Facets of low regularity in cross-diffusive systems

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Abstract

This work examines various systems of parabolic differential equations with nondiagonal diffusion matrices inter alia originating in biology. The destabilizing nature of the nondiagonal entries, the so-called cross-diffusion terms, is well-known; in fact, for none of the systems covered here, unconditional global existence results for classical solutions should be expected.

The low regularity of cross-diffusive systems can essentially be dealt with in two ways, both of which we explore for certain examples in this thesis. While the first one consists of rigorously showing that certain classical solutions blow up in finite time, thereby putting limits to the extent of potential global existence theorems, the second one aims to construct global solutions despite these challenges, either under certain additional assumptions (say, on the initial data) or in a more generalized sense.

In a second step, we then proceed to ask further qualitative and quantitative questions concerning these solutions. In particular, we describe their behavior at large times (if they are global-in-time) or near their blow-up time (if they are not).

Zusammenfassung

Die vorliegende Arbeit untersucht verschiedene Systeme parabolischer Differentialgleichungen mit nichtdiagonalen Diffusionsmatrizen, welche ihren Ursprung unter anderem in der Biologie haben. Der destabilisierende Effekt der Nichtdiagonal-Einträge, also der sogenannten Kreuz-Diffusions-Terme, ist wohlbekannt; insbesondere kann für keines der hier betrachteten Systeme ein bedingungsloses Globales-Existenz-Resultat erwartet werden.

Der Problematik geringer Regularität in kreuz-diffusiven Systemen lässt sich im Wesentlichen auf zwei Wegen nähern, welche wir beide für gewisse Beispiel-Probleme verfolgen. Während der erste aus dem Nachweis von in endlicher Zeit explodierenden Lösungen besteht, also daraus, Grenzen möglicher Resultate betreffend globaler Existenz aufzuzeigen, versucht der zweite nichtsdestotrotz globale Lösungen zu konstruieren, sowohl unter zusätzlichen Annahmen (beispielsweise an die Anfangsdaten) als auch in gewissen verallgemeinerten Sinnen.

In einem zweitem Schritt fragen wir dann nach weiteren qualitativen and quantitativen Eigenschaften dieser Lösungen. Insbesondere beschreiben wir deren Verhalten für große Zeiten (sofern sie global existieren) beziehungsweise nahe ihrer Explosionszeit (falls das nicht der Fall ist).

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1. Introduction

Parabolic differential equations can be used to describe a variety of natural phenomena such as pattern formation in *E. coli* colonies, embryogenesis or population dynamics, to just name a few examples from biology. Accordingly, there is a desire to understand these systems mathematically, both in order to be able to give predictions for the future and to validate the assumptions the models are based on.

Many of these equations approximatively take the form

$$U_t = \nabla \cdot (A(U)\nabla U) + F(U),$$

where U is a vector-valued (unknown) function whose components may for instance represent densities of biological organisms or chemical substances, U_t denotes its time derivative, A is the so-called *diffusion matrix* and F models local kinetics and external forces.

Let us discuss the influence of the diffusion matrix A. Positive diagonal entries model diffusive processes such as heat dissipation or random motion of organisms. If only these effects are present—that is, if A is a (positive definite) diagonal matrix—, the corresponding equations are known to be quite regular. For instance, in cases where F is bounded, global well-posedness of associated initial-boundary-value-problems follows from straightforwardly applying well-known energy methods.

However, many natural processes are not adequately described by such diffusion matrices. To give an example, apart from undergoing random motion, bacteria (with density U_1) may also be partly attracted by higher concentrations of a chemical substance (whose density is denoted by U_2). This can be modelled by the so-called *attractive taxis term* $-\nabla \cdot (U_1 \nabla U_2)$ in the first subequation; that is, by choosing a nontrivial nondiagonal entry in A, namely $A_{12}(U) \coloneqq -U_1$. Such an effect, where the gradient of a concentration influences the flux of *another* (chemical or biological) species, is called *cross-diffusion.*

While in certain situations desirable from a modeling perspective, cross-diffusive terms generally lead to lower regularity, making such systems particularly challenging to handle. Even questions of global existence are already quite delicate; in contrast to, say, systems with diagonal diffusion matrices, no general global existence theory seems to be available. Worse, for all problems treated in this thesis, unconditional global existence of classical solutions, that is, of functions solving the equations pointwise, cannot be expected.

There are two ways out of this apparent dilemma, both of which we explore for certain example problems in this thesis. The first one consists of embracing the low regularity and rigorously showing that global classical solutions may fail to exist. Accordingly, in the first part of this thesis, we concern ourselves with one the most drastic forms of pattern formation: *finite-time blow-up.* Classical solutions becoming unbounded in finite time are known to exist for instance for the *simplified Keller–Segel system*

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) \\ 0 = \Delta v - \overline{m}(t) + u, \quad \overline{m}(t) \coloneqq \frac{1}{|\Omega|} \int_\Omega u(\cdot, t) \end{cases}$$

with nondiagonal diffusion matrix

$$A\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}1 & -u\\0 & 1\end{pmatrix},$$

describing the behavior of bacteria attracted by a chemical they produce themselves. A (formerly) long-standing open question asks whether introducing a logistic term $\lambda u - \mu u^2$ with positive λ and μ in the first equation, modeling intrinsic bacteria growth and death, already guarantees that solutions remain locally bounded. Despite its brevity, Chapter 2 is able to provide a (negative) answer: For five and higher dimensional balls, $\lambda \geq 0$ and $\mu \in (0, \frac{n-4}{n})$, we are able to construct solutions blowing up in finite time.

In the next two chapters, we then concern ourselves with qualitative and quantitative properties of nonglobal solutions, including but not limited to the ones constructed in Chapter 2. Also allowing for nonlinear diffusion terms and taxis sensitivities, we show the existence of blow-up profiles for finite-time blow-up solutions of

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - S(u,v)\nabla v), \\ v_t = \Delta v - v + u \end{cases}$$

and similar systems in Chapter 3; that is, we prove that $(u(\cdot,t), v(\cdot,t))$ converges in a suitable sense as t approaches the maximal (finite) existence time. This result is accompanied by pointwise upper estimates of the first component (of both the solution and the blow-up profile), which not only quantitatively describe the behavior of such solutions blowing up in finite time, but also provide a useful tool for constructing them in the first place. Indeed, these findings have already been used as a key ingredient in further articles detecting finite-time blow-up, which we reference in the introduction of that chapter.

Moreover, in Chapter 4, we ask the question when one can guarantee optimality of the estimates obtained in Chapter 3. The main result states that if u is bounded in an optimal L^p space (a concept defined in the introduction of that chapter), then these upper estimates are also essentially optimal. In addition, we also provide estimates of similar flavor as in Chapter 3 for chemotaxis systems with nonlinear signal production.

The second way of dealing with low regularity in cross-diffusive systems reflects the desire to obtain global solutions (for instance, in order to be able to discuss the behavior at large times) even in situations where unconditional existence results for global classical solutions seem to be out of reach or are, as corresponding finite-time blow-up results show, impossible to obtain. This way again junctions into two further paths: One can aim to construct global solutions either under additional conditions or in some generalized sense.

With these ideas in mind, we analyze (variants of) the so-called *pursuit–evasion model*

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla v) + f_1(u,v), \\ v_t = \nabla \cdot (\nabla v + v\nabla u) + f_2(u,v) \end{cases}$$

in the second part of this thesis. This system describes the interaction between predators and their prey, whose densities are denoted by u and v, respectively. The key feature of this problem is that cross-diffusion is not only present in one but in both equations; that is, both nondiagonal entries of the diffusion matrix

$$A\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}1 & -u\\v & 1\end{pmatrix}$$

are nontrivial. While that is very sensible from a modeling perspective—after all, predators desire to move towards prey-rich regions *and* the prey seeks to avoid its predators—, from a mathematical point of view, such fully cross-diffusive systems pose even more challenges than their 'merely' single cross-diffusive counterparts.

Nonetheless, we are able to prove two global existence results. The first one is obtained in Chapter 5, where we establish the existence of global classical solutions for initial data being sufficiently close to homogeneous steady states. As already alluded to, global existence results can be seen as a prerequisite for analyzing the large time behavior and we are indeed able to go beyond these findings and additionally show convergence towards the equilibria and describe the rate of convergence.

In Chapter 6, we then extend this model to also allow for nonlinear diffusion and saturated taxis sensitivities. Under certain parameter assumptions, we are able to construct global weak solution for widely arbitrary (and possibly large or even unbounded) nonnegative initial data.

Further biological motivation for these systems, comparisons to as well as discussion of relevant literature and the precise statements of our theorems are given in the introductions of the corresponding chapters. While especially in these sections, we aim to avoid unnecessary repetition and instead refer to the introduction of another chapter whenever that is sensible, the avoidance of recurrences of similar arguments (which due to the rather different nature of the methods employed are quite rare in any case) is not taken to an extreme, allowing us to ensure that the chapters can still be read independently from each other.

1.1. Previous publications

Except for small modifications mainly in the introductory sections, the succeeding chapters coincide with the following publications. Accordingly, quotations from these works will not be marked separately.

Chapter 2:
[23]: Fuest, M.: *Approaching optimality in blow-up results for Keller–Segel systems with logistic-type dampening.* Nonlinear Differ. Equ. Appl. NoDEA, 28(2):16, 2021.

Chapter 3:
[20]: Fuest, M. *Blow-up profiles in quasilinear fully parabolic Keller–Segel systems.* Nonlinearity 33(5):2306–2334, 2020.

Chapter 4:
[25]: Fuest, M. *On the optimality of upper estimates near blow-up in quasilinear Keller–Segel systems.* Appl. Anal., to appear.

Chapter 5:
[22]: Fuest, M. *Global solutions near homogeneous steady states in a multidimensional population model with both predator- and prey-taxis.* SIAM J. Math. Anal., 52(6):5865–5891, 2020.

Chapter 6:
[24] Fuest, M.: *Global weak solutions to fully cross-diffusive systems with nonlinear diffusion and saturated taxis sensitivity.* Preprint, arXiv:2105.12619, 2021.

Contents

Part I.

Solutions blowing up in finite-time and their blow-up profiles

2. Approaching optimality in blow-up results for Keller–Segel systems with logistic-type dampening

2.1. Introduction

A considerable amount of the literature on chemotaxis systems deals with detecting critical parameters distinguishing between global existence and finite-time blow-up. Such a dichotomy is already present in the *minimal Keller–Segel system*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u \end{cases} \tag{2.1.1}$$

proposed by Keller and Segel to model chemotactic behavior of bacteria attracted by a chemical substance they produce themselves [46]. Considered in two-dimensional balls, the mass of u_0 is critical: If the initial datum u_0 is sufficiently regular, radially symmetric and satisfies $\int_\Omega u_0 < 8\pi$, then the corresponding solutions are global in time and bounded [68] while for any $m_0 > 8\pi$, there exists $u_0 \in C^0(\overline{\Omega})$ with $\int_\Omega u_0 = m_0$ leading to finite-time blow-up [34, 62]. (See also [66] for corresponding results in a parabolic–elliptic simplification of (2.1.1).) Let us note that this specific critical mass phenomenon is limited to the two-dimensional setting: While solutions to (2.1.1) are always global in time and bounded if considered in one-dimensional domains [71], in the spatially higher dimensional cases, finite-time blow-up has been detected even for arbitrary positive initial masses [97].

Other dichotomies between boundedness and blow-up include critical exponents both for nonlinear diffusion as well as nonlinear sensitivity [37] and nonlinear signal production [102]. Moreover, for a chemotaxis system with indirect signal production, another critical mass phenomenon has been detected in [84], this time distinguishing between boundedness and blow-up in infinite time. Instead of presenting these findings in detail here, we refer to the surveys [4] and [53] for a broader overview of chemotaxis systems and related results.

Aiming to further enhance our understanding of the exact strength of the destabilising taxis term, in this chapter, we present another critical parameter distinguishing between global existence and finite-time blow-up, namely the exponent $\kappa = 2$ in Keller–Segel systems with logistic-type degradation.

Before stating the main result of this chapter, let us introduce systems featuring such dampening terms and recall some of the corresponding results. That is, we will first consider the *Keller–Segel system with logistic source*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^2, \\ \tau v_t = \Delta v - v + u \end{cases} \tag{2.1.2}$$

in smooth, bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and given parameters $\lambda \in \mathbb{R}$, $\mu > 0$ and $\tau \geq 0$. (We note that in view of the global existence result for $\lambda = \mu = 0$ in one-dimensional domains mentioned above, at least for the question whether finite-time blow-up occurs, we may confine ourselves to the assumption $n \geq 2$.) The system (2.1.2) and variations thereof describe several biological processes such as population dynamics [36, 78], pattern formation [109] or embryogenesis [73] (see also [36] for an overview).

Already in 2007, Tello and Winkler showed that for $\tau = 0$, any $\lambda \in \mathbb{R}, \mu > \frac{n-2}{n}$ and any reasonably smooth initial data, the system (2.1.2) possesses global, bounded classical solutions [87]. Moreover, for $n \geq 3$ and $\mu = \frac{n-2}{n}$ (and again $\tau = 0$ and at least $\lambda \geq 0$) solutions to (2.1.2) are global in time [43], but to the best of our knowledge it is unknown whether these are also always bounded. For the parabolic–parabolic case, that is, for $\tau > 0$, the situation is similar: In the two-dimensional setting, assuming merely $\mu > 0$ suffices to guarantee global existence of classical solutions [70], even for dampening terms growing slightly slower then quadratically [116]. Moreover, for higher dimensional convex domains, global classical solutions have been constructed for $\mu > \mu_0$ for some $\mu_0 > 0$ in [95], where explicit upper bounds of μ_0 then have been derived in [65, 115] and the convexity assumption has been removed in [113] at the cost of worsening the condition on μ. In all these settings, however, the known upper bounds for μ_0 are larger than $\frac{n-2}{n}$.

However, if one resorts to more general solution concepts, further existence results are available. Under rather mild conditions, global weak solutions have been constructed in [87] and [50] for the cases $\tau = 0$ and $\tau = 1$, respectively. Moreover, if the degradation term $-\mu u^2$ in (2.1.2) is replaced by a weaker but still sufficiently strong superlinear dampening term, global generalized solutions have been obtained, again both for the parabolic–elliptic [92] and the fully parabolic case [105, 107, 117].

On the other hand, it has been observed that despite the presence of quadratic dampening terms, structures may form on intermediate time scales which even surpass so-called population thresholds to an arbitrary high extent (cf. [43, 49, 98] for the parabolic–elliptic and [100] for the parabolic–parabolic case).

While these findings already show that the destabilising effect of the chemotaxis term is strongly countered although not completely nullified by quadratic degradation terms, the question arises whether the most drastic form of spatial aggregation—finite-time blow-up—still occurs in Keller–Segel systems with superlinear degradation terms. A first partial (and affirmative) answer has been given in [96]: There, the compared to (2.1.2) with $\tau = 0$ slightly simplified system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, \\ 0 = \Delta v - \overline{m}(t) + u, \quad \overline{m}(t) \coloneqq \frac{1}{|\Omega|} \int_\Omega u(\cdot, t) \end{cases} \tag{2.1.3}$$

is considered in balls $\Omega \subset \mathbb{R}^n$, $n \geq 5$ and, for any $\lambda \geq 0$ and $\kappa \in (1, \frac{3}{2} + \frac{1}{2(n-1)})$, initial data leading to finite-time blow-up are constructed. The second important finding in this direction transfers this result to physically meaningful space dimensions. More concretely, [103] detects finite-time blow-up even in the system (2.1.2) with $\tau = 0$ (inter alia) for balls $\Omega \in \mathbb{R}^n$, $n \in \{3, 4\}$, $\lambda \geq 0$ and $\kappa \in (1, \frac{7}{6})$.

Recently, the regime of exponents allowing for finite-time blow-up in (2.1.3) has been further widened to $\kappa \in (1, \frac{4}{3})$ and $\kappa \in (1, \frac{3}{2})$ in the three- and four-dimensional settings, respectively [5]. Moreover, in planar domains, chemotactic collapse can be obtained if one replaces the term $-u^\kappa$ in (2.1.3) with certain heterogeneous dampening terms such as $-|x|^2 u^2$ [21].

Let us additionally note that similar finite-time blow-up results are also available for systems with nonlinear diffusion [5, 60, 80] or sublinear taxis sensitivity [80, 81].

Main results. At least in the four- and higher dimensional settings, the journey of detecting finite-time blow-up in (2.1.3) for ever increasing values of κ comes to an end with the results from the present chapter; we obtain the corresponding result up to (and for $n \geq 5$ even including) the optimal exponent $\kappa = 2$.

More precisely, the main result of this chapter reads

Theorem 2.1.1. *Suppose*

$$n \geq 3, \quad \kappa \in \left(1, \min\left\{2, \frac{n}{2}\right\}\right) \quad \text{and} \quad \mu > 0 \tag{2.1.4a}$$

$$\text{or} \quad n \geq 5, \quad \kappa = 2 \quad \text{and} \quad \mu \in \left(0, \frac{n-4}{n}\right). \tag{2.1.4b}$$

Moreover, let $R > 0$, $\Omega := B_R(0) \subset \mathbb{R}^n$, $m_0 > 0$, $m_1 \in (0, m_0)$ and $\lambda \geq 0$. Then there exists $r_1 \in (0, R)$ such that whenever

$$u_0 \in C^1(\overline{\Omega}) \quad \text{is positive, radially symmetric as well as radially decreasing} \tag{2.1.5}$$

and fulfills

$$\int_\Omega u_0 = m_0 \quad \text{as well as} \quad \int_{B_{r_1}(0)} u_0 \geq m_1, \tag{2.1.6}$$

the following holds: There exist $T_{\max} < \infty$ and a classical solution

$$(u, v) \in \left(C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\right)^2 \tag{2.1.7}$$

of

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa & \text{in } \Omega \times (0, T_{\max}), \\ 0 = \Delta v - \overline{m}(t) + u, \quad \overline{m}(t) := \frac{1}{|\Omega|} \int_\Omega u(\cdot, t) & \text{in } \Omega \times (0, T_{\max}), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.1.8}$$

which blows up at $T_{\max}$ in the sense that $\lim_{t \nearrow T_{\max}} u(0, t) = \infty$.

Main ideas. Following Jäger and Luckhaus [39], we rely on the mass accumulation function given by $w(s, t) := \int_0^{\sqrt[n]{s}} \rho^{n-1} u(\rho, t) \,\mathrm{d}\rho$, which transforms (2.1.8) to a scalar equation, see Lemma 2.3.1. The predecessors [5] and [103] of this chapter, which deal with (variations of) the system (2.1.3), then proceed to show that the function ϕ defined by

$$\phi(s_0, t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s, t) \,\mathrm{d}s \tag{2.1.9}$$

cannot, at least not for certain initial data, $s_0 \in (0, R^n)$ and $\gamma \in (0, 1)$, exist globally in time, implying that u must blow up in finite time. One of the most challenging terms to estimate arises from the degradation term; one essentially has to control the integral

$\int_0^{s_0} w_s^\kappa(s,t)\,\mathrm{d}s$. At this point, pointwise estimates for w_s come in handy, which due to the identity $w_s(s,t) = \frac{1}{n}u(s^{\frac{1}{n}},t)$ are available once pointwise estimates for u are known. These in turn can for instance be obtained by analyzing general parabolic equations in divergence form (cf. [106] and Chapter 3) or by arguments similar to Lemma 2.3.3 below. In fact, one of the main points in [5] is to discuss how pointwise upper estimates for u of the form $u(x) \le C|x|^{-p}$ influence the possibility to detect finite-time blow-up.

However, a natural limitation of this approach is the exponent $p = n$ since for fixed $C > 0$ and $p < n$, nonnegative functions $u_0 \in C^0(\overline{\Omega})$ with $u_0(x) \le C|x|^{-p}$ cannot have their mass concentrated arbitrary close to the origin; that is, depending on the value of C and p, none of these functions may fulfill (2.1.6). However, as seen in [5], even the choice $p = n$, implying an estimate of the form $w_s \le \frac{C}{s}$, 'only' yields finite-time blow-up in the system (2.1.3) for certain $\kappa < \frac{3}{2}$.

Thus, in the present chapter, where we handle exponents up to $\kappa = 2$, we choose a slightly different path. At the basis of our analysis stands Lemma 2.3.3: There, we derive the key estimate $w_s \le \frac{w}{s}$, which due to $w(0,\cdot) \equiv 0$ actually improves on $w_s \le \frac{C}{s}$. Its proof is surprisingly simple: As already observed in similar contexts (cf. [5, 21, 102]), for radially decreasing initial data, $w_s(\cdot,t)$ is decreasing for all times t, see Lemma 2.3.2. The desired estimate is then just a consequence of the mean value theorem.

Another major difference of our methods compared to [5] and [103] is that we do not limit our analysis of (2.1.9) to $\gamma \in (0,1)$ but also allow for parameters γ being larger than 1. In the five- and higher dimensional settings, this will then allow us to obtain finite-time blow-up even for $\kappa = 2$. (In 3D and 4D, the term stemming from the diffusion forces γ to be smaller than 1 and hence we cannot employ the same method as in higher dimensions.) We also note that the realization of the idea of taking $\gamma > 1$ is made possible by the new crucial estimate $w_s \le \frac{w}{s}$.

The remainder of the chapter is organized as follows: After stating some preliminary results in Section 2.2, in Section 2.3 we derive $w_s \le \frac{w}{s}$ in Lemma 2.3.3. Section 2.4 then starts with the definition of the function ϕ and a calculation of its derivative, see Lemma 2.4.1, Next, in the Lemma 2.4.2, we suitably estimate the term originating in the logistic source, before dealing with the remaining terms and the initial datum of ϕ in the subsequent lemmata. In Lemma 2.4.6, we then finally prove finiteness of the maximal existence time $T_{\max}$.

2.2. Preliminaries

In the remainder of the chapter, we henceforth fix $n \ge 3$, $R > 0$, $\Omega := B_R(0) \subset \mathbb{R}^n$, $\kappa \in (1,2]$, $\lambda \ge 0$ and $\mu > 0$.

Lemma 2.2.1. *Suppose that u_0 complies with (2.1.5). There exists $T_{\max} \in (0,\infty]$ and a unique pair (u,v) of regularity (2.1.7) which solves (2.1.8) classically and is such that if $T_{\max} < \infty$, then $\lim_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. Moreover, both u and v are radially symmetric and u is positive in $\overline{\Omega} \times [0,T_{\max})$.*

Proof. This is contained in [96, Lemma 1.1]. □

Given u_0 as in (2.1.5), we denote the solution given in Lemma 2.2.1 by (u,v) and its maximal existence time by $T_{\max}$. Moreover, we always set $\overline{m}(t) := \frac{1}{|\Omega|}\int_\Omega u(\cdot,t)$ for $t \in [0,T_{\max})$.

Since the zeroth order term in the first equation in (2.1.8), $\lambda u - \mu u^\kappa$, grows at most linearly in u, we can easily control the mass of the first solution component.

Lemma 2.2.2. *Suppose that u_0 satisfies* (2.1.5). *Then*

$$\int_\Omega u(\cdot,t) \le \mathrm{e}^{\lambda t} \int_\Omega u_0 \qquad \text{for all } t \in (0, T_{\max}).$$

Proof. This immediately follows from integrating the first equation in (2.1.8) and using that $\mu > 0$. ☐

As used multiple times in the sequel, let us also state the following elementary

Lemma 2.2.3. *Given $a > -1$, there is $B \in (0,\infty)$ such that for any $s_0 > 0$, the identity*

$$\int_0^{s_0} s^a (s_0 - s) \,\mathrm{d}s = B s_0^{a+2}$$

holds.

Proof. We substitute $s \mapsto s_0 s$ and take $B \coloneqq \int_0^1 s^a(1-s)\,\mathrm{d}s \in (0,\infty)$. ☐

2.3. The mass accumulation function w

Given u_0 as in (2.1.5) (and thus (u,v) as in Lemma 2.2.1), we denote the mass accumulation function by

$$w(s,t) \coloneqq \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t)\,\mathrm{d}\rho, \qquad (s,t) \in [0,R^n] \times [0,T_{\max}), \tag{2.3.1}$$

which has been introduced in the context of chemotaxis systems in [39]. In this section, we prove some of its properties, most notably the crucial estimate $w_s \le \frac{w}{s}$ in Lemma 2.3.3.

We start, however, by noting that w solves the following scalar equation.

Lemma 2.3.1. *For every u_0 satisfying* (2.1.5)*, the function w given by* (2.3.1) *belongs to* $C^0([0,R^n] \times [0,T_{\max})) \cap C^{2,1}([0,R^n] \times (0,T_{\max}))$ *and fulfills*

$$w_s(s,t) = \frac{u(s^{\frac{1}{n}},t)}{n} \qquad \text{for all } (s,t) \in [0,R^n] \times [0,T_{\max}) \tag{2.3.2}$$

as well as

$$w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + n w w_s - n\overline{m}(t) s w_s + \lambda w - n^{\kappa-1}\mu \int_0^s w_s^\kappa(\sigma,t)\,\mathrm{d}\sigma \tag{2.3.3}$$

in $(0,R^n) \times (0,T_{\max})$.

Proof. This can be seen by a direct calculation. In fact, the asserted regularity is a consequence of Lemma 2.2.1, the identity (2.3.2) follows from the chain rule, and [96, equation (1.4)] asserts that (2.3.3) holds. ☐

Next, as a major step towards proving $w_s \le \frac{w}{s}$, we show that for initial data fulfilling (2.1.5), the first solution component is radially decreasing throughout evolution.

Lemma 2.3.2. *Suppose u_0 complies with* (2.1.5)*. Then $u_r \le 0$ in $(0,R)\times(0,T_{\max})$.*

Proof. This can be shown as in [5, Lemma 5.1] or [21, Lemma 3.7] (which in turn both follow [102, Lemma 2.2]). However, due to the importance of this lemma for showing the crucial estimate $w_s \le \frac{w}{s}$ in the succeeding lemma, we choose to at least sketch the proof here. First, by an approximation argument as in [102, Lemma 2.2], we may assume $u_r \in C^0([0,R]\times[0,T_{\max})) \cap C^{2,1}((0,R)\times(0,T_{\max}))$.

Furthermore, the second equation in (2.1.8) asserts

$$r^{1-n}(r^{n-1}uv_r)_r = u_r v_r + ur^{1-n}(r^{n-1}v_r)_r = u_r v_r - u^2 + \overline{m}(t)u \qquad \text{in } (0,R)\times(0,T_{\max})$$

and hence, with $f(z) \coloneqq \lambda z - \mu z^\kappa$ for $z \ge 0$,

$$\begin{aligned} u_{rt} &= \left(r^{1-n}\left(r^{n-1}(u_r - uv_r)\right)_r + f(u)\right)_r \\ &= u_{rrr} + \frac{n-1}{r}u_{rr} - \frac{n-1}{r^2}u_r - u_{rr}v_r - u_r v_{rr} + 2uu_r - \overline{m}(t)u_r + f'(u)u_r \\ &= u_{rrr} + a(r,t)u_{rr} + b(r,t)u_r \qquad \text{in } (0,R)\times(0,T_{\max}), \end{aligned}$$

where

$$a(r,t) \coloneqq \frac{n-1}{r} - v_r(r,t) \quad \text{and} \quad b(r,t) \coloneqq -\frac{n-1}{r^2} - v_{rr}(r,t) + 2u(r,t) - \overline{m}(t) + f'(u(r,t))$$

for $(r,t) \in (0,R)\times(0,T_{\max})$.

As can be rapidly seen by writing the second equation in (2.1.8) in radial coordinates (and has been argued in more detail in [21, Lemma 3.6], for instance), $-v_{rr} \le u$ holds throughout $(0,R)\times(0,T_{\max})$, so that for fixed $T \in (0,T_{\max})$, we can estimate

$$\sup_{r\in(0,R),t\in(0,T)} b(r,t) \le 3\|u\|_{L^\infty((0,R)\times(0,T))} + \|f'\|_{L^\infty(0,\|u\|_{L^\infty((0,R)\times(0,T))})} < \infty.$$

An application of the maximum principle (cf. [75, Proposition 52.4]) then gives $u_r \le 0$ in $(0,R)\times(0,T)$, which upon taking $T \nearrow T_{\max}$ implies the statement. □

As already advertised multiple times, this lemma now allows us to rapidly obtain the important estimate $w_s \le \frac{w}{s}$.

Lemma 2.3.3. *Assume that u_0 satisfies* (2.1.5)*. For all $s \in [0,R^n]$ and $t \in [0,T_{\max})$,*

$$w_s(s,t) \le \frac{w(s,t)}{s} \le w_s(0,t) \tag{2.3.4}$$

holds. In particular, for all $t_0 \in (0,T_{\max})$ there is $C > 0$ such that

$$\frac{s}{C} \le w(s,t) \le Cs \qquad \text{for } s \in [0,R^n] \text{ and } t \in [0,t_0]. \tag{2.3.5}$$

Proof. For fixed $t \in [0,T_{\max})$ and $s \in [0,R^n]$, the mean value theorem provides us with $\xi \in (0,s)$ such that $w(s,t) = sw_s(\xi,t)$, which already proves (2.3.4) since w_s is decreasing by Lemma 2.3.2 and (2.3.2). Moreover, a consequence thereof is (2.3.5), since w_s is positive and bounded in $[0,R^n]\times[0,t_0]$ for any $t_0 \in (0,T_{\max})$ by Lemma 2.2.1 and (2.3.2). □

2.4. A supersolution to a superlinear ODE: finite-time blow-up

We will construct initial data leading to finite-time blow-up and hence prove Theorem 2.1.1 in this section. As already mentioned in the introduction of this chapter, our argument is based on constructing a function ϕ which cannot exist globally, implying that the solution of (2.1.8) also can only exist on a finite time interval. In fact, we define ϕ as in [5] or [103]; that is, for given u_0 as in (2.1.5) and $\gamma \in (0,2)$, we set

$$\phi(s_0,t) := \int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)\,\mathrm{d}s, \qquad s_0 \in (0,R^n), t \in (0,T_{\max}). \tag{2.4.1}$$

However, as the parameter γ herein may be larger than 1 (unlike as in [5] or [103]), some care is needed for calculating the time derivative of ϕ. This is done in the following

Lemma 2.4.1. *Suppose that u_0 complies with* (2.1.5)*. Let $\gamma \in (0,2)$ and ϕ be as in* (2.4.1)*. For every $s_0 \in (0,R^n)$, $\phi(s_0,\cdot)$ belongs to $C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$ and fulfills*

$$\begin{aligned}\phi_t(s_0,t) \ge n^2 &\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)w_{ss}\,\mathrm{d}s\\ &+n\int_0^{s_0} s^{-\gamma}(s_0-s)ww_s\,\mathrm{d}s\\ &-n\overline{m}(t)\int_0^{s_0} s^{1-\gamma}(s_0-s)w_s\,\mathrm{d}s\\ &-n^{\kappa-1}\mu\int_0^{s_0} s^{-\gamma}(s_0-s)\int_0^s w_s^\kappa(\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}s\\ &=: I_1(s_0,t)+I_2(s_0,t)+I_3(s_0,t)+I_4(s_0,t) \qquad \textit{for all } t \in (0,T_{\max}).\end{aligned} \tag{2.4.2}$$

Proof. We first fix $s_0 \in (0,R^n)$ and note that $\phi(s_0,\cdot) \in C^0([0,T_{\max}))$ because of (2.3.5) and $1-\gamma > -1$. Letting $0 < t_0 < t_1 < T_{\max}$, we then make use of Lemma 2.3.1 and Lemma 2.3.3 to obtain $c_1,c_2,c_3,c_4 > 0$ such that

$$w(s,t) \le c_1 s, \quad w_s(s,t) \le c_2, \quad |w_{ss}(s,t)| \le c_3 \quad \text{and} \quad \overline{m}(t) \le c_4$$

for $(s,t) \in [0,s_0] \times [t_0,t_1]$. Recalling (2.3.3), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(s^{-\gamma}(s_0-s)w\right)\\ &=\left(n^2 s^{2-\frac{2}{n}}w_{ss}+nww_s-n\overline{m}(t)sw_s+\lambda w-n^{\kappa-1}\mu\int_0^s w_s^\kappa(\sigma,t)\,\mathrm{d}\sigma\right)s^{-\gamma}(s_0-s)\end{aligned}$$

for $s \in (0,s_0)$ and $t \in (0,T_{\max})$, so that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(s^{-\gamma}(s_0-s)w(s,t)\right)\right| \le \left(n^2c_3s_0^{1-\frac{2}{n}}+nc_1c_2+nc_2c_4+\lambda c_1+n^{\kappa-1}\mu c_2^\kappa\right)s^{1-\gamma}(s_0-s)$$

for all $s \in (0,s_0)$ and $t \in (t_0,t_1)$. Again due to $1-\gamma > -1$, we therefore have $\phi(s_0,\cdot) \in C^1((0,T_{\max}))$ and

$$\phi_t(s_0,t) = I_1(s_0,t)+I_2(s_0,t)+I_3(s_0,t)+\lambda\int_0^{s_0} s^{-\gamma}(s_0-s)w\,\mathrm{d}s+I_4(s_0,t)$$

for all $t \in (0,T_{\max})$, which due to $\lambda \ge 0$ implies (2.4.2). ☐

Aiming to derive that ϕ is a supersolution to a superlinear ODE, we now estimate the terms $I_1, \dots, I_4$ in (2.4.2) and begin with I_4, the term stemming from the logistic source. In the following proof, we will crucially make use of the estimate (2.3.4) to improve on corresponding results obtained by the predecessors [5] and [103].

Lemma 2.4.2. *Let I_2 and I_4 be as in* (2.4.2)*.*

(i) If $\kappa = 2$, $\gamma > 1$ and u_0 fulfills (2.1.5)*, then*

$$I_4(s_0, t) \geq -\frac{\mu}{\gamma - 1} I_2(s_0, t) \qquad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, T_{\max}). \tag{2.4.3}$$

(ii) Let $\kappa \in (1, 2)$ and $\gamma \in (\frac{2(\kappa-1)}{\kappa}, 1)$. There exists $C_4 > 0$ such that whenever u_0 fulfills (2.1.5)*, then*

$$I_4(s_0, t) \geq C_4 s_0^{\frac{2-\kappa}{2}} I_2^{\frac{\kappa}{2}}(s_0, t) \qquad \text{for all } s_0 \in (0, \min\{1, R^n\}) \text{ and } t \in (0, T_{\max}). \tag{2.4.4}$$

Proof. We let $\gamma \in (0, \infty) \setminus \{1\}$ and also fix u_0 as in (2.1.5) but will make sure that C_4 can be taken independently of u_0. By Fubini's theorem, we first observe that

$$\begin{aligned}
I_4(s_0, t) &= -n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s) \int_0^s w_s^\kappa(\sigma, t) \,\mathrm{d}\sigma \,\mathrm{d}s \\
&= -n^{\kappa-1}\mu \int_0^{s_0} \left(\int_\sigma^{s_0} s^{-\gamma}(s_0 - s) \,\mathrm{d}s \right) w_s^\kappa(\sigma, t) \,\mathrm{d}\sigma \\
&\geq -n^{\kappa-1}\mu \int_0^{s_0} \left(\int_\sigma^{s_0} s^{-\gamma} \,\mathrm{d}s \right) (s_0 - \sigma) w_s^\kappa(\sigma, t) \,\mathrm{d}\sigma \\
&= -\frac{n^{\kappa-1}\mu}{1-\gamma} \int_0^{s_0} \left(s_0^{1-\gamma} - s^{1-\gamma} \right) (s_0 - s) w_s^\kappa(s, t) \,\mathrm{d}s
\end{aligned} \tag{2.4.5}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, T_{\max})$.

In the case of $\gamma > 1$ and $\kappa = 2$, we drop a positive term and employ (2.3.4) in calculating

$$\begin{aligned}
I_4(s_0, t) &\geq -\frac{n\mu}{\gamma - 1} \int_0^{s_0} s^{1-\gamma}(s_0 - s) w_s^2 \,\mathrm{d}s \\
&\geq -\frac{n\mu}{\gamma - 1} \int_0^{s_0} s^{-\gamma}(s_0 - s) w w_s \,\mathrm{d}s \\
&= -\frac{\mu}{\gamma - 1} I_2(s_0, t) \qquad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, T_{\max}),
\end{aligned}$$

which already implies (2.4.3).

If on the other hand $\gamma \in (0, 1)$ and $\kappa \in (1, 2)$, going back to (2.4.5) and making use of (2.3.4), we see that

$$\begin{aligned}
I_4(s_0, t) &\geq -\frac{n^{\kappa-1}\mu}{1-\gamma} s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa(s, t) \,\mathrm{d}s \\
&\geq -\frac{n^{\kappa-1}\mu}{1-\gamma} R^{n(1-\gamma)} \int_0^{s_0} s^{-\frac{\kappa}{2}} (s_0 - s)(w w_s)^{\frac{\kappa}{2}} \,\mathrm{d}s
\end{aligned} \tag{2.4.6}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, T_{\max})$. By Hölder's inequality (with exponents $\frac{2}{2-\kappa}, \frac{2}{\kappa}$), we have therein

$$\begin{aligned}
&\int_0^{s_0} s^{-\frac{\kappa}{2}}(s_0 - s)(ww_s)^{\frac{\kappa}{2}}\,\mathrm{d}s \\
=& \int_0^{s_0} s^{-\frac{(1-\gamma)\kappa}{2}}(s_0 - s)(s^{-\gamma}ww_s)^{\frac{\kappa}{2}}\,\mathrm{d}s \\
\le& \left(\int_0^{s_0} s^{-\frac{(1-\gamma)\kappa}{2-\kappa}}(s_0 - s)\,\mathrm{d}s\right)^{\frac{2-\kappa}{2}} \left(\int_\Omega s^{-\gamma}(s_0 - s)ww_s\,\mathrm{d}s\right)^{\frac{\kappa}{2}} \qquad (2.4.7)
\end{aligned}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, T_{\max})$. We assume now moreover that $\gamma > \frac{2(\kappa-1)}{\kappa}$ and hence $\gamma - 1 > \frac{\kappa-2}{\kappa}$ as well as $a \coloneqq \frac{(\gamma-1)\kappa}{2-\kappa} > -1$, so that applying Lemma 2.2.3 (with B as in that lemma) gives

$$\int_0^{s_0} s^{-\frac{(1-\gamma)\kappa}{2-\kappa}}(s_0 - s)\,\mathrm{d}s = Bs_0^{a+2} \le Bs_0 \qquad \text{for all } s_0 \in (0, \min\{1, R^n\}). \qquad (2.4.8)$$

Finally, combining (2.4.6)–(2.4.8) and the definition of I_2 yields (2.4.4) for some $C_4 > 0$ independent of u_0. □

The remaining integrals in (2.4.2) can be estimated as in [103] or [5]. However, at least for the statement concerning I_1, we would like to give a full proof here in order to show the basis of the restriction on κ in Theorem 2.1.1. Indeed, while in Lemma 2.4.2 above, γ has to be taken sufficiently large, for estimating I_1, we need γ to be suitably small. We will obtain finite-time blow-up precisely in the cases where the set of admissible γ for both these lemmata is nonempty. Moreover, compared to [103], the proof below makes use of the estimate (2.3.4) and is hence somewhat shorter.

Lemma 2.4.3. *Let $\gamma \in (0, 2 - \frac{4}{n})$. There is $C_1 > 0$ such that whenever u_0 satisfies* (2.1.5) *and I_1, I_2 are as in* (2.4.2)*, then*

$$I_1(s_0, t) \ge -C_1 s_0^{\frac{3-\gamma}{2} - \frac{2}{n}} I_2^{\frac{1}{2}}(s_0, t) \qquad \textit{for all } s_0 \in (0, R^n) \textit{ and } t \in (0, T_{\max}).$$

Proof. For convenience, we fix u_0 as in (2.1.5), albeit we emphasize that the constants below do not depend on u_0. An integration by parts gives

$$\begin{aligned}
&\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)w_{ss}\,\mathrm{d}s \\
=& -\left(2 - \frac{2}{n} - \gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)w_s\,\mathrm{d}s + \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}w_s\,\mathrm{d}s + \left[s^{2-\frac{2}{n}-\gamma}(s_0 - s)w_s\right]_0^{s_0}
\end{aligned}$$

in $(0, T_{\max})$. Herein, the second term on the right-hand side is positive and the last one is zero because of $\gamma < 2 - \frac{4}{n} < 2 - \frac{2}{n}$.

Setting $c_1 \coloneqq 2 - \frac{2}{n} - \gamma > 0$, we hence infer from (2.3.4) and Hölder's inequality that

$$\begin{aligned}
&\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)w_{ss}\,\mathrm{d}s \\
\ge& -\left(2 - \frac{2}{n} - \gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)w_s\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\geq -c_1 \int_0^{s_0} s^{\frac{1}{2}-\frac{2}{n}-\gamma}(s_0-s)(ww_s)^{\frac{1}{2}}\,\mathrm{d}s \\
&\geq -c_1 \left(\int_0^{s_0} s^{1-\gamma-\frac{4}{n}}(s_0-s)\,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^{s_0} s^{-\gamma}(s_0-s)ww_s\,\mathrm{d}s\right)^{\frac{1}{2}} \qquad \text{holds in } (0,T_{\max}).
\end{aligned}$$

Since $\gamma < 2-\frac{4}{n}$ and hence $a \coloneqq 1-\gamma-\frac{4}{n} > -1$, Lemma 2.2.3 asserts that (with B as in that lemma)

$$\left(\int_0^{s_0} s^{1-\gamma-\frac{4}{n}}(s_0-s)\,\mathrm{d}s\right)^{\frac{1}{2}} = B^{\frac{1}{2}} s_0^{\frac{3-\gamma}{2}-\frac{2}{n}} \qquad \text{for all } s_0 \in (0,R^n),$$

so that the statement follows by the definitions of I_1 and I_2. □

Next, for estimating the integrals I_2 and I_3 in (2.4.2), we basically recall the corresponding results from [103].

Lemma 2.4.4. *There exist $C_2, C_3 > 0$ such that for u_0 satisfying* (2.1.5)*, we have*

$$I_2(s_0,t) \geq C_2 s_0^{-(3-\gamma)}\phi^2(s_0,t) \qquad \textit{for all } s_0 \in (0,R^n) \textit{ and } t \in (0,T_{\max}) \tag{2.4.9}$$

and

$$I_3(s_0,t) \geq -C_3 \left(\int_\Omega u_0\right) s_0^{\frac{3-\gamma}{2}} I_2^{\frac{1}{2}}(s_0,t) \qquad \textit{for all } s_0 \in (0,R^n) \textit{ and } t \in (0,\hat{T}_{\max}), \tag{2.4.10}$$

where ϕ is in (2.4.1)*, I_2, I_3 are defined in* (2.4.2) *and $\hat{T}_{\max} \coloneqq \min\{1, T_{\max}\}$.*

Proof. Arguing as in [103, Lemma 4.4], we obtain

$$\phi(s_0,t) \leq c_1 s_0^{\frac{3-\gamma}{2}} I_2^{\frac{1}{2}}(s_0,t) \qquad \text{for all } s_0 \in (0,R^n) \text{ and } t \in (0,T_{\max}) \tag{2.4.11}$$

for some $c_1 > 0$ independent of u_0. Taking both the left- and the right-hand side therein to the power 2 already yields (2.4.9). Moreover, as

$$I_3(s_0,t) = -n\overline{m}(t)\int_0^{s_0} s^{1-\gamma}(s_0-s)w_s(s,t)\,\mathrm{d}s \geq -\frac{n}{|\Omega|}\left(\int_\Omega u_0\right)\mathrm{e}^{\lambda t}\phi(s_0,t)$$

for $(s_0,t) \in (0,R^n)\times(0,T_{\max})$ by Lemma 2.2.2 and (2.3.4), another consequence of (2.4.11) is (2.4.10). □

As a final preparation, we note that, under certain circumstances, $\phi(s_0,0)$ can be shown to be sufficiently large.

Lemma 2.4.5. *For every $m_1 > 0$, there exists $C_0 > 0$ with the following property: Let $s_0 \in (0,R^n)$, set $s_1 \coloneqq \frac{s_0}{4}$ as well as $r_1 \coloneqq s_1^{\frac{1}{n}}$ and suppose that u_0 fulfills* (2.1.5) *as well as $\int_{B_{r_1}(0)} u_0 \geq m_1$. Then $\phi(s_0,0) \geq C_0 s_0^{2-\gamma}$.*

Proof. See [103, estimate (5.5)]; the main idea is to use the monotonicity of w_0 which in turn is implied by nonnegativity of u_0. □

A combination of the results obtained above now reveals that for initial data whose mass is sufficiently concentrated near the origin, the corresponding solution cannot exist globally in time. Again, the argument is not too different from [96] or [103], but we choose to give it nonetheless in order to show that s_0 and u_0 can be chosen in such a way that ϕ would blow up in finite time if (u,v) were a global solution.

Lemma 2.4.6. *Let $m_0 > m_1 > 0$ and suppose that* (2.1.4) *holds. There exists $r_1 \in (0, R)$ such that whenever u_0 fulfills* (2.1.5) *and* (2.1.6)*, then $T_{\max} \le \frac{1}{2}$.*

Proof. Let us begin by fixing some parameters. If (2.1.4a) holds, then $\kappa \in (1, \frac{n}{2})$ and hence

$$\frac{2(\kappa-1)}{\kappa} - \left(2 - \frac{4}{n}\right) < \frac{2 \cdot \frac{n-2}{2}}{\frac{n}{2}} - \frac{2(n-2)}{n} = 0.$$

As additionally $\kappa < 2$, we may hence choose $\gamma \in (\frac{2(\kappa-1)}{\kappa}, \min\{2-\frac{4}{n}, 1\})$. We moreover fix an arbitrary $\varepsilon > 0$ and apply Lemma 2.4.2 (ii) as well as Young's inequality (with exponents $\frac{2}{2-\kappa}, \frac{2}{\kappa}$) to obtain $C_4' > 0$ with

$$I_4(s_0, t) \ge -\frac{\mu}{\mu+\varepsilon} I_2(s_0, t) - C_4' s_0 \tag{2.4.12}$$

for all $s_0 \in (0, \min\{1, R^n\})$ and $t \in (0, T_{\max})$, whenever u_0 satisfies (2.1.5) and where I_2 and I_4 are as in (2.4.2).

We now suppose that on the other hand (2.1.4b) holds. Because of $\mu \in (0, \frac{n-4}{n})$, we may then choose $\gamma \in (1+\mu, 2-\frac{4}{n})$. Setting moreover $\varepsilon \coloneqq \gamma - 1 - \mu > 0$, an application of Lemma 2.4.2 (i) reveals that (2.4.12) holds also in this case (with $C_4' \coloneqq 0$ and for all u_0 complying with (2.1.5)).

In both cases, the definition of γ entails $0 < \gamma < 2 - \frac{4}{n}$, hence by Lemma 2.4.1, Lemma 2.4.3, (2.4.10), (2.4.12), Young's inequality and (2.4.9), there are $c_1, c_2 > 0$ such that

$$\begin{aligned} \phi_t(s_0, t) &\ge I_1(s_0, t) + I_2(s_0, t) + I_3(s_0, t) + I_4(s_0, t) \\ &\ge \frac{\varepsilon}{\mu+\varepsilon} I_2(s_0, t) - \left(C_1 s_0^{\frac{3-\gamma}{2} - \frac{2}{n}} + C_3 m_0 s_0^{\frac{3-\gamma}{2}}\right) I_2^{\frac{1}{2}}(s_0, t) - C_4' s_0 \\ &\ge c_1 I_2(s_0, t) - c_2 s_0^{\min\{3-\gamma-\frac{4}{n}, 3-\gamma, 1\}} \\ &\ge C_2 c_1 s_0^{-(3-\gamma)} \phi^2(s_0, t) - c_2 s_0 \end{aligned} \tag{2.4.13}$$

for all $s_0 \in (0, \min\{1, R^n\})$, $t \in (0, \min\{1, T_{\max}\})$ and u_0 satisfying (2.1.5) as well as $\int_\Omega u_0 = m_0$, where $\phi, I_1, \dots, I_4$ are as in Lemma 2.4.1, C_1 is as in Lemma 2.4.3 and C_2, C_3 are as in Lemma 2.4.4.

For $s_0 > 0$, we set $c_3 \coloneqq C_2 c_1$,

$$d_1(s_0) \coloneqq c_3 s_0^{-(3-\gamma)}, \quad d_2(s_0) \coloneqq c_2 s_0, \quad d_3(s_0) \coloneqq \left(\frac{d_2(s_0)}{d_1(s_0)}\right)^{\frac{1}{2}} \quad \text{and} \quad \phi_0(s_0) \coloneqq C_0 s_0^{2-\gamma},$$

where C_0 is as in Lemma 2.4.5. We observe that $d_1(s_0) \to \infty$ for $s_0 \searrow 0$ since $3 - \gamma > 1 > 0$. Therefore, noting further that

$$\frac{1}{2}(1 + 3 - \gamma) = 2 - \frac{\gamma}{2} > 2 - \gamma,$$

we may also fix $s_0 \in (0, \min\{1, R^n\})$ so small that

$$\phi_0(s_0) \ge d_3(s_0) + \frac{2}{d_1(s_0)}. \tag{2.4.14}$$

Moreover, we now fix u_0 not only complying with (2.1.5) but also with (2.1.6) for $r_1 := (\frac{s_0}{4})^{\frac{1}{n}}$ and will show that the corresponding solution given by Lemma 2.2.1 blows up in finite time. From (2.4.13) and Lemma 2.4.5, we infer that $\phi(s_0, \cdot)$ satisfies

$$\begin{cases} \phi_t(s_0, t) \ge d_1(s_0)\phi^2(s_0, t) - d_2(s_0) & \text{for all } t \in (0, \min\{1, T_{\max}\}), \\ \phi(s_0, 0) \ge \phi_0(s_0). \end{cases} \tag{2.4.15}$$

Since (2.4.14) implies $\phi_0(s_0) \ge d_3(s_0)$ and because of $d_1(s_0)d_3(s_0)^2 - d_2(s_0) = 0$, the comparison principle and (2.4.15) assert $\phi(s_0, t) \ge d_3(s_0)$ for all $t \in (0, \min\{1, T_{\max}\})$, so that by (2.4.15), we have

$$\begin{aligned} \phi_t(s_0, t) &\ge d_1(s_0) \left(\phi^2(s_0, t) - d_3(s_0)^2\right) \\ &\ge d_1(s_0) \left(\phi(s_0, t) - d_3(s_0)\right)^2 \qquad \text{for all } t \in (0, \min\{1, T_{\max}\}). \end{aligned}$$

Dividing by the right-hand side therein yields upon an integration in time

$$t = \int_0^t 1 \,\mathrm{d}s \le \int_{\phi(s_0,0)}^{\phi(s_0,t)} \frac{\mathrm{d}\sigma}{d_1(s_0)(\sigma - d_3(s_0))^2} \le \left[-\frac{1}{d_1(s_0)(\sigma - d_3(s_0))}\right]_{\phi_0(s_0)}^{\infty} \le \frac{1}{2}$$

for all $t \in (0, \min\{1, T_{\max}\})$, implying $T_{\max} \le \frac{1}{2}$. □

Finally, we conclude that Theorem 2.1.1 is now merely a direct consequence of the lemmata above.

Proof of Theorem 2.1.1. Lemma 2.4.6 asserts that there is $r_1 \in (0, R)$ such that under the conditions of Theorem 2.1.1, the maximal existence time $T_{\max}$ is finite. By Lemma 2.3.2 and Lemma 2.2.1, this then implies $u(0, t) = \|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \nearrow T_{\max}$. □

3. Blow-up profiles in quasilinear fully parabolic Keller–Segel systems

3.1. Introduction

The possibility of (finite-time) blow-up constitutes one of the most striking features of the quasilinear system

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - S(u,v)\nabla v), & \text{in } \Omega \times (0,T), \\ v_t = \Delta v - v + u, & \text{in } \Omega \times (0,T), \\ (D(u,v)\nabla u - S(u,v)\nabla v) \cdot \nu = \partial_\nu v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0, & \text{in } \Omega, \end{cases} \tag{3.KS}$$

proposed by Keller and Segel [46] to model chemotaxis, that is, the directed movement of bacteria or cells towards a chemical signal, and attracting interest of mathematicians for nearly half a century (see for instance [4] for a recent survey).

Therein $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a smooth, bounded domain, $T \in (0,\infty]$ and $u_0, v_0 \colon \overline{\Omega} \to [0,\infty)$ as well as $D, S \colon [0,\infty]^2 \to [0,\infty)$ are sufficiently smooth given functions, the most classical choices being $D \equiv 1$ and $S(u,v) = u$.

For these selections, namely, solutions blowing up in finite time have been constructed in two- [34] and higher- [97] dimensional balls. On the other hand, if $n = 1$ [71], if $n = 2$ and $\int_\Omega u_0 < 4\pi$ (or $\int_\Omega u_0 < 8\pi$ in the radially symmetric setting) [68] or if $n \geq 3$ and $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} + \|v_0\|_{W^{1,n}(\Omega)}$ is sufficiently small [7], all solutions are global in time and remain bounded. We should also note that if one replaces the second equation in (3.KS) by a suitable elliptic counterpart, finite-time blow-up results have been achieved already in the 1990s [33, 39, 66].

Motivated inter alia by the desire to model volume-filling effects, it has been suggested to consider certain nonlinear functions $D \equiv D(u)$ and $S \equiv S(u)$ instead [36, 72, 110] and, in order to account for immotility in absence of bacteria [19, 57] or receptor-binding and saturation effects [36, 42], one might also (need to) choose functions D and S explicitly depending on v.

For the sake of exposition, we will for now confine ourselves with the choices $D(u,v) = (u+1)^{m-1}$ and $S(u,v) = u(u+1)^{q-1}$ for certain $m, q \in \mathbb{R}$, but remark that all the works cited below allow for more general functions D and S as well. From a mathematical point of view, these are the most prototypical choices, as they generalize $D \equiv 1, S(u,v) = u$, which are obtained upon setting $m = q = 1$, and since estimates of the form $D \geq u^{m-1}, |S| \leq u^q$, $u \geq 1$, come in handy at several places (see for instance the proofs of the present chapter). Moreover, even these prototypical functions directly appear in biologically motivated models;

by choosing $m > 1$ and $q = 1$ we arrive at (a nondegenerate version of) system (M5) in [36] while the choices $m = 1$ and $q = 0$ lead to model (M3b) in [36].

Regarding the question of global-in-time boundedness, the number $\frac{n-2}{n}$ is critical: If $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a smooth, bounded domain and $m - q > \frac{n-2}{n}$, then all solutions to (3.KS) are global in time and bounded [37, 38, 83]. (We also refer to [77] for earlier partial results in this direction and to [47] for existence results in the case of degenerate diffusion). Conversely, if $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a ball and $m-q < \frac{n-2}{n}$, there exist initial data such that the corresponding solution blows up in either finite or infinite time [37, 93].

If in addition to $m - q < \frac{n-2}{n}$ one assumes $n \ge 3$ as well as either $m \ge 1$ (and hence $q > \frac{2}{n} > 0$) or $m \in \mathbb{R}$ and $q \ge 1$, finite-time blow-up is possible [11–13], while for $q \le 0$ solutions are always global in time [104]. Whether solutions may blow up in finite time given $m - q < \frac{n-2}{2}$ and $q > 0$ but $q < 1$ or $m < 1$ is, to the best of our knowledge, still an open question. (For finite-time blow-up in the one-dimensional case, see [10].)

The picture is more complete if one replaces the second equation in (3.KS) with a suitable elliptic equation. Again solutions are global and bounded provided that $m-q > \frac{n-2}{2}$ and in the radial symmetric setting there exist unbounded solutions if $m - q < \frac{n-2}{2}$. Additionally, it is known for which parameters finite-time blow-up may occur: If $q \le 0$, these solutions are always global, while for $q > 0$ finite-time blow-up is possible [52, 108]. (Let us also reference the earlier partial results [6] and [14].) An obvious conjecture, stated for instance in [104], is that the same holds true for the fully parabolic system (3.KS).

Similar results are also available for functions D and S decaying exponentially fast in u (see [15] for boundedness in 2D, [93] for the existence of unbounded solutions and [101] for the possibility of infinite-time blow-up, for instance).

A natural next step is to examine the qualitative behavior of (finite- or infinite-time) blow-up solutions in more detail. While far from exhaustive, some results in this regard have been obtained for the classical Keller–Segel system, that is, for $D \equiv 1$ and $S(u, v) = u$.

In the two-dimensional settings some blow-up solutions collapse to a Dirac-type singularity (see [34, 67] or also [76] for similar results for the parabolic–elliptic case). Additionally, for all $n \ge 2$, temporal blow-up rates (even for $S(u, v) = u^q, q \in (0, 2)$) have been established [61] and it is known that $\{u^{\frac{n}{2}}(\cdot, t) \colon t \in (0, T_{\max})\}$ cannot be equi-integrable, where $T_{\max}$ denotes the blow-up time [8].

Quite recently, the questions whether spatial blow-up profiles exist, that is, whether $U := \lim_{t \nearrow T_{\max}} u(\cdot, t)$, $T_{\max}$ again denoting the blow-up time, is meaningful in some sense, and, if this is indeed the case, properties of U have been studied.

Choosing Ω to be a ball in two or more dimensions, $D \equiv 1$ and $S(u, v) = u$, it has been shown in [106] that for all nonnegative, radially symmetric solutions blowing up at $T_{\max} < \infty$ there exists a blow-up profile U in the sense that $u(\cdot, t) \to U$ in $C^2_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\})$ as $t \nearrow T_{\max}$. Moreover, an upper estimate for U is available: For any $\eta > 0$ one can find $C > 0$ with

$$U(x) \le C|x|^{-n(n-1)-\eta} \qquad \text{for all } x \in \Omega.$$

If one simplifies (3.KS) by not only setting $D \equiv 1$ and $S(u, v) = u$ but also replacing the second equation therein with $0 = \Delta v - \frac{1}{|\Omega|} \int_\Omega u_0 + u$, more detailed information is available. In [79], the authors consider $\Omega := B_R(0) \subset \mathbb{R}^n$, $R > 0, n \ge 3$, and construct a large class of initial data for which the corresponding solutions (u, v) blow up in finite time. The blow-up profile $U := \lim_{t \nearrow T_{\max}} u(\cdot, t)$ exists pointwise and

$$U(x) \le C|x|^{-2} \qquad \text{for all } x \in \Omega$$

holds for some $C > 0$, wherein the exponent 2 is optimal. Furthermore, the same paper also provides certain lower bounds for U.

Up to now, however, in the case of nonlinear diffusion there seems to be nearly no information available regarding behavior of finite-time blow-up solutions to (3.KS) at their blow-up time. The present chapter aims to be a first step towards closing this gap.

Main results. At first, we will deal with (a slight generalization of) the first subproblem in (3.KS) and derive pointwise estimates for its solutions.

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a smooth, bounded domain with $0 \in \Omega$ as well as*

$$m, q \in \mathbb{R}, K_{D,1}, K_{D,2}, K_S, K_f, M, L, \beta > 0, \theta > n, \mathbb{p} \ge 1 \tag{3.1.1}$$

be such that

$$m - q \in \left(\frac{\mathbb{p}}{\theta} - \frac{\mathbb{p}}{n}, \frac{\mathbb{p}}{\theta} + \frac{\beta\mathbb{p} - \mathbb{p}}{n} \right] \quad \text{and} \quad m > \frac{n - 2\mathbb{p}}{n}. \tag{3.1.2}$$

Then for any

$$\alpha > \frac{\beta}{m - q + \frac{\mathbb{p}}{n} - \frac{\mathbb{p}}{\theta}}, \tag{3.1.3}$$

we can find $C > 0$ with the following property:

Suppose that for some $T \in (0, \infty]$, the function $u \in C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$ is nonnegative, fulfills

$$\sup_{t \in (0,T)} \int_\Omega u^{\mathbb{p}} \le M \tag{3.1.4}$$

and is a classical solution of

$$\begin{cases} u_t \le \nabla \cdot (D(x, t, u)\nabla u + S(x, t, u)f(x, t)), & \text{in } \Omega \times (0, T), \\ (D(x, t, u)\nabla u + S(x, t, u)f) \cdot \nu \le 0, & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) \le u_0, & \text{in } \Omega, \end{cases} \tag{3.1.5}$$

where

$$D, S \in C^1(\overline{\Omega} \times (0, T) \times [0, \infty)), \quad f \in C^1(\overline{\Omega} \times (0, T); \mathbb{R}^n) \quad \text{and} \quad u_0 \in C^0(\overline{\Omega}) \tag{3.1.6}$$

satisfy (with $Q_T := \Omega \times (0, T)$)

$$\inf_{(x,t) \in Q_T} D(x, t, \rho) \ge K_{D,1}\rho^{m-1}, \tag{3.1.7}$$

$$\sup_{(x,t) \in Q_T} D(x, t, \rho) \le K_{D,2} \max\{\rho, 1\}^{m-1}, \tag{3.1.8}$$

$$\sup_{(x,t) \in Q_T} |S(x, t, \rho)| \le K_S \max\{\rho, 1\}^q \tag{3.1.9}$$

for all $\rho > 0$ and

$$\sup_{t\in(0,T)} \int_\Omega |x|^{\theta\beta}|f(x,t)|^\theta \,\mathrm{d}x \le K_f \tag{3.1.10}$$

as well as

$$u_0(x) \le L|x|^{-\alpha} \text{ for all } x \in \Omega. \tag{3.1.11}$$

Then

$$u(x,t) \le C|x|^{-\alpha} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T). \tag{3.1.12}$$

Remark 3.1.2. For $\mathbb{p} = 1$, the condition (3.1.4) in Theorem 3.1.1 can be replaced by

$$\int_\Omega u_0 \le M$$

as integrating the PDI in (3.1.5) over Ω and integrating by parts (all boundary terms are nonpositive because of the second condition in (3.1.5)) assert $\int_\Omega u(\cdot,t) \le \int_\Omega u_0$ for all $t \in (0, T_{\max})$.

As a second step, we then apply this result to radially symmetric solutions to (3.KS) and obtain

Theorem 3.1.3. *Let $n \ge 2$, $R > 0$ and $\Omega \coloneqq B_R(0)$ as well as*

$$m, q \in \mathbb{R}, K_{D,1}, K_{D,2}, K_S, > 0, M, L > 0 \tag{3.1.13}$$

such that

$$m - q \in \left(-\frac{1}{n}, \frac{n-2}{n}\right] \quad \text{and} \quad m > \frac{n-2}{n}. \tag{3.1.14}$$

For any

$$\alpha > \underline{\alpha} \coloneqq \frac{n(n-1)}{(m-q)n+1} \tag{3.1.15}$$

and any $\beta > n-1$, there exists $C > 0$ with the following property: Let $T \in (0,\infty]$. Any nonnegative and radially symmetric classical solution

$$(u,v) \in \left(C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T))\right)^2$$

of (3.KS) *fulfills* (3.1.12) *and $|\nabla v(x,t)| \le C|x|^{-\beta}$ for all $x \in \Omega$ and $t \in (0,T)$, provided*

$$D, S \in C^1([0,\infty)^2), \quad u_0 \in C^0(\overline{\Omega}) \quad \text{and} \quad v_0 \in W^{1,\infty}(\Omega) \tag{3.1.16}$$

satisfy

$$\inf_{\sigma\ge0} D(\rho,\sigma) \ge K_{D,1}\rho^{m-1}, \tag{3.1.17}$$

$$\sup_{\sigma\ge0} D(\rho,\sigma) \le K_{D,2}\max\{\rho,1\}^{m-1} \quad \text{and} \tag{3.1.18}$$

$$\sup_{\sigma\ge0} |S(\rho,\sigma)| \le K_S \max\{\rho,1\}^q \tag{3.1.19}$$

for all $\rho \ge 0$ as well as (3.1.11),

$$\int_\Omega u_0 \le M \quad \text{and} \quad \|v_0\|_{W^{1,\infty}(\Omega)} \le L. \tag{3.1.20}$$

Remark 3.1.4. (i) Let us briefly discuss the conditions in (3.1.14). On the one hand, observe that $m - q \searrow -\frac{1}{n}$ implies $\underline{\alpha} \nearrow \infty$. On the other hand, [83] proves that all solutions to (3.KS) for a large class of functions D, S are global in time and bounded, provided $m, q \in \mathbb{R}$ satisfy $m - q > \frac{n-2}{n}$. In both cases a statement of the form (3.1.12) would not be very interesting. (However, for $m - q > \frac{n-2}{n}$ the statement still holds if one sets $\underline{\alpha} \coloneqq n$ because if (3.1.9) is fulfilled for some $q \in \mathbb{R}$ then also for all larger q.)

The second condition in (3.1.14), however, is purely needed for technical reasons and we conjecture that Theorem 3.1.3 holds even without this restriction, albeit the constant C may then depend on T as well.

(ii) In [17, Corollary 2.3], it has been shown that (3.1.12) cannot hold for any

$$\alpha < \overline{\alpha} \coloneqq \min\left\{\frac{2}{(1+q-m)_+}, \frac{1}{(q-m)_+}\right\}.$$

As $m - q < \frac{n-2}{n}$ implies $\underline{\alpha} > n > \overline{\alpha}$, we do not know whether (3.1.15) is in general optimal. However, in the case of $m - q = \frac{n-2}{n}$ (and $m > \frac{n-2}{n}$) we have $\underline{\alpha} = n = \overline{\alpha}$, hence at least in this extremal case the condition $\alpha > \overline{\alpha}$ is, up to equality, optimal.

