Vector Bundles and Sheaf-Theoretic Matrix-Valued Kernel Functions

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Zusammenfassung

In dieser Arbeit untersuchen wir das Zusammenspiel bestimmter dreifacher Massey-Produkte mit assoziativen $r-$Matrizen und den sogenannten Szegö-Kernen. Im ersten Kapitel wiederholen wir die algebraisch-geometrische Theorie der assoziativen Yang-Baxter-Gleichung, die diese Identität mit der Untersuchung von Vektorbündeln über Kurven des arithmetischen Geschlecht 1 in Beziehung setzt. Ausgehend von bestimmten Vektorbündeln auf der kubischen Weierstraß-Kurve, berechnen wir im zweiten Kapitel Lösungen der oben genannten Gleichung, die sogenannten assoziativen $r-$Matrizen. Im dritten Kapitel stellen wir Szegö-Kerne vor, beweisen, dass sie schiefsymmetrisch sind, und zeigen, dass sie durch dreifache Massey-Produkte beschrieben werden können, die mit Vektorbündeln über Gorenstein-Kurven assoziiert sind. Darüber hinaus leiten wir zwei Identitäten ab, die solche Kernfunktionen erfüllen müssen, insbesondere sollte die Zweite als eine garbentheoretische Version der matrixwertigen Fay Identität betrachtet werden. Im letzten Teil dieser Arbeit wiederholen wir einige Ergebnisse bezüglich Linienbündeln über Riemannschen Flächen und Theta-Funktionen, um als besonderen Fall unserer Identität die trisekante Identität von Fay abzuleiten.

Abstract

In this thesis we investigate the interplay of certain triple Massey products with associative $r-$matrices and the so called Szegö kernels. In the first chapter we recall the algebro-geometric theory of the associative Yang-Baxter equation which relates this identity to the study of vector bundles over curves of arithmetic genus one. In the second chapter, starting with certain vector bundles over Weierstraß cubic curve, we compute solutions, the so called associative $r-$matrices, of the aforementioned equation. In chapter three we introduce Szegö kernels, we prove that they are skew-symmetric, and we show that they can be described through triple Massey products associated with vector bundles over Gorenstein curves. Moreover, we derive two identities which these kernel functions have to satisfy, in particular the second one should be considered as a sheaf-theoretic version of the matrix-valued Fay's identity. In the final part of this thesis we recall some results regarding line bundles over Riemann surfaces and theta functions in order

to deduce, as a particular case of our identity, the Fay's trisecant identity.

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To my Father

Introduction

In this thesis, we study tensor-valued functions satisfying the associative Yang-Baxter equation and their analogs. Such functions arise from appropriate triple Massey products in the derived category of coherent sheaves on Gorenstein curves or as Szegö kernels associated with vector bundles on such curves with vanishing cohomology. In the first part of this work, we study the algebro-geometric aspects of the associative Yang-Baxter equation and we compute solutions, also called associative $r-$matrices, of this identity. In the second part, we identify certain Massey products with the aforementioned Szegö kernels via canonical isomorphisms. We also investigate some identities that these kernel functions satisfy.

The associative Yang-Baxter equation (AYBE) appeared for the first time in a work of Fomin and Kirillov [23] in the context of intersection theory on flag varieties. Later, it was studied in connection with deformation theory of Hopf algebras by Aguiar in [1, 2]. Finally, Polishchuk [39], in the framework of A_∞-categories, introduced the version with spectral parameters as follows.
Let $r : (\mathbb{C}^4, 0) \to \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the germ of a meromorphic function, then the associative Yang-Baxter equation is the identity

$$r^{12}(m_1, m_2; z_1, z_2) r^{23}(m_1, m_3; z_2, z_3) =$$
$$= r^{13}(m_1, m_3; z_1, z_3) r^{12}(m_3, m_2; z_1, z_2) + r^{23}(m_2, m_3; z_2, z_3) r^{13}(m_1, m_2; z_1, z_3),$$

where $r^{ij} = \sigma^{ij} \circ r$ and σ^{ij} are appropriate embeddings of $\mathrm{Mat}_{n\times n}(\mathbb{C})^{\otimes 2}$ into $\mathrm{Mat}_{n\times n}(\mathbb{C})^{\otimes 3}$, for instance $\sigma^{12}(a \otimes b) = a \otimes b \otimes \mathbb{1}$. We shall study solutions of the AYBE which satisfy the property

$$r^{12}(m_1, m_2; z_1, z_2) = -r^{21}(m_2, m_1; z_2, z_1)$$

and that are invertible, for generic points $(m_1, m_2; z_1, z_2)$, viewed as elements in $\mathrm{End}(\mathrm{Mat}_{n\times n}(\mathbb{C})) \simeq \mathrm{Mat}_{n\times n}(\mathbb{C})^{\otimes 2}$; these $r-$matrices are called skew-symmetric and non-degenerate, respectively.
The approach of Polishchuk, which was further developed by Burban and Kreussler in [18], permits to obtain a skew-symmetric, non-degenerate and associative $r-$matrix from a pair of non-isomorphic, stable vector bundles over a Calabi-Yau curve and a pair of distinct points belonging to the same irreducible component. These solutions are obtained by computing appropriate triple Massey products in the bounded category of coherent sheaves on such curves. Irreducible Calabi-Yau curves are just Weierstraß cubic curves $W \subset \mathbb{P}^2(\mathbb{C})$, i.e. curves given by the equation $zu^2 = 4v^3 - a_1vz^2 - a_2z^3$; where $a_1, a_2 \in \mathbb{C}$.
The smoothness of these curves is controlled by the discriminant $\mathrm{Disc}(a_1, a_2) = a_1^3 - 27a_2^2$. In the case $a_1 = a_2 = 0$, the singularity of the corresponding Weierstraß cubic is a cusp; it is a node for any other non-trivial solution of the relation $\mathrm{Disc}(a_1, a_2) = 0$, whereas it is smooth, in particular it is an elliptic curve, when $\mathrm{Disc}(a_1, a_2) \neq 0$.
Using the theory of stable vector bundles over W, Burban and Kreussler obtained a more explicit description of the aforementioned Massey products which permits a direct computation of $r-$matrices. Inspired by this method, we obtain solutions of the AYBE as follows.
We know, from the theory of elliptic curves, that for any couple (a_1, a_2) such that $\mathrm{Disc}(a_1, a_2) \neq 0$, there exists a $\tau \in \mathbb{C}$, with the property $\Im(\tau) > 0$, such that the corresponding Weierstraß cubic is isomorphic to the $1-$dimensional complex torus $T_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Let $(n, d) \in \mathbb{N}^+ \times \mathbb{Z}$ be two coprime integers and $\mathcal{M}(n, d)$ be the moduli space of stable vector bundles of rank n and degree d over T_τ. Moreover, let $\mathcal{U}(n, d)$ be a universal family on $\mathcal{M}(n, d)$, take $\zeta = \exp(\frac{2\pi i d}{n})$ and define the matrices

$$B_1 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \zeta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix} \quad \text{and} \quad B_2 := \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

We can now state the first result of this thesis:

Theorem A. *Let $(n,d) \in \mathbb{N}^+ \times \mathbb{Z}$ be a pair of coprime integers. Then the following is a solution of the associative Yang-Baxter equation corresponding to the universal family $\mathcal{U}(n,d)$*

$$r(m;z) = \sum_{a,b=0}^{n-1} \exp\Big(\frac{-2\pi i d}{n} az\Big) \kappa\Big(\frac{d}{n}\Big(b - a\tau\Big) + \frac{m}{n}, z\Big) B_{ab}^* \otimes B_{ab},$$

where

$$\kappa(m,z) = \frac{\theta_1'(0|\tau)\theta_1(m+z|\tau)}{\theta_1(m|\tau)\theta_1(z|\tau)}$$

is the Kronecker function, $m = m_2 - m_1$, $z = z_2 - z_1$ and $B_{ab} = B_2^a B_1^{-b}$ whereas $B_{ab}^* = \frac{1}{n} B_1^b B_2^{-a}$.

In the case of nodal and cuspidal Weierstraß curves, using the description of stable vector bundles over singular Weierstraß cubic curves due to Bodnarchuk and Burban, see for instance [12, 15], we perform computations of associative $r-$matrices for $n = 3$ and $d = 1$.

As mentioned above, in the second part of this thesis we study Szegö kernels. They owe their name to the mathematician Gábor Szegö, who introduced them in the context of complex analysis, however their fame, in the framework of algebraic geometry, is due to Fay [20, 21]. Inspired by the paper of Polishchuk [41], which we elaborate and further develop, we relate these kernel functions to appropriate triple Massey products as well as, in particular cases, to solutions of the Yang-Baxter equation.

Let C be a Gorenstein, reduced, projective curve of positive arithmetic genus g and X be a non-empty regular irreducible subset of C. Let $\mathcal{E}$ be a vector bundle over C, we shall denote by $\mathcal{E}^*$ its dual and by $\mathcal{E}^\vee$ the bundle $\mathcal{E}^* \otimes \Omega_C \simeq \mathrm{Hom}(\mathcal{E}, \Omega_C)$, where Ω_C is the dualising sheaf of C. We shall also introduce the concept of a good triple, which is a tuple $(\mathcal{E}, x, y)$ such that: $\mathcal{E}$ is a vector bundle on C, $x \neq y \in X$, the canonical maps $H^0(C,\mathcal{E}) \to \mathcal{E}|_y$ and $H^0(C,\mathcal{E}^\vee) \to \mathcal{E}^\vee|_x$ are zero. In this setting one can define a triple Massey product

$$m_{x,y}^{\mathcal{E}} : \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x) \otimes \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}) \otimes \mathrm{Hom}(\mathcal{E}, \mathbb{C}_y) \to \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_y)$$

as well as a map $\mathbf{S}^{\mathcal{E}}(x,y)$

$$\begin{tikzcd} & H^0(C,\mathcal{E}(x)) \arrow[dl, two heads, "\mathrm{res}_x"'] \arrow[dr, "\mathrm{ev}_y"] & \\ (\mathcal{E}\otimes\Omega_C^*)|_x \arrow[rr, "\mathbf{S}^{\mathcal{E}}(x,y)"] & & \mathcal{E}|_y, \end{tikzcd}$$

where res_x and ev_y are canonical maps defined in Section 3.3. Using Serre Duality and applying some canonical maps which are explicitly described in Section 3.4, we obtain an isomorphism

$$\begin{aligned} \mathrm{Lin}(\mathrm{Hom}(\mathcal{O}_C,\mathbb{C}_x)\otimes \mathrm{Ext}^1(\mathbb{C}_x,\mathcal{E})\otimes \mathrm{Hom}(\mathcal{E},\mathbb{C}_y),\mathrm{Hom}(\mathcal{O}_C,\mathbb{C}_y)) \simeq \\ \simeq \mathrm{Lin}((\mathcal{E}\otimes\Omega_C^*)|_x,\mathcal{E}|_y). \end{aligned} \tag{1}$$

Theorem B. *For any good triple $(\mathcal{E},x,y)$, the map $\mathbf{S}^{\mathcal{E}}(x,y)$ is the image of the Massey product $m^{\mathcal{E}}_{x,y}$ via the isomorphism* (1)*.*

The existence of the aforementioned identification, already observed by Polishchuk in [41], is here proved in a new and more general form, since the underlying curve C is not required to be smooth. Moreover, if we further suppose $H^0(C,\mathcal{E})=0=H^1(C,\mathcal{E})$, there exists an isomorphism

$$H^0(C\times X,\mathcal{E}^\vee\boxtimes\mathcal{E}|_X(\Delta)) \xrightarrow{\mathrm{res}_\Delta} H^0(C,\mathrm{End}(\mathcal{E}|_X)),$$

where Δ is the image of the diagonal embedding $\delta : X\to C\times X$.
Thus we call Szegö kernel the unique element $\tilde{S}\in H^0(C\times X,\mathcal{E}^\vee\boxtimes\mathcal{E}|_X(\Delta))$ whose residue along the diagonal is the identity matrix; i.e. $\mathrm{res}_\Delta(\tilde{S})=\mathbb{1}_{\mathrm{End}(\mathcal{E}|_X)}$.
Using the Riemann-Roch theorem, one can easily see that the vanishing of cohomology immediately implies that $(\mathcal{E},x,y)$ is good for all distinct points $x,y\in X$.
We shall recall the existence of a canonical isomorphism

$$\mathcal{E}^\vee|_x\otimes\mathcal{E}|_y\simeq \mathrm{Lin}((\mathcal{E}\otimes\Omega_C^*)|_x,\mathcal{E}|_y)$$

so that we can state the following result.

Theorem C. *If $\mathcal{E}$ is a vector bundle over C such that $H^0(C,\mathcal{E})=0=H^1(C,\mathcal{E})$, then, for any $x,y\in X\times X\setminus\Delta$, the tensor $\tilde{S}(x,y)$ and the map $\mathbf{S}^{\mathcal{E}}(x,y)$ get identified via the canonical isomorphism above.*

Let us now introduce the map $\tau : X \times X \to X \times X$, which sends (x, y) to (y, x), and $\lambda : \mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}^\vee$, which permutes both factors of the product at the level of appropriate local sections. The following holds:

Theorem D. *The Szegö kernel $\tilde{S}$ is skew-symmetric, i.e.*

$$\lambda(\tilde{S}) = -\tau^*(\tilde{S})$$

viewed as a meromorphic section of $\mathcal{E}^\vee \boxtimes \mathcal{E}$.

The skew-symmetry of the Szegö kernel $\tilde{S}$, at least in the case of smooth curves, seems to be known, nevertheless we could not find a clear proof in the algebro-geometric framework.
At this point, an observation made by Polishchuk [41] is useful. Unfortunately it seems that the author does not provide any proof of this, so we have independently developed his proposal, obtaining the next result:

Theorem E. *Let $\mathcal{E}_1, \dots, \mathcal{E}_n$ be vector bundles over C. Let $T : \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n \to \Omega_C$ be a morphism of vector bundles and $x_1, \dots, x_n$ be points of X such that the triples $(\mathcal{E}_j, x_j, x_i)$ are good for any $i \neq j$. Then the following relation is true:*

$$\sum_{i=1}^{n} T_{x_i}(id_{\mathcal{E}_i \otimes \Omega^*_C|_{x_i}} \otimes \bigotimes_{j \neq i} \mathbf{S}^{\mathcal{E}_j}(x_j, x_i)) = 0.$$

Since the triple $(\mathcal{O}_C(y_1 - x_1), x_1, z)$ is good for all distinct points $x_1, y_1, z \in X$, the element $\mathbf{S}^{\mathcal{O}_C(y_1 - x_1)}(x_1, z)$ is well-defined, moreover it is an isomorphism. We denote it by $s(z)$ and we state the following result.

Theorem F. *Let $\mathcal{E}$ be a vector bundle such that $H^0(C, \mathcal{E}) = 0 = H^1(C, \mathcal{E})$. Then the following equality holds:*

$$\mathbf{S}^{\mathcal{E}(x_1 - y_1)}(x_0, y_0)s(x_0)s(y_0)^{-1} = \mathbf{S}^{\mathcal{E}}(x_0, y_0) - \mathbf{S}^{\mathcal{E}}(x_1, y_0) \cdot \mathbf{S}^{\mathcal{E}}(x_1, y_1)^{-1} \cdot \mathbf{S}^{\mathcal{E}}(x_0, y_1), \quad (2)$$

where $x_0, x_1, y_0, y_1 \in X$ are four distinct points.

The latter equation is a sheaf-theoretic version of the celebrated matrix-valued Fay's trisecant identity. This relation appeared for the first time in line bundle cases in [20], as an equality between theta functions, and it owes the name trisecant to Mumford [36], who used it to show that the family of trisecants of the Kummer variety of a Riemann surface of positive genus is four-dimensional. Later, it was generalized by Fay in [21], who developed the vector bundle case. In order to see that the equation (2) and the one in [21] are equivalent, we have to work over smooth curves, then, using the theory of theta functions, one obtains $s(x_0)s(y_0)^{-1} = \frac{E(x_0,x_1)}{E(x_0,y_1)}\frac{E(y_0,y_1)}{E(y_0,x_1)}$, where E is the prime form as in [36].
Under this identification, the equation in Theorem F reduces directly to the matrix-valued Fay's identity. As an ulterior corollary, in the case of line bundles, the equation in Theorem F immediately gives the one in [20]. Furthermore, when the genus of C is equal to one, the Szegö kernel $\tilde{S}$ is given by the Kronecker function mentioned in Theorem A. Moreover, a straightforward computation shows that in this case the Fay's identity and the associative Yang-Baxter equations are equivalent.

Organization of the material:

In Chapter one, we introduce the AYBE and we recall results of Polishchuk [39] as well as those of Burban and Kreussler [18] regarding triple Massey products and their description through residue and evaluation sequences.

In Chapter two, we summarize classical results about vector bundles over elliptic curves from [5, 37] and on their degenerations from [12, 13, 15, 18]. We then perform computations of $r-$matrices in particular cases of nodal and cuspidal cubics. Afterwards, we obtain the elliptic solutions cited in Theorem A above. These results can be found in Sections 2.7, 2.8 and 2.9. We conclude the chapter relating those solutions to other forms of Yang-Baxter equation, namely the quantum and the classical one.

In Chapter three, we start recalling some results from [25], which relate Szegö kernels to the classical Yang-Baxter equation. Then we proceed comparing the kernel function $\tilde{S}$ with $\mathbf{S}^{\mathcal{E}}$ and the triple Massey product $m^{\mathcal{E}}_{x,y}$, demonstrating Theorems

B, C and D; these are contained in Sections 3.3, 3.4 and 3.5. We conclude the chapter with Section 3.6 in which we prove the results stated in Theorems E and F.

In Chapter four, we recall some classical results about Riemann surfaces and theta functions, we also construct a universal line bundle over $C \times C \times Pic^{g-1}(C)$ due to which we can deduce Fay's identity and the scalar associative Yang-Baxter equation from Theorem F.

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NOTATIONS

- We denote by $W \subset \mathbb{P}^2(\mathbb{C})$ the Weierstraß cubic curve, i.e. a plane projective curve defined by the equation $zu^2 = 4v^3 - a_1vz^2 - a_2z^3$; where $a_1, a_2 \in \mathbb{C}$. It is irreducible and its arithmetic genus is 1. Moreover, it is smooth if $a_1^3 - 27a_2^2 \neq 0$, its singularity is cusp if $a_1 = a_2 = 0$, whereas it is a node in any other case if $a_1^3 - 27a_2^2 = 0$.

- We use the symbol $D^b_{Coh}(W)$ to denote the triangulated category of bounded complexes of $\mathcal{O}$-modules with coherent cohomology. Whereas $\mathrm{Perf}(W)$ is the full subcategory of $D^b_{Coh}(W)$ admitting a bounded locally free resolution.

- Let C be a reduced projective Gorenstein curve, $n : \tilde{C} \to C$ its normalization, Ω_C be its dualising sheaf and denote by $\mathbf{M}_{\tilde{C}}$ the sheaf of meromorphic 1-forms over $\tilde{C}$. Then Ω_C can be identified with the sheaf of regular 1-forms defined as follows. For any open set $U \subset C$ a regular 1-form over U is an element $\omega \in \Gamma(U, n_*\mathbf{M}_{\tilde{C}})$ such that for any $x \in C$ and any $f \in \mathcal{O}_C(U)$ one has
$$\sum_{x_i \in n^{-1}(x)} \mathrm{res}_{x_i}((f \circ n)\omega) = 0,$$
where res is the standard residue on the smooth curve $\tilde{C}$.
Observe that on any non-singular part of C, a regular 1-form is holomorphic.

- If $\mathcal{E}$ is a vector bundle over a curve C and $x \in C$, $\mathcal{E}|_x$ is the fiber of $\mathcal{E}$ over x. Whereas we denote by $\mathbb{C}_x$ the skyscraper sheaf supported at x. Moreover, we will denote by $Vec(C)$ the category of vector bundles over C.

- For any vector bundle $\mathcal{E}$ over a curve C the symbol $\mathcal{E}^*$ stands for the dual of $\mathcal{E}$ whereas $\mathcal{E}^\vee = \mathcal{E}^* \otimes \Omega_C \simeq \mathrm{Hom}(\mathcal{E}, \Omega_C)$.

- The notations $\mathrm{Hom}(-,-)$ and $\mathrm{Ext}(-,-)$ are mainly used for global morphisms and extensions between coherent sheaves or vector bundles, while $\mathrm{Lin}(-,-)$ denotes maps between vector spaces.
- We denote by $h^n(C,\mathcal{E})$ the dimension of the space $H^n(C,\mathcal{E})$ for all $n \in \mathbb{N}$.
- Especially in chapter four, we shall identify classes of line bundles over a Riemann surface with divisors modulo linear equivalence.
- Given a curve C and a vector bundle $\mathcal{E}$ we will denote by $\tilde{S}$ the associated Szegö kernel and by $\mathbf{S}^{\mathcal{E}}$ the morphism corresponding to $\tilde{S}$ via the identification described in Section 3.3.

Chapter 1

Associative Yang-Baxter equation

In this chapter we introduce the associative Yang-Baxter equation starting with the algebraic definition and giving its solution. We then introduce appropriate triple Massey products and recall the results of Polishchuk [39], Burban and Kreussler [18] in order to show how these products can be used to construct solutions of the associative Yang-Baxter equation starting with a geometric data. Continuing with such an approach, we conclude this chapter defining the residue and evaluation sequences, so that we will be able to give a different and more explicit description of the aforementioned Massey products. Such products will be related to the study of vector bundles over a Weierstraß cubic curve. This last description of the Massey products will be used in chapter two in order to compute solutions of the associative Yang-Baxter equation.

1.1 Associative Yang-Baxter equation

Let $M = \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the algebra of $n \times n$ matrices over $\mathbb{C}$, with $n \in \mathbb{N}$. Let $r : (\mathbb{C}^4, 0) \to M\otimes M$ be the germ of a meromorphic function and $\sigma^{ij} : M^{\otimes 2} \to M^{\otimes 3}$, $i \neq j$, $i, j \in \{1, 2, 3\}$, be the map which sends a simple tensor $A \otimes B \in M \otimes M$ to the element with A in i-th spot, B in the j-th spot and $\mathbb{1}$ in the remaining tensor factor; for instance $\sigma^{13}(A \otimes B) = A \otimes \mathbb{1} \otimes B$. If we define $r^{ij} = \sigma^{ij} \circ r$, then the most general version of the associative Yang-Baxter equation (AYBE) is the following

$$\begin{aligned} &r^{12}(m_1, m_2; z_1, z_2)r^{23}(m_1, m_3; z_2, z_3) = \\ &= r^{13}(m_1, m_3; z_1, z_3)r^{12}(m_3, m_2; z_1, z_2) + r^{23}(m_2, m_3; z_2, z_3)r^{13}(m_1, m_2; z_1, z_3). \end{aligned} \tag{1.1}$$

If r satisfies the latter relation, it is said to be a solution of the AYBE or, equivalently, an associative $r-$matrix or simply $r-$matrix.

Definition 1.1.1. Let $r(m_1, m_2; z_1, z_2)$ be an $r-$matrix.

(i) r is called non-degenerate if, under the canonical isomorphism $M \otimes M \simeq \text{End}(M)$ induced by the trace map, it is invertible for generic $(m_1, m_2; z_1, z_2)$.

(ii) Let $g : (\mathbb{C}^2, 0) \to GL_n(\mathbb{C})$ be the germ of a holomorphic function, then the function $r'(m_1, m_2; z_1, z_2)$ equal to

$$(g(m_1; z_1) \otimes g(m_2; z_2))r(m_1, m_2, z_1, z_2)(g(m_2; z_1)^{-1} \otimes g(m_1; z_2)^{-1})$$

is still an $r-$matrix, which is said to be gauge equivalent to r.

(iii) r is called skew-symmetric if $r^{12}(m_1, m_2; z_1, z_2) = -r^{21}(m_2, m_1; z_2, z_1)$. Observe that gauge equivalence preserves skew-symmetry.

In this thesis we will mainly work on skew-symmetric solutions of (1.1) that depend on the difference $m = m_1 - m_2$. Then, once we introduce the notation $r(m_1, m_2; z_1, z_2) = r(m, z_1, z_2)$, the AYBE can be rewritten as follows

$$\begin{aligned} &r^{12}(n; z_1, z_2)r^{23}(n + m; z_2, z_3) = \\ &= r^{13}(n + m; z_1, z_3)r^{12}(-m; z_1, z_2) + r^{23}(m; z_2, z_3)r^{13}(n; z_1, z_3). \end{aligned} \tag{1.2}$$

One can easily specialize the definitions of non-degeneracy, skew-symmetry and gauge equivalence to this case.

Example 1.1.2. We take $M = \text{Mat}_{2\times 2}(\mathbb{C})$ with the standard basis given by the vectors e_{jk}, $j, k \in \{1, 2\}$. Then the following expression is a solution of (1.2)

$$r(m; z) = \frac{1}{2m}\mathbb{1} \otimes \mathbb{1} + \frac{1}{z}\sum_{i,j=1}^{2} e_{ij} \otimes e_{ji}$$

which can be proved by a straightforward computation.

1.2 Triple Massey products and AYBE over irreducible Calabi-Yau curves

A Calabi-Yau curve is a reduced projective Gorenstein curve with trivial dualising sheaf. A complete classification of such curves is given by the following list:

(i) a generic configuration of $n \geq 3$ concurrent lines in $\mathbb{P}^{n-1}$,

(ii) a cuspidal plane cubic curve,

(iii) a tachnode cubic curve,

(iv) a cycle of $n \geq 1$ projective lines, that is, for $n = 1$, a nodal cubic,

(v) an elliptic curve.

A proof for the latter classification can be found in [47], Section 3. If we also assume our curve to be irreducible, then the Calabi-Yau curves are exactly the Weierstraß cubics that we are going to define now.
Let $W \subset \mathbb{P}^2(\mathbb{C})$ be a Weierstraß cubic curve, i.e. a plane projective curve defined by the equation $zu^2 = 4v^3 - a_1vz^2 - a_2z^3$, where $a_1, a_2 \in \mathbb{C}$. These curves are irreducible and they have arithmetic genus one. Let us define the discriminant of W as $\mathrm{Disc}(a_1, a_2) := a_1^3 - 27a_2^2$. It is well-known that W is smooth if $\mathrm{Disc}(a_1, a_2) \neq 0$, in this case W is an elliptic curve, it is a nodal curve (ordinary double point) if $\mathrm{Disc}(a_1, a_2) = 0$ and $(a_1, a_2) \neq (0, 0)$, whereas it is a cusp if $(a_1, a_2) = (0, 0)$.

Let W be a Weierstraß curve over $\mathbb{C}$. It is well-known that the choice of a non-zero element $\omega \in H^0(W, \Omega_W)$ induces an isomorphism $\Omega_W \simeq_\omega \mathcal{O}_W$, so that we can always identify these two sheaves. Let $z_1 \neq z_2 \in W$ be two distinct points and $\mathcal{E}_1 \not\cong \mathcal{E}_2$ be two non-isomorphic vector bundles such that $rank(\mathcal{E}_1) = rank(\mathcal{E}_2)$, moreover

$$\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) = 0 \quad \text{and} \quad \mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) = 0. \tag{1.3}$$

Observe that non-isomorphic simple vector bundles of same rank and degree satisfy the latter condition. We call triple Massey product the following map

$$
\begin{tikzcd}
\mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_1}) \otimes \mathrm{Ext}^1(\mathbb{C}_{z_1},\mathcal{E}_2) \otimes \mathrm{Hom}(\mathcal{E}_2,\mathbb{C}_{z_2}) \arrow[d, "m^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}"'] \\
\mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_2}),
\end{tikzcd}
\tag{1.4}
$$

where $\mathbb{C}_{z_1}$ and $\mathbb{C}_{z_2}$ are skyscraper sheaves and the map is defined as follows. Let $e \in \mathrm{Ext}^1(\mathbb{C}_{z_1},\mathcal{E}_2)$, $f \in \mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_1})$ and $g \in \mathrm{Hom}(\mathcal{E}_2,\mathbb{C}_{z_2})$. The element e can be represented by a short exact sequence

$$0 \longrightarrow \mathcal{E}_2 \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathbb{C}_{z_1} \longrightarrow 0\,.$$

The conditions (1.3) imply that there exist unique lifts of f and g to morphisms $\tilde{f} : \mathcal{E}_1 \to \mathcal{B}$ and $\tilde{g} : \mathcal{B} \to \mathbb{C}_{z_2}$. This gives the following commutative diagram:

$$
\begin{tikzcd}
 & & \mathcal{E}_1 \arrow[d, "\tilde{f}"'] \arrow[dr, "f"] & & \\
e : 0 \arrow[r] & \mathcal{E}_2 \arrow[r, "a"] \arrow[dr, "g"] & \mathcal{B} \arrow[r, "b"] \arrow[d, "\tilde{g}"] & \mathbb{C}_{z_1} \arrow[r] & 0 \\
 & & \mathbb{C}_{z_2} & &
\end{tikzcd}
$$

and the triple Massey product is defined as $m^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}(f \otimes e \otimes g) = \tilde{g}\tilde{f}$. We now observe that Serre Duality, for a curve with trivial dualising sheaf, assures the existence of a bifunctorial isomorphism:

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{E}) \simeq \mathrm{Hom}(\mathcal{E},\mathcal{F})^*, \tag{1.5}$$

for any vector bundle $\mathcal{E}$ and $\mathcal{F}$ over W. Using the isomorphism (1.5), we can rewrite $m^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}$ as follows

$$\tilde{m}^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2} : \mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_2,\mathbb{C}_{z_2}) \to \mathrm{Hom}(\mathcal{E}_2,\mathbb{C}_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_2}). \tag{1.6}$$

In fact

$$\mathrm{Lin}(\mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_1}) \otimes \mathrm{Ext}^1(\mathbb{C}_{z_1},\mathcal{E}_2) \otimes \mathrm{Hom}(\mathcal{E}_2,\mathbb{C}_{z_2}), \mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_2}) \simeq$$
$$\mathrm{Lin}(\mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_2,\mathbb{C}_{z_2}), \mathrm{Hom}(\mathcal{E}_2,\mathbb{C}_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_1,\mathbb{C}_{z_2})).$$

Observe now that $\tilde{m}^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}$ can be thought as a linear map

$$\tilde{m}^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2} : \mathcal{E}_1^*|_{z_1} \otimes \mathcal{E}_2^*|_{z_2} \longrightarrow \mathcal{E}_2^*|_{z_1} \otimes \mathcal{E}_1^*|_{z_2}.$$

Let ψ be the following canonical isomorphism of vector spaces

$$\psi : \mathrm{Hom}(\mathcal{E}_1^*|_{z_1} \otimes \mathcal{E}_2^*|_{z_2}, \mathcal{E}_2^*|_{z_1} \otimes \mathcal{E}_1^*|_{z_2}) \to \mathrm{Hom}(\mathcal{E}_2|_{z_1}, \mathcal{E}_1|_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_1|_{z_2}, \mathcal{E}_2|_{z_2}),$$

then we define

$$r^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2} := \psi(\tilde{m}^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}). \tag{1.7}$$

The following theorem is due to Polishchuk [39], for the case when W is smooth, and it has been generalized by Burban and Kreussler [18] to the singular cases.

