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Analysis of Singular Stochastic Systems: Two Classes of Examples

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1. Abstract

We study two classes of singular stochastic systems: stochastic spikes appearing as scaling limits of the solutions of specific stochastic differential equations with applications in quantum mechanics and Fleming-Viot-type particle systems used as approximation algorithm in mathematical statistics. In the case of the stochastic spikes we generalize results of [5] regarding the convergence to a homogeneous Poisson process using classical probabilistic tools and apply the obtained abstract criteria to two example classes. Fleming-Viot-type particle systems may exhibit singular behavior in the sense that there are possibly infinitely many particle jumps in finite time. Establishing structural results throughout the analysis of the problem we deduce a sufficient condition to decide whether this happens and are thereby able to give new insights in the case of three moving particles.

Zusammenfassung

Wir studieren zwei Klassen singulärer stochastischer Systeme: Stochastische Spikes, die als Skalierungslimiten von Lösungen gewisser stochastischer Differenzialgleichungen auftreten und in der Quantenmechanik Anwendung finden sowie Fleming-Viot-artige Partikelsysteme, die als Approximationsalgorithmen in der mathematischen Statistik verwendet werden. Im Falle der stochastischen Spikes verallgemeinern wir Ergebnisse von [5] bezüglich der Konvergenz gegen einen homogenen Poisson-Prozess unter Verwendung klassischer wahrscheinlichkeitstheoretischer Methoden und wenden die gefundenen abstrakten Kriterien auf zwei Beispielklassen an. Fleming-Viot-artige Partikelsysteme können in dem Sinne singuläres Verhalten aufweisen, dass es möglicherweise zu unendlich vielen Sprüngen in endlicher Zeit kommt. Während der Analyse des Problems stellen wir strukturelle Resultate auf und leiten eine hinreichende Bedingungen her, um zu entscheiden, ob dies passiert; dadurch sind wir in der Lage, neue Einsichten in den Fall von drei sich bewegender Partikel zu geben.

Contents

2. Introduction

We analyze random phenomena where different notions of singularities may apply. In the case of the first example class we consider solutions to stochastic differential equations (SDEs) parameterized by $\lambda < \infty$ and $\varepsilon > 0$. Loosely speaking, the limit processes $\lambda \to \infty$ and $\varepsilon \to 0$ may be thought of being competitive: The parameter λ acts as an acceleration in the time scale and increases the speed of the movement of the stochastic process; as long as $\varepsilon > 0$ is positive there is a drift enforcing the process to stay in the positive half-line – as ε becomes small so does the drift and near the origin the diffusion coefficient tends to 0 which may be seen as the singularity in this model. A game is initiated between wandering around quickly and sticking near the origin. On a suitable curve a non-trivial limiting process is exhibited.

In the case of the Fleming-Viot N-particle system the parameter $\nu \in \mathbb{R}$ being smaller corresponds to drifting the positive particles stronger towards the origin. Here, capturing the particles in the positive half-line is accomplished by discrete jumps as antagonist. We use generalized Bessel processes driven according to the SDE

$$dX_t = dB_t + \frac{\frac{\nu-1}{2}}{X_t}\, dt$$

where B_t denotes Brownian motion. Equivalently, one considers squared Bessel processes with differential

$$dX_t^2 = 2\, X_t\, dW_t + \nu\, dt$$

where W_t represents Brownian motion; note, that again the diffusion coefficient becomes small near the origin. This also connects the singularity phenomenon of the Fleming-Viot particle system to the one raised by the stochastic spikes. An important reason for investigating (generalized) Bessel processes is the anticipation of a phase transition: For the parameter ν sufficiently small a singularity may occur in the sense that there might be infinitely many jumps in finite time whereas this can not happen for ν large. For another work in which a particle system with singular behavior is studied exhibiting a phase transition similar to the case of Bessel diffusions we refer to [32].

2.1. Stochastic spikes

Motivated by applications in Quantum Mechanics Bauer and Bernard investigated in the recent contribution [5] scaling limits $\lambda \to \infty$ and $\varepsilon \to 0$ for classes of stochastic differential equations of the form

$$dX_t = \frac{\lambda^2}{2}(\varepsilon \cdot b_1(X_t) - b_2(X_t))\,dt + \lambda \cdot \sigma(X_t)\,dB_t \tag{2.1}$$

with functions b_1, b_2 and σ. More precisely, in case of constant $b_1 > 0$ and linear b_2 and σ, i.e. for stochastic differential equations of the form

$$dX_t = \frac{\lambda^2}{2}(\varepsilon - bX_t)\,dt + \lambda \cdot X_t\,dB_t \tag{2.2}$$

Bauer and Bernard rigorously study the non-trivial scaling limit of the process $(X_t)_{t\geq 0}$ in the regime $\lambda \to \infty$ and $\varepsilon \to 0$ such that $\lambda^2\varepsilon^{b+1}$ is constant and conjecture the validity of similar assertions for a larger class of stochastic differential equations of the type (2.1). In this scaling limit the first hitting time of a level z for the diffusion (2.2) started at $x < z$ converges in distribution to a mixture of a point mass in zero and an exponential distributed random variable.

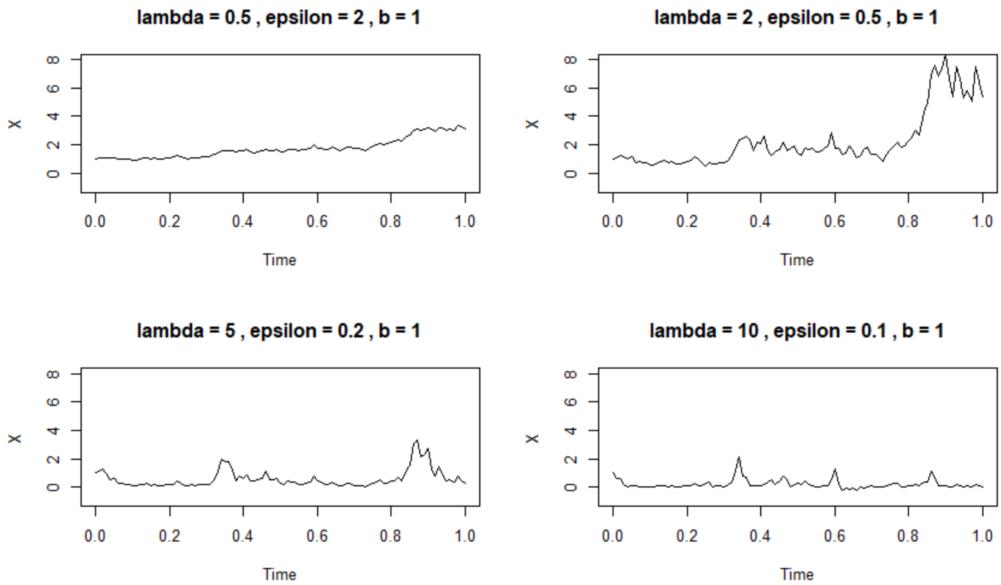


Figure 2.1.: Solution samples to the SDE (2.2) for different parameters obtained using the R package sde: `https://CRAN.R-project.org/package=sde`. The source code can be found in Section C.1 in the Appendix. The images should be thought of as conceptual visualization aids rather than robust numerical simulation results.

Related questions for a slightly different model have previously been physically motivated and then analyzed by Bauer, Bernard and Tilloy in [47] and [6]. Observe that the diffusion given by (2.2) is scale invariant, a fact which allows specific arguments and simplifies several calculations. Bauer and Bernard in particular proved that in the scaling limit $\lambda \to \infty$ and $\varepsilon \to 0$ with $\lambda^2 \varepsilon^{b+1} = J$ constant the first hitting time of a level z with start from $x < z$ converges in distribution to a convex combination of a exponential distributed random variable and the trivial random variable which is constant equal to zero. These results are called large noise limits. Furthermore, the authors also deduce a Poisson approximation for the number of hits above the level z. The analytic approach of Bauer and Bernard allows to cover also certain types of stochastic differential equations which are different from (2.2) but still share the property of scale invariance. Using non-rigorous arguments the authors of [5] come to the conjecture that the results will carry over to a larger rather general class of stochastic differential equations and they provide certain natural but not always precisely formulated conditions, under which the results are expected to hold. Our main aim is to provide a different rather elementary approach to the results of Bauer and Bernard, which allows to prove analogous results for general classes of stochastic differential equations, which do not necessarily satisfy a form of scale invariance. In particular we can extend the results to a 'linearized version' of the stochastic differential equation describing the homodyne detection of Rabi oscillations. The resulting stochastic differential equation has a clear quantum mechanical background which is in more detail described in [5]. As a fact we will mainly rely on classical methods from probability theory such as Poisson approximation and some further mainly basic properties of diffusion processes. This is in contrast to the tools used by Bauer and Bernard which are analytic i.e. based on analysis of differential equations and basic Itô theory for diffusions. Apart from extending the validity of the results to a larger class of stochastic differential equations we believe that our approach helps to put the results in a clear probabilistic perspective.

Let us stress that the results are related to known assertions about hitting times of large levels for diffusion processes such as e.g. [42] and [11]. There the authors consider the behavior of hitting times of a high level and deduce that in an appropriate scaling limit this hitting time is exponentially distributed. We want to point out that in the case of a non scale-invariant diffusion it does not seem possible to directly use known theorems concerning the extreme value behavior of hitting of large sets as given e.g. in [42] and [11]. In the case of equation (2.2) it is possible to connect the hitting of a fixed level z when started from ε into the question of hitting z/ε with start in 1. For this situation one can make direct use of known results on asymptotic exponentiality, see e.g. [11] and [42, paragraph 2, section V]. For start in a fixed point x and for more non scale-invariant equations this does not seem possible. In any case due to the connections to the theory of quantum systems under continuous measurement we believe that our results and methods - which might not be that well known in the physics community - are of sufficiently broad interest and are useful in order to derive results for the most interesting higher dimensional situation.

Let us also mention the preprint [8] containing further results concerning the fine structure of the spikes in large noise limits of general classes of SDE's.

For the sake of having a compact overview let us summarize our new results:

- We facilitate probabilistic tools and methods and recover e.g. [5, Corollary 3].
- With our approach we are able to generalize the assumptions on the SDE coefficient functions and give abstract criteria (A1) to (A3), (B1) to (B2) in order for the generalized scaling limit to exhibit a Poisson process. (Cf. Theorem 4.19.)
- We apply the theorem to two classes of SDEs. The first one is a narrower generalization of [5, Equation (3)] and the second one is motivated by a quantum mechanical problem.
- The results are also published as a journal article in [36].

2.2. Fleming Viot particle systems

In [15], the authors investigate the following model which was introduced in [16] and is related to many known ideas in probability and physics as laid out in [15, Section 3]: Consider a fixed open connected subset of the Euclidean space as domain and a system of particles starting at some deterministic point. As long as no particle hits the boundary of the domain they move independently according to Brownian motion. If a particle does hit the boundary, it jumps to the position of some independently uniformly randomly chosen other one keeping the number of particles constant. Then, they again move independently according to Brownian motion, and so on. In [15] different limit theorems for bounded domains are proven, among them one states, as the number of particles grows, the empirical measure at some finite time converges to the law of Brownian motion conditioned to stay inside the domain. Letting the time tend to infinity the distribution converges to the quasi-stationary distribution. Thus, the Fleming Viot particle system may also serve as a numerical approximation algorithm as e.g. in [39].
Crucial for the model to be well-defined is the particle system not jumping infinitely often in finite time. This is also referred to as *blow up* or *explosion* when viewed as Markov process or *extinction* if interpreted as particle system or population model. The proof of the corresponding assertion [15, Theorem 1.1] was delivered later on in [9, Theorem 5.4] with additional constraints on the domain's boundary. Another attempt can be found in [40, Theorem 7]. Sufficient criteria for non-extinction in different directions of generality concerning regularity conditions of the domain's boundary, the underlying random motion and the jumping mechanism not necessarily to be uniformly have been proven. They include [9, Theorem 5.4] for Lipschitz domains, [28, Theorem 1] with certain abstractly defined conditions on the domain, the underlying diffusion processes and the jumping mechanism and [48, Theorem 2.1] where the diffusion processes which drive the particles between the jumps may depend on the particles and their coefficients are not necessarily bounded.
As proposed in [10] we consider the case of generalized Bessel processes on $(0, \infty)$ with one common real parameter value ν and uniform jumping. [10, Theorem 1.1(i)] states that in the case of only $N = 2$ particles the system explodes almost surely if and only if $\nu < 0$. This indicates that the drift parameter really may impact the particle system's behavior

in terms of non-extinction. As a second part [10, Theorem 1.1(ii)] entails that for $\nu \geq 2/N$ there is no explosion almost surely. In light of the discussed case of two particles the bound $2/N$ is seen to be not sharp. Our aim is to achieve a better understanding in the situation of three or more particles and this gap. Arguments valid for general diffusion processes such as suitable couplings as in e.g. [48, Section 2.3], [28, Proposition 3] or [9, proof of Theorem 5.4] do not seem to suffice for a finer analysis. Instead, we use the model's symmetry due to the uniformly jumping mechanism and properties specific to generalized Bessel processes, especially the self similarity analogous to Brownian scaling, to obtain a structural result on the limit of jump times (cf. Corollary 5.29) implying a sufficient condition for that limit to be infinite almost surely and one for it to be finite almost surely (cf. Theorem 5.71). Then, in the application to the case of three moving particles, explicit calculations with the generalized Bessel transition probabilities are carried out to estimate a certain functional where the very influence of the jumps is rediscovered in the formulas (cf. remark 5.74). In view of [10, Open Problem 1.5] Theorem 5.78 in some sense gives an affirmative answer to the converse question: There exists a Fleming-Viot-type process with extinction almost surely, for the 2-particle system, but non-extinction with probability one for the 3-particle system. This illustrates that generally speaking adding another particle to the system potentially does cause non-extinction.
Models where the underlying random motion is a continuous time Markov chain on a countable state space have also been investigated, for instance in [26], [2], [3], [30] and [21]. In our setting the jump times are determined by the positions of the particles referred to as *hard killing*. In the *soft killing* case random times are invoked in order to realize the jump times. Fleming-Viot-type processes form an active research area. Recent works include: [20], [33], [18], [19], [29], [7] and [43].

Let us again give a list of our new contributions:

- We manipulate the problem and analyze the underlying structure exhibiting an ergodic Hidden Markov Model. (Cf. Proposition 5.65.)
- We manage to give sufficient conditions for explosion and non-explosion. (Cf. Theorem 5.71.)
- Using the criterion we infer that in the case of three moving particles there are negative values for the drift parameter such that there is no explosion almost surely. This particularly shows that even $\nu \geq 0$ is not a sharp boundary for general N.
- We submitted a manuscript of our work to the Electronic Journal of Probability.

3. One-dimensional time-homogeneous SDEs

The aim of the third chapter is to collect basic propositions on one dimensional time-homogeneous SDEs as can be found in the literature. We use a unified notation and later on we want to calculate certain expectation values of some functionals of hitting times. We mainly orient ourselves by the book [34] of Ioannis Karatzas and Steven E. Shreve. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ a filtrated probability space and let us start with a basic definition.

Definition 3.1. *A Brownian motion $(B_t)_{t\geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$* or *$\mathcal{F}_t$-Brownian motion* for short, is a Brownian motion adapted to $\mathcal{F}_t$ with the increment $B_t - B_s$ independent of $\mathcal{F}_s$ for all $0 \leq s < t$.

3.1. Notions of solution and uniqueness

We follow [34, Section 5.2] to lay out the theoretical framework. In order to develop the concept of *strong solution*, we choose a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as well as a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ with respect to its natural filtration on it. We assume also that this space is rich enough to accommodate a real random variable ξ independent of $\sigma(B_s; s \geq 0)$ with given distribution. We consider the left-continuous filtration

$$\mathcal{G}_t := \sigma(\xi, B_s; 0 \leq s \leq t); \quad 0 \leq t < \infty,$$

$$\mathcal{G}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{G}_t\right)$$

as well as the collection of null sets

$$\mathcal{N} := \{N \subseteq \Omega : \exists G \in \mathcal{G}_\infty \text{ with } N \subseteq G \text{ and } \mathbb{P}(G) = 0\}$$

and create the *augmented filtration*

$$\mathcal{F}_t := \sigma(\mathcal{G}_t \cup \mathcal{N}), \quad 0 \leq t < \infty; \quad \mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right). \tag{3.1}$$

The process B_t is a Brownian motion also with respect to $\mathcal{G}_t$ by the assumed independence of ξ and is furthermore a Brownian motion with respect to $\mathcal{F}_t$ since adding null sets does not disturb the independency of the increments. It can be shown that $\mathcal{F}_t$ is continuous and therefore satisfies the usual conditions.

Definition 3.2. (Cf. [34, Definition 5.2.1].) A *strong solution* of the stochastic differential equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \tag{3.2}$$

with Borel-measurable real functions $b(x)$ and $\sigma(x)$, on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with respect to the fixed Brownian motion B_t and initial condition ξ, is a process $(X_t)_{t\geq 0}$ with continuous sample paths and with the following properties:

(i) X_t is adapted to the filtration $\mathcal{F}_t$ of (3.1),

(ii) $X_0 = \xi$ holds $\mathbb{P}$-a.s.

(iii) $\int_0^t |b(X_s)| + \sigma^2(X_s)\,ds < \infty$ holds $\mathbb{P}$-a.s. for every $t \in [0, \infty)$ and

(iv) $X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s;\ \ \forall t \in [0, \infty)$ holds $\mathbb{P}$-a.s.

From now on we always assume the functions $b(x)$ and $\sigma(x)$ to be measurable.
We cite the following existence theorem from [34] where we also use the result of [34, Problem 5.2.12]. The proofs are omitted.

Theorem 3.3. *(Cf. [34, Theorem 5.2.9].) Suppose that the coefficients $b(x)$, $\sigma(x)$ satisfy the global Lipschitz and linear growth conditions*

$$\begin{aligned} |b(x) - b(y)| + |\sigma(x) - \sigma(y)| &\leq K|x - y|, \\ b(x)^2 + \sigma^2(x) &\leq K^2(1 + x^2), \end{aligned}$$

for every $x, y \in \mathbb{R}$, where K is a positive constant. On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let ξ be a real random variable, independent of the one-dimensional Brownian motion B_t with respect to its natural filtration. Let $\mathcal{F}_t$ be as in (3.1)*. Then there exists a continuous, $\mathcal{F}_t$-adapted process X_t which is a strong solution of equation* (3.2) *relative to B_t, with initial condition ξ.*

Definition 3.4. (Cf. [34, Definition 5.3.1].) A *weak solution* of equation (3.2) is a triple $(X_t, B_t), (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_t$, where

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{F}_t$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions,

(ii) X_t is a continuous, $\mathcal{F}_t$-adapted real valued process, B_t is a one-dimensional Brownian motion with respect to $\mathcal{F}_t$ and

(iii), (iv) of Definition 3.2 are satisfied.

The probability measure $\mathbb{P}(X_0 \in \cdot)$ is called the *initial distribution* of the solution.

Definition 3.5. (Cf. [34, Definition 5.3.2].) Suppose that whenever (X_t, B_t), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t$ and $(\tilde{X}_t, B_t)$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\tilde{\mathcal{F}}_t$ are weak solutions to (3.2) with common Brownian motion B_t (relative to possibly different filtrations) on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with common initial value, i.e. $\mathbb{P}(X_0 = \tilde{X}_0) = 1$, the two processes X_t and $\tilde{X}_t$ are indistinguishable: $\bigcap_{t\geq 0}\{X_t = \tilde{X}_t\}$ holds $\mathbb{P}$-a.s. We say then that *pathwise uniqueness holds for equation* (3.2).

The following uniqueness result is also taken from [34] taking into account [34, Remark 5.3.3]. Again, we omit the proofs.

Theorem 3.6. *(Cf. [34, Theorem 5.2.5].) Suppose that the coefficients $b(x)$ and $\sigma(x)$ are locally Lipschitz-continuous, i.e. for every integer $n \geq 1$ there exists a constant $K_n > 0$ such that for every $|x| \leq n$ and $|y| \leq n$:*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K_n|x - y|.$$

Then pathwise uniqueness holds for equation (3.2).

There is also the following relation. For a more detailed description and the proof we refer to [34].

Theorem 3.7. *(Cf. [34, Corollary 5.2.23].) The existence of a weak solution and pathwise uniqueness imply the existence of a strong solution on any sufficiently rich probability space.*

Definition 3.8. (Cf. [34, Definition 5.3.4].) We say that *uniqueness in the sense of probability law* holds for equation (3.2) if, for any two weak solutions (X_t, B_t), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t$ and $(\tilde{X}_t, \tilde{B}_t)$, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, $\tilde{\mathcal{F}}_t$ with the same initial distribution, i.e.

$$\mathbb{P}(X_0 \in \Gamma) = \tilde{\mathbb{P}}(\tilde{X}_0 \in \Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}),$$

the two processes (X_t) and $(\tilde{X}_t)$ have the same law.

We cite the following result due to Yamada and Watanabe (1971) and omit the proof which can be found in [34].

Theorem 3.9. *(Cf. [34, Proposition 5.3.20].) Pathwise uniqueness implies uniqueness in the sense of probability law.*

Notation 3.10. For $a, b \in \mathbb{R}$ we denote as $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ the minimum and maximum of a and b.

Definition 3.11. For $l < a < b < r$ we define the exit time

$$T_{a,b} := \inf\{t \geq 0 : X_t \notin (a, b)\}.$$

Definition 3.12. (Cf. [34, Definition 5.5.20].) A *weak solution in the interval $I = (l, r)$* of equation (3.2) is a triple (X_t, B_t), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t$, where

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\mathcal{F}_t$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions,

(ii) X_t is a continuous, $\mathcal{F}_t$-adapted, $[l, r]$-valued process with $X_0 \in I$ a.s., and B_t is a one-dimensional Brownian motion with respect to $\mathcal{F}_t$,

(iii) with $(l_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ strictly monotone sequences satisfying $l < l_n < r_n < r$, $\lim_{n\to\infty} l_n = l$, $\lim_{n\to\infty} r_n = r$ and

$$S_n := T_{l_n,r_n}, \quad n \in \mathbb{N},$$

it holds

$$\int_0^{t\wedge S_n} |b(X_s)| + \sigma^2(X_s)\, ds < \infty \quad \mathbb{P}\text{-a.s. for every } t \in [0, \infty), n \in \mathbb{N}$$

and for every $n \in \mathbb{N}$ it holds $\mathbb{P}$-a.s.

$$X_{t\wedge S_n} = X_0 + \int_0^t b(X_s)\mathbb{1}_{\{s\le S_n\}}\, ds + \int_0^t \sigma(X_s)\mathbb{1}_{\{s\le S_n\}}\, dB_s; \;\; \forall t \in [0, \infty)$$

We refer to

$$S := \inf\{t \ge 0 : X_t \notin (l, r)\} = \lim_{n\to\infty} S_n$$

as the *exit time* from I. A weak solution in the interval I with the property that $S = \infty$ holds $\mathbb{P}$-a.s. is a weak solution in the sense of Definition 3.4 with the feature that X_t is I-valued $\mathbb{P}$-a.s. In the case of $S < \infty$ we stipulate that the processes freezes from S on, that is $X_t := X_S$ for $t \in [S, \infty)$, so that the notion of uniqueness in law carries over to weak solutions in the interval (l, r) with solutions being $[l, r]$-valued.

There is the following important existence and uniqueness result. The assertion is a consequence of [34, Theorem 5.5.15] and the proof is omitted here.

Theorem 3.13. *Assume that $I = (l, r)$ is an interval and the coefficient functions $\sigma : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ of the SDE* (3.2) *fulfill the conditions*

(i) $\sigma^2(x) > 0$ for all $x \in I$.

(ii) For all $x \in I$ there exists $\varepsilon > 0$ such that $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|b(y)|}{\sigma^2(y)}\, dy < \infty$.

Then the equation (3.2) *has a weak solution in the interval I and this solution is unique in the sense of probability law.*

3.2. Scale function

The following discussion is based on [34, Section 5.5 B. The Method of Removal of Drift]. We consider equation (3.2) on an interval $I = (l, r) \subseteq (-\infty, \infty)$.

Definition 3.14. For real-valued, twice continuously differentiable functions $f \in C^2(I)$ on I we define the second order differential operator Lf as

$$(Lf)(x) \coloneqq \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x). \tag{3.3}$$

For a suitable subclass of functions $f \in C^2(I)$ the relation

$$\lim_{t\downarrow 0} \frac{\mathbb{E}_x[f(X_t) - f(x)]}{t} = (Lf)(x), \quad x \in I$$

holds and therefore the operator L is referred to as the *infinitesimal generator*. Later, we will not strictly require the function f to be twice continuously differentiable but that f has absolutely continuous derivative on compact subintervals of I. There, f'' exists almost everywhere and the Lebesgue-null set where it does not will not play a substantial role.

Lemma 3.15. *Let (X_t, B_t) be a weak solution in the interval I of equation (3.2) and let $f : I \to \mathbb{R}$ be a function whose derivative is absolutely continuous on any compact subinterval of I. Then for any $T_{a,b}$, $l < a < b < r$ it holds*

$$f(X_{t\wedge T_{a,b}}) = f(X_0) + \int_0^{t\wedge T_{a,b}} Lf(X_s)\,ds + \int_0^{t\wedge T_{a,b}} f'(X_s)\,\sigma(X_s)\,dB_s$$

in the sense that $Lf(x)$ may be chosen arbitrary on the Lebesgue-null set where f'' does not exist.

Proof. Even though f may fail to be twice continuously differentiable we may still apply a generalized version of Ito's formula (cf. [34, Problem 3.7.3]) and obtain

$$\begin{aligned} f(X_{t\wedge T_{a,b}}) &= f(X_0) + \int_0^{t\wedge T_{a,b}} f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s \\ &= f(X_0) + \int_0^{t\wedge T_{a,b}} f'(X_s)\,b(X_s)\,ds + \int_0^{t\wedge T_{a,b}} f'(X_s)\,\sigma(X_s)\,dB_s + \frac{1}{2}\int_0^{t\wedge T_{a,b}} f''(X_s)\,\sigma^2(X_s)\,ds \\ &= f(X_0) + \int_0^{t\wedge T_{a,b}} Lf(X_s)\,ds + \int_0^{t\wedge T_{a,b}} f'(X_s)\,\sigma(X_s)\,dB_s. \end{aligned}$$

□

In what follows we impose a *nondegeneracy* condition (ND) on the coefficients $\sigma : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$:

$$\sigma^2(x) > 0 \quad \text{for all } x \in I \tag{ND}$$

as well as a *local integrability* condition (LI) :

$$\text{for all } x \in I \text{ there exists } \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|b(y)|}{\sigma^2(y)}\,dy < \infty. \tag{LI}$$

These conditions are sometimes also referred to as the *Engelbert-Schmidt* conditions. We have already met them in Theorem 3.13.

Definition 3.16. Under the assumptions (ND) and (LI) we define for $c \in I$ the *scale function*

$$s_c(x) := \int_c^x \exp\left(-2\int_c^\xi \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)d\xi$$

for $x \in I$.

The scale function is strictly increasing and due to (LI) the integrand $\exp\left(-2\int_c^\xi \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)$ is continuous in the integration variable ξ. Therefore, the scale function is continuously differentiable with derivative

$$s_c'(x) = \exp\left(-2\int_c^x \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)$$

which is a strictly positive function on I and absolutely continuous on compact subintervals of I. The second derivative s_c'' exists almost everywhere on I and satisfies

$$s_c''(x) = \exp\left(-2\int_c^x \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)\cdot\left(-2\frac{b(x)}{\sigma^2(x)}\right) = -\frac{2b(x)}{\sigma^2(x)}s_c'(x).$$

Henceforth, whenever we write s_c'', we shall mean the function defined by

$$s_c''(x) := -\frac{2b(x)}{\sigma^2(x)}s_c'(x), \quad x \in I. \tag{3.4}$$

We further extend s_c to $[-l, r]$ by

$$s_c(l) := \lim_{x\downarrow l} s_c(x) \in \mathbb{R}\cup\{-\infty\} \quad \text{and} \quad s_c(r) := \lim_{x\uparrow r} s_c(x) \in \mathbb{R}\cup\{\infty\}.$$

Lemma 3.17. *(Cf. [34, Problem 5.5.12]) For $a, c \in I$ it holds*

$$s_a(x) = s_a(c) + s_a'(c)s_c(x).$$

Particularly, the finiteness neither of $s_c(l)$ nor of $s_c(r)$ does depend on the choice of c.

Proof. It holds

$$\begin{aligned} s_a'(c)\cdot s_c(x) &= \exp\left(-2\int_a^c \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)\cdot\int_c^x \exp\left(-2\int_c^\xi \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)d\xi \\ &= \int_c^x \exp\left(-2\int_a^\xi \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)d\xi \end{aligned}$$

and

$$s_a(x) - s_a(c) = \int_a^x \exp\left(-2\int_a^\xi \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)\,d\xi - \int_a^c \exp\left(-2\int_a^\xi \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)\,d\xi$$
$$= \int_c^x \exp\left(-2\int_a^\xi \frac{b(\zeta)}{\sigma^2(\zeta)}\,d\zeta\right)\,d\xi$$

which shows the asserted equation

$$s_a(x) = s_a(c) + s_a'(c)s_c(x).$$

Particularly, $s_a(x)$ is an affine transformation of $s_c(x)$ with positive coefficient $s_a'(c) > 0$ and therefore $s_c(l) = -\infty$ implies $s_c(a) = -\infty$ and also $s_c(r) = \infty$ implies $s_a(r) = \infty$. Since the choice of $c, a \in I$ was arbitrary the assertion follows. □

Lemma 3.18. *For all $c, x \in I$ it holds $(Ls_c)(x) = 0$ in the sense that the second derivative of s_c needed to apply the operator L is taken by* (3.4)*.*

Proof. By definition (3.4) it holds for $x \in I$

$$s_c''(x) := -\frac{2b(x)}{\sigma^2(x)}s_c'(x).$$

After multiplication with $\sigma^2(x)/2$ the equation reads

$$\frac{1}{2}\sigma^2(x)s_c''(x) + b(x)s_c'(x) = 0.$$

The left hand side is $(Ls_c)(x)$ and the assertion follows. □

Lemma 3.19. *For (X_t, B_t) a weak solution in the interval I of equation* (3.2) *with starting point $X_0 = x_0 \in I$ under (ND) and (LI) for any $T_{a,b}$, $l < a < b < r$ the process $(s_c(X_{t\wedge T_{a,b}}))_{t\geq 0}$ is a local martingale.*

Proof. By continuity of $s_c : I \to \mathbb{R}$ the paths $t \to s_c(X_{t\wedge T_{a,b}})$ are continuous. Combining Lemma 3.15 and Lemma 3.18 we obtain

$$s_c(X_{t\wedge T_{a,b}}) = s_c(x_0) + \int_0^{t\wedge T_{a,b}} s_c'(X_s)\,\sigma(X_s)\,dB_s \tag{3.5}$$

which corresponds to [34, Expression (5.5.49)]. The assertion follows, since Ito integrals with respect to Brownian motion are local martingales. □

Lemma 3.20. *For (X_t, B_t) a weak solution in the interval I of equation* (3.2) *with starting point $X_0 = x \in I$ under (ND) and (LI) for any $T_{a,b}$, $l < a < b < r$ the exit probabilities are given by*

$$\mathbb{P}_x(X_{T_{a,b}} = a) = \frac{s_c(b) - s_c(x)}{s_c(b) - s_c(a)} \quad \textit{and} \quad \mathbb{P}_x(X_{T_{a,b}} = b) = \frac{s_c(x) - s_c(a)}{s_c(b) - s_c(a)}. \tag{3.6}$$

These expressions do not depend on the particular choice of $c \in I$.

Proof. In the next subsection we will establish that under the given assumptions $\mathbb{E}_x[T_{a,b}] < \infty$ which particularly implies $T_{a,b} < \infty$ a.s. We will already use the latter fact in this proof and refer to Proposition 3.31. Let

$$\tau_n := \inf\left\{ t \geq 0 : \int_0^t \sigma^2(X_s)\, ds \geq n \right\}, \quad n \in \mathbb{N}. \tag{3.7}$$

As in equation (3.5) it holds for $n \in \mathbb{N}$

$$s_c(X_{t\wedge T_{a,b}\wedge\tau_n}) = s_c(x_0) + \int_0^{t\wedge T_{a,b}\wedge\tau_n} s_c'(X_s)\,\sigma(X_s)\,dB_s.$$

The function s_c' is continuous on the compact interval $[a, b]$ and therefore bounded and we deduce

$$\mathbb{E}_x\left[\int_0^{t\wedge T_{a,b}\wedge\tau_n} s_c'(X_s)\,\sigma(X_s)\,dB_s\right] = 0$$

whence $\mathbb{E}_x[s_c(X_{t\wedge T_{a,b}\wedge\tau_n})] = s_c(x_0)$ holds for any $n \in \mathbb{N}$. Since s_c is continuous and bounded as function on $[a, b]$ we may pass to the limit $n \to \infty$ and infer $\mathbb{E}_x[s_c(X_{t\wedge T_{a,b}})] = s_c(x_0)$. With the same argument and using $T_{a,b} < \infty$ a.s. (cf. Proposition 3.31) we may pass to the limit $t \to \infty$ so that finally

$$s_c(x_0) = \mathbb{E}_x[s_c(X_{T_{a,b}})] = \mathbb{P}_x(X_{T_{a,b}} = a)\cdot s_c(a) + \mathbb{P}_x(X_{T_{a,b}} = b)\cdot s_c(b).$$

It follows

$$\mathbb{P}_x(X_{T_{a,b}} = a) = \frac{s_c(b) - s_c(x)}{s_c(b) - s_c(a)} \quad \text{and} \quad \mathbb{P}_x(X_{T_{a,b}} = b) = \frac{s_c(x) - s_c(a)}{s_c(b) - s_c(a)}.$$

The independency of c is a consequence of Lemma 3.17. □

3.3. Speed measure and Green function

We still work under the framework of a weak solution (X_t, B_t) in an interval $I = (l, r) \subseteq \mathbb{R}$ of equation (3.2) assuming (ND) and (LI) to hold.

Definition 3.21. (Cf. [34, Expression (5.5.51)].) Under those assumptions we introduce the *speed measure*

$$m_c(dx) := \frac{2}{s_c'(x)\sigma^2(x)}\,dx$$

for $x \in I$.

The density $\frac{2}{s_c'(x)\sigma^2(x)}$ is locally integrable by the positivity and continuity of $2/s_c'(x)$ and the local integrability assumption on $1/\sigma^2(x)$.

Definition 3.22. (Cf. [34, Expression (5.5.52)].) Furthermore the *Green function on* $[a, b] \subsetneq I$ is given by

$$G_{a,b,c}(x, y) := \frac{(s_c(x\wedge y) - s_c(a))(s_c(b) - s_c(x\vee y))}{s_c(b) - s_c(a)}, \quad x, y \in [a, b].$$

Since the scale function is strictly increasing for all $x, y \in [a, b]$ the Green function $G_{a,b,c}(x, y) \geq 0$ is nonnegative.

Notation 3.23. We will consider locally integrable distributions f as generalized functions on (a, b) as usually by identifying them with the linear functional

$$\phi \mapsto \int_a^b f(x)\,\phi(x)\,dx$$

for infinitely differentiable and in (a, b) compactly supported supported test functions $\phi \in C_c^\infty(a, b)$.

In particular, if for two distributions f and g on (a, b) it holds

$$\int_a^b (f(x) - g(x))\,\phi(x)\,dx = 0, \quad \phi \in C_c^\infty(a, b),$$

we will say $f = g$ in the sense of distributions.

Definition 3.24. We denote δ for the Dirac Delta distribution such that

$$\int_a^b \delta(x - y)\,\phi(x)\,dx = \phi(y), y \in (a, b), \phi \in C_c^\infty(a, b).$$

Definition 3.25. The derivative f' of the distribution f is defined to be the distribution satisfying

$$\int_a^b f'(x)\phi(x)\,dx = -\int_a^b f(x)\,\phi'(x)\,dx, \quad \phi \in C_c^\infty(a, b).$$

Lemma 3.26. *Let $y \in (a, b)$. The mapping $x \mapsto x \wedge y$ on $x \in (a, b)$ has first derivative*

$$\frac{\partial}{\partial x}[x \wedge y](x, y) = \mathbb{1}_{x<y}(x, y) := \begin{cases} 1, & x < y, \\ 0, & x \geq y. \end{cases}$$

and second derivative

$$\frac{\partial^2}{\partial x^2}[x \wedge y](x, y) = \frac{\partial}{\partial x}\mathbb{1}_{x<y}(x, y) = -\delta(x - y)$$

in the sense of distributions.

Proof. The function $(a, b) \to (a, b)$, $x \mapsto x \wedge y$ is continuous and has classical derivatives 1 for $x < y$ and 0 for $x > y$. Therefore,

$$\begin{aligned} &-\int_a^b (x \wedge y)\phi'(x)\,dx = -\int_a^y x\phi'(x)\,dx - \int_y^b y\phi'(x)\,dx \\ &= \int_a^y \phi(x)\,dx - [x\phi(x)]_{x=a}^{x=y} - [y\phi(x)]_{x=y}^{x=b} = \int_a^y \phi(x)\,dx = \int_a^b \mathbb{1}_{x<y}(x, y)\,\phi(x)\,dx. \end{aligned}$$

For the second derivative

$$-\int_a^b \mathbb{1}_{x<y}(x,y)\,\phi'(x)\,dx = -\int_a^y \phi'(x)\,dx = -\phi(y) = -\int_a^b \delta(x-y)\,\phi(x)\,dx.$$

□

Lemma 3.27. *Let $y \in (a,b)$. In the sense of distributions the following equations hold*

$$\begin{cases} LG_{a,b,c}(\bullet,y)(x)\,m_c(dy) = -\delta(x-y)\,dy, & x \in (a,b), \\ G_{a,b,c}(a,y) = G_{a,b,c}(b,y) = 0. \end{cases}$$

Proof. By Lemma 3.26, in the sense of distributions

$$\frac{\partial}{\partial x}[s_c(x \wedge y) - s_c(a)](x,y) = s_c'(x \wedge y)\cdot \mathbb{1}_{x<y}(x,y) = s_c'(x)\mathbb{1}_{x<y}(x,y)$$

and

$$\frac{\partial}{\partial x}[s_c'(x)\cdot \mathbb{1}_{x<y}(x,y)](x,y) = s_c''(x)\cdot \mathbb{1}_{x<y}(x,y) - s_c'(x)\delta(x-y).$$

Since $x \vee y = x + y - (x \wedge y)$ it similarly holds

$$\frac{\partial}{\partial x}[s_c(b) - s_c(x \vee y)](x,y) = -s_c'(x \vee y)\cdot \mathbb{1}_{x>y}(x,y) = -s_c'(x)\mathbb{1}_{x>y}(x,y)$$

and

$$\frac{\partial}{\partial x}[s_c'(x)\cdot \mathbb{1}_{x>y}(x,y)](x,y) = s_c''(x)\cdot \mathbb{1}_{x>y}(x,y) + s_c'(x)\delta(x-y).$$

Therefore, the mapping $x \mapsto G_{a,b,c}(x,y)$ has derivative

$$\frac{\partial}{\partial x}G_{a,b,c}(x,y) = \frac{s_c'(x)\mathbb{1}_{x<y}(x,y)(s_c(b)-s_c(y)) - (s_c(y)-s_c(a))s_c'(x)\mathbb{1}_{x>y}(x,y)}{s_c(b)-s_c(a)}$$

and second derivative using equation (3.4)

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}G_{a,b,c}(x,y) &= \frac{1}{s_c(b)-s_c(a)} \\
&\quad \times \big((s_c''(x)\cdot \mathbb{1}_{x<y}(x,y) - s_c'(x)\delta(x-y))(s_c(b)-s_c(y)) \\
&\quad - (s_c(y)-s_c(a))(s_c''(x)\cdot \mathbb{1}_{x>y}(x,y) + s_c'(x)\delta(x-y))\big) \\
&= \frac{s_c''(x)\mathbb{1}_{x<y}(x,y)(s_c(b)-s_c(y)) - (s_c(y)-s_c(a))s_c''(x)\mathbb{1}_{x>y}(x,y)}{s_c(b)-s_c(a)} - s_c'(x)\delta(x-y) \\
&= -\frac{2b(x)}{\sigma^2(x)}\cdot\frac{\partial}{\partial x}G_{a,b,c}(x,y) - s_c'(x)\delta(x-y).
\end{aligned}$$

Again multiplying with $\sigma^2(x)/2$ yields

$$LG_{a,b,c}(\bullet, y)(x) = -\frac{s_c'(x)\sigma^2(x)}{2}\delta(x-y) = -\frac{s_c'(y)\sigma^2(y)}{2}\delta(x-y),$$

whence it holds

$$LG_{a,b,c}(\bullet, y)(x)m_c(dy) = -\frac{s_c'(y)\sigma^2(y)}{2}\delta(x-y)m_c(dy) = -\delta(x-y)\,dy.$$

The other assertions $G_{a,b,c}(a,y) = G_{a,b,c}(b,y) = 0$ are immediate. □

Definition 3.28. (Cf. [34, Expression (5.5.55)].) We also define for $[a,b] \subsetneq I$ and a bounded continuous function $f : [a,b] \to \mathbb{R}$ the *Green measure of f with start in $x \in [a,b]$* as

$$\begin{aligned} M_{a,b}^f(x) &:= \int_a^b f(y)\, G_{a,b,c}(x,y)\, m_c(dy) \\ &= 2\frac{s_c(b)-s_c(x)}{s_c(b)-s_c(a)} \int_a^x f(y)\, \frac{s_c(y)-s_c(a)}{s_c'(y)\sigma^2(y)}\, dy \\ &\quad + 2\frac{s_c(x)-s_c(a)}{s_c(b)-s_c(a)} \int_x^b f(y)\, \frac{s_c(b)-s_c(y)}{s_c'(y)\sigma^2(y)}\, dy. \end{aligned} \tag{3.8}$$

We omitted the notation of $c \in I$ in the definition of $M_{a,b}^f(x)$ since as consequence of Lemma 3.17 the value of $M_{a,b}^f(x)$ does not depend on the choice of c.

By the nonnegativity of the Green function also $M_{a,b}(x,y) \geq 0$ is nonnegative for all $x, y \in [a,b]$ provided that the function $f(x) \geq 0$ is nonnegative for all $x \in [a,b]$.

Lemma 3.29. *(Cf. [34, Expression (5.5.53) and Expression (5.5.54)].) In the sense of distributions the following equations hold*

$$\begin{cases} LM_{a,b}^f(x) = -f(x), & x \in (a,b), \\ M_{a,b}^f(a) = M_{a,b}^f(b) = 0. \end{cases}$$

Proof. By the previous Lemma it holds

$$\begin{aligned} LM_{a,b}^f(x) &= L\left(\int_a^b f(y)\, G_{a,b,c}(\bullet, y)\, m_c(dy)\right)(x) = \int_a^b f(y) LG_{a,b,c}(\bullet, y)(x)\, m_c(dy) \\ &= -\int_a^b f(y)\, \delta(x-y)\, dy = -f(x) \end{aligned}$$

and the assertions $M_{a,b}^f(a) = M_{a,b}^f(b) = 0$ follows from $G_{a,b,c}(a,y) = G_{a,b,c}(b,y) = 0$. □

Remark 3.30. In the classical sense, in view of equation (3.8) the function $M_{a,b}^f$ may fail to be differentiable since σ^2 is not necessarily continuous. Still, $M_{a,b}^f$ is absolutely continuous and therefore $M^f_{a,b}$ exists except possibly on a set of Lebesgue measure zero and satisfies

$$\begin{aligned}
&\frac{d}{dx}M^f_{a,b}(x)\\
&= -2\frac{s_c'(x)}{s_c(b)-s_c(a)}\int_a^x f(y)\,\frac{s_c(y)-s_c(a)}{s_c'(y)\sigma^2(y)}\,dy + 2\frac{s_c(b)-s_c(x)}{s_c(b)-s_c(a)}f(x)\frac{s_c(x)-s_c(a)}{s_c'(x)\sigma^2(x)}\\
&\quad + 2\frac{s_c'(x)}{s_c(b)-s_c(a)}\int_x^b f(y)\,\frac{s_c(b)-s_c(y)}{s_c'(y)\sigma^2(y)}\,dy - 2\frac{s_c(x)-s_c(a)}{s_c(b)-s_c(a)}f(x)\frac{s_c(b)-s_c(x)}{s_c'(x)\sigma^2(x)}\\
&= \frac{2s_c'(x)}{s_c(b)-s_c(a)}\left(\int_x^b f(y)\,\frac{s_c(b)-s_c(y)}{s_c'(y)\sigma^2(y)}\,dy - \int_a^x f(y)\,\frac{s_c(y)-s_c(a)}{s_c'(y)\sigma^2(y)}\,dy\right). \qquad (3.9)
\end{aligned}$$

Again, the second derivative $\frac{d^2}{dx^2}M^f_{a,b}$ exists except possibly on a set of Lebesgue measure zero and satisfies

$$\begin{aligned}
\frac{d^2}{dx^2}M^f_{a,b}(x) &= \frac{2s_c'(x)}{s_c(b)-s_c(a)}\left(\int_x^b f(y)\,\frac{s_c(b)-s_c(y)}{s_c'(y)\sigma^2(y)}\,dy - \int_a^x f(y)\,\frac{s_c(y)-s_c(a)}{s_c'(y)\sigma^2(y)}\,dy\right)\\
&\quad + \frac{2s_c'(x)}{s_c(b)-s_c(a)}\cdot\left(-f(x)\frac{s_c(b)-s_c(x)+s_c(x)-s_c(a)}{s_c'(y)\sigma^2(y)}\right) \qquad (3.10)\\
&= -\frac{2b(x)}{\sigma^2(x)}\frac{d}{dx}M^f_{a,b}(x) - \frac{2}{\sigma^2(x)}f(x).
\end{aligned}$$

In other words, we again see $LM^f_{a,b}(x) = -f(x)$ where the expressions are well defined and taking equation (3.9) and equation (3.10) as definition the assertion of the lemma holds everywhere. This is in the same spirit as we have done before in (3.4) and the way the authors of [34] follow.

The following proposition justifies the labeling of $M^f_{a,b}(x)$ to be the Green measure of f with start in $x \in [a,b]$.

Proposition 3.31. *(Cf. [34, Exercise 5.5.39].) Assume that (ND) and (LI) holds and let (X_t, B_t), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t$ be a weak solution in $I = (l,r)$ of (3.2) with nonrandom initial condition $X_0 = x \in I$. Then for $[a,b] \subsetneq I$ and a bounded continuous function $f : [a,b] \to \mathbb{R}$ we have the following stochastic representation:*

$$M^f_{a,b}(x) = \mathbb{E}_x\left[\int_0^{T_{a,b}} f(X_s)\,ds.\right]$$

Particularly, X_t exits from every compact subinterval of I in finite expected time.

Proof. As we did with equation (3.7) we define

$$\tau_n := \inf\left\{t \geq 0 : \int_0^t \sigma^2(X_s)\,ds \geq n\right\}, \quad n \in \mathbb{N}.$$

By a generalized version of Ito's lemma such as Lemma 3.15 combined with Lemma 3.29 it holds

$$M_{a,b}^f(X_{t\wedge\tau_n\wedge T_{a,b}}) = M_{a,b}^f(x) - (t\wedge\tau_n\wedge T_{a,b}) + \int_0^{t\wedge\tau_n\wedge T_{a,b}} \left(\frac{d}{dx}M_{a,b}^f(X_s)\right)\sigma(X_s)\,dB_s.$$

After taking expectations on both sides we deduce

$$\mathbb{E}_x[M_{a,b}^f(X_{t\wedge\tau_n\wedge T_{a,b}})] = M_{a,b}^f(x) - \mathbb{E}_x\left[\int_0^{t\wedge\tau_n\wedge T_{a,b}} f(X_s)\,ds\right].$$

We may pass to the limit $n\to\infty$ as we can use Lebesgue's theorem since $M_{a,b}^f$ and f are bounded continuous functions on $[a,b]$ and derive

$$\mathbb{E}_x[M_{a,b}^f(X_{t\wedge T_{a,b}})] = M_{a,b}^f(x) - \mathbb{E}_x\left[\int_0^{t\wedge T_{a,b}} f(X_s)\,ds\right]. \tag{3.11}$$

Applied to the function $f(x) := 1$ this entails rearranged the inequality

$$\begin{aligned}\mathbb{E}_x[t\wedge T_{a,b}] &= \int_a^b G_{a,b,c}(x,y)\,m_c(dy) - \mathbb{E}_x\left[\int_a^b G_{a,b,c}(X_{t\wedge T_{a,b}},y)\,m_c(dy)\right]\\ &\le \int_a^b G_{a,b,c}(x,y)\,m_c(dy) < \infty.\end{aligned}$$

From monotone convergence as $t\to\infty$ it follows that

$$\mathbb{E}_x[T_{a,b}] \le \int_a^b G_{a,b,c}(x,y)\,m_c(dy) < \infty.$$

Particularly, X_t exits from every compact subinterval of I in finite expected time. Consequently, $X_{T_{a,b}}$ exists and by Lebesgue's theorem combined with the boundary values $M_{a,b}^f(a) = M_{a,b}^f(b) = 0$ in Lemma 3.29 it follows

$$\lim_{t\to\infty}\mathbb{E}_x[M_{a,b}^f(X_{t\wedge T_{a,b}})] = \mathbb{E}_x[M_{a,b}^f(X_{T_{a,b}})] = 0.$$

In view of equation (3.11) together with the fact that

$$\left|\int_0^{t\wedge T_{a,b}} f(X_s)\,ds\right| \le \int_0^{T_{a,b}} |f(X_s)|\,ds \in L^1$$

this implies by Lebesgue's theorem

$$M_{a,b}^f(x) = \lim_{t\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge T_{a,b}} f(X_s)\,ds\right] = \mathbb{E}_x\left[\int_0^{T_{a,b}} f(X_s)\,ds\right]$$

and finishes the proof. □

3.4. Feller's Test for Explosions

In this section we conclude the discussion following [34, Section 5.5 C. Feller's Test for Explosions]. We still consider an interval $I = (l, r) \subseteq \mathbb{R}$ and impose conditions (ND) and (LI) on the coefficients of SDE (3.2). The next theorem gives a characterization to the question whether the event $\{S = \infty\}$ has full probability, i.e. there is no explosion in finite time a.s.

Definition 3.32. (Cf. [34, Expression (5.5.65)]) Let us define the *Feller test function* for $x \in I$ as

$$v_c(x) := \int_c^x s_c'(y) \int_c^y \frac{2}{s_c'(z)\sigma^2(z)}\, dz\, dy = \int_c^x \frac{2}{s_c'(z)\sigma^2(z)} \int_z^x s_c'(y)\, dy\, dz$$
$$= \int_c^x s_c(x) - s_c(z)\, m_c(dz).$$

Theorem 3.33 (Feller's test for explosions)**.** *(Cf. [34, Theorem 5.5.29]) In the setting of Proposition 3.31 it holds $\mathbb{P}(S = \infty) = 1$ or $\mathbb{P}(S = \infty) < 1$, according to whether*

$$v_c(l) := \lim_{x \downarrow l} v_c(x) = v_c(r) := \lim_{x \uparrow r} v_c(x) = \infty$$

or not. This in independent of the particular choice of $c \in I$.

Proof. The proof is omitted here and can be found in [34, Theorem 5.5.29] together with [34, Problem 5.5.28]. □

There is the following sufficient criterion.

Lemma 3.34. *(Cf. [34, Proposition 5.5.22(a)]) In the setting of Proposition 3.31 if*

$$s_c(l) = -\infty \quad \textit{and} \quad s_c(r) = \infty, \tag{3.12}$$

then

$$\mathbb{P}_x(S = \infty) = \mathbb{P}_x\left(\sup_{0 \le t < \infty} X_t = r\right) = \mathbb{P}_x\left(\inf_{0 \le t < \infty} X_t = l\right) = 1. \tag{3.13}$$

In particular, the process X_t is recurrent: for every $y \in I$, we have

$$\mathbb{P}_x(X_t = y \textit{ for some } t \in [0, \infty)) = 1. \tag{3.14}$$

Moreover, in the setting of Proposition 3.31 the equations (3.13) *already imply* (3.12)*.*

Proof. The proof is omitted here and can be found in the book of [34]. □

Remark 3.35. Diffusion processes with the property (3.14) are referred to as being *regular*. A boundary point where the scaling function is not finite is called *not attracting* or *repulsive* and is particularly *not attainable* anonymously *not accessible*, i.e. the Feller test function is infinite there and the diffusion process will not reach this point in finite time a.s. Boundary points may further be classified; the interested reader is referred to [35, Table 6.2] where the terminology due to Feller as well as the classification scheme due to Russian probabilists, e.g. Gikhman and Skorokhod, is summarized and compared.

Definition 3.36. Let us still assume (ND) and (LI) to hold. For $b, c \in (l, r)$ or $a, c \in (l, r)$, respectively, we define the *one sided Green functions*

$$G_{l,b,c}(x, y) := s_c(b) - s_c(x \vee y), \quad x, y \in (l, b)$$

or

$$G_{a,r,c}(x, y) := s_c(x \wedge y) - s_c(a), \quad x, y \in (a, r),$$

respectively. For $b, c \in (l, r)$ and $f \in C_b[l, b]$ under the hypothesis that

$$\int_l^b m_c(dy) < \infty$$

or for $a, c \in (l, r)$ and $f \in C_b[a, r]$ under the hypothesis that

$$\int_a^r m_c(dy) < \infty,$$

respectively, we define

$$M_{l,b}^f(x) := \int_l^b f(y)\, G_{l,b,c}(x, y)\, m_c(dy), \quad x \in (l, b)$$

or

$$M_{a,r}^f(x) := \int_a^r f(y)\, G_{a,r,c}(x, y)\, m_c(dy), \quad x \in (a, r),$$

respectively.

Definition 3.37. Let $T_z := \inf\{t \geq 0 : X_t = z\}$ the first hitting time of $z \in \mathbb{R}$.

Proposition 3.38. *In the setting of Proposition 3.31 and under the assumptions for defining $M_{l,b}^f$ or $M_{a,r}^f$, respectively, it holds assuming $\lim_{x \downarrow l} s_c(x) = -\infty$ for $x \in (l, b)$ the following analogues to Proposition 3.31:*

$$M_{l,b}^f(x) = \mathbb{E}_x\left[\int_0^{T_b} f(X_s)\, ds\right]$$

or assuming $\lim_{x \uparrow r} s_c(x) = \infty$ for $x \in (a, r)$ it holds

$$M_{a,r}^f(x) = \mathbb{E}_x\left[\int_0^{T_a} f(X_s)\, ds\right],$$

respectively.

Proof. Let us consider the case where $b, c \in (l, r)$, $f \in C_b[l, b]$, $\lim_{x \downarrow l} s_c(x) = -\infty$ and $x \in (l, b)$. The other assertion of the other case follows completely analogously. We have already seen in Proposition 3.31 that $M_{a,b}^f(x) = \mathbb{E}_x\left[\int_0^{T_{a,b}} f(X_s)\, ds\right]$, i.e.

$$\int_l^b \mathbb{1}_{(a,b)}(y)\, f(y)\, G_{a,b,c}(x, y)\, m_c(dy) = \mathbb{E}_x\left[\int_0^{T_b} \mathbb{1}_{(0,T_a)}(s)\, f(X_s)\, ds\right]. \tag{3.15}$$

The idea is to pass to the limit $a \downarrow l$ on both sides. The left hand side in (3.15) equals

$$\begin{aligned}
&\int_l^b \mathbb{1}_{(a,b)}(y)\, f(y)\, G_{a,b,c}(x,y)\, m_c(dy) \\
&= (s_c(b) - s_c(x)) \int_l^x f(y)\, \mathbb{1}_{(a,x)}(y)\, \frac{s_c(y) - s_c(a)}{s_c(b) - s_c(a)}\, m_c(dy) \\
&\quad + \frac{s_c(x) - s_c(a)}{s_c(b) - s_c(a)} \int_x^b f(y)\, (s_c(b) - s_c(y))\, m_c(dy). \qquad (3.16)
\end{aligned}$$

Since s_c is a increasing function it holds

$$0 \le \mathbb{1}_{(a,x)}(y)\, \frac{s_c(y) - s_c(a)}{s_c(b) - s_c(a)} \le 1$$

for $y \in (l, x)$ and by $f \in C_b[l, b]$ we may use $|f(y)| \le \sup_{x\in[l,b]} |f(x)|$ and combined with the assumption $\int_l^x m_c(dy) \le \int_l^b m_c(dy) < \infty$ Lebesgue's theorem implies

$$\lim_{a\downarrow l} \int_l^x f(y)\, \mathbb{1}_{(a,x)}(y)\, \frac{s_c(y) - s_c(a)}{s_c(b) - s_c(a)}\, m_c(dy) = \int_l^x f(y)\, \lim_{a\downarrow l} \mathbb{1}_{(a,x)}(y) \frac{s_c(y) - s_c(a)}{s_c(b) - s_c(a)}\, m_c(dy).$$

For $y \in (l, x)$ due to $\lim_{x\downarrow l} s_c(x) = -\infty$ it holds

$$\lim_{a\downarrow l} \mathbb{1}_{(a,x)}(y) \frac{s_c(y) - s_c(a)}{s_c(b) - s_c(a)} = 1$$

and therefore

$$\int_l^x f(y)\, \lim_{a\downarrow l} \mathbb{1}_{(a,x)}(y) \frac{s_c(y) - s_c(a)}{s_c(b) - s_c(a)}\, m_c(dy) = \int_l^x f(y)\, m_c(dy)$$

is the overall limit of the first integral in (3.16) as $a \downarrow l$. For the second summand in (3.16) we again use $\lim_{x\downarrow l} s_c(x) = -\infty$ to obtain

$$\begin{aligned}
&\lim_{a\downarrow l} \Big[(s_c(b) - s_c(x)) \int_l^x f(y)\, \mathbb{1}_{(a,x)}(y)\, \frac{s_c(y) - s_c(a)}{s_c(b) - s_c(a)}\, m_c(dy) \\
&\qquad + \frac{s_c(x) - s_c(a)}{s_c(b) - s_c(a)} \int_x^b f(y)\, (s_c(b) - s_c(y))\, m_c(dy) \Big] \\
&= (s_c(b) - s_c(x)) \int_l^x f(y)\, m_c(dy) + \int_x^b f(y)\, (s_c(b) - s_c(y))\, m_c(dy) \\
&= \int_l^b f(y)\, (s_c(b) - s_c(x \vee y))\, m_c(dy) = \int_l^b f(y)\, G_{l,b,c}(x,y)\, m_c(dy) = M^f_{l,b}(x) < \infty
\end{aligned}$$

since $f \in C_b[l, b]$ and $\int_l^b m_c(dy) < \infty$ and $\int_x^b |s_c(y)|\, m_c(dy) < \infty$. We now turn to the right hand side of (3.15). As preparation to apply Lebesgue's theorem we consider that as $a \downarrow l$ the random variables $T_a \wedge T_b$ are non-negative and non-decreasing. Therefore, by the monotone convergence theorem

$$\lim_{a\downarrow l} \mathbb{E}_x[T_a \wedge T_b] = \mathbb{E}_x\left[\lim_{a\downarrow l} T_a \wedge T_b\right] \in [0,\infty].$$

In light of Remark 3.35 the assumption $\lim_{x\downarrow l} s_c(x) = -\infty$ implies $\lim_{a\downarrow l} T_a = \infty$ with $\mathbb{P}_x$-probability 1. Using Proposition 3.31 and the already shown convergence of the left hand side of (3.15), i.e. $\lim_{a\downarrow l} M^f_{a,b}(x) = M^f_{l,b}(x)$, it follows

$$\mathbb{E}_x[T_b] = \lim_{a\downarrow l} \mathbb{E}_x[T_a \wedge T_b] = \lim_{a\downarrow l} \mathbb{E}_x[T_{a,b}] = \lim_{a\downarrow l} M^{x\mapsto 1}_{a,b}(x) = M^{x\mapsto 1}_{l,b}(x) < \infty.$$

Upon using the $\mathbb{P}_x$-a.s. domination

$$\left|\int_0^{T_b} \mathbb{1}_{(0,T_a)}(s)\, f(X_s)\, ds\right| \leq \sup_{x\in[l,b]} |f(x)|\, T_b \in L^1$$

it then follows by Lebesgue's theorem

$$\lim_{a\downarrow l} \mathbb{E}_x\left[\int_0^{T_b} \mathbb{1}_{(0,T_a)}(s)\, f(X_s)\, ds\right] = \mathbb{E}_x\left[\lim_{a\downarrow l}\int_0^{T_b} \mathbb{1}_{(0,T_a)}(s)\, f(X_s)\, ds\right].$$

We may use Lebesgue's theorem once again since

$$\left|\mathbb{1}_{(0,T_a)}(s)\, f(X_s)\right| \leq \sup_{x\in[l,b]} |f(x)| < \infty$$

and $\mathbb{E}_x[T_b] < \infty$ implies $\mathbb{P}_x(\int_0^{T_b} ds = T_b < \infty) = 1$ to obtain

$$\mathbb{E}_x\left[\lim_{a\downarrow l}\int_0^{T_b} \mathbb{1}_{(0,T_a)}(s)\, f(X_s)\, ds\right] = \mathbb{E}_x\left[\int_0^{T_b} \lim_{a\downarrow l} \mathbb{1}_{(0,T_a)}(s)\, f(X_s)\, ds\right] = \mathbb{E}_x\left[\int_0^{T_b} f(X_s)\, ds\right].$$

This shows the asserted convergence of the right hand side of Equation(3.15) and thereby finishes the proof. □

Theorem 3.39 (Generalized Kac's moment formula)**.** *(Cf. [41, Theorem 4.1].) In the setting of Proposition 3.31 with $[a,b] \subsetneq I$ and $c \in I$ if $f : [0,\infty) \to \mathbb{R}$ is a differentiable function such that the mapping $[a,b] \to \mathbb{R}$, $x \mapsto \mathbb{E}_x[f'(T_{a,b})]$ is continuous, it holds*

$$\mathbb{E}_x[f(T_{a,b})] = f(0) + \int_a^b \mathbb{E}_y[f'(T_{a,b})]\, G_{a,b,c}(x,y)\, m_c(dy).$$

Proof. Even though the setting and assumptions in [41] are sligthly different the assertion still applies to our situation. We refer to the [41, proof of Theorem 4.1]. □

Corollary 3.40. *(Cf. [41, Display between Equations (4.4) and (4.5)].) In the setting of Proposition 3.31 with $[a,b] \subsetneq I$ and $c \in I$ it holds*

$$\mathbb{E}_x[(T_{a,b})^2] = 2\int_a^b \mathbb{E}_y[T_{a,b}]\, G_{a,b,c}(x,y)\, m_c(dy).$$

Proof. Either by using $f(t) := t$ in Kac's moment formula or by Proposition 3.31 it holds

$$\mathbb{E}_x[T_{a,b}] = M_{a,b}^{x\mapsto 1}(x) = \int_a^b G_{a,b,c}(x,y)\, m_c(dy).$$

This quantity is continuous in $x \in [a,b]$ by the form of the Green function and the continuity of the scale function s_c and we may therefore use Kac's moment formula for $f(t) := t^2$ which shows the assertion. □

Proposition 3.41. *(Cf. [41, Proposition 4.2].) In the setting of Proposition 3.31 and under the assumptions for defining $M_{l,b}^f$ or $M_{a,r}^f$, respectively, it holds assuming $\lim_{x\downarrow l} s_c(x) = -\infty$ for $x \in (l,b)$ the following analogues to Corollary 3.40:*

$$\mathbb{E}_x[(T_b)^2] = 2\int_l^b \mathbb{E}_y[T_b]\, G_{l,b,c}(x,y)\, m_c(dy)$$

or assuming $\lim_{x\uparrow r} s_c(x) = \infty$ for $x \in (a,r)$ it holds

$$\mathbb{E}_x[(T_a)^2] = 2\int_a^r \mathbb{E}_y[T_a]\, G_{a,r,c}(x,y)\, m_c(dy),$$

respectively.

Proof. The assertion follows from Corollary 3.40 in the spirit as did Proposition 3.38 follow from Proposition 3.31: One considers $a \downarrow l$ or $b \uparrow r$, respectively, and uses Lebesgue's theorem. The details are omitted. □

3.5. Approach using Sturm-Liouville theory

There is also an approach using Sturm-Liouville theory which we want to sketch briefly. We follow [49, Chapter 13]. Let again $I = (l,r) \subseteq \mathbb{R}$ an open real interval. Throughout the section we assume conditions (ND) and (LI) to hold for the coefficients of SDE (3.2).

Definition 3.42. For $c \in I$ let

$$p_c(x) := \exp\left(2\int_c^x \frac{b(z)}{\sigma^2(z)}\, dz\right) > 0, \quad x \in I$$

and

$$r_c(x) := \frac{p_c(x)}{\sigma^2(x)} > 0, \quad x \in I.$$

It then holds $s_c(x) = \int_c^x \frac{1}{p_c(y)}\, dy$ and $m_c(dy) = 2r_c(y)\, dy$. Recall from equation (3.3) that the formal generator of the SDE is given by

$$(Lf)(x) = \frac{1}{2}\sigma^2(x) f''(x) + b(x) f'(x).$$

Since a.e.

$$p_c'(x) = \exp\left(2\int_c^x \frac{b(z)}{\sigma^2(z)}\,dz\right)\cdot 2\frac{b(x)}{\sigma^2(x)} = p_c(x)\cdot 2\frac{b(x)}{\sigma^2(x)}$$

it holds a.e.

$$\begin{aligned}(Lf)(x) &= \frac{1}{2r_c(x)}\left(p_c(x)f''(x) + 2\frac{b(x)}{\sigma^2(x)}p_c(x)f'(x)\right)\\ &= \frac{1}{2r_c(x)}\big(p_c(x)f''(x) + p_c'(x)f'(x)\big) = \frac{1}{2r_c(x)}\big(p_c(x)f'(x)\big)'.\end{aligned}$$

The latter is a *Sturm-Liouville (differential) expression* in the sense of [49, Section 13.1]. There, some assumptions are invoked which are particularly fulfilled in our situation due to (ND) and (LI). In [49] a *solution* f to $Lf = 0$ is demanded to be absolutely continuous with $x \mapsto p_c(x)f'(x)$ also absolutely continuous in order to ensure that f' and $(p_c f')'$ exist properly. If one considers the exit time of some interval $x \in [a,b] \subsetneq I$ solutions are given by

$$v(x) := \int_a^x \frac{1}{p_c(z)}\,dz = s_c(x) - s_c(a) \quad \text{and} \quad w(x) := \int_x^b \frac{1}{p_c(z)}\,dz = s_c(b) - s_c(x)$$

with the additional property that $v(a) = w(b) = 0$. Since as functions on $[a,b]$ both $v, w \in L^2(m_c)$ amongst other results [49, Satz 13.21] entails the assertion of Proposition 3.31. In the terminology of [49, Satz 13.18] the condition

$$\int_l^b s_c^2(x)\,m_c(dx) = \infty \tag{3.17}$$

ensures the Sturm-Liouville expression to be in the *limit-point case at l*. The same nomenclature applies respectively for the boundary point r. Together with the assumption $\int_l^b m_c(dy) < \infty$ condition (3.17) particularly implies $\lim_{x\downarrow l} s_c(x) = -\infty$. Indeed, the assertion of Proposition 3.38 is also covered by [49, Satz 13.21] and the solutions with proper boundary conditions $v(a) = 0$ or $w(b) = 0$, respectively, are given by

$$v(x) := 1 \quad \text{and} \quad w(x) := s_c(b) - s_c(x)$$

or

$$v(x) := s_c(x) - s_c(a) \quad \text{and} \quad w(x) := 1,$$

respectively. In either one of the three situations the general form of the Green function is given by

$$G(x,y) := \frac{v(x\wedge y)\,w(x\vee y)}{W_c(w,v)}$$

where

$$W_c(w,v) := \det\begin{pmatrix} w(x) & v(x)\\ p_c(x)\,w'(x) & p_c(x)\,v'(x)\end{pmatrix} = w(x)\,p_c(x)\,v'(x) - p_c(x)\,w'(x)\,v(x)$$

is independent of $x \in I$ and denotes the modified Wronskian. (Cf. [49, Proof of Korollar 13.3].) It holds

$$\begin{aligned}&W_c(s_c(b) - s_c(\bullet), s_c(\bullet) - s_c(a))\\&= (s_c(b) - s_c(x))\, p_c(x)\, \frac{1}{p_c(x)} - p_c(x) \left(-\frac{1}{p_c(x)}\right) (s_c(x) - s_c(a)) = s_c(b) - s_c(a)\end{aligned}$$

and $W_c(s_c(b) - s_c(\bullet), 1) = 0 - p_c(x) \left(-\frac{1}{p_c(x)}\right) = 1$ or $W_c(1, s_c(\bullet) - s_c(a)) = 1$, respectively.