The third and final step will then consist of proving that $\lim_{t \nearrow T_{\max}} u(\cdot, t)$ and $\lim_{t \nearrow T_{\max}} v(\cdot, t)$ exist in an appropriate sense provided the diffusion mechanism in the first equation in (3.KS) is nondegenerate.

Theorem 3.1.5. *Let $n \ge 2$, $R > 0$, $\Omega \coloneqq B_R(0)$ and suppose that the parameters in* (3.1.13) *and the functions in* (3.1.16) *comply with* (3.1.11), (3.1.14) *and* (3.1.17)–(3.1.20). *Furthermore, suppose also that there is $\eta > 0$ with*

$$D \ge \eta \qquad \text{in } [0, \infty)^2. \tag{3.1.21}$$

Then for any nonnegative and radially symmetric classical solution (u, v) blowing up in finite time in the sense that there is $T_{\max} < \infty$ such that

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty,$$

there exist nonnegative, radially symmetric $U, V \in C^2(\Omega \setminus \{0\})$ such that

$$u(\cdot, t) \to U \quad \text{and} \quad v(\cdot, t) \to V \qquad \text{in } C^2_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}) \text{ as } t \nearrow T_{\max}. \tag{3.1.22}$$

Moreover, for any $\alpha > \underline{\alpha}$ (with $\underline{\alpha}$ as in (3.1.15)*) and any $\beta > n - 1$, we can find $C > 0$ with the property that*

$$U(x) \le C|x|^{-\alpha} \quad \text{and} \quad |\nabla V(x)| \le C|x|^{-\beta} \qquad \text{for all } x \in \Omega. \tag{3.1.23}$$

Remark 3.1.6. Obviously, Theorem 3.1.5 is only of interest if, given S and D, there are indeed initial data leading to finite-time blow-up. Therefore, we stress that, for instance, the choices $D(\rho, \sigma) \coloneqq (\rho + 1)^{m-1}$ and $S(\rho, \sigma) \coloneqq \rho(\rho + 1)^{q-1}$ for $\rho, \sigma \ge 0$ and $m \in \mathbb{R}, q \ge 0$ satisfying (3.1.14) as well as $q \ge 1$ or $m \ge 1$ not only comply with (3.1.16)–(3.1.19) and (3.1.21) for certain parameters but also allow for finite-time blow-up [11, 13]. That is, there exist initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega)$ such that the corresponding solution to (3.KS) blows up in finite time. As (3.1.11) and (3.1.20) are then obviously fulfilled for certain $L, M > 0$, we may indeed apply Theorem 3.1.5.

Moreover, let us emphasize that our results can indeed be applied to models stemming from a biological motivation, for instance to (a nondegenerate version of) the system (M5) in [36], that is, to (3.KS) with $m > 1$ and $q = 0$. Furthermore, even the degenerate case is covered by Theorem 3.1.3.

Remark 3.1.7. Let us also point out that Theorem 3.1.5 includes the result in [106, Corollary 1.4], as in the case of $m = 1$ and $q = 1$, we have $\underline{\alpha} = n(n-1)$.

Remark 3.1.8. As already briefly mentioned in the introduction of Chapter 2, pointwise estimates of the above flavor may be used to detect finite-time blow-up in chemotaxis systems with logistic-type dampening. This has first been observed in [103], where pointwise estimates derived in [106] for the special case $m = q = 1$ have been employed.

A systematic analysis of the question how estimates of this type affect the possibility to establish finite-time blow-up also in quasilinear chemotaxis systems is then given in [5], where an important corollary combines the main result of that paper with Theorem 3.1.1 in order to indeed obtain finite-time blow-up for certain parameter ranges. In [80], these results have been further extended to systems with saturated taxis sensitivities, again making crucial use of Theorem 3.1.1.

Plan of the chapter. The reasoning from [106], where estimates on blow-up profiles to solutions to (3.KS) with $D \equiv 1$ and $S(u, v) = u$ have been derived, is to consider $w := \zeta^\alpha u$ with $\zeta(x) \approx |x|$ and to make use of semigroup arguments as well as L^p-L^q estimates in order to derive an L^∞ bound for w which in turn implies the desired estimate of the form (3.1.12) for u. However, through their mere nature, these methods are evidently inadequate to handle equations with nonlinear diffusion.

The present chapter is built upon the belief that, generally, an iterative testing procedure should be as strong as semigroup arguments. While the latter method may be quite elegant, the former has the distinct advantage of being applicable not only to equations with linear diffusion but also to (3.1.5).

Indeed, iteratively testing with $w^{p_j - 1}$ for certain $1 \le p_j \nearrow \infty$ allows us to obtain an L^∞ bound for w at the end of Section 3.2—provided the critical assumption (3.1.3) is fulfilled.

Applying Theorem 3.1.1 to solutions of (3.KS) mainly consists of adequately estimating $f := -\nabla v$. To that end, we may basically rely on the results in [106]. It probably should also be noted that this is the only part where we explicitly make use of the radially symmetric setting.

Finally, the existence of blow-up profiles is shown in Section 3.4 by considering global solutions $(u_\varepsilon, v_\varepsilon)$, $\varepsilon \in (0, 1)$, to suitably approximative problems which converge (along a subsequence) on all compact sets in $\overline{\Omega} \setminus \{0\} \times (0, \infty)$ to $(\widehat{u}, \widehat{v})$ for certain functions $\widehat{u}, \widehat{v} \colon \overline{\Omega} \times [0, \infty) \to [0, \infty)$. We then prove that these functions coincide which u and v on $\overline{\Omega} \times [0, T_{\max})$ such that we may set $U := \widehat{u}(\cdot, T_{\max})$ as well as $V := \widehat{v}(\cdot, T_{\max})$ and make use of regularity of $\widehat{u}$ and $\widehat{v}$.

In order to identify $(\widehat{u}, \widehat{v})$ with (u, v), we crucially need uniqueness of solutions to (3.KS) which we show in Lemma 3.5.1—provided that the first equation is nondegenerate. As this might potentially be of independent interest, we choose to prove uniqueness for a class of systems slightly generalizing (3.KS).

3.2. Pointwise estimates for subsolutions to equations in divergence form

Unless otherwise stated, we assume throughout this section that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a smooth, bounded domain with $0 \in \Omega$, set $R := \sup_{x\in\Omega} |x|$ and suppose that the parameters (all henceforth fixed) in (3.1.1) as well as α comply with (3.1.2) and (3.1.3). Moreover, we may also assume

$$(m - q)\alpha < \beta, \tag{3.2.1}$$

since whenever (3.1.10) is fulfilled for some $\beta > 0$, then also for all $\tilde{\beta} > \beta$ (provided one replaces K_f by $\max\{R, 1\}^{\tilde{\beta}-\beta} K_f$).

In order to simplify the notation, we also fix $T \in (0, \infty]$ and functions in (3.1.6) satisfying (3.1.4) and (3.1.7)–(3.1.11) as well as a nonnegative classical solution $u \in C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$ of (3.1.5), but emphasize that all constants below only depend on the parameters in (3.1.1) as well as on α.

Our goal, which will be achieved in Lemma 3.2.10 below, is to prove an L^∞ bound for the function

$$w\colon \overline{\Omega} \times [0, T) \to \mathbb{R}, \qquad (x, t) \mapsto |x|^\alpha u(x, t) \tag{3.2.2}$$

which in turn directly implies the desired estimate (3.1.12).

To this end, we will rely on a testing procedure to obtain L^p bounds for all $p \in (1, \infty)$. Due to an iteration technique, this will then be improved to an L^∞ bound—hence the constants in the following proofs need also to be independent of p.

In order to prepare for said testing procedure, we first state

Lemma 3.2.1. *Let $s \in \mathbb{R}$ and $0 \leq g \in C^0(\overline{\Omega} \times (0, T) \times (0, \infty))$ with*

$$\sup_{(x,t)\in\Omega\times(0,T)} g(x, t, \rho) \leq K_g \max\{\rho, 1\}^s \tag{3.2.3}$$

for all $\rho \geq 0$ and some $K_g > 0$.

For any $\mu \in \mathbb{R}$, $\gamma \in \mathbb{R}$ and $\kappa > 0$, there exist $p_0 \geq 1$ and $C > 0$ such that for all $p \geq p_0$, we have

$$\int_\Omega \left(g(x, t, u)|x|^\mu w^{p+\gamma}\right)^\kappa \leq C \left(1 + \int_\Omega \left(|x|^{\mu-\alpha s} w^{p+\gamma+s}\right)^\kappa\right) \qquad \text{in } (0, T). \tag{3.2.4}$$

Proof. For any $p > p_1 := -\gamma + \frac{|\mu|}{\alpha}$, all integrals in (3.2.4) are finite by (3.2.2).

As in the case of $s \leq 0$ the statement follows directly from (3.2.3) and (3.2.2) (for $p_0 := \max\{1, p_1\}$ and $C := K_g$), we may assume $s > 0$. Then (3.2.3) only implies

$$\int_\Omega \left(g(x, t, u)|x|^\mu w^{p+\gamma}\right)^\kappa \leq K_g \int_{\{u\geq1\}} \left(|x|^{\mu-\alpha s} w^{p+\gamma+s}\right)^\kappa + K_g \int_{\{u<1\}} \left(|x|^\mu w^{p+\gamma}\right)^\kappa$$

for all $p \geq p_1$ in $(0, T)$.

Since $s > 0$, we may therein employ Young's inequality (with exponents $\frac{p+\gamma+s}{p+\gamma}$, $\frac{p+\gamma+s}{s}$) to obtain

$$\int_{\{u<1\}} \left(|x|^{\mu} w^{p+\gamma}\right)^{\kappa} \leq \frac{p+\gamma}{p+\gamma+s} \int_{\Omega} \left(|x|^{\mu \cdot \frac{p+\gamma+s}{p+\gamma}} w^{p+\gamma+s}\right)^{\kappa} + \frac{s}{p+\gamma+s}|\Omega|$$

for all $p \geq p_1$ in $(0, T)$.

As

$$\lim_{p \nearrow \infty} \mu \cdot \frac{p+\gamma+s}{p+\gamma} = \mu > \mu - \alpha s$$

since $\alpha > 0$ and $s > 0$, we may find $p_2 > 1$ such that $\mu \cdot \frac{p+\gamma+s}{p+\gamma} > \mu - \alpha s$ for all $p > p_2$.

Therefore, for $x \in B_1(0)$ and $p \geq p_2$,

$$|x|^{(\mu \cdot \frac{p+\gamma+s}{p+\gamma})\kappa} \leq |x|^{(\mu - \alpha s)\kappa},$$

while for $x \in \overline{\Omega} \setminus B_1(0)$ and any $p > 1$,

$$|x|^{(\mu \cdot \frac{p+\gamma+s}{p+\gamma})\kappa} \leq \max\left\{1, R^{(\mu \cdot \frac{p+\gamma+s}{p+\gamma})\kappa}\right\} \leq c_1 \leq c_1 \max\{1, R^{-(\mu-\alpha s)\kappa}\}|x|^{(\mu-\alpha s)\kappa}$$

for some $c_1 > 0$.

Since $\frac{p+\gamma}{p+\gamma+s} \leq 1$ and $\frac{s}{p+\gamma+s} \leq 1$, we arrive at (3.2.4) by setting $p_0 := \max\{p_1, p_2\}$ and $C > 0$ appropriately. □

We may now initiate the aforementioned testing procedure and obtain a first estimate for the quantity $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^p$ in $(0, T)$.

Lemma 3.2.2. *There exist $C_1, C_2 > 0$ and $p_0 > 1$ such that for all $p \geq p_0$,*

$$\begin{aligned} &\frac{1}{p^2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^p + C_1 \int_\Omega |x|^{-(m-1)\alpha} w^{p+m-3} |\nabla w|^2 \\ &\leq C_2 \sum_{i=1}^{3} \left(\int_\Omega \left(|x|^{-\mu_i} w^{p+\gamma_i}\right)^{\kappa_i} \right)^{\frac{1}{\kappa_i}} + C_2 \qquad \text{in } (0, T), \end{aligned} \tag{3.2.5}$$

where

$$\mu_1 := (m-1)\alpha + 2, \qquad \mu_2 := (2q-m-1)\alpha + 2\beta, \qquad \mu_3 := (q-1)\alpha + 1 + \beta, \tag{3.2.6}$$

$$\gamma_1 := m-1, \qquad \gamma_2 := 2q-m-1, \qquad \gamma_3 := q-1, \tag{3.2.7}$$

$$\kappa_1 := 1, \qquad \kappa_2 := \frac{\theta}{\theta-2} \quad \text{and} \quad \kappa_3 := \frac{\theta}{\theta-1}. \tag{3.2.8}$$

Proof. As

$$\nabla u = \nabla(|x|^{-\alpha} w) = |x|^{-\alpha} \nabla w - \alpha |x|^{-\alpha-1} w \nabla |x|,$$

in $\overline{\Omega} \times (0,T)$, testing the PDI in (3.1.5) with $|x|^\alpha w^{p-1}$ and integrating by parts gives

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^p &= \int_\Omega w_t w^{p-1} \\
&= \int_\Omega u_t(|x|^\alpha w^{p-1}) \\
&\le -\int_\Omega (D(x,t,u)\nabla u + S(x,t,u)f)\cdot\nabla(|x|^\alpha w^{p-1}) \\
&\quad + \int_{\partial\Omega} |x|^\alpha w^{p-1}(D(x,t,u)\nabla u + S(x,t,u)f)\cdot\nu \qquad \text{in } (0,T),
\end{aligned}$$

wherein the boundary term is nonpositive because of the second line in (3.1.5). Therefore,

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^p \le &-(p-1)\int_\Omega D(x,t,u)w^{p-2}|\nabla w|^2 \\
&+\alpha(p-1)\int_\Omega D(x,t,u)|x|^{-1}w^{p-1}\nabla w\cdot\nabla|x| \\
&-\alpha\int_\Omega D(x,t,u)|x|^{-1}w^{p-1}\nabla w\cdot\nabla|x| \\
&+\alpha^2\int_\Omega D(x,t,u)|x|^{-2}w^p|\nabla|x||^2 \\
&-(p-1)\int_\Omega S(x,t,u)|x|^\alpha w^{p-2}f\cdot\nabla w \\
&-\alpha\int_\Omega S(x,t,u)|x|^{\alpha-1}w^{p-1}f\cdot\nabla|x| \qquad \text{in } (0,T).
\end{aligned}$$

Therein is by Young's inequality

$$\begin{aligned}
&\alpha(p-2)\int_\Omega D(x,t,u)|x|^{-1}w^{p-1}\nabla w\cdot\nabla|x| \\
\le\ &\frac{p-1}{2}\int_\Omega D(x,t,u)w^{p-2}|\nabla w|^2 + \frac{\alpha^2(p-2)^2}{2(p-1)}\int_\Omega D(x,t,u)|x|^{-2}w^p|\nabla|x||^2 \qquad \text{in } (0,T).
\end{aligned}$$

As $|\nabla|x|| = 1$ for all $x \in \Omega\setminus\{0\}$ and using (3.1.7), we may therefore find $c_1, c_2, c_3, c_4 > 0$ such that for all $p \ge 2$

$$\begin{aligned}
\frac{1}{p^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^p \le &-c_1\int_\Omega |x|^{-(m-1)\alpha}w^{p+m-3}|\nabla w|^2 \\
&+c_2\int_\Omega D(x,t,u)|x|^{-2}w^p \\
&+c_3\int_\Omega |S(x,t,u)||x|^\alpha w^{p-2}|f\cdot\nabla w| \\
&+c_4\int_\Omega |S(x,t,u)||x|^{\alpha-1}w^{p-1}|f| \qquad \text{holds in } (0,T).
\end{aligned} \tag{3.2.9}$$

By Lemma 3.2.1 (with $s = m-1$, $g = D$, $\mu = -2$, $\gamma = 0$, $\kappa = 1$) and (3.1.8) there are $c_5 > 0$ and $p_1 \ge 1$ such that

$$\int_\Omega D(x,t,u)|x|^{-2}w^p \le c_5\int_\Omega |x|^{-(m-1)\alpha-2}w^{p+m-1} + c_5 \tag{3.2.10}$$

for all $p \geq p_1$ in $(0, T)$.

Furthermore, by employing Young's inequality we may find $c_6 > 0$ such that

$$\begin{aligned} &\int_\Omega |S(x,t,u)||x|^\alpha w^{p-2} |f \cdot \nabla w| \\ &\leq \frac{c_1}{2c_3} \int_\Omega |x|^{-(m-1)\alpha} w^{p+m-3} |\nabla w|^2 + c_6 \int_\Omega |S(x,t,u)|^2 |x|^{(m+1)\alpha} w^{p-m-1} |f|^2 \end{aligned} \tag{3.2.11}$$

for all $p \geq 1$ in $(0, T)$. Therein is by Hölder's inequality (with exponents $\frac{\theta}{2}$, $\frac{\theta}{\theta-2}$; note that $\theta > n \geq 2$ by (3.1.1)) and (3.1.10)

$$\begin{aligned} &\int_\Omega |S(x,t,u)|^2 |x|^{(m+1)\alpha} w^{p-m-1} |f|^2 \\ &\leq K_f^{\frac{2}{\theta}} \left(\int_\Omega \left(|S(x,t,u)|^2 |x|^{(m+1)\alpha - 2\beta} w^{p-m-1} \right)^{\frac{\theta}{\theta-2}} \right)^{\frac{\theta-2}{\theta}} \end{aligned} \tag{3.2.12}$$

for all $p \geq 1$ in $(0, T)$.

Herein, we again make use of Lemma 3.2.1 (with $s = 2q$, $g = S^2$, $\mu = (m+1)\alpha - 2\beta$, $\gamma = -m-1$, $\kappa = \frac{\theta}{\theta-2}$) and (3.1.9) to obtain $p_2 \geq 1$ and $c_7 > 0$ such that

$$\begin{aligned} &\int_\Omega \left(|S(x,t,u)|^2 |x|^{(m+1)\alpha - 2\beta} w^{p-m-1} \right)^{\frac{\theta}{\theta-2}} \\ &\leq c_7 \int_\Omega \left(|x|^{-(2q-m-1)\alpha - 2\beta} w^{p+2q-m-1} \right)^{\frac{\theta}{\theta-2}} + c_7 \end{aligned} \tag{3.2.13}$$

holds for all $p \geq p_2$ in $(0, T)$.

Once more employing Hölder's inequality, (3.1.10), Lemma 3.2.1 (with $s = q$, $g = |S|$, $\mu = \alpha - 1 - \beta$, $\gamma = -1$, $\kappa = \frac{\theta}{\theta-1}$) and (3.1.9), we see that

$$\begin{aligned} \int_\Omega |S(x,t,u)||x|^{\alpha-1} w^{p-1} |f| &\leq K_f^{\frac{1}{\theta}} \left(\int_\Omega \left(|S(x,t,u)||x|^{\alpha-1-\beta} w^{p-1} \right)^{\frac{\theta}{\theta-1}} \right)^{\frac{\theta-1}{\theta}} \\ &\leq c_8 \left(\int_\Omega \left(|x|^{-[(q-1)\alpha+1+\beta]} w^{p+q-1} \right)^{\frac{\theta}{\theta-1}} \right)^{\frac{\theta-1}{\theta}} + c_8 \end{aligned} \tag{3.2.14}$$

holds for all $p \geq p_3$ in $(0, T)$ for certain $p_3 \geq 1$ and $c_8 > 0$.

Finally, by plugging (3.2.10)–(3.2.14) into (3.2.9), we obtain the desired estimate (3.2.5) for $p_0 := \max\{p_1, p_2, p_3\}$ and certain $C_1, C_2 > 0$. □

Before estimating the terms on the right-hand side of (3.2.5) against the dissipative term therein, we have a deeper look at the parameters in (3.2.6)–(3.2.8). Precisely due to (3.1.3), our condition on α, they allow for the following

Lemma 3.2.3. *Let $i \in \{1, 2, 3\}$ as well as μ_i and κ_i as in* (3.2.6) *and* (3.2.8)*, respectively. Then*

$$\lambda_i := \frac{\alpha \mathfrak{p}}{\kappa_i (\mu_i - (m-1)\alpha)_+} \tag{3.2.15}$$

fulfills

$$\lambda_i \in (1, \infty) \quad \textit{as well as} \quad \frac{2\kappa_i \lambda_i}{\lambda_i - 1} < \frac{2n}{n-2}.$$

Proof. Plugging (3.2.6) into (3.2.15) yields

$$\lambda_1 = \frac{\alpha \mathbb{p}}{2\kappa_1}, \quad \lambda_2 = \frac{\alpha \mathbb{p}}{\kappa_2(2\beta - 2(m-q)\alpha)_+} \quad \text{and} \quad \lambda_3 = \frac{\alpha \mathbb{p}}{\kappa_3(1+\beta-(m-q)\alpha)_+},$$

hence $\lambda_i < \infty$ since $(m-q)\alpha < \beta$ and $\kappa_i > 0$ by (3.2.1) and (3.2.8), respectively, for $i \in \{1,2,3\}$.

As $m - q \le \frac{\mathbb{p}}{\theta} + \frac{\beta\mathbb{p}-\mathbb{p}}{n}$ by (3.1.2), we furthermore have

$$\alpha > \frac{\beta}{m-q+\frac{\mathbb{p}}{n}-\frac{\mathbb{p}}{\theta}} \ge \frac{\beta}{\frac{\beta\mathbb{p}}{n}} = \frac{n}{\mathbb{p}}$$

by (3.1.3).

Since $\lambda_1 = \frac{\alpha\mathbb{p}}{2}$ and $\alpha > \frac{n}{\mathbb{p}}$, we immediately obtain $\lambda_1 > 1$ and

$$\frac{2\kappa_1\lambda_1}{\lambda_1 - 1} = \frac{2\alpha\mathbb{p}}{\alpha\mathbb{p}-2} < \frac{2n}{n-2}.$$

By (3.1.3), we have $\alpha > \frac{\beta}{m-q+\frac{\mathbb{p}}{n}-\frac{\mathbb{p}}{\theta}}$ and thus due to (3.1.2) also

$$(m-q)\alpha > \beta - \frac{\alpha\mathbb{p}}{n} + \frac{\alpha\mathbb{p}}{\theta}.$$

Therefore, we may further compute

$$\kappa_2\lambda_2 = \frac{\alpha\mathbb{p}}{2\beta - 2(m-q)\alpha} > \frac{\alpha\mathbb{p}}{2(\frac{\alpha\mathbb{p}}{n} - \frac{\alpha\mathbb{p}}{\theta})} = \frac{n\theta}{2(\theta-n)},$$

hence $\lambda_2 > \frac{(\theta-2)n}{2(\theta-n)} \ge \frac{2(\theta-2)}{2(\theta-2)} = 1$ since $n \ge 2$ and (as $(\kappa_2,\infty) \ni \xi \mapsto \frac{2\xi}{\frac{\xi}{\kappa_2}-1}$ is strictly decreasing)

$$\frac{2\kappa_2\lambda_2}{\lambda_2-1} = \frac{2\kappa_2\lambda_2}{\frac{\kappa_2\lambda_2}{\kappa_2}-1} < \frac{2n\theta}{\frac{\theta-2}{\theta}n\theta - 2(\theta-n)} = \frac{2n\theta}{n(\theta-2)-2\theta+2n} = \frac{2n}{n-2}.$$

Similarly, we see that

$$\kappa_3\lambda_3 = \frac{\alpha\mathbb{p}}{1+\beta-(m-q)\alpha} > \frac{\alpha\mathbb{p}}{1+\frac{\alpha\mathbb{p}}{n}-\frac{\alpha\mathbb{p}}{\theta}} > \frac{\alpha\mathbb{p}}{\frac{2\alpha\mathbb{p}}{n}-\frac{\alpha\mathbb{p}}{\theta}} = \frac{n\theta}{2\theta-n}$$

since $1 < \frac{\alpha\mathbb{p}}{n}$, thus $\lambda_3 > \frac{(\theta-1)n}{2\theta-n} \ge \frac{2\theta-2}{2\theta-2} = 1$ and

$$\frac{2\kappa_3\lambda_3}{\lambda_3-1} = \frac{2\kappa_3\lambda_3}{\frac{\kappa_3\lambda_3}{\kappa_3}-1} < \frac{2n\theta}{\frac{\theta-1}{\theta}n\theta-2\theta+n} = \frac{2n\theta}{n(\theta-1)-2\theta+n} = \frac{2n}{n-2}.$$

This clearly proves the lemma. □

Another important ingredient will be

Lemma 3.2.4. *Throughout* $(0, T)$,

$$\int_\Omega |x|^{-\alpha \mathbb{p}} w^{\mathbb{p}} \le M$$

holds.

Proof. This is an immediate consequence of (3.2.2) and (3.1.4). □

As further preparation, we state a quantitative Ehrling-type lemma. Since this will be also used in the proof of the quite general Lemma 3.5.1 below we neither require $n \ge 2$ nor $0 \in \Omega$.

Lemma 3.2.5. *Let* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, *be a smooth, bounded domain and* $0 < s < r < \frac{2n}{(n-2)_+}$. *Then there exist* $a \in (0,1)$ *and* $C > 0$ *such that for all* $\varepsilon > 0$, *we have*

$$\|\varphi\|_{L^r(\Omega)} \le \varepsilon \|\nabla \varphi\|_{L^2(\Omega)} + C \min\{1, \varepsilon\}^{-\frac{a}{1-a}} \|\varphi\|_{L^s(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Here and below, we set $\|\varphi\|_{L^q(\Omega)} := \left(\int_\Omega |\varphi|^q\right)^{\frac{1}{q}}$ *even for* $q \in (0,1)$.

Proof. The conditions $s < r < \frac{2n}{(n-2)_+}$ imply that

$$a := \frac{\frac{1}{s} - \frac{1}{r}}{\frac{1}{s} + \frac{1}{n} - \frac{1}{2}} = \frac{\frac{r-s}{rs}}{\frac{2n+2s-ns}{2ns}} = \frac{2nr - 2ns}{2nr + 2rs - nsr} = \frac{r-s}{r - \frac{n-2}{2n} \cdot rs}$$

satisfies $a \in (0,1)$.

Hence, we may invoke the Gagliardo–Nirenberg inequality (which holds even for $r, s \in (0,1)$, see for instance [58, Lemma 2.3]) to obtain $c_1 > 0$ with the property that

$$\|\varphi\|_{L^r(\Omega)} \le c_1 \|\nabla\varphi\|_{L^2(\Omega)}^a \|\varphi\|_{L^s(\Omega)}^{1-a} + c_1 \|\varphi\|_{L^s(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Therein we have by Young's inequality (with exponents $\frac{1}{a}, \frac{1}{1-a}$) for all $\varepsilon \in (0,1)$ and all $\varphi \in W^{1,2}(\Omega)$,

$$\begin{aligned} \|\nabla\varphi\|_{L^2(\Omega)}^a \|\varphi\|_{L^s(\Omega)}^{1-a} &= \left(\frac{\varepsilon}{ac_1} \|\nabla\varphi\|_{L^2(\Omega)}\right)^a \cdot \left(\left(\frac{\varepsilon}{ac_1}\right)^{-\frac{a}{1-a}} \|\varphi\|_{L^s(\Omega)}\right)^{1-a} \\ &\le \frac{\varepsilon}{c_1} \|\nabla\varphi\|_{L^2(\Omega)} + c_2 \varepsilon^{-\frac{a}{1-a}} \|\varphi\|_{L^s(\Omega)}, \end{aligned}$$

where $c_2 := (1-a)(ac_1)^{\frac{a}{1-a}}$.

This already implies the statement for $C := c_1(1 + c_2)$. □

In order to be able to apply Lemma 3.2.5, we first rewrite the dissipative term in (3.2.5).

Lemma 3.2.6. *There are* $c_1, c_2 > 0$ *and* $p_0 \ge 1$ *such that for all* $p \ge p_0$ *we have*

$$\left(\Omega \ni x \mapsto |x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}(x,t)\right) \in W^{1,2}(\Omega) \qquad \text{for all } t \in (0,T)$$

and

$$
\begin{aligned}
&-p^2\int_\Omega |x|^{-(m-1)\alpha} w^{p+m-3}|\nabla w|^2\\
&\le -c_1\int_\Omega \left|\nabla\left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)\right|^2 + c_2\left(\int_\Omega \left(|x|^{-\mu_1} w^{p+\gamma_1}\right)^{\kappa_1}\right)^{\frac{1}{\kappa_1}} \qquad \textit{in } (0,T),
\end{aligned}
$$

where μ_1, γ_1 and κ_1 are as in (3.2.6), (3.2.7) *and* (3.2.8), *respectively.*

Proof. We first note that for $x \in \Omega$ and $t \in (0,T)$, we have

$$
|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}(x,t) = |x|^{\frac{\alpha p}{2}} u^{\frac{p+m-1}{2}}(x,t)
$$

and hence

$$
\left(\Omega \ni x \mapsto |x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}(x,t)\right) \in C^1(\overline{\Omega}) \subset W^{1,2}(\Omega)
$$

for all $p > p_1 := \max\{\frac{2}{\alpha}, 3-m\}$ and all $t \in (0, T_{\max})$.

Thus, for $p \ge p_1$, making use of the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, $a, b \in \mathbb{R}$, we may calculate

$$
\begin{aligned}
&-\int_\Omega |x|^{-(m-1)\alpha} w^{p+m-3}|\nabla w|^2\\
&\le -\frac{2}{(p+m-1)^2}\int_\Omega \left|\nabla\left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)\right|^2\\
&\quad + \frac{((m-1)\alpha)^2}{(p+m-1)^2}\int_\Omega |x|^{-(m-1)\alpha-2} w^{p+m-1}|\nabla|x||^2 \qquad \text{in } (0,T).
\end{aligned}
$$

Because of $|\nabla|x|| \equiv 1$ in $\Omega \setminus \{0\}$ and by the definition of μ_1, γ_1 and κ_1, we have therein

$$
\int_\Omega |x|^{-(m-1)\alpha-2} w^{p+m-1}|\nabla|x||^2 = \left(\int_\Omega \left(|x|^{-\mu_1} w^{p+\gamma_1}\right)^{\kappa_1}\right)^{\frac{1}{\kappa_1}} \qquad \text{in } (0,T)
$$

for all $p \ge 1$.

Moreover, setting $p_2 := 2|m-1|$, we have $\frac{9}{4}p^2 \ge (p+m-1)^2 \ge \frac{1}{4}p^2$ for all $p \ge p_2$, so that the statement follows for $c_1 := \frac{8}{9}$, $c_2 := 4((m-1)\alpha)^2$ and $p_0 := \max\{1, p_1, p_2\} + 1$. ☐

A first application of Lemma 3.2.4 and Lemma 3.2.5 now shows that the dissipative term $\int_\Omega \left|\nabla\left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)\right|^2$ can be basically turned into $\int_\Omega w^p$. This is the only place where we (directly) need the second condition in (3.1.2), namely that $m > \frac{n-2\mathbb{p}}{n}$.

Lemma 3.2.7. *For given $\varepsilon > 0$ and $s \in (0,2)$, we may find $C > 0$ and $p_0 \ge 1$ such that*

$$
\int_\Omega w^p \le \varepsilon\int_\Omega \left|\nabla\left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)\right|^2 + C\left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)^s\right)^{\frac{2}{s}} + C \qquad (3.2.16)
$$

for all $p \ge p_0$ in $(0,T)$.

Proof. We fix $\varepsilon > 0$, $s \in (0,2)$ and p_0 as given by Lemma 3.2.6. We divide the proof in two parts.

Case 1: $m \geq 1$. Young's inequality and Lemma 3.2.5 (with $r = 2 < \frac{2n}{n-2}$) imply

$$\begin{aligned}\int_\Omega w^p &\leq \int_\Omega w^{p+m-1} + |\Omega| \\ &\leq R^{(m-1)\alpha} \int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)^2 + |\Omega| \\ &\leq \varepsilon \int_\Omega \left|\nabla \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)\right|^2 + c_1 \left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)^s\right)^{\frac{2}{s}} + |\Omega|\end{aligned}$$

in $(0,T)$ for some $c_1 > 0$ and thus (3.2.16) for $C := \max\{c_1, |\Omega|\}$.

Case 2: $m < 1$. Since (3.1.2) and $n \geq 2$ assert $m > \frac{n-2\mathbb{p}}{n} \geq 1 - \mathbb{p}$, we have $r := \frac{2\mathbb{p}}{m-1+\mathbb{p}} \in (2, \frac{2n}{n-2})$ and $\lambda := \frac{\mathbb{p}}{1-m} \in (1,\infty)$. We then obtain

$$\begin{aligned}\int_\Omega w^p &\leq \left(\int_\Omega |x|^{-\alpha\mathbb{p}} w^{\mathbb{p}}\right)^{\frac{1}{\lambda}} \left(\int_\Omega |x|^{\frac{\alpha\mathbb{p}}{\lambda-1}} w^{\frac{p\lambda-\mathbb{p}}{\lambda-1}}\right)^{\frac{\lambda-1}{\lambda}} \\ &\leq M^{\frac{1}{\lambda}} \left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)^r\right)^{\frac{\lambda-1}{\lambda}} \end{aligned} \tag{3.2.17}$$

for all $p \geq 1$ in $(0,T)$ by Hölder's inequality as well as Lemma 3.2.4 and because of

$$\frac{\alpha\mathbb{p}}{\lambda-1} \cdot \left(-\frac{2}{(m-1)\alpha r}\right) = \frac{(m-1)\alpha\mathbb{p}}{m-1+\mathbb{p}} \cdot \frac{m-1+\mathbb{p}}{(m-1)\alpha\mathbb{p}} = 1$$

as well as

$$\frac{p\lambda-\mathbb{p}}{\lambda-1} \cdot \frac{2}{(p+m-1)r} = \frac{(m-1)(\frac{p\mathbb{p}}{m-1} + \mathbb{p})}{m-1+\mathbb{p}} \cdot \frac{m-1+\mathbb{p}}{(p+m-1)\mathbb{p}} = 1.$$

Noting that $\frac{r(\lambda-1)}{\lambda} = 2$, we again employ Lemma 3.2.5 to see that

$$\begin{aligned}&\left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)^r\right)^{\frac{(\lambda-1)}{\lambda}} \\ &= \left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)^r\right)^{\frac{2}{r}} \\ &\leq \frac{\varepsilon}{M^{\frac{1}{\lambda}}} \int_\Omega \left|\nabla \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)\right|^2 + c_2 \left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}}\right)^s\right)^{\frac{2}{s}}\end{aligned} \tag{3.2.18}$$

holds in $(0,T)$ for some $c_2 > 0$. The desired estimate (3.2.16) is then a direct consequence of (3.2.17) and (3.2.18). □

We are now prepared to prove

Lemma 3.2.8. *For any $0 < s < s_0 := \min\{\frac{2n}{n-2}, \frac{1}{(m-1)_+}\}$, we can find $C > 0$, $p_0 > 1$ and $\nu \geq 1$ such that for all $p \geq p_0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^p + \int_\Omega w^p \leq Cp^\nu + Cp^\nu \left(\int_\Omega w^{(p+m-1)s-1}\right)^{\frac{1}{s}} \qquad \textit{in } (0,T). \tag{3.2.19}$$

Proof. By Lemma 3.2.2 and Lemma 3.2.6, there are $c_1, c_2 > 0$ and $p_1 > 1$ such that for all $p \geq p_1$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^p + c_1 \int_\Omega \left| \nabla \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}} \right) \right|^2 \\
&\leq c_2 p^2 \sum_{i=1}^{3} \left(\int_\Omega \left(|x|^{-\mu_i} w^{p+\gamma_i} \right)^{\kappa_i} \right)^{\frac{1}{\kappa_i}} + c_2 p^2
\end{aligned} \tag{3.2.20}$$

holds throughout $(0, T)$, where $\mu_i, \gamma_i, \kappa_i$, $i \in \{1, 2, 3\}$, are given by (3.2.6), (3.2.7) and (3.2.8), respectively.

Our goal is to estimate the terms on the right-hand side in (3.2.20) against the dissipative term therein. As a starting point, we use Hölder's inequality and Lemma 3.2.4 to compute for $\lambda > 1$, $p \geq 1$ and $i \in \{1, 2, 3\}$,

$$\begin{aligned}
&\left(\int_\Omega \left(|x|^{-\mu_i} w^{p+\gamma_i} \right)^{\kappa_i} \right)^{\frac{1}{\kappa_i}} \\
&= \left(\int_\Omega |x|^{-\frac{\alpha \mathbb{p}}{\lambda}} w^{\frac{\mathbb{p}}{\lambda}} \cdot |x|^{-\mu_i \kappa_i + \frac{\alpha \mathbb{p}}{\lambda}} w^{(p+\gamma_i)\kappa_i - \frac{\mathbb{p}}{\lambda}} \right)^{\frac{1}{\kappa_i}} \\
&\leq M^{\frac{1}{\kappa_i \lambda}} \left(\int_\Omega |x|^{\frac{-\mu_i \kappa_i \lambda + \alpha \mathbb{p}}{\lambda - 1}} w^{\frac{(p+\gamma_i)\kappa_i \lambda - \mathbb{p}}{\lambda - 1}} \right)^{\frac{\lambda - 1}{\kappa_i \lambda}} \qquad \text{in } (0, T).
\end{aligned} \tag{3.2.21}$$

For $p \in (1, \infty)$ and $i \in \{1, 2, 3\}$, we set

$$\lambda_i(p) \coloneqq \begin{cases} \dfrac{\alpha p \mathbb{p}}{\kappa_i [p(\mu_i - (m-1)\alpha) + (m-1)(\mu_i - \alpha\gamma_i)]_+}, & p < \infty, \\ \dfrac{\alpha \mathbb{p}}{\kappa_i (\mu_i - (m-1)\alpha)_+}, & p = \infty, \end{cases}$$

which entails that $\lim_{p \nearrow \infty} \lambda_i(p) = \lambda_i(\infty)$. Lemma 3.2.3 asserts $\lambda_i(\infty) \in (1, \infty)$, hence there is $p_2 \geq p_1$ such that also $\lambda_i(p) \in (1, \infty)$ for all $p \geq p_2$ and $i \in \{1, 2, 3\}$.

Setting furthermore

$$b_i(p) \coloneqq 2 \cdot \frac{\alpha p - (\mu_i - \alpha \gamma_i)}{\alpha p}, \qquad i \in \{1, 2, 3\},$$

and choosing $\lambda = \lambda_i(p)$ in (3.2.21), we obtain

$$\begin{aligned}
&\left(\int_\Omega \left(|x|^{-\mu_i} w^{p+\gamma_i} \right)^{\kappa_i} \right)^{\frac{1}{\kappa_i}} \\
&\leq \max\{M, 1\} \left(\int_\Omega \left(|x|^{\frac{-(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}} \right)^{\frac{\kappa_i \lambda_i(p)}{\lambda_i(p) - 1} b_i(p)} \right)^{\frac{\lambda_i(p) - 1}{\kappa_i \lambda_i(p)}} \qquad \text{in } (0, T)
\end{aligned} \tag{3.2.22}$$

for all $p \geq p_2$ and $i \in \{1, 2, 3\}$ since

$$\begin{aligned}
&\frac{-\mu_i \kappa_i \lambda_i(p) + \alpha \mathbb{p}}{\lambda_i(p) - 1} \cdot \frac{2(\lambda_i(p) - 1)}{-(m-1)\alpha \kappa_i \lambda_i(p) b_i(p)} \\
&= \frac{-\mu_i + \frac{\alpha \mathbb{p}}{\kappa_i \lambda_i(p)}}{-(m-1)\alpha} \cdot \frac{\alpha p}{\alpha p - (\mu_i - \alpha \gamma_i)} \\
&= \frac{-\mu_i \alpha p + \alpha[p(\mu_i - (m-1)\alpha) + (m-1)(\mu_i - \alpha \gamma_i)]}{-(m-1)\alpha(\alpha p - (\mu_i - \alpha \gamma_i))} = 1
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(p+\gamma_i)\kappa_i\lambda_i(p)-\mathbb{p}}{\lambda_i(p)-1}\cdot\frac{2(\lambda_i(p)-1)}{(p+m-1)\kappa_i\lambda_i(p)b_i(p)}\\
&=\frac{(p+\gamma_i)-\frac{\mathbb{p}}{\kappa_i\lambda_i(p)}}{p+m-1}\cdot\frac{\alpha p}{\alpha p-(\mu_i-\alpha\gamma_i)}\\
&=\frac{(p+\gamma_i)\alpha p-[p(\mu_i-(m-1)\alpha)+(m-1)(\mu_i-\alpha\gamma_i)]}{(p+m-1)(\alpha p-(\mu_i-\alpha\gamma_i))}=1
\end{aligned}$$

for all $p\ge p_2$ and $i\in\{1,2,3\}$.

Lemma 3.2.3 further asserts

$$\lim_{p\nearrow\infty}\frac{2\kappa_i\lambda_i(p)}{\lambda_i(p)-1}=\frac{2\kappa_i\lambda_i(\infty)}{\lambda_i(\infty)-1}<\frac{2n}{n-2}$$

for all $i\in\{1,2,3\}$. As moreover (3.2.6) and (3.2.7) entail

$$\mu_i-\alpha\gamma_i=\begin{cases}2, & i=1,\\ 2\beta, & i=2,\\ 1+\beta, & i=3\end{cases}$$

and hence $\beta_i(p)<2$ for all $p\ge 1$ and $i\in\{1,2,3\}$, we may choose $p_3\ge p_2$ and $r\in(s,\frac{2n}{n-2})$ such that still

$$\frac{\kappa_i\lambda_i(p)}{\lambda_i(p)-1}b_i(p)\le r$$

for all $i\in\{1,2,3\}$ and all $p\ge p_3$.

By Hölder's inequality and the elementary inequality $\xi^A\le 1+\xi^B$ for $\xi\ge 0$ and $0<A<B$, we have

$$\begin{aligned}
&\left(\int_\Omega\left(|x|^{-\frac{(m-1)\alpha}{2}}w^{\frac{p+m-1}{2}}\right)^{\frac{\kappa_i\lambda_i(p)}{\lambda_i(p)-1}b_i(p)}\right)^{\frac{\lambda_i(p)-1}{\kappa_i\lambda_i(p)}}\\
&\le\max\{|\Omega|,1\}\left(\int_\Omega\left(|x|^{-\frac{(m-1)\alpha}{2}}w^{\frac{p+m-1}{2}}\right)^r\right)^{\frac{b_i(p)}{r}}\\
&\le c_3+c_3\left(\int_\Omega\left(|x|^{-\frac{(m-1)\alpha}{2}}w^{\frac{p+m-1}{2}}\right)^r\right)^{\frac{2}{r}}\qquad\text{in }(0,T)
\end{aligned}\tag{3.2.23}$$

for all $p\ge p_3$ and $i\in\{1,2,3\}$, where $c_3:=\max\{|\Omega|,1\}$.

Herein, we may now finally apply Lemma 3.2.5 together with Young's inequality to obtain $c_4>0$ such that

$$\begin{aligned}
&\left(\int_\Omega\left(|x|^{-\frac{(m-1)\alpha}{2}}w^{\frac{p+m-1}{2}}\right)^r\right)^{\frac{2}{r}}\\
&\le\frac{c_1}{6c_2c_3p^2\max\{M,1\}}\int_\Omega\left|\nabla\left(|x|^{-\frac{(m-1)\alpha}{2}}w^{\frac{p+m-1}{2}}\right)\right|^2\\
&\quad+c_4p^{\frac{2a}{1-a}}\left(\int_\Omega\left(|x|^{-\frac{(m-1)\alpha}{2}}w^{\frac{p+m-1}{2}}\right)^s\right)^{\frac{2}{s}}\qquad\text{in }(0,T)
\end{aligned}\tag{3.2.24}$$

for all $p \geq p_3$.

By combining (3.2.20), (3.2.22)–(3.2.24) and Lemma 3.2.7 (with $\varepsilon = \frac{c_1}{2}$), we may find $c_5 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^p + \int_\Omega w^p \leq c_5 p^{\frac{4a}{1-a}} + c_5 p^{\frac{4a}{1-a}} \left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}} \right)^s \right)^{\frac{2}{s}} \qquad \text{in } (0,T) \quad (3.2.25)$$

for all $p \geq p_3$.

The assumption $s \leq \frac{1}{(m-1)_+}$ implies $\alpha - (m-1)\alpha s \geq 0$, thus again by Hölder's inequality and Lemma 3.2.4,

$$\begin{aligned}\left(\int_\Omega \left(|x|^{-\frac{(m-1)\alpha}{2}} w^{\frac{p+m-1}{2}} \right)^s \right)^2 &\leq \left(\int_\Omega |x|^{-\alpha} w \right) \left(\int_\Omega |x|^{-(m-1)\alpha s + \alpha} w^{(p+m-1)s-1} \right) \\ &\leq M^{\frac{1}{\mathbb{p}}} |\Omega|^{\frac{\mathbb{p}-1}{\mathbb{p}}} R^{\alpha-(m-1)\alpha s} \int_\Omega w^{(p+m-1)s-1} \qquad \text{in } (0,T),\end{aligned}$$

which together with (3.2.25) implies (3.2.19) for some $C > 0$, $p_0 \coloneqq p_3$ and $\nu \coloneqq \frac{4a}{1-a}$. □

A direct consequence thereof is

Lemma 3.2.9. *For all $p \in (1,\infty)$, there exists $C > 0$ such that*

$$\int_\Omega w^p(\cdot,t) \leq C \qquad \textit{for all } t \in (0,T). \qquad (3.2.26)$$

Proof. Let $p_0 > 1$ and $s_0 > 0$ be as in Lemma 3.2.8. By Hölder's inequality we may without loss of generality assume that $p > p_0$ with $(p+m-1)s_0 - 1 > 1$.

Choosing $s \in (0, s_0)$ such that $(p+m-1)s - 1 = 1$ and noting that

$$\int_\Omega w \leq R^\alpha \int_\Omega |x|^{-\alpha} w \leq R^\alpha \int_\Omega |x|^{-\alpha} w \leq R^\alpha M^{\frac{1}{\mathbb{p}}} |\Omega|^{\frac{\mathbb{p}-1}{\mathbb{p}}} \qquad \text{in } (0,T)$$

by Hölder's inequality and Lemma 3.2.6, we may apply Lemma 3.2.8 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^p \leq -\int_\Omega w^p + C_p \qquad \text{in } (0,T)$$

for some $C_p > 0$ and hence $\int_\Omega w^p \leq \max\{\int_\Omega w(\cdot,0)^p, C_p\}$. Since moreover $\int_\Omega w(\cdot,0)^p \leq |\Omega| \cdot \|w(\cdot,0)\|_{L^\infty(\Omega)}^p \leq |\Omega| L^p$ by (3.2.2) and (3.1.11), we may conclude (3.2.26). □

Due to a well-established Moser-type iteration technique (see [2] and [64] for early examples or also [83, Lemma A.1] for an application relevant to quasilinear Keller–Segel systems), we can also obtain an L^∞ bound for w.

Lemma 3.2.10. *There is $C > 0$ such that*

$$\|w\|_{L^\infty(\Omega\times(0,T))} < C. \qquad (3.2.27)$$

Proof. We set $s \coloneqq \frac{1}{2}\min\{\frac{1}{(m-1)_+}, 1\} < \frac{2n}{n-2}$. Then Lemma 3.2.8 asserts the existence of $\tilde{p} > 1$, $c_1 > 0$ and $\nu > 1$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^p + \int_\Omega w^p \leq c_1 p^\nu + c_1 p^\nu \left(\int_\Omega w^{(p+m-1)s-1} \right)^{\frac{1}{s}} \qquad \text{in } (0,T) \qquad (3.2.28)$$

for all $p \ge \tilde{p}$.

We further set

$$p_0 \coloneqq \max\{\tilde{p}, 1-(m-1)s\} \tag{3.2.29}$$

and

$$p_j \coloneqq \frac{p_{j-1}+1-(m-1)s}{s} \tag{3.2.30}$$

for $j \in \mathbb{N}\setminus\{0\}$.

As $s \le \frac{1}{(m-1)_+}$ and $s \le \frac{1}{2}$, a straightforward induction gives

$$p_j \ge \frac{p_{j-1}}{s} \ge \frac{p_0}{s^j} \ge 2^j p_0 \ge 2^j \qquad \text{for } j \in \mathbb{N}_0, \tag{3.2.31}$$

in particular the sequence $(p_j)_{j\in\mathbb{N}_0}$ is increasing. On the other hand, by (3.2.29) and another induction,

$$p_j \le \frac{p_{j-1}+p_0}{s} \le \frac{2p_{j-1}}{s} \le \left(\frac{2}{s}\right)^j p_0 \qquad \text{for } j \in \mathbb{N}. \tag{3.2.32}$$

Since (3.2.30) is equivalent to $p_{j-1} = (p_j+m-1)s-1$, $j \in \mathbb{N}$, an ODE comparison argument and (3.2.28) (with $p = p_j$) yield

$$\int_\Omega w^{p_j}(\cdot,t) \le \max\left\{\int_\Omega w^{p_j}(\cdot,0),\, c_1 p_j^\nu + c_1 p_j^\nu \sup_{\tau\in(0,T)} \left(\int_\Omega w^{p_{j-1}}(\cdot,\tau)\right)^{\frac{1}{s}}\right\}$$

for all $t \in (0,T]$ and all $j \in \mathbb{N}$. We note that Lemma 3.2.9 asserts finiteness of the right-hand side therein.

Therefore, $A_j \coloneqq \sup_{t\in(0,T)} \|w(\cdot,t)\|_{L^{p_j}(\Omega)}$, $j \in \mathbb{N}_0$, fulfills

$$A_j \le \max\left\{\|w(\cdot,0)\|_{L^{p_j}(\Omega)},\, (c_1 p_j^\nu)^{\frac{1}{p_j}} \left(1 + A_{j-1}^{\frac{p_{j-1}}{s}}\right)^{\frac{1}{p_j}}\right\} \qquad \text{for all } j \in \mathbb{N}.$$

To make sure that C will indeed only depend on the parameters in (3.1.1) and on α, similarly as in the proof of [26, Lemma 2.11], we also set $B_0 \coloneqq \max\{L,1\}\max\{|\Omega|,1\}$ and

$$B_j \coloneqq \max\left\{B_0,\, (c_1 p_j^\nu)^{\frac{1}{p_j}} \left(1 + B_{j-1}^{\frac{p_{j-1}}{s}}\right)^{\frac{1}{p_j}}\right\} \qquad \text{for all } j \in \mathbb{N}. \tag{3.2.33}$$

Since $\|w(\cdot,0)\|_{L^p(\Omega)} \le L|\Omega|^{\frac{1}{p}} \le B_0$ by by (3.1.11) and (3.2.2) for all $p \in [1,\infty)$, we conclude $A_0 \le B_0$ and, as $(0,\infty) \ni \xi \mapsto (c_1 p_j^\nu)^{\frac{1}{p_j}} \left(1+\xi^{\frac{p_{j-1}}{s}}\right)^{\frac{1}{p_j}}$ is increasing for all $j \in \mathbb{N}$, also $A_j \le B_j$ for all $j \in \mathbb{N}$.

We first suppose that there is a strictly increasing sequence $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $B_{j_k} \le B_0$ for all $k \in \mathbb{N}$. As then

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} = \lim_{k\to\infty} \|w(\cdot,t)\|_{L^{p_{j_k}}(\Omega)} \le \limsup_{k\to\infty} A_{j_k} \le \limsup_{k\to\infty} B_{j_k} \le B_0$$

for all $t \in (0,T)$ since $\lim_{k\to\infty} p_{j_k} = \infty$ by (3.2.31), this already implies (3.2.27) for $C := B_0$.

Hence, we now suppose that on the contrary there is $j_0 \in \mathbb{N}$ such that $B_j > B_0$ for all $j \geq j_0$. Since then also $B_j \geq 1$ for all $j \geq j_0$ and because of $\frac{p_j}{s} > 1$ for all $j \in \mathbb{N}_0$, we conclude from (3.2.33) that

$$B_j \leq (2c_1 p_j^\nu)^{\frac{1}{p_j}} B_{j-1}^{\frac{p_{j-1}}{p_j s}} \qquad \text{for all } j > j_0.$$

As (3.2.31) entails $\frac{p_{j-1}}{p_j s} \leq 1$, we further obtain

$$B_j \leq (c_2 p_j^\nu)^{\frac{1}{p_j}} B_{j-1} \qquad \text{for all } j > j_0,$$

where $c_2 := 2c_1$, and hence by induction and (3.2.32),

$$B_j \leq \left(\prod_{i=j_0+1}^{j} (c_2 p_i^\nu)^{\frac{1}{p_i}} \right) B_{j_0} \leq c_3^{\sum_{i=j_0+1}^{j} \frac{1}{p_i}} \cdot \left(\tfrac{2}{s}\right)^{\sum_{i=j_0+1}^{j} \frac{i\nu}{p_i}} \cdot B_{j_0} \qquad \text{for all } j > j_0$$

with $c_3 := c_2 p_0^\nu$.

As therein by (3.2.31),

$$\sum_{i=j_0+1}^{j} \frac{1}{p_i} \leq \sum_{i=j_0+1}^{j} \frac{i\nu}{p_i} \leq \sum_{i=1}^{\infty} \frac{i\nu}{2^i} =: c_4 < \infty \qquad \text{for all } j \geq j_0,$$

we conclude

$$\sup_{t\in(0,T)} \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq \limsup_{j\to\infty} A_j \leq \limsup_{j\to\infty} B_j \leq \left(\frac{2c_3}{s}\right)^{c_4} B_{j_0} < \infty,$$

which in turn directly implies the statement. □

The main result of this section now follows immediately.

Proof of Theorem 3.1.1. Combine Lemma 3.2.10 and (3.2.2). □

3.3. Pointwise estimates in quasilinear Keller–Segel systems

We suppose henceforth that $n \geq 2$, $R > 0$ and $\Omega := B_R(0)$.

In order to apply Theorem 3.1.1 to the system (3.KS)—and hence prove Theorem 3.1.3—, we need some integrability information about ∇v. This is provided by

Lemma 3.3.1. *Let $K, L, M > 0$, $\tilde{\alpha} > \beta > n-1$ and $\theta \in (1,\infty]$. Then there is $C > 0$ with the following property:*

Suppose that $T \in (0,\infty]$, $g \in C^0(\overline{\Omega} \times [0,T))$ is radially symmetric and nonnegative with

$$\|g(\cdot,t)\|_{L^1(\Omega)} \leq M \qquad \text{for all } t \in (0,T),$$

that $v_0 \in W^{1,\infty}(\Omega)$ *is radially symmetric and nonnegative with*

$$\|v_0\|_{W^{1,\infty}(\Omega)} \le L$$

and that, if $\theta = \infty$,

$$g(x,t) \le K|x|^{-\tilde{\alpha}} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T).$$

Then any classical, radially symmetric solution $v \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *to*

$$\begin{cases} v_t = \Delta v - v + g(x,t), & \text{in } \Omega \times (0,T), \\ \partial_\nu v = 0, & \text{in } \partial\Omega \times (0,T), \\ v(\cdot,0) = v_0, & \text{in } \Omega \end{cases}$$

fulfills

$$\sup_{t\in(0,T)} \int_\Omega |x|^{\theta\beta} |\nabla v(x,t)|^\theta \, \mathrm{d}x \le C$$

if $\theta < \infty$ *and*

$$\sup_{t\in(0,T)} |\nabla v(x,t)| \le C|x|^{-\beta} \qquad \text{for all } x \in \Omega$$

if $\theta = \infty$.

Proof. See [106, Lemma 3.4]. (Let us also remark that certain generalizations of this lemma will be proven in Chapter 4.) □

We are now indeed able to employ Theorem 3.1.1 in order to obtain pointwise estimates for solutions to systems slightly more general than (3.KS). (The generality is needed as the following lemma will be used not only to prove Theorem 3.1.3 but also in the proof of Lemma 3.4.3 below.)

Lemma 3.3.2. *Suppose that the parameters in* (3.1.13) *comply with* (3.1.14) *and set* $K_g > 0$. *Then for any* $\alpha > \underline{\alpha}$, *with* $\underline{\alpha}$ *as in* (3.1.15), *and any* $\beta > n - 1$, *there exists* $C > 0$ *with the following property:*

Given functions in (3.1.16) *and* $g \in C^0([0,\infty))$ *complying with* (3.1.11), (3.1.17)–(3.1.20) *and*

$$g(\rho) \le K_g \rho \qquad \text{for } \rho \ge 0, \tag{3.3.1}$$

any nonnegative and radially symmetric classical solution $(u,v) \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *of*

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - S(u,v)\nabla v), & \text{in } \Omega \times (0,T), \\ v_t = \Delta v - v + g(u), & \text{in } \Omega \times (0,T), \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0, & \text{in } \Omega \end{cases} \tag{3.3.2}$$

fulfills (3.1.12) *and* $|\nabla v(x,t)| \le C|x|^{-\beta}$ *for* $x \in \Omega$ *and* $t \in (0,T)$.

Proof. We fix such a solution (u, v) and functions in (3.1.16) as well as $g \in C^0([0,\infty))$, but emphasize that all constants below only depend on the parameters in (3.1.13) as well as on K_g, α and β.

Setting $\mathbb{p} := 1$ and noting that

$$\lim_{\tilde\beta \searrow n-1} \lim_{\theta \nearrow \infty} \frac{\tilde\beta}{m - q + \frac{\mathbb{p}}{n} - \frac{\mathbb{p}}{\theta}} = \frac{n(n-1)}{(m-q)n+1} = \underline{\alpha},$$

we can choose $\tilde\beta \in (n-1, \beta)$ small enough and $\theta > n$ large enough such that still

$$\alpha > \frac{\tilde\beta}{m - q + \frac{\mathbb{p}}{n} - \frac{\mathbb{p}}{\theta}}.$$

Setting

$$\tilde D(x,t,\rho) := D(\rho, v(x,t)), \quad \tilde S(x,t,\rho) := D(\rho, v(x,t)) \quad \text{and} \quad f(x,t) := -\nabla v(x,t)$$

for $\rho \ge 0, x \in \overline\Omega$ and $t \in (0,T)$, we see that (3.1.6)–(3.1.9) are satisfied (for $\tilde D, \tilde S$ instead of D, S), while (3.1.4) follows by (3.1.20) and Remark 3.1.2. Furthermore, the boundary conditions in (3.3.2) imply

$$\big(\tilde D(x,t,u)\nabla u + \tilde S(x,t,u) f\big) \cdot \nu = 0 \le 0 \qquad \text{on } \partial\Omega \times (0,T).$$

As also

$$K_f := \sup_{t\in(0,T)} \int_\Omega |x|^{\theta\tilde\beta} |f(x,t)|^\theta \,\mathrm{d}x = \sup_{t\in(0,T)} \int_\Omega |x|^{\theta\tilde\beta} |\nabla v(x,t)|^\theta \,\mathrm{d}x < \infty$$

by Lemma 3.3.1, we may indeed invoke Theorem 3.1.1 to obtain $C > 0$ such that (3.1.12) holds. Once more applying Lemma 3.3.1, now with $\theta = \infty$, yields

$$|\nabla v(x,t)| \le C'|x|^{-\tilde\beta} \le C' \max\{R,1\}^{\beta-\tilde\beta} |x|^{-\beta} \qquad \text{for } x \in \Omega \text{ and } t \in (0,T)$$

for some $C' > 0$. □

An immediate consequence thereof is Theorem 3.1.3.

Proof of Theorem 3.1.3. Choosing $g = \mathrm{id}$ (and, say, $K_g = 1$) in Lemma 3.3.2, we see that (3.3.2) reduces then to (3.KS). □

3.4. Existence of blow-up profiles

Throughout this section, we suppose $n \ge 2$, $R > 0$, $\Omega := B_R(0)$, and that (3.1.11) and (3.1.17)–(3.1.19) are fulfilled for certain parameters and functions in (3.1.13) and (3.1.16), respectively. In addition—and in contrast to the preceding sections—, we also assume (3.1.21), that is, that $D \ge \eta$ for some $\eta > 0$.

Furthermore, we fix $T_{\max} < \infty$ and a solution (u, v) of (3.KS) (with $T_{\max}$ instead of T) with the property $\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

We now examine whether and in which form $\lim_{t\nearrow T_{\max}} u(\cdot,t)$ and $\lim_{t\nearrow T_{\max}} v(\cdot,t)$ exist. To that end, we may moreover assume

$$u_0, v_0 \in C^2(\overline{\Omega}) \quad \text{as well as} \quad u, v \in C^{2,1}(\overline{\Omega} \times [0, T_{\max}))$$

since the behavior of (u,v) at $T_{\max}$ may be directly inferred from that of $(\tilde{u},\tilde{v})$ at $\frac{T_{\max}}{2}$, where

$$(\tilde{u},\tilde{v}) \coloneqq \left(u(\cdot,\cdot+\tfrac{T_{\max}}{2}), v(\cdot,\cdot+\tfrac{T_{\max}}{2})\right) \in \left(C^{2,1}(\overline{\Omega}\times[0,\tfrac{T_{\max}}{2}))\right)^2.$$

Furthermore, for $\varepsilon \in (0,1)$, we fix henceforth $G_\varepsilon \in C^\infty([0,\infty))$ satisfying $G_\varepsilon(\xi) = \xi$ for all $\xi \in [0, \frac{1}{\varepsilon}]$ and $0 \le G_\varepsilon(\xi) \le \frac{2}{\varepsilon}$ for all $\xi \ge 0$.

The main idea is to construct solutions $(u_\varepsilon, v_\varepsilon)$, $\varepsilon \in (0,1)$ to certain approximative problems which converge along a subsequence to, say, $(\widehat{u},\widehat{v})$. We will then see that these functions coincide with u and v in $\Omega\setminus\{0\}\times(0,T_{\max})$ such that, for instance, $\lim_{t\nearrow T_{\max}} u(\cdot,t) = \widehat{u}(\cdot,T_{\max})$.

Lemma 3.4.1. *For any $\varepsilon \in (0,1)$, there exists $T_{\max,\varepsilon}$ and a pair of nonnegative functions $(u_\varepsilon, v_\varepsilon)$ solving*

$$\begin{cases} u_{\varepsilon t} = \nabla\cdot(D(u_\varepsilon,v_\varepsilon)\nabla u_\varepsilon - S(G_\varepsilon(u_\varepsilon),v_\varepsilon)\nabla v_\varepsilon), & \text{in } \Omega\times(0,T_{\max,\varepsilon}),\\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + G_\varepsilon(u_\varepsilon), & \text{in } \Omega\times(0,T_{\max,\varepsilon}),\\ \partial_\nu u_\varepsilon = \partial_\nu v_\varepsilon = 0, & \text{on } \partial\Omega\times(0,T_{\max,\varepsilon}),\\ u_\varepsilon(\cdot,0)=u_0, v_\varepsilon(\cdot,0)=v_0, & \text{in } \Omega \end{cases} \tag{3.4.1}$$

classically and having the property that if $T_{\max,\varepsilon} < \infty$ then

$$\limsup_{t\nearrow T_{\max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Proof. Local existence and extensibility can be proved as in [51, Lemmata 2.1–2.4], which essentially relies on regularity theory for nondegenerate parabolic equations and Schauder's fixed point theorem—while nonnegativity follows by the maximum principle. ☐

For all $\varepsilon \in (0,1)$, we henceforth fix $u_\varepsilon, v_\varepsilon$ and $T_{\max,\varepsilon}$ as given by Lemma 3.4.1. Quite standard methods now allow us to conclude that the regularized solutions are global in time.

Lemma 3.4.2. *Let $\varepsilon \in (0,1)$. Then the solution $(u_\varepsilon, v_\varepsilon)$ constructed in Lemma 3.4.1 is global in time; that is, $T_{\max,\varepsilon} = \infty$.*

Proof. Since G_ε is bounded, L^p-L^q estimates (cf. [94, Lemma 1.3 (ii)]) rapidly yield

$$c_1 \coloneqq \sup_{t\in(0,T_{\max,\varepsilon})} \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} < \infty.$$

Testing the first equation in (3.4.1) with u_ε^{p-1}, $p > 2$, gives

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^p \\ &= -(p-1)\int_\Omega u_\varepsilon^{p-2} D(u_\varepsilon,v_\varepsilon)|\nabla u_\varepsilon|^2 + (p-1)\int_\Omega u_\varepsilon^{p-2} S(G(u_\varepsilon),v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\ &\le -\eta(p-1)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + c_1c_2(p-1)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon| \qquad \text{in } (0,T_{\max,\varepsilon}), \end{aligned} \tag{3.4.2}$$

where $c_2 := \|S\|_{L^\infty((0,\frac{2}{\varepsilon})\times(0,\infty))}$.

Therein is by Young's inequality

$$\begin{aligned}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon| &\le \frac{\eta}{4c_1c_2}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \frac{c_1c_2}{\eta}\int_\Omega u_\varepsilon^{p-2} \\ &\le \frac{\eta}{4c_1c_2}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \frac{c_1c_2}{2\eta}\int_\Omega u_\varepsilon^{p} + \frac{c_1c_2|\Omega|}{2\eta} \qquad \text{in } (0,T_{\max,\varepsilon}),\end{aligned}$$

so that integrating (3.4.2) along with an ODE comparison argument yields

$$\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^p(\Omega)} < \infty \qquad \text{for all finite } T\in(0,T_{\max,\varepsilon}]. \tag{3.4.3}$$

By [83, Lemma A.1], this implies (3.4.3) also for $p=\infty$, so that the extensibility criterion in Lemma 3.4.1 indeed asserts $T_{\max,\varepsilon}=\infty$. □

Parabolic regularity allows us to obtain the following

Lemma 3.4.3. *For each $\delta\in(0,R)$ and $0<\tau<T<\infty$, there exist $C>0$ and $\gamma\in(0,1)$ such that for all $\varepsilon\in(0,1)$*

$$\|u_\varepsilon\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(K)} \le C \quad \textit{and} \quad \|v_\varepsilon\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(K)} \le C, \tag{3.4.4}$$

where $K := \overline{\Omega}\setminus B_\delta(0)\times[\tau,T]$.