Theorem 1.2.1. *Let W be a Weierstraß cubic curve, then*

(i) $r^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}$ *satisfies the associative Yang-Baxter equation*

$$(r^{\mathcal{E}_1,\mathcal{E}_3}_{z_1,z_3})^{13}(r^{\mathcal{E}_3,\mathcal{E}_2}_{z_1,z_2})^{12} - (r^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2})^{12}(r^{\mathcal{E}_1,\mathcal{E}_3}_{z_2,z_3})^{23} + (r^{\mathcal{E}_2,\mathcal{E}_3}_{z_2,z_3})^{23}(r^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_3})^{13} = 0. \tag{1.8}$$

(ii) The tensor $r^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}$ *is skew-symmetric:*

$$\tau(r^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}) = -r^{\mathcal{E}_2,\mathcal{E}_1}_{z_2,z_1},$$

where τ is the map which flips the entries, i.e. $\tau(a \otimes b) = b \otimes a$.

(iii) Moreover, $r^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}$ *is non-degenerate.*

Proof. Let us give an idea of the proof.
(i) Let us denote by $\mathcal{D}^b_{coh}(W)$ the triangulated category of bounded complexes of $\mathcal{O}_W-$ modules with coherent cohomology and by $\mathrm{Perf}(W)$ its full-subcategory admitting a bounded locally free resolution. Since W is Gorenstein, we have

$$\begin{tikzcd} Perf(W) \arrow[r] \arrow[d, hook] & Hot^b_{Coh}(I(W)) \arrow[d, hook] \\ \mathcal{D}^b_{Coh}(W) \arrow[r] & Hot^{b+}_{coh}(I(W)), \end{tikzcd}$$

where the horizontal lines are isomorphisms and $Hot_{Coh}(I(W))$ is the sub-category of the homotopy theory, whose objects are complexes such that all cohomologies are coherent sheaves on W. As a consequence of the homological perturbation lemma of Kadeishvili [31], Perf(W) is an A_∞-category. This means that, for any collection of objects $Z_0, \ldots, Z_n \in Perf(W)$, we have linear maps

$$\begin{array}{c} \bigotimes_{s=1}^{n} \mathrm{Ext}^{i_s}(Z_{s-1}, Z_s) \\ \Big\downarrow{\scriptstyle m_n} \\ \mathrm{Ext}^{i_1+\cdots+i_n-(n-2)}(Z_0, Z_n), \end{array}$$

satisfying the identities

$$\sum_{i,j,l>0, i+j+l=n} (-1)^{i+jl} m_{i+j+l}(\mathbb{1}^{\otimes r} \otimes m_j \otimes \mathbb{1}^t) = 0.$$

By definition m_2 is the usual composition of morphisms. It is well-known that, in the sense of triangulated category for $\mathcal{E}_1, \mathcal{E}_2$ satisfying (1.3) and $z_1 \neq z_2 \in W$, m_3 is a map from $\mathrm{Hom}(\mathcal{E}_1, \mathbb{C}_{z_1}) \otimes \mathrm{Ext}^1(\mathbb{C}_{z_1}, \mathcal{E}_2) \otimes \mathrm{Hom}(\mathcal{E}_2, \mathbb{C}_{z_2})$ to $\mathrm{Hom}(\mathcal{E}_1, \mathbb{C}_{z_2})$ equal to $m_{z_1,z_2}^{\mathcal{E}_1,\mathcal{E}_2}$ defined in (1.4). The latter relation and the vanishing of Hom and Ext, which allows us to cancel all terms m_j, $j \neq 3$, lead to

$$(\tilde{m}_{z_1,z_2}^{\mathcal{E}_3,\mathcal{E}_2})^{12}(\tilde{m}_{z_1,z_3}^{\mathcal{E}_1,\mathcal{E}_3})^{13} - (\tilde{m}_{z_2,z_3}^{\mathcal{E}_1,\mathcal{E}_3})^{23}(\tilde{m}_{z_1,z_2}^{\mathcal{E}_1,\mathcal{E}_2})^{12} + (\tilde{m}_{z_1,z_3}^{\mathcal{E}_1,\mathcal{E}_2})^{13}(\tilde{m}_{z_2,z_3}^{\mathcal{E}_2,\mathcal{E}_3})^{23} = 0,$$

viewed as a map

$$\begin{aligned} &\mathrm{Hom}(\mathcal{E}_1, \mathbb{C}_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_2, \mathbb{C}_{z_2}) \otimes \mathrm{Hom}(\mathcal{E}_3, \mathbb{C}_{z_3}) \longrightarrow \\ &\longrightarrow \mathrm{Hom}(\mathcal{E}_2, \mathbb{C}_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_3, \mathbb{C}_{z_2}) \otimes \mathrm{Hom}(\mathcal{E}_1, \mathbb{C}_{z_3}). \end{aligned}$$

Applying (1.7) to this equation we get the statement.
(ii) The skew-symmetry remains to be proved. We now suppose W to be an elliptic curve and we take $a_1 \in \mathrm{Hom}(\mathcal{E}_1, \mathbb{C}_{z_1})$, $\beta_1 \in \mathrm{Ext}^1(\mathbb{C}_{z_1}, \mathcal{E}_2)$, $a_2 \in \mathrm{Hom}(\mathcal{E}_2, \mathbb{C}_{z_2})$, $\beta_2 \in \mathrm{Ext}^1(\mathcal{E}_2, \mathbb{C}_{z_1})$. Then the skew-symmetry follows from the compatibility of the A_∞ structure with the Serre pairing given by $p : \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \times \mathrm{Ext}^1(\mathcal{F}, \mathcal{E}) \to \mathbb{C}$. Namely:

$$p(m_3(a_1, \beta_1, a_2), \beta_2) = -p(a_1, m_3(\beta_1, a_2, \beta_2)) = -p(m_3(a_2, \beta_2, a_1), \beta_1).$$

It remains to apply the isomorphism ψ to get the statement. A generalization to the cuspidal and nodal case follows from the continuity of the Massey product with respect to a family of Weierstraß cubic curves, see [18, Section 6] for more

details.
(*iii*) One can prove, see [39, Theorems 3 and 4], that $\tilde{r}^{\mathcal{E}_1,\mathcal{E}_2}_{z_1,z_2}$ is non-degenerate if and only if $\mathrm{Ext}^1(\mathcal{E}_1(z_2), \mathcal{E}_2(z_1)) = 0$. The statement follows by conditions (1.3) and using Riemann-Roch theorem. □

Remark 1.2.2. Observe that the AYBE which appears in Theorem 1.2.1 is more abstract than the one introduced in Section 1.1. They match under a choice of a trivialization of the universal family of stable vector bundles. Details will be treated in the next chapter.

1.3 Residue and Evaluation sequences

Let W be a Weierstraß cubic curve, W_{reg} be the set of smooth points of W and Ω_W be the sheaf of global differential 1-forms on W. Recall that we have an isomorphism $\Omega_W \simeq_\omega \mathcal{O}_W$, where ω is a non-zero global section of Ω_W. We then fix such an ω and for any $p \in W_{reg}$ we define the following short exact sequence

$$0 \longrightarrow \Omega_W \longrightarrow \Omega_W(p) \xrightarrow{\mathrm{res}_p} \mathbb{C}_p \longrightarrow 0. \tag{1.9}$$

Lemma 1.3.1. *Let $\mathcal{E}_1, \mathcal{E}_2$ be two vector bundles over W such that*

$$\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) = 0 = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$$

and ω be as above, then for any point $p \in W_{reg}$ the following map is an isomorphism

$$\mathrm{res}^{\mathcal{E}_1,\mathcal{E}_2}_{p,\omega} : \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p)) \longrightarrow \mathrm{Lin}(\mathcal{E}_1|_p, \mathcal{E}_2|_p). \tag{1.10}$$

Proof. We recall the proof of Burban and Kreussler in [18].
We take the sequence (1.9), we use the isomorphism, which depends on the choice of ω, $\Omega_W \simeq \mathcal{O}_W$ to get the sequence

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_W(p) \xrightarrow{\mathrm{res}_{p,\omega}} \mathbb{C}_p \longrightarrow 0.$$

We now tensor the sequence above by $\mathcal{E}_2$ and we apply $\mathrm{Hom}(\mathcal{E}_1, -)$, so we get

$$0 \longrightarrow \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \to \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p)) \xrightarrow{\mathrm{Res}^{\mathcal{E}_1,\mathcal{E}_2}_{p,\omega}} \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2 \otimes \mathbb{C}_p) \longrightarrow \mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2).$$

By hypothesis $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) = 0 = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$, then

$$\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p)) \simeq \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2 \otimes \mathbb{C}_p).$$

We now conclude defining $\mathrm{res}_{p,\omega}^{\mathcal{E}_1,\mathcal{E}_2}$ as $\mathrm{Res}_{p,\omega}^{\mathcal{E}_1,\mathcal{E}_2}$ composed with the canonical isomorphism $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2 \otimes \mathbb{C}_p) \simeq \mathrm{Lin}(\mathcal{E}_1|_p, \mathcal{E}_2|_p)$.

□

We now introduce another short exact sequence. Let $q \in W_{reg}$, the evaluation sequence is the following

$$0 \longrightarrow \mathcal{O}(-q) \longrightarrow \mathcal{O} \xrightarrow{\mathrm{ev}_q} \mathbb{C}_q \longrightarrow 0. \tag{1.11}$$

Burban and Kreussler in [18] noticed that it induces another important isomorphism.

Lemma 1.3.2. *Let $\mathcal{E}_1, \mathcal{E}_2$ be two vector bundles over W such that*

$$\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2(p-q)) = 0 = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p-q)),$$

then for any two points $p, q \in W_{reg}$, such that $p \neq q$, the following map is an isomorphism

$$\mathrm{ev}_q^{\mathcal{E}_1,\mathcal{E}_2(p)} : \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p)) \longrightarrow \mathrm{Lin}(\mathcal{E}_1|_q, \mathcal{E}_2|_q). \tag{1.12}$$

Proof. We tensor the sequence (1.11) by $\mathcal{E}_2(p)$, then we get

$$0 \longrightarrow \mathcal{E}_2(p-q) \longrightarrow \mathcal{E}(p) \xrightarrow{\mathrm{Ev}_q} \mathcal{E}_2(p) \otimes \mathbb{C}_q \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}(\mathcal{E}_1, -)$ we get

$$0 \to \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p-q)) \to \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p)) \xrightarrow{\underline{\mathrm{ev}}_q^{\mathcal{E}_1,\mathcal{E}_2(p)}} \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p) \otimes \mathbb{C}_q) \to$$

$$\to \mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2(p-q)).$$

Since $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2(p-q))$ and $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p-q))$ are both 0 and $\mathcal{E}_2(p)|_q \simeq \mathcal{E}_2|_q$, the map in the statement is obtained composing $\underline{\mathrm{ev}}_q^{\mathcal{E}_1,\mathcal{E}_2(p)}$ with the canonical morphism $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p) \otimes \mathbb{C}_q) \simeq \mathrm{Lin}(\mathcal{E}_1|_q, \mathcal{E}_2|_q)$. □

Then, according to the previous two lemmas, the map

$$\tilde{R}_{p,q}^{\mathcal{E}_1,\mathcal{E}_2} = \mathrm{ev}_q^{\mathcal{E}_1,\mathcal{E}_2(p)} \circ (\mathrm{res}_{x,\omega}^{\mathcal{E}_1,\mathcal{E}_2})^{-1}$$

is well-defined, i.e.

$$\tilde{R}_{p,q}^{\mathcal{E}_1,\mathcal{E}_2} : \mathrm{Lin}(\mathcal{E}_1|_p, \mathcal{E}_2|_p) \xrightarrow{(\mathrm{res}_{x,\omega}^{\mathcal{E}_1,\mathcal{E}_2})^{-1}} \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2(p)) \xrightarrow{\mathrm{ev}_q^{\mathcal{E}_1,\mathcal{E}_2(p)}} \mathrm{Lin}(\mathcal{E}_1|_q, \mathcal{E}_2|_q). \tag{1.13}$$

Before stating the main theorem of this section, recall that we have the following isomorphism of vector spaces

$$\mathrm{Hom}(\mathcal{E}_2|_p, \mathcal{E}_1|_p) \otimes \mathrm{Hom}(\mathcal{E}_1|_q, \mathcal{E}_2|_q) \simeq \mathrm{Lin}(\mathrm{Lin}(\mathcal{E}_1|_p, \mathcal{E}_2|_p), \mathrm{Lin}(\mathcal{E}_1|_q, \mathcal{E}_2|_q)). \tag{1.14}$$

Theorem 1.3.3. *([18, Section 4]) The map $\tilde{R}_{p,q}^{\mathcal{E}_1,\mathcal{E}_2}$ defined as above is the image of $r_{p,q}^{\mathcal{E}_1,\mathcal{E}_2}$ (1.7) under the canonical isomorphism (1.14).*

The latter theorem allows us to compute the triple Massey products in terms of residue and evaluation sequences. In the next sections we will provide a trivialization of vector bundles over cubic curves and consequently also an explicit description of the maps which appear in (1.13). Once we fix a trivialization, we will have that the AYBE (1.8) corresponds to the associative Yang-Baxter equation (1.1), see [18] for further details.

Chapter 2

Vector bundles on irreducible Calabi-Yau curves

In this chapter we will study vector bundles over irreducible Calabi-Yau curves that, as already stated in Section 1.2, are just Weierstraß cubic curves. Why are we interested in studying those vector bundles? The answer is given by a theorem of Burban and Kreussler [18]. Let $\mathcal{M}(n,d)$ be the moduli space of stable vector bundles of rank n and degree d over a Weierstraß curve W, $\mathcal{U}$ be a universal family and $\phi = \{U_\alpha, \phi_\alpha\}$ be a trivialization of $\mathcal{U}$. Recall that $\mathcal{U}$ is a vector bundle over $W \times \mathcal{M}(n,d)$ such that, for any vector bundle $\mathcal{V}$ over W, there exists a unique $v \in \mathcal{M}(n,d)$ such that $\mathcal{V} \simeq \mathcal{U}|_{W\times\{v\}} := \mathcal{U}^v$. We take $z_1, z_2 \in W$, $m_1, m_2 \in \mathcal{M}(n,d)$ and we consider the corresponding vector bundles $\mathcal{U}^{m_1}, \mathcal{U}^{m_2}$ on W. Then, if we denote by $r_\phi(m_1,m_2,z_1,z_2) \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$ the image under the trivialization ϕ of the map $r^{\mathcal{U}^{m_1},\mathcal{U}^{m_2}}_{z_1,z_2} \in \mathrm{Hom}(\mathcal{E}_2|_{z_1}, \mathcal{E}_1|_{z_1}) \otimes \mathrm{Hom}(\mathcal{E}_1|_{z_2}, \mathcal{E}_2|_{z_2})$, which appears in equation (1.7), the following theorem holds.

Theorem 2.0.1. *[18] Provided that $gcd(n,d) = 1$, the tensor $r_\phi(m_1,m_2,z_1,z_2)$:*

(i) is an $r-$matrix in the sense of Section 1.1, i.e. it satisfies the associative Yang-Baxter equation (1.1);

(ii) is skew-symmetric;

(iii) is non-degenerate;

(iv) a different trivialization ψ provides a solution r_ψ gauge equivalent to r_ϕ.

Namely, one can associate to any coprime couple of numbers $(n, d) \in \mathbb{N} \times \mathbb{Z}$ an associative $r-$matrix with the properties stated above.

2.1 Category of triples and vector bundles on the projective line

In this and the next sections we will study vector bundles on singular cubic curves following the approach of [18]. In order to deal with them we have to introduce the category of triples.

Let C be a reduced singular projective curve. Consider the following commutative diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{\alpha}} & \tilde{C} \\ {\scriptstyle \tilde{n}}\downarrow & & \downarrow{\scriptstyle n} \\ A & \xrightarrow{\alpha} & C, \end{array}$$

where:

(1) $n : \tilde{C} \to C$ is the normalization of C;

(*ii*) $Con := \mathcal{H}om_{\mathcal{O}}(n_*(\mathcal{O}_{\tilde{C}}), \mathcal{O}_C) = Ann_{\mathcal{O}}(n_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C)$ is the so called conductor ideal sheaf;

(*iii*) $\alpha : A = V(Con) \hookrightarrow C$ is the closed artinian subspace defined by Con and supported at the singular points of C, whereas $\tilde{\alpha} : \tilde{A} \to \tilde{C}$ is its pull-back in $\tilde{A}$.

Let us denote $\nu = \alpha\tilde{n} = n\tilde{\alpha}$.

Definition 2.1.1. The category $Tpl(C)$ of triples is defined as follows.

(*i*) Objects: They are triples $(\tilde{\mathcal{E}}, \mathcal{V}, \tilde{\lambda})$, where $\tilde{\mathcal{E}}$ is a vector bundle over $\tilde{C}$, $N \in Vec(A)$ and

$$\tilde{\lambda} : \tilde{n}^*\mathcal{V} \to \tilde{\alpha}^*\tilde{\mathcal{E}}$$

is an isomorphism of $\mathcal{O}_{\tilde{A}}$ modules.

(*ii*) Morphisms: $\mathrm{Hom}_{Tpl(C)}((\tilde{\mathcal{E}}_1, \mathcal{V}_1, \tilde{\lambda}_1), (\tilde{\mathcal{E}}_2, \mathcal{V}_2, \tilde{\lambda}_2))$ consists of all pairs (f, g), where $f : \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_2$ and $g : \mathcal{V}_1 \to \mathcal{V}_2$ are morphisms of vector bundles such that the following diagram is commutative

$$\begin{tikzcd} \tilde{n}^*\mathcal{V}_1 \arrow[r, "\tilde{\lambda}_1"] \arrow[d, "\tilde{n}^*(g)"'] & \tilde{\alpha}^*\tilde{\mathcal{E}}_1 \arrow[d, "\tilde{\alpha}^*(f)"] \\ \tilde{n}^*\mathcal{V}_2 \arrow[r, "\tilde{\lambda}_2"] & \tilde{\alpha}^*\tilde{\mathcal{E}}_2. \end{tikzcd} \tag{2.1}$$

It is easy to see that $Tpl(C)$ has a natural interior tensor product

$$(\tilde{\mathcal{E}}_1, \mathcal{V}_1, \tilde{\lambda}_1) \otimes (\tilde{\mathcal{E}}_2, \mathcal{V}_2, \tilde{\lambda}_2) = (\tilde{\mathcal{E}}_1 \otimes \tilde{\mathcal{E}}_2, \mathcal{V}_1 \otimes \mathcal{V}_2, \tilde{\lambda}_1 \otimes \tilde{\lambda}_2).$$

The main theorem about category of triples and its relation with the category of vector bundles over C is the following.

Theorem 2.1.2. (*i*) *There exists an equivalence of categories*

$$\Phi : Vec(C) \to Tpl(C)$$

given by

$$\Phi(\mathcal{E}) = (n^*\mathcal{E}, \alpha^*\mathcal{E}, \tilde{\lambda}_{\mathcal{V}}),$$

where $\tilde{\lambda}_{\mathcal{E}} : \tilde{n}^*(\alpha^*\mathcal{E}) \to \tilde{\alpha}^*(n^*\mathcal{E})$ *is the canonical isomorphism. Furthermore,*

$$\Phi(\mathcal{E}_1 \otimes \mathcal{E}_2) \simeq \Phi(\mathcal{E}_1) \otimes \Phi(\mathcal{E}_2)$$

and the determinant commutes with the functor Φ*; i.e.* $\Phi \circ det \simeq det \circ \Phi$.

(*ii*) *The quasi-inverse*

$$\Psi : Tpl(C) \to Vec(C)$$

associates to a triple $(\tilde{\mathcal{E}}, \mathcal{V}, \tilde{\lambda})$ *the locally free coherent sheaf*

$$\mathcal{E} := Ker(n_*\tilde{\mathcal{E}} \oplus \alpha_*\mathcal{V} \xrightarrow{(c,m)} \nu_*\tilde{\alpha}^*\tilde{\mathcal{E}}),$$

where $c = c^{\tilde{\mathcal{E}}}$ *is the canonical morphism* $n_*\tilde{\mathcal{E}} \to \nu_*\tilde{\alpha}^*\tilde{\mathcal{E}}$ *and* λ *is the composition* $\alpha_*\mathcal{V} \xrightarrow{can} \nu_*\tilde{\alpha}^*\mathcal{V} \xrightarrow{\nu_*(\tilde{\lambda})} \nu_*\tilde{\alpha}^*\tilde{\mathcal{E}}$.

(iii) Let $\mathbb{T}_1 = (\tilde{\mathcal{E}}_i, \mathcal{V}_i, \tilde{\lambda}_i)$, $i \in \{1,2\}$, *be a pair of objects in* $Tpl(C)$ *and* $\mathcal{E}_i = \Psi(\mathbb{T}_i)$. *Then* $\Phi(\mathcal{H}om_C(\mathcal{E}_1, \mathcal{E}_2))$ *is isomorphic to*

$$(\mathcal{H}om_{\tilde{C}}(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2), \mathcal{H}om_A(\mathcal{E}_1, \mathcal{E}_2), h(\tilde{\lambda}_1, \tilde{\lambda}_2)),$$

where h *arises from the commutative diagram:*

$$\begin{tikzcd}
\tilde{n}^*\mathcal{H}om_A(\mathcal{E}_1, \mathcal{E}_2) \arrow[r, "h(\tilde{\lambda}_1)\text{,}\tilde{\lambda}_2"] \arrow[d, "can"'] & \tilde{\alpha}^*\mathcal{H}om_{\tilde{C}}(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2) \arrow[d, "can"] \\
\mathcal{H}om_{\tilde{A}}(\tilde{\alpha}^*\mathcal{E}_1, \tilde{n}^*\mathcal{E}_2) \arrow[r, "cnj(\tilde{\lambda}_1\text{,}\tilde{\lambda}_2)"] & \mathcal{H}om_{\tilde{A}}(\tilde{\alpha}^*\tilde{\mathcal{E}}_1, \tilde{\alpha}^*\tilde{\mathcal{E}}_2)
\end{tikzcd}$$

and $cnj(\tilde{\lambda}_1, \tilde{\lambda}_2)(\phi) = \tilde{\lambda}_2 \circ \phi \circ \tilde{\lambda}_1$.

The proof of the latter theorem can be found in [13, 15].
According to what was said so far, we have to describe the normalization of cubics we are working on. The next lemma is a classical result in algebraic geometry, we will follow the proof given in [38].

Lemma 2.1.3. *The normalization* $\tilde{C}$ *of the cubics* $W_1 = V(zu^2 - v^3)$ *and* $W_2 = V(zu^2 - v^3 - v^2 z)$ *is the projective line* $\mathbb{P}^1$. *Moreover, for any* $l \in \mathbb{C}$, *there exist bijections* $Pic^l(W_1) \simeq \mathbb{C}$ *and* $Pic^l(W_2) \simeq \mathbb{C}^*$.

Proof. Both curves have singularities just at $[0:0:1]$ and normalization can be computed locally. Thus we work over the affine curves $u^2 = v^3$, $u^2 = v^3 + v^2$ and we prove that their normalization is $\mathbb{A}^1$.
Regarding the cuspidal curve, consider the ring $R = \mathbb{C}[\overline{v}, \overline{u}] = \mathbb{C}[v,u]/(u^2 - v^3)$. Since the class of u is equal to the class of $(\frac{v}{u})^2$, then $\frac{\overline{v}}{\overline{u}}$ is integral over $\mathbb{C}[\overline{v}, \overline{u}]$ which is not integrally closed. Thus $\mathbb{C}[v,u]/(u^2 - v^3) \subset \mathbb{C}[\overline{v}, \overline{u}, \frac{\overline{v}}{\overline{u}}]$. Now observe that $\mathbb{C}[\overline{v}, \overline{u}, \frac{\overline{v}}{\overline{u}}] = \mathbb{C}[t]$, where $t = \frac{\overline{v}}{\overline{u}}$, in fact $\overline{v} = \overline{u}(\frac{\overline{v}}{\overline{u}})$. The proof follows observing that $\mathbb{C}[t]$ is integrally closed since it is a unique factorization domain. The proof for the nodal curve is analogous.
A straightforward computation shows that $Con(W_1) = \langle t^2, t^3 \rangle$ whereas $Con(W_2)$ is equal to $\langle t^2 - 1, t^3 - t \rangle$. Moreover $\mathbb{C}[t]^* = \mathbb{C}^*$, $(\mathbb{C}[t]/Con(W_1))^* \simeq \mathbb{C}^* \oplus \mathbb{C}$, $(R/Con(W_1))^* \simeq \mathbb{C}^*$. Furthermore, if we denote by $R' = \mathbb{C}[v,u]/(u^2 - v^3 - v^2)$, we also have $(R'/Con(W_1))^* \simeq \mathbb{C}^*$ and $(\mathbb{C}[t]/Con(W_2))^* \simeq \mathbb{C}^* \oplus \mathbb{C}^*$. We now need the following lemma, see [38] for a proof.

Lemma 2.1.4. *Let $S = \mathbb{C}[x,y]/p(x,y)$, where p is a polynomial, and N be the normalization of S, then the following is an exact sequence*

$$0 \to S^* \to N^* \times (S/Con)^* \to (S/Con)^* \to Pic(A) \to Pic(N) \oplus Pic(S/Con).$$

We now apply the latter lemma to the curves W_1 and W_2 and we observe that $Pic(\mathbb{C}[t]) = 0 = Pic(\mathbb{C})$. We get:

$$0 \to R^* \to \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \oplus \mathbb{C} \xrightarrow{f} Pic(R) \to 0$$

and

$$0 \to R'^* \to \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \oplus \mathbb{C}^* \xrightarrow{f'} Pic(R') \to 0.$$

We conclude that $Pic(R) \simeq (\mathbb{C}^* \times \mathbb{C})/ker(f)$ and $Pic(R') \simeq (\mathbb{C}^* \times \mathbb{C}^*)/ker(f')$, but in both cases a tedious computation shows that $ker(f) = ker(f') = \mathbb{C}^*$ In order to conclude the proof, we have to come back to projective versions of both curves. Due to exercise 6.9 in [27], one gets the short exact sequences

$$0 \to \mathbb{C} \to Pic(W_1) \to Pic(\mathbb{P}^1) \to 0$$

and

$$0 \to \mathbb{C} \to Pic(W_2) \to Pic(\mathbb{P}^1) \to 0.$$

The claim follows observing that, see next theorem, $Pic(\mathbb{P}^1) \simeq \mathbb{Z}$ and therefore

$$Pic(W_1) \simeq \mathbb{Z} \oplus \mathbb{C} \text{ and } Pic(W_2) \simeq \mathbb{Z} \oplus \mathbb{C}^*,$$

so, once we fix the degree, we have the thesis. □

The next step is to obtain an explicit description of stable vector bundles over a singular cubic curve W. We then take $C = W$ and its normalization $\tilde{C} = \mathbb{P}^1$. In order to understand the category of triples, we need a description of vector bundles over the projective line.

