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Pareto Efficiency in Weighted School Choice Problems

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Abstract

There are a number of school choice problems in which students are heterogeneous according to the number of seats they occupy at the school they are assigned to. We propose a weighted school choice problem by assigning each student a so-called weight and extend the top trading cycles algorithm to fit to this extension. We call the new mechanism the weighted TTC and show that it is strategy-proof and results in a Pareto efficient matching. Therefore, the TTC is robust towards the introduction of weights. Nevertheless, it is more complex to guarantee each student a seat at a school, as the extension introduces a trade-off between weights and priorities.

Keywords: Matching, School Choice, College Admission Problems, Top Trading Cycles, Pareto Efficiency, Strategy-Proofness

JEL: C78, D47

1 Introduction

Allocating children to daycare facilities is an important topic across Europe. In Germany, for example, since 2013 all children aged one year or older are entitled to daycare supervision.[1] Nevertheless, the execution of this law is difficult as there are not enough daycare spots for all children. Moreover, the allocation processes are mostly decentrally organized and lead to some unwanted consequences, such as parents having to register for a daycare spot even before their child is born (Carlsson & Thomsen, 2015). These issues in allocation processes lead to various social and economic problems. For example, parents, especially mothers, of children who have not found a spot yet, are not able to return to work (Chevalier & Viitanen, 2002). As childhood education is especially important for children from disadvantaged socio-economic backgrounds (Heckman & Masterov, 2007), not being allocated to a daycare spot might affect these children's future. Naturally, the question arises how to improve the allocation process to improve the resulting allocations. Therefore, we start by having a look at centralized allocation mechanisms that were introduced in the school choice literature.

In school choice problems, as introduced by Abdulkadiroğlu & Sönmez (2003), students are matched to schools. Students submit preferences of schools and schools compare students according to some priorities. This priority ordering is based on state and local laws. Thus, in school choice only the students are considered as agents with preferences while the schools are treated as (unstrategic) objects. A matching is then produced by a centralized mechanism. As Abdulkadiroğlu & Sönmez (2003) point out, the resulting matching can either be Pareto efficient for the students or stable. The Pareto efficient matching is found using the top trading cycles algorithm (TTC), the stable one by using the deferred acceptance algorithm (DA). Both algorithms are strategy-proof. Consequently, there is a trade-off between Pareto efficiency and stability (Abdulkadiroğlu & Sönmez, 2003). On the one hand, while the DA finds the student-optimal stable outcome that Pareto dominates all other

[1] You can find the particular law under §24 Abs. 2 SGB VIII.

stable matchings, it can still be Pareto dominated by some other unstable outcome. The TTC on the other hand does not take any fairness considerations into account and, thus, might not prevent students from justifiably envying students at other schools.[2]

Unfortunately, both mechanisms do not take into account that students might be heterogeneous, so they are not necessarily useful in our setting. In Germany, for example, staff-to-child ratios in daycare facilities differ with the age of children (younger children need more care than older ones); the same is true for Japan (Kamada & Kojima, 2019). The staff-to-child ratio might also differ with the health status (disabled children need more care in comparison to the others).[3] A daycare facility with a fixed amount of personnel thus has the possibility to admit a small number of very young children, a large number of older children or a medium number of children; some of them older, some of them younger. Furthermore, matching students to supervisors to write a final thesis is another example of a school choice problem with heterogenous agents, as the supervision of a Master thesis is costlier and more time consuming than the supervision of a Bachelor thesis. Again, the supervisors have a fixed overall capacity that can be filled with a mixture of Bachelor and Master students (Hoyer & Stroh-Maraun, 2020). To find Pareto efficient or stable outcomes with the help of matching mechanisms that deal with this heterogeneity we need to answer the following questions: how can this heterogeneity be incorporated into a matching market? Can we find an economically reasonable outcome with the help of a centralized matching mechanism if we allow agents to differ not only according to their preferences but also according to their need of care and supervision?

To find answers, we model a weighted school choice problem. Therefore, we extend the original model by introducing students' weights. We find that in this extended setting, a variant of the TTC still reaches Pareto efficient and strategy-proof outcomes for the students. Nevertheless, students with higher weights might have a disadvantage in comparison to students with smaller

[2] Please note that all Pareto efficiency considerations are just based on the students.

[3] More detailed data can be found in European Commission/EACEA/Eurydice/Eurostat (2014)

weights when being assigned to a preferred school.

So far, weighted matching problems are studied only in terms of weighted college admissions problems (Hoyer & Stroh-Maraun, 2019) or matching markets with sizes (Biró & McDermid, 2014). These models differ from weighted school choice problems mainly in the interpretation of the schools' role (Abdulkadiroğlu & Sönmez, 2003). In contrast to (weighted) school choice problems, (weighted) college admissions problems and also matching markets with sizes consider a two-sided matching market, where both sides are strategic agents with preferences (Abdulkadiroğlu & Sönmez, 2003). As a result, the analysis of weighted matching problems concentrates on stability aspects. A stable matching can always be found in a college admissions problem by using the deferred acceptance algorithm (Gale & Shapley, 1962). In contrast to this result, in a weighted college admissions problem, stability is not assured (McDermid & Manlove, 2010) and even if it does exist, a stable matching cannot be found by using the deferred acceptance algorithm (Hoyer & Stroh-Maraun, 2019). Dean et al. (2006) assign different sized jobs to machines and try to find the job-optimal stable outcome among all minimally congested stable matchings. Delacrétaz (2019) relaxes the stability notion in matching markets with sizes. Stability is also studied in matching markets with couples (Roth, 1984), where couples submit a joint preference list in a matching market for medical interns. Roth (1984) shows that a stable matching may not exist in such a setting. In contrast to our model, a couple need not be matched to the same school or hospital. Summing up these findings, by introducing heterogeneity on one side of the matching market, stability is no longer assured.

In contrast to these works we focus on a weighted school choice model where schools are not treated as strategic agents but as objects with a priority structure. Therefore, stability may no longer be the most important property of a matching in such a setting. Instead policy makers and economists face a trade-off between stability and Pareto efficiency and have to decide which of the two properties they want to focus on (Abdulkadiroğlu & Sönmez, 2003). To find a Pareto efficient matching, the TTC algorithm was introduced (Abdulkadiroğlu & Sönmez, 2003) which is based on Gale's top trading cycles. This algorithm finds the unique core allocation (Roth & Postlewaite, 1977) in the context of housing markets (Shapley & Scarf, 1974). In this work, we are focusing on finding a Pareto efficient outcome which is still possible with the help of a variant of the TTC in a weighted school choice problem.