3.6. h-transformation in the sense of Doob

We follow [46, Chapter 4.1 The h-transform]. Let again $I = (l, r) \subseteq \mathbb{R}$ an open real interval.

Definition 3.43. For $k \in \mathbb{N}_0$ and $\alpha \in (0, 1]$ let $C^{k,\alpha}(I)$ denote the space of functions which are continuously differentiable up to order k and the k-th derivative is α-Hölder continuous on any compact subinterval of I.

Definition 3.44. Let again

$$(Lf)(x) = \frac{1}{2}\sigma^2(x) f''(x) + b(x) f'(x)$$

the formal generator associated to the SDE (3.2) as in equation (3.3). Let $h \in C^{2,\alpha}(I)$ satisfy $h > 0$ and $Lh = 0$ on I. The operator L^h defined by

$$L^h f = \frac{1}{h} L(hf)$$

is called the *h-transform of the operator L*. Written out explicitly, one has

$$\begin{aligned}(L^h f)(x) &= \frac{L(hf)(x)}{h(x)} = \frac{(Lh)(x) \cdot f(x) + \sigma^2(x) h'(x) f'(x) + h(x) \cdot (Lf)(x)}{h(x)}\\&= (Lf)(x) + \sigma^2(x) \frac{h'(x)}{h(x)}\, f'(x).\end{aligned}$$

Remark 3.45. Suppose (ND) and (LI) holds and (X_t, B_t), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t$ is a weak solution in I of SDE (3.2) with nonrandom initial condition $X_0 = x \in (a, b) \subseteq I$ and additionally the coefficient functions $b, \sigma \in C^1(I)$ are continuously differentiable and $-\infty < s_c(a) < s_c(b) < \infty$ are finite. Then the function

$$h : [a, b] \to [0, 1], \quad h(x) := \mathbb{P}_x(T_{a,b} = a) = \frac{s_c(b) - s_c(x)}{s_c(b) - s_c(a)}$$

is stricly decreasing with $h(a) = 1$ and $h(b) = 0$ and three times continuously differentiable. Particularly it holds $h \in C^{2,\alpha}((a, b))$ in the sense of Definition 3.43 and $h(x) > 0$ and

$(Lh)(x) = 0$ on $x \in (a, b)$ since $Lh = \frac{-1}{s_c(b)-s_c(a)} Ls_c \equiv 0$ due to Lemma 3.18. The SDE corresponding to the h-transform L^h of the operator L reads

$$dX_t^h = \left(b(X_t) + \sigma^2(X_t) \frac{h'(X_t)}{h(X_t)} \right) dt + \sigma(X_t)\, dB_t,$$

fulfills the assumptions (ND) and (LI) and gives rise to the diffusion X_t^h up to time $T_{a,b}$ with law

$$\mathbb{P}_x(X_t^h \in \bullet) = \mathbb{P}_x(X_t \in \bullet \mid T_{a,b} = a).$$

The process X_t^h exhibits for $c, x \in (a, b)$ a scale function $s_c^h(x) = \int_c^x 1/p_c^h(y)\, dy$ and speed measure $m_c^h(dx) = 2r_c^h(x)\, dx$ accordingly with

$$\begin{aligned} p_c^h(x) &= \exp\left(2 \int_c^x \frac{b(z) + \sigma^2(z) \frac{h'(z)}{h(z)}}{\sigma^2(z)}\, dz \right) \\ &= \exp\left(2 \int_c^x \frac{b(z)}{\sigma^2(z)}\, dz + 2 \int_c^x \frac{h'(z)}{h(z)}\, dz \right) = p_c(x) \cdot \left(\frac{h(x)}{h(c)} \right)^2 \end{aligned}$$

and

$$r_c^h(x) = \frac{p_c^h(x)}{\sigma^2(x)} = \frac{p_c(x)}{\sigma^2(x)} \cdot \left(\frac{h(x)}{h(c)} \right)^2 = r_c(x) \cdot \left(\frac{h(x)}{h(c)} \right)^2.$$

Lemma 3.46. *In the setting of the previous Remark 3.45 if $c \in (a, b)$ and $\int_a^b m_c(dy) < \infty$ then $\int_a^b m_c^h(dy) < \infty$.*

Proof. According to the assumptions it holds

$$\int_a^b m_c^h(dy) = \int_a^b 2r_c(y) \left(\frac{h(y)}{h(c)} \right)^2 dy \leq \frac{1}{h(c)^2} \int_a^b m_c(dy) < \infty$$

which shows the assertion. □

Lemma 3.47. *In the setting of the previous Remark 3.45 if $c \in (a, b)$ then $s_c^h(b) = \infty$.*

Proof. According to the assumptions it holds

$$\begin{aligned} s_c^h(b) &= \int_c^b \frac{1}{p_c^h(z)}\, dz = \int_c^b \frac{1}{p_c(z)} \left(\frac{h(c)}{h(z)} \right)^2 dz = \int_c^b s_c'(z) \left(\frac{s_c(b) - s_c(c)}{s_c(b) - s_c(z)} \right)^2 dz \\ &= (s_c(b) - s_c(c))^2 \left[\frac{1}{s_c(b) - s_c(z)} \right]_{z=c}^{z=b} = \infty - (s_c(b) - s_c(c)) = \infty \end{aligned}$$

which shows the assertion. □

4. Stochastic Spikes

4.1. Notation

Let us give some basic definitions and notations. With parameters $\lambda, \varepsilon > 0$ and measurable functions $b_1(x), b_2(x), \sigma(x)$ on $x \in [0, \infty)$ we write $(X_t^\lambda)_{t\geq 0}$ for the solution to the SDE

$$dX_t^\lambda = \frac{\lambda^2}{2}(\varepsilon \cdot b_1(X_t^\lambda) - b_2(X_t^\lambda))\, dt + \lambda \cdot \sigma(X_t^\lambda)\, dB_t \tag{4.1}$$

upon employing as standing assumption:

(A1) For all sufficiently small $\varepsilon > 0$ and $\lambda = 1$ there exists a solution to the SDE (4.1) in the interval $I = (0, \infty)$ in the sense of Definition 3.12 with the boundary point 0 being not attainable and which is unique in law in the sense of Definition 3.8 for any deterministic starting value $X_0 \coloneqq x > 0$.

Definition 4.1. For $x > 0$, we denote as $\mathbb{P}_x$ the probability measure for the diffusion process conditioned to start at x and write $\mathbb{E}_x$ for the corresponding expectation.

4.2. Discussion of the parameter λ

The parameter λ acts as a constant time acceleration by the factor λ^2 in the following sense:

Lemma 4.2. *For arbitrary deterministic starting value $x > 0$ the process $(X_t^\lambda)_{t\geq 0}$ is distributed as $(X_{\lambda^2 t}^1)_{t\geq 0}$.*

Proof. Using the substitution $\lambda^2 r = s$ it holds

$$\int_0^{\lambda^2 t} \frac{1}{2}\left(\varepsilon\, b_1(X_s^1) - b_2(X_s^1)\right)\, ds = \lambda^2 \int_0^t \frac{1}{2}\left(\varepsilon\, b_1(X_{\lambda^2 r}^1) - b_2(X_{\lambda^2 r}^1)\right)\, dr.$$

By the time change formula for Itô integrals [44, Theorem 8.5.7] in accordance to [44, Expression (8.5.14)] the processes

$$\int_0^{\lambda^2 t} \sigma(X_s^1)\, dB_s$$

and

$$\int_0^t \sigma(X_{\lambda^2 r}^1)\, \sqrt{\lambda^2}\, dB_r$$

are equal in distribution. Therfore, it follows the equality of distribution of the processes

$$\begin{aligned} X^1_{\lambda^2 t} &= \int_0^{\lambda^2 t} \frac{1}{2} \left(\varepsilon\, b_1(X^1_s) - b_2(X^1_s)\right) \, ds + \int_0^{\lambda^2 t} \sigma(X^1_s)\, dB_s \\ &\overset{\mathcal{D}}{=} \lambda^2 \int_0^t \frac{1}{2} \left(\varepsilon\, b_1(X^1_{\lambda^2 r}) - b_2(X^1_{\lambda^2 r})\right) \, dr + \lambda \int_0^t \sigma(X^1_{\lambda^2 r})\, dB_r. \end{aligned}$$

The process $(X^1_{\lambda^2 t})_{t\geq 0}$ is the unique solution of the SDE

$$dX_t = \frac{\lambda^2}{2} \left(\varepsilon\, b_1(X_t) - b_2(X_t)\right) \, dt + \lambda\, \sigma(X_t)\, dB_t$$

and the assertion follows. □

In the following to deduce properties of X^λ_t it suffices to consider $X^1_{\lambda^2 t}$ which is the reason why we use the notation $X_t := X^1_t$.

Definition 4.3. We set $X_t := X^1_t$ for the diffusion process with parameter $\lambda = 1$ and the hitting time for the process $(X_t)_{t\geq 0}$ of some level $z > 0$ will be denoted as

$$T_z = \inf\{t \geq 0 : X_t = z\}.$$

4.3. An embedded approximate Poisson process

In [5], the distribution of the first hitting time T_z is deduced by calculating the Laplace transform of T_z, i.e. the expectation

$$\mathbb{E}_x\left[e^{-sT_z}\right], \quad s > 0,\, 0 < x < z$$

making use of the fact that they solve certain ordinary differential equations. Our approach has a somewhat different more probabilistic flavor. We are using the following rather classical strategy:

- Starting the diffusion near zero, we introduce stopping times, which decompose the path up to an arbitrary time T into cycles.
- During every cycle, the diffusion reaches with a small probability the level z.
- Counting only the hits of level z now up to a time $\lambda^2 T$ results in an approximate Poisson process.

Assumption 4.4. From now on we require the existence of two functions $0 < \alpha(\varepsilon) < \beta(\varepsilon)$ for small $\varepsilon > 0$ which are differentiable in 0 with $\lim_{\varepsilon\downarrow 0} \beta(\varepsilon) = \lim_{\varepsilon\downarrow 0} \alpha(\varepsilon) = 0$.

For the moment being we call a *cycle* a path from $\alpha(\varepsilon)$ to $\beta(\varepsilon)$ and back to $\alpha(\varepsilon)$ when λ is set to equal 1. Later we will alter the definition. If we speed up the time scale which is done by introducing the large time scale factor λ^2 we have many cycles in a time interval $[0, \lambda^2 T]$ and in each cycle we hit the level z with small probability. This is the standard situation, where the Poisson heuristic should apply. We recall that the Poisson distribution exactly occurs in a scaling limit of many independent trials each having small success probability. The waiting time until the first event in a simple Poisson process is exponential. This connects the Poisson heuristic to the assertion of the main Theorem 4.19 we are aiming for.

4.3.1. A thinned renewal process

Definition 4.5. For the motivation to be more formal we introduce the following quantities:

$$\sigma_0 := 0,\ \tau_1 := \inf\{t \geq 0 : X_t = \beta(\varepsilon)\},\ \sigma_1 := \inf\{t \geq \tau_1 : X_t = \alpha(\varepsilon)\};$$

furthermore, for $i \geq 2$ we define:

$$\tau_i := \inf\{t \geq \sigma_{i-1} : X_t = \beta(\varepsilon)\},\ \sigma_i := \inf\{t \geq \tau_i : X_t = \alpha(\varepsilon)\}.$$

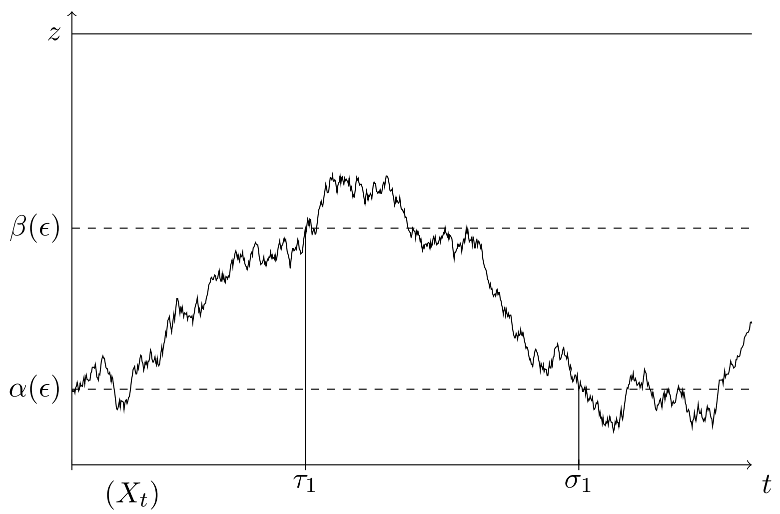


Figure 4.1.: Illustration of the stopping times τ_i and σ_i. The image is not the result of an exact simulation but a conceptual visualization only and is due to Gabriel Cicek.

Definition 4.6. Let us furthermore define the counting variable

$$N(T) := \max\{i \in \mathbb{N}_0 : \sigma_i \leq T\}.$$

The involved quantities do depend on $\varepsilon > 0$ even though this is not explicit in the notation.

Definition 4.7. Moreover, we consider the probability

$$p_{\varepsilon,z} := \mathbb{P}_{\beta(\varepsilon)}(T_z < T_{\alpha(\varepsilon)})$$

of reaching the level $z > 0$ before falling back to $\alpha(\varepsilon)$ when started at $\beta(\varepsilon) < z$.

The quantity $N(T)$ encodes the number of cycles completed up to time T. For given $z > 0$ we are actually not interested in the number of completed cycles up to time T but in the number of cycles up to time T, which do cross the level z. Thus we have to delete those cycles which do not cross the level z and we observe that this happens with probability $1 - p_{\varepsilon,z}$.

As a first motivation we consider the thinned-rescaled point process obtained by retaining every point of N with probability $p_{\varepsilon,z}$ independently of the other points and of the point process $N(T)$, and then replacing the retained point at time instant t_i by a point at $\lambda^{-2} \cdot t_i$. This gives rise to a counting process

$$\mathfrak{N}_{p_{\varepsilon,z},\lambda}(T), \quad T \geq 0$$

with

$$\mathfrak{N}_{p_{\varepsilon,z},\lambda}(T) = \xi_1 + \cdots + \xi_{N(\lambda^2 T)},$$

where the random variables $\xi_1, \xi_2, \ldots$ are independent and identically distributed with $\mathrm{P}(\xi_i = 1) = p_{\varepsilon,z}$, $\mathrm{P}(\xi_i = 0) = 1 - p_{\varepsilon,z}$ and independent of the process $(N(t))_{t \geq 0}$. This thinned counting process converges to a Poisson process in the total variation norm, but the independent thinning does not precisely describe what we are really interested in: The number of completed cycles in some given time interval is not independent of the success probability to cross the level z.

4.3.2. Poisson Limits in the high noise regime

We investigate the probability

$$\mathbb{P}_{\alpha(\varepsilon)}(T_z > \lambda^2 T) = \mathbb{P}_{\alpha(\varepsilon)}\left(\sup_{0 \leq t \leq \lambda^2 T} X_t < z\right)$$

that starting at a small positive value the process does not cross level z in the time interval at all. Since it holds $\lambda^2 T \in [\sigma_{N(\lambda^2 T)}, \sigma_{N(\lambda^2 T)+1})$ we yield an adequate approximation by truncating the time interval to the last fully completed cycle, i.e. we consider

$$\mathbb{P}_{\alpha(\varepsilon)}\left(\sup_{0 \leq t \leq \sigma_{N(\lambda^2 T)}} X_t < z\right) = \sum_{k=0}^{\infty} \mathbb{P}_{\alpha(\varepsilon)}(N(\lambda^2 T) = k, \forall 1 \leq i \leq k : \sup_{\tau_i \leq t \leq \sigma_i} X_t < z)$$
$$= \sum_{k=0}^{\infty} \mathbb{P}_{\alpha(\varepsilon)}(N(\lambda^2 T) = k, \forall 1 \leq i \leq k : \sup_{\tau_i \leq t \leq \sigma_i} X_t < z).$$

For $k \in \mathbb{N}_0$ we may decompose as

$$\begin{aligned}&\mathbb{P}_{\alpha(\varepsilon)}(N(\lambda^2 T) = k, \forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z) \\ &= \mathbb{P}_{\alpha(\varepsilon)}(\forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z) \cdot \mathbb{P}_{\alpha(\varepsilon)}(N(\lambda^2 T) = k \mid \forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z).\end{aligned}$$

Using the strong Markov property along the sequence $(\tau_i)_{i=1}^k$ of stopping times we deduce

$$\mathbb{P}_{\alpha(\varepsilon)}(\forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z) = \prod_{i=1}^k \mathbb{P}_{X_{\tau_i}}(T_{\alpha(\varepsilon)} < T_z) = \mathbb{P}_{\beta(\varepsilon)}(T_{\alpha(\varepsilon)} < T_z)^k.$$

It is now for the conditional probability where the stochastic dependency enters the framework.

Definition 4.8. Starting at $\beta(\varepsilon) < z$ we define the stochastic process X_t^h as the process obtained by conditioning the process X_t to hit $\alpha(\varepsilon)$ prior to z. This is performed by a h-transform in the sense of Doob as outlined in Section 3.6 using $h(x) \coloneqq \mathbb{P}_x(T_{\alpha(\varepsilon)} < T_z)$.

Definition 4.9. We can now adjust the definition of the cycles. Without loss of generality we may only consider $\varepsilon > 0$ small so that $\beta(\varepsilon) < z$. For X_t always starting at $\alpha(\varepsilon)$ and X_t^h always starting at $\beta(\varepsilon)$ let

$$\widetilde{\sigma}_0 \coloneqq 0,\ \widetilde{\tau}_1 \coloneqq \inf\{t \ge 0 : X_t = \beta(\varepsilon)\},\ \widetilde{\sigma}_1 \coloneqq \inf\{t \ge \widetilde{\tau}_1 : X_t^h = \alpha(\varepsilon)\};$$

furthermore, for $i \ge 2$ we define:

$$\widetilde{\tau}_i \coloneqq \inf\{t \ge \widetilde{\sigma}_{i-1} : X_t = \beta(\varepsilon)\},\ \widetilde{\sigma}_i \coloneqq \inf\{t \ge \widetilde{\tau}_i : X_t^h = \alpha(\varepsilon)\}.$$

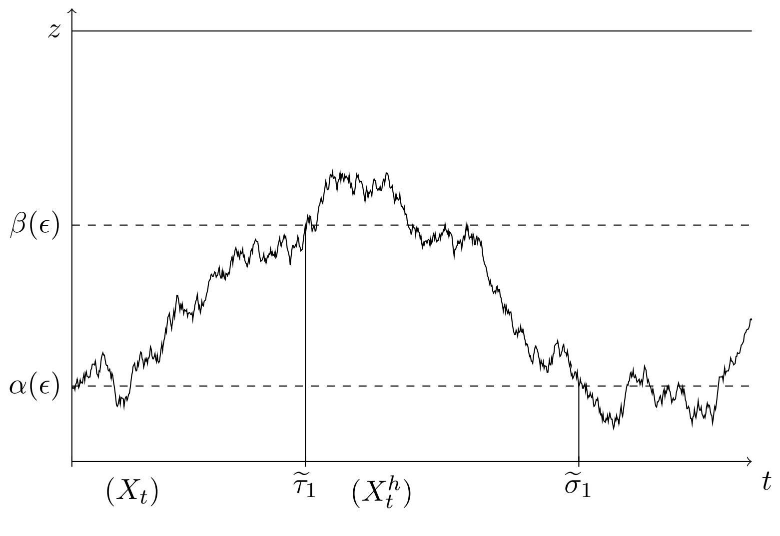


Figure 4.2.: Illustration of the cycle decomposition given by $\widetilde{\tau}_i$ and $\widetilde{\sigma}_i$.

Now we run the process starting from $\alpha(\varepsilon)$ until it reaches $\beta(\varepsilon)$. Then, we condition to fall back to $\alpha(\varepsilon)$ again before reaching the level $z > 0$. Then, the first cycle is completed and again starting from $\alpha(\varepsilon)$ the second cycle starts by letting the unconditioned process X_t run and so on. The i-th cycle starts at random time σ_{i-1}, lasts until time σ_i and consists of a first phase lasting from σ_{i-1} to τ_i where no condition is put on the process and a second phase lasting from τ_i to σ_i where the process is forced to never reach z.

Definition 4.10. The process counting the number of completed cycles up to time T becomes

$$\widetilde{N}(T) \coloneqq \max\{i \in \mathbb{N}_0 : \widetilde{\sigma}_i \le T\}.$$

The construction is built to allow for writing

$$\mathbb{P}_{\alpha(\varepsilon)}(N(\lambda^2 T) = k \mid \forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z) = \mathbb{P}_{\alpha(\varepsilon)}\big(\widetilde{N}(\lambda^2 T) = k\big)$$

Now, by the considerations in the beginning of the section it holds:

$$\begin{aligned}
&\mathbb{P}_{\alpha(\varepsilon)}\left(\sup_{0 \le t \le \sigma_{N(\lambda^2 T)}} X_t < z\right) = \sum_{k=0}^{\infty} \mathbb{P}_{\alpha(\varepsilon)}(N(\lambda^2 T) = k,\, \forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z) \\
&= \sum_{k=0}^{\infty} \mathbb{P}_{\alpha(\varepsilon)}(\forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z) \cdot \mathbb{P}_{\alpha(\varepsilon)}(N(\lambda^2 T) = k \mid \forall 1 \le i \le k : \sup_{\tau_i \le t \le \sigma_i} X_t < z) \\
&= \sum_{k=0}^{\infty} \mathbb{P}_{\beta(\varepsilon)}(T_{\alpha(\varepsilon)} < T_z)^k \cdot \mathbb{P}_{\alpha(\varepsilon)}\big(\widetilde{N}(\lambda^2 T) = k\big).
\end{aligned}$$

Defining

$$\widetilde{\mathfrak{N}}_{p_{\varepsilon,z},\lambda}(T) \coloneqq \widetilde{\xi}_1 + \cdots + \widetilde{\xi}_{\widetilde{N}(\lambda^2 T)}$$

with $(\widetilde{\xi}_i)_{i \ge 1}$ being an independent family of Bernoulli distributed random variables with $\mathrm{P}(\widetilde{\xi}_1 = 1) = p_{\varepsilon,z}$ and independent of the counting process $\widetilde{N}(T)$ we may rewrite as

$$\sum_{k=0}^{\infty} \mathbb{P}_{\beta(\varepsilon)}(T_{\alpha(\varepsilon)} < T_z)^k \cdot \mathbb{P}_{\alpha(\varepsilon)}\big(\widetilde{N}(\lambda^2 T) = k\big) = \mathrm{P}(\widetilde{\mathfrak{N}}_{p_{\varepsilon,z},\lambda}(T) = 0).$$

Again, since $\widetilde{N}$ grows linearly we are perfectly in the picture of classical Poisson approximation and the appropriate curve on which we scale $\lambda \to \infty$ and $p_{\varepsilon,z} \to 0$ should be the one with $\lambda^2 \cdot p_{\varepsilon,z}$ equal to some positive finite constant. We fix this formally in the following.

Definition 4.11. We call the *(generalized) scaling limit* the limit process of letting $\lambda \to \infty$ and $\varepsilon \to 0$ such that $\lambda^2 \cdot p_{\varepsilon,z} = J \in (0, \infty)$. This particularly entails that $\varepsilon \to 0$ must imply $p_{\varepsilon,z} \to 0$ for the scaling limit to be well defined. When useful we write $\lim_{scaling} \big[\,\cdot\,\big]$ as shorthand notation for $\lim_{\substack{\lambda\to\infty \\ \varepsilon\to 0 \\ \lambda^2 \cdot p_{\varepsilon,z} = J}} \Big[\,\cdot\,\Big]$.

Let us pose two additional assumptions.

(A2) The expected cycle duration converges to some positive real number independent of z:

$$\mathbb{E}_{\alpha(\varepsilon)}[\widetilde{\sigma}_1] \underset{\varepsilon\downarrow 0}{\longrightarrow} \kappa^{-1} \in (0,\infty).$$

(A3) For small $\varepsilon > 0$ the cycles have finite second moment uniformly in ε:

$$\limsup_{\varepsilon\downarrow 0} \mathbb{E}_{\alpha(\varepsilon)}[\widetilde{\sigma}_1^2] < \infty.$$

Remark 4.12. Technically, (A3) may be weakened by $\limsup_{\varepsilon\downarrow 0} \mathbb{E}_{\alpha(\varepsilon)}[\widetilde{\sigma}_1^{1+\rho}] < \infty$ for some positive $\rho > 0$. That is, only $(1+\rho)$-th moment is actually needed. The conditions (A1), (A2) and (A3) are rather natural and not too restrictive.

Assumption (A2) and (A3) are used in order to control the renewal structure uniformly in $\epsilon > 0$.

Proposition 4.13. *Under the assumptions (A1) to (A3) for fixed $T > 0$ it holds*

$$\lim_{scaling} \mathbb{P}_{\alpha(\varepsilon)}\left(\sup_{0\le t\le \sigma_{N(\lambda^2 T)}} X_t < z\right) = e^{-\kappa JT}.$$

Proof. In view of

$$\mathbb{P}_{\alpha(\varepsilon)}\left(\sup_{0\le t\le \sigma_{N(\lambda^2 T)}} X_t < z\right) = \mathrm{P}(\widetilde{\mathfrak{N}}_{p_{\varepsilon,z},\lambda}(T) = 0)$$

we may apply standard results on Poisson approximation as e.g. [50, Equation (23)] that state

$$d_{TV}(\widetilde{\mathfrak{N}}_{p_{\varepsilon,z},\lambda}(T), \mathrm{Poi}_{\kappa JT}) \le \frac{p_{\varepsilon,z}}{2\sqrt{1-p_{\varepsilon,z}}} + \mathbb{E}_{\alpha(\varepsilon)}\left[\left|p_{\varepsilon,z}\widetilde{N}(\lambda^2 T) - \kappa JT\right|\right]$$

where

$$d_{TV}(X,Y) = \sup_A |\mathbb{P}(X\in A) - \mathbb{P}(Y\in A)| = \frac{1}{2}\sum_{k=0}^{\infty} |\mathbb{P}(X=k) - \mathbb{P}(Y=k)|$$

denotes the total variation distance of measures essentially supported in $\mathbb{N}_0$. Therefore, in order to conclude the assertion it suffices to show the convergence

$$\lim_{scaling} \mathbb{E}_{\alpha(\varepsilon)}\left[\left|p_{\varepsilon,z}\widetilde{N}(\lambda^2 T) - \kappa JT\right|\right] = 0.$$

With $\kappa_\varepsilon := 1/\mathbb{E}_{\alpha(\varepsilon)}[\widetilde{\sigma}_1] \in (0,\infty)$ we obtain

$$\lim_{scaling} \mathbb{E}_{\alpha(\varepsilon)}\left[\left|p_{\varepsilon,z}\widetilde{N}(\lambda^2 T) - \kappa JT\right|\right] = J\cdot \lim_{scaling} \mathbb{E}_{\alpha(\varepsilon)}\left[\left|\lambda^{-2}\widetilde{N}(\lambda^2 T) - \kappa T\right|\right]$$

and

$$\mathbb{E}_{\alpha(\varepsilon)}\big[\big|\lambda^{-2}\widetilde{N}(\lambda^2 T) - \kappa T\big|\big] \leq \mathbb{E}_{\alpha(\varepsilon)}\big[\big|\lambda^{-2}\widetilde{N}(\lambda^2 T) - \kappa_\varepsilon T\big|\big] + |\kappa - \kappa_\varepsilon| \cdot T.$$

The vanishing of $|\kappa - \kappa_\varepsilon| \to 0$ is a reformulation of (A2) and due to (A2) together with (A3) we can apply a suitable version of the uniform renewal theorem such as [37, Theorem 10] in order to conclude for $\delta > 0$ sufficiently small such that $\sup_{\delta > \varepsilon > 0} \mathbb{E}_{\alpha(\varepsilon)}[\widetilde{\sigma}_1^2] < \infty$

$$\begin{aligned} &\lim_{scaling} \mathbb{E}_{\alpha(\varepsilon)}\big[\big|\lambda^{-2}\widetilde{N}(\lambda^2 T) - \kappa_\varepsilon T\big|\big] \leq T \cdot \limsup_{\lambda\to\infty} \sup_{\delta>\varepsilon>0} \mathbb{E}_{\alpha(\varepsilon)}\big[\big|\frac{\widetilde{N}(\lambda^2 T)}{\lambda^2 T} - \kappa_\varepsilon\big|\big] \\ &= T \cdot \limsup_{\lambda\to\infty} \sup_{\delta>\varepsilon>0} \mathbb{E}_{\alpha(\varepsilon)}\big[\big|\frac{\widetilde{N}(\lambda)}{\lambda} - \kappa_\varepsilon\big|\big] = 0. \end{aligned}$$

This finishes the proof. □

In order to describe the event $\{T_z > T\}$ we need to consider the events

$$\{N(\lambda^2 T) = k, \sup_{\sigma_k \leq t \leq T} X_t < z\},$$

this means we also have to ensure that during the cycle started before time T but not completed before this time the level z has not been hit.

Corollary 4.14. *Under the assumptions (A1) to (A3) for fixed $T > 0$ it holds*

$$\lim_{scaling} \mathbb{P}_{\alpha(\varepsilon)}\left(T_z > \lambda^2 T\right) = e^{-\kappa J T}.$$

Proof. From the fact $\sigma_{N(\lambda^2 T)} \leq \lambda^2 T < \sigma_{N(\lambda^2 T)+1}$ we see

$$\mathbb{P}_{\alpha(\varepsilon)}(T_z > \sigma_{N(\lambda^2 T)+1}) < \mathbb{P}_{\alpha(\varepsilon)}(T_z > \lambda^2 T) \leq \mathbb{P}_{\alpha(\varepsilon)}(T_z > \sigma_{N(\lambda^2 T)})$$

and by the previous Proposition 4.13 the upper bound has the asserted scaling limit. For the lower, we may define

$$\widetilde{\mathfrak{N}}^+_{p_\varepsilon,z,\lambda}(T) := \widetilde{\xi}^+_1 + \cdots + \widetilde{\xi}^+_{\widetilde{N}(\lambda^2 T)+1}$$

with $(\widetilde{\xi}^+_i)_{i\geq 1}$ being an independent family of Bernoulli distributed random variables with $\mathrm{P}(\widetilde{\xi}^+_1 = 1) = p_{\varepsilon,z}$ and independent of the counting process $\widetilde{N}(T)$ and repeat the argumentation as in the proof of Proposition 4.13:

$$\begin{aligned} &\sum_{k=0}^{\infty} \mathbb{P}_{\alpha(\varepsilon)}\left(N(\lambda^2 T) = k, \forall 1 \leq i \leq k+1 : \sup_{\tau_i \leq t \leq \sigma_i} X^1_t < z\right) \\ &= \sum_{k=0}^{\infty} \mathbb{P}_{\beta(\varepsilon)}(T_\varepsilon < T_z)^{k+1}\, \mathbb{P}_{\alpha(\varepsilon)}(\widetilde{N}(\lambda^2 T) = k) = \mathrm{P}(\widetilde{\mathfrak{N}}^+_{p_\varepsilon,z,\lambda}(T) = 0). \end{aligned}$$

Then, in the scaling limit

$$\lim_{scaling} \mathbb{E}_{\alpha(\varepsilon)}\left[\left|p_{\varepsilon,z}\left(\widetilde{N}(\lambda^2 T)+1\right)-\kappa JT\right|\right]=0$$

still holds and the assertion is shown.

□

Let us now start the diffusion process from a point $0 < x < z$ and derive the law of T_z with respect to $\mathbb{P}_x$. Starting at x there are two cases to consider:

- The diffusion reaches $\alpha(\varepsilon)$ before hitting z.
- The process hits z before visiting $\alpha(\varepsilon)$.

Definition 4.15. Let

$$T_z^\lambda = \inf\{t \geq 0 : X_t^\lambda = z\}$$

denote the stopping time of the first hitting of level $z > 0$ for the diffusion process X_t^λ. In our notation it holds $T_z^1 = T_z$ which corresponds to $X_t^1 = X_t$. By Lemma 4.2, the hitting time T_z^λ is distributed as $\lambda^{-2} T_z$.

The previous result entails the exponential behavior of the hitting of a fixed level z, when started very close to zero. In order to deduce the result when started from a fixed level $0 < x < z$ we assume the following conditions:

(B1) In the generalized scaling limit for any $z > 0$ and $0 < x < z$, under $\mathbb{P}_x$,

$$T_{\alpha(\varepsilon)}^\lambda \wedge T_z^\lambda \xrightarrow[scaling]{\mathcal{D}} \delta_0$$

converges in distribution to the point mass in zero.

(B2) Furthermore, the limit

$$\mathbb{P}_x(T_{\alpha(\varepsilon)} < T_z) \xrightarrow[\varepsilon\to 0]{} \alpha_{x,z} \in (0,1)$$

exists for all $z > 0$, $0 < x < z$.

Remark 4.16. Our assumptions (A1) – (A3) and (B1) – (B2) are natural and related but not fully comparable to the conditions formulated by Bauer and Bernard. We point at some similarities. [5, Condition ix)] essentially corresponds to (A3) and the assumption $p_{\varepsilon,z} \to 0$ is related to [5, Condition ii)]. [5, Condition i)] is encoded in the example below as (E2) and (E3).

Lemma 4.17. *Assumptions (B1) and (B2) imply that for any $z > 0$ and $0 < x < z$ in the generalized scaling limit the law of $T_{\alpha(\varepsilon)}^\lambda$ under $\mathbb{P}_x(\bullet \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda)$ converges to the point mass in zero:*

$$\mathbb{P}_x(T_{\alpha(\varepsilon)}^\lambda \in \bullet \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda) \xrightarrow[scaling]{\mathcal{D}} \delta_0.$$

Proof. Lemma 4.2 implies for $\alpha(\varepsilon) < x < z$ to hold

$$\begin{aligned}&\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z) = \mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)}((X^\lambda_t)_t) < T^\lambda_z((X^\lambda_t)_t)) = \mathbb{P}_x(T_{\alpha(\varepsilon)}((X_{\lambda^2 t})_t) < T_z((X_{\lambda^2 t})_t)) \\ &= \mathbb{P}_x(\lambda^{-2} T_{\alpha(\varepsilon)}((X_t)_t) < \lambda^{-2} T_z((X_t)_t)) = \mathbb{P}_x(T_{\alpha(\varepsilon)} < T_z). \end{aligned} \tag{4.2}$$

Convergence in distribution to the Dirac measure of the constant 0 is equivalent to convergence in probability to the constant random variable 0. Let $t > 0$ arbitrarily small. Then by equation (4.2), (B1) and (B2)

$$\begin{aligned}\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} > t \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z) &= \frac{\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} > t, T^\lambda_{\alpha(\varepsilon)} \wedge T^\lambda_z = T^\lambda_{\alpha(\varepsilon)})}{\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z)} \\ = \frac{\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} \wedge T^\lambda_z > t, T^\lambda_{\alpha(\varepsilon)} \wedge T^\lambda_z = T^\lambda_{\alpha(\varepsilon)})}{\mathbb{P}_x(T_{\alpha(\varepsilon)} < T_z)} &\leq \frac{\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} \wedge T^\lambda_z > t)}{\mathbb{P}_x(T_{\alpha(\varepsilon)} < T_z)} \xrightarrow[\varepsilon\to 0]{} \frac{0}{\alpha_{x,z}} = 0\end{aligned}$$

which shows that $T^\lambda_{\alpha(\varepsilon)}$ converges to 0 in probability with respect to the probability measure $\mathbb{P}_x(\bullet \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z)$. Therefore, the assertion follows. □

Due to assumption (B1) the probability of neither having hit $\alpha(\varepsilon)$ nor z by some finite time $T > 0$ vanishes in the scaling limit. Assumption (B2) on the other hand allows us to control $\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z)$ in the scaling limit. Let us give another Lemma as preparation for the main result.

Lemma 4.18. *Let $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ sequences of probability measures on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with*

$$\mu_n \xrightarrow[n\to\infty]{\mathcal{D}} \mu$$

and

$$\nu_n \xrightarrow[n\to\infty]{\mathcal{D}} \nu,$$

both μ and ν probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as well. Then, the convolution of measures

$$\mu_n * \nu_n \xrightarrow[n\to\infty]{\mathcal{D}} \mu * \nu$$

converges in distribution to the convolution of the limit objects.

Proof. The essential proof idea is to use Levy's continuity theorem: Let

$$\varphi_\mu : \mathbb{R} \to \mathbb{C}, \quad \varphi_\mu(t) := \int_{-\infty}^{\infty} e^{itx}\, \mu(dx)$$

denote the characteristic function of a real probability measure μ. From the assumed convergence $\mu_n \to \mu$ it follows $\varphi_{\mu_n}(t) \to \varphi_\mu(t)$ for any $t \in \mathbb{R}$ and likewise $\varphi_{\nu_n}(t) \to \varphi_\nu(t)$. Since it generally holds

$$\varphi_{\mu_n * \nu_n}(t) = \int_{-\infty}^{\infty} e^{itx}\, (\mu_n * \nu_n)(dx) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(x+y)} \mu_n(dx)\, \nu_n(dy)$$

$$= \int_{-\infty}^{\infty} e^{itx}\,\mu_n(dx) \cdot \int_{-\infty}^{\infty} e^{ity}\,\nu_n(dy) = \varphi_{\mu_n}(t) \cdot \varphi_{\nu_n}(t)$$

from the assumed convergences $\varphi_{\mu_n}(t) \to \varphi_\mu(t)$ and $\varphi_{\nu_n}(t) \to \varphi_\nu(t)$ if follows

$$\varphi_{\mu_n * \nu_n}(t) = \varphi_{\mu_n}(t) \cdot \varphi_{\nu_n}(t) \to \varphi_\mu(t) \cdot \varphi_\nu(t) = \varphi_{\mu * \nu}(t).$$

This in return implies $\mu_n * \nu_n \xrightarrow[n\to\infty]{\mathcal{D}} \mu * \nu$ which finishes the proof. □

We are now ready to state our main result.

Theorem 4.19. *Assume that the conditions (A1) to (A3), (B1) and (B2) are satisfied, then in the scaling limit $\lambda \to \infty$, $\varepsilon \to 0$ with $\lambda^2 p_{\varepsilon,z} = J \in (0,\infty)$ the law of the hitting time T_z^λ when started at $0 < x < z$ equals*

$$(1 - \alpha_{x,z})\,\delta_0 + \alpha_{x,z}\,\mathrm{Exp}_{J\kappa}\,.$$

Remark 4.20. In the special case of equation (2.2) this result corresponds to [5, Corollary 3].

Proof of Theorem 4.19. Let $0 < x < z$, $T > 0$ and $\varepsilon > 0$ sufficiently small. It holds

$$\mathbb{P}_x(T_z^\lambda > T) = \mathbb{P}_x(T_{\alpha(\varepsilon)}^\lambda \wedge T_z^\lambda > T) + \mathbb{P}_x(T_z^\lambda > T \geq T_{\alpha(\varepsilon)}^\lambda).$$

From (B1) it follows, that the first summand vanishes in the scaling limit and writing

$$\mathbb{P}_x(T_z^\lambda > T \geq T_{\alpha(\varepsilon)}^\lambda) = \mathbb{P}_x(T_z^\lambda > T_{\alpha(\varepsilon)}^\lambda) \cdot \mathbb{P}_x(T_z^\lambda > T \geq T_{\alpha(\varepsilon)}^\lambda \mid T_z^\lambda > T_{\alpha(\varepsilon)}^\lambda)$$

we see by equation (4.2) in combination with (B2) that the first factor of this product has the scaling limit

$$\lim_{scaling} \mathbb{P}_x(T_z^\lambda > T_{\alpha(\varepsilon)}^\lambda) = \lim_{scaling} \mathbb{P}_x(T_{\alpha(\varepsilon)} < T_z) = \alpha_{x,z}.$$

For the second factor

$$\mathbb{P}_x(T_z^\lambda > T \geq T_{\alpha(\varepsilon)}^\lambda \mid T_z^\lambda > T_{\alpha(\varepsilon)}^\lambda)$$

we use the strong Markov property at the first hitting time of level $\alpha(\varepsilon)$ under the condition $\{T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda\}$ and obtain

$$\mathbb{P}_x(T_z^\lambda > T \geq T_{\alpha(\varepsilon)}^\lambda \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda) = \int_{\{T_{\alpha(\varepsilon)}^\lambda \leq T\}} \mathbb{P}_{\alpha(\varepsilon)}(T_z^\lambda > T - T_{\alpha(\varepsilon)}^\lambda(\omega))\,\mathbb{P}_x(d\omega \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda)$$

$$= \int \mathbb{P}_{\alpha(\varepsilon)}(T_z^\lambda > T - T_{\alpha(\varepsilon)}^\lambda(\omega))\,\mathbb{P}_x(d\omega \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda) - \int_{\{T_{\alpha(\varepsilon)}^\lambda > T\}} \mathbb{P}_x(d\omega \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda)$$

$$= \int \mathbb{P}_{\alpha(\varepsilon)}(T_{\alpha(\varepsilon)}^\lambda(\omega) + T_z^\lambda > T)\,\mathbb{P}_x(d\omega \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda) - \mathbb{P}_x(T_{\alpha(\varepsilon)}^\lambda > T \mid T_{\alpha(\varepsilon)}^\lambda < T_z^\lambda). \quad (4.3)$$

Since the strong Markov property entails the independence of the future from the past given the present at time $T^\lambda_{\alpha(\varepsilon)}(\omega)$ the last integral in equation (4.3) may be seen as a probability of the convolution

$$\int \mathbb{P}_{\alpha(\varepsilon)}(T^\lambda_{\alpha(\varepsilon)}(\omega) + T^\lambda_z > T)\,\mathbb{P}_x(d\omega \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z)$$
$$= \Big[\Big(\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} \in \bullet \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z)\Big) * \Big(\mathbb{P}_{\alpha(\varepsilon)}(T^\lambda_z \in \bullet)\Big)\Big]\,((T,\infty)).$$

Lemma 4.17 states the convergence

$$\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} \in \bullet \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z) \xrightarrow[\textit{scaling}]{\mathcal{D}} \delta_0$$

and by Corollary 4.14

$$\mathbb{P}_{\alpha(\varepsilon)}(T^\lambda_z \in \bullet) = \mathbb{P}_{\alpha(\varepsilon)}(\lambda^{-2} T_z \in \bullet) \xrightarrow[\textit{scaling}]{\mathcal{D}} \mathrm{Exp}_{J\kappa}\,.$$

As a consequence of the previous Lemma 4.18 it holds

$$\lim_{\textit{scaling}} \Big[\Big(\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} \in \bullet \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z)\Big) * \Big(\mathbb{P}_{\alpha(\varepsilon)}(T^\lambda_z \in \bullet)\Big)\Big]\,((T,\infty)) = e^{-\kappa J T}.$$

Again by Lemma 4.17 it holds for the remaining last term in equation (4.3)

$$\lim_{\textit{scaling}} \mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} > T \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z) = 0.$$

Summarizing, we have shown

$$\begin{aligned}
&\lim_{\textit{scaling}} \mathbb{P}_x(T^\lambda_z > T) = \lim_{\textit{scaling}} \Big(\mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} \wedge T^\lambda_z > T) + \mathbb{P}_x(T^\lambda_z > T \geq T^\lambda_{\alpha(\varepsilon)})\Big)\\
&= 0 + \lim_{\textit{scaling}} \mathbb{P}_x(T^\lambda_z > T^\lambda_{\alpha(\varepsilon)}) \cdot \mathbb{P}_x(T^\lambda_z > T \geq T^\lambda_{\alpha(\varepsilon)} \mid T^\lambda_z > T^\lambda_{\alpha(\varepsilon)})\\
&= \alpha_{x,z} \cdot \lim_{\textit{scaling}} \mathbb{P}_x(T^\lambda_z > T \geq T^\lambda_{\alpha(\varepsilon)} \mid T^\lambda_{\alpha(\varepsilon)} < T^\lambda_z)\\
&= \alpha_{x,z} \cdot \lim_{\textit{scaling}} \Big(\int \mathbb{P}_{\alpha(\varepsilon)}(T^\lambda_{\alpha(\varepsilon)}(\omega) + T^\lambda_z > T)\,\mathbb{P}_x(d\omega \mid T^\lambda_{\alpha(\varepsilon)} < T_z)\\
&\qquad\qquad\qquad - \mathbb{P}_x(T^\lambda_{\alpha(\varepsilon)} > T \mid T^\lambda_{\alpha(\varepsilon)} < T_z)\Big)\\
&= \alpha_{x,z} \cdot \Big(e^{-\kappa J T} - 0\Big) = \alpha_{x,z} e^{-\kappa J T}
\end{aligned}$$

which finishes the proof. □

Theorem 4.19 can be interpreted in the following way, which has also been observed in [5]. If the diffusion process starts at the point x and wants to reach level z then there are two options: Either the process reaches level z without coming close to zero and in the scaling limit this takes no time or it first reaches a neighborhood of zero. Once it has reached the neighborhood of 0 it needs many trials to get up to level z and each trial has low success probability (see e.g. [14]). The latter follows from the form of the stochastic differential equation; the drift is weak near zero and the diffusion is slowed down near zero.

The asymptotic of $p_{\varepsilon,z} \to 0$, as $\varepsilon \to 0$ may depend on the value of z. We want to consider the z-free scaling limit.

Definition 4.21. For $z \in (0, \infty)$ define

$$q(z) := \begin{cases} 1 - \alpha_{z,1}, & z \leq 1, \\ (1 - \alpha_{1,z})^{-1}, & z > 1. \end{cases}$$

Corollary 4.22. *Assume that all conditions (A1) to (A3), (B1) and (B2) are satisfied, then in the z-free scaling limit* $\lambda \to \infty$, $\varepsilon \to 0$ *with* $\lambda^2 p_{\varepsilon,1} = J \in (0, \infty)$ *the law of the hitting time* T_z^λ *when started at* $0 < x < z$ *equals*

$$(1 - \alpha_{x,z})\, \delta_0 + \alpha_{x,z} \operatorname{Exp}_{\kappa J/q(z)} .$$

Remark 4.23. The choice of the normalized level one in $p_{\varepsilon,1}$ and in the definition of $q(z)$ respectively is of course rather arbitrary.

Proof. As always, we assume $\beta(\varepsilon) < z$. By the strong Markov property, it holds in the case $z \geq 1$

$$\mathbb{P}_{\beta(\varepsilon)}(T_z < T_{\alpha(\varepsilon)}) = \mathbb{P}_{\beta(\varepsilon)}(T_1 < T_{\alpha(\varepsilon)}) \cdot \mathbb{P}_1(T_z < T_{\alpha(\varepsilon)})$$

and in the case $z \leq 1$ it holds

$$\mathbb{P}_{\beta(\varepsilon)}(T_z < T_{\alpha(\varepsilon)}) = \mathbb{P}_{\beta(\varepsilon)}(T_z < T_{\alpha(\varepsilon)}) \cdot \mathbb{P}_z(T_1 < T_{\alpha(\varepsilon)}).$$

In summary, we obtain

$$\mathbb{P}_{\beta(\varepsilon)}(T_{1\vee z} < T_{\alpha(\varepsilon)}) = \mathbb{P}_{\beta(\varepsilon)}(T_{1\wedge z} < T_{\alpha(\varepsilon)}) \cdot \mathbb{P}_{1\wedge z}(T_{1\vee z} < T_{\alpha(\varepsilon)}).$$

Using again the notation $p_{\varepsilon,z} = \mathbb{P}_{\beta(\varepsilon)}(T_z < T_{\alpha(\varepsilon)})$ we find

$$p_{\varepsilon,1\vee z} = p_{\varepsilon,1\wedge z} \cdot (1 - \mathbb{P}_{1\wedge z}(T_{\alpha(\varepsilon)} < T_{1\vee z})).$$

For $\varepsilon > 0$ sufficiently small it follows from condition (B2)

$$\begin{aligned} \frac{p_{\varepsilon,z}}{p_{\varepsilon,1}} &= \begin{cases} (1 - \mathbb{P}_z(T_{\alpha(\varepsilon)} < T_1))^{-1}, & z \leq 1, \\ 1 - \mathbb{P}_1(T_{\alpha(\varepsilon)} < T_z), & z > 1, \end{cases} \\ &\xrightarrow[\varepsilon \to 0]{} \begin{cases} (1 - \alpha_{z,1})^{-1}, & z \leq 1, \\ 1 - \alpha_{1,z}, & z > 1, \end{cases} = q^{-1}(z). \end{aligned} \tag{4.4}$$

We will now use equation (4.4) to deduce the assertion. For $T > 0$ arbitrary it holds

$$\lim_{\substack{\lambda \to \infty \\ \varepsilon \to 0 \\ \lambda^2 p_{\varepsilon,1} = J}} \mathbb{P}_x(T_z^\lambda > T) = \lim_{\substack{\lambda \to \infty \\ \varepsilon \to 0 \\ \lambda^2 p_{\varepsilon,1} = J}} \mathbb{P}_x(\lambda^{-2} T_z > T) = \lim_{\varepsilon \to 0} \mathbb{P}_x \left(\frac{p_{\varepsilon,z}}{J} T_z > \frac{p_{\varepsilon,z}}{p_{\varepsilon,1}} T \right)$$

$$= \lim_{\substack{\lambda\to\infty\\ \varepsilon\to 0\\ \lambda^2 p_{\varepsilon,z}=J}} \mathbb{P}_x\left(T_z^\lambda > \frac{p_{\varepsilon,z}}{p_{\varepsilon,1}}T\right) = \lim_{scaling} \mathbb{P}_x\left(T_z^\lambda > \left(q^{-1}(z) + \frac{p_{\varepsilon,z}}{p_{\varepsilon,1}} - q^{-1}(z)\right)T\right).$$

In view of the already proven convergence $\frac{p_{\varepsilon,z}}{p_{\varepsilon,1}} - q^{-1}(z) \xrightarrow[\varepsilon\to 0]{} 0$ as consequence of Theorem 4.19 this shows for $\delta > 0$ arbitrarily small

$$\limsup_{\substack{\lambda\to\infty\\ \varepsilon\to 0\\ \lambda^2 p_{\varepsilon,1}=J}} \mathbb{P}_x(T_z^\lambda > T) \le \alpha_{x,z} e^{-\kappa J T (q^{-1}(z)-\delta)}$$

and

$$\liminf_{\substack{\lambda\to\infty\\ \varepsilon\to 0\\ \lambda^2 p_{\varepsilon,1}=J}} \mathbb{P}_x(T_z^\lambda > T) \ge \alpha_{x,z} e^{-\kappa J T (q^{-1}(z)+\delta)}.$$

Therefore, it must hold

$$\lim_{\substack{\lambda\to\infty\\ \varepsilon\to 0\\ \lambda^2 p_{\varepsilon,1}=J}} \mathbb{P}_x(T_z^\lambda > T) = \alpha_{x,z} e^{-\kappa J T q^{-1}(z)}$$

which shows the assertion. □

Remark 4.24. In both examples worked out below the scaling limit relation $\lambda^2 p_{\varepsilon,1} = const$ is essentially (meaning up to some arbitrary positive multiplicative constant) equivalent to choosing the curve $\lambda^2 Z_\varepsilon = const$ which is used in [5] in order to formulate the general conjecture. There,

$$Z_\varepsilon := \int_0^\infty \frac{1}{x^4} \exp\left(-\frac{\varepsilon}{3}\frac{1}{x^3} + \frac{b}{2}\frac{1}{x^2}\right) dx.$$

denotes the total mass of some invariant measure, cf. [5, Condition vii) in Section 5.1 main conjectures]. Also, $q(z)$ in that article is the same as our $q(z)$ here if the limit in (B2) has the form as in the examples. Note, that our main result corresponds to [5, Conjecture B (i) and (ii)].

4.4. Applications

In the section we present two important classes of examples which illustrate our approach. We recall that the general SDE for the process $(X_t)_{t\ge 0}$, $X_0 := x \in (0,\infty)$ has the form

$$dX_t = \frac{1}{2}(\varepsilon \cdot b_1(X_t) - b_2(X_t))\,dt + \sigma(X_t)\,dB_t \tag{4.5}$$

where $b_1, b_2, \sigma : [0,\infty) \to [0,\infty)$ are measurable functions. In the following the first example class is related to [5, Equation (3)] and the second example is motivated by a specific quantum mechanic situation.

4.4.1. Asymptotic linear stochastic differential equations

As first example class let us pose the following assumptions.

(E1) The function b_1 is continuously differentiable and $a := b_1(0) > 0$.

(E2) The function b_2 is twice continuously differentiable with $b_2(0) = 0$ and $b := b_2'(0) > 0$.

(E3) The function σ is twice continuously differentiable with

a) $\sigma(x) = 0 \Leftrightarrow x = 0$,

b) $\sigma := \sigma'(0) > 0$.

This example class can be viewed as generalization of the specification

$$b_1(x) := 1, \qquad b_2(x) := b \cdot x, \qquad \sigma(x) := x \tag{4.6}$$

in the sense that at the origin the coefficients exhibit the same behavior.

Remark 4.25.

- Due to (E3) a) condition (ND) is fulfilled.
- Also (LI) holds, since the function

$$y \mapsto \frac{1 + \frac{1}{2}|\varepsilon b_1(y) - b_2(y)|}{\sigma^2(y)}$$

is continuous on $y \in (0, \infty)$ and therefore bounded on compact subintervals of $I = (0, \infty)$ and particularly integrable.

- The scale function for $c, x \in (0, \infty)$ is given by

$$s_c(x) = \int_c^x \exp\left(-\int_c^y \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) dy.$$

- The speed measure for $c, x \in (0, \infty)$ is given by

$$m_c(dx) = \frac{2}{\sigma^2(x)\, s_c'(x)} = \frac{2}{\sigma^2(x)} \exp\left(\int_c^x \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right).$$

- As in Definition 3.42 we set for $c, x \in (0, \infty)$

$$p_c(x) = \exp\left(\int_c^x \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right)$$

and

$$r_c(x) = \frac{1}{\sigma^2(x)} \exp\left(\int_c^x \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right).$$

- By Taylor's theorem, there is $\delta_0, M > 0$ such that

$$|b_1(x) - a| \le Mx, \;\; |b_2(x) - bx| \le Mx^2, \;\; |\sigma^2(x) - \sigma^2 x^2| \le Mx^3 \tag{4.7}$$

  for all $0 \le x \le \delta_0$.

For the rest of the discussion of the example class (E1) to (E3) we now fix some arbitrarily chosen level $z \in (0, \infty)$. We always assume $\varepsilon > 0$ to be sufficiently small such that $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ exist with $\beta(\varepsilon) < z$.

Definition 4.26. With $\delta_0, M > 0$ from the last remark (4.7) we set

$$\delta := \delta_0 \wedge \frac{a \wedge b \wedge \sigma^2}{2M} \wedge \frac{z}{2}.$$

For all $x \in [0, \delta]$ it holds

$$b_1(x) \in (a/2, 3a/2), \;\; b_2(x) \in (bx/2, 3bx/2), \;\; \sigma^2(x) \in (\sigma^2 x^2/2, 3\sigma^2 x^2/2).$$

As it will turn out, close to the origin, in many aspects the process exhibits the same qualitative behavior as the process belonging to the SDE

$$dX_t = \frac{1}{2}(\varepsilon a - bX_t)\, dt + \sigma X_t \, dB_t. \tag{4.8}$$

Lemma 4.27. *It holds $\int_0^z m_\delta(dy) < \infty$.*

Proof. By the continuity of the integrand it holds

$$\int_\delta^z m_\delta(dy) = \int_\delta^z \frac{2}{\sigma^2(y)} \exp\left(\int_\delta^y \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)} \, dl \right) dy < \infty.$$

Using Lemma A.1 in the appendix we bound

$$\begin{aligned}
&\int_0^\delta \frac{2}{\sigma^2(y)} \exp\left(\int_\delta^y \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)} \, dl \right) dy = \int_0^\delta \frac{2}{\sigma^2(y)} \exp\left(-\int_y^\delta \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)} \, dl \right) dy \\
&\le \int_0^\delta \frac{2}{\frac{\sigma^2 y^2}{2}} \exp\left(-\int_y^\delta \varepsilon \frac{\frac{1}{2}a}{\frac{3}{2}\sigma^2 l^2} - \frac{\frac{3}{2}bl}{\frac{1}{2}\sigma^2 l^2} \, dl \right) dy \\
&= \int_0^\delta \frac{4}{\sigma^2 y^2} \exp\left(\varepsilon \frac{a}{3\sigma^2} \left(\frac{1}{\delta} - \frac{1}{y} \right) \right) \cdot \left(\frac{\delta}{y} \right)^{\frac{3b}{\sigma^2}} dy \\
&= \frac{4\delta^{3b/\sigma^2}}{\sigma^2} \exp\left(\frac{a\varepsilon}{3\sigma^2 \cdot \delta} \right) \int_0^\delta \frac{1}{y^{2+3b/\sigma^2}} \exp\left(-\frac{a\varepsilon}{3\sigma^2 \cdot y} \right) dy \\
&\le \frac{4\delta^{3b/\sigma^2}}{\sigma^2} \exp\left(\frac{a\varepsilon}{3\sigma^2 \cdot \delta} \right) \int_0^\infty \frac{1}{y^{2+3b/\sigma^2}} \exp\left(-\frac{a\varepsilon}{3\sigma^2 \cdot y} \right) dy \\
&= \frac{4\delta^{3b/\sigma^2}}{\sigma^2} \exp\left(\frac{a\varepsilon}{3\sigma^2 \cdot \delta} \right) \left(\frac{3\sigma^2}{a\varepsilon} \right)^{1+3b/\sigma^2} \Gamma(1 + 3b/\sigma^2) < \infty.
\end{aligned}$$

This finishes the proof. $\square$

Lemma 4.28. *For the example class (E1) to (E3) it holds (A1), i.e. the boundary point 0 is not attainable.*

Proof. We show that 0 is not even attracting by verifying $s_\delta(0) = -\infty$:

$$\begin{aligned}
s_\delta(0) &= \int_\delta^0 \exp\left(-\int_\delta^y \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) dy = -\int_0^\delta \exp\left(\int_y^\delta \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) dy \\
&\leq -\int_0^\delta \exp\left(\int_y^\delta \varepsilon \frac{\frac{1}{2}a}{\frac{3}{2}\sigma^2 l^2} - \frac{\frac{3}{2}bl}{\frac{1}{2}\sigma^2 l^2}\, dl\right) dy = -\int_0^\delta \exp\left(\varepsilon \frac{a}{3\sigma^2}\left(\frac{1}{y} - \frac{1}{\delta}\right)\right) \cdot \left(\frac{y}{\delta}\right)^{\frac{3b}{\sigma^2}} dy \\
&= -\exp\left(-\frac{a\varepsilon}{3\sigma^2 \cdot \delta}\right) \delta^{-3b/\sigma^2} \int_0^\delta y^{3b/\sigma^2} \exp\left(\frac{a\varepsilon}{3\sigma^2 \cdot y}\right) dy \\
&= -\exp\left(-\frac{a\varepsilon}{3\sigma^2 \cdot \delta}\right) \delta^{-3b/\sigma^2} \left(\frac{a\varepsilon}{3\sigma^2}\right)^{1+3b/\sigma^2} \int_{\frac{a\varepsilon}{3\sigma^2 \cdot \delta}}^\infty \frac{e^s}{s^{2+3b/\sigma^2}}\, ds = -\infty.
\end{aligned}$$

In the second last equation we used the substitution $s = \frac{a\varepsilon}{3\sigma^2 \cdot y}$. Alternatively, one can use the estimate

$$\exp\left(\frac{a\varepsilon}{3\sigma^2 \cdot y}\right) = \sum_{k=0}^\infty \frac{\left(\frac{a\varepsilon}{3\sigma^2 \cdot y}\right)^k}{k!} \geq \frac{\left(\frac{a\varepsilon}{3\sigma^2 \cdot y}\right)^{\lceil 3b/\sigma^2 \rceil + 1}}{(\lceil 3b/\sigma^2 \rceil + 1)!}$$

to show the infiniteness of the integral. The proof is finished. □

In the situation of (4.8) a strong form of scale invariance holds, i.e. $Y_t := X_t/\varepsilon$ fulfills the SDE

$$\begin{aligned}
dY_t &= \frac{1}{\varepsilon}\, dX_t = \frac{1}{\varepsilon}\left(\frac{1}{2}(\varepsilon a - b \cdot X_t)\, dt + \sigma X_t\, dB_t\right) = \frac{1}{2}\big(a - b \cdot \frac{X_t}{\varepsilon}\big)\, dt + \sigma \frac{X_t}{\varepsilon}\, dB_t \\
&= \frac{1}{2}\left(a - b \cdot Y_t\right)\, dt + \sigma Y_t\, dB_t
\end{aligned}$$

making it plausible to choose $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ of linear order. Our next goal is to perform the needed calculations for showing (A2), (A3), (B1) and (B2) where we set $\alpha(\varepsilon) := \varepsilon$ and $\beta(\varepsilon) := 2\varepsilon$.

Definition 4.29. Throughout the discussion of this example class let us fix $\alpha(\varepsilon) := \varepsilon$ and $\beta(\varepsilon) := 2\varepsilon$.

Lemma 4.30. *For $\varepsilon \to 0$ the limit $p_{\varepsilon,z} \xrightarrow[\varepsilon \to 0]{} 0$ holds.*

Proof. In view of Lemma 3.20 it holds

$$p_{\varepsilon,z} = \mathbb{P}_{2\varepsilon}(T_z < T_\varepsilon) = \frac{s_\delta(2\varepsilon) - s_\delta(\varepsilon)}{s_\delta(z) - s_\delta(\varepsilon)} = \frac{\int_\varepsilon^{2\varepsilon} 1/p_\delta(y)\, dy}{\int_\varepsilon^z 1/p_\delta(y)\, dy}. \tag{4.9}$$

In the following we investigate numerator and denominator separately to see that the former vanishes whereas the latter is bounded from below.

It holds for $\varepsilon < \delta/2$

$$\begin{aligned}
\int_\varepsilon^{2\varepsilon} 1/p_\delta(y)\,dy &= \int_\varepsilon^{2\varepsilon} \exp\left(-\varepsilon \int_\delta^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \exp\left(\int_\delta^y \frac{b_2(l)}{\sigma^2(l)}\,dl\right) dy \\
&\le \int_\varepsilon^{2\varepsilon} \exp\left(-\varepsilon \int_\delta^y \frac{3a/2}{\sigma^2 l^2/2}\,dl\right) \exp\left(\int_\delta^y \frac{bl/2}{3\sigma^2 l^2/2}\,dl\right) dy \\
&= \int_\varepsilon^{2\varepsilon} \exp\left(3\,\varepsilon\, a/\sigma^2 \cdot (1/y - 1/\delta)\right) (y/\delta)^{b/(3\sigma^2)}\,dy \\
&= \frac{1}{\delta^{b/(3\sigma^2)}} \exp\left(\frac{-3\,a}{\sigma^2\delta}\,\varepsilon\right) \int_\varepsilon^{2\varepsilon} \exp\left(3\,\varepsilon\, a/\sigma^2 \cdot 1/y\right) y^{b/(3\sigma^2)}\,dy \\
&= \frac{1}{\delta^{b/(3\sigma^2)}} \exp\left(\frac{-3\,a}{\sigma^2\delta}\,\varepsilon\right) \varepsilon^{1+b/(3\sigma^2)} \int_1^2 \exp\left(3\,a/\sigma^2 \cdot 1/y\right) y^{b/(3\sigma^2)}\,dy \xrightarrow[\varepsilon\to 0]{} 0.
\end{aligned} \tag{4.10}$$

In the denominator for $\varepsilon < \delta$ we have

$$\begin{aligned}
\int_\varepsilon^{z} 1/p_\delta(y)\,dy &= \int_\varepsilon^{z} \exp\left(-\varepsilon \int_\delta^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \exp\left(\int_\delta^y \frac{b_2(l)}{\sigma^2(l)}\,dl\right) dy \\
&\ge \int_\delta^{z} \exp\left(-\varepsilon \int_\delta^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \exp\left(\int_\delta^y \frac{b_2(l)}{\sigma^2(l)}\,dl\right) dy \\
&\ge \int_\delta^{z} \exp\left(-\delta \int_\delta^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \exp\left(\int_\delta^y \frac{b_2(l)}{\sigma^2(l)}\,dl\right) dy > 0.
\end{aligned} \tag{4.11}$$

□

In order to prove the validity of (A2) we investigate $\mathbb{E}_\varepsilon[\widetilde{\sigma}_1]$ for small $\varepsilon > 0$, i.e. the expected amount of time it takes to complete a cycle. To do so, we split the cycle path into two pieces: The first one is the path of X_t starting at $\alpha(\varepsilon) = \varepsilon$ until the first time hitting the level $\beta(\varepsilon) = 2\varepsilon$ and the subsequent second piece of the cycle path of returning back to ε is produced under the condition of not reaching the fixed level z. Formally,

$$\mathbb{E}_\varepsilon[\widetilde{\sigma}_1] = \mathbb{E}_\varepsilon[T_{2\varepsilon}] + \mathbb{E}_{2\varepsilon}[T_\varepsilon \mid T_\varepsilon < T_z].$$

For later usage we want to be slightly more general and consider

$$\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}] + \mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon} \mid T_{\alpha\varepsilon} < T_z]$$

with positive finite constants $\infty > \beta > \alpha > 0$.

Lemma 4.31. *It holds for $0 < \alpha < \beta < \infty$ the following formula:*

$$\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}] = 2\varepsilon^2 \int_0^\alpha \int_\alpha^\beta r_\delta(y\varepsilon)/p_\delta(w\varepsilon)\,dw\,dy + 2\varepsilon^2 \int_\alpha^\beta \int_y^\beta r_\delta(y\varepsilon)/p_\delta(w\varepsilon)\,dw\,dy.$$

Proof. Using the function $f : [0, \beta\varepsilon] \to \mathbb{R}$, $f(x) \coloneqq 1$ by Proposition 3.38 we may use the one sided Green function to deduce

$$\begin{aligned}
\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}] &= \mathbb{E}_{\alpha\varepsilon}\left[\int_0^{T_{\beta\varepsilon}} ds\right] = \mathbb{E}_{\alpha\varepsilon}\left[\int_0^{T_{\beta\varepsilon}} f(X_s)\, ds\right] = M^f_{0,\beta\varepsilon}(\alpha\varepsilon) \\
&= \int_0^{\beta\varepsilon} f(y)\, G_{\alpha\varepsilon,\beta\varepsilon,\delta}(\alpha\varepsilon, y)\, m_\delta(dy) = \int_0^{\beta\varepsilon} (s_\delta(\beta\varepsilon) - s_\delta((\alpha\varepsilon) \vee y))\, m_\delta(dy) \\
&= \int_0^{\beta\varepsilon} \int_{(\alpha\varepsilon)\vee y}^{\beta\varepsilon} 1/p_\delta(w)\, dw\, 2r_\delta(y)\, dy = 2\int_0^{\beta\varepsilon} \int_{(\alpha\varepsilon)\vee y}^{\beta\varepsilon} r_\delta(y)/p_\delta(w)\, dw\, dy \\
&= 2\varepsilon^2 \int_0^{\beta} \int_{\alpha\vee y}^{\beta} r_\delta(y\varepsilon)/p_\delta(w\varepsilon)\, dw\, dy \\
&= 2\varepsilon^2 \int_0^{\alpha} \int_{\alpha}^{\beta} r_\delta(y\varepsilon)/p_\delta(w\varepsilon)\, dw\, dy + 2\varepsilon^2 \int_\alpha^{\beta} \int_{y}^{\beta} r_\delta(y\varepsilon)/p_\delta(w\varepsilon)\, dw\, dy.
\end{aligned}$$

□

In order to handle the quantity $\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon} \mid T_{\alpha\varepsilon} < T_z]$ where $\beta\varepsilon < z$ we consider the process X^h_t obtained as h-transform of X_t in the interval $(\alpha\varepsilon, z) \subseteq (0, \infty)$ using

$$h : [\alpha\varepsilon, z] \to [0, 1], \quad h(x) \coloneqq \mathbb{P}_x(T_{\alpha\varepsilon} < T_z) = \int_x^z 1/p_\delta(y)\, dy \Big/ \int_{\alpha\varepsilon}^z 1/p_\delta(y)\, dy. \tag{4.12}$$

We are in the situation of Remark 3.45 with $X^h_0 = \beta\varepsilon$ and

$$dX^h_t = \left(\frac{1}{2}(\varepsilon b_1(X^h_t) - b_2(X^h_t)) + \sigma^2(X^h_t)\frac{h'(X^h_t)}{h(X^h_t)}\right) dt + \sigma(X^h_t)\, dB_t.$$

The coefficient functions $x \mapsto \left(\frac{1}{2}(\varepsilon b_1(x) - b_2(x)) + \sigma^2(x)\frac{h'(x)}{h(x)}\right)$ and $x \mapsto \sigma^2(x)$ fulfill (ND) and (LI) and exhibit for $c \in (\alpha\varepsilon, z)$ scale function s^h_c, speed measure m^h_c with p^h_c and r^h_c accordingly. Furthermore it holds $\int_{\alpha\varepsilon}^z m^h_c(dy) < \infty$ due to Lemma 3.46 and $s_c(z) = \infty$ due to Lemma 3.47. As analog to Lemma 4.31 it holds the following formula for the expected time of the second cycle phase.

