection $\pi_{TG}\colon TG \to G$. (Note that the definitions of $\dot\eta$ and $\delta(\eta)$ do not depend on the choice of the partition and charts.)

Lemma 4.3.2. *Let G be a Lie group modeled on a sequentially complete locally convex space E and $p \in [1,\infty]$. If $\eta \in AC_{L^p}([a,b],G)$, then $\delta(\eta) \in L^p([a,b],\mathfrak{g})$.*

Proof. By definition, there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ and there exist charts $\varphi_j \colon U_j \to V_j$ for G such that $\eta([t_{j-1},t_j]) \subseteq U_j$ and $\eta_j := \varphi_j \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],E)$ for every $j \in \{1,\ldots,n\}$. We denote $[\gamma_j] := \eta_j'$ and $[\gamma] := \dot\eta$ and see that

$$\omega_l \circ \gamma|_{[t_{j-1},t_j]} = \omega_l \circ T\varphi_j^{-1} \circ (\eta_j,\gamma_j) \in \mathcal{L}^p([t_{j-1},t_j],\mathfrak{g})$$

by Lemma 4.1.23, since $\omega_l \circ T\varphi_j^{-1} \colon V_j \times E \to \mathfrak{g}$ is continuous and linear in the second argument. From Lemma 4.1.21, it follows that $\delta(\eta) = [\omega_l \circ \gamma] \in L^p([a,b],\mathfrak{g})$. □

Recall that the tangent bundle TG of a Lie group G can be considered as a Lie group. We identify $g \in G$ with $0_g \in T_gG$.

Lemma 4.3.3. *Let G be a Lie group modeled on a sequentially complete locally convex space, let $p \in [1,\infty]$. For η, $\zeta \in AC_{L^p}([a,b],G)$ with $\dot\eta = [\gamma]$, $\dot\zeta = [\xi]$ we have*

$$(\eta \cdot \zeta)^{\cdot} = [t \mapsto \gamma(t).\zeta(t) + \eta(t).\xi(t)] \tag{4.14}$$

and

$$(\eta^{-1})^{\cdot} = [t \mapsto -\eta(t)^{-1}.\gamma(t).\eta(t)^{-1}]. \tag{4.15}$$

Further, if $f \colon G \to H$ is a smooth function between Lie groups modeled on sequentially complete locally convex spaces, then

$$(f \circ \eta)^{\cdot} = [Tf \circ \gamma]. \tag{4.16}$$

Proof. We prove the last equation (4.16) first. Consider a partition $a = t_0 < t_1 < \ldots < t_n = b$ and charts $\varphi_j \colon U_j \to V_j$, $\psi_j \colon P_j \to Q_j$ for G and H, respectively, such that

$$\varphi_j \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1}, t_j], E),$$
$$\psi_j \circ f \circ \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1}, t_j], F),$$

where E and F are the model spaces of G and H. Denote

$$[\gamma_j] := (\varphi_j \circ \eta|_{[t_{j-1},t_j]})' \in L^p([t_{j-1}, t_j], E),$$
$$[\xi_j] := (\psi_j \circ f \circ \eta|_{[t_{j-1},t_j]})' \in L^p([t_{j-1}, t_j], F).$$

Then (using (4.8)) we have

$$[\xi_j] = ((\psi_j \circ f \circ \varphi_j^{-1}) \circ (\varphi_j \circ \eta|_{[t_{j-1},t_j]}))' = [d(\psi_j \circ f \circ \varphi_j^{-1})(\varphi_j \circ \eta|_{[t_{j-1},t_j]}, \gamma_j)].$$

Therefore, for $[\delta] := (f \circ \eta)^{\cdot}$ and almost all $t \in [t_{j-1}, t_j[$ we have

$$\begin{aligned}
\delta(t) &= T\psi_j^{-1}((\psi_j \circ f \circ \eta)(t), d(\psi_j \circ f \circ \varphi_j^{-1})((\varphi_j \circ \eta)(t), \gamma_j(t))) \\
&= T\psi_j^{-1}((\psi_j \circ f \circ \varphi_j^{-1} \circ \varphi_j \circ \eta)(t), d(\psi_j \circ f \circ \varphi_j^{-1})((\varphi_j \circ \eta)(t), \gamma_j(t))) \\
&= (T\psi_j^{-1} \circ T(\psi_j \circ f \circ \varphi_j^{-1}))((\varphi_j \circ \eta)(t), \gamma_j(t)) \\
&= (Tf \circ T\varphi_j^{-1})((\varphi_j \circ \eta)(t), \gamma_j(t)) \\
&= (Tf \circ \gamma)(t).
\end{aligned}$$

Now,

$$(\eta \cdot \zeta)^{\cdot} = (m_G \circ (\eta, \zeta))^{\cdot} = [Tm_G \circ (\gamma, \xi)] = [t \mapsto \eta(t).\xi(t) + \gamma(t).\zeta(t)]$$

and

$$(\eta^{-1})^{\cdot} = (j_G \circ \eta)^{\cdot} = [Tj_G \circ \gamma] = [t \mapsto -\eta(t)^{-1}.\gamma(t).\eta(t)^{-1}].$$

□

Lemma 4.3.4. *Let η, $\zeta \in AC_{L^p}([a,b], G)$ and denote $\delta(\eta) = [\gamma]$, $\dot{\eta} = [\bar{\gamma}]$, $\delta(\zeta) = [\xi]$, $\dot{\zeta} = [\bar{\xi}]$. Then the following holds.*

(i) We have

$$\delta(\eta \cdot \zeta) = [t \mapsto \zeta(t)^{-1}.\gamma(t).\zeta(t) + \xi(t)], \tag{4.17}$$

and

$$\delta(\eta^{-1}) = [t \mapsto -\bar{\gamma}(t).\eta(t)^{-1}]. \tag{4.18}$$

(ii) We have $\delta(\eta) = 0$ if and only if η is constant.