While weights have not been studied in the context of school choice there are some related problems. Combe (2018) formulates a teacher assignment problem where some teachers have initial assignments. Hamada et al. (2017) focus on a school choice model with minimum quotas and initial endowments of students. Dur & Wiseman (2019) define school choice with neighbors. Here, students have preferences over schools when they are matched to these schools alone and over schools when they are matched to these schools together with a neighbor. All three works show that while in these cases stability is also no longer assured, variations of the TTC still yield a Pareto efficient and strategy-proof outcome. While the last of the three models is closer to ours than the other two, there are still some striking differences. A student together with the neighbor can be interpreted as one student with a higher weight. In this case the student-neighbor pair cannot be separated by schools or schools' priorities. The pair then has to submit a joint preference list and is not able to be matched to a school without their neighbor. This is in contrast to the neighbors model in which both students can be assigned to different schools. Also the preferences of the student and neighbor can differ from one another, which is not the case in the weighted case. Thus, our model is more restrictive on the one hand. On the other hand, it is more flexible as weights are not restricted to 1 or 2 but a student can have a weight of any real number. Our model is also connected to the so-called type-specific quotas that were introduced by Abdulkadiroğlu & Sönmez (2003) to account for controlled choice, which should maintain the ethnical balance in the United States. They introduce additional constraints to ensure that a minimum amount of each school's quota is filled by a certain ethnical group although this might introduce inefficiencies. The authors show that the TTC is strategy-proof in this case and still results in a constrained efficient matching.

The paper is organized as follows. In Section 2 we introduce the weighted school choice problem before showing the results in Section 3 by firstly considering a special case (Section 3.1) before introducing the weighted top trading cycles mechanism and its properties (Section 3.2). We discuss the consequences of this adjustment in Section 4 and conclude briefly in Section 5.

2 Weighted School Choice Model

Our model relies on the school choice model as introduced by Abdulkadiroğlu & Sönmez (2003). Let $I = \{i_1, \ldots, i_m\}$ and $S = \{s_1, \ldots, s_n\}$ be finite, disjoint sets, and let us call them students and schools, respectively. Each school s has a quota or capacity q_s with $q = (q_{s_1}, \ldots, q_{s_n})$ up to which it can accept students. Furthermore, each student $i \in I$ has strict preferences P_i over the set $S \cup \{i\}$ of schools and the possibility of staying unmatched. sP_is' indicates that s is strictly preferred over s', iP_is' indicates that school s' is unacceptable, i.e., student i prefers staying unmatched over being matched to school s'. Schools have strict priorities π over students. Each school s has a complete, irreflexive and transitive binary priority π_s over $I \cup \{s\}$. Thus, $i\pi_si'$ means that i has strictly higher priority than i' at school s.[4] Additionally, each student has a weight $p_i \in \mathbb{R}$. We call $p = (p_{i_1}, \ldots, p_{i_m})$ the vector of weights associated with the students $i_1, \ldots, i_m$. In the classical model, all students are assumed to have the same weight, where each student takes up exactly one slot at a school, $p_i = 1 \;\; \forall\, i \in I$. We assume that the student $\underline{i}$ with the smallest weight takes up exactly one slot at a given college ($p_{\underline{i}} = 1$), student $\bar{i}$ has the highest weight and takes up $p_{\bar{i}} = a \cdot p_{\underline{i}}$ slots at any given college, where $a \in \mathbb{R}_{>1}$. Furthermore, if two students i and j have the same weight $p_i = p_j$, we say they are of the same type. There are d different weights or types, $T = \{1, \ldots, d\}$ with $d \leq m$. The family of sets $T^1, \ldots, T^T$ is a partition of the set of students I in such a way that all students i that are elements of T^t are of type t and have the same weight, denoted as $p^t = p_i \,\forall\, i \in T^t$. A weighted school choice market is now defined as $M = (I, S, P, \pi, p, q)$. An outcome of M is called matching and assigns each student to at most one school and each school is matched

[4] Schools are not treated as agents but as objects with some priority ordering derived from some objective criteria, such as walking distance from school or the fact whether a student already has siblings studying at the school. Thus, we do not need to consider the schools' preferences over sets but just use the binary priorities.

to a number of students whose aggregated weights do not exceed the school's quota. More precisely, a matching is a mapping[5] μ on $I \cup S$ with $\mu(s) \subseteq I$ for all $s \in S$ and $|\mu(s)| \leq q_s$, $\mu(i) \in S \cup \{i\}$ for all $i \in I$, and $\mu(i) = s$ if and only if $i \in \mu(s)$. The remaining capacity of a school s in a matching μ is denoted by q_s^μ, indicating the total amount of quota that is not assigned to a student in this matching.

Each student i may compare his or her matches in two different matchings, μ and ν. Either she is indifferent as she is matched to the same school in both matchings, denoted $\mu(i) \succsim_i \nu(i)$ if and only if $\mu(i) P_i \nu(i)$ or $\nu(i) = \mu(i)$.

To answer questions about the revelation of students' true preferences, we define the school choice problem as a preference revelation game induced by a particular matching mechanism, in this case the weighted TTC, where the students I are the players with a set of feasible strategies Q. Thus, Q_i denotes a player i's set of strategies in the game $\Gamma(\succsim)$. More precisely, a student can state a preference list. Formally, each student i is stating some strict preference ordering Q_i. The set of stated preference lists by all students is denoted by $Q = (Q_{i_1}, \ldots, Q_{i_m})$. The outcome is the matching $\mu = h(Q)$, where h describes the outcome of the matching algorithm given the stated preferences Q and $\succsim$ describe the players' preferences over matchings. Thus, the game is defined as $\Gamma(\succsim) = (I, \{Q_i\}, h, \succsim)$.

We define Q_{-i} as the set of choices by all players except for i. The set of all choices then would be $Q = (Q_{-i}, Q_i)$. Thus, Q collects all stated preferences in the preference revelation game consisting of the stated preferences by all students except for i and student i's stated preferences.

In the following we define the different properties that a matching and the mechanism that finds it might have. Stability, Pareto efficiency and strategy-proofness are widely discussed in the matching literature. As we are dealing with a weighted matching problem here, these properties need to be reconsidered.

Definition 1 (Individual Rationality (Roth & Sotomayor, 1990))**.** *A match $\mu(i) = s$ is individual rational if i and s find each other acceptable.*

[5]Note that $\mu(i) \in S \cup I$ and $\mu(s) \subseteq I \cup S$.

Individual rationality ensures that both sides can only gain from taking part in the matching process as no student is matched to a school, although he or she would prefer to be unmatched than being matched to this particular school and no school is matched to a student it does not want to be matched to.