Lemma 4.32. *It holds for $0 < \alpha < \beta < \infty$ and $\varepsilon < \delta/\beta$ the following formula:*

$$\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon} \mid T_{\alpha\varepsilon} < T_z] = 2\varepsilon^2 \int_\alpha^\beta \int_\alpha^y r^h_\delta(y\varepsilon)/p^h_\delta(w\varepsilon)\, dw\, dy + 2\varepsilon^2 \int_\beta^{z/\varepsilon} \int_\alpha^\beta r^h_\delta(y\varepsilon)/p^h_\delta(w\varepsilon)\, dw\, dy.$$

Proof. We again use Proposition 3.38, this time with the function $f : [\alpha\varepsilon, z] \to \mathbb{R}$, $f(x) \coloneqq 1$ and the other one sided Green function to deduce that for

$$T^h_{\alpha\varepsilon} \coloneqq \inf\{t \geq 0 : X^h_t = \alpha\varepsilon\} = T_{\alpha\varepsilon}((X^h_t)_t)$$

using the suggestive notations

$$G^h_{\alpha\varepsilon,z,\delta}(\beta\varepsilon, y) \coloneqq s^h_\delta((\beta\varepsilon) \wedge y) - s^h_\delta(\alpha\varepsilon), \quad y \in (\alpha\varepsilon, z)$$

and

$$M^{f,h}_{\alpha\varepsilon,z}(\beta\varepsilon) := \int_{\alpha\varepsilon}^{\beta\varepsilon} f(y)\, G^h_{\alpha\varepsilon,z,\delta}(\beta\varepsilon, y)\, m^h_\delta(dy), \quad f \in C_b[\alpha\varepsilon, z]$$

it holds

$$\begin{aligned}
&\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon} \mid T_{\alpha\varepsilon} < T_z] = \mathbb{E}_{\beta\varepsilon}[T^h_{\alpha\varepsilon}] = \mathbb{E}_{\beta\varepsilon}\left[\int_0^{T^h_{\alpha\varepsilon}} f(X^h_s)\, ds\right] = M^{f,h}_{\alpha\varepsilon,z}(\beta\varepsilon) \\
&= \int_{\alpha\varepsilon}^{z} (s^h_\delta((\beta\varepsilon)\wedge y) - s^h_\delta(\alpha\varepsilon))\, m^h_\delta(dy) = 2\varepsilon^2 \int_\alpha^{z/\varepsilon}\int_\alpha^{\beta\wedge y} r^h_\delta(y\varepsilon)/p^h_\delta(w\varepsilon)\, dw\, dy \\
&= 2\varepsilon^2 \int_\alpha^{\beta}\int_\alpha^{y} r^h_\delta(y\varepsilon)/p^h_\delta(w\varepsilon)\, dw\, dy + 2\varepsilon^2 \int_\beta^{z/\varepsilon}\int_\alpha^{\beta} r^h_\delta(y\varepsilon)/p^h_\delta(w\varepsilon)\, dw\, dy.
\end{aligned}$$

□

As preparation and for later use we collect some explicit estimates

Lemma 4.33. *For $c > 0$ the following assertions hold true:*

a) For $0 < y\varepsilon \le w\varepsilon < \delta$ the following estimates hold

$$\varepsilon^2 \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} \begin{matrix}\ge\\ \le\end{matrix} \frac{1}{\sigma^2 \cdot y^2 \pm M y^3 \varepsilon}\left(\frac{\sigma^2 \mp M w\varepsilon}{\sigma^2 \mp M y\varepsilon}\cdot\frac{y}{w}\right)^{\pm\varepsilon M(a+\sigma^2)/\sigma^4} \times$$
$$\times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right)\left(\frac{w}{y}\right)^{b/\sigma^2}\left(\frac{\sigma^2 \pm M y\varepsilon}{\sigma^2 \pm M w\varepsilon}\right)^{1+b/\sigma^2}$$

b) For $0 < w\varepsilon \le y\varepsilon < \delta$ it holds

$$\varepsilon^2 \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} \begin{matrix}\ge\\ \le\end{matrix} \frac{1}{\sigma^2 \cdot y^2 \pm M y^3 \varepsilon}\left(\frac{\sigma^2 \pm M w\varepsilon}{\sigma^2 \pm M y\varepsilon}\cdot\frac{y}{w}\right)^{\mp\varepsilon M(a+\sigma^2)/\sigma^4} \times$$
$$\times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right)\left(\frac{w}{y}\right)^{b/\sigma^2}\left(\frac{\sigma^2 \mp M y\varepsilon}{\sigma^2 \mp M w\varepsilon}\right)^{1+b/\sigma^2}$$

c) For $0 < y\varepsilon \le w\varepsilon < \delta$ and for $0 < w\varepsilon \le y\varepsilon < \delta$ we have

$$\lim_{\varepsilon\to 0} \varepsilon^2 \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} = \frac{1}{\sigma^2} e^{\frac{a}{\sigma^2}(1/w - 1/y)} \frac{w^{b/\sigma^2}}{y^{b/\sigma^2+2}}.$$

Proof. In the case a) we obtain the estimates

$$\begin{aligned}
\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} &= \frac{1}{\sigma^2(y\varepsilon)} \exp\left(-\int_c^{y\varepsilon} \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) \Big/ \exp\left(-\int_c^{w\varepsilon} \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) \\
&= \frac{1}{\sigma^2(y\varepsilon)} \exp\left(-\varepsilon \int_{y\varepsilon}^{w\varepsilon} \frac{b_1(l)}{\sigma^2(l)}\, dl\right) \exp\left(\int_{y\varepsilon}^{w\varepsilon} \frac{b_2(l)}{\sigma^2(l)}\, dl\right) \qquad (4.13)
\end{aligned}$$

$$\geq \frac{1}{\sigma^2 \cdot (y\varepsilon)^2 + M(y\varepsilon)^3} \exp\left(\varepsilon \int_{w\varepsilon}^{y\varepsilon} \frac{a+Ml}{\sigma^2 l^2 - Ml^3}\, dl\right) \exp\left(\int_{y\varepsilon}^{w\varepsilon} \frac{bl - Ml^2}{\sigma^2 l^2 + Ml^3}\, dl\right)$$

and analogously

$$\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} \leq \frac{1}{\sigma^2 \cdot (y\varepsilon)^2 - M(y\varepsilon)^3} \exp\left(\varepsilon \int_{w\varepsilon}^{y\varepsilon} \frac{a-Ml}{\sigma^2 l^2 + Ml^3}\, dl\right) \exp\left(\int_{y\varepsilon}^{w\varepsilon} \frac{b+Ml}{\sigma^2 l - Ml^2}\, dl\right).$$

By assumption it holds $Mw\varepsilon < M\delta < \sigma^2/2$; we therefore do not run into problems regarding singularities in the denominators. We now employ the technique of partial fraction decomposition to solve for the integrals:

$$\begin{aligned}
&\exp\left(\varepsilon \int_{w\varepsilon}^{y\varepsilon} \frac{a \pm Ml}{\sigma^2 l^2 \mp Ml^3}\, dl\right) \\
&= \exp\left(\varepsilon \int_{w\varepsilon}^{y\varepsilon} \frac{M^2(a+\sigma^2)}{\sigma^4(\sigma^2 \mp Ml)} \pm \frac{M(a+\sigma^2)}{\sigma^4 l} + \frac{a}{\sigma^2 l^2}\, dl\right) \\
&= \exp\left(\varepsilon \left(\mp \frac{M(a+\sigma^2)}{\sigma^4} \left[\ln(\sigma^2 \mp Ml)\right]_{l=w\varepsilon}^{l=y\varepsilon} \pm \frac{M(a+\sigma^2)}{\sigma^4} \ln \frac{y}{w}\right) + \frac{a}{\sigma^2}(1/w - 1/y)\right) \\
&= \left(\frac{\sigma^2 \mp My\varepsilon}{\sigma^2 \mp Mw\varepsilon}\right)^{\mp\varepsilon M(a+\sigma^2)/\sigma^4} \left(\frac{y}{w}\right)^{\pm\varepsilon M(a+\sigma^2)/\sigma^4} \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \\
&\xrightarrow{\varepsilon \downarrow 0} \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right),
\end{aligned} \tag{4.14}$$

and for the other

$$\begin{aligned}
\exp\left(\int_{y\varepsilon}^{w\varepsilon} \frac{b \mp Ml}{\sigma^2 l \pm Ml^2}\, dl\right) &= \exp\left(\int_{y\varepsilon}^{w\varepsilon} \frac{b}{\sigma^2 l} \mp \frac{M(b+\sigma^2)}{\sigma^2(\sigma^2 \pm Ml)}\, dl\right) \\
= \left(\frac{w}{y}\right)^{b/\sigma^2} \left(\frac{\sigma^2 \pm My\varepsilon}{\sigma^2 \pm Mw\varepsilon}\right)^{1+b/\sigma^2} &\xrightarrow{\varepsilon \downarrow 0} \left(\frac{w}{y}\right)^{b/\sigma^2}.
\end{aligned} \tag{4.15}$$

Put together this shows part a) of the lemma and combined with

$$\frac{\varepsilon^2}{\sigma^2 \cdot y^2 \cdot \varepsilon^2 \pm My^3\varepsilon^3} = \frac{1}{\sigma^2 \cdot y^2 \pm My^3\varepsilon} \xrightarrow[\varepsilon \to 0]{} \frac{1}{\sigma^2 \cdot y^2}$$

the assertion of part c) is derived for the case $0 < y \leq w < \delta/\varepsilon$.

The other case of $0 < w \leq y < \delta/\varepsilon$ is treated in a very similar fashion. Firstly, in view of the expression in the second line of equation (4.13) the directions of the integration domain are switched; we must estimate in the opposite directions to attain

$$\varepsilon^2 \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} \begin{matrix}\geq \\ \leq\end{matrix} \frac{\varepsilon^2}{\sigma^2 \cdot (y\varepsilon)^2 \pm M(y\varepsilon)^3} \exp\left(\varepsilon \int_{w\varepsilon}^{y\varepsilon} \frac{a \mp Ml}{\sigma^2 l^2 \pm Ml^3}\, dl\right) \exp\left(\int_{y\varepsilon}^{w\varepsilon} \frac{b \pm Ml}{\sigma^2 l \mp Ml^2}\, dl\right).$$

Next, we may reuse the integral formulas (4.14) and (4.15) with M having opposite sign; namely,

$$\begin{aligned}&\exp\left(\varepsilon\int_{w\varepsilon}^{y\varepsilon}\frac{a\mp Ml}{\sigma^2l^2\pm Ml^3}\,dl\right)\\&=\left(\frac{\sigma^2\pm My\varepsilon}{\sigma^2\pm Mw\varepsilon}\right)^{\pm\varepsilon M(a+\sigma^2)/\sigma^4}\left(\frac{y}{w}\right)^{\mp\varepsilon M(a+\sigma^2)/\sigma^4}\exp\left(\frac{a}{\sigma^2}(1/w-1/y)\right)\end{aligned}\tag{4.16}$$

and

$$\exp\left(\int_{y\varepsilon}^{w\varepsilon}\frac{b\pm Ml}{\sigma^2l\mp Ml^2}\,dl\right)=\left(\frac{w}{y}\right)^{b/\sigma^2}\left(\frac{\sigma^2\mp My\varepsilon}{\sigma^2\mp Mw\varepsilon}\right)^{1+b/\sigma^2}.\tag{4.17}$$

Putting the calculations together yields

$$\begin{aligned}\varepsilon^2\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\,\substack{\geq\\\leq}\,&\frac{\varepsilon^2}{\sigma^2\cdot(y\varepsilon)^2\pm M(y\varepsilon)^3}\left(\frac{\sigma^2\pm My\varepsilon}{\sigma^2\pm Mw\varepsilon}\cdot\frac{w}{y}\right)^{\pm\varepsilon M(a+\sigma^2)/\sigma^4}\times\\&\times\exp\left(\frac{a}{\sigma^2}(1/w-1/y)\right)\left(\frac{w}{y}\right)^{b/\sigma^2}\left(\frac{\sigma^2\mp My\varepsilon}{\sigma^2\mp Mw\varepsilon}\right)^{1+b/\sigma^2}\\&\xrightarrow[\varepsilon\to0]{}\frac{1}{\sigma^2\cdot y^2}\exp\left(\frac{a}{\sigma^2}(1/w-1/y)\right)\left(\frac{w}{y}\right)^{b/\sigma^2}.\end{aligned}$$

This shows part b) of the lemma and also affirms the same limit for $\varepsilon\to0$ in the case $0<w\leq y<\delta/\varepsilon$ as posed in part c) and thereby finishes the proof. □

We are now ready to investigate the limit $\varepsilon\to0$ and thereby showing that (A1) holds true.

Proposition 4.34 (implying (A2) under (E1) to (E3))**.** *Let $0<\alpha<\beta<\infty$. Under (E1) to (E3) the expected time of going from $\alpha\varepsilon$ to $\beta\varepsilon$ and back again without hitting z is well behaved in the sense, that*

$$\begin{aligned}\lim_{\varepsilon\downarrow0}\mathbb{E}_{\alpha\varepsilon}[\widetilde{\sigma}_1]&=\lim_{\varepsilon\downarrow0}\Big(\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}]+\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}\mid T_{\alpha\varepsilon}<T_z]\Big)=\lim_{\varepsilon\downarrow0}\Big(\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}]+\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}]\Big)\\&=\frac{2}{\sigma^2}\left[\int_0^\infty\int_\alpha^\beta\exp\left(\frac{a}{\sigma^2}(1/w-1/y)\right)\frac{w^{b/\sigma^2}}{y^{b/\sigma^2+2}}\,dw\,dy\right]\in(0,\infty).\end{aligned}$$

Proof. Throughout the proof we use the formulas given in Lemma 4.31 and Lemma 4.32, facilitate the upper bounds in Lemma 4.33 a) and b) in order to apply Lebesgue's theorem to deduce the limit $\varepsilon\to0$ with Lemma 4.33 c).
Let us start by citing Lemma 4.31: It holds

$$\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}]=2\varepsilon^2\left[\int_0^\alpha\int_\alpha^\beta\frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}\,dw\,dy+\int_\alpha^\beta\int_y^\beta\frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}\,dw\,dy\right].$$

Lemma 4.33 part a) entails for $\varepsilon < \delta/\beta$ the upper bound

$$\varepsilon^2 \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)} \leq \frac{1}{\sigma^2 \cdot y^2 - My^3\varepsilon} \left(\frac{\sigma^2 + My\varepsilon}{\sigma^2 + Mw\varepsilon} \cdot \frac{w}{y} \right)^{\varepsilon M(a+\sigma^2)/\sigma^4} \times \\ \times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y) \right) \left(\frac{w}{y} \right)^{b/\sigma^2} \left(\frac{\sigma^2 - My\varepsilon}{\sigma^2 - Mw\varepsilon} \right)^{1+b/\sigma^2}$$

on $0 < y \leq w < \delta/\varepsilon$, which particularly covers

$$(y, w) \in D := (0, \alpha) \times (\alpha, \beta) \cup \{(y, w) : \alpha < y < \beta,\, y \leq w < \beta\}.$$

To find an integrable majorant on $0 < y \leq w < \delta/\varepsilon$ we may assume $\varepsilon < \delta/\beta$ and use the estimates

$$\frac{1}{\sigma^2 \cdot y^2 - My^3\varepsilon} \leq \frac{1}{\sigma^2 \cdot y^2 - My^2\delta} \leq \frac{2}{\sigma^2 \cdot y^2},$$

$$\frac{\sigma^2 + My\varepsilon}{\sigma^2 + Mw\varepsilon} \leq \frac{\sigma^2 + Mw\varepsilon}{\sigma^2 + Mw\varepsilon} = 1,$$

$$\varepsilon M(a + \sigma^2)/\sigma^4 \leq \delta M(a + \sigma^2)/(\beta\sigma^4) \leq (a + \sigma^2)/(2\beta\sigma^2)$$

and

$$\frac{\sigma^2 - My\varepsilon}{\sigma^2 - Mw\varepsilon} \leq \frac{\sigma^2 - 0}{\sigma^2/2} = 2$$

to conclude

$$\begin{aligned} 2\varepsilon^2 \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)} &\leq \frac{2}{\sigma^2 \cdot y^2 - My^3\varepsilon} \left(\frac{\sigma^2 + My\varepsilon}{\sigma^2 + Mw\varepsilon} \cdot \frac{w}{y} \right)^{\varepsilon M(a+\sigma^2)/\sigma^4} \times \\ &\quad \times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y) \right) \left(\frac{w}{y} \right)^{b/\sigma^2} \left(\frac{\sigma^2 - My\varepsilon}{\sigma^2 - Mw\varepsilon} \right)^{1+b/\sigma^2} \\ &\leq \frac{4}{\sigma^2 \cdot y^2} \left(\frac{w}{y} \right)^{(a+\sigma^2)/(2\beta\sigma^2)} \exp\left(\frac{a}{\sigma^2}(1/w - 1/y) \right) \left(\frac{w}{y} \right)^{b/\sigma^2} 2^{1+b/\sigma^2}. \end{aligned} \tag{4.18}$$

In this upper bound (4.18) the limit variable ε does not appear anymore. For it to be a proper majorant we need to show integrability. For this to end we introduce the shorthand notations $c := 2^{3+b/\sigma^2}/\sigma^2 > 0$, $d := (a + \sigma^2)/(2\beta\sigma^2) + b/\sigma^2 > 0$ and $q := a/\sigma^2 > 0$; using those abbreviations we need to show that the integral

$$\int_0^\beta \int_y^\beta \frac{4}{\sigma^2 \cdot y^2} \left(\frac{w}{y} \right)^{(a+\sigma^2)/(2\beta\sigma^2)} \exp\left(\frac{a}{\sigma^2}(1/w - 1/y) \right) \left(\frac{w}{y} \right)^{b/\sigma^2} 2^{1+b/\sigma^2} \, dw \, dy$$

$$= \int_0^\beta \int_y^\beta c \cdot \frac{w^d}{y^{d+2}} \exp(q/w - q/y)\, dw\, dy$$

is indeed finite. Since the domain $y \in [0, \beta]$ is a compact space and the mapping $(0, \beta] \to [0, \infty)$, $y \mapsto c \int_y^\beta \frac{w^d}{y^{d+2}} \exp(q/w - q/y)\, dw$ is continuous it suffices to show that the limit

$$\lim_{y \downarrow 0} \int_y^\beta \frac{w^d}{y^{d+2}} \exp\left(\frac{q}{w} - \frac{q}{y}\right) dw$$

exists and is finite. Following L'Hôpital's rule to analyze the limiting behavior of the fraction

$$\int_y^\beta \frac{w^d}{y^{d+2}} \exp\left(\frac{q}{w} - \frac{q}{y}\right) dw = \frac{\int_y^\beta w^d e^{q/w}\, dw}{y^{d+2} e^{q/y}},$$

we investigate numerator and denominator separately:

$$\lim_{y \downarrow 0} \int_y^\beta w^d e^{q/w}\, dw = \lim_{y \downarrow 0} \int_{q/\beta}^{q/y} (q/w)^d e^w \frac{q}{w^2} dw = q^{d+1} \int_{q/\beta}^\infty \frac{e^w}{w^{d+2}} = \infty$$

and

$$\frac{\partial}{\partial y} \left[\int_y^\beta w^d e^{q/w}\, dw \right] = -y^d e^{q/y};$$

$$\lim_{y \downarrow 0} y^{d+2} e^{q/y} = \infty$$

and

$$\frac{\partial}{\partial y} \left[y^{d+2} e^{q/y} \right] = (d+2) y^{d+1} e^{q/y} + y^{d+2} e^{q/y} \cdot (-q/y^2) = y^d e^{q/y} \cdot ((d+2)y - q).$$

It follows

$$\lim_{y \downarrow 0} \frac{\int_y^\beta w^d e^{q/w}\, dw}{y^{d+2} e^{q/y}} = \lim_{y \downarrow 0} \frac{-y^d e^{q/y}}{y^d e^{q/y} \cdot ((d+2)y - q)} = 1/q < \infty$$

exists and is finite. By the pursued argumentation, we have now shown that the majorant given in (4.18) is integrable over $0 < y \le w < \beta$ and particularly on the domain D and we therefore may apply Lebesgue's theorem. The pointwise limit in the last line of the following equations is given by Lemma 4.33 part c) to be

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}] &= \lim_{\varepsilon \to 0} \int_D \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}\, d(y, w) = \int_D \lim_{\varepsilon \to 0} \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}\, d(y, w) \\ &= \int_D \frac{2}{\sigma^2} e^{\frac{a}{\sigma^2}(1/w - 1/y)} \frac{w^{b/\sigma^2}}{y^{b/\sigma^2 + 2}}\, d(y, w) < \infty, \end{aligned} \tag{4.19}$$

where the finiteness in the last calculation step is a direct consequence of the existence of an integrable majorant.

We now turn to $\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon} \mid T_{\alpha\varepsilon} < T_z]$. Recalling Lemma 4.32, we are equipped with

$$\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon} \mid T_{\alpha\varepsilon} < T_z] = 2\varepsilon^2 \left[\int_\alpha^\beta \int_\alpha^y \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\, dw\, dy + \int_\beta^{z/\varepsilon} \int_\alpha^\beta \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\, dw\, dy \right]. \tag{4.20}$$

Here we have

$$\begin{aligned} \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)} &= \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)} \cdot \left(\frac{h(y\varepsilon)}{h(w\varepsilon)} \right)^2 \\ &= \frac{1}{\sigma^2(y\varepsilon)} \exp\left(\varepsilon \int_{w\varepsilon}^{y\varepsilon} \frac{b_1(l)}{\sigma^2(l)}\, dl \right) \exp\left(\int_{y\varepsilon}^{w\varepsilon} \frac{b_2(l)}{\sigma^2(l)}\, dl \right) \left(\frac{h(y\varepsilon)}{h(w\varepsilon)} \right)^2. \end{aligned}$$

Since on the integration domains the relation $w \leq y$ holds and since by (4.12) the harmonic function h is non-increasing, it holds

$$\left(\frac{h(y\varepsilon)}{h(w\varepsilon)} \right)^2 \leq 1 \tag{4.21}$$

uniformly on all of the integration domain. For $\alpha < w \leq y \leq z/\varepsilon$ one obtains

$$\frac{h(y\varepsilon)}{h(w\varepsilon)} \geq \frac{h(y\varepsilon)}{h(\alpha\varepsilon)} = \frac{\int_{y\varepsilon}^{z} 1/p_\delta(l)\, dl}{\int_{\alpha\varepsilon}^{z} 1/p_\delta(l)\, dl} = 1 - \frac{\int_{\alpha\varepsilon}^{y\varepsilon} 1/p_\delta(l)\, dl}{\int_{\alpha\varepsilon}^{z} 1/p_\delta(l)\, dl}.$$

Now, similarly to (4.10) and (4.11) we can infer for $\varepsilon < \delta/y$

$$\begin{aligned} &\int_{\alpha\varepsilon}^{y\varepsilon} 1/p_\delta(l)\, dl \\ &\leq \frac{1}{\delta^{b/(3\sigma^2)}} \exp\left(\frac{-3\,a}{\sigma^2\delta}\,\varepsilon \right) \varepsilon^{1+b/(3\sigma^2)} \int_\alpha^y \exp\left(3\,a/\sigma^2 \cdot 1/l\right) l^{b/(3\sigma^2)}\, dl \xrightarrow[\varepsilon\to 0]{} 0 \end{aligned}$$

an in the denominator for $\varepsilon < (\delta/\alpha) \wedge \delta$

$$\int_{\alpha\varepsilon}^{z} 1/p_\delta(l)\, dl \geq \int_\delta^z \exp\left(-\delta \int_\delta^l \frac{b_1(m)}{\sigma^2(m)}\, dm \right) \exp\left(\int_\delta^l \frac{b_2(m)}{\sigma^2(m)}\, dm \right) dl > 0.$$

This shows the convergence

$$\frac{h(y\varepsilon)}{h(w\varepsilon)} \xrightarrow[\varepsilon\to 0]{} 1 \tag{4.22}$$

for arbitrary $\alpha < w \leq y \leq z/\varepsilon$.

The expression (4.20) is a sum of two integrals. The first one may be handled by means as is in the first cycle phase. More explicitly, by Lemma 4.33 part b) for $\varepsilon < \delta/\beta$ there is the bound

$$2\varepsilon^2 \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)} \leq 2\varepsilon^2 \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}$$

$$\leq \frac{2}{\sigma^2 \cdot y^2 - My^3\varepsilon} \left(\frac{\sigma^2 - Mw\varepsilon}{\sigma^2 - My\varepsilon} \cdot \frac{y}{w} \right)^{\varepsilon M(a+\sigma^2)/\sigma^4} \times$$
$$\times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \left(\frac{w}{y}\right)^{b/\sigma^2} \left(\frac{\sigma^2 + My\varepsilon}{\sigma^2 + Mw\varepsilon}\right)^{1+b/\sigma^2}.$$

On the domain $\alpha < w \leq y < \beta$ for $\varepsilon < \delta/\beta$ we may combine the estimates

$$\frac{2}{\sigma^2 \cdot y^2 - My^3\varepsilon} \leq \frac{4}{\sigma^2 \cdot y^2},$$

$$\frac{\sigma^2 - Mw\varepsilon}{\sigma^2 - My\varepsilon} \leq \frac{\sigma^2}{\sigma^2/2} = 2,$$

$$\varepsilon M(a + \sigma^2)/\sigma^4 \leq (a + \sigma^2)/(2\beta\sigma^2)$$

and

$$\frac{\sigma^2 + My\varepsilon}{\sigma^2 + Mw\varepsilon} \leq \frac{\sigma^2 + \sigma^2/2}{\sigma^2} = 3/2$$

to obtain the majorant as upper bound in

$$\frac{2}{\sigma^2 \cdot y^2 - My^3\varepsilon} \left(\frac{\sigma^2 - Mw\varepsilon}{\sigma^2 - My\varepsilon} \cdot \frac{y}{w} \right)^{\varepsilon M(a+\sigma^2)/\sigma^4} \times$$
$$\times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \left(\frac{w}{y}\right)^{b/\sigma^2} \left(\frac{\sigma^2 + My\varepsilon}{\sigma^2 + Mw\varepsilon}\right)^{1+b/\sigma^2}$$
$$\leq \frac{4}{\sigma^2 \cdot y^2} \left(\frac{2y}{w}\right)^{(a+\sigma^2)/(2\beta\sigma^2)} \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \left(\frac{w}{y}\right)^{b/\sigma^2} (3/2)^{(1+b/\sigma^2)}.$$

The majorant is integrable as a continuous function over the compact domain $\alpha \leq w \leq y \leq \beta$. Another application of Lebesgue's theorem thus yields incorporating the pointwise limits (4.22) and Lemma 4.33 part c)

$$\begin{aligned} \lim_{\varepsilon \to 0} 2\varepsilon^2 \int_\alpha^\beta \int_\alpha^y \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\, dw\, dy &= \lim_{\varepsilon \to 0} 2\varepsilon^2 \int_\alpha^\beta \int_\alpha^y \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}\, dw\, dy \\ &= \frac{2}{\sigma^2}\left[\int_\alpha^\beta \int_\alpha^y \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \frac{w^{b/\sigma^2}}{y^{b/\sigma^2+2}}\, dw\, dy\right] < \infty. \end{aligned} \tag{4.23}$$

For the second integral in (4.20) the main difficulty is not in the fraction of h-functions but in the fact the integration variable y reaching values up to z. While the functions are continuous so are their compositions and the domain $(\alpha\varepsilon, z]$ might be splitted in $(\alpha\varepsilon, \delta)$ where we have good estimation controls and in $[\delta, z]$ where we then have uniform continuity; e.g. $1/\sigma^2(y)$ attains its finite maximum and positive minimum. But what we would really like to see, is a limit independent of the point z. This accounts for the fact that starting from $\beta\varepsilon$ the chance of hitting z before $\alpha\varepsilon$ tends to 0. That is to say, e.g. for $\alpha = 1, \beta = 2$: $p_{\varepsilon,z} \to 0$ as $\varepsilon \to 0$.

Therefore, we decompose the second integral in (4.20) into two parts

$$2\varepsilon^2 \int_\beta^{z/\varepsilon} \int_\alpha^\beta \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\, dw\, dy = I_1(\varepsilon) + I_2(\varepsilon) \tag{4.24}$$

where

$$\begin{aligned} I_1(\varepsilon) &:= 2\varepsilon^2 \int_\beta^{\delta/\varepsilon} \int_\alpha^\beta \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\, dw\, dy, \\ I_2(\varepsilon) &:= 2\varepsilon^2 \int_{\delta/\varepsilon}^{z/\varepsilon} \int_\alpha^\beta \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\, dw\, dy. \end{aligned}$$

In view of Lemma 4.33 part b) the integrand in

$$I_1(\varepsilon) = \int_\beta^\infty \int_\alpha^\beta \mathbb{1}_{\{y<\delta/\varepsilon\}}\, 2\varepsilon^2 \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)} \left(\frac{h(y\varepsilon)}{h(w\varepsilon)}\right)^2 dw\, dy$$

is bounded on its integration domain for $\varepsilon < \sigma^4/(M(a+\sigma^2))$ by

$$\begin{aligned} &\mathbb{1}_{\{y<\delta/\varepsilon\}}\, 2\varepsilon^2 \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)} \left(\frac{h(y\varepsilon)}{h(w\varepsilon)}\right)^2 \le \mathbb{1}_{\{y<\delta/\varepsilon\}}\, 2\varepsilon^2 \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)} \\ &\le \mathbb{1}_{\{y<\delta/\varepsilon\}} \frac{2}{\sigma^2\cdot y^2 - My^3\varepsilon} \left(\frac{\sigma^2 - Mw\varepsilon}{\sigma^2 - My\varepsilon}\cdot\frac{y}{w}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4} \times \\ &\qquad \times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \left(\frac{w}{y}\right)^{b/\sigma^2} \left(\frac{\sigma^2 + My\varepsilon}{\sigma^2 + Mw\varepsilon}\right)^{1+b/\sigma^2} \\ &\le \mathbb{1}_{\{y<\delta/\varepsilon\}} \frac{2}{\sigma^2\cdot y^2 - y^2\sigma^2/2} \left(\frac{\sigma^2 - 0}{\sigma^2 - \sigma^2/2}\,\frac{y}{w}\right)^1 \times \\ &\qquad \times \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \left(\frac{w}{y}\right)^{b/\sigma^2} \left(\frac{\sigma^2 + \sigma^2/2}{\sigma^2 + 0}\right)^{1+b/\sigma^2} \\ &\le \frac{8}{y^2\sigma^2}\,\frac{y}{w} \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \left(\frac{w}{y}\right)^{b/\sigma^2} \left(\frac{3}{2}\right)^{1+b/\sigma^2}. \end{aligned}$$

The last estimate is an integrable majorant due to

$$\begin{aligned} &\int_\beta^\infty \int_\alpha^\infty \frac{8}{y^2\sigma^2}\,\frac{y}{w} \exp\left(\frac{a}{\sigma^2}(1/w - 1/y)\right) \left(\frac{w}{y}\right)^{b/\sigma^2} \left(\frac{3}{2}\right)^{1+b/\sigma^2} dw\, dy \\ &= \frac{8}{\sigma^2}\left(\frac{3}{2}\right)^{1+b/\sigma^2} \int_\alpha^\beta w^{-1+b/\sigma^2} \exp\left(\frac{a}{\sigma^2\cdot w}\right) dw \cdot \int_\beta^\infty y^{-1-b/\sigma^2} \exp\left(-\frac{a}{\sigma^2\cdot y}\right) dy \end{aligned}$$

and

$$\int_\beta^\infty y^{-1-b/\sigma^2} \exp\left(-\frac{a}{\sigma^2\cdot y}\right) dy \le \int_\beta^\infty y^{-1-b/\sigma^2}\, dy = \frac{\beta^{-b/\sigma^2}}{b/\sigma^2} < \infty.$$

From Lebesgue's Theorem it follows using $\mathbb{1}_{\{y<\delta/\varepsilon\}} \xrightarrow[\varepsilon\to 0]{} \mathbb{1}_{\{y<\infty\}}$, part c) of Lemma 4.33 and (4.22)

$$\begin{aligned}\lim_{\varepsilon\to 0} I_1(\varepsilon) &= \int_\beta^\infty \int_\alpha^\beta \lim_{\varepsilon\to 0} \mathbb{1}_{\{y<\delta/\varepsilon\}}\, 2\varepsilon^2 \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)} \left(\frac{h(y\varepsilon)}{h(w\varepsilon)}\right)^2 dw\, dy \\ &= \int_\beta^\infty \int_\alpha^\beta \frac{2}{y^2\sigma^2} \exp\left(\frac{a}{\sigma^2}\left(\frac{1}{w}-\frac{1}{y}\right)\right)\left(\frac{w}{y}\right)^{b/\sigma^2} dw\, dy.\end{aligned} \tag{4.25}$$

We still need to show $I_2(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$. As first step we bound for $\varepsilon < 1 \wedge (\delta/\beta)$

$$\begin{aligned} I_2(\varepsilon) &= 2\varepsilon^2 \int_{\delta/\varepsilon}^{z/\varepsilon} \int_\alpha^\beta \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\, dw\, dy = 2\int_\delta^z \int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{r_\delta^h(y)}{p_\delta^h(w)}\, dw\, dy \\ &\le 2\int_\delta^z \int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{r_\delta(y)}{p_\delta(w)}\, dw\, dy \\ &= \int_\delta^z \frac{2}{\sigma^2(y)} \exp\left(\int_\delta^y \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\, dl\right) dy \cdot \int_{\alpha\varepsilon}^{\beta\varepsilon} \exp\left(\int_w^\delta \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\, dl\right) dw \\ &\le \int_\delta^z \frac{2}{\sigma^2(y)} \exp\left(\int_\delta^y \frac{b_1(l)-b_2(l)}{\sigma^2(l)}\, dl\right) dy \cdot \int_{\alpha\varepsilon}^{\beta\varepsilon} \exp\left(\varepsilon\int_w^\delta \frac{b_1(l)}{\sigma^2(l)}\, dl\right) dw \end{aligned}$$

as product where the first factor is finite and constant. For the second factor, we further estimate the innermost integral according to

$$\begin{aligned}\int_w^\delta \frac{b_1(l)}{\sigma^2(l)}\, dl \le \int_w^\delta \frac{a+Ml}{\sigma^2\cdot l^2 - Ml^3}\, dl &\le \int_w^\delta \frac{a+Ml}{\sigma^2/2\, l^2}\, dl \\ &= \frac{a}{\sigma^2/2}(1/w - 1/\delta) + \frac{M}{\sigma^2/2}(\ln\delta - \ln w).\end{aligned}$$

This implies for the outer integral

$$\begin{aligned}&\int_{\alpha\varepsilon}^{\beta\varepsilon} \exp\left(\varepsilon\int_w^\delta \frac{b_1(l)}{\sigma^2(l)}\, dl\right) dw \\ &\le \int_{\alpha\varepsilon}^{\beta\varepsilon} \exp\left(\varepsilon\left(\frac{a}{\sigma^2/2} 1/(\alpha\varepsilon) + \frac{M}{\sigma^2/2}(\ln\delta - \ln(\alpha\varepsilon))\right)\right) dw \\ &= \exp\left(\frac{a}{\sigma^2/2}1/\alpha + \varepsilon\left(\frac{M}{\sigma^2/2}(\ln\delta - \ln\alpha)\right) - \varepsilon\ln\varepsilon\cdot\frac{M}{\sigma^2/2}\right)\cdot(\beta-\alpha)\varepsilon \xrightarrow[\varepsilon\to 0]{} 0\end{aligned}$$

since $\varepsilon\ln\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$.

With the decomposition in (4.24) recalling (4.25) we have proven

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} 2\varepsilon^2 \int_\beta^{z/\varepsilon}\int_\alpha^\beta \frac{r_\delta^h(y\varepsilon)}{p_\delta^h(w\varepsilon)}\,dw\,dy \\
&= \int_\beta^\infty \int_\alpha^\beta \frac{2}{y^2\sigma^2}\exp\left(\frac{a}{\sigma^2}\left(\frac{1}{w}-\frac{1}{y}\right)\right)\left(\frac{w}{y}\right)^{b/\sigma^2} dw\,dy \\
&= \lim_{\varepsilon\to 0} 2\varepsilon^2 \int_\beta^{z/\varepsilon}\int_\alpha^\beta \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}\,dw\,dy.
\end{aligned}
$$

In combination with the result (4.23) in view of (4.20) we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}\mid T_{\alpha\varepsilon}<T_z] = \lim_{\varepsilon\to 0}\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}] \\
&= \frac{2}{\sigma^2}\left[\int_\alpha^\beta\int_\alpha^y \exp\left(\frac{a}{\sigma^2}(1/w-1/y)\right)\frac{w^{b/\sigma^2}}{y^{b/\sigma^2+2}}\,dw\,dy\right] \\
&\qquad + \int_\beta^\infty\int_\alpha^\beta \frac{2}{y^2\sigma^2}\exp\left(\frac{a}{\sigma^2}\left(\frac{1}{w}-\frac{1}{y}\right)\right)\left(\frac{w}{y}\right)^{b/\sigma^2} dw\,dy.
\end{aligned}
$$

Together with (4.19) it follows

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\Big(\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}]+\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}\mid T_{\alpha\varepsilon}<T_z]\Big) = \lim_{\varepsilon\downarrow 0}\Big(\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}]+\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}]\Big) \\
&= \frac{2}{\sigma^2}\left[\int_0^\infty\int_\alpha^\beta \exp\left(\frac{a}{\sigma^2}(1/w-1/y)\right)\frac{w^{b/\sigma^2}}{y^{b/\sigma^2+2}}\,dw\,dy\right] \in (0,\infty).
\end{aligned}
$$

□

This gives the required property of the first moment of the cycle duration. As preparation to formulate the uniform boundedness of the second moment we will thereafter establish we introduce the notation of time shifts.

Definition 4.35. For $s\in\mathbb{R}$ and a random process $(X_t)_{t\geq 0}$ let $\theta_s : (X_t)_{t\geq 0}\mapsto (X_{s+t})_{t\geq 0}$ denote the time shift.

Proposition 4.36 (A3)**.** *The cycle durations have finite second moment uniformly in $\varepsilon>0$: For arbitrary $0<\alpha<\beta<\infty$ it holds*

$$
\limsup_{\varepsilon\downarrow 0}\mathbb{E}_{\alpha\varepsilon}[(\widetilde{\sigma}_1)^2] = \limsup_{\varepsilon\downarrow 0}\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon}+T^h_{\alpha\varepsilon}\circ\theta_{T_{\beta\varepsilon}})^2] < \infty.
$$

Proof. For the rest of the proof let us fix some arbitrarily chosen $\alpha,\beta>0$. Since for $x,y\in\mathbb{R}$ the inequality

$$
(x+y)^2 = 2x^2+2y^2-(x-y)^2 \leq 2\,(x^2+y^2)
$$

holds, we may estimate for $\varepsilon>0$ using the strong Markov property

$$
\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon}+T^h_{\alpha\varepsilon}\circ\theta_{T_{\beta\varepsilon}})^2] \leq 2\left(\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon})^2]+\mathbb{E}_{\alpha\varepsilon}[\big(T^h_{\alpha\varepsilon}\circ\theta_{T_{\beta\varepsilon}}\big)^2]\right) = 2\left(\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon})^2]+\mathbb{E}_{\beta\varepsilon}[\big(T^h_{\alpha\varepsilon}\big)^2]\right).
$$

It therefore suffices to show the finiteness of both $\limsup_{\varepsilon\downarrow 0}\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon})^2]$ and $\limsup_{\varepsilon\downarrow 0}\mathbb{E}_{\beta\varepsilon}[\left(T^h_{\alpha\varepsilon}\right)^2]$ separately. (And the finiteness of both of them is also even necessary.)

With the help of Proposition 3.41 we infer

$$\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon})^2] = 2\int_0^{\beta\varepsilon} \mathbb{E}_y[T_{\beta\varepsilon}]\, G_{0,\beta\varepsilon,\delta}(\alpha\varepsilon, y)\, m_\delta(dy).$$

Together with reusing Lemma 4.31 for $0 < y < \beta\varepsilon$

$$\begin{aligned}
\mathbb{E}_y[T_{\beta\varepsilon}] &= 2\varepsilon^2\left[\int_0^{y/\varepsilon}\int_{y/\varepsilon}^{\beta} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y} + \int_{y/\varepsilon}^{\beta}\int_{\widehat{y}}^{\beta} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y}\right] \\
&\le 2\varepsilon^2\left[\int_0^{y/\varepsilon}\int_{y/\varepsilon}^{\beta} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y} + \int_{y/\varepsilon}^{\beta}\int_{\widehat{y}}^{\beta} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y} + \int_0^{y/\varepsilon}\int_{\widehat{y}}^{y/\varepsilon} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y}\right] \\
&= 2\varepsilon^2\int_0^{\beta}\int_{\widehat{y}}^{\beta} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y}
\end{aligned}$$

which is a bound independent of y this yields reprocessing Proposition 3.38

$$\begin{aligned}
\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon})^2] &= 2\int_0^{\beta\varepsilon} \mathbb{E}_y[T_{\beta\varepsilon}]\, G_{0,\beta\varepsilon,\delta}(\alpha\varepsilon, y)\, m_\delta(dy) \\
&\le 2\varepsilon^2\int_0^{\beta}\int_{\widehat{y}}^{\beta} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y}\cdot 2\int_0^{\beta\varepsilon} G_{0,\beta\varepsilon,\delta}(\alpha\varepsilon, y)\, m_\delta(dy) \\
&= 2\varepsilon^2\int_0^{\beta}\int_{\widehat{y}}^{\beta} \frac{r_\delta(\widehat{y}\varepsilon)}{p_\delta(\widehat{w}\varepsilon)}\, d\widehat{w}\, d\widehat{y}\cdot 2\mathbb{E}_{\alpha\varepsilon}[T_{\beta\varepsilon}] \\
&\le 2\left[2\varepsilon^2\int_0^{\beta}\int_{y}^{\beta} \frac{r_\delta(y\varepsilon)}{p_\delta(w\varepsilon)}\, dw\, dy\right]^2 \\
&\xrightarrow[\varepsilon\downarrow 0]{} 2\left[\frac{2}{\sigma^2}\int_0^{\beta}\int_{y}^{\beta} \exp\left(\frac{a}{\sigma^2}\,(1/w - 1/y)\right)\frac{w^{b/\sigma^2}}{y^{b/\sigma^2+2}}\, dw\, dy\right]^2 < \infty,
\end{aligned}$$

where the application of Lebesgue's theorem in the last step is justified by reusing majorant (4.18).
Estimating the quotients of h-functions by 1, the second moment of the second cycle phase is bounded by

$$\begin{aligned}
&\mathbb{E}_{\beta\varepsilon}[(T^h_{\alpha\varepsilon})^2] = \mathbb{E}_{\beta\varepsilon}[(T_{\alpha\varepsilon})^2 \mid T_{\alpha\varepsilon} < T_z] \le \mathbb{E}_{\beta\varepsilon}[(T_{\alpha\varepsilon})^2] \\
&= 4\left[\int_{\alpha\varepsilon}^{\beta\varepsilon}\int_{\alpha\varepsilon}^{y} \frac{r_\delta(y)}{p_\delta(w)}\mathbb{E}_y[T_{\alpha\varepsilon}]\, dw\, dy + \int_{\beta\varepsilon}^{z}\int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{r_\delta(y)}{p_\delta(w)}\mathbb{E}_y[T_{\alpha\varepsilon}]\, dw\, dy\right]. \qquad (4.26)
\end{aligned}$$

The factor 4 in the formula above stems from the fact that the density function $m_\delta(dy) = 2r_\delta(y)\, dy$ of the speed measure contributes a factor 2 and so does Kac's moment formula. Since on $\alpha\varepsilon \le y \le \beta\varepsilon$

$$\mathbb{E}_y[T_{\alpha\varepsilon}] = 2\left[\int_{\alpha\varepsilon}^{y}\int_{\alpha\varepsilon}^{\widehat{y}} \frac{r_\delta(\widehat{y})}{p_\delta(\widehat{y})}\, d\widehat{w}\, d\widehat{y} + \int_{y}^{z}\int_{\alpha\varepsilon}^{y} \frac{r_\delta(\widehat{y})}{p_\delta(\widehat{w})}\, d\widehat{w}\, d\widehat{y}\right] \qquad (4.27)$$

$$
\begin{aligned}
&= 2\left[\int_{\alpha\varepsilon}^{y}\int_{\alpha\varepsilon}^{\widehat{y}} \frac{r_\delta(\widehat{y})}{p_\delta(\widehat{y})}\,d\widehat{w}\,d\widehat{y} + \int_{y}^{\beta\varepsilon}\int_{\alpha\varepsilon}^{y} \frac{r_\delta(\widehat{y})}{p_\delta(\widehat{y})}\,d\widehat{w}\,d\widehat{y}\right] + 2\left[\int_{\beta\varepsilon}^{z}\int_{\alpha\varepsilon}^{y} \frac{r_\delta(\widehat{y})}{p_\delta(\widehat{y})}\,d\widehat{w}\,d\widehat{y}\right] \\
&\le 2\int_{\alpha\varepsilon}^{\beta\varepsilon}\int_{\alpha\varepsilon}^{\widehat{y}} \frac{r_\delta(\widehat{y})}{p_\delta(\widehat{y})}\,d\widehat{w}\,d\widehat{y} + 2\int_{\beta\varepsilon}^{z}\int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{r_\delta(\widehat{y})}{p_\delta(\widehat{y})}\,d\widehat{w}\,d\widehat{y} = \mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}]
\end{aligned}
$$

it holds

$$
4\int_{\alpha\varepsilon}^{\beta\varepsilon}\int_{\alpha\varepsilon}^{y} \frac{r_\delta(y)}{p_\delta(w)}\mathbb{E}_y[T_{\alpha\varepsilon}]\,dw\,dy \le 2\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}]\cdot 2\int_{\alpha\varepsilon}^{\beta\varepsilon}\int_{\alpha\varepsilon}^{y} \frac{r_\delta(y)}{p_\delta(w)}\,dw\,dy \le 2\mathbb{E}_{\beta\varepsilon}[T_{\alpha\varepsilon}]^2.
$$

We have already seen that the last expectation remains bounded as $\varepsilon \downarrow 0$. Therefore the first integral in the right hand side of (4.26) is finite uniformly in $\varepsilon > 0$. On $\alpha\varepsilon \le w \le y$ which particularly covers the domain of the second double integral in (4.26) the following inequality holds:

$$
\frac{r_\delta(y)}{p_\delta(w)} = \sigma^{-2}(y)\exp\left(\int_w^y \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\,dl\right) \le \sigma^{-2}(y)\exp\left(\varepsilon\int_{\alpha\varepsilon}^{y} \frac{b_1(l)}{\sigma^2(l)}\,dl\right).
$$

On $\alpha\varepsilon \le w \le y \le z$ there are the two cases $y < \delta$ and $y \ge \delta$. In the first case $\alpha\varepsilon \le y < \delta$ we estimate in the spirit of (4.10) and obtain

$$
\begin{aligned}
&\sigma^{-2}(y)\exp\left(\varepsilon\int_{\alpha\varepsilon}^{y} \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \le \frac{1}{\sigma^2\cdot y^2/2}\exp\left(\varepsilon\int_{\alpha\varepsilon}^{\delta} \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \\
&\le \frac{2}{y^2\sigma^2}\exp\left(\varepsilon\int_{\alpha\varepsilon}^{\delta} \frac{3a/2}{\sigma^2\cdot l^2/2}\,dl\right) = \frac{2}{y^2\sigma^2}\exp\left(\varepsilon\frac{3a}{\sigma^2}\left(\frac{1}{\alpha\varepsilon} - \frac{1}{\delta}\right)\right) \le \frac{2}{y^2\sigma^2}\exp\left(\frac{3a}{\alpha\sigma^2}\right).
\end{aligned}
$$

In the second case $\delta \le y \le z$ assuming $\varepsilon < (\delta/\alpha) \wedge 1$ we obtain

$$
\begin{aligned}
&\sigma^{-2}(y)\exp\left(\varepsilon\int_{\alpha\varepsilon}^{y} \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \le \sup_{u\in[\delta,z]} \sigma^{-2}(u)\cdot\exp\left(\varepsilon\int_{\alpha\varepsilon}^{\delta} \frac{b_1(l)}{\sigma^2(l)}\,dl\right)\exp\left(\varepsilon\int_{\delta}^{z} \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \\
&\le \sup_{u\in[\delta,z]} \sigma^{-2}(u)\exp\left(\frac{3a}{\sigma^2\alpha}\right)\exp\left(\int_{\delta}^{z} \frac{b_1(l)}{\sigma^2(l)}\,dl\right).
\end{aligned}
$$

Merging both cases together, we compactly write on $\alpha\varepsilon \le w \le y \le z$ for $\varepsilon < (\delta/\alpha) \wedge 1$

$$
0 < \frac{r_\delta(y)}{p_\delta(w)} \le f(y) := \begin{cases} c_1 y^{-2} & \text{for } y < \delta, \\ c_2 & \text{for } y \in [\delta, z], \end{cases} \tag{4.28}
$$

with positive finite constants

$$
\begin{aligned}
c_1 &:= \frac{2}{\sigma^2}\exp\left(\frac{3a}{\sigma^2\alpha}\right), \\
c_2 &:= \sup_{u\in[\delta,z]} \sigma^{-2}(u)\exp\left(\frac{3a}{\sigma^2\alpha}\right)\exp\left(\int_{\delta}^{z} \frac{b_1(l)}{\sigma^2(l)}\,dl\right).
\end{aligned}
$$

Turning back to the second integral in (4.26) we attain

$$\int_{\beta\varepsilon}^{z}\int_{\alpha\varepsilon}^{\beta\varepsilon}\frac{r_\delta(y)}{p_\delta(w)}\mathbb{E}_y[T_{\alpha\varepsilon}]\,dw\,dy \leq \int_{\beta\varepsilon}^{z}\int_{\alpha\varepsilon}^{\beta\varepsilon} f(y)\mathbb{E}_y[T_{\alpha\varepsilon}]\,dw\,dy$$
$$= (\beta-\alpha)\cdot\varepsilon\int_{\beta\varepsilon}^{z} f(y)\mathbb{E}_y[T_{\alpha\varepsilon}]\,dy. \qquad (4.29)$$

Applying the estimate (4.28) on the formula (4.27) for the expectation $\mathbb{E}_{y\varepsilon}[T_{\alpha\varepsilon}]$ yields for $\alpha\varepsilon \leq y \leq z$

$$\mathbb{E}_y[T_{\alpha\varepsilon}] = 2\left[\int_{\alpha\varepsilon}^{y}\int_{\alpha\varepsilon}^{\widehat{y}}\frac{r_\delta(\widehat{y})}{p_\delta(\widehat{y})}\,d\widehat{w}\,d\widehat{y} + \int_{y}^{z}\int_{\alpha\varepsilon}^{y}\frac{r_\delta(\widehat{y})}{p_\delta(\widehat{w})}\,d\widehat{w}\,d\widehat{y}\right]$$
$$\leq 2\left[\int_{\alpha\varepsilon}^{y}\int_{\alpha\varepsilon}^{\widehat{y}} f(\widehat{y})\,d\widehat{w}\,d\widehat{y} + \int_{y}^{z}\int_{\alpha\varepsilon}^{y} f(\widehat{y})\,d\widehat{w}\,d\widehat{y}\right]$$
$$= 2\left[\int_{\alpha\varepsilon}^{y}(\widehat{y}-\alpha\varepsilon)\,f(\widehat{y})\,d\widehat{y} + \int_{y}^{z}(y-\alpha\varepsilon)\,f(\widehat{y})\,d\widehat{y}\right] \leq 2\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\,f(\widehat{y})\,d\widehat{y} + \int_{y}^{z} y\,f(\widehat{y})\,d\widehat{y}\right].$$

Plugging in expression (4.29) leads to

$$(\beta-\alpha)\cdot\varepsilon\int_{\beta\varepsilon}^{z} f(y)\mathbb{E}_y[T_{\alpha\varepsilon}]\,dy \leq 2(\beta-\alpha)\varepsilon\int_{\beta\varepsilon}^{z} f(y)\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\,f(\widehat{y})\,d\widehat{y} + \int_{y}^{z} y\,f(\widehat{y})\,d\widehat{y}\right]dy$$
$$= 2(\beta-\alpha)\varepsilon\Bigg(\int_{\delta}^{z} f(y)\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\,f(\widehat{y})\,d\widehat{y} + \int_{y}^{z} y\,f(\widehat{y})\,d\widehat{y}\right]dy+ \qquad (4.30)$$
$$+\int_{\beta\varepsilon}^{\delta} f(y)\int_{\delta}^{z} y\,f(\widehat{y})\,d\widehat{y}\,dy+ \qquad (4.31)$$
$$+\int_{\beta\varepsilon}^{\delta} f(y)\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\,f(\widehat{y})\,d\widehat{y} + \int_{y}^{\delta} y\,f(\widehat{y})\,d\widehat{y}\right]dy\Bigg) \qquad (4.32)$$

In the following we will for all of (4.30), (4.31) and (4.32) show that the expressions stay finite as $\varepsilon \to 0$. For $y \geq \delta$ we bound according to definition given in (4.28) by

$$f(y)\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\,f(\widehat{y})\,d\widehat{y} + \int_{y}^{z} y\,f(\widehat{y})\,d\widehat{y}\right] \leq c_2\left[\int_{\alpha\varepsilon}^{\delta}\widehat{y}\,f(\widehat{y})\,d\widehat{y} + \int_{\delta}^{z} y\,f(\widehat{y})d\widehat{y}\right]$$
$$\leq c_2\left[\int_{\alpha\varepsilon}^{\delta}\widehat{y}\,\frac{c_1}{\widehat{y}^2}\,d\widehat{y} + \int_{\delta}^{z} y\,c_2\,d\widehat{y}\right] = c_2c_1\ln\left(\frac{\delta}{\alpha\varepsilon}\right) + c_2^2y(z-\delta) \leq c_2c_1\ln\left(\frac{\delta}{\alpha\varepsilon}\right) + c_2^2zy.$$

It follows for (4.30)

$$2(\beta-\alpha)\varepsilon\int_{\delta}^{z} f(y)\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\,f(\widehat{y})\,d\widehat{y} + \int_{y}^{z} y\,f(\widehat{y})\,d\widehat{y}\right]dy$$
$$\leq 2(\beta-\alpha)\varepsilon\int_{\delta}^{z} c_2c_1\ln\left(\frac{\delta}{\alpha\varepsilon}\right) + c_2^2zy\,dy$$

$$\leq 2(\beta-\alpha)\varepsilon\left(c_2 c_1 \ln\left(\frac{\delta}{\alpha\varepsilon}\right) z + c_2^2 z\, z^2/2\right) \xrightarrow[\varepsilon\to 0]{} 0.$$

Analogously for (4.31)

$$2(\beta-\alpha)\varepsilon\int_{\beta\varepsilon}^{\delta} f(y)\int_{\delta}^{z} y\, f(\widehat{y})\, d\widehat{y}\, dy \leq 2(\beta-\alpha)\varepsilon\int_{\beta\varepsilon}^{\delta}\frac{c_1}{y^2}\int_{\delta}^{z} y\, c_2\, d\widehat{y}\, dy$$
$$\leq 2(\beta-\alpha)\varepsilon\int_{\beta\varepsilon}^{\delta}\frac{c_1}{y^2}\, y\, c_2\, z\, dy = 2(\beta-\alpha)\varepsilon\, c_1\, c_2\, z\, \ln\left(\frac{\delta}{\beta\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} 0.$$

On the other hand we have for (4.32)

$$2(\beta-\alpha)\varepsilon\int_{\beta\varepsilon}^{\delta} f(y)\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\, f(\widehat{y})\, d\widehat{y} + \int_{y}^{\delta} y\, f(\widehat{y})\, d\widehat{y}\right] dy$$
$$\leq 2(\beta-\alpha)\varepsilon\int_{\beta\varepsilon}^{\delta}\frac{c_1}{y^2}\left[\int_{\alpha\varepsilon}^{y}\widehat{y}\,\frac{c_1}{\widehat{y}^2}\, d\widehat{y} + \int_{y}^{\delta} y\,\frac{c_1}{\widehat{y}^2}\, d\widehat{y}\right] dy$$
$$= 2(\beta-\alpha)\varepsilon\int_{\beta\varepsilon}^{\delta}\frac{c_1}{y^2}\left[c_1\ln\frac{y}{\alpha\varepsilon} + y c_1\left(\frac{1}{y^3}-\frac{1}{\delta^3}\right)\right] dy \leq 2(\beta-\alpha)c_1\varepsilon\int_{\beta\varepsilon}^{\delta}\frac{1}{y^2}\left[\ln\frac{y}{\alpha\varepsilon}+1\right] dy$$
$$= 2(\beta-\alpha)c_1\int_{\beta}^{\delta/\varepsilon}\frac{1}{y^2}\left[\ln\frac{y}{\alpha}+1\right] dy \leq 2(\beta-\alpha)c_1\int_{\beta}^{\infty}\frac{1}{y^2}\left[\ln\frac{y}{\alpha}+1\right] dy < \infty,$$

which is a finite bound independent of ε. Summing up, we have shown that for the second integral in (4.26) it holds

$$\limsup_{\varepsilon\downarrow 0}\int_{\beta\varepsilon}^{z}\int_{\alpha\varepsilon}^{\beta\varepsilon}\frac{r_\delta(y)}{p_\delta(w)}\mathbb{E}_y[T_{\alpha\varepsilon}]\, dw\, dy < \infty.$$

It follows $\limsup_{\varepsilon\downarrow 0}\mathbb{E}_{\beta\varepsilon}[(T^h_{\alpha\varepsilon})^2] < \infty$ and $\limsup_{\varepsilon\downarrow 0}\mathbb{E}_{\alpha\varepsilon}[(T_{\beta\varepsilon}+T^h_{\alpha\varepsilon}\circ\theta_{T_{\beta\varepsilon}})^2] < \infty$. This was to be proven. □

It remains to consider an arbitrary starting point $x > 0$. In other words: proving (B1) and (B2). We will show the asserted convergence in (B1) even in the corresponding $L^1(\mathbb{P}_x)$ spaces which particularly implies the convergence in distribution. Let us start with the following preparation:

Lemma 4.37. *In the scaling limit $\lambda\to\infty$, $\varepsilon\to 0$ with $\lambda^2 p_{\varepsilon,z} = J \in (0,\infty)$ we have*

$$p_{\varepsilon,z}\in O(\varepsilon^{b/\sigma^2+1}).$$

In particular, $\lim\limits_{scaling}\frac{\ln\varepsilon}{\lambda^2} = 0.$

Proof. By the calculations (4.11) in the fraction (4.9) given by

$$p_{\varepsilon,z} = \mathbb{P}_{2\varepsilon}(T_z < T_\varepsilon) = \frac{\int_\varepsilon^{2\varepsilon} 1/p_\delta(y)\, dy}{\int_\varepsilon^{2\varepsilon} 1/p_\delta(y)\, dy}$$

the denominator is bounded away from 0 as ε approaches 0. For the numerator, assuming $\varepsilon < \delta/2$ we are entirely in the regime, where the approximations of coefficient functions given in equation (4.7) hold. It follows

$$\begin{aligned}\int_{\varepsilon}^{2\varepsilon} 1/p_\delta(y)\,dy =& \varepsilon \int_1^2 \exp\left(\varepsilon \int_{y\varepsilon}^{\delta} \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \exp\left(-\int_{y\varepsilon}^{\delta} \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy\\ &\le \varepsilon \int_1^2 \exp\left(\varepsilon \int_{y\varepsilon}^{\delta} \frac{a+Ml}{\sigma^2 l^2 - Ml^3}\,dl\right) \exp\left(-\int_{y\varepsilon}^{\delta} \frac{b-Ml}{\sigma^2 l + Ml^2}\,dl\right)\,dy.\end{aligned}$$

With the integral formulas (4.16) and (4.17) steaming from the computations carried out in (4.14) and (4.15) this evaluates to

$$\begin{aligned}&\varepsilon \int_1^2 \exp\left(\varepsilon \int_{y\varepsilon}^{\delta} \frac{a+Ml}{\sigma^2 l^2 - Ml^3}\,dl\right) \exp\left(-\int_{y\varepsilon}^{\delta} \frac{b-Ml}{\sigma^2 l + Ml^2}\,dl\right)\,dy\\ &= \varepsilon \int_1^2 \left(\frac{\sigma^2 - My\varepsilon}{\sigma^2 - M\delta}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4} \left(\frac{\delta}{y\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\\ &\qquad \times \exp\left(\frac{a}{\sigma^2}\left(\frac{1}{y} - \frac{\varepsilon}{\delta}\right)\right) \left(\frac{y\varepsilon}{\delta}\right)^{b/\sigma^2} \left(\frac{\sigma^2 + M\delta}{\sigma^2 + My\varepsilon}\right)^{b/\sigma^2+1}\,dy.\end{aligned}$$

By the identity $\ln x \le x - 1$ for $x > 0$ it follows for $\varepsilon > 0$

$$\varepsilon^{-\varepsilon} = \exp\left(\varepsilon \ln \frac{1}{\varepsilon}\right) \le \exp\left(\varepsilon\left(\frac{1}{\varepsilon} - 1\right)\right) = \exp(1-\varepsilon) \le e. \tag{4.33}$$

We can therefore estimate still assuming $\varepsilon < \delta/2$

$$\begin{aligned}&\varepsilon \int_1^2 \left(\frac{\sigma^2 - My\varepsilon}{\sigma^2 - M\delta}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4} \left(\frac{\delta}{y\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\\ &\qquad \times \exp\left(\frac{a}{\sigma^2}\left(\frac{1}{y} - \frac{\varepsilon}{\delta}\right)\right) \left(\frac{y\varepsilon}{\delta}\right)^{b/\sigma^2} \left(\frac{\sigma^2 + M\delta}{\sigma^2 + My\varepsilon}\right)^{b/\sigma^2+1}\,dy.\\ &\le \varepsilon^{1+b/\sigma^2} \int_1^2 \left(\frac{\sigma^2 - 0}{\sigma^2 - \sigma^2/2}\right)^{\delta M(a+\sigma^2)/(2\sigma^4)} \left(\frac{\delta}{y} \vee 1\right)^{\delta M(a+\sigma^2)/(2\sigma^4)} e^{\delta M(a+\sigma^2)/(2\sigma^4)}\\ &\qquad \times \exp\left(\frac{a}{\sigma^2} \cdot \frac{1}{y}\right) \left(\frac{y}{\delta}\right)^{b/\sigma^2} \left(\frac{\sigma^2 + M\delta}{\sigma^2 + 0}\right)^{b/\sigma^2+1}\,dy.\end{aligned}$$

This indeed shows $p_{\varepsilon,z} \in O(\varepsilon^{b/\sigma^2+1})$. It particularly follows

$$\lim_{\substack{\lambda\to\infty\\ \varepsilon\to 0\\ \lambda^2 p_{\varepsilon,z} = J}} \frac{\ln \varepsilon}{\lambda^2} = \lim_{\varepsilon\to 0} \frac{p_{\varepsilon,z} \ln \varepsilon}{J} = 0.$$

All assertions are shown and the proof is finished. □

Proposition 4.38 (B1). *It holds*

$$\lim_{scaling} \mathbb{E}_x[T_\varepsilon^\lambda \wedge T_z^\lambda] = 0 \text{ for } 0 < x < z.$$

Proof. For the rest of the proof let us fix some arbitrarily chosen $0 < x < z$. Again exploiting the Green function approach, this time with

$$G_{\varepsilon,z,\delta}(x,y) = \frac{2}{K\lambda^2} u(x \wedge y) v(x \vee y).$$

where $K := \int_\varepsilon^z \frac{1}{p_\delta(w)}\,dw$, $u(x) := \int_\varepsilon^x 1/p_\delta(w)\,dw$ and $v(x) := \int_x^z 1/p_\delta(w)\,dw$ Proposition 3.31 states that

$$\begin{aligned}
\mathbb{E}_x[T_\varepsilon^\lambda \wedge T_z^\lambda] &= \frac{1}{\lambda^2}\mathbb{E}_x[T_\varepsilon \wedge T_z] = \frac{1}{K\lambda^2}\int_\varepsilon^z G_{\varepsilon,z,\delta}(x,y)\, m_\delta(dy) \\
&= \frac{2}{K\lambda^2}\left[v(x)\int_\varepsilon^x \int_\varepsilon^y \frac{r_\delta(y)}{p_\delta(w)}\,dw\,dy + u(x)\int_x^z \int_y^z \frac{r_\delta(y)}{p_\delta(w)}\,dw\,dy\right] \\
&\le \frac{2}{\lambda^2}\left[\int_\varepsilon^x \int_\varepsilon^y \frac{r_\delta(y)}{p_\delta(w)}\,dw\,dy + \int_x^z \int_y^z \frac{r_\delta(y)}{p_\delta(w)}\,dw\,dy\right]
\end{aligned}$$

with integrand

$$\frac{r_\delta(y)}{p_\delta(w)} = \frac{1}{\sigma^2(y)} \exp\left(\varepsilon \int_w^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \exp\left(\int_y^w \frac{b_2(l)}{\sigma^2(l)}\,dl\right).$$

Since on the integration domain of the second integral $w \ge y$ holds, the exponential with the ε term in it is bounded by $\exp(0) = 1$ and the integral is overall bounded by

$$\int_x^z \int_y^z \frac{r_\delta(y)}{p_\delta(w)}\,dw\,dy \le \int_x^z \int_y^z \frac{1}{\sigma^2(y)} \exp\left(\int_y^w \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dw\,dy$$

which is a constant independent of ε and λ. It follows that the second integral will vanish in the scaling limit due to the multiplication with the time scaling factor λ^{-2}. The first integral may be decomposed in

$$\begin{aligned}
&\int_\varepsilon^x \int_\varepsilon^y \frac{r_\delta(y)}{p_\delta(w)}\,dw\,dy = \int_\varepsilon^x \int_\varepsilon^y \frac{1}{\sigma^2(y)} \exp\left(\varepsilon \int_w^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right) \exp\left(\int_y^w \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dw\,dy \\
&\le \int_\varepsilon^x \int_\varepsilon^y \frac{1}{\sigma^2(y)} \exp\left(\varepsilon \int_w^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right)\,dw\,dy \\
&= \int_\varepsilon^{\delta\wedge x} \int_\varepsilon^y \frac{1}{\sigma^2(y)} \exp\left(\varepsilon \int_w^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right)\,dw\,dy + \\
&\qquad + \int_{\delta\wedge x}^x \int_\varepsilon^{\delta\wedge x} \frac{1}{\sigma^2(y)} \exp\left(\varepsilon \int_w^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right)\,dw\,dy + \\
&\qquad + \int_{\delta\wedge x}^x \int_{\delta\wedge x}^y \frac{1}{\sigma^2(y)} \exp\left(\varepsilon \int_w^y \frac{b_1(l)}{\sigma^2(l)}\,dl\right)\,dw\,dy
\end{aligned} \tag{4.34}$$

The last term in (4.34) is bounded, since the ε-exponential is monotonically decreasing:

$$\int_{\delta\wedge x}^{x}\int_{\delta\wedge x}^{y}\frac{1}{\sigma^2(y)}\exp\left(\varepsilon\int_w^y\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy\leq\int_{\delta\wedge x}^{x}\int_{\delta\wedge x}^{y}\frac{1}{\sigma^2(y)}\exp\left(\int_w^y\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy$$

for $\varepsilon<1$ which again is a constant independent of ε. Therefore,

$$\lim_{scaling}\frac{2}{\lambda^2}\int_{\delta\wedge x}^{x}\int_{\delta\wedge x}^{y}\frac{1}{\sigma^2(y)}\exp\left(\varepsilon\int_w^y\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy=0.$$

For the second term in (4.34) we estimate

$$\begin{aligned}&\int_{\delta\wedge x}^{x}\int_{\varepsilon}^{\delta\wedge x}\frac{1}{\sigma^2(y)}\exp\left(\varepsilon\int_w^y\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy\\&\leq\sup_{z\in[\delta\wedge x,x]}\sigma^{-2}(z)\cdot\int_{\delta\wedge x}^{x}\int_{\varepsilon}^{\delta\wedge x}\exp\left(\varepsilon\int_w^x\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy\\&=\sup_{z\in[\delta\wedge x,x]}\sigma^{-2}(z)\cdot\exp\left(\varepsilon\int_{\delta\wedge x}^x\frac{b_1(l)}{\sigma^2(l)}\,dl\right)(x-(\delta\wedge x))\int_{\varepsilon}^{\delta\wedge x}\exp\left(\varepsilon\int_w^{\delta\wedge x}\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw.\end{aligned}$$

Again, the factor $\exp\left(\varepsilon\int_{\delta\wedge x}^x\frac{b_1(l)}{\sigma^2(l)}\,dl\right)$ is monotonically decreasing in ε. Assuming $\varepsilon<\delta\wedge x$ and applying integral formula (4.16) we furthermore by using the fact

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^\varepsilon}=\lim_{\varepsilon\downarrow 0}\exp\left(-\varepsilon\ln\varepsilon\right)=\exp(-0)=1$$

find the bound

$$\begin{aligned}&\int_{\varepsilon}^{\delta\wedge x}\exp\left(\varepsilon\int_w^{\delta\wedge x}\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\leq\int_{\varepsilon}^{\delta\wedge x}\exp\left(\varepsilon\int_w^{\delta\wedge x}\frac{a+Ml}{\sigma^2\cdot l^2-Ml^3}\,dl\right)dw\\&=\int_{\varepsilon}^{\delta\wedge x}\left(\frac{\sigma^2-Mw}{\sigma^2-M(\delta\wedge x)}\cdot\frac{\delta\wedge x}{w}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\exp\left(\frac{a}{\sigma^2}\left(\frac{\varepsilon}{w}-\frac{\varepsilon}{\delta\wedge x}\right)\right)dw\\&\leq\int_{\varepsilon}^{\delta\wedge x}\left(\frac{\sigma^2-0}{\sigma^2-\sigma^2/2}\cdot\frac{\delta\wedge x}{w}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\exp\left(\frac{a}{\sigma^2}\left(\frac{\varepsilon}{w}-\frac{\varepsilon}{\delta\wedge x}\right)\right)dw\\&\leq\left(2\cdot\frac{\delta\wedge x}{\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\exp\left(\frac{a}{\sigma^2}\right)\cdot((\delta\wedge x)-\varepsilon)\xrightarrow[\varepsilon\to 0]{}(\delta\wedge x)\exp(a/\sigma^2)<\infty.\end{aligned}$$

It follows by the boundedness of the second term in (4.34) the convergence with respect to the scaling limit

$$\begin{aligned}&\lim_{scaling}\frac{2}{\lambda^2}\int_{\delta\wedge x}^{x}\int_{\varepsilon}^{\delta\wedge x}\frac{1}{\sigma^2(y)}\exp\left(\varepsilon\int_w^y\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy\\&\leq\lim_{scaling}\frac{2}{\lambda^2}\left(2\cdot\frac{\delta\wedge x}{\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\exp\left(\frac{a}{\sigma^2}\right)\cdot((\delta\wedge x)-\varepsilon)=0.\end{aligned}$$

The first term in (4.34) is not bonded but the scaling limit will still vanish. Since for $\varepsilon < \delta \wedge x$ we are entirely in the approximation regime we may again use integral formula (4.16) and estimate according to

$$
\begin{aligned}
&\int_{\varepsilon}^{\delta\wedge x}\int_{\varepsilon}^{y}\frac{1}{\sigma^2(y)}\exp\left(\varepsilon\int_w^y\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy\\
&\leq \int_1^{(\delta\wedge x)/\varepsilon}\int_1^y\frac{\varepsilon^2}{\sigma^2\cdot(y\varepsilon)^2-M(y\varepsilon)^3}\exp\left(\varepsilon\int_{w\varepsilon}^{y\varepsilon}\frac{a+Ml}{\sigma^2 l^2-Ml^3}\,dl\right)dw\,dy\\
&=\int_1^{(\delta\wedge x)/\varepsilon}\int_1^y\frac{1}{\sigma^2-My\varepsilon}\frac{1}{y^2}\left(\frac{\sigma^2-Mw\varepsilon}{\sigma^2-My\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\\
&\qquad\times\left(\frac{y}{w}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\exp\left(a/\sigma^2\left(\frac{1}{w}-\frac{1}{y}\right)\right)dw\,dy\\
&\leq \int_1^{(\delta\wedge x)/\varepsilon}\int_1^y\frac{1}{\sigma^2-M(\delta\wedge x)}\frac{1}{y^2}\left(\frac{\sigma^2}{\sigma^2-M(\delta\wedge x)}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\\
&\qquad\times\left(\frac{y}{w}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\exp\left(a/\sigma^2\left(\frac{1}{w}-\frac{1}{y}\right)\right)dw\,dy\\
&\leq\frac{2}{\sigma^2}e^{a/\sigma^2}\int_1^{(\delta\wedge x)/\varepsilon}\int_1^y\frac{1}{y^2}\left(\frac{2(\delta\wedge x)}{\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}dw\,dy\\
&=\frac{2}{\sigma^2}e^{a/\sigma^2}\left(\frac{2(\delta\wedge x)}{\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\int_1^{(\delta\wedge x)/\varepsilon}\frac{y-1}{y^2}\,dy.
\end{aligned}
$$

Recalling (4.33) we may further estimate still assuming $\varepsilon < \delta \wedge x$

$$
\begin{aligned}
&\frac{2}{\sigma^2}e^{a/\sigma^2}\left(\frac{2(\delta\wedge x)}{\varepsilon}\right)^{\varepsilon M(a+\sigma^2)/\sigma^4}\int_1^{(\delta\wedge x)/\varepsilon}\frac{y-1}{y^2}\,dy\\
&\leq\frac{2}{\sigma^2}\exp\left(a/\sigma^2+M(a+\sigma^2)/\sigma^4\right)\left(2(\delta\wedge x)\vee 1\right)^{(\delta\wedge x)M(a+\sigma^2)/\sigma^4}\left(\ln(\delta\wedge x)-\ln\varepsilon\right).
\end{aligned}
$$

By the last part of the previous lemma it holds $\lim_{scaling}\frac{\ln\varepsilon}{\lambda^2}=0$. It therefore follows for the first term in (4.34)

$$
\begin{aligned}
&\lim_{scaling}\frac{2}{\lambda^2}\int_{\varepsilon}^{\delta\wedge x}\int_{\varepsilon}^{y}\frac{1}{\sigma^2(y)}\exp\left(\varepsilon\int_w^y\frac{b_1(l)}{\sigma^2(l)}\,dl\right)dw\,dy\\
&\leq\lim_{scaling}\frac{2}{\lambda^2}\cdot\frac{2}{\sigma^2}\exp\left(a/\sigma^2+M(a+\sigma^2)/\sigma^4\right)\left(2(\delta\wedge x)\vee 1\right)^{(\delta\wedge x)M(a+\sigma^2)/\sigma^4}\left(\ln(\delta\wedge x)-\ln\varepsilon\right)\\
&=0.
\end{aligned}
$$

Summing up, we have shown $\lim_{scaling}\mathbb{E}_x[T_\varepsilon\wedge T_z]=0$ which was to be proven.