(iii) We have $\delta(\eta) = \delta(\zeta)$ if and only if $\eta = g\zeta$ for some $g \in G$.

Proof. (i) Using Equations (4.14) and (4.15), we get

$$\begin{aligned}\delta(\eta \cdot \zeta) &= [t \mapsto (\eta(t)\zeta(t))^{-1}.(\bar{\gamma}(t).\zeta(t) + \eta(t).\bar{\xi}(t))]\\ &= [t \mapsto (\zeta(t)^{-1}\eta(t)^{-1}).\bar{\gamma}(t).\zeta(t) + (\zeta(t)^{-1}\eta(t)^{-1}).\eta(t).\bar{\xi}(t)]\\ &= [t \mapsto \zeta(t)^{-1}.\gamma(t).\zeta(t) + \xi(t)],\end{aligned}$$

and

$$\delta(\eta^{-1}) = [t \mapsto \eta(t).(-\eta(t)^{-1}.\bar{\gamma}(t).\eta(t)^{-1})] = [t \mapsto -\bar{\gamma}(t).\eta(t)^{-1}].$$

(ii) Now, we assume that $\delta(\eta) = 0$, that is, $[t \mapsto \eta(t)^{-1}.\bar{\gamma}(t)] = 0 \in L^p([a,b], \mathfrak{g})$. In other words, $\eta(t)^{-1}.\bar{\gamma}(t) = 0 \in \mathfrak{g}$ for a.e. $t \in [a,b]$. Let $a = t_0 < t_1 < \ldots < t_n = b$, charts φ_j and $[\gamma_j]$ be as in Definition 4.3.1. Then for $\bar{\gamma}(t) \in T_{\eta(t)}G$ we have $d\varphi_j(\bar{\gamma}(t)) = 0 \in E$ for a.e. $t \in [t_{j-1}, t_j]$. On the other hand, we have $d\varphi_j(\bar{\gamma}(t)) = \gamma_j(t)$ for a.e. $t \in [t_{j-1}, t_j]$, thus $[\gamma_j] = 0 \in L^p([t_{j-1}, t_j], E)$. That means, that $\varphi \circ \eta|_{[t_{j-1},t_j]}$ is constant, whence $\eta|_{[t_{j-1},t_j]}$ is constant, whence η is constant.

Conversely, assume $\eta(t) = g \in G$ for all $t \in [a,b]$. Then for some chart φ around g we have

$$\varphi(g) = \varphi(\eta(t)) = \varphi(g) + \int_a^t \gamma_g(s)\, ds$$

for every $t \in [a,b]$, thus $\gamma_g(s) = 0$ for a.e. $s \in [a,b]$, by Lemma 4.2.6, in other words, $(\varphi \circ \eta)' = 0 \in L^p([a,b], E)$. Therefore,

$$\bar{\gamma}(t) = T\varphi^{-1}(\varphi(\eta(t)), 0) = T\varphi^{-1}(\varphi(g), 0)$$

a.e., whence

$$\delta(\eta) = [t \mapsto \eta(t)^{-1}.T\varphi^{-1}(\varphi(g), 0)] = [t \mapsto g^{-1}.T\varphi^{-1}(\varphi(g), 0)] = 0 \in L^p([a,b], \mathfrak{g}).$$

(iii) Now, assume $[\gamma] = \delta(\eta) = \delta(\zeta) = [\xi]$, then (using Equations (4.17) and (4.18))

$$\begin{aligned}\delta(\eta \cdot \zeta^{-1}) &= [t \mapsto \zeta(t).\gamma(t).\zeta(t)^{-1} - \bar{\xi}(t).\zeta(t)^{-1}]\\ &= [t \mapsto \zeta(t).\xi(t).\zeta(t)^{-1} - \bar{\xi}(t).\zeta(t)^{-1}]\\ &= [t \mapsto \bar{\xi}(t).\zeta(t)^{-1} - \bar{\xi}(t).\zeta(t)^{-1}] = 0 \in L^p([a,b], \mathfrak{g}).\end{aligned}$$

Then, by the above, the curve $\eta \cdot \zeta^{-1}$ is constant, say $\eta \cdot \zeta^{-1} = g \in G$, thus $\eta = g\zeta$.

Conversely, assume $\eta = g\zeta$. We define $\eta_g \colon [a,b] \to G, t \mapsto g$ in $AC_{L^p}([a,b], G)$, then $[\gamma_g] = \delta(\eta_g) = 0 \in L^p([a,b], \mathfrak{g})$ (by the above), whence

$$\delta(\eta) = \delta(\eta_g \cdot \zeta) = [t \mapsto \xi(t)] = \delta(\zeta),$$

using Equation (4.17). □

The following proposition (a version of [17, Lemma 5.10]) will be useful.

Proposition 4.3.5. *Let G be a smooth Lie group, let E, F be locally convex spaces and $f\colon G\times E\to F$ be a C^{k+1}-function (for some $k\in\mathbb{N}_0\cup\{\infty\}$) which is linear in the second argument. Then for $p\in[1,\infty]$ the function*

$$C([a,b],G)\times L^p([a,b],E)\to L^p([a,b],F),\quad (\eta,[\gamma])\mapsto[f\circ(\eta,\gamma)] \tag{4.19}$$

is C^k.

Proof. The function is defined by Lemma 4.1.23. We fix some $\bar\eta\in C([a,b],G)$ and some open identity neighborhood $U\subseteq G$. Then U contains some open identity neighborhood W such that $WW\subseteq U$. The function in (4.19) will be C^k if the restriction

$$Q\times L^p([a,b],E)\to L^p([a,b],F),\quad (\eta,[\gamma])\mapsto[f\circ(\eta,\gamma)] \tag{4.20}$$

is C^k, where $Q:=\{\zeta\in C([a,b],G):\bar\eta^{-1}\cdot\zeta\in C([a,b],W)\}$ is an open neighborhood of $\bar\eta$.