Definition 2 (Weighted Justified Envy)**.** *Giving a matching μ, a student i has justified envy towards some students $i'_1, \ldots, i'_k$ who are currently matched to school s if*
(a) $\mu(i) \neq s$ and $sP_i\mu(i)$, and
(b) $t\, i\pi_s i'_l$ for all $l \in \{1, \ldots, k\}$ where $\{i'_1, \ldots, i'_k\} \subseteq \mu(s)$ with $p_i \leq q_s^\mu - \sum_{l=1}^{k} p_{i'_l}$.

Justified envy occurs if a student and a school are currently not matched but prefer each other over parts of their current matches. This means, the student prefers the school over his or her current match. The school, on the other hand, has a higher priority for the student in comparison to a set of students currently matched to the school and it would have enough remaining capacity without this set for the higher ranked student to be placed in the school.

Definition 3 (Weighted Non-wastefulness)**.** *A matching is weighted non-wasteful if for all students $i \in I$ and all schools $s \in S$ it holds that $sP_i\mu(i)$ implies $q_s^\mu < p_i$.*

Non-wastefulness means that if a student is not matched to a preferred school, this school has not enough free capacity left to accept this student. Therefore, if a matching is non-wasteful, a student who is eligible for an empty seat at a preferred school is matched to this school (Abdulkadiroglu & Sönmez, 2013).

Definition 4 (Weighted Stability)**.** *A matching is weighted stable if it is individual rational, weighted non-wasteful and no student has justified envy.*

As McDermid & Manlove (2010) have shown, weighted college admissions problems need not have stable outcomes. Moreover, even if a weighted stable matching exists it cannot be found with the deferred acceptance algorithm

(Hoyer & Stroh-Maraun, 2019). As school choice problems differ from college admissions problems mainly in the interpretation of the schools' role, these results carry over. A weighted stable matching might not exist in a weighted school choice problem.

Definition 5 (Pareto efficiency (Abdulkadiroğlu et al., 2017))**.** *A matching* μ *Pareto dominates a matching* ν *if* $\mu(i) \succsim_i \nu(i)$ *for all* $i \in I$ *and* $\mu(i) P_i \nu(i)$ *for some* $i \in I$*. A matching is Pareto efficient if it is not Pareto dominated by any other matching.*

A matching is Pareto efficient if it is not possible to improve a student's allocation without making another student worse off. Note that schools are not taken into account in Pareto efficiency considerations as they are considered as objects. This is the major difference to college admissions problems where the focus lies on stability to account for both sides of the market.

Whereas so far we have focused on properties of the outcomes we now want to focus on the properties of the mechanism that finds the outcome by defining the school choice problem as a strategic game.

Definition 6 (Dominant Strategy (Roth & Sotomayor, 1990))**.** *A strategy* Q_i^* *is a best response of* i *to* Q_{-i} *if* $h(Q_{-i}, Q_i^*) \succsim_i h(Q_{-i}, Q_i)$ *where* Q_i *is any other strategy* i *could play. A dominant strategy for agent* i *is a strategy* Q_i^* *that is a best response to all possible sets of strategy choices* Q_{-i} *by the other agents.*

If a player has a dominant strategy, she can never gain by deviating from it.

Definition 7 (Strategy-proofness (Roth & Sotomayor, 1990))**.** *A mechanism is strategy-proof if it makes it a dominant strategy for each player* i *to state the true preferences* P_i *in the strategic game* $\Gamma(\succsim)$*.*

If a mechanism is strategy-proof all players, here students, are able to state their true preferences without being harmed.

Summing up, the important properties of a matching and the mechanism to find it in a weighted school choice problem are weighted stability, Pareto efficiency and strategy-proofness. As it has already been shown that weighted stability can no longer be ensured in this setting, we will focus on the existence of the other two by introducing the weighted top trading cycles mechanism.

3 Pareto Efficiency in Weighted School Choice

The top trading cycles (TTC) mechanism was introduced for school choice problems by Abdulkadiroğlu & Sönmez (2003). In the classical model it yields the Pareto efficient outcome for students. In this mechanism, the schools' priorities are not interpreted as schools' preferences but as students' opportunities to be assigned to a school; a higher priority means a better opportunity for a student to be assigned to this school in comparison to another student with the same preferences that has a lower priority. As we will see in the analysis, weights can be similarly interpreted. A higher weight decreases the opportunity for a student to be assigned to this school holding all other things equal. The original TTC mechanism works as follows. For each school a counter $c_s(k)$ counts the number of still available seats in round k. In the beginning, it is equal to the school's quota. In each round all students point to their most preferred school that still has some free capacity and each school points to its most prioritized student that is still unmatched. If a subset of students and a subset of schools form a cycle, the students are matched to the school they are pointing to, the students leave the market and the schools' quotas are reduced by one. If a quota is reduced to zero, the school leaves the market as well. The algorithm terminates when all students are matched or no schools have any available seats. Note that there might be some students who are unmatched in the final outcome.

Unfortunately, the TTC does not take heterogeneous students into account. Therefore, it cannot be directly used in a weighted school choice problem. To assess the problem, in Section 3.1 we first have a look at a special case where all schools prefer students type-wise before concentrating on the general case in Section 3.2.

3.1 Type-Specific Priorities

Before having a look at the general problem, we focus our analysis on a special case. Imagine all schools prioritize students of a specific type over students with another specific type and these students over another student type. Thus, the schools have a rank ordering over the weights and within each type they have priorities over students. If one type is ranked over another type a school prioritizes a student of the favored type always higher than a student of the less favored type. The clearinghouse that is described in Hoyer & Stroh-Maraun (2020) actually uses this priority structure. In this particular application at a German university, students who want to write their Bachelor or Master theses are assigned to supervisors in a centralized clearinghouse. Master students have a higher weight than Bachelor students as it is assumed that supervising a Master thesis is more time consuming than supervising a Bachelor thesis. We may assume that supervisors always give Master students a higher priority over Bachelor students. Thus, all supervisors favor one student type over the other. To phrase it differently, a Bachelor student always has a lower priority than a Master student, independent of the individual students. To capture these priorities of student types we assume that schools have a common type priority. Therefore, we assume that each school s has original priorities which are independent of the students' types π_s and a priority ordering of the types t. We assume w.l.o.g. that students of type 1 with weights p^1 are the most preferred type, type 2 with weights p^2 are the second most preferred student type, and students of type j are the j most preferred type for $j \geq 1$.