□

We now complete the discussion of the example class with

Proposition 4.39 (B2)**.** *It holds*

$$\mathbb{P}_x(T_\varepsilon < T_z) \xrightarrow[\varepsilon\to 0]{} \frac{\int_x^z \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy}{\int_0^z \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy} \qquad \text{for } 0 < x < z.$$

Proof. For the rest of the proof let us fix some arbitrarily chosen $0 < x < z$. We first recall that by the scale function approach Lemma 3.20 it holds

$$\mathbb{P}_x(T_\varepsilon < T_z) = \frac{\int_x^z \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy}{\int_\varepsilon^z \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy}.$$

By Lebesgue's theorem

$$\lim_{\varepsilon\downarrow 0}\int_x^z \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy = \int_x^z \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy. \tag{4.35}$$

Similarly it holds

$$\lim_{\varepsilon\downarrow 0}\int_\delta^z \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy = \int_\delta^z \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy. \tag{4.36}$$

Since for $\varepsilon < \delta$

$$\begin{aligned}
\mathbb{1}_{(\varepsilon,\delta)}(y)\,\exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right) &\le \mathbb{1}_{(\varepsilon,\delta)}(y)\,\exp\left(\int_y^z \frac{\varepsilon b_1(l)}{\sigma^2(l)}\,dl\right)\\
&= \mathbb{1}_{(\varepsilon,\delta)}(y)\,\exp\left(\int_y^\delta \frac{\varepsilon b_1(l)}{\sigma^2(l)}\,dl\right)\exp\left(\int_\delta^z \frac{\varepsilon b_1(l)}{\sigma^2(l)}\,dl\right)\\
&\le ((2\delta\wedge 1)\cdot e)^{\delta M(a+\sigma^2)/\sigma^4}\exp(a/\sigma^2)\exp\left(\delta\int_\delta^z \frac{b_1(l)}{\sigma^2(l)}\,dl\right) < \infty
\end{aligned}$$

we may also apply Lebesgue's theorem for the lower part of the denominator's integration domain:

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\delta \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy = \int_0^\delta \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy. \tag{4.37}$$

The limits (4.36) and (4.37) may be combined to

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\varepsilon^z \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy\\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^\delta \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy + \lim_{\varepsilon\to 0}\int_\delta^z \exp\left(\int_y^z \frac{\varepsilon b_1(l)-b_2(l)}{\sigma^2(l)}\,dl\right)\,dy\\
&= \int_0^\delta \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy + \int_\delta^z \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy\\
&= \int_0^z \exp\left(-\int_y^z \frac{b_2(l)}{\sigma^2(l)}\,dl\right)\,dy.
\end{aligned} \tag{4.38}$$

Both assertions (4.35) and (4.38) together imply the Proposition. □

4.4.2. Homodyne detection of Rabi oscillation

As is carefully described in [5] an analysis of homodyne detection of Rabi oscillations leads to the following stochastic differential equation on the state space $\Theta = (0, 2\pi)$

$$d\theta_t = -\lambda^2 \sin\theta_t \big(1 - \cos\theta_t\big)\, dt + \lambda\big(1 - \cos\theta_t\big) dB_t. \tag{4.39}$$

Following a suggestion of [5, sections 2.3, 5.2] we investigate a 'linearized' version of (4.39), i.e. the case where in (4.5)

$$b_1(x) = 1, \qquad b_2(x) = b \cdot x, \qquad \sigma(x) = x^2,$$

with $b > 0$ some positive real number. Note, in this model $\sigma^2(x) = x^4$. The corresponding SDE reads

$$dX_t = \frac{1}{2}(\varepsilon - bX_t)\, dt + (X_t)^2\, dB_t. \tag{4.40}$$

Remark 4.40.
- Conditions (ND) and (LI) are fulfilled.
- The scale function for $c, x \in (0, \infty)$ is given by

$$\begin{aligned} s_c(x) &= \int_c^x \exp\left(-\int_c^y \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) dy = \int_c^x \exp\left(-\int_c^y \frac{\varepsilon - bl}{l^4}\, dl\right) dy \\ &= \int_c^x \exp\left(\frac{\varepsilon}{3}\left(\frac{1}{y^3} - \frac{1}{c^3}\right) - \frac{b}{2}\left(\frac{1}{y^2} - \frac{1}{c^2}\right)\right) dy \\ &= \exp\left(\frac{b}{2c^2} - \frac{\varepsilon}{3c^3}\right) \int_c^x \exp\left(\frac{\varepsilon}{3y^3} - \frac{b}{2y^2}\right) dy. \end{aligned}$$

- The speed measure for $c, x \in (0, \infty)$ is given by

$$\begin{aligned} m_c(dx) &= \frac{2}{\sigma^2(x)\, s_c'(x)} = \frac{2}{\sigma^2(x)} \exp\left(\int_c^x \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) \\ &= \frac{2}{x^4} \exp\left(\frac{\varepsilon}{3}\left(\frac{1}{c^3} - \frac{1}{x^3}\right) - \frac{b}{2}\left(\frac{1}{c^2} - \frac{1}{x^2}\right)\right). \end{aligned}$$

- As in Definition 3.42 we set for $c, x \in (0, \infty)$

$$p_c(x) = \exp\left(\int_c^x \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) = \exp\left(\frac{\varepsilon}{3}\left(\frac{1}{c^3} - \frac{1}{x^3}\right) - \frac{b}{2}\left(\frac{1}{c^2} - \frac{1}{x^2}\right)\right).$$

  and

$$\begin{aligned} r_c(x) &= \frac{1}{\sigma^2(x)} \exp\left(\int_c^x \frac{\varepsilon b_1(l) - b_2(l)}{\sigma^2(l)}\, dl\right) \\ &= \frac{1}{x^4} \exp\left(\frac{\varepsilon}{3}\left(\frac{1}{c^3} - \frac{1}{x^3}\right) - \frac{b}{2}\left(\frac{1}{c^2} - \frac{1}{x^2}\right)\right). \end{aligned}$$

Lemma 4.41. *For any deterministic starting value* $X_0 := x > 0$ *and all* $\varepsilon > 0$ *there exists a strong solution to the SDE* (4.40) *in the sense of Definition 3.2 which is pathwise unique in the sense of Definition 3.5 and does neither attain* 0 *nor* ∞ *in finite time. Particularly Assumption (A1) is met.*

Proof. By Theorem 3.6 pathwise uniqueness holds. Since (ND) and (LI) hold in $I = (0, \infty)$ by Theorem 3.13 there is a weak solution of (4.40) in the interval I in the sense of Definition 3.12 which is unique in the sense of probability law by means of Definition 3.8. (Uniqueness in the sense of probability law is generally implied by pathwise uniqueness as stated in Theorem 3.9.) We still need to show that indeed neither 0 nor ∞ are attainable. It will then follow that any weak solution in I is actually a weak solution in the sense of Definition 3.4 and Theorem 3.7 even shows the existence of a strong solution in the sense of Definition 3.2. In the following we even show the repulsiveness of the boundary points by verifying $s_c(0) = -\infty$ and $s_c(\infty) = \infty$. For $c = \varepsilon/(3b)$ it holds

$$
\begin{aligned}
s_c(0) &= \exp\left(\frac{b}{2c^2} - \frac{\varepsilon}{3c^3}\right) \int_c^0 \exp\left(\frac{\varepsilon}{3y^3} - \frac{b}{2y^2}\right) dy \\
&= -\exp\left(\frac{b}{2c^2} - \frac{\varepsilon}{3c^3}\right) \int_0^c \exp\left(\frac{b}{2y^2}\left(\frac{2\varepsilon}{3by} - 1\right)\right) dy \\
&\le -\exp\left(\frac{b}{2c^2} - \frac{\varepsilon}{3c^3}\right) \int_0^c \exp\left(\frac{b}{2y^2}\right) dy = -\exp\left(\frac{b}{2c^2} - \frac{\varepsilon}{3c^3}\right) \int_{\frac{b}{2c^2}}^\infty \frac{be^y}{(2y)^{3/2}}\, dy = -\infty
\end{aligned}
$$

since $\frac{be^y}{(2y)^{3/2}} \xrightarrow[y\to\infty]{} \infty$ and

$$
s_c(\infty) = \exp\left(\frac{b}{2c^2} - \frac{\varepsilon}{3c^3}\right) \int_c^\infty \exp\left(\frac{\varepsilon}{3y^3} - \frac{b}{2y^2}\right) dy = \infty
$$

since $\frac{\varepsilon}{3y^3} - \frac{b}{2y^2} \xrightarrow[y\to\infty]{} 0$. This finishes the proof. □

Lemma 4.42. *It holds* $\int_0^\infty m_c(dy) < \infty$.

Proof. For $c > 0$ it holds

$$
\begin{aligned}
\int_0^\infty m_c(dy) &= \int_0^\infty \frac{2}{y^4} \exp\left(\frac{\varepsilon}{3}\left(\frac{1}{c^3} - \frac{1}{y^3}\right) - \frac{b}{2}\left(\frac{1}{c^2} - \frac{1}{y^2}\right)\right) dy \\
&= 2\exp\left(\frac{\varepsilon}{3c^3} - \frac{b}{2c^2}\right) \int_0^\infty \frac{\exp\left(\frac{b}{2y^2} - \frac{\varepsilon}{3y^3}\right)}{y^4}\, dy \\
&= 2\exp\left(\frac{\varepsilon}{3c^3} - \frac{b}{2c^2}\right) \int_0^\infty y^2 \exp\left(y^2\left(\frac{b}{2} - \frac{\varepsilon}{3}y\right)\right) dy \\
&= 2\exp\left(\frac{\varepsilon}{3c^3} - \frac{b}{2c^2}\right) \left(\int_0^{1\vee\frac{3}{\varepsilon}(\frac{b}{2}+1)} y^2 \exp\left(y^2\left(\frac{b}{2} - \frac{\varepsilon}{3}y\right)\right) dy \right. \\
&\qquad\qquad \left. + \int_{1\vee\frac{3}{\varepsilon}(\frac{b}{2}+1)}^\infty y^2 \exp\left(y^2\left(\frac{b}{2} - \frac{\varepsilon}{3}y\right)\right) dy \right).
\end{aligned}
$$

The prefactor $0 < 2\exp\left(\frac{\varepsilon}{3c^3} - \frac{b}{2c^2}\right) < \infty$ is just some positive finite constant as is

$$0 < \int_0^{1\vee\frac{3}{\varepsilon}(\frac{b}{2}+1)} y^2 \exp\left(y^2\left(\frac{b}{2} - \frac{\varepsilon}{3}y\right)\right) dy < \infty.$$

For the second integral we estimate using the gamma function

$$\int_{1\vee\frac{3}{\varepsilon}(\frac{b}{2}+1)}^{\infty} y^2 \exp\left(y^2\left(\frac{b}{2} - \frac{\varepsilon}{3}y\right)\right) dy \leq \int_{1\vee\frac{3}{\varepsilon}(\frac{b}{2}+1)}^{\infty} y^2 e^{-y}\, dy \leq \Gamma(3) = 2.$$

This finishes the proof. □

For the rest of the discussion of the example class we now fix some arbitrarily chosen level $z \in (0, \infty)$. We always assume $\varepsilon > 0$ to be sufficiently small such that $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ exist with $\beta(\varepsilon) < z$.

Remark 4.43 (Heuristics for the choice of α and β)**.** One way to guess the form of the functions α and β appearing in the cycle decomposition is the following. First it is of course natural to assume that the point, where the drift changes sign does play a specific role. Therefore, let us define $\alpha(\varepsilon) := \varepsilon/b$. In order to get an idea, of how to choose $\beta(\varepsilon)$ one can e.g. first transform the stochastic differential equation using a transformation going back at least to Feller (compare [25, Expressions (7.1), (7.8) and (7.9)]). We replace X_t by $Y_t := F(X_t)$, where $F(x) := \int_\infty^x \frac{1}{\sigma(u)}\, du = -1/x$. According to Itô's lemma the SDE then becomes using $F'(x) = 1/x^2$ and $F''(x) = -2/x^3$

$$\begin{aligned} dY_t &= F'(X_t)\, dX_t + \frac{1}{2}F''(X_t)\, d\langle X\rangle_t = \frac{1}{X_t^2}\left(\frac{1}{2}(\varepsilon - bX_t)\, dt + X_t^2 dB_t\right) + \frac{1}{2}\frac{-2}{X_t^3} X_t^4\, dt \\ &= \left(\frac{\varepsilon - bX_t}{2X_t^2} - X_t\right) dt + dB_t = \left(\frac{1}{2}Y_t(b + \varepsilon Y_t) + \frac{1}{Y_t}\right) dt + dB_t. \end{aligned}$$

Thus we end up with diffusion process with unit diffusion coefficient. For the diffusion X_t started from $\alpha(\varepsilon)$ to complete a cycle it has to get from $\alpha(\varepsilon) = \varepsilon/b$ to $\beta(\varepsilon) > \alpha\varepsilon$ and back which translates for the diffusion process Y_t into traveling from $F(\alpha(\varepsilon))$ to $F(\beta(\varepsilon))$ and back. During a downcrossing from $F(\beta(\varepsilon))$ to $F(\alpha(\varepsilon))$ one takes advantage of the fact that the drift always points in towards $F(\alpha(\varepsilon))$ and it turns out that the deterministic part is strong enough to get finite expectation for this phase of the cycle. The diffusion Y_t also has to perform an upcrossing from $F(\alpha(\varepsilon))$ to $F(\beta(\varepsilon))$ during a cycle of X_t. During such an upcrossing the drift in the equation of Y_t is of order ε near $F(\alpha(\varepsilon))$ since

$$\frac{1}{2}\left(-\frac{b}{\varepsilon}\right)\left(b + \varepsilon\left(-\frac{b}{\varepsilon}\right)\right) - \frac{\varepsilon}{b} = -\frac{\varepsilon}{b}$$

and therefore the Brownian part has to be essential to complete this part of the cycle sufficiently fast. Therefore, it seems reasonable to take $\beta(\varepsilon) = \alpha(\varepsilon) + \varepsilon^2$ as this results in a distance of heights of

$$F(\beta(\varepsilon)) - F(\alpha(\varepsilon)) = -\frac{1}{\frac{\varepsilon}{b} + \varepsilon^2} + \frac{b}{\varepsilon} = \frac{1}{\varepsilon}\left(b - \frac{b}{1 + b\varepsilon}\right) = \frac{1}{\varepsilon}\frac{b + b^2\varepsilon - b}{1 + b\varepsilon} = \frac{b^2}{1 + \varepsilon\, b} \leq b^2$$

to overcome. The exit times of Brownian motion from bounded sets have moments of all order, thus this might be a reasonable first guess. Working with $\beta(\varepsilon) = \alpha(\varepsilon) + \varepsilon$ in contrast leads to a distance

$$F(\beta(\varepsilon)) - F(\alpha(\varepsilon)) = \frac{1}{\varepsilon}\left(b - \frac{b}{1+b}\right) = \frac{b^2}{\varepsilon(1+b)}$$

which is of order ε^{-1} and therefore the expected time to complete this phase of the cycle can then be expected to diverge with $\varepsilon \to 0$.

We now show, that Theorem 4.19 applies to this situation, which means that we need to prove $(A2), (A3), (B1)$ and $(B2)$ for

$$\alpha(\varepsilon) := \varepsilon/b, \qquad \beta(\varepsilon) := \varepsilon/b + \varepsilon^2.$$

Before we start we show two preparatory lemmas.

Lemma 4.44. *For $x \geq 1/b$ it holds*

$$\frac{1}{3x^3} - \frac{b}{2x^2} \leq -\frac{b^3}{6} + \frac{b^5}{2}\left(x - \frac{1}{b}\right)^2 \tag{4.41}$$

and

$$\frac{1}{3x^3} - \frac{b}{2x^2} \geq -\frac{b^3}{6} + \frac{b^5}{2}\left(x - \frac{1}{b}\right)^2 - \frac{4b^6}{3}\left(x - \frac{1}{b}\right)^3. \tag{4.42}$$

Proof. We observe that for positive $x > 0$ the function

$$f(x) := \frac{1}{3x^3} - \frac{b}{2x^2}$$

has k-th derivative

$$f^{(k)}(x) = (-1)^k \frac{1}{3}\frac{(k+2)!}{2x^{k+3}} + (-1)^{k+1}\frac{1}{2}\frac{(k+1)!}{x^{k+2}}$$

and the series expansion at $x = 1/b$ computes to

$$\begin{aligned}\frac{1}{3x^3} - \frac{b}{2x^2} &= \sum_{k=0}^{\infty}(-1)^k(k^2-1)\frac{b^{3+k}}{6}\left(x - \frac{1}{b}\right)^k \\ &= -\frac{b^3}{6} + \frac{1}{2}b^5\left(x - \frac{1}{b}\right)^2 - \frac{4}{3}b^6\left(x - \frac{1}{b}\right)^3 + O\left(\left(x - \frac{1}{b}\right)^4\right).\end{aligned}$$

By Taylor's theorem we may rewrite

$$\frac{1}{3x^3} - \frac{b}{2x^2} = -\frac{b^3}{6} + \frac{1}{2}b^5\left(x - \frac{1}{b}\right)^2 + \int_{1/b}^{x} \frac{(x-t)^2}{2}\left(-\frac{20}{t^6} + \frac{12b}{t^5}\right) dt.$$

This provides an upper bound for $x \geq 1/b$ as by partial integration technique

$$\int_{1/b}^{x} \frac{(x-t)^2}{2}\left(-\frac{20}{t^6}+\frac{12b}{t^5}\right)dt = -\frac{(x-1/b)^2}{2}(4b^5-3b^5)+\int_{1/b}^{x}(x-t)\left(\frac{4}{t^5}-\frac{3b}{t^4}\right)dt$$
$$= -\frac{(x-1/b)^2}{2}b^5+\int_{1/b}^{x}\left(-\frac{1}{t^4}+\frac{b}{t^3}\right)dt = -\frac{(x-1/b)^2}{2}b^5+\frac{1}{3x^3}-\frac{b}{2x^2}+\frac{b^3}{6}$$
$$= -\frac{3(xb)^5-6(xb)^4+2(xb)^2+3xb-2}{6x^3} = -\frac{(x-1/b)^3b^3(2+3bx+3b^2x^2)}{6x^3} \leq 0,$$

hence

$$\frac{1}{3x^3}-\frac{b}{2x^2} \leq -\frac{b^3}{6}+\frac{1}{2}b^5\left(x-\frac{1}{b}\right)^2.$$

Analogously

$$\frac{1}{3x^3}-\frac{b}{2x^2} \geq -\frac{b^3}{6}+\frac{1}{2}b^5\left(x-\frac{1}{b}\right)^2-\frac{4}{3}b^6\left(x-\frac{1}{b}\right)^3$$

since

$$\frac{1}{3x^3}-\frac{b}{2x^2} = -\frac{b^3}{6}+\frac{1}{2}b^5\left(x-\frac{1}{b}\right)^2-\frac{4}{3}b^6\left(x-\frac{1}{b}\right)^3+\int_{1/b}^{x}\frac{(x-t)^3}{6}\left(\frac{120}{t^7}-\frac{60b}{t^6}\right)dt$$

and by the calculation carried out earlier

$$\int_{1/b}^{x}\frac{(x-t)^3}{6}\left(\frac{120}{t^7}-\frac{60b}{t^6}\right)dt = -\frac{(x-1/b)^3}{6}\cdot(-8b^6)+\int_{1/b}^{x}\frac{(x-t)^2}{2}\left(-\frac{20}{t^6}+\frac{12b}{t^5}\right)dt$$
$$= \frac{(x-1/b)^3b^3(8(xb)^3-3(bx)^2-3bx-2)}{6x^3} = \frac{(x-1/b)^4b^4(2+5bx+8b^2x^2)}{6x^3} \geq 0.$$

□

As additional preparation for the following proofs we show:

Lemma 4.45. *For $l \geq 0$ it holds*

$$\mathbb{P}_{\varepsilon/b+l\varepsilon^2}(T_z < T_{\varepsilon/b}) \sim \varepsilon^2 \exp\left(-\frac{1}{\varepsilon^2}\frac{b^3}{6}\right)\frac{\int_0^l \exp\left(\frac{b^5}{2}x^2\right)dx}{\int_0^z \exp\left(-\frac{b}{2x^2}\right)dx} \qquad \textit{as } \varepsilon \downarrow 0.$$

Proof. For the rest of the proof let us fix some arbitrarily chosen $l \geq 0$. It holds

$$\mathbb{P}_{\varepsilon/b+l\varepsilon^2}(T_z < T_{\varepsilon/b}) = \frac{\int_{\varepsilon/b}^{\varepsilon/b+l\varepsilon^2} 1/p_c(x)\,dx}{\int_{\varepsilon/b}^{z} 1/p_c(x)\,dx},$$

where

$$\frac{1}{p_c(x)} = \exp\left(\varepsilon\int_x^c \frac{1}{t^4}dt\right)\exp\left(b\int_c^x \frac{1}{t^3}dt\right)$$

$$= \exp\left(\frac{\varepsilon}{3}\left(\frac{1}{x^3} - \frac{1}{c^3}\right)\right) \exp\left(\frac{b}{2}\left(\frac{1}{c^2} - \frac{1}{x^2}\right)\right).$$

Plugging in and reducing the fraction yields

$$\begin{aligned}\frac{\int_{\varepsilon/b}^{\varepsilon/b+l\varepsilon^2} 1/p_c(x)\,dx}{\int_{\varepsilon/b}^{z} 1/p_c(x)\,dx} &= \frac{\int_{\varepsilon/b}^{\varepsilon/b+l\varepsilon^2} \exp\left(\frac{\varepsilon}{3x^3} - \frac{b}{2x^2}\right)\,dx}{\int_{\varepsilon/b}^{z} \exp\left(\frac{\varepsilon}{3x^3} - \frac{b}{2x^2}\right)\,dx} \\ &= \varepsilon \cdot \frac{\int_{1/b}^{1/b+l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3x^3} - \frac{b}{2x^2}\right)\right)\,dx}{\int_{\varepsilon/b}^{z} \exp\left(\frac{\varepsilon}{3x^3} - \frac{b}{2x^2}\right)\,dx}. \end{aligned} \tag{4.43}$$

For the numerator in the last fraction (4.43) we use the estimates (4.41) and (4.42):

$$\begin{aligned}&\int_{1/b}^{1/b+l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3x^3} - \frac{b}{2x^2}\right)\right)\,dx \le \int_{1/b}^{1/b+l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} + \frac{1}{2}b^5\left(x - \frac{1}{b}\right)^2\right)\right)\,dx \\ &= \exp\left(-\frac{1}{\varepsilon^2}\frac{b^3}{6}\right)\int_0^{l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\frac{1}{2}b^5x^2\right)\,dx = \exp\left(-\frac{1}{\varepsilon^2}\frac{b^3}{6}\right)\varepsilon\int_0^{l} \exp\left(\frac{b^5}{2}x^2\right)\,dx\end{aligned} \tag{4.44}$$

and for the opposite direction

$$\begin{aligned}&\int_{1/b}^{1/b+l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3x^3} - \frac{b}{2x^2}\right)\right)\,dx \\ &\ge \int_{1/b}^{1/b+l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} + \frac{1}{2}b^5\left(x - \frac{1}{b}\right)^2 - \frac{4}{3}b^6\left(x - \frac{1}{b}\right)^3\right)\right)\,dx \\ &= \int_0^{l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} + \frac{1}{2}b^5x^2 - \frac{4}{3}b^6x^3\right)\right)\,dx \\ &= \exp\left(-\frac{1}{\varepsilon^2}\frac{b^3}{6}\right)\varepsilon\int_0^{l} \exp\left(\frac{1}{2}b^5x^2 - \varepsilon\frac{4}{3}b^6x^3\right)\,dx \sim \exp\left(-\frac{1}{\varepsilon^2}\frac{b^3}{6}\right)\varepsilon\int_0^{l} \exp\left(\frac{1}{2}b^5x^2\right)\,dx.\end{aligned} \tag{4.45}$$

The asymptotic estimates (4.44) and (4.45) together imply

$$\int_{1/b}^{1/b+l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3x^3} - \frac{b}{2x^2}\right)\right)\,dx \sim \exp\left(-\frac{1}{\varepsilon^2}\frac{b^3}{6}\right)\varepsilon\int_0^{l} \exp\left(\frac{1}{2}b^5x^2\right)\,dx. \tag{4.46}$$

For the denominator of the aforementioned fraction (4.43) we may write

$$\int_{\varepsilon/b}^{z} \exp\left(\frac{\varepsilon}{3x^3} - \frac{b}{2x^2}\right)\,dx = \int_0^{z} \mathbb{1}_{\{x>\varepsilon/b\}} \exp\left(\frac{\varepsilon}{3x^3} - \frac{b}{2x^2}\right)\,dx.$$

Due to

$$x \mapsto \frac{\varepsilon}{3x^3} - \frac{b}{2x^2} = \frac{1}{6x^3}\cdot(2\varepsilon - 3bx)$$

being non-positive for $x \geq 2\varepsilon/(3b)$ which particularly covers $\varepsilon/b < x < z$, the integrand is bounded in-between 0 and 1 allowing to apply Lebesgue's theorem which results in

$$\lim_{\varepsilon \to 0} \int_0^z \mathbb{1}_{\{x > \varepsilon/b\}} \exp\left(\frac{\varepsilon}{3x^3} - \frac{b}{2x^2}\right) dx = \int_0^z \exp\left(-\frac{b}{2x^2}\right) dx.$$

In combination with the already analyzed asymptotic (4.46) of the numerator in (4.43) this yields

$$\frac{\int_{1/b}^{1/b+l\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3x^3} - \frac{b}{2x^2}\right)\right) dx}{\int_{\varepsilon/b}^{z} \exp\left(\frac{\varepsilon}{3x^3} - \frac{b}{2x^2}\right) dx} \sim \frac{\exp\left(-\frac{1}{\varepsilon^2}\frac{b^3}{6}\right) \varepsilon \int_0^l \exp\left(\frac{b^5}{2}x^2\right) dx}{\int_0^z \exp\left(-\frac{b}{2x^2}\right) dx}$$

as $\varepsilon \downarrow 0$, finishing the proof. □

Proposition 4.46 (implying (A2)). *It holds*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}_{\varepsilon/b}[\widetilde{\sigma}_1] = \lim_{\varepsilon \downarrow 0} \mathbb{E}_{\varepsilon/b}[\sigma_1] = 4b^4 \int_0^\infty \int_0^1 \exp\left(\frac{b^5}{2}\left(w^2 - y^2\right)\right) dw\, dy.$$

Proof. Using again the approach with the corresponding Green function we infer analogously to Lemma 4.31

$$\mathbb{E}_{\varepsilon/b}[T_{\varepsilon/b+\varepsilon^2}] = 2\varepsilon^2 \left[\int_0^{1/b} \int_{1/b}^{1/b+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} dw\, dy + \int_{1/b}^{1/b+\varepsilon} \int_y^{1/b+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} dw\, dy\right] \tag{4.47}$$

where we can explicitly write

$$\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} = \frac{1}{(y\varepsilon)^4} \exp\left(\frac{1}{\varepsilon^2}\left(\left(\frac{1}{3w^3} - \frac{b}{2w^2}\right) - \left(\frac{1}{3y^3} - \frac{b}{2y^2}\right)\right)\right). \tag{4.48}$$

Observe that the right hand side of expression (4.48) factorizes in a function of w and a function of y. To calculate the limit $\varepsilon \to 0$ of the first term in (4.47), we consider the asymptotic behavior of both factors given by the integral with respect to y and w, respectively. We have

$$\begin{aligned}
&2\varepsilon^2 \int_0^{1/b} \int_{1/b}^{1/b+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} dw\, dy \\
&= 2\varepsilon^2 \int_0^{1/b} \int_{1/b}^{1/b+\varepsilon} \frac{1}{(y\varepsilon)^4} \exp\left(\frac{1}{\varepsilon^2}\left(\left(\frac{1}{3w^3} - \frac{b}{2w^2}\right) - \left(\frac{1}{3y^3} - \frac{b}{2y^2}\right)\right)\right) dw\, dy \\
&= \frac{2}{\varepsilon^2} \int_0^{1/b} \frac{1}{y^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{1}{3y^3} + \frac{b}{2y^2}\right)\right) dy \cdot \int_{1/b}^{1/b+\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3} - \frac{b}{2w^2}\right)\right) dw.
\end{aligned} \tag{4.49}$$

Upon using the substitution $u := \frac{1/y - b}{\varepsilon}$ implying $du = -\frac{dy}{\varepsilon y^2}$ it follows

$$\int_0^{\frac{1}{b}} \frac{1}{y^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{1}{3y^3} + \frac{b}{2y^2}\right)\right) dy$$

$$= \exp\left(\frac{b^3}{6\varepsilon^2}\right)\varepsilon\int_0^\infty (u\varepsilon+b)^2\exp\left(-\frac{b}{2}u^2-\frac{\varepsilon}{3}u^3\right)\,du.$$

We may apply Lebesgue's theorem since assuming $\varepsilon < 1$ the majorant

$$(u+b)^2\exp\left(-\frac{b}{2}u^2\right) \le 2(u^2+b^2)\exp\left(-\frac{b}{2}u^2\right)$$

is integrable: With substitution $t := \frac{b}{2}u^2$ we observe

$$\int_0^\infty 2(u^2+b^2)\exp\left(-\frac{b}{2}u^2\right)\,du = \int_0^\infty \frac{2\left(\frac{2t}{b}+b^2\right)}{\sqrt{2bt}}\exp(-t)\,dt$$
$$= \left(\frac{2}{b}\right)^{3/2}\Gamma(3/2)+\sqrt{2b^3}\,\Gamma(1/2) < \infty.$$

We therefore find

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_0^\infty (u\varepsilon+b)^2\exp\left(-\frac{b}{2}u^2-\frac{\varepsilon}{3}u^3\right)\,du = \int_0^\infty b^2\exp\left(-\frac{b}{2}u^2\right)\,du\\ &= \int_0^\infty b^4\exp\left(-\frac{b^5}{2}u^2\right)\,du.\end{aligned}\tag{4.50}$$

Making use of (4.46) with $l := 1$ inserted shows

$$\int_{1/b}^{1/b+\varepsilon}\exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3}-\frac{b}{2w^2}\right)\right)\,dw \sim \exp\left(-\frac{b^3}{6\varepsilon^2}\right)\varepsilon\int_0^1\exp\left(\frac{b^5w^2}{2}\right)\,dw.\tag{4.51}$$

In view of the factorization in (4.49) the assertions (4.50) and (4.51) imply

$$2\varepsilon^2\int_0^{1/b}\int_{1/b}^{1/b+\varepsilon}\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\,dw\,dy \underset{\varepsilon\downarrow 0}{\longrightarrow} 2b^4\int_0^\infty\int_0^1\exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right)\,dw\,dy.\tag{4.52}$$

To handle the second summand in expression (4.47), the integral

$$\begin{aligned}&2\varepsilon^2\int_{1/b}^{1/b+\varepsilon}\int_y^{1/b+\varepsilon}\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\,dw\,dy\\ &= \frac{2}{\varepsilon^2}\int_{1/b}^{1/b+\varepsilon}\frac{1}{y^4}\exp\left(\frac{1}{\varepsilon^2}\left(-\frac{1}{3y^3}+\frac{b}{2y^2}\right)\right)\int_y^{1/b+\varepsilon}\exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3}-\frac{b}{2w^2}\right)\right)\,dw\,dy\end{aligned}$$

we proceed in a similar fashion. Note, that taking account for the first exponential term, the corresponding Taylor series has just opposite sign. We derive

$$\begin{aligned}&\frac{2}{\varepsilon^2}\int_{1/b}^{1/b+\varepsilon}\frac{1}{y^4}\exp\left(\frac{1}{\varepsilon^2}\left(-\frac{1}{3y^3}+\frac{b}{2y^2}\right)\right)\int_y^{1/b+\varepsilon}\exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3}-\frac{b}{2w^2}\right)\right)\,dw\,dy\\ &\le \frac{2}{\varepsilon^2}\int_0^{\varepsilon}\frac{1}{(y+1/b)^4}\exp\left(\frac{1}{\varepsilon^2}\left(\frac{b^3}{6}-\frac{1}{2}b^5y^2+\frac{4}{3}b^6y^3\right)\right)\int_y^{\varepsilon}\exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6}+\frac{1}{2}b^5w^2\right)\right)\,dw\,dy\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^1 \frac{1}{(y\varepsilon+1/b)^4}\exp\left(-\frac{1}{2}b^5y^2+\varepsilon\frac{4}{3}b^6y^3\right)\int_y^1 \exp\left(\frac{1}{2}b^5w^2\right)\,dw\,dy\\
&\xrightarrow[\varepsilon\to 0]{} 2b^4\int_0^1\int_y^1 \exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right)\,dw\,dy
\end{aligned}
$$

and accordingly

$$
\begin{aligned}
&\frac{2}{\varepsilon^2}\int_{1/b}^{1/b+\varepsilon}\frac{1}{y^4}\exp\left(\frac{1}{\varepsilon^2}\left(-\frac{1}{3y^3}+\frac{b}{2y^2}\right)\right)\int_y^{1/b+\varepsilon}\exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3}-\frac{b}{2w^2}\right)\right)\,dw\,dy\\
&\geq \frac{2}{\varepsilon^2}\int_0^{\varepsilon}\frac{1}{(y+1/b)^4}\exp\left(\frac{1}{\varepsilon^2}\left(\frac{b^3}{6}-\frac{1}{2}b^5y^2\right)\right)\int_y^{\varepsilon}\exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6}+\frac{1}{2}b^5w^2-\frac{4}{3}b^6w^3\right)\right)\,dw\,dy\\
&= 2\int_0^1\frac{1}{(y\varepsilon+1/b)^4}\exp\left(-\frac{1}{2}b^5y^2\right)\int_y^1\exp\left(\frac{1}{2}b^5w^2-\varepsilon\frac{4}{3}b^6w^3\right)\,dw\,dy\\
&\xrightarrow[\varepsilon\to 0]{} 2b^4\int_0^1\int_y^1\exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right)\,dw\,dy.
\end{aligned}
$$

In summary, combined with the already deduced limit (4.52) of the first summand in (4.47) we obtain

$$
\begin{aligned}
\mathbb{E}_{\varepsilon/b}[T_{\varepsilon/b+\varepsilon^2}] \xrightarrow[\varepsilon\downarrow 0]{} &2b^4\int_0^{\infty}\int_0^1\exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right)\,dw\,dy\\
&+2b^4\int_0^1\int_y^1\exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right)\,dw\,dy<\infty.
\end{aligned}
\tag{4.53}
$$

To infer the expected cycle duration of the second phase, where the process starts from $\beta(\varepsilon)=\varepsilon/b+\varepsilon^2$ and is conditioned to hit $\alpha(\varepsilon)=\varepsilon/b$ prior to some arbitrary level $z>\beta(\varepsilon)$, we will again use a h-transform in the sense of Doob in order find the dynamics of the conditioned process. We find

$$
\mathbb{E}_{\frac{\varepsilon}{b}+\varepsilon^2}[T_{\varepsilon/b}\mid T_{\varepsilon/b}<T_z]=2\varepsilon^2\left[\int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon}\int_{\frac{1}{b}}^{y}\frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\,dw\,dy+\int_{\frac{1}{b}+\varepsilon}^{\frac{z}{\varepsilon}}\int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon}\frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\,dw\,dy\right],
\tag{4.54}
$$

where the integrand is given by

$$
\frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}=\left(\frac{h(y\varepsilon)}{h(w\varepsilon)}\right)^2\cdot\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}.
$$

We recall that the harmonic function under consideration is $h(s):=\mathbb{P}_s(T_{\varepsilon/b}<T_z)$. Let us start with the first summand. Because on the integration domain $w\leq y$ holds, the estimate

$$
\frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\leq\frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}
$$

allows us use a strategy very similar to the situation of the first cycle phase. In particular, we have

$$
\begin{aligned}
&2\varepsilon^2 \int_{1/b}^{1/b+\varepsilon} \int_{1/b}^{y} \frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\, dw\, dy \leq 2\varepsilon^2 \int_{1/b}^{1/b+\varepsilon} \int_{1/b}^{y} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\, dw\, dy \\
&= \frac{2}{\varepsilon^2} \int_{1/b}^{1/b+\varepsilon} \frac{1}{y^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{1}{3y^3}+\frac{b}{2y^2}\right)\right) \int_{1/b}^{y} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3}-\frac{b}{2w^2}\right)\right) dw\, dy \\
&\leq \frac{2}{\varepsilon^2} \int_{0}^{\varepsilon} \frac{1}{(y+1/b)^4} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{b^3}{6}-\frac{1}{2}b^5y^2+\frac{4}{3}b^6y^3\right)\right) \int_{0}^{y} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6}+\frac{1}{2}b^5w^2\right)\right) dw\, dy \\
&= 2\int_0^1 \frac{1}{(y\varepsilon+1/b)^4} \exp\left(-\frac{1}{2}b^5y^2+\varepsilon\frac{4}{3}b^6y^3\right) \int_0^y \exp\left(\frac{1}{2}b^5w^2\right) dw\, dy \\
&\xrightarrow[\varepsilon\to 0]{} 2b^4 \int_0^1 \int_0^y \exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right) dw\, dy.
\end{aligned}
$$

In order to derive a matching result for the opposite direction of approximation we use our standard estimates to find

$$
\begin{aligned}
&2\varepsilon^2 \int_{1/b}^{1/b+\varepsilon} \int_{1/b}^{y} \frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\, dw\, dy \\
&\geq 2\int_0^1 \frac{1}{(y\varepsilon+1/b)^4} \exp\left(-\frac{1}{2}b^5y^2\right) \int_0^y \left(\frac{h((y\varepsilon+1/b)\varepsilon)}{h((w\varepsilon+1/b)\varepsilon)}\right)^2 \exp\left(\frac{1}{2}b^5w^2-\varepsilon\frac{4}{3}b^6y^3\right) dw\, dy.
\end{aligned}
$$

By the bounded convergence theorem we can interchange the limit and the integrals and using Lemma 4.45 to observe $h((l\varepsilon+1/b)\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{} 1$, $l \in [0,1]$ we conclude

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0} 2\varepsilon^2 \int_{1/b}^{1/b+\varepsilon} \int_{1/b}^{y} \frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\, dw\, dy &= \lim_{\varepsilon\downarrow 0} 2\varepsilon^2 \int_{1/b}^{1/b+\varepsilon} \int_{1/b}^{y} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\, dw\, dy \\
&= 2b^4 \int_0^1 \int_0^y \exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right) dw\, dy.
\end{aligned} \tag{4.55}
$$

We now consider the second term in equation (4.54). We rewrite this term as

$$
\begin{aligned}
&2\varepsilon^2 \int_{1/b+\varepsilon}^{z/\varepsilon} \int_{1/b}^{1/b+\varepsilon} \frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\, dw\, dy = 2\varepsilon^4 \int_1^{z/\varepsilon^2-1/(b\varepsilon)} \int_0^1 \frac{r_c^h((y\varepsilon+1/b)\varepsilon)}{p_c^h((w\varepsilon+1/b)\varepsilon)}\, dw\, dy \\
&= 2\int_1^\infty \int_0^1 \mathbb{1}_{\{y<z/\varepsilon^2-1/(b\varepsilon)\}} \frac{h((y\varepsilon+1/b)\varepsilon)^2}{h((w\varepsilon+1/b)\varepsilon)^2} \frac{1}{(y\varepsilon+1/b)^4} \\
&\quad \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3(w\varepsilon+1/b)^3}-\frac{b}{2(w\varepsilon+1/b)^2}-\frac{1}{3(y\varepsilon+1/b)^3}+\frac{b}{2(y\varepsilon+1/b)^2}\right)\right) dw\, dy.
\end{aligned} \tag{4.56}
$$

Elementary algebra with results in

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\left(\frac{1}{3(w\varepsilon+1/b)^3}-\frac{b}{2(w\varepsilon+1/b)^2}-\frac{1}{3(y\varepsilon+1/b)^3}+\frac{b}{2(y\varepsilon+1/b)^2}\right)\\
&=\frac{1}{\varepsilon^2}\left(\frac{2(y\varepsilon+1/b)^3-3b(y\varepsilon+1/b)^3(w\varepsilon+1/b)-2(w\varepsilon+1/b)^3+3b(w\varepsilon+1/b)^3(y\varepsilon+1/b)}{6(w\varepsilon+1/b)^3(y\varepsilon+1/b)^3}\right)\\
&=\frac{b^3}{\varepsilon^2}\left(\frac{2(by\varepsilon+1)^3-3(by\varepsilon+1)^3(bw\varepsilon+1)-2(bw\varepsilon+1)^3+3(bw\varepsilon+1)^3(by\varepsilon+1)}{6(bw\varepsilon+1)^3(by\varepsilon+1)^3}\right)\\
&=\frac{b^3}{\varepsilon^2}\left(\frac{(by\varepsilon+1)^3(2-3bw\varepsilon-3)+(bw\varepsilon+1)^3(3by\varepsilon+3-2)}{6(bw\varepsilon+1)^3(by\varepsilon+1)^3}\right)\\
&=\frac{b^3}{\varepsilon^2}\left(\frac{(bw\varepsilon+1)^3(3by\varepsilon+1)-(by\varepsilon+1)^3(3bw\varepsilon+1)}{6(bw\varepsilon+1)^3(by\varepsilon+1)^3}\right).
\end{aligned} \tag{4.57}$$

Using the abbreviating notations $s \coloneqq bw\varepsilon$ and $t \coloneqq by\varepsilon$ the numerator in (4.57) may be written as

$$\begin{aligned}
&(bw\varepsilon+1)^3(3by\varepsilon+1)-(by\varepsilon+1)^3(3bw\varepsilon+1)\\
&=(s+1)^3(3t+1)-(t+1)^3(3s+1)\\
&=3s^3t+9s^2t+9st+3t+s^3+3s^2+3s+1-3t^3s-9t^2s-9st-3s-t^3-3t^2-3t-1\\
&=3s^3t+9s^2t+s^3+3s^2-3t^3s-9t^2s-t^3-3t^2\\
&=3st(s^2-t^2+3s-3t)+s^3-t^3+3(s^2-t^2).
\end{aligned}$$

If follows turning back to expression (4.57)

$$\begin{aligned}
&\frac{b^3}{\varepsilon^2}\left(\frac{(by\varepsilon+1)^3(2-3bw\varepsilon-3)+(bw\varepsilon+1)^3(3by\varepsilon+3-2)}{6(bw\varepsilon+1)^3(by\varepsilon+1)^3}\right)\\
&=\frac{b^3}{\varepsilon^2}\left(\frac{3st(s^2-t^2+3s-3t)+s^3-t^3+3(s^2-t^2)}{6(s+1)^3(t+1)^3}\right)\\
&=\frac{b^5(3b^2w^3\varepsilon^2y-3b^2w\varepsilon^2y^3+bw^3\varepsilon+9bw^2\varepsilon y-9bw\varepsilon y^2-b\varepsilon y^3+3w^2-3y^2)}{6(bw\varepsilon+1)^3(b\varepsilon y+1)^3}.
\end{aligned} \tag{4.58}$$

We observe that on the domain of integration in (4.56) we have $0 < w \leq y$ and therefore $0 < s \leq t$. It follows $3st(s^2-t^2+3s-3t) \leq 0$, $s^3-t^3 \leq 0$ and $(s+1)^3(t+1)^3 \geq 1$ implying for the integrand in (4.56)

$$\begin{aligned}
&\mathbb{1}_{\{y<z/\varepsilon^2-1/(b\varepsilon)\}}\frac{h((y\varepsilon+1/b)\varepsilon)^2}{h((w\varepsilon+1/b)\varepsilon)^2}\frac{1}{(y\varepsilon+1/b)^4}\times\\
&\quad\times\exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3(w\varepsilon+1/b)^3}-\frac{b}{2(w\varepsilon+1/b)^2}-\frac{1}{3(y\varepsilon+1/b)^3}+\frac{b}{2(y\varepsilon+1/b)^2}\right)\right)\\
&\leq\frac{1}{(1/b)^4}\exp\left(\frac{b^3}{\varepsilon^2}\left(\frac{3st(s^2-t^2+3s-3t)+s^3-t^3+3(s^2-t^2)}{6(s+1)^3(t+1)^3}\right)\right)
\end{aligned}$$

$$\le b^4 \exp\left(\frac{b^3}{\varepsilon^2}\left(\frac{3(s^2-t^2)}{6(s+1)^3(t+1)^3}\right)\right) \le b^4 \exp\left(\frac{b^3}{2\varepsilon^2}(s^2-t^2)\right) = b^4 \exp\left(\frac{b^5}{2}(w^2-y^2)\right).$$

Since this is an integrable majorant we may apply Lebesgue's dominated convergence theorem on the integral (4.56). Consequently, using the representation (4.58)

$$\lim_{\varepsilon\downarrow 0} 2\varepsilon^2 \int_{1/b+\varepsilon}^{z/\varepsilon}\int_{1/b}^{1/b+\varepsilon} \frac{r_c^h(y\varepsilon)}{p_c^h(w\varepsilon)}\,dw\,dy = \lim_{\varepsilon\downarrow 0} 2\varepsilon^2 \int_{1/b+\varepsilon}^{z/\varepsilon}\int_{1/b}^{1/b+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\,dw\,dy$$

$$= 2\int_1^\infty \int_0^1 \frac{1}{(1/b)^4}$$

$$\lim_{\varepsilon\downarrow 0}\exp\left(\frac{b^5(3b^2w^3\varepsilon^2 y - 3b^2w\varepsilon^2y^3 + bw^3\varepsilon + 9bw^2\varepsilon y - 9bw\varepsilon y^2 - b\varepsilon y^3 + 3w^2 - 3y^2)}{6(bw\varepsilon+1)^3(b\varepsilon y+1)^3}\right) dw\,dy$$

$$= 2b^4 \int_1^\infty \int_0^1 \exp\left(\frac{b^5}{2}\left(w^2-y^2\right)\right) dw\,dy.$$

This gives together with (4.55) the required limit for the cycle phase and adding (4.53) therefore finishes the proof. □

Proposition 4.47 (A3)**.** *It holds*

$$\limsup_{\varepsilon\downarrow 0} \mathbb{E}_{\varepsilon/b}[(\widetilde{\sigma}_1)^2] < \infty.$$

Proof. Analogously to the proof of Proposition 4.36 we use Kac's moment formula and start with showing $\limsup_{\varepsilon\downarrow 0}\mathbb{E}_{\varepsilon/b}[(T_{\varepsilon/b+\varepsilon^2})^2] < \infty$.

On the second double integral in

$$\mathbb{E}_{\varepsilon/b}[T^2_{\varepsilon/b+\varepsilon^2}] = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon$$
$$= 4\varepsilon^2\left[\int_0^{\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\mathbb{E}_{y\varepsilon}[T_{\frac{\varepsilon}{b}+\varepsilon^2}]\,dw\,dy + \int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon}\int_y^{\frac{1}{b}+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\mathbb{E}_{y\varepsilon}[T_{\frac{\varepsilon}{b}+\varepsilon^2}]\,dw\,dy\right]$$

we may estimate $\mathbb{E}_{y\varepsilon}[T_{\frac{\varepsilon}{b}+\varepsilon^2}] \le \mathbb{E}_{\frac{\varepsilon}{b}}[T_{\frac{\varepsilon}{b}+\varepsilon^2}]$ which is justified by an enlargement of the integration domain as in (4.27) and therefore finiteness follows by the convergence of the first moment shown in Proposition 4.46 as by using (4.47)

$$\limsup_{\varepsilon\downarrow 0} \mathrm{II}_\varepsilon \le \limsup_{\varepsilon\downarrow 0} 2\left(\mathbb{E}_{\varepsilon/b}[T_{\varepsilon/b+\varepsilon^2}]\right)^2 < \infty. \tag{4.59}$$

For the first integral I_ε we need to show

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^4 \int_0^{\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon}\int_0^y\int_y^{\frac{1}{b}+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\frac{r_c(\widetilde{y}\varepsilon)}{p_c(\widetilde{w}\varepsilon)}\,d\widetilde{w}\,d\widetilde{y}\,dw\,dy < \infty \tag{4.60}$$

and

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^4 \int_0^{\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon}\int_y^{\frac{1}{b}+\varepsilon}\int_{\widetilde{y}}^{\frac{1}{b}+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)}\frac{r_c(\widetilde{y}\varepsilon)}{p_c(\widetilde{w}\varepsilon)}\,d\widetilde{w}\,d\widetilde{y}\,dw\,dy < \infty. \tag{4.61}$$

Using (4.48) and (4.46) with $l := 1$ and writing $f(x) := \frac{1}{3x^3} - \frac{b}{2x^2}$ we conclude

$$
\begin{aligned}
&\varepsilon^4 \int_0^{\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon} \int_0^y \int_y^{\frac{1}{b}+\varepsilon} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} \frac{r_c(\widetilde{y}\varepsilon)}{p_c(\widetilde{y}\varepsilon)} \, d\widetilde{w}\, d\widetilde{y}\, dw\, dy \\
&= \varepsilon^4 \int_0^{\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon} \int_0^y \int_y^{\frac{1}{b}+\varepsilon} \\
&\frac{1}{y^4 \widetilde{y}^4 \varepsilon^8} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3} - \frac{b}{2w^2} - \frac{1}{3y^3} + \frac{b}{2y^2} + \frac{1}{3\widetilde{w}^3} - \frac{b}{2\widetilde{w}^2} - \frac{1}{3\widetilde{y}^3} + \frac{b}{2\widetilde{y}^2}\right)\right) d\widetilde{w}\, d\widetilde{y}\, dw\, dy \\
&= \int_{1/b}^{1/b+\varepsilon} \exp\left(\frac{1}{\varepsilon^2}\left(\frac{1}{3w^3} - \frac{b}{2w^2}\right)\right) \cdot \varepsilon^4 \int_0^{\frac{1}{b}} \int_0^y \int_y^{\frac{1}{b}+\varepsilon} \\
&\frac{1}{y^4 \widetilde{y}^4 \varepsilon^8} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{1}{3y^3} + \frac{b}{2y^2} + \frac{1}{3\widetilde{w}^3} - \frac{b}{2\widetilde{w}^2} - \frac{1}{3\widetilde{y}^3} + \frac{b}{2\widetilde{y}^2}\right)\right) d\widetilde{y}\, dw\, dy \\
&\sim \int_0^1 \exp\left(\frac{b^5}{2} w^2\right) dw \times \\
&\times \frac{1}{\varepsilon^3} \int_0^{\frac{1}{b}} \int_0^y \int_y^{\frac{1}{b}+\varepsilon} \frac{1}{(y\widetilde{y})^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f(y) + f(\widetilde{w}) - f(\widetilde{y})\right)\right) d\widetilde{w}\, d\widetilde{y}\, dy.
\end{aligned}
$$

Substituting every variable to the reciprocal, translating by b and enlarging the integration domain accordingly to the fact $-\frac{b^2\varepsilon}{1+b\varepsilon} \geq -b^2\varepsilon$ implies

$$
\begin{aligned}
&\frac{1}{\varepsilon^3} \int_0^{\frac{1}{b}} \int_0^y \int_y^{\frac{1}{b}+\varepsilon} \frac{1}{(y\widetilde{y})^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f(y) + f(\widetilde{w}) - f(\widetilde{y})\right)\right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&= \frac{1}{\varepsilon^3} \int_b^\infty \int_y^\infty \int_{\frac{b}{1+b\varepsilon}}^y \left(\frac{y\widetilde{y}}{\widetilde{w}}\right)^2 \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f\left(\frac{1}{y}\right) + f\left(\frac{1}{\widetilde{w}}\right) - f\left(\frac{1}{\widetilde{y}}\right)\right)\right) d\widetilde{r}\, d\widetilde{y}\, dy \\
&= \frac{1}{\varepsilon^3} \int_0^\infty \int_y^\infty \int_{-\frac{b^2\varepsilon}{1+b\varepsilon}}^y \left(\frac{(y+b)(\widetilde{y}+b)}{\widetilde{w}+b}\right)^2 \\
&\qquad \times \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f\left(\frac{1}{y+b}\right) + f\left(\frac{1}{\widetilde{w}+b}\right) - f\left(\frac{1}{\widetilde{y}+b}\right)\right)\right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&\leq \frac{1}{\varepsilon^3} \int_0^\infty \int_y^\infty \int_{-b^2\varepsilon}^y \left(\frac{(y+b)(\widetilde{y}+b)}{\widetilde{w}+b}\right)^2 \\
&\qquad \times \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f\left(\frac{1}{y+b}\right) + f\left(\frac{1}{\widetilde{w}+b}\right) - f\left(\frac{1}{\widetilde{y}+b}\right)\right)\right) d\widetilde{w}\, d\widetilde{y}\, dy.
\end{aligned} \tag{4.62}
$$

By the identity

$$
\begin{aligned}
&f\left(\frac{1}{y+b}\right) = \frac{1}{3}(y+b)^3 - \frac{b}{2}(y+b)^2 = \frac{1}{3}y^3 + y^2 b + yb^2 + \frac{1}{3}b^3 - \frac{b}{2}y^2 - yb^2 - \frac{b^3}{2} \\
&= \frac{1}{3}y^3 + \frac{b}{2}y^2 - \frac{1}{6}b^3 = f\left(-\frac{1}{y}\right) - \frac{1}{6}b^3
\end{aligned}
$$

it follows

$$
\begin{aligned}
&-\frac{b^3}{6} - f\left(\frac{1}{y+b}\right) + f\left(\frac{1}{\widetilde{w}+b}\right) - f\left(\frac{1}{\widetilde{y}+b}\right) = f\left(-\frac{1}{y}\right) - f\left(-\frac{1}{\widetilde{w}}\right) + f\left(-\frac{1}{\widetilde{y}}\right) \\
&= \frac{1}{3}\left(\widetilde{w}^3 - y^3 - \widetilde{y}^3\right) + \frac{b}{2}\left(\widetilde{w}^2 - y^2 - \widetilde{y}^2\right).
\end{aligned}
$$

For $-b^2\varepsilon \leq \widetilde{w} \leq y$ and $y, \widetilde{y} \geq 0$ it holds

$$
\widetilde{w}^3 - y^3 \leq 0, \qquad -\widetilde{y}^3 \leq 0 \quad \text{and} \quad \widetilde{w}^2 - y^2 \leq b^4\varepsilon^2.
$$

Consequently,

$$
-\frac{b^3}{6} - f\left(\frac{1}{y+b}\right) + f\left(\frac{1}{\widetilde{w}+b}\right) - f\left(\frac{1}{\widetilde{y}+b}\right) \leq -\frac{b}{2}\widetilde{y}^2 + \frac{b^5}{2}\varepsilon^2.
$$

We continue our estimation of (4.62) with extending the integration domain as

$$
\begin{aligned}
\{(y,\widetilde{w}) : 0 < y < \infty, -b^2 < \widetilde{w} \leq y\} &\subseteq \{(y,\widetilde{w}) : -b^2 < y < \infty, -b^2 < \widetilde{w} \leq y\} \\
&= \{(y,\widetilde{w}) : -b^2 < \widetilde{w} \leq y < \infty\}
\end{aligned}
$$

and using Fubini's and Lebesgue's theorem to deduce

$$
\begin{aligned}
&\frac{1}{\varepsilon^3}\int_0^\infty \int_y^\infty \int_{-b^2\varepsilon}^y \left(\frac{(y+b)(\widetilde{y}+b)}{\widetilde{w}+b}\right)^2 \\
&\qquad \times \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f\left(\frac{1}{y+b}\right) + f\left(\frac{1}{\widetilde{w}+b}\right) - f\left(\frac{1}{\widetilde{y}+b}\right)\right)\right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&\leq \frac{1}{\varepsilon^3}\int_0^\infty \int_y^\infty \int_{-b^2\varepsilon}^y \left(\frac{(y+b)(\widetilde{y}+b)}{\widetilde{w}+b}\right)^2 \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b}{2}\widetilde{y}^2 + \frac{b^5}{2}\varepsilon^2\right)\right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&= \exp\left(\frac{b^5}{2}\right)\int_0^\infty \int_y^\infty \int_{-b^2}^y \left(\frac{(y\varepsilon+b)(\widetilde{y}\varepsilon+b)}{\widetilde{w}\varepsilon+b}\right)^2 \exp\left(-\frac{b}{2}\widetilde{y}^2\right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&= \exp\left(\frac{b^5}{2}\right)\int_0^\infty \int_0^\infty \int_{-b^2}^y \left(\frac{(y\varepsilon+b)(\widetilde{y}\varepsilon+y\varepsilon+b)}{\widetilde{w}\varepsilon+b}\right)^2 \exp\left(-\frac{b}{2}\left(\widetilde{y}+y\right)^2\right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&\leq \exp\left(\frac{b^5}{2}\right)\int_{-b^2}^\infty \int_0^\infty \int_{\widetilde{w}}^\infty \left(\frac{(y\varepsilon+b)(\widetilde{y}\varepsilon+y\varepsilon+b)}{\widetilde{w}\varepsilon+b}\right)^2 \exp\left(-\frac{b}{2}\left(\widetilde{y}+y\right)^2\right) dy\, d\widetilde{y}\, d\widetilde{w} \\
&= \exp\left(\frac{b^5}{2}\right)\int_{-b^2}^\infty \int_0^\infty \int_0^\infty \left(\frac{(y\varepsilon+\widetilde{w}\varepsilon+b)(\widetilde{y}\varepsilon+y\varepsilon+\widetilde{w}\varepsilon+b)}{\widetilde{w}\varepsilon+b}\right)^2 \exp\left(-\frac{b}{2}\left(\widetilde{y}+y+\widetilde{w}\right)^2\right) \\
&\qquad\qquad dy\, d\widetilde{y}\, d\widetilde{w} \\
&\xrightarrow{\varepsilon\to 0} \exp\left(\frac{b^5}{2}\right) b^2 \int_{-b^2}^\infty \int_0^\infty \int_0^\infty \exp\left(-\frac{b}{2}\left(\widetilde{y}+y+\widetilde{w}\right)^2\right) dy\, d\widetilde{y}\, d\widetilde{w} < \infty
\end{aligned}
$$

which shows (4.60). Proving (4.61) can be performed very similar to (4.60): Reusing the transformation $x \mapsto \frac{1}{x+b}$ yields the bound

$$
\int_0^\infty \int_{-b^2}^y \int_{-b^2}^{\widetilde{y}} \left(\frac{(y\varepsilon+b)(\widetilde{y}\varepsilon+b)}{\widetilde{w}\varepsilon+b}\right)^2 \exp\left(\frac{b}{2}\left(\widetilde{w}^2 - \widetilde{y}^2 - y^2\right)\right) d\widetilde{w}\, d\widetilde{y}\, dy.
$$

Noting $\widetilde{w}^2 - \widetilde{y}^2 \leq b^4$ and again using Fubini's theorem on the extended integration domain we end up with the same expression with y and $\widetilde{y}$ switched which is the same quantity.