Consider a partition $a=t_0<t_1<\ldots<t_n=b$ such that

$$\bar\eta(t_{j-1})^{-1}\bar\eta([t_{j-1},t_j])\subseteq W.$$

From Lemma 4.1.21, it follows that the above function will be C^k if

$$\begin{aligned} Q\times L^p([a,b],E)&\to\prod_{j=1}^n L^p([t_{j-1},t_j],F),\\ (\eta,[\gamma])&\mapsto\Big([f\circ(\eta,\gamma)|_{[t_{j-1},t_j]}]\Big)_{j=1,\ldots,n}\end{aligned}$$

is C^k, which will be the case if each component

$$Q\times L^p([a,b],E)\to L^p([t_{j-1},t_j],F),\quad (\eta,[\gamma])\mapsto[f\circ(\eta,\gamma)|_{[t_{j-1},t_j]}] \tag{4.21}$$

is C^k.

Now, by Lemmas 4.1.21 and 4.2.31, the function

$$\begin{aligned} Q\times L^p([a,b],E)&\to C([t_{j-1},t_j],G)\times L^p([t_{j-1},t_j],E),\\ (\eta,[\gamma])&\mapsto(\bar\eta(t_{j-1})^{-1}\eta|_{[t_{j-1},t_j]},[\gamma|_{[t_{j-1},t_j]}])\end{aligned}$$

is smooth; for $\eta\in Q$ and $t\in[t_{j-1},t_j]$ we have

$$\bar\eta(t_{j-1})^{-1}\eta(t)=\bar\eta(t_{j-1})^{-1}\bar\eta(t)\bar\eta(t)^{-1}\eta(t)\in WW\subseteq U.$$

Thus

$$\begin{aligned} Q\times L^p([a,b],E)&\to C([t_{j-1},t_j],V)\times L^p([t_{j-1},t_j],E),\\ (\eta,[\gamma])&\mapsto(\varphi\circ\bar\eta(t_{j-1})^{-1}\eta|_{[t_{j-1},t_j]},[\gamma|_{[t_{j-1},t_j]}])\end{aligned}$$

is smooth if $\varphi\colon U \to V$ is a chart for G. We define

$$g\colon V \times E \to F, \quad (x,y) \mapsto f(\bar{\eta}(t_{j-1})\varphi^{-1}(x), y),$$

which is C^{k+1} and linear in the second argument, and we use Proposition 4.1.25 to obtain a C^k-map

$$\begin{aligned} &Q \times L^p([a,b],E) \to L^p([t_{j-1},t_j],F),\\ &\qquad (\eta,[\gamma]) \mapsto [g \circ (\varphi \circ \bar{\eta}(t_{j-1})^{-1}\eta|_{[t_{j-1},t_j]}, \gamma|_{[t_{j-1},t_j]})] = [f \circ (\eta,\gamma)|_{[t_{j-1},t_j]}], \end{aligned}$$

which is exactly the required function from (4.21). □

We prove a version of [17, Lemma 5.29]:

Lemma 4.3.6. *Let G be a smooth Lie group modeled on a sequentially complete locally convex space E. The function*

$$\delta\colon AC_{L^p}([a,b],G) \to L^p([a,b],\mathfrak{g}), \quad \eta \mapsto \delta(\eta)$$

is smooth.

Proof. First we prove that the restriction $\delta|_{\tilde{U}}$ is smooth for some open identity neighborhood $\tilde{U} \subseteq AC_{L^p}([a,b],G)$. Let $U \subseteq G$ be an open e-neighborhood and $\varphi\colon U \to V$ be a chart for G. Then $\tilde{U} := AC_{L^p}([a,b],U)$ is an open identity neighborhood with chart $\tilde{\varphi} := AC_{L^p}([a,b],\varphi)\colon AC_{L^p}([a,b],U) \to AC_{L^p}([a,b],V)$. We have

$$\delta(\varphi^{-1} \circ \eta) = [\omega_l \circ T\varphi^{-1} \circ (\eta,\gamma)],$$

for $\eta \in AC_{L^p}([a,b],V)$, $\eta' = [\gamma]$. Now, the function

$$AC_{L^p}([a,b],V) \to C([a,b],V) \times L^p([a,b],E), \quad \eta \mapsto (\eta,\eta')$$

is smooth (see Lemma 4.2.9), as well as the function

$$C([a,b],V) \times L^p([a,b],E) \to L^p([a,b],\mathfrak{g}), \quad (\eta,[\gamma]) \mapsto [\omega_l \circ T\varphi^{-1} \circ (\eta,\gamma)],$$

since $\omega_l \circ T\varphi^{-1}\colon V \times E \to \mathfrak{g}$ is smooth and linear in the second argument (see Proposition 4.1.25). Consequently, the function

$$\delta|_{\tilde{U}} \circ \tilde{\varphi}^{-1}\colon AC_{L^p}([a,b],V) \to L^p([a,b],\mathfrak{g}), \quad \eta \mapsto \delta(\varphi^{-1} \circ \eta)$$

is smooth, thus the restriction $\delta|_{\tilde{U}}$ is smooth.