Proof. This can be shown as in [106, Lemma 4.3]. We briefly recall the main idea.

We start by fixing a cutoff function $\zeta\in C^\infty(\overline{\Omega}\times[0,\infty))$ such that

$$\begin{aligned}\zeta &= 1 \qquad \text{in } K,\\ \zeta &= 0 \qquad \text{in } \left(\overline{B}_{\frac{\delta}{2}}(0)\times[0,\infty)\right)\cup\left(\overline{\Omega}\times[0,\tfrac{\tau}{2}]\right) \text{ and}\\ \partial_\nu\zeta &= 0 \qquad \text{on } \partial\Omega\times[0,\infty)\end{aligned}$$

and set, for $\varepsilon\in(0,1)$,

$$w_\varepsilon := \zeta u_\varepsilon \quad \text{as well as} \quad z_\varepsilon := \zeta v_\varepsilon.$$

By Lemma 3.3.2, there exist $c_1,\alpha,\beta>0$ such that

$$|u_\varepsilon(x,t)| \le c_1|x|^{-\alpha} \quad \text{and} \quad |\nabla v_\varepsilon(x,t)| \le c_1|x|^{-\beta}$$

for all $x\in\Omega$, $t\in(0,T+1)$ and $\varepsilon\in(0,1)$. In particular,

$$\sup_{\varepsilon\in(0,1)}\left(\|w_\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])} + \|z_\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])}\right) < \infty.$$

Basically, the statement follows then by parabolic regularity theory, applied to w_ε and z_ε for $\varepsilon\in(0,1)$. We sketch the main steps.

At first, [74, Theorem 1.3] gives $\tau_1\in(0,\tau)$ and $\gamma_1\in(0,1)$ such that

$$\sup_{\varepsilon\in(0,1)}\|w_\varepsilon\|_{C^{\gamma_1,\frac{\gamma_1}{2}}(\overline{\Omega}\times[\tau_1,T])} < \infty.$$

In a second step, one uses this information along with [48, Theorem IV.5.3] to obtain

$$\sup_{\varepsilon\in(0,1)} \|z_\varepsilon\|_{C^{2+\gamma_2,1+\frac{\gamma_2}{2}}(\overline{\Omega}\times[\tau_2,T])} < \infty$$

for some $\tau_2 \in (\tau_1, \tau)$ and $\gamma_2 \in (0, \gamma_1)$.

Finally, by employing first [59, Theorem 1.1] and then again [48, Theorem IV.5.3], we may find $\tau_2 < \tau_3 < \tau_4 < \tau$ and $0 < \gamma_4 < \gamma_3 < \gamma_2$ such that

$$\sup_{\varepsilon\in(0,1)} \|w_\varepsilon\|_{C^{1+\gamma_3,\frac{1+\gamma_3}{2}}(\overline{\Omega}\times[\tau_3,T])} < \infty$$

and

$$\sup_{\varepsilon\in(0,1)} \|w_\varepsilon\|_{C^{2+\gamma_4,1+\frac{\gamma_4}{2}}(\overline{\Omega}\times[\tau_4,T])} < \infty.$$

Going back to u_ε and v_ε, this indeed gives (3.4.4). □

Lemma 3.4.4. *There exist $\widehat{u}, \widehat{v} \in C^2(\overline{\Omega} \setminus \{0\} \times (0,\infty))$ and a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as well as*

$$u_{\varepsilon_j} \to \widehat{u} \quad \textit{and} \quad v_{\varepsilon_j} \to \widehat{v} \qquad \textit{in } C^2_{loc}(\overline{\Omega} \setminus \{0\} \times (0,\infty)) \textit{ as } j \to \infty.$$

Proof. This follows directly from Lemma 3.4.3, the Arzelà–Ascoli theorem and a diagonalization argument. □

Lemma 3.4.5. *There exists $\varepsilon_0 > 0$ such that*

$$T_\varepsilon := \sup\left\{T \in (0, T_{\max}) \colon u \le \frac{1}{\varepsilon} \textit{ in } \overline{\Omega} \times [0,T]\right\}$$

is well-defined for all $\varepsilon \in (0, \varepsilon_0)$ and for all $\varepsilon \in (0, \varepsilon_0)$

$$u_\varepsilon = u \quad \textit{and} \quad v_\varepsilon = v \qquad \textit{holds in } \overline{\Omega} \times [0, T_\varepsilon).$$

Proof. As $u_0 \equiv 0$ would imply $u \equiv 0$ by Lemma 3.5.1, we may without loss of generality assume $u_0 \not\equiv 0$. Then $\varepsilon_0 := \frac{1}{2\|u_0\|_{L^\infty(\Omega)}}$ is positive, and as u is continuous, T_ε is indeed well-defined for all $\varepsilon \in (0, \varepsilon_0)$.

Let $\varepsilon \in (0, \varepsilon_0)$. In $\overline{\Omega} \times [0, T_\varepsilon)$, both (u, v) and $(u_\varepsilon, v_\varepsilon)$ are solutions to (3.KS) with $T = T_\varepsilon$, such that the statement follows due to uniqueness, see Lemma 3.5.1 below. □

With these preparations at hand, we may now prove Theorem 3.1.5.

Proof of Theorem 3.1.5. Let $\widehat{u}, \widehat{v}$ be given by Lemma 3.4.4. Since also $u_\varepsilon \to u$ and $v_\varepsilon \to v$ pointwise (as $\varepsilon \searrow 0$) by Lemma 3.4.5, we have $u = \widehat{u}$ and $v = \widehat{v}$ in $\overline{\Omega} \setminus \{0\} \times [0, T_{\max})$.

Because of $\widehat{u}, \widehat{v} \in C^0([0, T_{\max}]; C^2_{\text{loc}}(\Omega \setminus \{0\}))$, a consequence thereof is (3.1.22) if we set $U := \widehat{u}(\cdot, T_{\max})$ and $V := \widehat{v}(\cdot, T_{\max})$. Finally, (3.1.23) follows by Theorem 3.1.3. □

3.5. Uniqueness in nondegenerate quasilinear Keller–Segel systems

In this section, we prove the uniqueness result used in Lemma 3.4.5 above. As most of the works on quasilinear Keller–Segel systems cited in the introduction of this chapter do not state whether the solution is unique, a uniqueness result for quite general systems, also accounting for cell proliferation or consumption of chemicals, for instance, might be of independent interest.

Since these generalizations do not drastically complicate or enlarge the proof, we choose to prove a version slightly more general than actually needed for our purposes.

Lemma 3.5.1. *Suppose $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a smooth, bounded domain. Let $\eta > 0$, $p > \max\{2, n\}$, $T \in (0, \infty]$ as well as $D, S, f, g \in C^1([0,\infty)^2)$ with $D \geq \eta$. Furthermore, assume also that $u_0, v_0 \in W^{1,p}(\Omega)$ are nonnegative.*

Then there exists at most one pair of nonnegative functions

$$(u, v) \in \left(C^{2,1}(\overline{\Omega} \times (0,T)) \cap C^0([0,T); W^{1,p}(\Omega))\right)^2$$

solving

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - S(u,v)\nabla v) + f(u,v), & \text{in } \Omega \times (0,T), \\ v_t = \Delta v + g(u,v), & \text{in } \Omega \times (0,T), \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0, & \text{in } \Omega \end{cases}$$

classically.

Proof. We suppose that (u_1, v_1) and (u_2, v_2) are two such solutions and let $T' \in (0,T)$. Due to the supposed regularity and the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, we can find $L > 0$ such that $u_1, u_2, v_1, v_2 \leq L$ in $\overline{\Omega} \times [0, T']$.

As then

$$\begin{aligned} (u_1 - u_2)_t &= \nabla \cdot (D(u_1,v_1)\nabla u_1 - S(u_1,v_1)\nabla v_1) + f(u_1,v_1) \\ &\quad -\nabla \cdot D(u_2,v_2)\nabla u_2 + S(u_2,v_2)\nabla v_2) - f(u_2,v_2) \\ &= \nabla \cdot (D(u_1,v_1)\nabla(u_1-u_2)) + \nabla \cdot ((D(u_1,v_1) - D(u_2,v_2))\nabla u_2) \\ &\quad -\nabla \cdot (S(u_1,v_1)\nabla(v_1-v_2)) - \nabla \cdot ((S(u_1,v_1) - S(u_2,v_2))\nabla v_2) \\ &\quad +f(u_1,v_1) - f(u_2,v_2) \qquad \text{in } \Omega \times (0,T'), \end{aligned}$$

testing with $u_1 - u_2$ and integrating by parts gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_1-u_2)^2 = &-\int_\Omega D(u_1,v_1)|\nabla(u_1-u_2)|^2 \\ &-\int_\Omega [D(u_1,v_1) - D(u_2,v_2)]\nabla u_2 \cdot \nabla(u_1-u_2) \\ &+\int_\Omega S(u_1,v_1)\nabla(v_1-v_2)\cdot\nabla(u_1-u_2) \\ &+\int_\Omega [S(u_1,v_1) - S(u_2,v_2)]\nabla v_2 \cdot \nabla(u_1-u_2) \end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega [f(u_1, v_1) - f(u_2, v_2)](u_1 - u_2) \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 \qquad \text{in } (0, T').
\end{aligned}
$$

Therein we make first use of the nondegeneracy, that is, the crucial assumption that $D \ge \eta$, to see that

$$
I_1 \le -\eta \int_\Omega |\nabla(u_1 - u_2)|^2 \qquad \text{holds in } (0, T').
$$

Also, by Young's inequality

$$
I_3 \le \frac{\eta}{4} \int_\Omega |\nabla(u_1 - u_2)|^2 + c_1 \int_\Omega |\nabla(v_1 - v_2)|^2 \qquad \text{holds in } (0, T'),
$$

where $c_1 := \frac{\|S\|^2_{C^0([0,L]^2)}}{\eta}$.

By the mean value theorem, we can find $\xi_1, \xi_2 \colon \Omega \times (0, T') \to [0, L]$ such that

$$
\begin{aligned}
|D(u_1, v_1) - D(u_2, v_2)| &\le |D(u_1, v_1) - D(u_2, v_1)| + |D(u_2, v_1) - D(u_2, v_2)| \\
&= |D_u(\xi_1, v_1)(u_1 - u_2)| + |D_v(u_2, \xi_2)(v_1 - v_2)| \\
&\le \|D\|_{C^1([0,L]^2)} \left(|u_1 - u_2| + |v_1 - v_2|\right) \qquad \text{in } \Omega \times (0, T'),
\end{aligned}
$$

where $\|\varphi\|_{C^1([0,L]^2)} := \max\{\|\varphi\|_{C^0([0,L]^2)}, \|\varphi_u\|_{C^0([0,L]^2)}, \|\varphi_v\|_{C^0([0,L]^2)}\}$ for $\varphi \in C^1([0, L]^2)$.

Thus, by Young's and Hölder's inequalities (with exponents $\frac{p}{2}, \frac{p}{p-2}$),

$$
\begin{aligned}
I_2 &\le \frac{\eta}{8} \int_\Omega |\nabla(u_1 - u_2)|^2 + c_2 \left(\int_\Omega |\nabla u_2|^2 (u_1 - u_2)^2 + \int_\Omega |\nabla u_2|^2 (v_1 - v_2)^2 \right) \\
&\le \frac{\eta}{8} \int_\Omega |\nabla(u_1 - u_2)|^2 + c_3 \left(\int_\Omega (u_1 - u_2)^{\frac{2p}{p-2}} \right)^{\frac{p-2}{p}} + c_3 \left(\int_\Omega (v_1 - v_2)^{\frac{2p}{p-2}} \right)^{\frac{p-2}{p}}
\end{aligned}
$$

in $(0, T')$ with $c_2 := \frac{4\|D\|^2_{C^1([0,L]^2)}}{\eta}$ and $c_3 := c_2 \|\nabla u_2\|^2_{L^\infty((0,T'); W^{1,p}(\Omega))}$.

As our assumptions on p imply $r := \frac{2p}{p-2} < \frac{2n}{(n-2)_+}$, we may invoke Lemma 3.2.5 to find $c_4 > 0$ with the property that

$$
\left(\int_\Omega |\varphi|^r \right)^{\frac{2}{r}} \le \frac{\eta}{8c_3} \int_\Omega |\nabla \varphi|^2 + c_4 \int_\Omega \varphi^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),
$$

hence

$$
I_2 \le \frac{\eta}{4} \int_\Omega |\nabla(u_1 - u_2)|^2 + \frac{\eta}{8} \int_\Omega |\nabla(v_1 - v_2)|^2 + c_5 \int_\Omega (u_1 - u_2)^2 + c_5 \int_\Omega (v_1 - v_2)^2
$$

in $(0, T')$, where $c_5 := c_3 c_4$.

Similarly, we see that

$$
I_4 \le \frac{\eta}{4} \int_\Omega |\nabla(u_1 - u_2)|^2 + \frac{\eta}{8} \int_\Omega |\nabla(v_1 - v_2)|^2 + c_6 \int_\Omega (u_1 - u_2)^2 + c_6 \int_\Omega (v_1 - v_2)^2
$$

in $(0, T')$ for some $c_6 > 0$.

As again by the mean value theorem

$$|f(u_1, v_1) - f(u_2, v_2)| \leq \|f\|_{C^1([0,L]^2)} \left(|u_1 - u_2| + |v_1 - v_2|\right) \qquad \text{in } \Omega \times (0, T'),$$

we conclude

$$I_5 \leq c_7 \int_\Omega (u_1 - u_2)^2 + c_8 \int_\Omega (v_1 - v_2)^2 \qquad \text{in } (0, T'),$$

where $c_7 := \frac{3}{2}\|f\|_{C^1([0,L]^2)}$ and $c_8 := \frac{1}{2}\|f\|_{C^1([0,L]^2)}$.

Moreover,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v_1 - v_2)^2 \leq -\int_\Omega |\nabla(v_1 - v_2)|^2 + \int_\Omega (g(u_1, v_1) - g(u_2, v_2))(v_1 - v_2)$$

in $(0, T')$. Therein we make once more use of the mean value theorem to see that

$$|g(u_1, v_1) - g(u_2, v_2)| \leq \|g\|_{C^1([0,L]^2)} \left(|u_1 - u_2| + |v_1 - v_2|\right) \qquad \text{in } \Omega \times (0, T'),$$

hence

$$\int_\Omega (g(u_1, v_1) - g(u_2, v_2))(v_1 - v_2) \leq c_9 \int_\Omega (u_1 - u_2)^2 + c_{10} \int_\Omega (v_1 - v_2)^2 \qquad \text{in } (0, T'),$$

where $c_9 := \frac{1}{2}\|g\|_{C^1([0,L]^2)}$ and $c_{10} := \frac{3}{2}\|g\|_{C^1([0,L]^2)}$.

By combining the above estimates, we obtain with $\lambda := c_1 + \frac{\eta}{4}$ and some $c_{11} > 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega (u_1 - u_2)^2 + \lambda \int_\Omega (v_1 - v_2)^2\right) \leq c_{11}\left(\int_\Omega (u_1 - u_2)^2 + \lambda \int_\Omega (v_1 - v_2)^2\right)$$

in $(0, T')$ and thus

$$\int_\Omega (u_1 - u_2)^2(\cdot, t) + \lambda \int_\Omega (v_1 - v_2)^2(\cdot, t) \leq \mathrm{e}^{c_{11} t}\left(\int_\Omega (u_0 - u_0)^2 + \lambda \int_\Omega (v_0 - v_0)^2\right) = 0.$$

for $t \in [0, T']$ by Grönwall's inequality.

Since $u_1, u_2, v_1, v_2 \in C^0(\overline{\Omega} \times [0, T'])$, this implies $u_1 \equiv u_2$ and $v_1 \equiv v_2$ in $\overline{\Omega} \times [0, T']$. The statement follows upon taking $T' \nearrow T$. $\square$

4. On the optimality of upper estimates near blow-up in quasilinear Keller–Segel systems

4.1. Introduction

Inter alia motivated by the desire to improve on the pointwise estimates derived in the preceding chapter, we now have a closer look at (generalizations of) the second subproblem in (3.KS). That is, in the first and main part of the present chapter, we establish pointwise upper gradient estimates for solutions to

$$\begin{cases}\tau v_t = \Delta v - v + g & \text{in } \Omega \times (0,T),\\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,T),\\ v(\cdot,0) = v_0 \text{ if } \tau > 0 & \text{in } \Omega,\end{cases} \tag{4.P}$$

where $\Omega = B_R(0)$, $R > 0$, is an n-dimensional ball, $\tau \geq 0$, $T \in (0,\infty)$ and v_0 and g are sufficiently smooth given functions on Ω and $\Omega \times (0,T)$, respectively.

Elliptic or parabolic regularity theory (cf. Lemma 4.2.1 and Lemma 4.4.1 below) and embedding theorems warrant that, if g is uniformly-in-time bounded $L^{\mathbb{q}}(\Omega)$ for some $\mathbb{q} \in [1,n]$, then v is uniformly-in-time bounded in $W^{1,p}(\Omega)$ for all $p \in [1, \frac{n\mathbb{q}}{n-\mathbb{q}})$.

An estimate of the form

$$|\nabla v(x,t)| \leq C_\beta |x|^{-\beta} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T) \tag{4.1.1}$$

for some $\beta < \frac{n-\mathbb{q}}{\mathbb{q}}$ would imply

$$\sup_{t\in(0,T)} \int_\Omega |\nabla v(\cdot,t)|^p \leq C_\beta^p \omega_{n-1} \int_0^R r^{n-1-p\beta} \,\mathrm{d}r < \infty$$

for all $p \in (0, \frac{n}{\beta})$ and hence in particular for $p = \frac{\frac{n}{\beta} + \frac{n\mathbb{q}}{n-\mathbb{q}}}{2} > \frac{n\mathbb{q}}{n-\mathbb{q}}$. Thus, assuming that the uniform-in-time bounds discussed above are optimal, such an estimate should not be obtainable if one only requires $\sup_{t\in(0,T)} \|g(\cdot,t)\|_{L^{\mathbb{q}}(\Omega)}$ to be finite. However, we achieve (4.1.1) for all $\beta > \frac{n-\mathbb{q}}{\mathbb{q}}$. We conjecture that this estimate, possibly up to equality regarding the exponent therein, is optimal.

In the elliptic case, the corresponding proof is quite short: In Section 4.2, we first derive an $L^{\mathbb{q}}$ bound for Δv and then make use of the symmetry assumption to obtain the following result.

Proposition 4.1.1. *Let $n \geq 2$, $R > 0$, $\Omega \coloneqq B_R(0) \subset \mathbb{R}^n$, $M > 0$, $\mathfrak{q} \in [1, n]$ and $\beta \geq \frac{n-\mathfrak{q}}{\mathfrak{q}}$. There is $C > 0$ such that whenever $g \in C^0(\overline{\Omega})$ is a radially symmetric function fulfilling*

$$\|g\|_{L^{\mathfrak{q}}(\Omega)} \leq M \tag{4.1.2}$$

and $v \in C^2(\overline{\Omega})$ solves

$$\begin{cases} 0 = \Delta v - v + g & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.3}$$

then

$$|\nabla v(x)| \leq C|x|^{-\beta} \qquad \text{for all } x \in \overline{\Omega}. \tag{4.1.4}$$

In principle, one could argue similarly in the parabolic setting, although one would at least need to require $v_0 \in W^{2,\mathfrak{q}}(\Omega)$ with $\partial_\nu v_0 = 0$ on $\partial\Omega$ in the sense of traces—or v cannot be uniformly-in-time bounded in $W^{2,\mathfrak{q}}(\Omega)$. Not wanting to impose such an unnatural requirement, we argue differently and rely on various semigroup estimates, which are introduced in Section 4.3, instead.

For $\mathfrak{q} \in (1, \frac{n}{2}]$, we can follow [106, Section 3], where corresponding estimates have been derived for $\mathfrak{q} = 1$. The main idea is to notice that $z \coloneqq \zeta^\beta v$, where $\zeta(x) \approx |x|$, solves a certain initial boundary value problem and then make use of several semigroup estimates to obtain an L^∞ bound for ∇z—which in turn together with pointwise upper bounds for v (cf. Lemma 4.4.2) implies (4.1.1).

However, these arguments rely in several places on the fact that $\mathfrak{q} \in (1, \frac{n}{2}]$ and $\beta > \frac{n-\mathfrak{q}}{\mathfrak{q}}$ imply $\beta > 1$ and hence $\zeta^\beta \in C^1(\overline{\Omega})$. Switching to radial notation, this for instance means that $z_r(0, \cdot) \equiv 0$. For $\mathfrak{q} \in (\frac{n}{2}, n]$ and thus possibly $\beta \in (0, 1)$, this is no longer the case. We overcome this problem by considering (for $\mathfrak{q} \in (\frac{n}{2}, n]$)

$$z(x, t) \coloneqq \zeta^\beta(x)(v(x, t) - v(0, t)), \qquad (x, t) \in \overline{\Omega} \times [0, T), \tag{4.1.5}$$

instead. Due to uniform-in-time Hölder bounds (see Lemma 4.4.3), we then obtain $z_r(0, \cdot) \equiv 0$ and an L^∞ bound for ∇z again implies (4.1.1). On the other hand, compared to $\zeta^\beta v$, a new problem arises for z defined as in (4.1.5): The time derivative of z now additionally includes $\zeta^\beta v_t(0, \cdot)$. In order to handle this term, we first derive time Hölder bounds for v in Lemma 4.4.5 and then apply more subtle semigroup arguments as in the case of $\mathfrak{q} \in (1, \frac{n}{2}]$ in Lemma 4.4.6.

Finally, we arrive at

Theorem 4.1.2. *Let $n \geq 2$, $R > 0$, $\Omega \coloneqq B_R(0) \subset \mathbb{R}^n$. For every $M > 0$, $\mathfrak{q} \in (1, n]$, $\beta > \frac{n-\mathfrak{q}}{\mathfrak{q}}$ and $p_0 > \max\{\frac{n}{\beta}, 1\}$, there is $C > 0$ with the following property: Suppose $\tau > 0$, $T \in (0, \infty]$ and that*

$$\begin{gathered} v_0 \in C^0(\overline{\Omega}) \text{ is radially symmetric and nonnegative with} \\ \|v_0\|_{W^{1,p_0}(\Omega)} + \||x|^\beta \nabla v_0\|_{L^\infty(\Omega)} \leq M \end{gathered} \tag{4.1.6}$$

as well as

$$g \in C^0(\overline{\Omega} \times [0, T)) \text{ is radially symmetric with } \sup_{t \in (0,T)} \|g(\cdot, t)\|_{L^{\mathfrak{q}}(\Omega)} \leq M. \tag{4.1.7}$$

Then

$$|\nabla v(x,t)| \le C|x|^{-\beta} \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in [0,T), \tag{4.1.8}$$

provided $v \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *is a nonnegative classical solution of*

$$\begin{cases} \tau v_t = \Delta v - v + g & \text{in } \Omega \times (0,T), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,T), \\ v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases} \tag{4.1.9}$$

Remark 4.1.3. (i) In [106, Lemma 3.4], corresponding estimates have been derived for $\tau = 1$ and $\mathbb{q} = 1$ (provided that in addition to (4.1.7), certain pointwise upper estimates of $|g|$ are known). This is the reason why we concern ourselves only with $\mathbb{q} > 1$ in Theorem 4.1.2.

(ii) The constant C in Theorem 4.1.2 evidently needs at least to depend on $\||x|^\beta \nabla v_0\|_{L^\infty(\Omega)}$ and we avoid further dependencies on the initial data as much as possible; in particular, we do neither rely on a $W^{2,\mathbb{q}}(\Omega)$ bound nor on fulfillment of certain boundary conditions. For technical reasons, however, we need to require (4.1.6), which is nearly optimal in the sense that a bound of $\||x|^\beta \nabla v_0\|_{L^\infty(\Omega)}$ implies bounds for $\|\nabla v_0\|_{L^p(\Omega)}$ for all $p \in [1, \frac{n}{\beta})$.

Next, we apply Proposition 4.1.1 and Theorem 4.1.2 to the solutions (or, more precisely, to their second components) of the quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - S(u,v)\nabla v), & \text{in } \Omega \times (0,T), \\ \tau v_t = \Delta v - v + f(u,v), & \text{in } \Omega \times (0,T), \\ (D(u,v)\nabla u - S(u,v)\nabla v) \cdot \nu = \partial_\nu v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \\ v(\cdot,0) = v_0 \text{ if } \tau > 0, & \text{in } \Omega, \end{cases} \tag{4.KS}$$

where again Ω is an n-dimensional ball, $\tau \ge 0$, $T \in (0,\infty]$ and u_0, v_0, D, S, f are given functions. Such systems aim to describe chemotaxis, the partially directed movement of organisms u towards a chemical stimulus v and have (for certain choices of parameters) first been proposed by Keller and Segel [46]. In certain biological settings, the functions D and S need to be nonlinear—accounting for volume-filling effects [36, 72, 110], immotility of the attracted organisms [19, 57] or saturation of the chemotactic sensitivity [42], for instance.

For known results regarding (4.KS), especially concerning questions of boundedness, global existence and finite-time blow-up, we refer to the introduction of Chapter 3. Let us also recall that Theorem 3.1.3, proved in the preceding chapter, contains the following statement: In n-dimensional balls, $n \ge 2$, and for arbitrary $m > \frac{n-2}{n}$, $m - q \in (-\frac{1}{n}, \frac{n-2}{n}]$, $\alpha > \frac{n(n-1)}{(m-q)n+1}$ and $\beta > n-1$, solutions (u,v) of (4.KS) with $\tau = 1$ blowing up at $T_{\max} \in (0,\infty)$ fulfill

$$u(x,t) \le C|x|^{-\alpha} \quad \text{and} \quad v(x,t) \le C|x|^{-\beta} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{\max})$$

for some $C > 0$. (The special case $m = q = 1$ has already been treated in [106].) Apart from certain corner cases, however, it is to the best of our knowledge not known whether the exponents α and β therein are (essentially) optimal.

However, we now apply Proposition 4.1.1 and Theorem 4.1.2 in order to improve on these estimates—provided that the first solution component is uniformly-in-time bounded in $L^{\mathbb{p}}(\Omega)$ for some $\mathbb{p} > 1$.

Theorem 4.1.4. *Let $n \geq 2$, $R > 0$, $\Omega := B_R(0) \subset \mathbb{R}^n$ and*

$$m, q \in \mathbb{R}, s > 0, \tau \geq 0, K_{D,1}, K_{D,2}, K_S, K_f > 0, M > 0, \mathbb{p} \in [\max\{s, 1\}, ns]$$

be such that

$$m - q \in \left(-\frac{\mathbb{p}}{n}, \frac{ns - 2\mathbb{p}}{n}\right] \quad \text{and} \quad m > \frac{n - 2\mathbb{p}}{n}. \tag{4.1.10}$$

For any

$$\alpha > \underline{\alpha} := \frac{n(ns - \mathbb{p})}{[(m-q)n + \mathbb{p}]\mathbb{p}} \quad \text{and} \quad \beta > \frac{ns - \mathbb{p}}{\mathbb{p}}, \tag{4.1.11}$$

we can find $C > 0$ such that whenever $(u, v) \in \left(C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))\right)^2$, $T \in (0, \infty]$, with

$$\sup_{t \in (0,T)} \|u(\cdot, t)\|_{L^{\mathbb{p}}(\Omega)} \leq M \tag{4.1.12}$$

is a nonnegative, radially symmetric solution of (4.KS), *where*

$$D, S \in C^1([0, \infty)^2), \quad f \in C^0([0, \infty)^2), \quad 0 \leq u_0 \in C^0(\overline{\Omega}) \quad \text{and} \quad 0 \leq v_0 \in C^0(\overline{\Omega})$$

fulfill

$$\begin{aligned}
&\inf_{\sigma \geq 0} D(\rho, \sigma) \geq K_{D,1} \rho^{m-1}, \\
&\sup_{\sigma \geq 0} D(\rho, \sigma) \leq K_{D,2} \max\{\rho, 1\}^{m-1} \\
&\sup_{\sigma \geq 0} |S(\rho, \sigma)| \leq K_S \max\{\rho, 1\}^q \quad \text{and} \\
&\sup_{\sigma \geq 0} |f(\rho, \sigma)| \leq K_f \max\{\rho, 1\}^s
\end{aligned}$$

for all $\rho \geq 0$ as well as

$$u_0(x) \leq M|x|^{-\alpha} \text{ for all } x \in \Omega \quad \text{and} \quad \|v_0\|_{W^{1,\infty}(\Omega)} \leq M,$$

then

$$u(x, t) \leq C|x|^{-\alpha} \quad \text{and} \quad |\nabla v(x, t)| \leq C|x|^{-\beta} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T). \tag{4.1.13}$$

As a first application of Theorem 4.1.4, let us state

Remark 4.1.5. To the best of our knowledge, the results above give the first estimates of type (4.1.13) for chemotaxis systems with nonlinear signal production. For instance, letting $u_0 \in C^0(\overline{\Omega})$, $v_0 \in W^{1,\infty}(\Omega)$, $m = q = 1$, $\tau \geq 0$, $\mathbb{p} = 1$, $s \in (\frac{2}{n}, 1]$ and $\varepsilon > 0$, solutions of

$$\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla v), & \text{in } \Omega \times (0, T), \\
\tau v_t = \Delta v - v + u^s, & \text{in } \Omega \times (0, T), \\
\partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times (0, T), \\
u(\cdot, 0) = u_0, & \text{in } \Omega, \\
v(\cdot, 0) = v_0 \text{ if } \tau > 0, & \text{in } \Omega
\end{cases}$$

fulfill

$$u(x, t) \leq C|x|^{-n(ns-1)-\varepsilon} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T)$$

for some $C > 0$.

Next, we show that Theorem 4.1.4 implies a certain (essentially) conditional optimality for pointwise upper estimates of solutions to (4.KS).

Remark 4.1.6. Suppose $s = 1$ and

$$m - q \in \left(-1, \frac{n-2}{n}\right] \quad \text{as well as} \quad q > 0 \tag{4.1.14}$$

and that (4.1.12) holds for

$$\mathbb{p} = \frac{n}{2}(1 - (m-q)) \in [1, ns). \tag{4.1.15}$$

Then

$$m - q = \frac{n - 2\mathbb{p}}{n} \in \left(-\frac{\mathbb{p}}{n}, \frac{n-2\mathbb{p}}{n}\right],$$

hence (4.1.10) is fulfilled. This implies that for $\underline{\alpha}$ (4.1.11), we have

$$\underline{\alpha} = \frac{n}{\mathbb{p}} \cdot \frac{n - \mathbb{p}}{(m-q)n + \mathbb{p}} = \frac{n}{\mathbb{p}} \cdot \frac{\frac{n}{2} + \frac{(m-q)n}{2}}{\frac{n}{2} + \frac{(m-q)n}{2}} = \frac{n}{\mathbb{p}} = \frac{2}{1-(m-q)}$$

so that [17, Corollary 2.3] asserts that condition (4.1.11) is (up to equality) optimal. Furthermore, we note that requiring (4.1.12) for any $\mathbb{p} > \frac{n}{2}(1-(m-q))$ already implies global existence (cf. [17, Theorem 2.2]), while, to the best of our knowledge, even a solution blowing up in finite time might fulfill (4.1.12) for $\mathbb{p} = \frac{n}{2}(1-(m-q))$.

To sum up, *optimal L^p bounds imply essentially optimal pointwise upper estimates.*

Notation. Henceforth, we fix $n \geq 2$, $R > 0$ and $\Omega \coloneqq B_R(0)$. Moreover, with the usual slight abuse of notation, we switch to radial coordinates whenever convenient and thus write for instance $v(|x|)$ for $v(x)$.

4.2. Pointwise estimates for ∇v: the elliptic case

We first deal with the much simpler elliptic case; that is, we set $\tau \coloneqq 0$ in this section. As a starting point, we obtain an $L^{\mathbb{q}}$ bound for Δv by a straightforward testing procedure. For the parabolic case, which we will deal with in Section 4.4, one cannot expect a similar result to hold if one only wants to assume that the initial datum satisfies (4.1.6) and not, say, $v_0 \in W^{2,2}(\Omega)$ with $\partial_\nu v_0 = 0$ in the sense of traces and $\|v_0\|_{W^{2,2}(\Omega)} \leq M$.

Lemma 4.2.1. *Let $M > 0$ and $\mathbb{q} \in [1, \infty)$. If g is as in* (4.1.2) *and $v \in C^2(\overline{\Omega})$ is a classical solution of* (4.1.3)*, then*

$$\|\Delta v\|_{L^{\mathbb{q}}(\Omega)} \leq 2M.$$

Proof. Testing (4.1.3) with $v^{\mathbb{q}-1}$ and making use of Young's inequality gives

$$\int_\Omega v^{\mathbb{q}} = \int_\Omega v^{\mathbb{q}-1}\Delta v + \int_\Omega v^{\mathbb{q}-1} g \leq -(\mathbb{q}-1)\int_\Omega v^{\mathbb{q}-2}|\nabla v|^2 + \frac{\mathbb{q}-1}{\mathbb{q}}\int_\Omega v^{\mathbb{q}} + \frac{1}{\mathbb{q}}\int_\Omega g^{\mathbb{q}}$$

and hence

$$\int_\Omega v^{\mathbb{q}} \le \int_\Omega g^{\mathbb{q}} \le M^{\mathbb{q}}.$$

For $\mathbb{q} = 1$, this already implies

$$\int_\Omega |\Delta v| \le \int_\Omega (|v| + |g|) \le 2M,$$

while for $\mathbb{q} > 1$, we further test (4.1.3) with $-\Delta v|\Delta v|^{\mathbb{q}-2}$ and use Young's inequality to obtain

$$\int_\Omega |\Delta v|^{\mathbb{q}} \le \int_\Omega (|v| + |g|)|\Delta v|^{\mathbb{q}-1} \le \frac{\mathbb{q}-1}{\mathbb{q}} \int_\Omega |\Delta v|^{\mathbb{q}} + \frac{2^{\mathbb{q}-1}}{\mathbb{q}} \int_\Omega |v|^{\mathbb{q}} + \frac{2^{\mathbb{q}-1}}{\mathbb{q}} \int_\Omega |g|^{\mathbb{q}},$$

which also implies

$$\int_\Omega |\Delta v|^{\mathbb{q}} \le 2^{\mathbb{q}-1} \int_\Omega |v|^{\mathbb{q}} + 2^{\mathbb{q}-1} \int_\Omega |g|^{\mathbb{q}} \le 2^{\mathbb{q}} M^{\mathbb{q}},$$

as desired. □

Making crucial use of the radial symmetry, we now show that the bound obtained in Lemma 4.2.1 implies the desired estimate (4.1.4).

Lemma 4.2.2. *Let $M > 0$, $\mathbb{q} \in [1, n)$ and $\beta \ge \frac{n-\mathbb{q}}{\mathbb{q}}$. There is $C > 0$ such that if g satisfies* (4.1.2) *and $v \in C^2(\overline{\Omega})$ is as a classical solution of* (4.1.3)*, then* (4.1.4) *holds.*

Proof. By the fundamental theorem of calculus, Hölder's inequality and Lemma 4.2.1, we may calculate

$$\begin{aligned} r^{n-1}|v_r(r)| &= \left| \int_0^r \rho^{\frac{n-1}{\mathbb{q}}} \rho^{1-n} (\rho^{n-1} v_r)_r \cdot \rho^{-(n-1)\frac{1-\mathbb{q}}{\mathbb{q}}} \,\mathrm{d}\rho \right| \\ &\le \frac{\|\Delta v\|_{L^{\mathbb{q}}(\Omega)}}{\sqrt[\mathbb{q}]{\omega_{n-1}}} \left(\int_0^r \rho^{n-1} \,\mathrm{d}\rho \right)^{\frac{\mathbb{q}-1}{\mathbb{q}}} \le \frac{2Mn^{-\frac{\mathbb{q}-1}{\mathbb{q}}}}{\sqrt[\mathbb{q}]{\omega_{n-1}}} \cdot r^{n-\frac{n}{\mathbb{q}}} \end{aligned} \tag{4.2.1}$$

for all $r \in (0, R)$. In view of $r^{n-\frac{n}{\mathbb{q}}-(n-1)} = r^{-\frac{n-\mathbb{q}}{\mathbb{q}}} \le R^{\beta-\frac{n-\mathbb{q}}{\mathbb{q}}} r^{-\beta}$ for $r \in (0, R)$, dividing by r^{n-1} on both the left- and the right-hand side in (4.2.1) implies (4.1.4) for an appropriately chosen $C > 0$. □

4.3. Intermission: semigroup estimates

The proof of a parabolic counterpart to the preceding section will in multiple places rely on certain semigroup estimates, which we collect here for convenience. As we will apply them in both Ω and $(0, R)$, we consider arbitrary smooth bounded domains $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$, in this section.

Lemma 4.3.1. *Let $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a smooth bounded domain, and $p \in (1,\infty)$. Set*

$$W_N^{2,p}(G) := \left\{ \varphi \in W^{2,p}(G) : \partial_\nu \varphi = 0 \text{ on } \partial\Omega \text{ in the sense of traces} \right\}$$

and define the operator A on $L^p(G)$ by

$$A\varphi := A_p\varphi := -\Delta\varphi + \varphi \qquad \text{for } \varphi \in \mathcal{D}(A) := W_N^{2,p}(G).$$

Define moreover the fractional powers A^μ, $\mu \in (0,1)$, of the operator above as in [88, Section 1.15]. Then there are $C_1, C_2 > 0$ such that

$$\|\varphi\|_{W^{2\mu,p}(G)} \le C_1 \|A^\mu \varphi\|_{L^p(G)} \qquad \text{for all } \varphi \in \mathcal{D}(A^\mu) \text{ and all } \mu \in (0,1)$$

and

$$\|A^\mu \varphi\|_{L^p(G)} \le C_2 \|\varphi\|_{W^{2\mu,p}(G)} \qquad \text{for all } \varphi \in W^{2\mu,p}(G) \text{ and all } \mu \in \left(0, \frac{1+\frac{1}{p}}{2}\right).$$

Proof. Let $\mu \in (0,1)$. From [88, Theorems 1.15.3 and 4.3.3], we infer that $\mathcal{D}(A^\mu) = [L^p(G), W_N^{2,p}(G)]_\mu \subset H_p^{2\mu}(G)$ with equality if $2\mu < 1 + \frac{1}{p}$. (Herein, $[\cdot,\cdot]_\mu$ and $H_p^{2\mu}(G)$ are as in [88, Convention 1.9.2] and [88, Definition 4.2.1], respectively.) Since G is smooth, [88, Theorem 4.6.1 (d)] moreover asserts that $H_p^{2\mu}(G)$ coincides with $W^{\mu,p}(G)$. Thus, we obtain the desired estimates by noting that A^μ is an isomorphism between $D(A^\mu)$ and $L^p(G)$ (cf. [88, Theorem 1.15.2 (e)]). □

Lemma 4.3.2. *Let $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a smooth bounded domain.*

(i) Suppose $\sigma \in \{0,1\}$, $\mu \in \mathbb{R}$, $q \in (1,\infty)$, $p \in [q,\infty]$ and

$$s \begin{cases} \ge \frac{N}{q} - \frac{N}{p}, & p < \infty, \\ > \frac{N}{q}, & p = \infty \end{cases}$$

are such that $\mu + \frac{\sigma+s}{2} \ge 0$. For any $\lambda \in [0, \mu + \frac{\sigma+s}{2}] \cap [0, \frac{1}{2} + \frac{1}{2q})$ and $\delta \in (0,1)$, we can then find $C > 0$

$$\|\nabla^\sigma A^\mu \mathrm{e}^{-tA} \varphi\|_{L^p(G)} \le C t^{\lambda-\mu-\frac{\sigma+s}{2}} \mathrm{e}^{-\delta t} \|\varphi\|_{W^{2\lambda,q}(G)}$$

for all $t > 0$ and $\varphi \in W^{2\lambda,q}(G)$, where $A = A_q$ is as in Lemma 4.3.1. (Here and below, $\nabla^0 = \mathrm{id}$ and $\nabla^1 = \nabla$.)

(ii) In particular, for any $\sigma \in \{0,1\}$, $\mu \in \mathbb{R}$ with $\mu + \frac{\sigma}{2} \ge 0$, $\lambda \in [0, \mu + \frac{\sigma}{2}] \cap [0, \frac{1}{2})$, $\delta \in (0,1)$ and $\varepsilon \in (0, 2N)$, there is $C' > 0$ such that

$$\|\nabla^\sigma A^\mu \mathrm{e}^{-tA} \varphi\|_{L^\infty(G)} \le C' t^{\lambda-\mu-\frac{\sigma}{2}-\varepsilon} \mathrm{e}^{-\delta t} \|\varphi\|_{C^{2\lambda}(\overline{G})}$$

for all $t > 0$ and $\varphi \in C^{2\lambda}(\overline{G})$, where $A = A_q$ for a certain $q \in (1,\infty)$ is again as in Lemma 4.3.1.

Proof. Let us first prove part (i) for $s < 1$. To that end, we begin by fixing some constants: By [88, Theorem 4.6.1 (c) and (e)], there is $c_1 > 0$ such that

$$\|\psi\|_{L^p(G)} \le c_1 \|\psi\|_{W^{s,q}(G)} \qquad \text{for all } \psi \in W^{s,q}(G).$$

Moreover, noting that $\sigma + s < 2$, $2\lambda < 1 + \frac{1}{q}$ and $q \in (1, \infty)$, Lemma 4.3.1 asserts that we can find $c_2, c_3 > 0$ with

$$\|\psi\|_{W^{\sigma+s,q}(G)} \le c_2 \|A^{\frac{\sigma+s}{2}}\psi\|_{L^q(G)} \qquad \text{for all } \psi \in \mathcal{D}(A^{\frac{\sigma+s}{2}})$$

as well as

$$\|A^\lambda \psi\|_{L^q(G)} \le c_3 \|\psi\|_{W^{2\lambda,q}(G)} \qquad \text{for all } \psi \in W^{2\lambda,q}(G)$$

and [32, Theorem 1.4.3] provides us with $c_4 > 0$ such that

$$\|A^\gamma \mathrm{e}^{tA}\psi\|_{L^q(G)} \le c_4 t^{-\gamma} \mathrm{e}^{-\delta t} \|\psi\|_{L^q(G)} \qquad \text{for all } \psi \in L^q(G),$$

where $\gamma := -\lambda + \mu + \frac{\sigma+s}{2} \ge 0$ by the assumption on λ.

Moreover noting that $A^\mu \mathrm{e}^{-tA}\varphi = \mathrm{e}^{-\frac{t}{2}A} A^\mu \mathrm{e}^{-\frac{t}{2}A}\varphi \in \mathcal{D}(A^{\frac{\sigma+s}{2}}) \cap W^{s,q}(G)$ for all $\varphi \in L^p(G)$, we may therefore estimate

$$\begin{aligned}
\|\nabla^\sigma A^\mu \mathrm{e}^{-tA}\varphi\|_{L^p(G)} &\le c_1 \|\nabla^\sigma A^\mu \mathrm{e}^{-tA}\varphi\|_{W^{s,q}(G)} \\
&\le c_1 \|A^\mu \mathrm{e}^{-tA}\varphi\|_{W^{\sigma+s,q}(G)} \\
&\le c_1 c_2 \|A^{\frac{\sigma+s}{2}+\mu} \mathrm{e}^{-tA}\varphi\|_{L^q(G)} \\
&= c_1 c_2 \|A^{-\lambda+\mu+\frac{\sigma+s}{2}} \mathrm{e}^{-tA} A^\lambda \varphi\|_{L^q(G)} \\
&\le c_1 c_2 c_4 t^{-\gamma} \|A^\lambda \varphi\|_{L^q(G)} \\
&\le c_1 c_2 c_3 c_4 t^{-\gamma} \|\varphi\|_{W^{2\lambda,q}(G)} \qquad \text{for all } t > 0 \text{ and } \varphi \in W^{2\lambda,q}(G),
\end{aligned}$$

which proves part (i) if $s < 1$. If $s \in [1, \infty)$ and $p < \infty$, we fix $k \in \mathbb{N}$ and $p = p_0 \ge p_1 \ge \dots \ge p_k = q$ such that $s_j := \frac{N}{p_j} - \frac{N}{p_{j-1}} < 1$. Furthermore, we set

$$\mu_j := \begin{cases} -\frac{s_j}{2}, & j < k, \\ \mu + \sum_{i=1}^{k-1} \frac{s_i}{2}, & j = k \end{cases} \qquad \text{for } j \in \{1, \dots, k\}$$

and choose λ to be $\frac{\sigma}{2}$ or 0 (depending on whether the operator ∇^σ is involved) in first $k-1$ steps below. By the case already proven, we obtain then $c_5 > 0$ such that

$$\begin{aligned}
&\left\|\nabla^\sigma A^\mu \mathrm{e}^{-tA}\varphi\right\|_{L^p(G)} \\
&= \left\|\nabla^\sigma \prod_{j=1}^k \left(A^{\mu_j} \mathrm{e}^{-\frac{t}{k}A}\right)\varphi\right\|_{L^p(G)} \\
&\le c_5 \mathrm{e}^{-\frac{\delta}{k}t} \left\|\prod_{j=2}^k \left(A^{\mu_j} \mathrm{e}^{-\frac{t}{k}A}\right)\varphi\right\|_{W^{\sigma,p_1}(G)} \\
&\le c_5 \mathrm{e}^{-\frac{\delta}{k}t} \left(\left\|\nabla^\sigma \prod_{j=2}^k \left(A^{\mu_j} \mathrm{e}^{-\frac{t}{k}A}\right)\varphi\right\|_{L^{p_1}(G)} + \left\|\prod_{j=2}^k \left(A^{\mu_j} \mathrm{e}^{-\frac{t}{k}A}\right)\varphi\right\|_{L^{p_1}(G)} \right) \\
&\le c_5^{k-1} \mathrm{e}^{-\frac{(k-1)\delta}{k}t} \left(\left\|\nabla^\sigma A^{\mu_k} \mathrm{e}^{-\frac{t}{k}A}\varphi\right\|_{L^{p_{k-1}}(G)} + (k-1)\|A_{p_{k-1}}^{-\frac{\sigma}{2}}\| \left\|A^{\mu_k+\frac{\sigma}{2}} \mathrm{e}^{-\frac{t}{k}A}\varphi\right\|_{L^{p_{k-1}}(G)} \right) \\
&\le c_5^k (1 + (k-1)\|A_{p_{k-1}}^{-\frac{\sigma}{2}}\|) t^{\lambda-\mu_k-\frac{\sigma+s_k}{2}} \mathrm{e}^{-\delta t} \|\varphi\|_{W^{\sigma,p_k}(G)} \\
&= c_5^k (1 + (k-1)\|A_{p_{k-1}}^{-\frac{\sigma}{2}}\|) t^{\lambda-\mu-\frac{\sigma+s}{2}} \mathrm{e}^{-\delta t} \|\varphi\|_{W^{\sigma,q}(G)} \qquad \text{for all } t > 0 \text{ and } \varphi \in W^{2\lambda,q}(G),
\end{aligned}$$

where in the last two steps we have made use of $\mu + \frac{\sigma+s}{2} = \mu_k + \frac{\sigma+s_k}{2}$. Finally, for $s \in [1,\infty)$ and $p = \infty$, the desired estimate follows from a similar iterative argument.

Ad (ii): Due to $\varepsilon \in (0, 2N)$, we have $q \coloneqq \frac{2N}{\varepsilon} \in (1,\infty)$ and hence $s \coloneqq \frac{2N}{q} = \varepsilon$. We set moreover $p \coloneqq \infty$ and $\tilde\lambda \coloneqq \lambda - \frac{\varepsilon}{2}$. Then the statement follows from part (i) (with λ replaced by $\tilde\lambda$) and the embedding $W^{2\lambda,q}(G) \hookrightarrow C^{2\lambda+\varepsilon}(\overline{G})$, which in turn directly follows from the fact that $\|\cdot\|_{W^{2\lambda,q}(G)}$ is equivalent to the norm given in [88, 4.4.1 (8)]. $\square$

While Lemma 4.3.2 is quite general, its main shortcoming is the lack of L^∞-L^∞ estimates. These are provided by the following lemma, at least for the special case $\mu = \lambda = 0$.

Lemma 4.3.3. *Letting $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a smooth bounded domain and defining the operator A as in Lemma 4.3.1, we can find $C > 0$ such that*

$$\|\nabla^\sigma \mathrm{e}^{-tA}\varphi\|_{L^\infty(G)} \le C\mathrm{e}^{-t}\|\nabla^\sigma\varphi\|_{L^\infty(G)} \qquad \text{for all } t \ge 0,\ \varphi \in W^{\sigma,\infty}(G) \text{ and } \sigma \in \{0,1\}.$$

Proof. This immediately follows from the maximum principle and [63, formula (2.39)]. $\square$

4.4. Pointwise estimates for ∇v: the parabolic case

In this section, we deal with the remaining case $\tau > 0$ and first argue that we may without loss of generality assume $\tau = 1$. If $v \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ is a classical solution of (4.1.9) for some $\tau > 0$, $T \in (0,\infty]$, $v_0 \in C^0(\overline{\Omega})$ and $g \in C^0(\overline{\Omega} \times [0,T))$, then the function $\tilde v$ defined by $\tilde v(x,t) \coloneqq v(x, \frac{t}{\tau})$ for $(x,t) \in \overline{\Omega} \times [0, T\tau)$ solves

$$\begin{cases} \tilde v_t = \Delta \tilde v - \tilde v + \tilde g & \text{in } \Omega \times (0, T\tau), \\ \partial_\nu \tilde v = 0 & \text{on } \partial\Omega \times (0, T\tau), \\ \tilde v(\cdot, 0) = v_0 & \text{in } \Omega \end{cases}$$

classically, where $\tilde g(x,t) \coloneqq g(x, \frac{t}{\tau})$ for $(x,t) \in \overline{\Omega} \times [0, T\tau)$. Since Theorem 4.1.2 requires C to be independent of T and $\sup_{t\in(0,T\tau)} \|\tilde g(\cdot,t)\|_{L^{\mathfrak{q}}(\Omega)} = \sup_{t\in(0,T)} \|g(\cdot,t)\|_{L^{\mathfrak{q}}(\Omega)}$ for all $\mathfrak{q} \ge 1$, we may thus henceforth indeed fix $\tau = 1$ and prove Theorem 4.1.2 only for this special case.

Moreover, given $M > 0$, let us abbreviate $X \coloneqq C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ and

$$\begin{cases} v_0 \text{ and } g \text{ comply with (4.1.6) and (4.1.7)}, \\ v \in X \text{ is a nonnegative classical solution of (4.1.9).} \end{cases} \tag{4.4.1}$$

Before proving Theorem 4.1.2 in Lemma 4.4.6 below, we first collect several estimates, starting with an $W^{1,p}(\Omega)$ bound for certain $p > 1$.

Lemma 4.4.1. *Let $M > 0$, $\mathfrak{q} \in [1, n]$, $p_0 > 1$ and $p \in (1, \frac{n\mathfrak{q}}{n-\mathfrak{q}}) \cap (1, p_0]$. There is $C > 0$ such that if (4.4.1) holds, then*

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \text{for all } t \in (0,T). \tag{4.4.2}$$

PROOF. Letting A be as in Lemma 4.3.1, we apply Lemma 4.3.2 (with $\sigma := 1$, $\mu := 0$, $q := p$, $s := 0$, $\lambda := \frac{1}{2}$ and $\sigma := 1$, $\mu := 0$, $q := \mathfrak{q}$, $s := \frac{n}{q} - \frac{n}{p}$, $\lambda := 0$) to obtain $c_1, c_2 > 0$ and $\delta > 0$ such that

$$\|\nabla e^{-tA}\varphi\|_{L^p(\Omega)} \le c_1 e^{-\delta t}\|\nabla\varphi\|_{L^p(\Omega)} \qquad \text{for all } t > 0 \text{ and } \varphi \in W^{1,p}(\Omega),$$

and

$$\|\nabla e^{-tA}\varphi\|_{L^p(\Omega)} \le c_2 t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mathfrak{q}}-\frac{1}{p})} e^{-\delta t}\|\varphi\|_{L^{\mathfrak{q}}(\Omega)} \qquad \text{for all } t > 0 \text{ and } \varphi \in L^{\mathfrak{q}}(\Omega).$$

Hence, assuming (4.4.1), we make use of the variation-of-constants formula, (4.1.6) and (4.1.7) to see that

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^p(\Omega)} &\le \left\|\nabla e^{-tA}v_0\right\|_{L^p(\Omega)} + \int_0^t \left\|e^{-(t-s)A}g(\cdot,s)\right\|_{L^p(\Omega)} ds \\
&\le c_1 e^{-\delta t}\|\nabla v_0\|_{L^p(\Omega)} + c_2\|g\|_{L^\infty((0,T);L^{\mathfrak{q}}(\Omega))}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mathfrak{q}}-\frac{1}{p})} e^{-\delta(t-s)} ds \\
&\le Mc_1|\Omega|^{\frac{p_0}{p_0-p}} + Mc_2\int_0^\infty s^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mathfrak{q}}-\frac{1}{p})} e^{-\delta s} ds \qquad \text{for all } t \in (0,T).
\end{aligned}$$

The last integral therein is finite because the assumption $p < \frac{n\mathfrak{q}}{n-\mathfrak{q}}$ warrants

$$-\frac{1}{2} - \frac{n}{2}\left(\frac{1}{\mathfrak{q}} - \frac{1}{p}\right) > -\frac{1}{2} - \frac{n}{2}\left(\frac{n}{n\mathfrak{q}} - \frac{n-\mathfrak{q}}{n\mathfrak{q}}\right) = -1. \qquad \square$$

If $\mathfrak{q} \in [1, \frac{n}{2}]$, then the gradient bound obtained in Lemma 4.4.1 implies certain pointwise upper bounds for v. For the special case $\mathfrak{q} = 1$, this has already been proven (similarly as below) in [97, Lemma 3.2].

Lemma 4.4.2. *Given $M > 0$, $\mathfrak{q} \in [1, \frac{n}{2}]$, $p_0 > 1$ and $\kappa \in (-\infty, -\frac{n-2\mathfrak{q}}{\mathfrak{q}}) \cap (-\infty, -\frac{n-p_0}{p_0}]$, there is $C > 0$ with the following property: If $T \in (0,\infty]$ and (4.4.1) holds, then*

$$v(x,t) \le C|x|^\kappa \qquad \textit{for all } x \in \overline{\Omega} \textit{ and } t \in (0,T).$$

PROOF. For fixed $\kappa \le -\frac{n-p_0}{p_0}$ with

$$\kappa < -\frac{n-2\mathfrak{q}}{\mathfrak{q}} = -\frac{(n-\mathfrak{q})-\mathfrak{q}}{\mathfrak{q}} = -\frac{n-\frac{n\mathfrak{q}}{n-\mathfrak{q}}}{\frac{n\mathfrak{q}}{n-\mathfrak{q}}},$$

we may choose $p \in (1, \frac{n\mathfrak{q}}{n-\mathfrak{q}}) \cap (1, p_0]$ such that $\kappa \le -\frac{n-p}{p}$. Then Lemma 4.4.1 warrants that there is $c_1 > 0$ such that (4.4.2) (with C replaced by c_1) is fulfilled whenever (4.4.1) holds. Moreover, we let

$$c_2 := M\max\left\{|\Omega|^{\frac{p_0-1}{p_0}}, |\Omega|^{\frac{\mathfrak{q}-1}{\mathfrak{q}}}\right\} \quad \text{as well as} \quad c_3 := \frac{c_2}{\left|B_R(0)\setminus B_{\frac{R}{2}}(0)\right|}$$

and now assume (4.4.1). Since

$$\|v_0\|_{L^1(\Omega)} \le |\Omega|^{\frac{p_0-1}{p_0}}\|v_0\|_{W^{1,p_0}(\Omega)} \le c_2$$

and

$$\|g\|_{L^\infty((0,T);L^1(\Omega))} \le |\Omega|^{\frac{\mathfrak{q}-1}{\mathfrak{q}}} \|g\|_{L^\infty((0,T);L^{\mathfrak{q}}(\Omega))} \le c_2$$

by (4.1.6), (4.1.7) and the definition of c_2, the comparison principle asserts $\int_\Omega v(\cdot,t) \le c_2$ for all $t \in [0,T)$.

Thus, assuming that there is $t \in [0,T)$ such that $v(r,t) > c_3$ for all $r \in (\frac{R}{2}, R)$ would lead to the contradiction

$$c_2 \ge \int_\Omega v(\cdot,t) \ge \int_{B_R(0)\setminus B_{\frac{R}{2}}(0)} v(\cdot,t) > \int_{B_R(0)\setminus B_{\frac{R}{2}}(0)} c_3 = c_2,$$

and therefore, for all $t \in [0,T)$, we may choose $r_0(t) \in (\frac{R}{2}, R)$ with $v(r_0(t),t) \le c_3$. We then calculate

$$\begin{aligned} v(r,t) - v(r_0(t),t) &= \int_{r_0(t)}^r \rho^{\frac{n-1}{p}} v_r(\rho,t) \cdot \rho^{-\frac{n-1}{p}} \,\mathrm{d}\rho \\ &\le \frac{\|\nabla v(\cdot,t)\|_{L^p(\Omega)}}{\sqrt[p]{\omega_{n-1}}} \left| \int_{r_0(t)}^r \rho^{-\frac{n-1}{p-1}} \right|^{\frac{p-1}{p}} \\ &\le \frac{c_1}{\sqrt[p]{\omega_{n-1}}} \left| \int_{r_0(t)}^r \rho^{-\frac{n-1}{p-1}} \right|^{\frac{p-1}{p}} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T). \end{aligned}$$

As $p \in (1,n)$ because of $\mathfrak{q} \le \frac{n}{2}$ and $\frac{n\mathfrak{q}}{n-\mathfrak{q}} \le n$ and since $r_0(t) > \frac{R}{2} \ge \frac{r}{2}$ for all $r \in (0,R)$ and $t \in (0,T)$, we have therein

$$\begin{aligned} \left| \int_{r_0(t)}^r \rho^{-\frac{n-1}{p-1}} \right|^{\frac{p-1}{p}} &\le \left(\int_{\min\{r,r_0(t)\}}^\infty \rho^{-\frac{n-p}{p-1}-1} \right)^{\frac{p-1}{p}} \\ &= \left(\frac{p-1}{n-p} \right)^{\frac{p-1}{p}} \min\{r, r_0(t)\}^{-\frac{n-p}{p}} \\ &\le 2^{\frac{n-p}{p}} \left(\frac{p-1}{n-p} \right)^{\frac{p-1}{p}} r^{-\frac{n-p}{p}} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T). \end{aligned}$$

Moreover noting that $v(r_0(t),t) \le c_3 \le c_3 R^{\frac{n-p}{p}} r^{-\frac{n-p}{p}}$ for all $r \in (0,R)$ and $t \in (0,T)$, we obtain the statement. □

Since $\mathfrak{q} > \frac{n}{2}$ implies $\frac{2\mathfrak{q}-n}{\mathfrak{q}} > 0$, one cannot expect that Lemma 4.4.2 holds for any $\mathfrak{q} > \frac{n}{2}$. However, we have the following analogon of said lemma.

Lemma 4.4.3. *For $M > 0$, $\mathfrak{q} \in (\frac{n}{2}, n]$, $p_0 > 1$ and $\kappa \in (0, \frac{2\mathfrak{q}-n}{\mathfrak{q}}) \cap (0, \frac{p_0-n}{p_0}]$, there is $C > 0$ such that if $T \in (0,\infty]$ and* (4.4.1) *holds, then*

$$|v(x,t) - v(0,t)| \le C|x|^\kappa \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in [0,T).$$

Proof. Let $\kappa \in (0, \frac{2\mathfrak{q}-n}{\mathfrak{q}})$. The assumption $\mathfrak{q} \in (\frac{n}{2}, n]$ implies $\kappa \in (0,1)$, hence $p := \frac{n}{1-\kappa} \in (1, \frac{n\mathfrak{q}}{n-\mathfrak{q}}) \cap (1, p_0]$. Thus, the statement follows from Lemma 4.4.1 and Morrey's inequality, which because of $\kappa = 1 - \frac{n}{p}$ asserts that $W^{1,p}(\Omega)$ embeds into $C^\kappa(\overline{\Omega})$. □

Lemma 4.4.3 now allows us to show that a function resembling $|x|^\beta v$ solves a suitable initial boundary value problem. In Lemma 4.4.6 below, we then apply semigroup arguments to obtain certain gradient bounds for this function implying (4.1.8).

Lemma 4.4.4. *Let $M > 0$, $\mathfrak{q} \in [1, n]$, $\beta > \frac{n-\mathfrak{q}}{\mathfrak{q}}$,*

$$\zeta \in C^\infty([0,R]) \text{ with } \zeta(r) = r \text{ for all } r \in [0, \tfrac{R}{2}],\ \zeta_r \ge 0 \text{ in } (0,R) \text{ and } \zeta_r(R) = 0 \tag{4.4.3}$$

and

$$p_0 > \begin{cases} 1, & \mathfrak{q} \in [1, \frac{n}{2}], \\ \frac{n}{\min\{1,\beta\}}, & \mathfrak{q} \in (\frac{n}{2}, n]. \end{cases}$$

There exist $b_1, b_2, b_3 \in C^\infty((0,R))$ and $C > 0$ such that

$$|b_1(r)| \le Cr^{\beta-2}, \qquad |b_2(r)| \le Cr^{\beta-1} \qquad \text{and} \qquad |b_3(r)| \le Cr^\beta \tag{4.4.4}$$

for all $r \in (0,R)$ and, moreover, the following holds: Let $T \in [0,\infty)$, v_0, g, v be as in (4.4.1) and

$$\tilde v(r,t) := \begin{cases} v(r,t), & \mathfrak{q} \in [1, \frac{n}{2}], \\ v(r,t) - v(0,t), & \mathfrak{q} \in (\frac{n}{2}, n] \end{cases} \qquad \text{for } r \in [0,R] \text{ and } t \in [0,T). \tag{4.4.5}$$

Then the function $z := \zeta^\beta \tilde v$ belongs to $C^0([0,R] \times [0,T))$, $C^{1,1}([0,R] \times (0,T))$ as well as $C^{2,1}((0,R) \times (0,T))$ and solves

$$\begin{cases} z_t = z_{rr} - z + b_1 \tilde v + b_2 v_r + b_3 g - [\operatorname{sign}(\mathfrak{q} - \frac{n}{2})]_+ \zeta^\beta v_t(0,t), & \text{in } (0,R) \times (0,T), \\ z_r = 0, & \text{in } \{0,R\} \times (0,T), \\ z(\cdot,0) = \zeta^\beta \tilde v(\cdot,0) & \text{in } (0,R) \end{cases} \tag{4.4.6}$$

classically. (Here and below, $[\operatorname{sign}\xi]_+ = 1$ for $\xi > 0$ and $[\operatorname{sign}\xi]_+ = 0$ for $\xi \le 0$.)

Proof. Since the assumptions on ζ warrant $\|\zeta\|_{C^2([0,R])} < \infty$ and $\sup_{r\in(0,R)} \frac{\zeta(r)}{r} < \infty$, there is $C > 0$ such that the functions

$$\begin{aligned} b_1 &:= -\beta(\beta-1)\zeta^{\beta-2}\zeta_r^2 - \beta\zeta^{\beta-1}\zeta_{rr} \\ b_2 &:= -2\beta\zeta^{\beta-1}\zeta_r + \frac{n-1}{r}\zeta^\beta \quad \text{and} \\ b_3 &:= \zeta^\beta \end{aligned}$$

comply with (4.4.4). As direct calculations give

$$\begin{aligned} z_r &= \beta\zeta^{\beta-1}\zeta_r\tilde v + \zeta^\beta v_r, \\ z_{rr} &= [\beta(\beta-1)\zeta^{\beta-2}\zeta_r^2 + \beta\zeta^{\beta-1}\zeta_{rr}]\tilde v + 2\beta\zeta^{\beta-1}\zeta_r v_r + \zeta^\beta v_{rr} \quad \text{and} \\ v_t &= v_{rr} + \frac{n-1}{r}v_r - v + g \end{aligned}$$

in $(0,R) \times (0,T)$, we moreover obtain

$$\begin{aligned} \zeta^\beta v_t &= \zeta^\beta v_{rr} + \frac{n-1}{r}\zeta^\beta v_r - \zeta^\beta v + \zeta^\beta g \\ &= z_{rr} - \left[\beta(\beta-1)\zeta^{\beta-2}\zeta_r^2 + \beta\zeta^{\beta-1}\zeta_{rr}\right]\tilde v + \left[-2\beta\zeta^{\beta-1}\zeta_r + \frac{n-1}{r}\zeta^\beta\right] v_r - z + \zeta^\beta g \end{aligned}$$

in $(0,R)\times(0,T)$. Thus,

$$z_t(r,t) = \zeta^\beta(r)v_t(r,t) - \left[\operatorname{sign}\left(\mathfrak{q}-\frac{n}{2}\right)\right]_+ \zeta^\beta(r)v_t(0,t) \qquad \text{for all } (r,t)\in[0,R)\times[0,T),$$

implying that the first equation in (4.4.6) holds.