Theorem 2.1.5. *(Birkhoff-Grothendieck) Every vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles, i.e. $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_j)$. Furthermore, there exists an isomorphism $deg : Pic(\mathbb{P}^1) \xrightarrow{\cong} \mathbb{Z}$.*

Proof. We proceed by induction on the rank. If $rank(\mathcal{E}) = 1$ then the statement is trivial. Let us suppose we have proved the theorem for rank r vector bundles and suppose we have a bundle $\mathcal{E}$ of rank $r+1$. According to classical results of Serre regarding ample line bundles and Serre duality, there exists a unique $s_0 \in \mathbb{N}$ such that $H^0(\mathbb{P}^1, \mathcal{E}(s_0)) \neq 0$ and $H^0(\mathbb{P}^1, \mathcal{E}(s)) = 0$ for any $s < s_0$. Then we have a map $\sigma : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}(s_0)$ which provides a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}(s_0) \to \mathcal{E}' \to 0,$$

where the third term is locally free. In fact the quotient $\mathcal{E}(s_0)/Im(\sigma)$ is torsion free, otherwise $\mathcal{O}_{\mathbb{P}^1}(D) \hookrightarrow \mathcal{E}(s_0)$ for some effective divisor D. This would lead to a contradiction, in fact in that case we would get $H^0(\mathbb{P}^1, \mathcal{E}(s_0)(-D)) = H^0(\mathbb{P}^1, \mathcal{E}(s_0)(-deg(D))) \neq 0$. On $\mathcal{E}'$ we can use the inductive hypothesis, then it splits. It remains to prove that the latter short exact sequence splits, then, tensoring it by $\mathcal{O}_{\mathbb{P}^1}(-s_0)$ we would get the statement. We can tensor the short exact sequence above by $\mathcal{O}_{\mathbb{P}^1}(-1)$ and we can pass to cohomology to get $H^0(\mathbb{P}^1, \mathcal{E}'(-1)) = H^0(\mathbb{P}^1, \mathcal{E}(s_0 - 1)) = 0$, where the last equality follows from the definition of s_0. We now use the induction on $\mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n)$ and we obtain $d_i < 0$ for any i. We can conclude $\mathrm{Ext}^1(\mathcal{E}', \mathcal{O}_{\mathbb{P}^1}) \simeq H^1(\mathbb{P}^1, \mathcal{E}') \simeq \bigoplus_i H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_i - 2))^* = 0$.
Uniqueness. Suppose that we have two different splitting $\mathcal{E} \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(d_i) \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(d'_i)$ and, without loss of generalities, let d_b be the first integer such that $d_b \neq d'_b$ and $d_b > d'_b$. Then if we tensor both compositions by $\mathcal{O}_{\mathbb{P}^1}(d_b)$, we have two isomorphic vector bundles with different number of holomorphic sections which is not possible. $\square$

Thus it follows that if $(\tilde{\mathcal{E}}, \mathcal{V}, \tilde{\lambda})$ is a triple on a Weierstraß cubic curve, we have

(i) $\tilde{\mathcal{E}} = \oplus_{l \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n_l)^{k_l}$.

(ii) $\mathcal{V} = \mathcal{O}_A^{rank(\mathcal{E})}$.

(iii) $\sum_{l \in \mathbb{Z}} k_l = rank(\mathcal{E})$

Something more can be said about the structure of a bundle of the form $n^*\mathcal{E}$.

Lemma 2.1.6. *Let $\mathcal{E}$ be a vector bundle over a singular Weierstraß cubic curve W and $n : \mathbb{P}^1 \to W$ be its normalization, then we have:*

(*i*) *$\mathcal{E}$ is stable if and only if it is simple;*

(*ii*) *The bundles $\mathcal{E}$ and $n^*\mathcal{E}$ have the same degree;*

(*iii*) *there exist $(a, r_1, r_2) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ such that $n^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{r_1} \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)^{r_2}$ and $r_1 + r_2 = rank(\mathcal{E})$.*

Proof. The proof can be found in [18, Section 9]. □

Observe that, due to the latter theorem, we now know that, for a singular Weierstraß curve, simple is equivalent to stable, then we will work on stable vector bundles.
According to what we said so far, we know how to describe the first two terms of a triple, it remains to describe $\tilde{\lambda}$ in terms of matrices. However this will be done in two separated steps, one for the cuspidal curve and the other one for the nodal cubic.

2.2 Stable vector bundles over a cuspidal cubic curve

According to the definition the map $\tilde{\lambda}$ is an isomorphism of $\mathcal{O}_{\tilde{A}}$-modules, however, in order to write $\tilde{\lambda}$ as a matrix in $GL(\mathcal{O}_{\tilde{A}})$, some choices have to be done. In fact while $\tilde{n}^*\mathcal{V}$ is canonically isomorphic to $\mathcal{O}^n_{\tilde{A}}$, where $n = rank(\tilde{\mathcal{E}})$, we need a trivialization to get $\alpha^*(\tilde{\mathcal{E}}) \simeq \mathcal{O}^n_{\tilde{A}}$. First of all we have to fix coordinates as follows. Let W_1 be the cuspidal curve described by the equation $zu^2 = v^3$ and $n : \mathbb{P}^1 \to W_1$ be its normalization $n([z_0 : z_1]) = [z_0^2 z_1 : z_0^3 : z_1^3]$. Once we have chosen such a normalization, we have that the preimage of the singular point $s = [0 : 0 : 1]$ is given by $n^{-1}(s) = [0 : 1] = \infty$. Using these coordinates we have

$$(supp(A), \mathcal{O}_A) = (\{s\}, \mathbb{C}_s) \quad \text{and} \quad (supp(\tilde{A}), \mathcal{O}_{\tilde{A}}) = (\{\infty\}, \mathbb{C}[t]/t^2).$$

Then, for any section $\sigma \in H^0(\mathbb{P}^1, \mathcal{O}(l))$, $l \in \mathbb{Z}$, and for any open set $U \subset \mathbb{P}^1$ which does not contain the point $[1 : 1]$, we define

$$\phi(\sigma) = \left.\frac{\sigma}{(z_0 - z_1)^l}\right|_{\tilde{A}}.$$

Thus $\tilde{\lambda}$ is represented by

$$\mu = \mu_0 + t\mu_t \in GL_n(\mathcal{O}_{\tilde{A}}),$$

where μ_0 and μ_t are matrices in $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Moreover, due to the fact that μ is invertible and acting with elementary transformations, one can prove that μ_0 can be taken equal to $\mathbb{1}_n$ and μ_t is upper triangular. We would like to prove that a certain subcategory of triples is equivalent to an appropriate category of matrices. In order to obtain such an equivalence we have to prove that a morphism between triples induces a function between matrices in $GL_n(\mathcal{O}_{\tilde{A}})$. Observe that, if (f, g) is a morphism of triples, then $n^*(g)$ has a natural identification with a matrix in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ whereas f is a matrix with coefficients in $\mathbb{C}[z_0, z_1]^{j-i} \simeq \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(i), \mathcal{O}_{\mathbb{P}^1}(j))$. Thus, after choosing the trivialization ϕ, if q is a homogeneous polynomial of degree $j-i$ of the form $q = a_0 z_0^{j-i} + a_1 z_0^{j-i-1} z_1 + \dots a_{j-i} z_1^{j-i}$, then

$$\alpha^*(q) = a_{j-i-1} + t a_{j-i} \in \mathrm{Mat}(\mathcal{O}_{\tilde{A}}).$$

Moreover, f is presented by a lower block triangular matrix.

Definition 2.2.1. Let $(r_1, r_2) \in \mathbb{N}^+ \times \mathbb{N}$ such that $r_1 + r_2 = n$, then the category $Bl_c(W_1)$ is defined as follows.

(i) Objects are matrices

$$B = \left(\begin{array}{c|c} B_{00} & B_{01} \\ \hline * & B_{11} \end{array}\right)$$

where any block B_{ij} is a matrix of size $r_i \times r_j$ with coefficients in $\mathbb{C}$. Here $*$ is an empty block.

(ii) Morphisms between two objects B and B' are matrices

$$F = \left(\begin{array}{c|c} F_{00} & * \\ \hline F_{10} & F_{11} \end{array}\right)$$

with blocks of the same size of B and B' such that $FB = B'F$. The composition of two morphisms is given by the standard product between matrices.

Let us denote by $Vec^{0,1}(W_1)$ the full subcategory of vector bundles $\mathcal{E}$ over W_1 such that their pull-back over $\mathbb{P}^1$ splits into a sum of terms of the form $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(1)$. Similarly, let $Tpl^{0,1}(W_1)$ be the corresponding subcategory of the category $Tpl(W_1)$. Then the following proposition holds.

Proposition 2.2.2. *There exists an equivalence of categories $\tilde{\Phi}$ given by the following commutative diagram*

$$\begin{tikzcd} Vec^{0,1}(W_1) \arrow[rr, "\tilde{\Phi}"] \arrow[dr, "\Phi"] & & Bl_c(W_1) \\ & Tpl^{0,1}(W_1), \arrow[ur, "\Phi'"] & \end{tikzcd}$$

where Φ' is the functor induced by the trivialization described above and Φ is as in Theorem 2.1.2. Moreover, if $B \in Bl_c(W_1)$ is simple, then the block B_{01} is a full rank matrix.

Proof. A proof can be found in [13]. □

Example 2.2.3. Any element of $Pic^1(W_1)$ can be written as $\mathcal{O}_{W_1}(p)$ for a point $p \in \mathbb{P}^1 \setminus \{\infty\} \simeq \mathbb{C}$, here we identify the regular part of W_1 with its image in $\mathbb{P}^1$ via the normalization map. Moreover, a straightforward computation shows that Φ sends $\mathcal{O}_{W_1}(p)$ to the triple $(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, 1 - pt)$.

2.3 Stable vector bundles over a nodal cubic curve

The case of the nodal curve W_2 defined by the equation $zu^2 - v^3 - v^2 z$ is similar to the previous one. Let us fix some coordinates such that, $n^{-1}(s) = \{0 : \infty\} \in \mathbb{P}^1$, where $s = [0 : 0 : 1]$ is the singular point of W_2. It is easy to check that one has

$$(supp(A), \mathcal{O}_A) = (\{s\}, \mathbb{C}_s) \quad \text{and} \quad (supp(\tilde{A}), \mathcal{O}_{\tilde{A}}) = (\{0, \infty\}, \mathbb{C}^0 \oplus \mathbb{C}^\infty).$$

Then, similarly to what was already done in the previous section, for any section $\sigma \in H^0(\mathbb{P}^1, \mathcal{O}(l))$, $l \in \mathbb{Z}$, and for any open set $U \subset \mathbb{P}^1$ which does not contain the point $[1, 1]$, we define

$$\phi(\sigma) = \left. \frac{\sigma}{(z_0 - z_1)^l} \right|_{\tilde{A}}.$$

Therefore $\tilde{\lambda}$ is represented by a couple of invertible matrices μ^0 and μ^∞ over $\mathbb{C}$. Moreover, acting with elementary transformations of matrices, one can always suppose that $\mu^\infty = \mathbb{1}$.

Again, proceeding as in the previous section, we want to prove that a certain subcategory of triples is equivalent to that one of appropriate matrices. Then we have to describe morphisms of triples. Similarly to the cuspidal case, if (f, g) is a morphism between triples, then g can be naturally viewed as a matrix in $\mathrm{Mat}(\mathbb{C})$ whereas f can be described as follows. After choosing the trivialization ϕ, if q is a homogeneous polynomial of degree $j-i$ of the form $q = a_0 z_0^{j-i} + a_1 z_0^{j-i-1} z_1 + \dots a_{j-i} z_1^{j-i}$, then

$$\alpha^*(q) = ((-1)^{j-i} a_0, a_{j-i}) \in \mathrm{Mat}(\mathbb{C}) \times \mathrm{Mat}(\mathbb{C}).$$

Moreover, f is presented by a lower block triangular matrix.

Definition 2.3.1. Let $(r_1, r_2) \in \mathbb{N}^+ \times \mathbb{N}$ such that $r_1 + r_2 = n$ then the category $Bl_n(W_2)$ is defined as follows.

(i) Objects are invertible matrices

$$B = \left(\begin{array}{c|c} B_{00} & B_{01} \\ \hline B_{10} & B_{11} \end{array}\right)$$

where any block B_{ij} is a matrix of size $r_i \times r_j$ with coefficients in $\mathbb{C}$.

(ii) Morphisms between two objects B and B' are couples (F, G) of block matrices

$$F = \begin{pmatrix} D_1 & 0 \\ D' & D_2 \end{pmatrix} \qquad G = \begin{pmatrix} D_1 & 0 \\ D'' & D_2 \end{pmatrix}$$

such that $FB = B'G$. Here F and G have blocks of the same size of B and B'. The composition of two morphisms is given by the usual product between matrices.

Let us denote by $Vec^{0,1}(W_2)$ the full subcategory of vector bundles $\mathcal{E}$ over W_2 such that their pull-back over $\mathbb{P}^1$ splits into a sum of terms of the form $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(1)$. Let $Tpl^{0,1}(W_2)$ the corresponding subcategory of the category $Tpl(W_2)$. Similarly to the previous case, the following proposition holds.

Proposition 2.3.2. *There exists an equivalence of categories $\tilde{\Phi}$ given by the following commutative diagram*

$$\begin{tikzcd} Vec^{0,1}(W_2) \arrow[rr, "\tilde{\Phi}"] \arrow[dr, "\Phi"'] & & Bl_n(W_2) \\ & Tpl^{0,1}(W_2), \arrow[ur, "\Phi'"'] & \end{tikzcd}$$

where Φ' is the functor induced by the trivialization described above and Φ is as in Theorem 2.1.2. Moreover, if $B \in Bl_n(W_2)$ is simple, then the block B_{01} is a full rank matrix.

Analogously to the previous section, a proof can be found in [13].

Example 2.3.3. We conclude this section with the following example. Any element of $Pic^1(W_2)$ can be written as $\mathcal{O}_{W_2}(p)$ for a point $p \in \mathbb{P}^1 \setminus \{0, \infty\} \simeq \mathbb{C}^*$. Here we identify the regular part of W_2 with its image in $\mathbb{P}^1$ via the normalization map. Moreover, a straightforward computation shows that Φ sends $\mathcal{O}_{W_2}(p)$ to the triple $(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, (p, 1))$.

2.4 Vector bundles on elliptic curves

Let $\tau \in \mathbb{C}$ such that $\Im(\tau) > 0$ and consider the full rank lattice $\Gamma = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{R}^2 \simeq \mathbb{C}$. We call (1-dimensional) complex torus the quotient space

$$T = T_\tau = \mathbb{C}/\Gamma.$$

One can prove that W_τ, defined by the equation $zu^2 = 4v^3 - a_1(\Gamma)vz^2 - a_2(\Gamma)z^3$, is an elliptic curve; where

$$a_1(\Gamma) = 60 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^4} \qquad \text{and} \qquad a_2(\Gamma) = 140 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^6}.$$

Conversely, for any elliptic curve W, one can find a lattice Γ' such that the corresponding elliptic curve $W_{\tau'}$ is isomorphic to W. Let us denote by $\wp$ the Weierstraß function

$$\wp(x, \Gamma) = \frac{1}{x^2} + \sum_{\gamma \neq 0} \left(\frac{1}{(x-\gamma)^2} - \frac{1}{\gamma}^2 \right),$$

then the following holds.

Theorem 2.4.1. *The complex torus T_τ and the elliptic curve W are isomorphic both as manifolds and as groups. The isomorphism is given by:*

$$T \to W \subset \mathbb{P}^2(\mathbb{C})$$

$$t \to [\wp(t, \Gamma), \wp'(t, \Gamma), 1].$$

A proof of this classical result can be found in [46].
We can then identify a complex torus with an elliptic curve. The latter isomorphism will be strongly used in what follows, in fact the classification of stable vector bundles over elliptic curves is known (see [5]) and an explicit description of vector bundles over a torus can be given through automorphy factors.

Definition 2.4.2. The category of automorphy factors associated with $\tau \in \mathbb{C}$, $\Im(\tau) > 0$, is defined as follows.

(*i*) Objects: They are couples (F, n), where $n \in \mathbb{N}^+$ is a positive integer and $F : \Gamma \times \mathbb{C} \to GL_n(\mathbb{C})$ is a holomorphic function such that

$$F(\gamma_1 + \gamma_2, z') = F(\gamma_1, z' + \gamma_2)F(\gamma_2, z'), \qquad \forall \gamma_1, \gamma_2 \in \Gamma, z' \in \mathbb{C}.$$

(*ii*) Let (F, n_1), (G, n_2) be two automorphy factors, then morphisms are described by holomorphic functions $M : \mathbb{C} \to \mathrm{Mat}_{n_1 \times n_2}(\mathbb{C})$ such that

$$M(z' + \tau)G(\gamma, z') = F(\gamma, z')M(z')$$

and the composition is given by the usual product of matrices. Two automorphy factors (F, n_1) and (G, n_2) are isomorphic if and only if $n_1 = n_2$ and G is related to F as follows

$$G(\gamma, z') = M^{-1}(z' + \gamma)F(\gamma, z')M(z').$$

Lemma 2.4.3. *There exists an equivalence of categories of automorphy factors and vector bundles over a complex torus T. In fact:*

(*i*) *An automorphy factor (F, n) defines a vector bundle*

$$\mathcal{E}_F := (\mathbb{C} \times V)/\sim$$

over T, where V is a vector space of dimension n and

$$(z', v) \sim (z' + \gamma, F(\gamma, z')v), \forall (\gamma, z', v) \in \Gamma \times \mathbb{C} \times V.$$

(*ii*) *Conversely, for any vector bundle $\mathcal{E}$ over T there exists an automorphy factor (F, n) such that $\mathcal{E} \simeq \mathcal{E}_F$ as in point (i).*

Proof. A proof of the correspondence between automorphy factors and vector bundles can be found in [30]. $\square$

In order to simplify computations, we impose the conditions

$$F(0, z') = F(1, z') = \mathbb{1}_n,$$

so it is sufficient to study the behaviour of an automorphy factor along the generator τ.
Since our goal is to describe vector bundles over an elliptic curve with coprime rank and degree, we need a way to produce automorphy factors. We start with the next theorem which provides a way to explicitly describe the Picard group of T.

Theorem 2.4.4. *Let W be an elliptic curve and $\mathcal{E}$ be a vector bundle over W such that rank($\mathcal{E}$) and deg($\mathcal{E}$) are coprime. Then:*

(i) $\mathcal{E}$ is univocally defined by the triple $(rank(\mathcal{E}), deg(\mathcal{E}), det(\mathcal{E})) \in \mathbb{N} \times \mathbb{Z} \times Pic^l(W)$,

(ii) $e(z') = e(\tau, z') = \exp(-\pi i \tau - 2\pi i z')$ is an automorphy factor of dimension 1. Let $w \in W$ be a point and $D = [p]$ be the divisor of degree 1 supported at p, then there exists an isomorphism:

$$\mathcal{O}_W(D) \simeq \mathcal{E}_{e(z'-p+\frac{1+\tau}{2})}.$$

(iii) Furthermore, a divisor of degree l can be written as $D = (l-1)[q] + [q-p]$. Then, if we define $f(z') = e(z' - p + \frac{1+\tau}{2})e^l(z')$, we have $\mathcal{O}_W(D) \simeq \mathcal{E}_f$.

Proof. The first statement is a result from Atiyah [5]. The last two statements can be found in [18]. $\square$

We now know how to describe all possible line bundles over a torus T. We then would like to describe all possible vector bundles over T. An algorithm which provides such a description will be given in Section 2.6.

2.5 Algorithm: stable vector bundles over singular Weierstraß cubics

In this section we present an algorithm, established in [12, 18], which permits to describe the universal family of stable vector bundles over W_1 and W_2 in terms of block matrices. The theory developed in this section is valid for simple vector bundles. However, as already stated, for a Weierstraß cubic curve simplicity is equivalent to stability.

Theorem 2.5.1. *Let W be a singular Weierstraß cubic curve and $Sim(n, d)$ be the set of all isomorphism classes of simple vector bundles of rank n and degree d. If $gcd(n, d) \neq 1$, then*

$$Sim(n, d) = \emptyset.$$

Otherwise the map

$$det : Sim(n, d) \to Pic^d(W)$$

is a bijection .

A proof can be found in [18, Sections 9.2 and 9.3.].
We are then obliged to suppose $gcd(n, d) = 1$, the question is: how do we describe the family $Sim(n, d)$? According to Lemma 2.1.6 and supposing, without loss of generality, that $0 \leq d < n$, we can always decompose a bundle in order to reduce it to the form $\mathcal{O}_{\mathbb{P}^1}^{r_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r_2}$. In fact we can simply take $r_1 = n - d$ and $r_2 = d$. For any pair of positive coprime integers (r_1, r_2) we define the following objects of Bl_c.

(i) We generate a sequence of pairs of natural numbers replacing at each step (r_1, r_2) by $(r_1 - r_2, r_2)$ if $r_1 > r_2$ and by $(r_1, r_2 - r_1)$ if $r_2 > r_1$. We iterate the process until we get $(1, 1)$, then it terminates.

(ii) We now start, motivated by Proposition 2.2.2, with the matrix $M_{1,1}(m) = \begin{pmatrix} 0 & 1 \\ 0 & m \end{pmatrix}$ and we produce a sequence of matrices as follows.
If we consider

$$M^c_{r_1,r_2}(m) = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right),$$

then we place

$$M^c_{r_1+r_2,r_2}(m) = \left(\begin{array}{cc|c} B_1 & B_2 & 0 \\ \hline 0 & B_4 & \mathbb{1}_{r_2} \\ \hline 0 & 0 & 0 \end{array}\right)$$

if we go from (r_1, r_2) to $(r_1 + r_2, r_2)$.

If instead we go from (r_1, r_2) to $(r_1, r_1 + r_2)$, we take

$$M^c_{r_1,r_1+r_2}(m) = \left(\begin{array}{c|cc} 0 & I_{r_1} & 0 \\ \hline 0 & B_1 & B_2 \\ 0 & 0 & B_4 \end{array}\right).$$

(*iii*) Let $M^{c'}(m)$ be the matrix we obtain after the last step, we then replace all diagonal entries of $M^{c'}(m)$ by $\frac{m}{n}$.

(*iv*) If $M^c(m)$ is the matrix we get at the end of this algorithm, then the correspondent $\tilde{\lambda}$ in the category of triple is presented by $\mathbb{1}_n + tM(m)$.

Remark 2.5.2. The step (*iii*) is needed in order to make the family of vector bundles compatible with the action of $Pic^0(W_1) \simeq \mathbb{C}$, i.e. $M^c(n\beta + m) = \beta\mathbb{1}_n + M^c(m)$. Such a choice will provide a solution of the associative Yang-Baxter equation which depends just on the differences of the moduli space parameters.

We now describe a similar algorithm for the nodal cubic curve.

(*i*) We produce a sequence of integers (r_1, r_2) as in the previous algorithm.

(*ii*) We now start, motivated by Proposition 2.3.2, with the matrix $M_{1,1}(m) = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ and we produce a sequence of matrices as follows.

If we go from (r_1, r_2) to $(r_1 + r_2, r_2)$, we send the matrix

$$M^{nod}_{r_1,r_2}(m) = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right)$$

to

$$M^{nod}_{r_1+r_2,r_2}(m) = \left(\begin{array}{cc|c} B_1 & B_2 & 0 \\ \hline 0 & 0 & \mathbb{1}_{r_2} \\ \hline B_3 & B_4 & 0 \end{array}\right).$$

If instead we go from (r_1, r_2) to $(r_1, r_1 + r_2)$, we take

$$M^{nod}_{r_1, r_1+r_2}(m) = \left(\begin{array}{c|cc} 0 & \mathbb{1}_{r_1} & 0 \\ \hline B_1 & 0 & B_2 \\ B_3 & 0 & B_4 \end{array}\right).$$

(*iii*) For the same reason of Remark 2.5.2, we replace all the non-zero entries of the final matrix by m.

(*iv*) We then take $\mu^\infty = \mathbb{1}_n$ and $\mu^0 = M^{nod}(m)$, where $M^{nod}(m)$ is the matrix resulting from the first three steps.

Example 2.5.3. The universal family of stable vector bundles of rank 3 and degree 1 over W_1 is given by $(\mathcal{O}^2_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^3_s, \mu)$, where

$$\mu(m) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} m & 1 & 0 \\ 0 & m & 1 \\ 0 & 0 & m \end{pmatrix}, m \in \mathbb{C}.$$

Example 2.5.4. The universal family of vector bundles of rank 3 and degree 1 over W_2 is given by the triple $(\mathcal{O}^2_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^3_s, \mu)$, where

$$\mu^0(m) = \left(\begin{array}{cc|c} 0 & m & 0 \\ 0 & 0 & m \\ \hline m & 0 & 0 \end{array}\right), m \in \mathbb{C}^* \quad \text{and} \quad \mu^\infty = \mathbb{1}_3.$$

The final step is to return to the category of vector bundles. One can prove, see [18], that, if a curve C is either W_1 or W_2, then there exists a universal family $\mathcal{U}$ of stable vector bundles of rank n and degree d such that, for any point m of the moduli space, the corresponding vector bundle $\mathcal{U}^m$ is given by

$$\Psi(\mathcal{O}^{n-d}_{\mathbb{P}^1} \oplus \mathcal{O}^d_{\mathbb{P}^1}(1), \mathbb{C}_s, \mu).$$

Recall that Ψ is the functor defined in (2.1.2) and μ is the gluing map obtained, respectively for the cuspidal and nodal cubic, from the algorithms described above.

2.6 Algorithm: stable vector bundles on the complex torus.

In order to describe vector bundles over a 1-dimensional complex torus T, we have to recall an important result from Oda [37].

Theorem 2.6.1. *Let $\sigma_n : T_{n\tau} \to T_\tau$ be an ètale covering of degree n and $\mathcal{E} \to T_\tau$ be a stable vector bundle of rank n and degree l such that $gcd(n, l) = 1$. Then, there exists a line bundle $\mathcal{L} \to T_{n\tau}$ such that $\mathcal{E} \simeq \sigma_{n*}(\mathcal{L})$. Conversely, for any line bundle $\mathcal{L} \to T_{n\tau}$ of degree l, the bundle $\sigma_{n*}\mathcal{L} \simeq \mathcal{E}$ is stable of rank n and degree l.*

The last theorem, the description of $Pic^l(W)$ established in (2.4.4) and a straightforward computation imply that we can describe any vector bundles of rank n and degree l, $gcd(n, l) = 1$, through an automorphy factor (F, n) as follows.

The automorphy factor which represents the set of stable vector bundles of rank n and degree l can be defined by the algorithm:

(i) We start with the one dimensional automorphy factor

$$\tilde{e}(z') = \tilde{e}(\tau, z') = \exp\left(\pi i l \tau - \frac{2\pi i}{n} z'\right);$$

(ii) We use Oda's theorem to get

$$\tilde{F}(z') = \exp\left(\frac{-2\pi i m}{n}\right)\begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \tilde{e}^n(z') & 0 & \dots & 0 \end{pmatrix}; \tag{2.2}$$

(iii) We define $F(0, z') = \mathbb{1}_n$ and for any $j \in \mathbb{N}^+$ we place

$$F(j\tau, z') = \tilde{F}(z' + (j-1)\tau) \dots \tilde{F}(z') \text{ and } F(-j\tau, z') = F(j\tau, z' - j\tau)^{-1};$$

(iv) Moreover, we also set the condition:

$$F(n_1\tau + n_2, z') = F(n_1\tau, z'),$$

for any $n_1, n_2 \in \mathbb{Z}$.

Finally, if we use the isomorphism $\mathcal{M}(n,d) \simeq T$, then we have that $\mathcal{U}^m \simeq \mathcal{E}_{F(m,\gamma,z')}$, where $F(m,\gamma,z')$ is the automorphy factor defined through (2.2) and the latter algorithm. Observe that now we are considering m as a variable. It is the moduli parameter coming from the description of the Picard group in Section 2.4.

2.7 Associative $r-$matrix obtained from a cuspidal cubic curve

Once we know how to represents vector bundles over the cubic curve $zu^2 = v^3$, we are ready to perform the computation of an $r-$matrix. Let $\mathcal{U}$ be the universal family of stable vector bundles of rank n and degree d over W_1, $gcd(n,d) = 1$. According to what was said so far, we can always map a bundle $\mathcal{U}^{m_1}$ on W_1 to its corresponding triple via the functor Φ as in Theorem 2.1.2. Moreover, for any triple $(\tilde{\mathcal{E}}, \mathcal{V}, \tilde{\lambda})$, let $\mathbb{F} : Tpl(W_1) \to Vec(\mathbb{P}^1)$ be the functor $\mathbb{F}(\tilde{\mathcal{E}}, \mathcal{V}, \tilde{\lambda}) = \tilde{\mathcal{E}}$ and let $\mathcal{U}^{m_1}$, $\mathcal{U}^{m_2}$ be two non-isomorphic vector bundles over W_1. We define the space $\mathcal{S}ol_c := Im\Bigg(\mathrm{Hom}_{Tpl(W_1)}(\Phi(\mathcal{U}^{m_1}), \Phi(\mathcal{U}^{m_2})) \xrightarrow{\mathbb{F}} \mathrm{Hom}_{\mathbb{P}^1}(\tilde{B}, \tilde{B}(1)) \Bigg)$, where $\tilde{B} := \mathcal{O}_{\mathbb{P}^1}^{n-d} \oplus \mathcal{O}_{\mathbb{P}^1}^{d}(1)$.