Now, we fix $\zeta \in AC_{L^p}([a,b],G)$ and show that $\delta|_{\tilde{U}\cdot\zeta}$ is smooth. Using Lemma 4.3.4, for $\eta \in \tilde{U}\cdot\zeta$ we have

$$\delta(\eta) = \delta((\eta\cdot\zeta^{-1})\cdot\zeta) = \zeta^{-1}.\delta(\eta\cdot\zeta^{-1}).\zeta + \delta(\zeta).$$

Clearly, the function

$$\tau\colon \tilde{U}\cdot\zeta \to \tilde{U}, \quad \eta \mapsto \eta\cdot\zeta^{-1}$$

is smooth, and so is the function

$$\delta|_{\tilde{U}} \circ \tau\colon \tilde{U}\cdot\zeta \to L^p([a,b],\mathfrak{g}), \quad \eta \mapsto \delta(\eta\cdot\zeta^{-1}),$$

by the previous step. Now, as

$$G\times\mathfrak{g} \to \mathfrak{g}, \quad (x,v) \mapsto x^{-1}.v.x$$

is smooth and linear in the second argument, the function

$$C([a,b],G)\times L^p([a,b],\mathfrak{g}) \to L^p([a,b],\mathfrak{g}), \quad (\eta,\gamma) \mapsto \eta^{-1}.\gamma.\eta$$

is smooth, by Proposition 4.3.5. Therefore, the function

$$L^p([a,b],\mathfrak{g}) \to L^p([a,b],\mathfrak{g}), \quad \gamma \mapsto \zeta^{-1}.\gamma.\zeta$$

is smooth. Altogether, using the smoothness of

$$L^p([a,b],\mathfrak{g}) \to L^p([a,b],\mathfrak{g}), \quad \gamma \mapsto \gamma+\delta(\zeta),$$

we conclude that

$$\delta|_{\tilde{U}\cdot\zeta}\colon \tilde{U}\cdot\zeta \to L^p([a,b],\mathfrak{g}), \quad \eta \mapsto \delta(\eta) = \zeta^{-1}.\delta(\eta\cdot\zeta^{-1}).\zeta+\delta(\zeta)$$

is smooth. Thus δ is smooth on $AC_{L^p}([a,b],G)$ and the proof is finished. □

Definition 4.3.7. Let G be a smooth Lie group modeled on a sequentially complete locally convex space. For $p\in[1,\infty]$, the Lie group G is called *L^p-semiregular* if for every $\gamma\in L^p([0,1],\mathfrak{g})$ the initial value problem

$$\delta(\eta)=\gamma, \quad \eta(0)=e \tag{4.22}$$

has a solution $\eta_\gamma \in AC_{L^p}([a,b],G)$ (which is unique, by Lemma 4.3.4).

An L^p-semiregular Lie group G is called *L^p-regular* if the function

$$\mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to AC_{L^p}([a,b],G), \quad \gamma \mapsto \eta_\gamma \tag{4.23}$$

is smooth.

Remark 4.3.8. As in [17, Remark 5.18], we note that if a Lie group G is L^p-regular, then the function

$$\mathrm{evol}\colon L^p([0,1],\mathfrak{g}) \to G, \quad \gamma \mapsto \mathrm{Evol}(\gamma)(1)$$

is smooth, since so is the evaluation map $\mathrm{ev}_1\colon AC_{L^p}([0,1],G)\to G, \eta\mapsto\eta(1)$ (see Lemma 4.2.30).

Consider a very useful property of the function Evol ([17, Proposition 5.20]).

Theorem 4.3.9. *Let G be an L^p-semiregular Lie group. Then the function* Evol *is smooth if and only if* Evol *is smooth as a function to $C([0,1],G)$.*

Proof. First assume that $\mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to AC_{L^p}([0,1],G)$ is smooth. As the inclusion map $\mathrm{incl}\colon AC_{L^p}([0,1],G) \to C([0,1],G)$ is smooth (see Lemma 4.2.29), the composition $\mathrm{incl} \circ \mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to C([0,1],G)$ is smooth.

Conversely, assume that $\mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to C([0,1],G)$ is smooth; for some fixed $\bar\gamma \in L^p([0,1],\mathfrak{g})$ we are going to find some open neighborhood P of $\bar\gamma$ such that the restriction $\mathrm{Evol}|_P\colon P \to AC_{L^p}([0,1],G)$ is smooth.

To this end, let $U \subseteq G$ be an open identity neighborhood and $\varphi\colon U \to V$ be a chart. Then U contains some open identity neighborhood W such that $WW \subseteq U$. For $\eta_{\bar\gamma} := \mathrm{Evol}(\bar\gamma)$, the subset

$$Q := \{\zeta \in C([0,1],G) : \eta_{\bar\gamma}^{-1} \cdot \zeta \in C([0,1],W)\}$$

is an open neighborhood of $\eta_{\bar\gamma}$. Set

$$P := \mathrm{Evol}^{-1}(Q).$$

Now, we want to show that the function

$$P \to AC_{L^p}([0,1],G), \quad \gamma \mapsto \eta_\gamma := \mathrm{Evol}(\gamma) \tag{4.24}$$

is smooth.

As $\eta_{\bar\gamma}$ is continuous, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\eta_{\bar\gamma}(t_j)^{-1}\eta_{\bar\gamma}([t_{j-1},t_j]) \subseteq W$ for each $j \in \{1,\ldots,n\}$. Using the function Γ_G from Lemma 4.2.31, the map in (4.24) will be smooth if

$$P \mapsto \prod_{j=1}^{n} AC_{L^p}([t_{j-1},t_j],G), \quad \gamma \mapsto \left(\eta_\gamma|_{[t_{j-1},t_j]}\right)_{j=1,\ldots,n} \tag{4.25}$$

is smooth, which will be the case if each of the components

$$P \mapsto AC_{L^p}([t_{j-1},t_j],G), \quad \gamma \mapsto \eta_\gamma|_{[t_{j-1},t_j]} \tag{4.26}$$

is smooth. As left translations on the Lie group $AC_{L^p}([t_{j-1},t_j],G)$ are smooth diffeomorphisms, the function in (4.26) will be smooth if

$$P \mapsto AC_{L^p}([t_{j-1},t_j],G), \quad \gamma \mapsto \eta_{\bar\gamma}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]} \tag{4.27}$$

is a smooth map.