Since the third equation in (4.4.6) is a direct consequence of the definition of z, and as $\zeta_r(R)=0$ and $v_r(R,\cdot)\equiv 0$ and $\zeta(R)>0$ imply $z_r(R,\cdot)\equiv 0$, it only remains to be shown that $z_r(0,\cdot)\equiv 0$ in $(0,T)$. For $\mathfrak{q}\in[1,\frac{n}{2}]$ and hence $\beta>1$, this holds because then $\lim_{r\searrow 0}\zeta^{\beta-1}(r)=0$. Thus, we suppose now that $\mathfrak{q}\in(\frac{n}{2},n]$. Due to $\frac{2\mathfrak{q}-n}{\mathfrak{q}}>\max\{1-\beta,0\}$ and $\frac{p_0-n}{p_0}>\max\{1-\beta,0\}$, we may choose $\kappa\in(\max\{1-\beta,0\},\min\{\frac{2\mathfrak{q}-n}{\mathfrak{q}},\frac{p_0-n}{p_0}\})$ and apply Lemma 4.4.3 to obtain $c_1>0$ such that $|v(r,t)-v(0,t)|\le c_1r^\kappa$ for all $(r,t)\in(0,R)\times(0,T)$. Thus, $|\zeta^{\beta-1}(r)\tilde{v}(r,t)|\le c_1r^{\beta-1+\kappa}\to 0$ as $\frac{R}{2}\ge r\searrow 0$. □

For $\mathfrak{q}\in(\frac{n}{2},n]$, we need to handle the term $\zeta^\beta v_t(0,\cdot)$ in the first equation in (4.4.6) if we want to apply semigroup arguments to the problem (4.4.6). To that end, we argue similar as in [91, Lemma 3.4] and derive sufficiently strong time regularity in

Lemma 4.4.5. *Suppose $M>0$, $\mathfrak{q}\in(\frac{n}{2},n]$, $p_0>n$ and $\theta\in(0,\min\{\frac{2\mathfrak{q}-n}{2\mathfrak{q}},\frac{p_0-n}{2p_0}\})$. Then there exists $C>0$ such that for $T\in(0,\infty]$ and v_0,g,v complying with (4.4.1), we have*

$$|v(0,t_1)-v(0,t_2)|\le C|t_1-t_2|^\theta \qquad \textit{for all } t_1,t_2\in[0,T). \tag{4.4.7}$$

Proof. Since $0<\theta<\frac{1}{2}-\frac{n}{2p_0}$, we can choose $p\in(1,p_0)$ and $\varepsilon>0$ such that $\theta=\frac{1}{2}-\frac{n}{2p}-\varepsilon$. Letting A be as in Lemma 4.3.1, by Lemma 4.3.1 and Lemma 4.3.2 (i) (with $\sigma\coloneqq 0$, $\mu\coloneqq\frac{1}{2}$, $q\coloneqq p$, $p\coloneqq\infty$, $s\coloneqq\frac{n}{q}+\varepsilon$, $\lambda\coloneqq 0$), we find $c_1,c_2>0$ such that

$$\|A^{\frac{1}{2}}\varphi\|_{L^p(\Omega)}\le c_1\|\varphi\|_{W^{1,p}(\Omega)} \qquad \text{for all } \varphi\in W^{1,p}(\Omega) \tag{4.4.8}$$

and

$$\left\|A^{\frac{1}{2}}\mathrm{e}^{-tA}\varphi\right\|_{L^\infty(\Omega)}\le c_2t^{-\frac{1}{2}-\frac{n}{2p}-\varepsilon}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0 \text{ and } \varphi\in L^p(\Omega). \tag{4.4.9}$$

For $\mu\in\{1,2\}$, we may again employ Lemma 4.3.2 (i) (with $\sigma\coloneqq 0$, $\mu\coloneqq\mu$, $q\coloneqq\mathfrak{q}$, $p\coloneqq\infty$, $s\coloneqq 2(1-\theta)$, $\lambda\coloneqq 0$, noting that $2(1-\theta)>\frac{n}{\mathfrak{q}}$) in order to obtain $c_3>0$ with

$$\left\|A^\mu\mathrm{e}^{-tA}\varphi\right\|_{L^\infty(\Omega)}\le c_3t^{-\mu-(1-\theta)}\|\varphi\|_{L^{\mathfrak{q}}(\Omega)} \qquad \text{for all } t>0 \text{ and } \varphi\in L^{\mathfrak{q}}(\Omega). \tag{4.4.10}$$

Henceforth fixing $0\le t_1<t_2<T$ and assuming (4.4.1), we then obtain by the variation-of-constants formula

$$\begin{aligned}
&\|v(\cdot,t_2)-v(\cdot,t_1)\|_{L^\infty(\Omega)}\\
&\le \left\|\mathrm{e}^{-t_2A}v_0-\mathrm{e}^{-t_1A}v_0\right\|_{L^\infty(\Omega)}+\left\|\int_0^{t_2}\mathrm{e}^{-(t_2-s)A}g(\cdot,s)\,\mathrm{d}s-\int_0^{t_1}\mathrm{e}^{-(t_1-s)A}g(\cdot,s)\,\mathrm{d}s\right\|_{L^\infty(\Omega)}\\
&\le \left\|\mathrm{e}^{-t_2A}v_0-\mathrm{e}^{-t_1A}v_0\right\|_{L^\infty(\Omega)}\\
&\quad+\int_{t_1}^{t_2}\left\|\mathrm{e}^{-(t_2-s)A}g(\cdot,s)\right\|_{L^\infty(\Omega)}\mathrm{d}s+\int_0^{t_1}\left\|\left[\mathrm{e}^{-(t_2-s)A}-\mathrm{e}^{-(t_1-s)A}\right]g(\cdot,s)\right\|_{L^\infty(\Omega)}\mathrm{d}s\\
&\eqqcolon I_1+I_2+I_3.
\end{aligned}$$

Firstly, due to the fundamental theorem of calculus, since $A^{\frac{1}{2}}\mathrm{e}^{-tA} = \mathrm{e}^{-tA}A^{\frac{1}{2}}$ on $\mathcal{D}(A)$ for all $t \geq 0$, and because of (4.4.9), (4.4.8), the definition of θ and (4.1.6), we have therein

$$\begin{aligned}
I_1 &= \left\| \int_{t_1}^{t_2} A\mathrm{e}^{-sA} v_0 \,\mathrm{d}s \right\|_{L^\infty(\Omega)} \\
&\leq \int_{t_1}^{t_2} \left\| A^{\frac{1}{2}}\mathrm{e}^{-sA} A^{\frac{1}{2}} v_0 \right\|_{L^\infty(\Omega)} \,\mathrm{d}s \\
&\leq c_2 \|A^{\frac{1}{2}} v_0\|_{L^p(\Omega)} \int_{t_1}^{t_2} s^{-\frac{1}{2}-\frac{n}{2p}-\varepsilon} \,\mathrm{d}s \\
&\leq \frac{c_1 c_2 \|v_0\|_{W^{1,p}(\Omega)}}{\theta} (t_2-t_1)^\theta \leq \frac{M c_1 c_2 |\Omega|^{\frac{p_0-p}{p_0}}}{\theta} (t_2-t_1)^\theta,
\end{aligned}$$

secondly, (4.4.10), the fundamental theorem of calculus and (4.1.7) imply

$$I_2 = \int_{t_1}^{t_2} \left\| \mathrm{e}^{-(t_2-s)A} g(\cdot,s) \right\|_{L^\infty(\Omega)} \,\mathrm{d}s \leq c_3 \int_{t_1}^{t_2} (t_2-s)^{\theta-1} \|g(\cdot,s)\|_{L^{\mathbb{q}}(\Omega)} \,\mathrm{d}s \leq \frac{Mc_3}{\theta}(t_2-t_1)^\theta$$

and thirdly, from (4.4.10), the fundamental theorem of calculus, (4.1.7) and the fact that $t_2 > t_1$, we infer

$$\begin{aligned}
I_3 &= \int_0^{t_1} \int_{t_1}^{t_2} \left\| A\mathrm{e}^{-(\sigma-s)A} g(\cdot,s) \right\|_{L^\infty(\Omega)} \,\mathrm{d}\sigma \,\mathrm{d}s \\
&\leq c_3 \int_0^{t_1} \int_{t_1}^{t_2} (\sigma-s)^{\theta-2} \|g(\cdot,s)\|_{L^{\mathbb{q}}(\Omega)} \,\mathrm{d}\sigma \,\mathrm{d}s \\
&\leq -\frac{Mc_3}{1-\theta} \int_0^{t_1} \left[(t_2-s)^{\theta-1} - (t_1-s)^{\theta-1} \right] \mathrm{d}s \\
&= \frac{Mc_3}{\theta(1-\theta)} \left[(t_2-t_1)^\theta - t_2^\theta + t_1^\theta \right] \leq \frac{Mc_3}{\theta(1-\theta)} (t_2-t_1)^\theta.
\end{aligned}$$

Together, this implies (4.4.7). □

We now combine the estimates gathered above to prove Theorem 4.1.2.

Lemma 4.4.6. *Let $M > 0$, $\mathbb{q} \in (1, n]$, $\beta > \frac{n-\mathbb{q}}{\mathbb{q}}$ and $p_0 > \max\{\frac{n}{\beta}, 1\}$. There exists $C > 0$ such that whenever $T \in (0, \infty]$ and v_0, g, v satisfy* (4.4.1)*, then* (4.1.8) *holds.*

Proof. For $\mathbb{q} \in (1, \frac{n}{2}]$ and $\mathbb{q} \in (\frac{n}{2}, n]$, we assume without loss of generality $\beta \in [1, n)$ and $\beta \in (0, 1)$, respectively. Moreover, the assumptions on the parameters allow us to choose $\bar{p} \in (\max\{\frac{n}{\beta}, 1\}, \min\{\frac{n\mathbb{q}}{n-\mathbb{q}}, p_0\})$ and

$$\kappa \in \left(1-\beta, \min\left\{ \frac{2\mathbb{q}-n}{\mathbb{q}}, \frac{p_0-n}{p_0} \right\} \right). \tag{4.4.11}$$

Noting that $\bar{p} > \max\{\frac{n}{\beta}, 1\}$ and hence

$$\frac{(\beta-1)\bar{p}-(n-1)}{\bar{p}-1} > \frac{1-\bar{p}}{\bar{p}-1} = -1$$

hold, that $\kappa > 1-\beta$ implies $\beta-2+\kappa > -1$ and that the main assumption, $\beta > \frac{n-\mathrm{q}}{\mathrm{q}}$, asserts

$$\frac{\beta\mathrm{q}-(n-1)}{\mathrm{q}-1} > \frac{-(\mathrm{q}-1)}{\mathrm{q}-1} = -1,$$

we can find $p \in (1, \min\{\bar{p}, \mathrm{q}\})$ such that still

$$\lambda_1 := (\beta-2+\kappa)p > -1, \tag{4.4.12}$$

$$\lambda_2 := \frac{[(\beta-1)\bar{p}-(n-1)]p}{\bar{p}-p} > -1 \quad \text{and} \tag{4.4.13}$$

$$\lambda_3 := \frac{[\beta\mathrm{q}-(n-1)]p}{\mathrm{q}-p} > -1. \tag{4.4.14}$$

Letting now A be as in Lemma 4.3.1 with $G := (0,R)$ and setting $\gamma_1 := -\frac{1}{2} - \frac{p+1}{4p}$, Lemma 4.3.3 and Lemma 4.3.2 (i) allow us to fix $c_1, c_2 > 0$ and $\delta_1 > 0$ such that

$$\|\partial_r \mathrm{e}^{-\tau A}\varphi\|_{L^\infty((0,R))} \le c_1 \mathrm{e}^{-\tau}\|\varphi_r\|_{L^\infty((0,R))} \qquad \text{for all } \varphi \in W^{1,\infty}((0,R)) \tag{4.4.15}$$

and

$$\|\partial_r \mathrm{e}^{-\tau A}\varphi\|_{L^\infty((0,R))} \le c_2 \tau^{\gamma_1}\mathrm{e}^{-\delta_1\tau}\|\varphi\|_{L^p((0,R))} \qquad \text{for all } \varphi \in L^p((0,R)), \tag{4.4.16}$$

and $\tau > 0$. (We note that $\frac{p+1}{2p} > \frac{1}{p}$ because of $p > 1$ so that Lemma 4.3.2 is indeed applicable.) Since $p > 1$, we have $\gamma_1 > -1$ and hence

$$c_3 := \sup_{t\in(0,\infty)} \int_0^t (t-s)^{\gamma_1}\mathrm{e}^{-\delta_1(t-s)}\,\mathrm{d}s = \int_0^\infty s^{\gamma_1}\mathrm{e}^{-\delta_1 s}\,\mathrm{d}s < \infty. \tag{4.4.17}$$

Moreover, by Lemma 4.4.1, Lemma 4.4.2 and Lemma 4.4.3, there are $c_4, c_5 > 0$ such that

$$\|\nabla v(\cdot,t)\|_{L^{\bar{p}}(\Omega)} \le c_4 \quad \text{and} \quad |\tilde{v}(x,t)| \le c_5|x|^\kappa \qquad \text{for all } x \in \Omega \text{ and } t \in [0,T), \tag{4.4.18}$$

whenever (4.4.1) is fulfilled and where $\tilde{v}$ is given by (4.4.5).

If $\mathrm{q} \in (\frac{n}{2}, n]$, due to $\frac{2\mathrm{q}-n}{2\mathrm{q}} + \frac{\beta}{2} > \frac{2\mathrm{q}-n}{2\mathrm{q}} + \frac{n-\mathrm{q}}{2\mathrm{q}} = \frac{1}{2}$, we may also choose $\varepsilon \in (0,2)$ and $\theta \in (0, \frac{2\mathrm{q}-n}{2\mathrm{q}})$ sufficiently small and large, respectively, such that

$$\gamma_2 := \theta + \frac{\beta}{2} - \frac{3}{2} - \varepsilon > -1.$$

Since $\mathrm{q} \in (\frac{n}{2}, n]$ implies $\beta \in (0,1)$, an application of Lemma 4.3.2 (ii) then yields $c_6 > 0$ and $\delta_2 > 0$ such that for $\mu \in \{0,1\}$,

$$\|\partial_r A^\mu \mathrm{e}^{-\tau A}\varphi\|_{L^\infty((0,R))} \le c_6 \tau^{\frac{\beta}{2}-\mu-\frac{1}{2}-\varepsilon}\mathrm{e}^{-\delta_2\tau}\|\varphi\|_{C^\beta([0,R])} \tag{4.4.19}$$

for all $\varphi \in W^{1,\infty}((0,R))$ and all $\tau > 0$. Furthermore, again only in the case $\mathrm{q} \in (\frac{n}{2}, n]$, Lemma 4.4.5 allows us to fix $c_7 > 0$ such that

$$|v(0,t_2) - v(0,t_1)| \le c_7|t_2-t_1|^\theta \qquad \text{for all } t_1, t_2 \in (0,T) \tag{4.4.20}$$

and (provided $\mathfrak{q} \in (\frac{n}{2}, n])$ we set

$$c_8 := \int_0^\infty s^{\gamma_2} \mathrm{e}^{-\delta_2 s} \,\mathrm{d}s + \sup_{t \in (0,\infty)} t^{\gamma_2+1} \mathrm{e}^{-\delta_2 t} < \infty. \tag{4.4.21}$$

As a last preparation, regardless of the sign of $\mathfrak{q} - \frac{n}{2}$, we fix an arbitrary ζ as in (4.4.3). Hence there are $c_9, c_{10}, c_{11} > 0$ with

$$\frac{r}{c_9} \le \zeta(r) \le c_9 r, \quad |\zeta_r(r)| \le c_{10} \quad \text{and} \quad \|\zeta^\beta\|_{C^\beta([0,R])} \le c_{11} \qquad \text{for all } r \in (0,R) \tag{4.4.22}$$

and, by Lemma 4.4.4, there is moreover $c_{12} > 0$ such that (4.4.4) holds (with C replaced by c_{12}), where b_1, b_2, b_3 are also given by Lemma 4.4.4.

We now suppose (4.4.1). Noting that $\beta > \frac{n-\mathfrak{q}}{\mathfrak{q}}$, we may infer from Lemma 4.4.4 that $z := \zeta^\beta \tilde{v}$ is a classical solution of (4.4.6). By the variation-of-constants formula, we may therefore write

$$\begin{aligned}
\|z_r(\cdot,t)\|_{L^\infty((0,R))} &\le \|\partial_r \mathrm{e}^{-tA} z(\cdot,0)\|_{L^\infty((0,R))} \\
&\quad + \int_0^t \|\partial_r \mathrm{e}^{-(t-s)A}[b_1 \tilde{v}(\cdot,s) + b_2 v_r(\cdot,s) + b_3 g(\cdot,s)]\|_{L^\infty((0,R))} \,\mathrm{d}s \\
&\quad + \left[\operatorname{sign}\left(\mathfrak{q} - \frac{n}{2}\right)\right]_+ \int_0^t \|\partial_r \mathrm{e}^{-(t-s)A} \zeta^\beta v_t(0,s)\|_{L^\infty((0,R))} \,\mathrm{d}s \\
&=: I_1(t) + I_2(t) + I_3(t) \qquad \text{for } t \in (0,T).
\end{aligned}$$

Next, we estimate the terms I_1–I_3 therein. Starting with the first one, we apply (4.4.15), (4.4.22), (4.4.18) (4.1.6) and (4.4.11) to obtain

$$\begin{aligned}
I_1(t) &\le c_1 \mathrm{e}^{-t} \|(\zeta^\beta \tilde{v}(\cdot,0))_r\|_{L^\infty((0,R))} \\
&\le c_1 \left(\|\zeta^\beta v_{0r}\|_{L^\infty((0,R))} + \beta \|\zeta^{\beta-1} \zeta_r \tilde{v}(\cdot,0)\|_{L^\infty((0,R))}\right) \\
&\le c_1 \left(c_9 \|r^\beta v_{0r}\|_{L^\infty((0,R))} + c_5 c_9^{|\beta-1|} c_{10} \beta \|r^{\beta-1+\kappa}\|_{L^\infty((0,R))}\right) \\
&\le c_1 \left(c_9 M + c_5 c_9^{|\beta-1|} c_{10} \beta R^{\beta+\kappa-1}\right) \qquad \text{for } t \in (0,T).
\end{aligned} \tag{4.4.23}$$

By (4.4.16), we moreover have

$$I_2(t) \le c_2 \int_0^t (t-s)^{\gamma_1} \mathrm{e}^{-(t-s)\delta_1} \|b_1 \tilde{v}(\cdot,s) + b_2 v_r(\cdot,s) + b_3 g(\cdot,s)\|_{L^p((0,R))} \,\mathrm{d}s \tag{4.4.24}$$

for $t \in (0,T)$. Therein are

$$\begin{aligned}
&\|b_1 \tilde{v}(\cdot,s)\|_{L^p((0,R))}^p \\
&\le c_{12}^p \int_0^R r^{(\beta-2)p} (\tilde{v})^p(r,s) \,\mathrm{d}r \le c_5^p c_{12}^p \int_0^R r^{\lambda_1} \,\mathrm{d}r = c_5^p c_{12}^p \frac{R^{\lambda_1+1}}{\lambda_1+1} < \infty,
\end{aligned} \tag{4.4.25}$$

$$\begin{aligned}
&\|b_2 v_r(\cdot,s)\|_{L^p((0,R))}^p \\
&\le c_{12}^p \int_0^R \left(r^{n-1} |v_r(r,s)|^{\bar{p}}\right)^{\frac{p}{\bar{p}}} r^{\frac{[(\beta-1)\bar{p}-(n-1)]p}{\bar{p}}} \,\mathrm{d}r \\
&\le \frac{c_{12}^p \|\nabla v(\cdot,s)\|_{L^{\bar{p}}(\Omega)}^p}{\omega_{n-1}} \left(\int_0^R r^{\lambda_2} \,\mathrm{d}r\right)^{\frac{\bar{p}-p}{\bar{p}}} \le \frac{c_4^p c_{12}^p}{\omega_{n-1}} \left(\frac{R^{\lambda_2+1}}{\lambda_2+1}\right)^{\frac{\bar{p}-p}{\bar{p}}} < \infty
\end{aligned} \tag{4.4.26}$$

and

$$\begin{aligned}
&\|b_3 g(\cdot,s)\|^p_{L^p((0,R))}\\
&\le c_{12}^p \int_0^R \left(r^{n-1} g^{\mathfrak{q}}(r,s)\right)^{\frac{p}{\mathfrak{q}}} r^{\frac{[\beta\mathfrak{q}-(n-1)]p}{\mathfrak{q}}}\,\mathrm{d}r\\
&\le \frac{c_{12}^p\|g(\cdot,s)\|^p_{L^{\mathfrak{q}}(\Omega)}}{\omega_{n-1}}\left(\int_0^R r^{\lambda_3}\,\mathrm{d}r\right)^{\frac{\mathfrak{q}-p}{\mathfrak{q}}} \le \frac{M^p c_{12}^p}{\omega_{n-1}}\left(\frac{R^{\lambda_3+1}}{\lambda_3+1}\right)^{\frac{\mathfrak{q}-p}{\mathfrak{q}}} < \infty
\end{aligned}\tag{4.4.27}$$

for all $s \in (0,T)$ by (4.4.4), (4.4.18), (4.1.7) and (4.4.12)–(4.4.14). Combining (4.4.24) with (4.4.17) and (4.4.25)–(4.4.27) then yields

$$I_2(t) \le c_2c_3c_{12}\left(c_5\left(\frac{R^{\lambda_1+1}}{\lambda_1+1}\right)^{\frac{1}{p}} + \frac{c_4}{\sqrt[p]{\omega_{n-1}}}\left(\frac{R^{\lambda_2+1}}{\lambda_2+1}\right)^{\frac{\bar{p}-p}{p\bar{p}}} + \frac{M}{\sqrt[p]{\omega_{n-1}}}\left(\frac{R^{\lambda_3+1}}{\lambda_3+1}\right)^{\frac{\mathfrak{q}-p}{p\mathfrak{q}}}\right)\tag{4.4.28}$$

for all $t \in (0,T)$.

Moreover, as $[\operatorname{sign}(\mathfrak{q}-\frac{n}{2})]_+ = 0$ for $\mathfrak{q} \le \frac{n}{2}$, for estimating I_3 we may assume $\mathfrak{q} > \frac{n}{2}$ (and hence make use of (4.4.19)–(4.4.21)). Using linearity of $\mathrm{e}^{\tau A}$ for $\tau > 0$, integrating by parts and applying (4.4.20), (4.4.19) and (4.4.21), we then obtain

$$\begin{aligned}
I_3(t) &= \left\|\int_0^t \partial_r \mathrm{e}^{-(t-s)A}\left(\zeta^\beta \partial_s v(0,s)\right)\,\mathrm{d}s\right\|_{L^\infty((0,R))}\\
&= \left\|\int_0^t \partial_s[v(0,s)-v(0,t)]\partial_r \mathrm{e}^{-(t-s)A}\zeta^\beta\,\mathrm{d}s\right\|_{L^\infty((0,R))}\\
&\le \left\|\int_0^t [v(0,s)-v(0,t)]\partial_r\partial_s \mathrm{e}^{-(t-s)A}\zeta^\beta\,\mathrm{d}s\right\|_{L^\infty((0,R))}\\
&\quad + \left\|\left[[v(0,s)-v(0,t)]\partial_r \mathrm{e}^{-(t-s)A}\zeta^\beta\right]_{s=0}^{s=t}\right\|_{L^\infty((0,R))}\\
&\le c_7\int_0^t (t-s)^\theta \left\|\partial_r A \mathrm{e}^{-(t-s)A}\zeta^\beta\right\|_{L^\infty((0,R))}\,\mathrm{d}s + c_7 t^\theta \left\|\partial_r \mathrm{e}^{-tA}\zeta^\beta\right\|_{L^\infty((0,R))}\\
&\le c_6c_7\left(\int_0^t s^{\theta+\frac{\beta}{2}-\frac{3}{2}-\varepsilon}\mathrm{e}^{-\delta_2 s}\,\mathrm{d}s + t^{\theta+\frac{\beta}{2}-\frac{1}{2}-\varepsilon}\mathrm{e}^{-\delta_2 t}\right)\|\zeta^\beta\|_{C^\beta([0,R])}\\
&\le c_6c_7c_8c_{11} \qquad \text{for all } t \in (0,T).
\end{aligned}\tag{4.4.29}$$

Combining (4.4.23), (4.4.28) and (4.4.29) shows that $\|z\|_{L^\infty((0,R)\times(0,T))} \le c_{13}$ for some $c_{13} > 0$ only depending on Ω, M, $\mathfrak{q}$, β and p_0. Thus, due to the definitions of $\tilde{v}$ and z, (4.4.18), (4.4.22) and (4.4.11),

$$\begin{aligned}
|v_r(r,t)| &= |\tilde{v}_r(r,t)|\\
&= |\zeta^{-\beta}(r)z_r(r,t) - \beta\zeta^{-\beta-1}(r)\zeta_r(r)z(r,t)|\\
&\le \zeta^{-\beta}(r)|z_r(r,t)| + \beta\zeta^{-1}(r)|\zeta_r(r)||\tilde{v}(r,t)|\\
&\le c_9^\beta c_{13} r^{-\beta} + c_5c_9c_{10}\beta r^{-1+\kappa}\\
&\le \left(c_9^\beta c_{13} + c_5c_9c_{10}\beta R^{\beta+\kappa-1}\right) r^{-\beta} \qquad \text{holds for all } (r,t) \in (0,R)\times(0,T),
\end{aligned}$$

so that we finally arrive at (4.1.8). □

4.5. Proofs of the main theorems

Finally, let us prove Proposition 4.1.1, Theorem 4.1.2 and Theorem 4.1.4.

Proof of Proposition 4.1.1 and Theorem 4.1.2. The corresponding statements have been shown in Lemma 4.2.2 and Lemma 4.4.6. □

Proof of Theorem 4.1.4. For $\mathbb{p} = 1$, this has already been shown in Theorem 3.1.3. Moreover, in the case of $\mathbb{p} > 1$, we set $\mathbb{q} := \frac{\mathbb{p}}{s}$ as well as $g(x,t) := f(u(x,t), v(x,t))$ for $x \in \Omega$ and $t \in (0,T)$ and, for $\alpha > \frac{n(ns-\mathbb{p})}{[(m-q)n+\mathbb{p}]\mathbb{p}} = \frac{\frac{n-\mathbb{q}}{\mathbb{q}}}{m-q+\frac{\mathbb{p}}{n}}$, we choose $\tilde{\beta} > \frac{n-\mathbb{q}}{\mathbb{q}} = \frac{ns-\mathbb{p}}{\mathbb{p}}$ as well as $\theta > n$ such that $\alpha \geq \frac{\tilde{\beta}}{(m-q)+\frac{\mathbb{p}}{n}-\frac{\mathbb{p}}{\theta}}$ and $m - q \in (\frac{p}{\theta} - \frac{p}{n}, \frac{p}{\theta} + \frac{\tilde{\beta}\mathbb{p}-\mathbb{p}}{n}]$. Since we may without loss generality assume $\beta \leq \tilde{\beta}$, the statement follows immediately from Theorem 4.1.2 and Theorem 3.1.1. □

Part II.

Global existence in fully cross-diffusive systems

5. Global solutions near homogeneous steady states in a multi-dimensional population model with both predator- and prey-taxis

5.1. Introduction

In the second part of this thesis, we move away from solutions blowing up in finite time and instead aim to construct global-in-time solutions despite the challenges indicated by finite-time blow-up results such as the one proven in Chapter 2. In particular, we concern ourselves with global solvability of (variants of) the fully cross-diffusive system

$$\begin{cases} u_t = D_1 \Delta u + \nabla \cdot (S_1(u,v)\nabla v) + f(u,v), \\ v_t = D_2 \Delta v + \nabla \cdot (S_2(u,v)\nabla u) + g(u,v), \end{cases} \tag{5.1.1}$$

which describes migration-influenced interaction between predators and prey whose densities are denoted by u and v, respectively.

Apart from growth, death or intra-species competition, the functions f and g model predation: Encounters are beneficial for the predators and harmful to the prey. Moreover, the species are not only assumed to move around randomly (terms $D_1\Delta u$ and $D_2\Delta v$), but also to be able to direct their movement toward (attractive taxis, negative S_i) or away from (repulsive taxis, positive S_i) higher concentration of the other species.

The relevance of attractive prey-taxis ('predators move towards their prey', negative S_1) has first been biologically verified in [44]. It has been observed that such an effect may actually reduce effective biocontrol, contradicting intuitive assumptions [54]. Moreover, the presence of (sufficiently strong) prey-taxis may actually lead to a lack of pattern formation [55].

Among systems of the form (5.1.1), those with only attractive prey- but no predator-taxis ($S_1 < 0$ and $S_2 \equiv 0$), have been studied most extensively—perhaps because they resemble attractive chemotaxis systems from a mathematical point of view, which in turn have been studied in comparatively great detail; see for instance the survey [4].

For $S_1(u,v) = -\chi u$ and several f, g, namely, the existence of globally bounded classical solutions to (5.1.1) has been proved in [111], provided $\chi > 0$ is sufficiently small. In two space dimensions, the smallness condition on χ is, again for various choices of f and g, not necessary [40, 114], while in the three-dimensional setting, one may overcome this restriction by either assuming the prey-taxis to be saturated at larger predator quantities [30, 82] or by considering weak solutions instead [99].

Moreover, a repulsive predator-taxis mechanism ('prey moves away from their predators, positive S_2) has, for instance, been detected for crayfish seeking shelter [27, 35, 54].

While less extensively studied than those with prey-taxis, such systems have been mathematically examined as well: Now without any smallness assumptions on χ, globally bounded classical solutions to (5.1.1) have been constructed for $S_1 \equiv 0$, $S_2(u, v) = \chi v$ and certain f, g in [112]. The same article also considered pattern formation and shows that a strong taxis mechanism (large χ) leads to the absence of stable nonconstant steady states.

Combining both these effects ($S_1 < 0$, $S_2 > 0$) leads to the study of so-called pursuit–evasion models which have been proposed in [89] (see also [28, 90] for the modeling of related systems featuring different taxis mechanisms). There, propagating waves differing from those in taxis-free predator–prey systems have been detected numerically.

Main results. In the present chapter, we handle a system including both predator- and prey-taxis and take the prototypical choices $S_1(u, v) = -\chi_1 u$, $S_2(u, v) = \chi_2 v$, $f(u, v) = u(\lambda_1 - \mu_1 u + a_1 v)$ and $g(u, v) = v(\lambda_2 - \mu_2 v - a_1 u)$ for $u, v \geq 0$ in (5.1.1). That is, we consider

$$\begin{cases} u_t = D_1 \Delta u - \chi_1 \nabla \cdot (u \nabla v) + u(\lambda_1 - \mu_1 u + a_1 v) & \text{in } \Omega \times (0, \infty), \\ v_t = D_2 \Delta v + \chi_2 \nabla \cdot (v \nabla u) + v(\lambda_2 - \mu_2 v - a_2 u) & \text{in } \Omega \times (0, \infty), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0, v(\cdot, 0) = v_0 & \text{in } \Omega \end{cases} \tag{5.P}$$

in smooth, bounded domains Ω for $D_1, D_2, \chi_1, \chi_2 > 0$ and $\lambda_1, \lambda_2, \mu_1, \mu_2, a_1, a_2 \geq 0$.

From a mathematical point of view, such systems are much more challenging than those containing a taxis term in 'only' one equation, which are in turn already highly nontrivial. For instance, if $\chi_2 = 0$ then the L^∞-L^1 bound for the first equation obtained by integrating a suitable linear combination of the first two equations in (5.P) can be used to obtain certain a priori estimates even for the gradient of the second equation by straightforward semigroup arguments. However, for (5.P), bounds for one of the first two equations therein generally do not 'automatically' imply bounds for the other one. As another example, suppose that one could derive L^∞ estimates for both solution components (ignoring for a moment the fact that these are definitely not easy to obtain): How does one then proceed to obtain, say, Hölder bounds? At least, classical results for scalar parabolic equations are not applicable.

We also remark that, apart from (5.P), several other fully cross-diffusive systems have been examined, of which the one proposed by Shigesada, Kawasaki and Teramoto to model spatial segregation [78] probably has gained the most attention among mathematicians (at least if one limits the systems of interest to those where the diffusion matrix does not give raise to a monotone operator). Indeed, there is some quite general global solution theory for such cross-diffusive systems available, both for weak [41] and renormalized [9, 16] solutions. Unfortunately, however, the results obtained there are not applicable to the system (5.P) (with $\chi_1, \chi_2 > 0$), the main reason being that although (5.P) allows for an entropy-like inequality (cf. the introduction of Chapter 6 and especially (6.1.6)), stronger versions thereof would be needed.

Instead, constructing weak solutions for variants of (5.P) with nonlinear diffusion and saturated sensitivity via alternative methods will be the topic of Chapter 6, where we extend on results in the spatially one-dimensional setting derived in [85, 86]. Apart from these findings, however, no global existence results regarding (5.P) appear to be available, which in turn further indicates the difficulty of that problem.

In order to overcome the obstacles outlined above, we thus need to substantially make use of the special structure in (5.P). To that end, we carefully design certain functionals in such a way that, in calculating their derivatives, favourable cancellations occur. We will introduce them in a moment, but before we would like to state the main result of the present chapter. Making a first step towards extending the knowledge about such systems also in the higher dimensional setting, we analyze the stability of homogeneous steady states for (5.1.1) and obtain

Theorem 5.1.1. *Suppose $\Omega \subset \mathbb{R}^n$, $n \in \{1,2,3\}$, is a smooth, bounded domain, and let*

$$D_1, D_2, \chi_1, \chi_2 > 0 \quad \textit{and} \quad m_1, m_2 \geq 0 \tag{5.1.2}$$

Suppose either

$$\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = a_1 = a_2 = 0 \tag{5.H1}$$

or

$$\lambda_1, \lambda_2 \geq 0 \quad \textit{and} \quad a_1, a_2, \mu_1, \mu_2 > 0. \tag{5.H2}$$

Then there exist $\varepsilon > 0$ and $K_1, K_2 > 0$ with the following properties: For any

$$0 \leq u_0, v_0 \in W^{2,2}_N(\Omega) \quad \textit{with } \int_\Omega u_0 = m_1 \textit{ and } \int_\Omega v_0 = m_2 \textit{ if } (5.\text{H1}) \textit{ holds}, \tag{5.1.3}$$

where

$$W^{2,2}_N(\Omega) := \{\, \varphi \in W^{2,2}(\Omega) \colon \partial_\nu \varphi = 0 \textit{ in the sense of traces}\}, \tag{5.1.4}$$

and fulfilling

$$\|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0 - v_\star\|_{W^{2,2}(\Omega)} < \varepsilon, \tag{5.1.5}$$

where

$$(u_\star, v_\star) := \begin{cases} \left(\frac{m_1}{|\Omega|}, \frac{m_2}{|\Omega|}\right), & \textit{if } (5.\text{H1}) \textit{ holds}, \\ \left(\frac{\lambda_1\mu_2 + \lambda_2 a_1}{\mu_1\mu_2 + a_1 a_2}, \frac{\lambda_2\mu_1 - \lambda_1 a_2}{\mu_1\mu_2 + a_1 a_2}\right), & \textit{if } (5.\text{H2}) \textit{ holds and } \lambda_2\mu_1 > \lambda_1 a_2, \\ \left(\frac{\lambda_1}{\mu_1}, 0\right), & \textit{if } (5.\text{H2}) \textit{ holds and } \lambda_2\mu_1 \leq \lambda_1 a_2, \end{cases} \tag{5.1.6}$$

there exist a unique pair

$$(u, v) \in \left(C^0([0,\infty); W^{2,2}_N(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,\infty))\right)^2$$

solving (5.P) classically. Moreover, each solution component is nonnegative and (u,v) converges to $(u_\star, v_\star)$ in the sense that

$$\begin{aligned} &\|u(\cdot,t) - u_\star\|_{W^{2,2}(\Omega)} + \|v(\cdot,t) - v_\star\|_{W^{2,2}(\Omega)} \\ &\leq \begin{cases} (\frac{1}{K_1\varepsilon} + K_2 t)^{-1}, & \textit{if } (5.\text{H2}) \textit{ holds and } \lambda_2\mu_1 = \lambda_1 a_2, \\ K_1\varepsilon \mathrm{e}^{-K_2 t}, & \textit{else} \end{cases} \end{aligned} \tag{5.1.7}$$

for all $t > 0$.

Remark 5.1.2. Let us give some heuristic arguments why we believe that the rates in (5.1.7) are, up to the values of K_1 and K_2 therein, optimal.

For the heat equation, convergence is exponentially fast (take for instance an eigenfunction as initial datum) and adding taxis terms (but no terms of zeroth order) should not dramatically speed up the convergence. Moreover, in the around $(u_\star, v_\star)$ linearized ODE system, $(u_\star, v_\star)$ is a stable fixed point, provided (5.H2) with $\lambda_2\mu_1 \neq \lambda_1 a_2$ holds. Hence, also here, 'only' an exponential convergence rate can be expected.

The case (5.H2) with $\lambda_2\mu_1 = \lambda_1 a_2$ is different. As u converges to $\frac{\lambda_1}{\mu_1}$, one might expect that v behaves similarly as the solution $\tilde{v}$ to

$$\tilde{v}' = \tilde{v}\left(\lambda_2 - \mu_2\tilde{v} - a_2 \cdot \frac{\lambda_1}{\mu_1}\right) = -\mu_2(\tilde{v})^2,$$

which is given by

$$\tilde{v}(t) = \frac{1}{\frac{1}{\tilde{v}(0)} + \mu_1 t}, \qquad t \geq 0.$$

Main ideas. After obtaining local-in-time solutions by Amann's theory in Lemma 5.2.1, we will focus our analysis on estimates holding in $\overline{\Omega} \times [0, T_\eta)$ for $\eta > 0$ to be fixed later, where $T_\eta \in [0, \infty]$ is the maximal time up to which $\|u - u_\star\|_{L^\infty(\Omega)} + \|v - v_\star\|_{L^\infty(\Omega)} < \eta$.

In the case of (5.H1), that is, without any cell proliferation, one formally computes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u - u_\star)^2 + D_1\int_\Omega |\nabla u|^2 = \chi_1\int_\Omega u\nabla u \cdot \nabla v \qquad \text{in } (0, T_{\max}).$$

The key idea is that one can rewrite the problematic term on the right-hand side as

$$\chi_1\int_\Omega u\nabla u \cdot \nabla v = \chi_1\int_\Omega (u - u_\star)\nabla u \cdot \nabla v + \chi_1 u_\star\int_\Omega \nabla u \cdot \nabla v \qquad \text{in } (0, T_{\max})$$

and note that, as the signs for the taxis terms in (5.P) are opposite, two problematic terms cancel out in calculating

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\chi_2 v_\star}{2}\int_\Omega (u - u_\star)^2 + \frac{\chi_1 u_\star}{2}\int_\Omega (v - v_\star)^2\right) + \chi_2 D_1 v_\star\int_\Omega |\nabla u|^2 + \chi_1 D_2 u_\star\int_\Omega |\nabla v|^2 \\
&= \chi_1\chi_2 v_\star\int_\Omega (u - u_\star)\nabla u \cdot \nabla v - \chi_1\chi_2 u_\star\int_\Omega (v - v_\star)\nabla u \cdot \nabla v \qquad \text{in } (0, T_{\max}).
\end{aligned}$$

If $\eta > 0$ is chosen small enough, the remaining terms on the right-hand side can be absorbed by the dissipative terms—at least in $(0, T_\eta)$.

Fortunately, for higher order terms, one can proceed similarly and thus see that the sum of (norms equivalent to) the $W^{2,2}(\Omega)$ norms of both solution components is decreasing, which implies $T_\eta = T_{\max}$, provided $\eta > 0$ is small enough and assuming $T_\eta > 0$, which can be achieved by choosing $\varepsilon > 0$ in Theorem 5.1.1 sufficiently small. Due to the blow-up criterion in Lemma 5.2.1, one then also sees that $T_{\max} = \infty$. Convergence to the mean $(\overline{u}_0, \overline{v}_0)$ as well as the convergence rate are then merely corollaries of the estimates already gained.

For (5.H2), however, this idea alone is insufficient. For instance, if $u_\star > 0$ and $v_\star > 0$, arguing similarly as above, for any $A_1, A_2 > 0$ there is $\eta > 0$ such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_1}{2}\int_\Omega (u-u_\star)^2 + \frac{A_2}{2}\int_\Omega (v-v_\star)^2\right) \\
&\quad + \frac{A_1\mu_1}{2}\int_\Omega (u-u_\star)^2 + \frac{A_2\mu_2}{2}\int_\Omega (v-v_\star)^2 + \frac{A_1D_1}{2}\int_\Omega |\nabla u|^2 + \frac{A_2D_2}{2}\int_\Omega |\nabla v|^2 \\
&\le (A_1a_1u_\star - A_2a_2v_\star)\int_\Omega (u-u_\star)(v-v_\star) + (A_1\chi_1u_\star - A_2\chi_2v_\star)\int_\Omega \nabla u\cdot\nabla v
\end{aligned} \tag{5.1.8}$$

in $(0, T_\eta)$, see Lemma 5.3.2 and (the proof of) Lemma 5.4.3.

For the special case that $(a_1, a_2) = \gamma(\chi_1, \chi_2)$ for some $\gamma \ge 0$, taking $A_1 \coloneqq \chi_2 v_\star$ and $A_2 \coloneqq \chi_1 u_\star$ already implies that the right-hand side in (5.1.8) is zero. Alternatively, if D_1 and D_2 are sufficiently large compared to $a_1, a_2, \chi_1, \chi_2, u_\star$ and $v_\star$, the dissipative terms in (5.1.8) can be used to absorb the terms on the right-hand side. In both these special cases, higher order terms can be handled similarly again so that we can conclude as above.

For arbitrary parameter values, such shortcuts are apparently unavailable and hence we need to argue differently. Actually, this is the reason for considering (5.P) with so many parameters: We want to emphasize that our approach does not rely on certain relationships between them.

Quite miraculously, appropriately choosing positive linear combinations of the six functionals

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-u_\star)^2, \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta u|^2, \\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v-v_\star)^2, \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta v|^2
\end{aligned} \tag{5.1.9}$$

still allows for a cancellation of all problematic terms, see Lemma 5.4.3.

The remaining case, (5.H2) with $\lambda_2\mu_1 \le \lambda_1 a_2$, is handled in Subsection 5.4.2. In a desire to keep the introduction of this chapter at reasonable length, we just note here that the proofs also rely on the functionals in (5.1.9), albeit in a somewhat different fashion as in the first case, and refer for a more detailed discussion to (the beginning of) Subsection 5.4.2. Moreover, the in some sense degenerate case (5.H2) with $\lambda_2\mu_1 = \lambda_1 a_2$ deserves additional special treatment. We introduce a new functional in Lemma 5.4.6 and discuss directly beforehand why that seems to be necessary.

As a last step, in Lemma 5.5.1, we bring all these estimates together and prove global existence as well as convergence to $(u_\star, v_\star)$. Moreover, we discuss possible generalizations of Theorem 5.1.1 in Section 5.6.

Finally, we collect certain Gagliardo–Nirenberg-type inequalities used throughout the present chapter in Section 5.7. They may potentially be of independent interest and differentiate themselves from more often seen inequalities in two ways: Firstly, although we assume Ω to be bounded, we get rid of the additional additive term on the right-hand side. Secondly, instead of $\|D^2\varphi\|_{L^p(\Omega)}$ and $\|D^3\varphi\|_{L^p(\Omega)}$, our versions only contain $\|\Delta\varphi\|_{L^p(\Omega)}$ and $\|\nabla\Delta\varphi\|_{L^p(\Omega)}$ (for certain values of $p \in (1,\infty)$).

5.2. Preliminaries

Local existence. For systems with a taxis term in just one equation, for instance, for (5.P) with either $\chi_1 = 0$ or $\chi_2 = 0$, it suffices to make use of parabolic regularity theory for *scalar* equations (see for instance [37] and [51], which rely on the concept of mild solutions as well as Banach's fixed point theorem and on Schauder's fixed point theorem, respectively). Apparently, for fully cross-diffusive systems such as (5.P) this is no longer fruitful—at least if we want to consider both arbitrary nonnegative parameters and large initial data. Therefore, we resort to the abstract existence theory by Amann instead.

Lemma 5.2.1. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a smooth, bounded domain, and let $D_1, D_2, \chi_1, \chi_2 > 0$ as well as $\lambda_1, \lambda_2, \mu_1, \mu_2, a_1, a_2 \geq 0$. Moreover, let $p > n$ and $u_0, v_0 \in W^{1,p}(\Omega)$ be nonnegative.*

Then there exist $T_{\max} \in (0, \infty]$ and uniquely determined nonnegative

$$u, v \in C^0([0, T_{\max}); W^{1,p}(\Omega)) \cap C^\infty(\overline{\Omega} \times (0, T_{\max})) \tag{5.2.1}$$

such that (u, v) is a classical solution of (5.P) and, if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \left(\|u(\cdot, t)\|_{C^\alpha(\Omega)} + \|v(\cdot, t)\|_{C^\alpha(\Omega)} \right) = \infty \qquad \text{for all } \alpha \in (0, 1). \tag{5.2.2}$$

Moreover, this solution further satisfies

$$u, v \in C^0([0, T_{\max}); W^{2,2}_N(\Omega)), \tag{5.2.3}$$

provided u_0, v_0 satisfy (5.1.3).

Proof. We will construct a solution U to

$$\begin{cases} U_t = \nabla \cdot (A(U)\nabla U) + F(U) & \text{in } \Omega \times (0, T_{\max}), \\ \nu \cdot A(U)\nabla U = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\ U(\cdot, 0) = U_0 & \text{in } \Omega, \end{cases} \tag{5.2.4}$$

where

$$A\begin{pmatrix} u \\ v \end{pmatrix} \coloneqq \begin{pmatrix} D_1 & -\chi_1 u \\ \chi_2 v & D_2 \end{pmatrix}, \quad F\begin{pmatrix} u \\ v \end{pmatrix} \coloneqq \begin{pmatrix} u(\lambda_1 - \mu_1 u + a_1 v) \\ v(\lambda_2 - \mu_2 v - a_2 u) \end{pmatrix} \quad \text{and} \quad U_0 \coloneqq \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

for $u, v \in \mathbb{R}$. Here and below, $\nabla(u, v)^T \coloneqq (\nabla u, \nabla v)^T$, $\nu \cdot (a, b)^T \coloneqq (\nu \cdot a, \nu \cdot b)^T$ etc. for, say, $u, v \in C^1(\overline{\Omega})$ and $a, b \in \mathbb{R}^n$.

If $u, v \geq 0$, then $\operatorname{tr} A((u, v)^T) = D_1 + D_2 > 0$ and $\det A((u, v)^T) = D_1 D_2 + \chi_1 \chi_2 uv > 0$, hence by continuity of the trace and the determinant, we may fix an (open) neighborhood D_0 of $[0, \infty)^2$ in $\mathbb{R}^2$ such that the real parts of all eigenvalues of $A((u, v)^T)$ are still positive for all $(u, v) \in D_0$. Thus, defining the operators $\mathcal{A}, \mathcal{B}$ by $\mathcal{A}(\eta)U \coloneqq \nabla \cdot (A(\eta)\nabla U)$ and $\mathcal{B}(\eta) \coloneqq \nu \cdot A(\eta)\nabla U$ for $\eta \in D_0$ and $U \in (W^{2,p}(\Omega))^2$, we see that $(\mathcal{A}(\eta), \mathcal{B}(\eta))$ are of separated divergence form and hence normally elliptic for all η in D_0 (cf. [3, Example 4.3(e)]).

Therefore, we may apply [3, Theorem 14.4, Theorem 14.6 and Corollary 14.7] to obtain $T_{\max} > 0$ and a unique $U \in C^0([0, T_{\max}); (W^{1,p}(\Omega))^2) \cap (C^\infty(\overline{\Omega} \times (0, T_{\max})))^2$ solving (5.2.4) classically. Moreover, since both components of U are nonnegative by the maximum principle (for scalar equations), [3, Theorem 15.3] asserts that in the case of $T_{\max} < \infty$ we have

$$\limsup_{t \nearrow T_{\max}} \|U(\cdot, t)\|_{(C^\alpha(\overline{\Omega}))^2} = \infty \qquad \text{for all } \alpha \in (0, 1).$$

Thus, $(u, v) := U^T$ satisfies the first, second and fourth equations in (5.P), if $T_{\max} < \infty$, then (5.2.2) holds and, moreover, $D_1 \partial_\nu u = \chi_1 u \partial_\nu v$ and $D_2 \partial_\nu v = -\chi_2 v \partial_\nu u$ on $\partial\Omega \times (0, T_{\max})$. As u and v are nonnegative, $\partial_\nu u = \frac{\chi_1}{D_1} u \partial_\nu v = -\frac{\chi_1 \chi_2}{D_1 D_2} uv \partial_\nu u$ on $\partial\Omega \times (0, T_{\max})$ implies $\partial_\nu u \equiv 0$ on $\partial\Omega \times (0, T_{\max})$. Analogously, we also obtain $\partial_\nu v \equiv 0$ on $\partial\Omega \times (0, T_{\max})$, hence (u, v) is the unique solution of regularity (5.2.1) to (5.P) in $\overline{\Omega} \times [0, T_{\max})$.

Since [3, Theorem 4.1] further asserts that, for all $t \in (0, T_{\max})$, the operator $\mathcal{A}(U(t))$ in $(L^2(\Omega))^2$ with $\mathcal{D}(\mathcal{A}(U(t))) = (W_N^{2,2}(\Omega))^2$ generates an analytical semigroup on $(L^2(\Omega))^2$, we may employ [3, Theorem 10.1] to obtain (5.2.3) for $u_0, v_0 \in W_N^{2,2}(\Omega)$. $\square$

Fixing parameters. In what follows, we fix $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$, parameters as in (5.1.2) and (5.H1) or (5.H2), and define $(u_\star, v_\star)$ as in (5.1.6). Moreover, we henceforth set $\overline{\varphi} := \frac{1}{|\Omega|} \int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$.

As we will see later in the proofs of Lemma 5.4.1 and Lemma 5.4.4, $W^{2,2}(\Omega)$ continuity of both solution components up to $t = 0$ will turn out to be crucial. By Lemma 5.2.1, this can be achieved if one supposes that u_0, v_0 satisfy (5.1.3). Given such initial data, we will denote the solution to (5.P) constructed in Lemma 5.2.1 by $(u(\cdot, \cdot; u_0, v_0), v(\cdot, \cdot; u_0, v_0))$ and its maximal existence time by $T_{\max}(u_0, v_0)$. After fixing (u_0, v_0), we will often for the sake of brevity write (u, v) and $T_{\max}$, respectively, instead. We also note that all constants below (for instance the c_i, $i \in \mathbb{N}$, in several proofs) depend only on the parameters fixed above, not on u_0 and v_0.

The functions f and g. Furthermore, we abbreviate

$$f(u, v) := u(\lambda_1 - \mu_1 u + a_1 v) \quad \text{and} \quad g(u, v) := v(\lambda_2 - \mu_2 v - a_2 u) \qquad \text{for } u, v > 0.$$

We note that $f(u_\star, v_\star) = 0 = g(u_\star v_\star)$ and

$$\begin{pmatrix} f_u(u, v) & f_v(u, v) \\ g_u(u, v) & g_v(u, v) \end{pmatrix} = \begin{pmatrix} \lambda_1 - 2\mu_1 u + a_1 v & a_1 u \\ -a_2 v & \lambda_2 - 2\mu_2 v - a_2 u \end{pmatrix} \qquad \text{for } u, v \geq 0,$$

that is,

$$\begin{pmatrix} f_u(u_\star, v_\star) & f_v(u_\star, v_\star) \\ g_u(u_\star, v_\star) & g_v(u_\star, v_\star) \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if (5.H1) holds,} \\ \begin{pmatrix} -\mu_1 u_\star & a_1 u_\star \\ -a_2 v_\star & -\mu_2 v_\star \end{pmatrix}, & \text{if (5.H2) holds and } \lambda_2 \mu_1 > \lambda_1 a_2, \\ \begin{pmatrix} -\lambda_1 & a_1 u_\star \\ 0 & \lambda_2 - \frac{\lambda_1 a_2}{\mu_1} \end{pmatrix}, & \text{if (5.H2) holds and } \lambda_2 \mu_1 \leq \lambda_1 a_2. \end{cases}$$

Thus,

$$f_u(u_\star, v_\star) \le 0 \quad \text{as well as} \quad g_v(u_\star, v_\star) \le 0 \tag{5.2.5}$$

and

$$\text{if (5.H2) holds and } \lambda_2\mu_1 \neq \lambda_1 a_2, \text{ then } f_u(u_\star, v_\star) < 0 \text{ as well as } g_v(u_\star, v_\star) < 0. \tag{5.2.6}$$

5.3. Estimates within $[0, T_\eta)$

For u_0, v_0 satisfying (5.1.3) and $\eta > 0$, we set

$$E(t; u_0, v_0) \coloneqq \|u(\cdot, t; u_0, v_0) - u_\star\|_{L^\infty(\Omega)} + \|v(\cdot, t; u_0, v_0) - v_\star\|_{L^\infty(\Omega)}$$

and

$$T_\eta(u_0, v_0) \coloneqq \sup \left\{ t \in (0, T_{\max}(u_0, v_0)) : E(\tilde t; u_0, v_0) < \eta \text{ for all } \tilde t \in (0, t) \right\} \tag{5.3.1}$$

(with the convention $\sup \emptyset \coloneqq -\infty$). When confusion seems unlikely, we abbreviate $T_\eta \coloneqq T_\eta(u_0, v_0)$.

We now derive several estimates within $(0, T_\eta)$. Obviously, if $(0, T_\eta) = \emptyset$, the statements below are trivially true. Thus upon reading the proofs, the reader might as well always assume that $(0, T_\eta)$ is not empty. The only exception is Lemma 5.5.1, where we finally choose $\varepsilon > 0$ in (5.1.5) sufficiently small and guarantee positivity of T_η for certain $\eta > 0$.

We note that $T_{\eta_1} \le T_{\eta_2}$ for $\eta_1 \le \eta_2$. Moreover,

$$\begin{aligned}\|u - \overline{u}\|_{L^\infty(\Omega)} &\le \|u - u_\star\|_{L^\infty(\Omega)} + \|\overline{u} - u_\star\|_{L^\infty(\Omega)} \\ &= \|u - u_\star\|_{L^\infty(\Omega)} + \frac{1}{|\Omega|}\left|\int_\Omega (u - u_\star)\right| \le 2\eta \qquad \text{in } (0, T_\eta)\end{aligned} \tag{5.3.2}$$

and likewise

$$\|v - \overline{v}\|_{L^\infty(\Omega)} \le 2\eta \qquad \text{in } (0, T_\eta) \tag{5.3.3}$$

for all $\eta > 0$, where $(u, v, T_{\max}) = (u(u_0, v_0), v(u_0, v_0), T_{\max}(u_0, v_0))$ for any u_0, v_0 complying with (5.1.3).

In the remainder of this section, we derive estimates in $(0, T_\eta)$ for positive linear combinations of

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u - u_\star)^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v - v_\star)^2, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^2 \qquad \text{as well as} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta u|^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta v|^2.\end{aligned} \tag{5.3.4}$$

We begin by treating the first pair in

Lemma 5.3.1. *There is $\eta_0 > 0$ such that if u_0, v_0 comply with (5.1.3) and $(u, v) = (u(u_0, v_0), v(u_0, v_0))$ denotes the corresponding solution, then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-u_\star)^2 + \frac{3D_1}{4}\int_\Omega |\nabla u|^2 + (-f_u(u_\star, v_\star) - \eta(a_1+\mu_1))\int_\Omega (u-u_\star)^2$$
$$\le a_1 u_\star \int_\Omega (u-u_\star)(v-v_\star) + \chi_1 u_\star \int_\Omega \nabla u\cdot\nabla v + \frac{\eta\chi_1}{2}\int_\Omega |\nabla v|^2 \tag{5.3.5}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v-v_\star)^2 + \frac{3D_2}{4}\int_\Omega |\nabla v|^2 + (-g_v(u_\star, v_\star) - \eta(a_2+\mu_2))\int_\Omega (v-v_\star)^2$$
$$\le -a_2 v_\star \int_\Omega (u-u_\star)(v-v_\star) - \chi_2 u_\star \int_\Omega \nabla u\cdot\nabla v + \frac{\eta\chi_2}{2}\int_\Omega |\nabla u|^2 \tag{5.3.6}$$

hold in $(0, T_\eta)$ for all $\eta \in (0, \eta_0)$, where T_η is given by (5.3.1).

Proof. We let

$$\eta_0 := \frac{1}{2}\min\left\{\frac{D_1}{\chi_1}, \frac{D_2}{\chi_2}\right\}. \tag{5.3.7}$$

Fixing u_0, v_0 satisfying (5.1.3), by a direct calculation, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-u_\star)^2 + D_1\int_\Omega |\nabla u|^2 = \chi_1\int_\Omega u\nabla u\cdot\nabla v + \int_\Omega f(u,v)(u-u_\star)$$

holds in $(0, T_{\max})$.

For any $\eta > 0$, we have therein by Young's inequality

$$\chi_1\int_\Omega u\nabla u\cdot\nabla v = \chi_1 u_\star\int_\Omega \nabla u\cdot\nabla v + \chi_1\int_\Omega (u-u_\star)\nabla u\cdot\nabla v$$
$$\le \chi_1 u_\star\int_\Omega \nabla u\cdot\nabla v + \frac{\eta\chi_1}{2}\int_\Omega |\nabla u|^2 + \frac{\eta\chi_1}{2}\int_\Omega |\nabla v|^2 \qquad \text{in } (0, T_\eta).$$

Moreover, as $f(u_\star, v_\star) = 0$,

$$\int_\Omega f(u,v)(u-u_\star) = \int_\Omega f(u,v_\star)(u-u_\star) + a_1\int_\Omega u(v-v_\star)(u-u_\star)$$
$$= f_u(u_\star, v_\star)\int_\Omega (u-u_\star)^2 + \frac{f_{uu}(u_\star, v_\star)}{2}\int_\Omega (u-u_\star)^3$$
$$+ a_1\int_\Omega (u-u_\star)^2(v-v_\star) + a_1 u_\star\int_\Omega (u-u_\star)(v-v_\star) \qquad \text{in } (0, T_{\max}).$$

Since $f_{uu}(u_\star, v_\star) = -2\mu_1$, we may further estimate

$$\frac{f_{uu}(u_\star, v_\star)}{2}\int_\Omega (u-u_\star)^3 \le \eta\mu_1\int_\Omega (u-u_\star)^2 \qquad \text{in } (0, T_\eta) \text{ for all } \eta > 0$$

and

$$a_1\int_\Omega (u-u_\star)^2(v-v_\star) \le \eta a_1\int_\Omega (u-u_\star)^2 \qquad \text{in } (0, T_\eta) \text{ for all } \eta > 0.$$

Noting that (5.3.7) implies $D_1 - \frac{\eta_0\chi_1}{2} \ge \frac{3}{4}D_1$, we may combine these estimates to obtain (5.3.5), while (5.3.6) follows from an analogous computation. ☐

For sufficiently small η and suitable linear combinations of (5.3.5) and (5.3.6), the terms $\frac{\eta\chi_1}{2}\int_\Omega |\nabla v|^2$ and $\frac{\eta\chi_2}{2}\int_\Omega |\nabla u|^2$ can be absorbed by the dissipative terms therein.

Lemma 5.3.2. *For any $A_1, A_2 > 0$, there is $\eta_0 > 0$ such that whenever u_0, v_0 satisfy* (5.1.3), *then the corresponding solution $(u,v) = (u(u_0,v_0), v(u_0,v_0))$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_1}{2}\int_\Omega (u-u_\star)^2 + \frac{A_2}{2}\int_\Omega (v-v_\star)^2\right) + \frac{A_1D_1}{2}\int_\Omega |\nabla u|^2 + \frac{A_2D_2}{2}\int_\Omega |\nabla v|^2$$
$$+A_1\left(-f_u(u_\star,v_\star) - \eta(a_1+\mu_1)\right)\int_\Omega (u-u_\star)^2 + A_2\left(-g_v(u_\star,v_\star) - \eta(a_2+\mu_2)\right)\int_\Omega (v-v_\star)^2$$
$$\le (A_1a_1u_\star - A_2a_2v_\star)\int_\Omega (u-u_\star)(v-v_\star) + (A_1\chi_1u_\star - A_2\chi_2v_\star)\int_\Omega \nabla u\cdot\nabla v \tag{5.3.8}$$

in $(0,T_\eta)$ for all $\eta < \eta_0$, where T_η is as in (5.3.1).

Proof. Lemma 5.3.1 allows us to choose η_1 such that (5.3.5) and (5.3.6) hold in $(0,T_{\eta_1})$. We let moreover $A_1, A_2 > 0$, fix $\eta_2 > 0$ sufficiently small such that

$$\frac{A_2\eta_2\chi_2}{2} \le \frac{A_1D_1}{4} \quad\text{and}\quad \frac{A_1\eta_2\chi_1}{2} \le \frac{A_2D_2}{4}$$

and set $\eta_0 := \min\{\eta_1,\eta_2\}$.

The statement then immediately follows upon multiplying (5.3.5) and (5.3.6) with A_1 and A_2, respectively, and adding these inequalities together. □

Next, we handle the second pair in (5.3.4), this time only in a coupled version.

Lemma 5.3.3. *Let $B_1, B_2 > 0$. There is $\eta > 0$ such that for any u_0, v_0 complying with* (5.1.3) *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{B_1}{2}\int_\Omega |\nabla u|^2 + \frac{B_2}{2}\int_\Omega |\nabla v|^2\right) + \frac{B_1D_1}{2}\int_\Omega |\Delta u|^2 + \frac{B_2D_2}{2}\int_\Omega |\Delta v|^2$$
$$\le (B_1a_1u_\star - B_2a_2v_\star)\int_\Omega \nabla u\cdot\nabla v + (B_1\chi_1u_\star - B_2\chi_2v_\star)\int_\Omega \Delta u\Delta v \qquad \text{in } (0,T_\eta),$$

where again $(u,v) := (u(u_0,v_0), v(u_0,v_0))$ and $T_\eta := T_\eta(u_0,v_0))$ is given by (5.3.1).