Theorem 2.7.1. *Let ω be the rational 1-form over $\mathbb{P}^1$ given by dp, where p is a coordinate in $\mathbb{P}^1 \setminus \{\infty\}$. Let $\mathcal{U}^{m_1}$ and $\mathcal{U}^{m_2}$ be two non-isomorphic vector bundles over W_1 and z_1, z_2 be two distinct points of W_1. Then the map*

$$\mathrm{Lin}(\mathcal{U}^{m_1}|_{z_1}, \mathcal{U}^{m_2}|_{z_1}) \xrightarrow{(\mathrm{res}_{z_1,\omega}^{\mathcal{U}^{m_1},\mathcal{U}^{m_2}})^{-1}} \mathrm{Hom}(\mathcal{U}^{m_1}, \mathcal{U}^{m_2}(z_1)) \xrightarrow{\mathrm{ev}_{z_2}^{\mathcal{U}^{m_1},\mathcal{U}^{m_2}(z_1)}} \mathrm{Lin}(\mathcal{U}^{m_1}|_{z_2}, \mathcal{U}^{m_2}|_{z_2})$$

defined in Section 1.3 can be identified, under the trivialization introduced in Section 2.2, with the map

$$\mathrm{Mat}_{n\times n}(\mathbb{C}) \xrightarrow{\mathrm{res}_{z_1}^{-1}} \mathcal{S}ol_c \xrightarrow{\mathrm{ev}_{z_2}} \mathrm{Mat}_{n\times n}(\mathbb{C}). \tag{2.3}$$

Here we put

$$\mathrm{res}_{z_1}(F) = F(1, z_1) \quad \textit{and} \quad \mathrm{ev}_{z_1}(F) = \tfrac{F(1,z_2)}{z_2 - z_1}.$$

Moreover, let $r(m_1, m_2; z_1, z_2)$ be the image of $\mathrm{ev}_{z_2} \circ \mathrm{res}_{z_1}^{-1}$ under the canonical map

$$\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Lin}(\mathrm{Mat}_{n\times n}(\mathbb{C}), \mathrm{Mat}_{n\times n}(\mathbb{C})) \tag{2.4}$$

induced by the trace, i.e. $X \otimes Y$ *is sent to* $Z \to Tr(XZ)Y$.

Then the tensor-valued function $r(m_1, m_2; z_1, z_2)$ *is a solution of the AYBE*

$$\begin{gathered} r^{12}(n; z_1, z_2) r^{23}(n+m; z_2, z_3) = \\ = r^{13}(n+m; z_1, z_3) r^{12}(-m; z_1, z_2) + r^{23}(m; z_2, z_3) r^{13}(n; z_1, z_3). \end{gathered} \tag{2.5}$$

Moreover, $r(m_1, m_2; z_1, z_2)$ *is skew-symmetric and non-degenerate.*

We now perform the computation of the associative $r-$matrix obtained from the universal family of vector bundles described in example (2.5.3), namely

$$(\mathcal{O}^2_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^3_s, \mu) \quad \text{and} \quad \mu = \mathbb{1}_3 + t\mathcal{J}_3(m)$$

where $\mathcal{J}_3(m)$ is the Jordan block with m along the diagonal. This computation is the first original result of this thesis.

A morphism F in $\mathrm{Hom}(\mathcal{O}^{n-d}_{\mathbb{P}^1} \oplus \mathcal{O}^{d}_{\mathbb{P}^1}(1), \mathcal{O}^{n-d}_{\mathbb{P}^1}(1) \oplus \mathcal{O}^{d}_{\mathbb{P}^1}(2))$ is a matrix whose entries are polynomials of degree at most two that has to be evaluated on $\tilde{A}$ in the way described in Section 2.2. We get

$$F|_{\tilde{A}} = \left(\begin{array}{cc|c} a'' + a'\varepsilon & b'' + b'\varepsilon & h \\ c'' + c'\varepsilon & d' + d''\varepsilon & s \\ \hline e''' + e''\varepsilon & f''' + f''\varepsilon & g'' + g'\varepsilon \end{array}\right).$$

Moreover the diagram (2.1) which appears in point (ii) in the definition of the category of triples has to be commutative, which, due to our trivialization, gives:

$$\begin{gathered} \left(\begin{array}{cc|c} a' & b' & 0 \\ c' & d' & 0 \\ \hline e'' & f'' & g' \end{array}\right) + \left(\begin{array}{cc|c} a'' & b'' & h \\ c'' & d'' & s \\ \hline e''' & f''' & g'' \end{array}\right) \begin{pmatrix} m_1 & 1 & 0 \\ 0 & m_1 & 1 \\ 0 & 0 & m_1 \end{pmatrix} = \\ = \begin{pmatrix} m_2 - z_1 & 1 & 0 \\ 0 & m_2 - z_1 & 1 \\ 0 & 0 & m_2 - z_1 \end{pmatrix} \left(\begin{array}{cc|c} a'' & b'' & h \\ c'' & d'' & s \\ \hline e''' & f''' & g'' \end{array}\right). \end{gathered}$$

The latter condition leads to the system $(\star)$:

$$\left(\begin{array}{cc|c} a' = (m - z_1)a'' + c'' & b' = (m - z_1)b'' + d'' - a'' & b'' = (m - z_1)h + s \\ c' = (m - z_1)c'' + e''' & d' = (m - z_1)d'' + f''' - c'' & d'' = (m - z_1)s + g'' \\ \hline e'' = (m - z_1)e''' & f'' = (m - z_1)f''' - e''' & g' = (m - z_1)g'' - f''' \end{array}\right),$$

where $m = m_2 - m_1$. We now have to compute $\mathrm{Res}_{z_1}(F) = F(1, z_1)$, that gives

$$\left(\begin{array}{cc|c} a' + a''z_1 & b' + b''z_1 & h \\ c' + c''z_1 & d' + d''z_1 & s \\ \hline e' + e''z_1 + e'''z_1^2 & f' + f''z_1 + f'''z_1^2 & g' + g''z_1 \end{array}\right) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}.$$

Using $(\star)$, we can solve the last system and write every entrance, on the left hand side, in terms of α_{lm}. We explicitly write down some of them:

$$\begin{cases} h = \alpha_{13} \\ s = \alpha_{23} \\ b'' = (m - z_1)\alpha_{13} + \alpha_{23} \\ b' = \alpha_{12} - z_1(m - z_1)\alpha_{13} - z_1\alpha_{23} \\ a'' = \frac{1}{3m}(\alpha_{11} + \alpha_{22} + \alpha_{33}) - \frac{2}{3}\alpha_{12} + \frac{2m}{3}(m - z_1)\alpha_{13} + (m - \frac{z_1}{3})\alpha_{23} \\ a' = \alpha_{11} - z_1\frac{1}{3m}(\alpha_{11} + \alpha_{22} + \alpha_{33}) + \frac{2}{3}z_1\alpha_{12} - \frac{2m}{3}z_1(m - z_1)\alpha_{13} - z_1(m - \frac{z_1}{3})\alpha_{23} \\ c'' = \alpha_{11} - \frac{1}{3}(\alpha_{11} + \alpha_{22} + \alpha_{33}) + \frac{2m}{3}\alpha_{12} - \frac{2m^2}{3}(m - z_1)\alpha_{13} - m(m - \frac{z_1}{3})\alpha_{23} \\ c' = \alpha_{21} - z_1c'' \\ e''' = \alpha_{21} - m\alpha_{11} + \frac{m}{3}(\alpha_{11} + \alpha_{22} + \alpha_{33}) - \frac{2m^2}{3}\alpha_{12} + \\ + \frac{2m^3}{3}(m - z_1)\alpha_{13} + m^2(m - \frac{z_1}{3})\alpha_{23} \\ e'' = (m - z_1)e''' \\ e' = \alpha_{31} - z_1me''' \\ g'' = \frac{1}{3m}(\alpha_{11} + \alpha_{22} + \alpha_{33}) + \frac{1}{3}\alpha_{12} - \frac{m}{3}(m - z_1)\alpha_{13} + (\frac{2}{3}z_1 - m)\alpha_{23} \\ g' = \alpha_{33} - g''z_1 \\ d'' = \frac{1}{3}\alpha_{12} + \frac{1}{3m}(\alpha_{11} + \alpha_{22} + \alpha_{33}) - \frac{1}{3}m(m - z_1)\alpha_{13} - \frac{z_1}{3}\alpha_{23} \\ d' = \alpha_{22} - d''z_1 \\ f''' = mg'' - \alpha_{33} \\ f'' = (m - z_1)f''' - e''' \\ f' = \alpha_{32} - f''z_1 - f'''z_1^2. \end{cases}$$

We are now ready to compute $\mathrm{ev}_{z_2}(F) = \frac{1}{z_2 - z_1}\{\beta_{lm}\}_{l,m\in\{1,2,3\}} \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$. We get

$$\frac{1}{z_2-z_1}\left(\begin{array}{cc|c} a'+a''z_2 & b'+b''z_2 & h \\ c'+c''z_2 & d'+d''z_2 & s \\ \hline e'+e''z_2+e'''z_2^2 & f'+f''z_2+f'''z_2^2 & g'+g''z_2 \end{array}\right) = \frac{1}{z}\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix},$$

where $z = z_2 - z_1$. According to what we said so far, we can write any β_{lm} in terms of α-coefficients:

$$\begin{cases} \beta_{13} = \alpha_{13} \\ \beta_{23} = \alpha_{23} \\ \beta_{12} = \alpha_{12} + z[(m-z_1)\alpha_{13} + \alpha_{23}] \\ \beta_{11} = \alpha_{11} + z[\frac{1}{3m}(\alpha_{11}+\alpha_{22}+\alpha_{33}) - \frac{2}{3}\alpha_{12} + \frac{2m}{3}(m-z_1)\alpha_{13} + (m-\frac{z_1}{3})\alpha_{23}] \\ \beta_{21} = \alpha_{21} + z[\alpha_{11} - \frac{1}{3}(\alpha_{11}+\alpha_{22}+\alpha_{33}) + \frac{2m}{3}\alpha_{12} - \frac{2m^2}{3}(m-z_1)\alpha_{13} + \\ -m(m-\frac{z_1}{3})\alpha_{23}] \\ \beta_{22} = \alpha_{22} + z[\frac{1}{3m}(\alpha_{11}+\alpha_{22}+\alpha_{33}) + \frac{1}{3}\alpha_{12} - \frac{m}{3}(m-z_1)\alpha_{13} - \frac{1}{3}z_1\alpha_{23}] \\ \beta_{33} = \alpha_{33} + z[\frac{1}{3m}(\sum \alpha_{ll}) + \frac{1}{3}\alpha_{12} - \frac{m}{3}(m-z_1)\alpha_{13} + (\frac{2}{3}z_1 - m)\alpha_{23}] \\ \beta_{31} = \alpha_{31} + z(m+z_2)[\alpha_{21} - m\alpha_{11} + \frac{m}{3}(\alpha_{11}+\alpha_{22}+\alpha_{33}) + \\ -\frac{2m^2}{3}\alpha_{12} + \frac{2m^3}{3}(m-z_1)\alpha_{13} + m^2(m-\frac{z_1}{3})\alpha_{23}] \\ \beta_{32} = \alpha_{32} + z(m+z_2)[-\alpha_{33} + \frac{1}{3}\sum \alpha_{ll} + \frac{m}{3}\alpha_{12} + \\ -\frac{1}{3}m^2(m-z_1)\alpha_{13} + (\frac{2mz_1}{3} - m^2)\alpha_{23}] - z[\alpha_{21} - m\alpha_{11} + \frac{m}{3}\sum \alpha_{ll} - \frac{2m^2}{3}\alpha_{12} + \\ +\frac{2m^3}{3}(m-z_1)\alpha_{13} + m^2(m-\frac{z_1}{3})\alpha_{23}]. \end{cases}$$

We conclude using the isomorphism

$$\mathrm{Lin}(\mathrm{Mat}_{3\times3}(\mathbb{C}), \mathrm{Mat}_{3\times3}(\mathbb{C})) \to \mathrm{Mat}_{3\times3}(\mathbb{C}) \otimes \mathrm{Mat}_{3\times3}(\mathbb{C})$$

which sends the linear map $e_{ij} \to \gamma_{ik}^{kl} e_{kl}$ to the tensor $\gamma_{ij}^{kl} e_{ji} \otimes e_{kl}$.

The last isomorphism gives the associative $r-$matrix:

$$
\begin{aligned}
r(m; z_1, z_2) = &+ \frac{1}{3m}(\mathbb{1} \otimes \mathbb{1}) - \frac{2}{3}e_{21} \otimes e_{11} + \frac{1}{3}e_{21} \otimes (e_{22} + e_{33}) \\
&+ \frac{2m}{3}(m - z_1)e_{31} \otimes e_{11} + (m - \frac{z_1}{3})e_{32} \otimes e_{11} + \frac{1}{z}\sum_{k,l=1}^{3} e_{kl} \otimes e_{lk} + \\
&+ (m - z_1)e_{31} \otimes e_{12} + e_{32} \otimes e_{12} + \frac{2}{3}e_{11} \otimes e_{21} - \frac{1}{3}e_{22} \otimes e_{21} - \frac{1}{3}e_{33} \otimes e_{21} \\
&+ \frac{2m}{3}e_{21} \otimes e_{21} - \frac{2m^2}{3}(m - z_1)e_{31} \otimes e_{21} - m(m - \frac{z_1}{3})e_{32} \otimes e_{21} + \\
&- \frac{m}{3}(m - z_1)e_{31} \otimes e_{22} - \frac{1}{3}z_1 e_{32} \otimes e_{22} + \\
&- \frac{m}{3}(m - z_1)e_{31} \otimes e_{33} + (\frac{2z_1}{3} - m)e_{32} \otimes e_{33} + (m + z_2)e_{12} \otimes e_{31} + \\
&- \frac{2m}{3}(m + z_2)e_{11} \otimes e_{31} + \frac{1}{3}m(m + z_2)(e_{22} + e_{33}) \otimes e_{31} + \\
&- \frac{2m^2}{3}(m + z_2)e_{21} \otimes e_{31} + \frac{2m^3}{3}(m - z_1)(m + z_2)e_{31} \otimes e_{31} + \\
&+ m^2(m - \frac{z_1}{3})(m + z_2)e_{32} \otimes e_{31} - \frac{2}{3}(m + z_2)e_{33} \otimes e_{32} + \\
&+ \frac{1}{3}(m + z_2)(e_{11} + e_{22}) \otimes e_{32} + \frac{m}{3}(m + z_2)e_{21} \otimes e_{32} + \\
&- \frac{m^2}{3}(m - z_1)(m + z_2)e_{31} \otimes e_{32} + \\
&+ (\frac{2mz_1}{3} - m^2)(m + z_2)e_{32} \otimes e_{32} - e_{12} \otimes e_{32} + \\
&+ \frac{2m}{3}e_{11} \otimes e_{32} - \frac{m}{3}(e_{22} + e_{33}) \otimes e_{32} + \frac{2m^2}{3}e_{21} \otimes e_{32} + \\
&- \frac{2m^3}{3}(m - z_1)e_{31} \otimes e_{32} - m^2(m - \frac{z_1}{3})e_{32} \otimes e_{32}.
\end{aligned}
$$

2.8 Associative $r-$matrix obtained from a nodal cubic curve

Here, similarly to the previous section, we perform a computation for the curve $W_2 = V(zu^2 - v^3 - v^2 z)$. Let $\mathcal{U}$ be the universal family of stable vector bundles of rank n and degree d, with $gcd(n, d) = 1$. Let Φ be the functor defined in Theorem 2.1.2 and let $\mathbb{F} : Tpl(W_2) \to Vec(\mathbb{P}^1)$ be the functor whose image is the first term of a triple. Then, analogously to the previous section, we define the space $\mathcal{S}ol_{nd} := Im\left(\mathrm{Hom}_{Tpl(W_2)}(\Phi(\mathcal{U}^{m_1}), \Phi(\mathcal{U}^{m_2})) \xrightarrow{\mathbb{F}} \mathrm{Hom}_{\mathbb{P}^1}(\tilde{B}, \tilde{B}(1))\right)$, where $\tilde{B} := \mathcal{O}_{\mathbb{P}^1}^{n-d} \oplus \mathcal{O}_{\mathbb{P}^1}^{d}(1)$.

Theorem 2.8.1. *Let $\omega = \frac{dp}{p}$ be the rational 1-form over $\mathbb{P}^1$, where p is a coordinate in $\mathbb{P}^1 \setminus \{0, \infty\}$. Let $\mathcal{U}^{m_1}$ and $\mathcal{U}^{m_2}$ be two non-isomorphic vector bundles over W_2 and z_1, z_2 be two distinct points of W_2. Then the map*

$$\mathrm{Lin}(\mathcal{U}^{m_1}|_{z_1}, \mathcal{U}^{m_2}|_{z_1}) \xrightarrow{(\mathrm{res}^{\mathcal{U}^{m_1}, \mathcal{U}^{m_2}}_{z_1, \omega})^{-1}} \mathrm{Hom}(\mathcal{U}^{m_1}, \mathcal{U}^{m_2}(z_1)) \xrightarrow{\mathrm{ev}^{\mathcal{U}^{m_1}, \mathcal{U}^{m_2}(z_1)}_{z_2}} \mathrm{Lin}(\mathcal{U}^{m_1}|_{z_2}, \mathcal{U}^{m_2}|_{z_2})$$

defined in Section 1.3 can be identified, under the trivialization introduced in Section 2.3, with the map

$$\mathrm{Mat}_{n\times n}(\mathbb{C}) \xrightarrow{\mathrm{res}_{z_1}^{-1}} \mathcal{S}ol_{nd} \xrightarrow{\mathrm{ev}_{z_2}} \mathrm{Mat}_{n\times n}(\mathbb{C}). \tag{2.6}$$

Here we put

$$\mathrm{res}_{z_1}(F) = \frac{1}{z_1} F(1, z_1) \quad \textit{and} \quad \mathrm{ev}_{z_1}(F) = \frac{F(1,z_2)}{z_2 - z_1}.$$

Moreover, let $r(m_1, m_2; z_1, z_2)$ be the image of $\mathrm{ev}_{z_2} \circ \mathrm{res}_{z_1}^{-1}$ under the map (2.4)

$$\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Lin}(\mathrm{Mat}_{n\times n}(\mathbb{C}), \mathrm{Mat}_{n\times n}(\mathbb{C})).$$

Then the tensor-valued function $r(m_1, m_2; z_1, z_2)$ is a skew-symmetric and non-degenerate solution of the AYBE (2.5).

We now compute the associative $r-$matrix derived from the family described in example (2.5.4). The triple is given by $(\mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s^3, \mu))$, where μ^∞ is the identity matrix whereas μ^0 is:

$$\left(\begin{array}{cc|c} 0 & m & 0 \\ 0 & 0 & m \\ \hline m & 0 & 0 \end{array}\right).$$

As in the previous case we have to compute $F|_{\tilde{A}}$, where F is a morphism in $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}^{n-d} \oplus \mathcal{O}_{\mathbb{P}^1}^{d}(1), \mathcal{O}_{\mathbb{P}^1}^{n-d}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{d}(2))$. A straightforward computation gives

$$F^0 = \left(\begin{array}{cc|c} -a' & -b' & t \\ -c' & -d' & s \\ \hline e' & f' & -g' \end{array}\right)$$

and

$$F^{\infty} = \left(\begin{array}{cc|c} a'' & b'' & t \\ e'' & d'' & s \\ \hline e''' & f''' & g''' \end{array}\right).$$

Moreover, it has to makes commute the diagram (2.1) in the definition of triples in Section 2.1. The latter condition leads to:

$$\left(\begin{array}{cc|c} t & -a' & -b' \\ s & -c' & -d' \\ \hline -g' & +e' & f' \end{array}\right) = z_1 m \left(\begin{array}{cc|c} c'' & d'' & s \\ e''' & f''' & g'' \\ \hline a'' & b'' & t \end{array}\right), \qquad (\star)$$

where $m = \frac{m_2}{m_1}$. Computing the residue we obtain:

$$z_1 \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \left(\begin{array}{cc|c} a' + a''z_1 & b' + b''z_1 & t \\ c' + c''z_1 & d' + d''z_1 & s \\ \hline e' + e''z_1 + e'''z_1^2 & f' + f''z_1 + f'''z_1^2 & g' + g''z_1 \end{array}\right).$$

If we now solve the last system using $(\star)$, we find the relations $(\star\star)$:

$$\begin{cases} t = z_1\alpha_{13} \\ f' = mz_1^2\alpha_{13} \\ c'' = \frac{\alpha_{13}}{m} \\ c' = z_1\alpha_{21} - \frac{z_1}{m}\alpha_{13} \\ f''' = \frac{\alpha_{13}}{m^2} - \frac{\alpha_{21}}{m} \\ f'' = \alpha_{32} - z_1(m + \frac{1}{m^2})\alpha_{13} + \frac{z_1}{m}\alpha_{21} \end{cases} \qquad \begin{cases} s = z_1\alpha_{23} \\ e''' = \frac{\alpha_{23}}{m} \\ b' = -mz_1^2\alpha_{23} \\ b'' = \alpha_{12} + mz_1\alpha_{23} \\ e' = mz_1\alpha_{12} + m^2z_1^2\alpha_{23} \\ e'' = \alpha_{31} - m\alpha_{12} - z_1(m^2 + \frac{1}{m})\alpha_{23} \end{cases}$$

and

$$\begin{cases} a' = -\frac{z_1}{1-m^3}(m^3\alpha_{11} + m^2\alpha_{33} + m\alpha_{22}) \\ a'' = \frac{1}{1-m^3}(\alpha_{11} + m\alpha_{22} + m^2\alpha_{33}) \\ g' = -\frac{z_1 m}{1-m^3}(\alpha_{11} + m\alpha_{22} + m^2\alpha_{33}) \\ g'' = +\frac{1}{1-m^3}(m\alpha_{11} + \alpha_{33} + m^2\alpha_{22}) \\ d' = -\frac{z_1}{1-m^3}(m^2\alpha_{11} + m\alpha_{33} + m^3\alpha_{22}) \\ d'' = +\frac{1}{1-m^3}(m^2\alpha_{11} + m\alpha_{33} + \alpha_{22}). \end{cases} \qquad (\star\star\star)$$

We now have to compute the evaluation map $\mathrm{ev}_{z_2}(F) = \frac{1}{z_2-z_1}(\beta_{ij})_{i,j\in\{1,2,3\}}$, namely:

$$\frac{1}{z_2-z_1}\left(\begin{array}{cc|c} a'+a''z_2 & b'+b''z_2 & t \\ c'+c''z_2 & d'+d''z_2 & s \\ \hline e'+e''z_2+e'''z_2^2 & f'+f''z_2+f'''z_2^2 & g'+g''z_2 \end{array}\right) = \frac{1}{z}\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix},$$

where $z = z_2 - z_1$. We now solve the last system and observe that, due to $(\star\,\star)$ and $(\star\,\star\,\star)$, we can write every β_{ij} as linear combinations of α_{kl}. We then end up with:

$$\begin{cases} \beta_{12} = z_1 z m \alpha_{23} + z_2\alpha_{12} \\ \beta_{23} = z_1\alpha_{23} \\ \beta_{31} = -zm\alpha_{12} + z_2\alpha_{31} + \\ +z_1z_2(-m^2 - \frac{1}{m} + \frac{z_1}{z_2}m^2 + \frac{z_2}{z_1 m})\alpha_{23} \end{cases} \qquad \begin{cases} \beta_{13} = z_1\alpha_{13} \\ \beta_{21} = z_1\alpha_{21} + \frac{z}{m}\alpha_{31} \\ \beta_{32} = z_2\alpha_{32} + -z_2\frac{z}{m}\alpha_{21} + \\ +z_1z_2(\frac{z_1}{z_2}m - m - \frac{1}{m^2} + \frac{z_2}{z_1m^2})\alpha_{13} \end{cases}$$

and

$$\begin{cases} \beta_{11} = \frac{z_2 - z_1m^3}{1-m^3}\alpha_{11} + \frac{zm}{1-m^3}\alpha_{22} + \frac{zm^2}{1-m^3}\alpha_{33} \\ \beta_{22} = +\frac{zm^2}{1-m^3}\alpha_{11} + \frac{z_2-z_1m^3}{1-m^3}\alpha_{22} + \frac{zm}{1-m^3}\alpha_{33} \\ \beta_{33} = +\frac{zm}{1-m^3}\alpha_{11} + \frac{zm^2}{1-m^3}\alpha_{22} + \frac{z_2-z_1m^3}{1-m^3}\alpha_{33}. \end{cases}$$

Similarly to the previous case we have to apply the isomorphism

$$\mathrm{Lin}(\mathrm{Mat}_{3\times3}(\mathbb{C}), \mathrm{Mat}_{3\times3}(\mathbb{C})) \to \mathrm{Mat}_{3\times3}(\mathbb{C}) \otimes \mathrm{Mat}_{3\times3}(\mathbb{C})$$

which sends the linear map $e_{ij} \to \gamma_{ik}^{kl} e_{kl}$ to the tensor $\gamma_{ij}^{kl} e_{ji} \otimes e_{kl}$. We then get:

$$\begin{aligned} r(m; z_1, z_2) =& \frac{z_2 - z_1m^3}{z(1-m^3)}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{33}\otimes e_{33}) \\ &+ \frac{m}{1-m^3}(e_{22}\otimes e_{11} + e_{33}\otimes e_{22} + e_{11}\otimes e_{33}) + \\ &+ \frac{m^2}{1-m^3}(e_{33}\otimes e_{11} + e_{11}\otimes e_{22} + e_{22}\otimes e_{33}) + \\ &+ \frac{z_2}{z}(e_{21}\otimes e_{12} + e_{13}\otimes e_{31} + e_{23}\otimes e_{32}) + \\ &+ \frac{z_1}{z}(e_{32}\otimes e_{23} + e_{31}\otimes e_{13} + e_{12}\otimes e_{21}) + \\ &+ \frac{1}{m}e_{13}\otimes e_{21} - me_{21}\otimes e_{31} + z_1me_{32}\otimes e_{12} + \\ &+ \frac{z_1z_2}{z}(-m^2 - \frac{1}{m} + \frac{z_1}{z_2}m^2 + \frac{z_2}{z_1m})e_{32}\otimes e_{31} - \frac{z_2}{m}e_{12}\otimes e_{32} + \\ &+ \frac{z_1z_2}{z}(\frac{z_1}{z_2}m - m - \frac{1}{m^2} + \frac{z_2}{z_1m^2})e_{31}\otimes e_{32}. \end{aligned}$$

2.9 Associative $r-$matrices obtained from elliptic curves

As we have seen in Section 2.4 we can identify an elliptic curve with a 1-dimensional torus. Let $\tau \in \mathbb{C}$ be a complex number such that $\Im(\tau) > 0$ and $T = T_\tau \simeq \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be the corresponding complex torus. Let $\mathcal{U}$ be the universal family of stable vector bundles of rank n and degree d, such that $gcd(n,d) = 1$. Recall that $\mathcal{U} \simeq T$ and let $\mathcal{U}^m$ be the corresponding vector bundle on T at the point m.
Let A be the automorphy factor defined by $A(1,z') = diag(1,\zeta,\dots,\zeta^{n-1}) = B_1$, where $\zeta = \exp(\frac{2\pi id}{n})$ and $A(\tau,z') = \tilde{e}(z')\exp\left(\frac{-2\pi im}{n}\right)B_2$, with $(B_2)_{i,j} = \delta_{i-j+1\equiv_n 0}$. Here $\tilde{e}(z')$ is the function in (2.2). Namely

$$B_1 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \zeta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix} \quad \text{and} \quad B_2 := \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{2.7}$$

Definition 2.9.1. We denote by $\mathcal{S}$ the space of holomorphic functions

$$G : \mathbb{C} \to M = \mathrm{Mat}_{n\times n(\mathbb{C})}$$

such that

$$\mathcal{S} = \left\{ G : \mathbb{C} \to M \middle| G(z'+l) = \chi(l)B_l G(z') B_l^{-1} \right\}.$$

Here we put

$$\chi(l) = \begin{cases} 1 & \text{if } \ l = 1 \\ \exp\left(\frac{-2\pi im}{n}\right)h(z') & \text{if } \ l = \tau \end{cases}$$

and $h(z') = -\exp(-2\pi i(z'+\tau - z_1))$, $z_1 \in \mathbb{C}$.

Theorem 2.9.2. *Let $\omega = dz'$ be a nowhere vanishing global 1-form over T. Let $\mathcal{U}^{m_1}$ and $\mathcal{U}^{m_2}$ be two non-isomorphic vector bundles over T and z_1, z_2 be two distinct points of T. Then the map*

$$\mathrm{Lin}(\mathcal{U}^{m_1}|_{z_1}, \mathcal{U}^{m_2}|_{z_1}) \xrightarrow{(\mathrm{res}_{z_1,\omega}^{\mathcal{U}^{m_1},\mathcal{U}^{m_2}})^{-1}} \mathrm{Hom}(\mathcal{U}^{m_1}, \mathcal{U}^{m_2}(z_1)) \xrightarrow{\mathrm{ev}_{z_2}^{\mathcal{U}^{m_1},\mathcal{U}^{m_2}(z_1)}} \mathrm{Lin}(\mathcal{U}^{m_1}|_{z_2}, \mathcal{U}^{m_2}|_{z_2})$$

defined in Section 2.6 can be identified, under the trivialization introduced in Section 2.4, with the map

$$\mathrm{Mat}_{n\times n}(\mathbb{C}) \xrightarrow{\mathrm{res}_{z_1}^{-1}} \mathcal{S} \xrightarrow{\mathrm{ev}_{z_2}} \mathrm{Mat}_{n\times n}(\mathbb{C}). \tag{2.8}$$

Here for any $G \in \mathcal{S}$ we put

$$\mathrm{res}_{z_1}(G) = \frac{G(z_1)}{\theta_3'(\frac{1+\tau}{2}|\tau)}, \qquad \mathrm{ev}_{z_2}(G) = \frac{G(z_2)}{\theta_3(z_2 - z_1 + \frac{1+\tau}{2}|\tau)},$$

with

$$\theta_3(z'|\tau) = \sum_{l\in\mathbb{Z}} \exp(\pi i l^2\tau + 2\pi i l z'). \tag{2.9}$$

Moreover, let $r(m_1, m_2; z_1, z_2)$ be the image of $\mathrm{ev}_{z_2} \circ \mathrm{res}_{z_1}^{-1}$ under the canonical map (2.4)

$$\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Lin}(\mathrm{Mat}_{n\times n}(\mathbb{C}), \mathrm{Mat}_{n\times n}(\mathbb{C})).$$

Then the tensor-valued function $r(m_1, m_2; z_1, z_2)$ is a skew-symmetric non-degenerate solution of the AYBE

$$r^{12}(n; z)r^{23}(n+m; w) = r^{13}(n+m; z+w)r^{12}(-m; z) + r^{23}(m; w)r^{13}(n; z+w),$$

that is a particular case of (1.1) when $r(m_1, m_2; z_1, z_2) = r(m_1 - m_2, z_1 - z_2)$.