Now, for every $t \in [t_{j-1},t_j]$ we have

$$\eta_{\bar\gamma}(t_j)^{-1}\eta_\gamma(t) = \eta_{\bar\gamma}(t_j)^{-1}\eta_{\bar\gamma}(t)\eta_{\bar\gamma}(t)^{-1}\eta_\gamma(t) \in WW \subseteq U,$$

in other words, $\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],U)$. Thus the smoothness of (4.27) will follow from the smoothness of

$$P \to AC_{L^p}([t_{j-1},t_j],E), \quad \gamma \mapsto \varphi \circ \eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]}. \tag{4.28}$$

Using the definition of the topology on $AC_{L^p}([t_{j-1},t_j],E)$ (see Definition 4.2.8), we will show that

$$P \to E \times L^p([t_{j-1},t_j],E), \quad \gamma \mapsto (\varphi(\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma(t_{j-1})), (\varphi \circ \eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]})')$$

is smooth.

Using the assumed smoothness of $P \to C([0,1],G), \gamma \mapsto \eta_\gamma$, we see that the first component of the above function is smooth. Therefore, it remains to show that

$$P \to L^p([t_{j-1},t_j],E), \quad \gamma \mapsto (\varphi \circ \eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]})' \tag{4.29}$$

is smooth.

Identifying equivalence classes with functions, we have

$$(\varphi \circ \eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]})' = d\varphi \circ (\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]})^{\cdot}.$$

Consider the smooth function

$$\sigma\colon G \times \mathfrak{g} \to TG, \quad (g,v) \mapsto g.v.$$

We have

$$\begin{aligned} &d\varphi \circ (\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]})^{\cdot} \\ &\qquad = d\varphi \circ \sigma \circ (\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]}, \delta(\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]})) \\ &\qquad = d\varphi \circ \sigma \circ (\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]}, \delta(\eta_\gamma|_{[t_{j-1},t_j]})) \\ &\qquad = d\varphi \circ \sigma \circ (\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]}, \gamma|_{[t_{j-1},t_j]}), \end{aligned}$$

using (iii) from Lemma 4.3.4. Hence the map in (4.29) will be smooth if

$$P \to L^p([t_{j-1},t_j],E), \quad \gamma \mapsto d\varphi \circ \sigma \circ (\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]}, \gamma|_{[t_{j-1},t_j]}) \tag{4.30}$$

is smooth. But this is true, the function being a composition of the smooth functions

$$P \to C([t_{j-1},t_j],U) \times L^p([t_{j-1},t_j],E), \quad \gamma \mapsto (\eta_{\bar{\gamma}}(t_j)^{-1}\eta_\gamma|_{[t_{j-1},t_j]}, \gamma|_{[t_{j-1},t_j]})$$

and

$$C([t_{j-1},t_j],U) \times L^p([t_{j-1},t_j],\mathfrak{g}) \to L^p([t_{j-1},t_j],\mathfrak{g}), \quad (\eta,\gamma) \mapsto d\varphi \circ \sigma \circ (\eta,\gamma),$$

(the smoothness of the last function holds by Proposition 4.3.5, as the composition $d\varphi \circ \sigma\colon G \times \mathfrak{g} \to E$ is linear in the second argument). □

As in [17, Corollary 5.21], we obtain the following result.

Theorem 4.3.10. *Let G be a Lie group and $p, q \in [1, \infty]$ with $q \geq p$. If G is L^p-regular, then G is L^q-regular. Furthermore, in this case G is C^0-regular.*

Proof. Assume that G is L^p-regular and $q \geq p$. Since $L^q([0,1], \mathfrak{g}) \subseteq L^p([0,1], \mathfrak{g})$ with a smooth inclusion map (Remark 4.1.18), the Lie group G is L^q-semiregular and the function $L^q([0,1], \mathfrak{g}) \to C([0,1], G), \gamma \mapsto \mathrm{Evol}(\gamma)$ is smooth. From Theorem 4.3.9, it follows that $L^q([0,1], \mathfrak{g}) \to AC_{L^q}([0,1], G), \gamma \mapsto \mathrm{Evol}(\gamma)$ is smooth, whence G is L^q-regular.

Further, since $C([0,1], \mathfrak{g}) \subseteq L^p([0,1], \mathfrak{g})$, the Lie group is C^0-semiregular. Since the inclusion map $\mathrm{incl}\colon C([0,1], \mathfrak{g}) \to L^p([0,1], \mathfrak{g})$ is smooth, as well as the evaluation map $\mathrm{ev}_1\colon C([0,1], G) \to G$, the composition $C([0,1], \mathfrak{g}) \to G, \gamma \mapsto \mathrm{Evol}(\gamma)(1)$ is smooth, whence G is C^0-regular. □

The following results will enable us to show that it suffices for a Lie group G to be L^p-regular, if it is merely *locally L^p-regular* (see [17, Definition 5.19, Proposition 5.25]).

Lemma 4.3.11. *For $c < d$ in $\mathbb{R}$ and $a \leq \alpha < \beta \leq b$ define*

$$f\colon [c,d] \to [a,b], \quad f(t) := \alpha + \frac{t-c}{d-c}(\beta - \alpha).$$

Let G be a Lie group modeled on a sequentially complete locally convex space E. Then the following holds:

(i) If $\gamma \in \mathcal{L}^p([a,b], E)$, then $\gamma \circ f \in \mathcal{L}^p([c,d], E)$ and the function

$$\mathcal{L}^p(f, E)\colon \mathcal{L}^p([a,b], E) \to \mathcal{L}^p([c,d], E), \quad \gamma \mapsto \gamma \circ f$$

is continuous and linear.