Definition 8. *We construct school's s's type priority π_s^t from its original priorities π_s as follows: For all $i, i' \in I$ with $i \in T^t$, $i' \in T^{t'}$*

$$i \pi_s^t i' \quad \textit{if and only if} \quad \textit{(a) } t < t' \textit{ or (b) } t = t' \textit{ and } i \pi_s i'. \tag{0.1}$$

As the ordering of types is the same for all schools, the schools have a common type priority.

In this case, we can regard the different student types as separate matching problems, starting with matching the most prioritized type before matching the second prioritized type, and so on. This is continued until either all students are matched or no school has capacity left to accept any student.

Proposition 1. *Given is a weighted school choice problem with students* $I = (i_1, \ldots, i_m)$ *of types* t *and corresponding weights* p *and schools* $S = (s_1, \ldots, s_n)$ *with quota* q. *If schools have a common type priority, a Pareto efficient matching is obtained by application of the TTC mechanism.*

Proof. Without loss of generality we assume that students of type 1 with weights p^1 are the most preferred type, type 2 with weights p^2 are the second most preferred student type, and students of type j are the j most preferred type for $j \geq 1$. We can apply the TTC per type, starting with students of T^1. From Abdulkadiroğlu & Sönmez (2003) we know that the TTC yields a Pareto efficient allocation for the students of type 1. Let us call the resulting matching ν. After the first TTC with students of type 1, the schools' capacities are now reduced by the aggregated weights of students that are already matched accordingly. If a school s has some free capacity left after this first TTC, the quota of school s is either big enough that also at least one student of the remaining $(d-1)$ types with the smallest weight $p_{\underline{i}}$ might still be accepted, thus $q_s^\nu \geq p_{\underline{i}}$, or it is not, thus $q_s^\nu < p_{\underline{i}}$. If the quota is not large enough the school completely exits the mechanism; otherwise it remains with its reduced quota. A next run of the TTC can then be run on students of type 2 where all schools participate that have enough remaining capacity left after the previous round to accept a student of type 2. The TTC is applied over and over again for each student type until we have applied the TTC over the least preferred type. Any student who leaves the market is assigned to her top choice among those schools that still offer a capacity that is large enough. As the preferences are strict, the student cannot be better off without hurting a student who was matched to a school in a prior round or even prior instance. □

The functioning of the TTC is not disturbed in this special case of weighted school choice. The TTC is applied several times. As every application of the TTC is strategy-proof, so is the overall procedure. But what happens if we allow for arbitrary priorities? How do we find an efficient matching? And is the mechanism used also strategy-proof? To answer these questions, we will now introduce the weighted top trading cycles mechanism which incorporates arbitrary priorities. In the case of type-specific priorities it yields the same result as the application of various independent runs of the TTC.

3.2 The Weighted Top Trading Cycles Mechanism

If schools do not favor students according to their types, it is no longer possible to find a Pareto efficient matching with the original TTC mechanism. Therefore, we modify the TTC mechanism and term it *WTTC*, the weighted top trading cycles mechanism, to be applicable to arbitrary weighted school choice problems.

Functioning of the Modified Algorithm

The WTTC functions similar to the TTC but explicitly incorporates the students' weights. Note that it results in the same matching as the TTC if all students have the same weight. The WTTC works as follows: in addition to the counter $c_s(k)$ which counts the amount of still available capacity in round k, we assign to each school s a stopping rule $r_s^{p^t}(k)$ for each student's weight p^t. It is assigned zero in round k if the school's remaining quota is large enough to accept at least one student of type t, $r_s^{p^t}(k) = 0$. If the remaining quota is too small to accept a student of type t in round k', it is assigned one from that round on, $r_s^{p^t}(k') = 1$. The introduction of a binary marker is a similar approach to the one by Abdulkadiroğlu & Sönmez (2003). They use additional counters to account for what they call type-specific quotas to maintain the ethnical balance at schools.

WTTC

Round 1:
Each student i points to his or her most preferred school s. Each school points to its most prioritized student. Since there is a finite number of students and schools, there is at least one cycle. Moreover, each school and each student are part of at most one cycle as they are pointing to a single student or school respectively. Within each cycle the students are matched to the school they

are pointing to, the students leave the market and the schools' quotas are reduced by the assigned students' weights. If a quota is reduced to zero, the school leaves the market. Additionally, all stopping conditions are updated. If the counter is smaller than weight p^t, the stopping rule for round 2 is set to one, $r_s^{p^t}(2) = 1$. If all stopping rules for round 2 are equal to one, the school leaves the market immediately. All schools and students that are not part of a cycle are not affected.

Round k:
In each round k each remaining student $i \in T^t$ points to their most preferred school s that still has still enough free capacity to accept student $i \in T^t$, $r_s^{p^t}(k) = 0$. Each school points to its most prioritized student $j \in T^{t'}$ that is still unmatched and where $r_s^{p^{t'}}(k) = 0$ as well. Thus, students and schools are allowed to point to a certain counterpart only if the corresponding stopping rule is equal to zero. Consequently, there is at least one cycle. Within each cycle the students are matched to the school they are pointing to, the students leave the market and the schools' quotas are reduced by the assigned students' weights. If a quota is reduced to zero, the school leaves the market. All stopping conditions are updated. If the counter is smaller than weight p^t, the stopping rule is set to one in the next round, $r_s^{p^t}(k+1) = 1$. All students with weight p^t are no longer able to point to this school and the school is no longer able to point to students with weight p^t. If all stopping rules are equal to one, the school leaves the market as well. The algorithm terminates when all students are matched, no schools are left in the market or no student is able to point to a school anymore. Note that the number of rounds does not exceed the number of students, as at least one student is matched in every round.

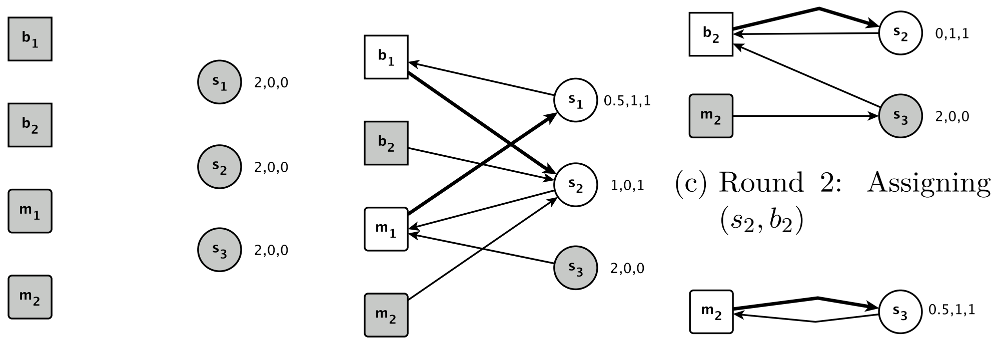
Before looking at the WTTC and its properties in greater detail we illustrate the functioning of the algorithm by the following example.