Proof. One can easily see that $\mathcal{S}$ represents $\mathrm{Hom}(\mathcal{U}^{m_1}, \mathcal{U}^{m_2}(z_1))$ under the trivialization induced by the automorphy factor $A = A(m, \gamma, z')$ above. In fact $\mathcal{S}$ is $\mathrm{Hom}(A(m_1, \gamma, z'), h(z')A(m_2, \gamma, z'))$, where $h(z') = h_{z_1}(z')$ is the automorphy factor of $\mathcal{O}_T(z_1)$. The theorem is then proved as in [18], see in particular Section 8.2. In that article the statement was demonstrated for the space

$$\mathcal{S}' = \left\{ G : \mathbb{C} \to M \middle| G(z'+l) = \chi(l) B_l G(z') B_l^{-1} \right\}$$

where $B_1 = \mathbb{1}_n$ and $B_2 = F(z')$ as in (2.2). A straightforward computation shows that the automorphy factor in (2.2) can be reduced, up to the rescaling by a constant matrix, to that one in (2.7). In fact one can obtain the automorphy factor (2.7) from (2.2) just acting with the matrix $diag(\tilde{e}^{n-1}, \ldots, \tilde{e}, 1)$. Therefore the argument in [18] is automatically adapted to this new description. □

If we define the function $\theta_1(z', \tau) = \frac{\theta_3(z' + \frac{1+\tau}{2}|\tau)}{i\exp(-\pi i z' - \pi i \frac{\tau}{4})}$, we introduce the notations

$$B_{ab} = B_2^a B_1^{-b}, \qquad B^*_{ab} = \tfrac{1}{n} B_1^b B_2^{-a}, \qquad a, b \in \{0, \dots, n-1\},$$

and the Kronecker elliptic function [49]

$$\kappa(m', z') = \frac{\theta_1'(0|\tau)\theta_1(m' + z'|\tau)}{\theta_1(m'|\tau)\theta_1(z'|\tau)},$$

we are ready to state the main result of this section.

Theorem 2.9.3. *The following is a solution of the associative Yang-Baxter equation given by the universal family of stable vector bundles $\mathcal{U}$ under the trivialization described at the beginning of this section:*

$$r(m; z) = \sum_{a,b=0}^{n-1} \exp\Big(\frac{-2\pi i d}{n} a z\Big) \kappa\Big(\frac{d}{n}\Big(b - a\tau\Big) + \frac{m}{n}, z\Big) B^*_{ab} \otimes B_{ab}, \tag{2.10}$$

with $m = m_2 - m_1$ and $z = z_2 - z_1$.

Proof. (*i*) First of all we observe that the set B_{ab} form a basis for the vector space $M = \mathrm{Mat}_{n \times n}(\mathbb{C})$, in particular any element in $G \in \mathcal{S}$ can be written as $G(z') = \sum g_{ab}(z') B_{kl}$.

(*ii*) A straightforward computation shows that

$$B_1 B_{ab}^{-1} B_1^{-1} = \zeta^a B_{ab},$$

while

$$B_2 B_{ab}^{-1} B_2^{-1} = \zeta^b B_{ab}.$$

Moreover the conjugations, $Ad_{B_j}(-) = B_l - B_j^{-1}$, $j \in \{1, 2\}$, commute.

(*iii*) Then $g_{a,b}(z') \in \mathcal{S}$ if and only if

$$g_{ab}(z' + 1) = \zeta^a g_{a,b}(z'), \qquad \text{and} \qquad g_{ab}(z' + \tau) = \zeta^l \exp\big(\tfrac{-2\pi i m}{n}\big) h(z') g_{ab}(z'),$$

with $h(z') = -\exp(-2\pi i(z' + \tau - z_1))$.

(*iv*) We observe that, from the theory of theta functions [35], the unique solution of the system

$$\begin{cases} g_{ab}(z' + 1) = g_{ab}(z') \\ g_{ab}(z' + \tau) = h(z') g_{ab}(z') \end{cases}$$

is given by the function $\theta_3(z'+\frac{1+\tau}{2}-z_1|\tau)$. Then we can adapt this machinery to solve the system which appears in (iv), therefore

$$g_{ab}(z') = \exp\left(\frac{-2\pi i d}{n} a z'\right)\theta_3\left(z' - z_1 + \frac{1+\tau}{2} - \frac{d}{n}(a\tau - b)\middle|\tau\right).$$

(v) A straightforward computation shows that, once we apply $\mathrm{res}_{z_1}^{-1}$ followed by ev_{z_2}, we get

$$\tilde{g}(m,z) = \exp\left(\frac{-2\pi i d}{n} a z\right)\frac{\theta_3'(\frac{1+\tau}{2}|\tau)\theta_3(z + \frac{m}{n} + \frac{1+\tau}{2} - \frac{d}{n}(a\tau - b)|\tau)}{\theta_3(\frac{m}{n} - \frac{d}{n}(a\tau - b) + \frac{1+\tau}{2}|\tau)\theta_3(z + \frac{1+\tau}{2}|\tau)}.$$

(vi) We conclude the proof using the relation $\theta_1(z,\tau) = \frac{\theta_3(z+\frac{1+\tau}{2}|\tau)}{i\exp(-\pi i z - \pi i \frac{\tau}{4})}$ and observing that the canonical morphism (2.4)

$$\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Lin}(\mathrm{Mat}_{n\times n}(\mathbb{C}), \mathrm{Mat}_{n\times n}(\mathbb{C})),$$

acts as follows

$$can(B_{ab}^* \otimes B_{ab})(B_{a'b'}) = \begin{cases} B_{ab} & \text{if } (a',b') = (a,b). \\ 0 & \text{otherwise.} \end{cases}$$

□

2.10 Classical and Quantum Yang-Baxter equations

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra and $r : (\mathbb{C}^2, 0) \to \mathfrak{g}\otimes\mathfrak{g}$ be the germ of a meromorphic function in a neighbourhood of 0. Then the classical Yang-Baxter equation (CYBE) is defined by the relation

$$[r^{12}(z_1,z_2), r^{23}(z_2,z_3)] + [r^{12}(z_1,z_2), r^{13}(z_1,z_3)] + [r^{13}(z_1,z_3), r^{23}(z_2,z_3)] = 0, \quad (2.11)$$

where r^{ij} are the appropriate embeddings of $\mathfrak{g}\otimes\mathfrak{g} \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ which send $g_1 \otimes g_2$ to the element who has g_1 in the $i-th$ spot, g_2 in the $j-th$ spot and the identity in the remaining one. A solution of the above relation is called classical $r-$matrix.

Example 2.10.1. A straightforward computation shows that the simplest solution of the CYBE which depends just on the difference $z = z_2 - z_1$ is given by

$$r(z) = \frac{1}{z}\left[\frac{1}{2}(e_{22} - e_{22}) \otimes (e_{11} - e_{22}) + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right],$$

where $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

The definitions of non-degeneracy, skew-symmetry and gauge equivalence from (1.1.1) can be easily adapted to this version of the Yang-Baxter equation.

Definition 2.10.2. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ be the simple lie algebra of traceless matrices. Then we will say that a classical $r-$matrix has infinitesimal symmetries if and only if there exists $g \in \mathfrak{sl}_n(\mathbb{C})$, $g \neq 0$, such that

$$[r(z_1, z_2), g \otimes \mathbb{1} + \mathbb{1} \otimes g] = 0.$$

Classical r-matrices were classified by Belavin and Drinfeld, such a classification is given by a certain lattice of poles.

Theorem 2.10.3. *([8, 9, 10])*
Let $r(z_1, z_2)$ be a non-degenerate classical $r-$matrix, then:

(1) *$r(z_1, z_2)$ is gauge equivalent, after a change of variables, to a solution which depends only on the difference $z = z_1 - z_2$ and it is skew-symmetric.*

(2) $r(z)$ extends to a meromorphic function on the whole complex plane $\mathbb{C}$.

(3) The set of poles of $r(z)$ forms a lattice $\Gamma \subset \mathbb{C}$ and rank(Γ) specifies the type of r. In fact one of the following holds:

  (i) if rank$(\Gamma) = 2$, then $r(z)$ is called elliptic, this means that it is equivalent to a linear combination of elliptic functions. Moreover, such a solution exists if and only if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

  (ii) If rank$(\Gamma) = 1$, then r is said to be trigonometric, i.e. there exists a rational function $f(z)$ and $\lambda \in \mathbb{C}$ such that $r(z)$ is equivalent to $f(\exp(\lambda z))$.

  (iii) If rank$(\Gamma) = 0$, then $r(z)$ is equivalent to a rational function $f(z)$, in this case the solution is called rational.

For instance the $r-$matrix in example (2.10.1) is rational.
A natural question is: what is the interplay between the two Yang-Baxter equations we introduced so far?
First of all observe that we have a natural projection $Prj : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$, given by $M \to M - \frac{trace(M)}{n}\mathbb{1}$. Then, if $r(m; z_1, z_2)$ is an associative $r-$matrix, it is well-defined the tensor $Prj^{\otimes 2}(r(m, z))$ and the following proposition holds.

Proposition 2.10.4. *Let $r(m; z_1, z_2)$ be a skew-symmetric associative $r-$matrix and suppose the limit*

$$\lim_{m\to 0} Prj^{\otimes 2}(r(m; z_1, z_2)) = r_c(z_1, z_2),$$

exists.
Then the tensor $r_c(z_1, z_2)$ in $\mathfrak{sl}_n(\mathbb{C})^{\otimes 2}$ satisfies the classical Yang-Baxter equation. Furthermore, suppose that $r(m; z_1, z_2)$ has a Laurent series given by

$$r(m; z_1, z_2) = \frac{\mathbb{1}\otimes\mathbb{1}}{m} + \sum_{i=0}^{\infty} m^i r_i(z_1, z_2). \tag{2.12}$$

In that case:

(i) Any other solution gauge equivalent to r is of the form (2.12) and the corresponding solutions of the CYBE are gauge equivalent to each other;

(ii) If $r(m; z_1, z_2)$ is non-degenerate, $r_c(z_1, z_2)$ is also non-degenerate;

(iii) Suppose that $r(m; z_1, z_2)$ is non-degenerate and that r_c is either elliptic or trigonometric or has no infinitesimal symmetries. Then, for fixed $m_0 \in \mathbb{C}$, $m_0 \neq 0$, the tensor $r(m_0; z_1, z_2) = r_{m_0}(z_1, z_2)$ solves the quantum Yang-Baxter equation (QYBE), i.e. the following relation

$$r^{12}_{m_0}(z_1, z_2)r^{13}_{m_0}(z_1, z_3)r^{23}_{m_0}(z_2, z_3) = r^{23}_{m_0}(z_2, z_3)r^{13}_{m_0}(z_1, z_3)r^{12}_{m_0}(z_1, z_2). \tag{2.13}$$

The first part of this theorem can be found in [17], Lemma 1.2. The second part was proved by Polishchuk and generalized by Henrich see for instance [28].

Example 2.10.5. (*i*) Clearly the solution obtained in (2.10) is of the form (2.12) and it is elliptic. Thus, for fixed $m = m_0$, it satisfies the QYBE (2.13) and gives the classical $r-$matrix:

$$r(z) = \sum_{a,b=0,(a,b)\neq(0,0)}^{n-1} \exp\Big(\frac{-2\pi id}{n}az\Big)\kappa\Big(\frac{d}{n}\Big(b-a\tau\Big),z\Big)B^*_{ab}\otimes B_{ab}.$$

Observe that we obtain the same solution computed in [17].

(*ii*) The associative $r-$matrix computed at the end of Section 2.7 is of the form (2.12) and gives the rational solution:

$$\begin{aligned} r_c(z_1,z_2) = &+\frac{1}{z}\sum_{k,l=1,k\neq l}^{3} e_{kl}\otimes e_{lk} + \frac{1}{z}\sum_{i=1}^{3}(-1)^{i+1}v_i\otimes v_i \\ &-z_1e_{32}\otimes v_2 - e_{21}\otimes v_1 - e_{12}\otimes e_{32} \\ &-z_1e_{31}\otimes e_{12} + e_{32}\otimes e_{12} + v_1\otimes e_{21} \\ &+z_2e_{12}\otimes e_{31} + z_2v_2\otimes e_{32}, \end{aligned}$$

where $v_1 = \frac{1}{3}diag(2,-1,-1)$, $v_2 = \frac{1}{3}diag(1,1,-2)$ and $v_3 = v_2 - v_1$. Moreover, a straightforward computation shows that it has no-infinitesimal symmetry, thus the corresponding associative $r-$matrix solves the quantum Yang-Baxter equation. We finally observe that r_c is the same computed in [17].

(*iii*) One can show that

$$r(m;z) = \frac{1}{2m}\mathbb{1}\otimes\mathbb{1} + \frac{1}{z}\sum_{i,j=1}^{2} e_{ij}\otimes e_{ji}$$

is an associative $r-$matrix who also solves (2.13) and the CYBE, however any $g\in\mathfrak{sl}_n(\mathbb{C})$, $g\neq 0$, is an infinitesimal symmetry of the corresponding classical solution.

Chapter 3

Szegö kernels

In this chapter we introduce the so called Szegö kernels. These kernel functions are closely related to solutions of a specific form of the Yang-Baxter equation that we shall introduce. Therefore, they are also related to the Yang-Baxter relations described in chapter two. Moreover, inspired by the ideas in [41] and following the approach of the first chapter of this thesis, we will prove that Szegö kernels have an alternative description both via appropriate triple Massey products as well as through residue and evaluation sequences. A first difference with respect to the topics treated in the first two chapters is given by the genus of the curve, which can also be taken to be greater than one. In the last part of this chapter, after we proved the skew-symmetry of the aforementioned kernel functions, we also demonstrate some identities that Szegö kernels have to satisfy. In particular we will obtain a sheaf-theoretic version of the matrix-valued Fay's identity.

3.1 Szegö kernels and classical Yang-Baxter equation

First of all we have to introduce a new form of the Yang-Baxter equation.

Definition 3.1.1. Let r be a meromorphic function $r : \mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional complex Lie algebra. The generalized classical Yang-Baxter equation (GCYBE) is the following:

$$[r^{12}(z_1, z_2), r^{23}(z_2, z_3)] + [r^{12}(z_1, z_2), r^{13}(z_1, z_3)] + [r^{32}(z_3, z_2), r^{13}(z_1, z_3)] = 0, \quad (3.1)$$

where r^{lm} are the same as in equation (2.11).
A solution of the GCYBE is called skew-symmetric if $r^{12}(z,w) = -r^{21}(w,z)$ for all z, w where r is defined.

Remark 3.1.2. It is clear that a skew-symmetric solution the of GCYBE is also a classical r-matrix, i.e. it solves the CYBE

$$[r^{12}(z_1,z_2), r^{23}(z_2,z_3)] + [r^{12}(z_1,z_2), r^{13}(z_1,z_3)] + [r^{13}(z_1,z_3), r^{23}(z_2,z_3)] = 0.$$

Example 3.1.3. Let $\mathfrak{g}$ be a simple complex Lie algebra. Then a straightforward computation shows that the following function is a non-degenerate solution of the *GCYBE* which does not satisfy the classical one:

$$r(z,w) = \frac{w}{z-w}\gamma.$$

Here γ is the Casimir element of $\mathfrak{g}$. It is easy to see that the latter solution is not skew-symmetric.

Let C be a reduced, complex, projective Gorenstein curve, X be an irreducible, affine open subset of C such that $\Omega_X \simeq \mathcal{O}_X$. Then (see either [16] or [25]) there exists a short exact sequence

$$0 \to \mathcal{O}_{C\times X} \to \mathcal{O}_{C\times X}(\Delta) \to \delta_*(\mathrm{Hom}_X(\Omega_X, \mathcal{O}_X)) \to 0,$$

where $\delta : X \to C \times X$ is the diagonal embedding and $\Delta = Im(\delta)$. The existence of such a sequence will be proved in the next section, see formula (3.2). We now use the fact that $\Omega_X \simeq_\omega \mathcal{O}_X$, recall that such a trivialization is given by the choice of a non-zero section ω of $H^0(C, \Omega_C)$. We obtain:

$$0 \to \mathcal{O}_{C\times X} \to \mathcal{O}_{C\times X}(\Delta) \to \mathcal{O}_\Delta \to 0.$$

If we now tensor everything by $\mathcal{G} \boxtimes \mathcal{G}$, we end up with

$$0 \to \mathcal{G} \boxtimes \mathcal{G}|_X \to \mathcal{G} \boxtimes \mathcal{G}|_X(\Delta) \to \delta_*(\mathcal{G}|_X \otimes \mathcal{G}|_X) \to 0,$$

where $\mathcal{G}$ is a sheaf of Lie algebras satisfying the following conditions.

(*i*) $\mathcal{G}$ is a coherent $\mathcal{O}_C$-module.

(*ii*) $H^0(C, \mathcal{G}) = 0 = H^1(C, \mathcal{G})$.

(iii) For all $p \in X$ we have $\mathcal{G}|_p \simeq \mathfrak{g}$.

Then, using the vanishing of cohomology of $\mathcal{G}$, applying the global sections functor and due to the Künneth formula, we get an isomorphism

$$\mathrm{res}^{\omega}_{\Delta} : H^0(C \times X, \mathcal{G} \boxtimes \mathcal{G}|_X(\Delta)) \to H^0(X, \mathcal{G}|_X \otimes \mathcal{G}|_X).$$

The latter isomorphism in particular implies the existence of a unique section $r^{\omega} \in H^0(C \times X, \mathcal{G} \boxtimes \mathcal{G}|_X(\Delta))$ such that $\mathrm{res}^{\omega}_{\Delta}(r^{\omega}) = \gamma$. Here we denote by γ the unique element of $H^0(C \times X, \mathcal{G} \boxtimes \mathcal{G}|_X(\Delta))$ which is sent to the identity by the isomorphism $H^0(X, \mathcal{G}|_X \otimes \mathcal{G}|_X) \simeq \mathrm{End}_C(\mathcal{G}|_X)$.

Definition 3.1.4. The unique element $r := r^{\omega}$ is called Szegö kernel or equivalently geometric $r-$matrix.

Remark 3.1.5. We are only interested whether the Szegö kernel r solves the GCYBE or not, in this case it is possible to suppress its dependence on ω, see [25] for more details.

One can prove that the following theorem holds.

Theorem 3.1.6. *[25] For any three distinct points $x, y, z \in X$ the Szegö kernel r is a non-degenerate solution of the generalized classical Yang-Baxter equation*

$$[r^{12}(z_1, z_2), r^{23}(z_2, z_3)] + [r^{12}(z_1, z_2), r^{13}(z_1, z_3)] + [r^{32}(z_3, z_2), r^{13}(z_1, z_3)] = 0.$$

In particular, if we work with locally free sheaves and if there exists a global nowhere vanishing one form $\omega \in \Omega_C$, the Szegö kernel r is skew-symmetric. Therefore it also solves the classical Yang-Baxter equation.

3.2 Curves of positive genus and Szegö kernels

Let C be a reduced, projective Gorenstein curve of (arithmetic) genus $g > 0$, $X \neq \emptyset$ be a regular irreducible subset of C and $\delta : X \to C \times X$ be the diagonal embedding, i.e. $\Delta = Im(\delta)$. Recall that, if C is smooth, its dualising sheaf Ω_C is isomorphic to the sheaf of holomorphic 1-forms, otherwise one has to introduce the sheaf of regular 1-forms.

Definition 3.2.1. Let C be a reduced projective Gorenstein curve, $n : \tilde{C} \to C$ its normalization and denote by $\mathbf{M}_{\tilde{C}}$ the sheaf of meromorphic 1-forms over $\tilde{C}$. Then, for any open set $U \subset C$ a regular 1-form over U is an element $\omega \in \Gamma(U, n_*\mathbf{M}_{\tilde{C}})$ such that, for any $x \in C$ and any $f \in \mathcal{O}_C(U)$, one has

$$\sum_{x_i \in n^{-1}(x)} \mathrm{res}_{x_i}((f \circ n)\omega) = 0,$$

where res is the standard residue on the smooth curve $\tilde{C}$.

Remark 3.2.2. Observe that a regular 1-form is holomorphic on any smooth part of C. One can prove, see for instance [6], that the dualising sheaf Ω_C of a reduced projective Gorenstein curve can be identify with the sheaf of regular 1-forms.

Recall Serre duality for a Gorenstein curve, see [27].

Theorem 3.2.3. *For any $\mathcal{V}, \mathcal{W} \in Perf(C)$ we have a non-degenerate bilinear form*

$$b(-,-)_s : \mathrm{Hom}(\mathcal{V}, \mathcal{W}) \times \mathrm{Ext}^1(\mathcal{W}, \mathcal{V} \otimes \Omega_C) \to \mathbb{C}.$$

In particular there exists a bifunctorial isomorphism

$$\mathrm{Ext}^1(\mathcal{W}, \mathcal{V} \otimes \Omega_C) \simeq \mathrm{Hom}(\mathcal{V}, \mathcal{W})^*.$$

Proposition 3.2.4. *For any vector bundle $\mathcal{E}$ over C the following residue sequence on $C \times X$ is exact:*

$$0 \to \pi_1^*\mathcal{E} \to \pi_1^*\mathcal{E}(\Delta) \xrightarrow{\mathrm{res}_\Delta} \delta_*(\mathcal{E} \otimes \Omega_C^{-1}) \to 0, \tag{3.2}$$

where $\pi_1 : C \times X \to C$.

Proof. We start demonstrating that the sequence

$$0 \to \mathcal{O}_{C \times X} \to \mathcal{O}_{C \times X}(\Delta) \to \delta_*(\Omega_C^{-1}) \to 0 \tag{3.3}$$

is exact.

The first map is trivially defined, we have to describe the second one.

Let $B = C \times X$, $x \in X$ and $U_x \subset B$ be an open neighbourhood of (x, x). We observe that any section $s \in H^0(U, \mathcal{O}_{C \times X}(\Delta))$ can be locally written as $\frac{s'(p,q)}{p-q}$, with $s' \in H^0(U, \mathcal{O}_{C \times X})$. We take any $\omega \in H^0(\delta^{-1}(U_x \cap \Delta), \Omega_C)$, clearly ω can be locally written as $\omega = \omega'(p)dp$, with ω' holomorphic on $X \simeq \Delta$. We observe that $\Omega_C^{-1} \simeq \mathrm{Hom}(\Omega_C, \mathcal{O}_C)$, then the residue map is the map which takes $z \in \delta^{-1}(U_x \cap \Delta)$ and sends it to $\omega'(z)s'(z,z)$. The statement follows taking the tensor product of the sequence (3.3) with $\pi_1^*\mathcal{E}$. $\square$

Corollary 3.2.5. *In particular, if we take $\mathcal{E}$ equal to Ω_C, we obtain the following sequence:*

$$0 \to \pi_1^*\Omega_C \to \pi_1^*\Omega_C(\Delta) \xrightarrow{\mathrm{res}_\Delta} \mathcal{O}_\Delta \to 0. \tag{3.4}$$

We now consider both projection maps π_1, π_2

$$\begin{tikzcd} & C \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ C & & X \end{tikzcd}$$

and we observe that, if we tensor the sequence (3.4) by $\pi_1^*\mathcal{E}^* \otimes \pi_2^*\mathcal{E}|_X := \mathcal{E}^* \boxtimes \mathcal{E}|_X$, we get

$$0 \to \mathcal{E}^\vee \boxtimes \mathcal{E}|_X \to \mathcal{E}^\vee \boxtimes \mathcal{E}|_X(\Delta) \to \delta_*(\mathrm{End}(\mathcal{E}|_X)) \to 0. \tag{3.5}$$

Here $\mathcal{E}$ is a vector bundle on C and we are using the notation $\mathcal{E}^\vee := \mathcal{E}^* \otimes \Omega_C$.

Lemma 3.2.6. *Let $\mathcal{E}$ be a vector bundle over C such that*

$$H^0(C,\mathcal{E}) = 0 = H^1(C,\mathcal{E}),$$

then

$$H^0(C \times X, \mathcal{E}^\vee \boxtimes \mathcal{E}|_X) = 0 = H^1(C \times X, \mathcal{E}^\vee \boxtimes \mathcal{E}|_X).$$

Moreover, we get an isomorphism of vector spaces

$$H^0(C \times X, \mathcal{E}^\vee \boxtimes \mathcal{E}|_X(\Delta)) \xrightarrow{\mathrm{res}_\Delta} H^0(C, \mathrm{End}(\mathcal{E}|_X)).$$

Proof. Observe that applying Riemann-Roch formula we immediately see that $H^0(C,\mathcal{E}) = 0 = H^1(C,\mathcal{E})$ if and only if $H^0(C,\mathcal{E}^\vee) = 0 = H^1(C,\mathcal{E}^\vee)$.
The first statement follows easily using Künneth formula. In fact we multiply any factor of the decomposition by an element of the form $H^*(C,\mathcal{E}^\vee) = 0$.
The last part follows from the sequence (3.5) applying the global sections functor. $\square$

Similarly to the previous section we can introduce the next definition.

Definition 3.2.7. The unique section $\tilde{S} \in H^0(C \times X, \mathcal{E}^\vee \boxtimes \mathcal{E}|_X(\Delta))$ such that $\mathrm{res}_\Delta(\tilde{S}) = \mathbb{1}_{\mathrm{End}(\mathcal{E}|_X)}$ is called Szegö kernel.

The name kernel is due to the sequence (3.5).

3.3 Residue and Evaluation sequences

Let C, X be as in the previous section, $\mathcal{E}$ be a vector bundle over C. We say that $p \in X$ is a base point for $\mathcal{E}$ if the map $H^0(C, \mathcal{E}) \to \mathcal{E}|_p$ is zero.

Definition 3.3.1. Let x, y be two points of X. We say that a triple $(\mathcal{E}, x, y)$ is good when:

(i) $x \neq y$;

(ii) x is a base point for $\mathcal{E}^\vee$;

(iii) y is a base point for $\mathcal{E}$.

Remark 3.3.2. Note that if $H^0(C, \mathcal{E}) = 0 = H^1(C, \mathcal{E})$, then, using Serre duality or Riemann-Roch theorem, one can easily see that the triple $(\mathcal{E}, x, y)$ is good for any $x, y \in X$ as long as $x \neq y$.

Lemma 3.3.3. *Let $\mathcal{E}$ be a vector bundle over C and $x \in X$, then the following are equivalent:*

(i) *x is a base point for $\mathcal{E}^\vee$;*

(ii) $H^0(C, \mathcal{E}^\vee(-x)) \simeq H^0(C, \mathcal{E}^\vee)$;

(iii) $H^1(C, \mathcal{E}) \simeq H^1(C, \mathcal{E}(x))$,

Moreover, if $x \in X$ is a base point for $\mathcal{E}^\vee$, the following relation is satisfied:

$$h^0(C, \mathcal{E}(x)) = rank(\mathcal{E}) + h^0(C, \mathcal{E}).$$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_C(-x) \to \mathcal{O}_C \xrightarrow{\mathrm{ev}_x} \mathbb{C}_x \to 0.$$

We tensor it by $\mathcal{E}^\vee$ and apply the functor $H^0(C,-)$, we get

$$0 \to H^0(C,\mathcal{E}^\vee(-x)) \to H^0(C,\mathcal{E}^\vee) \to \mathcal{E}^\vee|_x \to H^1(C,\mathcal{E}^\vee(-x)) \to H^1(C,\mathcal{E}^\vee) \to 0.$$

The base point hypothesis immediately implies point (ii). In fact the map $H^0(C,\mathcal{E}^\vee) \to \mathcal{E}^\vee|_x$ is zero. Conversely, if the first two terms of the latter sequence are isomorphic, then $Kern(H^0(C,\mathcal{E}^\vee) \to \mathcal{E}^\vee|_x) = H^0(C,\mathcal{E}^\vee)$, giving the equivalence between the points (i) and (ii).
The second part, $(ii) \iff (iii)$, follows from a chain of Serre isomorphisms: $H^0(C,\mathcal{E}^\vee(-x)) \simeq \mathrm{Hom}_C(\mathcal{E}(x),\Omega) \simeq \mathrm{Ext}^1(\mathcal{O}_C,\mathcal{E}(x))^* \simeq H^1(C,\mathcal{E}(x))^*$. A similar chain of morphisms for $H^0(C,\mathcal{E}^\vee)$ and the isomorphism (ii) lead to the thesis.