We now move on to the second cycle phase, i.e. proving

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}_{\varepsilon/b+\varepsilon^2}[T_{\varepsilon/b}^2 \mid T_{\varepsilon/b} < T_z] < \infty.$$

In the spirit of (4.59) it reduces to consider one summand and the analoga to (4.60) and (4.61) are

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^4 \int_{\frac{1}{b}+\varepsilon}^{\frac{z}{\varepsilon}} \int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon} \int_{\frac{1}{b}}^{y} \int_{\frac{1}{b}}^{\widetilde{y}} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} \frac{r_c(\widetilde{y}\varepsilon)}{p_c(\widetilde{y}\varepsilon)} \, d\widetilde{w} \, d\widetilde{y} \, dw \, dy < \infty.$$

and

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^4 \int_{\frac{1}{b}+\varepsilon}^{\frac{z}{\varepsilon}} \int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon} \int_{y}^{\frac{z}{\varepsilon}} \int_{\frac{1}{b}}^{y} \frac{r_c(y\varepsilon)}{p_c(w\varepsilon)} \frac{r_c(\widetilde{y}\varepsilon)}{p_c(\widetilde{y}\varepsilon)} \, d\widetilde{w} \, d\widetilde{y} \, dw \, dy < \infty.$$

Again using (4.46) and enlarging the domain of integration it is sufficient to show with familiar abbreviation $f(x) := \frac{1}{3x^3} - \frac{b}{2x^2}$

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^3} \int_{\frac{1}{b}}^{\infty} \int_{\frac{1}{b}}^{y} \int_{\frac{1}{b}}^{\widetilde{y}} \frac{1}{(y\widetilde{y})^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f(y) + f(\widetilde{w}) - f(\widetilde{y})\right)\right) d\widetilde{w} \, d\widetilde{y} \, dy < \infty \tag{4.63}$$

and

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^3} \int_{\frac{1}{b}}^{\infty} \int_{y}^{\infty} \int_{\frac{1}{b}}^{y} \frac{1}{(y\widetilde{y})^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f(y) + f(\widetilde{w}) - f(\widetilde{y})\right)\right) d\widetilde{w} \, d\widetilde{y} \, dy < \infty. \tag{4.64}$$

We use similar techniques resulting in

$$\begin{aligned}
&\frac{1}{\varepsilon^3} \int_{\frac{1}{b}}^{\infty} \int_{\frac{1}{b}}^{y} \int_{\frac{1}{b}}^{\widetilde{y}} \frac{1}{(y\widetilde{y})^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f(y) + f(\widetilde{w}) - f(\widetilde{y})\right)\right) d\widetilde{w} \, d\widetilde{y} \, dy \\
&= \frac{1}{\varepsilon^3} \int_{0}^{b} \int_{y}^{b} \int_{\widetilde{y}}^{b} \left(\frac{y\widetilde{y}}{\widetilde{w}}\right)^2 \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f\left(\frac{1}{y}\right) + f\left(\frac{1}{\widetilde{w}}\right) - f\left(\frac{1}{\widetilde{y}}\right)\right)\right) d\widetilde{r} \, d\widetilde{y} \, dy \\
&= \frac{1}{\varepsilon^3} \int_{-b}^{0} \int_{y}^{0} \int_{\widetilde{y}}^{0} \left(\frac{(y+b)(\widetilde{y}+b)}{\widetilde{w}+b}\right)^2 \\
&\qquad \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} - f\left(\frac{1}{y+b}\right) + f\left(\frac{1}{\widetilde{w}+b}\right) - f\left(\frac{1}{\widetilde{y}+b}\right)\right)\right) d\widetilde{r} \, d\widetilde{y} \, dy \\
&= \frac{1}{\varepsilon^3} \int_{-b}^{0} \int_{y}^{0} \int_{\widetilde{y}}^{0} \left(\frac{(y+b)(\widetilde{y}+b)}{\widetilde{w}+b}\right)^2 \exp\left(\frac{1}{\varepsilon^2}\left(f\left(-\frac{1}{y}\right) - f\left(-\frac{1}{\widetilde{w}}\right) + f\left(-\frac{1}{\widetilde{y}}\right)\right)\right) d\widetilde{r} \, d\widetilde{y} \, dy
\end{aligned}$$

$$= \frac{1}{\varepsilon^3} \int_0^b \int_0^y \int_0^{\widetilde{y}} \left(\frac{(b-y)(b-\widetilde{y})}{b-\widetilde{w}} \right)^2 \exp\left(\frac{1}{\varepsilon^2} \left(f\left(\frac{1}{y}\right) - f\left(\frac{1}{\widetilde{w}}\right) + f\left(\frac{1}{\widetilde{y}}\right) \right) \right) d\widetilde{w}\, d\widetilde{y}\, dy.$$

Due to f being monotonously increasing on $[1/b, \infty)$ which is seen by calculating the derivative to be

$$f'(x) = -\frac{1}{x^4} + \frac{b}{x^3} = \frac{bx-1}{x^4} \tag{4.65}$$

the difference $f(1/\widetilde{y}) - f(1/\widetilde{w}) \leq 0$ is non-positive which combined with the fact

$$f(1/y) = y^3/3 - by^2/2 \leq by^2/3 - by^2/2 = -by^2/6$$

provides for the estimate

$$\begin{aligned}
&\frac{1}{\varepsilon^3} \int_0^b \int_0^y \int_0^{\widetilde{y}} \left(\frac{(b-y)(b-\widetilde{y})}{b-\widetilde{w}} \right)^2 \exp\left(\frac{1}{\varepsilon^2} \left(f\left(\frac{1}{y}\right) - f\left(\frac{1}{\widetilde{w}}\right) + f\left(\frac{1}{\widetilde{y}}\right) \right) \right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&\leq \frac{1}{\varepsilon^3} \int_0^b \int_0^y \int_0^{\widetilde{y}} \left(\frac{(b-y)(b-\widetilde{y})}{b-\widetilde{w}} \right)^2 \exp\left(\frac{1}{\varepsilon^2} \cdot \left(-\frac{b}{6} y^2 \right) \right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&= \int_0^{b/\varepsilon} \int_0^y \int_0^{\widetilde{y}} \left(\frac{(b-y\varepsilon)(b-\widetilde{y}\varepsilon)}{b-\widetilde{w}\varepsilon} \right)^2 \exp\left(-\frac{b}{6} y^2 \right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&\leq \int_0^{b/\varepsilon} \int_0^y \int_0^{\widetilde{y}} \left(\frac{(b-y\varepsilon)(b-\widetilde{y}\varepsilon)}{b-\widetilde{y}\varepsilon} \right)^2 \exp\left(-\frac{b}{6} y^2 \right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&= \int_0^{b/\varepsilon} \int_0^y \widetilde{y}\, (b-y\varepsilon)^2 \exp\left(-\frac{b}{6} y^2 \right) d\widetilde{y}\, dy = \int_0^{b/\varepsilon} \frac{y^2}{2} (b-y\varepsilon)^2 \exp\left(-\frac{b}{6} y^2 \right) dy \\
&\leq \frac{b^2}{2} \int_0^\infty y^2 \exp\left(-\frac{b}{6} y^2 \right) dy < \infty.
\end{aligned}$$

showing (4.63).

Analogously for (4.64)

$$\begin{aligned}
&\frac{1}{\varepsilon^3} \int_{\frac{1}{b}}^\infty \int_y^\infty \int_{\frac{1}{b}}^y \frac{1}{(y\widetilde{y})^4} \exp\left(\frac{1}{\varepsilon^2} \left(-\frac{b^3}{6} - f(y) + f(\widetilde{w}) - f(\widetilde{y}) \right) \right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&= \frac{1}{\varepsilon^3} \int_0^b \int_y^b \int_0^y \left(\frac{(b-y)(b-\widetilde{y})}{b-\widetilde{w}} \right)^2 \exp\left(\frac{1}{\varepsilon^2} \left(f\left(\frac{1}{y}\right) - f\left(\frac{1}{\widetilde{w}}\right) + f\left(\frac{1}{\widetilde{y}}\right) \right) \right) \\
&\leq \int_0^{b/\varepsilon} \int_y^{b/\varepsilon} \int_0^y \left(\frac{(b-y\varepsilon)(b-\widetilde{y}\varepsilon)}{b-\widetilde{w}\varepsilon} \right)^2 \exp\left(-\frac{b}{6} \widetilde{y}^2 \right) d\widetilde{w}\, d\widetilde{y}\, dy \\
&= \int_0^{b/\varepsilon} \int_y^{b/\varepsilon} ((b-y\varepsilon)(b-\widetilde{y}\varepsilon))^2 \exp\left(-\frac{b}{6} \widetilde{y}^2 \right) \left[\frac{1/\varepsilon}{b-\widetilde{w}\varepsilon} \right]_{\widetilde{w}=0}^{\widetilde{w}=y} d\widetilde{y}\, dy \\
&= \int_0^{b/\varepsilon} \int_y^{b/\varepsilon} ((b-y\varepsilon)(b-\widetilde{y}\varepsilon))^2 \exp\left(-\frac{b}{6} \widetilde{y}^2 \right) \frac{1}{\varepsilon} \left(\frac{1}{b-y\varepsilon} - \frac{1}{b} \right) d\widetilde{y}\, dy
\end{aligned}$$

$$= \int_0^{b/\varepsilon} \int_y^{b/\varepsilon} (b - y\varepsilon)(b - \widetilde{y}\varepsilon)^2 \exp\left(-\frac{b}{6}\widetilde{y}^2\right) \frac{y}{b}\, d\widetilde{y}\, dy$$
$$\leq b^2 \int_0^{\infty} \int_y^{\infty} y \cdot \exp\left(-\frac{b}{6}\widetilde{y}^2\right) d\widetilde{y}\, dy.$$

As by Hospital's rule

$$\lim_{y\to\infty} y \int_y^{\infty} \exp\left(-\frac{b}{6}\widetilde{y}^2\right) d\widetilde{y} = \lim_{y\to\infty} \frac{-\exp\left(-\frac{b}{6}y^2\right)}{-1/y^2} = 0$$

we may further compute with partial integration

$$b^2 \int_0^{\infty} \int_y^{\infty} y \cdot \exp\left(-\frac{b}{6}\widetilde{y}^2\right) d\widetilde{y}\, dy = -b^2 \int_0^{\infty} \frac{y^2}{2} \cdot \left(-\exp\left(-\frac{b}{6}y^2\right)\right) dy < \infty.$$

This finishes the proof. □

Proposition 4.48 (B1)**.** *It holds*

$$\lim_{scaling} \mathbb{E}_x[T^\lambda_{\varepsilon/b} \wedge T^\lambda_z] = 0 \text{ for } 0 < x < z.$$

Proof. As in the first example class we are in the situation

$$\mathbb{E}_x[T^\lambda_{\varepsilon/b} \wedge T^\lambda_z] \leq \frac{2}{\lambda^2}\left[\int_{\varepsilon/b}^{x} \int_{\varepsilon/b}^{y} \frac{r_c(y)}{p_c(w)}\, dw\, dy + \int_x^z \int_y^z \frac{r_c(y)}{p_c(w)}\, dw\, dy\right]$$

with the second integral being bounded. By Lemma 4.45, for $\varepsilon > 0$ sufficiently small it holds in the scaling limit

$$\lambda^{-2} = p_{\varepsilon,z}/J \leq \varepsilon^2,$$

whence

$$\frac{2}{\lambda^2} \int_{\varepsilon/b}^{x} \int_{\varepsilon/b}^{y} \frac{r_c(y)}{p_c(w)}\, dw\, dy \leq 2\varepsilon^2 \int_{\varepsilon/b}^{x} \int_{\varepsilon/b}^{y} \frac{r_c(y)}{p_c(w)}\, dw\, dy.$$

Using equation (4.48) we infer

$$2\varepsilon^2 \int_{\varepsilon/b}^{x} \int_{\varepsilon/b}^{y} \frac{r_c(y)}{p_c(w)}\, dw\, dy = 2 \int_{1/b}^{x/\varepsilon} \int_{1/b}^{y} \frac{1}{y^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} + f(w) - f(y)\right)\right) dw\, dy$$

with notation

$$f : [1/b, \infty) \to \mathbb{R}, \quad f(x) := \frac{1}{3x^3} - \frac{b}{2x^2}.$$

By $f' \geq 0$ due to equation (4.65) it follows the function f is monotonously increasing. Therefore, it holds

$$2 \int_{1/b}^{x/\varepsilon} \int_{1/b}^{y} \frac{1}{y^4} \exp\left(\frac{1}{\varepsilon^2}\left(-\frac{b^3}{6} + f(w) - f(y)\right)\right) dw\, dy$$

$$\leq 2\int_{1/b}^{x/\varepsilon}\int_{1/b}^{y}\frac{1}{y^4}\exp\left(-\frac{b^3}{6}\frac{1}{\varepsilon^2}\right)dw\,dy = 2\exp\left(-\frac{b^3}{6}\frac{1}{\varepsilon^2}\right)\int_{1/b}^{x/\varepsilon}\frac{y-1/b}{y^4}\,dy$$
$$\leq 2\exp\left(-\frac{b^3}{6}\frac{1}{\varepsilon^2}\right)\int_{1/b}^{\infty}\frac{1}{y^3}\,dy = \exp\left(-\frac{b^3}{6}\frac{1}{\varepsilon^2}\right)b^2 \xrightarrow[\varepsilon\to 0]{} 0.$$

This finishes the proof. □

Proposition 4.49 (B2)**.** *It holds*

$$\mathbb{P}_x(T_{\varepsilon/b} < T_z) \xrightarrow[\varepsilon\to 0]{} \frac{\int_x^z \exp\left(-\frac{b}{2y^2}\right)\,dy}{\int_0^z \exp\left(-\frac{b}{2y^2}\right)\,dy} \qquad \text{for } 0 < x < z.$$

Proof. The scale function approach leads to

$$\mathbb{P}_x(T_{\varepsilon/b} < T_z) = \frac{\int_x^z \exp\left(\frac{\varepsilon}{3y^3} - \frac{b}{2y^2}\right)\,dy}{\int_{\varepsilon/b}^z \exp\left(\frac{\varepsilon}{3y^3} - \frac{b}{2y^2}\right)\,dy}; \tag{4.66}$$

dominated convergence theorem may be applied to numerator and denominator separately finishing the proof. For the denominator observe that

$$\frac{\varepsilon}{3y^3} - \frac{b}{2y^2} \leq \frac{\varepsilon}{3y^2\varepsilon/b} - \frac{b}{2y^2} = -\frac{b}{6y^2}$$

holds. □

4.5. Conclusion

This work was mainly motivated by [5] of M. Bauer and D. Bernard. Using a clear probabilistic heuristic we prove a version of Conjecture B under general abstract conditions and demonstrate their usability in the example sections. We believe that the approach presented above is flexible enough to cover most one-dimensional examples of interest. As already discussed in [5] the natural question of extending the results to multi-dimensional situations remains unanswered, even though numerical simulations seem very promising in the sense that a point process could be obtained in an appropriate scaling regime. The tools and key concepts used throughout our approach appear relatively general and it would be clearly interesting to see, whether the approach of this work can be extended to higher dimensional situations.

5. Fleming-Viot particle Systems

In this chapter we investigate a system of particles of constant size. Each particle evolves independently from each other according to a generalized Bessel process with one common drift parameter until one of them hits the origin. At this time the particle at the origin jumps uniformly distributed to one of the other positive particles and afterwards the particles evolve independently from each other again until one particle hits the origin and so on. Our overall goal is to understand for which particle sizes and drift parameters there are infinitely many jumps in finite time almost surely.

5.1. Notation and basic properties

In this section we use the observation that we only need to know the positions of the particles at jump times and how long it took for the next jump to occur. Without loss of generality the positive positions of the particles not jumping may be indexed in ascending order. Next, by a scaling property of generalized Bessel processes, we may transform the problem to polar coordinates and see that the next position only depends on the angles of the old positions. The dependency structure is expressible as Hidden Markov Model and entails us to give an alternative expression for the extinction probability. This will imply an abstract criterion in Section 4.

Throughout the constructions and transformations we will perform, different labeled processes will occur. Even though some of them live on different state space for readability and convenience we use in most cases the same notation $\mathbb{P}$ for the underlying probability measure and $\mathbb{E}_x$ for the expectation, correspondingly. In the context of Markov chains $(\mathbb{P}_x)_x$ denotes the family of probability measures with $\mathbb{P}_x$ obtained by conditioning the corresponding process X_n, Y_n, Z_n, etc. to start in x: $\mathbb{P}_x \coloneqq \mathbb{P}(\cdot \mid X_0 = x)$ or $\mathbb{P}_x \coloneqq \mathbb{P}(\cdot \mid Y_0 = x)$ etc.

5.1.1. Problem formulation

Let us start by giving a more formal description of the problem under investigation: We consider a system $(X_t)_{t\geq 0} = (X_t^1, \dots, X_t^N)_{t\geq 0}$ of $N \in \mathbb{N}, N \geq 2$ particles starting in $X_0 = x_0 \in (0,\infty)^N$. We will generally use the superindex to distinguish components and denote the time in the subindex. $\mathbb{P}_{x_0}(\cdot)$ and $\mathbb{E}_{x_0}[\cdot]$ represent probabilities and expectations regarding events and functionals of X_t starting in x_0. As long as no particle reaches 0, they move independently according to the generalized Bessel processes with parametrization

$$dX_t^j = dB_t^j + \frac{(\nu - 1)/2}{X_t^j}\,dt, \quad j = 1, \dots, N. \tag{5.1}$$

Here, $(B_t^j)_{t \geq 0}$ are independent Brownian motions. The stopping time

$$\tau_1 := \min_{j=1}^{N} \inf\{t > 0 : \lim_{s \uparrow t} X_s^j = 0\}$$

denotes the first time any of the particles would hit 0. Note, that $\mathbb{P}_{x_0}(\tau_1 < \infty) = 1$ if and only if $\nu < 2$ and in that case there is some unique j with $\tau_1 = \inf\{t > 0 : \lim_{s \uparrow t} X_s^j = 0\}$ almost surely. This particle with superindex j will be set independently and uniformly to the position of one of the other $N-1$ particles at time τ_1. Therefore, the system stays in the state space $(0, \infty)^N$ and the jump is implemented in a fashion, the paths of all particles being càdlàg. After the jump the particles again move as independent generalized Bessel processes until time τ_2 where $\tau_n := \tau_{n-1} + \tau_1 \circ \theta_{\tau_{n-1}}$ for $n \geq 2$ and θ_s denotes the time shift $\theta_s((X_t)_{t \geq 0}) := (X_{s+t})_{t \geq 0}$ with the process $X_t(\omega) := \omega(t)$ assumed to be in the canonical representation. The mechanism is repeated inductively; the system (X_t) is a Markovian process in continuous time $0 \leq t < \lim_{n \to \infty} \tau_n$ and state space $(0, \infty)^N$ with càdlàg paths.

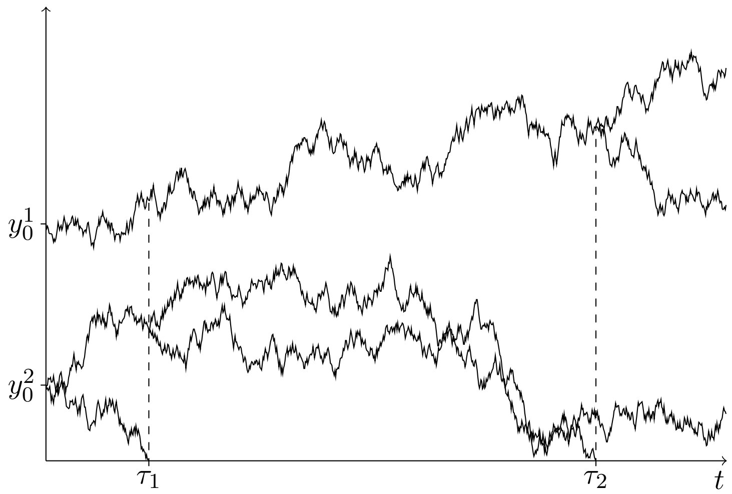


Figure 5.1.: Symbolic illustration of a sample path of the particle system X_t started in $X_0 = (y_0^1, y_0^2, y_0^2)$ with jump times τ_1 and τ_2.

Since there is no natural way to define the process for time points $t \geq \lim_{n \to \infty} \tau_n =: \tau_\infty$ the question arises if that limit in fact diverges (i.e. no extinction occurs) almost surely. Without loss of generality we may assume $\nu < 2$ since otherwise almost surely the origin is never reached, c.f. [34, Example 5.5.25].

Problem 5.1. For which particle sizes $N \in \{3, 4, \dots\}$ and drift parameters $\nu < 2$ does it hold

$$\tau_\infty = \infty$$

almost surely?

We are able to deduce the following partial answers:

- We will give a criterion for $\tau_\infty = \infty$ to hold almost surely, respectively, for $\tau_\infty < \infty$ to hold almost surely (Theorem 5.71). In order to fully exhaust the abstract criterion and to obtain the numerical results for general N and ν one needs to work with a density function of some stationary distribution of which the author is not able to infer an analytical explicit closed form expression.

- Still, the method will prove strong enough to explicitly exhibit slightly negative values of $\nu < 0$ such that for $N = 3$ it almost surely holds $\tau_\infty = \infty$ (Theorem 5.78). This is in contrast to the result [10, Theorem 1.1 (i)] for the case of $N = 2$ where $\nu < 0$ implies $\tau_\infty < \infty$ almost surely.

5.1.2. Symmetry of the model

In what follows we want to take advantage of the symmetry that is inherent in the model. Before we start, let us give some basic intuition and motivation.
Despite the underlying stochastic process X_t having continuous time parameter $t \in [0, \tau_\infty)$ the random sequence τ_n under investigation has discrete parameter $n \in \mathbb{N}_0$. For arbitrary $n \in \mathbb{N}_0$, given the positions

$$\lim_{s \uparrow \tau_n} X_s \in \{x \in [0, \infty)^N : x^j = 0 \text{ for exactly one } j\}$$

of the particles immediately ahead of a jump there is a Markovian transition to the positions immediately ahead of the next jump. We are interested in the random time $\tau_{n+1} - \tau_n$ it takes for the next jump to occur and we must keep track of the positions of the particles immediately ahead of jump times. But we do not need to know the whole time continuous trajectories in between the jumps, only the positions at jump times matter to us. In this sense, the problem is reduced to the analysis of a Markov chain in discrete time $n \in \mathbb{N}_0$.
There is more redundant information to abandon. Given the positions $\lim_{s \uparrow \tau_n} X_s$ as a N-tuple actually only the values $\lim_{s \uparrow \tau_n} X_s^j$, $j \in \{1, \dots, N\}$ are relevant to us but not the way they are ordered since the drift parameter ν is constant and identical for all particles. So we only need to know the ordered values

$$\lim_{s \uparrow \tau_n} X_s^{(1)} \geq \lim_{s \uparrow \tau_n} X_s^{(2)} \geq \ldots \geq \lim_{s \uparrow \tau_n} X_s^{(N)}$$

of the particles immediately ahead of jump times τ_n to retrieve the distribution of the increment $\tau_{n+1} - \tau_n$. Since a priori in this ordering $\lim_{s \uparrow \tau_n} X_s^{(j)} > 0$ for $j \in \{1, \dots, N-1\}$

are the positive values while $\lim_{s\uparrow\tau_n} X_s^{(N)} = 0$, all relevant information can be recovered from the $N-1$ first values.
In order to give an overview of the notations of the (Markov) processes constructed in the course of this procedure there is the Table 5.1. The full definitions of the processes will be given later in the next subsections (cf. Definition 5.5, Definition 5.15, where $\leftrightsquigarrow$ denotes the equivalence relation induced by the permutations of the coordinates and $A/\leftrightsquigarrow$ is the corresponding set of equivalence classes and Definition 5.16, where $A^{\downarrow}$ is the set of elements of A put in ascending order).

Name	**Notation**	**State Space**
Fleming-Viot particle system	$(X_t)_{t\geq 0}$	$(0,\infty)^N$
jump time chain (Definition 5.5)	$(\mathring{X}_n)_{n\in\mathbb{N}_0} \coloneqq (\lim_{s\uparrow\tau_n} X_s)_{n\in\mathbb{N}_0}$	$H \coloneqq \{x \in [0,\infty)^N :$ $x^j = 0$ for exactly one $j\}$
jump time chain modulo permutations (Definition 5.15)	$(\mathring{Z}_n)_{n\in\mathbb{N}_0} \coloneqq (\mathring{X}_n/\leftrightsquigarrow)_{n\in\mathbb{N}_0}$	$H/\leftrightsquigarrow$
positive ascending jump time chain (Definition 5.16)	$(Y_n)_{n\in\mathbb{N}_0}$ $\coloneqq ((\mathring{X}_n^{\downarrow})^1,\ldots,(\mathring{X}_n^{\downarrow})^{N-1})_{n\in\mathbb{N}_0}$	$(0,\infty)^{N-1\downarrow}$

Table 5.1.: Markov processes constructed in this subsection for adapting to the relevant information to Problem 5.1.

Embedded Jump Time Chain

Notation 5.2. Let $X_{t-} \coloneqq \lim_{s\uparrow t} X_s$ denote the limit from the left.

Definition 5.3. We set

$$\begin{aligned} H &\coloneqq \{x \in [0,\infty)^N : x^j = 0 \text{ for exactly one } j\} \\ &= \{x \in [0,\infty)^N : |\{j \in \{1,\ldots,N\} : x^j = 0\}| = 1\} \\ &= \dot{\bigcup_{j=1}^{N}} (0,\infty)^{j-1} \times \{0\} \times (0,\infty)^{N-j} \end{aligned} \tag{5.2}$$

so that it holds $X_{\tau_n-} \in H$ for $n \in \mathbb{N}$.

In the definition of H we use the convention $A^0 = \emptyset$ for sets A, i.e. specifically in our situation

$$(0,\infty)^0 \times \{0\} \times (0,\infty)^{N-1} = \{0\} \times (0,\infty)^{N-1}$$

and

$$(0,\infty)^{N-1}\times\{0\}\times(0,\infty)^0=(0,\infty)^{N-1}\times\{0\}.$$

Convention 5.4. For the sake of concise and consistent notation let us formally set $X_{0-}\coloneqq x_{0-}\in H$ for some deterministic starting configuration $x_{0-}\in H$ infinitesimally ahead of the 0-th jump meaning that if $j\in\{1,\dots,N\}$ is the unique index with $x_{0-}^j=0$ it holds for all $l\in\{1,\dots,N\}\setminus\{j\}$

$$\mathbb{P}_{x_{0-}}\Big(X_0=(x_{0-}^1,\dots,x_{0-}^{j-1},x_{0-}^l,x_{0-}^{j+1},\dots,x_{0-}^N)\Big)=1/(N-1).$$

Definition 5.5. Upon setting $\tau_0\coloneqq 0$ we may consider the *jump time chain*

$$(\mathring{X}_n)_{n\in\mathbb{N}_0}\coloneqq(X_{\tau_n-})_{n\in\mathbb{N}_0}$$

as discrete time Markov chain on state space H.

Permuting the Coordinates

In the following we want to neglect the order of the particle's indices which essentially is no loss of information.

Definition 5.6. For $x\in H$ and a permutation $\pi\in S_N$ on $\{1,\dots,N\}$ let us introduce the notation

$$x^\pi\coloneqq(x^{\pi(1)},\dots,x^{\pi(N)})$$

which means $(x^\pi)^j=x^{\pi(j)}$ and for subsets $A\subseteq H$ let us define

$$A^\pi\coloneqq\{a^\pi : a\in A\}.$$

Furthermore, π^{-1} denotes the permutation inverse to π, i.e. $\pi^{-1}\circ\pi=\pi\circ\pi^{-1}=\mathrm{id}_{\{1,\dots,N\}}$.

Example 5.7. For $N=3$, $\pi(1)=2$, $\pi(2)=3$, $\pi(3)=1$ it holds $(x^2,x^1,0)^\pi=(x^1,0,x^2)$.

For general $N\geq 2$, if $\pi^1,\pi^2,\dots,\pi^N\in S_N$ are permutations satisfying $\pi^j(j)\coloneqq N$, $j\in\{1,\dots,N\}$, it holds

$$H=\dot{\bigcup_{j=1}^{N}}\left((0,\infty)^{N-1}\times\{0\}\right)^{\pi^j}.$$

Definition 5.8. For $x\in H$ there is an uniquely defined $m(x)\in\{1,\dots,N\}$ with $x^m=0$. Furthermore let for $x\in H$ and $l\in\{1,\dots,N\}\setminus\{m(x)\}$

$$y(x,l)\coloneqq(x^1,\dots,x^{m(x)-1},x^l,x^{m(x)+1},\dots,x^N)\in(0,\infty)^N. \tag{5.3}$$

Example 5.9. For $N=3$ and $x^1,x^2\in(0,\infty)$ it holds $y((x^2,x^1,0),1)=(x^2,x^1,x^2)$ which is in general not the same as (x^2,x^1,x^1).

Lemma 5.10. *The mapping $x \mapsto y(x,l)$ is positively homogeneous in the sense that for $x \in H$, $l \in \{1,\dots,N\} \setminus \{m(x)\}$ and $\alpha > 0$ it holds*

$$y(\alpha \cdot x, l) = \alpha \cdot y(x,l).$$

Proof. It holds $m(\alpha \cdot x) = m(x)$ and by definition

$$\begin{aligned} y(\alpha \cdot x, l) &= ((\alpha \cdot x)^1, \dots, (\alpha \cdot x)^{m(x)-1}, (\alpha \cdot x)^l, (\alpha \cdot x)^{m(x)+1}, \dots, (\alpha \cdot x)^N) \\ &= \alpha \cdot (x^1, \dots, x^{m(x)-1}, x^l, x^{m(x)+1}, \dots, x^N) = \alpha \cdot y(x,l). \end{aligned}$$

□

Lemma 5.11. *For $x_0 \in H$, $l \in \{1,\dots,N\} \setminus \{m(x_0)\}$, $y_0 := y(x_0,l)$, $A \in \mathcal{B}(H)$ and $\pi \in S_N$ it holds*

$$\mathbb{P}_{y_0}(X_{\tau-} \in A) = \mathbb{P}_{y_0^\pi}(X_{\tau-}^{\pi^{-1}} \in A). \tag{5.4}$$

Proof. It holds

$$\mathbb{P}_{y_0}(X_{\tau-} \in A) = \sum_{j=1}^N \mathbb{P}_{y_0}(X_{\tau-} \in A, \tau = \tau^{\pi(j)}) = \sum_{j=1}^N \mathbb{P}_{y_0}\left(X_{\tau-} \in A, \bigcap_{\substack{k=1 \\ k \neq \pi(j)}}^N \{\tau^k > \tau^{\pi(j)}\} \right)$$

$$= \sum_{j=1}^N \int_0^\infty \mathbb{P}_{y_0}\left((X_t^1, \dots, X_t^{\pi(j)-1}, 0, X_t^{\pi(j)+1}, \dots, X_t^N) \in A, \bigcap_{\substack{k=1 \\ k \neq \pi(j)}}^N \{\tau^k > t\} \right) \mathbb{P}_{y_0}(\tau^{\pi(j)} \in dt).$$

Now $\mathbb{P}_{y_0}(\tau^{\pi(j)} \in dt)$ is the density of the 0-hitting time of a one-dimensional Bessel process with start in

$$y_0^{\pi(j)} = (y_0^\pi)^j,$$

thereby identical to $\mathbb{P}_{y_0^\pi}(\tau^j \in dt)$. Similarly, for $k \in \{1,\dots,N\}$ and $A^k \in \mathcal{B}((0,\infty))$ it holds

$$\mathbb{P}_{y_0}\left(X_t^k \in A^k, \tau^k > t \right) = \mathbb{P}_{y_0^\pi}\left(X_t^{\pi^{-1}(k)} \in A^k, \tau^{\pi^{-1}(k)} > t \right).$$

In view of Carathéodory's extension theorem we may assume $A = \times_{m=1}^N A^m$ to be a product set since they constitute a $\bigcap$-stable generator of $\mathcal{B}(H)$. Then, the integrand is given by

$$\mathbb{P}_{y_0}\left((X_t^1, \dots, X_t^{\pi(j)-1}, 0, X_t^{\pi(j)+1}, \dots, X_t^N) \in A, \bigcap_{\substack{k=1 \\ k \neq \pi(j)}}^N \{\tau^k > t\} \right)$$

$$= \mathbb{1}_{A^{\pi(j)}}(0) \prod_{\substack{k=1 \\ k \neq \pi(j)}}^N \mathbb{P}_{y_0}\left(X_t^k \in A^k, \tau^k > t \right) = \mathbb{1}_{A^{\pi(j)}}(0) \prod_{\substack{k=1 \\ k \neq \pi(j)}}^N \mathbb{P}_{y_0^\pi}\left(X_t^{\pi^{-1}(k)} \in A^k, \tau^{\pi^{-1}(k)} > t \right)$$

$$= \mathbb{P}_{y_0^\pi}\left((X_t^{\pi^{-1}(1)}, \dots, X_t^{\pi^{-1}(\pi(j)-1)}, 0, X_t^{\pi^{-1}(\pi(j)+1)}, \dots, X_t^{\pi^{-1}(N)}) \in A, \bigcap_{\substack{k=1\\k\neq j}}^N \{\tau^k > t\} \right),$$

whence

$$\sum_{j=1}^N \int_0^\infty \mathbb{P}_{y_0}\left((X_t^1, \dots, X_t^{\pi(j)-1}, 0, X_t^{\pi(j)+1}, \dots, X_t^N) \in A, \bigcap_{\substack{k=1\\k\neq \pi(j)}}^N \{\tau^k > t\} \right) \mathbb{P}_{y_0}(\tau^{\pi(j)} \in dt)$$

$$= \sum_{j=1}^N \int_0^\infty \mathbb{P}_{y_0^\pi}\left((X_t^{\pi^{-1}(1)}, \dots, X_t^{\pi^{-1}(\pi(j)-1)}, 0, X_t^{\pi^{-1}(\pi(j)+1)}, \dots, X_t^{\pi^{-1}(N)}) \in A, \bigcap_{\substack{k=1\\k\neq j}}^N \{\tau^k > t\} \right) \mathbb{P}_{y_0^\pi}(\tau^j \in dt)$$

$$= \sum_{j=1}^N \mathbb{P}_{y_0^\pi}\left(X_{\tau-}^{\pi^{-1}} \in A, \bigcap_{\substack{k=1\\k\neq j}}^N \{\tau^k > \tau^j\} \right) = \mathbb{P}_{y_0^\pi}(X_{\tau-}^{\pi^{-1}} \in A).$$

This shows equation (5.4).

□

Lemma 5.12. *For starting values $x_0 \in H$, permutations $\pi \in S_N$, measurable sets $A \in \mathcal{B}(H)$ and time indices $n \in \mathbb{N}_0$ it holds*

$$\mathbb{P}_{x_0}\left(\mathring{X}_n \in A\right) = \mathbb{P}_{x_0^\pi}\left(\mathring{X}_n^{\pi^{-1}} \in A\right) = \mathbb{P}_{x_0^\pi}\left(\mathring{X}_n \in A^\pi\right).$$

Proof. Let us employ the technique of induction over $n \in \mathbb{N}_0$. As base cases we show the assertion for $n = 0$ and $n = 1$ and then perform the induction step $n-1 \to n$ for $n \geq 2$. For $n = 0$ it holds

$$\begin{aligned}&\mathbb{P}_{x_0}(\mathring{X}_n \in A) = \mathbb{P}_{x_0}(\mathring{X}_0 \in A) = \mathbb{1}_A(x_0) = \mathbb{1}_A((x_0^\pi)^{\pi^{-1}}) = \mathbb{P}_{x_0^\pi}(\mathring{X}_0^{\pi^{-1}} \in A)\\ &= \mathbb{P}_{x_0^\pi}(\mathring{X}_n^{\pi^{-1}} \in A) = \mathbb{P}_{x_0^\pi}((\mathring{X}_n^{\pi^{-1}})^\pi \in A^\pi) = \mathbb{P}_{x_0^\pi}(\mathring{X}_n \in A^\pi).\end{aligned}$$

The last two equalities hold for all $n \in \mathbb{N}_0$ showing the second equality of the assertion.

Now we consider the case $n = 1$: Let us specially denote $m \coloneqq m(x_0)$.
Due to $x_0^{\pi(m(x_0^\pi))} = (x_0^\pi)^{m(x_0^\pi)} = 0$ it holds $\pi(m(x_0^\pi)) = m(x_0) = m$ which we may write as

$$m(x_0^\pi) = \pi^{-1}(m). \tag{5.5}$$

Since for $y_0 = y(x_0, l)$ with $l \in \{1, \dots, N\} \setminus \{m\}$ it holds

$$(y_0^\pi)^j = y_0^{\pi(j)} = \begin{cases} x_0^{\pi(j)}, & \pi(j) \neq m \\ x_0^l, & \pi(j) = m \end{cases} = \begin{cases} x_0^{\pi(j)}, & j \neq \pi^{-1}(m) \\ x_0^l, & j = \pi^{-1}(m) \end{cases},$$

it follows

$$
\begin{aligned}
&y(x_0,l)^\pi = y_0^\pi = (x_0^{\pi(1)},\dots,x_0^{\pi(\pi^{-1}(m)-1)},x_0^l,x_0^{\pi(\pi^{-1}(m)+1)},\dots,x_0^{\pi(N)})\\
&= ((x_0^\pi)^1,\dots,(x_0^\pi)^{\pi^{-1}(m)-1},(x_0^\pi)^{\pi^{-1}(l)},(x_0^\pi)^{\pi^{-1}(m)+1},\dots,(x_0^\pi)^N) = y(x_0^\pi,\pi^{-1}(l)).
\end{aligned}
\tag{5.6}
$$

Using equations (5.6) and (5.5) it follows with equation (5.4) the assertion for $n = 1$:

$$
\begin{aligned}
&\mathbb{P}_{x_0}(\mathring{X}_1 \in A) = \frac{1}{N-1}\sum_{\substack{l=1\\ l\neq m}}^N \mathbb{P}_{y(x_0,l)}(X_{\tau-} \in A) = \frac{1}{N-1}\sum_{\substack{l=1\\ l\neq m}}^N \mathbb{P}_{y(x_0,l)^\pi}(X_{\tau-}^{\pi^{-1}} \in A)\\
&= \frac{1}{N-1}\sum_{\substack{l=1\\ l\neq m}}^N \mathbb{P}_{y(x_0^\pi,\pi^{-1}(l))}(X_{\tau-}^{\pi^{-1}} \in A) = \frac{1}{N-1}\sum_{\substack{l=1\\ l\neq \pi^{-1}(m)}}^N \mathbb{P}_{y(x_0^\pi,l)}(X_{\tau-}^{\pi^{-1}} \in A)\\
&= \frac{1}{N-1}\sum_{\substack{l=1\\ l\neq m(x_0^\pi)}}^N \mathbb{P}_{y(x_0^\pi,l)}(X_{\tau-}^{\pi^{-1}} \in A) = \mathbb{P}_{x_0^\pi}(\mathring{X}_1^{\pi^{-1}} \in A).
\end{aligned}
$$

Let us turn to the induction step $n-1 \to n$: By the Markov property it follows

$$
\begin{aligned}
&\mathbb{P}_{x_0}(\mathring{X}_n \in A) = \mathbb{E}_{x_0}\left[\mathbb{P}_{\mathring{X}_1}(\mathring{X}_{n-1} \in A)\right] = \mathbb{E}_{x_0^\pi}\left[\mathbb{P}_{\mathring{X}_1^{\pi^{-1}}}(\mathring{X}_{n-1} \in A)\right]\\
&= \mathbb{E}_{x_0^\pi}\left[\mathbb{P}_{\left(\mathring{X}_1^{\pi^{-1}}\right)^\pi}(\mathring{X}_{n-1}^{\pi^{-1}} \in A)\right] = \mathbb{E}_{x_0^\pi}\left[\mathbb{P}_{\mathring{X}_1}(\mathring{X}_{n-1}^{\pi^{-1}} \in A)\right] = \mathbb{P}_{x_0^\pi}(\mathring{X}_n^{\pi^{-1}} \in A).
\end{aligned}
$$

This finishes the proof. □

The result of the previous lemma gives rise to the following definitions:

Definition 5.13. For $x, y \in H$ let the equivalence relation $x \leftrightsquigarrow y$ hold, if and only if $y = x^\pi$ for some $\pi \in S_N$. Let $[x] := \{x^\pi : \pi \in S_n\}$ denote the equivalence class of $x \in H$. For subsets $A \subseteq H$ let $A/\leftrightsquigarrow := \{[a] : a \in A\}$ the corresponding set of equivalence classes.

The set $\mathcal{B}(H)/\leftrightsquigarrow := \{A/\leftrightsquigarrow : A \in \mathcal{B}(H)\}$ is a σ-field. To ensure the well-definedness of the Markov kernel on the quotient sets the following two assertions will help us.

Lemma 5.14. *(i) From $A/\leftrightsquigarrow = B/\leftrightsquigarrow$ it follows $\bigcup_{\pi\in S_N} A^\pi = \bigcup_{\pi\in S_N} B^\pi$.*

(ii) From $[x] = [y]$ it follows $\kappa(x, \bigcup_{\pi\in S_N} A^\pi) = \kappa(y, \bigcup_{\pi\in S_N} A^\pi)$ where

$$\kappa : H \times (\mathcal{B}(H)) \to [0,1], \quad \kappa(x,A) = \mathbb{P}_x\left(\mathring{X}_1 \in A\right)$$

denotes the one step transition distribution of the jump time chain $(\mathring{X}_n)_{n\in\mathbb{N}_0}$.

Proof. (i) Let $x \in \bigcup_{\pi \in S_N} A^\pi$. Then, there is $a \in A$ and $\pi \in S_n$ with $x = a^\pi$. This means $[x] = [a] \in A/\leftrightsquigarrow = B/\leftrightsquigarrow$. Consequently, it must exist $b \in B$ with $[b] = [x]$. This implies the existence of $\sigma \in S_N$ with $x = b^\sigma \in \bigcup_{\pi \in S_N} B^\pi$. Since $x \in \bigcup_{\pi \in S_N} A\pi$ was arbitrary it follows $x \in \bigcup_{\pi \in S_N} A^\pi \subseteq \bigcup_{\pi \in S_N} B^\pi$. Upon relabeling the other direction of argumentation follows as well.

(ii) For $y = x^\sigma$ it holds according to Lemma 5.12

$$\mathbb{P}_{x^\sigma}\left(\mathring{X}_1 \in \bigcup_{\pi \in S_N} A^\pi\right) = \mathbb{P}_x\left(\mathring{X}_1 \in \left(\bigcup_{\pi \in S_N} A^\pi\right)^{\sigma^{-1}}\right) = \mathbb{P}_x\left(\mathring{X}_1 \in \bigcup_{\pi \in S_N} A^\pi\right)$$

as desired.

□

Definition 5.15. We can therefore define the Markov chain

$$(\mathring{Z}_n)_{n \in \mathbb{N}_0} := (\mathring{X}_n/\leftrightsquigarrow)_{n \in \mathbb{N}_0}$$

on state space $H/\leftrightsquigarrow$ with probability measures $(Q_x)_{x \in H/\leftrightsquigarrow}$ via

$$Q_{[x0]}\left(\mathring{Z}_n \in A/\leftrightsquigarrow\right) := \mathbb{P}_{x_0}\left(\mathring{X}_n \in \bigcup_{\pi \in S_N} A^\pi\right)$$

as *jump time chain modulo permutations.*

Neglecting the 0

For $x \in H$ let $x^\downarrow$ the uniquely determined representative of $[x]$ with

$$x^1 \geq x^2 \geq \ldots \geq x^N = 0$$

and for $A \subseteq H$ let $A^\downarrow := \{a^\downarrow : a \in A\}$. Using this notation it holds

$$Q_{[x_0]}\left(\mathring{Z}_n \in A/\leftrightsquigarrow\right) = \mathbb{P}_{x_0}\left(\mathring{X}_n \in \bigcup_{\pi \in S_N} A^\pi\right) = \mathbb{P}_{x_0^\downarrow}\left(\mathring{X}_n^\downarrow \in A^\downarrow\right).$$

We might as well consider the Markov chain $(\mathring{X}_n^\downarrow)_{n \in \mathbb{N}_0}$ on $\{x \in (0,\infty)^{N-1} \times \{0\} : x^1 \geq x^2 \geq \ldots x^N = 0\}$. This suggests to neglect the redundant 0 in the last N-th component. Finally, we may define

$$\begin{aligned}
&Q_{[z_0]}(Z_n \in A/\leftrightsquigarrow) := Q_{[(z_0^1,\ldots,z_0^{N-1},0)]}(\mathring{Z}_n \in \{[(a^1,\ldots,a^{N-1},0)] : a \in A\}) \\
&= \mathbb{P}_{(z_0^1,\ldots,z_0^{N-1},0)}(\mathring{X}_n \in \{(a^1,\ldots,a^{N-1},0)^\pi : a \in A, \pi \in S_N\}) \\
&= \frac{1}{N-1}\sum_{l=1}^{N-1} \mathbb{P}_{(z_0^1,\ldots,z_0^{N-1},z_0^l)}(X_{\tau_n -} \in \{(a^1,\ldots,a^{N-1},0)^\pi : a \in A, \pi \in S_N\}).
\end{aligned}$$

This leads to

Definition 5.16. Let us call the Markov chain

$$(Y_n)_{n\in\mathbb{N}_0} := ((X^{\downarrow}_{\tau_n-})^1, \dots, (X^{\downarrow}_{\tau_n-})^{N-1})_{n\in\mathbb{N}_0}$$

on the state space $(0,\infty)^{N-1\downarrow}$ the *positive ascending jump time chain.*

5.1.3. Self-similarity of generalized Bessel processes

Notation 5.17. For an Euclidean vector element $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, let

$$\|x\| := \left(\sum_{j=1}^{n} \left(x^j\right)^2 \right)^{1/2}$$

denote the Euclidean norm of x.

Conceptual idea

Let us describe some of the basic ideas in this subsection. The precise definitions will formally be repeated later. In what follows we substantially want to exploit the scaling property of generalized Bessel processes stopped at the origin which behaves like Brownian scaling:

If (X_t) is a generalized Bessel process and $c > 0$ a real number,
then $\left(c \cdot X_{t/c^2}\right)$ is again a generalized Bessel process.

This distributional invariance will imply that certain random variables do not depend on the Euclidean magnitude $\|X_t\| = (\sum_{j=1}^{N}(X_t^j)^2)^{1/2}$ of the Fleming-Viot particle system read as Euclidean vector element. More specifically, let us for example consider one Markov transition of the chain Y_n given in Definition 5.16. Let us suppose it holds $Y_0 = y_0 \in (0,\infty)^{N-1\downarrow}$ which in correspondence with Convention 5.4 means for some $x_{0-} \in H$ with $((x^{\downarrow}_{0-})^1, \dots, (x^{\downarrow}_{0-})^{N-1}) = y_0$ it holds $X_{0-} := x_{0-}$ and X_0 denotes the random positions after the 0-th jump, i.e. the 0-element in x_{0-} is uniformly replaced by one of the $N-1$ positive values. Let us consider the particle system $\overline{X}_t = (\overline{X}^1_t, \dots, \overline{X}^N_t)$ with start in $\overline{X}_{0-} := x_{0-}$ where this time $\overline{X}_0$ is obtained by a jump based on the scaled value $\frac{x_{0-}}{\|x_{0-}\|}$, i.e. the 0-element in $\frac{x_{0-}}{\|x_{0-}\|}$ is uniformly replaced by one of the $N-1$ positive values of $\frac{x_{0-}}{\|x_{0-}\|}$. Then again, given $\overline{X}_0$, the particles of $\overline{X}_t$ move as independent generalized Bessel process until random time

$$\overline{\tau} := \min_{j=1}^{N} \overline{\tau}^j := \min_{j=1}^{N} \inf\{t \geq 0 : \overline{X}^j_{t-} = 0\}$$

when the first one of them touches the origin. By Lemma 5.10 it holds the equality in distribution $X_0 \overset{\mathcal{D}}{=} \|x_{0-}\|\,\overline{X}_0$ and moreover with the scaling property of Bessel processes we will show the equality in distribution of the processes

$$\left(\mathbb{1}_{[0,\tau)}(t)\, X_t\right) \overset{\mathcal{D}}{=} \left(\mathbb{1}_{[0,\|x_{0-}\|^2\,\overline{\tau})}(t)\, \|x_{0-}\|\, \overline{X}_{t/\|x_{0-}\|^2}\right).$$

This particularly implies the equalities in distribution

$$X_{\tau-} \overset{\mathcal{D}}{=} \|x_{0-}\| \, \overline{X}_{\overline{\tau}-}$$

and

$$\tau \overset{\mathcal{D}}{=} \|x_{0-}\|^2 \, \overline{\tau}$$

from which we see that the distribution of the direction

$$\frac{Y_1}{\|Y_1\|} = \frac{\Big((X^{\downarrow}_{\tau-})^1, \dots, (X^{\downarrow}_{\tau-})^{N-1}\Big)}{\Big\|\Big((X^{\downarrow}_{\tau-})^1, \dots, (X^{\downarrow}_{\tau-})^{N-1}\Big)\Big\|} \overset{\mathcal{D}}{=} \frac{\Big((\overline{X}^{\downarrow}_{\overline{\tau}-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}-})^{N-1}\Big)}{\Big\|\Big((\overline{X}^{\downarrow}_{\overline{\tau}-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}-})^{N-1}\Big)\Big\|} =: \frac{\overline{Y}_1}{\|\overline{Y}_1\|}$$

and of the factor by which the magnitude changes

$$\frac{\|Y_1\|}{\|Y_0\|} = \frac{\|X_{\tau-}\|}{\|x_{0-}\|} \overset{\mathcal{D}}{=} \frac{\big\|\|x_{0-}\| \, \overline{X}_{\overline{\tau}-}\big\|}{\|x_{0-}\|} = \Big\|\Big((\overline{X}^{\downarrow}_{\overline{\tau}-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}-})^{N-1}\Big)\Big\| =: \|\overline{Y}_1\|$$

does not depend on the value of $\|Y_0\| = \|y_0\|$ given $\frac{Y_0}{\|Y_0\|} = \frac{y_0}{\|y_0\|}$. In other words, given

$$U_0 := \frac{\overline{Y}_0}{\|\overline{Y}_0\|} := \frac{\Big((\overline{X}^{\downarrow}_{0-})^1, \dots, (\overline{X}^{\downarrow}_{0-})^{N-1}\Big)}{\Big\|\Big((\overline{X}^{\downarrow}_{0-})^1, \dots, (\overline{X}^{\downarrow}_{0-})^{N-1}\Big)\Big\|} = \frac{y_0}{\|y_0\|},$$

the random variables $U_1 := \frac{\overline{Y}_1}{\|\overline{Y}_1\|}$, $R_1 := \|\overline{Y}_1\|$ and $T_1 := \overline{\tau} - \overline{\tau}_0 = \overline{\tau}$ are independent of $R_0 := \|\overline{Y}_0\|$ and $T_0 := \overline{\tau}_0 - 0 = 0$ and they can be appropriately scaled back in order to retrieve the distribution of $\frac{Y_1}{\|Y_1\|}$, $\frac{\|Y_1\|}{\|Y_0\|}$ and $\tau - \tau_0 = \tau$. The dependency structure is illustrated in Figure 5.3.
Due to the underlying Markovian structure of the jump time chain we essentially may repeat the multiplicative scaling at each jump by letting $\overline{X}_t$ the particle system where the n-th jump at time $\overline{\tau}_n$ is based on the quantity $\frac{\overline{X}_{\overline{\tau}_n-}}{\|\overline{X}_{\overline{\tau}_n-}\|}$ and use the random scaling factor $\prod_{j=1}^n \|\overline{X}_{\overline{\tau}_{j-1}-}\|$. More explicitly, we find

$$X_{\tau_n-} \overset{\mathcal{D}}{=} \prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_n-}$$

and

$$\tau_n - \tau_{n-1} \overset{\mathcal{D}}{=} \prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\|^2 (\overline{\tau}_n - \overline{\tau}_{n-1}) := \prod_{j=1}^{n} R^2_{j-1} T_n.$$

Denoting the Markov chain

$$(U_n)_{n\in\mathbb{N}_0} := \left(\frac{\overline{Y}_n}{\|\overline{Y}_n\|}\right)_{n\in\mathbb{N}_0} := \left(\frac{((\overline{X}^{\downarrow}_{\overline{\tau}_n-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_n-})^{N-1})}{\|((\overline{X}^{\downarrow}_{\overline{\tau}_n-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_n-})^{N-1})\|}\right)_{n\in\mathbb{N}_0}$$

on state space

$$\mathsf{S} := \{x \in (0,\infty)^{N-1^{\downarrow}} : \|x\| = 1\}$$

the Markov transitions from (U_n, R_n, T_n) to $(U_{n+1}, R_{n+1}, T_{n+1})$ rely on U_n only.

Example 5.18. To give an illustration let us consider the case of $N = 3$ moving particles.

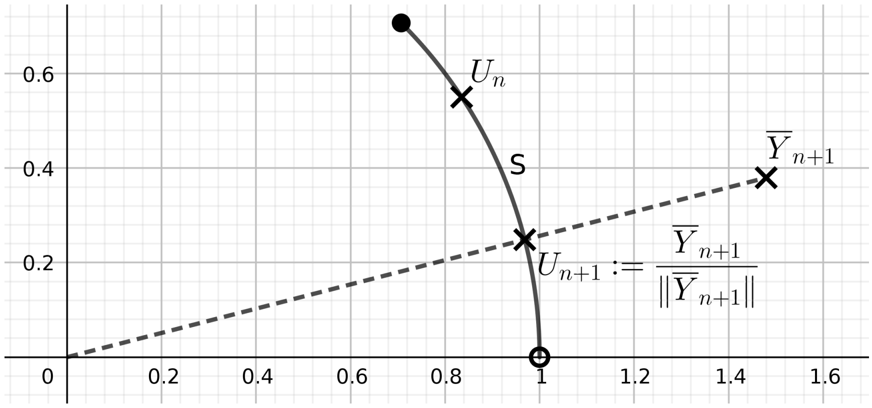


Figure 5.2.: Illustration of the Markov transition from U_n to U_{n+1}. The image was created using GeoGebra: `https://www.geogebra.org/`.

Suppose the chain is located somewhere at $U_n \in \mathsf{S}$. Then the three particles of $\overline{X}_{\overline{\tau}_n}$ will have the values U_n^1 and U_n^2, with equal probability either one of them taken for two particles:

$$\mathbb{P}(\overline{X}^{\downarrow}_{\overline{\tau}_n} = (U_n^1, U_n^1, U_n^2)) = \mathbb{P}(\overline{X}^{\downarrow}_{\overline{\tau}_n} = (U_n^1, U_n^2, U_n^2)) = \frac{1}{2}.$$

Next, the three particles of $\overline{X}_t$ run independently as generalized Bessel processes until one of them touches zero. The two positions of the surviving particles in ascending order build up $\overline{Y}_{n+1} = (\overline{Y}^1_{n+1}, \overline{Y}^2_{n+1})$, and the projection on S is the next value $U_{n+1} := \frac{\overline{Y}_{n+1}}{\|\overline{Y}_{n+1}\|}$. Further, we obtain R_{n+1} as the magnitude $\|\overline{Y}_{n+1}\|$ (length of the dashed line in the picture) and $T_{n+1} = \overline{\tau}_{n+1} - \overline{\tau}_n$ as time how long this procedure took.
The state space

$$\mathsf{S} = \{(x^1, x^2) \in (0,\infty)^2 : x^1 \geq x^2, (x^1)^2 + (x^2)^2 = 1\}$$

of directions may be identified with the state space $(0, \pi/4)$ of angles. This parametrization is actually used later when further analyzing the case $N = 3$.

In the case $N = 2$ the state space S is the singleton $\{1\}$ and the idea of our construction is in essence the same as the one given in the beginning of the proof of [10, Theoream 1.1 (i)].

Formal implementation

We are now ready to be more formal. Let again $H := \dot{\bigcup}_{j=1}^{N} (0,\infty)^{j-1} \times \{0\} \times (0,\infty)^{N-j}$ as in (5.2) denote the subspace of $[0,\infty)^N$ where exactly one component equals 0 and let $m : H \to \{1, \ldots, N\}$, $m(x) := \arg\{j : x^j = 0\}$ the mapping picking that component as in Definition 5.8 and moreover, recall that for $x \in H$ and $l \in \{1, \dots, N\} \setminus \{m(x)\}$ we defined in equation (5.3)

$$y(x,l) \coloneqq (x^1, \dots, x^{m(x)-1}, x^l, x^{m(x)+1}, \dots, x^N) \in (0,\infty)^N.$$

We now describe given $N \in \mathbb{N}$, $N \geq 2$, a "normed" Fleming-Viot N-particle process $(\overline{X}_t)_{t\geq 0}$ to the parameter $\nu < 2$ on the state space $(0,\infty)^N$. In a nutshell, ahead of each jump time, we normalize the $(N-1)$ positive particles by dividing by their norm $\|\overline{X}_{\overline{\tau}_n-}\|$. Just after the jump, their norm will then be strictly larger than 1.

Definition 5.19. Let $(\overline{V}_n)_{n\in\mathbb{N}_0}$ a family of independent random variables uniformly distributed on $\{1, \dots, N-1\}$ independently from anything else. Let

$$\overline{\tau}_0 = 0, \qquad \overline{\tau}_n = \min_{j=1}^{N} \inf\{t > \overline{\tau}_{n-1} : \overline{X}^j_{t-} = 0\} \text{ for } n \in \mathbb{N}.$$

For $\overline{X}_{0-} \coloneqq x_{0-} \in H$ denoting $\overline{b}_n : \{1, \dots, N-1\} \to \{1, \dots, N\} \setminus \{m(\overline{X}_{\overline{\tau}_n-})\}$ for the unique order preserving bijection we set

$$\overline{X}_{\overline{\tau}_n} \coloneqq y\left(\frac{\overline{X}_{\overline{\tau}_n-}}{\|\overline{X}_{\overline{\tau}_n-}\|}, \overline{b}_n(\overline{V}_n)\right), \quad n \in \mathbb{N}_0.$$

During time points $t \in (\overline{\tau}_{n-1}, \overline{\tau}_n)$, $n \in \mathbb{N}$, the components $(\overline{X}^j_t)$, $j = 1, \dots, N$ move independently from (V_n) and independently from each other according to a Bessel process starting in $\overline{X}^j_{\overline{\tau}_{n-1}}$ and global parameter $\nu < 2$ in the sense of (5.1).

Remark 5.20. The difference to the original process X_t is the normalizing at jump times $\overline{\tau}_n$. For $n \in \mathbb{N}_0$ neither $\overline{X}_{\overline{\tau}_n-}$ nor $\overline{X}_{\overline{\tau}_n}$ must have unit norm, the latter in fact has magnitude strictly larger than one. Letting $(V_n)_{n\in\mathbb{N}_0}$ a family of independent random variables uniformly distributed on $\{1, \dots, N-1\}$ independently from anything else and denoting $b_n : \{1, \dots, N\} \to \{1, \dots, N\} \setminus \{m(X_{\tau_n-})\}$ for the unique order preserving bijection we could have constructed

$$X_{\tau_n} = y(X_{\tau_n-}, b_n(V_n))$$

for $n \in \mathbb{N}_0$. In the following we use this notation.

In order to facilitate an alternative problem formulation we must relate the normed Fleming-Viot process $\overline{X}_t$ with the original process X_t, i.e. we must be able to properly scale it back so we have no loss of information:

Definition 5.21. We define for $n \in \mathbb{N}_0$ the *backscaled* (series of) *jump times*

$$\tau^{\text{bs}}_n \coloneqq \sum_{k=1}^{n} \prod_{j=1}^{k} \|\overline{X}_{\overline{\tau}_{j-1}-}\|^2 \cdot (\overline{\tau}_k - \overline{\tau}_{k-1})$$

and the *backscaled process*

$$X^{\text{bs}}_t \coloneqq \sum_{n=1}^{\infty} \mathbb{1}_{[\tau^{\text{bs}}_{n-1}, \tau^{\text{bs}}_n)}(t) \left(\prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_{n-1} + (t-\tau^{\text{bs}}_{n-1})/\prod_{j=1}^{n}\|\overline{X}_{\overline{\tau}_{j-1}-}\|^2}\right).$$

Example 5.22. The first backscaled jump times are

$$\tau_0^{\mathrm{bs}} = 0, \quad \tau_1^{\mathrm{bs}} = \|\overline{X}_{0-}\|^2 \overline{\tau}_1, \quad \tau_2^{\mathrm{bs}} = \|\overline{X}_{0-}\|^2 \overline{\tau}_1 + \|\overline{X}_{0-}\|^2 \|\overline{X}_{\overline{\tau}_1-}\|^2 (\overline{\tau}_2 - \overline{\tau}_1)$$

and the definition of the backscaled process starts with

$$X_t^{\mathrm{bs}} = \begin{cases} \|\overline{X}_{0-}\| \cdot \overline{X}_{t/\|\overline{X}_{0-}\|^2}, & 0 \le t < \|\overline{X}_{0-}\|^2 \overline{\tau}_1 \\ \|\overline{X}_{0-}\| \|\overline{X}_{\overline{\tau}_1-}\| \cdot \overline{X}_{\overline{\tau}_1 + \frac{t - \|\overline{X}_{0-}\|^2 \overline{\tau}_1}{\|\overline{X}_{0-}\|^2 \|\overline{X}_{\overline{\tau}_1-}\|^2}}, & \|\overline{X}_{0-}\|^2 \overline{\tau}_1 \le t < \|\overline{X}_{0-}\|^2 \overline{\tau}_1 + \\ & \quad + \|\overline{X}_{0-}\|^2 \|\overline{X}_{\overline{\tau}_1-}\|^2 (\overline{\tau}_2 - \overline{\tau}_1) \\ \vdots, & \vdots \end{cases}$$

Remark 5.23. Just ahead of the n-th jump, $n \in \mathbb{N}_0$, the backscaled process is located at

$$X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}} = \prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\left(\overline{\tau}_{n-1} + (\tau_n^{\mathrm{bs}} - \tau_{n-1}^{\mathrm{bs}}) / \prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\|^2\right)-} = \prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_n-}$$

and jumps to

$$X_{\tau_n^{\mathrm{bs}}}^{\mathrm{bs}} = \prod_{j=1}^{n+1} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_n}.$$

Due to

$$\prod_{j=1}^{n+1} \|\overline{X}_{\overline{\tau}_{j-1}-}\| = \prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \|\overline{X}_{\overline{\tau}_n-}\| = \|X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}}\|$$

and

$$\begin{aligned} \overline{X}_{\overline{\tau}_n} &= y\left(\frac{\overline{X}_{\overline{\tau}_n-}}{\|\overline{X}_{\overline{\tau}_n-}\|}, \overline{b}_n(\overline{V}_n)\right) = y\left(\frac{\prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_n-}}{\prod_{j=1}^{n+1} \|\overline{X}_{\overline{\tau}_{j-1}-}\|}, \overline{b}_n(\overline{V}_n)\right) \\ &= y\left(\frac{X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}}}{\|X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}}\|}, \overline{b}_n(\overline{V}_n)\right) \end{aligned}$$

it holds

$$X_{\tau_n^{\mathrm{bs}}}^{\mathrm{bs}} = \prod_{j=1}^{n+1} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_n} = \|X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}}\| \cdot y\left(\frac{X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}}}{\|X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}}\|}, \overline{b}_n(\overline{V}_n)\right) = y\left(X_{\tau_n^{\mathrm{bs}}-}^{\mathrm{bs}}, \overline{b}_n(\overline{V}_n)\right). \tag{5.7}$$

The jump dynamics of the backscaled process (X_t^{bs}) is therefore the same as the one of the original process (X_t).

Following the Brownian scaling property of generalized Bessel processes the law of (X_t) and (X_t^{bs}) also agree with each other in between the jumps and we attain:

Proposition 5.24. *(Cf. the alternative construction of the Fleming-Viot process in the beginning of the proof of [10, Theorem 1.1 (i)].) Let $X_{0-} := x_{0-} \in H$ arbitrary and set $\overline{X}_{0-} := x_{0-}$. Then the processes (X_t) and (X_t^{bs}) are identically distributed.*

Proof. Let us give a step by step explanation on how we apply the scaling property of generalized Bessel processes stopped at the origin.
Application to a generalized Bessel process with deterministic scaling factor and starting value. As direct consequence of the scaling property, for all $c, x_0 \in (0,\infty)$ it holds for X_t and $\overline{X}_t$ independent generalized Bessel processes starting from $X_0 = x_0$ and $\overline{X}_0 = x_0/c$ and

$$\tau := \inf\{t \geq 0 : X_t = 0\}$$

and

$$\overline{\tau} := \inf\{t \geq 0 : \overline{X}_t = 0\}$$

denoting their respective hitting times of the origin that the processes $(\mathbb{1}_{[0,\tau)}(t)\, X_t)$ and $(\mathbb{1}_{[0,c^2\cdot\overline{\tau})}(t)\, c \cdot \overline{X}_{t/c^2})$ are identically distributed, i.e.

$$\mathbb{P}\left((\mathbb{1}_{[0,\tau)}(t)\, X_t) \in \bullet \mid X_0 = x_0\right) = \mathbb{P}\left((\mathbb{1}_{[0,c^2\,\overline{\tau})}(t)\, c\, \overline{X}_{t/c^2}) \in \bullet \mid \overline{X}_0 = x_0/c\right).$$

Application to N independent generalized Bessel processes with deterministic scaling factor and starting value. Since the minimum operator complies with a scaling factor as multiplicative constant in the sense that for $c > 0$ and $x = (x^1, \dots, x^N)$ it holds

$$\min_{j=1}^{N} \left(c^2\, x^j\right) = c^2 \min_{j=1}^{N} x^j$$

we may apply the scaling property to N Bessel processes simultaneously as follows. Let again $c \in (0,\infty)$ and let $x_0 = (x_0^1, \dots, x_0^N) \in (0,\infty)^N$ and $X_t = (X_t^1, \dots, X_t^N)$ denote N independent generalized Bessel processes with starting values $x_0 \in (0,\infty)^N$ and independent of $\overline{X}_t$ consisting of N independent generalized Bessel processes starting in $x_0/c = (x_0^1/c, \dots, x_0^N/c)$. Then writing $\tau^j := \inf\{t \geq 0 : X_{t-}^j = 0\}$ and $\overline{\tau} := \inf\{t \geq 0 : \overline{X}_{t-}^j = 0\}$ for

$$\tau := \min_{j=1}^{N} \tau^j = \min_{j=1}^{N} \inf\{t \geq 0 : X_{t-}^j = 0\}$$

and

$$\overline{\tau} := \min_{j=1}^{N} \overline{\tau}^j = \min_{j=1}^{N} \inf\{t \geq 0 : \overline{X}_{t-}^j = 0\}$$

it holds

$$\begin{aligned}
&\mathbb{P}\left((\mathbb{1}_{[0,\tau)}(t)\, X_t) \in \bullet \mid X_0 = x_0\right) = \mathbb{P}\left((\mathbb{1}_{[0,\min_{j=1}^{N} \tau^j)}(t)\, X_t) \in \bullet \mid X_0 = x_0\right) \\
&= \mathbb{P}\left((\mathbb{1}_{[0,\min_{j=1}^{N} c^2\, \overline{\tau}^j)}(t)\, \overline{X}_t) \in \bullet \mid \overline{X}_0 = x_0/c\right) = \mathbb{P}\left((\mathbb{1}_{[0,c^2\,\overline{\tau})}(t)\, \overline{X}_t) \in \bullet \mid \overline{X}_0 = x_0/c\right).
\end{aligned}$$

Application to N independent generalized Bessel process with random scaling factor and starting value. Let us now suppose again the processes X_t and $\overline{X}_t$ are independent each consisting of N generalized Bessel processes which are independent given their starting values X_0 and $\overline{X}_0$, this time X_0 following some random distribution supported on $(0,\infty)^N$ and $(C,\overline{X}_0)$ following some random distribution supported on $(0,\infty)\times(0,\infty)^N$ independently from the process X_t and such that given $\overline{X}_0$ the process $\overline{X}_t$ is independent of C and the equality in distribution $C\cdot\overline{X}_0 \overset{\mathcal{D}}{=} X_0$ holds. Then

$$\begin{aligned}
&\mathbb{P}\left((\mathbb{1}_{[0,\tau)}(t)\,X_t)\in\bullet\right)=\int_{(0,\infty)^N}\mathbb{P}\left((\mathbb{1}_{[0,\tau)}(t)\,X_t)\in\bullet\mid X_0=x_0\right)\,\mathbb{P}(X_0\in dx_0)\\
&=\int_{(0,\infty)^N}\mathbb{P}\left((\mathbb{1}_{[0,\tau)}(t)\,X_t)\in\bullet\mid X_0=x_0\right)\,\mathbb{P}(C\cdot\overline{X}_0\in dx_0)\\
&=\int_{(0,\infty)\times(0,\infty)^N}\mathbb{P}\left((\mathbb{1}_{[0,\tau)}(t)\,X_t)\in\bullet\mid X_0=c\,\overline{x}_0\right)\,\mathbb{P}((C,\overline{X}_0)\in d(c,\overline{x}_0))\\
&=\int_{(0,\infty)\times(0,\infty)^N}\mathbb{P}\left((\mathbb{1}_{[0,c^2\,\overline{\tau})}(t)\,c\,\overline{X}_{t/c^2})\in\bullet\mid \overline{X}_0=\overline{x}_0\right)\,\mathbb{P}((C,\overline{X}_0)\in d(c,\overline{x}_0))\\
&=\int_{(0,\infty)^N}\int_0^\infty\mathbb{P}\left((\mathbb{1}_{[0,c^2\,\overline{\tau})}(t)\,c\,\overline{X}_{t/c^2})\in\bullet\mid \overline{X}_0=\overline{x}_0\right)\,\mathbb{P}(C\in dc\mid \overline{X}_0=\overline{x}_0)\,\mathbb{P}(\overline{X}_0\in dx_0)\\
&=\int_{(0,\infty)^N}\mathbb{P}\left((\mathbb{1}_{[0,C^2\cdot\overline{\tau})}(t)\,C\cdot\overline{X}_{t/C^2})\in\bullet\mid \overline{X}_0=\overline{x}_0\right)\,\mathbb{P}(\overline{X}_0\in dx_0)\\
&=\mathbb{P}\left((\mathbb{1}_{[0,C^2\cdot\overline{\tau})}(t)\,C\cdot\overline{X}_{t/C^2})\in\bullet\right).
\end{aligned}$$

Application to the Fleming-Viot particle system. The jump dynamic complies with a scaling factor as multiplicative constant in the sense of Lemma 5.10: For $c>0$, $x\in H$ and $l\in\{1,\dots,N\}\setminus\{m(x)\}$ it holds

$$y(c\,x,l)=c\,y(x,l).$$

In order to finish the proof let us inductively show for $M\in\mathbb{N}$:

$$\left(\sum_{n=1}^{M}\mathbb{1}_{[\tau^{\text{bs}}_{n-1},\tau^{\text{bs}}_n)}(t)X^{\text{bs}}_t\right)_{t\geq 0}\overset{d}{=}\left(\sum_{n=1}^{M}\mathbb{1}_{[\tau_{n-1},\tau_n)}(t)X_t\right)_{t\geq 0}.$$

Start of induction ($M=1$): Firstly, by definition

$$\begin{aligned}
X^{\text{bs}}_0&=X^{\text{bs}}_{\tau^{\text{bs}}_0}=\|\overline{X}_{\overline{\tau}_0-}\|\cdot\overline{X}_{\overline{\tau}_0}=\|\overline{X}_{0-}\|\cdot y(\overline{X}_{0-}/\|\overline{X}_{0-}\|,\overline{b}_0(\overline{V}_0))\\
&=\|x_{0-}\|\cdot y\left(x_{0-}/\|x_{0-}\|,\overline{b}_0(\overline{V}_0)\right)=y(x_{0-},\overline{b}_0(\overline{V}_0))\overset{\mathcal{D}}{=}y(x_{0-},b_0(V_0))=X_0.
\end{aligned}$$

Since

$$m(\overline{X}_{\overline{\tau}_0-})=m(\overline{X}_{0-})=m(x_{0-})=m(X_{0-})=m(X_{\tau_0-})$$

the mappings $\overline{b}$ and b coincide. Thus,

$$\overline{X}_0=\overline{X}_{\overline{\tau}_0}=y(\overline{X}_{\overline{\tau}_0-}/\|\overline{X}_{\overline{\tau}_0-}\|,\overline{b}_0(\overline{V}_0))=y(\overline{X}_{0-}/\|\overline{X}_{0-}\|,\overline{b}_0(\overline{V}_0))$$

$$= y(x_{0-}/\|x_{0-}\|, \overline{b}_0(\overline{V}_0)) = y(x_{0-}, b_0(\overline{V}_0))/\|x_{0-}\|.$$

It therefore holds

$$\begin{aligned}\overline{X}_0 &= y(x_{0-}, b_0(\overline{V}_0))/\|x_{0-}\| \overset{\mathcal{D}}{=} y(x_{0-}, b_0(V_0))/\|x_{0-}\| = y(X_{0-}, b_0(V_0))/\|x_{0-}\| \\ &= y(X_{\tau_0-}, b_0(V_0))/\|x_{0-}\| = X_{\tau_0}/\|x_{0-}\| = X_0/\|x_{0-}\|.\end{aligned}$$

As a consequence, we may apply the scaling property with $c \coloneqq \|x_{0-}\| > 0$ to obtain

$$\begin{aligned}&\left(\mathbb{1}_{[\tau_0^{\mathrm{bs}}, \tau_1^{\mathrm{bs}})}(t)\, X_t^{\mathrm{bs}}\right)_{t\geq 0} = \left(\mathbb{1}_{[0, \|\overline{X}_{0-}\|^2 \overline{\tau}_1)}(t)\, \|\overline{X}_{0-}\| \cdot \overline{X}_{t/\|\overline{X}_{0-}\|^2}\right)_{t\geq 0} \\ &= \left(\mathbb{1}_{[0, \|x_{0-}\|^2 \overline{\tau}_1)}(t)\, \|x_{0-}\| \cdot \overline{X}_{t/\|x_{0-}\|^2}\right)_{t\geq 0} \overset{\mathcal{D}}{=} \left(\mathbb{1}_{[\tau_0, \tau_1)}(t)\, X_t\right)_{t\geq 0}.\end{aligned}$$

Inductive step ($M \to M+1$): By induction hypothesis, $\tau_M^{\mathrm{bs}} \overset{\mathcal{D}}{=} \tau_M$ and $X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-} \overset{\mathcal{D}}{=} X_{\tau_M -}$. Hence using equation (5.7) it holds

$$X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}} = y\left(X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}, \overline{b}_M(\overline{V}_M)\right) \overset{\mathcal{D}}{=} y\left(X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}, \overline{b}_M(V_M)\right) \overset{\mathcal{D}}{=} y\left(X_{\tau_M -}, b_M(V_M)\right) = X_{\tau_M}$$

and again by scaling with $C \coloneqq \|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\| > 0$ and proportional starting distributions $X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}} = C \cdot \overline{X}_{\overline{\tau}_M} \overset{\mathcal{D}}{=} X_{\tau_M}$ we finally attain

$$\begin{aligned}&\left(\mathbb{1}_{[\tau_M^{\mathrm{bs}}, \tau_{M+1}^{\mathrm{bs}})}(t)\, X_t^{\mathrm{bs}}\right) = \left(\mathbb{1}_{[\tau_M^{\mathrm{bs}}, \tau_{M+1}^{\mathrm{bs}})}(t)\, \|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\| \cdot \overline{X}_{\overline{\tau}_M + (t-\tau_M^{\mathrm{bs}})/\|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\|^2}\right) \\ &\left(\mathbb{1}_{[\tau_M^{\mathrm{bs}}, \tau_M^{\mathrm{bs}} + C^2\cdot(\overline{\tau}_{M+1} - \overline{\tau}_M))}(t)\, C \cdot \overline{X}_{\overline{\tau}_M + (t-\tau_M^{\mathrm{bs}})/C^2}\right) \overset{\mathcal{D}}{=} \left(\mathbb{1}_{[\tau_M, \tau_{M+1})}(t)\, X_t\right).\end{aligned}$$

This shows

$$\left(\sum_{n=1}^{M+1} \mathbb{1}_{[\tau_{n-1}^{\mathrm{bs}}, \tau_n^{\mathrm{bs}})}(t) X_t^{\mathrm{bs}}\right)_{t\geq 0} \overset{d}{=} \left(\sum_{n=1}^{M+1} \mathbb{1}_{[\tau_{n-1}, \tau_n)}(t) X_t\right)_{t\geq 0}$$

and concludes the induction finishing the proof. □

Remark 5.25. Definition 5.19 and Definition 5.21 may be regarded to as generalization of the construction given in the beginning of the proof of [10, Theoream 1.1 (i)] where the scaling invariance of generalized Bessel processes was used as well. Demanding $\|\overline{X}_{0-}\| = 1$ in the notation of the paper's authors α_i corresponds to $\|\overline{X}_{\overline{\tau}_i -}\|$ and σ_j corresponds to $\overline{\tau}_j - \overline{\tau}_{j-1}$. We use a slightly different normalization scheme which does not entail a principal difference. Our construction can be seen as straight generalization; in our setting $N > 2$ the sequence of random variables $(\overline{X}^{\downarrow}_{\overline{\tau}_n -}/\|\overline{X}^{\downarrow}_{\overline{\tau}_n -}\|)_{n\in\mathbb{N}_0}$ are random directions that are not deterministically equal to $(1, 0)$ anymore.