Proof. We begin by fixing some parameters: By the Gagliardo–Nirenberg inequality 5.7.3, there is $c_1 > 0$ such that

$$\int_\Omega |\nabla\varphi|^4 \le c_1\|\varphi-\overline{\varphi}\|^2_{L^\infty(\Omega)}\int_\Omega |\Delta\varphi|^2 \qquad \text{for all } \varphi\in C^2(\overline{\Omega}) \text{ with } \partial_\nu\varphi = 0 \text{ on } \partial\Omega. \tag{5.3.9}$$

We choose $\eta > 0$ so small that

$$M_1(\eta) := \frac{B_1\eta\chi_1}{2} + \frac{B_2\eta\chi_2}{2} + \frac{2B_1\eta^2\chi_1^2c_1}{D_1} + \frac{2B_2\eta^2\chi_2^2c_1}{D_2}$$
$$+B_1C_{\mathrm{P}}\eta(2\mu_1+a_1) + \frac{B_1C_{\mathrm{P}}a_1\eta}{2} + \frac{B_2C_{\mathrm{P}}a_2\eta}{2}$$

and

$$M_2(\eta) := \frac{B_1\eta\chi_1}{2} + \frac{B_2\eta\chi_2}{2} + \frac{2B_1\eta^2\chi_1^2c_1}{D_1} + \frac{2B_2\eta^2\chi_2^2c_1}{D_2}$$
$$+B_2C_{\mathrm{P}}\eta(2\mu_2+a_2) + \frac{B_1C_{\mathrm{P}}a_1\eta}{2} + \frac{B_2C_{\mathrm{P}}a_2\eta}{2},$$

where C_{P} is as in Lemma 5.7.1, fulfill

$$M_1(\eta) < \frac{B_1 D_1}{4} \quad \text{and} \quad M_2(\eta) < \frac{B_2 D_2}{4}. \tag{5.3.10}$$

Fixing u_0, v_0 as in (5.1.3), we calculate

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 + D_1 \int_\Omega |\Delta u|^2 \\
&= \chi_1 \int_\Omega u \Delta u \Delta v + \chi_1 \int_\Omega \nabla u \cdot \nabla v \Delta u + \int_\Omega f_u(u,v)|\nabla u|^2 + a_1 \int_\Omega u \nabla u \cdot \nabla v \\
&=: I_1 + I_2 + I_3 + I_4 \qquad \text{in } (0, T_{\max}).
\end{aligned}$$

Therein is

$$\begin{aligned}
I_1 &= \chi_1 u_\star \int_\Omega \Delta u \Delta v + \chi_1 \int_\Omega (u - u_\star) \Delta u \Delta v \\
&\le \chi_1 u_\star \int_\Omega \Delta u \Delta v + \frac{\eta \chi_1}{2} \int_\Omega |\Delta u|^2 + \frac{\eta \chi_1}{2} \int_\Omega |\Delta v|^2 \qquad \text{in } (0, T_\eta).
\end{aligned}$$

Furthermore, by (5.3.9), (5.3.2) and Young's inequality,

$$\begin{aligned}
I_2 &\le \frac{D_1}{4} \int_\Omega |\Delta u|^2 + \frac{\chi_1^2}{D_1} \int_\Omega |\nabla u|^2 |\nabla v|^2 \\
&\le \frac{D_1}{4} \int_\Omega |\Delta u|^2 + \frac{\chi_1^2}{2D_1} \int_\Omega |\nabla u|^4 + \frac{\chi_1^2}{2D_1} \int_\Omega |\nabla v|^4 \\
&\le \frac{D_1}{4} \int_\Omega |\Delta u|^2 + \frac{2\eta^2 \chi_1^2 c_1}{D_1} \int_\Omega |\Delta u|^2 + \frac{2\eta^2 \chi_1^2 c_1}{D_1} \int_\Omega |\Delta v|^2 \qquad \text{in } (0, T_\eta).
\end{aligned}$$

Moreover, due to (5.2.5), by the mean value theorem, as $f_{uu} \equiv 2\mu_1$ and $f_{uv} \equiv a_1$ and because of the Poincaré inequality 5.7.1 (with $C_{\mathrm{P}} > 0$ as in that lemma),

$$\begin{aligned}
I_3 &\le \int_\Omega (f_u(u,v) - f_u(u_\star, v_\star))|\nabla u|^2 \\
&\le \int_\Omega \left(\|f_{uu}\|_{L^\infty((0,\infty)^2)} |u - u_\star| + \|f_{uv}\|_{L^\infty((0,\infty)^2)} |v - v_\star| \right) |\nabla u|^2 \\
&\le \eta(2\mu_1 + a_1) C_{\mathrm{P}} \int_\Omega |\Delta u|^2 \qquad \text{in } (0, T_\eta).
\end{aligned}$$

Finally, by Young's inequality and the Poincaré inequality 5.7.1 (again with $C_{\mathrm{P}} > 0$ as in that lemma),

$$\begin{aligned}
I_4 &= a_1 u_\star \int_\Omega \nabla u \cdot \nabla v + a_1 \int_\Omega (u - u_\star) \nabla u \cdot \nabla v \\
&\le a_1 u_\star \int_\Omega \nabla u \cdot \nabla v + \frac{\eta a_1 C_{\mathrm{P}}}{2} \left(\int_\Omega |\Delta u|^2 + \int_\Omega |\Delta v|^2 \right) \qquad \text{in } (0, T_\eta).
\end{aligned}$$

Along with an analogous computation for v, these estimates imply

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{B_1}{2}\int_\Omega |\nabla u|^2 + \frac{B_2}{2}\int_\Omega |\nabla v|^2\right)\\&\quad+\left(\frac{3B_1D_1}{4}-M_1(\eta)\right)\int_\Omega |\Delta u|^2 + \left(\frac{3B_2D_2}{4}-M_2(\eta)\right)\int_\Omega |\Delta v|^2\\&\le (B_1a_1u_\star - B_2a_2v_\star)\int_\Omega \nabla u\cdot\nabla v + (B_1\chi_1u_\star - B_2\chi_2v_\star)\int_\Omega \Delta u\Delta v \qquad \text{in } (0,T_\eta).\end{aligned}$$

The statement follows due to (5.3.10). □

At last, we deal with the third pair in (5.3.4).

Lemma 5.3.4. *For any $C_1, C_2 > 0$, there exists $\eta > 0$ such that with T_η as defined in (5.3.1), $(u, v, T_\eta) := (u(u_0, v_0), v(u_0, v_0), T_\eta(u_0, v_0))$ satisfies*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{C_1}{2}\int_\Omega |\Delta u|^2 + \frac{C_2}{2}\int_\Omega |\Delta v|^2\right) + \frac{C_1D_1}{2}\int_\Omega |\nabla\Delta u|^2 + \frac{C_1D_2}{2}\int_\Omega |\nabla\Delta v|^2\\&\le (C_1a_1u_\star - C_2a_2v_\star)\int_\Omega \Delta u\Delta v + (C_1\chi_1u_\star - C_2\chi_2v_\star)\int_\Omega \nabla\Delta u\cdot\nabla\Delta v \qquad \textit{in } (0,T_\eta),\end{aligned}$$

provided u_0, v_0 fulfill (5.1.3).

Proof. Fix $C_1, C_2 > 0$. Let us again begin by fixing some constants: By Lemma 5.7.4 and Lemma 5.7.2, there is $c_1 > 0$ such that

$$\begin{aligned}&6\max\left\{\frac{\chi_1^2}{D_1}, \frac{\chi_2^2}{D_2}\right\}\int_\Omega |\nabla\varphi|^6\\&\le c_1\|\varphi-\overline{\varphi}\|_{L^\infty(\Omega)}^4\int_\Omega |\nabla\Delta\varphi|^2 \qquad \text{for all } \varphi\in C^3(\overline{\Omega}) \text{ with } \partial_\nu\varphi = 0 \text{ on } \partial\Omega \qquad (5.3.11)\end{aligned}$$

as well as

$$\begin{aligned}&12\max\left\{\frac{\chi_1^2}{D_1}, \frac{\chi_2^2}{D_2}\right\}\int_\Omega |D^2\varphi|^3\\&\le c_1\|\varphi-\overline{\varphi}\|_{L^\infty(\Omega)}\int_\Omega |\nabla\Delta\varphi|^2 \qquad \text{for all } \varphi\in C^3(\overline{\Omega}) \text{ with } \partial_\nu\varphi = 0 \text{ on } \partial\Omega \qquad (5.3.12)\end{aligned}$$

and Lemma 5.7.3 provides us with $c_2 \ge 1$ such that

$$\int_\Omega |\nabla\varphi|^4 \le c_2\|\varphi-\overline{\varphi}\|_{L^\infty(\Omega)}^2\int_\Omega |\Delta\varphi|^2 \qquad \text{for all } \varphi\in C^2(\overline{\Omega}) \text{ with } \partial_\nu\varphi = 0 \text{ on } \partial\Omega. \quad (5.3.13)$$

We fix furthermore C_{P} as in Lemma 5.7.1 and choose $\eta > 0$ so small that

$$\begin{aligned}M_1(\eta) :=& \frac{C_1\eta\chi_1}{2} + \frac{C_2\eta\chi_2}{2} + (C_1+C_2)c_1(2\eta+16\eta^4)\\&+\frac{C_1C_{\mathrm{P}}c_2\eta(9a_1+14\mu_1)}{2} + \frac{5C_2C_{\mathrm{P}}a_2c_2\eta}{2}\end{aligned}$$

and

$$\begin{aligned}M_2(\eta) :=& \frac{C_1\eta\chi_1}{2} + \frac{C_2\eta\chi_2}{2} + (C_1+C_2)c_1(2\eta+16\eta^4)\\&+\frac{C_2C_{\mathrm{P}}c_2\eta(9a_2+14\mu_2)}{2} + \frac{5C_1C_{\mathrm{P}}a_1c_2\eta}{2}\end{aligned}$$

satisfy

$$M_1(\eta) < \frac{C_1 D_1}{4} \quad \text{and} \quad M_2(\eta) < \frac{C_2 D_1}{4}. \tag{5.3.14}$$

We also fix u_0, v_0 complying with (5.1.3). Since $\partial_\nu u = 0$ on $\partial\Omega \times (0, T_{\max})$ implies $(\partial_\nu u)_t = 0$ on $\partial\Omega \times (0, T_{\max})$ and as $|\Delta\varphi| \le \sqrt{n}|D^2\varphi|$ for all $\varphi \in C^2(\overline{\Omega})$, we may calculate

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta u|^2 \\
&= -\int_\Omega \nabla u_t \cdot \nabla\Delta u + \int_{\partial\Omega} (\partial_\nu u)_t \Delta u \\
&= -D_1 \int_\Omega |\nabla\Delta u|^2 + \chi_1 \int_\Omega \nabla(u\Delta v + \nabla u \cdot \nabla v) \cdot \nabla\Delta u - \int_\Omega \nabla(f(u,v)) \cdot \nabla\Delta u \\
&\le -D_1 \int_\Omega |\nabla\Delta u|^2 - \int_\Omega \nabla(f(u,v)) \cdot \nabla\Delta u \\
&\quad + \chi_1 \int_\Omega u\nabla\Delta u \cdot \nabla\Delta v + \chi_1 \int_\Omega (|D^2 u||\nabla v| + (1+\sqrt{n})|D^2 v||\nabla u|)|\nabla\Delta u|
\end{aligned} \tag{5.3.15}$$

in $(0, T_{\max})$. Herein is by Young's inequality,

$$\begin{aligned}
&\chi_1 \int_\Omega u\nabla\Delta v \cdot \nabla\Delta u \\
&= \chi_1 u_\star \int_\Omega \nabla\Delta v \cdot \nabla\Delta u + \chi_1 \int_\Omega (u - u_\star)\nabla\Delta v \cdot \nabla\Delta u \\
&= \chi_1 u_\star \int_\Omega \nabla\Delta v \cdot \nabla\Delta u + \frac{\eta\chi_1}{2}\int_\Omega |\nabla\Delta u|^2 + \frac{\eta\chi_1}{2}\int_\Omega |\nabla\Delta v|^2 \qquad \text{in } (0, T_\eta).
\end{aligned}$$

Again by Young's inequality combined with $\sqrt{n} \le 2$, (5.3.11), (5.3.12), (5.3.2) and (5.3.3), we further estimate

$$\begin{aligned}
&\chi_1 \int_\Omega \big(|D^2 u||\nabla v| + (1+\sqrt{n})|D^2 v||\nabla u|\big)\, |\nabla\Delta u| \\
&\le \frac{D_1}{4}\int_\Omega |\nabla\Delta u|^2 + \frac{2\chi_1^2}{D_1}\int_\Omega |D^2 u|^2|\nabla v|^2 + \frac{18\chi_1^2}{D_1}\int_\Omega |D^2 v|^2 |\nabla u|^2 \\
&\le \frac{D_1}{4}\int_\Omega |\nabla\Delta u|^2 + \frac{4\chi_1^2}{3D_1}\int_\Omega |D^2 u|^3 + \frac{2\chi_1^2}{3D_1}\int_\Omega |\nabla v|^6 + \frac{12\chi_1^2}{D_1}\int_\Omega |D^2 v|^3 + \frac{6\chi_1^2}{D_1}\int_\Omega |\nabla u|^6 \\
&\le \left(\frac{D_1}{4} + 2c_1\eta + 16c_1\eta^4\right)\int_\Omega |\nabla\Delta u|^2 + \big(2c_1\eta + 16c_1\eta^4\big)\int_\Omega |\nabla\Delta v|^2 \qquad \text{in } (0, T_\eta).
\end{aligned}$$

(We note that we estimated $\sqrt{n} \le 2$ only to keep the expressions as simple as possible. After possibly enlarging certain constants, the same estimates also hold in the higher dimensional settings; that is, no restriction on the dimension is imposed here.)

Regarding the remaining term in (5.3.15), we first note that

$$D^2 f(u,v) = \begin{pmatrix} -2\mu_1 & a_1 \\ a_1 & 0 \end{pmatrix} \qquad \text{in } (0, T_{\max})$$

and that (5.2.5) implies

$$\begin{aligned} f_u(u,v) &= f_u(u,v_\star) + a_1(v-v_\star) \\ &= f_u(u_\star,v_\star) + f_{uu}(u_\star,v_\star)(u-u_\star) + a_1(v-v_\star) \\ &\le -2\mu_1(u-u_\star) + a_1(v-v_\star) \qquad \text{in } (0,T_{\max}). \end{aligned}$$

Therefore, an integration by parts and applications of Young's inequality as well as Poincaré's inequality 5.7.1 yield

$$\begin{aligned} &-\int_\Omega \nabla(f(u,v))\cdot\nabla\Delta u \\ &= -\int_\Omega f_u(u,v)\nabla u\cdot\nabla\Delta u - \int_\Omega f_v(u,v)\nabla v\cdot\nabla\Delta u \\ &= \int_\Omega f_u(u,v)|\Delta u|^2 + \int_\Omega f_{uu}(u,v)|\nabla u|^2\Delta u + 2\int_\Omega f_{uv}(u,v)\nabla u\cdot\nabla v\Delta u \\ &\quad + \int_\Omega f_v(u,v)\Delta u\Delta v + \int_\Omega f_{vv}(u,v)|\nabla v|^2\Delta u \\ &\le \eta(a_1+2\mu_1)\int_\Omega|\Delta u|^2 + 2\mu_1\int_\Omega|\nabla u|^2|\Delta u| + 2a_1\int_\Omega\nabla u\cdot\nabla v\Delta u \\ &\quad + a_1\int_\Omega(u-u_\star)\Delta u\Delta v + a_1u_\star\int_\Omega\Delta u\Delta v \\ &\le C_{\mathrm{P}}\eta(a_1+2\mu_1)\int_\Omega|\nabla\Delta u|^2 + a_1u_\star\int_\Omega\Delta u\Delta v \\ &\quad + \eta\mu_1\int_\Omega|\Delta u|^2 + \frac{\mu_1}{\eta}\int_\Omega|\nabla u|^4 \\ &\quad + a_1\eta\int_\Omega|\Delta u|^2 + \frac{a_1}{2\eta}\int_\Omega|\nabla u|^4 + \frac{a_1}{2\eta}\int_\Omega|\nabla v|^4 \\ &\quad + \frac{a_1\eta}{2}\int_\Omega|\Delta u|^2 + \frac{a_1\eta}{2}\int_\Omega|\Delta v|^2 \\ &\le \frac{C_{\mathrm{P}}\eta(5a_1+6\mu_1)}{2}\int_\Omega|\nabla\Delta u|^2 + \frac{C_{\mathrm{P}}a_1\eta}{2}\int_\Omega|\nabla\Delta v|^2 + a_1u_\star\int_\Omega\Delta u\Delta v \\ &\quad + \frac{2\mu_1+a_1}{2\eta}\int_\Omega|\nabla u|^4 + \frac{a_1}{2\eta}\int_\Omega|\nabla v|^4 \qquad \text{in } (0,T_\eta). \end{aligned}$$

Herein we make use of (5.3.13), (5.3.2) and Poincaré's inequality 5.7.1 to further conclude

$$\int_\Omega|\nabla u|^4 \le c_2\|u-\overline{u}\|_{L^\infty(\Omega)}^2\int_\Omega|\Delta u|^2 \le 4C_{\mathrm{P}}c_2\eta^2\int_\Omega|\nabla\Delta u|^2 \qquad \text{in } (0,T_\eta)$$

and, likewise, now using (5.3.3) instead of (5.3.2),

$$\int_\Omega|\nabla v|^4 \le 4C_{\mathrm{P}}c_2\eta^2\int_\Omega|\nabla\Delta v|^2 \qquad \text{in } (0,T_\eta).$$

Thus, due to $c_2 \ge 1$,

$$\begin{aligned} &-\int_\Omega\nabla(f(u,v))\cdot\nabla\Delta u \\ &\le \frac{C_{\mathrm{P}}c_2\eta(9a_1+14\mu_1)}{2}\int_\Omega|\nabla\Delta u|^2 + \frac{5C_{\mathrm{P}}a_1c_2\eta}{2}\int_\Omega|\nabla\Delta v|^2 + a_1u_\star\int_\Omega\Delta u\Delta v \end{aligned}$$

holds in $(0, T_\eta)$.

As usual, we now combine the estimates above with analogous computations for v to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{C_1}{2}\int_\Omega |\Delta u|^2 + \frac{C_2}{2}\int_\Omega |\Delta v|^2\right) \\
&+\left(\frac{3C_1D_1}{4} - M_1(\eta)\right)\int_\Omega |\nabla\Delta u|^2 + \left(\frac{3C_2D_2}{4} - M_2(\eta)\right)\int_\Omega |\nabla\Delta v|^2 \\
\le\ &(C_1a_1u_\star - C_2a_2v_\star)\int_\Omega \Delta u\Delta v + (C_1\chi_1u_\star - C_2\chi_2v_\star)\int_\Omega \nabla\Delta u\cdot\nabla\Delta v \qquad \text{in } (0, T_\eta),
\end{aligned}$$

which in virtue of (5.3.14) implies the statement. □

5.4. Deriving $W^{2,2}(\Omega)$ bounds for u and v

In this section, we will make use of the estimates gained in the previous section to eventually obtain $W^{2,2}(\Omega)$ bounds for both solution components. That is, we aim to bound the quantity $\|u - u_\star\|_{W^{2,2}(\Omega)} + \|v - v_\star\|_{W^{2,2}(\Omega)}$ by, say, $\frac{\eta}{2}$ in $(0, T_\eta)$ (for a certain $\eta > 0$), as then $T_\eta = T_{\max} = \infty$ can be concluded—provided $T_\eta > 0$ which in turn can be achieved by requiring $\|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0 - v_\star\|_{W^{2,2}(\Omega)}$ to be sufficiently small.

In what follows, we will distinguish between multiple cases. More concretely, we will handle

- (5.H1) in Lemma 5.4.2,
- (5.H2) with $\lambda_2\mu_1 > \lambda_1a_2$ in Lemma 5.4.3,
- (5.H2) with $\lambda_2\mu_1 < \lambda_1a_2$ in Lemma 5.4.4 and Lemma 5.4.5
- (5.H2) with $\lambda_2\mu_1 = \lambda_1a_2$ and $\lambda_1 > 0$ in Lemma 5.4.7 (ii) and Lemma 5.4.8 as well as
- (5.H2) with $\lambda_1 = \lambda_2 = 0$ in Lemma 5.4.9.

These five cases can be divided into two groups, the first of which we deal with in the following subsection.

5.4.1. The cases (5.H1) and (5.H2) with $\lambda_2\mu_1 > \lambda_1a_2$

If either (5.H1) holds with $m_1, m_2 > 0$ or (5.H2) holds with $\lambda_2\mu_1 > \lambda_1a_2$, then $u_\star$ and $v_\star$ are positive—which is the reason these cases can be handled in a similar fashion. In both cases, we aim to apply the following elementary lemma.

Lemma 5.4.1. *For $A, B, C > 0$ and $\varphi \in W^{2,2}(\Omega)$, set*

$$\phi_{A,B,C}(\varphi) := \frac{A}{2}\int_\Omega \varphi^2 + \frac{B}{2}\int_\Omega |\nabla\varphi|^2 + \frac{C}{2}\int_\Omega |\Delta\varphi|^2 \tag{5.4.1}$$

and let $A_1, A_2, B_1, B_2, C_1, C_2 > 0$, $\eta > 0$ and $K_2 > 0$.

There is $K_1 > 0$ such that, if u_0, v_0 comply with (5.1.3), T_η is as in (5.3.1) and

$$y\colon [0, T_\eta) \to \mathbb{R}, \quad t \mapsto \phi_{A_1,B_1,C_1}(u(\cdot, t) - u_\star) + \phi_{A_2,B_2,C_2}(v(\cdot, t) - v_\star) \tag{5.4.2}$$

fulfills

$$y'(t) \le -2K_2(t) \qquad \text{in } (0,T_\eta), \tag{5.4.3}$$

then

$$\begin{aligned} &\|u(\cdot,t)-u_\star\|_{W^{2,2}(\Omega)} + \|v(\cdot,t)-v_\star\|_{W^{2,2}(\Omega)} \\ &\le K_1 \mathrm{e}^{-K_2 t}\left(\|u_0-u_\star\|_{W^{2,2}(\Omega)} + \|v_0-v_\star\|_{W^{2,2}(\Omega)}\right) \qquad \text{for all } t \in (0,T_\eta). \end{aligned} \tag{5.4.4}$$

Proof. As $W^{2,2}(\Omega)$ continuity of u and v up to $t=0$ is ensured by (5.2.3), we may make use of an ODE comparison argument to obtain

$$y(t) \le \mathrm{e}^{-2K_2 t} y(0) \qquad \text{for all } t \in (0,T_\eta).$$

The statement then follows by taking square roots on both sides and noting that $\|\varphi\| := \sqrt{\phi_{A,B,C}(\varphi)}$ defines for $A,B,C>0$ a norm on $W^{2,2}_N(\Omega)$, which is equivalent to the usual one by Lemma 5.7.2. □

For both cases covered in this subsection, we now choose $A_1,A_2,B_1,B_2,C_1,C_2>0$ appropriately so that Lemma 5.4.1 is applicable.

Lemma 5.4.2. *Suppose* (5.H1). *Then there are* $\eta>0$ *and* $K_1,K_2>0$ *such that* (5.4.4) *holds for all* u_0,v_0 *satisfying* (5.1.3).

Proof. In the case of (5.H1) with $m_1=0$ or $m_2=0$, that is, if at least one of the initial data is trivial, the uniqueness statement in Lemma 5.2.1 asserts that one solution component is constantly zero while the other solves the heat equation. As in that case the statement becomes trivial, we may assume $m_1>0$ and $m_2>0$.

Then $u_\star,v_\star>0$ and hence $A_1=B_1=C_1:=\chi_2 v_\star$ as well as $A_2=B_2=C_2:=\chi_1 u_\star$ are positive as well. Because of

$$A_1\chi_1u_\star - A_2\chi_2v_\star = 0, \quad B_1\chi_1u_\star - B_2\chi_2v_\star = 0, \quad C_1\chi_1u_\star - C_2\chi_2v_\star = 0$$

and (5.H1), Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.4 assert that there is $\eta>0$ such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_{A_1,B_1,C_1}(u(\cdot,t)-u_\star) + \phi_{A_2,B_2,C_2}(v(\cdot,t)-v_\star)\Big) \\ &\quad + \frac{C_1D_1}{2}\int_\Omega |\nabla\Delta u|^2 + \frac{C_2D_2}{2}\int_\Omega |\nabla\Delta v|^2 \\ &\le (A_1\chi_1u_\star - A_2\chi_2v_\star)\int_\Omega \nabla u\cdot\nabla v + (B_1\chi_1u_\star - B_2\chi_2v_\star)\int_\Omega \Delta u\Delta v \\ &\quad + (C_1\chi_1u_\star - C_2\chi_2v_\star)\int_\Omega \nabla\Delta u\cdot\nabla\Delta v \\ &= 0 \qquad \text{in } (0,T_\eta), \end{aligned}$$

whenever u_0,v_0 comply with (5.1.3), where ϕ and T_η are as in (5.4.1) and (5.3.1), respectively.

As integrating the first two equations in (5.P) implies $u_\star=\overline{u}_0=\overline{u}$ and $v_\star=\overline{v}_0=\overline{v}$ in $(0,T_{\max})$, we further obtain by Poincaré's inequality 5.7.1 that (5.4.3) is fulfilled for some $K_2>0$, hence the statement follows from Lemma 5.4.1. □

Somewhat surprisingly, also in the case (5.H2) with $\lambda_2\mu_1 > \lambda_1 a_2$, suitably choosing A_1, A_2, B_1, B_2, C_1, C_2 in Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.4 allows for a cancellation of all problematic terms.

Lemma 5.4.3. *Suppose* (5.H2) *holds with* $\lambda_2\mu_1 > \lambda_1 a_2$. *Then we can find* $\eta > 0$ *and* $K_1, K_2 > 0$ *with the property that* (5.4.4) *holds whenever* u_0, v_0 *satisfy* (5.1.3).

Proof. Positivity of $u_\star$ and $v_\star$ implies that the constants

$$\begin{aligned} A_1 &\coloneqq a_2 v_\star, & A_2 &\coloneqq a_1 u_\star, \\ B_1 &\coloneqq (a_2+\chi_2)v_\star, & B_2 &\coloneqq (a_1+\chi_1)u_\star, \\ C_1 &\coloneqq \chi_2 v_\star \quad \text{and} & C_2 &\coloneqq \chi_1 u_\star \end{aligned}$$

are all positive, so that we may apply Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.4 to obtain $\eta_1 > 0$ such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_{A_1,B_1,C_1}(u(\cdot,t)-u_\star) + \phi_{A_2,B_2,C_2}(v(\cdot,t)-v_\star)\Big) \\ &+\frac{C_1 D_1}{2}\int_\Omega |\nabla\Delta u|^2 + \frac{C_2 D_2}{2}\int_\Omega |\nabla\Delta v|^2 \\ &+A_1\left(-f_u(u_\star,v_\star) - \eta(a_1+\mu_1)\right)\int_\Omega (u-u_\star)^2 + A_2\left(-g_v(u_\star,v_\star)-\eta(a_2+\mu_2)\right)\int_\Omega (v-v_\star)^2 \\ \le\ &(A_1 a_1 u_\star - A_2 a_2 v_\star)\int_\Omega (u-u_\star)(v-v_\star) \\ &+[(A_1\chi_1 + B_1 a_1)u_\star - (A_2\chi_2 + B_2 a_2)v_\star]\int_\Omega \nabla u\cdot\nabla v \\ &+[(B_1\chi_1 + C_1 a_1)u_\star - (B_2\chi_2 + C_2 a_2)v_\star]\int_\Omega \Delta u\Delta v \\ &+(C_1\chi_1 u_\star - C_2\chi_2 v_\star)\int_\Omega \nabla\Delta u\cdot\nabla\Delta v \qquad \text{holds in } (0,T_\eta) \text{ for all } \eta \le \eta_1, \end{aligned}$$

provided u_0, v_0 satisfy (5.1.3), where again ϕ and T_η are defined in (5.4.1) and (5.3.1), respectively.

Setting further $\eta_2 \coloneqq \min\left\{\frac{-f_u(u_\star,v_\star)}{2(a_1+\mu_1)}, \frac{-g_u(u_\star,v_\star)}{2(a_2+\mu_2)}\right\}$, which is positive by (5.2.6), and noting that

$$\begin{aligned} A_1 a_1 u_\star - A_2 a_2 v_\star &= 0, \\ (A_1\chi_1 + B_1 a_1)u_\star - (A_2\chi_2 + B_2 a_2)v_\star &= 0, \\ (B_1\chi_1 + C_1 a_1)u_\star - (B_2\chi_2 + C_2 a_2)v_\star &= 0 \quad \text{as well as} \\ C_1\chi_1 u_\star - C_2\chi_2 v_\star &= 0, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_{A_1,B_1,C_1}(u(\cdot,t)-u_\star) + \phi_{A_2,B_2,C_2}(v(\cdot,t)-v_\star)\Big) \\ &+\frac{C_1 D_1}{2}\int_\Omega |\nabla\Delta u|^2 + \frac{C_2 D_2}{2}\int_\Omega |\nabla\Delta v|^2 \\ &-\frac{A_1 f_u(u_\star,v_\star)}{2}\int_\Omega (u-u_\star)^2 - \frac{A_2 g_v(u_\star,v_\star)}{2}\int_\Omega (v-v_\star)^2 \\ \le\ &0 \qquad \text{in } (0,T_\eta) \end{aligned}$$

for $\eta \coloneqq \min\{\eta_1, \eta_2\}$, provided u_0, v_0 comply with (5.1.3).

In virtue of Poincaré's inequality 5.7.1, this first asserts (5.4.3) for some $K_2 > 0$ and then also (5.4.4) for some $K_1 > 0$ by Lemma 5.4.1. □

5.4.2. The case (5.H2) with $\lambda_2\mu_1 \le \lambda_1 a_2$

The condition (5.H2) with $\lambda_2\mu_1 \le \lambda_1 a_2$ implies $v_\star = 0$, hence for any choice of A_1, A_2, B_1, B_2, C_1, $C_2 > 0$ in Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.4, unlike as in the previous subsection, no cancellation of problematic terms can occur (except if also $u_\star = 0$, but then we will rely on a different functional, see Lemma 5.4.9 below).

However, the disappearance of $v_\star$ can also be used to our advantage. As the coefficients of the problematic terms no longer depend on A_2, B_2 and C_2, we can choose (one of) these parameters comparatively large and thus obtain stronger dissipative terms. This idea first manifests itself in the following

Lemma 5.4.4. *Suppose* (5.H2) *holds with $\lambda_2\mu_1 \le \lambda_1 a_2$. There are $\eta > 0$ as well as $K > 0$ and $C_2 > 0$ such that whenever u_0, v_0 comply with* (5.1.3) *and T_η is as in* (5.3.1),

$$\int_\Omega |\Delta u(\cdot,t)|^2 + C_2 \int_\Omega |\Delta v(\cdot,t)|^2 \le \mathrm{e}^{-Kt} \left(\int_\Omega |\Delta u_0|^2 + C_2 \int_\Omega |\Delta v_0|^2 \right) \qquad \textit{for all } t \in (0, T_\eta).$$

Proof. Set $K \coloneqq \frac{\min\{D_1, D_2\}}{2} > 0$, $C_1 \coloneqq 1$ and

$$C_2 \coloneqq \frac{16 \max\{C_{\mathrm{P}}^2 a_1^2, \chi_1^2\}(u_\star + 1)^2}{D_1 D_2} > 0,$$

where $C_{\mathrm{P}} > 0$ denotes the constant given by Lemma 5.7.1.

By Lemma 5.3.4, there is $\eta > 0$ with the property that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega |\Delta u|^2 + C_2 \int_\Omega |\Delta v|^2 \right) + D_1 \int_\Omega |\nabla \Delta u|^2 + C_2 D_2 \int_\Omega |\nabla \Delta v|^2 \\ &\le 2 a_1 u_\star \int_\Omega \Delta u \Delta v + 2\chi_1 u_\star \int_\Omega \nabla \Delta u \cdot \nabla \Delta v \qquad \text{in } (0, T_\eta), \end{aligned}$$

provided the (henceforth fixed) initial data u_0, v_0 satisfy (5.1.3).

Therein are by Young's inequality and Poincaré's inequality 5.7.1, with $C_{\mathrm{P}} > 0$ as in that lemma,

$$\begin{aligned} 2 a_1 u_\star \int_\Omega \Delta u \Delta v &\le \frac{D_1}{4 C_{\mathrm{P}}} \int_\Omega |\Delta u|^2 + \frac{4 C_{\mathrm{P}} a_1^2 u_\star^2}{D_1} \int_\Omega |\Delta v|^2 \\ &\le \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{C_2 D_2}{4} \int_\Omega |\nabla \Delta v|^2 \qquad \text{in } (0, T_{\max}) \end{aligned}$$

and, again by Young's inequality,

$$\begin{aligned} 2\chi_1 u_\star \int_\Omega \nabla \Delta u \cdot \nabla \Delta v &\le \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{4 \chi_1^2 u_\star^2}{D_1} \int_\Omega |\nabla \Delta v|^2 \\ &\le \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{C_2 D_2}{4} \int_\Omega |\nabla \Delta v|^2 \qquad \text{in } (0, T_{\max}). \end{aligned}$$

Thus, the statement follows upon an integration over $(0, T_\eta)$ due to (5.2.3), the $W^{2,2}(\Omega)$ continuity of u and v up to $t = 0$. $\square$

In the case (5.H2) with $\lambda_2\mu_1 < \lambda_1 a_2$, by a similar argument, we also obtain bounds for $\int_\Omega (u - u_\star)^2$ and $\int_\Omega v^2$.

Lemma 5.4.5. *If* (5.H2) *holds with* $\lambda_2\mu_1 < \lambda_1 a_2$*, then there are* $\eta > 0$*,* $K > 0$ *and* $A_2 > 0$ *such that*

$$\int_\Omega (u - u_\star)^2 + A_2 \int_\Omega v^2 \le \mathrm{e}^{-Kt} \left(\int_\Omega (u_0 - u_\star)^2 + A_2 \int_\Omega v_0^2 \right) \qquad \text{for all } t \in (0, T_\eta).$$

provided u_0, v_0 *satisfy* (5.1.3) *and* T_η *is as in* (5.3.1).

Proof. Since $\lambda_2\mu_1 < \lambda_1 a_2$, both $f_u(u_\star, v_\star)$ and $g_v(u_\star, v_\star)$ are negative according to (5.2.6), hence there is $\eta_1 > 0$ such that

$$K \coloneqq \min\left\{ -f_u(u_\star, v_\star) - \eta_1(a_1 + \mu_1), -g_v(u_\star, v_\star) - \eta_1(a_2 + \mu_2) \right\} > 0.$$

Set moreover $A_1 \coloneqq 1$ and

$$A_2 \coloneqq \max\left\{ \frac{a_1^2}{K^2}, \frac{\chi_1^2}{D_1 D_2} \right\} u_\star^2 > 0.$$

Then Lemma 5.3.2 provides us with $\eta \in (0, \eta_1)$ such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega (u - u_\star)^2 + A_2 \int_\Omega v^2 \right) \\ &+ D_1 \int_\Omega |\nabla u|^2 + A_2 D_2 \int_\Omega |\nabla v|^2 \\ &+ 2K \int_\Omega (u - u_\star)^2 + 2A_2 K \int_\Omega v^2 \\ \le\ & 2a_1 u_\star \int_\Omega (u - u_\star) v + 2\chi_1 u_\star \int_\Omega \nabla u \cdot \nabla v \qquad \text{in } (0, T_\eta), \end{aligned}$$

whenever u_0, v_0 comply with (5.1.3).

Henceforth fixing such initial data, two applications of Young's inequality give

$$2a_1 u_\star \int_\Omega (u - u_\star) v \le K \int_\Omega (u - u_\star)^2 + \frac{a_1^2 u_\star^2}{K} \int_\Omega v^2 \le K \int_\Omega (u - u_\star)^2 + A_2 K \int_\Omega v^2$$

and

$$2\chi_1 u_\star \int_\Omega \nabla u \cdot \nabla v \le D_1 \int_\Omega |\nabla u|^2 + \frac{\chi_1^2 u_\star^2}{D_1} \int_\Omega |\nabla v|^2 \le D_1 \int_\Omega |\nabla u|^2 + A_2 D_2 \int_\Omega |\nabla v|^2$$

in $(0, T_{\max})$, so that the statement follows by the comparison principle for ordinary differential equations. $\square$

The case (5.H2) with $\lambda_2\mu_1 = \lambda_1 a_2$ cannot be handled in a similar fashion as then $g_v(u_\star, v_\star)$ vanishes, resulting in the term $A_2(-g_v(u_\star, v_\star) - \eta(a_2 + \mu_2)) \int_\Omega v^2$ in (5.3.8) having an unfavorable sign. Similarly, if $\lambda_1 = 0$, then $f_u(u_\star, v_\star) = 0$ and $A_1(-f_u(u_\star, v_\star) - \eta(a_1 + \mu_1)) < 0$. Thus, we introduce an additional functional to counter these terms.

Lemma 5.4.6. *Suppose that u_0, v_0 comply with* (5.1.3)*. If $\lambda_1 = 0$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = -\mu_1 \int_\Omega u^2 + a_1 \int_\Omega uv \qquad \text{in } (0, T_{\max}) \tag{5.4.5}$$

and if (5.H2) *holds with $\lambda_2\mu_1 = \lambda_1 a_2$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v = -\mu_2 \int_\Omega v^2 - a_2 \int_\Omega (u - u_\star)v \qquad \text{in } (0, T_{\max}). \tag{5.4.6}$$

Proof. The first statement immediately follows by integrating the first equation in (5.P). Furthermore, the assumptions (5.H2) and $\lambda_2\mu_1 = \lambda_1 a_2$ imply $(u_\star, v_\star) = (\frac{\lambda_1}{\mu_1}, 0) = (\frac{\lambda_2}{a_2}, 0)$ and hence

$$\begin{aligned} g(u,v) &= v(\lambda_2 - \mu_2 v - a_2 u) \\ &= v(\lambda_2 - \mu_2 v - a_2 u_\star) + a_2(u_\star - u)v \\ &= -\mu_2 v^2 - a_2(u - u_\star)v \qquad \text{in } (0, T_{\max}). \end{aligned}$$

Thus, the second statement follows also due to integrating. □

With the help of this lemma, we can now handle the remaining case, namely (5.H2) with $\lambda_2\mu_1 = \lambda_1 a_2$. The proof is split into three lemmata; before dealing with the (in some sense) fully degenerate case, in the following two lemmata, we first handle the half-degenerate case, where at least $u_\star > 0$ and $f_u(u_\star, v_\star) > 0$.

Lemma 5.4.7. *Suppose* (5.H2)*, $\lambda_2\mu_1 = \lambda_1 a_2$ as well as $\lambda_1 > 0$ and, for $\eta > 0$, let T_η be as in* (5.3.1)*.*

(i) There are $\eta > 0$ and $K_1, K_2 > 0$ such that

$$\|v(\cdot,t)\|_{L^1(\Omega)} \le \left(K_1 \left(\|u_0 - u_\star\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)}\right)^{-1} + K_2 t\right)^{-1}$$

for all $t \in (0, T_\eta)$, whenever u_0, v_0 are such that (5.1.3) *holds.*

(ii) We can find $\eta' > 0$ and $K_1', K_2' > 0$ such that

$$\|v(\cdot,t)\|_{W^{2,2}(\Omega)} \le \left(K_1' \left(\|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}\right)^{-1} + K_2' t\right)^{-1}$$

for all $t \in (0, T_{\eta'})$ if u_0, v_0 comply with (5.1.3)*.*

Proof. Setting $A_1 := 1$, $X_2 := \frac{a_1 u_\star}{a_2} > 0$, $A_2 := \frac{\chi_1^2 u_\star^2}{D_1 D_2} > 0$, by Lemma 5.3.2 and Lemma 5.4.6, we find $\eta_0 > 0$ such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_1}{2}\int_\Omega (u - u_\star)^2 + \frac{A_2}{2}\int_\Omega v^2 + X_2 \int_\Omega v\right) \\ &+\frac{A_1 D_1}{2}\int_\Omega |\nabla u|^2 + \frac{A_2 D_2}{2}\int_\Omega |\nabla v|^2 \\ &+ (-A_1 f_u(u_\star, v_\star) - A_1\eta(a_1 + \mu_1))\int_\Omega (u - u_\star)^2 + (X_2\mu_2 - A_2\eta(a_2 + \mu_2))\int_\Omega v^2 \\ \le{}& (A_1 a_1 u_\star - X_2 a_2)\int_\Omega (u - u_\star)v + A_1\chi_1 u_\star \int_\Omega \nabla u \cdot \nabla v \end{aligned} \tag{5.4.7}$$

in $(0,T_\eta)$ for all $\eta \le \eta_0$, whenever u_0, v_0 comply with (5.1.3).

We set $c_1 := \frac{A_1 f_u(u_\star,v_\star)}{2} > 0$, $c_2 := \frac{X_2\mu_2}{2} > 0$, $c_3 := \min\left\{\frac{4c_1}{3A_1^2}, \frac{2c_2}{3A_2^2}, \frac{c_2}{6X_2^2|\Omega|}\right\} > 0$ as well as

$$\eta := \min\left\{1, \eta_0, |\Omega|^{-\frac{1}{2}}, \frac{c_1}{A_1(a_1+\mu_1)}, \frac{c_2}{A_2(a_2+\mu_2)}\right\} > 0$$

and fix u_0, v_0 satisfying (5.1.3).

As the term $A_1a_1u_\star - X_2a_2$ vanishes due to the definitions of A_1 and X_2, and Young's inequality as well as the definition of A_2 imply

$$A_1\chi_1u_\star \int_\Omega \nabla u\cdot\nabla v \le \frac{A_1D_1}{2}\int_\Omega |\nabla u|^2 + \frac{A_2D_2}{2}\int_\Omega |\nabla u|^2 \qquad \text{in } (0,T_{\max}),$$

we may conclude from (5.4.7) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_1}{2}\int_\Omega (u-u_\star)^2 + \frac{A_2}{2}\int_\Omega v^2 + X_2\int_\Omega v\right) \\
&\le -c_1\int_\Omega (u-u_\star)^2 - c_2\int_\Omega v^2 \qquad \text{holds in } (0,T_\eta).
\end{aligned}$$

Since $\eta \le |\Omega|^{-\frac{1}{2}}$ implies $\int_\Omega (u-u_\star)^2 \le 1$ as well as $\int_\Omega v^2 \le 1$ in $(0,T_\eta)$ and due to Hölder's inequality as well as the elementary inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ for $a,b,c \in \mathbb{R}$, we further obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_1}{2}\int_\Omega (u-u_\star)^2 + \frac{A_2}{2}\int_\Omega v^2 + X_2\int_\Omega v\right) \\
&\le -c_1\int_\Omega (u-u_\star)^2 - \frac{c_2}{2}\int_\Omega v^2 - \frac{c_2}{2}\int_\Omega v^2 \\
&\le -c_1\left(\int_\Omega (u-u_\star)^2\right)^2 - \frac{c_2}{2}\left(\int_\Omega v^2\right)^2 - \frac{c_2}{2|\Omega|}\left(\int_\Omega v\right)^2 \\
&\le -c_3\left(\frac{A_1}{2}\int_\Omega (u-u_\star)^2 + \frac{A_2}{2}\int_\Omega v^2 + X_2\int_\Omega v\right)^2 \qquad \text{in } (0,T_\eta).
\end{aligned}$$

Because of $\eta \le 1$ and since without loss of generality both $\|u_0-u_\star\|_{L^\infty(\Omega)}$ and $\|v_0\|_{L^\infty(\Omega)}$ are smaller than η, this implies

$$\begin{aligned}
&X_2\|v(\cdot,t)\|_{L^1(\Omega)} \\
&\le \left(\left(\frac{A_1}{2}\int_\Omega (u_0-u_\star)^2 + \frac{A_2}{2}\int_\Omega v_0^2 + X_2\int_\Omega v_0\right)^{-1} + c_3t\right)^{-1} \\
&\le \left(\left(\frac{A_1}{2}\int_\Omega |u_0-u_\star| + \left(\frac{A_2}{2}+X_2\right)\int_\Omega v_0\right)^{-1} + c_3t\right)^{-1} \qquad \text{for all } t \in (0,T_\eta)
\end{aligned}$$

and hence proves part (i) for certain $K_1, K_2 > 0$.

Part (ii) follows then from Lemma 5.4.4, part (i) and the observation that

$$\begin{aligned}
\|v\|_{W^{2,2}(\Omega)} &\le \|v-\overline{v}\|_{W^{2,2}(\Omega)} + \|\overline{v}\|_{L^2(\Omega)} \\
&\le C\|\Delta v\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}}\|v\|_{L^1(\Omega)} \qquad \text{holds in } (0,T_{\max})
\end{aligned}$$

due to Lemma 5.7.2 (with $C > 0$ as in that lemma). □

Next, we proceed to gain similar estimates also for the first equation.

Lemma 5.4.8. *Assume* (5.H2) *holds and* $\lambda_2\mu_1 = \lambda_1 a_2$ *as well as* $\lambda_1 > 0$*. Then there are* $\eta > 0$ *and* $K_1, K_2 > 0$ *such that*

$$\|u(\cdot,t) - u_\star\|_{W^{2,2}(\Omega)} \le \Big(K_1\left(\|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}\right)^{-1} + K_2 t\Big)^{-1}$$

for all $t \in (0, T_\eta)$ *if* u_0, v_0 *satisfy* (5.1.3) *and* T_η *is as in* (5.3.1).

Proof. We choose $\eta_1 > 0$ so small that $c_1 \coloneqq \lambda_1 - (a_1 + \mu_1)\eta_1 > 0$ and set

$$c_2 \coloneqq \max\left\{\frac{a_1^2 u_\star^2}{c_1}, \frac{2\chi_1^2 u_\star^2}{3D_1} + \chi_1\right\}.$$

By Lemma 5.3.1 and Lemma 5.4.7, there are moreover $\eta_2, \eta_3 > 0$ and $c_3, c_4 > 0$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-u_\star)^2 + \frac{3D_1}{2}\int_\Omega |\nabla u|^2 + 2\left(-f_u(u_\star, v_\star) - \eta(a_1+\mu_1)\right)\int_\Omega (u-u_\star)^2 \\ &\le 2a_1 u_\star \int_\Omega (u-u_\star)v + 2\chi_1 u_\star \int_\Omega \nabla u \cdot \nabla v + \eta\chi_1 \int_\Omega |\nabla v|^2 \qquad \text{in } (0, T_\eta) \text{ for all } \eta \in (0, \eta_2]\end{aligned}$$

and

$$\|v(\cdot,t)\|^2_{W^{2,2}(\Omega)} \le \Big(\sqrt{c_2}c_3\left(\|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}\right)^{-1} + \sqrt{c_2}c_4 t\Big)^{-2}$$

in $(0, T_{\eta_3})$, provided u_0, v_0 comply with (5.1.3).

Thus, fixing $\eta \coloneqq \min\{\eta_1, \eta_2, \eta_3, 1\}$ as well as u_0, v_0 satisfying (5.1.3) and noting that $f_u(u_\star, v_\star) = -\lambda_1$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-u_\star)^2 \\ &\le -\frac{3D_1}{2}\int_\Omega |\nabla u|^2 - 2c_1 \int_\Omega (u-u_\star)^2 + 2a_1 u_\star \int_\Omega (u-u_\star)^2 v \\ &\quad + 2\chi_1 u_\star \int_\Omega \nabla u \cdot \nabla v + \eta\chi_1 \int_\Omega |\nabla u|^2 \\ &\le -c_1 \int_\Omega (u-u_\star)^2 + \frac{a_1^2 u_\star^2}{c_1}\int_\Omega v^2 + \left(\frac{2\chi_1^2 u_\star^2}{3D_1} + \chi_1\right)\int_\Omega |\nabla v|^2 \\ &\le -c_1 \int_\Omega (u-u_\star)^2 + c_2\left(\int_\Omega v^2 + \int_\Omega |\nabla v|^2\right) \\ &\le -c_1 \int_\Omega (u-u_\star)^2 + \Big(c_3\left(\|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}\right)^{-1} + c_4 t\Big)^{-2} \qquad \text{in } (0, T_\eta),\end{aligned}$$

which by the variation-of-constants formula implies

$$\begin{aligned}&\int_\Omega (u-u_\star)^2(\cdot,t) \\ &\le \mathrm{e}^{-c_1 t}\int_\Omega (u_0 - u_\star)^2(\cdot,t) + \int_0^t \mathrm{e}^{-c_1(t-s)}(c_3 I_0^{-1} + c_4 s)^{-2}\,\mathrm{d}s \qquad \text{for all } t \in (0, T_\eta),\end{aligned}$$

where we abbreviated $I_0 := \|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}$. Noting that $[0,\infty) \ni s \mapsto (c_b I_0^{-1} + c_c s)^{-2}$ is decreasing, we further calculate

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-c_1(t-s)}(c_3 I_0^{-1} + c_4 s)^{-2}\,\mathrm{d}s \\
&= \int_0^{t/2} \mathrm{e}^{-c_1(t-s)}(c_3 I_0^{-1} + c_4 s)^{-2}\,\mathrm{d}s + \int_{t/2}^t \mathrm{e}^{-c_1(t-s)}(c_3 I_0^{-1} + c_4 s)^{-2}\,\mathrm{d}s \\
&\le \frac{I_0^2}{c_3^2}\int_{t/2}^t \mathrm{e}^{-c_1 s}\,\mathrm{d}s + \left(c_3 I_0^{-1} + \frac{c_4 t}{2}\right)^{-2}\int_0^{t/2} \mathrm{e}^{-s}\,\mathrm{d}s \\
&\le \frac{I_0^2}{c_1 c_3^2}\mathrm{e}^{-\frac{c_1}{2}t} + \frac{1}{(\sqrt{c_1}c_3 I_0^{-1} + \frac{\sqrt{c_1}c_4 t}{2})^2} \qquad \text{for all } t \in (0, T_\eta).
\end{aligned}$$

Combining these estimates with Lemma 5.4.4 and Lemma 5.7.2 yields the statement for certain $K_1, K_2 > 0$. □

Finally, we deal with the aforementioned fully degenerate case.

Lemma 5.4.9. *Suppose* (5.H2) *and* $\lambda_1 = \lambda_2 = 0$. *Then there are* $\eta > 0$ *and* $K_1, K_2 > 0$ *such that*

$$\begin{aligned}
&\|u(\cdot,t)\|_{W^{2,2}(\Omega)} + \|v(\cdot,t)\|_{W^{2,2}(\Omega)} \\
&\le \left(K_1\left(\|u_0\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}\right)^{-1} + K_2 t\right)^{-1}
\end{aligned} \tag{5.4.8}$$

for all $t \in (0, T_\eta)$, *where* T_η *is defined in* (5.3.1), *provided* u_0, v_0 *satisfy* (5.1.3).

Proof. We set $c_1 := \frac{\min\{\mu_1,\mu_2\}}{2}$ and fix u_0, v_0 complying with (5.1.3).

By multiplying (5.4.5) and (5.4.6) with a_2 and a_1, respectively, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a_2\int_\Omega u + a_1\int_\Omega v\right) = -\mu_1 a_2\int_\Omega u^2 - \mu_2 a_1 \int_\Omega v^2 \qquad \text{in } (0, T_{\max}).$$

Hence, along with Hölder's inequality this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a_2\int_\Omega u + a_1\int_\Omega v\right) \le -c_1\left(a_2\int_\Omega u + a_1\int_\Omega v\right)^2 \qquad \text{in } (0, T_{\max}),$$

which upon integrating results in

$$a_2\int_\Omega u(\cdot,t) + a_1\int_\Omega v(\cdot,t) \le \left(\left(a_2\int_\Omega u_0 + a_1\int_\Omega v_0\right)^{-1} + c_1 t\right)^{-1} \tag{5.4.9}$$

for all $t \in (0, T_{\max})$.

As in the proof of Lemma 5.4.7, we now apply Lemma 5.7.2 (with $C > 0$ as in that lemma) to see that

$$\|\varphi\|_{W^{2,2}(\Omega)} \le \|\varphi - \overline{\varphi}\|_{W^{2,2}(\Omega)} + \|\overline{\varphi}\|_{L^2(\Omega)} \le C\|\Delta\varphi\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}}\|\varphi\|_{L^1(\Omega)}$$

for all $\varphi \in C^2(\overline{\Omega})$ with $\partial_\nu\varphi = 0$, which when applied to $\varphi = u$ and $\varphi = v$ and combined with (5.4.9) and Lemma 5.4.4 implies (5.4.8) for certain $K_1, K_2 > 0$ and $\eta > 0$. □

5.5. Proof of Theorem 5.1.1

The various lemmata from Section 5.4 allow us now to find $\varepsilon > 0$ such that if u_0, v_0 satisfy ((5.1.3) and) (5.1.5), then $T_{\max} = \infty$ and (u, v) converges to $(u_\star, v_\star)$.

Lemma 5.5.1. *For $\varepsilon > 0$ and $K_1, K_2 > 0$, define*

$$y_{\varepsilon,K_1,K_2} \colon [0,\infty) \to \mathbb{R}, \quad t \mapsto \begin{cases} (\frac{1}{K_1\varepsilon} + K_2 t)^{-1}, & \textit{if } (5.\mathrm{H2}) \textit{ holds and } \lambda_2\mu_1 = \lambda_1 a_2, \\ K_1\varepsilon \mathrm{e}^{-K_2 t}, & \textit{else.} \end{cases}$$

Then there are $\varepsilon > 0$ and $K_1, K_2 > 0$ such that $T_{\max}(u_0, v_0) = \infty$,

$$\|(u(u_0,v_0))(\cdot,t) - u_\star\|_{W^{2,2}(\Omega)} + \|(v(u_0,v_0))(\cdot,t) - v_\star\|_{W^{2,2}(\Omega)} \le y_{\varepsilon,K_1,K_2}(t)$$

for all $t \ge 0$, whenever u_0, v_0 satisfy (5.1.3) *and* (5.1.5).

Proof. Lemma 5.4.2, Lemma 5.4.3, Lemma 5.4.4, Lemma 5.4.5, Lemma 5.4.7 (ii), Lemma 5.4.8 and Lemma 5.4.9 imply that there are $\eta > 0$ and $K_1, K_2 > 0$ with the following property: Let $\varepsilon' > 0$. If u_0, v_0 comply with (5.1.3) and (5.1.5) with ε replaced by ε', then

$$\|u(\cdot,t) - u_\star\|_{W^{2,2}(\Omega)} + \|v(\cdot,t) - v_\star\|_{W^{2,2}(\Omega)} \le y_{\varepsilon',K_1,K_2}(t) \tag{5.5.1}$$

for all $t \in [0, T_\eta)$, where $(u, v) \coloneqq (u(u_0, v_0), v(u_0, v_0))$ and $T_\eta \coloneqq T_\eta(u_0, v_0)$ is as in (5.3.1).

Thanks to the restriction $n \le 3$, Sobolev's embedding theorem asserts that there are $\alpha \in (0, 1)$ and $c_1 > 0$ such that

$$\|\varphi\|_{C^\alpha(\overline{\Omega})} \le c_1\|\varphi\|_{W^{2,2}(\Omega)} \qquad \text{for all } \varphi \in W^{2,2}(\Omega).$$

Fix an arbitrary $\varepsilon \in (0, \frac{\eta}{c_1 \max\{K_1,1\}})$ and u_0, v_0 complying not only with (5.1.3) but also with (5.1.5). As then

$$\begin{aligned} &\|u_0 - u_\star\|_{L^\infty(\Omega)} + \|v_0 - v_\star\|_{L^\infty(\Omega)} \\ &\le c_1 \left(\|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0 - v_\star\|_{W^{2,2}(\Omega)}\right) \le c_1\varepsilon < \eta, \end{aligned}$$

we infer $T_\eta > 0$ from $u, v \in C^0(\overline{\Omega} \times [0, T_{\max}))$. Moreover,

$$\begin{aligned} &\|u(\cdot,t) - u_\star\|_{L^\infty(\Omega)} + \|v(\cdot,t) - u_\star\|_{L^\infty(\Omega)} \\ &\le \|u(\cdot,t) - u_\star\|_{C^\alpha(\overline{\Omega})} + \|v(\cdot,t) - u_\star\|_{C^\alpha(\overline{\Omega})} \\ &\le c_1 \left(\|u(\cdot,t) - u_\star\|_{W^{2,2}(\Omega)} + \|v(\cdot,t) - v_\star\|_{W^{2,2}(\Omega)}\right) \\ &\le c_1 y_{\varepsilon,K_1,K_2}(t) \\ &\le c_1 y_{\varepsilon,K_1,K_2}(0) \\ &= K_1 c_1 \varepsilon < \eta \qquad \text{for all } t \in (0, T_\eta), \end{aligned} \tag{5.5.2}$$

hence the definition (5.3.1) of T_η asserts $T_\eta = T_{\max}$. In that case, (5.5.2) further implies $T_{\max} = \infty$ because of the blow-up criterion (5.2.2). Finally, as then $T_\eta = T_{\max} = \infty$, the statement is equivalent to (5.5.1). □

Theorem 5.1.1 is now a direct consequence of already proved lemmata.

Proof of Theorem 5.1.1. Local existence and the regularity statements were already part of Lemma 5.2.1, while global extensibility, convergence to $(u_\star, v_\star)$ as well as the claimed convergence rates were the subject of Lemma 5.5.1. □

5.6. Possible generalizations of Theorem 5.1.1

Having proven Theorem 5.1.1, let us discuss whether the methods used in this chapter could potentially be used to derive more general versions thereof.

Remark 5.6.1. We recall that the limitation on the space dimension, namely that $n \in \{1, 2, 3\}$, has only been used at one place: In the proof of Lemma 5.5.1 we made use of the embedding $W^{2,2}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ (for some $\alpha \in (0,1)$), which only holds in said space dimensions. Thus, it is conceivable that replacing $W^{2,2}(\Omega)$ by $W^{m,2}(\Omega)$ for suitable $m \in \mathbb{N}$ in Theorem 5.1.1 allows for certain generalizations of the main result of the present chapter.

Indeed, if $n = 1$, Theorem 5.1.1 remains correct if one replaces $W^{2,2}(\Omega)$ by $W^{1,2}(\Omega)$ in all occurrences (and $W^{2,2}_N(\Omega)$ also by $W^{1,2}(\Omega)$). This can be seen by a straightforward modification of the proofs above: Combine Lemma 5.3.2 only with Lemma 5.3.3 and not also with Lemma 5.3.4. However, a detailed proof would lead to either a considerably longer or a unreasonably more complicated exposition (or to both) and is hence omitted.

At first glance, similar arguments as above appear to imply an analogon of Theorem 5.1.1 (with $W^{2,2}(\Omega)$ replaced by $W^{m,2}(\Omega)$ for sufficiently large $m \in \mathbb{N}$) even for higher dimensions. The main problem, however, is, that during the computations several boundary terms would appear, which apparently cannot be dealt with easily. Let us emphasize that the question whether (a suitably modified version of) Theorem 5.1.1 holds also in the higher dimensional setting is purely of mathematical interest. The biologically relevant dimensions are covered in Theorem 5.1.1.

Remark 5.6.2. The prototypical choices of S_1, S_2, f and g in (5.1.1) are mainly made for simplicity. We leave it to further research to determine more general conditions on these functions allowing for a theorem of the form of Theorem 5.1.1.

Still, the methods employed should be robust enough to also allow for (certain) nonlinear taxis sensitivities, for instance. At least for the case (5.H2) with $\lambda_2\mu_1 > \lambda_1 a_2$, however, the signs of S_1 and S_2 are important: Our approach demands, that, roughly speaking, predators move towards their prey and the prey flees from them.

The case (5.H2) with $\lambda_2\mu_1 \le \lambda_1 a_2$ is even less sensitive to such changes. In fact, as the proofs above clearly show, the conclusion of Theorem 5.1.1 remains true for different signs of χ_1, χ_2 (with the exception that for $\chi_1 > 0 > \chi_2$ or $\chi_1 < 0 < \chi_2$, one has to do some additional work at the level of local existence).

Likewise, the methods presented here should, in general, also work for different functional responses. Again, there is one caveat: The species moving towards (away from) the other one needs to benefit from (be harmed by) inter-species encounters.

5.7. Gagliardo–Nirenberg inequalities

At last, we prove various inequalities which have been used several times in the proof of Theorem 5.1.1. Throughout this section, we fix a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, define $\overline{\varphi} := \fint_\Omega \varphi$ for $\varphi \in L^1(\Omega)$ and set $W^{m,p}_N(\Omega) := \overline{\{\, \varphi \in C^\infty(\overline{\Omega}) : \partial_\nu \varphi = 0 \text{ on } \partial\Omega \,\}}^{\|\cdot\|_{W^{m,p}(\Omega)}}$ for $m \in \mathbb{N}$ and $p \in [1, \infty)$. (As can be easily seen, for $m = p = 2$, this definition is consistent with the definition of $W^{2,2}_N(\Omega)$ given in (5.1.4).)

We begin by stating the Poincaré inequality and straightforward consequences thereof.

Lemma 5.7.1. *There exists $C_{\mathrm{P}} > 0$ such that*

$$\begin{aligned}
\int_\Omega (\varphi - \overline{\varphi})^2 &\le C_{\mathrm{P}} \int_\Omega |\nabla \varphi|^2 && \textit{for all } \varphi \in W^{1,2}(\Omega) \\
\int_\Omega |\nabla \varphi|^2 &\le C_{\mathrm{P}} \int_\Omega |\Delta \varphi|^2 && \textit{for all } \varphi \in W^{2,2}_N(\Omega) \quad \textit{and} \\
\int_\Omega |\Delta \varphi|^2 &\le C_{\mathrm{P}} \int_\Omega |\nabla \Delta \varphi|^2 && \textit{for all } \varphi \in W^{3,2}_N(\Omega).
\end{aligned}$$

Proof. Since Ω is assumed to be smooth and bounded, Poincaré's inequality (cf. [56, Corollary 12.28]) asserts that there is $C_{\mathrm{P}} > 0$ such that

$$\int_\Omega (\varphi - \overline{\varphi})^2 \le C_{\mathrm{P}} \int_\Omega |\nabla \varphi|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{5.7.1}$$

By straightforward approximation and normalization arguments, it is sufficient to prove the remaining two inequalities for all $\varphi \in C^\infty(\overline{\Omega})$ with $\int_\Omega \varphi = 0$ and $\partial_\nu \varphi = 0$ on $\partial\Omega$. Thus, we fix such a φ.

An integration by parts, Hölder's inequality and (5.7.1) give

$$\begin{aligned}
\int_\Omega |\nabla \varphi|^2 &= - \int_\Omega \varphi \Delta \varphi + \int_{\partial\Omega} \varphi \partial_\nu \varphi \\
&\le \left(\int_\Omega \varphi^2 \right)^{\frac12} \left(\int_\Omega |\Delta \varphi|^2 \right)^{\frac12} + 0 \\
&\le \left(C_{\mathrm{P}} \int_\Omega |\nabla \varphi|^2 \right)^{\frac12} \left(\int_\Omega |\Delta \varphi|^2 \right)^{\frac12},
\end{aligned}$$

hence, in both cases $\int_\Omega |\nabla \varphi|^2 = 0$ and $\int_\Omega |\nabla \varphi|^2 > 0$,

$$\int_\Omega |\nabla \varphi|^2 \le C_{\mathrm{P}} \int_\Omega |\Delta \varphi|^2.$$

Similarly, we have

$$\begin{aligned}
\int_\Omega |\Delta \varphi|^2 &= - \int_\Omega \nabla \varphi \cdot \nabla \Delta \varphi + \int_\Omega \Delta \varphi \partial_\nu \varphi \\
&\le \left(C_{\mathrm{P}} \int_\Omega |\Delta \varphi|^2 \right)^{\frac12} \left(\int_\Omega |\nabla \Delta \varphi|^2 \right)^{\frac12} + 0 \\
&\le C_{\mathrm{P}} \int_\Omega |\nabla \Delta \varphi|^2.
\end{aligned}$$

□

The following lemma should also be well-known. However, failing to find a suitable reference, we choose to give a short proof.

Lemma 5.7.2. *Let $p \in (1, \infty)$. There exists $C > 0$ such that*

$$\|\varphi - \overline{\varphi}\|_{W^{2,p}(\Omega)} \le C \|\Delta \varphi\|_{L^p(\Omega)} \qquad \textit{for all } \varphi \in W^{2,p}_N(\Omega).$$

Proof. Suppose this is not the case. By an approximation/normalization argument, there would exist $(\varphi_k)_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ with $\int_\Omega \varphi_k = 0$ as well as $\partial_\nu \varphi_k = 0$ on $\partial\Omega$ and

$$\|\varphi_k\|_{W^{2,p}(\Omega)} > k\|\Delta\varphi_k\|_{L^p(\Omega)} \qquad \text{for all } k \in \mathbb{N}.$$

Without loss of generality, we may assume $\|\varphi_k\|_{W^{2,p}(\Omega)} = 1$ for all $k \in \mathbb{N}$. Thus, there are $\varphi_\infty \in W^{2,p}(\Omega)$ and $(k_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ with $k_j \to \infty$ for $j \to \infty$ such that

$$\varphi_{k_j} \rightharpoonup \varphi_\infty \qquad \text{in } W^{2,p}(\Omega) \text{ as } j \to \infty.$$

Since $W^{2,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$, this implies

$$\varphi_{k_j} \to \varphi_\infty \qquad \text{in } L^p(\Omega) \text{ as } j \to \infty$$

and thus also $\int_\Omega \varphi_\infty = 0$.

As

$$\begin{aligned}\left|\int_\Omega \nabla\varphi_\infty \cdot \nabla\psi\right| &= \lim_{j\to\infty}\left|\int_\Omega \nabla\varphi_{k_j} \cdot \nabla\psi\right| = \lim_{j\to\infty}\left|\int_\Omega \Delta\varphi_{k_j}\psi\right| \\ &\le \limsup_{j\to\infty} \frac{1}{k_j}\|\psi\|_{L^{\frac{p}{p-1}}(\Omega)} = 0 \qquad \text{for all } \psi \in C^\infty(\overline{\Omega})\end{aligned}$$

by Hölder's inequality, we further conclude that φ_∞ is constant and because of $\int_\Omega \varphi_\infty = 0$ we have $\varphi_\infty \equiv 0$.

However, as [18, Theorem 19.1] asserts

$$\|\psi\|_{W^{2,p}(\Omega)} \le C\|\Delta\psi\|_{L^p(\Omega)} + C\|\psi\|_{L^p(\Omega)} \qquad \text{for all } \psi \in C^2(\overline{\Omega}) \text{ with } \partial_\nu\psi = 0 \text{ on } \partial\Omega$$

for some $C > 0$, we derive

$$1 = \lim_{j\to\infty}\|\varphi_{k_j}\|_{W^{2,p}(\Omega)} \le C\limsup_{j\to\infty}\left(\|\Delta\varphi_{k_j}\|_{L^p(\Omega)} + \|\varphi_{k_j}\|_{L^p(\Omega)}\right) = 0,$$

a contradiction. □

These lemmata immediately imply the following version of the Gagliardo–Nirenberg inequality.