(ii) If $\eta \in AC_{L^p}([a,b], E)$, then $\eta \circ f \in AC_{L^p}([c,d], E)$ and

$$(\eta \circ f)' = \frac{\beta - \alpha}{d-c}[\gamma \circ f],$$

where $[\gamma] = \eta'$. Furthermore, the function

$$AC_{L^p}(f, E)\colon AC_{L^p}([a,b], E) \to AC_{L^p}([c,d], E), \quad \eta \mapsto \eta \circ f$$

is continuous and linear.

(iii) If $\eta \in AC_{L^p}([a,b], G)$, then $\eta \circ f \in AC_{L^p}([c,d], G)$ and

$$\delta(\eta \circ f) = \frac{\beta - \alpha}{d-c}[\gamma \circ f], \tag{4.31}$$

where $[\gamma] = \delta(\eta)$. Furthermore, the function

$$AC_{L^p}(f, G)\colon AC_{L^p}([a,b], G) \to AC_{L^p}([c,d], G), \quad \eta \mapsto \eta \circ f$$

is a smooth homomorphism.

Proof. (*i*) As f is continuous and $\lambda(f^{-1}(N)) = 0$ for every Borel set N with $\lambda(N) = 0$ (see [39, Theorem 3]), we can easily see that the composition $\gamma \circ f$ is measurable.

Assume first $p < \infty$. By [3, Satz 19.4], the function $q^p \circ (\gamma \circ f)$ is p-integrable for each continuous seminorm q on E, and

$$\int_c^d q(\gamma(f(t)))^p \, dt = \frac{d-c}{\beta-\alpha} \int_{f(c)}^{f(d)} q(\gamma(t))^p \, dt < \infty, \tag{4.32}$$

hence $\gamma \circ f \in \mathcal{L}^p([c,d], E)$. Furthermore, we see that

$$\|\gamma \circ f\|_{\mathcal{L}^p,q} \leq \left(\frac{d-c}{\beta-\alpha}\right)^{\frac{1}{p}} \|\gamma\|_{\mathcal{L}^p,q}, \tag{4.33}$$

whence the linear function $\mathcal{L}^p(f, E)$ is continuous.

Now, assume $p = \infty$. Then for every continuous seminorm q on E we have

$$\operatorname{ess\,sup}_{t\in[c,d]} q(\gamma(f(t))) \leq \operatorname{ess\,sup}_{t\in[a,b]} q(\gamma(t)) < \infty,$$

that is, $\gamma \circ f \in \mathcal{L}^\infty([c,d], E)$ and

$$\|\gamma \circ f\|_{\mathcal{L}^\infty,q} \leq \|\gamma\|_{\mathcal{L}^\infty,q}, \tag{4.34}$$

hence the linear map $\mathcal{L}^\infty(f, E)$ is continuous.

(*ii*) For $t \in [c,d]$ we have

$$\eta(f(t)) - \eta(f(c)) = \int_{f(c)}^{f(t)} \gamma(s) \, ds.$$

Then for any continuous linear functional $\mathcal{A}$ on E we have

$$\int_{f(c)}^{f(t)} \mathcal{A}(\gamma(s)) \, ds = \frac{\beta-\alpha}{d-c} \int_c^t \mathcal{A}(\gamma(f(s))) \, ds$$

(see [3, 19.4 Satz]), whence

$$\eta(f(t)) - \eta(f(c)) = \frac{\beta-\alpha}{d-c} \int_c^t \gamma(f(s)) \, ds,$$

in other words, $\eta \circ f \in AC_{L^p}([c,d], E)$ with $(\eta \circ f)' = \frac{\beta-\alpha}{d-c}[\gamma \circ f]$.

To prove the continuity of the linear function $AC_{L^p}(f, E)$, we show that

$$AC_{L^p}([a,b], E) \to E \times L^p([c,d], E), \quad \eta \mapsto (\eta(f(c)), (\eta \circ f)')$$

is continuous (where we used the isomorphism from Definition 4.2.8). The first component

$$AC_{L^p}([a,b], E) \to E, \quad \eta \mapsto \operatorname{ev}_{f(c)}(\eta)$$

is continuous, by Remark 4.2.11. Further, the map

$$\Psi\colon L^p([a,b],E) \to L^p([c,d],E), \quad [\gamma] \mapsto \frac{\beta-\alpha}{d-c}[\gamma \circ f]$$

is continuous, hence the second component

$$AC_{L^p}([a,b],E) \to L^p([c,d],E), \quad \eta \mapsto \Psi(\eta') = (\eta \circ f)'$$

is continuous.

(*iii*) As $\eta \circ f$ is a continuous curve, there exists a partition $c = t_0 < t_1 < \ldots < t_n = d$ and for every $j \in \{1,\ldots,n\}$ there is a chart $\varphi_j\colon U_j \to V_j$ for G such that $\eta(f([t_{j-1},t_j])) \subseteq U_j$. But $f([t_{j-1},t_j]) = [f(t_{j-1}), f(t_j)]$ is an interval and from Lemma 4.2.21, it follows that

$$\varphi_j \circ \eta|_{[f(t_{j-1}),f(t_j)]} \in AC_{L^p}([f(t_{j-1}),f(t_j)],V_j).$$

We have

$$\varphi_j \circ \eta \circ f|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1},t_j],V_j),$$

that is, $\eta \circ f \in AC_{L^p}([c,d],G)$.