We are now ready to consider the normed process written with descending ordering and introduce some new letters for sequences of random variables to be used later on.

Definition 5.26. Let us change to polar coordinates upon defining

$$(\overline{Y}_n, T_n)_{n\in\mathbb{N}_0} := (((\overline{X}^{\downarrow}_{\overline{\tau}_n-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_n-})^{N-1}), \overline{\tau}_n - \overline{\tau}_{(n-1)\vee 0})_{n\in\mathbb{N}_0}$$
$$\simeq \left(\frac{\overline{Y}_n}{\|\overline{Y}_n\|}, \|\overline{Y}_n\|, T_n\right)_{n\in\mathbb{N}_0} =: (U_n, R_n, T_n)_{n\in\mathbb{N}_0}.$$

Corollary 5.27. *Under the assumptions of Proposition 5.24 the following picture illustrates the dependency structure implied by Proposition 5.24: Given the direction U_n of the particles immediately ahead of a jump, there is a Markovian transition kernel to the direction U_{n+1} immediately ahead of the next jump, the factor R_{n+1} by which the magnitude will change and the scaled amount of time T_{n+1} it will take. The marginals of $(U_{n+1}, R_{n+1}, T_{n+1})$ given U_n are not independent.*

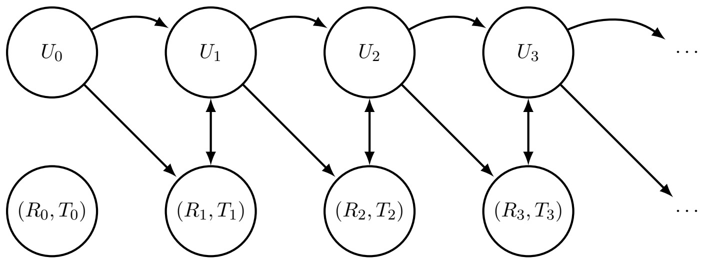


Figure 5.3.: Illustration of the dependency structure of the chain (U_n, R_n, T_n).

Proof. For $n \in \mathbb{N}_0$ it holds

$$U_n = \frac{((\overline{X}^{\downarrow}_{\overline{\tau}_n-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_n-})^{N-1})}{\|((\overline{X}^{\downarrow}_{\overline{\tau}_n-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_n-})^{N-1})\|} = \frac{((X^{\mathrm{bs}\downarrow}_{\tau^{\mathrm{bs}}_n-})^1, \dots, (X^{\mathrm{bs}\downarrow}_{\tau^{\mathrm{bs}}_n-})^{N-1})}{\|((X^{\mathrm{bs}\downarrow}_{\tau^{\mathrm{bs}}_n-})^1, \dots, (X^{\mathrm{bs}\downarrow}_{\tau^{\mathrm{bs}}_n-})^{N-1})\|}$$
$$\overset{\mathcal{D}}{=} \frac{((X^{\downarrow}_{\tau_n-})^1, \dots, (X^{\downarrow}_{\tau_n-})^{N-1})}{\|((X^{\downarrow}_{\tau_n-})^1, \dots, (X^{\downarrow}_{\tau_n-})^{N-1})\|}$$

and for $n \in \mathbb{N}$ it holds

$$R_n = \|((\overline{X}^{\downarrow}_{\overline{\tau}_n-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_n-})^{N-1})\| = \|\overline{X}_{\overline{\tau}_n-}\| = \frac{\|X^{\mathrm{bs}}_{\tau^{\mathrm{bs}}_n-}\|}{\|X^{\mathrm{bs}}_{\tau^{\mathrm{bs}}_{n-1}-}\|} \overset{\mathcal{D}}{=} \frac{\|X_{\tau_n-}\|}{\|X_{\tau_{n-1}-}\|}$$

and

$$T_n = \overline{\tau}_n - \overline{\tau}_{n-1} = \frac{\tau^{\mathrm{bs}}_n - \tau^{\mathrm{bs}}_{n-1}}{\|X^{\mathrm{bs}}_{\tau^{\mathrm{bs}}_{n-1}-}\|^2} \overset{\mathcal{D}}{=} \frac{\tau_n - \tau_{n-1}}{\|X_{\tau_{n-1}-}\|^2}$$

justifying the labels *direction*, *magnitude changing factor* and *scaled amount of time* for U_n, R_n and T_n, respectively. By definition for $n \in \mathbb{N}_0$

$$\overline{X}_{\overline{\tau}_n} = y\left(\frac{\overline{X}_{\overline{\tau}_n-}}{\|\overline{X}_{\overline{\tau}_n-}\|}, \overline{b}_n(\overline{V}_n)\right)$$

meaning that at time $\overline{\tau}_n$ the process $\overline{X}_t$ loses the information about its previous norm $\|\overline{X}_{\overline{\tau}_n-}\|$. Thus,

$$((\overline{X}^{\downarrow}_{\overline{\tau}_{n+1}-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_{n+1}-})^{N-1}, \overline{\tau}_{n+1} - \overline{\tau}_n) = (R_{n+1} \cdot U_{n+1}, T_{n+1})$$

is conditionally independent of $(\|\overline{X}_{\overline{\tau}_n-}\|, \overline{\tau}_n - \overline{\tau}_{(n-1)\vee 0}) = (R_n, T_n)$ given U_n. Since (U_{n+1}, R_{n+1}) is a measurable function of $R_{n+1} \cdot U_{n+1}$ this shows conditional independence of $(U_{n+1}, R_{n+1}, T_{n+1})$ and (R_n, T_n) given U_n.

□

Corollary 5.28. *Under the assumptions of Proposition 5.24 the bivariate discrete time process*

$$(M_n)_{n\in\mathbb{N}_0} \coloneqq ((U_n, U_{n+1}), (R_{n+1}, T_{n+1}))_{n\in\mathbb{N}_0}$$

is a hidden Markov model (HMM) with hidden chain (U_n, U_{n+1}) and observed chain (R_{n+1}, T_{n+1}) on the product space $\mathsf{S}^2 \times (0,\infty)^2$ endowed with the associated Borel σ-algebra where

$$\mathsf{S} \coloneqq \{x = (x^1, \dots, x^{N-1}) \in (0,\infty)^{N-1\downarrow} : \|x\| = 1\}. \tag{5.8}$$

Proof. In the construction of M_n the dependency structure is altered to:

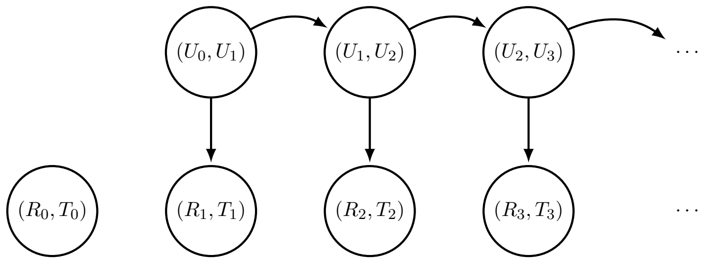


Figure 5.4.: Illustration of the dependency structure of $((U_n, U_{n+1}), (R_{n+1}, T_{n+1}))$.

This complies with the notion of HMMs, cf. [17, Fig 1.1 and Subsection 2.2.1]: Conditionally on the hidden chain $(U_n, U_{n+1})_{n\in\mathbb{N}_0}$, the observations (R_{n+1}, T_{n+1}) are independent, and for each n the conditional distribution of (R_{n+1}, T_{n+1}) depends on (U_n, U_{n+1}) only. □

Corollary 5.29. *Under the assumptions of Proposition 5.24 it holds*

$$\tau_\infty \overset{\mathcal{D}}{=} R_0^2 \sum_{k=1}^{\infty} \prod_{j=1}^{k-1} R_j^2 \cdot T_k.$$

Proof. We directly compute

$$\tau_\infty = \lim_{n\to\infty} \tau_n \overset{\mathcal{D}}{=} \lim_{n\to\infty} \tau_n^{\mathrm{bs}} = \lim_{n\to\infty} \sum_{k=1}^{n} \prod_{j=1}^{k} \|\overline{X}_{\overline{\tau}_{j-1}-}\|^2 \cdot (\overline{\tau}_k - \overline{\tau}_{k-1})$$

$$= \sum_{k=1}^{\infty} \|\overline{X}_{\overline{\tau}_0 -}\|^2 \prod_{j=1}^{k-1} \|\overline{X}_{\overline{\tau}_j -}\|^2 \cdot (\overline{\tau}_k - \overline{\tau}_{k-1}) = R_0^2 \sum_{k=1}^{\infty} \prod_{j=1}^{k-1} R_j^2 \cdot T_k.$$

□

5.2. Transition densities

This chapter is devoted to the computation of several density functions for later usage. Some of the technical aspects are outsourced to the appendix.

Lemma 5.30. *The Markov chain (Y_n) from Definition 5.16 admits a density function $h_{y_0}(y)$ of the form*

$$h_{y_0}(y) = \frac{2}{N-1} \sum_{i,j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(j) \le \pi(i)}} \int_0^{\infty} g_{y_0^j}(t)\, f_t(y_0^i, y^{\pi(j)}) \prod_{\substack{s=1 \\ s \ne j}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dt,$$

where

$$g_x(t) := \mathbb{P}_x(\inf\{t > 0 : X_t^1 = 0\} \in dt) \tag{5.9}$$

is the density function of the hitting time of 0 *of a generalized Bessel process and*

$$f_t(x, y) := \mathbb{P}_x(X_t^1 \in dy, \min\{X_s : s \le t\} > 0) \tag{5.10}$$

is the transition density of a generalized Bessel process stopped in the origin starting at $x > 0$ and moving to $y > 0$ at time $t > 0$.

Proof. Let $y_0 \in (0,\infty)^{N-1\downarrow}$ and $A \in \mathcal{B}\left((0,\infty)^{N-1\downarrow}\right)$ arbitrary. By independence the transition density of $((X_t^{\downarrow})^1, \dots, (X_t^{\downarrow})^{N-1})$ up to the first jump is given by

$$\mathbb{P}_{y_0}(Y_1 \in A) = \frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{P}_{(y_0^1, \dots, y_0^{N-1}, y_0^i)}(X_{\tau-}^{\downarrow} \in A)$$

$$= \frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{P}_{(y_0^1, \dots, y_0^{N-1}, y_0^i)}(X_{\tau-} \in \{(a^1, \dots, a^{N-1}, 0)^{\pi} : a \in A, \pi \in S_N\})$$

$$= \frac{1}{N-1} \sum_{i=1}^{N-1} \Bigg(\int_0^{\infty} \int_{\bigcup_{\pi \in S_{N-1}} A^{\pi}} \prod_{s=1}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^i}(t)\, dt$$

$$+ \sum_{j=1}^{N-1} \int_0^{\infty} \int_{\bigcup_{\pi \in S_{N-1}} A^{\pi}} f_t(y_0^i, y^j) \prod_{\substack{s=1 \\ s \ne j}}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^j}(t)\, dt \Bigg)$$

$$= \frac{1}{N-1}\sum_{i=1}^{N-1}\Bigg(2\int_0^\infty \int_{\bigcup_{\pi\in S_{N-1}} A^\pi} \prod_{s=1}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^i}(t)\, dt \\ + \sum_{\substack{j=1\\ j\neq i}}^{N-1} \int_0^\infty \int_{\bigcup_{\pi\in S_{N-1}} A^\pi} f_t(y_0^i, y^j) \prod_{\substack{s=1\\ s\neq j}}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^j}(t)\, dt\Bigg). \tag{5.11}$$

For $\pi_1 \neq \pi_2 \in S_{N-1}$ the set

$$A^{\pi_1} \cap A^{\pi_2} \subseteq (0,\infty)^{N-1\downarrow \pi_1} \cap (0,\infty)^{N-1\downarrow \pi_2} \subseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{N-1} \{a \in (0,\infty)^{N-1} : a^i = a^j\}$$

is a null set with respect to the $(N-1)$-dimensional Lebesgue measure and therefore it holds for arbitrary $t > 0$

$$\int_{\bigcup_{\pi\in S_{N-1}} A^\pi} \prod_{s=1}^{N-1} f_t(y_0^s, y^s)\, dy = \sum_{\pi\in S_{N-1}} \int_{A^\pi} \prod_{s=1}^{N-1} f_t(y_0^s, y^s)\, dy \\ = \sum_{\pi\in S_{N-1}} \int_A \prod_{s=1}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy \tag{5.12}$$

and for additional arbitrary $i \neq j \in \{1,\ldots,N-1\}$ using that $\pi(i) \neq \pi(j)$ for $\pi \in S_{N-1}$

$$\int_{\bigcup_{\pi\in S_{N-1}} A^\pi} f_t(y_0^i, y^j) \prod_{\substack{s=1\\ s\neq j}}^{N-1} f_t(y_0^s, y^s)\, dy = \sum_{\pi\in S_{N-1}} \int_{A^\pi} f_t(y_0^i, y^j) \prod_{\substack{s=1\\ s\neq j}}^{N-1} f_t(y_0^s, y^s)\, dy \\ = \sum_{\pi\in S_{N-1}} \int_A f_t(y_0^i, y^{\pi(j)}) \prod_{\substack{s=1\\ s\neq j}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy \\ = \sum_{\pi\in S_{N-1}} \int_A f_t(y_0^i, y^{\pi(j)}) f_t(y_0^i, y^{\pi(i)}) \prod_{\substack{s=1\\ s\notin\{i,j\}}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy \\ = \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)<\pi(j)}} \int_A f_t(y_0^i, y^{\pi(j)}) f_t(y_0^i, y^{\pi(i)}) \prod_{\substack{s=1\\ s\notin\{i,j\}}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy \\ + \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)>\pi(j)}} \int_A f_t(y_0^i, y^{\pi(j)}) f_t(y_0^i, y^{\pi(i)}) \prod_{\substack{s=1\\ s\notin\{i,j\}}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy. \tag{5.13}$$

Each $\pi \in S_{N-1}$ with $\pi(i) < \pi(j)$ can be mapped to $\widetilde{\pi} := \widetilde{\pi}(\pi) \in S_{N-1}$ via

$$\widetilde{\pi}(s) = \begin{cases} \pi(s), & \text{for } s \notin \{i,j\} \\ \pi(j), & \text{for } s = i, \\ \pi(i), & \text{for } s = j \end{cases} \tag{5.14}$$

and it holds

$$\{\pi \in S_{N-1} : \pi(i) > \pi(j)\} = \{\widetilde{\pi} \in S_{N-1} : \pi(i) < \pi(j)\}. \tag{5.15}$$

As a consequence,

$$\begin{aligned}
&\sum_{\substack{\pi \in S_{N-1} \\ \pi(i) > \pi(j)}} \int_A f_t(y_0^i, y^{\pi(j)}) f_t(y_0^i, y^{\pi(i)}) \prod_{\substack{s=1 \\ s \notin \{i,j\}}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy \\
&= \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) < \pi(j)}} \int_A f_t(y_0^i, y^{\widetilde{\pi}(j)}) f_t(y_0^i, y^{\widetilde{\pi}(i)}) \prod_{\substack{s=1 \\ s \notin \{i,j\}}}^{N-1} f_t(y_0^s, y^{\widetilde{\pi}(s)})\, dy \\
&= \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) < \pi(j)}} \int_A f_t(y_0^i, y^{\pi(i)}) f_t(y_0^i, y^{\pi(j)}) \prod_{\substack{s=1 \\ s \notin \{i,j\}}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy.
\end{aligned}$$

In view of the sum in the last expression of (5.13) this implies

$$\begin{aligned}
&\int_{\bigcup_{\pi \in S_{N-1}} A^\pi} f_t(y_0^i, y^j) \prod_{\substack{s=1 \\ s \neq j}}^{N-1} f_t(y_0^s, y^s)\, dy \\
&= 2 \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) < \pi(j)}} \int_A f_t(y_0^i, y^{\pi(j)}) f_t(y_0^i, y^{\pi(i)}) \prod_{\substack{s=1 \\ s \notin \{i,j\}}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy.
\end{aligned} \tag{5.16}$$

Plugging equations (5.12) and (5.16) into expression (5.11) finally yields

$$\begin{aligned}
&\frac{1}{N-1} \sum_{i=1}^{N-1} \Bigg(2 \int_0^\infty \int_{\bigcup_{\pi \in S_{N-1}} A^\pi} \prod_{s=1}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^i}(t)\, dt \\
&\qquad + \sum_{\substack{j=1 \\ j \neq i}}^{N-1} \int_0^\infty \int_{\bigcup_{\pi \in S_{N-1}} A^\pi} f_t(y_0^i, y^j) \prod_{\substack{s=1 \\ s \neq j}}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^j}(t)\, dt \Bigg) \\
&= \frac{1}{N-1} \sum_{i=1}^{N-1} \Bigg(2 \int_0^\infty \sum_{\pi \in S_{N-1}} \int_A \prod_{s=1}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dy\, g_{y_0^i}(t)\, dt
\end{aligned}$$

$$+\sum_{\substack{j=1\\j\neq i}}^{N-1}\int_0^\infty 2\sum_{\substack{\pi\in S_{N-1}\\\pi(i)<\pi(j)}}\int_A f_t(y_0^i,y^{\pi(j)})f_t(y_0^i,y^{\pi(i)})\prod_{\substack{s=1\\s\notin\{i,j\}}}^{N-1} f_t(y_0^s,y^{\pi(s)})\,dy\,g_{y_0^j}(t)\,dt\Bigg)$$

$$=\frac{2}{N-1}\sum_{i=1}^{N-1}\Bigg(\int_0^\infty\sum_{\pi\in S_{N-1}}\int_A f_t(y_0^i,y^{\pi(i)})\prod_{\substack{s=1\\s\neq i}}^{N-1} f_t(y_0^s,y^{\pi(s)})\,dy\,g_{y_0^i}(t)\,dt$$

$$+\sum_{\substack{j=1\\j\neq i}}^{N-1}\int_0^\infty\sum_{\substack{\pi\in S_{N-1}\\\pi(i)<\pi(j)}}\int_A f_t(y_0^i,y^{\pi(j)})\prod_{\substack{s=1\\s\neq j}}^{N-1} f_t(y_0^s,y^{\pi(s)})\,dy\,g_{y_0^j}(t)\,dt\Bigg)$$

$$=\frac{2}{N-1}\sum_{i,j=1}^{N-1}\int_0^\infty\sum_{\substack{\pi\in S_{N-1}\\\pi(j)\leq\pi(i)}}\int_A f_t(y_0^i,y^{\pi(j)})\prod_{\substack{s=1\\s\neq j}}^{N-1} f_t(y_0^s,y^{\pi(s)})\,dy\,g_{y_0^j}(t)\,dt.$$

An application of Fubini's theorem finishes the proof.

□

Remark 5.31.
- Accounting for the time length and using polar coordinates the Markov transition from (U_n) to $(U_{n+1},R_{n+1},T_{n+1})$ as in Definition 5.26 is thereby given by the density

$$\widetilde{h}_{u_0}(u,r,t)=\frac{2r^{N-2}}{N-1}\sum_{i,j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\\pi(i)\leq\pi(j)}} g_{u_0^j}(t)\,f_t(u_0^i,r\cdot u^{\pi(j)})\prod_{\substack{s=1\\s\neq j}}^{N-1} f_t(u_0^s,r\cdot u^{\pi(s)})\quad(5.17)$$

  for $u_0,u\in\mathsf{S}$ as in Definition 5.8 and $r,t\in(0,\infty)$ with the factor r^{N-2} accounting for the functional determinant of the coordinate transformation.

- The index i accounts for the position y_0^i that is replicated in the beginning and the index j accounts for the position y_0^j from which the one (out of possibly two if $j=i$) particle starts from that will be the first to be killed upon touching 0.

- Writing

$$\delta_{i,j}:=\begin{cases}1, & \text{for } i=j\\ 0, & \text{for } i\neq j\end{cases}$$

  for the Kronecker-Delta the representation

$$h_{y_0}(y)=\frac{1}{N-1}\int_0^\infty\sum_{\pi\in S_{N-1}}\sum_{i,j=1}^{N-1}(1+\delta_{i,j})\frac{g_{y_0^j}(t)\,f_t(y_0^i,y^{\pi(j)})}{f_t(y_0^j,y^{\pi(j)})}\prod_{s=1}^{N-1} f_t(y_0^s,y^{\pi(s)})\,dt$$

  inherits the fact

$$h_{y_0^{\pi_0}}(y^\pi)=h_{y_0}(y)$$

for any $y_0, y \in (0,\infty)^{N-1\downarrow}$ and $\pi_0, \pi \in S_{N-1}$. (Since we knew this by the previous analysis and construction we actually defined the transition kernel h_{y_0} for $y_0 \in (0,\infty)^{N-1\downarrow}$ and on $(0,\infty)^{N-1\downarrow}$ only.)

Definition 5.32. Let $0 < w := 1 - \nu/2$ an alternative representation of the drift parameter in the generalized Bessel processes.

Lemma 5.33. *(Cf. [45, Theorem 8 (ii)].) Using the parameterization with $w > 0$ the density function $h_{y_0}(y)$ in Lemma 5.30 may be computed to*

$$h_{y_0}(y) = \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s!\right]} \times \right.$$
$$\left. \times \prod_{s=1}^{N-1} (y^s)^{2k_s} \sum_{i=1}^{N-1} \frac{\left(y_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(y_0^{\pi(s)}\right)^{2k_s} \right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1} \left[(y_0^s)^2 + (y^s)^2 \right] \right)^{(w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s}} \right].$$

Proof. According to [12, Appenix 1.21] for $w > 0$ the transition density of a generalized Bessel process stopped in the origin starting at $x > 0$ and moving to $y > 0$ at time $t > 0$ is given by

$$\begin{aligned} &\mathbb{P}_x(X_t^1 \in dy, \min\{X_s : s \le t\} > 0) \\ &=: f_t(x,y) = \frac{x^{2w} y}{2^w t^{w+1}} \exp\left(-\frac{x^2 + y^2}{2t}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{xy}{2t}\right)^{2k}}{k!\,\Gamma(k+1+w)} \end{aligned} \tag{5.18}$$

and according to [38, Proposition 2.9] or to [27, Expression (15)] which is true for $\nu < 0$ also, the density of the hitting time of 0 is given by

$$\mathbb{P}_x(\inf\{t > 0 : X_t^1 = 0\} \in dt) =: g_x(t) = \frac{x^{2w}}{2^w t^{w+1} \Gamma(w)} \exp\left(-\frac{x^2}{2t}\right). \tag{5.19}$$

Using monotone convergence theorem we generally rearrange expressions of the form $\int_0^\infty \prod_{j=1}^{N-1} f_t\left(x^j, y^j\right) g_{x^N}(t)\, dt$ with all variables positive:

$$\int_0^\infty \prod_{j=1}^{N-1} f_t\left(x^j, y^j\right) g_{x^N}(t)\, dt = \frac{\left(\prod_{j=1}^N x^j\right)^{2w} \prod_{j=1}^{N-1} y^j}{2^{Nw}\Gamma(w)} \times$$
$$\int_0^\infty t^{-N(w+1)} \exp\left(-\frac{\sum_{j=1}^N \left(x^j\right)^2 + \sum_{j=1}^{N-1} \left(y^j\right)^2}{2t}\right) \prod_{j=1}^{N-1} \sum_{k=0}^{\infty} \frac{\left(\frac{x^j y^j}{2t}\right)^{2k}}{k!\Gamma(k+w+1)}\, dt$$
$$= \frac{\left(\prod_{j=1}^N x^j\right)^{2w} \prod_{j=1}^{N-1} y^j}{2^{Nw}\Gamma(w)} \int_0^\infty \left[t^{-N(w+1)} \exp\left(-\frac{\sum_{j=1}^N \left(x^j\right)^2 + \sum_{j=1}^{N-1} \left(y^j\right)^2}{2t}\right) \times \right.$$

$$\times \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \prod_{j=1}^{N-1} \frac{\left(\frac{x^j y^j}{2t}\right)^{2k_j}}{k_j!\Gamma(k_j+w+1)} \Bigg] dt$$

$$= \frac{\left(\prod_{j=1}^{N} x^j\right)^{2w} \prod_{j=1}^{N-1} y^j}{2^{Nw}\Gamma(w)} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \Bigg[\left(\prod_{j=1}^{N-1} \frac{\left(\frac{x^j y^j}{2}\right)^{2k_j}}{k_j!\Gamma(k_j+w+1)} \right) \times$$

$$\times \int_0^{\infty} t^{-N(w+1)-2\sum_{j=1}^{N-1} k_j} \exp\left(-\frac{\sum_{j=1}^{N} \left(x^j\right)^2 + \sum_{j=1}^{N-1} \left(y^j\right)^2}{2t} \right) dt \Bigg].$$

Applying Lemma A.1 results for the last integral in

$$\int_0^{\infty} t^{-N(w+1)-2\sum_{j=1}^{N-1} k_j} \exp\left(-\frac{\sum_{j=1}^{N} \left(x^j\right)^2 + \sum_{j=1}^{N-1} \left(y^j\right)^2}{2t} \right) dt$$

$$= \Gamma\left(-1 + N(w+1) + 2\sum_{j=1}^{N-1} k_j \right) \cdot \left(\frac{2}{\sum_{j=1}^{N} \left(x^j\right)^2 + \sum_{j=1}^{N-1} \left(y^j\right)^2} \right)^{-1+N(w+1)+2\sum_{j=1}^{N-1} k_j},$$

whence

$$\frac{\left(\prod_{j=1}^{N} x^j\right)^{2w} \prod_{j=1}^{N-1} y^j}{2^{Nw}\Gamma(w)} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \Bigg[\left(\prod_{j=1}^{N-1} \frac{\left(\frac{x^j y^j}{2}\right)^{2k_j}}{k_j!\Gamma(k_j+w+1)} \right) \times$$

$$\times \int_0^{\infty} t^{-N(w+1)-2\sum_{j=1}^{N-1} k_j} \exp\left(-\frac{\sum_{j=1}^{N} \left(x^j\right)^2 + \sum_{j=1}^{N-1} \left(y^j\right)^2}{2t} \right) dt \Bigg]$$

$$= \frac{2^{N-1}\left(\prod_{j=1}^{N} x^j\right)^{2w} \prod_{j=1}^{N-1} y^j}{\Gamma(w)} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \Bigg[\frac{\Gamma\left(-1+N(w+1)+2\sum_{j=1}^{N-1} k_j\right)}{\prod_{j=1}^{N-1} \Gamma(k_j+w+1) k_j!} \times$$

$$\times \frac{\prod_{j=1}^{N-1} \left(x^j y^j\right)^{2k_j}}{\left(\sum_{j=1}^{N} (x^j)^2 + \sum_{j=1}^{N-1} (y^j)^2\right)^{-1+N(w+1)+2\sum_{j=1}^{N-1} k_j}} \Bigg].$$

In view of Lemma 5.30 this implies for the transition density using the identity $\prod_{s=1}^{N-1} y^{\pi(s)} = \prod_{s=1}^{N-1} y^s$ for arbitrary $\pi \in S_{N-1}$

$$h_{y_0}(y) = \frac{2}{N-1} \sum_{i,j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \int_0^{\infty} f_t(y_0^i, y^{\pi(j)}) \prod_{\substack{s=1 \\ s \ne j}}^{N-1} f_t(y_0^s y^{\pi(s)})\, g_{y_0^j}(t)\, dt$$

$$= \frac{2}{N-1} \sum_{i,j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \frac{2^{N-1}\left(y_0^i \prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^{\pi(s)}}{\Gamma(w)} \times$$

$$\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \times \right.$$
$$\left. \times \frac{\left(y_0^i y^{\pi(j)}\right)^{2k_j} \prod_{\substack{s=1\\ s\neq j}}^{N-1} \left(y_0^s y^{\pi(s)}\right)^{2k_s}}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2 + \left(y^{\pi(s)}\right)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \right]$$
$$= \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \times$$
$$\times \sum_{i,j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \times \right.$$
$$\left. \times \frac{\left(y_0^i\right)^{2w}\left[\left(y_0^i/y_0^j\right)^{2k_j} \prod_{s=1}^{N-1} \left(y_0^s y^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \right]. \quad (5.20)$$

Permuting the order of $\{k_1,\dots,k_{N-1}\}$ by $\pi\in S_{N-1}$ features

$$\sum_{i,j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \times \right.$$
$$\left. \times \frac{\left(y_0^i\right)^{2w}\left[\left(y_0^i/y_0^j\right)^{2k_j} \prod_{s=1}^{N-1} \left(y_0^s y^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \right]$$
$$= \sum_{i,j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_{\pi(s)}\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_{\pi(s)}+w+1)k_{\pi(s)}!\right]} \times \right.$$
$$\left. \times \frac{\left(y_0^i\right)^{2w}\left[\left(y_0^i/y_0^j\right)^{2k_{\pi(j)}} \prod_{s=1}^{N-1} \left(y_0^s y^{\pi(s)}\right)^{2k_{\pi(s)}}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_{\pi(s)}}} \right]$$
$$= \sum_{i,j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \times \right.$$

$$
\times \frac{\left(y_0^i\right)^{2w} \left[\left(y_0^i\right)^{2k_{\pi(j)}} \prod_{\substack{s=1\\ s\neq j}}^{N-1} (y_0^s)^{2k_{\pi(s)}} \cdot \prod_{s=1}^{N-1} (y^s)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}}\Bigg]
$$

$$
= \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \prod_{s=1}^{N-1} (y^s)^{2k_s} \times \right.
$$

$$
\left. \times \sum_{i=1}^{N-1} \frac{\left(y_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi(j)}} \prod_{\substack{s=1\\ s\neq\pi(j)}}^{N-1} \left(y_0^{\pi^{-1}(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}}\right]. \tag{5.21}
$$

Again using definition (5.14) it holds for $i, j \in \{1, \ldots, N-1\}$

$$
\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)>\pi(j)}} \left[\left(y_0^i\right)^{2k_{\widetilde{\pi}(j)}} \prod_{\substack{s=1\\ s\neq\widetilde{\pi}(j)}}^{N-1} \left(y_0^{\widetilde{\pi}^{-1}(s)}\right)^{2k_s}\right] = \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)>\pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi(i)}} \prod_{\substack{s=1\\ s\neq\pi(i)}}^{N-1} \left(y_0^{\widetilde{\pi}^{-1}(s)}\right)^{2k_s}\right]
$$

$$
= \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)>\pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi(i)}} \left(y_0^i\right)^{2k_{\pi(j)}} \prod_{\substack{s=1\\ s\notin\{\pi(i),\pi(j)\}}}^{N-1} \left(y_0^{\pi^{-1}(s)}\right)^{2k_s}\right]
$$

$$
= \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)>\pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi(j)}} \prod_{\substack{s=1\\ s\neq\pi(j)}}^{N-1} \left(y_0^{\pi^{-1}(s)}\right)^{2k_s}\right]
$$

and therefore by the identity (5.15) it follows

$$
\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi(j)}} \prod_{\substack{s=1\\ s\neq\pi(j)}}^{N-1} \left(y_0^{\pi^{-1}(s)}\right)^{2k_s}\right]
$$

$$
= \sum_{\pi\in S_{N-1}} \frac{1+\delta_{i,j}}{2} \left[\left(y_0^i\right)^{2k_{\pi(j)}} \prod_{\substack{s=1\\ s\neq\pi(j)}}^{N-1} \left(y_0^{\pi^{-1}(s)}\right)^{2k_s}\right]
$$

$$
= \sum_{\pi\in S_{N-1}} \frac{1+\delta_{i,j}}{2} \left[\left(y_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(y_0^{\pi(s)}\right)^{2k_s}\right]
$$

$$= \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(y_0^{\pi(s)}\right)^{2k_s} \right]. \tag{5.22}$$

Turning back to expression (5.20) and inserting equations (5.21) and (5.22) we can finally transform to

$$\begin{aligned}
h_{y_0}(y) = {} & \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \times \\
& \times \sum_{i,j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \Bigg[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s!\right]} \times \\
& \times \frac{\left(y_0^i\right)^{2w} \left[\left(y_0^i / y_0^j\right)^{2k_j} \prod_{s=1}^{N-1} \left(y_0^s y^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1} \left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \Bigg] \\
= {} & \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \Bigg[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s!\right]} \times \\
& \times \prod_{s=1}^{N-1} (y^s)^{2k_s} \sum_{i=1}^{N-1} \frac{\left(y_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi(j)}} \prod_{\substack{s=1 \\ s \neq \pi(j)}}^{N-1} \left(y_0^{\pi^{-1}(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1} \left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \Bigg] \\
= {} & \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \Bigg[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s!\right]} \times \\
& \times \prod_{s=1}^{N-1} (y^s)^{2k_s} \sum_{i=1}^{N-1} \frac{\left(y_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(y_0^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1} \left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \Bigg].
\end{aligned}$$

□

Remark 5.34. Let us quote [45, Theorem 8]: "Let $Z_1, Z_2, \dots, Z_n$ be independent BESQ processes of dimensions $-\theta_1, \dots, -\theta_n$, where each $\theta_i \ge 0$. Assume that $Z_i(0) = z_i(0) > 0$, for every i. The distribution of $(\tau, Z(\tau))$ is supported on the set $(0, \infty) \times \cup_{i=1}^n H_i$, where H_i is the subspace orthogonal to the ith canonical basis vector e_i. That is,

$$H_i = \{(y_1, y_2, \dots, y_n) : y_i = 0\}.$$

(i) Let G_i, $i = 1, 2, \dots, n$ be independent Gamma random variables with parameters $\theta_i/2 + 1$, $i = 1, 2, \dots, n$. The law of τ is the same as that of $\min_i \frac{z_i}{2G_i}$ and

$$\mathbb{P}(\tau = T_i) = \mathbb{P}\left(\frac{G_i}{z_i} > \frac{G_j}{z_j} \text{ for all } j \neq i\right),$$

where T_i is the first hitting time of H_i.

(ii) The restriction of the law of the random vector $Z(\tau)$, restricted to the hyperplane H_i, admits a density with respect to all the variables y_j's, $j \neq i$, which is given by

$$\begin{aligned} =&\frac{S^{1-\theta_0/2-2n}}{\Gamma(\theta_i/2+1)} \prod_{j=1}^{n} z_j^{\theta_j/2+1} \sum_{N=0}^{\infty} \Gamma(\theta_0/2 + 2n + 2N - 1) S^{-2N} \\ &\times \sum_{\sum_{j\neq i} k_j = N} \prod_{j \neq i} \frac{(y_j z_j)^{k_j}}{k_j! \Gamma(\theta_j/2 + 2 + k_j)}. \end{aligned} \tag{5.23}$$

Here

$$S = \sum_{i=1}^{n} (y_i + z_i), \quad y_i = 0, \quad \theta_0 = \sum_{i=1}^{n} \theta_i.\text{''}$$

For reference we may call the restricted density (5.23) given in (ii)

$$q_{(z_1,\dots,z_n)}^{i,(\theta_1,\dots,\theta_n)}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n);$$

in the notation of the quoted theorem it holds

$$\begin{aligned} &\sum_{i=1}^{n} \int_{H_i \cap [0,\infty)^n} q_{(z_1,\dots,z_n)}^{i,(\theta_1,\dots,\theta_n)}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)\, dy_n \cdots dy_{i+1}\, \delta_0(dy_i)\, dy_{i-1} \cdots dy_1 \\ &= \sum_{i=1}^{n} \int_{(0,\infty)^{n-1}} q_{(z_1,\dots,z_n)}^{i,(\theta_1,\dots,\theta_n)}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)\, dy_n \cdots dy_{i+1}\, dy_{i-1} \cdots dy_1 = 1. \end{aligned}$$

The author of [45] considered squared Bessel processes, so we transform

$$g_{(z_1,\dots,z_n)}^{i,(\theta_1,\dots,\theta_n)}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) := q_{(z_1^2,\dots,z_n^2)}^{i,(\theta_1,\dots,\theta_n)}(y_1^2, \dots, y_{i-1}^2, y_{i+1}^2, \dots, y_n^2) \prod_{\substack{s=1 \\ s \neq i}}^{n} (2y_s).$$

Accounting for the change of parametrization in Definition 5.32 to $-\theta = -\nu = 2(w-1)$ it can be verified by explicit calculation that it holds

$$h_{y_0}(y) = \frac{1}{N-1} \sum_{i=1}^{N-1} \sum_{j=1}^{N} \sum_{\pi \in S_{N-1}} g_{(y_0^1,\dots,y_0^{N-1},y_0^i)}^{j,(2(w-1),\dots,2(w-1))}(y^\pi),$$

which is carried out in the appendix.

Definition 5.35. Let $\sigma := \sigma^{N-2}$ the $(N-2)$-dimensional Riemannian measure on the sphere as $(N-2)$-dimensional manifold in $\mathbb{R}^{N-1}$.

Lemma 5.36. *The transition density of the chain $(U_n)_{n\in\mathbb{N}_0}$ from Definition 5.26 with respect to the Riemannian measure σ may be expressed as*

$$p(u_0,u) := \mathbb{P}_{u_0}(U_1 \in du)/d\sigma(u_0) = \int_0^\infty \int_0^\infty \widetilde{h}_{u_0}(u,r,t)\,dt\,dr = \int_0^\infty r^{N-2} h_{u_0}(r\cdot u)\,dr$$

$$= \frac{2^{N-1}\left(\prod_{s=1}^{N-1} u_0^s\right)^{2w} \prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times$$

$$\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left(\sum_{s=1}^{N-1} k_s + Nw\right)\left(\sum_{s=1}^{N-1} k_s + N - 2\right)!}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \prod_{s=1}^{N-1} (u^s)^{2k_s} \times \right.$$

$$\left. \times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s}\right]}{\left(1+\left(u_0^i\right)^2\right)^{\sum_{s=1}^{N-1} k_s + Nw}} \right].$$

Proof. By applying Lemma A.5 in the Appendix we attain

$$\int_0^\infty r^{N-2} h_{u_0}(r\cdot u)\,dr$$

$$= \int_0^\infty r^{N-2} \frac{2^N\left(\prod_{s=1}^{N-1} u_0^s\right)^{2w} \prod_{s=1}^{N-1}(r\cdot u^s)}{(N-1)\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \times \right.$$

$$\left. \times \prod_{s=1}^{N-1} (r\cdot u^s)^{2k_s} \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(u_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(u_0^s)^2 + (r\cdot u^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \right] dr$$

$$= \frac{2^N\left(\prod_{s=1}^{N-1} u_0^s\right)^{2w} \prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times$$

$$\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \prod_{s=1}^{N-1} (u^s)^{2k_s} \times \right.$$

$$\times \sum_{i=1}^{N-1} \left[\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s}\right] \times \right.$$

$$\left.\left. \times \int_0^\infty \frac{r^{2N-3+2\sum_{s=1}^{N-1} k_s}}{\left(\left(u_0^i\right)^2 + 1 + r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}}\,dr \right]\right]$$

$$= \frac{2^{N-1}\left(\prod_{s=1}^{N-1} u_0^s\right)^{2w} \prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times$$
$$\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left(\sum_{s=1}^{N-1} k_s + Nw\right)\left(\sum_{s=1}^{N-1} k_s + N-2\right)!}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \prod_{s=1}^{N-1} (u^s)^{2k_s} \times \right.$$
$$\left. \times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \ne \pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s} \right]}{\left(1+\left(u_0^i\right)^2\right)^{\sum_{s=1}^{N-1} k_s + Nw}} \right].$$

□

Remark 5.37. By construction, the index $i \in \{1, \dots, N-1\}$ indicates which particle replicates and from the position u_0^j one (out of at most two if $j = i$) particle dies. Particularly, the mappings

$$p^i : (u_0, u) \mapsto \mathbb{P}_{u_0}(U_1 \in d\sigma(u), \text{particle } i \text{ replicates})/d\sigma(u) = \frac{2^{N-1}\left(\prod_{s=1}^{N-1} u_0^s\right)^{2w} \prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times$$
$$\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left(\sum_{s=1}^{N-1} k_s + Nw\right)\left(\sum_{s=1}^{N-1} k_s + N-2\right)!}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \prod_{s=1}^{N-1} (u^s)^{2k_s} \times \right.$$
$$\left. \times \frac{\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \ne \pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s} \right]}{\left(1+\left(u_0^i\right)^2\right)^{\sum_{s=1}^{N-1} k_s + Nw}} \right]$$

define transition densities to subkernels with total mass $1/(N-1)$: For $u, u_0 \in \mathsf{S}$ it holds

$$p(u_0, u) = \sum_{i=1}^{N-1} p^i(u_0, u)$$

and for $u_0 \in \mathsf{S}$ and $i \in \{1, \dots, N-1\}$ it holds

$$\int_{\mathsf{S}} p^i(u_0, u)\, du = 1/(N-1).$$

Remark 5.38. If we define given indexes $i, j \in \{1, \dots, N-1\}$, a permutation $\pi \in S_{N-1}$ and $u_0 \in \mathsf{S}$ the symbol $\widetilde{u_0}_{\pi,j} := \widetilde{u_0}_{\pi,j,i}(u_0)$ by

$$\widetilde{u_0}^s_{\pi,j,i} := \begin{cases} u_0^{\pi(s)}, & s \ne \pi^{-1}(j), \\ u_0^i, & s = \pi^{-1}(j), \end{cases}$$

and additionally given $u \in \mathsf{S}$ define $x_{\pi,j,i} := x_{\pi,j,i}(u_0, u)$ by

$$x^s_{\pi,j,i} := \frac{(u^s)^2}{1 + \left(u_0^i\right)^2} \cdot \left(\widetilde{u_0}^s_{\pi,j,i}\right)^2$$

and

$$F(x) = F(x^1, \ldots, x^{N-1}) := F_C^{(N-1)}(Nw, N-1, w+1, w+1, \ldots, w+1; (x^s)_{s=1}^{N-1})$$

where $F_C^{(N-1)}$ denotes the C-type Lauricella hypergeometric series as exploited in Definition A.8 in the appendix, we find the alternative representation

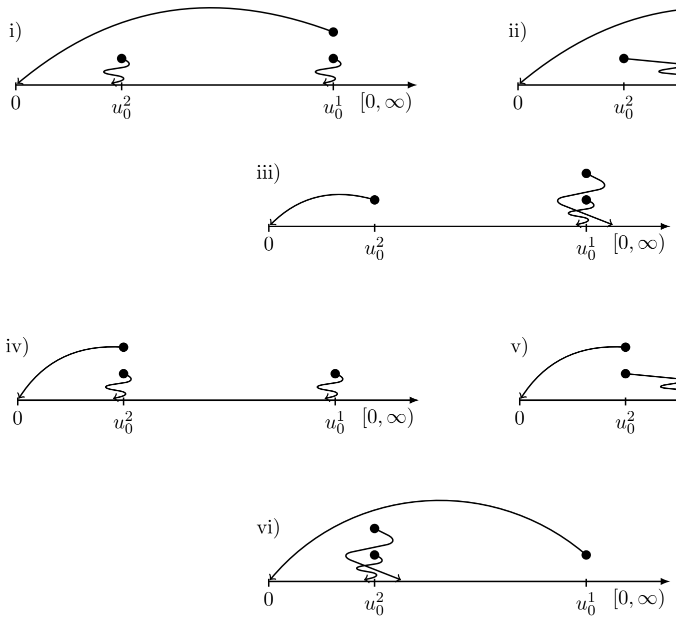
$$\begin{aligned} p(u_0, u) = {} & \frac{2^{N-1}\Gamma(Nw)\Gamma(N-1)}{(N-1)\Gamma(w)\Gamma(w+1)^{N-1}} \left(\prod_{s=1}^{N-1} u_0^s\right)^{2w} \prod_{s=1}^{N-1} u^s \\ & \times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w}}{\left(1 + \left(u_0^i\right)^2\right)^{Nw}} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} F(x_{\pi,j,i}(u_0, u)). \end{aligned}$$

We see that p is a positive continuous function in $(u_0, u) \in \mathsf{S}^2$ as F is continuous on $\{x \in \mathbb{R}^{N-1} : \sum_{s=1}^{N-1} \sqrt{|x^s|} < 1\}$. The latter is discussed in Proposition A.11 in the appendix.

Example 5.39. In the case of $N = 3$ the expression in Lemma 5.36 reads

$$\begin{aligned} p(u_0, u) = {} & \frac{2\left(u_0^1\right)^{2w}\left(u_0^1\right)^{2w} u^1 u^2}{(N-1)\Gamma(w)} \times \\ & \times \sum_{k_1,k_2=0}^{\infty} \left[\frac{\Gamma(k_1 + k_2 + 3w) \cdot (k_1 + k_2 + 1)!}{\Gamma(k_1 + w + 1)\,\Gamma(k_2 + w + 1)} \frac{\left(u^1\right)^{2k_1}}{k_1!} \frac{\left(u^2\right)^{2k_2}}{k_2!} \times \right. \\ & \times \left(\frac{\left(u_0^1\right)^{2w} \left[\left(u_0^1\right)^{2k_1} \left(u_0^2\right)^{2k_2} \boldsymbol{+} \left(u_0^2\right)^{2k_1} \left(u_0^1\right)^{2k_2} \boldsymbol{+} \left(u_0^1\right)^{2k_1} \left(u_0^1\right)^{2k_2} \right]}{\left(1 + \left(u_0^1\right)^2\right)^{k_1 + k_2 + 3w}} \right. \\ & \left.\left. + \frac{\left(u_0^2\right)^{2w} \left[\left(u_0^1\right)^{2k_1} \left(u_0^2\right)^{2k_2} \boldsymbol{+} \left(u_0^2\right)^{2k_1} \left(u_0^1\right)^{2k_2} \boldsymbol{+} \left(u_0^2\right)^{2k_1} \left(u_0^2\right)^{2k_2} \right]}{\left(1 + \left(u_0^2\right)^2\right)^{k_1 + k_2 + 3w}} \right) \right]. \end{aligned}$$

These six summands with their plus sign bold faced correspond to the following cases:



The figures in the first line illustrate $i = j = 1$ where in the left one the surviving particles preserve their order and on the right they switch. The figure in the second line of the display shows the situation $i = 1, j = 2$. In the third line, the particle from position u_0^2 replicated ($i = 2$) and also one of the two particles starting in u_0^2 dies ($j = 2$). Finally, in last figure, $i = 2$, $j = 1$.

5.3. Markov chains in general state space

We work in the framework of Markov chains in uncountable state space as laid out in [22, Chapters 9 to 11, 15]. From there we collect some definitions and theorems for later reference. We omit the proofs and refer to the book [22]. Some of the notions and concepts the reader may be familiar with in the context of (discrete time) Markov chains on finite or countable state spaces can be adjusted to properly fit to our situation of a more general space.

Notation 5.40. (Cf. [22, Definition 3.1.8, Definition 3.2.2 and Definition 4.2.1].) In this section we write $(\mathsf{X}, \mathcal{X})$ for an arbitrary measurable space, $\theta : \mathsf{X}^{\mathbb{N}_0} \to \mathsf{X}^{\mathbb{N}_0}$ for the shift operator

$$\omega = (\omega_0, \omega_1, \omega_2, \ldots) \mapsto \theta(\omega) = (\omega_1, \omega_2, \ldots)$$

and for $A \in \mathcal{X}$, the first hitting time τ_A, the return time σ_A and the number of visits N_A to the set A by the process $(X_n)_{n\in\mathbb{N}_0}$ are defined, respectively, by

$$\tau_A := \inf\{n \in \mathbb{N}_0 : X_n \in A\},$$

$$\sigma_A := \inf\{n \in \mathbb{N} : X_n \in A\}$$

and

$$N_A := \sum_{k=0}^{\infty} \mathbb{1}_A(X_k).$$

The definition of a set to be called accessible can be transferred literally.

Definition 5.41. (Cf. [22, Definition 3.5.1].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$.

(i) A set $A \in \mathcal{X}$ is said to be *accessible* if $\mathbb{P}_x(\sigma_A < \infty) > 0$ for all $x \in \mathcal{X}$.

(ii) The collection of all accessible sets is denoted by $\mathcal{X}_P^+$.

In order to apply Birkhoff's ergodic theorem it is desirable to have a sufficient criterion for ergodicity which is given by the next theorem. As remarked in [22, page 107] the uniqueness of the invariant probability measure is a sufficient but not a necessary condition for ergodicity (see [22, Exercise 5.8]).

Theorem 5.42. *(Cf. [22, Theorem 5.2.6].) Let P be a Markov kernel on $\mathsf{X}\times\mathcal{X}$ admitting a unique invariant probability measure π. Then the dynamical system $(\mathsf{X}^{\mathbb{N}_0}, \mathcal{X}^{\otimes \mathbb{N}_0}, \mathbb{P}_\pi, \theta)$ is ergodic.*

With further integrability assumptions we can finally formulate Birkhoff's theorem for Markov chains (on general state space) yielding a law of large numbers.

Theorem 5.43 (Birkhoff's theorem for Markov chains)**.** *(Cf. [22, Theorem 5.2.9].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$ and assume P admits an invariant probability measure π such that $(\mathsf{X}^{\mathbb{N}_0}, \mathcal{X}^{\otimes \mathbb{N}_0}, \mathbb{P}_\pi, \theta)$ is ergodic. Let $Y \in L^1(\mathbb{P}_\pi)$. Then for π-almost all $x \in \mathcal{X}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} Y \circ \theta_k = \mathbb{E}_\pi[Y] \qquad \mathbb{P}_x - a.s.$$

If a set has positive measure under the unique invariant probability it should be visited infinitely often almost surely from almost every start point. The next theorem states this precisely.

Theorem 5.44. *(Cf. [22, Theorem 5.2.13].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$ admitting a unique invariant probability measure π. Let $A \in \mathcal{X}$ be such that $\pi(A) > 0$. Then,*

$$\mathbb{P}_x(N_A = \infty) = 1,$$

for π-almost every $x \in \mathcal{X}$.

We need the following concept which generalizes the notation of what sometimes is referred to as an *atom*. It gives some regeneration control so that upon entering certain sets we do not need to know where we are exactly within this set.

Definition 5.45. (Cf. [22, Definition 9.1.1 (small set)].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$. A set $C \in \mathcal{X}$ is called a *small set* if there exist $m \in \mathbb{N}$ and a nonzero measure μ on $(\mathsf{X}, \mathcal{X})$ such that for all $x \in C$ and $A \in \mathcal{X}$,

$$P^m(x, A) \geq \mu(A). \tag{5.24}$$

The set C is then said to be an (m, μ)*-small set.*

As outlined in [22, page 191], the definition entails that μ is a finite measure and $0 < \mu(\mathsf{X}) \leq 1$. Hence it can be written $\mu = \mu(\mathsf{X})\,\nu$ and ν is a probability measure. If $\mu(\mathsf{X}) = 1$, then equality must hold in equation (5.24).
For small sets we introduce the following.

Definition 5.46. (Cf. [22, Definition 9.1.2].) An (m, μ)-small set C is said to be

- *strongly aperiodic* if $m = 1$ and $\mu(C) > 0$;
- *positive* if $\mathbb{E}_x[\sigma_C] < \infty$ for all $x \in C$.

In analogy as one would define irreducibility for Markov chains on discrete spaces we may replace accessible states by accessible small sets.

Definition 5.47. (Cf. [22, Definition 9.2.1 (irreducible kernel)].) A Markov kernel P on $\mathsf{X} \times \mathcal{X}$ is said to be *irreducible* if it admits an accessible small set.

As described in [22, page 194], the assumption of irreducibility may seem to be weak, but has some important consequences. The definition guarantees that a small set is always reached by a chain with some positive probability from any starting point.
There is also an equivalent characterization of irreducibility in terms of measures. It will turn out to justify the following nomenclature.

Definition 5.48. (Cf. [22, Defintion 9.2.2 (irreducibility measure)].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$. Let ϕ be a nontrivial σ-finite measure.

- ϕ is said to be an *irreducibility measure* if $\phi(A) > 0$ implies $A \in \mathcal{X}_P^+$.
- ϕ is said to be a *maximal irreducibility measure* if ϕ is an irreducibility measure and $A \in \mathcal{X}_P^+$ implies $\phi(A) > 0$.

As mentioned above there is the following result in the theory.

Theorem 5.49. *(Cf. [22, Theorem 9.2.4].) From any given irreducibility measure a maximal irreducibility measure can explicitly be constructed. All irreducibility measures are absolutely continuous with respect to every maximal irreducibility measure, and all maximal irreducibility measures are equivalent.*

There is also the following important result which has the subsequent corollary.

Theorem 5.50. *(Cf. [22, Theorem 9.2.15].) Let P be an irreducible Markov kernel. An invariant probability measure for P is a maximal irreducibility measure.*

Corollary 5.51. *(Cf. [22, Corollary 9.2.16].) If P is irreducible, then it admits at most one invariant probability measure.*

For irreducible Markov kernels we can now adapt the notion of periodicity to our situation.

Definition 5.52. (Cf. [22, Defintion 9.3.5 (period, aperiodicity, strong aperiodicity)].) Let P be an irreducible Markov kernel on $\mathsf{X} \times \mathcal{X}$.

- The common period of all accessible small sets is called the *period* of P.
- If the period is equal to one, the kernel is said to be *aperiodic*.
- If there exists an accessible $(1, \mu)$-small set C with $\mu(C) > 0$, the kernel is said to be *strongly aperiodic*.

The concept of small sets may even further be generalized as follows.

Definition 5.53. (Cf. [22, Definition 9.4.1 (petite set) and Definition 1.2.10 (Sampled kernel)].) A set $C \in \mathcal{X}$ is called *petite* if there exists a probability a on $\mathbb{N}_0$, that is, a sequence $(a_n)_{n\in\mathbb{N}_0}$ such that $a_n \geq 0$ for all $n \in \mathbb{N}_0$ and $\sum_{k=0}^{\infty} a_k = 1$, and a nonzero measure μ such that for all $x \in C$ and $A \in \mathcal{X}$

$$\sum_{n=0}^{\infty} a_n P^n(x, A) \geq \mu(A).$$

The set C is then said to be an (a, μ)-*petite* set.

An (m, μ)-small set is a (δ_m, μ)-petite set where δ_m is the Dirac delta distribution in $m \in \mathbb{N}$.
We now collect some results.

Lemma 5.54. *(Cf. [22, Lemma 9.4.8 (ii)].) Let P be an irreducible Markov kernel on $\mathsf{X} \times \mathcal{X}$, and C a petite set. Let r be a nonnegative increasing sequence such that $\lim_{n\to\infty} r_n = \infty$. Then every set $B \in \mathcal{X}$ such that $\sup_{x\in B} \mathbb{E}_x[r_{\tau_C}] < \infty$ is petite.*

Theorem 5.55. *(Cf. [22, Theorem 9.4.10].) If P is irreducible and aperiodic, then every petite set is small.*

As it will turn out later, the following notion of recurrence is well suited to our situation. For general state spaces one can consistently define the concept of recurrence but it turns out that this only implies an infinite number of visits to a set when started from this set *in expectation*. We use a stronger definition by requiring the infinite number of visits even *almost surely*.

Definition 5.56. (Cf. [22, Definition 10.2.1 (Harris recurrence)].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$.

(i) A set $A \in \mathcal{X}$ is said to be *Harris recurrent* if for all $x \in A$, $\mathbb{P}_x(N_A = \infty) = 1$.

(ii) The kernel P is said to be *Harris recurrent* if all accessible sets are Harris recurrent.

There is the following sufficient criterion for Harris recurrence which will turn out to be useful for us.

Proposition 5.57. *(Cf. [22, Proposition 10.2.4].) Let P be an irreducible Markov kernel on $\mathsf{X} \times \mathcal{X}$. If there exists a petite set C such that $\mathbb{P}_x(\sigma_C < \infty) = 1$ for all $x \notin C$, then P is Harris recurrent.*

We now state an important existence and uniqueness result for invariant measures of irreducible and recurrent (particularly Harris recurrent) Markov chains.

Theorem 5.58. *(Cf. [22, Proposition 11.2.5].) Let P be an irreducible and recurrent Markov kernel on $\mathsf{X} \times \mathcal{X}$. Then P admits a nonzero invariant measure λ, unique up to multiplication by a positive constant and such that $\lambda(C) < \infty$ for all petite sets C.*

We have already seen in Theorem 5.50 that an invariant probability measure is a maximal irreducibility measure. This property extends to possibly nonfinite measures.

Corollary 5.59. *(Cf. [22, Corollary 11.2.6].) Let P be an irreducible and recurrent Markov kernel on $\mathsf{X} \times \mathcal{X}$. Then an invariant measure is a maximal irreducibility measure.*

In the following we address the existence of an invariant *probability* measure instead of merely an invariant measure. We begin with a definition.

Definition 5.60. (Cf. [22, Definition 11.2.7 (positive and null Markov kernel].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$. If P is irreducible and admits an invariant probability measure π, the Markov kernel P is called *positive*. If P does not admit such a measure, then we call P *null*.

Theorem 5.61. *(Cf. [22, Corollary 11.2.9].) If P is an irreducible Markov kernel on $\mathsf{X} \times \mathcal{X}$ and if there exists a petite set C such that*

$$\sup_{x \in C} \mathbb{E}_x[\sigma_C] < \infty,$$

then P is positive.

One may also consider the rate of convergence. In our situation the convergence will turn out to be quite robust and fast. To be more precisely, we introduce to following definition. The subsequent theorem will be used later on.

Definition 5.62. (Cf. [22, Definition 15.2.1 (uniform geometric ergodicity)].) Let P a Markov kernel on $\mathsf{X} \times \mathcal{X}$.

(i) The Markov kernel P is said to be *uniformly ergodic* if P admits an invariant probability measure π such that there exists a nonnegative sequence $(\zeta_n)_{n \in \mathbb{N}_0}$ such that $\lim_{n \to \infty} \zeta_n = 0$ and

$$\sup_{x \in \mathsf{X}} d_{TV}(P^n(x, \cdot), \pi) \leq \zeta_n$$

in the total variation distance.

(ii) The Markov kernel P is said to be *uniformly geometrically ergodic* if P is uniformly ergodic and there exists constants $\zeta < \infty$ and $\beta > 1$ such that for all $n \in \mathbb{N}_0$,

$$\zeta_n \leq \zeta\, \beta^{-n}.$$

Theorem 5.63. *(Cf. [22, Theorem 15.3.1].) Let P be a Markov kernel on $\mathsf{X} \times \mathcal{X}$ with invariant probability π. The following statements are equivalent:*

(i) P is uniformly geometrically ergodic.

(ii) P is a positive aperiodic Markov kernel, and there exists a small set C and $\beta > 1$ such that

$$\sup_{x \in \mathsf{X}} \mathbb{E}_x[\beta^{\sigma_C}] < \infty.$$

(iii) The state space X is small.

5.4. (Non-)extinction criterion

The main result of this chapter is Theorem 5.71 which may be seen as simplification of the original problem in question. It would be highly desirable to compute or approximate the invariant probability measure η in Definition 5.66 in order to state more precise results.

5.4.1. Markov chain analysis

The basic proof idea for Proposition 5.64 is that in the critical regime of the state space in question, at least one component is close to zero. But then by the model under consideration the time-continuous moving has only a small amount of time to emerge and the process is dominated by the jump mechanism. This allows at least for a positive probability $1/(N-1)$ to make one small particle large by jumping to the largest of the $(N-1)$ others, which is at least at $1/\sqrt{N-1}$. Iterating this $N-2$ times ensures all particles to be sufficiently large even at a geometric rate.
Technically, the transition function p of the chain of directions U_n on S may be written in terms of Lauricella series (cf. Definition A.8 in the Appendix). Starting at $u_0 \in \mathsf{S}$ with u_0^{N-1} close to zero means the particles are scarcely given time to evolve and the discontinuous jump mechanism dominates. The density $p(u_0, \cdot)$ becoming singular corresponds to the arguments of the Lauricella series approaching the boundary of the domain of convergence.

Intuitive reasoning for three particles

To illustrate our approach let us consider the case of $N = 3$ moving particles. As depicted in Figure 5.2, for the compactification of the set

$$\mathsf{S} = \{(x^1, x^2) \in (0,\infty)^2 : x^1 \geq x^2, (x^1)^2 + (x^2)^2 = 1\}$$

only the singelton $\{(1,0)\}$ is missing in S. If U_n has a value near this boundary point this means $\overline{Y}_n = (\overline{Y}_n^1, \overline{Y}_n^2)$ is a 2-tuple with $\overline{Y}_n^1 \in (0,\infty)$ a relatively large value and $\overline{Y}_n^2 \in (0,\infty)$ a relatively small value. With probability $1/2$ the large value $\overline{Y}_n^1$ is duplicated. In this scenario, after the jump, the system $\overline{X}_{\overline{\tau}_n}$ consists of one particle relatively close to the origin and two particles starting equally from a high value. Shortly after, most likely the particle close to the origin will even touch the origin and the other two particles will have values which are not too far away from each other. The situation is illustrated in the figure in Example 5.39, case iii). This prevents the chain U_n from escaping to the boundary point $(1,0)$. Instead, in light of Proposition 5.57 this indicates the chain U_n to be Harris recurrent. As it will turn out, the argument even shows that for a given neighborhood of $(1/\sqrt{2}, 1/\sqrt{2})$ there is a universal positive probability $\kappa > 0$ independently of the current state of the chain to jump to this neighborhood. Particularly, Theorem 5.61 can be applied and U_n will be shown to be positive.

More specifically, still having fixed $N := 3$, we may set

$$C_0 := \left\{ u \in \mathsf{S} : u^2 \geq \frac{1}{2\sqrt{2}}/\sqrt{2} \right\}$$

which is a compact set implying that the continuous density function p of U_n is positive on C_0. Moreover, C_0 will be shown to be a small set, particularly the set C_0 is petite. The Markov kernel of the chain U_n will be shown to be irreducible with the spherical measure σ on the Euclidean unit sphere restricted to S as maximal irreducibility measure. The kernel will even be shown to be positive by considering the set

$$C_1^\delta = \{u \in \mathsf{S} : u^2 < \delta\}$$

for small $\delta > 0$. Starting from $u_0 \in C_1^\delta$ corresponds to one particle starting from at most δ. Replicating the particle at the position at least at $1-\delta$ has probability $1/2$ and gives rise to a subkernel p^1 as described in Remark 5.37. One can show that for sufficiently small $\delta > 0$ it holds

$$\sup_{u_0 \in C_1^\delta} p^1(u_0, \mathsf{S} \setminus C_0) < \frac{1}{2}/2.$$

It then follows

$$\kappa = \inf_{u_0 \in \mathsf{S}} \mathbb{P}_{u_0}(\sigma_{C_0} = 1) > 0$$

which means that there is a universal positive probability $\kappa > 0$ uniformly in the starting point to return to C_0. The return time σ_{C_0} is consequently dominated by a geometric random variable with success parameter κ and specifically has finite first moment.