Lemma 5.7.3. *Let $j \in \{0,1\}$ and suppose $p, q \in [1,\infty], r \in (1,\infty)$ are such that*

$$\theta := \frac{\frac{1}{p} - \frac{j}{n} - \frac{1}{q}}{\frac{1}{r} - \frac{2}{n} - \frac{1}{q}} \in \left[\frac{j}{2}, 1\right).$$

Then there exists $C > 0$ such that

$$\|\varphi - \overline{\varphi}\|_{W^{j,p}(\Omega)} \le C\|\Delta\varphi\|_{L^r(\Omega)}^\theta \|\varphi - \overline{\varphi}\|_{L^q(\Omega)}^{1-\theta} \qquad \text{for all } \varphi \in W_N^{2,r}(\Omega). \tag{5.7.2}$$

In particular, for any $r \in (1,\infty)$, we may find $C' > 0$ such that

$$\|\nabla\varphi\|_{L^{2r}(\Omega)}^{2r} \le C'\|\Delta\varphi\|_{L^r(\Omega)}^r \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)}^r \qquad \text{for all } \varphi \in W_N^{2,r}(\Omega). \tag{5.7.3}$$

Proof. The usual Gagliardo–Nirenberg inequality [69] gives $c_1 > 0$ such that

$$\|\varphi - \overline{\varphi}\|_{W^{j,p}(\Omega)} \le c_1 \|D^2\varphi\|_{L^r(\Omega)}^{\theta} \|\varphi - \overline{\varphi}\|_{L^q(\Omega)}^{1-\theta} + c_1 \|\varphi - \overline{\varphi}\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{2,r}(\Omega).$$

As Hölder's inequality asserts

$$\|\psi\|_{L^1(\Omega)} \le c_2 \|\psi\|_{L^r(\Omega)}^{\theta} \|\psi\|_{L^q(\Omega)}^{1-\theta} \qquad \text{for all } \psi \in L^r(\Omega) \cap L^q(\Omega)$$

for some $c_2 > 0$, we find $c_3 > 0$ such that

$$\|\varphi - \overline{\varphi}\|_{W^{j,p}(\Omega)} \le c_3 \|\varphi - \overline{\varphi}\|_{W^{2,r}(\Omega)}^{\theta} \|\varphi - \overline{\varphi}\|_{L^q(\Omega)}^{1-\theta} \qquad \text{for all } \varphi \in W^{2,r}(\Omega).$$

In conjunction with Lemma 5.7.2, this proves (5.7.2).

Moreover, for any $r \in (1, \infty)$, letting $j \coloneqq 1$, $p \coloneqq 2r$ and $q \coloneqq \infty$, we see that

$$\frac{\frac{1}{p} - \frac{j}{n} - \frac{1}{q}}{\frac{1}{r} - \frac{2}{n} - \frac{1}{q}} = \frac{\frac{1}{2r} - \frac{1}{n}}{\frac{1}{r} - \frac{2}{n}} = \frac{1}{2} \in \left[\frac{j}{2}, 1\right).$$

Hence, (5.7.3) follows from (5.7.2). □

In order to avoid any discussions how $\int_\Omega |D^3\varphi|^2$ and $\int_\Omega |\nabla\Delta\varphi|^2$ relate for $\varphi \in W_N^{3,2}(\Omega)$, we choose to prove the following Gagliardo–Nirenberg-type inequalities, which have been used in the proof of Lemma 5.3.4, by hand.

Lemma 5.7.4. *There exists $C > 0$ such that for all $\varphi \in W_N^{3,2}(\Omega)$ the estimates*

$$\int_\Omega |\nabla\varphi|^6 \le C \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)}^4 \int_\Omega |\nabla\Delta\varphi|^2$$

and

$$\int_\Omega |\Delta\varphi|^3 \le C \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \int_\Omega |\nabla\Delta\varphi|^2$$

hold.

Proof. By Lemma 5.7.3, there is $c_1 > 0$ such that

$$\int_\Omega |\nabla\varphi|^6 \le c_1 \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)}^3 \int_\Omega |\Delta\varphi|^3 \qquad \text{for all } \varphi \in W_N^{2,3}(\Omega). \tag{5.7.4}$$

Let $\varphi \in C^3(\overline{\Omega})$ with $\partial_\nu\varphi = 0$ on $\partial\Omega$. Noting that $(|\xi|\xi)' = 2|\xi|$ for $\xi \in \mathbb{R}$, by an integration by parts, Hölder's inequality and (5.7.4) we obtain

$$\begin{aligned}
\int_\Omega |\Delta\varphi|^3 &= \int_\Omega |\Delta\varphi|\Delta\varphi\Delta\varphi \\
&= -\int_\Omega \nabla(|\Delta\varphi|\Delta\varphi) \cdot \nabla\varphi \\
&= -2\int_\Omega |\Delta\varphi|\nabla\varphi \cdot \nabla\Delta\varphi \\
&\le 2\left(\int_\Omega |\Delta\varphi|^3\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla\varphi|^6\right)^{\frac{1}{6}} \left(\int_\Omega |\nabla\Delta\varphi|^2\right)^{\frac{1}{2}} \\
&\le 2c_1^{\frac{1}{6}} \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \left(\int_\Omega |\Delta\varphi|^3\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\Delta\varphi|^2\right)^{\frac{1}{2}},
\end{aligned}$$

hence

$$\int_\Omega |\Delta\varphi|^3 \leq c_2\|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \int_\Omega |\nabla\Delta\varphi|^2,$$

where $c_2 \coloneqq 4c_1^{\frac{1}{3}}$. Plugging this into (5.7.4) yields

$$\int_\Omega |\nabla\varphi|^6 \leq c_1 c_2\|\varphi - \overline{\varphi}\|^4_{L^\infty(\Omega)} \int_\Omega |\nabla\Delta\varphi|^2.$$

The statement follows by an approximation procedure and by setting $C \coloneqq \max\{c_1, c_1c_2\}$. □

6. Global weak solutions to fully cross-diffusive systems with nonlinear diffusion and saturated taxis sensitivity

6.1. Introduction

In the present chapter, we continue our study of variants of the so-called *pursuit–evasion model*

$$\begin{cases} u_t = \nabla \cdot (d_1 \nabla u - \chi_1 u \nabla v) + f_1(u,v), \\ v_t = \nabla \cdot (d_2 \nabla v + \chi_2 v \nabla u) + f_2(u,v), \end{cases} \tag{6.1.1}$$

which has been proposed in [89] for modeling predator–prey interactions. Herein, u and v correspond to the predator and prey densities, $d_1, d_2, \chi_1, \chi_2 > 0$ are given parameters and f_1, f_2 relate to certain functional responses.

While systems of ordinary differential equations essentially assume a spatially homogeneous setting, the simplest way to account for nontrivial spatial behavior is to assume that the species move around randomly. However, sufficiently intelligent predators and prey may also partially orient their movement towards or away from higher concentrations of the other species—abilities intended to be captured by positive values of χ_1 and χ_2.

For further motivation regarding the system (6.1.1) and especially for an overview of results treating the single cross-diffusive cases, that is, when either $\chi_1 = 0$ or $\chi_2 = 0$, we refer to the introduction of Chapter 5.

Nonlinear diffusion and saturated taxis sensitivities. We now extend the system (6.1.1) to also allow for nonlinear diffusion and saturated taxis sensitives. By doing so, we follow various precedents regarding the minimal Keller–Segel system

$$\begin{cases} u_t = \nabla \cdot (d_1 \nabla u - \chi_1 u \nabla v), \\ v_t = d_2 \Delta v - v + u, \end{cases} \tag{6.1.2}$$

which has been proposed in [45] to describe the behavior of the slime molds *Dictyostelium discoideum* u, which are attracted by the chemical substance v they produce themselves, and variants of which we have analyzed in the first part of this thesis.

Indeed, among the various modifications proposed for (6.1.2), particular prominent examples include replacing the linear diffusion term with a quasilinear one and allowing for saturated taxis sensitivities (see also [36] for a (non-exhaustive) list of further possible changes). While in part this has already been suggested by Keller and Segel in [45], the need for these adjustments has been further emphasized by the desire to account for volume-filling in [72] (see also [36, 110]).

Apart from biological motivations, suitable nonlinearities may also improve the regularity of the system, as already thoroughly discussed in the introduction of Chapter 3. Accordingly, we transfer these ideas to the model (6.1.1) and consider the system

$$\begin{cases} u_t = \nabla \cdot (D_1(u)\nabla u - S_1(u)\nabla v) + f_1(u,v) & \text{in } \Omega \times (0,\infty), \\ u_t = \nabla \cdot (D_2(v)\nabla v + S_2(v)\nabla u) + f_2(u,v) & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \tag{6.P}$$

in smooth bounded domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Although the methods established below would allow for more general choices, mainly for the sake of clarity we confine ourselves to certain prototypical functions in (6.P); that is, we set

$$D_i(s) \coloneqq d_i(s+1)^{m_i-1} \quad \text{and} \quad S_i(s) \coloneqq \chi_i s(s+1)^{q_i-1} \tag{6.1.3}$$

for $s \geq 0$ and $i \in \{1,2\}$, and where the parameters therein are such that

$$d_1, d_2, \chi_1, \chi_2 > 0,\ m_1, m_2 \in \mathbb{R},\ q_1, q_2 \in (-\infty, 1]. \tag{6.1.4}$$

Moreover, we choose to either neglect zeroth order kinetics altogether or assume a typical Lotka–Voltera-type predator–prey interaction; that is, we further set

$$f_i(s_1, s_2) \coloneqq \lambda_i s_i - \mu_i s_i^2 + (-1)^{i+1} a_i s_1 s_2 \tag{6.1.5}$$

for $s_1, s_2 \geq 0$ and $i \in \{1,2\}$, where

$$\text{either} \qquad \lambda_1, \lambda_2, \mu_1, \mu_2, a_1, a_2 = 0 \tag{6.H1}$$

$$\text{or} \qquad \lambda_1, \lambda_2, \mu_1, \mu_2, a_1, a_2 > 0. \tag{6.H2}$$

The entropy-like identity. Our goal is to construct global weak solutions of (6.P) for widely arbitrary initial data. Thus, conditional estimates valid only as long u and v are close to certain steady states (such as those derived in Chapter 5) are evidently insufficient for our purposes. Instead, we will rely on the following unconditional entropy-like identity which has already been made use of in [85, 86] for related systems. Setting

$$G_i(s) \coloneqq \int_1^s \int_1^\rho \frac{1}{S_i(\sigma)} \,\mathrm{d}\sigma \,\mathrm{d}\rho \qquad \text{for } s \geq 0 \text{ and } i \in \{1,2\},$$

a sufficiently smooth and positive global solution (u,v) to (6.P) satisfies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega G_1(u) + \int_\Omega G_2(v)\right) + \int_\Omega \frac{D_1(u)}{S_1(u)}|\nabla u|^2 + \int_\Omega \frac{D_2(v)}{S_2(v)}|\nabla v|^2 \\ &= \int_\Omega \left(\frac{S_1(u)}{S_1(u)} - \frac{S_2(v)}{S_2(v)}\right)\nabla u \cdot \nabla v + \int_\Omega G_1'(u) f_1(u,v) + \int_\Omega G_2'(v) f_2(u,v) \end{aligned} \tag{6.1.6}$$

in $(0,\infty)$. This functional inequality constitutes the main—if not essentially the only—source for a priori estimates. In order to indeed gain any useful bounds from (6.1.6), however, we have to control the right-hand side therein. Evidently, the first term there just vanishes; the functions G_1 and G_2 have been chosen precisely to guarantee a cancellation of the cross-diffusive contributions.

Moreover, the last two summands on the right-hand side in (6.1.6) also simply vanish if (6.H1) holds and they can be easily controlled if there are $C_1, C_2 > 0$ such that

$$G_1'(s_1)f_1(s_1,s_2) + G_2'(s_2)f_2(s_1,s_2) \le -C_1 s_1^2 \ln s_1 - C_1 s_2^2 \ln s_2 + C_2 \tag{6.F1}$$

for all $s_1, s_2 \ge 1$. (We note that, while for bounding the right-hand side in (6.1.6) it would suffice to take $C_1 = 0$, positive values of C_1 guarantee uniform integrability of $f_i(u,v)$ which in turn will allow us to undertake certain limit processes in approximative problems.) Unfortunately, (6.F1) cannot hold unconditionally. Indeed, suppose $q_1 = q_2 = q \le 1$ and that (6.F1) holds for $C_1 = 0$ and some $C_2 > 0$. Taking $s_1 = s_2 = s \ge 1$ in (6.F1) then implies

$$\begin{aligned} C_2 &\ge G_1'(s)(\lambda_1 s - \mu_1 s^2 + a_1 s^2) + G_2'(s)(\lambda_2 s - \mu_2 s^2 - a_2 s^2) \\ &\ge \int_1^s \frac{(\sigma+1)^{1-q}}{\sigma} \,\mathrm{d}\sigma \left(\frac{-\mu_1 + a_1}{\chi_1} + \frac{-\mu_2 - a_2}{\chi_2} \right) s^2, \end{aligned}$$

where the right-hand side diverges to ∞ as $s \nearrow \infty$, provided $\frac{a_1}{\chi_1} > \frac{\mu_1}{\chi_1} + \frac{\mu_2}{\chi_2} + \frac{a_2}{\chi_2}$. Still, in the case of $q_1 = q_2 = q \le 1$, Young's inequality shows that (6.F1) holds provided a_1 is sufficiently small or χ_1 is sufficiently large compared to the other parameters, for instance.

Of course, instead of (6.F1) one may also rely on the dissipative terms in (6.1.6) for controlling the right-hand side in (6.1.6) and this idea will allow us to derive another sufficient condition for bounding the right-hand side in (6.1.6). As integrating certain linear combinations of the first two equations in (6.P) provides us with a locally uniform-in-time $L^1(\Omega)$ bound for both u and v, combining the Gagliardo–Nirenberg and Young inequalities shows that requiring

$$m_1 > \frac{2n-2}{n} + \frac{(3-q_2)(2-q_1) - (3-q_1)(2-q_2)}{2-q_2} \quad \text{or} \quad m_2 > \frac{2n-2}{n} + (q_2 - q_1) \tag{6.F2}$$

suffices to estimate the right-hand side in (6.1.6) against the dissipative terms therein (cf. Lemma 6.4.8). We note that if $q_1 = q_2$, then (6.F2) is equivalent to $\max\{m_1, m_2\} > \frac{2n-2}{n}$.

Next, one could discuss more refined approaches and for instance also make use of the L^2 space-time bounds (which in the case of (6.H2) result as a by-product when obtaining $L^1(\Omega)$ bounds). However, here we confine ourselves to the conditions (6.F1) and (6.F2), mainly because treating the most general case possible would lead to several technical difficulties which we would like to rather avoid here. Still, the important special cases that either a_1 is small or χ_1 is large (condition (6.F1)) or m_1 or m_2 are large (condition (6.F2)) are included in our analysis and, as the examples above show, at least qualitatively, these conditions seem to be optimal.

Obtaining further a priori estimates. With the right-hand side of (6.1.6) under control, we then make use of (a corollary of) the Gagliardo–Nirenberg inequality to obtain space-time bounds for $u, v, \nabla u$ and ∇v. That is, assuming

$$m_i - q_i > -1 \qquad \text{for } i \in \{1,2\}, \tag{6.1.7}$$

we can obtain estimates in L^{p_1}, L^{p_2}, L^{r_1} and L^{r_2}, respectively, where

$$p_i := \begin{cases} \max\{m_i + 1 - q_i + \frac{2(2-q_i)}{n}, 2 - q_i\}, & \text{if (6.H1) holds} \\ \max\{m_i + 1 - q_i + \frac{2(2-q_i)}{n}, 3 - q_i\}, & \text{if (6.H2) holds} \end{cases} \quad \text{for } i \in \{1,2\} \tag{6.1.8}$$

and

$$r_i := \min\left\{\frac{2p_i}{p_i - (m_i - q_i - 1)}, 2\right\}, \qquad \text{for } i \in \{1,2\}, \tag{6.1.9}$$

see Lemma 6.4.11 and Lemma 6.4.12. Lacking any other sources of helpful a priori bounds, these estimates need to be strong enough to inter alia assert convergence of the corresponding approximative terms to

$$\int_0^\infty \int_\Omega S_1(u)\nabla v \cdot \nabla\varphi \quad \text{and} \quad \int_0^\infty \int_\Omega S_2(v)\nabla u \cdot \nabla\varphi, \qquad \varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty)).$$

This is the case when p_i and r_i are sufficiently large. More precisely, we need to require

$$\begin{cases} \frac{1}{r_{3-i}} < 1, & q_i \le 0, \\ \frac{q_i}{p_i} + \frac{1}{r_{3-i}} < 1, & 0 < q_i < 1, \\ \frac{1}{p_i} + \frac{1}{r_{3-i}} \le 1, & q_i = 1, \end{cases} \qquad \text{for } i \in \{1,2\} \tag{6.1.10}$$

(In the case of $q_i = 1$, we obtain slightly stronger bounds than outlined above so that equality in (6.1.10) is sufficient for that case.) We remark that if $m_i = m \in \mathbb{R}$ and $q_i = q \in (-\infty, 1]$ for $i \in \{1,2\}$, then $q \le 0$ implies (6.1.10) while for $q \in (0,1)$ and if (6.H1) holds, (6.1.10) is equivalent to

$$m > \min\left\{\frac{(2n+1)q - 2}{n}, 4q - 1\right\} \tag{6.1.11}$$

Moreover, in the case of (6.H2) (and again $q \in (0,1)$), (6.1.10) is not only implied by (6.1.11) but also by $m > 4q - 2$.

Under these assumptions, we are then finally able to construct global weak solutions of the problem (6.P).

Theorem 6.1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a smooth, bounded domain. Suppose that* (6.1.3), (6.1.4), (6.1.5), (6.1.7), *either* (6.H1) *or* (6.H2), (6.F1) *or* (6.F2), *as well as* (6.1.10) *(with p_i and r_i as in* (6.1.8) *and* (6.1.9), *respectively) hold and that*

$$u_0, v_0 \in \begin{cases} L^{2-q_i}(\Omega), & q_i < 1, \\ L\log L(\Omega), & q_i = 1 \end{cases} \quad \textit{are nonnegative a.e.} \tag{6.1.12}$$

Then there exists a global nonnegative weak solution (u,v) of (6.P) *in the sense of Definition 6.5.1.*

Structure of the chapter. A challenge not yet addressed is the construction of global solutions to certain approximative problems. For systems similar to (6.P) but where either $S_1 \equiv 0$ or $S_2 \equiv 0$, this is usually a straightforward task. For the fully cross-diffusive system (6.P), however, even if all given functions are assumed to be bounded, the question of global existence is already highly nontrivial, even for a weak solution concept.

Thus, Section 6.2 is devoted to the construction of so-called weak $W^{1,2}$-solutions to systems suitably approximating (6.P). The corresponding proof then relies on an additional approximation; we make use of fourth-order regularization terms. The general strategy is described more thoroughly at the beginning of Section 6.2, so we do not go into much more detail at this point. However, it seems worth emphasizing that apart from obtaining these solutions, we also prove a corresponding version of the entropy-like identity (6.1.6).

Next, in Section 6.3, we fix the final approximation functions used and rely on the results in the preceding section to obtain a global weak $W^{1,2}$-solution fulfilling a certain entropy-like inequality, see Lemma 6.3.2.

Section 6.4 then makes use of this inequality and the hypotheses of Theorem 6.1.1 in order to guarantee sufficiently strong convergence towards a function pair (u, v), which in Section 6.5 is then finally seen to be a weak solution of (6.P).

Notation. Throughout the chapter, we fix $n \in \mathbb{N}$ and a smooth bounded domain $\Omega \subset \mathbb{R}^n$. For $p \in (1, \infty)$, we set $W^{2,p}_N(\Omega) := \{ \varphi \in W^{2,p}(\Omega) : \partial_\nu \varphi = 0 \text{ in the sense of traces} \}$.

Additionally, we use the following notation for Sobolev spaces involving evolution triples. For an interval $I \subset \mathbb{R}$ and an evolution triple $V \hookrightarrow H \hookrightarrow V^\star$, we set $W^{1,2}(I; V, H) := \{ \varphi \in L^2(I; V) : \varphi_t \in L^2(I; V^\star) \}$ and $W^{1,2}_{\mathrm{loc}}(I; V, H) := \bigcup_{[a,b] \subset I} W^{1,2}([a, b]; V, H)$. Also, we abbreviate $W^{1,2}_{(\mathrm{loc})}(I; W^{1,2}(\Omega)) := W^{1,2}_{(\mathrm{loc})}(I; W^{1,2}(\Omega), L^2(\Omega))$.

Moreover, for a set X, a function $\varphi \colon X \to \mathbb{R}$ and $A \in \mathbb{R}$, we abbreviate $\{ x \in X : \varphi(x) \le A \}$ by $\{\varphi \le A\}$, the set X being implied by the context. Similarly for other order relations.

6.2. Global weak $W^{1,2}$-solutions to approximative systems

In this section, we prove the following quite general global existence theorem, which we will then use in Section 6.3 to obtain solutions to certain approximate problems. In contrast to the hypotheses of Theorem 6.1.1, here we also assume that all given functions are bounded. That is, in this section, we do not need to assume any of the conditions introduced in the introduction of this chapter but instead require that (6.2.1)–(6.2.6) below are fulfilled.

Theorem 6.2.1. *Suppose that, for $i \in \{1, 2\}$,*

$$D_i \in C^0([0, \infty)) \cap L^\infty((0, \infty)), \tag{6.2.1}$$

$$S_i \in C^1([0, \infty)) \cap W^{1,\infty}((0, \infty)) \quad \text{and} \tag{6.2.2}$$

$$f_i \in C^0([0, \infty)^2) \cap L^\infty((0, \infty)^2) \tag{6.2.3}$$

fulfill

$$\inf_{s \in [0,\infty)} D_i(s) > 0, \quad \inf_{s \in (0,1)} \frac{S_i(s)}{s} > 0, \quad \inf_{s \in [1,\infty)} S_i(s) > 0 \quad \text{and} \quad S_i(0) = 0 \tag{6.2.4}$$

as well as

$$\lim_{s_1 \searrow 0} \sup_{s_2 \ge 0} |f_1(s_1, s_2) \ln s_1| = 0 \quad \text{and} \quad \lim_{s_2 \searrow 0} \sup_{s_1 \ge 0} |f_2(s_1, s_2) \ln s_2| = 0 \tag{6.2.5}$$

and assume that

$$u_0, v_0 \in C^\infty(\overline{\Omega}) \quad \text{are positive in } \overline{\Omega}. \tag{6.2.6}$$

Then there exists a global nonnegative weak $W^{1,2}$-solution (u, v) of (6.P)*, meaning that u and v belong to the space $W^{1,2}_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$, satisfy*

$$u(\cdot, 0) = u_0 \quad \text{as well as} \quad v(\cdot, 0) = v_0 \qquad \text{a.e. in } \Omega \tag{6.2.7}$$

and fulfill

$$\int_0^\infty \int_\Omega u_t \varphi = -\int_0^\infty \int_\Omega D_1(u) \nabla u \cdot \nabla \varphi + \int_0^\infty \int_\Omega S_1(u) \nabla v \cdot \nabla \varphi + \int_0^\infty \int_\Omega f_1(u, v) \varphi \tag{6.2.8}$$

as well as

$$\int_0^\infty \int_\Omega v_t \varphi = -\int_0^\infty \int_\Omega D_2(u) \nabla v \cdot \nabla \varphi - \int_0^\infty \int_\Omega S_2(u) \nabla u \cdot \nabla \varphi + \int_0^\infty \int_\Omega f_2(u, v) \varphi \tag{6.2.9}$$

for all $\varphi \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$.

In what follows, we fix D_i, S_i, f_i, $i \in \{1, 2\}$ fulfilling (6.2.1)–(6.2.4) as well as u_0, v_0 as in (6.2.6).

As already alluded to in the introduction of this chapter, a cornerstone for gaining a priori bounds for these solutions is the following theorem, which shows that the solutions constructed in Theorem 6.2.1 fulfill an inequality reminiscent of (6.1.6).

Theorem 6.2.2. *Denote the weak $W^{1,2}$-solution of* (6.P) *given by Theorem 6.2.1 by (u, v) and let*

$$G_i(s) \coloneqq \int_1^s \int_1^\rho \frac{1}{S_i(\sigma)} \,\mathrm{d}\sigma \,\mathrm{d}\rho \qquad \text{for } s \in \mathbb{R} \text{ and } i \in \{1, 2\}$$

as well as

$$\begin{aligned}
\mathcal{E}(t) &\coloneqq \int_\Omega G_1(u(\cdot, t)) + \int_\Omega G_2(v(\cdot, t)), \\
\mathcal{D}(t) &\coloneqq \int_\Omega \frac{D_1(u(\cdot, t))}{S_1(u(\cdot, t))} |\nabla u(\cdot, t)|^2 + \int_\Omega \frac{D_2(v(\cdot, t))}{S_2(v(\cdot, t))} |\nabla v(\cdot, t)|^2 \quad \text{and} \\
\mathcal{R}(t) &\coloneqq \int_\Omega G_1'(u(\cdot, t)) f_1(u(\cdot, t), v(\cdot, t)) + \int_\Omega G_2'(v(\cdot, t)) f_2(u(\cdot, t), v(\cdot, t))
\end{aligned}$$

for $t \in [0,\infty)$. (We remark that $\mathcal{D}$ and $\mathcal{R}$ are to be understood as functions in $L^0((0,\infty))$; that is, they are only well-defined up to modifications on null sets.) Then

$$\mathcal{E}(T)\zeta(T) + \int_0^T \mathcal{D}(t)\zeta(t) \,\mathrm{d}t \le \mathcal{E}(0)\zeta(0) + \int_0^T \mathcal{R}(t)\zeta(t) \,\mathrm{d}t + \int_0^T \mathcal{E}(t)\zeta'(t) \,\mathrm{d}t \tag{6.2.10}$$

for all $T \in (0,\infty)$ and $0 \le \zeta \in C^\infty([0, T])$.

Next, we describe our approach of proving the theorems above. Similar to [85, 86], where one-dimensional relatives of (6.P) have been studied, our general approach is approximation by a fourth order regularization. That is, for $\varepsilon, \delta \in (0,1)$, we will first construct global solutions to

$$\begin{cases} u_{\varepsilon\delta t} = \nabla \cdot F_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) + f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) & \text{in } \Omega \times (0,\infty), \\ v_{\varepsilon\delta t} = \nabla \cdot F_{2\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) + f_{2\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) & \text{in } \Omega \times (0,\infty), \\ \partial_\nu \Delta u_{\varepsilon\delta} = \partial_\nu u_{\varepsilon\delta} = \partial_\nu \Delta v_{\varepsilon\delta} = \partial_\nu v_{\varepsilon\delta} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u_{\varepsilon\delta}(\cdot,0) = u_0, v_{\varepsilon\delta}(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \tag{6.P$_{\varepsilon\delta}$}$$

with fluxes

$$F_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) \coloneqq -\varepsilon S_{1\delta}(u_{\varepsilon\delta}) \nabla \Delta u_{\varepsilon\delta} + D_1(|u_{\varepsilon\delta}|) \nabla u_{\varepsilon\delta} - S_{1\delta}(u_{\varepsilon\delta}) \nabla v_{\varepsilon\delta}) \quad \text{and}$$
$$F_{2\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) \coloneqq -\varepsilon S_{2\delta}(v_{\varepsilon\delta}) \nabla \Delta v_{\varepsilon\delta} + D_2(|v_{\varepsilon\delta}|) \nabla v_{\varepsilon\delta} + S_{2\delta}(v_{\varepsilon\delta}) \nabla u_{\varepsilon\delta})$$

and where

$$S_{i\delta}(s) \coloneqq S_i(|s|) + \delta \qquad \text{for } s \in \mathbb{R},\ \delta \in (0,1) \text{ and } i \in \{1,2\} \tag{6.2.11}$$

and

$$f_{i\delta}(s_1, s_2) \coloneqq f_i((s_1)_+, (s_2)_+) \qquad \text{for } s_1, s_2 \in \mathbb{R},\ \delta \in (0,1) \text{ and } i \in \{1,2\}. \tag{6.2.12}$$

We note that (6.2.5) entails $f_1(0,\cdot) \equiv 0$ and hence $f_{1\delta}(\rho,\sigma) = 0$ for all $\rho \le 0$ and $\sigma \in \mathbb{R}$. Likewise, $f_{2\delta}(\rho,\sigma) = 0$ for all $\rho \in \mathbb{R}$ and $\sigma \le 0$.

For convenience, let us introduce several abbreviations. For $i \in \{1,2\}$, we set

$$\overline{D}_i \coloneqq \|D_i\|_{L^\infty((0,\infty))}, \quad \overline{S}_i \coloneqq \|S_i\|_{L^\infty((0,\infty))} + 1 \quad \text{and} \quad \overline{S}_i' \coloneqq \|S_i'\|_{L^\infty((0,\infty))}$$

as well as

$$\underline{D}_i \coloneqq \inf_{s \in [0,\infty)} D_i(s) \quad \text{and} \quad \underline{S}_i \coloneqq \inf_{s \in (0,\infty)} S_i(s)[(\tfrac{1}{s} - 1)\mathbb{1}_{(0,1)}(s) + 1].$$

Due to continuity of S_i up to 0, the definition of $\underline{S}_i$ entails that $S_i(s) \ge \underline{S}_i s$ for all $s \in [0,1)$, $i \in \{1,2\}$.

The rest of this section is organized as follows. The first step towards proving Theorem 6.2.1 and Theorem 6.2.2 consists of constructing solutions to (6.P$_{\varepsilon\delta}$) and is achieved by a Galerkin approach. To that end, non-degeneracy of the fourth order terms in (6.P$_{\varepsilon\delta}$) is of crucial importance, which is the reason for introducing the parameter δ.

A general problem for equations of fourth-order is the lack of a maximum principle; that is, $u_{\varepsilon\delta}, v_{\varepsilon\delta}$ might become negative even for strictly positive initial data. Following [29], however, we see in Subsection 6.2.2 that suitably constructed limit functions $u_\varepsilon, v_\varepsilon$ are indeed nonnegative. Here, degeneracy for $\delta = 0$ actually comes in handy.

In contrast to Section 6.4, where we aim to argue similarly but only assume the hypotheses of Theorem 6.1.1, the assumptions (6.2.3) and (6.2.5) allow us to rather easily obtain certain a priori bounds from a version of the entropy-like identity (6.1.6). These allow us to so finally let $\varepsilon \searrow 0$ in Subsection 6.2.3 and then to prove Theorem 6.2.1 and Theorem 6.2.2.

6.2.1. The limit process $k \to \infty$: existence of weak solutions to $(6.\mathrm{P}_{\varepsilon\delta})$ by a Galerkin method

To prepare the Galerkin approach used below for constructing solutions to $(6.\mathrm{P}_{\varepsilon\delta})$, we briefly state the well-known

Lemma 6.2.3. *There exists an orthonormal basis $\{\, \varphi_j : j \in \mathbb{N} \,\}$ of $L^2(\Omega)$ consisting of smooth eigenfunctions of $-\Delta$ with homogeneous Neumann boundary conditions.*

Proof. The existence of an orthonormal basis consisting of eigenfunctions of $-\Delta$ with homogeneous Neumann boundary conditions is given by [31, Theorem 1.2.8] and their smoothness is proved by iteratively applying [18, Theorem 19.1]. □

For the Galerkin approach, we first construct local-in-time solutions to certain finite-dimensional problems.

Lemma 6.2.4. *Let $(\varphi_j)_{j\in\mathbb{N}}$ be as in Lemma 6.2.3 and set $X_k \coloneqq \operatorname{span}\{\, \varphi_j : 1 \le j \le k \,\}$ for $k \in \mathbb{N}$. For $\varepsilon, \delta \in (0,1)$ and $k \in \mathbb{N}$, there exist $T_{\max,\varepsilon\delta k} \in (0,\infty]$ and functions*

$$u_{\varepsilon\delta k}, v_{\varepsilon\delta k} \in C^\infty(\overline{\Omega} \times [0, T_{\max,\varepsilon\delta k})) \tag{6.2.13}$$

with

$$\partial_\nu u_{\varepsilon\delta k} = \partial_\nu \Delta u_{\varepsilon\delta k} = \partial_\nu v_{\varepsilon\delta k} = \partial_\nu \Delta v_{\varepsilon\delta k} = 0 \tag{6.2.14}$$

fulfilling

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{\varepsilon\delta k}\psi = \varepsilon \int_\Omega S_{1\delta}(u_{\varepsilon\delta k})\nabla\Delta u_{\varepsilon\delta k}\cdot\nabla\psi - \int_\Omega D_1(|u_{\varepsilon\delta k}|)\nabla u_{\varepsilon\delta k}\cdot\nabla\psi \\ + \int_\Omega S_{1\delta}(u_{\varepsilon\delta k})\nabla v_{\varepsilon\delta k}\cdot\nabla\psi + \int_\Omega f_{1\delta}(u_{\varepsilon\delta k}, v_{\varepsilon\delta k})\psi\end{aligned} \tag{6.2.15}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_{\varepsilon\delta k}\psi = \varepsilon \int_\Omega S_{2\delta}(v_{\varepsilon\delta k})\nabla\Delta v_{\varepsilon\delta k}\cdot\nabla\psi - \int_\Omega D_2(|v_{\varepsilon\delta k}|)\nabla v_{\varepsilon\delta k}\cdot\nabla\psi \\ - \int_\Omega S_{2\delta}(v_{\varepsilon\delta k})\nabla u_{\varepsilon\delta k}\cdot\nabla\psi + \int_\Omega f_{2\delta}(u_{\varepsilon\delta k}, v_{\varepsilon\delta k})\psi\end{aligned} \tag{6.2.16}$$

in $(0, T_{\max,\varepsilon\delta k})$ for all $\psi \in X_k$ as well as

$$\int_\Omega u_{\varepsilon\delta k}(\cdot,0)\psi = \int_\Omega u_0\psi \quad \text{and} \quad \int_\Omega v_{\varepsilon\delta k}(\cdot,0)\psi = \int_\Omega v_0\psi \qquad \text{for all } \psi \in X_k. \tag{6.2.17}$$

Additionally, if $T_{\max,\varepsilon\delta k} < \infty$, then

$$\limsup_{t \nearrow T_{\max,\varepsilon\delta k}} \left(\|u_{\varepsilon\delta k}(\cdot,t)\|_{L^2(\Omega)} + \|v_{\varepsilon\delta k}(\cdot,t)\|_{L^2(\Omega)} \right) = \infty. \tag{6.2.18}$$

Proof. We fix $\varepsilon, \delta \in (0,1)$ and $k \in \mathbb{N}$. For $w, z \in \mathbb{R}^k$, we define $F_1(w,z), F_2(w,z) \in \mathbb{R}^k$ by

$$\begin{aligned}
(F_1(w,z))_i \coloneqq \varepsilon \int_\Omega & S_{1\delta}\left(\textstyle\sum_{j=1}^k w_j\varphi_j\right) \nabla\Delta \left(\textstyle\sum_{j=1}^k w_j\varphi_j\right) \cdot \nabla\varphi_i \\
&- \int_\Omega D_1\left(\left|\textstyle\sum_{j=1}^k w_j\varphi_j\right|\right) \nabla \left(\textstyle\sum_{j=1}^k w_j\varphi_j\right) \cdot \nabla\varphi_i \\
&+ \int_\Omega S_{1\delta}\left(\textstyle\sum_{j=1}^k w_j\varphi_j\right) \nabla \left(\textstyle\sum_{j=1}^k z_j\varphi_j\right) \cdot \nabla\varphi_i \\
&+ \int_\Omega f_{1\delta}\left(\textstyle\sum_{j=1}^k w_j\varphi_j, \sum_{j=1}^k z_j\varphi_j\right) \varphi_i
\end{aligned}$$

and

$$\begin{aligned}
(F_2(w,z))_i \coloneqq \varepsilon \int_\Omega & S_{2\delta}\left(\textstyle\sum_{j=1}^k z_j\varphi_j\right) \nabla\Delta \left(\textstyle\sum_{j=1}^k z_j\varphi_j\right) \cdot \nabla\varphi_i \\
&- \int_\Omega D_2\left(\left|\textstyle\sum_{j=1}^k z_j\varphi_j\right|\right) \nabla \left(\textstyle\sum_{j=1}^k z_j\varphi_j\right) \cdot \nabla\varphi_i \\
&- \int_\Omega S_{2\delta}\left(\textstyle\sum_{j=1}^k z_j\varphi_j\right) \nabla \left(\textstyle\sum_{j=1}^k w_j\varphi_j\right) \cdot \nabla\varphi_i \\
&+ \int_\Omega f_{2\delta}\left(\textstyle\sum_{j=1}^k w_j\varphi_j, \sum_{j=1}^k z_j\varphi_j\right) \varphi_i
\end{aligned}$$

for $i \in \{1, \dots, k\}$.

As F_1 and F_2 are locally Lipschitz continuous, the Picard–Lindelöf theorem asserts the existence of $T_{\max,\varepsilon\delta k} \in (0,\infty]$ and $w, z \in C^0([0, T_{\max,\varepsilon\delta k})) \cap C^1((0, T_{\max,\varepsilon\delta k}))$ which solve

$$\begin{cases}
w' = F_1(w,z) & \text{in } (0, T_{\max,\varepsilon\delta k}), \\
z' = F_2(w,z) & \text{in } (0, T_{\max,\varepsilon\delta k}), \\
w(0) = \int_\Omega u_{0\varepsilon}\varphi, \\
z(0) = \int_\Omega v_{0\varepsilon}\varphi
\end{cases}$$

classically and, if $T_{\max,\varepsilon\delta k} < \infty$, then

$$\limsup_{t \nearrow T_{\max,\varepsilon\delta k}} (|w(t)| + |z(t)|) = \infty. \tag{6.2.19}$$

According to Lemma 6.2.3, the functions

$$u_{\varepsilon\delta k}(x,t) \coloneqq \sum_{j=1}^k w_j(t)\varphi_j(x) \quad \text{and} \quad v_{\varepsilon\delta k}(x,t) \coloneqq \sum_{j=1}^k z_j(t)\varphi_j(x), \qquad x \in \overline{\Omega}, t \in [0, T_{\max,\varepsilon\delta k}),$$

satisfy (6.2.13) and (6.2.14). Moreover, they fulfill

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_{\varepsilon\delta k}\varphi_i = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\textstyle\sum_{j=1}^k w_j\varphi_j\right) \varphi_i = \sum_{j=1}^k w_j' \int_\Omega \varphi_i\varphi_j = w_i' = (F_1(w,z))_i$$

in $(0, T_{\max,\varepsilon\delta k})$ for $i \in \{1, \dots, k\}$. Thus, (6.2.15) is fulfilled for $\psi = \varphi_i$ for all $i \in \{1, \dots, k\}$ and, due to linearity, also for all $\psi \in X_k$, as desired. Likewise, we obtain that (6.2.16) is also fulfilled for all $\psi \in X_k$.

From $\int_\Omega \varphi_i \varphi_j = \delta_{ij}$ for $i, j \in \{1, \dots, k\}$, we further infer

$$\sum_{j=0}^k w_{\varepsilon\delta kj}^2 = \sum_{j=0}^k \int_\Omega w_{\varepsilon\delta kj}^2 \varphi_j^2 = \int_\Omega \left(\sum_{j=0}^k w_{\varepsilon\delta kj} \varphi_j \right)^2 = \int_\Omega u_{\varepsilon\delta k}^2 \quad \text{in } (0, T_{\max,\varepsilon\delta k})$$

and, likewise,

$$\sum_{j=0}^k z_{\varepsilon\delta kj}^2 = \int_\Omega u_{\varepsilon\delta k}^2 \quad \text{in } (0, T_{\max,\varepsilon\delta k}).$$

Thus, if (6.2.18) is not fulfilled, then (6.2.19) is also not satisfied, implying $T_{\max,\varepsilon\delta k} = \infty$. □

In the following lemma, we show that the solutions $(u_{\varepsilon\delta k}, v_{\varepsilon\delta k})$ constructed in Lemma 6.2.4 are global in time. Moreover, in order to prepare the application of certain compactness theorems, we also collect several k-independent a priori estimates.

As opposed to [29], however, these bounds may depend on δ, the reason being that in our situation the terms stemming from the possibly nonlinear diffusion terms D_1 and D_2 can no longer be controlled independently of δ, at least not in all situations covered by Theorem 6.2.1. This problem will then be circumvented by deriving appropriate δ-independent estimates in Lemma 6.2.11 below, which are, however, weaker than those obtained in the present subsection.

Lemma 6.2.5. *For all $\varepsilon, \delta \in (0, 1)$ and $k \in \mathbb{N}$, let $(u_{\varepsilon\delta}, v_{\varepsilon\delta})$ and $T_{\max,\varepsilon\delta k}$ be as given by Lemma 6.2.4. Then $T_{\max,\varepsilon\delta k} = \infty$ for all $\varepsilon, \delta \in (0, 1)$ and $k \in \mathbb{N}$ and, moreover, for all $\varepsilon, \delta \in (0, 1)$ and all $T \in (0, \infty)$, there exists $C > 0$ such that for all $k \in \mathbb{N}$, the estimates*

$$\sup_{t\in(0,T)} \int_\Omega u_{\varepsilon\delta k}^2(\cdot, t) + \sup_{t\in(0,T)} \int_\Omega v_{\varepsilon\delta k}^2(\cdot, t) \le C, \tag{6.2.20}$$

$$\sup_{t\in(0,T)} \int_\Omega |\nabla u_{\varepsilon\delta k}(\cdot, t)|^2 + \sup_{t\in(0,T)} \int_\Omega |\nabla v_{\varepsilon\delta k}(\cdot, t)|^2 \le C \quad \textit{and} \tag{6.2.21}$$

$$\int_0^T \int_\Omega |\nabla \Delta u_{\varepsilon\delta k}|^2 + \int_0^T \int_\Omega |\nabla \Delta v_{\varepsilon\delta k}|^2 \le C \tag{6.2.22}$$

hold.

Proof. According to the Poincaré inequality (cf. Lemma 5.7.1), there is $C_{\mathrm{P}} > 0$ such that

$$\int_\Omega |\Delta \psi|^2 \le C_{\mathrm{P}} \int_\Omega |\nabla \Delta \psi|^2 \qquad \text{for all } \psi \in W_N^{3,2}(\Omega). \tag{6.2.23}$$

We then fix $\varepsilon, \delta \in (0, 1)$, take $u_{\varepsilon\delta k}$ as test function in (6.2.15) and apply Young's inequality to obtain

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_{\varepsilon\delta k}^2 \\
&= \varepsilon \int_\Omega S_{1\delta}(u_{\varepsilon\delta k}) \nabla \Delta u_{\varepsilon\delta k} \cdot \nabla u_{\varepsilon\delta k} - \int_\Omega D_1(|u_{\varepsilon\delta k}|) |\nabla u_{\varepsilon\delta k}|^2 \\
&\quad + \int_\Omega S_{1\delta}(u_{\varepsilon\delta k}) \nabla u_{\varepsilon\delta k} \cdot \nabla v_{\varepsilon\delta k} + \int_\Omega f_{1\delta}(u_{\varepsilon\delta k}, v_{\varepsilon\delta k}) u_{\varepsilon\delta k}
\end{aligned}$$

$$\le \frac{\varepsilon}{4}\int_\Omega S_{1\delta}(u_{\varepsilon\delta k})|\nabla\Delta u_{\varepsilon\delta k}|^2 + \left(\varepsilon\overline{S}_1 - \underline{D}_1 + \frac{\overline{S}_1}{2}\right)\int_\Omega |\nabla u_{\varepsilon\delta k}|^2$$
$$+\frac{\overline{S}_1}{2}\int_\Omega |\nabla v_{\varepsilon\delta k}|^2 + \frac{1}{2}\int_\Omega u_{\varepsilon\delta k}^2 + \frac{|\Omega|\|f_1\|^2_{L^\infty([0,\infty)^2)}}{2}$$

in $(0, T_{\max,\varepsilon\delta k})$ for all $k \in \mathbb{N}$. Moreover, as the Laplacian leaves the space X_k defined in Lemma 6.2.4 invariant, we may also use $-\Delta u_{\varepsilon\delta k} \in X_k$ as a test function in (6.2.15), which when combined with Young's inequality, (6.2.12), (6.2.23) and (6.2.11) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_{\varepsilon\delta k}|^2$$
$$= -\varepsilon\int_\Omega S_{1\delta}(u_{\varepsilon\delta k})|\nabla\Delta u_{\varepsilon\delta k}|^2 + \int_\Omega D_1(|u_{\varepsilon\delta k}|)\nabla\Delta u_{\varepsilon\delta k}\cdot\nabla u_{\varepsilon\delta k}$$
$$-\int_\Omega S_{1\delta}(u_{\varepsilon\delta k})\nabla\Delta u_{\varepsilon\delta k}\cdot\nabla v_{\varepsilon\delta k} - \int_\Omega f_{1\delta}(u_{\varepsilon\delta k}, v_{\varepsilon\delta k})\Delta u_{\varepsilon\delta k}$$
$$\le -\frac{3\varepsilon}{4}\int_\Omega S_{1\delta}(u_{\varepsilon\delta k})|\nabla\Delta u_{\varepsilon\delta k}|^2 + \frac{\varepsilon\delta}{8}\int_\Omega |\nabla\Delta u_{\varepsilon\delta k}|^2 + \frac{\varepsilon\delta}{8C_{\mathrm{P}}}\int_\Omega |\Delta u_{\varepsilon\delta k}|^2$$
$$+\frac{2\overline{D}_1^2}{\varepsilon\delta}\int_\Omega |\nabla u_{\varepsilon\delta k}|^2 + \frac{\overline{S}_1}{\varepsilon}\int_\Omega |\nabla v_{\varepsilon\delta k}|^2 + \frac{2C_{\mathrm{P}}|\Omega|}{\varepsilon\delta}\|f_1\|^2_{L^\infty([0,\infty)^2)}$$
$$\le -\frac{\varepsilon}{4}\int_\Omega S_{1\delta}(u_{\varepsilon\delta k})|\nabla\Delta u_{\varepsilon\delta k}|^2 - \frac{\varepsilon\delta}{4}\int_\Omega |\nabla\Delta u_{\varepsilon\delta k}|^2$$
$$+\frac{2\overline{D}_1^2}{\varepsilon\delta}\int_\Omega |\nabla u_{\varepsilon\delta k}|^2 + \frac{\overline{S}_1}{\varepsilon}\int_\Omega |\nabla v_{\varepsilon\delta k}|^2 + \frac{2C_{\mathrm{P}}|\Omega|}{\varepsilon\delta}\|f_1\|^2_{L^\infty([0,\infty)^2)}$$

in $(0, T_{\max,\varepsilon\delta k})$ for all $k \in \mathbb{N}$.

Along with analogous computations for the second equation, we see that there are $c_1, c_2 > 0$ such that for all $k \in \mathbb{N}$, the function

$$y(t) \coloneqq \frac{1}{2}\int_\Omega u_{\varepsilon\delta k}^2 + \frac{1}{2}\int_\Omega |\nabla u_{\varepsilon\delta k}|^2 + \frac{1}{2}\int_\Omega v_{\varepsilon\delta k}^2 + \frac{1}{2}\int_\Omega |\nabla v_{\varepsilon\delta k}|^2, \qquad t \in [0, T_{\max,\varepsilon\delta k}),$$

solves the ODI

$$y'(t) \le -c_1\int_\Omega |\nabla\Delta u_{\varepsilon\delta k}|^2 - c_1\int_\Omega |\nabla\Delta v_{\varepsilon\delta k}|^2 + c_2 y + c_2 \qquad \text{in } (0, T_{\max,\varepsilon\delta k}).$$

According to Grönwall's inequality and as $y(0)$ is finite and bounded independently of k by (6.2.6), the estimates (6.2.20)–(6.2.22) are then valid for all finite $T \in (0, T_{\max,\varepsilon\delta k}]$ and certain $C > 0$ (depending on ε, δ and T but not on k). Due to the extensibility criterion (6.2.18), this then implies $T_{\max,\varepsilon\delta k} = \infty$ for all $k \in \mathbb{N}$ and then that (6.2.20)–(6.2.22) indeed hold for all $T \in (0, \infty)$ (and corresponding $C > 0$). ☐

Having an application of the Aubin–Lions lemma in mind, we next collect a priori estimates for the time derivatives.

Lemma 6.2.6. *For $\varepsilon, \delta \in (0,1)$ and $k \in \mathbb{N}$, we denote the solution given by Lemma 6.2.4 by $(u_{\varepsilon\delta k}, v_{\varepsilon\delta k})$. For all $\varepsilon, \delta \in (0,1)$ and $T \in (0,\infty)$, there exist $C_1, C_2 > 0$ such that*

$$\|u_{\varepsilon\delta kt}\|_{L^2((0,T);(W^{1,2}(\Omega))^\star)} + \|v_{\varepsilon\delta kt}\|_{L^2((0,T);(W^{1,2}(\Omega))^\star)} \le C_1 \tag{6.2.24}$$

and

$$\|\nabla u_{\varepsilon\delta kt}\|_{L^2((0,T);(W_N^{2,2}(\Omega))^\star)} + \|\nabla v_{\varepsilon\delta kt}\|_{L^2((0,T);(W_N^{2,2}(\Omega))^\star)} \le C_2 \tag{6.2.25}$$

for all $k \in \mathbb{N}$.

Proof. Let $\varepsilon, \delta \in (0,1)$ and $T \in (0,\infty)$. Letting X_k be as in Lemma 6.2.4, we denote the orthogonal projection from $W^{1,2}(\Omega)$ onto X_k by P_k. Applying Lemma 6.2.4 and Hölder's inequality shows that

$$\begin{aligned}
&\left|\int_\Omega u_{\varepsilon\delta kt}\varphi\right| = \left|\int_\Omega u_{\varepsilon\delta kt} P_k\varphi\right| \\
&\le \varepsilon\left|\int_\Omega S_{1\delta}(u_{\varepsilon\delta k})\nabla\Delta u_{\varepsilon\delta k}\cdot\nabla P_k\varphi\right| + \left|\int_\Omega D_1(|u_{\varepsilon\delta k}|)\nabla u_{\varepsilon\delta k}\cdot\nabla P_k\varphi\right| \\
&\quad + \left|\int_\Omega S_{1\delta}(u_{\varepsilon\delta k})\nabla v_{\varepsilon\delta k}\cdot\nabla P_k\varphi\right| + \left|\int_\Omega f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta})P_k\varphi\right| \\
&\le \Big(\varepsilon\overline{S}_1\|\nabla\Delta u_{\varepsilon\delta k}\|_{L^2(\Omega)} + \overline{D}_1\|\nabla u_{\varepsilon\delta k}\|_{L^2(\Omega)} \\
&\quad + \overline{S}_1\|\nabla v_{\varepsilon\delta k}\|_{L^2(\Omega)} + (\|f_1\|_{L^\infty([0,\infty)^2)})|\Omega|^{\frac{1}{2}}\Big)\|P_k\varphi\|_{W^{1,2}(\Omega)}
\end{aligned}$$

for all $\varphi \in W^{1,2}(\Omega)$ and $k \in \mathbb{N}$. Upon integrating this inequality over $(0,T)$ and in conjunction with an analogous argument for $v_{\varepsilon\delta kt}$, we then infer (6.2.24) from (6.2.22), (6.2.21) and (6.2.3).

Since for all $\varphi \in W_N^{2,2}(\Omega;\mathbb{R}^n)$ and $k \in \mathbb{N}$, we have

$$\begin{aligned}
\left|\int_\Omega \nabla u_{\varepsilon\delta kt}\cdot\varphi\right| &= \left|\int_\Omega u_{\varepsilon\delta kt}\nabla\cdot\varphi\right| \\
&\le \|u_{\varepsilon\delta kt}\|_{(W^{1,2}(\Omega))^\star}\|\nabla\cdot\varphi\|_{W^{1,2}(\Omega)} \\
&\le \|u_{\varepsilon\delta kt}\|_{(W^{1,2}(\Omega))^\star}\|\varphi\|_{W^{2,2}(\Omega;\mathbb{R}^n)}
\end{aligned}$$

(and likewise for $\nabla v_{\varepsilon\delta kt}$), a consequence thereof is (6.2.25). □

The bounds obtained above now allow us to obtain convergences of $u_{\varepsilon\delta k}$ and $v_{\varepsilon\delta k}$ along certain subsequences of $(k)_{k\in\mathbb{N}}$.

Lemma 6.2.7. *For all* $\varepsilon, \delta \in (0,1)$, *there exist a subsequence* $(k_j)_{j\in\mathbb{N}}$ *of* $(k)_{k\in\mathbb{N}}$ *and functions*

$$u_{\varepsilon\delta}, v_{\varepsilon\delta} \in W^{1,2}_{\mathrm{loc}}([0,\infty); W_N^{2,2}(\Omega), W^{1,2}(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty); W^{3,2}(\Omega)) \cap C^0([0,\infty); W^{1,2}(\Omega))$$

such that

$$\begin{aligned}
u_{\varepsilon\delta k_j} &\to u_{\varepsilon\delta} &&\text{and} & v_{\varepsilon\delta k_j} &\to v_{\varepsilon\delta} &&\text{pointwise a.e.,} && (6.2.26)\\
u_{\varepsilon\delta k_j} &\to u_{\varepsilon\delta} &&\text{and} & v_{\varepsilon\delta k_j} &\to v_{\varepsilon\delta} &&\text{in } C^0([0,\infty); L^2(\Omega)), && (6.2.27)\\
\nabla u_{\varepsilon\delta k_j} &\to \nabla u_{\varepsilon\delta} &&\text{and} & \nabla v_{\varepsilon\delta k_j} &\to \nabla v_{\varepsilon\delta} &&\text{in } L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty);\mathbb{R}^n), && (6.2.28)\\
\nabla\Delta u_{\varepsilon\delta k_j} &\rightharpoonup \nabla\Delta u_{\varepsilon\delta} &&\text{and} & \nabla\Delta v_{\varepsilon\delta k_j} &\rightharpoonup \nabla\Delta v_{\varepsilon\delta} &&\text{in } L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty);\mathbb{R}^n), && (6.2.29)\\
u_{\varepsilon\delta k_j t} &\rightharpoonup u_{\varepsilon\delta t} &&\text{and} & v_{\varepsilon\delta k_j t} &\rightharpoonup v_{\varepsilon\delta t} &&\text{in } L^2_{\mathrm{loc}}([0,\infty); (W^{1,2}(\Omega))^\star) && (6.2.30)
\end{aligned}$$

as $j \to \infty$.

Proof. As the claims for the second solution component can be shown analogously, it suffices to prove (6.2.26)–(6.2.30) for the first one. According to (6.2.20)–(6.2.22), (6.2.24) and (6.2.25), the sequence $(u_{\varepsilon\delta k})_{k\in\mathbb{N}}$ is bounded in the space $W^{1,2}_{\mathrm{loc}}([0,\infty); W^{2,2}(\Omega), W^{1,2}(\Omega))$ so that by a diagonalization argument, we obtain a sequence $(k_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ with $k_j \to \infty$ and a function $u_{\varepsilon\delta} \in W^{1,2}_{\mathrm{loc}}([0,\infty); W^{2,2}(\Omega), W^{1,2}(\Omega))$ such that

$$u_{\varepsilon\delta k_j} \rightharpoonup u_{\varepsilon\delta} \qquad \text{in } W^{1,2}_{\mathrm{loc}}([0,\infty); W^{2,2}_N(\Omega), W^{1,2}(\Omega)) \text{ as } j \to \infty,$$

which directly implies (6.2.29) and (6.2.30) and together with the Aubin–Lions lemma also (6.2.28).

Thanks to (6.2.21) and (6.2.24), another application of the Aubin–Lions lemma yields (6.2.27) and thus also (6.2.26), possibly after switching to subsequences. ☐

We conclude this subsection by showing that the pair $(u_{\varepsilon\delta}, v_{\varepsilon\delta})$ constructed in Lemma 6.2.7 indeed solves (6.P$_{\varepsilon\delta}$) in a weak sense.

Lemma 6.2.8. *Let $\varepsilon, \delta \in (0,1)$. The tuple $(u_{\varepsilon\delta}, v_{\varepsilon\delta})$ constructed in Lemma 6.2.7 is a weak solution of (6.P$_{\varepsilon\delta}$) in the sense that*

$$u_{\varepsilon\delta}(\cdot,0) = u_0 \quad \textit{as well as} \quad v_{\varepsilon\delta}(\cdot,0) = v_0 \qquad \textit{hold a.e. in } \Omega \times (0,\infty), \tag{6.2.31}$$

and, for all $T \in (0,\infty)$ and $\varphi \in L^2((0,T); W^{1,2}(\Omega))$, we have

$$\begin{aligned}\int_0^T \int_\Omega u_{\varepsilon\delta t}\varphi = \varepsilon \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta})\nabla\Delta u_{\varepsilon\delta} \cdot \nabla\varphi - \int_0^T \int_\Omega D_1(|u_{\varepsilon\delta}|)\nabla u_{\varepsilon\delta} \cdot \nabla\varphi \\ + \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta})\nabla v_{\varepsilon\delta} \cdot \nabla\varphi + \int_0^T \int_\Omega f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta})\varphi\end{aligned} \tag{6.2.32}$$

as well as

$$\begin{aligned}\int_0^T \int_\Omega v_{\varepsilon\delta t}\varphi = \varepsilon \int_0^T \int_\Omega S_{2\delta}(v_{\varepsilon\delta})\nabla\Delta v_{\varepsilon\delta} \cdot \nabla\varphi - \int_0^T \int_\Omega D_2(|v_{\varepsilon\delta}|)\nabla v_{\varepsilon\delta} \cdot \nabla\varphi \\ - \int_0^T \int_\Omega S_{2\delta}(v_{\varepsilon\delta})\nabla u_{\varepsilon\delta} \cdot \nabla\varphi + \int_0^T \int_\Omega f_{2\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta})\varphi.\end{aligned} \tag{6.2.33}$$

Proof. We fix $T \in (0,\infty)$ as well as $\varphi \in L^2((0,T); W^{1,2}(\Omega))$, denote the orthogonal projection on X_k by P_k (where X_k is as in Lemma 6.2.4) and set $(P_k\varphi)(x,t) \coloneqq (P_k\varphi(\cdot,t))(x)$ for $(x,t) \in \Omega \times (0,T)$. Moreover, let $(u_{\varepsilon\delta}, v_{\varepsilon\delta k})$ and $(k_j)_{j\in\mathbb{N}}$ be as given by Lemma 6.2.7. According to Lemma 6.2.4 and Lemma 6.2.5, we then have

$$\begin{aligned}\int_0^T \int_\Omega u_{\varepsilon\delta kt}P_k\varphi = \varepsilon \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta k})\nabla\Delta u_{\varepsilon\delta k} \cdot \nabla P_k\varphi - \int_0^T \int_\Omega D_1(|u_{\varepsilon\delta k}|)\nabla u_{\varepsilon\delta k} \cdot \nabla P_k\varphi \\ + \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta k})\nabla v_{\varepsilon\delta k} \cdot \nabla P_k\varphi + \int_0^T \int_\Omega f_{1\delta}(u_{\varepsilon\delta k}, v_{\varepsilon\delta k})P_k\varphi\end{aligned}$$

for all $k \in \mathbb{N}$. Since $P_k\varphi \to \varphi$ in $L^2((0,T); W^{1,2}(\Omega))$ for $k \to \infty$, we infer

$$\lim_{j\to\infty} \int_0^T \int_\Omega u_{\varepsilon\delta k_j t}P_{k_j}\varphi = \int_0^T \int_\Omega u_{\varepsilon\delta t}\varphi$$

from (6.2.30). Moreover, as $f_{1\delta}$ is bounded, (6.2.26) asserts $f_{1\delta}(u_{\varepsilon\delta k_j}, v_{\varepsilon\delta k_j}) \to f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta})$ in $L^2(\Omega \times (0,T))$ as $j \to \infty$ and hence

$$\lim_{j\to\infty} \int_0^T \int_\Omega f_{1\delta}(u_{\varepsilon\delta k_j}, v_{\varepsilon\delta k_j}) P_{k_j}\varphi = \int_0^T \int_\Omega f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta})\varphi.$$

Boundedness of $S_{1\delta}$, (6.2.26) and Lebesgue's theorem imply

$$\begin{aligned} &\|S_{1\delta}(u_{\varepsilon\delta k_j})\nabla P_{k_j}\varphi - S_{1\delta}(u_{\varepsilon\delta k_j})\nabla\varphi\|_{L^2(\Omega\times(0,T))} \\ \le\ &\|[S_{1\delta}(u_{\varepsilon\delta k_j}) - S_{1\delta}(u_{\varepsilon\delta})]\nabla\varphi\|_{L^2(\Omega\times(0,T))} + \|S_{1\delta}(u_{\varepsilon\delta k_j})\nabla[P_{k_j}\varphi - \varphi]\|_{L^2(\Omega\times(0,T))} \to 0 \end{aligned}$$

as $j \to \infty$ and hence

$$\lim_{j\to\infty} \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta k_j})\nabla\Delta u_{\varepsilon\delta k_j} \cdot \nabla P_{k_j}\varphi = \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta})\nabla\Delta u_{\varepsilon\delta} \cdot \nabla\varphi$$

due to (6.2.29).

A similar reasoning, relying on (6.2.28) instead of (6.2.29), gives

$$\lim_{j\to\infty} \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta k_j})\nabla u_{\varepsilon\delta k_j} \cdot \nabla P_{k_j}\varphi = \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta})\nabla u_{\varepsilon\delta} \cdot \nabla\varphi$$

and

$$\lim_{j\to\infty} \int_0^T \int_\Omega D_1(|u_{\varepsilon\delta k_j}|)\nabla u_{\varepsilon\delta k_j} \cdot \nabla P_{k_j}\varphi = \int_0^T \int_\Omega D_1(|u_{\varepsilon\delta k_j}|)\nabla u_{\varepsilon\delta} \cdot \nabla\varphi$$

so that indeed (6.2.32) holds, while (6.2.33) can be derived analogously.

Finally, we note that (6.2.27) implies $u_{\varepsilon\delta k_j}(\cdot,0) \to u_{\varepsilon\delta}(\cdot,0)$ in $L^2(\Omega)$ as $j \to \infty$ so that (6.2.17) asserts

$$\int_\Omega u_{\varepsilon\delta}(\cdot,0)\psi = \lim_{j\to\infty}\int_\Omega u_{\varepsilon\delta k_j}(\cdot,0)P_{k_j}\psi = \int_\Omega u_0\psi \qquad \text{for all } \psi \in L^2(\Omega).$$

This implies $u_{\varepsilon\delta}(\cdot,0) = u_0$ a.e. and, by combining this with an analogous argument for the second solution component, we arrive at (6.2.31). □

6.2.2. The limit process $\delta \searrow 0$: guaranteeing nonnegativity

As opposed to the problem solved by $(u_{\varepsilon\delta k}, v_{\varepsilon\delta k})$ for $k \in \mathbb{N}$, where (6.2.15) and (6.2.16) require that $\varphi(\cdot,t) \in X_k$ for all $t \in (0,\infty)$, in the weak formulation for the problem (6.P$_{\varepsilon\delta}$), (6.2.32) and (6.2.33), all $\varphi \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ are admissible test functions. In particular, we may now test with anti-derivatives of $\frac{1}{S_{1\delta}(u_{\varepsilon\delta})}$ and $\frac{1}{S_{2\delta}(v_{\varepsilon\delta})}$, allowing us to obtain estimates independent of both ε and δ in Lemma 6.2.10. These bounds not only form the basis for the limit processes $\delta \searrow 0$ and $\varepsilon \searrow 0$ (which are finally performed in Lemma 6.2.14 and Lemma 6.2.17, respectively) but are also important for showing that the later obtained limit functions $u_\varepsilon, v_\varepsilon$ are nonnegative (see Lemma 6.2.15).

To further prepare these testing procedures, we state the following lemma which should essentially be well-known.

Lemma 6.2.9. *Let $T \in (0,\infty)$, $w, z \in W^{1,2}([0,T]; W^{1,2}(\Omega))$ and $\varphi \in C^1([0,T])$.*

(i) For $H \in C^2(\mathbb{R}^2)$ with $D^2 H \in L^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$, the functions $H_w(w,z)$ and $H_z(w,z)$ belong to $L^2((0,T); W^{1,2}(\Omega))$ and

$$\begin{aligned} &-\int_0^T \int_\Omega H(w,z)\varphi_t + \int_\Omega H(w(\cdot,T), z(\cdot,T))\varphi(\cdot,T) - \int_\Omega H(w(\cdot,0), z(\cdot,0))\varphi(\cdot,0) \\ &= \int_0^T \int_\Omega w_t H_w(w,z)\varphi + \int_0^T \int_\Omega z_t H_z(w,z)\varphi \end{aligned} \tag{6.2.34}$$

holds.

(ii) Let $\widetilde{H} \in C^2(\mathbb{R})$ with $\widetilde{H}'' \in L^\infty(\mathbb{R})$. Then $\widetilde{H}'(w) \in L^2((0,T); W^{1,2}(\Omega))$ and

$$-\int_0^T \int_\Omega \widetilde{H}(w)\varphi_t + \int_\Omega \widetilde{H}(w(\cdot,T)) - \int_\Omega \widetilde{H}(w(\cdot,0)). = \int_0^T \int_\Omega w_t \widetilde{H}'(w)\varphi$$

Proof. We first fix $(w_\ell)_{\ell\in\mathbb{N}}, (z_\ell)_{\ell\in\mathbb{N}} \subset C^\infty(\overline{\Omega}\times[0,T])$ with $w_\ell \to w$ and $z_\ell \to z$ in $W^{1,2}([0,T]; W^{1,2}(\Omega))$ as $j \to \infty$. Hence, for $X := L^2((0,T); W^{1,2}(\Omega))$ and thus $X^\star = L^2((0,T); (W^{1,2}(\Omega))^\star)$, we have $w_\ell \to w$ and $z_\ell \to z$ in X, $w_{\ell t} \to w_t$ and $z_{\ell t} \to z_t$ in $X^\star$ as well as $w_\ell \to w$ and $z_\ell \to z$ in $C^0([0,T]; L^2(\Omega))$.