Example 1. *Imagine some students want to write their theses with different supervisors at a university. We can formulate the following school choice problem with two different student types, Bachelor students b_1 and b_2 who are of type b and Master students m_1 and m_2 who are of type m with weights $p^b = 1$ and $p^m = 1.5$ respectively, and three schools or more precisely supervisors, in* *this example* s_1, s_2, s_3 *with quotas* $q = (2, 2, 2)$.

$$\begin{array}{lll} P_{b_1} : s_2,\, s_1,\, s_3 & P_{m_1} : s_1,\, s_2,\, s_3 & \pi_{s_1} : b_1,\, m_1,\, b_2,\, m_2 \\ P_{b_2} : s_2,\, s_1,\, s_3 & P_{m_2} : s_2,\, s_3,\, s_1 & \pi_{s_2} : m_1,\, b_2,\, m_2,\, b_1 \\ & & \pi_{s_3} : m_1,\, b_2,\, m_2,\, b_2 \end{array}$$

In Figure 1 you can see how the three rounds of the WTTC work. You can find the different counters behind the supervisors' circles denoted by $c_s(k), r_s^{p^b}(k), r_s^{p^m}(k)$ *where* $c_s(k)$ *indicates the remaining capacity,* $r_s^{p^b}(k)$ *is the stopping rule for Bachelor students and* $r_s^{p^m}(k)$ *is the one for Master students (cf. Figure 1a). Throughout the mechanism, students and schools point to their most preferred or prioritized counterpart that still offers seats for them. In round 1,* m_1 *points to* s_1*,* s_1 *points to* b_1*,* b_1 *points to* s_2 *and* s_2 *points back to* m_1*. Thus, we find a cycle between the two students and two schools. The students are matched to the schools they are pointing to and leave the market. The schools' quotas are reduced by the students' weights.* s_2*'s remaining quota is reduced to* 1*, meaning that the school does not have enough remaining capacity to be matched to a Master student. Thus, the stopping rule is set to one,* $r_{s_2}^{p^m}(2) = 1$*. Also* s_1*'s remaining capacity is decreased to* 0.5*. As all students' weights are greater than* 0.5*, all stopping rules are set equal to one and* s_1 *is not able to point to a student anymore. Furthermore, no student is able to point to* s_1 *anymore. Therefore,* s_1 *leaves the market (cf. Figure 1b). In the next round,* b_2 *and* s_2 *are pointing to each other and form a match. Student* b_2 *leaves the market, the school's remaining quota is decreased to 0 and* s_1 *leaves the market. Please note that* m_2 *and* s_2 *are no longer allowed to point to each other because the stopping rule is equal to one,* $r_{s_2}^{p^m}(3) = 1$ *(cf. Figure 1c). Therefore, in round 3,* m_2 *and* s_3 *are the only two remaining participants pointing to each other. They are matched and leave the market (cf. Figure 1d). The algorithm terminates as all students and schools have left the market. The resulting matching is* μ *with* (s_1, m_1), (s_2, b_1), (s_2, b_2), (s_3, m_2)*. It is easy to verify that* μ *is Pareto efficient.*

Example 1 already gives an intuition that this algorithm works similar to the original top trading cycles algorithm. Next we show that this intuition



(a) Students and schools that take part in the WTTC

(b) Round 1: Assigning (s_1, m_1) and (s_2, b_1)

(c) Round 2: Assigning (s_2, b_2)

(d) Round 3: Assigning (s_3, m_2)

Figure 1: Functioning of the WTTC in Example 1

is also supported by the theoretical results. The main properties of the TTC carry over, Pareto efficiency and strategy-proofness.

Theorem 1. *The WTTC algorithm yields a Pareto efficient outcome.*

Proof. Consider the weighted top trading cycles algorithm. Any student who leaves the market in the first round is matched to her top choice and thus cannot be made better off. In a proceeding round, any student who leaves the market is assigned to her top choice among those schools that still offer a capacity that is large enough. As the preferences are strict, the student cannot be better off without hurting a student who was matched to a school in a prior round. □

The incorporation of weights does not affect the Pareto efficiency of the TTC. The TTC algorithm is robust to this modification of the original school choice model as it also holds that the weighted TTC algorithm is strategy-proof. To show strategy-proofness, we use the following Lemma 1 from Abdulkadiroğlu & Sönmez (2003).

Lemma 1. *[Abdulkadiroğlu & Sönmez (2003)] Fix the announced preferences of all students except i at Q_{-i}. Suppose that in the algorithm, student i leaves at round k under some Q_i and at round k' under some Q_i'. W.l.o.g. $k \leq k'$.*

Then the remaining students and schools at the beginning of round k are the same whether student i announces Q_i or Q'_i.

Proof. Student i is not part of a cycle in both cases prior to round k. As all the other students do not change their preferences, the same cycles are formed prior to round k in both cases. The students always point to their most preferred school that still offers enough capacity and the schools point to the student with the highest priority and a weight that fits into the schools' capacity. Thus, the same students and schools are removed. □

Theorem 2. *The WTTC algorithm is strategy-proof.*

Proof. The following proof is similar to the original one by Abdulkadiroğlu & Sönmez (2003). We now consider that student i has true preferences P_i. We fix the stated preferences of all other students Q_{-i}. If i states any other preferences Q_i, she is assigned to a school s at round k where she is – without loss of generality – part of a cycle $(s, i_1, s_1, \ldots, s_h, i)$, which means s is pointing to i_1, i_1 is pointing to s_1 and so on until s_h which is pointing to i and i is pointing to s. Thus, i is matched to s, $\mu(i) = s$. Furthermore, let k^* be the round in which i is assigned to a school $\mu^*(i)$ when she states her true preferences P_i. We only have to consider two cases and show that in both cases i is weakly better off when stating her true preferences P_i.

Case 1: $k^* < k$

According to Lemma 1 we know that at round k^* the same schools and students remain in the market independent of i's stated preferences. If i states her true preferences, she is matched in k^* to the best available school $\mu^*(i)$. Thus, she weakly prefers this match over s, where she would be matched to under Q_i in some later round k, $\mu^*(i) \succsim \mu(i)$.