Let us now return to the general case of $N \geq 3$ moving particles.

Proposition 5.64. *The Markov chain (U_n) is irreducible with the $(N-2)$-dimensional spherical measure σ as maximal irreducibility measure, strongly aperiodic, positive Harris and uniformly geometrically ergodic. The unique invariant probability measure admits a density with respect to σ which is strictly positive $\sigma-$ a.e.*

Proof. In view of Lemma 5.36 the transition density $p : \mathsf{S}^2 \to (0,\infty)$ is a positive continuous mapping and therefore has a positive minimum on the compact set

$$C_0 \coloneqq \left\{ u \in \mathsf{S} : u^{N-1} \geq \left(\frac{1}{2\sqrt{2}}\right)^{N-2} / \sqrt{N-1} \right\};$$

i.e.

$$\min_{u_0, u \in C_0} p(u_0, u) =: \delta > 0.$$

Consequently, C_0 is an $(1,\xi)$-small set for the kernel of the Markov chain (U_n) in the sense of Definition 5.45 where

$$\xi : \mathcal{B}(\mathsf{S}) \to [0,\infty), \quad \xi(A) \coloneqq \delta\, \sigma(A \cap C_0).$$

Due to $\xi(C_0) = \delta\sigma(C_0) > 0$ we further see, that C_0 is strongly aperiodic in the sense of Definition 5.46. Moreover, any set $A \in \mathcal{B}(\mathsf{S})$ with $\sigma(A) > 0$ is accessible in the sense of Definition 5.41 since for arbitrary $u_0 \in \mathsf{S}$ for the return time $\sigma_A \coloneqq \inf\{n \in \mathbb{N} : U_n \in A\}$ it holds

$$\mathbb{P}_{u_0}(\sigma_A < \infty) \geq \mathbb{P}_{u_0}(U_1 \in A) = \int_A p(u_0, u)\, d\sigma(u) > 0.$$

Particularly, the set C_0 is accessible and the kernel of (U_n) is seen to be irreducible by means of Definition 5.47. The argument also exhibits σ to be an irreducibility measure in the sense of Definition 5.48. Conversely, consider a set $A \in \mathcal{B}(\mathsf{S})$ with $\sigma(A) = 0$. Then,

$$\begin{aligned} \mathbb{P}_{u_0}(\sigma_A < \infty) &= \mathbb{P}_{u_0}\left(\bigcup_{n=1}^{\infty} \{U_n \in A\}\right) \leq \sum_{n=1}^{\infty} \mathbb{P}_{u_0}(U_n \in A) \leq \sum_{n=1}^{\infty} \sup_{u_0' \in \mathsf{S}} \mathbb{P}_{u_0'}(U_1 \in A) \\ &= \sum_{n=1}^{\infty} \sup_{u_0' \in \mathsf{S}} \int_A \mathbb{P}_{u_0'}(U_1 \in du) = \sum_{n=1}^{\infty} \sup_{u_0' \in \mathsf{S}} \int_A p(u_0', u)\, d\sigma(u) = 0 \end{aligned}$$

for $u_0 \in \mathsf{S}$. This shows that σ is a maximal irreducibility measure, that is, the set of accessible sets is given by $\{A \in \mathcal{B}(\mathsf{S}) : \sigma(A) > 0\}$. Following Definition 5.52 not only the set C_0 but also the kernel of (U_n) is strongly aperiodic.
We now turn to Harris recurrence and positivity properties of (U_n). Observe, that for compact sets $K \subseteq \mathsf{S}$, it holds

$$\inf_{u_0 \in K} \mathbb{P}_{u_0}(U_1 \in C_0) = \min_{u_0 \in K} \int_{C_0} p(u_0, u)\, d\sigma(u) > 0, \tag{5.25}$$

since the mapping $\mathsf{S} \to (0,1], u_0 \mapsto \int_{C_0} p(u_0, u)\, d\sigma(u)$ is continuous and p is strictly positive. The space $K \coloneqq \mathsf{S}$ is not compact, but still we will show

$$\inf_{u_0 \in \mathsf{S}} \mathbb{P}_{u_0}(\cup_{n=1}^{N-2} \{U_n \in C_0\}) > 0.$$

For this to end, define given an index $j \in \{1, \dots, N-1\}$, a permutation $\pi \in S_{N-1}$ and $u_0 \in \mathsf{S}$ the symbol $\widetilde{u_0}_{\pi,j} := \widetilde{u_0}_{\pi,j}(u_0)$ by

$$\widetilde{u_0}^s_{\pi,j} := \begin{cases} u_0^{\pi(s)}, & s \neq \pi^{-1}(j), \\ u_0^1, & s = \pi^{-1}(j), \end{cases}$$

and additionally given $u \in \mathsf{S}$ define $x_{\pi,j} := x_{\pi,j}(u_0, u)$ by

$$x^s_{\pi,j} := \frac{(u^s)^2}{1 + \left(u_0^1\right)^2} \cdot \left(\widetilde{u_0}^s_{\pi,j}\right)^2.$$

Further, for $k \in \{0, \dots, N-2\}$ we consider the sets

$$C_k := \left\{ u \in \mathsf{S} : u^{N-1-k} \geq \left(\frac{1}{2\sqrt{2}}\right)^{N-2-k} / \sqrt{N-1} \right\}.$$

Claim 1: For all $j \in \{1, \dots, N-1\}$, $\pi \in S_{N-1}$ and $k \in \{1, \dots, N-2\}$ it holds

$$\sup_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} \sum_{s=1}^{N-1} \left(x^s_{\pi,j}\right)^{1/2} < 1.$$

Proof of Claim 1: Let $\langle x, y \rangle := \sum_{s=1}^{N-1} x^s \cdot y^s$ denote the scalar product on the Euclidean space $\mathbb{R}^{N-1}$. By the classical rearrangement inequality [31, Theorem 368]

$$\max_{j=1}^{N-1} \max_{\pi \in S_{N-1}} \sum_{s=1}^{N-1} \left(x^s_{\pi,j}\right)^{1/2} = \max_{j=1}^{N-1} \max_{\pi \in S_{N-1}} \left\langle \frac{\widetilde{u_0}^{\downarrow}_{\pi,j}}{\sqrt{1 + \left(u_0^1\right)^2}}, u \right\rangle$$
$$= \max_{j=1}^{N-1} \left\langle \frac{(u_0^1, u_0^1, \dots, u_0^{j-1}, u_0^{j+1}, \dots, u_0^{N-1})}{\sqrt{1 + \left(u_0^1\right)^2}}, u \right\rangle.$$

In the case $j \leq N-1-k$ we find using $\langle x, y \rangle \leq \|x\| \cdot \|y\|$

$$\sup_{u_0 \in C_k} \left\langle \frac{(u_0^1, u_0^1, \dots, u_0^{j-1}, u_0^{j+1}, \dots, u_0^{N-1})}{\sqrt{1 + \left(u_0^1\right)^2}}, u \right\rangle \leq \sup_{u_0 \in C_k} \frac{\|(u_0^1, u_0^1, \dots, u_0^{j-1}, u_0^{j+1}, \dots, u_0^{N-1})\|}{\sqrt{1 + \left(u_0^1\right)^2}}$$
$$= \sup_{u_0 \in C_k} \left(\frac{1 + \left(u_0^1\right)^2 - \left(u_0^j\right)^2}{1 + \left(u_0^1\right)^2} \right)^{1/2} \leq \left(1 - \frac{\left(\frac{1}{2\sqrt{2}}\right)^{N-2-k} / \sqrt{N-1}}{2} \right)^{1/2} < 1$$

and in the case $j \geq N-k$ we find using $\langle x, y \rangle = (\|x\|^2 + \|y\|^2 - \|x-y\|^2)/2$

$$\sup_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} \left\langle \frac{(u_0^1, u_0^1, \dots, u_0^{j-1}, u_0^{j+1}, \dots, u_0^{N-1})}{\sqrt{1 + \left(u_0^1\right)^2}}, u \right\rangle$$

$$\leq \sup_{\substack{u_0\in C_k\\ u\in \mathsf{S}\setminus C_{k-1}}} \left\langle \frac{(u_0^1,u_0^1,\ldots,u_0^{j-1},u_0^{j+1},\ldots,u_0^{N-1})}{\|(u_0^1,u_0^1,\ldots,u_0^{j-1},u_0^{j+1},\ldots,u_0^{N-1})\|}, u\right\rangle$$
$$= \sup_{\substack{u_0\in C_k\\ u\in \mathsf{S}\setminus C_{k-1}}} \left[1-\left\|\frac{(u_0^1,u_0^1,\ldots,u_0^{j-1},u_0^{j+1},\ldots,u_0^{N-1})}{\|(u_0^1,u_0^1,\ldots,u_0^{j-1},u_0^{j+1},\ldots,u_0^{N-1})\|}-u\right\|^2/2\right]$$
$$\leq 1-\inf_{\substack{u_0\in C_k\\ u\in \mathsf{S}\setminus C_{k-1}}} \left(\frac{u_0^{N-k-1}}{\sqrt{1+\left(u_0^1\right)^2-\left(u_0^j\right)^2}}-u^{N-k}\right)^2/2$$
$$\leq 1-\left(\frac{\left(\frac{1}{2\sqrt{2}}\right)^{N-2-k}/\sqrt{N-1}}{\sqrt{2}}-\left(\frac{1}{2\sqrt{2}}\right)^{N-1-k}/\sqrt{N-1}\right)^2/2$$
$$= 1-\left(\left(\frac{1}{2\sqrt{2}}\right)^{N-1-k}/\sqrt{N-1}\right)^2/2<1.$$

This finishes the proof of Claim 1. ■

Let further

$$F(x)=F(x^1,\ldots,x^{N-1}) \coloneqq F_C^{(N-1)}(Nw,N-1,w+1,w+1,\ldots,w+1;(x^s)_{s=1}^{N-1})$$

with the C-type Lauricella series as in Definition A.8.
Claim 2: For all $j\in\{1,\ldots,N-1\}$, $\pi\in S_{N-1}$, $k\in\{1,\ldots,N-2\}$ it holds

$$\sup_{\substack{u_0\in C_k\\ u\in \mathsf{S}\setminus C_{k-1}}} F(x_{\pi,j}(u_0,u))<\infty.$$

Proof of Claim 2: The mapping $\alpha: x\mapsto \sum_{s=1}^{N-1}|x^s|^{1/2}$ is continuous. Therefore, the set

$$D\coloneqq \{x_{\pi,j}(u_0,u): u_0\in C_k, u\in \mathsf{S}\setminus C_{k-1}\}\subseteq \mathbb{R}^{N-1}$$

is contained in the compact space $D\subseteq K\coloneqq \alpha^{-1}([0,\sup_{x\in D}\alpha(x)])$ where by the previous claim, $\sup_{x\in D}\alpha(x)<1$. Because $x\mapsto F(x)$ is continuous on $\alpha^{-1}([0,1))\supseteq K$ (Proposition A.11) it follows

$$\sup_{\substack{u_0\in C_k\\ u\in \mathsf{S}\setminus C_{k-1}}} F(x_{\pi,j}(u_0,u))=\sup_{x\in D}F(x)\leq \sup_{x\in K}F(x)=\max_{x\in K}F(x)<\infty.$$

This finishes the proof of Claim 2. ■

For $k\in\{1,\ldots,N-2\}$ and $\delta>0$ let us introduce the sets

$$C_k^\delta \coloneqq \{u\in C_k: u^{N-1}<\delta\}.$$

Claim 3: For $k \in \{1, \ldots, N-2\}$ it holds

$$\lim_{\delta \downarrow 0} \sup_{u_0 \in C_k^\delta} \int_{\mathsf{S} \setminus C_{k-1}} p^1(u_0, u)\, d\sigma(u) = 0.$$

Proof of Claim 3: Using the constant

$$c_{w,N} := \frac{2^{N-1}\Gamma(Nw)\Gamma(N-1)}{(N-1)\Gamma(w)\Gamma(w+1)^{N-1}}$$

depending on w and N only, we find the representation

$$p^1(u_0, u) = c_{w,N} \left(\prod_{s=1}^{N-1} u_0^s \right)^{2w} \prod_{s=1}^{N-1} u^s \cdot \frac{\left(u_0^1\right)^{2w}}{\left(1 + \left(u_0^1\right)^2\right)^{Nw}} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(1) \leq \pi(j)}} F(x_{\pi,j}(u_0, u))$$

$$\leq c_{w,N} \left(u_0^{N-1} \right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(1) \leq \pi(j)}} F(x_{\pi,j}(u_0, u)).$$

It follows by the previous claim

$$\sup_{u_0 \in C_k^\delta} \int_{C_{k-1}} p^1(u_0, u)\, d\sigma(u) \leq \sigma(C_{k-1}) \sup_{\substack{u_0 \in C_k^\delta \\ u \in \mathsf{S} \setminus C_{k-1}}} p^1(u_0, u)$$

$$\leq \sigma(C_{k-1})\, c_{w,N}\, \delta^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(1) \leq \pi(j)}} \sup_{\substack{u_0 \in C_k^\delta \\ u \in \mathsf{S} \setminus C_{k-1}}} F(x_{\pi,j}(u_0, u))$$

$$\leq \sigma(C_{k-1})\, c_{w,N}\, \delta^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(1) \leq \pi(j)}} \sup_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} F(x_{\pi,j}(u_0, u)) \xrightarrow[\delta \downarrow 0]{} 0.$$

This finishes the proof of Claim 3. ■

Claim 4: For $k \in \{1, \ldots, N-2\}$ it holds

$$\inf_{u_0 \in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) > 0.$$

Proof of Claim 4: By the previous claim, there exists $\delta > 0$ such that

$$\sup_{u_0 \in C_k^\delta} \int_{\mathsf{S} \setminus C_{k-1}} p^1(u_0, u)\, d\sigma(u) < \frac{1}{N-1}/2.$$

We then split

$$\inf_{u_0 \in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) = \inf_{u_0 \in C_k \setminus C_k^\delta} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) \wedge \inf_{u_0 \in C_k^\delta} P_{u_0}(U_1 \in C_{k-1}).$$

The first infimum is positive since $C_k \setminus C_k^\delta$ is compact and we may apply (5.25). For the second infimum, recalling Remark 5.37,

$$\begin{aligned}\inf_{u_0\in C_k^\delta} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) &\geq \inf_{u_0\in C_k^\delta} \int_{C_{k-1}} p^1(u_0,u)\,d\sigma(u)\\ &= \frac{1}{N-1} - \sup_{u_0\in C_k^\delta} \int_{\mathsf{S}\setminus C_{k-1}} p^1(u_0,u)\,d\sigma(u) \geq \frac{1}{N-1}/2 > 0.\end{aligned}$$

This finishes the proof of Claim 4. ■

Claim 5: For $k \in \{1,\dots,N-2\}$ it holds

$$\inf_{u_0\in C_k} \mathbb{P}_{u_0}(\sigma_{C_0} \leq k) > 0.$$

Proof of Claim 5: Let us show the claim by induction over $k \in \{1,\dots,N-2\}$. The base case $k=1$ is a consequence of Claim 4. Assuming the assertion for $k-1 \in \{1,\dots,N-3\}$ implies

$$\begin{aligned}\inf_{u_0\in C_k} \mathbb{P}_{u_0}(\sigma_{C_0} \leq k) &\geq \inf_{u_0\in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) \cdot \inf_{u_0\in C_k} \mathbb{P}_{u_0}(\sigma_{C_0} \mid U_1 \in C_{k-1})\\ &\geq \inf_{u_0\in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) \cdot \inf_{u_0\in C_{k-1}} \mathbb{P}_{u_0}(\sigma_{C_0} \leq k-1),\end{aligned}$$

where the first factor is positive by Claim 4 and the second factor is positive by induction hypothesis. This finishes the proof of Claim 5. ■

As a consequence of Claim 5, i.e. specifying to the case of $k = N-2$,

$$\kappa := \inf_{u_0\in\mathsf{S}} \mathbb{P}_{u_0}(\sigma_{C_0} \leq N-2) > 0;$$

for each trial of $N-2$ consecutive transitions, there is at least probability $\kappa > 0$ to return to C_0 during this time period. In other words, for $G \sim \mathrm{Geo}(\kappa)$ geometrically distributed supported on $\mathbb{N}$ with success parameter κ, i.e. $\mathbb{P}(G = l) = \kappa(1-\kappa)^{l-1}$ for $l \in \mathbb{N}$, it holds

$$\sup_{u_0\in\mathsf{S}} \mathbb{E}_{u_0}[\sigma_{C_0}] \leq \mathbb{E}[(N-2)\cdot G] = (N-2)/\kappa < \infty. \tag{5.26}$$

Following Proposition 5.57, we deduce that the Markov kernel to (U_n) is Harris recurrent and in view of Theorem 5.61 (or by Theorem 5.58) is further positive. Therefore, by Theorem 5.50 (but we could have used Corollary 5.59) in conjunction with Corollary 5.51 the unique invariant probability measure is a maximal irreducibility measure which are all equivalent. (Theorem 5.49) We infer, that it admits a positive density function with respect to σ. By (5.26), Lemma 5.54 shows that the state space S is petite. Since our kernel is irreducible and aperiodic, every petite set is small (Theorem 5.55). Hence, also S is small and according to Theorem 5.63 the kernel is uniformly geometrically ergodic in the sense of Definition 5.62. (Alternatively, one can directly check Theorem 5.63 (ii) as for $\beta := ((1+1/(1-\kappa))/2)^{1/(N-2)} > 1$ it holds $\sup_{u_0\in\mathsf{S}} \mathbb{E}_{u_0}[\beta^{\sigma_{C_0}}] \leq \mathbb{E}[\beta^{(N-2)G}] = 1 + 1/(1-\kappa) < \infty$.) □

We use the formalism to transfer Proposition 5.64 to the enlarged state space applicable to the HMM.

Proposition 5.65. *The Markov kernel K of the HMM $M = ((U_n, U_{n+1}), (R_{n+1}, T_{n+1}))_{n\in\mathbb{N}_0}$ is irreducible with maximal irreducibility measure $\sigma\big|^{\otimes 2}_{\mathcal{B}(\mathsf{S})} \otimes \mathrm{Leb}\,\big|^{\otimes 2}_{\mathcal{B}((0,\infty))}$, aperiodic, positive Harris recurrent and uniformly geometrically ergodic. The invariant probability has positive density $\widetilde{h}_{u_1}(u_2, r_0, t_0)\, p_\eta(u_1)$ with $\widetilde{h}$ from equation* (5.17) *and p_η the invariant density of (U_n).*

Proof. Let $\delta_a(x)$ denote the Dirac-Delta distribution and $d\sigma(u_1, u_2) := d\sigma(u_2)\, d\sigma(u_1)$ for

$$u = (u_1, u_2) = \left(\left(u_1^1, \dots, u_1^{N-1}\right), \left(u_2^1, \dots, u_2^{N-1}\right)\right) \in \mathsf{S}^2.$$

The Markov kernel Q of the hidden chain $(U_n, U_{n+1})_{n\in\mathbb{N}_0}$ on $(\mathsf{S}^2, \mathcal{B}(\mathsf{S}^2))$ is given by

$$Q(u, V) = Q\left((u_1, u_2), V\right) = \int_V p(u_2, v_2)\, \delta_{u_2}(dv_1)\, d\sigma(v_2),$$

where $u \in \mathsf{S}^2$ and $V \in \mathcal{B}(\mathsf{S}^2)$. The transition is independent of u_1. Consider the kernel G from $(\mathsf{S}^2, \mathcal{B}(\mathsf{S}^2))$ to $((0,\infty)^2, \mathcal{B}((0,\infty)^2))$:

$$G((u_1, u_2), A) = \frac{\int_A \widetilde{h}_{u_1}(u_2, r, t)\, d(r,t)}{\int_{(0,\infty)^2} \widetilde{h}_{u_1}(u_2, r, t)\, d(r,t)} = \frac{\int_A \widetilde{h}_{u_1}(u_2, r, t)\, d(r,t)}{p(u_1, u_2)}.$$

The Markov kernel of the HMM $M = ((U_n, U_{n+1}), (R_{n+1}, T_{n+1}))_{n\in\mathbb{N}_0}$ is given by

$$K(((u_1, u_2), (r_0, t_0)), D) = \int_D G((v_1, v_2), d(r,t))\, Q((u_1, u_2), dv)$$
$$= \int_D \frac{\widetilde{h}_{v_1}(v_2, r, t)\, d(r,t)}{p(v_1, v_2)}\, \delta_{u_2}(dv_1)\, p(u_2, v_2)\, d\sigma(v_2) = \int_D \widetilde{h}_{v_1}(v_2, r, t)\, d(r,t)\, \delta_{u_2}(dv_1)\, d\sigma(v_2)$$

as in [17, Equation (2.14)] and is independent of u_1, r_0, t_0.
We have seen in the proof of Proposition 5.64 that the set

$$C_0 := \left\{ u \in \mathsf{S} : u^{N-1} \geq \left(\frac{1}{2\sqrt{2}}\right)^{N-2} / \sqrt{N-1} \right\}$$

is small for the kernel of (U_n) because $\delta := \min_{u_0, u\in C_0} p(u_0, u) > 0$ is positive. The measure

$$\mu(D) := \delta \int_{D\cap(C_0\times\mathsf{S}\times(0,\infty)^2)} \widetilde{h}_{w_1}(w_2, \rho, s)\, d(\rho, s)\, d\sigma(w_1, w_2)$$

on $\mathcal{B}(\mathsf{S}^2 \times (0,\infty)^2)$ is nonzero as

$$\mu(C_0 \times \mathsf{S} \times (0,\infty)^2) = \delta \int_{C_0\times\mathsf{S}} p(w_1, w_2)\, d\sigma(w_1, w_2) = \delta \int_{C_0} d\sigma(w_1) = \delta\, \sigma(C_0) > 0$$

and the set $\mathsf{S} \times C_0 \times (0,\infty)^2$ is $(2,\mu)$-small with respect to K: For arbitrary $x = (u_1, u_2, r_0, t_0) \in \mathsf{S} \times C_0 \times (0,\infty)^2$ and $D \in \mathcal{B}(\mathsf{S}^2 \times (0,\infty)^2)$:

$$
\begin{aligned}
&K^2(x,D) \geq K^2(x, D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2)) \\
&= \int_{\mathsf{S}^2 \times (0,\infty)^2} K(y, D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2))\, K(x,dy) \\
&= \int_{\mathsf{S}^2 \times (0,\infty)^2} K((v_1,v_2,\rho,s), D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2))\, K((u_1,u_2,r_0,t_0), d(v_1,v_2,\rho,s)) \\
&= \int_{\mathsf{S}^2 \times (0,\infty)^2} \int_{D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2)} \widetilde{h}_{w_1}(w_2,\rho,s)\, d(\rho,s)\, \delta_{v_2}(dw_1)\, d\sigma(w_2)\, \widetilde{h}_{v_1}(v_2,r,t)\, d(r,t)\, \delta_{u_2}(dv_1)\, d\sigma(v_2) \\
&= \int_{\mathsf{S} \times (0,\infty)^2} \int_{D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2)} \widetilde{h}_{w_1}(w_2,\rho,s)\, d(\rho,s)\, \delta_{v_2}(dw_1)\, d\sigma(w_2)\, \widetilde{h}_{u_2}(v_2,r,t)\, d(r,t)\, d\sigma(v_2) \\
&= \int_{\mathsf{S}} \int_{D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2)} \widetilde{h}_{w_1}(w_2,\rho,s)\, d(\rho,s)\, \delta_{v_2}(dw_1)\, d\sigma(w_2)\, p(u_2,v_2)\, d\sigma(v_2) \\
&= \int_{D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2)} \widetilde{h}_{w_1}(w_2,\rho,s)\, d(\rho,s) \int_{\mathsf{S}} \delta_{v_2}(dw_1)\, p(u_2,v_2)\, d\sigma(v_2)\, d\sigma(w_2) \\
&= \int_{D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2)} \widetilde{h}_{w_1}(w_2,\rho,s)\, d(\rho,s)\; p(u_2,w_1)\, d\sigma(w_1,w_2) \geq \mu(D).
\end{aligned}
$$

Next, we show, that sets $D \in \mathcal{B}(\mathsf{S}^2 \times (0,\infty)^2)$ with $\int_D d(\rho,s)\, d\sigma(w_1,w_2) > 0$ are accessible: Let $x = (u_1,u_2,r_0,t_0) \in \mathsf{S}^2 \times (0,\infty)^2$ arbitrary. Then

$$
\begin{aligned}
\mathbb{P}_x(\sigma_D < \infty) &\geq \mathbb{P}_x(\sigma_D \in \{1,2\}) \geq K^2(x,D) \\
&= \int_D \widetilde{h}_{w_1}(w_2,\rho,s)\, d(\rho,s)\, p(u_2,w_1)\, d\sigma(w_1,w_2).
\end{aligned}
$$

We must check, that the last quantity indeed is positive. Since for arbitrary $u_2, w_1 \in \mathsf{S}$, the value $p(u_2,w_1) > 0$ is positive and for arbitrary $w_1, w_2 \in \mathsf{S}$ and $\rho, s \in (0,\infty)$, the value $\widetilde{h}_{w_1}(w_2,\rho,s) > 0$ is positive and by assumption D has positive measure $\phi(D) > 0$ with respect to the nontrivial σ-finite measure $\phi := \sigma\big|_{\mathcal{B}(\mathsf{S})}^{\otimes 2} \otimes \mathrm{Leb}\,\big|_{\mathcal{B}((0,\infty))}^{\otimes 2}$ we may proceed as in the proof of [17, Proposition 14.3.1]: Letting $D_m^{u_2} := \{(w_1,w_2,\rho,s) \in D : \widetilde{h}_{w_1}(w_2,\rho,s)\, p(u_2,w_1) \geq 1/m\}$ we have

$$
\begin{aligned}
D &= \{(w_1,w_2,\rho,s) \in D : \widetilde{h}_{w_1}(w_2,\rho,s)\, p(u_2,w_1) > 0\} \\
&= \bigcup_{m=1}^{\infty} \{(w_1,w_2,\rho,s) \in D : \widetilde{h}_{w_1}(w_2,\rho,s)\, p(u_2,w_1) \geq 1/m\} = \bigcup_{m=1}^{\infty} D_m^{u_2}.
\end{aligned}
$$

If there was not an $m \in \mathbb{N}$ with $\phi(D_m^{u_2}) > 0$ it would follow $\phi(D) = \phi(\cup_{m=1}^{\infty} D_m^{u_2}) \leq \sum_{m=1}^{\infty} \phi(D_m^{u_2}) = 0$. Therefore, there is. Then

$$
\int_D \widetilde{h}_{w_1}(w_2,\rho,s)\, d(\rho,s)\, p(u_2,w_1)\, d\sigma(w_1,w_2)
$$

$$\geq \int_{D_m^{u_2}} \widetilde{h}_{w_1}(w_2,\rho,s)\, p(u_2,w_1)\, d(\rho,s)\, d\sigma(w_1,w_2) \geq \phi(D_m^{u_2})/m > 0.$$

The calculation shows, that ϕ is an irreducibility measure and particularly, the small set $\mathsf{S} \times C_0 \times (0,\infty)^2$ is accessible since

$$\phi(\mathsf{S} \times C_0 \times (0,\infty)^2) = \sigma(\mathsf{S}) \cdot \sigma(C_0) \cdot \mathrm{Leb}((0,\infty))^2 = \infty > 0.$$

Therefore K is seen to be irreducible. Let us now consider a set $D \in \mathcal{B}(\mathsf{S}^2 \times (0,\infty)^2)$ with $\phi(D) = 0$. From Fubini's theorem it follows with $D_{v_1,r,t} \coloneqq \{v_2 \in \mathsf{S} : (v_1,v_2,r,t) \in D\}$

$$0 = \phi(D) = \int_D d(r,t)\, d\sigma(v_1,v_2) = \int_{(0,\infty)^2} \int_{\mathsf{S}} \int_{D_{v_1,r,t}} d\sigma(v_2)\, d\sigma(v_1)\, d(r,t)$$
$$= \int_{(0,\infty)^2} \int_{\mathsf{S}} \sigma(D_{v_1,r,t})\, d\sigma(v_1)\, d(r,t).$$

We have previously seen that integrating a positive function over a domain with positive measure implies a positive integral value. By contraposition, there must exist $u_2 \in \mathsf{S}, r_0, t_0 \in (0,\infty)$ such that $\sigma(D_{u_2,r,t}) = 0$. Then for $x = (u_1,u_2,r_0,t_0)$ with $u_1 \in \mathsf{S}$ arbitrary

$$\mathbb{P}_x(\tau_D < \infty) \leq K(x,D) + \sum_{n=2}^{\infty} \sup_{x \in \mathsf{S}^2 \times (0,\infty)^2} K^2(x,D).$$

Both summands vanish because for the first one

$$K(x,D) = \int_D \widetilde{h}_{v_1}(v_2,r,t)\, d(r,t)\, \delta_{u_2}(dv_1)\, d\sigma(v_2)$$
$$= \int_{(0,\infty)^2} \int_{\mathsf{S}} \mathbb{1}_D\left((u_2,v_2,r,t)\right) \widetilde{h}_{u_2}(v_2,r,t)\, d\sigma(v_2)\, d(r,t)$$
$$= \int_{(0,\infty)^2} \int_{D_{u_2,r,t}} \widetilde{h}_{u_2}(v_2,r,t)\, d\sigma(v_2)\, d(r,t)$$

where the inner integral is equal to zero since $\sigma(D_{u_2,r,t}) = 0$ and for the second summand

$$\sup_{x \in \mathsf{S}^2 \times (0,\infty)^2} K^2(x,D) = \sup_{x \in \mathsf{S}^2 \times (0,\infty)^2} \int_D \widetilde{h}_{w_1}(w_2,\rho,s)\, p(u_2,w_1)\, d(\rho,s)\, d\sigma(w_1,w_2) = 0$$

since $\phi(D) = 0$. In summary, we have computed ϕ to be a maximal irreducibility measure for K. Next, we want to show that K is aperiodic. For $x = (u_1,u_2,r_0,t_0) \in \mathsf{S} \times C_0 \times (0,\infty)^2$ arbitrary we will have $K(x, \mathsf{S} \times C_0 \times (0,\infty)^2) \geq \mu(C_0 \times \mathsf{S} \times (0,\infty)^2) > 0$:

$$K(x, \mathsf{S} \times C_0 \times (0,\infty)^2) = \int_{\mathsf{S} \times C_0 \times (0,\infty)^2} \widetilde{h}_{v_1}(v_2,r,t)\, d(r,t)\, \delta_{u_2}(dv_1)\, d\sigma(v_2)$$
$$= \int_{C_0} p(u_2,v_2)\, d\sigma(v_2) \geq \delta \cdot \sigma(C_0) > 0.$$

As a consequence, $\inf_{x\in \mathsf{S}\times C_0\times(0,\infty)^2} K(x, \mathsf{S}\times C_0\times(0,\infty)^2) \geq \delta\sigma(C_0) > 0$ which proves the asserted aperiodicity.
Writing

$$\sigma_D\left((M_n)_{n\in\mathbb{N}_0}\right) = \inf\{n\in\mathbb{N} : M_n\in D\}$$

for $D\in\mathcal{B}(\mathsf{S}^2\times(0,\infty)^2)$ to emphasize that σ_D is a return time for the process $(M_n)_{n\in\mathbb{N}_0}$ the random values $\sigma_{\mathsf{S}\times C_0\times(0,\infty)^2}\left((M_n)_{n\in\mathbb{N}_0}\right)$ and $\sigma_{C_0}\left((U_n)_{n\in\mathbb{N}_0}\right)$ relate in an easy way:

$$\mathbb{P}_{(u_{0,1},u_{0,2},r_0,t_0)}\left(\sigma_{\mathsf{S}\times C_0\times(0,\infty)^2}((M_n)_n)\in\bullet\right) = \mathbb{P}_{u_{0,2}}\left(\sigma_{C_0}((U_n)_n)\in\bullet\right); \tag{5.27}$$

the straightforward calculation will be carried out below. This then enables us to use the same machinery as in the proof of Proposition 5.64 to deduce the HMM M to be positive Harris, uniformly geometrically ergodic with the invariant probability having positive density with respect to $\sigma\big|^{\otimes 2}_{\mathcal{B}(\mathsf{S})}\otimes\mathrm{Leb}\big|^{\otimes 2}_{\mathcal{B}((0,\infty))}$. Moreover, if η denotes the invariant probability measure of the kernel of (U_n) and p_η its density such that $\eta(du) = p_\eta(u)\,d\sigma(u)$ for $u\in\mathsf{S}$, the invariant probability measure for K has density $\widetilde{h}_{u_1}(u_2,r_0,t_0)\,p_\eta(u_1)$, since for $D\in\mathcal{B}(\mathsf{S}^2\times(0,\infty)^2)$

$$\begin{aligned}
&\int_{\mathsf{S}^2\times(0,\infty)^2} K(((u_1,u_2),(r_0,t_0)),D)\,\widetilde{h}_{u_1}(u_2,r_0,t_0)\,p_\eta(u_1)\,d(r_0,t_0)\,d\sigma(u_1,u_2)\\
&= \int_{\mathsf{S}^2\times(0,\infty)^2}\int_D \widetilde{h}_{v_1}(v_2,r,t)\,\delta_{u_2}(dv_1)\,d(r,t)\,d\sigma(v_2)\,\widetilde{h}_{u_1}(u_2,r_0,t_0)\,d(r_0,t_0)\,d\sigma(u_2)\,\eta(du_1)\\
&= \int_{\mathsf{S}^2}\int_D \widetilde{h}_{v_1}(v_2,r,t)\,\delta_{u_2}(dv_1)\,d(r,t)\,d\sigma(v_2)\,p(u_1,u_2)\,d\sigma(u_2)\,\eta(du_1)
\end{aligned}$$

and by Fubini's theorem and

$$\begin{aligned}
&\int_{\mathsf{S}^2}\delta_{u_2}(dv_1)\,p(u_1,u_2)\,d\sigma(u_2)\,\eta(du_1) = \int_{\mathsf{S}}\delta_{u_2}(dv_1)\int_{\mathsf{S}} p(u_1,u_2)\,\eta(du_1)\,d\sigma(u_2)\\
&= \int_{\mathsf{S}}\delta_{u_2}(dv_1)\,\eta(du_2) = \int_{\mathsf{S}}\delta_{u_2}(dv_1)\,p_\eta(u_2)\,d\sigma(u_2) = p_\eta(v_1)\int_{\mathsf{S}}\delta_{u_2}(dv_1)\,d\sigma(u_2) = p_\eta(v_1)\,d\sigma(v_1)
\end{aligned}$$

it follows

$$\begin{aligned}
&\int_{\mathsf{S}^2}\int_D \widetilde{h}_{v_1}(v_2,r,t)\,\delta_{u_2}(dv_1)\,d(r,t)\,d\sigma(v_2)\,p(u_1,u_2)\,d\sigma(u_2)\,\eta(du_1)\\
&= \int_D \widetilde{h}_{v_1}(v_2,r,t)\,p_\eta(v_1)\,d(r,t)\,d\sigma(v_1,v_2).
\end{aligned}$$

To show equation (5.27), let $x_0 = (u_{0,1},u_{0,2},r_0,t_0)\in\mathsf{S}^2\times(0,\infty)^2$ and $k\in\mathbb{N}$. Then

$$\begin{aligned}
&\mathbb{P}_{x_0}(\sigma_{\mathsf{S}\times C_0\times(0,\infty)^2}((M_n)) = k)\\
&= \mathbb{P}_{x_0}(M_1\in(\mathsf{S}\times C_0\times(0,\infty)^2)^c,\ldots,M_{k-1}\in(\mathsf{S}\times C_0\times(0,\infty)^2)^c, M_k\in\mathsf{S}\times C_0\times(0,\infty)^2)\\
&= \mathbb{P}_{x_0}(M_1\in\mathsf{S}\times C_0^c\times(0,\infty)^2,\ldots,M_{k-1}\in\mathsf{S}\times C_0^c\times(0,\infty)^2, M_k\in\mathsf{S}\times C_0\times(0,\infty)^2)\\
&= \int_{\mathsf{S}\times C_0^c\times(0,\infty)^2}\cdots\int_{\mathsf{S}\times C_0^c\times(0,\infty)^2}\int_{\mathsf{S}\times C_0\times(0,\infty)^2} K(x_{k-1},dx_k)\,K(x_{k-2},dx_{k-1})\cdots K(x_0,dx_1)
\end{aligned}$$

$$= \int_{\mathsf{S}\times C_0^c\times(0,\infty)^2} \cdots \int_{\mathsf{S}\times C_0^c\times(0,\infty)^2} \int_{\mathsf{S}\times C_0\times(0,\infty)^2} \widetilde{h}_{u_{k-1,2}}(u_{k,2}, r_k, t_k)\, d(r_k,t_k)\, \delta_{u_{k-1,2}}(u_{k,1})\, d\sigma(u_{k,1}, u_{k,2})$$
$$K(x_{k-2}, dx_{k-1}) \cdots K(x_0, dx_1)$$
$$= \int_{\mathsf{S}\times C_0^c\times(0,\infty)^2} \cdots \int_{\mathsf{S}\times C_0^c\times(0,\infty)^2} \int_{C_0} p(u_{k-1,2}, u_{k,2})\, d\sigma(u_{k,2})\, K(x_{k-2}, dx_{k-1}) \cdots K(x_0, dx_1)$$
$$= \int_{\mathsf{S}\times C_0^c\times(0,\infty)^2} \cdots \int_{\mathsf{S}\times C_0^c\times(0,\infty)^2} \int_{C_0^c} \int_{C_0} p(u_{k-1,2}, u_{k,2})\, d\sigma(u_{k,2})\, p(u_{k-2,2}, u_{k-1,2})\, d\sigma(u_{k-1,2})$$
$$K(x_{k-3}, dx_{k-2}) \cdots K(x_0, dx_1)$$
$$= \ldots = \int_{C_0^c} \cdots \int_{C_0^c} \int_{C_0} p(u_{k-1,2}, u_{k,2})\, d\sigma(u_{k,2})\, p(u_{k-2,2}, u_{k-1,2})\, d\sigma(u_{k-1,2}) \cdots p(u_{0,2}, u_{1,2})\, d\sigma(u_{1,2})$$
$$= \mathbb{P}_{u_{0,2}}(U_1 \notin C_0, \ldots, U_{k-1} \notin C_0, U_k \in C) = \mathbb{P}_{u_{0,2}}(\sigma_{C_0}((U_n)) = k).$$

□

Definition 5.66. Let η denote the invariant probability of U_n on S and μ denote the invariant probability of M_n on $\mathsf{S}^2 \times (0,\infty)^2$.

5.4.2. Integrability of the ergodic elements

We use some calculation techniques already used before to show some quantities under consideration are in $L^1(\mu)$. Together with Lemma 5.69 this can be seen as preparation in order to use Birkhoff's ergodic theorem in the proof of this chapter's main result Theorem 5.71.

Lemma 5.67. *The expectation $\mathbb{E}_\mu[|\ln R_1|] < \infty$ is finite and for arbitrary $u_0 \in \mathsf{S}$ the expectation $\mathbb{E}_{u_0}[|\ln R_1|] < \infty$ is finite.*

Proof. Define the interval $I := [1/2, 2\sqrt{2}]$ and $I^c := (0,\infty) \setminus I = (0, 1/2) \cup (2\sqrt{2}, \infty)$. Then

$$\begin{aligned}\mathbb{E}_\mu[|\ln R_1|] &= \int_{\mathsf{S}^2\times(0,\infty)^2} |\ln r|\, \widetilde{h}_{u_0}(u, r, t)\, p_\eta(u_0)\, d(r,t)\, d\sigma(u_0, u)\\ &= \int_{\mathsf{S}^2\times(0,\infty)} |\ln r|\, r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, p_\eta(u_0)\, d\sigma(u_0, u)\\ &= \int_{\mathsf{S}^2} \int_{1/2}^{2\sqrt{2}} |\ln r|\, r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, p_\eta(u_0)\, d\sigma(u_0, u)\\ &\quad + \int_{\mathsf{S}^2} \int_{I^c} |\ln r|\, r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, p_\eta(u_0)\, d\sigma(u_0, u). \end{aligned} \tag{5.28}$$

The first summand in equation (5.28) is finite since

$$\int_{\mathsf{S}^2} \int_{1/2}^{2\sqrt{2}} |\ln r|\, r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, p_\eta(u_0)\, d\sigma(u_0, u)$$

$$
\begin{aligned}
&\leq \max_{1/2 \leq r \leq 2\sqrt{2}} |\ln r| \int_{\mathsf{S}^2} \int_{1/2}^{2\sqrt{2}} r^{N-2} \, h_{u_0}(r \cdot u) \, dr \, p_\eta(u_0) \, d\sigma(u_0, u) \\
&\leq \max_{1/2 \leq r \leq 2\sqrt{2}} |\ln r| \int_{\mathsf{S}^2} \int_0^\infty r^{N-2} \, h_{u_0}(r \cdot u) \, dr \, p_\eta(u_0) \, d\sigma(u_0, u) \\
&\leq \max_{1/2 \leq r \leq 2\sqrt{2}} |\ln r| < \infty.
\end{aligned}
$$

Let us turn to the second summand in equation (5.28). For $u_0, u \in \mathsf{S}$, $r > 0$ by Lemma 5.33,

$$
\begin{aligned}
h_{u_0}(r \cdot u) = &\frac{r^{N-1} 2^N \left(\prod_{s=1}^{N-1} u_0^s\right)^{2w} \prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times \\
&\times \sum_{k_1, \ldots, k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1} [\Gamma(k_s + w + 1) k_s!]} \prod_{s=1}^{N-1} (r \cdot u^s)^{2k_s} \times \right. \\
&\left. \times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s} \right]}{\left(\left(u_0^i\right)^2 + 1 + r^2\right)^{(w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s}} \right].
\end{aligned}
$$

By the Legendre duplication formula for the Gamma function,

$$
\begin{aligned}
&\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right) \\
&= \frac{2^{(w+1)N - 2 + 2\sum_{s=1}^{N-1} k_s}}{\sqrt{\pi}} \Gamma\left(\frac{(w+1)N - 1}{2} + \sum_{s=1}^{N-1} k_s\right) \Gamma\left(\frac{(w+1)N}{2} + \sum_{s=1}^{N-1} k_s\right).
\end{aligned}
$$

Given indices $i, j \in \{1, \ldots, N-1\}$, a permutation $\pi \in S_{N-1}$ and $u_0 \in \mathsf{S}$, we define the symbol $\widetilde{u_0}_{\pi,j,i} \coloneqq \widetilde{u_0}_{\pi,j,i}(u_0)$ by

$$
\widetilde{u_0}^s_{\pi,j,i} \coloneqq \begin{cases} u_0^{\pi(s)}, & s \neq \pi^{-1}(j), \\ u_0^i, & s = \pi^{-1}(j), \end{cases}
$$

and additionally given $r > 0$ and $u \in \mathsf{S}$ define $x_{\pi,j,i,r} \coloneqq x_{\pi,j,i,r}(u_0, u)$ by

$$
x^s_{\pi,j,i,r} \coloneqq \left(\frac{2r \, \widetilde{u_0}^s_{\pi,j,i} \, u^s}{\left(u_0^i\right)^2 + 1 + r^2} \right)^2
$$

and let

$$
F\left((x^s)_{s=1}^{N-1}\right) \coloneqq F_C^{(N-1)}\left(\frac{(w+1)N - 1}{2}, \frac{(w+1)N}{2}, w+1, \ldots, w+1, (x^s)_{s=1}^{N-1}\right)
$$

the corresponding Lauricella series to write

$$h_{u_0}(r\cdot u) = \frac{r^{N-1}2^N\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}\prod_{s=1}^{N-1}u^s}{(N-1)\,\Gamma(w)}\sum_{i=1}^{N-1}\frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1}}\times$$

$$\times\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)\prod_{s=1}^{N-1}\frac{\left(\frac{r\cdot\widetilde{u_0}_{\pi,j,i}\cdot u^s}{\left(\left(u_0^i\right)^2+1+r^2\right)}\right)^{2k_s}}{[\Gamma(k_s+w+1)k_s!]}$$

$$=\frac{r^{N-1}2^{(w+2)N-2}\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}\prod_{s=1}^{N-1}u^s}{\sqrt{\pi}\,(N-1)\,\Gamma(w)}\sum_{i=1}^{N-1}\frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1}}\times$$

$$\times\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\Gamma\left(\frac{(w+1)N-1}{2}+\sum_{s=1}^{N-1}k_s\right)\Gamma\left(\frac{(w+1)N}{2}+\sum_{s=1}^{N-1}k_s\right)\times$$

$$\times\prod_{s=1}^{N-1}\frac{\left(\frac{2r\cdot\widetilde{u_0}_{\pi,j,i}\cdot u^s}{\left(\left(u_0^i\right)^2+1+r^2\right)}\right)^{2k_s}}{[\Gamma(k_s+w+1)k_s!]}$$

$$=\frac{r^{N-1}2^{(w+2)N-2}\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}\prod_{s=1}^{N-1}u^s\,\Gamma\left(\frac{(w+1)N-1}{2}\right)\,\Gamma\left(\frac{(w+1)N}{2}\right)}{\sqrt{\pi}\,(N-1)\,\Gamma(w)\,\Gamma(w+1)^{N-1}}\times$$

$$\times\sum_{i=1}^{N-1}\frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1}}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}F\left(x_{\pi,j,i,r}(u_0,u)\right).$$

Abbreviating

$$C_{w,N}\coloneqq\frac{2^{(w+2)N-2}\,\Gamma\left(\frac{(w+1)N-1}{2}\right)\,\Gamma\left(\frac{(w+1)N}{2}\right)}{\sqrt{\pi}\,(N-1)\,\Gamma(w)\,\Gamma(w+1)^{N-1}}$$

we bound the second summand in equation (5.28) according to

$$\int_{\mathsf{S}^2}\int_{I^c}|\ln r|\,r^{N-2}\,h_{u_0}(r\cdot u)\,dr\,p_\eta(u_0)\,d\sigma(u_0,u)$$

$$\le C_{w,N}\int_{\mathsf{S}^2}\int_{I^c}\frac{|\ln r|\,r^{2N-3}}{(1+r^2)^{N-1+Nw}}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}F\left(x_{\pi,j,i,r}(u_0,u)\right)dr\,p_\eta(u_0)\,d\sigma(u_0,u).$$

Note,

$$\sup\left\{\sum_{s=1}^{N-1}\left(x^s_{\pi,j,i,r}(u_0,u)\right)^{1/2}:u_0,u\in\mathsf{S},\pi\in S_{N-1},r\in I^c,i,j=1,\dots,N-1\right\}$$

$$= \sup \left\{ \frac{2r}{\left(u_0^i\right)^2 + 1 + r^2} \left\langle \widetilde{u_0}_{\pi,j,i}, u \right\rangle : u_0, u \in \mathsf{S}, \pi \in S_{N-1}, r \in I^c, i, j = 1, \ldots, N-1 \right\}$$
$$\leq \sup \left\{ \frac{2r\sqrt{1+\xi}}{\xi + 1 + r^2} : \xi \in [0,1], r \in I^c \right\}.$$

Since the derivative

$$\frac{\partial}{\partial \xi} \left[\frac{2r\sqrt{1+\xi}}{\xi + 1 + r^2} \right] = \frac{\frac{r}{\sqrt{1+\xi}}(\xi + 1 + r^2) - 2r\sqrt{1+\xi}}{\left(\xi + 1 + r^2\right)^2} = \frac{r\left(r^2 - 1 - \xi\right)}{\sqrt{1+\xi}\left(\xi + 1 + r^2\right)^2} \neq 0$$

does not change its sign for any fixed $r \in I^c$ on the interval $\xi \in [0,1]$ due to $1 + \xi \in [1,2] \not\ni r^2$ it follows

$$\sup \left\{ \frac{2r\sqrt{1+\xi}}{\xi + 1 + r^2} : \xi \in [0,1], r \in I^c \right\} \leq \max \left\{ \sup_{r \in I^c} \frac{2r}{1 + r^2}, \sup_{r \in I^c} \frac{2\sqrt{2}r}{2 + r^2} \right\}.$$

For $a \in \{1, 2\}$ it holds

$$\frac{d}{dr} \left[\frac{r}{a + r^2} \right] = \frac{(a + r^2) - 2r^2}{(a + r^2)^2} = \frac{a - r^2}{(a + r^2)^2} \neq 0,$$

with unique roots $r = \sqrt{a}$ outside the domain, $\lim_{r \downarrow 0} \frac{r}{a+r^2} = \lim_{r \uparrow \infty} \frac{r}{a+r^2} = 0$ and $\frac{r}{a+r^2} \geq 0$ and $\frac{2 \cdot 1}{1+1^2} = \frac{2\sqrt{2} \cdot \sqrt{2}}{2 + \sqrt{2}^2} = 1$.
Consequently,

$$\max \left\{ \sup_{r \in I^c} \frac{2r}{1 + r^2}, \sup_{r \in I^c} \frac{2\sqrt{2}r}{2 + r^2} \right\} < 1.$$

Turning back to the integral, we finally achieve using the argument along the lines of the proof of Claim 2 in the proof of Proposition 5.64

$$C_{w,N} \int_{\mathsf{S}^2} \int_{I^c} \frac{|\ln r|\, r^{2N-3}}{\left(1 + r^2\right)^{N-1+Nw}} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} F\left(x_{\pi,j,i,r}(u_0, u)\right) dr\, p_\eta(u_0)\, d\sigma(u_0, u)$$
$$\leq \text{const} \cdot C_{w,N} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} \int_{\mathsf{S}^2} \int_{I^c} \frac{|\ln r|\, r^{2N-3}}{\left(1 + r^2\right)^{N-1+Nw}} dr\, p_\eta(u_0)\, d\sigma(u_0, u)$$
$$\leq \text{const} \cdot C_{w,N} \cdot \frac{(N-1)^2}{2} \int_0^\infty \frac{|\ln r|\, r^{2N-3}}{\left(1 + r^2\right)^{N-1+Nw}} dr < \infty.$$

This shows the finiteness of the second summand in equation (5.28) also, and therefore finishes the proof of the first assertion $\mathbb{E}_\mu[|\ln R_1|] < \infty$. The second assertion $\mathbb{E}_{u_0}[|\ln R_1|] < \infty$ for $u_0 \in \mathsf{S}$ arbitrary, follows along the lines. $\square$

Definition 5.68. Let us denote the mapping

$$\ln^+ : (0,\infty) \to [0,\infty), \quad x \mapsto \ln^+(x) := \max\{0, \ln x\} = \begin{cases} 0, & x \le 1, \\ \ln x, & x > 1, \end{cases}$$

as positive part of the natural logarithm.

Lemma 5.69. *The expectation $\mathbb{E}_\mu[\ln^+ T_1] < \infty$ is finite.*

Proof. According to [10, Equation (2.1)], the hitting time of the origin of a Bessel process started at $x > 0$ is distributed as $\frac{\sqrt{x}}{2G}$ where $G \sim \text{Gamma}(w)$ is Gamma-distributed, that is

$$\mathbb{P}(G \in dt) = \frac{1}{\Gamma(w)} t^{w-1} e^{-t}\, dt, \quad t > 0.$$

Letting $(G, G_0, \dots, G_{N-1})$ independent and $\Gamma(w)$-distributed we bound

$$\mathbb{E}_\mu[\ln^+ T_1] \le \int_{\mathsf{S}} \ln^+ \left(\min \left\{ \frac{\sqrt{u_0^1}}{2G_0}, \frac{\sqrt{u_0^1}}{2G_1}, \dots, \frac{\sqrt{u_0^{N-1}}}{2G_{N-1}} \right\} \right) \eta(du_0) \le \mathbb{E}\left[\ln^+ \frac{1}{2G} \right]$$
$$= \int_0^\infty \ln^+ \left(\frac{1}{2t} \right) \frac{1}{\Gamma(w)} t^{w-1} e^{-t}\, dt = \frac{1}{2^w \Gamma(w)} \int_0^\infty s \cdot \exp\left(-s \cdot w - e^{-s}/2 \right) ds < \infty.$$

□

5.4.3. Application of Birkhoff's ergodic theorem

Definition 5.70. Let $\mathbb{P}_{u_0}$ denote the probability measure associated to the density $\widetilde{h}_{u_0}$ from equation (5.17) and $\mathbb{E}_{u_0}$ the corresponding expectation. Let $\mathbb{E}_\eta[\cdot] = \int_{\mathsf{S}} \mathbb{E}_{u_0}[\cdot]\, \eta(du_0)$.

Theorem 5.71. *(for the second part cf. [13, Theorem 9.1.1].) If $\mathbb{E}_\eta[\ln R_1] > 0$ then $\tau_\infty = \infty$ a.s. and if $\mathbb{E}_\eta[\ln R_1] < 0$ then $\tau_\infty < \infty$ a.s.*

Proof. In view of Corollary 5.29 it is desirable to use Cauchy's root test on the series. By Proposition 5.64 combined with Theorem 5.42, Lemma 5.67 and Lemma 5.69 tell us, that we may apply Birkhoff's theorem for Markov chains Theorem 5.43 on the the HMM M to deduce for μ-almost all $x = (u_1, u_2, r, t) \in \mathsf{S}^2 \times (0,\infty)^2$

$$\frac{1}{k} \sum_{j=1}^{k} \ln R_j \to \mathbb{E}_\mu[\ln R_1] = \int_{\mathsf{S}^2 \times (0,\infty)^2} \ln r \cdot \widetilde{h}_{u_0}(u_1, r, t)\, d(r,t)\, d\sigma(u_1)\, \eta(du_0)$$
$$= \int_{\mathsf{S}} \mathbb{E}_{u_0}[\ln R_1]\, \eta(du_0) = \mathbb{E}_\eta[\ln R_1]$$

$\mathbb{P}_x$-a.s. Similarly,

$$\frac{1}{k} \sum_{j=1}^{k} \ln^+ T_j \xrightarrow[k\to\infty]{\mathbb{P}_x - a.s.} \mathbb{E}_\mu[\ln^+ T_1] < \infty.$$

Thus, if $\mathbb{E}_\eta[\ln R_1] < 0$ then a.s.

$$\begin{aligned}
\limsup_{k\to\infty}\left(T_k\prod_{j=1}^{k-1}R_j^2\right)^{1/k} &= \exp\left(\limsup_{k\to\infty}\left(\frac{\ln T_k}{k}+\frac{2}{k}\sum_{j=1}^{k-1}\ln R_j\right)\right)\\
&\le \exp\left(\limsup_{k\to\infty}\left(\frac{\ln^+ T_k}{k}+\frac{2}{k}\sum_{j=1}^{k-1}\ln R_j\right)\right)\\
&= \exp\left(\limsup_{k\to\infty}\left(\frac{1}{k}\sum_{j=1}^{k}\ln^+ T_j-\frac{1}{k}\sum_{j=1}^{k-1}\ln^+ T_j+\frac{2}{k}\sum_{j=1}^{k-1}\ln R_j\right)\right)\\
&= \exp\left(2\mathbb{E}_\eta[\ln R_1]\right) < 1,
\end{aligned}$$

which by Cauchy's root test implies $\tau_\infty < \infty$ a.s.
On the other hand, Theorem 5.44 implies that for μ-almost all $x \in \mathsf{S}^2 \times (0,\infty)^2$, $\mathbb{P}_x$-almost surely, for $A := \{l \in \mathbb{N} : T_l \ge 1\}$ it holds $|A| = \infty$. In other words, there exists an increasing subsequence $(k_l)_{l\in\mathbb{N}}$ such that $\{k_l : l \in \mathbb{N}\} = A$ and we infer in the case $\mathbb{E}_\eta[\ln R_1] > 0$

$$\begin{aligned}
\limsup_{k\to\infty}\left(T_k\prod_{j=1}^{k-1}R_j^2\right)^{1/k} &= \exp\left(\limsup_{k\to\infty}\left(\frac{\ln T_k}{k}+\frac{2}{k}\sum_{j=1}^{k-1}\ln R_j\right)\right)\\
&\ge \exp\left(\limsup_{l\to\infty}\left(\frac{\ln T_{k_l}}{k_l}+\frac{2}{k_l}\sum_{j=1}^{k_l-1}\ln R_j\right)\right) \ge \exp\left(\limsup_{l\to\infty}\left(\frac{2}{k_l}\sum_{j=1}^{k_l-1}\ln R_j\right)\right)\\
&= \exp\left(2\mathbb{E}_\eta[\ln R_1]\right) > 1.
\end{aligned}$$

This finishes the proof. □

5.4.4. Computation of the integrand $\ln R_1$ in Theorem 5.71

With the intention to use Theorem 5.71 more explicitly, let us give an explicit formula for $\mathbb{E}_{u_0}[\ln R_1]$. We will then rederive [10, Theorem 1.1 (ii)] and in chapter 5 we will show that for $\nu \ge -0.03$ the particle system does almost surely not become extinct in Theorem 5.78. As in Definition A.6 in the appendix let ψ denote the Digamma function.

Lemma 5.72. *The integrand in Theorem 5.71 is given by*

$$\begin{aligned}
\mathbb{E}_{u_0}[\ln R_1] = &\frac{\sum_{s=1}^{N-1}\ln\left(1+(u_0^s)^2\right)}{2\,(N-1)}+\\
&+\frac{2\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\,(N-1)!\,\Gamma(w)}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\left[\frac{\Gamma\left(wN+\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\Gamma(k_s+w+1)}\times\right.
\end{aligned}$$

$$\times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1\right)^{wN+\sum_{s=1}^{N-1}k_s}} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s}\right] \times$$
$$\times \frac{\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2}\Bigg].$$

Proof. Applying Lemma 5.33 and Fubini's theorem results in

$$\mathbb{E}_{u_0}[\ln R_1] = \int_{\mathsf{S}\times(0,\infty)} \ln r\cdot r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, d\sigma(u)$$
$$= \int_{\mathsf{S}\times(0,\infty)} \ln r\cdot r^{2N-3}\, \frac{2^N\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}\prod_{s=1}^{N-1}u^s}{(N-1)\Gamma(w)}$$
$$\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\Bigg[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]}\prod_{s=1}^{N-1}(r\cdot u^s)^{2k_s}\times$$
$$\times\sum_{i=1}^{N-1}\frac{\left(u_0^i\right)^{2w}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\Bigg]\, dr\, d\sigma(u).$$
$$= \frac{2^N\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\Gamma(w)}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\Bigg[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]}\int_{\mathsf{S}}\prod_{s=1}^{N-1}(u^s)^{2k_s+1}\, d\sigma(u)\times$$
$$\times\sum_{i=1}^{N-1}\left(u_0^i\right)^{2w}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]\times$$
$$\times\int_0^\infty\frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}\ln r}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\, dr\Bigg]. \tag{5.29}$$

By the constructed symmetry and [4, Equation (8)] it holds

$$\int_{\mathsf{S}}\prod_{s=1}^{N-1}(u^s)^{2k_s+1}\, d\sigma(u) = \frac{1}{(N-1)!}\int_{\{u\in(0,\infty)^{N-1}:\|u\|=1\}}\prod_{s=1}^{N-1}(u^s)^{2k_s+1}\, d\sigma(u)$$
$$= \frac{1}{2^{N-1}(N-1)!}\int_{\{u\in\mathbb{R}^{N-1}:\|u\|=1\}}\prod_{s=1}^{N-1}(u^s)^{2k_s+1}\, d\sigma(u)$$
$$= \frac{\prod_{s=1}^{N-1}(k_s!)}{2^{N-2}(N-1)!\,(N-2+\sum_{s=1}^{N-1}k_s)!}. \tag{5.30}$$

By Lemma A.7 in the Appendix

$$\int_0^\infty \frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}\ln r}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr$$

$$= \frac{\ln\left(\left(u_0^i\right)^2+1\right)+\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{4\left(\left(u_0^i\right)^2+1\right)^{wN+\sum_{s=1}^{N-1}k_s}}\times$$

$$\times\frac{\Gamma\left(N-1+\sum_{s=1}^{N-1}k_s\right)\Gamma\left(wN+\sum_{s=1}^{N-1}k_s\right)}{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}$$

$$= \frac{\ln\left(\left(u_0^i\right)^2+1\right)+\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2}\times$$

$$\times\int_0^\infty \frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr$$

$$= \frac{\ln\left(\left(u_0^i\right)^2+1\right)}{2}\int_0^\infty \frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr+$$

$$+\frac{\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2}\times$$

$$\times\frac{\Gamma\left(N-1+\sum_{s=1}^{N-1}k_s\right)\Gamma\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2\left(\left(u_0^i\right)^2+1\right)^{wN+\sum_{s=1}^{N-1}k_s}\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}. \tag{5.31}$$

Plugging (5.30) and (5.31) into (5.29), recalling Remark 5.37, this implies

$$\frac{2^N\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\Gamma(w)}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\left[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]}\int_{\mathsf{S}}\prod_{s=1}^{N-1}(u^s)^{2k_s+1}\,d\sigma(u)\times\right.$$

$$\times\sum_{i=1}^{N-1}\left(u_0^i\right)^{2w}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]\times$$

$$\left.\times\int_0^\infty \frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}\ln r}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr\right]$$

$$=\frac{\sum_{s=1}^{N-1}\ln\left(1+(u_0^s)^2\right)}{2\,(N-1)}+\frac{2\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\,(N-1)!\,\Gamma(w)}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\left[\frac{\Gamma\left(wN+\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\Gamma(k_s+w+1)}\times\right.$$

$$\times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1\right)^{wN+\sum_{s=1}^{N-1} k_s}} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s}\right] \times$$
$$\times \frac{\psi\left(N-1+\sum_{s=1}^{N-1} k_s\right) - \psi\left(wN+\sum_{s=1}^{N-1} k_s\right)}{2}\Bigg].$$

□

As first application, we immediately obtain

Corollary 5.73. *For $\nu \geq 2/N$ the particle system does almost surely not become extinct.*

Proof. In view of Theorem 5.71 it suffices to show $\mathbb{E}_{u_0}[\ln R_1] > 0$ uniformly for all $u_0 \in \mathsf{S}$. This is the case by Lemma 5.72: Since the digamma function restricted to $(0, \infty)$ is strictly monotonously increasing, for $\nu \geq 2/N \Leftrightarrow w \leq (N-1)/N$ the difference

$$\frac{\psi\left(N-1+\sum_{s=1}^{N-1} k_s\right) - \psi\left(wN+\sum_{s=1}^{N-1} k_s\right)}{2} \geq 0$$

is non negative and it holds

$$\inf_{u_0 \in \mathsf{S}} \mathbb{E}_{u_0}[\ln R_1] \geq \inf_{u_0 \in \mathsf{S}} \frac{\sum_{s=1}^{N-1} \ln\left(1+(u_0^s)^2\right)}{2\,(N-1)} = \frac{\ln 2}{2\,(N-1)} > 0.$$

□

Remark 5.74. Corollary 5.73 recovers [10, Theorem 1.1 (ii)]. There, the argumentation suffices if each particle is only reflected upon hitting 0 instead of performing an actual jump. The basic idea is to use the fact that for $Z = (Z^1, \dots, Z^N)$ consisting of $N-1$ independent Bessel processes of parameter ν, the process $\|Z\|$ is a Bessel process with parameter $N \cdot \nu$. Setting $\nu := 2/N$ leading to a Bessel process with parameter $N \cdot 2/N = 2$ has the law of $\|(B^1, B^2)\|$ with two independent Brownian motions and thereby never hits 0. Observe, also by Lemma 5.72 for $\nu := 2/N$ it holds with notation $X_{0-} := u_0$

$$\mathbb{E}_{u_0}[\ln R_1] = \frac{1}{N-1} \sum_{s=1}^{N-1} \ln\|(u_0^1, \dots, u_0^s, u_0^s, \dots, u_0^{N-1})\| = \mathbb{E}_{u_0}[\ln\|X_0\|] > 0$$

as expected since by the explanation above $\ln\|X_t\| \overset{\mathcal{D}}{=} \ln\|(B^1, B^2)\|$ for $\nu = 2/N$ is a local martingale.

If the particles actually do jump (and therefore interact) this generally may be taken into account by considering $\inf_{u_0 \in \mathsf{S}} \mathbb{E}_{u_0}[\ln R_1]$ instead where the contributions of different p^i's might outweigh each other. This approach still neglects the ratios of the particles infinitesimally ahead of the jump times τ_n so we do not need to know the stationary distribution η any more explicit.

5.5. Three particles

In this section we fix $N := 3$ and in the same fashion as in the proof of Corollary 5.73 we want to find regimes of parameter values ν with $\inf_{u_0 \in \mathsf{S}} \mathbb{E}_{u_0}[\ln R_1] > 0$. Then Criterion 5.71 implies non-extinction. For this purpose we may without loss of generality assume $w \geq 2/3$ in what follows. Lemma 5.72 specifies for $N = 3$ using the angle parametrization $\cos\varphi_0 = u_0^1$ and $\sin\varphi_0 = u_0^2$ to

$$\begin{aligned}
\mathbb{E}_{\varphi_0}[\ln R_1] &= \frac{\ln(1+\cos^2\varphi_0) + \ln(1+\sin^2\varphi_0)}{4} + \\
&+ \frac{(\cos\varphi_0 \sin\varphi_0)^{2w}}{2\,\Gamma(w)} \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(3w+k_1+k_2)}{\Gamma(k_1+w+1)\Gamma(k_2+w+1)} \times \\
&\times \Bigg(\frac{\cos^{2w}(\varphi_0)}{(1+\cos^2\varphi_0)^{3w+k_1+k_2}} \cdot \frac{\psi(2+k_1+k_2) - \psi(3w+k_1+k_2)}{2} \cdot \\
&\qquad \cdot \Big(2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0 + \cos^{2k_1}\varphi_0 \cos^{2k_2}\varphi_0 \Big) + \\
&+ \frac{\sin^{2w}(\varphi_0)}{(1+\sin^2\varphi_0)^{3w+k_1+k_2}} \cdot \frac{\psi(2+k_1+k_2) - \psi(3w+k_1+k_2)}{2} \cdot \\
&\qquad \cdot \Big(2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0 + \sin^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0 \Big) \Bigg).
\end{aligned}$$

Let us recall Example 5.39; e.g. the summand with $2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0$ in it corresponds to the situation $i = j = 1$, the one with $\cos^{2k_1}\varphi_0 \cos^{2k_2}\varphi_0$ to $i = 1$ and $j = 2$.
Since the first term $\frac{\ln(1+\cos^2\varphi_0)+\ln(1+\sin^2\varphi_0)}{4} \geq 0$ is non-negative which corresponds to the particle system performing a jump, we can allow the remainder to be slightly negative accordingly which corresponds to the continuous drift to be more negative.

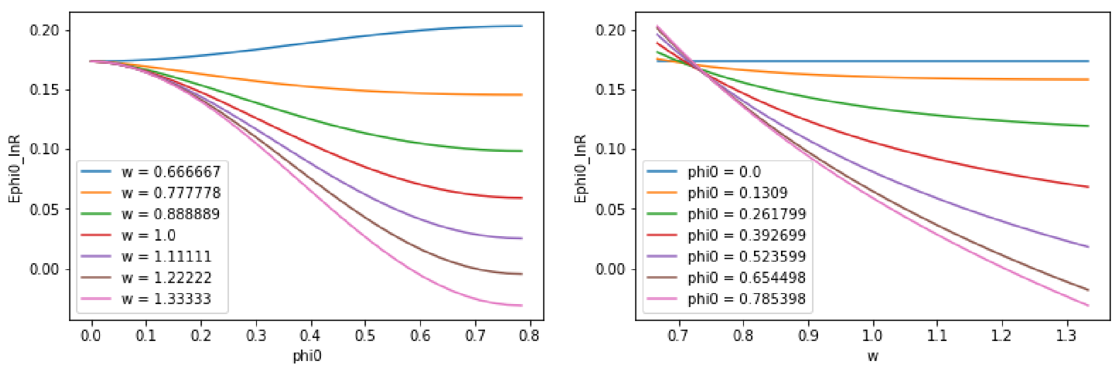


Figure 5.5.: The functional $\mathbb{E}_{\varphi_0}[\ln R_1]$ is plotted for different values of $\varphi_0 \in (0, \pi/4]$ and $w \in [2/3, 4/3]$. The exact numerical values may be inaccurate but the coarse shapes seem to be reasonable. The python code can be found in Section C.2 in the Appendix.