Then

$$\begin{aligned} &-\int_0^T \int_\Omega H(w_\ell, z_\ell)\varphi_t + \left[\int_\Omega H(w_\ell(\cdot,t), z_\ell(\cdot,t))\varphi(\cdot,t)\right]_{t=0}^{t=T} \\ &= \int_0^T \int_\Omega [H(w_\ell, z_\ell)]_t \varphi = \int_0^T \int_\Omega w_{\ell t} H_w(w_\ell, z_\ell)\varphi + \int_0^T \int_\Omega z_{\ell t} H_z(w_\ell, z_\ell)\varphi. \end{aligned}$$

By Taylor's theorem for multivariate functions and Young's inequality, we obtain

$$\begin{aligned} &\left|\int_\Omega H(w(\cdot,t), z(\cdot,t)) - \int_\Omega H(w_\ell(\cdot,t), z_\ell(\cdot,t))\right| \\ &\le \sum_{|\alpha|=1} \int_\Omega \frac{|D^\alpha H(w(\cdot,t), z(\cdot,t))|}{\alpha!} [w(\cdot,t) - w_\ell(\cdot,t), z(\cdot,t) - z_\ell(\cdot,t)]^\alpha \\ &\quad + \sum_{|\alpha|=2} \frac{\max_{|\beta|=|\alpha|} \|D^\beta H\|_{L^\infty(\mathbb{R}^2)}}{\alpha!} \int_\Omega [w(\cdot,t) - w_\ell(\cdot,t), z(\cdot,t) - z_\ell(\cdot,t)]^\alpha \\ &\le |H_w(w(\cdot,t), z(\cdot,t))| \int_\Omega |w(\cdot,t) - w_\ell(\cdot,t)| + |H_z(w(\cdot,t), z(\cdot,t))| \int_\Omega |z(\cdot,t) - z_\ell(\cdot,t)| \\ &\quad + \|D^2 H\|_{L^\infty(\mathbb{R}^{2\times 2})} \int_\Omega (w(\cdot,t) - z(\cdot,t))^2 + \|D^2 H\|_{L^\infty(\mathbb{R}^{2\times 2})} \int_\Omega (z(\cdot,t) - z_\ell(\cdot,t))^2 \\ &\to 0 \qquad \text{as } \ell \to \infty \text{ for all } t \in [0,T]. \end{aligned}$$

Since moreover

$$\begin{aligned} &\|H_w(w,z) - H_w(w_\ell, z_\ell)\|_{L^2(\Omega\times(0,T))} \\ &\le \|H_w(w, z_\ell) - H_w(w_\ell, z_\ell)\|_{L^2(\Omega\times(0,T))} + \|H_w(w,z) - H_w(w, z_\ell)\|_{L^2(\Omega\times(0,T))} \\ &\le \|H_{ww}\|_{L^\infty(\mathbb{R}^2)} \|w - w_\ell\|_{L^2(\Omega\times(0,T))} + \|H_{wz}\|_{L^\infty(\mathbb{R}^2)} \|z - z_\ell\|_{L^2(\Omega\times(0,T))} \\ &\to 0 \qquad \text{as } \ell \to \infty \end{aligned} \tag{6.2.35}$$

by the mean value theorem and

$$\begin{aligned}
&\sup_{\ell\in\mathbb{N}} \int_0^T \int_\Omega |\nabla H_w(w_\ell, z_\ell)|^2 \\
&= \sup_{\ell\in\mathbb{N}} \int_0^T \int_\Omega |H_{ww}(w_\ell, z_\ell)\nabla w_\ell + H_{wz}(w_\ell, z_\ell)\nabla z_\ell|^2 \\
&\le \sup_{\ell\in\mathbb{N}} \left(2\|H_{ww}\|_{L^\infty(\mathbb{R}^2)}^2 \int_0^T \int_\Omega |\nabla w_\ell|^2 + 2\|H_{wz}\|_{L^\infty(\mathbb{R}^2)}^2 \int_0^T \int_\Omega |\nabla z_\ell|^2 \right) < \infty
\end{aligned}$$

by the chain rule, we conclude $\sup_{\ell\in\mathbb{N}} \|H_w(w_\ell, z_\ell)\|_X^2 < \infty$. Therefore, after switching to subsequences if necessary, we have

$$H_w(w_\ell, z_\ell) \rightharpoonup \tilde{w} \qquad \text{in } X \text{ as } \ell \to \infty \tag{6.2.36}$$

for some $\tilde{w} \in X$. From (6.2.35), we infer $\tilde{w} = H_w(w, z)$ so that (6.2.36) and the convergence $w_{\ell t} \to w_t$ in $X^\star$ imply

$$\int_0^T \int_\Omega w_{\ell t} H_w(w_\ell, z_\ell)\varphi \to \int_0^T \int_\Omega w_t H_w(w, z)\varphi \qquad \text{as } \ell \to \infty.$$

Likewise, we obtain

$$\int_0^T \int_\Omega z_{\ell t} H_z(w_\ell, z_\ell)\varphi \to \int_0^T \int_\Omega z_t H_z(w, z)\varphi \qquad \text{as } \ell \to \infty$$

and thus (6.2.34).

Finally, the second part follows from the first one by setting $H(\rho, \sigma) = \widetilde{H}(\rho)$ for $\rho, \sigma \in \mathbb{R}$. □

With Lemma 6.2.9 at hand, we are now able to prove an analogue to the entropy-like inequality (6.2.10).

Lemma 6.2.10. *Let $\varepsilon, \delta \in (0, 1)$ and $u_{\varepsilon\delta}, v_{\varepsilon\delta}$ be as in Lemma 6.2.7. Set moreover*

$$G_{i\delta}(s) := \int_1^s \int_1^\rho \frac{1}{S_{i\delta}(\sigma)} \,\mathrm{d}\sigma \,\mathrm{d}\rho \qquad \textit{for } i \in \{1, 2\}$$

as well as

$$\begin{aligned}
\mathcal{E}_{\varepsilon\delta}(t) &:= \int_\Omega G_{1\delta}(u_{\varepsilon\delta}(\cdot, t)) + \int_\Omega G_{2\delta}(v_{\varepsilon\delta}(\cdot, t)), \\
\mathcal{D}_{\varepsilon\delta}(t) &:= \varepsilon \int_\Omega |\Delta u_{\varepsilon\delta}(\cdot, t)|^2 + \varepsilon \int_\Omega |\Delta v_{\varepsilon\delta}(\cdot, t)|^2 \\
&\quad + \int_\Omega \frac{D_1(|u_{\varepsilon\delta}(\cdot, t)|)}{S_{1\delta}(u_{\varepsilon\delta}(\cdot, t))} |\nabla u_{\varepsilon\delta}(\cdot, t)|^2 + \int_\Omega \frac{D_2(|v_{\varepsilon\delta}(\cdot, t)|)}{S_{2\delta}(v_{\varepsilon\delta}(\cdot, t))} |\nabla v_{\varepsilon\delta}(\cdot, t)|^2 \quad \textit{and} \\
\mathcal{R}_{\varepsilon\delta}(t) &:= \int_\Omega G'_{1\delta}(u_{\varepsilon\delta}(\cdot, t)) f_{1\delta}(u_{\varepsilon\delta}(\cdot, t), v_{\varepsilon\delta}(\cdot, t)) + \int_\Omega G'_{2\delta}(v_{\varepsilon\delta}(\cdot, t)) f_{2\delta}(u_{\varepsilon\delta}(\cdot, t), v_{\varepsilon\delta}(\cdot, t))
\end{aligned}$$

for $t \in [0, \infty)$. (Here, similarly as on Theorem 6.2.2, $\mathcal{D}_{\varepsilon\delta}$ and $\mathcal{R}_{\varepsilon\delta}$ are to be understood as functions in $L^0((0, \infty))$.) Then

$$\begin{aligned}
&\mathcal{E}_{\varepsilon\delta}(T)\zeta(T) + \int_0^T \mathcal{D}_{\varepsilon\delta}(t)\zeta(t) \,\mathrm{d}t \\
&\le \mathcal{E}_{\varepsilon\delta}(0)\zeta(0) + \int_0^T \mathcal{R}_{\varepsilon\delta}(t)\zeta(t) \,\mathrm{d}t + \int_0^T \mathcal{E}_{\varepsilon\delta}(t)\zeta'(t) \,\mathrm{d}t
\end{aligned} \tag{6.2.37}$$

holds for any $T \in (0, \infty)$ and $0 \le \zeta \in C^\infty([0, T])$.

Proof. As $\frac{1}{S_{1\delta}}$ is continuous, positive and bounded, we may apply Lemma 6.2.9 (ii) and Lemma 6.2.8 to obtain

$$\begin{aligned}
&\int_\Omega G_{1\delta}(u_{\varepsilon\delta}(\cdot,T))\zeta(T) - \int_\Omega G_{1\delta}(u_0)\zeta(0) - \int_0^T \int_\Omega G_{1\delta}(u_{\varepsilon\delta})\zeta' \\
&= \int_0^T \int_\Omega u_{\varepsilon\delta t} G'_{1\delta}(u_{\varepsilon\delta})\zeta \\
&= -\varepsilon \int_0^T \int_\Omega |\Delta u_{\varepsilon\delta}|^2 \zeta - \int_0^T \int_\Omega \frac{D_1(|u_{\varepsilon\delta}|)}{S_{1\delta}(u_{\varepsilon\delta})} |\nabla u_{\varepsilon\delta}|^2 \zeta \\
&\quad + \int_0^T \int_\Omega \nabla u_{\varepsilon\delta} \cdot \nabla v_{\varepsilon\delta} \zeta + \int_0^T \int_\Omega G'_{1\delta}(u_{\varepsilon\delta}) f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta})\zeta
\end{aligned} \tag{6.2.38}$$

for all $\varepsilon, \delta \in (0,1)$. Since the signs of the cross-diffusive terms in the first two equations in (6.$P_{\varepsilon\delta}$) are opposite, (6.2.38) and a corresponding identity for the second solution component already yield (6.2.37). □

Aiming to derive (ε,δ)-independent a priori estimates from (6.2.37) with $\zeta \equiv 1$, we next estimate the right-hand side therein and obtain

Lemma 6.2.11. *Let $T \in (0,\infty)$ and $G_{i\delta}$, $\delta \in (0,1)$ $i \in \{1,2\}$ be as in Lemma 6.2.10. Then there is $C > 0$ such that*

$$\sup_{t\in(0,T)} \left(\int_\Omega G_{1\delta}(u_{\varepsilon\delta}(\cdot,t)) + \int_\Omega G_{2\delta}(v_{\varepsilon\delta}(\cdot,t)) \right) \le C, \tag{6.2.39}$$

$$\varepsilon \int_0^T \int_\Omega |\Delta u_{\varepsilon\delta}|^2 + \varepsilon \int_0^T \int_\Omega |\Delta v_{\varepsilon\delta}|^2 \le C, \tag{6.2.40}$$

$$\int_0^T \int_\Omega |\nabla u_{\varepsilon\delta}|^2 + \int_0^T \int_\Omega |\nabla v_{\varepsilon\delta}|^2 \le C \quad \textit{and} \tag{6.2.41}$$

$$\int_0^T \int_\Omega u_{\varepsilon\delta}^2 + \int_0^T \int_\Omega v_{\varepsilon\delta}^2 \le C \tag{6.2.42}$$

for all $\varepsilon, \delta \in (0,1)$, where $u_{\varepsilon\delta}$ and $v_{\varepsilon\delta}$ are as in Lemma 6.2.7.

Proof. Since the definition of $\underline{S}_1$ entails that

$$S_{1\delta}(s) \ge S_1(|s|) + \delta \ge \begin{cases} \underline{S}_1 s, & |s| < 1, \\ \underline{S}_1, & |s| \ge 1 \end{cases} \qquad \text{for all } s \ge 0 \text{ and } \delta \in (0,1),$$

we may estimate

$$|G'_{1\delta}(u_{\varepsilon\delta})| = \int_{u_{\varepsilon\delta}}^1 \frac{\mathrm{d}\sigma}{S_{1\delta}(\sigma)} \le \frac{|\ln u_{\varepsilon\delta}|}{\underline{S}_1} \qquad \text{in } \{0 < u_{\varepsilon\delta} \le 1\} \text{ for all } \varepsilon, \delta \in (0,1)$$

and

$$|G'_{1\delta}(u_{\varepsilon\delta})| = \int_1^{u_{\varepsilon\delta}} \frac{\mathrm{d}\sigma}{S_{1\delta}(\sigma)} \le \frac{u_{\varepsilon\delta} - 1}{\underline{S}_1} \qquad \text{in } \{1 < u_{\varepsilon\delta}\} \text{ for all } \varepsilon, \delta \in (0,1).$$

Due to $f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) = 0$ in $\{u_{\varepsilon\delta} \le 0\}$ and because of (6.2.3) and (6.2.5), we thus obtain $c_1 > 0$ such that

$$\int_0^t \int_\Omega f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon\delta}) G'_{1\delta}(u_{\varepsilon\delta}) \le c_1 \int_0^t \int_\Omega (1 + u_{\varepsilon\delta}^2) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon, \delta \in (0,1).$$

Moreover, positivity of u_0 and v_0 implies finiteness of

$$\sup_{\delta\in(0,1)}\left(\int_\Omega G_{1\delta}(u_0)+\int_\Omega G_{2\delta}(v_0)\right).$$

As $\frac{D_i(|s|)}{S_{i\delta}(s)}\ge\frac{\underline{D}_i}{\overline{S}_i}$, $i\in\{1,2\}$, for all $s\in\mathbb{R}$, along with an analogous computation for the second solution component and choosing $\zeta\equiv 1$ in (6.2.37), we obtain $c_2>0$ such that

$$\begin{aligned}&\int_\Omega G_{1\delta}(u_{\varepsilon\delta}(\cdot,t))+\int_\Omega G_{2\delta}(v_{\varepsilon\delta}(\cdot,t))\\&+\varepsilon\int_0^t\int_\Omega|\Delta u_{\varepsilon\delta}|^2+\varepsilon\int_0^t\int_\Omega|\Delta v_{\varepsilon\delta}|^2+\frac{\underline{D}_1}{\overline{S}_1}\int_0^t\int_\Omega|\nabla u_{\varepsilon\delta}|^2+\frac{\underline{D}_2}{\overline{S}_2}\int_0^t\int_\Omega|\nabla v_{\varepsilon\delta}|^2\\&\le c_2+c_2\int_0^t\int_\Omega u_{\varepsilon\delta}^2+c_2\int_0^t\int_\Omega v_{\varepsilon\delta}^2\qquad\text{for all } t\in(0,T)\text{ and }\varepsilon,\delta\in(0,1).\end{aligned}\tag{6.2.43}$$

Since

$$\begin{aligned}G_{1\delta}(u_{\varepsilon\delta})&=\int_1^{u_{\varepsilon\delta}}\int_1^\rho\frac{1}{S_{1\delta}(\sigma)}\,\mathrm{d}\sigma\,\mathrm{d}\rho\ge\frac{1}{\overline{S}_1}\int_1^{u_{\varepsilon\delta}}(\rho-1)\,\mathrm{d}\rho\\&=\frac{1}{\overline{S}_1}\left(\frac12u_{\varepsilon\delta}^2-\frac12-(u_{\varepsilon\delta}-1)\right)\ge\frac{1}{\overline{S}_1}\left(\frac14u_{\varepsilon\delta}^2-\frac12\right)\end{aligned}$$

in $\Omega\times(0,T)$ for all $\varepsilon,\delta\in(0,1)$ and hence

$$\int_0^t\int_\Omega u_{\varepsilon\delta}^2\le 4\overline{S}_1\int_0^t\int_\Omega G_{1\delta}(u_{\varepsilon\delta})+2|\Omega|T\qquad\text{for all } t\in(0,T)\text{ and }\varepsilon,\delta\in(0,1),\tag{6.2.44}$$

a consequence of (6.2.43) is

$$\begin{aligned}&\int_\Omega G_{1\delta}(u_{\varepsilon\delta}(\cdot,t))+\int_\Omega G_{2\delta}(v_{\varepsilon\delta}(\cdot,t))\\&\le c_3+4c_2\max\{\overline{S}_1,\overline{S}_2\}\int_0^t\left(\int_\Omega G_{1\delta}(u_{\varepsilon\delta})+\int_\Omega G_{2\delta}(v_{\varepsilon\delta})\right)\end{aligned}$$

for all $t\in(0,T)$, $\varepsilon,\delta\in(0,1)$ and $c_3\coloneqq c_2+4c_2|\Omega|T$. Grönwall's inequality thus asserts that

$$\int_\Omega G_{1\delta}(u_{\varepsilon\delta}(\cdot,t))+\int_\Omega G_{1\delta}(v_{\varepsilon\delta}(\cdot,t))\le c_3\mathrm{e}^{4c_2\max\{\overline{S}_1,\overline{S}_2\}T}$$

holds for all $t\in(0,T)$ and $\varepsilon,\delta\in(0,1)$, implying (6.2.39). Finally, (6.2.40)–(6.2.42) follow from (6.2.43), (6.2.44) and (6.2.39). □

Again seeking to apply the Aubin–Lions lemma, we complement the bounds (6.2.39)–(6.2.41) by estimates for the time derivatives in the next two lemmata. However, in contrast to Lemma 6.2.6 and owing to the fourth-order regularization terms, we have to settle for bounds in $L^2((0,T);(W^{n+1,2}(\Omega))^\star)$ instead of $L^2((0,T);(W^{1,2}(\Omega))^\star)$.

Lemma 6.2.12. *For $T\in(0,\infty)$, there exists $C>0$ such that*

$$\begin{aligned}&\left|\int_0^T\int_\Omega u_{\varepsilon\delta t}\varphi+\int_0^T\int_\Omega D_1(|u_{\varepsilon\delta}|)\nabla u_{\varepsilon\delta}\cdot\nabla\varphi\right.\\&\qquad\left.-\int_0^T\int_\Omega S_1(u_{\varepsilon\delta})\nabla v_{\varepsilon\delta}\cdot\nabla\varphi-\int_0^T\int_\Omega f_1(u_{\varepsilon\delta},v_{\varepsilon\delta})\varphi\right|\\&\le C\varepsilon^{\frac12}\|\varphi\|_{L^2((0,T);W^{n+1,2}(\Omega))}\end{aligned}\tag{6.2.45}$$

and

$$\begin{aligned}
&\bigg| \int_0^T \int_\Omega v_{\varepsilon\delta t}\varphi + \int_0^T \int_\Omega D_2(|v_{\varepsilon\delta}|)\nabla v_{\varepsilon\delta} \cdot \nabla\varphi \\
&\quad + \int_0^T \int_\Omega S_2(v_{\varepsilon\delta})\nabla u_{\varepsilon\delta} \cdot \nabla\varphi - \int_0^T \int_\Omega f_2(u_{\varepsilon\delta}, v_{\varepsilon\delta})\varphi \bigg| \\
&\le C\varepsilon^{\frac{1}{2}} \|\varphi\|_{L^2((0,T);W^{n+1,2}(\Omega))}
\end{aligned} \tag{6.2.46}$$

for all $\varepsilon, \delta \in (0,1)$ *and* $\varphi \in L^2((0,T);W^{n+1,2}(\Omega))$*, where* $u_{\varepsilon\delta}$ *and* $v_{\varepsilon\delta}$ *are as in Lemma 6.2.7.*

Proof. Since $n+1 > \frac{n}{2}+1$, Sobolev's embedding theorem allows us to fix $c_1 > 0$ with

$$\|\nabla\varphi\|_{L^\infty(\Omega)} \le c_1 \|\varphi\|_{W^{n+1,2}(\Omega)} \qquad \text{for all } \varphi \in W^{n+1,2}(\Omega).$$

Moreover, we fix $T \in (0,\infty)$ and choose $c_2 > 0$ such that (6.2.40) and (6.2.41) hold (with C replaced by c_2^2). Then

$$\begin{aligned}
&\left| \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta})\nabla\Delta u_{\varepsilon\delta} \cdot \nabla\varphi \right| \\
&\le \left| \int_0^T \int_\Omega S_{1\delta}'(u_{\varepsilon\delta})\Delta u_{\varepsilon\delta}\nabla u_{\varepsilon\delta} \cdot \nabla\varphi \right| + \left| \int_0^T \int_\Omega S_{1\delta}(u_{\varepsilon\delta})\Delta u_{\varepsilon\delta}\Delta\varphi \right| \\
&\le \|\Delta u_{\varepsilon\delta}\|_{L^2(\Omega\times(0,T))} \left(\overline{S}_1' \|\nabla u_{\varepsilon\delta}\|_{L^2(\Omega\times(0,T))} \|\nabla\varphi\|_{L^\infty(\Omega\times(0,T))} + \overline{S}_1 \|\Delta\varphi\|_{L^2(\Omega\times(0,T))} \right) \\
&\le \varepsilon^{-\frac{1}{2}} \cdot c_2(c_1c_2\overline{S}_1' + \overline{S}_1)\|\varphi\|_{L^2((0,T);W^{n+1,2}(\Omega))} \qquad \text{for all } \varepsilon, \delta \in (0,1).
\end{aligned}$$

Combined with (6.2.32), this already implies (6.2.45), while (6.2.46) can be shown analogously. □

Lemma 6.2.13. *Let* $\varepsilon \in (0,1)$*,* $T \in (0,\infty)$ *and* $(u_{\varepsilon\delta}, v_{\varepsilon\delta})$ *be as in Lemma 6.2.7 for* $\delta \in (0,1)$*. Then there exists* $C > 0$ *such that*

$$\|u_{\varepsilon\delta t}\|_{L^2((0,T);(W^{n+1,2}(\Omega))^\star)} + \|v_{\varepsilon\delta t}\|_{L^2((0,T);(W^{n+1,2}(\Omega))^\star)} \le C \qquad \text{for all } \delta \in (0,1). \tag{6.2.47}$$

Proof. This immediately follows from Lemma 6.2.12 and the bounds provided by Lemma 6.2.11. □

With the estimates above at hand, we are now able to obtain convergence of certain subsequences of $(u_{\varepsilon\delta}, v_{\varepsilon\delta})$.

Lemma 6.2.14. *Let* $\varepsilon \in (0,1)$*. For* $\delta \in (0,1)$*, let* $u_{\varepsilon\delta}, v_{\varepsilon\delta}$ *be as given by Lemma 6.2.7. There are functions* $u_\varepsilon, v_\varepsilon \colon \Omega \times (0,\infty) \to \mathbb{R}$ *and a null sequence* $(\delta_j)_{j\in\mathbb{N}} \subset (0,1)$ *along which*

$$\begin{aligned}
u_{\varepsilon\delta_j} &\to u_\varepsilon &&\text{and} & v_{\varepsilon\delta_j} &\to v_\varepsilon &&\text{pointwise a.e.,} && (6.2.48)\\
u_{\varepsilon\delta_j} &\to u_\varepsilon &&\text{and} & v_{\varepsilon\delta_j} &\to v_\varepsilon &&\text{in } L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)), && (6.2.49)\\
u_{\varepsilon_j}(\cdot,t) &\to u(\cdot,t) &&\text{and} & v_{\varepsilon_j}(\cdot,t) &\to v(\cdot,t) &&\text{in } L^2(\Omega) \text{ for a.e. } t \in (0,\infty), && (6.2.50)\\
\nabla u_{\varepsilon\delta_j} &\rightharpoonup \nabla u_\varepsilon &&\text{and} & \nabla v_{\varepsilon\delta_j} &\rightharpoonup \nabla v_\varepsilon &&\text{in } L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty);\mathbb{R}^n), && (6.2.51)\\
u_{\varepsilon\delta_j t} &\rightharpoonup u_{\varepsilon t} &&\text{and} & v_{\varepsilon\delta_j t} &\rightharpoonup v_{\varepsilon t} &&\text{in } L^2_{\mathrm{loc}}([0,\infty);(W^{n+1,2}(\Omega))^\star), && (6.2.52)
\end{aligned}$$

as $j \to \infty$.

Proof. Due to the bounds in (6.2.41), (6.2.42) and (6.2.47), by means of the Aubin–Lions lemma and a diagonalization argument, we can obtain a null sequence $(\delta_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions $u_\varepsilon, v_\varepsilon \colon \Omega \times (0,\infty) \to \mathbb{R}$ such that (6.2.49), (6.2.51) and (6.2.52) hold. Upon switching to subsequences, if necessary, (6.2.48) and (6.2.50) follow then from (6.2.49). □

As already alluded to, the main reason for introducing the parameter δ in (6.P$_{\varepsilon\delta}$) is to be able to establish a.e. nonnegativity of the functions u_ε and v_ε constructed in Lemma 6.2.14. This will inter alia assure that each component of the solution (u, v) to (6.P) obtained in Subsection 6.2.3 below is nonnegative and hence may be interpreted as a population density.

Lemma 6.2.15. *For all $\varepsilon \in (0,1)$, $u_\varepsilon \ge 0$ and $v_\varepsilon \ge 0$ a.e. in $\Omega \times (0,\infty)$, where u_ε and v_ε are given by Lemma 6.2.14.*

Proof. This can be shown similarly as in [29, pages 554–555]. However, since the solutions considered there fulfill regularity properties going beyond those stated in Lemma 6.2.14, we give a (slightly different) proof here.

Let us fix $\varepsilon \in (0,1)$ as well as $T \in (0,\infty)$ and for the sake of contradiction assume that (a henceforth fixed representative of) u_ε is not nonnegative a.e. in $\Omega \times (0,T)$. That is, $|\{u_\varepsilon < 0\}| > 0$ so that by the sigma additivity of the Lebesgue measure, there is $\eta > 0$ such that $A \coloneqq \{\, (x,t) \in \Omega \times (0,T) : u_\varepsilon(x,t) \le -\eta \,\}$ has positive measure.

For $\delta \in (0,1)$, we now let $u_{\varepsilon\delta}$ and $G_{1\delta}$ be as in Lemma 6.2.7 and Lemma 6.2.10, respectively, and denote by $(\delta_j)_{j\in\mathbb{N}}$ the sequence given by Lemma 6.2.14. Thanks to (6.2.48) and Egorov's theorem, we then obtain a measurable $A' \subset A$ with $|A \setminus A'| < \frac{|A|}{2}$ such that $u_{\varepsilon\delta_j} \to u_\varepsilon$ uniformly in A' as $j \to \infty$; in particular, there is $j_0 \in \mathbb{N}$ with $u_{\varepsilon\delta_j}(x,t) \le -\frac{\eta}{2}$ for all $(x,t) \in A'$ and $j \ge j_0$.

Thanks to nonnegativity of $S_{1\delta}$, since $S_1(|s|) \le -\overline{S}_1' s$ for $s \le 0$ (due to the mean value theorem and as $S_1(0) = 0$ by (6.2.4)) and by Fatou's lemma (we note that $\lim_{\delta\searrow 0}(-\ln\delta + \ln(-\rho+\delta)) = \infty$ for all $\rho < 0$), we then have

$$
\begin{aligned}
\liminf_{j\to\infty} \int_0^T \int_\Omega G_{1\delta_j}(u_{\varepsilon\delta_j}) &\ge \liminf_{j\to\infty} \int_{A'} \int_1^{u_{\varepsilon\delta_j}(x,t)} \int_1^\rho \frac{1}{S_{1\delta_j}(\sigma)} \,\mathrm{d}\sigma \,\mathrm{d}\rho \,\mathrm{d}(x,t) \\
&\ge \liminf_{j\to\infty} |A'| \int_{-\frac{\eta}{2}}^1 \int_\rho^1 \frac{1}{S_1(|\sigma|) + \delta_j} \,\mathrm{d}\sigma \,\mathrm{d}\rho \\
&\ge \liminf_{j\to\infty} \frac{|A'|}{\max\{\overline{S}_1', 1\}} \int_{-\frac{\eta}{2}}^0 \int_\rho^0 \frac{1}{-\sigma + \delta_j} \,\mathrm{d}\sigma \,\mathrm{d}\rho \\
&= \liminf_{j\to\infty} \frac{|A'|}{\max\{\overline{S}_1', 1\}} \int_{-\frac{\eta}{2}}^0 (-\ln\delta_j + \ln(-\rho + \delta_j)) \,\mathrm{d}\rho = \infty,
\end{aligned}
$$

contradicting (6.2.39). The same argument is also applicable for the second solution component. □

Let us close this subsection by discussing in which way the pair $(u_\varepsilon, v_\varepsilon)$ obtained in Lemma 6.2.14 can be seen as a solution to the problem obtained by formally setting $\delta = 0$ in (6.P$_{\varepsilon\delta}$). Within a similar context, in [29, pages 552–553] it is shown that the limit functions solve the corresponding problem in a certain generalized sense. However, as already remarked in the preceding subsection, due to the possibly nonlinear diffusion terms D_1 and

D_2, the convergences obtained in Lemma 6.2.14 are slightly weaker than those established in [29]; that is, the methods developed in [29] are not directly applicable to our situation.

Nonetheless, we are able to prove that $(u_\varepsilon, v_\varepsilon)$ is up to an error term of order $\varepsilon^{\frac{1}{2}}$ a *weak* solution of that problem, which, having the limit process $\varepsilon \searrow 0$ in mind, turns out to be more convenient for our purposes in any case.

Lemma 6.2.16. *Let $\varepsilon \in (0,1)$, $u_\varepsilon, v_\varepsilon$ be as in Lemma 6.2.14 and $T \in (0,\infty)$. Then there is $C > 0$ such that*

$$\begin{aligned}&\bigg| -\int_0^T \int_\Omega u_\varepsilon \varphi_t - \int_\Omega u_0 \varphi(\cdot,0) + \int_0^T \int_\Omega D_1(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi \\ &\quad - \int_0^T \int_\Omega S_1(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi - \int_0^T \int_\Omega f_1(u_\varepsilon, v_\varepsilon) \varphi \bigg| \\ &\le C \varepsilon^{\frac{1}{2}} \|\varphi\|_{L^2((0,T);W^{n+1,2}(\Omega))}\end{aligned} \tag{6.2.53}$$

and

$$\begin{aligned}&\bigg| -\int_0^T \int_\Omega v_\varepsilon \varphi_t - \int_\Omega v_0 \varphi(\cdot,0) + \int_0^T \int_\Omega D_2(v_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi \\ &\quad + \int_0^T \int_\Omega S_2(v_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi - \int_0^T \int_\Omega f_2(u_\varepsilon, v_\varepsilon) \varphi \bigg| \\ &\le C \varepsilon^{\frac{1}{2}} \|\varphi\|_{L^2((0,T);W^{n+1,2}(\Omega))}\end{aligned} \tag{6.2.54}$$

for all $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$.

Proof. For $\delta \in (0,1)$, we let $(u_{\varepsilon\delta}, v_{\varepsilon\delta})$ be as in Lemma 6.2.7 and we denote the null sequence given by Lemma 6.2.14 by $(\delta_j)_{j\in\mathbb{N}}$. The convergences (6.2.49), (6.2.51) and (6.2.48) imply that

$$\begin{aligned}&\bigg| -\int_0^T \int_\Omega u_\varepsilon \varphi_t - \int_\Omega u_0 \varphi(\cdot,0) + \int_0^T \int_\Omega D_1(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi \\ &\quad - \int_0^T \int_\Omega S_1(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi - \int_0^T \int_\Omega f_1(u_\varepsilon, v_\varepsilon) \varphi \bigg| \\ &= \lim_{j\to\infty} \bigg| -\int_0^T \int_\Omega u_{\varepsilon\delta_j} \varphi_t - \int_\Omega u_0 \varphi(\cdot,0) + \int_0^T \int_\Omega D_1(|u_{\varepsilon\delta_j}|) \nabla u_{\varepsilon\delta_j} \cdot \nabla \varphi \\ &\quad - \int_0^T \int_\Omega S_{1\delta_j}(u_{\varepsilon\delta_j}) \nabla v_{\varepsilon\delta_j} \cdot \nabla \varphi - \int_0^T \int_\Omega f_{1\delta_j}(u_{\varepsilon\delta_j}, v_{\varepsilon\delta_j}) \varphi \bigg|\end{aligned}$$

for all $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$. As Lemma 6.2.9 (ii) and (6.2.31) assert

$$-\int_0^T \int_\Omega u_{\varepsilon\delta_j} \varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = \int_0^T \int_\Omega u_{\varepsilon\delta_j t} \varphi \qquad \text{for all } \varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty)) \text{ and } j \in \mathbb{N},$$

we see that (6.2.53) (with C as in Lemma 6.2.12) follows from (6.2.45). An analogous argumentation yields (6.2.54). □

6.2.3. The limit process $\varepsilon \searrow 0$: proofs of Theorem 6.2.1 and Theorem 6.2.2

Since Lemma 6.2.11 and Lemma 6.2.13 already contain ε-independent estimates, there are no further preparations necessary in order to undertake the final limit process of this section, namely $\varepsilon \searrow 0$.

Lemma 6.2.17. *Let $u_\varepsilon, v_\varepsilon$ be as in Lemma 6.2.14. There are nonnegative functions $u, v \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ and a null sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that*

$$u_{\varepsilon_j} \to u \quad \text{and} \quad v_{\varepsilon_j} \to v \quad \text{pointwise a.e.,} \tag{6.2.55}$$

$$u_{\varepsilon_j} \to u \quad \text{and} \quad v_{\varepsilon_j} \to v \quad \text{in } L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)), \tag{6.2.56}$$

$$u_{\varepsilon_j}(\cdot,t) \to u(\cdot,t) \quad \text{and} \quad v_{\varepsilon_j}(\cdot,t) \to v(\cdot,t) \quad \text{in } L^2(\Omega) \text{ for a.e. } t\in(0,\infty), \tag{6.2.57}$$

$$\nabla u_{\varepsilon_j} \rightharpoonup \nabla u \quad \text{and} \quad \nabla v_{\varepsilon_j} \rightharpoonup \nabla v \quad \text{in } L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty);\mathbb{R}^n) \tag{6.2.58}$$

as $j\to\infty$.

Proof. As the estimates (6.2.41) and (6.2.42) do not depend on ε and the right-hand sides in (6.2.45) and (6.2.46) are bounded in ε, the existence of $u, v \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ and a null sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that (6.2.55)–(6.2.58) hold can be shown as in Lemma 6.2.14. Moreover, nonnegativity of u and v follow from Lemma 6.3.2 and (6.2.55). □

Next, we show that the convergences asserted by Lemma 6.2.17 are sufficiently strong to imply that the pair (u,v) constructed in that lemma at least solves (6.P) in the following sense, which is yet somewhat weaker than the solution concept imposed by Theorem 6.2.1.

Lemma 6.2.18. *The pair (u,v) constructed in Lemma 6.2.17 fulfills*

$$\begin{aligned}&-\int_0^\infty\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0)\\ &= -\int_0^\infty\int_\Omega D_1(u)\nabla u\cdot\nabla\varphi + \int_0^\infty\int_\Omega S_1(u)\nabla v\cdot\nabla\varphi + \int_0^\infty\int_\Omega f_1(u,v)\varphi\end{aligned} \tag{6.2.59}$$

and

$$\begin{aligned}&-\int_0^\infty\int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0)\\ &= -\int_0^\infty\int_\Omega D_2(u)\nabla v\cdot\nabla\varphi - \int_0^\infty\int_\Omega S_2(u)\nabla u\cdot\nabla\varphi + \int_0^\infty\int_\Omega f_2(u,v)\varphi\end{aligned} \tag{6.2.60}$$

for all $\varphi \in C_c^\infty(\overline{\Omega}\times[0,\infty))$.

Proof. Since D_i, S_i and f_i, $i\in\{1,2\}$, are bounded, the statement immediately follows from Lemma 6.2.16 and Lemma 6.2.17. □

In order to prove Theorem 6.2.1, in addition to Lemma 6.2.18, we need to make sure that u, v are sufficiently regular; that is, that they belong to $W^{1,2}_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$. To that end, the ε-independent estimates of the time derivatives obtained in Lemma 6.2.13 are insufficient. However, we can obtain the desired regularity by testing directly at the $\varepsilon = 0$ level.

Lemma 6.2.19. *The functions u, v constructed in Lemma 6.2.17 are contained in the space $W^{1,2}_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ and satisfy (6.2.8) and (6.2.9) for all $\varphi \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$.*

PROOF. We fix $T \in (0,\infty)$. From Lemma 6.2.18 and Hölder's inequality, we infer that

$$\begin{aligned}
&\left|\int_0^T \int_\Omega u_t\varphi\right| = \left|\int_0^T \int_\Omega u\varphi_t\right| \\
\le& \left|\int_0^T \int_\Omega D_1(u)\nabla u\cdot\nabla\varphi\right| + \left|\int_0^T \int_\Omega S_1(u)\nabla v\cdot\nabla\varphi\right| + \left|\int_0^T \int_\Omega f_1(u,v)\varphi\right| \\
\le& \left(\overline{D_1}\|\nabla u\|_{L^2(\Omega\times(0,T))} + \overline{S_1}\|\nabla v\|_{L^2(\Omega\times(0,T))} + \|f_1\|_{L^\infty([0,\infty)^2)}(|\Omega|T)^{\frac12}\right)\|\varphi\|_{L^2((0,T);W^{1,2}(\Omega))}
\end{aligned}$$

for all $\varphi \in C_c^\infty(\overline{\Omega}\times(0,T))$, so that since $u, v \in L^2((0,T);W^{1,2}(\Omega))$ by Lemma 6.2.17 and as $C_c^\infty(\overline{\Omega}\times(0,T))$ is dense in $L^2((0,T);W^{1,2}(\Omega))$, we can conclude $u_t \in (L^2((0,T);W^{1,2}(\Omega))^\star = L^2((0,T);(W^{1,2}(\Omega))^\star)$. Thus, u, and by the same reasoning also v, indeed belongs to $W^{1,2}([0,T];W^{1,2}(\Omega))$.

As therefore

$$\int_0^1\int_\Omega u_t\varphi = -\int_0^1\int_\Omega u\varphi_t - \int_\Omega u(\cdot,0)\varphi(\cdot,0) \qquad \text{for all } \varphi\in C_c^\infty(\overline{\Omega}\times[0,1)) \tag{6.2.61}$$

by Lemma 6.2.9 (ii), we infer from (6.2.59) and the regularity of u and v that there is $c_1 > 0$ such that

$$\begin{aligned}
&\left|\int_\Omega (u(\cdot,0)-u_0)\varphi(\cdot,0)\right| \\
\le& \big(\|u_t\|_{L^2((0,1);(W^{1,2}(\Omega))^\star)} + \|D_1(u)\nabla u - S_1(u)\nabla v\|_{L^2(\Omega\times(0,1))} \\
&+\|f_1(u,v)\|_{L^2(\Omega\times(0,1))}\big)\|\varphi\|_{L^2((0,1);W^{1,2}(\Omega))} \\
\le& c_1\|\varphi\|_{L^2((0,1);W^{1,2}(\Omega))} \qquad \text{for all } \varphi\in C_c^\infty(\overline{\Omega}\times[0,1)).
\end{aligned}$$

Taking here φ supported near $\overline{\Omega}\times\{0\}$, we further conclude

$$\int_\Omega (u(\cdot,0)-u_0)\psi = 0 \qquad \text{for all } \psi\in C^\infty(\overline{\Omega}),$$

which due to density of $C^\infty(\overline{\Omega})$ in $L^2(\Omega)$ implies $u(\cdot,0) = u_0$ a.e. in Ω and hence the first assertion in (6.2.7). Therefore, (6.2.8) follows from (6.2.59) and (6.2.61); first for all $\varphi\in C_c^\infty(\overline{\Omega}\times[0,\infty))$ and thus by a density argument also for all $\varphi\in L^2_{\mathrm{loc}}([0,\infty);W^{1,2}(\Omega))$. The remaining statements for the second solution component can be derived analogously. ☐

Finally, we show that an analogue to the entropy-type inequality (6.2.37) also holds for the limit functions u, v.

Lemma 6.2.20. *Let G_i, $i\in\{1,2\}$, $\mathcal{E}$, $\mathcal{D}$, $\mathcal{R}$ be as in Theorem 6.2.2, $T\in(0,\infty)$ and $0\le\zeta\in C^\infty([0,T])$. The functions u, v given by Lemma 6.2.3 then satisfy* (6.2.10).

PROOF. For $\varepsilon,\delta\in(0,1)$, we denote the pairs constructed in Lemma 6.2.7 and Lemma 6.2.14 by $(u_{\varepsilon\delta}, v_{\varepsilon\delta})$ and $(u_\varepsilon, v_\varepsilon)$, respectively, and let the sequences $(\varepsilon_j)_{j\in\mathbb{N}}$ and $(\delta_{j'})_{j'\in\mathbb{N}}$ be as in Lemma 6.2.14 and Lemma 6.2.17. Moreover, again for $\varepsilon,\delta\in(0,1)$, we let $G_{i\delta}$, $i\in\{1,2\}$, $\mathcal{E}_{\varepsilon\delta}$, $\mathcal{D}_{\varepsilon\delta}$ and $\mathcal{R}_{\varepsilon\delta}$ be as in Lemma 6.2.10.

In order to prove (6.2.10), we essentially need to ensure that the inequality (6.2.37) survives the limit processes $\varepsilon=\varepsilon_j\searrow 0$ and $\delta=\delta_j\searrow 0$. To that end, we first note that for any

$\eta > 0$, the family

$$\left(\frac{D_1(|u_{\varepsilon_j \delta_{j'}}|)}{S_{1\delta_{j'}}(u_{\varepsilon_j \delta_{j'}}) + \eta} \zeta \right)_{j,j' \in \mathbb{N}}$$

is bounded in $L^\infty(\Omega \times (0,T))$ and, as first $j \to \infty$ and then $j' \to \infty$, converges a.e. in $\Omega \times (0,T)$ to $\frac{D_1(u)}{S_1(u)+\eta}\zeta$, thanks to (6.2.48) and (6.2.55). Thus, combined with (6.2.51) and (6.2.58), we see that

$$\left(\frac{D_1(|u_{\varepsilon_j \delta_{j'}}|)}{S_{1\delta_{j'}}(u_{\varepsilon_j \delta_{j'}}) + \eta} \zeta \right)^{\frac{1}{2}} \nabla u_{\varepsilon_j \delta_{j'}} \rightharpoonup \left(\frac{D_1(u)}{S_1(u) + \eta} \zeta \right)^{\frac{1}{2}} \nabla u$$

in $L^2(\Omega \times (0,T); \mathbb{R}^n)$ as first $j' \to \infty$ and then $j \to \infty$ for all $\eta > 0$. Consequently,

$$\liminf_{j \to \infty} \liminf_{j' \to \infty} \int_0^T \int_\Omega \frac{D_1(|u_{\varepsilon_j \delta_{j'}}|)}{S_{1\delta_{j'}}(u_{\varepsilon_j \delta_{j'}}) + \eta} |\nabla u_{\varepsilon_j \delta_{j'}}|^2 \zeta \geq \int_0^T \int_\Omega \frac{D_1(u)}{S_1(u) + \eta} |\nabla u|^2 \zeta$$

for all $\eta > 0$ by the weakly lower semicontinuity of the norm. Since $\eta > 0$ and by Fatou's lemma, we can conclude that

$$\liminf_{j \to \infty} \liminf_{j' \to \infty} \int_0^T \int_\Omega \frac{D_1(|u_{\varepsilon_j \delta_{j'}}|)}{S_{1\delta_{j'}}(u_{\varepsilon_j \delta_{j'}})} |\nabla u_{\varepsilon_j \delta_{j'}}|^2 \zeta \geq \int_0^T \int_\Omega \frac{D_1(u)}{S_1(u)} |\nabla u|^2 \zeta.$$

Next, we show that

$$\lim_{j \to \infty} \lim_{j' \to \infty} \int_0^T \int_\Omega G'_{1\delta_{j'}}(u_{\varepsilon_j \delta_{j'}}) f_1((u_{\varepsilon_j \delta_{j'}})_+, (v_{\varepsilon_j \delta_{j'}})_+) \zeta = \int_0^T \int_\Omega G(u) f_1(u,v) \zeta. \tag{6.2.62}$$

To that end, we first establish pointwise a.e. convergence to 0 of the integrand; that is, we prove that

$$\lim_{j \to \infty} \lim_{j' \to \infty} G'_{1\delta_{j'}}(u_{\varepsilon_j \delta_{j'}}) f_1((u_{\varepsilon_j \delta_{j'}})_+, (v_{\varepsilon_j \delta_{j'}})_+) = G(u) f_1(u,v) \tag{6.2.63}$$

a.e. in $\Omega \times (0, \infty)$. We first prove convergence on the set

$$A := \left\{ (x,t) \in \Omega \times (0,\infty) : \lim_{j \to \infty} \lim_{j' \to \infty} u_{\varepsilon_j \delta_{j'}}(x,t) = u(x,t) > 0 \right\}.$$

For $(x,t) \in A$ and arbitrary $\eta \in (0, \frac{u(x,t)}{2})$, there is $j_0 \in \mathbb{N}$ such that for $j \geq j_0$, we can find $j'_0(j) \in \mathbb{N}$ with the property that $|u_{\varepsilon_j \delta_{j'}}(x,t) - u(x,t)| < \eta$ and hence $u_{\varepsilon_j \delta_{j'}}(x,t) > \frac{u(x,t)}{2}$ for all $j' \geq j'_0(j)$ and $j \geq j_0$. Since $\frac{1}{S_1}$ is bounded on $(\frac{u(x,t)}{2}, \infty)$, Lebesgue's theorem gives

$$\begin{aligned}
\lim_{j \to \infty} \lim_{j' \to \infty} G'_{1\delta}(u_{\varepsilon_j \delta_{j'}}(x,t)) &= \lim_{j \to \infty} \lim_{j' \to \infty} \int_0^\infty \frac{\mathbb{1}_{(1, u_{\varepsilon_j \delta_{j'}}(x,t))}(\sigma) - \mathbb{1}_{(u_{\varepsilon_j \delta_{j'}}(x,t), 1)}(\sigma)}{S_1(\sigma) + \delta_j} \,\mathrm{d}\sigma \\
&= \int_0^\infty \frac{\mathbb{1}_{(1, u(x,t))}(\sigma) - \mathbb{1}_{(u(x,t), 1)}(\sigma)}{S_1(\sigma)} \,\mathrm{d}\sigma = G'_1(u(x,t)).
\end{aligned}$$

As f_1 is continuous and $u, v \geq 0$, we thus obtain (6.2.63) for all points in A. Next, we consider points in space-time where u vanishes and set

$$B := \left\{ (x,t) \in \Omega \times (0,\infty) : \lim_{j \to \infty} \lim_{j' \to \infty} u_{\varepsilon_j \delta_{j'}}(x,t) = u(x,t) = 0 \right\}.$$

Similarly as above, we can see that $|u_{\varepsilon_j\delta_{j'}}| < 1$ for sufficiently large $j, j' \in \mathbb{N}$. Since

$$|G'_{1\delta}(u_{\varepsilon\delta})| = \left| \int_1^{u_{\varepsilon\delta}} \frac{1}{S_1(\sigma)+\delta} \, \mathrm{d}\sigma \right| \le \frac{1}{\underline{S}_1} \int_{u_{\varepsilon\delta}}^1 \frac{1}{\sigma} \, \mathrm{d}\sigma = \frac{1}{\underline{S}_1} |\ln(u_{\varepsilon\delta})| \qquad \text{in } \{0 < u_{\varepsilon\delta} \le 1\}$$

for all $\varepsilon, \delta \in (0,1)$, the assumption (6.2.5) and the fact that $f_1((u_{\varepsilon\delta})_+, (v_{\varepsilon\delta})_+) = 0$ in $\{u_{\varepsilon\delta} \le 0\}$ imply that (6.2.63) also holds for points in B. As (6.2.48), (6.2.55) and the nonnegativity of u assert that $(\Omega \times (0,\infty)) \setminus (A \cup B)$ is a null set, we indeed obtain (6.2.63) a.e. in $\Omega \times (0,\infty)$.

Again thanks to (6.2.5), there is $c_1 > 0$ such that

$$\begin{aligned}
&|G'_{1\delta}(u_{\varepsilon\delta}) f_{1\delta}(u_{\varepsilon\delta}, v_{\varepsilon_j\delta_{j'}})\zeta| \\
\le{}& \frac{\|\zeta\|_{L^\infty(\Omega\times(0,T))}}{\underline{S}_1} \Big(|\ln(u_{\varepsilon\delta}) f_1((u_{\varepsilon\delta})_+, (v_{\varepsilon_j\delta_{j'}})_+)| \mathbb{1}_{\{0<u_{\varepsilon\delta}\le 1\}} \\
&+ \|f_1\|_{L^\infty([0,\infty)^2)} (u_{\varepsilon\delta} - 1) \mathbb{1}_{\{1<u_{\varepsilon\delta}\}} \Big) \\
\le{}& c_1(1 + |u_{\varepsilon\delta}|) \qquad \text{in } \Omega \times (0,T) \text{ for all } \varepsilon, \delta \in (0,1)
\end{aligned}$$

so that (6.2.63), Vitali's theorem as well as the bound (6.2.42) assert (6.2.62).

As moreover $0 \le G_{1\delta}(u_{\varepsilon\delta}) \le c_2(1 + u_{\varepsilon\delta}^2)$ in $\Omega \times (0,T))$ for all $\varepsilon, \delta \in (0,1)$ and some $c_2 > 0$ and since $\lim_{j\to\infty} \lim_{j'\to\infty} (1 + u_{\varepsilon_j\delta_{j'}}^2) = (1 + u^2)$ in $L^1(\Omega \times (0,T))$ is contained in (6.2.49) and (6.2.56), Pratt's lemma asserts that

$$\lim_{j\to\infty} \lim_{j'\to\infty} \int_0^T \int_\Omega G_{1\delta_{j'}}(u_{\varepsilon_j\delta_{j'}})\zeta' = \int_0^T \int_\Omega G_1(u)\zeta' \qquad \text{for all } T \in (0,\infty).$$

Likewise, now relying on (6.2.50) and (6.2.57) instead of (6.2.49) and (6.2.56), we also obtain

$$\lim_{j\to\infty} \lim_{j'\to\infty} \int_\Omega G_{1\delta_{j'}}(u_{\varepsilon_j\delta_{j'}}(\cdot,T))\zeta(\cdot,T) = \int_\Omega G_1(u(\cdot,T))\zeta(\cdot,T) \qquad \text{for a.e. } T \in (0,\infty).$$

Finally,

$$G_{1\delta}(u_0) = \int_0^{u_0} \int_0^\rho \frac{1}{S_1(\sigma)+\delta} \, \mathrm{d}\sigma \, \mathrm{d}\rho \to \int_0^{u_0} \int_0^\rho \frac{1}{S_1(\sigma)} \, \mathrm{d}\sigma \, \mathrm{d}\rho = G_1(u_0) \qquad \text{as } \delta \searrow 0$$

by Beppo Levi's theorem so that according to Lebesgue's theorem,

$$\int_\Omega G_{1\delta}(u_0)\zeta(0) \to \int_\Omega G_1(u_0)\zeta(0) \qquad \text{as } \delta \searrow 0.$$

Combined with analogous arguments for the second solution component, these convergences show that (6.2.10) holds for a.e. $T \in (0,\infty)$. Since $u, v \in C^0([0,\infty); L^2(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ by Lemma 6.2.19, the inequality (6.2.10) holds indeed for all $T \in (0,\infty)$. □

Finally, we note that the previous two lemmata already contain the main results of this section.

Proof of Theorem 6.2.1 and Theorem 6.2.2. Theorem 6.2.1 and Theorem 6.2.2 are direct consequences of Lemma 6.2.19 and Lemma 6.2.20, respectively. □

6.3. Approximative solutions to (6.P)

In the remainder of the chapter, we will construct global weak solutions (in the sense of Definition 6.5.1 below) of (6.P). To that end, we henceforth suppose that (6.1.4), either (6.H1) or (6.H2), (6.F1) or (6.F2), (6.1.7), (6.1.10) (with p_i and r_i, $i \in \{1,2\}$, as in (6.1.8) and (6.1.9)) as well as (6.1.12) hold and that D_i, S_i, f_i, $i \in \{1,2\}$ are as in (6.1.3) and (6.1.5).

Sections 6.3–6.5 are organized as follows. In the present section, we will define approximations of D_i, S_i, f_i, $i \in \{1,2\}$ as well as of u_0 and v_0 so that Theorem 6.2.1, which has been proven in the preceding section, becomes applicable and thus provides us with global weak solutions (u_α, v_α), $\alpha \in (0,1)$, to the corresponding approximative problems.

The main part of Section 6.4 then consists of deriving α-independent bounds from the entropy-like inequality given by Theorem 6.2.2. This will then allow us to obtain solution candidates (u,v) of (6.P) in Lemma 6.4.14. Finally, in Section 6.5, we show that under the hypotheses of Theorem 6.1.1, these convergences are sufficiently strong to conclude that (u,v) is indeed a global weak solution of (6.P).

Having an application of Theorem 6.2.1 in mind, we now define approximative functions for each henceforth fixed $\alpha \in (0,1)$. We begin by setting

$$D_{i\alpha}(s) \coloneqq d_i \left(\frac{s+1}{1+\alpha(s+1)}\right)^{m_i-1} + \alpha \quad \text{and} \quad S_{i\alpha}(s) \coloneqq \frac{\chi_i s(s+1)^{q_i-1}}{(1+\alpha(s+1))^{q_i}}$$

for $s \ge 0$ and $i \in \{1,2\}$.

We also fix $\xi \in C^\infty(\mathbb{R})$ with $\xi(s) = 1$ for $s \le 0$ and $\xi(s) = 0$ for $s \ge 1$ and set

$$f_{i\alpha}(s_1,s_2) \coloneqq f_i(s_1,s_2)\xi_{1\alpha}(s_1)\xi_{2\alpha}(s_2)$$

where $\xi_{i\alpha}(s) \coloneqq \xi(\alpha^{\frac{1}{4-\min\{q_1,q_2\}}} s - 1)$ for $s \in \mathbb{R}$; in particular,

$$\xi_{i\alpha}(s) = \begin{cases} 1, & s \le \alpha^{-\frac{1}{4-\min\{q_1,q_2\}}}, \\ 0, & s \ge 2\alpha^{-\frac{1}{4-\min\{q_1,q_2\}}} \end{cases} \qquad \text{for all } \alpha \in (0,1) \text{ and } i \in \{1,2\}. \tag{6.3.1}$$

As a last yet undefined component, let us construct initial data $u_{0\alpha}, v_{0\alpha}$ approximating u_0, v_0 in a suitable sense as $\alpha \searrow 0$.

Lemma 6.3.1. *There are families $(u_{0\alpha})_{\alpha\in(0,1)}, (v_{0\alpha})_{\alpha\in(0,1)} \subset C^\infty(\overline{\Omega})$ such that $u_{0\alpha} > 0$ and $v_{0\alpha} > 0$ in $\overline{\Omega}$ for all $\alpha \in (0,1)$, $(\int_\Omega u_0)(\int_\Omega u_{0\alpha}) = (\int_\Omega u_0)^2$ and $(\int_\Omega v_0)(\int_\Omega v_{0\alpha}) = (\int_\Omega v_0)^2$ for all $\alpha \in (0,1)$,*

$$(u_{0\alpha}, v_{0\alpha}) \to (u_0, v_0) \qquad \text{a.e. and in } X_1 \times X_2 \text{ as } \alpha \searrow 0, \tag{6.3.2}$$

where $X_i \coloneqq L^{2-q_i}(\Omega)$ if $q_i < 1$ and $X_i \coloneqq L\log L(\Omega)$ if $q_i = 1$ for $i \in \{1,2\}$, as well as

$$\lim_{\alpha\searrow 0} \alpha\|u_{0\alpha}\|^p_{L^p(\Omega)} = 0 \quad \text{and} \quad \lim_{\alpha\searrow 0} \alpha\|v_{0\alpha}\|^p_{L^p(\Omega)} = 0, \tag{6.3.3}$$

where $p \coloneqq 3 - \min\{q_1,q_2\}$.

Proof. As $C^\infty(\overline{\Omega})$ is dense in X_1 (cf. [1, Theorem 8.21] for $X_1 = L\log L(\Omega)$), and since u_0 belongs to X_1 and is nonnegative by (6.1.12), there is a sequence of nonnegative functions $(\tilde{u}_{0j})_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ with $\tilde{u}_{0j} \to u_0$ in X_1 as $j \to \infty$. Since we may without loss of generality assume that $u_0 \not\equiv 0$, $\gamma_j := (\int_\Omega u_0)(\int_\Omega(\tilde{u}_{0j}+\frac{1}{j}))^{-1}$ is positive for all $j \in \mathbb{N}$ so that the functions $u_{0j} := \gamma_j(\tilde{u}_{0j} + \frac{1}{j})$ not only fulfill $u_{0j} \to u_0$ in X_1 as $j \to \infty$ but also $\int_\Omega u_{0j} = \int_\Omega u_0$ and $u_{0j} \ge \frac{\gamma_j}{j} > 0$ for $j \in \mathbb{N}$. Since $X_1 \hookrightarrow L^1(\Omega)$, after switching to a subsequence if necessary, we may without loss of generality also assume that $\tilde{u}_{0j} \to u_0$ a.e. as $j \to \infty$.

For $\alpha \in (0,1)$, we observe then that

$$A_\alpha := \left\{ j \in \mathbb{N} : j \le \frac{1}{\alpha} \text{ and } \|u_{0j}\|_{L^p(\Omega)}^{p+1} \le \frac{1}{\alpha} \right\} \cup \{1\}$$

is nonempty and finite, so that

$$j_\alpha := \max A_\alpha \quad \text{and} \quad u_{0\alpha} := u_{0j_\alpha}, \qquad \alpha \in (0,1),$$

are well-defined. Because $j_\alpha \to \infty$ as $\alpha \searrow 0$ and $\alpha\|u_{0j_\alpha}\|_{L^p(\Omega)}^p \le \alpha^{1-\frac{p}{p+1}}$ for all $\alpha \in (0,1)$ with $j_\alpha > 1$, we obtain the statement given an analogous definition of and argumentation for $(v_{0\alpha})_{\alpha\in(0,1)}$. □

With these preparations at hand, we are now able to apply Theorem 6.2.1 to obtain global weak $W^{1,2}$-solutions of certain approximative problems.

Lemma 6.3.2. *Let $\alpha \in (0,1)$, $D_{i\alpha}, S_{i\alpha}, f_{i\alpha}$, $i \in \{1,2\}$ be as defined above and $u_{0\alpha}, v_{0\alpha}$ be as given by Lemma 6.3.1. Then there exists a global nonnegative weak $W^{1,2}$-solution (in the sense of Theorem 6.2.1) $(u_\alpha, v_\alpha) \in (W^{1,2}_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)))^2$ to*

$$\begin{cases} u_{\alpha t} = \nabla\cdot(D_{1\alpha}(u_\alpha)\nabla u_\alpha - S_{1\alpha}(u_\alpha)\nabla v_\alpha) + f_{1\alpha}(u_\alpha, v_\alpha) & \text{in } \Omega\times(0,\infty) \\ v_{\alpha t} = \nabla\cdot(D_{2\alpha}(v_\alpha)\nabla v_\alpha + S_{2\alpha}(v_\alpha)\nabla u_\alpha) + f_{2\alpha}(u_\alpha, v_\alpha) & \text{in } \Omega\times(0,\infty) \\ \partial_\nu u_\alpha = \partial_\nu v_\alpha = 0 & \text{on } \partial\Omega\times(0,\infty) \\ u_\alpha(\cdot,0) = u_{0\alpha}, v_\alpha(\cdot,0) = v_{0\alpha} & \text{in } \Omega. \end{cases} \tag{6.3.4}$$

Setting

$$G_{i\alpha}(s) := \int_1^s \int_1^\rho \frac{1}{S_{i\alpha}(\sigma)}\,\mathrm{d}\sigma\,\mathrm{d}\rho \qquad \text{for } s \ge 0 \text{ and } i \in \{1,2\}, \tag{6.3.5}$$

this solution moreover satisfies

$$\begin{aligned} &\int_\Omega G_{1\alpha}(u_\alpha(\cdot,T)) + \int_\Omega G_{2\alpha}(v_\alpha(\cdot,T)) + \int_0^T\int_\Omega \frac{D_{1\alpha}(u_\alpha)}{S_{1\alpha}(u_\alpha)}|\nabla u_\alpha|^2 + \int_0^T\int_\Omega \frac{D_{2\alpha}(u_\alpha)}{S_{2\alpha}(u_\alpha)}|\nabla v_\alpha|^2 \\ \le &\int_\Omega G_{1\alpha}(u_{0\alpha}) + \int_\Omega G_{2\alpha}(v_{0\alpha}) + \int_0^T\int_\Omega G'_{1\alpha}(u_\alpha)f_{1\alpha}(u_\alpha,v_\alpha) + \int_0^T\int_\Omega G'_{2\alpha}(v_\alpha)f_{2\alpha}(u_\alpha,v_\alpha) \end{aligned} \tag{6.3.6}$$

for all $T \in (0,\infty)$.

Proof. As $u_{0\alpha}, v_{0\alpha}$ belong to $C^\infty(\overline{\Omega})$ and are positive in $\overline{\Omega}$ by Lemma 6.3.1, the statement follows from Theorem 6.2.1 and Theorem 6.2.2 (with $\zeta \equiv 1$) once we have shown that (6.2.1)–(6.2.5) hold for D_i, S_i, f_i replaced by $D_{i\alpha}, S_{i\alpha}, f_{i\alpha}$, $i \in \{1,2\}$.

Indeed, by definition $D_{i\alpha}, S_{i\alpha}$ belong to $C^\infty([0,\infty))$ with

$$\alpha \le D_{i\alpha}(s) \le d_i \begin{cases} \alpha^{1-m_i} + \alpha, & m_i > 1, \\ (\frac{1}{1+\alpha})^{m_i-1} + \alpha, & m_i \le 1 \end{cases} \quad \text{and} \quad 0 \le S_{i\alpha}(s) \le \chi_i \begin{cases} \alpha^{-q_i}, & q_i > 0, \\ (\frac{1}{1+\alpha})^{q_i}, & q_i \le 0 \end{cases}$$

as well as

$$\begin{aligned} \frac{|S_{i\alpha}'(s)|}{\chi_i} &\le \frac{(1+|q_i-1|)(s+1)^{q_i-1}}{(1+\alpha(s+1))^{q_i}} + \frac{|q_i|\alpha(s+1)^{q_i}}{(1+\alpha(s+1))^{q_i+1}} \\ &\le (1+|q_i-1|+|q_i|\alpha) \begin{cases} \alpha^{-q_i}, & q_i > 0, \\ (\frac{1}{1+\alpha})^{q_i}, & q_i \le 0 \end{cases} \end{aligned}$$

for $s \ge 0$ and $i \in \{1,2\}$. Also, for $i \in \{1,2\}$, the function

$$[0,1] \ni s \mapsto \frac{S_{i\alpha}(s)}{s} = \frac{\chi_i (s+1)^{q_i-1}}{(1+\alpha(s+1))^{q_i}}$$

is continuous and positive and, as $s \ge \frac{s+1}{2}$ for all $s \ge 1$,

$$\inf_{s\ge1} S_{i\alpha}(s) \ge \frac{\chi_i}{2} \inf_{s\ge1} \left(\frac{s+1}{1+\alpha(s+1)} \right)^{q_i} \ge \frac{\chi_i}{2} \begin{cases} (\frac{2}{1+2\alpha})^{q_i}, & q_i > 0, \\ \alpha^{-q_i}, & q_i \le 0 \end{cases} \qquad \text{for } i \in \{1,2\}.$$

That is, (6.2.1), (6.2.2) and (6.2.4) hold.

As $f_{i\alpha}$ is continuous with $\operatorname{supp} f_{i\alpha} \subset [0, 2\alpha^{-\frac{1}{4-\min\{q_1,q_2\}}}]^2 =: K$, $\|f_{i\alpha}\|_{L^\infty((0,\infty)^2)}$ equals $\|f_{i\alpha}\|_{C^0(K)}$ and is thus finite, implying that (6.2.3) is fulfilled for $i \in \{1,2\}$. Moreover, the definitions of f_1 and $f_{1\alpha}$ entail that $[0,\infty)^2 \ni (s_1,s_2) \mapsto \frac{f_{1\alpha}(s_1,s_2)}{s_1}$ is also continuous and supported in K, implying $\lim_{s_1 \searrow 0} \sup_{s_2 \ge 0} |f_{1\alpha}(s_1,s_2) \ln s_1| = 0$. The second statement in (6.2.5) follows analogously. □

6.4. The limit process $\alpha \searrow 0$: obtaining solution candidates

Apart from assumptions made at the beginning of the preceding section, throughout this section, for $\alpha \in (0,1)$, we also let $D_{i\alpha}, S_{i\alpha}, \xi_{i\alpha}$, $i \in \{1,2\}$, as introduced in Section 6.3, $u_{0\alpha}, v_{0\alpha}$ as well as u_α, v_α as given by Lemma 6.3.1 and Lemma 6.3.2, respectively, and $G_{i\alpha}$, $i \in \{1,2\}$, as in (6.3.5).

In order to prepare taking the limit $\alpha \searrow 0$, we collect several a priori estimates. As already alluded to in the introduction of this chapter, the main ingredient will be an entropy-like inequality; that is, we will heavily rely on (6.3.6).