Case 2: $k^* \geq k$

Assume that student i states the true preferences P_i. Consider round k. By Lemma 1 we know that the same students and schools remain in the market at the beginning of this round independent of student i's stated preferences. Thus, up to round k there is no difference in the algorithm whether i states some preferences Q_i or the true preferences P_i. We know that s is pointing to i_1, i_1 is pointing to s_1 and so on until school s_h which is pointing to i. Actually, as long as i stays in the market, they will remain doing this and s will continue to have enough remaining capacity left for i. As i states her preferences truthfully, she points to the best remaining school in k. As soon as i is part of a cycle, she is assigned to the school she points to, $\mu^*(i)$, and leaves the market. She is either part of the cycle $(s, i_1, s_1, \ldots, s_h, i)$ or she can achieve a better option. Thus, her matching must be at least as good as s, $\mu^*(i) \succsim \mu(i)$. □

Thus, the functioning of the TTC algorithm is robust to the introduction of weights regarding truth-telling and Pareto efficiency. Note that the WTTC reduces to the TTC if all students have the same weight, as the stopping rule does not affect the functioning of the TTC then. If schools favor students according to their types, the WTTC yields the same result as the application of various independent rounds of the TTC, as shown above in Section 3.1. This is due to the fact that students and schools of a specific type t keep pointing to each other until all students of this type are matched.

Impact of Weights on the Outcome of the WTTC

Let us have a closer look at the implications of the incorporation of weights. First of all the schools' priorities and the students' weights both affect the opportunities of students to be matched to a school. While on the one hand a higher priority increases a students' chances to be matched to a school, the student's weight decreases the chances. To discuss this trade-off and also some additional implications, we assume from now on that schools and students always find each other acceptable. Thus, a student would prefer being matched to a school over staying unassigned. Also, each school has a higher priority to fill a seat than to leave a slot vacant.

Proposition 2. *Given two school choice markets $M = (I, S, P, \pi, p, q)$ and $M' = (I, S, P, \pi, p', q)$ with $p = (p_{i_1}, p_{i_2}, \ldots, p_{i_m})$ and $p' = (p'_{i_1}, p_{i_2}, \ldots, p_{i_m})$ where p and p' just differ in the weight of student i_1 with $p'_{i_1} > p_{i_1}$ w.l.o.g., i_1 is either matched in the same round in both markets or in a later round. Thus, a student with a higher weight has smaller chances to be matched to* *a particular school in comparison to a student with a smaller weight ceteris paribus.*

Proof. Let us have a closer look at the functioning of the WTTC where M yields the outcome μ and M' yields the outcome μ'. Thus, we have to consider two cases.

Case 1: In Round k, student i_1 is matched in M and M'

As the student i_1 is matched in the same rounds in both markets, i_1's outcome does not differ. i_1 is matched to the same school s in both markets, $\mu(i_1) = \mu'(i_1)$. Especially the weight does not affect i_1 directly.

Case 2: In Round k, student i_1 is matched in only one of the two markets M and M'

M and M' differ only in the weight of student i_1. This means that i_1 fits to the school s in one market but not in the other. More precisely, his or her weight is too big in M' but not in M. If i_1 is matched to a school s' in a later round $k^* > k$, this means that $\mu(i_1) \succsim_{i_1} \mu'(i_1)$ as $sP_{i_1}s'$. Therefore, the higher weight in M' actually makes i_1 worse off by matching him or her to a less preferred school.

Summing up, a higher weight leads to a smaller chance to be matched to a certain school. □

If a student is matched in some round with a high weight, she would be matched in the same round at latest with a smaller weight. Thus, the incorporation of weights introduces a trade-off between priorities and weights. Although a student and a school express their wishes to be matched to each other, they might end up being matched to other, less preferred partners. This problem occurs because the student's weight is too large to fit in the school, whereas there is enough capacity left for a student with a smaller weight.

Example 1 continued. *In Example 1, school s_2 prefers m_2 over b_1 and also m_2 ranks school s_2 as the highest preference. Nevertheless, they are not matched as after the first round there is not enough remaining capacity for m_2 at s_2. Instead, b_1 and s_2 are matched.*

If the matching market size increases by increasing the quotas of every school and the numbers of students, this trade-off might vanish as it mainly comes into play if a school has enough remaining capacity to accept a small student but not a large one. The problem only affects a fraction of students. This fraction decreases if the market gets larger, as more and more students enter it and the schools' capacities increase as long as the amount of schools do not change. Thus, the problem described in Proposition 2 diminishes in large markets.

Additionally, in contrast to the original school choice model it is not sufficient that the sum of all capacities is equal to the sum of all students multiplied by their weights to guarantee a seat for all students.

Proposition 3. *Although overall there is enough capacity to match every student, some students might end up unmatched.*

Proof. Schools might leave the market although they have some remaining capacity left. This is the case if all relevant stopping rules are set to one, as all students' weights are larger than the school's remaining capacity. □

You can observe in Example 1 that schools might actually leave the market with some remaining capacity left. By increasing the weights, the overall amount of capacity needed to match all students is increased over-proportionately. Otherwise, students might end up unmatched. But can we assure that there is an outcome where every student is matched in a weighted school choice market where the WTTC is used? To motivate the answer here, let us again have a closer look at Example 1.

Example 1 continued. *Let us have a look at Example 1 again. Each school has a quota of 2. This means we can either fill the slots with two Bachelor students or one Master student. In the later case, the school has a free capacity of* 0.5*. This remaining capacity is too small to accept another student. Thus, we need more capacity than the sum of all weights of the students (two Bachelor and two Master students need an overall capacity of at least 5) to guarantee a match for everyone. In this example, the overall amount of capacity of 6 is sufficient to guarantee the matches, given the preferences as they are. Now assume that student* b_2 *prefers* s_3 *over* s_2 *and* s_1*.*

$$P_{b_2} : s_3,\ s_2,\ s_1.$$

In this case, after the WTTC is used student m_2 would stay unmatched as m_1 is matched to s_1, b_1 is matched to s_2 and b_2 is matched to s_3. Thus, s_2 and s_3 both have remaining capacities of 1, which means that m_2 does not fit in either of the two.

Thus, the required amount of seats depends on two factors. On the one hand, the amount might be higher than the overall sum of weights as schools have remaining capacities. On the other hand, the required amount of seats also depends on the students' preferences. A capacity structure might be sufficient under one special preference structure but not under another, as you have seen above in Example 1. The question now is whether we can assure that in the solution of the WTTC all students are matched without any requirements concerning the preferences. The answer is yes, in the following way.