Next, consider a partition $c = t_0 < t_1 < \ldots < t_n = d$ and charts $\varphi\colon U_j \to V_j$ with $\eta(f([t_{j-1},t_j])) \subseteq U_j$. Write

$$f_j := f|_{[t_{j-1},t_j]}, \quad \eta_j := \eta|_{[f(t_{j-1}),f(t_j)]}.$$

Identifying equivalence classes with functions, we obtain

$$\begin{aligned}
\delta(\eta \circ f)|_{[t_{j-1},t_j]} &= \omega_l \circ T\varphi_j^{-1} \circ (\varphi_j \circ \eta \circ f_j, (\varphi_j \circ \eta \circ f_j)') \\
&= \omega_l \circ T\varphi_j^{-1} \circ (\varphi_j \circ \eta \circ f_j, \frac{\beta-\alpha}{d-c}(\varphi_j \circ \eta_j)' \circ f_j) \\
&= \omega_l \circ T\varphi_j^{-1} \circ (\varphi \circ \eta_j, \frac{\beta-\alpha}{d-c}(\varphi_j \circ \eta_j)') \circ f_j \\
&= \frac{\beta-\alpha}{d-c}\Big(\omega_l \circ T\varphi_j^{-1} \circ (\varphi \circ \eta_j, (\varphi_j \circ \eta_j)') \circ f_j\Big) \\
&= \frac{\beta-\alpha}{d-c}(\delta(\eta) \circ f|_{[t_{j-1},t_j]}),
\end{aligned}$$

using the formula in (*ii*) and the linearity of $\omega_l \circ T\varphi_j^{-1}$ in its second argument.

Finally, for any open identity neighborhood $U \subseteq G$ and any chart $\varphi\colon U \to V$ for G the function

$$\begin{aligned}
&AC_{L^p}([a,b],V) \to AC_{L^p}([c,d],V), \\
&\qquad \zeta \mapsto (AC_{L^p}([c,d],\varphi) \circ AC_{L^p}(f,G) \circ AC_{L^p}([a,b],\varphi)^{-1})(\zeta) = \zeta \circ f
\end{aligned}$$

is smooth, hence the group homomorphism $AC_{L^p}(f,G)$ is smooth. □

The following lemma shows that the L^p-spaces have the *subdivision property* [17, Lemma 5.26].

Lemma 4.3.12. *Let E be a locally convex space, let $\gamma \in \mathcal{L}^p([0,1],E)$. For $n \in \mathbb{N}$ and $k \in \{0,\dots,n-1\}$ define*

$$\gamma_{n,k}\colon [0,1] \to E, \quad \gamma_{n,k}(t) := \frac{1}{n}\gamma\left(\frac{k+t}{n}\right). \tag{4.35}$$

Then $\gamma_{n,k} \in \mathcal{L}^p([0,1],E)$ for every n,k and

$$\lim_{n\to\infty} \max_{k\in\{0,\dots,n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^p,q} = 0$$

for each continuous seminorm q on E.

More generally, the same holds for $\gamma \in \mathcal{L}^p([a,b],E)$ and

$$\gamma_{n,k}\colon [a,b] \to E, \quad \gamma_{n,k}(t) := \frac{1}{n}\gamma\left(a + \frac{k(b-a)+t-a}{n}\right).$$

Proof. The functions $f_{n,k}\colon [0,1] \to [k/n, k+1/n]$, $f_{n,k}(t) := k+t/n$ are as in Lemma 4.3.11, hence $\gamma_{n,k} = 1/n(\gamma \circ f_{n,k}) \in \mathcal{L}^p([0,1],E)$.

Further, for fixed $n \in \mathbb{N}$ and $p = \infty$ we have

$$\|\gamma_{n,k}\|_{\mathcal{L}^\infty,q} = \frac{1}{n}\|\gamma \circ f_{n,k}\|_{\mathcal{L}^\infty,q} \leq \frac{1}{n}\|\gamma\|_{\mathcal{L}^\infty,q}$$

for every continuous seminorm q on E and every $k \in \{0,\dots,n-1\}$, by (4.34). Hence

$$\max_{k\in\{0,\dots,n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^\infty,q} \leq \frac{1}{n}\|\gamma\|_{\mathcal{L}^\infty,q} \to 0$$

as $n \to \infty$.

Now, if $2 \leq p < \infty$, then for $n \in \mathbb{N}$ and a continuous seminorm q on E we have

$$\|\gamma_{n,k}\|_{\mathcal{L}^p,q} = \frac{1}{n}\|\gamma \circ f_{n,k}\|_{\mathcal{L}^p,q} \leq \frac{n^{\frac{1}{p}}}{n}\|\gamma\|_{\mathcal{L}^p,q} = n^{\frac{1}{p}-1}\|\gamma\|_{\mathcal{L}^p,q},$$

for each $k \in \{0,\dots,n-1\}$, by (4.33). Hence

$$\max_{k\in\{0,\dots,n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^p,q} \leq n^{\frac{1}{p}-1}\|\gamma\|_{\mathcal{L}^p,q} \to 0$$

as $n \to \infty$.

Finally, let $p = 1$. Fix some $\varepsilon > 0$ and a continuous seminorm q on E. Each of the sets

$$A_m := \{t \in [a,b] : q(\gamma(t)) > m$$

are in $\mathcal{B}([0,1])$ and

$$\lim_{m\to\infty}\int_{A_m} q(\gamma(t))\,dt = \int_{\bigcap_{m\in\mathbb{N}} A_m} q(\gamma(t))\,dt = 0,$$

because $(A_m)_{m\in\mathbb{N}}$ is a decreasing sequence and $\bigcap_{m\in\mathbb{N}} A_m = \emptyset$. Therefore, for some $m \in \mathbb{N}$ we have

$$\int_{A_m} q(\gamma(t))\,dt < \frac{\varepsilon}{2}.$$

We fix some $N \in \mathbb{N}$ such that $m/N < \varepsilon/2$ and for every $n \geq N$ we define