The last statement is a straightforward computation. One has to write down the Riemann-Roch relations for both $\mathcal{E}(x)$ and $\mathcal{E}$, then solve the system using (ii) and Serre duality, such a process leads to

$$h^0(C,\mathcal{E}(x)) = deg(\mathcal{E}) + rank(\mathcal{E}) + h^0(C,\mathcal{E}) - deg(\mathcal{E}).$$

□

The previous lemma assure that, for any natural number $r = rank(\mathcal{E})$, there exists always at least a non-trivial section of $\mathcal{E}(x)$ which does not belong to $\mathcal{E}$.

Remark 3.3.4. Observe that the residue sequence (3.6), on the subspace $C \times \{x\}$, $x \in X$, can be identified with the following sequence:

$$0 \to \Omega_C \to \Omega_C(x) \xrightarrow{\mathrm{res}_x} \mathbb{C}_x \to 0. \tag{3.6}$$

In fact $\pi_1^*\Omega_C$ over C is just Ω_C, $\mathcal{O}_C(\Delta)$ reduces to $\mathcal{O}_C(x)$ and everything on the right hand side, since it is a restriction on a fiber, is trivial. Note that $\mathrm{res}_x(\phi)$ is just the classical residue for any local meromorphic 1-form ϕ with at most a simple pole at x.

We now tensor the sequence (3.6) by $\mathcal{E}\otimes\Omega_C^*$ and we pass to cohomology. Then we get

$$0 \to H^0(C,\mathcal{E}) \to H^0(C,\mathcal{E}(x)) \xrightarrow{\mathrm{res}_x} (\mathcal{E}\otimes\Omega_C^*)|_x \to H^1(C,\mathcal{E}) \to H^1(C,\mathcal{E}(x)) \to 0.$$

Using Lemma 3.3.3 we see that $\mathrm{res}_x : H^0(C,\mathcal{E}(x)) \to (\mathcal{E}\otimes\Omega_C^*)|_x$ is surjective and its kernel is $H^0(C,\mathcal{E})$. Here we use the same symbol for a morphism of sheaves and for the map of global sections.

We now introduce the evaluation sequence, let $y \in X \subset C$, then the following is a short exact sequence:

$$0 \to \mathcal{O}_C(-y) \to \mathcal{O}_C \xrightarrow{\mathrm{ev}_y} \mathbb{C}_y \to 0. \tag{3.7}$$

If we tensor the latter sequence by $\mathcal{E}(x)$ and we pass to cohomology, we get

$$0 \to H^0(C, \mathcal{E}(x-y)) \to H^0(C, \mathcal{E}(x)) \xrightarrow{\mathrm{ev}_y} \mathcal{E}(x)|_y \to H^1(C, \mathcal{E}(x-y)) \dots . \tag{3.8}$$

We observe that $\mathcal{E}(x)|_y \simeq \mathcal{E}|_y$ as long as $x \neq y$, then, due to the good triple hypothesis, $\mathrm{ev}_y : H^0(C, \mathcal{E}(x)) \to \mathcal{E}|_y$ vanishes on $H^0(C, \mathcal{E})$, that is the kernel of res_x. Therefore, due to homomorphism theorem, there exists a unique linear map $\mathbf{S}^{\mathcal{E}}(x, y)$ which makes the following diagram commutative:

$$\begin{tikzcd} & H^0(C, \mathcal{E}(x)) \arrow[dl, two heads, "\mathrm{res}_x"'] \arrow[dr, "\mathrm{ev}_y"] & \\ (\mathcal{E} \otimes \Omega_C^*)|_x \arrow[rr, "\mathbf{S}^{\mathcal{E}}(x,y)"] & & \mathcal{E}|_y . \end{tikzcd} \tag{3.9}$$

Remark 3.3.5. Observe that the sequence (3.8) implies that y is a base point for $\mathcal{E}(x)$ if and only if $H^0(C, \mathcal{E}(x-y)) \simeq H^0(C, \mathcal{E}(x))$, which is equivalent, due to the Lemma 3.3.3, to $h^0(C, \mathcal{E}(x-y)) = rank(\mathcal{E}) + h^0(C, \mathcal{E}))$.

Remark 3.3.6. Note that, if we suppose $H^0(C, \mathcal{E}) = 0 = H^1(C, \mathcal{E})$, the maps res_x and ev_y become isomorphisms. In fact, due to the semi-continuity theorem (see [27], Theorem 12.8), $H^*(C, \mathcal{E}(x-y))$ are zero in a Zaritsky open subset of C. Hence $\mathbf{S}^{\mathcal{E}}(x, y) = \mathrm{ev}_y \circ \mathrm{res}_x^{-1}$.

Example 3.3.7. We claim that the triple $(\mathcal{O}_C(z-x), x, y)$ is good for any $x, y, z \in X$ such that $x \neq z$ and $x \neq y$. Observe that since $H^0(C, \mathcal{O}_C(z-x)) = 0$ the condition for y is already fulfilled. Moreover, consider the canonical sequence

$$0 \to \mathcal{O}_C(z-x) \to \mathcal{O}_C(z) \to \mathbb{C}_x \to 0$$

and pass to cohomology. We obtain

$$0 \to H^0(C, \mathcal{O}_C(z)) \xrightarrow{\simeq} \mathbb{C} \to H^1(C, \mathcal{O}_C(z-x)) \to H^1(C, \mathcal{O}_C(z)) \to 0.$$

The latter sequence implies $H^1(C, \mathcal{O}_C(z-x)) \simeq H^1(C, \mathcal{O}_C(z))$, the statement follows from Lemma 3.3.3.

Example 3.3.8. Let x, y, z be three distinct points of C and $\mathcal{E} = \mathcal{O}(z - x)$, then it is easy to see that $\mathbf{S}^{\mathcal{E}}(x, y)$ is an isomorphism.
If instead we take $y = z$, then $\mathbf{S}^{\mathcal{E}}(x, y) = 0$. In fact, due to Remark 3.3.5, y is a base point of $\mathcal{E}(x)$, which immediately implies ev_y is the zero map.

We now wish to investigate the relation between $\mathbf{S}$ and the Szegö kernel $\tilde{S}$ defined in the previous section, see Definition 3.2.7. For this purpose, observe that, for an arbitrary bundle $\mathcal{W}$, there exists a morphism $tr : \mathcal{W}^* \otimes \mathcal{W} \to \mathcal{O}_C$, given by $f \otimes w \to f(w)$. Then for any vector bundle $\mathcal{V}$, there exists an isomorphism

$$\mathcal{V}^\vee|_x \otimes \mathcal{V}|_y \simeq \mathrm{Lin}((\mathcal{V} \otimes \Omega_C^*)|_x, \mathcal{V}|_y), \tag{3.10}$$

where the action of a tensor $g \otimes v$ on the element $u \in (\mathcal{V} \otimes \Omega_C^*)|_x$ is given by $(g \otimes v)(u) = g(u)v$.

Lemma 3.3.9. *Let $\mathcal{E}$ be a vector bundle over C such that $H^*(C, \mathcal{E}) = 0$. For any $x \neq y$ $x, y \in X$, consider the linear map $\mathbf{S}^{\mathcal{E}}(x, y) = \mathrm{ev}_y \circ \mathrm{res}_x^{-1}$*

$$(\mathcal{E} \otimes \Omega_C^*)|_x \xrightarrow{\mathrm{res}_x^{-1}} H^0(C, \mathcal{E}(x)) \xrightarrow{\mathrm{ev}_y} \mathcal{E}|_y.$$

Then the value of the Szegö kernel $\tilde{S}$ at the point (x, y), i.e. $\tilde{S}(x, y) \in \mathcal{E}^\vee|_x \otimes \mathcal{E}|_y$, is the image of $\mathbf{S}^{\mathcal{E}}(x, y)$ under the isomorphism (3.10).

Proof. Consider the bi-orthogonal system, with respect to the trace map, given by $\{e_1, \dots, e_n\}$ basis of $\mathcal{E} \otimes \Omega_C^*|_x$ and $\{e_1^*, \dots, e_n^*\}$ basis of $\mathcal{E}^* \otimes \Omega_C|_x$. We can also fix a basis $\{E_1, \dots, E_n\}$ of $H^0(C, \mathcal{E}(x))$ determined by the relations $\mathrm{res}_x(E_i) = e_i$. We can then write $\tilde{S}|_{\{x\} \times X} = \sum \tilde{e}_i^* \otimes E_i$, where any element of $\tilde{e}_i^*$ is a linear combination of the basis $\{e_1^*, \dots, e_n^*\}$, i.e. $\tilde{e}_i^* = \sum h_{ij} e_j{}^*$. We conclude that $\tilde{e}_j^* = e_j^*$ from the fact that $\mathrm{res}_x(E_i) = e_i$ and $\sum e_i^* \otimes e_i = \mathbb{1}$. It remains to observe that both $\mathbf{S}$ and $\tilde{S}$, under the isomorphism (3.10), send e_i to E_i evaluated in y.

□

3.4 Szegö kernels and triple Massey products

In this section, similarly to what was done in the first chapter, we define a certain triple Massey product. We prove that for a good triple $(\mathcal{E}, x, y)$ such a product is related by a canonical isomorphism to the map $\mathbf{S}^{\mathcal{E}}(x, y)$ defined in the previous section. Let C be a reduced, projective Gorenstein curve of arithmetic genus $g > 0$, X be a non-empty regular irreducible subset of C. Let $x, y \in X$ be two distinct points and $\mathcal{E}$ be a vector bundle over C such that the triple $(\mathcal{E}, x, y)$ is good. Then we define the triple Massey product

$$\begin{array}{c} \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x) \otimes \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}) \otimes \mathrm{Hom}(\mathcal{E}, \mathbb{C}_y) \\ \Big\downarrow{\scriptstyle m^{\mathcal{E}}_{x,y}} \\ \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_y) \end{array} \tag{3.11}$$

as follows.
Let $f \in \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x)$, $w \in \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E})$ and $g \in \mathrm{Hom}(\mathcal{E}, \mathbb{C}_y)$. Observe that by definition w is represented by

$$0 \to \mathcal{E} \to \mathcal{W} \to \mathbb{C}_x \to 0.$$

Moreover, since f and g are given, we can write the following diagram:

$$\begin{tikzcd} & & \mathcal{O}_C \arrow[d, "\tilde{f}"'] \arrow[dr, "f"] & & \\ 0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[dr, "g"] & \mathcal{W} \arrow[r] \arrow[d, "\tilde{g}"] & \mathbb{C}_x \arrow[r] & 0 \\ & & \mathbb{C}_y, & & \end{tikzcd} \tag{3.12}$$

where $\tilde{f} : \mathcal{O}_C \to W$ and $\tilde{g} : W \to \mathbb{C}_y$, if they exist, are lifts of f and g. The triple Massey product is then defined by $m^{\mathcal{E}}_{x,y}(f \otimes w \otimes g) = \tilde{g}\tilde{f}$. It remains to prove that the definition is well-posed.

Lemma 3.4.1. *If the triple $(\mathcal{E}, x, y)$ is good, then the triple Massey product $m^{\mathcal{E}}_{x,y}$ is well-defined and uni-valued.*

Proof. First of all, we have to prove the existence of the maps $\tilde{f}$ and $\tilde{g}$. We take the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{W} \to \mathbb{C}_x \to 0$$

and we apply the functor $\mathrm{Hom}(-,\mathbb{C}_y)$. We get

$$0 \to \mathrm{Hom}(\mathbb{C}_x,\mathbb{C}_y) \to \mathrm{Hom}(\mathcal{W},\mathbb{C}_y) \to \mathrm{Hom}(\mathcal{E},\mathbb{C}_y) \to \mathrm{Ext}^1(\mathbb{C}_x,\mathbb{C}_y) \to \dots.$$

Using the properties of skyscraper sheaves supported at the distinct points x and y, one can easily see that $\mathrm{Hom}(\mathbb{C}_x,\mathbb{C}_y) = 0 = \mathrm{Ext}^1(\mathbb{C}_x,\mathbb{C}_y)$. The latter conditions imply $\mathrm{Hom}(W,\mathbb{C}_y) \simeq \mathrm{Hom}(\mathcal{E},\mathbb{C}_y)$, therefore there exists a unique $\tilde{g}$.
We now repeat the procedure applying the functor $\mathrm{Hom}(\mathcal{O}_C,-)$, we obtain

$$0 \to \mathrm{Hom}(\mathcal{O}_C,\mathcal{E}) \to Hom(\mathcal{O}_C,W) \to \mathrm{Hom}(\mathcal{O}_C,\mathbb{C}_x) \to \mathrm{Ext}^1(\mathcal{O}_C,\mathcal{E}) \to \dots.$$

Observe that if $H^0(C,\mathcal{E}) = 0 = H^1(C,\mathcal{E})$ the proof is similar to the previous case, moreover $\tilde{f}$ is unique. In the general case, we have to prove that the map $\mathrm{Hom}(\mathcal{O}_C,\mathcal{W}) \to \mathrm{Hom}(\mathcal{O}_C,\mathbb{C}_x)$ is surjective. Consider the pull-back

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{W}' \arrow[r] \arrow[d] & \mathcal{O}_C \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{W} \arrow[r] & \mathbb{C}_x \arrow[r] & 0.
\end{tikzcd}$$

If the map $F : \mathrm{Ext}^1(\mathbb{C}_x,\mathcal{E}) \otimes \mathrm{Hom}(\mathcal{O}_C,\mathbb{C}_x) \to \mathrm{Ext}^1(\mathcal{O}_C,\mathcal{E})$ sends $w \otimes f$ to the zero extension, we get the map $\tilde{f}$ we are searching for. In fact the upper sequence in the previous diagram would split. Observe that using Serre duality we have $\mathrm{Ext}^1(\mathbb{C}_x,\mathcal{E}) \simeq (\mathcal{E} \otimes \Omega_C^*)|_x$ as well as $\mathrm{Ext}^1(\mathcal{O}_C,\mathcal{E}) \simeq H^0(\mathcal{E}^\vee)^*$. Due to the isomorphism $\mathrm{Hom}(\mathcal{O}_C,\mathbb{C}_x) \simeq \mathbb{C}$, one can see that the map F can be rewritten as $F' : (\mathcal{E} \otimes \Omega_C^*)|_x \to H^0(\mathcal{E}^\vee)^*$ that is dual to $\mathrm{ev}_x : H^0(\mathcal{E}^\vee) \to \mathcal{E}^\vee|_x$. F' is zero due to the base point hypothesis, thus we have a surjection $\mathrm{Hom}(\mathcal{O}_C,\mathcal{W}) \to \mathrm{Hom}(\mathcal{O}_C,\mathbb{C}_x)$ and so we get $\tilde{f}$.
We conclude the proof observing that the triple Massey product is uni-valued if and only if the triple $(\mathcal{E},x,y)$ is good. In fact any factorization of the form

$$\begin{tikzcd}
\mathcal{O}_C \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathbb{C}_y \\
& \mathcal{E}|_y \arrow[ur] &
\end{tikzcd}$$

gives no contribution to the diagram (3.12). $\square$

We are now ready to state the main result of this section.

Theorem 3.4.2. *Let C and X be as at the beginning of this section, let $\mathcal{E}$ be a vector bundle over C and $x, y \in C$ such that $(\mathcal{E}, x, y)$ is a good triple. Then the Massey product $m^{\mathcal{E}}_{x,y}$ in (3.11) is the image of the map $\mathbf{S}^{\mathcal{E}}(x, y)$, defined by the commutative diagram (3.9), under the isomorphism:*

$$\mathrm{Lin}(\mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x) \otimes \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}) \otimes \mathrm{Hom}(\mathcal{E}, \mathbb{C}_y), \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_y)) \simeq$$
$$\simeq \mathrm{Lin}((\mathcal{E} \otimes \Omega^*_C)|_x, \mathcal{E}|_y).$$

Proof. Before proving the main theorem we need some preliminary steps.

Lemma 3.4.3. *There exists an isomorphism of functors*

$$\alpha_x : \mathrm{Hom}(\mathbb{C}_x, - \otimes \Omega^*_C \otimes \mathbb{C}_x) \to \mathrm{Ext}^1(\mathbb{C}_x, -),$$

between vector bundles over C and complex vector spaces.

Proof. Let $\mathcal{E}$ be a bundle over C such that $rank(\mathcal{E}) = r$ and $x \in C$. Consider the short exact sequence

$$0 \to \mathcal{E} \otimes \Omega_C \to \mathcal{E} \otimes \Omega_C(x) \xrightarrow{\mathrm{res}_x} \mathcal{E} \otimes \mathbb{C}_x \to 0$$

and apply the functor $\mathrm{Hom}(\mathbb{C}_x, -\otimes\Omega^*_C)$. Using the vanishing of the first two terms due to Serre duality, we get

$$0 \to \mathrm{Hom}(\mathbb{C}_x, \mathcal{E} \otimes \Omega^*_C \otimes \mathbb{C}_x) \xrightarrow{\alpha_x} \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}) \to \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}(x)) \to$$

$$\to \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E} \otimes \Omega^*_C \otimes \mathbb{C}_x) \to 0.$$

We conclude the proof observing that the last two terms are both of dimension r. In fact: $\mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}(x)) \simeq H^0(\mathcal{E}xt^1(\mathbb{C}_x, \mathcal{E}(x)))$, whereas $\mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E} \otimes \Omega^*_C|_x)$ is isomorphic, due to Serre duality, to the r dimensional vector space $\mathcal{E} \otimes \Omega^{-2}_C|_x$. Therefore α_x is an isomorphism. □

We now observe that, using the residue sequence (3.6) and for $w \in \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E})$ as in diagram (3.12), we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r, "a"] \arrow[d, "id"] & \mathcal{W} \arrow[r] \arrow[d] & \mathbb{C}_x \arrow[r] \arrow[d, "\alpha_x^{-1}(w)"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}(x) \arrow[r] & \mathcal{E} \otimes \Omega^*_C \otimes \mathbb{C}_x \arrow[r] & 0.
\end{tikzcd}$$

The latter diagram, together with the definition of Massey product (3.12), can be extended to:

$$\begin{tikzcd}
 & \mathcal{O}_C \arrow[r] \arrow[d, "\tilde{f}"] \arrow[drr, "f"] & \mathcal{O}_C \otimes \mathbb{C}_x \arrow[dr, phantom] & \\
0 \arrow[r] & \mathcal{E} \arrow[r, "a"] \arrow[d, "id"] & \mathcal{W} \arrow[r] \arrow[d, "\lambda"] & \mathbb{C}_x \arrow[r] \arrow[d, "\alpha_x^{-1}(w)"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}(x) \arrow[r] & \mathcal{E} \otimes \Omega_C^* \otimes \mathbb{C}_x \arrow[r] & 0.
\end{tikzcd}$$

Remember that by definition $\mathrm{res}_x : H^0(C, \mathcal{E}(x)) \to (\mathcal{E} \otimes \Omega_C^*)|_x \simeq \mathcal{E} \otimes \Omega_C^* \otimes \mathbb{C}_x$ and recall that $\mathrm{Hom}(\mathcal{O}_C, -) \simeq H^0(C, -)$. Thus we have

$$\alpha_x^{-1}(w) f_x = \mathrm{res}_x(\lambda \tilde{f}).$$

Note that, if we tensor everything by $\mathbb{C}_y$, we also get

$$\begin{tikzcd}
 & \mathcal{O}_C \otimes \mathbb{C}_y \arrow[d, "\tilde{f}_y"] \\
\mathcal{E} \otimes \mathbb{C}_y \arrow[r, "a_y"] \arrow[d] & \mathcal{W} \otimes \mathbb{C}_y \arrow[d, "\lambda_y"] \\
\mathcal{E} \otimes \mathbb{C}_y \arrow[r] & \mathcal{E}(x) \otimes \mathbb{C}_y,
\end{tikzcd}$$

therefore

$$\mathrm{ev}_y(\lambda \tilde{f}) = a_y^{-1} \tilde{f}_y.$$

If we now recall the definition of the triple Massey product, we immediately get

$$(m_{x,y}^{\mathcal{E}})(f \otimes w \otimes g)_y = g_y \circ a_y^{-1} \circ \tilde{f}_y = g_y \circ \mathbf{S}_{x,y}^{\mathcal{E}}(\alpha_x^{-1}(w) f_x).$$

In order to conclude the proof, we have to describe $m_{x,y}^{\mathcal{E}}$ under a sequence of isomorphisms. We claim that the following relation holds:

$$\gamma_1(f \otimes w \otimes g) = \gamma_2(f \otimes s(w))(g).$$

Here $s : \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}) \simeq \mathrm{Hom}(\mathcal{E} \otimes \Omega_C^*, \mathbb{C}_x)^*$ is the isomorphism given by the Serre pairing and

$$\gamma_1 \in \mathrm{Lin}(\mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x) \otimes \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}) \otimes \mathrm{Hom}(\mathcal{E}, \mathbb{C}_y), \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_y)),$$

whereas

$$\gamma_2 \in \mathrm{Lin}(\mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x) \otimes \mathrm{Hom}(\mathcal{E} \otimes \Omega_C^*, \mathbb{C}_x)^*, \mathrm{Lin}(\mathrm{Hom}(\mathcal{E}, \mathbb{C}_y), \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_y))).$$

Clearly there exists a canonical isomorphism between the spaces in which the two elements γ_1 and γ_2 live.
We now claim that $s(w)$ is sent to $\alpha_x^{-1}(w) \in \mathrm{Hom}(\mathbb{C}_x, (\mathcal{E} \otimes \Omega_C^*)|_x)$.

Lemma 3.4.4. *The following is a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E}) \arrow[r, "s"] \arrow[d, "\alpha_x^{-1}"'] & \mathrm{Hom}(\mathcal{E} \otimes \Omega_C^*, \mathbb{C}_x)^* \arrow[d] \\
\mathrm{Hom}(\mathbb{C}_x, \mathcal{E} \otimes \Omega_C^* \otimes \mathbb{C}_x) \arrow[r, "\tau"] & \mathrm{Hom}(\mathcal{E} \otimes \Omega^* \otimes \mathbb{C}_x, \mathbb{C}_x)^*,
\end{tikzcd}$$

where the map τ is induced by the canonical isomorphism of vector spaces $\mathrm{Hom}(U, V)^* \simeq \mathrm{Hom}(V, U)$.

Proof. Given a vector bundle $\mathcal{W}$ over C, using the residue sequence (3.6) and applying $\mathrm{Hom}(\mathcal{W}, -)$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{W}|_x, \mathcal{W}|_x) \arrow[r] \arrow[d, "tr_x"'] & \mathrm{Hom}(\mathcal{W}, \mathcal{W} \otimes \mathbb{C}_x) \arrow[r, "\nu_x"] \arrow[d, "\beta^1_{\mathcal{W}}"'] & \mathrm{Ext}^1(\mathcal{W}, \mathcal{W} \otimes \Omega_C) \arrow[d, "\beta^2_{\mathcal{W}}"'] \\
\mathbb{C} \arrow[r] \arrow[dr] & H^0(C, \mathbb{C}_x) \arrow[r, "\nu_x"] \arrow[d] & H^1(C, \Omega_C) \arrow[dl, "\tilde{t}"] \\
& \mathbb{C} &
\end{tikzcd}$$

Here we are adopting the following notation: tr_x is the usual trace map, $\tilde{t}$ is the "trace" on a Gorenstein variety G of dimension r_1, i.e. $\tilde{t} : H^{r_1}(G, \Omega_G) \to \mathbb{C}$ (see [27], III.7); ν_x is the connecting morphism, $\beta^1_{\mathcal{W}}$, $\beta^2_{\mathcal{W}}$ are again trace maps and all the non-labeled functions are canonical morphisms.
The commutativity of the last diagram implies that, for any $f \in \mathrm{Hom}(\mathcal{W}, \mathcal{W} \otimes \mathbb{C}_x)$, one has

$$\tilde{t}(\beta^1_{\mathcal{W}}(\nu_x(f))) = tr(f_x).$$

We conclude the proof observing that:

$$tr(\theta \circ \alpha^{-1}(w')) = tr(\alpha^{-1}(w') \circ \theta) = \tilde{t}(\beta^1_{\mathcal{V}}(\nu_x(\alpha_x^{-1}(w') \circ \theta))) = s(w')(\theta)$$

for any $\theta \in \mathrm{Hom}(\mathcal{E} \otimes \Omega_C^*, \mathbb{C}_x)$ and $w' \in \mathrm{Ext}^1(\mathbb{C}_x, \mathcal{E})$.

□

We now come back to the proof of the main statement. Observe that $\alpha^{-1}(w)$ can be viewed, up to isomorphism, as an element in $(\mathcal{E} \otimes \Omega_C^*)|_x$. Any map γ_2, as above, is then sent to a map

$$\gamma_3 \in \mathrm{Lin}(\mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x) \otimes (\mathcal{E} \otimes \Omega_C^*)|_x, \mathrm{Lin}(\mathrm{Hom}(\mathcal{E}, \mathbb{C}_y), \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_y)))$$

such that $\gamma_3(f_x \otimes \tilde{\alpha}_x^{-1}(w))(g_y) = \gamma_1(f \otimes w \otimes g)_y$.
If we now observe that $\mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_x) \simeq \mathbb{C} \simeq \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}_y)$ and that $\mathrm{Hom}(\mathcal{E}, \mathbb{C}_y)$ is isomorphic to $\mathcal{E}^*|_y$, we deduce that γ_3 is sent to a linear map γ_4 in $\mathrm{Lin}(\mathcal{E} \otimes \Omega_C^*|_x, \mathcal{E}|_y)$ such that the following diagram is commutative

$$\begin{tikzcd}
(\mathcal{E} \otimes \Omega^*)|_x \arrow[rr, "\gamma_4(\tilde{\alpha}_x^{-1}(w) f_x)"] \arrow[dr, "\gamma_1(f \otimes w \otimes g)_y"'] & & \mathcal{E}|_y \arrow[dl, "g_y"] \\
& \mathbb{C} &
\end{tikzcd}$$

The theorem is proved observing that $\gamma_4 = \mathbf{S}_{x,y}^{\mathcal{E}}$ while $\gamma_1 = m_{x,y}^{\mathcal{E}}$. $\square$

Corollary 3.4.5. *If moreover $H^0(C, \mathcal{E}) = 0 = H^1(C, \mathcal{E})$, then the triple Massey product $m_{x,y}^{\mathcal{E}}$ is isomorphic, in the sense of Lemma 3.3.9, to the Szegö kernel $\tilde{S}(x, y)$, for any $x, y \in C$, $x \neq y$.*

3.5 Skew-symmetry

In this section we prove that Szegö kernels are skew-symmetric, but before proceeding with this intent we need some remarks.
Let C be a reduced, projective Gorenstein curve of arithmetic genus $g > 0$, $X \neq \emptyset$ be a regular irreducible subset of C and $\mathcal{E}$ be a vector bundle over C with vanishing cohomology. In particular $deg(\mathcal{E}) = (g-1)rank(\mathcal{E})$. As already stated several times, using Riemann-Roch theorem, one can see that $\mathcal{E}^\vee$ has the same properties of $\mathcal{E}$. In fact

$$H^0(C, \mathcal{E}) = 0 = H^1(C, \mathcal{E}^\vee) \quad \text{and} \quad H^1(C, \mathcal{E}) = 0 = H^0(C, \mathcal{E}^\vee).$$

The latter conditions imply that the residue map can be defined for the bundle $\mathcal{E}^\vee$. We then have an isomorphism

$$\mathrm{res}_x : H^0(C, \mathcal{E}^\vee(x)) \to \mathcal{E}^*|_x.$$

Moreover, we can define

$$\mathbf{S}^{\mathcal{E}^\vee}(x,y) := \mathrm{ev}_y \circ \mathrm{res}_x^{-1} : \mathcal{E}^*|_x \to (\mathcal{E}^* \otimes \Omega_C)|_y.$$

Let $\tau : \mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}^\vee$ be the morphism which switches the factors on the level of appropriated local sections and $\lambda : X \times X \to X \times X$ be the map which flips the entries. Recall that $\tilde{S}$ is the Szegö kernel from Definition 3.2.7.

Definition 3.5.1. The Szegö kernel $\tilde{S}$ is skew-symmetric if and only if

$$\tau(\tilde{S}) = -\lambda^*(\tilde{S})$$

for any $x, y \in X \times X \setminus \Delta$, where $\tilde{S}$ is viewed by restriction as a meromorphic section of $\mathcal{E}^\vee \boxtimes \mathcal{E}$.

Recall that $\mathbf{S}^{\mathcal{E}}(x,y) : \mathcal{E} \otimes \Omega_C^*|_x \to \mathcal{E}|_y$ and that we have trace map

$$tr : \mathcal{E}^* \otimes \mathcal{E} \simeq \mathrm{End}(\mathcal{E}) \to \mathbb{C}.$$

Lemma 3.5.2. *The following are equivalent:*

(*i*) *The Szegö kernel $\tilde{S}$ is skew-symmetric;*

(*ii*) $tr_x(\mathrm{res}_x(s_2), \mathrm{ev}_x(s_1)) = -tr_y(\mathrm{res}_y(s_1), \mathrm{ev}_y(s_2))$ *for any couple of distinct points $x, y \in X$ and for all $s_1 \in H^0(C, \mathcal{E}^\vee(y))$, $s_2 \in H^0(C, \mathcal{E}(x))$.*

Proof. Observe that, due to isomorphism in Lemma 3.3.9, that uses the trace map, the first condition can be rewritten as

$$tr_x(\mathbf{S}^{\mathcal{E}^\vee}(y,x)(a), b) = -tr_y(a, \mathbf{S}^{\mathcal{E}}(x,y)(b)), \tag{3.13}$$

for all $(a,b) \in \mathcal{E}^*|_y \times (\mathcal{E} \otimes \Omega_C^*)|_x$.
We now consider $(s_1, s_2) \in H^0(X, \mathcal{E}^\vee(y)) \times H^0(C, \mathcal{E}(x))$ such that $a = \mathrm{res}_y(s_1)$ and $b = \mathrm{res}_x(s_2)$. Then, using the symmetry of tr, equation (3.13) can be rewritten as

$$tr_x(\mathrm{res}_x(s_2), \mathrm{ev}_x(s_1)) = -tr_y(\mathrm{res}_y(s_1), \mathrm{ev}_y(s_2)),$$

for any $(s_1, s_2) \in H^0(C, \mathcal{E}^\vee(y)) \times H^0(C, \mathcal{E}(x))$. $\square$

Applying $-\otimes \Omega_C$ to $tr : \mathcal{E}^* \otimes \mathcal{E} \to \mathcal{O}_C$ yields

$$tr^{\Omega_C} : \mathcal{E}^\vee \otimes \mathcal{E} \to \Omega_c,$$

which allows us to demonstrate the next result.