Proposition 4. *Consider a weighted school choice model with m and n as the number of students and schools, respectively, and students' weights p up to $p_{\bar{i}}$. To guarantee a seat for every student independent of preferences and priorities it is sufficient that*

1. *the amount of seats equals the amount of seats that need to be available if all students have the highest weight which is $\sum_{s=1}^{n} q_s = m \cdot p_{\bar{i}}$, and*
2. *each school's capacity q_s must be divisible by the highest weight, $q_s = p_{\bar{i}} \cdot r_s$ where $r_s \in \mathbb{N}$ for all $s = (s_1, \ldots, s_n)$.*

Proof. We know that each student's weight can be at most $p_{\bar{i}}$. If all students have the highest weight, the required amount of seats is $\sum_{s=1}^{n} q_s = m \cdot p_{\bar{i}}$. As a student can be matched to a school only if there is enough remaining capacity, each school's capacity must be divisible by the highest weight, $q_{s_j} = p_{\bar{i}} \cdot r_j$ where $r_j \in \mathbb{N}$ for all $j = (1, \ldots, n)$.

This also holds true if a student has a smaller weight than $p_{\bar{i}}$ as this just decreases the amount of seats that is actually needed. A student with a smaller weight also fits into a school that has enough capacity to accept a student with the higher weight $p_{\bar{i}}$. □

To illustrate the consequences of Proposition 4 let us again have a look at Example 1.

Example 1 continued. *Considering Example 1 again, we see that only the first condition of Proposition 4 is fulfilled. In the example there are four students with* $p_{\bar{i}} = p^m = 1.5$*. Therefore, the overall amount of capacity needed is* $\sum_{s=1}^{3} q_s = 4 \cdot 1.5 = 6$*. This is fulfilled as* $q = (2, 2, 2)$*. Nevertheless, the second condition is not fulfilled as the schools' capacities are not divisible by* p^m*. Thus, we cannot guarantee that every student is matched after the use of the WTTC, although we see that under the original preferences* P *all four students actually have found a match: namely* μ *where the matches are* (s_1, m_1), (s_2, b_1), (s_2, b_2), (s_3, m_2)*. To guarantee a match we would require that one of the schools has a quota of* 3 *and the other two schools have quotas of* 1.5*. But only if* s_2 *has enough quota to accept two Bachelor students do we actually obtain* μ*. Again we see that the outcome is crucially affected by the existence of weights.*

We observe two things by looking again at the example. First, there is a trade-off between weights and priorities. Therefore, students with large weights might have disadvantages in comparison to students with smaller ones. Second, depending on the preferences and priorities the actually required sum of capacities can be smaller. Nevertheless, guaranteeing a slot for every student might lead to very large capacities that are not necessarily needed in the actual weighted school choice problem if students differ a lot in their weights. This can be interpreted as a worst-case scenario and guarantees that all students can be matched to a school. Can we quantify the damage here? In the original school choice model without weights, to guarantee a match for every student it is sufficient that the overall amount of capacity is equal to the amount of students, $\sum_{s=1}^{n} q_s = m$. No remaining capacity is left after the (Pareto efficient) matching is found.

Definition 9. *If a Pareto efficient matching* μ *is found using the WTTC, there might be some capacity left. The overall amount of remaining capacity is called* waste $\omega(\mu) = \sum_{s=1}^{n} q_s^{\mu}$*. It can be calculated by subtracting the accumulated weight of all matched students from the overall capacity.*

If no remaining capacity is left after we have found matching μ, we have zero waste: $\omega(\mu) = 0$. If we now apply Proposition 4 to the concept of waste, we see the following.

Corollary 1. *To guarantee that each student is matched in the WTTC, the overall capacity needed is* $\sum_{s=1}^{n} q_s = m \cdot p_{\bar{i}}$*. The sum of all weights, as everyone is matched, is* $\sum_{i=1}^{m} p_i$ *the amount of waste is given as the difference between these two numbers,* $\omega(\mu) = \sum_{s=1}^{n} q_s \mu = m \cdot p_{\bar{i}} - \sum_{i=1}^{m} p_i$*.*

We identify two main determinants of the amount of waste here. The waste increases with the weight of the largest student or with the share of small students. Thus, we identify two determinants of *heterogeneity*: the absolute heterogeneity (by comparing the weights of two single students) and the relative heterogeneity (by comparing the shares of large and small students). We see that the first source of heterogeneity diminishes in large markets. The second source of heterogeneity might already vanish in small markets if the amount of large students is quite large. Summing up, guaranteeing that every student, especially the students with a large weight, finds a match might lead to extremely large amounts of remaining capacity, especially if the students' weights are very heterogeneous. Note also that just guaranteeing a match for every student does not mean that every student actually receives a favorable match. This is of course also the case in the original school choice model but the introduction of weights tighten the problem here.

If the amount of waste is extremely high in a particular market, it is perhaps not possible to offer such large capacities that each student is guaranteed a match. Can we in this case quantify how many students are left unmatched in advance? Again, the answer should be independent of the preferences. We analyze the situation when the overall capacity is just as large as needed in the ideal case, namely if the overall amount of capacity is equal to the sum of all weights, $\sum_{s=1}^{n} q_s = \sum_{i=1}^{m} p_i$.

Proposition 5. *There is a weighted school choice model where the schools offer a total capacity that exceeds the total amount of students' weights,* $\sum_{s=1}^{n} q_s \geq \sum_{i=1}^{m} p_i$*, and the schools' quotas are integers. Additionally, all students of* $T^{\underline{t}}$ *with the smallest weight* $p_{\underline{i}} = p^{\underline{t}} = 1$ *have an overall weight of* c*. With the*

WTTC a matching is found where at least a subset of students with an overall weight of c are matched.

Proof. We know that if all students have the same weight of $p_i = 1$ for all $i \in (1, \ldots, m)$, as in the original school choice model, it is sufficient that $\sum_{s=1}^{n} q_s = \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} 1 = m$. There are c students with $p_{\underline{i}}$. As each capacity is an integer, all students with the weight of $p_{\underline{i}} = 1$ fit into at least one school in the beginning as long as no other student is matched. If another student is matched, his or her weight is at least as large as $p_{\underline{i}} = 1$. Therefore, a subset of students with an overall weight of c is accepted during the WTTC. □

Additionally, there are cases where just a subset of students with an overall amount of weight that equals exactly b is matched. Imagine the following Example 2.