6.4.1. Preliminary observations

To streamline later arguments, in this subsection, we first collect several elementary statements regarding the parameters and nonlinearities involved in Theorem 6.1.1.

Lemma 6.4.1. *Set $\beta_i := m_i - q_i - 1$ for $i \in \{1,2\}$. Then the inequalities*

$$2\left(m_i - 1 - \frac{\beta_i}{2}\right) < p_i, \quad \beta_i > -2 \quad \textit{and} \quad p_i > 0 \tag{6.4.1}$$

hold.

Proof. Recalling that $p_i \ge \beta_i + 2 + \frac{2(2-q_i)}{n}$ by (6.1.8) and $q_i \le 1 < 2$ by (6.1.4), we have

$$2\left(m_i - 1 - \frac{\beta_i}{2}\right) = \beta_i + 2q_i < \beta_i + 2q_i + \frac{2(2-q_i)}{n} \le p_i \qquad \text{for } i \in \{1,2\},$$

which shows that the first inequality in (6.4.1) is fulfilled. The second one therein is equivalent to the assumption (6.1.7), upon which the third one follows by the definition of p_i as $q_i < 2$. □

As further preparation, we estimate the functions $G_{i\alpha}$ defined in (6.3.5) and their derivatives both from above and from below.

Lemma 6.4.2. *Set*

$$L_q(s) := \begin{cases} 1, & q < 1, \\ \ln s, & q = 1 \end{cases} \qquad \textit{for } s \ge 0 \textit{ and } q \le 1. \tag{6.4.2}$$

and let $G_{i\alpha}$ be as in (6.3.5) for $i \in \{1,2\}$ and $\alpha \in (0,1)$. Then there are $C_1, C_2, C_3, C_4, > 0$ such that

$$G_{i\alpha}(s) \begin{cases} \ge C_1 \left(\frac{s+1}{1+\alpha(s+1)}\right)^{2-q_i} L_{q_i}(s+\mathrm{e}) - C_2 \\ \le C_2 s^{2-q_i} L_{q_i}(s) + C_2 \alpha s^{3-q_i} + C_2 \end{cases} \qquad \textit{for } s \ge 0, \tag{6.4.3}$$

$$G'_{i\alpha}(s) \begin{cases} \ge C_3 \ln s \\ \le 0 \end{cases} \qquad \textit{for } s \in (0,1) \quad \textit{and} \tag{6.4.4}$$

$$G'_{i\alpha}(s) \begin{cases} \ge G'_i(s) - C_4 \alpha s^{2-q_i} \\ \le G'_i(s) + C_4 \alpha s^{2-q_i} \end{cases} \qquad \textit{for } s \ge 1 \tag{6.4.5}$$

for $\alpha \in (0,1)$ and $i \in \{1,2\}$.

Proof. We fix $i \in \{1,2\}$. Since

$$\left| \frac{\partial}{\partial \alpha} [1 + \alpha(s+1)]^{q_i} \right| = |q_i| [1 + \alpha(s+1)]^{q_i - 1} (s+1) \le |q_i|(s+1)$$

for all $s \ge 0$ and $\alpha \in (0,1)$, the mean value theorem implies that

$$\begin{aligned} \chi_i |G'_{i\alpha}(s) - G'_i(s)| &= \operatorname{sign}(s-1) \int_1^s \frac{[1+\alpha(\sigma+1)]^{q_i} - 1}{\sigma(\sigma+1)^{q_i - 1}} \,\mathrm{d}\sigma \\ &\le \alpha |q_i| \operatorname{sign}(1-s) \int_s^1 \frac{(\sigma+1)^{2-q_i}}{\sigma} \,\mathrm{d}\sigma \end{aligned}$$

for all $s > 0$ and $\alpha \in (0,1)$. Estimating here $\sigma + 1 \le 2$ and $\sigma + 1 \le 2\sigma$ for $\sigma \in (0,1)$ and $\sigma \ge 1$, respectively, for all $\alpha \in (0,1)$, we obtain

$$\chi_i |G'_{i\alpha}(s) - G'_i(s)| \le 2^{2-q_i} \alpha |q_i| \int_s^1 \frac{1}{\sigma} \,\mathrm{d}\sigma = 2^{2-q_i} \alpha |q_i| |\ln s| \qquad \text{for all } s \in (0,1)$$

and

$$\chi_i |G'_{i\alpha}(s) - G'_i(s)| \le 2^{2-q_i}\alpha|q_i| \int_1^s \sigma^{1-q_i}\,\mathrm{d}\sigma \le \frac{2^{2-q_i}\alpha|q_i|}{2-q_i} s^{2-q_i} \qquad \text{for all } s \ge 1.$$

As moreover $G'_{i\alpha}(s) \le 0$ for $s \in (0,1)$ and $\alpha \in (0,1)$ and

$$\chi_i |G'_i(s)| = \int_s^1 \frac{(\sigma+1)^{1-q_i}}{\sigma}\,\mathrm{d}\sigma \le 2^{1-q_i}|\ln s| \qquad \text{for all } s \in (0,1) \text{ and } \alpha \in (0,1),$$

consequences thereof are (6.4.4) and (6.4.5) for a certain $C_3, C_4 > 0$.

Furthermore, again making use of the fact that $s+1 \le 2s$ for $s \ge 1$, a direct computation shows that

$$\begin{aligned}
\chi_i G_i(s) &= \int_1^s \int_1^\rho \frac{(\sigma+1)^{1-q_i}}{\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\rho \\
&\le 2^{1-q_i} \int_1^s \int_1^\rho \sigma^{-q_i}\,\mathrm{d}\sigma\,\mathrm{d}\rho \\
&\le \frac{2^{1-q_i}}{1-q_i \mathbb{1}_{\{q_i<1\}}} \int_1^s \rho^{1-q_i} L_{q_i}(\rho)\,\mathrm{d}\rho \\
&\le \frac{2^{1-q_i}}{(2-q_i)(1-q_i\mathbb{1}_{\{q_i<1\}})} s^{2-q_i} L_{q_i}(s) \qquad \text{for } s \ge 1.
\end{aligned} \tag{6.4.6}$$

In a similar vein, we obtain $c_1, c_2 > 0$ such that

$$\begin{aligned}
\chi_i G_{i\alpha}(s) &= \int_1^s \int_1^\rho \frac{(1+\alpha(\sigma+1))^{q_i}}{\sigma(\sigma+1)^{q_i-1}}\,\mathrm{d}\sigma\,\mathrm{d}\rho \\
&\ge (1+\alpha(s+1))^{\min\{q_i,0\}} \int_1^s \int_1^\rho \sigma^{-q_i}\,\mathrm{d}\sigma\,\mathrm{d}\rho \\
&\ge \frac{c_1 s^{2-q_i} L_{q_i}(s)}{(1+\alpha(s+1))^{\max\{-q_i,0\}}} - c_2 s \\
&\ge \frac{c_1 s^{2-q_i} L_{q_i}(s+\mathrm{e})}{(1+\alpha(s+1))^{\max\{-q_i,0\}}} - \frac{c_1 \ln(1+\mathrm{e}) \mathbb{1}_{\{q_i<1\}} s^{2-q_i}}{(1+\alpha(s+1))^{\max\{-q_i,0\}}} - c_2 s
\end{aligned}$$

for $s \ge 1$ and $\alpha \in (0,1)$, where in the last step we have made use of the fact that $\ln(s+\mathrm{e}) - \ln s = \ln \frac{s+\mathrm{e}}{s} \le \ln(1+\mathrm{e})$ for $s \ge 1$. Since the first term on the right-hand side herein grows faster than the other two, there is moreover $c_3 > 0$ such that

$$\begin{aligned}
\chi_i G_{i\alpha}(s) &\ge \frac{c_1 s^{2-q_i} L_{q_i}(s+\mathrm{e})}{2(1+\alpha(s+1))^{\max\{-q_i,0\}}} - c_3 \\
&\ge \frac{c_1}{2}\left(\frac{s}{1+\alpha(s+1)}\right)^{2-q_i} L_{q_i}(s+\mathrm{e}) - c_3 \\
&\ge \frac{c_1}{2^{3-q_i}}\left(\frac{s+1}{1+\alpha(s+1)}\right)^{2-q_i} L_{q_i}(s+\mathrm{e}) - c_3 \qquad \text{for } s \ge 1 \text{ and } \alpha \in (0,1),
\end{aligned}$$

which, when combined with (6.4.4)–(6.4.6), implies the existence of $C_1, C_2 > 0$ such that (6.4.3) holds for all $s \ge 1$ and $\alpha \in (0,1)$.

Finally, by integrating (6.4.4), we see that there is $c_4 > 0$ such that $-c_4 \le G_{i\alpha}(s) \le c_4$ for all $s \in [0,1)$ and $\alpha \in (0,1)$ so that, possibly after enlarging C_1 and C_2, (6.4.3) is indeed valid for all $s \ge 0$ and $\alpha \in (0,1)$. □

The estimates obtained in Lemma 6.4.2 and the definitions of $D_{i\alpha}$ and $S_{i\alpha}$ now allow us to infer the following from the entropy-like inequality (6.3.6).

Lemma 6.4.3. *Let $T \in (0, \infty)$. Then there exists $C_1, C_2 > 0$ such that*

$$\begin{aligned} &C_1 \int_\Omega B_\alpha^{2-q_1}(u_\alpha(\cdot,t)) L_{q_1}(u_\alpha(\cdot,t) + \mathrm{e}) + C_1 \int_\Omega B_\alpha^{2-q_2}(v_\alpha(\cdot,t)) L_{q_2}(v_\alpha(\cdot,t) + \mathrm{e}) \\ &+ \frac{d_1}{\chi_1} \int_0^t \int_\Omega B_\alpha^{m_1-q_1-1}(u_\alpha) |\nabla u_\alpha|^2 + \frac{d_2}{\chi_2} \int_0^t \int_\Omega B_\alpha^{m_2-q_2-1}(v_\alpha) |\nabla v_\alpha|^2 \\ \leq\ & C_2 + \int_0^t \int_\Omega G_{1\alpha}'(u_\alpha) f_{1\alpha}(u_\alpha, v_\alpha) + \int_0^t \int_\Omega G_{2\alpha}'(v_\alpha) f_{2\alpha}(u_\alpha, v_\alpha) \end{aligned} \tag{6.4.7}$$

for all $t \in (0, T)$ and all $\alpha \in (0, 1)$, where L_{q_i} is as in (6.4.2) *and*

$$B_\alpha(s) \coloneqq \frac{s+1}{1+\alpha(s+1)}, \qquad s \geq 0, \alpha \in (0,1). \tag{6.4.8}$$

Proof. As according to (6.3.2) and (6.3.3), there is $c_1 > 0$ such that

$$\int_\Omega u_{0\alpha}^{2-q_1} L_{q_1}(u_{0\alpha}) + \alpha \int_\Omega u_{0\alpha}^{3-q_1} + \int_\Omega v_{0\alpha}^{2-q_2} L_{q_2}(v_{0\alpha}) + \alpha \int_\Omega v_{0\alpha}^{3-q_2} \leq c_1 \qquad \text{for all } \alpha \in (0,1),$$

an application of (6.4.3) gives $c_2 > 0$ such that

$$\int_\Omega G_{1\alpha}(u_{0\alpha}) + \int_\Omega G_{2\alpha}(v_{0\alpha}) \leq c_2 \qquad \text{for all } \alpha \in (0,1),$$

Moreover,

$$D_{i\alpha}(s) \geq d_i B_\alpha^{m_i-1}(s) \quad \text{and} \quad S_{i\alpha}(s) \leq \chi_i B_\alpha^{q_i}(s)$$

and hence $\frac{D_{i\alpha}(s)}{S_{i\alpha}(s)} \geq \frac{d_i}{\chi_i} B_\alpha^{m_i-q_i-1}(s)$ for $s \geq 0$, $\alpha \in (0,1)$ and $i \in \{1,2\}$. Also making use of the first inequality in (6.4.3), we can then infer (6.4.7) from (6.3.6) for certain $C_1, C_2 > 0$. ☐

6.4.2. Controlling the right-hand side of (6.4.7)

In order to obtain α-independent a priori estimates from (6.4.7), we need to obtain an upper bound for the terms on the right-hand side therein. Restricted to the set where u_α and v_α are at least 1, we will bound the corresponding integrand using one of the assumptions (6.F1) and (6.F2). This is complemented by the following observation essentially showing we may indeed focus on that regime.

Lemma 6.4.4. *There is $C > 0$ such that*

$$\begin{aligned} &G_{1\alpha}'(u_\alpha) f_{1\alpha}(u_\alpha, v_\alpha) + G_{2\alpha}'(v_\alpha) f_{2\alpha}(u_\alpha, v_\alpha) \\ &\leq C + \left(G_1'(u_\alpha) f_1(u_\alpha, v_\alpha) + G_2'(v_\alpha) f_2(u_\alpha, v_\alpha)\right) \xi_{1\alpha}(u_\alpha) \xi_{2\alpha}(v_\alpha) \mathbb{1}_{\{u_\alpha \geq 1\} \cap \{v_\alpha \geq 1\}} \end{aligned} \tag{6.4.9}$$

a.e. in $\Omega \times (0, \infty)$ for all $\alpha \in (0, 1)$.

Proof. For $\alpha \in (0,1)$, we fix representatives of u_α and v_α in $L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ so that sets such as $\{u_\alpha < 1\}$ or $\{v_\alpha < 1\}$ are well-defined.

According to (6.4.4), there is $c_1 > 0$ such that

$$c_1 \ln s \le G'_{i\alpha}(s) \le 0 \qquad \text{for all } s \in (0,1),\ \alpha \in (0,1) \text{ and } i \in \{1,2\}.$$

Recalling the definition of $f_{i\alpha}$ and that u_α, v_α are nonnegative, this implies

$$\begin{aligned} G'_{1\alpha}(u_\alpha) f_{1\alpha}(u_\alpha, v_\alpha) &\le c_1 |\ln u_\alpha| \mu_1 u_\alpha^2 \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) && \text{in } \{u_\alpha < 1\} \quad \text{and} \\ G'_{2\alpha}(v_\alpha) f_{2\alpha}(u_\alpha, v_\alpha) &\le c_1 |\ln v_\alpha| (\mu_2 v_\alpha^2 + a_2 u_\alpha v_\alpha) \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) && \text{in } \{v_\alpha < 1\} \end{aligned}$$

for all $\alpha \in (0,1)$. Since $(0,1) \ni s \mapsto s \ln s$ is bounded, there is $c_2 > 0$ such that

$$G'_{1\alpha}(u_\alpha) f_{1\alpha}(u_\alpha, v_\alpha) \le c_2 \qquad \text{in } \{u_\alpha < 1\} \quad \text{and} \tag{6.4.10}$$

$$G'_{2\alpha}(v_\alpha) f_{2\alpha}(u_\alpha, v_\alpha) \le c_2 + c_2 u_\alpha \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \qquad \text{in } \{v_\alpha < 1\}. \tag{6.4.11}$$

for all $\alpha \in (0,1)$ and thus (6.4.9) holds on the set $\{u_\alpha < 1\} \cap \{v_\alpha < 1\}$ for some $C > 0$.

Moreover, by (6.4.5), there is $c_3 > 0$ such that

$$|G'_{i\alpha}(s) - G'_i(s)| \le c_3 \alpha s^{2-q_i} \qquad \text{for all } s \ge 1,\ \alpha \in (0,1) \text{ and } i \in \{1,2\}.$$

As (6.3.1) entails that u_α and v_α are bounded by $2\alpha^{-\frac{1}{4-\min\{q_1,q_2\}}}$ on $\operatorname{supp} \xi_{1\alpha}(u_\alpha)$ and $\operatorname{supp} \xi_{2\alpha}(v_\alpha)$, respectively, and hence

$$\begin{aligned} \alpha u_\alpha^{2-q_1} |f_{1\alpha}(u_\alpha, v_\alpha)| &\le \alpha u_\alpha^{2-q_1} (\lambda_1 u_\alpha + \mu_1 u_\alpha^2 + a_1 u_\alpha v_\alpha) \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \\ &\le 2^{4-q_1}(\lambda_1 + \mu_1 + a_1) =: c_4 \qquad \text{in } \{u_\alpha \ge 1\} \text{ for all } \alpha \in (0,1), \end{aligned}$$

we can conclude

$$G'_{1\alpha}(u_\alpha) f_{1\alpha}(u_\alpha, v_\alpha) \le G'_1(u_\alpha) f_1(u_\alpha, v_\alpha) \xi_\alpha(v_\alpha)\xi_\alpha(v_\alpha) + c_3 c_4 \qquad \text{in } \{u_\alpha \ge 1\} \tag{6.4.12}$$

for all $\alpha \in (0,1)$. Likewise, there is $c_5 > 0$ such that

$$G'_{2\alpha}(v_\alpha) f_{2\alpha}(u_\alpha, v_\alpha) \le G'_2(v_\alpha) f_2(u_\alpha, v_\alpha) \xi_\alpha(v_\alpha)\xi_\alpha(v_\alpha) + c_3 c_5 \qquad \text{in } \{v_\alpha \ge 1\} \tag{6.4.13}$$

for all $\alpha \in (0,1)$. Therefore, after enlarging C if necessary, (6.4.9) holds also in the regime $\{u_\alpha \ge 1\} \cap \{v_\alpha \ge 1\}$.

Furthermore,

$$f_1(u_\alpha, v_\alpha) \le u_\alpha \left(\lambda_1 - \frac{\mu_1}{2} u_\alpha + a_1\right) - \frac{\mu_1}{2} u_\alpha^2 \le -\frac{\mu_1}{2} u_\alpha^2 \qquad \text{in } \left\{\frac{u_\alpha}{2} \ge \frac{\lambda_1 + a_1}{\mu_1}\right\} \cap \{v_\alpha < 1\}$$

for all $\alpha \in (0,1)$ so that since $G'_1(s) \ge 0$ for $s \ge 1$, we have

$$G'_1(u_\alpha) f_1(u_\alpha, v_\alpha) \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \le c_6 - \frac{\mu_1}{2} u_\alpha^2 \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \qquad \text{in } \{u_\alpha \ge 1\} \cap \{v_\alpha < 1\}$$

for all $\alpha \in (0,1)$, wherein $c_6 := \|G'_1 f_1(\cdot, 1)\|_{L^\infty(0, \frac{2(\lambda_1 + a_1)}{\mu_1})}$ is finite as $G'_1 f_1(\cdot, 1)$ is continuous on $[0,\infty)$. Combined with (6.4.11) and (6.4.12), and possibly after further enlarging C, this shows that (6.4.9) holds also on the set $\{u_\alpha \ge 1\} \cap \{v_\alpha < 1\}$,

Finally, for the remaining subset $\{u_\alpha < 1\} \cap \{v_\alpha \ge 1\}$ of $\Omega \times (0,\infty)$, we can argue similarly as above. □

If (6.F1) holds, the preceding lemma immediately allows us to bound the integrands on the right-hand side of (6.4.7).

Lemma 6.4.5. *Let $T \in (0,\infty)$ and suppose that* (6.F1) *holds. Then we can find $C_1, C_2 > 0$ such that*

$$\int_\Omega G'_{1\alpha}(u_\alpha) f_{1\alpha}(u_\alpha, v_\alpha) + \int_\Omega G'_{2\alpha}(v_\alpha) f_{2\alpha}(u_\alpha, v_\alpha) \\ \le C_1 - C_2 \int_0^T \int_\Omega u_\alpha^2 \ln u_\alpha - C_2 \int_0^T \int_\Omega v_\alpha^2 \ln v_\alpha \tag{6.4.14}$$

a.e. in $\Omega \times (0,\infty)$ for all $\alpha \in (0,1)$.

Proof. This directly follows from combining (6.F1), (6.4.7) and (6.4.9). □

In the majority of the remainder of this subsection, we will show that (6.4.14) also holds if we assume (6.F2) instead of (6.F1). To that end, we may assume that (6.H2) holds since the right-hand side of (6.4.7) is trivially bounded in the case of (6.H1). The key ingredient to the corresponding proof will be the Gagliardo–Nirenberg inequality whose application we prepare by obtaining locally uniform-in-time $L^1(\Omega)$ bounds in the following

Lemma 6.4.6. *Let $T \in (0,\infty)$ and suppose that* (6.H2) *holds. There is $C > 0$ such that*

$$\int_\Omega u_\alpha(\cdot,t) + \int_\Omega v_\alpha(\cdot,t) \le C \qquad \text{for all } t \in (0,T) \text{ and } \alpha \in (0,1). \tag{6.4.15}$$

Proof. Testing the first equation in (6.3.4) with the constant function $a_2 > 0$, recalling the definition of $f_{1\alpha}$ and applying Young's inequality give

$$\begin{aligned} &a_2 \int_\Omega u_\alpha(\cdot,t) - a_2 \int_\Omega u_{0\alpha} = a_2 \int_0^t \int_\Omega u_{\alpha t} \\ &= a_2\lambda_1 \int_0^t \int_\Omega u_\alpha \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) - a_2\mu_1 \int_0^t \int_\Omega u_\alpha^2 \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \\ &\quad + a_1a_2 \int_0^t \int_\Omega u_\alpha v_\alpha \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \\ &\le \frac{a_2\lambda_1^2}{4\mu_1}|\Omega|T + a_1a_2 \int_0^t \int_\Omega u_\alpha v_\alpha \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \qquad \text{for } t \in (0,T) \text{ and } \alpha \in (0,1). \end{aligned}$$

As likewise

$$a_1 \int_\Omega v_\alpha(\cdot,t) - a_1 \int_\Omega v_{0\alpha} \le \frac{a_1\lambda_2^2}{4\mu_2}|\Omega|T - a_1a_2 \int_0^t \int_\Omega u_\alpha v_\alpha \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha)$$

for $t \in (0,T)$ and $\alpha \in (0,1)$, we conclude

$$a_2 \int_\Omega u_\alpha(\cdot,t) + a_1 \int_\Omega v_\alpha(\cdot,t) \le a_2 \int_\Omega u_{0\alpha} + a_1 \int_\Omega u_{0\alpha} + \frac{a_2\lambda_1^2}{4\mu_1}|\Omega|T + \frac{a_1\lambda_2^2}{4\mu_2}|\Omega|T$$

for $t \in (0,T)$ and $\alpha \in (0,1)$. In view of (6.3.2), this implies (6.4.15) for a certain $C > 0$. □

Lemma 6.4.7. *Let $T \in (0,\infty)$, $\eta > 0$, $\beta_i := m_i - q_i - 1$ for $i \in \{1,2\}$ and suppose that* (6.H2) *holds. For $p \in (0, \frac{(\beta_1+2)n+2}{n})$, there is $C_1 > 0$ such that*

$$\int_\Omega u_\alpha^p(\cdot,t)\xi_{1\alpha}(u_\alpha(\cdot,t)) \le \eta \int_\Omega B_\alpha^{\beta_1}(u_\alpha(\cdot,t))|\nabla u_\alpha(\cdot,t)|^2 + C_1 \tag{6.4.16}$$

for all $t \in (0,T)$ *and* $\alpha \in (0,1)$ *and, for* $p \in (0, \frac{(\beta_2+2)n+2}{n})$*, there is* $C_2 > 0$ *such that*

$$\int_\Omega v_\alpha^p(\cdot,t)\xi_{2\alpha}(v_\alpha(\cdot,t)) \le \eta \int_\Omega B_\alpha^{\beta_2}(v_\alpha(\cdot,t))|\nabla v_\alpha(\cdot,t)|^2 + C_2 \tag{6.4.17}$$

for all $t \in (0,T)$ *and* $\alpha \in (0,1)$*, where* B_α *is as in* (6.4.8)*; that is,* $B_\alpha(s) = \frac{s+1}{1+\alpha(s+1)}$ *for* $s \ge 0$ *and* $\alpha \in (0,1)$*.*

Proof. As $B_\alpha(s) \le s+1$ for all $s \ge 0$ and $\alpha \in (0,1)$, Lemma 6.4.6 allows us to fix $c_1 > 0$ such that

$$\int_\Omega B_\alpha(u_\alpha(\cdot,t)) + \int_\Omega B_\alpha(v_\alpha(\cdot,t)) \le c_1 \qquad \text{for all } t \in (0,T) \text{ and } \alpha \in (0,1).$$

The definitions $\tilde p_1 := \frac{2((\beta_1+2)n+2)}{(\beta_1+2)n}$ and $\tilde q_1 := \frac{2}{\beta_1+2}$ imply

$$\begin{aligned} b &:= \frac{\frac{1}{\tilde q_1} - \frac{1}{\tilde p_1}}{\frac{1}{\tilde q_1} - \frac{n-2}{2n}} = \frac{(\beta_1+2)((\beta_1+2)n+2)n - (\beta_1+2)n^2}{(\beta_1+2)((\beta_1+2)n+2)n - (n-2)((\beta_1+2)n+2)} \\ &= \frac{(\beta_1+2)n((\beta_1+1)n+2)}{((\beta_1+1)n+2))((\beta_1+2)n+2)} = \frac{(\beta_1+2)n}{(\beta_1+2)n+2} \in (0,1). \end{aligned}$$

Since $\frac{\tilde p_1 b}{2} = 1$, an application of the Gagliardo–Nirenberg inequality (cf. [58, Lemma 2.3] for a version allowing for merely positive $\tilde q_1$) gives $c_2 > 0$ such that

$$\int_\Omega \varphi^{\tilde p_1} \le c_2 \left(\int_\Omega |\nabla\varphi|^2\right)\left(\int_\Omega \varphi^{\frac{2}{\beta_1+2}}\right)^{\frac{\tilde p_1(1-b)}{\tilde q_1}} + c_2\left(\int_\Omega \varphi^{\frac{2}{\beta_1+2}}\right)^{\frac{\tilde p_1}{\tilde q_1}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Thus, setting $c_3 := \frac{(\beta+1)^2}{4} c_1^{\frac{\tilde p_1(1-b)}{\tilde q_1}} c_2$ and noting that $|B_\alpha'|(s) = \frac{1}{(1+\alpha(s+1))^2} \le 1$ for $s \ge 0$ and $\alpha \in (0,1)$, we conclude

$$\begin{aligned} &\int_\Omega B_\alpha^{\frac{(\beta_1+2)n+2}{n}}(u_\alpha(\cdot,t)) \\ &= \int_\Omega \left(B_\alpha^{\frac{\beta_1+2}{2}}(u_\alpha(\cdot,t))\right)^{\tilde p_1} \\ &\le c_2 \int_\Omega \left|\nabla B_\alpha^{\frac{\beta_1+2}{2}}(u_\alpha(\cdot,t))\right|^2 \left(\int_\Omega B_\alpha(u_\alpha(\cdot,t))\right)^{\frac{\tilde p_1(1-b)}{\tilde q_1}} + c_2\left(\int_\Omega B_\alpha(u_\alpha(\cdot,t))\right)^{\frac{\tilde p_1}{\tilde q_1}} \\ &\le c_3 \int_\Omega B_\alpha^{\beta_1}(u_\alpha(\cdot,t))|B_\alpha'(u_\alpha(\cdot,t))|^2|\nabla u_\alpha(\cdot,t)|^2 + c_1^{\frac{\tilde p_1}{\tilde q_1}} c_2 \\ &\le c_3 \int_\Omega B_\alpha^{\beta_1}(u_\alpha(\cdot,t))|\nabla u_\alpha(\cdot,t)|^2 + c_1^{\frac{\tilde p_1}{\tilde q_1}} c_2 \qquad \text{for all } t \in (0,T) \text{ and } \alpha \in (0,1). \end{aligned}$$

We now fix $\eta > 0$ and $p \in (0, \frac{(\beta_1+2)n+2}{n})$. By Young's inequality, we then obtain $c_4 > 0$ such that

$$\int_\Omega B_\alpha^p(u_\alpha(\cdot,t)) \le \frac{\eta}{4^p}\int_\Omega B_\alpha^{\beta_1}(u_\alpha(\cdot,t))|\nabla u_\alpha(\cdot,t)|^2 + c_4 \tag{6.4.18}$$

for all $t \in (0,T)$ and $\alpha \in (0,1)$. Moreover, as for $\alpha \in (0,1)$ and $s \in \operatorname{supp} \xi_{1\alpha} \subset [0, 2\alpha^{-1/(4-\min\{q_1,q_2\})}] \subset [0, 2\alpha^{-1}]$, we have

$$s = (1+\alpha(s+1))\frac{s}{1+\alpha(s+1)} \le 4B_\alpha(s),$$

the monotonicity of $[0,\infty) \ni s \mapsto s^p$ asserts

$$\int_\Omega \left(\frac{u_\alpha(\cdot,t)}{4}\right)^p \xi_{1\alpha}(u_\alpha(\cdot,t)) \le \int_\Omega B_\alpha^p(u_\alpha(\cdot,t)) \qquad \text{for all } t \in (0,T) \text{ and } \alpha \in (0,1).$$

Together with (6.4.18), this implies (6.4.16) for $C_1 := 4^p c_4$. By an analogous argumentation, we also obtain the corresponding statement for the second solution component. ☐

If β_1 and β_2 are sufficiently large compared to q_1 and q_2, one might hope that the estimates obtained in Lemma 6.4.7 are strong enough to control the right-hand side of (6.4.7). This idea can be quantified as follows.

Lemma 6.4.8. *Let $T \in (0,\infty)$ and suppose that (6.F2) and (6.H2) hold. Then there are $C_1, C_2 > 0$ such that (6.4.14) holds.*

Proof. We will crucially rely on the assumption (6.F2) which asserts that $m_1 > \underline{m}_1$ or $m_2 > \underline{m}_2$, where

$$\underline{m}_1 := \frac{2n-2}{n} + \frac{(3-q_2)(2-q_1)-(3-q_1)(2-q_2)}{2-q_2} \quad \text{and} \quad \underline{m}_2 := \frac{2n-2}{n} + (q_2-q_1).$$

Setting again $\beta_i := m_i - q_i - 1$ for $i \in \{1,2\}$, these definitions imply

$$\frac{(\beta_1+2)n+2}{n} > \underline{m}_1 - q_1 + 1 + \frac{2}{n} = \frac{(3-q_2)(2-q_1)}{2-q_2} \qquad \text{if } m_1 > \underline{m}_1 \quad \text{and}$$
$$\frac{(\beta_2+2)n+2}{n} > \underline{m}_2 - q_2 + 1 + \frac{2}{n} = 3-q_1 \qquad \text{if } m_2 > \underline{m}_2,$$

whence there is $\eta \in (0,1)$ such that still

$$\frac{(\beta_1+2)n+2}{n} > \frac{(3-q_2)(2-q_1+\eta)}{2-q_2} \qquad \text{if } m_1 > \underline{m}_1 \quad \text{and} \tag{6.4.19}$$
$$\frac{(\beta_2+2)n+2}{n} > \frac{3-q_1}{1-\eta} \qquad \text{if } m_2 > \underline{m}_2. \tag{6.4.20}$$

For $s \ge 1$, we have $\frac{s+1}{s} \in [1,2]$ and hence $s^{1-q_i} \le (s+1)^{1-q_i} \le 2^{1-q_i}s^{1-q_i}$ for $i \in \{1,2\}$ which due to $\chi_i G_i'(s) = \int_1^s \frac{(\sigma+1)^{1-q_i}}{\sigma} \,\mathrm{d}\sigma$ for $s \ge 1$ and $i \in \{1,2\}$ implies that

$$\frac{s^{1-q_i}L_{q_i}(s)}{1-q_i\mathbb{1}_{\{q_i<1\}}} \le \chi_i G_i'(s) \le \frac{2^{1-q_i}s^{1-q_i}L_{q_i}(s)}{1-q_i\mathbb{1}_{\{q_i<1\}}} \qquad \text{for } s \ge 1 \text{ and } i \in \{1,2\}.$$

(We recall that $L_q(s) = \mathbb{1}_{\{q<1\}} + \mathbb{1}_{\{q=1\}} \ln s$ for $s \ge 1$ and $q \le 1$ by (6.4.2).) Combined with the facts that $\ln(s+\mathrm{e}) - \ln s = \ln \frac{s+\mathrm{e}}{s} \le \ln(1+\mathrm{e})$ and $\ln s \le s^\eta$ for $s \ge 1$ and Young's inequality, we thus obtain $c_1, c_2 > 0$ such that

$$\begin{aligned}
&[G_1'(u_\alpha)f_1(u_\alpha,v_\alpha) + G_2'(v_\alpha)f_2(u_\alpha,v_\alpha)]\,\xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \\
&\le \left[c_1 u_\alpha^{2-q_1+\eta}v_\alpha - 2c_2 u_\alpha^{3-q_1}L_{q_1}(u_\alpha+\mathrm{e}) - 2c_2 v_\alpha^{3-q_2}L_{q_2}(v_\alpha+\mathrm{e})\right]\xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) + c_1
\end{aligned}$$

in $\{u_\alpha \ge 1\} \cap \{v_\alpha \ge 1\}$ for all $\alpha \in (0,1)$.

We now distinguish between the cases $m_1 > \underline{m}_1$ and $m_2 > \underline{m}_2$. In the former one, we first employ Young's inequality to obtain $c_3 > 0$ such that

$$\begin{aligned} &c_1 u_\alpha^{2-q_1+\eta} v_\alpha \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \\ &\le c_3 u_\alpha^{\frac{(3-q_2)(2-q_1+\eta)}{2-q_2}} \xi_{1\alpha}(u_\alpha) + c_2 v_\alpha^{3-q_2}\xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \qquad \text{in } \Omega\times(0,T) \text{ for all } \alpha\in(0,1) \end{aligned}$$

and then make use of the assumption $m_1 > \underline{m}_1$ which allows us to apply (6.4.19) and Lemma 6.4.7 to obtain $c_4 > 0$ such that

$$c_3 \int_0^T \int_\Omega u_\alpha^{\frac{(3-q_2)(2-q_1+\eta)}{2-q_2}} \xi_\alpha(u_\alpha) \le \frac{d_1}{2\chi_1}\int_0^T\int_\Omega B_\alpha^{\beta_1}(u_\alpha)|\nabla u_\alpha|^2 + c_4 \qquad \text{for all } \alpha\in(0,1),$$

If on the other hand $m_2 > \underline{m}_2$, then we again make first use of Young's inequality to obtain $c_5 > 0$ such that

$$c_1 u_\alpha^{2-q_1+\eta} v_\alpha \xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \le c_2 u_\alpha^{3-q_1}\xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) + c_5 v_\alpha^{\frac{3-q_1}{1-\eta}}\xi_{2\alpha}(v_\alpha)$$

in $\Omega\times(0,T)$ for all $\alpha\in(0,1)$. According to Lemma 6.4.7 (which is applicable thanks to (6.4.20)), there is then $c_6 > 0$ such that

$$c_5\int_0^T\int_\Omega v_\alpha^{\frac{3-q_1}{1-\eta}}\xi_{2\alpha}(v_\alpha) \le \frac{d_2}{2\chi_2}\int_0^T\int_\Omega B_\alpha^{\beta_2}(v_\alpha)|\nabla v_\alpha|^2 + c_6 \qquad \text{for all } \alpha\in(0,1).$$

In both cases $m_1 > \underline{m}_1$ and $m_2 > \underline{m}_2$, we then conclude from the estimates above that there is $c_7 > 0$ such that

$$\begin{aligned} &\int_0^T\int_\Omega \left[G_1'(u_\alpha)f_1(u_\alpha,v_\alpha) + G_2'(v_\alpha)f_2(u_\alpha,v_\alpha)\right]\xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha)\mathbb{1}_{\{u_\alpha\ge1\}\cap\{v_\alpha\ge1\}} \\ \le\ &\frac{d_1}{2\chi_1}\int_0^T\int_\Omega B_\alpha^{\beta_1}(u_\alpha)|\nabla u_\alpha|^2 + \frac{d_2}{2\chi_2}\int_0^T\int_\Omega B_\alpha^{\beta_2}(v_\alpha)|\nabla v_\alpha|^2 \\ &-c_2\int_0^T\int_\Omega u_\alpha^{3-q_1}L_{q_1}(u_\alpha+\mathrm{e})\xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) - c_2\int_0^T\int_\Omega v_\alpha^{3-q_2}L_{q_2}(u_\alpha+\mathrm{e})\xi_{1\alpha}(u_\alpha)\xi_{2\alpha}(v_\alpha) \\ &+c_7 \qquad \text{for all } \alpha\in(0,1), \end{aligned}$$

which in conjunction with (6.4.7) and (6.4.9) gives the claim. ☐

This concludes our journey of controlling the right-hand side in (6.4.7). As a consequence, we obtain the following a priori bounds.

Lemma 6.4.9. *Let $T\in(0,\infty)$. There is $C_1 > 0$ such that*

$$\begin{aligned} \sup_{t\in(0,T)} \bigg(&\int_\Omega B_\alpha^{2-q_1}(u_\alpha(\cdot,t))L_{q_1}(u_\alpha(\cdot,t)+\mathrm{e}) \\ &+ \int_\Omega B_\alpha^{2-q_1}(v_\alpha(\cdot,t))L_{q_2}(v_\alpha(\cdot,t)+\mathrm{e})\bigg) \le C_1 \end{aligned} \tag{6.4.21}$$

and

$$\int_0^T\int_\Omega B_\alpha^{\beta_1}(u_\alpha)|\nabla u_\alpha|^2 + \int_0^T\int_\Omega B_\alpha^{\beta_2}(v_\alpha)|\nabla v_\alpha|^2 \le C_1 \tag{6.4.22}$$

for all $\alpha \in (0,1)$, where again $\beta_i := m_i - q_i - i$ for $i \in \{1,2\}$, and L_{q_i} and B_α are as in (6.4.2) *and* (6.4.8)*, respectively. Moreover, if* (6.H2) *holds, then we can find $C_2 > 0$ with the property that*

$$\int_0^T \int_\Omega u_\alpha^2 \ln(u_\alpha + \mathrm{e}) + \int_0^T \int_\Omega v_\alpha^2 \ln(v_\alpha + \mathrm{e}) \le C_2 \qquad \textit{for all } \alpha \in (0,1). \tag{6.4.23}$$

Proof. According to Lemma 6.4.3, Lemma 6.4.5 and Lemma 6.4.8, there are $c_1, c_2 > 0$ and $c_3 \ge 0$ such that c_3 is positive if (6.H2) holds and

$$\begin{aligned} &c_1 \int_\Omega B_\alpha^{2-q_1}(u_\alpha(\cdot,t)) L_{q_1}(u_\alpha(\cdot,t)+\mathrm{e}) + c_1 \int_\Omega B_\alpha^{2-q_2}(v_\alpha(\cdot,t)) L_{q_2}(v_\alpha(\cdot,t)+\mathrm{e}) \\ &+ \frac{d_1}{2\chi_1} \int_0^T \int_\Omega B_\alpha^{\beta_1}(u_\alpha)|\nabla u_\alpha|^2 + \frac{d_2}{2\chi_2} \int_0^T \int_\Omega B_\alpha^{\beta_2}(v_\alpha)|\nabla v_\alpha|^2 \\ \le{}& c_2 - c_3 \int_0^T \int_\Omega u_\alpha^2 \ln(u_\alpha + \mathrm{e}) - c_3 \int_0^T \int_\Omega v_\alpha^2 \ln(v_\alpha + \mathrm{e}) \qquad \text{for } t \in (0,T) \text{ and } \alpha \in (0,1), \end{aligned}$$

as desired. □

6.4.3. Space-time bounds and the limit process

As a next step, we derive further space-time bounds from (6.4.21) and (6.4.22). To that end, we make use of the following interpolation inequality which is both a refinement and a consequence of the Gagliardo–Nirenberg inequality and has been proven by Tao and Winkler in [86].

Lemma 6.4.10. *Let $0 < q < p < \frac{2n}{(n-2)_+}$ and suppose that $\Lambda \in C^0(\mathbb{R})$ fulfills $\Lambda \ge 1$ on $\mathbb{R}$. Then there exist $C > 0$ and $\theta \in (0,1]$ such that*

$$\int_\Omega |\varphi|^p \Lambda^\theta(\varphi) \le C \left(\int_\Omega |\nabla \varphi| \right)^{\frac{pb}{2}} \left(\int_\Omega |\varphi|^q \Lambda(\varphi) \right)^{\frac{p(1-b)}{q}} + C \left(\int_\Omega |\varphi|^q \Lambda(\varphi) \right)^{\frac{p}{q}}$$

for all $\varphi \in W^{1,2}(\Omega)$, where

$$b := \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{n} - \frac{1}{2}} \in (0,1).$$

Proof. This is a direct consequence of [86, Lemma 7.5]. □

Lemma 6.4.11. *For all $T \in (0,\infty)$, there are $C > 0$ and $\theta_1, \theta_2 \in (0,1]$ such that*

$$\int_0^T \int_\Omega B_\alpha^{p_1}(u_\alpha) L_{q_1}^{\theta_1}(B_\alpha(u_\alpha) + \mathrm{e}) + \int_0^T \int_\Omega B_\alpha^{p_2}(v_\alpha) L_{q_2}^{\theta_2}(B_\alpha(v_\alpha) + \mathrm{e}) \le C \tag{6.4.24}$$

for all $\alpha \in (0,1)$, where p_1 and p_2 are as in (6.1.8)*, and L_{q_i} and B_α are as in* (6.4.2) *and* (6.4.8)*, respectively.*

Proof. We fix $T \in (0,\infty)$. As usual, it suffices to show the statement for the first solution component.

Let us first assume $p_1 = 3 - q_1$ and that (6.H2) holds. Then (6.4.23) already contains (6.4.24). Moreover, if $p_1 = 2 - q_1$, then (6.4.21) and an integration in time also show (6.4.24). According to (6.1.8), it remains to be shown that (6.4.24) also holds for $2 - q_1 < p_1 = \beta_1 + 2 + \frac{2(2-q_1)}{n}$, where again $\beta_1 \coloneqq m_1 - q_1 - 1$. As already alluded to, the main ingredients for this proof are (6.4.21) and (6.4.22) which assert that there are $c_1, c_2 > 0$ such that

$$\sup_{t\in(0,T)} \int_\Omega B_\alpha^{2-q_1}(u_\alpha(\cdot,t))L_{q_1}(u_\alpha(\cdot,t)+\mathrm{e}) \le c_1 \quad \text{and} \quad \int_0^T\int_\Omega B_\alpha^{\beta_1}(u_\alpha)|\nabla u_\alpha|^2 \le c_2.$$

Preparing an application of Lemma 6.4.10, we set $\tilde q_1 \coloneqq \frac{2(2-q_1)}{\beta_1+2}$, which is positive as $\beta_1 > -2$ is contained in (6.4.1). Moreover,

$$\tilde p_1 \coloneqq \frac{2(n+\tilde q_1)}{n} = 2\left(1 + \frac{\tilde q_1}{n}\right) = \frac{2(\beta_1 + 2 + \frac{2(2-q_1)}{n})}{\beta_1+2} = \frac{2p_1}{\beta_1+2} > \tilde q_1$$

thanks to $p_1 > 2 - q_1$. Thus, $\tilde p_1 < \frac{2(n+\tilde p_1)}{n}$ and hence $\frac{n-2}{n}\tilde p_1 < 2$ which in turn implies $\tilde p_1 < \frac{2n}{(n-2)_+}$. Therefore, we may indeed apply Lemma 6.4.10 to obtain $c_3 > 0$, $\theta_1 \in (0,1]$ and $b \in (0,1)$ such that with $\Lambda(s) \coloneqq L_{q_1}(s^{\frac{2}{\beta+2}} + \mathrm{e})$, $s \ge 0$,

$$\int_\Omega \varphi^{\tilde p_1}\Lambda^{\theta_1}(\varphi) \le c_3 \left(\int_\Omega |\nabla\varphi|^2\right)^{\frac{\tilde p_1 b}{2}} \left(\int_\Omega \varphi^{\frac{2(2-q_1)}{\beta_1+2}}\Lambda(\varphi)\right)^{\frac{\tilde p_1(1-b)}{\tilde q_1}} + c_3\left(\int_\Omega \varphi^{\frac{2(2-q_1)}{\beta_1+2}}\Lambda(\varphi)\right)^{\frac{\tilde p_1}{\tilde q_1}}$$

for all nonnegative $\varphi \in W^{1,2}(\Omega)$. Taking here $\varphi = B_\alpha^{p_1}(u_\alpha(\cdot,t))$, $t \in (0,T)$, and integrating in time yield

$$\begin{aligned}
&\int_0^T\int_\Omega B_\alpha^{p_1}(u_\alpha)L_{q_1}^{\theta_1}(B_\alpha(u_\alpha)+\mathrm{e}) \\
&= \int_0^T\int_\Omega \left(B_\alpha^{\frac{\beta_1+2}{2}}(u_\alpha)\right)^{\frac{2p_1}{\beta_1+2}} \Lambda^{\theta_1}(B_\alpha^{\frac{\beta_1+2}{2}}(u_\alpha)) \\
&\le c_3\int_0^T \left(\int_\Omega \left|\nabla B_\alpha^{\frac{\beta_1+2}{2}}(u_\alpha)\right|^2\right)^{\frac{\tilde p_1 b}{2}} \left(\int_\Omega B_\alpha^{2-q_1}(u_\alpha)L_{q_1}(u_\alpha+\mathrm{e})\right)^{\frac{\tilde p_1(1-b)}{\tilde q_1}} \\
&\quad + c_3\int_0^T\left(\int_\Omega B_\alpha^{2-q_1}(u_\alpha)L_{q_1}(u_\alpha+\mathrm{e})\right)^{\frac{\tilde p_1}{\tilde q_1}} \\
&\le \frac{c_1^{\tilde p_1/\tilde q_1}c_3(\beta_1+2)^2}{4}\int_0^T\int_\Omega B_\alpha^{\beta_1}(u_\alpha)|B'(u_\alpha)|^2|\nabla u_\alpha|^2 + Tc_1^{\tilde p_1/\tilde q_1}c_3 \\
&\le \frac{c_1^{\tilde p_1/\tilde q_1}c_2c_3(\beta_1+2)^2}{4} + Tc_1^{\tilde p_1/\tilde q_1}c_3 \qquad \text{for all } \alpha \in (0,1),
\end{aligned}$$

where in the last step we have used that $|B_\alpha'(s)| = \frac{1}{(1+\alpha(s+1))^2} \le 1$ for $s \ge 0$ and $\alpha \in (0,1)$. Thus, (6.4.24) indeed holds in all cases treated by this lemma. □

As an application of Young's inequality reveals, (6.4.22) and (6.4.24) allow us to also obtain gradient space-time bounds.

Lemma 6.4.12. *Let $T \in (0,\infty)$ and r_1, r_2 be as in* (6.1.9)*. Then there is $C > 0$ such that*

$$\int_0^T\int_\Omega |\nabla u_\alpha|^{r_1} + \int_0^T\int_\Omega |\nabla v_\alpha|^{r_2} \le C \qquad \text{for all } \alpha \in (0,1). \tag{6.4.25}$$

Proof. Again, it suffices to prove the bound only for u_α, $\alpha \in (0,1)$. We first assume that $r_1 < 2$ and hence $r_1 = \frac{2p_1}{p_1-\beta_1}$ by (6.1.9), where $\beta_1 := m_1 - q_1 - 1$. With B_α as in (6.4.8), we then make use of Young's inequality to obtain

$$\begin{aligned}\int_0^T \int_\Omega |\nabla u_\alpha|^{r_1} &= \int_0^T \int_\Omega B_\alpha^{\frac{\beta_1 r_1}{2}}(u_\alpha)|\nabla u_\alpha|^{r_1} B_\alpha^{-\frac{\beta_1 r_1}{2}}(u_\alpha) \\ &\le \frac{r_1}{2} \int_0^T \int_\Omega B_\alpha^{\beta_1}(u_\alpha)|\nabla u_\alpha|^2 + \frac{2-r_1}{2} \int_0^T \int_\Omega B_\alpha^{-\frac{\beta_1 r_1}{2-r_1}}(u_\alpha)\end{aligned}$$

for all $\alpha \in (0,1)$ which due to (6.4.24) and

$$-\frac{\beta_1 r_1}{2-r_1} = \frac{-\beta_1}{\frac{2}{r_1}-1} = \frac{-\beta_1}{\frac{p_1-\beta_1}{p_1}-1} = \frac{-\beta_1}{\frac{-\beta_1}{p_1}} = p_1$$

implies (6.4.25) for some $C > 0$.

If, on the other hand $r_1 \ge 2$ and hence $r_1 = 2 \le \frac{2p_1}{p_1-\beta_1}$ by (6.1.9), then $\beta_1 \ge 0$ since positivity of p_1 is contained in (6.4.1). Thus, in this case the estimate (6.4.22) directly implies (6.4.25). □

As a last preparation before obtaining limit functions u and v by applying several compactness theorems—in particular, the Aubin–Lions lemma—, we derive estimates for the time derivatives $u_{\alpha t}$ and $v_{\alpha t}$, $\alpha \in (0,1)$.

Lemma 6.4.13. *Let $T \in (0,\infty)$. Then there exists $C > 0$ such that*

$$\|u_{\alpha t}\|_{L^1((0,T);(W^{n+1,2}(\Omega))^\star)} + \|v_{\alpha t}\|_{L^1((0,T);(W^{n+1,2}(\Omega))^\star)} \le C \qquad \text{for all } \alpha \in (0,1). \quad (6.4.26)$$

Proof. Since $u_\alpha \in L^2((0,T);W^{1,2}(\Omega))$ by Lemma 6.3.2, the weak formulation (6.2.8) entails that

$$\begin{aligned}\int_\Omega u_{\alpha t}(\cdot,t)\psi = &-\int_\Omega D_{1\alpha}(u_\alpha(\cdot,t))\nabla u_\alpha(\cdot,t)\cdot\nabla\psi + \int_\Omega S_{1\alpha}(u_\alpha(\cdot,t))\nabla v_\alpha(\cdot,t)\cdot\nabla\psi \\ &+ \int_\Omega f_{1\alpha}(u_\alpha(\cdot,t),v_\alpha(\cdot,t))\psi\end{aligned}$$

for a.e. $t \in (0,T)$, all $\psi \in W^{1,2}(\Omega)$ and all $\alpha \in (0,1)$. Thus, recalling that $D_{1\alpha}(u_\alpha) \le d_1 B_\alpha^{m_1-1}(u_\alpha) + 1$ and $S_{1\alpha}(u_\alpha) \le \chi_1 B_\alpha^{q_1}(u_\alpha)$ for $\alpha \in (0,1)$ if B_α as in (6.4.8), we may estimate

$$\begin{aligned}&\left|\int_\Omega u_{\alpha t}(\cdot,t)\psi\right| \\ &\le \left|\int_\Omega (D_{1\alpha}(u_\alpha(\cdot,t))\nabla u_\alpha(\cdot,t)\cdot\nabla\psi\right| + \left|\int_\Omega S_{1\alpha}(u_\alpha(\cdot,t))\nabla v_\alpha(\cdot,t)\cdot\nabla\psi\right| \\ &\quad + \left|\int_\Omega f_{1\alpha}(u_\alpha(\cdot,t),v_\alpha(\cdot,t))\psi\right| \\ &\le d_1 \int_\Omega \left(B_\alpha^{m_1-1-\frac{\beta_1}{2}}(u_\alpha(\cdot,t)) + 1\right)^2 \|\nabla\psi\|_{L^\infty(\Omega)} \\ &\quad + d_1 \int_\Omega \left(\left(B_\alpha^{\frac{\beta_1}{2}}(u_\alpha(\cdot,t)) + 1\right)|\nabla u_\alpha(\cdot,t)|\right)^2 \|\nabla\psi\|_{L^\infty(\Omega)}\end{aligned}$$

$$+\chi_1 \left(\int_\Omega (B_\alpha^{q_1}(u_\alpha(\cdot,t)))^{\frac{r_2}{r_2-1}} + \int_\Omega |\nabla v_\alpha(\cdot,t)|^{r_2} \right) \|\nabla\psi\|_{L^\infty(\Omega)}$$
$$+ \left(\int_\Omega |f_{1\alpha}(u_\alpha(\cdot,t), v_\alpha(\cdot,t))| \right) \|\psi\|_{L^\infty(\Omega)}$$

for a.e. $t \in (0,T)$, all $\psi \in W^{1,\infty}(\Omega)$ and all $\alpha \in (0,1)$, wherein as usual $\beta_1 \coloneqq m_1 - q_1 - 1$. As according to (6.4.1) and (6.1.10), both $2(m_1 - 1 - \frac{\beta_1}{2})$ and $\frac{\max\{q_1,0\}r_2}{r_2-1}$ are at most p_1, the bounds (6.4.24), (6.4.22), (6.4.25) and (6.4.23) along with the embeddings $W^{n+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ and an integration in time yield $c_1 > 0$ such that

$$\int_0^T \sup_{\substack{\psi \in W^{n+1,2}(\Omega)\\ \|\psi\|_{W^{n+1,2}(\Omega)} \le 1}} \left| \int_\Omega u_{\alpha t}\psi \right| \le c_1 \qquad \text{for all } \alpha \in (0,1),$$

which together with analogous considerations regarding $v_{\alpha t}$ implies (6.4.26). □

The a priori bounds gained in the lemmata above now allow us to conclude that (u_α, v_α) converge in certain spaces along some null sequence $(\alpha_j)_{j\in\mathbb{N}}$.

Lemma 6.4.14. *Set*

$$\mathcal{P}_i \coloneqq \begin{cases} [1,p_i), & q_i < 1,\\ [1,p_i], & q_i = 1. \end{cases}$$

Then there exists a null sequence $(\alpha_j)_{j\in\mathbb{N}} \subset (0,1)$ and nonnegative $u, v \in L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ such that

$$u_{\alpha_j} \to u \qquad \textit{pointwise a.e.,} \tag{6.4.27}$$
$$v_{\alpha_j} \to v \qquad \textit{pointwise a.e.,} \tag{6.4.28}$$
$$B_\alpha(u_{\alpha_j}) \to u+1 \qquad \textit{in } L^p_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)) \textit{ for all } p \in \mathcal{P}_1, \tag{6.4.29}$$
$$B_\alpha(v_{\alpha_j}) \to v+1 \qquad \textit{in } L^p_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)) \textit{ for all } p \in \mathcal{P}_2, \tag{6.4.30}$$
$$u_{\alpha_j} \rightharpoonup u \qquad \textit{in } L^{r_1}_{\mathrm{loc}}([0,\infty); W^{1,r_1}(\Omega)), \tag{6.4.31}$$
$$v_{\alpha_j} \rightharpoonup v \qquad \textit{in } L^{r_1}_{\mathrm{loc}}([0,\infty); W^{1,r_1}(\Omega)), \tag{6.4.32}$$
$$f_{1\alpha}(u_\alpha, v_\alpha) \to f_1(u,v) \qquad \textit{in } L^1_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)) \quad \textit{and} \tag{6.4.33}$$
$$f_{2\alpha}(u_\alpha, v_\alpha) \to f_2(u,v) \qquad \textit{in } L^1_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)) \tag{6.4.34}$$

as $j \to 0$, where B_α is as in (6.4.8) *for $\alpha \in (0,1)$.*

Proof. Thanks to (6.4.25) and (6.4.26), the Aubin–Lions lemma (along with a diagonalization argument) provides us with a null sequence $(\alpha_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions $u, v \in L^1_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty))$ such that $u_{\alpha_j} \to u$ and $v_{\alpha_j} \to v$ in $L^1_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty))$ as $j \to \infty$. After switching to a subsequence if necessary, we may thus assume that (6.4.27) and (6.4.28) hold. Thus, nonnegativity of u and v is inherited from nonnegativity of u_{α_j} and v_{α_j}, $j \in \mathbb{N}$, which in turn is asserted by Lemma 6.3.2. Due to the bound (6.4.24), and because $B_\alpha(u_\alpha) \to u_\alpha+1$ and $B_\alpha(v_\alpha) \to v_\alpha + 1$ pointwise a.e. as $\alpha \searrow 0$ by (6.4.27) and (6.4.28), Vitali's theorem asserts that (6.4.29) and (6.4.30) hold.

Moreover, possibly after switching to further subsequences, (6.4.31) and (6.4.32) follow from (6.4.25). (We note that (6.4.27) and (6.4.28) guarantee that the corresponding limit functions coincide.)

Finally, additional consequences of (6.4.27) and (6.4.28) are (6.4.33) and (6.4.34): For fixed $T \in (0, \infty)$, the complement of

$$A \coloneqq \left\{ (x,t) \in \Omega \times (0,T) : u(x,t), v(x,t) < \infty \text{ and } \lim_{j\to\infty} (u_{\alpha_j}, v_{\alpha_j})(x,t) = (u,v)(x,t) \right\}$$

in $\Omega \times (0,T)$ is a null set (since the inclusions $u, v \in L^1(\Omega \times (0,T))$ imply $u, v < \infty$ a.e.). Given $(x,t) \in A$, there is $M > 0$ with $\max\{u(x,t), v(x,t)\} < M$. Thus, we can find $j_1 \in \mathbb{N}$ such that $\max\{u_{\alpha_j}(x,t), v_{\alpha_j}(x,t)\} < 2M$ for all $j \geq j_1$. Taking moreover $j_2 \in \mathbb{N}$ so large that $2M \leq \alpha_j^{-1/(4-\min\{q_1,q_2\})}$ for all $j \geq j_2$, we see that $\xi_{\alpha_j}(u(x,t)) = \xi_{\alpha_j}(v(x,t)) = 1$ and hence $f_{1\alpha_j}(u_{\alpha_j}(x,t), v_{\alpha_j}(x,t)) = f_1(u_{\alpha_j}(x,t), v_{\alpha_j}(x,t))$ for all $j \geq \max\{j_1, j_2\}$ so that

$$f_{1\alpha_j}(u_{\alpha_j}(x,t), v_{\alpha_j}(x,t)) \to f_1(u(x,t), v(x,t)) \qquad \text{as } j \to \infty$$

by the continuity of f_1. Since $(x,t) \in A$ was arbitrary, $f_{1\alpha_j}(u_{\alpha_j}, v_{\alpha_j}) \to f_1(u,v)$ a.e. as $j \to \infty$. In the case of (6.H1), (6.4.33) is trivially true while for (6.H2), we make first use of Young's inequality to obtain $c_1 > 0$ such that $|f_{1\alpha}(s_1, s_2)| \leq c_1(s_1^2 + s_2^2 + 1)$ for all $s_1, s_2 \geq 0$ and $\alpha \in (0,1)$ and then employ Vitali's theorem along with (6.4.23) and the just obtained pointwise convergence of $f_{1\alpha}$ to also obtain (6.4.33) in that case. As usual, (6.4.34) can be shown analogously. □

6.5. Existence of global weak solutions to (6.P): proof of Theorem 6.1.1

In this final section, we show that the pair (u,v) constructed in Lemma 6.4.14 is a solution to (6.P) in the following sense.

Definition 6.5.1. A pair $(u,v) \in L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ is called a *global nonnegative weak solution* of (6.P) if $u, v \geq 0$,

$$D_1(u)\nabla u, S_1(u)\nabla v, D_2(u)\nabla v, S_2(v)\nabla u, f_1, f_2 \in L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$$

and

$$\begin{aligned} &-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) \\ &= -\int_0^\infty \int_\Omega D_1(u)\nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega S_1(u)\nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega f_1(u,v)\varphi \end{aligned} \tag{6.5.1}$$

as well as

$$\begin{aligned} &-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0 \varphi(\cdot,0) \\ &= -\int_0^\infty \int_\Omega D_2(u)\nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega S_2(u)\nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega f_2(u,v)\varphi \end{aligned} \tag{6.5.2}$$

hold for all $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$.

Lemma 6.5.2. *The tuple (u,v) constructed in Lemma 6.4.14 is a weak solution of* (6.P) *in the sense of Definition 6.5.1.*

Proof. Both the required regularity and nonnegativity of u and v are contained in Lemma 6.4.14.

In order show that (6.5.1) holds, we first fix $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$. For all $\alpha \in (0,1)$, the pair (u_α, v_α) given by Lemma 6.3.2 solves (6.3.4) weakly so that by (6.2.8) and an integration by parts,

$$\begin{aligned}
&I_{1\alpha} + I_{2\alpha} \\
&:= -\int_0^\infty \int_\Omega u_\alpha \varphi_t - \int_\Omega u_{0\alpha}\varphi(\cdot,0) \\
&= -\int_0^\infty \int_\Omega D_{1\alpha}(u_\alpha)\nabla u_\alpha \cdot \nabla\varphi + \int_0^\infty \int_\Omega S_{1\alpha}(u_\alpha)\nabla v_\alpha \cdot \nabla\varphi + \int_0^\infty \int_\Omega f_{1\alpha}(u_\alpha, v_\alpha)\varphi \\
&=: I_{3\alpha} + I_{4\alpha} + I_{5\alpha} \qquad \text{for all } \alpha \in (0,1).
\end{aligned} \tag{6.5.3}$$

Mainly relying on the convergences provided by Lemma 6.4.14, we now take the limit $\alpha = \alpha_j \searrow 0$ in each term herein. First,

$$I_{2\alpha_j} \to -\int_\Omega u_0\varphi(\cdot,0) \quad \text{and} \quad I_{5\alpha_j} \to \int_0^\infty \int_\Omega f_1(u,v)\varphi \qquad \text{as } j \to \infty$$

are direct consequences of (6.3.2) and (6.4.33). Moreover, as $r_1 > 1$ by (6.1.10), we infer from (6.4.31) that $u_{\alpha_j} \to u$ in $L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ and thus

$$I_{1\alpha_j} \to -\int_0^\infty \int_\Omega u\varphi_t \qquad \text{as } j \to \infty.$$

Regarding $I_{3\alpha}$, we first note that in the case of $m_1 \le 1$,

$$B_{\alpha_j}^{m_1-1}(u_{\alpha_j}) \to (u+1)^{m_1-1} \qquad \text{in } L_{\mathrm{loc}}^{\frac{r_1}{r_1-1}}(\overline{\Omega} \times [0,\infty)) \text{ as } j \to \infty \tag{6.5.4}$$

by Lebesgue's theorem and (6.4.27), where B_α is as in (6.4.8) for $\alpha \in (0,1)$. We now show that (6.5.4) also holds for $m_1 > 1$. If additionally $r_1 = \frac{2p_1}{p_1-\beta_1}$ with $\beta_1 := m_1 - q_1 - 1$, then $(m_1-1)\frac{r_1}{r_1-1} = (m_1-1)\frac{2p_1}{p_1+\beta_1} < p_1$ since $0 < 2(m_1-1) < p_1 + \beta_1$ is entailed in (6.4.1). If on the other hand ($m_1 > 1$ and) $r_1 \neq \frac{2p_1}{p_1-\beta_1}$ and thus $r_1 = 2 > \frac{2p_1}{p_1-\beta_1}$ by (6.1.9), then $\beta_1 < 0$ so that (6.4.1) asserts $2(m_1-1) < p_1$ and hence also $(m_1-1)\frac{r_1}{r_1-1} < p_1$. Therefore, (6.4.29) asserts that (6.5.4) indeed also holds for $m_1 > 1$. Combined with (6.4.31), (6.5.4) then implies

$$\int_0^\infty \int_\Omega B_{\alpha_j}^{m_1-1}(u_{\alpha_j})\nabla u_{\alpha_j} \cdot \nabla\varphi \to \int_0^\infty \int_\Omega (u+1)^{m_1-1}\nabla u \cdot \nabla\varphi \qquad \text{as } j \to \infty,$$

and since additionally $\alpha_j \int_0^\infty \int_\Omega \nabla u_{\alpha_j} \cdot \nabla\varphi \to 0$ as $j \to \infty$ by (6.4.31), we conclude

$$I_{3\alpha_j} \to -\int_0^\infty \int_\Omega D_1(u)\nabla u \cdot \nabla\varphi \qquad \text{as } j \to \infty.$$

Finally, we concern ourselves with the term stemming from the cross-diffusion: Precisely due to our main condition (6.1.10), we can choose $p > 1$ such that

$$\frac{1}{p} + \frac{1}{r_1} = 1 \quad \text{and} \quad p \in \begin{cases} [1,\infty), & q_1 \le 0, \\ [1, \frac{p_1}{q_1}), & 0 < q_1 < 1, \\ [1, p_1], & q_1 = 1. \end{cases}$$

As also $0 \le S_{1\alpha}(s) \le \chi_1 B_\alpha^{q_1}(s)$ for all $s \ge 0$ and $\alpha \in (0,1)$ as well as $S_{1\alpha_j}(u_{\alpha_j}) \to S_1(u_{\alpha_j})$ a.e. as $\alpha \searrow 0$, Pratt's lemma and (6.4.29) assert that $S^p_{1\alpha_j}(u_{\alpha_j}) \to S^p_1(u)$ in $L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ as $j \to \infty$, provided that $q_1 \ge 0$. For $q_1 < 0$, the same conclusion can be reached by Lebesgue's theorem. Combined with (6.4.32), this entails that $S_{1\alpha_j}(u_{\alpha_j})\nabla v_{\alpha_j} \rightharpoonup S_1(u)\nabla v$ in $L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ as $j \to \infty$ and thus

$$I_{4\alpha_j} \to \int_0^\infty \int_\Omega S_1(u)\nabla v \cdot \nabla\varphi \qquad \text{as } j \to \infty.$$

In combination, these convergences and (6.5.3) prove (6.5.1), and since (6.5.2) can be shown analogously, (u,v) is indeed a weak solution of (6.P). □

This lemma already contains the main theorem of the present chapter.

Proof of Theorem 6.1.1. All claims have been proven in Lemma 6.5.2. □

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