$$A_{n,k} := \{t \in [0,1] : f_{n,k}(t) \in A_m\}$$

Then

$$\int_{A_{n,k}} q(\gamma_{n,k}(t))\,dt = \frac{1}{n}\int_{A_{n,k}} q(\gamma(f_{n,k}(t)))\,dt = \int_{f_{n,k}(A_{n,k})} q(\gamma(t))\,dt,$$

by Equation (4.32). Since $f_{n,k}(A_{n,k}) = A_m \cap [k/n, k+1/n]$, we obtain

$$\int_{f_{n,k}(A_{n,k})} q(\gamma(t))\,dt \leq \int_{A_m} q(f(t))\,dt < \frac{\varepsilon}{2},$$

by the choice of m. Further

$$\|\gamma_{n,k}\|_{\mathcal{L}^1,q} = \int_0^1 q(\gamma_{n,k}(t))\,dt = \int_{A_{n,k}} q(\gamma_{n,k}(t))\,dt + \int_{[0,1]\setminus A_{n,k}} q(\gamma_{n,k}(t))\,dt < \varepsilon,$$

because $q(\gamma_{n,k}(t)) = 1/n\, q(\gamma(f_{n,k}(t))) \leq m/n < \varepsilon/2$ for $t \in [0,1] \setminus A_{n,k}$. Consequently,

$$\max_{k\in\{0,\ldots,n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^1,q} < \varepsilon,$$

in other words, $\max_{k\in\{0,\ldots,n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^1,q} \to 0$ as $n \to \infty$, as required. □

Finally, we prove that a Lie group is L^p-regular if it is *locally L^p-regular* [17, Proposition 5.25].

Theorem 4.3.13. *Let G be a Lie group modeled on a sequentially complete locally convex space E, let $\mathfrak{g}$ denote the Lie algebra of G. Let $\Omega \subseteq L^p([0,1],\mathfrak{g})$ be an open 0-neighbourhood. If for every $\gamma \in \Omega$ the initial value problem (4.22) has a (necessarily unique) solution $\eta_\gamma \in AC_{L^p}([0,1],G)$, then G is L^p-semiregular. If, in addition, the function* $\mathrm{Evol}\colon \Omega \to AC_{L^p}([0,1],G), \gamma \mapsto \eta_\gamma$ *is smooth, then G is L^p-regular.*

Proof. First, fix some $\gamma \in L^p([0,1],\mathfrak{g})$ and for $n \in \mathbb{N}$, $k \in \{0,\ldots,n-1\}$ define $\gamma_{n,k} \in L^p([0,1],\mathfrak{g})$ as in (4.35). Let Q be a continuous seminorm on $L^p([0,1],\mathfrak{g})$ such that

$B_1^Q(0) \subseteq \Omega$. By Lemma 4.3.12, there exists some $n \in \mathbb{N}$ such that $\gamma_{n,k} \in \Omega$ for $k \in \{0, \ldots, n-1\}$. We set $\eta_{n,k} := \mathrm{Evol}(\gamma_{n,k}) \in AC_{L^p}([0,1], G)$ and define $\eta_\gamma \colon [0,1] \to G$ via

$$\eta_\gamma(t) := (\eta_{n,0} \circ f_{n,0})(t), \quad \text{for } t \in [0, 1/n], \tag{4.36}$$

and

$$\eta_\gamma(t) := \eta_{n,0}(1) \cdots \eta_{n,k-1}(1)(\eta_{n,k} \circ f_{n,k})(t), \quad \text{for } t \in [k/n, k+1/n], \tag{4.37}$$

where

$$f_{n,k} \colon [k/n, k+1/n] \to [0,1], \quad f_{n,k}(t) := nt - k.$$

Then we easily verify that the function η_γ is continuous and from Lemma 4.3.11, it follows that $\eta_\gamma|_{[k/n, k+1/n]} \in AC_{L^p}([k/n, k+1/n], G)$, whence $\eta_\gamma \in AC_{L^p}([0,1], G)$. Furthermore, $\eta_\gamma(0) = e$ and $\delta(\eta_\gamma) = \gamma$. Consequently, $\mathrm{Evol}(\gamma) := \eta_\gamma$ solves the initial value problem in (4.22) for γ, whence G is L^p-semiregular.

Now, assume that $\mathrm{Evol} \colon \Omega \to AC_{L^p}([0,1], G)$ is smooth; we will show the smoothness of Evol on some open neighborhood of γ. From the continuity of each

$$\pi_{n,k} \colon L^p([0,1], \mathfrak{g}) \to L^p([0,1], \mathfrak{g}), \quad \xi \mapsto \xi_{n,k},$$

(see Lemma 4.3.11), it follows that there exists an open neighborhood $W \subseteq L^p([0,1], \mathfrak{g})$ of γ such that $\pi_{n,k}(W) \subseteq \Omega$ for every $k \in \{0, \ldots, n-1\}$. Then

$$\mathrm{Evol} \colon W \to AC_{L^p}([0,1], G), \quad \xi \mapsto \eta_\xi$$

is defined, where η_ξ is as in (4.36) and (4.37). It will be smooth if we show (using Lemma 4.2.31) that each

$$W \to AC_{L^p}([k/n, k+1/n], G), \quad \xi \mapsto \eta_\xi|_{[k/n, k+1/n]} \tag{4.38}$$

is smooth. But, by construction, we have

$$\eta_\xi|_{[0, 1/n]} = \mathrm{Evol}(\xi_{n,0}) \circ f_{n,0}$$

and

$$\eta_\xi|_{[k/n, k+1/n]} = \mathrm{evol}(\xi_{n,0}) \cdots \mathrm{evol}(\xi_{n,k-1}) \, \mathrm{Evol}(\xi_{n,k}) \circ f_{n,k},$$

so the smoothness of (4.38) follows from Lemma 4.3.11 and Remark 4.3.8. □

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