Example 2. *Imagine some students want to write their theses with different supervisors at a university. We can formulate the following school choice problem with two different student types, one Bachelor student b_1 who is of type b and two Master students m_1 and m_2 who are of type m with weights $p^b = 1$ and $p^m = 1.5$ respectively, and three schools or more precisely supervisors, in this example s_1, s_2, s_3 with quotas $q = (2, 1, 1)$.*

$$
\begin{array}{lll}
P_{b_1} : s_1,\, s_2,\, s_3 & P_{m_1} : s_1,\, s_2,\, s_3 & \pi_{s_1} : b_1,\, m_1,\, m_2 \\
 & P_{m_2} : s_2,\, s_3,\, s_1 & \pi_{s_2} : m_1,\, m_2,\, b_1 \\
 & & \pi_{s_3} : m_1,\, m_2,\, b_1
\end{array}
$$

In the first round of the WTTC all schools point to b_1. She is favored by school s_1 and the other schools do not have enough capacity to accept the Master students. All students point to school s_1 for similar reasons. s_1 and b_1 form a match. Afterwards, no school has enough capacity to accept any of the Master students. Thus, the accumulated sum of weights that is matched to a school equals the amount of students with the smallest weight $p_{\underline{i}} = p^b = 1$.

This problem illustrated in Example 2 is an extreme case. Here, the market is so small that only one out of the three schools would actually offer a slot that is large enough for the students with a higher weight. The waste in this case can be particularly high as well. In the worst case in a matching μ where only the small students are matched, the waste equals the following amount: $\omega(\mu) = \sum_{i=1}^{m} p_i - c$. Thus, the waste is less than the overall amount of students' weights in the market. If the market size increases as additional seats are offered by the schools and an additional number of students enter the market, it is less likely that a school only offers a slot for a few students with a small weight. Thus, this problem becomes smaller, when the market size increases. Summing up the discussion on weights and priorities, we find that the introduction of heterogeneous students introduces new problems in the market. The main implications are twofold. First, a new trade-off between the schools' priorities and the weights is introduced as the impact of weights can overturn the schools' priorities. This trade-off results in disadvantages for students with higher weights. Second, weights themselves introduce new restrictions. Fractions of the schools' capacities might be left unused as the students' weights might be too large to fit in.

Although the incorporation of weights also introduces new problems to the functioning of the (weighted) TTC, the impact seems negligible in at least some markets as the impact is crucial, especially in small markets. In large markets where schools' capacities become larger the consequences diminish at least partly.

4 Discussion

Our results support using a variant of the TTC instead of the DA in school choice problems. Abdulkadiroğlu & Sönmez (2003) formulate the trade-off between Pareto efficiency and stability without favoring one of the two properties. We find that under some natural extensions of the school choice problem, it is worth looking at the (weighted) TTC and Pareto efficiency instead of the DA and stability.

The deferred acceptance algorithm is not only not robust to the introduction of a weighted college admissions problem. It is actually not robust to various small extensions of a school choice model, such as the introduction of indifferences (Erdil & Ergin, 2008), of reciprocal preferences (Haake & Stroh-Maraun, 2018) or of constrained rank ordered lists (Haeringer & Klijn, 2009). These adjustments directly disturb the functioning of the deferred acceptance algorithm. While a stable matching still can be found with the help of the DA, either the finding is not the optimal stable outcome anymore or the procedure is not strategy-proof. The introduction of weights (Hoyer & Stroh-Maraun, 2019) even prevents the DA from finding any stable outcome at all. In contrast to this finding, the introduction of weights does not disturb the functioning of the (weighted) TTC as we show in this work. This is in line with the findings by Combe (2018) who formulates a teacher assignment problem where some teachers have initial assignments, Hamada et al. (2017) who introduce a school choice model with minimum quotas and initial endowments of students, and Dur & Wiseman (2019) who define school choice with neighbors. They all find that the TTC is quite robust to the extensions and still yields a Pareto efficient and strategy-proof outcome. Thus, it is worth considering it as a matching mechanism in a school choice setting.

The TTC actually fulfills some additional interesting properties. Although it is not stable, it is still fair in a weaker sense. For example, Abdulkadiroğlu et al. (2017) show that the TTC minimizes justified envy among all Pareto efficient and strategy-proof mechanisms when each school has one seat. Unfortunately, this result does not carry over to our model, as we assume that schools have more than one seat. Abdulkadiroglu & Che (2010), Dur (2012) and Morrill (2013) characterize the TTC as the only strategy-proof, Pareto efficient outcome that fulfills some additional weaker fairness conditions.

This is especially interesting as until today there are several real-life examples where a version of the DA is used to match students to schools (Roth, 2008), but no implementation of a TTC as far as publicly known (Abdulkadiroğlu et al., 2017), although both mechanisms have theoretically desirable properties and the TTC was actually recommended by policy makers in Boston and San Francisco (Hakimov & Kesten, 2018). In New Orleans, the TTC was adopted in 2012 but abandoned one year later (Abdulkadiroğlu et al., 2017). This lack of actual implementation might be at least partially due to the fact that the DA is easier to understand (Li, 2017; Ashlagi & Gonczarowski, 2018).

Nevertheless, experimental studies confirmed that the theoretical properties carry over to the lab as truth-telling can be observed there (Chen & Sönmez, 2006; Chen et al., 2016). Thus, the TTC and its variants are interesting alternatives for implementation in applications, such as kindergarten matching. It fulfills desirable properties, are robust to the introduction of weights and other extensions, and also worked quite well in the laboratory.

5 Conclusion

Students might be heterogenous regarding the amount of seats they require at a school in a (weighted) school choice problem. This heterogeneity occurs quite often in reality, e.g. in kindergarten matching where young children need more care than older ones. Thus, child-care facilities can accept a smaller number of young children in comparison to older ones. Nevertheless, it is not necessarily the case that the facilities know a priori how many older and younger children they want to accept. Instead, they have an overall capacity that can be filled by a combination of younger and older children. We model a weighted school choice problem to incorporate this heterogeneity. More precisely, we assign each student an individual weight to represent the different needs for care. We show that under this extension we can still find Pareto efficient outcomes with the help of the weighted version of the TTC algorithm. Furthermore, the WTTC algorithm is still strategy-proof. Unfortunately, we introduce new problems, especially a trade-off between weights and priorities into the solution concept, particularly in smaller markets, as a student might be accepted by a preferred school only if his or her weight is small enough to fit in. The problems become smaller, when the market size increases.

Despite this newly arising tension, the WTTC is quite robust to the extension of heterogenous agents. This is in contrast to the other matching algorithm that is widely used to solve school choice problems, the deferred acceptance algorithm. It is not robust to the extension to heterogenous agents as it was shown before that a stable matching may not exist anymore and the deferred acceptance algorithm cannot be applied. Therefore, the (weighted) top trading cycles algorithm might be a valuable alternative to the widely used deferred acceptance algorithm to be used in applications, such as a centralized kindergarten matching.

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