Theorem 3.5.3. *The Szegö kernel $\tilde{S}$ is skew-symmetric.*

Proof. First of all we prove that the Szegö kernel $\tilde{S}$ is skew-symmetric if and only if

$$\mathrm{res}_x(tr^{\Omega_C}(s_1, s_2)) + \mathrm{res}_y(tr^{\Omega_C}(s_1, s_2)) = 0,$$

for any $s_1 \in H^0(C, \mathcal{E}^\vee(y))$, $s_2 \in H^0(C, \mathcal{E}(x))$ and for all $(x, y) \in X \times X \setminus \Delta$. We observe that we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{E}^\vee(y) \otimes \mathcal{E}(x) \arrow[r] \arrow[d, "\mathrm{res}_y \otimes \mathrm{ev}_y"'] & \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{O}_C(x+y) \arrow[dd, "tr^{\Omega_C}"] \\
\mathcal{E}^*|_y \otimes \mathcal{E}|_y \arrow[d, "tr"] & \\
\mathbb{C}_y & \Omega_C(y) \otimes \mathcal{O}_C(x). \arrow[l, "\mathrm{res}_y"]
\end{tikzcd}$$

In fact a straightforward computation shows that in both ways we get

$$tr_y(\mathrm{ev}_y(s_1), \mathrm{ev}_y(s_2)) \cdot \mathrm{ev}_y(\mu) \cdot \mathrm{res}_y(\lambda),$$

where $\lambda \in \Omega_C(y)$ and $\mu \in \mathcal{O}_X(x)$ are local sections. Observe that here occurs an abuse of notation. In fact the vertical residue at y is the map between vector bundles already defined in diagram (3.9), whereas the lowest one is the classical residue over a curve.

The statement at the beginning of this proof follows using the latter diagram and the previous lemma; i.e. $\mathrm{res}_y(tr^{\Omega_C}(-,-)) = tr_y(-,-)$. Using the symmetry of the trace map we get also $\mathrm{res}_x(tr^{\Omega_C}(-,-)) = tr_x(-.-)$. The proof of the lemma follows applying the residue theorem. In fact the residual sum has to vanish, see for instance [3], over all closed points of C, but it obviously vanishes on any point that is neither x nor y. $\square$

3.6 Identities for Szegö kernels

Let C and X be as in the previous sections and $(\mathcal{E}, x, y)$ be a good triple. We prove some relations satisfied by the function $\mathbf{S}^{\mathcal{E}}(x, y)$. In particular, we compute a sheaf-theoretic version of the matrix-valued Fay's trisecant identity. The first result of this section is the following:

Theorem 3.6.1. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be vector bundles on C. Let $T : \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n \to \Omega_C$ be a morphism of vector bundles and $x_1, \ldots, x_n$ be points of X such that the triples $(\mathcal{E}_j, x_j, x_i)$ are good for any $i \neq j$. Then the following relation is true:*

$$\sum_{i=1}^{n} T_{x_i}(id_{\mathcal{E}_i \otimes \Omega_C^*|_{x_i}} \otimes \bigotimes_{j \neq i} \mathbf{S}^{\mathcal{E}_j}(x_j, x_i)) = 0. \tag{3.14}$$

Proof. For any $j \in \{1, \ldots, n\}$, we can choose $s_j \in H^0(C, \mathcal{E}_j(x_j)) \setminus H^0(C, \mathcal{E}_j)$ with the properties $\mathrm{res}_{x_j}(s_j) = a_j \in (\mathcal{E}_j \otimes \Omega_C^*)|_{x_j}$. Observe that one has the following commutative diagram

$$\begin{tikzcd}
\bigotimes_{l=1}^{n} H^0(C, \mathcal{E}_l(x_l)) \arrow[rr] \arrow[d, "\mathrm{res}_{x_1} \otimes \mathrm{ev}_{x_1} \otimes \cdots \otimes \mathrm{ev}_{x_1}"'] & & H^0(C, \bigotimes_{l=1}^{n} \mathcal{E}_l(x_l)) \arrow[d, "T"] \\
(\Omega_C^* \otimes \bigotimes_{l=1}^{n} \mathcal{E})|_{x_1} \arrow[d, "T_{x_1}"] & & H^0(C, \Omega_C(x_1 + \cdots + x_n)) \arrow[d, "\mathrm{res}_{x_1}"] \\
\mathbb{C}_{x_1} & & \mathbb{C}_{x_1} \arrow[ll]
\end{tikzcd}$$

which implies $\mathrm{res}_{x_1}(T(s_1 \otimes \cdots \otimes s_n)) = T_{x_1}(\mathrm{res}_{x_1} \otimes \mathrm{ev}_{x_1} \otimes \ldots \mathrm{ev}_{x_1}(s_1 \otimes \cdots \otimes s_n))$. Writing down the same commutative diagram for any x_i we get:

$$\mathrm{res}_{x_i}(T(s_1 \otimes \cdots \otimes s_n)) = T_{x_i}(\mathrm{ev}_{x_i}^{\otimes(i-1)} \otimes \mathrm{res}_{x_i} \otimes \mathrm{ev}_{x_i}^{\otimes(n-i)}(s_1 \otimes \cdots \otimes s_n)).$$

Applying residue theorem we end up with:

$$\sum_{i=1}^{n} T_{x_i}(\mathrm{ev}_{x_i}^{\otimes(i-1)} \otimes \mathrm{res}_{x_i} \otimes \mathrm{ev}_{x_i}^{\otimes(n-i)}(s_1 \otimes \cdots \otimes s_n)) = \sum_{i=1}^{n} \mathrm{res}_{x_i}(T(s_1 \otimes \cdots \otimes s_n)) = 0.$$

We now observe that, using the properties $\mathrm{res}_{x_j}(s_j) = a_j \in (\mathcal{E}_j \otimes \Omega_C^*)|_{x_j}$, we have $\mathrm{res}_{x_i}(s_i) = id_{\mathcal{E}_i \otimes \Omega_C^*|_{x_i}}$ and $\mathrm{ev}_{x_i}(s_j) = \mathbf{S}^{\mathcal{E}_j}(x_j, x_i)$. Thus the statement follows. $\square$

Remark 3.6.2. Note that relation (3.14) reduces to the skew-symmetry once we take $n = 2$. In fact we can set: $\mathcal{E}_1 = \mathcal{E}$, $\mathcal{E}_2 = \mathcal{E}^\vee$, with the properties $H^0(C,\mathcal{E}) = 0 = H^1(C,\mathcal{E})$, and $T := tr^{\Omega_C} : \mathcal{E}^\vee \otimes \mathcal{E} \to \Omega_C$ as in Section 3.5. .

As already mentioned in Section 3.3, if one considers a vector bundle $\mathcal{E}$ with vanishing cohomology and takes two distinct points $x_1, y_1 \in X$, then the bundle $\mathcal{E}(x_1 - y_1)$ has vanishing cohomology. Moreover the maps $\mathbf{S}^{\mathcal{E}}$ and $\mathbf{S}^{\mathcal{E}^\vee}$ are locally invertible. Recall from Remark 3.3.7 that $(\mathcal{O}_C(y_1 - x_1), x_1, z)$ is a good triple for any three distinct points $x_1, y_1, z \in X$. Let us denote by $s(*)$ the element $\mathbf{S}^{\mathcal{O}_C(y_1-x_1)}(x_1, *)$, which we know to be an isomorphism (Remark 3.3.8), we can now prove the following result.

Theorem 3.6.3. *Let $\mathcal{E}$ be a vector bundle such that $H^0(C,\mathcal{E}) = 0 = H^1(C,\mathcal{E})$, let $x_0, x_1, y_0, y_1 \in X$ be four distinct points. Then the following equality holds*

$$\mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, y_0)s(x_0)s(y_0)^{-1} = \mathbf{S}^{\mathcal{E}}(x_0, y_0) - \mathbf{S}^{\mathcal{E}}(x_1, y_0)\mathbf{S}^{\mathcal{E}}(x_1, y_1)^{-1}\mathbf{S}^{\mathcal{E}}(x_0, y_1). \tag{3.15}$$

Proof. **Step I**
We apply relation (3.15) to the bundles $\mathcal{E}_1 = \mathcal{E}^\vee$, $\mathcal{E}_2 = \mathcal{E}(x_1 - y_1)$ and $\mathcal{E}_3$ equals to $\mathcal{O}_C(y_1 - x_1)$. Observe that in this case the map T is naturally defined via the trace map, i.e. $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \simeq \mathcal{E}^* \otimes \Omega_C \otimes \mathcal{E} \to \Omega_C$. We then compute everything at the distinct points y_1, x_0, x_1. If we denote $\mathbf{S}^{\mathcal{O}_C(y_1-x_1)}(x_1, z) = I_{x_1,z}$, we immediately get

$$T_{y_1}(id_{\mathcal{E}_1\otimes\Omega_C^*|_{y_1}} \otimes \mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, y_1) \otimes I_{x_1,y_1}) + T_{x_0}(\mathbf{S}^{\mathcal{E}^\vee}(y_1, x_0) \otimes id_{\mathcal{E}_2\otimes\Omega_C^*|_{x_0}} \otimes I_{x_1,x_0}) + \\ + T_{x_1}(\mathbf{S}^{\mathcal{E}^\vee}(y_1, x_1) \otimes \mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, x_1) \otimes Id_{\mathcal{E}_3\otimes\Omega_C^*|_{x_1}}) = 0.$$

We recall that $\mathbf{S}^{\mathcal{O}_C(y_1-x_1)}(x_1, y_1) = 0$ (see Remark 3.3.8). Thus we obtain

$$tr_{x_0}(\mathbf{S}^{\mathcal{E}^\vee}(y_1, x_0)(a), b \cdot I_{x_1,x_0}) + tr_{x_1}(\mathbf{S}^{\mathcal{E}^\vee}(y_1, x_1)(a), \mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, x_1)(b) \cdot c) = 0,$$

where $(a, b, c) \in \mathcal{E}^*|_{y_1} \times (\mathcal{E}(x_1 - y_1) \otimes \Omega_C^*)|_{x_0} \times (\mathcal{O}_C(y_1 - x_1) \otimes \Omega_C^*)|_{x_1}$. Using skew-symmetry and non-degeneracy of the trace map as well as the fact the inverse of $\mathbf{S}^{\mathcal{E}}$ is well-defined, we end up with

$$\mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, x_1) = -(\mathbf{S}^{\mathcal{E}}(x_1, y_1))^{-1}\mathbf{S}^{\mathcal{E}}(x_0, y_1)\mathbf{S}^{\mathcal{O}_C(y_1-x_1)}(x_1, x_0).$$

Step II
We repeat the step I one in a slightly different way. We take the bundles $\mathcal{E}_1 = \mathcal{E}^\vee$, $\mathcal{E}_2 = \mathcal{E}(x_1 - y_1)$ and $\mathcal{E}_3 = \mathcal{O}_C(y_1 - x_1)$. We compute everything at the distinct points y_0, x_0, x_1. We can write

$$tr_{y_0}(a, \mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, y_0)(b) I_{x_1,y_0}(c)) + tr_{x_0}(\mathbf{S}^{\mathcal{E}^\vee}(y_0, x_0)(a), b \cdot I_{x_1,x_0}(c))$$
$$+tr_{x_1}(\mathbf{S}^{\mathcal{E}^\vee}(y_0, x_1)(a), \mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, x_1)(b) \cdot c) = 0,$$

with $(a, b, c) \in \mathcal{E}^*|_{y_0} \times (\mathcal{E}(x_1 - y_1) \otimes \Omega_C^*)|_{x_0} \times (\mathcal{O}_C(y_1 - x_1) \otimes \Omega_C^*)|_{x_1}$. The latter equation, proceeding as in the previous step, gives the relation

$$\mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, y_0) \cdot \mathbf{S}^{\mathcal{O}_C(y_1-x_1)}(x_1, y_0) - \mathbf{S}^{\mathcal{E}}(x_0, y_0) \cdot \mathbf{S}^{\mathcal{O}_C(y_1-x_1)}(x_1, x_0) +$$
$$-\mathbf{S}^{\mathcal{E}}(x_1, y_0) \cdot \mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0, x_1) = 0.$$

Step III
We substitute the equation obtained in the first step inside that one got in the previous one, the statement follows reordering terms. □

The latter can be considered as a generalization in the case of Gorenstein curves of the celebrated Fay's identity which appears in [20, 21]. In order to reduce it to that identity, we shall work over Riemann surfaces. Details are treated in the next chapter.

Chapter 4

Line bundles over complex tori and Fay's identity

This chapter is devoted to the study of line bundles over smooth projective curves of positive genus, i.e. Riemann surfaces. In order to study sections of those bundles we have to investigate the theory of complex tori and their line bundles. We have to introduce the Jacobian of a Riemann surface, that is a specific torus which can be used to describe many properties of the surface itself. In particular we will define a Poincaré bundle over the Jacobian and we will pull it back to the curve. We conclude this thesis describing the identities obtained at the end of the third chapter in terms of theta functions.

4.1 Line bundles over complex tori

Before studying line bundles over a $g-$dimensional complex torus, we need a preliminary and well-known lemma.

Lemma 4.1.1. *Any line bundle over $\mathbb{C}^g$ is trivial.*

Let V be a complex vector space of dimension g and Γ be a full rank lattice in V. We call the complex, compact and connected g-dimensional manifold $T = V/\Gamma$ complex torus of dimension g. Observe that T has a natural structure of an abelian group. We denote by $\pi : V \to T$ the canonical projection and we observe that, due to the previous lemma, $\pi^*\mathcal{L}$ is a trivial bundle for any line bundle $\mathcal{L}$ over T. Thus we can write $\pi^*\mathcal{L} \simeq V \times \mathbb{C}$. Moreover the natural action of Γ over V can be lifted to an action on $V \times \mathbb{C}$ in order to have $\mathcal{L}$ isomorphic to $V \times \mathbb{C}$ modulo such an action. An element $\gamma \in \Gamma$ acts on the fibers as follows:

$$\gamma(v, z) = (v + \gamma, a_\gamma(v)z),$$

for any $v \in V$, $z \in \mathbb{C}$ and where a_γ is a holomorphic invertible function on V. The latter equation defines a group action of Γ on $V \times \mathbb{C}$ if and only if it satisfies the cocycle relation, i.e.

$$a_{\gamma_1+\gamma_2}(z) = a_{\gamma_1}(z + \gamma_2)a_{\gamma_2}(z).$$

A function a_γ as above is called automorphy factor and any line bundle on the torus is defined by a family of automorphy factors.

Definition 4.1.2. Let $(a_\gamma)_{\gamma\in\Gamma}$ be a family of automorphy factors, then a holomorphic function $\theta : V \to \mathbb{C}$, which satisfies the relation

$$\theta(v + \gamma) = a_\gamma(v)\theta(v)$$

for all $v \in V$ and $\gamma \in \Gamma$, is called theta function.

Lemma 4.1.3. *Let $\mathcal{L}$ be a line bundle on T and $(a_\gamma)_{\gamma\in\Gamma}$ be a family of automorphy factors for $\mathcal{L}$. Then the space of theta functions for $(a_\gamma)_{\gamma\in\Gamma}$ is canonically isomorphic to the space of sections of $H^0(T, \mathcal{L})$.*

Proof. We recall a proof from [7].
Any section σ of $\mathcal{L}$ lifts to a section σ' of $\pi^*\mathcal{L}$ given by $\sigma'(v) = (v, \sigma(\pi(v)))$. One can easily check that σ' is invariant under the action of Γ, in fact $\sigma'(v+\gamma) = \gamma\sigma'(v)$. This means that it is a theta function. Conversely, any section of $\pi^*\mathcal{L}$, which is invariant under the action of Γ, is the lift of a section of $\mathcal{L}$. Moreover, using the trivialization of the pull-back bundle, i.e. $\pi^*\mathcal{L} \simeq V \times \mathbb{C}$, we can see that any section of such a bundle is of the form $v \to (v, f(v))$, where f is a holomorphic function from V to $\mathbb{C}$. We conclude the proof observing that those kind of sections are invariant, under the action of Γ, if and only if f is a theta function. $\square$

Thus, in terms of sheaves, for any $\mathcal{L} \in Pic(T)$ and for any U open set of T we have

$$\mathcal{L}(U) = \{\theta : \tilde{U} \to \mathbb{C} | \theta(\gamma + v) = a_\gamma(v)\theta(v), \forall \gamma, v \in \Gamma \times \tilde{U}\},$$

where $\tilde{U} = \pi^{-1}(U)$.
We would like to give a description of factors of automorphy.
Let $H : V \times V \to \mathbb{C}$ be a Hermitian form anti-linear in the first variable and linear in the second one. Let $z, w \in \mathbb{C}$, we define $R(z,w) = \Re(H(z,w))$ and $I(z,w) = \Im(H(z,w))$. Observe that R and I are symmetric and skew-symmetric real bilinear forms, respectively. Moreover, if one of H, R, I is non-degenerate, then all of them are non-degenerate. Let H be a Hermitian form such that $I(\gamma_1, \gamma_2) \in \mathbb{Z}$ for all $\gamma_1, \gamma_2 \in \Gamma$. Let f be a function from Γ to $\mathcal{S}^1 = \{z \in \mathbb{C} | |z| = 1\}$ such that $f(\gamma_1 + \gamma_2) = f(\gamma_1)f(\gamma_2)(-1)^{I(\gamma_1,\gamma_2)}$ and denote by $\mathcal{H}$ the set of all pairs (H, f) with the properties just stated, then we are ready to state the next theorem.

Theorem 4.1.4. *(Appel-Humbert)[33]*
There exists a group isomorphism $\mathcal{H} \to Pic(T)$, where:

(i) The group structure of $\mathcal{H}$ is defined by

$$(H_1, f_1) \cdot (H_2, f_2) = (H_1 + H_2, f_1 f_2);$$

(ii) whereas the isomorphism is given by $(H, f) \to \mathcal{L}(H, f)$, with the line bundle $\mathcal{L}(H, f)$ defined by the automorphy factors

$$a_\gamma(v) = f(\gamma) \exp(\pi(H(\gamma, v) + \frac{1}{2}H(\gamma, \gamma)).$$

We further suppose H to be positive definite and we observe that $I = \Im(H)$ can be considered as a non-degenerate, skew-symmetric form from $\Gamma \times \Gamma$ to $\mathbb{Z}$. Then, due to Frobenius lemma [24], there exists a basis of Γ such that the matrix which represents I is of the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = diag(d_1, \dots, d_g)$ is a diagonal matrix whose elements are $d_i \in \mathbb{Z}$ such that d_i divides d_{i+1}.

Theorem 4.1.5. *The dimension of the space $H^0(T, \mathcal{L}(H, f))$ is equal to $\prod_{i=1}^{g} d_i$.*

Proof. This is a well-known result, a proof can be found either in [7] or in [33]. $\square$

4.2 Poincaré bundle

Given a complex torus $T = V/\Gamma$, we consider the space $\overline{V} = \mathrm{Hom}_a(V, \mathbb{C})$ of anti-linear maps $L : V \to \mathbb{C}$. Observe that there exists an isomorphism of real vector spaces between $\overline{V}$ and $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ which provides a non-degenerate bilinear form

$$(-,-) : \overline{V} \times V \to \mathbb{R}$$

defined by $(L, v) := \Im(L(v))$.
Therefore, the dual lattice $\Gamma^* = \{L \in \overline{V} | (L, v) \subset \mathbb{Z}\}$ is a lattice in $\overline{V}$. Thus we can define the dual torus $T^* = \overline{V}/\Gamma^*$. It satisfies the property $(T^*)^* = T$. Observe that the non-degeneracy of the bilinear form $(-,-)$ implies that the map $\alpha : \overline{V} \to \mathrm{Hom}(\Gamma, \mathcal{S}^1)$, given by $L \to \exp(2\pi i(L, -))$, is surjective. Moreover, it is not hard to see that the kernel of the latter map is precisely Γ^*.

Lemma 4.2.1. *The map α induces an isomorphism $T^* \simeq Pic^0(T)$.*

Proof. The map α gives an isomorphism between $T^* \simeq \mathrm{Hom}(\Gamma, \mathcal{S}^1)$. Moreover, using the Appel-Humbert theorem, one can prove that there exists an isomorphism $\mathrm{Hom}(\Gamma, \mathcal{S}^1) \simeq Pic^0(T)$, see [33] for more details. □

We are then ready to prove the following theorem.

Theorem 4.2.2. *There exists a unique holomorphic line bundle $\mathcal{P}$ over $T \times T^*$ such that:*

(i) For any line bundle $\mathcal{L} \in T^*$ one has $\mathcal{P}|_{T\times\{\mathcal{L}\}} \simeq \mathcal{L}$;

(ii) $\mathcal{P}|_{\{0\}\times T^*} \simeq \mathcal{O}_{T^*}$.

Moreover, if $\mathbf{A}$ is any normal analytic space and $\mathcal{L}'$ is any line bundle on $T \times \mathbf{A}$ satisfying the conditions:

$$\mathcal{L}'|_{T\times\{x\}} \in Pic^0(T) \text{ for any } x \in \mathbf{A} \quad \text{and} \quad \mathcal{L}'|_{\{0\}\times\mathbf{A}} \text{ is trivial.}$$

Then there exists a unique holomorphic map $\nu : \mathbf{A} \to T^*$ such that $\mathcal{L}' \simeq (id\times\nu)^*\mathcal{P}$.

Proof. We give a sketch of the proof from [33].
We have to define a Hermitian form. We then pose

$$H : (V \times \overline{V})^{\times 2} \to \mathbb{C},$$

where $H((v_1, L_1), (v_2, L_2)) = \overline{L_2(v_1)} + L_1(v_2)$. Observe that by definition H, restricted to $(\Gamma \times \Gamma^*)^{\times 2}$, takes integer values. We also define $f : \Gamma \times \Gamma^* \to \mathcal{S}^1$ as $f(\gamma, L') = \exp(\pi i \cdot \Im(L'(\gamma)))$. According to Theorem 4.1.4, we have just defined a line bundle, that we will denote by $\mathcal{P}$, over $T \times T^*$. The automorphy factors associated with such a bundle is given by

$$a_\gamma((\gamma, L'), (v, L)) = f(\gamma, L') \exp\left(\pi H((v, L), (\gamma, L')) + \frac{1}{2}\pi H\big((\gamma, L'), (\gamma, L')\big)\right).$$

We immediately observe that $a_\gamma((0, L'), (0, L)) = 1$ for all $L \in \Gamma^*$, $L \in \overline{V}$, so the property (ii) is proved. The first one is easy to prove, in fact we take a line bundle $\mathcal{L} \in T^* \simeq Pic^0(T)$, defined by a certain $L \in \Gamma^*$, i.e. $\mathcal{L} = \mathcal{L}(\exp(2\pi i \Im(L)), 0)$. A straightforward computation shows that $\exp(2\pi i \Im(L)$ is equivalent, up to a non-zero holomorphic map, to $a_\gamma((\gamma, 0), (v, L))$, which proves (i). The uniqueness follows easily from the See-saw Theorem 4.8.2.
The final part of the statement follows again from the See-saw Theorem 4.8.2. In fact, define $\nu : \mathbf{A} \to T^*$ by $x \mapsto \mathcal{L}'|_{T \times \{x\}}$, the map is well-defined and unique by the See-saw theorem. It remains to prove that ν is holomorphic, however it is not hard to see that graph(ν) is bi-holomorphic to $\mathbf{A}$, therefore ν is holomorphic. □

Thus, we shall identify $x \in T^*$ with the line bundle $\mathcal{P}|_{T \times \{x\}}$.

4.3 Jacobian of a Riemann surface

In this section we connect the theory of line bundles over a complex torus to that one of locally free sheaves over Riemann surfaces. Let C be a smooth projective curve of genus $g > 0$, i.e. a Riemann surface of genus g. It is well-known that $H_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ and $H^0(C, \Omega_C) \simeq \mathbb{C}^g$. Moreover, as a consequence of Poincaré and De Rham duality, there exists a well-defined non-degenerate intersection form $\langle -, - \rangle$ between elements in $H_1(C, \mathbb{Z})$. Using such an intersection form one can choose a basis $\{a_i, b_i\}_{i=1}^g$ of the first homology group $H_1(C, \mathbb{Z})$ such that

$$\langle a_i, a_j \rangle = 0 = \langle b_i, b_j \rangle \quad \text{and} \quad \langle a_i, b_j \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker delta function.
Once we fix a basis of $H_1(C, \mathbb{Z})$ as above, that is said to be canonical, we can use the Gram-Schmidt algorithm to get a basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(C, \Omega_C)$ such that

$$\int_{a_i} \omega_j = \delta_{ij}.$$

Theorem 4.3.1. *Let C be a compact Riemann surface of genus $g > 0$, $\{a_i, b_j\}_{i=1}^g$ be a canonical basis of $H_1(C, \mathbb{Z})$ and let $\{\omega_1, \dots, \omega_g\}$ be a basis of $H^0(C, \Omega_C)$ such that*

$$\left(\int_{a_i} \omega_j \right)_{i,j=1,\dots,g} = \delta_{ij}.$$

Then the matrix $\mathcal{B} = \left(\int_{b_i} \omega_j \right)_{i,j=1,\dots,g}$ is symmetric and its imaginary part is positive definite. The matrix $\mathcal{B}$ is called period matrix of C.

As a consequence of the latter classical theorem, whose proof can be found for example in [11], we can define the full-rank lattice $\Gamma_C = \mathbb{Z}^g + \mathcal{B}\mathbb{Z}^g$.
We then call Jacobian of C the complex torus

$$J = J(C) := \mathbb{C}^g / \Gamma_C.$$

In order to relate the curve C to its Jacobian we need the next definition.

Definition 4.3.2. Let p_0 be a point of C. We call Abel map with base point p_0 the map $\mathcal{A} : C \to J(C)$ defined by

$$\mathcal{A}(p) = \left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \right) = \left(\int_{p_0}^{p} \vec{\omega} \right).$$

The integration is taken on any path from p_0 to p, since ω_i are holomorphic, therefore closed, the integral depends just on the class of the path in $H_1(C, \mathbb{Z})$. Thus, since we are working on the quotient by the lattice generated by $\{a_i, b_1\}_{i=1}^g$, the definition is well-posed.
The Abel map has the following remarkable properties.

Theorem 4.3.3. *The Abel map is an embedding. Moreover, it induces an isomorphism* $\alpha : Pic^0(C) \to J$*,*

$$\mathcal{O}_C(p_1 + \cdots + p_n - q_1 - \cdots - q_n) \to \sum_{i=1}^{n} (\mathcal{A}(p_i) - \mathcal{A}(q_i)),$$

for any $p_1, \ldots, p_n, q_1, \ldots, q_n \in C$*. The function* α *is called Abel-Jacobi map.*

Proof. This result is well-known and a reference for the proof can be found in [11]. □

4.4 Theta functions and theta divisor

Let $\mathcal{B}'$ be a symmetric $g \times g$ matrix with complex entries and such that its imaginary part is positive definite. We call the holomorphic function $\theta : \mathbb{C}^g \to \mathbb{C}$ defined as follows:

$$\theta(z, \mathcal{B}') = \theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \mathcal{B}' n + 2\pi i n^t z),$$

theta function associated with $\mathcal{B}'$ or simply theta function. One can prove that this series converges uniformly on compact sets and thus it defines a holomorphic function on $\mathbb{C}^g$.
According to the previous section's discussion, we can define the theta function associated with the period matrix $\mathcal{B}$ of a Riemann surface C. Let $\Gamma_C = Z^g + \mathcal{B}\mathbb{Z}^g$, then θ transforms in the following way:

$$\theta(z + \gamma) = \theta(z)$$

$$\theta(z + \mathcal{B}\gamma) = \exp(-\pi i \gamma^t \mathcal{B} \gamma - 2\pi i \gamma z)\theta(z)$$

for any $\gamma \in \mathbb{Z}^g$. So θ is quasi-periodic with respect to the lattice Γ_C.
Let us denote

$$\Theta := \{z \in J(C) | \theta(z) = 0\},$$

such a zero locus is called theta divisor. In fact it is a $(g-1)$-dimensional subvariety of $J = J(C)$. Observe that despite the quasi-periodicity of θ, the variety Θ is well-defined. Moreover, according to Theorem 4.1.5, the function θ is the only section of $\mathcal{O}_J(\Theta)$. In fact one can easily see that in this case $d_1 = \cdots = d_g = 1$. The variety Θ can be described, via the Abel map, as follows.