For w not much larger than $2/3$ the negative drift is rather small so starting more balanced results in $\ln R_1$ having larger expectation since then the conditioning on survival heavily affects the distribution of the particles. If w is sufficiently large the plotted functional $\varphi_0 \mapsto \mathbb{E}_{\varphi_0}[\ln R_1]$ should be monotonously decreasing as from a more balanced start the negative drift has more time to take effect. Unfortunately, the author was not able to give a concise proof. For $\varphi_0 \downarrow 0$ the expectation must be $\lim_{\varphi\downarrow 0} \mathbb{E}_{\varphi_0}[\ln R_1] = \frac{1}{2} \cdot \ln \sqrt{2} = \frac{\ln 2}{4} \approx 0.173287$ for any $w > 0$. For $\varphi_0 > 0$ small which corresponds to an highly unbalanced ratio there are the competing effects that a more negative drift generally shrinks the particles more but on the other hand shortens the time for the drift to take affect. Thus, the expectation $\mathbb{E}_{\varphi_0}[\ln R_1]$ as a function of w has a minimum somewhere strictly in between $w \in (2/3, 4/3)$. For φ_0 sufficiently large the time shortening effect of the more negative drift is negligible and $w \mapsto \mathbb{E}_{\varphi_0}[\ln R_1]$ is decreasing. Numerical approximations suggest that the root of $w \mapsto \mathbb{E}_{\pi/4}[\ln R_1]$ is located in the interval $(1.20355, 1.20365)$ and therefore the particle system does almost surely not become extinct for $\nu \geq -0.4071$. Our analytical treatment is not quite this fine but still we are able to deduce almost sure non-extinction for slightly negative values of ν.

Lemma 5.75. *For $\nu \geq 0.2404$ the particle system does almost surely not become extinct.*

Proof. Because the derivative of the digamma function, the trigamma function, is strictly decreasing when restricted to $(0, \infty)$, the difference $\psi'(3w + k) - \psi'(2 + k)$ is negative for all $k \geq 0$. Thereby the function $k \mapsto \psi(3w + k) - \psi(2 + k)$ is recognized to be strictly decreasing yielding the uniform bound

$$\psi(2 + k_1 + k_2) - \psi(3w + k_1 + k_2) \geq \psi(2) - \psi(3w)$$

whence

$$\inf_{\varphi_0} \mathbb{E}_{\varphi_0}[\ln R_1] \geq \inf_{\varphi_0} \frac{\ln(2 + \sin^2 \varphi_0 \cos^2 \varphi_0)}{4} + \frac{\psi(2) - \psi(3w)}{2} \geq \frac{\ln 2}{4} + \frac{\psi(2) - \psi(3w)}{2}.$$

Here, in spirit of Remark 5.37, we have used that

$$\begin{aligned}
&\frac{(\cos \varphi_0 \sin \varphi_0)^{2w}}{2\,\Gamma(w)} \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(3w + k_1 + k_2)}{\Gamma(k_1 + w + 1)\Gamma(k_2 + w + 1)} \times \\
&\quad \times \left(\frac{\cos^{2w}(\varphi_0)}{(1 + \cos^2 \varphi_0)^{3w+k_1+k_2}} \cdot \left(2 \cos^{2k_1} \varphi_0 \sin^{2k_2} \varphi_0 + \cos^{2k_1} \varphi_0 \cos^{2k_2} \varphi_0\right)\right) \\
&= \frac{(\cos \varphi_0 \sin \varphi_0)^{2w}}{2\,\Gamma(w)} \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(3w + k_1 + k_2)}{\Gamma(k_1 + w + 1)\Gamma(k_2 + w + 1)} \times \\
&\quad \times \left(\frac{\sin^{2w}(\varphi_0)}{(1 + \sin^2 \varphi_0)^{3w+k_1+k_2}} \cdot \left(2 \cos^{2k_1} \varphi_0 \sin^{2k_2} \varphi_0 + \sin^{2k_1} \varphi_0 \sin^{2k_2} \varphi_0\right)\right) = \frac{1}{2}.
\end{aligned}$$

This shows the assertion for $w \leq 0.8798$, respectively, $\nu = 2 \cdot (1 - w) \geq 0.2404$. □

In order to achieve finer estimates we split the domain of summation $(k_1, k_2) \in \mathbb{N}_0^2$ into $\{(0,0)\}$ and $\mathbb{N}_0^2 \setminus \{(0,0)\}$. With the following definition we measure the contribution $B(w, \sin^2\varphi_0)$ induced by the term with $k_1 = k_2 = 0$.

Definition 5.76. Let

$$B : (0,\infty) \times [0, 1/2] \to [0,1];$$
$$B(w,\xi) \coloneqq \frac{3\,\Gamma(3w)}{2\,\Gamma(w)\,\Gamma(w+1)^2} \cdot (\xi(1-\xi))^w \cdot \left(\left(\frac{1-\xi}{(2-\xi)^3}\right)^w + \left(\frac{\xi}{(1+\xi)^3}\right)^w \right).$$

Lemma 5.77. *For $\nu \geq 0$ the particle system does almost surely not become extinct.*

Proof. Firstly according to Definition 5.76 and by using

$$\psi(2 + k_1 + k_2) - \psi(3w + k_1 + k_2) \geq \psi(3) - \psi(3w+1)$$

for $k_1 + k_2 \geq 1$ and for the equality the identity $\psi(x+1) - \psi(x) = 1/x$ for positive $x > 0$:

$$\begin{aligned} \mathbb{E}_{\varphi_0}[\ln R_1^2] = 2\,\mathbb{E}_{\varphi_0}[\ln R_1] &\geq \frac{\ln(2+\cos^2\varphi_0 \sin^2\varphi_0)}{2} + \\ &+ B(w,\sin^2\varphi_0)\cdot(\psi(2) - \psi(3w)) + (1 - B(w,\sin^2\varphi_0))\cdot(\psi(3) - \psi(3w+1)) \\ &= \frac{\ln(2+\cos^2\varphi_0 \sin^2\varphi_0)}{2} + \psi(3) - \psi(3w+1) + B(w,\sin^2\varphi_0)\cdot\left(\frac{1}{3w} - \frac{1}{2}\right). \end{aligned} \tag{5.32}$$

By introducing the abbreviation $\xi \coloneqq \sin^2\varphi_0$ and by using the generalized Bernoulli's inequality $(1+x)^r \leq 1 + r\cdot x$ for $x > -1, 0 \leq r \leq 1$ it follows

$$\begin{aligned} B(w,\xi)\cdot\left(\frac{1}{3w} - \frac{1}{2}\right) &= -(3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\left(\left(\frac{(1-\xi)^2\xi}{(2-\xi)^3}\right)^w + \left(\frac{(1-\xi)\xi^2}{(1+\xi)^3}\right)^w\right) \\ &\geq -(3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\left(2 - 2w + w\cdot\left(\frac{(1-\xi)^2\xi}{(2-\xi)^3} + \frac{(1-\xi)\xi^2}{(1+\xi)^3}\right)\right). \end{aligned} \tag{5.33}$$

Writing $\zeta \coloneqq (1-\xi)\cdot\xi = \cos^2\varphi_0 \sin^2\varphi_0$ we infer

$$\begin{aligned} &\frac{(1-\xi)^2\xi}{(2-\xi)^3} + \frac{(1-\xi)\xi^2}{(1+\xi)^3} = (1-\xi)\xi\cdot\frac{(1-\xi)\cdot(1+\xi)^3 + \xi\cdot(2-\xi)^3}{((2-\xi)\cdot(1+\xi))^3} \\ &= (1-\xi)\xi\cdot\frac{1 + 3\xi + 3\xi^2 + \xi^3 - \xi - 3\xi^2 - 3\xi^3 - \xi^4 + 8\xi - 12\xi^2 + 6\xi^3 - \xi^4}{(2 + 2\xi - \xi - \xi^2)^3} \\ &= (1-\xi)\xi\cdot\frac{1 + 10\xi - 12\xi^2 + 4\xi^3 - 2\xi^4}{(2+\xi-\xi^2)^3} = \zeta\cdot\frac{1 + 10\zeta - 2\zeta^2}{(2+\zeta)^3} \\ &= \frac{-2(\zeta^3 + 6\zeta^2 + 12\zeta + 8) + 22(\zeta^2 + 4\zeta + 4) - 63(\zeta + 2) + 54}{(\zeta+2)^3} \end{aligned} \tag{5.34}$$

$$= \frac{-2(\zeta+2)^3 + 22(\zeta+2)^2 - 63(\zeta+2) + 54}{(\zeta+2)^3},$$

whence

$$\mathbb{E}_{\varphi_0}[\ln R^2] \geq \frac{\ln(2+\zeta)}{2} + \psi(3) - \psi(3w+1) -$$
$$- (3w-2) \cdot \frac{\Gamma(3w)}{4\,\Gamma(w+1)^3} \left(2 - 2w + w \cdot \frac{-2(\zeta+2)^3 + 22(\zeta+2)^2 - 63(\zeta+2) + 54}{(\zeta+2)^3}\right) =: h(\zeta). \tag{5.35}$$

In the relevant domain $w \in [0.8798, 1], \zeta \in [0, 1/4]$ differentiating with respect to ζ computes to

$$\frac{d}{d\zeta} h(\zeta) = \frac{1}{2(2+\zeta)} - (3w-2) \cdot w \cdot \frac{\Gamma(3w)}{4\,\Gamma(w+1)^3} \left(\frac{-22(\zeta+2)^2 + 126(\zeta+2) - 162}{(\zeta+2)^4}\right)$$
$$= \frac{1}{2(2+\zeta)} \left(1 - (3w-2) \cdot w \cdot \frac{\Gamma(3w)}{\Gamma(w+1)^3} \left(\frac{-11(\zeta+2)^2 + 63(\zeta+2) - 81}{(\zeta+2)^3}\right)\right). \tag{5.36}$$

It holds

$$0 < (3w-2) \cdot w \leq 1 \tag{5.37}$$

and by

$$\frac{d}{dw} \frac{\Gamma(3w)}{\Gamma(w+1)^3}$$
$$= \frac{3\,\Gamma'(3w)\,\Gamma(w+1)^3 - 3\,\Gamma(3w)\,\Gamma(w+1)^2\,\Gamma'(w+1)}{\Gamma(w+1)^6} = \frac{3\,\Gamma(3w)}{\Gamma(w+1)^3} \cdot (\psi(3w) - \psi(w+1)) > 0$$

also

$$\frac{\Gamma(3w)}{\Gamma(w+1)^3} \leq \frac{\Gamma(3 \cdot 1)}{\Gamma(1+1)^3} = 2. \tag{5.38}$$

Furthermore, due to

$$\frac{d}{d\zeta} \frac{-11(\zeta+2)^2 + 63(\zeta+2) - 81}{(\zeta+2)^3} = \frac{11(\zeta+2)^2 - 126(\zeta+2) + 243}{(\zeta+2)^4}$$
$$= \frac{11}{(\zeta+2)^4} \left(\zeta^2 + 4\zeta + 4 - \frac{126}{11}\zeta - \frac{252}{11} + \frac{243}{11}\right) = \frac{11}{(\zeta+2)^4} \left(\frac{5}{11} - \zeta\right)(7 - \zeta) > 0$$

we achieve the estimate

$$\frac{-11(\zeta+2)^2 + 63(\zeta+2) - 81}{(\zeta+2)^3} \leq \frac{-11 \cdot (9/4)^2 + 63 \cdot 9/4 - 81}{(9/4)^3}$$
$$= -11 \cdot 4/9 + 7 \cdot 4^2/9 - 4^3/9 = 4/9. \tag{5.39}$$

Using (5.37), (5.38) and (5.39) in (5.36) we deduce

$$\frac{d}{d\zeta}h(\zeta) = \frac{1}{2(2+\zeta)}\left(1-(3w-2)\cdot w\cdot\frac{\Gamma(3w)}{\Gamma(w+1)^3}\left(\frac{-11(\zeta+2)^2+63(\zeta+2)-81}{(\zeta+2)^3}\right)\right)$$
$$\geq \frac{1}{2(2+\zeta)}\left(1-2\cdot 4/9\right) > 0$$

and therefore by recalling the bound given in (5.35) and equation (5.34)

$$\mathbb{E}_{\varphi_0}[\ln R^2]$$
$$\geq \frac{\ln(2+\zeta)}{2}+\psi(3)-\psi(3w+1)-(3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\left(2-2w+w\cdot\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}\right)$$
$$\geq \ln(2)/2+\psi(3)-\psi(3w+1)-(3w-2)\cdot(1-w)\cdot\frac{\Gamma(3w)}{2\,\Gamma(w+1)^3}.$$

Because the gamma function $\Gamma(x)$ is strictly increasing for values larger than $x > 1.46163..$, the unique positive root of the digamma function, it holds $\Gamma(3w)/(2\Gamma(w+1)^3) \leq \Gamma(1.8798)^{-3}$ and we may further estimate:

$$\mathbb{E}_{\varphi_0}[\ln R_1^2] \geq \ln(2)/2+\psi(3)-\psi(3w+1)-(3w-2)\cdot(1-w)\,/\Gamma(1.8798)^3 =: \alpha(w).$$

Differentiating with respect to w leads to

$$\alpha'(w) = -3\psi'(3w+1)-(3\cdot(1-w)-(3w-2))/\Gamma(1.8798)^3 = 3\left(\frac{2w-5/3}{\Gamma(1.8798)^3}-\psi'(3w+1)\right).$$

Differentiating once again results in

$$\alpha''(w) = 9\cdot\left(\frac{2}{3\,\Gamma(1.8798)^3}-\psi'(3w+1)\right) > 0,$$

because $\psi''(x) < 0$ for all $x > 0$. Due to $\alpha'(0.8798) \approx -0.627691 < 0 < 0.296638 \approx \alpha'(1)$ in the interval $w \in [0.8798, 1]$ there is an unique global minimum of α in the interior $(0.8798, 1)$. By $\alpha'(0.9611) \approx -0.000299177 < 0 < 0.000466669 \approx \alpha'(0.9612)$ we can narrow down more and estimate

$$\alpha(w) \geq \ln(2)/2+\psi(3)-\psi(3\cdot 0.9612)-(3\cdot 0.9612-2)\cdot(1-0.9611)/\Gamma(1.8798)^3$$
$$\approx 0.00736878 > 0.$$

We have now shown $\mathbb{E}_{\varphi_0}[\ln R_1^2] > 0$ uniformly in $(w,\varphi_0) \in [0.8798, 1]\times[0,\pi/4]$, which shows the assertion. □

In the case of $N = 2$ particles there is a critical parameter value $\nu = 0$ (cf. [9, Theorem 1.1 (i)]). This is not longer true for $N = 3$ particles as the following main result shows. The assumption $\nu \geq 0$ in the preceding lemma was of technical nature only; we may replace Bernoulli's inequality adequately by the estimate $x^w + y^w \leq (x+y)^w$ valid for $w \geq 1$ and $x, y > 0$.

Theorem 5.78. *For $\nu \geq -0.03$ the particle system does almost surely not become extinct.*

Proof. We want to recycle a few computations from the proof of Lemma 5.77 and again write $\xi := \sin^2 \varphi_0$ and $\zeta := \xi \cdot (1-\xi)$.

Firstly, again by (5.32) and (5.33)

$$\mathbb{E}_{\varphi_0}[\ln R_1^2] = \mathbb{E}_{\arcsin\sqrt{\xi}}[\ln R_1^2] \geq \frac{\ln(2+\cos^2\varphi_0 \sin^2\varphi_0)}{2} + \psi(3) - \psi(3w+1) - \\ - (3w-2) \cdot \frac{\Gamma(3w)}{4\,\Gamma(w+1)^3} \left(\left(\frac{(1-\xi)^2\xi}{(2-\xi)^3} \right)^w + \left(\frac{(1-\xi)\xi^2}{(1+\xi)^3} \right)^w \right).$$

For $w \geq 1$ we may now apply the inequality $x^w + y^w \leq (x+y)^w$ for $x, y > 0$ and attain with calculation (5.34)

$$\left(\frac{(1-\xi)^2\xi}{(2-\xi)^3} \right)^w + \left(\frac{(1-\xi)\xi^2}{(1+\xi)^3} \right)^w \leq \left(\frac{(1-\xi)^2\xi}{(2-\xi)^3} + \frac{(1-\xi)\xi^2}{(1+\xi)^3} \right)^w = \left(\zeta \cdot \frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3} \right)^w.$$

Due to $\zeta \cdot \frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3} \leq \frac{1}{4} \cdot \frac{1+10/4}{(2+0)^3} = 7/64 < 1$ we further estimate

$$\left(\zeta \cdot \frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3} \right)^w \leq \zeta \cdot \frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3} \leq w \cdot \zeta \cdot \frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}$$

and altogether

$$\mathbb{E}_{\varphi_0}[\ln R_1^2] \geq \frac{\ln(2+\zeta)}{2} + \psi(3) - \psi(3w+1) - \\ - (3w-2) \cdot \frac{\Gamma(3w)}{4\,\Gamma(w+1)^3} \cdot w \cdot \zeta \cdot \frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}.$$

This expression read with respect to ζ is the same as the already analyzed bound (5.35) up to some additive constant; we can directly transfer, that the minimum is attained in $\zeta = 0$:

$$\mathbb{E}_{\varphi_0}[\ln R_1^2] \geq \ln(2)/2 + \psi(3) - \psi(3w+1).$$

The unique root is located at $w \approx 1.01565264025354...$. □

5.6. Open problems

It is clear that the bound of Theorem 5.78 can not be sharp. With the same method of requiring $\ln R_1$ to be positive uniformly for all $\varphi_0 \in [0, \pi/4]$, numerical approximations suggest that the critical value is at $w \approx 1.20360229090196$ which corresponds to $\nu \approx -0.40720458180392$; there, the minimum is attained for $\varphi_0 = \pi/4$. There are several related directions of possible further studies and conjectures.

The almost surely extinction for ν sufficiently small. Even for the case of only $N = 3$ particles the almost sure explosion for extremely negative drift parameters has not been proven. Intuitively, it seems very plausible that there exists $\nu_\star$ sufficiently small, such that the particle system becomes extinct almost surely for all $\nu \leq \nu_\star$. For the criterion Theorem 5.71 to be useful in proving so, since the integrand $\ln R_1$ will be negative only in some regime of S near $(1, 0)$ and positive near $(1/\sqrt{2}, 1/\sqrt{2})$, it seems to be a reasonable strategy to partition $\mathsf{S} = \biguplus_j \mathsf{S}_j$ properly and to show bounds for $\ln R_1$ with $u_0 \in \mathsf{S}_j$ and for $\eta(\mathsf{S}_j)$. The latter might be achieved even in considering only one-step transition probabilities. For general N, there should at least exist such $\nu_\star(N)$ possibly depending on N.

A threshold value $\nu_\star = \nu_\star(N)$. One would guess that the problem of extinction in finite time is monotone in the drift parameter in the sense that if for given ν it holds $\tau_\infty = \infty$ almost surely, respectively, $\tau_\infty < \infty$ almost surely, it must also hold $\tau_\infty = \infty$ almost surely, respectively, $\tau_\infty < \infty$ almost surely for greater, respectively, smaller values of ν. This montonicity property is less obvious as one might think at a first glance. The more negative drift will impact the system in jumping more often which results in being pushed further away from the origin at least temporarily and thus it seems difficult to achieve proper coupling arguments. Despite the technical involvement, in analogy to the case of $N = 2$ moving particles it is reasonable to conjecture the existence of some sharp threshold value $\nu_\star(N)$ depending on N with $\nu \geq \nu_\star(N)$ implying non-extinction almost surely and $\nu < \nu_\star(N)$ implying extinction almost surely.

The limit case $N \to \infty$. Another concern addresses asymptotics for $N \to \infty$. The result Theorem 5.78 shows that adding particles potentially really enlarges the domain of parameter values where non-extinction occurs almost surely. It would be very interesting to know, whether for all $\nu \in \mathbb{R}$ there exists N_ν, such that for all $N \geq N_\nu$ the particle system does not become extinct almost surely. So far, it is only known that the answer is affirmative for positive values of ν.

A. Properties of Hypergeometric Functions

In this Appendix we collect some explicit properties of special functions which are used during this thesis.
The *Gamma function* $\Gamma(x)$ for $x > 0$ may be represented by the integral $\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$. as in [1, Expression 6.1.1]. This implies the following integral formula:

Lemma A.1. *For $a > 0$, $b > 1$ it holds*

$$\int_0^\infty t^{-b}\, e^{-a/t}\, dt = \Gamma(b-1)/a^{b-1}.$$

Proof. Substituting $s := a/t$, we derive

$$\int_0^\infty t^{-b}\, e^{-a/t}\, dt = \int_0^\infty (s/a)^b\, e^{-s}\, a/s^2\, ds = \Gamma(b-1)/a^{b-1}.$$

□

Definition A.2. For $x > 0$ we denote by

$$(x)_k := \prod_{j=0}^{k-1}(x+j) = \Gamma(x+k)/\Gamma(x), \quad k \in \mathbb{N}_0$$

the *Pochhammer symbol* (rising factorial) as in [1, Expression 6.1.22].

Definition A.3. For $a, b, c > 0$ the *Gausian hypergeometric function* (cf. [1, Expression 15.1.1]) is defined for $|z| < 1$ as

$${}_2F_1(a,b;c;z) := \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

Lemma A.4. *The hypergeometric function obeys for $a, b > 0$ and $x < 1$ the identity*

$${}_2F_1(a,b;a;x) = (1-x)^{-b} \tag{A.1}$$

and according to [23, 2.12 (5)] has the following integral representation:

$${}_2F_1(a,b;c;1-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty s^{b-1}(1+s)^{a-c}(1+sz)^{-a}\, ds, \quad c > b > 0. \tag{A.2}$$

Lemma A.5. *For $a, c, \beta > 0, \gamma > \frac{1+\delta}{\beta} > 0$ it holds*

$$\int_0^\infty \frac{y^\delta}{(c + a \cdot y^\beta)^\gamma}\, dy = \beta^{-1}\, c^{-\gamma} \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}} \cdot \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)}.$$

Proof. By (A.2) and (A.1)

$$\begin{aligned}\int_0^\infty \frac{y^\delta}{(c + a \cdot y^\beta)^\gamma}\, dy &= \frac{c^{-\gamma}}{\beta} \int_0^\infty \frac{z^{\frac{1+\delta-\beta}{\beta}}}{\left(1 + \frac{a}{c} \cdot z\right)^\gamma}\, dz \\ &= \frac{c^{-\gamma}}{\beta} \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \cdot {}_2F_1\left(\gamma, \frac{1+\delta}{\beta}; \gamma; 1 - \frac{a}{c}\right) \\ &= \frac{c^{-\gamma}}{\beta} \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \cdot \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}}.\end{aligned}$$

□

Definition A.6. Let

$$\psi = (\ln \circ \Gamma)' = \Gamma'/\Gamma$$

denote the *Digamma function* (Psi function) as in [1, Expression 6.3.1].

Lemma A.7. *For $a, c, \beta > 0, \gamma > \frac{1+\delta}{\beta} > 0$ it holds*

$$\begin{aligned}&\int_0^\infty \frac{y^\delta \cdot \ln y}{(c + a \cdot y^\beta)^\gamma}\, dy \\ &= \beta^{-2}\, c^{-\gamma} \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}} \cdot \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \left(\ln \frac{c}{a} + \psi\left(\frac{1+\delta}{\beta}\right) - \psi\left(\gamma - \frac{1+\delta}{\beta}\right)\right) \\ &= \frac{\ln \frac{c}{a} + \psi\left(\frac{1+\delta}{\beta}\right) - \psi\left(\gamma - \frac{1+\delta}{\beta}\right)}{\beta} \int_0^\infty \frac{y^\delta}{(c + a \cdot y^\beta)^\gamma}\, dy.\end{aligned}$$

Proof. Recalling Lemma A.5, we attain as immediate consequence

$$\begin{aligned}&\int_0^\infty \frac{y^\delta \cdot \ln y}{(c + a \cdot y^\beta)^\gamma}\, dy = \frac{\partial}{\partial \delta} \int_0^\infty \frac{y^\delta}{(c + a \cdot y^\beta)^\gamma}\, dy \\ &= \beta^{-2}\, c^{-\gamma} \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}} \cdot \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \left(\ln \frac{c}{a} + \psi\left(\frac{1+\delta}{\beta}\right) - \psi\left(\gamma - \frac{1+\delta}{\beta}\right)\right).\end{aligned}$$

□

Definition A.8. For $a, b, c_1, \ldots, c_n > 0$ let

$$F_C^{(n)}(a, b, c_1, \ldots, c_n, x_1, \ldots, x_n) := \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(a)_{k_1 + \ldots + k_n} (b)_{k_1 + \ldots + k_n}}{(c_1)_{k_1} \cdots (c_n)_{k_n} k_1! \cdots k_n!}\, x_1^{k_1} \cdots x_n^{k_n}$$

denote the *C-type Lauricella hypergeometric series* as can be found e.g. in [24, Equation (2.1.3)].

Definition A.9. We call $f(x)$ and $g(x)$ *asymptotically equivalent* (with respect to the limit process $x \to x_0$), written $f(x) \sim g(x)$, if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1;$$

we require $g(x) \neq 0$ in a neighborhood of x_0. If the the limit point x_0 is clear from context we may omit the explicit statement of $x \to x_0$ and just write $f(x) \sim g(x)$.

Lemma A.10. *For $\alpha, \beta \in \mathbb{R}$ real numbers it holds*

$$\Gamma(\alpha + k)/\Gamma(\beta + k) \sim k^{\alpha - \beta} \quad \textit{as } k \to \infty.$$

Proof. Stirling's formula for the Gamma function states that

$$\Gamma(z+1) \sim \sqrt{2\pi z} \left(\frac{z}{e}\right)^z$$

as $z \to \infty$. It further holds

$$\lim_{k \to \infty} \frac{\alpha + k}{\beta + k} = 1 + \lim_{k \to \infty} \frac{\alpha - \beta}{\beta + k} = 1$$

and

$$\exp(x) = \lim_{k \to \infty} \left(1 + \frac{x}{k}\right)^k$$

for $x \in \mathbb{R}$. We combine these facts to deduce

$$\begin{aligned}
\frac{\Gamma(\alpha+1+k)}{\Gamma(\beta+1+k)} &\sim \frac{\sqrt{2\pi(\alpha+k)}((\alpha+k)/e)^{\alpha+k}}{\sqrt{2\pi(\beta+k)}((\beta+k)/e)^{\beta+k}} = \sqrt{\frac{\alpha+k}{\beta+k}}\frac{e^\beta}{e^\alpha}\frac{(\alpha+k)^{\alpha+k}}{(\beta+k)^{\beta+k}} \\
&\sim \frac{e^\beta}{e^\alpha}\frac{(\alpha+k)^{\alpha+k}}{(\beta+k)^{\beta+k}} \sim \frac{(1+\beta/k)^k}{(1+\alpha/k)^k}\frac{(\alpha+k)^{\alpha+k}}{(\beta+k)^{\beta+k}} = \frac{(k+\beta)^k}{(k+\alpha)^k}\frac{(\alpha+k)^{\alpha+k}}{(\beta+k)^{\beta+k}} = \frac{(\alpha+k)^\alpha}{(\beta+k)^\beta}.
\end{aligned}$$

It holds $\frac{(\alpha+k)^\alpha}{(\beta+k)^\beta} \sim k^{\alpha-\beta}$ from which the assertion follows since

$$\frac{(\alpha+k)^\alpha}{(\beta+k)^\beta}/k^{\alpha-\beta} = \frac{\left(1+\frac{\alpha}{k}\right)^\alpha}{\left(1+\frac{\beta}{k}\right)^\beta} \xrightarrow[k \to \infty]{} 1.$$

□

Proposition A.11. *(Cf. [24, Section 2.2 Convergence of the Lauricella Series] The C-type Lauricella series converges absolutely on*

$$\{|x_1|^{1/2} + \ldots + |x_n|^{1/2} < 1\}$$

and is continuous on this domain.

Proof. Let $a, b, c_1, \ldots, c_n > 0$ arbitrary positive real numbers. In order to apply the previous lemma we transform according to

$$\begin{aligned}\frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}k_1!\cdots k_n!} &= \frac{\Gamma(a+k_1+\ldots+k_n)\Gamma(b+k_1+\ldots+k_n)\Gamma(c_1)\cdots\Gamma(c_n)}{\Gamma(a)\Gamma(b)\Gamma(c_1+k_1)\cdots\Gamma(c_n+k_n)k_1!\cdots k_n!} \\ &= \frac{\Gamma(c_1)\cdots\Gamma(c_n)}{\Gamma(a)\Gamma(b)}\cdot\frac{\Gamma(a+k_1+\ldots+k_n)}{\Gamma(1+k_1+\ldots+k_n)}\cdot\frac{\Gamma(b+k_1+\ldots+k_n)}{\Gamma(1+k_1+\ldots+k_n)} \\ &\quad\times\frac{\Gamma(1+k_1)}{\Gamma(c_1+k_1)}\cdots\frac{\Gamma(1+k_n)}{\Gamma(c_n+k_n)}\cdot\left(\frac{(k_1+\ldots+k_n)!}{k_1!\cdots k_n!}\right)^2.\end{aligned}$$

If either of $k_1, \ldots, k_n$ tends to infinity, so does the sum $k \coloneqq k_1 + \ldots + k_n$ and we find

$$\frac{\Gamma(a+k_1+\ldots+k_n)}{\Gamma(1+k_1+\ldots+k_n)} \sim (k_1+\ldots+k_n)^{a-1}$$

as $k_1 + \ldots + k_n \to \infty$. Therefore, there must exist $A \in \mathbb{N}$ such that for all $i \in \{1, \ldots, n\}$ the relation $k_i \geq A$ implies

$$\frac{\Gamma(a+k_1+\ldots+k_n)}{\Gamma(1+k_1+\ldots+k_n)} \leq 2(k_1+\ldots+k_n)^{a-1}.$$

Analogously, there is $B \in \mathbb{N}$ such that

$$\frac{\Gamma(b+k_1+\ldots+k_n)}{\Gamma(1+k_1+\ldots+k_n)} \leq 2(k_1+\ldots+k_n)^{b-1}$$

as soon as $k_i \geq B$ for any $i \in \{1, \ldots, n\}$. Furthermore, there are constants M_{c_i} for $i = 1, \ldots, n$ such that

$$\frac{\Gamma(1+k_i)}{\Gamma(c_i+k_i)} \leq 2k_i^{1-c_i}$$

for $k_i \geq C_i$. We now define $M \coloneqq \max\{A, B, C_1, \ldots, C_n\}$ to be the maximum of the constants we have found. In the sum

$$F_C^{(n)}(a, b, c_1, \ldots, c_n, |x_1|, \ldots, |x_n|) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}k_1!\cdots k_n!}|x_1|^{k_1}\cdots|x_n|^{k_n}$$

each summand is positive and by Fubini's theorem we may add up in arbitrary order. Defining $A_0 \coloneqq \{0, \ldots, M-1\}$ and $A_1 \coloneqq \{M, M+1, \ldots\}$ we may partition $\mathbb{N}_0 = A_0 \uplus A_1$ in two disjunctive sets. Regarding all of the n sums we then partition according to

$$\mathbb{N}_0^n = (A_0 \uplus A_1)^n = \biguplus_{z=(z_1,\ldots,z_n)\in\{0,1\}^n} A_{z_1} \times \cdots \times A_{z_n}$$

resulting in

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}k_1!\cdots k_n!}|x_1|^{k_1}\cdots|x_n|^{k_n}$$

$$= \sum_{z_1,\ldots,z_n=0}^{1} \sum_{k_1\in A_{z_1},\ldots,k_n\in A_{z_n}} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}k_1!\cdots k_n!}|x_1|^{k_1}\cdots|x_n|^{k_n}$$

$$= \sum_{\substack{r\in\{0,\ldots,n\}\\ \{1,\ldots,n\}=\{i_1,\ldots,i_{n-r},j_1,\ldots,j_r\}\\ i_1<i_2<\ldots<i_{n-r}\\ j_1<j_2<\ldots<j_r}} \sum_{k_{i_1},\ldots,k_{i_{n-r}}=0}^{M-1} \sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}k_1!\cdots k_n!}|x_1|^{k_1}\cdots|x_n|^{k_n}$$

$$= \sum_{\substack{r\in\{0,\ldots,n\}\\ \{1,\ldots,n\}=\{i_1,\ldots,i_{n-r},j_1,\ldots,j_r\}\\ i_1<i_2<\ldots<i_{n-r}\\ j_1<j_2<\ldots<j_r}} \sum_{k_{i_1},\ldots,k_{i_{n-r}}=0}^{M-1} \frac{1}{(c_{i_1})_{k_{i_1}}\cdots(c_{i_{n-r}})_{k_{i_{n-r}}}}$$

$$\times \sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_{j_1})_{k_{j_1}}\cdots(c_{j_r})_{k_{j_r}}k_1!\cdots k_n!}|x_1|^{k_1}\cdots|x_n|^{k_n}.$$

In the following we show that each of the sums

$$\sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_{j_1})_{k_{j_1}}\cdots(c_{j_r})_{k_{j_r}}k_1!\cdots k_n!}|x_1|^{k_1}\cdots|x_n|^{k_n}$$

with fixed $k_{i_1},\ldots,k_{i_{n-r}}$ is finite. Since there are only finitely many of them the overall sum still remains finite. By construction of $M\in\mathbb{N}$ it holds

$$\sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_{j_1})_{k_{j_1}}\cdots(c_{j_r})_{k_{j_r}}k_1!\cdots k_n!}|x_1|^{k_1}\cdots|x_n|^{k_n}$$

$$= \sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} \frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)}\cdot\frac{\Gamma(a+k_1+\ldots+k_n)}{\Gamma(1+k_1+\ldots+k_n)}\cdot\frac{\Gamma(b+k_1+\ldots+k_n)}{\Gamma(1+k_1+\ldots+k_n)}$$

$$\times \frac{\Gamma(1+k_{j_1})}{\Gamma(c_{j_1}+k_{j_1})}\cdots\frac{\Gamma(1+k_{j_r})}{\Gamma(c_{j_r}+k_{j_r})}\cdot\left(\frac{(k_1+\ldots+k_n)!}{k_1!\cdots k_n!}\right)^2|x_1|^{k_1}\cdots|x_n|^{k_n}$$

$$\leq \frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)} \sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} 2(k_1+\ldots+k_n)^{a-1}\cdot 2(k_1+\ldots+k_n)^{b-1}$$

$$\times (2k_{j_1}^{1-c_{j_1}})\cdots(2k_{j_r}^{1-c_{j_r}})\cdot\left(\frac{(k_1+\ldots+k_n)!}{k_1!\cdots k_n!}\right)^2|x_1|^{k_1}\cdots|x_n|^{k_n}$$

$$= 2^{2+r}\frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)} \sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} (k_1+\ldots+k_n)^{a+b-2}$$

$$\times (k_{j_1}^{1-c_{j_1}})\cdots(k_{j_r}^{1-c_{j_r}})\cdot\left(\frac{(k_1+\ldots+k_n)!}{k_1!\cdots k_n!}\right)^2|x_1|^{k_1}\cdots|x_n|^{k_n}.$$

To obtain further estimates we consider the factors $(k_{j_1}^{1-c_{j_1}})\cdots(k_{j_r}^{1-c_{j_r}})$ and $\left(\frac{(k_1+\ldots+k_n)!}{k_1!\cdots k_n!}\right)^2$ separately. Since $c_{j_1},\ldots,c_{j_r}>0$ are all positive it holds

$$(k_{j_1}^{1-c_{j_1}})\cdots(k_{j_r}^{1-c_{j_r}}) \leq k_{j_1}\cdots k_{j_r} \leq (k_{j_1}+\ldots+k_{j_r})^r \leq (k_1+\ldots+k_n)^r.$$

For the second factor using the notation of binomial coefficients we may expand according to

$$\left(\frac{(k_1+\ldots+k_n)!}{k_1!\cdots k_n!}\right)^2 = \frac{(k_1+\ldots+k_n)!^2}{k_1!^2(k_2+\ldots+k_n)!^2}\cdot\frac{(k_2+\ldots+k_n)!^2}{k_2!^2(k_3+\ldots+k_n)!^2}\cdots\frac{(k_{n-1}+k_n)!^2}{k_{n-1}!^2k_n!^2}\cdot\frac{k_n!^2}{k_n!^2}$$
$$= \binom{k_1+\ldots+k_n}{k_1}^2\cdot\binom{k_2+\ldots+k_n}{k_2}^2\cdots\binom{k_{n-1}+k_n}{k_{n-1}}^2.$$

With the convention $\binom{x}{y}=0$ whenever $x<y$, Vandermonde's identity states that for any $u,v,m\in\mathbb{N}_0$ it holds

$$\binom{u+v}{m} = \sum_{s=0}^{m}\binom{u}{s}\binom{v}{m-s}.$$

This implies for $u,t\in\mathbb{N}_0$ the inequality

$$\binom{u}{t}^2 = \binom{u}{t}\cdot\binom{u}{2t-t} \leq \sum_{s=0}^{2t}\binom{u}{s}\binom{u}{2t-s} = \binom{u+u}{2t} = \binom{2u}{2t}.$$

Piecewisely applied to the product we are given the bound

$$\binom{k_1+\ldots+k_n}{k_1}^2\cdot\binom{k_2+\ldots+k_n}{k_2}^2\cdots\binom{k_{n-1}+k_n}{k_{n-1}}^2$$
$$\leq \binom{2(k_1+\ldots+k_n)}{2k_1}\cdot\binom{2(k_2+\ldots+k_n)}{2k_2}\cdots\binom{2(k_{n-1}+k_n)}{2(k_{n-1})}$$
$$= \frac{(2(k_1+\ldots+k_n))!}{(2k_1)!(2(k_2+\ldots+k_n))!}\cdot\frac{(2(k_2+\ldots+k_n))!}{(2k_2)!(2(k_3+\ldots+k_n))!}\cdots\frac{(2(k_{n-1}+k_n))!}{(2k_{n-1})!(2k_n)!}$$
$$= \frac{(2(k_1+\ldots+k_n))!}{(2k_1)!\cdots(2k_n)!}.$$

Now turning back to the overall expression we achieve

$$2^{2+r}\frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)}\sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty}(k_1+\ldots+k_n)^{a+b-2}$$
$$\times(k_{j_1}^{1-c_{j_1}})\cdots(k_{j_r}^{1-c_{j_r}})\cdot\left(\frac{(k_1+\ldots+k_n)!}{k_1!\cdots k_n!}\right)^2|x_1|^{k_1}\cdots|x_n|^{k_n}$$
$$\leq 2^{2+r}\frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)}\sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty}(k_1+\ldots+k_n)^{a+b+r-2}$$

$$\times \frac{(2(k_1+\ldots+k_n))!}{(2k_1)!\cdots(2k_n)!}|x_1|^{k_1}\cdots|x_n|^{k_n}.$$

As next step we want to use the multinomial theorem: For $m \in \mathbb{N}$ real numbers $z_1,\ldots,z_m \in \mathbb{R}$ it holds for any nonnegative integer power $k \in \mathbb{N}_0$

$$(z_1+\ldots+z_m)^k = \sum_{\substack{k_1,\ldots,k_m\in\mathbb{N}_0\\ k_1+\ldots+k_m=k}} \frac{k!}{k_1!\cdots k_m!} z_1^{k_1}\cdots z_m^{k_m}.$$

As preparation we first enlarge the summation domain twice.

$$\begin{aligned}
&\sum_{k_{j_1},\ldots,k_{j_r}=M}^{\infty} (k_1+\ldots+k_n)^{a+b+r-2}\frac{(2(k_1+\ldots+k_n))!}{(2k_1)!\cdots(2k_n)!}|x_1|^{k_1}\cdots|x_n|^{k_n}\\
&\leq \sum_{k_1,\ldots,k_n=0}^{\infty} (k_1+\ldots+k_n)^{a+b+r-2}\frac{(2(k_1+\ldots+k_n))!}{(2k_1)!\cdots(2k_n)!}|x_1|^{k_1}\cdots|x_n|^{k_n}\\
&= \sum_{k=0}^{\infty}\sum_{\substack{k_1,\ldots,k_n\in\mathbb{N}_0\\ k_1+\ldots+k_n=k}} (k_1+\ldots+k_n)^{a+b+r-2}\frac{(2(k_1+\ldots+k_n))!}{(2k_1)!\cdots(2k_n)!}|x_1|^{k_1}\cdots|x_n|^{k_n}\\
&= \sum_{k=0}^{\infty} k^{a+b+r-2}\sum_{\substack{k_1,\ldots,k_n\in\mathbb{N}_0\\ 2k_1+\ldots+2k_n=2k}} \frac{(2k)!}{(2k_1)!\cdots(2k_n)!}|x_1|^{k_1}\cdots|x_n|^{k_n}\\
&\leq \sum_{k=0}^{\infty} k^{a+b+r-2}\sum_{\substack{k_1,\ldots,k_n\in\mathbb{N}_0\\ k_1+\ldots+k_n=2k}} \frac{(2k)!}{(k_1)!\cdots(k_n)!}|x_1|^{k_1/2}\cdots|x_n|^{k_n/2}\\
&= \sum_{k=0}^{\infty} k^{a+b+r-2}(|x_1|^{1/2}+\ldots+|x_n|^{1/2})^{2k}.
\end{aligned}$$

The last sum is finite if $|x_1|^{1/2}+\ldots+|x_n|^{1/2}<1$ since exponential decay rules out polynomial growth. To summarize, we have seen that for $|x_1|^{1/2}+\ldots+|x_n|^{1/2}<1$ it holds

$$\begin{aligned}
&F_C^{(n)}(a,b,c_1,\ldots,c_n,x_1,\ldots,x_n) \leq F_C^{(n)}(a,b,c_1,\ldots,c_n,|x_1|,\ldots,|x_n|)\\
&\leq \sum_{\substack{r\in\{0,\ldots,n\}\\ \{1,\ldots,n\}=\{i_1,\ldots,i_{n-r},j_1,\ldots,j_r\}\\ i_1<i_2<\ldots<i_{n-r}\\ j_1<j_2<\ldots<j_r}} \sum_{k_{i_1},\ldots,k_{i_{n-r}}=0}^{M-1} \frac{1}{(c_{i_1})_{k_{i_1}}\cdots(c_{i_{n-r}})_{k_{i_{n-r}}}}\\
&\qquad \times 2^{2+r}\frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty} k^{a+b+r-2}(|x_1|^{1/2}+\ldots+|x_n|^{1/2})^{2k}\\
&= \sum_{k=0}^{\infty} F_C^{(n),k}(a,b,c_1,\ldots,c_n,|x_1|,\ldots,|x_n|) < \infty
\end{aligned}$$

where

$$F_C^{(n),k}(a,b,c_1,\dots,c_n,x_1,\dots,x_n) \coloneqq \sum_{\substack{r\in\{0,\dots,n\}\\ \{1,\dots,n\}=\{i_1,\dots,i_{n-r},j_1,\dots,j_r\}\\ i_1<i_2<\dots<i_{n-r}\\ j_1<j_2<\dots<j_r}} \sum_{k_{i_1},\dots,k_{i_{n-r}}=0}^{M-1} \frac{1}{(c_{i_1})_{k_{i_1}}\cdots(c_{i_{n-r}})_{k_{i_{n-r}}}} \times 2^{2+r}\frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)}k^{a+b+r-2}(x_1^{1/2}+\dots+x_n^{1/2})^{2k}.$$

In the following we further want to show continuity of the Lauricella series. We denote the mapping $\rho:\mathbb{R}^n\to[0,\infty)$, $(x_1,\dots,x_n)\mapsto\sqrt{|x_1|}+\dots+\sqrt{|x_n|}$. Now let

$$z=(z_1,\dots,z_n)\in\{x\in\mathbb{R}^n:\sqrt{|x_1|}+\dots+\sqrt{|x_n|}<1\}=\rho^{-1}([0,1))$$

arbitrary. For all $k\in\mathbb{N}_0$ and uniformly on $x\in D\coloneqq\rho^{-1}([0,(\rho(z)+1)/2])$ it holds

$$\begin{aligned}&|F_C^{(n),k}(a,b,c_1,\dots,c_n,x_1,\dots,x_n)|\\ &\le \sum_{\substack{r\in\{0,\dots,n\}\\ \{1,\dots,n\}=\{i_1,\dots,i_{n-r},j_1,\dots,j_r\}\\ i_1<i_2<\dots<i_{n-r}\\ j_1<j_2<\dots<j_r}} \sum_{k_{i_1},\dots,k_{i_{n-r}}=0}^{M-1} \frac{1}{(c_{i_1})_{k_{i_1}}\cdots(c_{i_{n-r}})_{k_{i_{n-r}}}}\\ &\qquad\times 2^{2+r}\frac{\Gamma(c_{j_1})\cdots\Gamma(c_{j_r})}{\Gamma(a)\Gamma(b)}k^{a+b+r-2}((\rho(z)+1)/2)^{2k} \eqqcolon M_k.\end{aligned}$$

Due to $\sum_{k=0}^{\infty}M_k<\infty$ by the Weierstrass M-test the series

$$F_C^{(n)}(a,b,c_1,\dots,c_n,x_1,\dots,x_n)=\sum_{k=0}^{\infty}F_C^{(n),k}(a,b,c_1,\dots,c_n,x_1,\dots,x_n)$$

converges absolutely and uniformly on D. Because for each of $k\in\mathbb{N}_0$ the function $x\mapsto F_C^{(n),k}(a,b,c_1,\dots,c_n,x_1,\dots,x_n)$ is continuous the finite sum

$$\sum_{k=0}^{K}F_C^{(n),k}(a,b,c_1,\dots,c_n,x_1,\dots,x_n)$$

is a continuous function in x as well. Together with the uniform convergence as $K\to\infty$ the uniform limit theorem implies the function $F_C^{(n)}(a,b,c_1,\dots,c_n,x_1,\dots,x_n)$ to be continuous on $x\in D$. Particularly the Lauricella series is seen to be continuous in $z\in D$. Since z was arbitrarily chosen in $\rho^{-1}([0,1))$ the continuity on this domain follows. □

B. Proof omitted in Remark 5.34

This part of the appendix is devoted to the proof of a statement posed in Remark 5.34. This is needed for verifying that a density we calculated earlier complies with the formula given in [45, Theorem 8 (ii)]. There, the authors considered squared Bessel processes with individual drift parameters. More specifically, the theorem entails the following statement.

Let $Z = (Z^1, Z^2, \dots, Z^n)$ independent squared Bessel processes with drift parameters $-\theta_1, \dots, -\theta_n \le 0$ and assume they start deterministically at $Z_0 = z_0 \in (0,\infty)^n$. Let

$$\tau := \min_{i=1}^{n} \inf\{t > 0 : Z_t^i = 0\}$$

the first hitting time of the origin by any one of the n components. Then for $i \in \{1, \dots, n\}$ the law of the random vector Z_τ restricted to the event $\{Z_\tau^i = 0\}$ admits a density for all the variables y_j, $j \in \{1, \dots, n\} \setminus \{i\}$ which is given by

$$\begin{aligned} &q^{i,(\theta_1,\dots,\theta_n)}_{(z_1,\dots,z_n)}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) = \mathbb{P}_{z_0}(Z_\tau \in d(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n), Z_\tau^i = 0) \\ &= \frac{S^{1-\theta_0/2-2n}}{\Gamma(\theta_i/2+1)} \prod_{j=1}^{n} z_j^{\theta_j/2+1} \sum_{N=0}^{\infty} \Gamma(\theta_0/2 + 2n + 2N - 1) S^{-2N} \\ &\qquad \times \sum_{\sum_{j\neq i} k_j = N} \prod_{j \neq i} \frac{(y_j z_j)^{k_j}}{k_j! \Gamma(\theta_j/2 + 2 + k_j)} \end{aligned} \tag{B.1}$$

where

$$S := \sum_{i=1}^{n} (y_i + z_i), \quad y_i := 0, \quad \text{and} \quad \theta_0 := \sum_{i=1}^{n} \theta_i.$$

Since the authors of [45] considered squared Bessel processes instead of Bessel processes we use the transformation

$$g^{i,(\theta_1,\dots,\theta_n)}_{(z_1,\dots,z_n)}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) := q^{i,(\theta_1,\dots,\theta_n)}_{(z_1^2,\dots,z_n^2)}(y_1^2, \dots, y_{i-1}^2, y_{i+1}^2, \dots, y_n^2) \prod_{\substack{s=1 \\ s\neq i}}^{n} (2y_s).$$

Recall from Lemma 5.33 that the transition density function $h_{y_0}(y)$ of the Markov chain (Y_n) given in Definition 5.16 may be expressed as

$$h_{y_0}(y) = \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1} [\Gamma(k_s + w + 1) k_s!]} \times \right.$$

$$\times \prod_{s=1}^{N-1} (y^s)^{2k_s} \sum_{i=1}^{N-1} \frac{\left(y_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(y_0^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2 + (y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\Bigg].$$

What we want to show is formulated in the following lemma. It may be regarded as a check if our calculations are correct.

Lemma B.1. *In the previous notations it holds*

$$h_{y_0}(y) = \frac{1}{N-1} \sum_{i=1}^{N-1} \sum_{j=1}^{N} \sum_{\pi\in S_{N-1}} g^{j,(2(w-1),\ldots,2(w-1))}_{(y_0^1,\ldots,y_0^{N-1},y_0^i)}(y^\pi).$$

Proof. It generally holds

$$\sum_{N=0}^{\infty} \Gamma(\theta_0/2 + 2n + 2N - 1) S^{-2N} \sum_{\sum_{j\neq i} k_j = N} \prod_{j\neq i} \frac{(y_j z_j)^{k_j}}{k_j!\Gamma(\theta_j/2 + 2 + k_j)}$$
$$= \sum_{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n=0}^{\infty} \Gamma(\theta_0/2 + 2n + 2N - 1) S^{-2N} \prod_{j\neq i} \frac{(y_j z_j)^{k_j}}{k_j!\Gamma(\theta_j/2 + 2 + k_j)}$$

where in the second expression $N = \sum_{j\neq i} k_j$. It follows

$$g^{j,(2(w-1),\ldots,2(w-1))}_{(y_0^1,\ldots,y_0^{N-1},y_0^i)}(y^\pi)$$
$$= \frac{\left((y_0^i)^2 + \sum_{s=1}^{N-1}((y_0^s)^2 + (y^{\pi(s)})^2)\right)^{1-N(w-1)-2N}}{\Gamma(w)} \prod_{s=1}^{N-1} (y_0^s)^{2w} \cdot (y_0^i)^{2w} \times$$
$$\times \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \frac{\Gamma(N(w-1) + 2N + 2\sum_{s=1}^{N-1} k_s - 1)}{\left((y_0^i)^2 + \sum_{s=1}^{N-1}((y_0^s)^2 + (y^{\pi(s)})^2)\right)^{2\sum_{s=1}^{N-1} k_s}} \prod_{s=1}^{N-1} (2y^{\pi(s)}) \times$$
$$\times \begin{cases} \prod_{s=1}^{j-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \prod_{s=j}^{N-2} \frac{(y_0^{s+1} y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \frac{(y_0^i y^{\pi(N-1)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})}, & j\le N-1 \\ \prod_{s=1}^{N-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}, & j = N \end{cases}$$
$$= \frac{2^{N-1}}{\Gamma(w)} \prod_{s=1}^{N-1} (y_0^s)^{2w} \cdot (y_0^i)^{2w} \times$$
$$\times \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \frac{\Gamma((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s)}{\left((y_0^i)^2 + \sum_{s=1}^{N-1}((y_0^s)^2 + (y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \prod_{s=1}^{N-1} (y^s) \times$$
$$\times \begin{cases} \prod_{s=1}^{j-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \prod_{s=j}^{N-2} \frac{(y_0^{s+1} y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \frac{(y_0^i y^{\pi(N-1)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})}, & j\le N-1 \\ \prod_{s=1}^{N-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}, & j = N \end{cases}.$$

For the sum it follows

$$\sum_{j=1}^{N}\sum_{\pi\in S_{N-1}} g^{j,(2(w-1),\dots,2(w-1))}_{(y_0^1,\dots,y_0^{N-1},y_0^i)}(y^\pi) = \frac{2^{N-1}}{\Gamma(w)}\prod_{s=1}^{N-1}(y_0^s)^{2w}\cdot(y_0^i)^{2w}\times$$
$$\times\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}(y^s)\times$$
$$\times\sum_{\pi\in S_{N-1}}\left(\sum_{j=1}^{N-1}\prod_{s=1}^{j-1}\frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\prod_{s=j}^{N-2}\frac{(y_0^{s+1}y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\frac{(y_0^i y^{\pi(N-1)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})}\right.$$
$$\left.+\prod_{s=1}^{N-1}\frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\right). \tag{B.2}$$

Defining $\widetilde{\pi} \coloneqq \widetilde{\pi}(\pi)\in S_{N-1}$ by

$$\widetilde{\pi}(s) = \begin{cases}\pi(s), & s\le j-1,\\ \pi(s+1), & j\le s\le N-2,\\ \pi(j), & s=N-1,\end{cases}$$

we see that

$$\{\widetilde{\pi}:\pi\in S_{N-1}\} = S_{N-1}$$

and finally that $\pi\mapsto\widetilde{\pi}$ is a bijection on S_{N-1}. We may thus alter the summation according to

$$\sum_{\pi\in S_{N-1}}\sum_{j=1}^{N-1}\prod_{s=1}^{j-1}\frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\prod_{s=j}^{N-2}\frac{(y_0^{s+1}y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\frac{(y_0^i y^{\pi(N-1)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})}$$
$$=\sum_{\widetilde{\pi}\in S_{N-1}}\sum_{j=1}^{N-1}\prod_{s=1}^{j-1}\frac{(y_0^s y^{\widetilde{\pi}(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\prod_{s=j}^{N-2}\frac{(y_0^{s+1}y^{\widetilde{\pi}(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\frac{(y_0^i y^{\widetilde{\pi}(N-1)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})}$$
$$=\sum_{\pi\in S_{N-1}}\sum_{j=1}^{N-1}\prod_{s=1}^{j-1}\frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\prod_{s=j}^{N-2}\frac{(y_0^{s+1}y^{\pi(s+1)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\frac{(y_0^i y^{\pi(j)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})}.$$

By Fubini's theorem and the commutativity of summation,

$$\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}(y^s)\times$$
$$\times\sum_{\pi\in S_{N-1}}\sum_{j=1}^{N-1}\prod_{s=1}^{j-1}\frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\prod_{s=j}^{N-2}\frac{(y_0^{s+1}y^{\pi(s+1)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\cdot\frac{(y_0^i y^{\pi(j)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})}$$

$$= \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}} \prod_{s=1}^{N-1}(y^s)\times$$

$$\times \sum_{\pi\in S_{N-1}} \sum_{j=1}^{N-1}\prod_{s=1}^{j-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \prod_{s=j}^{N-2} \frac{(y_0^{s+1} y^{\pi(s+1)})^{2k_{s+1}}}{k_{s+1}!\Gamma(w+1+k_{s+1})} \cdot \frac{(y_0^i y^{\pi(j)})^{2k_j}}{k_j!\Gamma(w+1+k_j)}.$$

Using

$$\prod_{s=1}^{j-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \prod_{s=j}^{N-2} \frac{(y_0^{s+1} y^{\pi(s+1)})^{2k_{s+1}}}{k_{s+1}!\Gamma(w+1+k_{s+1})} \cdot \frac{(y_0^i y^{\pi(j)})^{2k_j}}{k_j!\Gamma(w+1+k_j)}$$
$$= \prod_{\substack{s=1\\ s\neq j}}^{N-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \frac{(y_0^i y^{\pi(j)})^{2k_j}}{k_j!\Gamma(w+1+k_j)}$$

it follows

$$\sum_{j=1}^{N-1}\prod_{s=1}^{j-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \prod_{s=j}^{N-2} \frac{(y_0^{s+1} y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \frac{(y_0^i y^{\pi(N-1)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})} + \prod_{s=1}^{N-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}$$
$$= \sum_{j=1}^{N-1}\prod_{\substack{s=1\\ s\neq j}}^{N-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \frac{(y_0^i y^{\pi(j)})^{2k_j}}{k_j!\Gamma(w+1+k_j)} + \prod_{s=1}^{N-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}$$
$$= \prod_{s=1}^{N-1} \frac{1}{k_s!\Gamma(w+1+k_s)} \cdot \left(\sum_{j=1}^{N-1}\prod_{\substack{s=1\\ s\neq j}}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \cdot (y_0^i y^{\pi(j)})^{2k_j} + \prod_{s=1}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \right)$$
$$= \prod_{s=1}^{N-1} \frac{1}{k_s!\Gamma(w+1+k_s)} \cdot \left(\sum_{\substack{j=1\\ j\neq i}}^{N-1}\prod_{\substack{s=1\\ s\neq j}}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \cdot (y_0^i y^{\pi(j)})^{2k_j} + 2\prod_{s=1}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \right).$$

Turning back to equation (B.2) this shows using the fact that $\pi(i)\neq\pi(j)$ implies $i\neq j$

$$\sum_{j=1}^{N}\sum_{\pi\in S_{N-1}} g^{j,(2(w-1),\dots,2(w-1))}_{(y_0^1,\dots,y_0^{N-1},y_0^i)}(y^\pi) = \frac{2^{N-1}}{\Gamma(w)} \prod_{s=1}^{N-1} (y_0^s)^{2w}\cdot (y_0^i)^{2w}\times$$

$$\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}} \prod_{s=1}^{N-1}(y^s)\times$$

$$\times \sum_{\pi\in S_{N-1}} \left(\sum_{j=1}^{N-1}\prod_{s=1}^{j-1} \frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \prod_{s=j}^{N-2} \frac{(y_0^{s+1} y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)} \cdot \frac{(y_0^i y^{\pi(N-1)})^{2k_{N-1}}}{k_{N-1}!\Gamma(w+1+k_{N-1})} \right.$$

$$+\prod_{s=1}^{N-1}\frac{(y_0^s y^{\pi(s)})^{2k_s}}{k_s!\Gamma(w+1+k_s)}\Bigg)$$

$$=\frac{2^N}{\Gamma(w)}\prod_{s=1}^{N-1}(y_0^s)^{2w}\cdot(y_0^i)^{2w}$$

$$\times\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}$$

$$\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\sum_{\pi\in S_{N-1}}\left(\frac{1}{2}\sum_{\substack{j=1\\j\neq i}}^{N-1}\prod_{\substack{s=1\\s\neq j}}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\cdot(y_0^i y^{\pi(j)})^{2k_j}+\prod_{s=1}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\right)$$

$$=\frac{2^N}{\Gamma(w)}\prod_{s=1}^{N-1}(y_0^s)^{2w}\cdot(y_0^i)^{2w}\times$$

$$\times\Bigg(\sum_{j=1}^{N-1}\Bigg(\sum_{\substack{\pi\in S_{N-1}\\\pi(i)<\pi(j)}}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}$$

$$\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\frac{1}{2}\prod_{\substack{s=1\\s\neq j}}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\cdot(y_0^i y^{\pi(j)})^{2k_j}$$

$$+\sum_{\substack{\pi\in S_{N-1}\\\pi(i)>\pi(j)}}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}$$

$$\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\frac{1}{2}\prod_{\substack{s=1\\s\neq j}}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\cdot(y_0^i y^{\pi(j)})^{2k_j}\Bigg)$$

$$+\sum_{\pi\in S_{N-1}}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}$$

$$\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\prod_{s=1}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\Bigg).\tag{B.3}$$

Reusing the definition (5.14) and the identity (5.15) it holds switching k_i and k_j

$$\sum_{\substack{\pi\in S_{N-1}\\\pi(i)>\pi(j)}}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\times$$

$$\times \frac{1}{2}\prod_{\substack{s=1\\s\neq j}}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\cdot(y_0^i y^{\pi(j)})^{2k_j}$$

$$= \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)<\pi(j)}}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\times$$

$$\times \frac{1}{2}\prod_{\substack{s=1\\s\neq j}}^{N-1}(y_0^s y^{\widetilde{\pi}(s)})^{2k_s}\cdot(y_0^i y^{\widetilde{\pi}(j)})^{2k_j}$$

$$= \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)<\pi(j)}}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\times$$

$$\times \frac{1}{2}\prod_{\substack{s=1\\s\notin\{i,j\}}}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\cdot(y_0^i y^{\pi(j)})^{2k_i}\cdot(y_0^i y^{\pi(i)})^{2k_j}$$

$$= \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)<\pi(j)}}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\times$$

$$\times \frac{1}{2}\prod_{\substack{s=1\\s\notin\{i,j\}}}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\cdot(y_0^i y^{\pi(j)})^{2k_j}\cdot(y_0^i y^{\pi(i)})^{2k_i}$$

$$= \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)<\pi(j)}}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}\times$$

$$\times \frac{1}{2}\prod_{\substack{s=1\\s\neq j}}^{N-1}(y_0^s y^{\pi(s)})^{2k_s}\cdot(y_0^i y^{\pi(j)})^{2k_j}.$$

With respect to eqution (B.3) we can therefore reunite the sums to obtain

$$\sum_{j=1}^{N}\sum_{\pi\in S_{N-1}} g^{j,(2(w-1),\dots,2(w-1))}_{(y_0^1,\dots,y_0^{N-1},y_0^i)}(y^\pi)$$

$$= \frac{2^N}{\Gamma(w)}\prod_{s=1}^{N-1}(y_0^s)^{2w}\cdot(y_0^i)^{2w}\Bigg(\sum_{j=1}^{N-1}\Bigg(\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)<\pi(j)}}\sum_{k_1,\dots,k_{N-1}=0}^{\infty}$$

$$\frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1}k_s)}{\left((y_0^i)^2+\sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\prod_{s=1}^{N-1}\frac{y^s}{k_s!\Gamma(w+1+k_s)}$$

$$\times \frac{1}{2} \prod_{\substack{s=1 \\ s\neq j}}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \cdot (y_0^i y^{\pi(j)})^{2k_j}$$

$$+ \sum_{\substack{\pi \in S_{N-1} \\ \pi(i)>\pi(j)}} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1} k_s)}{\left((y_0^i)^2 + \sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \prod_{s=1}^{N-1} \frac{y^s}{k_s!\Gamma(w+1+k_s)} \times$$

$$\times \frac{1}{2} \prod_{\substack{s=1 \\ s\neq j}}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \cdot (y_0^i y^{\pi(j)})^{2k_j} \Bigg)$$

$$+ \sum_{\pi \in S_{N-1}} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1} k_s)}{\left((y_0^i)^2 + \sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \prod_{s=1}^{N-1} \frac{y^s}{k_s!\Gamma(w+1+k_s)} \times$$

$$\times \prod_{s=1}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \Bigg)$$

$$= \frac{2^N}{\Gamma(w)} \prod_{s=1}^{N-1} (y_0^s)^{2w} \cdot (y_0^i)^{2w} \times$$

$$\times \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i)\leq\pi(j)}} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \frac{\Gamma((w+1)N-1+2\sum_{s=1}^{N-1} k_s)}{\left((y_0^i)^2 + \sum_{s=1}^{N-1}((y_0^s)^2+(y^s)^2)\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}}$$

$$\prod_{s=1}^{N-1} \frac{y^s}{k_s!\Gamma(w+1+k_s)} \prod_{\substack{s=1 \\ s\neq j}}^{N-1} (y_0^s y^{\pi(s)})^{2k_s} \cdot (y_0^i y^{\pi(j)})^{2k_j}.$$

Matching expression (5.20) this shows

$$\frac{1}{N-1} \sum_{i=1}^{N-1} \sum_{j=1}^{N} \sum_{\pi \in S_{N-1}} g^{j,(2(w-1),\ldots,2(w-1))}_{(y_0^1,\ldots,y_0^{N-1},y_0^i)}(y^\pi)$$

$$= \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \times$$

$$\times \sum_{i,j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i)\leq\pi(j)}} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1}[\Gamma(k_s+w+1)k_s!]} \times \right.$$

$$\left. \times \frac{\left(y_0^i\right)^{2w} \left[\left(y_0^i/y_0^j\right)^{2k_j} \prod_{s=1}^{N-1} \left(y_0^s y^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1}\left[(y_0^s)^2+(y^s)^2\right]\right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \right]$$

$$= h_{y_0}(y)$$

what was to be shown. □

C. Source Codes

C.1. Large noise limits

The following is the R code which was used to produce the SDE samples in Figure 2.1.

```
# We want to simulate the SDE
#
#      dX_t = lambda^2/2*(epsilon-b*x) dt + lambda B_t dB_t
#
# with start in X_0 = 1 as toy model.
#
#
# To do so, we use the CRAN package sde; c.f.
#
#      https://CRAN.R-project.org/package=sde

library(sde)



# Let us define a generic function

plotToySDE <- function (lambda, epsilon, b) {
set.seed(12345)

d <- as.expression(bquote(.(lambda)**2/2*(.(epsilon)-.(b)*x)))
s <- as.expression(bquote(.(lambda)*x))

sde.sim(X0=1, drift=d, sigma=s) -> X
plot(X,main=paste("lambda =", lambda, ", epsilon =", epsilon, ", b =", b),
     ylim=c(-1,8))
}



# We want to exhibit the scaling limit on the curve
#
#      lambda^2 * epsilon^(b+1) = J.
```

```
#
# Let us fix b := 1 so that lambda and epsilon
# must be inversely proportional.



par(mfrow=c(2,2))

plotToySDE(1/2,2,1)
plotToySDE(2,1/2,1)
plotToySDE(5,1/5,1)
plotToySDE(10,1/10,1)
```

C.2. Plot of $\mathbb{E}_{\varphi_0}[\ln R_1]$

The following is the python code which was used to produce the plots in Figure 5.5.

```
# We want to plot the expectation E_{phi_0}[ln R_1] for
# parameters 0 < phi_0 <= pi/4 and 1/3 <= w <= 4/3


import mpmath as mp
import matplotlib.pyplot as plt



def summand(w, phi0, k1, k2):
    factor1 = mp.gamma(3*w+k1+k2)/mp.gamma(k1+w+1)/mp.gamma(k2+w+1)

    f2s1factor1 = mp.cos(phi0)**(2*w)/(1+mp.cos(phi0)**2)**(3*w+k1+k2)
    f2s1factor2 = (mp.digamma(2+k1+k2) - mp.digamma(3*w+k1+k2))/2
    f2s1factor3 = 2 * mp.cos(phi0)**(2*k1) * mp.sin(phi0)**(2*k2)
                  + mp.cos(phi0)**(2*k1)*mp.cos(phi0)**(2*k2)

    f2s2factor1 = mp.sin(phi0)**(2*w)/(1+mp.sin(phi0)**2)**(3*w+k1+k2)
    f2s2factor2 = (mp.digamma(2+k1+k2) - mp.digamma(3*w+k1+k2))/2
    f2s2factor3 = 2 * mp.cos(phi0)**(2*k1) * mp.sin(phi0)**(2*k2)
                  + mp.sin(phi0)**(2*k1)*mp.sin(phi0)**(2*k2)

    f2sum1 = f2s1factor1 * f2s1factor2 * f2s1factor3
    f2sum2 = f2s2factor1 * f2s2factor2 * f2s2factor3

    factor2 = f2sum1 + f2sum2
```

```
    return(factor1 * factor2)



def Ephi0_lnR(w,phi0):
    summand1 = (mp.ln(1+mp.cos(phi0)**2) + mp.ln(1+mp.sin(phi0)**2))/4

    factorDoubleSum = (mp.cos(phi0)*mp.sin(phi0))**(2*w)/(2*mp.gamma(w))
    doubleSum = mp.nsum(lambda k1,k2: summand(w, phi0, k1,k2), [0,mp.inf], [0,mp.inf])

    return(summand1 + factorDoubleSum * doubleSum)




# for several fixed w, plot with respect to phi0:
Phi0 = mp.linspace(0, mp.pi/4, 32)
plt.xlabel('phi0')
plt.ylabel('Ephi0_lnR')

legend = []

for w in mp.linspace(2/3,4/3,7):
    E = [Ephi0_lnR(w,phi0) for phi0 in Phi0]
    plt.plot(Phi0,E)
    legend.append('w = ' + mp.nstr(w))


plt.legend(legend)
plt.show()




# for several fixed phi0, plot with respect to w:
W = mp.linspace(2/3, 4/3, 32)
plt.xlabel('w')
plt.ylabel('Ephi0_lnR')

legend = []

for phi0 in mp.linspace(0,mp.pi/4,7):
    E = [Ephi0_lnR(w,phi0) for w in W]
```

```
    plt.plot(W,E)
    legend.append('phi0 = ' + mp.nstr(phi0))


plt.legend(legend)
plt.show()
```

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