Theorem 4.4.1. *(Riemann's theorem)([11, 33]) A point* $z \in J$ *belongs to* Θ, *i.e.* $\theta(z) = 0$, *if and only if there exist* $g-1$ *points* $p_1, \dots, p_{g-1} \in C$ *such that*

$$z = K + \sum_{i=1}^{g-1} \mathcal{A}(p_i),$$

where $\mathcal{A}$ *is the Abel map and* $K \in J$ *is the Riemann point. Such a point depends on the choice of the basis of* $H_1(C, \mathbb{Z})$ *and on the base point of the Abel map. However the sum on the right hand side of this theorem does not depend on the choice of the base point of the Abel map.*
Moreover, there exists a unique $\mathcal{K} \in Pic^{g-1}(C)$ *such that* $\mathcal{K}^2 \simeq \Omega_C$ *and, if* $\mathcal{K}$ *is equal to* $\mathcal{O}_C(q_1 + \dots + q_{g-1})$ *for some points* $q_i \in C$, *then* $K = -\sum_{i=1}^{g-1} \mathcal{A}(q_i)$.

We wish to give a different formulation of the latter theorem. Observe that, if we fix an element $\mathcal{L}_d \in Pic^d(C)$, we have an isomorphism

$$\mathcal{L}_d \otimes - : Pic^0(C) \to Pic^d(C))$$

induced by the tensor product. In particular, we get a commutative diagram

$$\begin{tikzcd} & J & \\ Pic^0(c) \arrow[ur, "\alpha"] \arrow[rr, "\mathcal{K}\otimes-"] & & Pic^{g-1}(C), \arrow[ul, "\alpha_K"'] \end{tikzcd}$$

where α_K is the map $\mathcal{O}_C(D_{g-1}) \to -K + \mathcal{A}(D_{g-1})$ and where D_{g-1} is a divisor of degree $g-1$. Moreover, the map α_K gives another commutative diagram

$$\begin{tikzcd} Pic^{g-1}(C) \arrow[r, "\alpha_K"] \arrow[d, "i^\vee"'] & J \arrow[d, "i"] \\ Pic^{g-1}(C) \arrow[r, "\alpha_K"] & J, \end{tikzcd}$$

where $i^\vee(\mathcal{L}) = \mathcal{L}^\vee$ and $i(z) = -z$.
Furthermore, if we denote

$$W = \{\mathcal{L} \in Pic^{g-1}(C) | h^0(C, \mathcal{L}) > 0\},$$

the Riemann's theorem can be translated as follows

Lemma 4.4.2. *W is a divisor in $Pic^{g-1}(C)$ and $\alpha_K(W) = \Theta$.*

We conclude this section observing that we also get an isomorphism

$$\{\mathcal{L} \in Pic^{g-1}(C) | \mathcal{L}^2 \simeq \Omega_C\} \xrightarrow{\alpha_K} \{z \in J | 2z = 0\}$$

which sends $\mathcal{K}$ to 0. Elements of both sets are called theta characteristic.

4.5 Poincaré bundle over a Riemann surface

We wish to use the Poincaré bundle over a torus in order to define a universal line bundle over a Riemann surface C. We need some preliminary results.

Proposition 4.5.1. *There exists an isomorphism $\Phi_\Theta : J \to J^* = Pic^0(J)$ defined by*

$$x \to t_x^*(\mathcal{O}_j(\Theta)) \otimes \mathcal{O}_J(-\Theta),$$

where $t_x : J \to J$ is given by $c \to x + c$.

Proof. See [7, Theorem 2.8] or [33, Section 2.4]. □

We also need the following:

Theorem 4.5.2. *(i) The Abel map $\mathcal{A} : C \to J$ induces an isomorphism*

$$\mathcal{A}^* : Pic^0(J) \to Pic^0(C).$$

(ii) The following diagram is commutative

$$\begin{tikzcd} Pic^0(C) \arrow[r, "(\mathcal{A}^*)^{-1}"] \arrow[d, "\alpha"'] & J^* \\ J \arrow[r, "i"] & J. \arrow[u, "\Phi_\Theta"'] \end{tikzcd}$$

(iii) For any $x \in J$ let $\mathcal{L}_x \in Pic^0(C)$ be the corresponding element via the Abel-Jacobi map. Then we have

$$\mathcal{A}^*(t_{-x}^*\mathcal{O}_J(\Theta)) \simeq \mathcal{L}_x^{-1} \otimes \mathcal{K} \otimes \mathcal{O}_C(p_0),$$

where p_0 is the base point of the Abel map.

Proof. See [33] for a proof of these classical results. □

The last theorem implies that $\mathcal{Q} := (\mathcal{A} \times \mathcal{A}^{\#})^* \mathcal{P}$, where $\mathcal{A}^{\#} = (\mathcal{A}^*)^{-1}$, is a universal family over $C \times Pic^0(C)$ such that:

(i) $\mathcal{Q}|_{\{p\} \times Pic^0(C)} \simeq \mathcal{O}_{Pic^0(C)}$;

(ii) For any $\mathcal{L} \in Pic^0(C)$ we denote $\tilde{\mathcal{L}} = \mathcal{A}^{\#}(\mathcal{L})$. Then

$$\mathcal{A}^*(\mathcal{P}|_{\tilde{\mathcal{L}} \times J}) = \mathcal{A}^*(\tilde{\mathcal{L}}) \simeq \mathcal{L}.$$

Similarly, we get a universal family $\mathcal{R} := \mathcal{Q} \otimes \pi_C^*(\mathcal{K})$ over $C \times Pic^{g-1}(C)$, where π_C is the canonical projection along the first term.
Denote $\mathcal{R}^\vee := \mathcal{H}om(\mathcal{R}, \pi_C^* \Omega_C)$, take $\delta(x,y) : C \times C \to J$ defined by $\delta(x,y) = \int_y^x \vec{\omega}$ and denote $\tilde{\Delta} = \Delta \times Pic^{g-1}(C)$. Here Δ is the diagonal in $C \times C$. Finally, recall that $i : J \to J$ is the involution $i(v) = -v$, then the following holds.

Theorem 4.5.3. *There exists an isomorphism*

$$\mathcal{R}^\vee \boxtimes \mathcal{R}(\tilde{\Delta}) \simeq (\delta \times \alpha_{\mathcal{K}})^* \mathbf{Q}, \tag{4.1}$$

where $\mathbf{Q} := (id \times i)^*(id \times \Phi_\Theta)^* \mathcal{P} \otimes \pi_1^*(\mathcal{O}_J(\Theta))$ *and* $\pi_1 : J \times J \to J$. *Moreover, denote* $C^4 = C \times C \times C \times C$ *and consider*

$$C^4 \times Pic^{g-1}(C) \xrightarrow{\delta^{(2)} \times \alpha_{\mathcal{K}}} J \times J,$$

where $\delta^{(2)}(x_1, x_2, y_1, y_2) = \int_{y_1+y_2}^{x_1+x_2} \vec{\omega}$. *Then*

$$(\mathcal{R}^\vee \boxtimes \mathcal{R})^{\otimes 2}(\tilde{\Delta}^2) \simeq (\delta^{(2)} \times \alpha_{\mathcal{K}})^* \mathbf{Q}, \tag{4.2}$$

where $\tilde{\Delta}^2 := Pic^{g-1}(C) \times (\Delta_{12} + \Delta_{14} + \Delta_{23} + \Delta_{34} - \Delta_{13} - \Delta_{24})$ *and* Δ_{ij} *is the divisor in* C^4 *where the entries* i *and* j *coincide.*

Proof. The results stated in this theorem seem to be known, however we could not find a good reference in the literature, so we provide a proof.
First of all observe that

(i) Since $\mathbf{Q}$ comes from the Poincaré bundle of a torus, we have

$$\mathbf{Q}|_{\{0\} \times J} \simeq \mathcal{O}_J.$$

(*ii*) Moreover, using Proposition 4.5.1, we deduce

$$\mathbf{Q}|_{J\times\{\xi\}} \simeq t^*_{-\xi}(\mathcal{O}_J(\Theta))$$

for any $\xi \in J$.

Now we recall a variant of the See-Saw Theorem from [34, chapter III, Section 10].

Theorem 4.5.4. *Let X_1, X_2 and X_3 be varieties such that X_1 and X_2 are complete whereas the last one is connected. Let L be a line bundle over the product $X_1 \times X_2 \times X_3$ and suppose there exists a triple $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ such that the bundles*

$$L|_{\{x_1\}\times X_2\times X_3}, \quad L|_{X_1\times\{x_2\}\times X_3} \quad \text{and} \quad L|_{X_1\times X_2\times\{x_3\}}$$

are trivial. Then $L \simeq \mathcal{O}_{X_1\times X_2\times X_3}$.

We use the latter theorem to prove the equation (4.1).
(1) Let $\tau \in Pic^{g-1}(C)$ be a non-singular odd characteristic, i.e. $\tau^2 \simeq \Omega_C$, $h^0(C,\tau) = 1$ and $\lambda := \alpha(\tau - K) \in J$ is such that $2\lambda = 0$ (or equivalently $\lambda = -\lambda$). Mumford in [36] proved that such a τ always exists. Clearly the LHS of (4.1) gives

$$(\mathcal{R}^\vee \boxtimes \mathcal{R}(\tilde{\Delta}))|_{C\times C\times\{\tau\}} \simeq (\mathcal{L}_\tau \boxtimes \mathcal{L}_\tau)(\Delta),$$

where $\mathcal{L}_\lambda$ is the line bundle corresponding to τ via the universal family $\mathcal{R}$. The RHS of (4.1) gives, according to what we said at the beginning of this proof,

$$\delta^*(t^*_{-\lambda}(\mathcal{O}_J(\Theta)).$$

Therefore, observing that $\lambda = -\lambda$ and $\mathcal{L}_\tau \simeq \mathcal{L}_\tau^\vee$, equation (4.1) give

$$\delta^*(t^*_{\lambda}(\mathcal{O}_J(\Theta)) \simeq (\mathcal{L}_\tau \boxtimes \mathcal{L}_\tau)(\Delta).$$

We are allow to write they are isomorphic according to what we will see regarding the prime form in Section 4.6.
In fact we will prove that $\theta[\tau](x-y)$ has $D \times C + C \times D + \Delta$ as divisors of zeroes. Here D is a divisor such that $\tau \simeq \mathcal{O}_C(D)$ and $\theta[\tau]$ is the translated of theta by the vector $\lambda \in J \simeq Pic^0(C))$. The statement follows from See-saw Theorem 4.8.2 applied to $\{p\} \times C$ and $C \times \{p\}$.
(2) We now consider the restriction to $C \times \{p\} \times Pic^{g-1}(C)$. One can easily see

that, since $\mathcal{R}|_{\{p\}\times Pic^{g-1}(C)}$ is trivial, the LHS is isomorphic to $\mathcal{R}^\vee \otimes \pi_C^*(\mathcal{O}_C(p))$. Observe that we have a diagram

$$\begin{tikzcd} C \times C \times Pic^{g-1}(C) \arrow[r, "\delta\times\alpha_K"] & J \times J \\ C \times Pic^{g-1}(C) \arrow[u, hook] \arrow[r, "\mathcal{A}\times\alpha_K"] & J \times J. \arrow[u, hook, "id\times id"'] \end{tikzcd}$$

We have to prove that

$$(\mathcal{A} \times \alpha_K)^*(\mathbf{Q}|_{C\times J}) \simeq \mathcal{R}^\vee \otimes \pi_C^*(\mathcal{O}_C(p)).$$

Clearly a further restriction to $\{q\} \times Pic^{g-1}$ gives the trivial line bundle on both sides. We then take $\zeta \in Pic^{g-1}(C)$, so that $\mathcal{R}^\vee|_{C\times\{\zeta\}} = \zeta^{-1} \otimes \Omega_C$. We can now write $\zeta = \mathcal{K} \otimes \nu$, where $\nu \in Pic^0(C)$. Thus $\mathcal{R}^\vee|_{C\times\{\zeta\}} = \nu^{-1} \otimes \mathcal{K}$. Therefore, using Theorem 4.5.2, we get

$$\mathcal{A}^*(t_{-\nu}^*(\mathcal{O}_J(\Theta))) \simeq \mathcal{L}_\nu^{-1} \otimes \mathcal{K} \otimes \mathcal{O}_C(p).$$

The latter condition implies the result once we use the See-saw Theorem 4.8.2.
(3) The third point follows from the commutativity of the diagram:

$$\begin{tikzcd} C \times C \times Pic^{g-1}(C) \arrow[r, "\delta\times\alpha_K"] \arrow[d, "\tau\times i^\vee"'] & J \times J \arrow[d, "i\times i"] \\ C \times C \times Pic^{g-1}(C) \arrow[r, "\delta\times\alpha_K"] & J \times J, \end{tikzcd}$$

where $i(x) = -x$, $\tau(x, y) = (y, x)$ and $i^\vee(\mathcal{L}) = \mathcal{L}^\vee$. In fact the Poincaré bundle $\mathcal{P}$ on $J \times J^*$ satisfies, due to the self-duality of $(J^*)^*$, both relation $(i \times id)^*\mathcal{P} \simeq \mathcal{P}^*$ and $(id \times i)^*\mathcal{P} \simeq \mathcal{P}^*$. Therefore

$$i^*(\mathcal{O}_J(\Theta)) \simeq \mathcal{O}_J(\Theta).$$

So the third point can be reduced to the second one.
The equation (4.2) can be proved using the See-Saw Theorem 4.8.2 over the space $C^4 = C^2 \times C^2$ and using relation (4.1).

$\square$

Remark 4.5.5. A remark about the notation is needed. For any $\beta \in Pic^{g-1}(C)$, we have the corresponding $\beta' \in Pic^0(C)$, given by $\mathcal{K} \otimes \beta' = \beta$, as well as $\beta'' \in J$ defined by $\alpha(\beta') = \beta''$. The last theorem implies that

$$(\mathcal{R}^\vee \boxtimes \mathcal{R})^{\otimes 2}(\tilde{\Delta}^2)|_{\{\beta\}\times C\times C\times\{t_0\}\times\{t_0\}} \simeq \mathcal{L}_\beta^\vee \boxtimes \mathcal{L}_\beta(\Delta).$$

Moreover

$$\mathcal{L}_\beta^\vee \boxtimes \mathcal{L}_\beta(\Delta) \simeq \delta^*(t^*_{-\beta''}\mathcal{O}_J(\Theta)), \tag{4.3}$$

where $\mathcal{L}_\beta$ and $\mathcal{L}_\beta^\vee$ are line bundles corresponding to β via the universal families $\mathcal{R}$ and $\mathcal{R}^\vee$, respectively.
Then we denote by $\theta[\beta'']$ the unique (up to scalar) section of the bundle $t^*_{\beta''}(\mathcal{O}_J(\Theta))$. Observe that any β'' can be written as $\beta'' = \mathcal{B}a_1 + a_2$, with $a_1, a_2 \in \mathbb{R}$. Therefore, after a straightforward computation, we have

$$\theta[\beta''](z) = \theta[a_1, a_2](z) = \exp(\pi i a_1 \mathcal{B} a_1 + 2\pi i a_1(z + a_2))\theta(z + \mathcal{B}a_1 + a_2).$$

We pose, with an abuse of notation, $\theta[\beta] := \theta[-\beta'']$; so that $\theta[\beta]$ is considered as a section of $\delta^*(t^*_{-\beta''}\mathcal{O}_J(\Theta)) \simeq \mathcal{L}_\beta^\vee \boxtimes \mathcal{L}_\beta(\Delta)$.

4.6 Prime form

Let $\overline{P_1}, \ldots, \overline{P}_{g-1}$ be points of C such that $h^0(C, \tau) = 1$ and $\tau^2 \simeq \Omega_C$, where $\tau = \mathcal{O}_C(\overline{P_1} + \cdots + \overline{P}_{g-1})$. Mumford [36] proved that such a τ always exists.

Lemma 4.6.1. *If τ as above, then the divisor of zeroes of*

$$\theta[\tau](x - y)$$

is given by

$$(\overline{P_1} + \cdots + \overline{P}_{g-1}) \times C + C \times (\overline{P_1} + \cdots + \overline{P}_{g-1}) + \Delta,$$

where Δ is the diagonal in $C \times C$

Proof. We follow [36] for this proof.
According to Riemann's theorem, $\theta[\tau](0)$ if and only if $h^0(C, \tau(y - x)) > 0$ or equivalently if and only if $\tau(y - x) \in W$. If $h^0(C, \tau) = 1$, then $h^0(C, \tau(y))$ is either 1 or 2.

(i) If $h^0(C,\tau(y)) = 1$, then there exists a section whose zeroes are $y, \overline{P_1}, \dots, \overline{P}_{g-1}$. Since $h^0(C,\tau(y-x)) > 0$ still has to hold, we deduce that such a section has to vanish in x. Thus $x \in \{y, \overline{P_1}, \dots, \overline{P}_{g-1}\}$; which gives part of the thesis.

(ii) Suppose $h^0(C,\tau(y)) = 2$. Then, using Riemann-Roch theorem and the isomorphism $\tau^2 \simeq \Omega_C$, we deduce $h^0(C,\tau(-y)) = 1$. Hence the unique section of τ has to vanish in y, therefore $y \in \{\overline{P_1}, \dots, \overline{P}_{g-1}\}$.

□

Remark 4.6.2. The previous argument fills the missing details in the proof of Theorem 4.5.3.

Observe that, due to Theorem 4.5.3 and following its notation, we can consider $\theta[\tau](x-y)$ as a section of the bundle $\mathcal{L}_\tau \boxtimes \mathcal{L}_\tau(\Delta)$. We now take $s_\tau \in H^0(C,\mathcal{L}_\tau)$, observe that, since $\mathcal{L}_\tau \simeq \mathcal{L}_\tau^\vee$, one has $s_\tau \in H^0(C,\mathcal{L}_\tau^\vee)$. Moreover s_τ has $\overline{P_1}, \dots, \overline{P}_{g-1}$ as divisor of zeroes. Using Künneth formula $s_\tau(-) \cdot s_\tau(-)$ can be considered as a section of $\mathcal{L}_\tau \boxtimes \mathcal{L}_\tau$. Thus $\frac{\theta[\tau](x-y)}{s_\tau(x)s_\tau(y)}$ can be regarded as a holomorphic section of $\mathcal{O}(\Delta)$ on $C \times C$.

Definition 4.6.3. We call prime form the expression

$$E(x,y) = \frac{\theta[\tau](x-y)}{s_\tau(x)s_\tau(y)}. \tag{4.4}$$

Remark 4.6.4. Observe that since $\tau^2 \simeq \Omega_C$ the element s_τ^2 is a global 1-form. Thus

$$\operatorname{res}_\Delta \frac{1}{E(x,y)}$$

is well-defined. Furthermore, we suppose the prime form to be normalized in order to have $\operatorname{res}_{x=y} \frac{1}{E(x,y)} = 1$.

Let $\tau \in Pic^{g-1}(C)$ such that $h^0(C,\tau) = 1$ and $\tau^2 \simeq \Omega_C$, then the ratio

$$\frac{E(t,x)}{E(t,y)} \tag{4.5}$$

can be considered a meromorphic section of the line bundle $\mathcal{O}_C(x)$ whose only pole is in y, for any $x, y \in C$.

Corollary 4.6.5. *Let $\mathcal{E}$ be a vector bundle over C such that $H^0(C,\mathcal{E}) = 0$ and $H^1(C,\mathcal{E}) = 0$. Let $x_0, x_1, y_0, y_1 \in C$ be four distinct points. Then the following relation is true:*

$$\mathbf{S}^{\mathcal{E}(x_1-y_1)}(x_0,y_0)\frac{E(x_0,x_1)}{E(x_0,y_1)}\frac{E(y_0,y_1)}{E(y_0,x_1)} = \mathbf{S}^{\mathcal{E}}(x_0,y_0) - \mathbf{S}^{\mathcal{E}}(x_1,y_0)\mathbf{S}^{\mathcal{E}}(x_1,y_1)^{-1}\mathbf{S}^{\mathcal{E}}(x_0,y_1).$$

Proof. It follows from Theorem 3.6.3 observing that $s(x_0)s(y_0)^{-1}$ is equal to $\frac{E(x_0,x_1)}{E(x_0,y_1)}\frac{E(y_0,y_1)}{E(y_0,x_1)}$. In fact $\mathrm{ev}_{x_0}(\mathrm{res}_{x_1}^{-1}(\frac{E(t,x)}{E(t,y)})) = \frac{E(x_0,x_1)}{E(x_0,y_1)E(x_1,y_1)}$. Similarly we have $\mathrm{ev}_{y_0}(\mathrm{res}_{x_1}^{-1}(\frac{E(t,x)}{E(t,y)})) = \frac{E(y_0,x_1)}{E(y_0,y_1)E(x_1,y_1)}$. □

The relation above is equivalent to the matrix-valued Fay's trisecant identity which appears in [21, 22, 43].

4.7 Fay's trisecant identity

Let $\lambda \in Pic^{g-1}(C)$ and $\mathcal{L}_\lambda$ be the corresponding line bundle, under the isomorphism (4.1) restricted to $\{x\} \times C \times \{\lambda\}$, $x \in C$, such that $H^0(C,\mathcal{L}_\lambda) = 0$. Then, using Riemann-Roch theorem, one can easily see that $H^1(C,\mathcal{L}_\lambda) = 0$. Therefore the element $\mathbf{S}^\lambda(x,y)$ as in (3.9) is well-defined. Moreover, according to Remark 4.5.5, we can consider the function $\theta[\lambda](x-t)$ as a section of the bundle $\mathcal{L}_\lambda(x)$. Thus, due to what was said about the prime form (4.4), the element

$$s(t) = \frac{\theta[\lambda](x-t)}{E(x,t)} \tag{4.6}$$

can be thought as a, necessarily meromorphic, section of the bundle $\mathcal{L}_\lambda$ whose divisor of poles is supported at x. In particular we can compute the residue map of $s(t)$. In fact, due to Remark 4.6.4 and the fact that $\theta[\lambda](0) = 0$ if and only if $h^0(C,\mathcal{L}_\lambda) > 0$, we have

$$\mathrm{res}_x(s(t)) = \theta[\lambda](0) \neq 0.$$

Thus, the element $\mathbf{S}^\lambda(x,y) = \mathrm{ev}_y(\mathrm{res}_x^{-1}(s))$ is given by

$$\mathbf{S}^\lambda(x,y) = \frac{\theta[\lambda](x-y)}{E(x,y)}\frac{1}{\theta[\lambda](0)}. \tag{4.7}$$

In a similar way one can take the universal family (4.2) in order to get

$$\mathbf{S}^{\lambda(x_1-y_1)}(x,y) = \frac{\theta[\lambda](x-y+x_1-y_1)}{E(x,y)}\frac{1}{\theta[\lambda](x_1-y_1)}.$$

Corollary 4.7.1. *Let $\lambda \in Pic^{g-1} \setminus W$ and $x, y, x_1, y_1 \in C$. Then the following relation holds true*

$$E(x,y)E(x_1,y_1)\theta[\lambda](x-y_1)\theta[\lambda](x_1-y)+E(x,y_1)E(y,x_1)\theta[\lambda](x-y)\theta[\lambda](x_1-y_1) =$$

$$= E(x,x_1)E(y,y_1)\theta[\lambda](x-y+x_1-y_1)\theta[\lambda](0)$$

Proof. The proof follows from (4.6.5) taking $\mathbf{S}^\lambda$ as in (4.7), reordering terms and using $E(p,q) = -E(q,p)$ for all $p, q \in C$. □

The latter is the Fay's trisecant identity, it appeared for the first time in [20].

4.8 The case $g = 1$ and the AYBE

We now suppose $g = 1$ so that C is an elliptic curve. Moreover, the choice of a basis of $H_1(C,\mathbb{Z})$ fixes a nowhere vanishing element ω of $H^0(C,\Omega_C)$ and therefore an isomorphism $\Omega_C \simeq_\omega \mathcal{O}_C$. In this setting the isomorphism (4.3), for $\tau \in Pic^0(C)$ such that $\mathcal{L}_\tau^2 \simeq \mathcal{O}_C$ and $H^0(C,\mathcal{L}_\tau) = 1$, reduces to

$$\delta^*(\mathcal{O}_J(\Theta)) \simeq \mathcal{O}_C(\Delta).$$

Over a complex torus, the unique τ with the properties above gives the theta function

$$-\theta_1(z) = -2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} sin(2(n+1)\pi z),$$

where $q = \exp(\pi i \mathcal{B})$ and $\mathcal{B}$ is the period matrix which reduces to a complex number whose imaginary part is positive.
Thus the prime form is given by

$$E(x,y) = \frac{\theta_1(x-y)}{\theta_1'(0)},$$

where the denominator is due to the normalization with respect to the residue as in Remark 4.6.4.
Since the theta divisor is sent to the zero element in $Pic^0(C)$, that one whose section is given by θ_1, any element $\lambda \in Pic^0(C)$, $\lambda \neq 0$, gives a line bundle with

vanishing cohomology. Thus we can compute the element $\mathbf{S}^\lambda$. Observe that, since $\theta_1(0) = 0$, a section of $t^*_{-\lambda}(\Theta)$ is given by $\theta_1(x-y-\lambda)$ so that:

$$\mathbf{S}^\lambda(x,y) = \frac{\theta_1(x-y-\lambda)\theta_1'(0)}{\theta_1(-\lambda)\theta_1(x-y)}.$$

We now observe that from Theorem 4.5.3, which in particular implies that θ_1 is an odd function, we have

$$\frac{\theta_1(x-y-\lambda)\theta_1'(0)}{\theta_1(-\lambda)\theta_1(x-y)} = -\frac{\theta_1(y-x+\lambda)\theta_1'(0)}{\theta_1(\lambda)\theta_1(y-x)}.$$

Therefore the *RHS* of the equation above is equal to the Kronecker function κ which appears in Theorem 2.10. A straightforward computation shows that in this case the relation in Corollary 4.7.1 reduces to

$$\frac{\theta_1(x-x_1)\theta_1(y-y_1)}{\theta_1(x-y_1)\theta_1(y-x_1)}\theta_1(\lambda)\theta_1(\lambda+y-x+y_1-x_1) - \theta_1(\lambda+y_1-x_1)\theta_1(\lambda+y-x)+$$
$$+\frac{\theta_1(y_1-x_1)\theta_1(y-x)}{\theta_1(y_1-x)\theta_1(y-x_1)}\theta_1(\lambda+y_1-z)\theta_1(\lambda+y-x_1) = 0.$$

Moreover, the latter identity is the scalar Yang-Baxter equation

$$\kappa(u;v)\kappa(u+u';v') = \kappa(u+u';v+v')\kappa(-u';v) + \kappa(u';v')\kappa(u;v+v'),$$

where $u = y-x$, $v = \lambda$, $u' = x-y_1$ and $v' = y_1 - x_1$.

Remark 4.8.1. In the case of line bundles over elliptic curves, the function $\mathbf{S}$ and the associative $r-$matrix as in (2.10) can be both identify with the Kronecker function κ. Moreover, the Fay's identity reduces to the scalar associative Yang-Baxter equation.

SEE-SAW THEOREM

In this appendix we provide a proof, slightly readjusted, of the See-Saw theorem which appears in [33].

Theorem 4.8.2. *Let V and W be complex varieties such that V is complete and let $\mathcal{L}$ be a line bundle over $V \times W$. Then:*

(i) the set

$$W_{\mathcal{L}} = \{w \in W | \mathcal{L}|_{V\times\{w\}} \textit{is trivial}\}$$

is Zaritsky closed in W;

(ii) there exists a line bundle $\mathcal{N}$ over $W_{\mathcal{L}}$ such that $\mathcal{L}|_{V\times W_{\mathcal{L}}} \simeq \pi_2^*\mathcal{N}$, where $\pi_2 : V \times W \to W$;

(iii) If V and W are compact complex manifolds and $\mathcal{L}$ is a holomorphic line bundle such that $\mathcal{L}|_{V\times\{w\}}$ and $\mathcal{L}|_{\{v\}\times W}$ are trivial respectively for any $v \in V$ and $w \in W$. Then $\mathcal{L}$ is trivial.

Proof. (i) Observe that on any complete variety M a line bundle L is trivial if and only $h^0(M, L)$ and $h^0(M, L^*)$ are both greater than 0. In particular

$$W_{\mathcal{L}} = \{w \in W | h^0(\mathcal{L}|_{V\times\{w\}}) > 0 \ \text{ and } \ h^0(\mathcal{L}^*|_{V\times\{w\}}) > 0\}.$$

Using the semi-continuity theorem for cohomology, one immediately deduces that $W_{\mathcal{L}}$ has to be closed.

(ii) We now consider $\mathcal{N} = \pi_{2*}(\mathcal{L}|_{X\times W_{\mathcal{L}}})$. For any $w \in W$ there exists an isomorphism, provided by the base change theorem, $f : R^0\mathcal{N}(w) \to H^0(\mathcal{L}|_{X\times\{w\}}) \simeq \mathbb{C}$. We now observe that, since f is surjective, the function

$$H^0(\pi_{2*}\mathcal{L}|_{X\times\{w\}}) \to H^0(\mathcal{L}|_{X\times\{w\}})$$

is surjective for all $w \in W$ as well. The second part of the theorem is now proved, i.e.

$$\pi_2^*(\mathcal{N}) \simeq \mathcal{L}|_{X \times W_{\mathcal{L}}}.$$

(iii) The last point is a simple corollary of the first two points.

□

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