Approximation of a super-Brownian motion with point source

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Abstract

We study a measure-valued process X^α with $(1+\beta)$-branching, which is related to the self-adjoint extensions of the Laplacian $-\Delta_\alpha$, $\alpha \geq 0$, via a nonlinear partial differential equation. This can be understood as a super-Brownian motion with point source at the origin. Existence of this process was shown by Fleischmann and Mueller 2004. We show that the process X^α can be approximated with a family of processes $X^{\alpha,\varepsilon}$, $\varepsilon \in (0,1)$, related to nonlinear equations involving suitably scaled perturbations of the Laplacian $-\Delta + V_{\alpha,\varepsilon}$. This is done for dimension $d = 3$ and $0 \leq \beta < \frac{1}{3}$. The strategy is mainly analytic, as we prove convergence of solutions of the nonlinear equations in a weighted Lebesgue space. Norm estimates for the resolvents of $-\Delta_\alpha$ and $-\Delta + V_{\alpha,\varepsilon}$ and the associated semigroups are developed in order to control these semigroups in the weighted space. Furthermore, we study basic properties of the approximating processes, such as path regularity.

Zusammenfassung

Wir untersuchen einen maßwertigen Prozess X^α mit $(1+\beta)$-Verzweigung, der mit den selbstadjungierten Erweiterungen des Laplace-Operators $-\Delta_\alpha$, $\alpha \geq 0$, über eine nichtlineare partielle Differentialgleichung in Beziehung steht. Der Prozess kann als super-Brownsche Bewegung mit Punktquelle im Ursprung aufgefasst werden. Seine Existenz wurde von Fleischmann und Mueller 2004 bewiesen. Wir zeigen, dass X^α mit einer Familie von Prozessen $X^{\alpha,\varepsilon}$, $\varepsilon \in (0,1)$ approximiert werden kann, die über nichtlineare Gleichungen mit passend skalierten Störungen des Laplace-Operators $-\Delta + V_{\alpha,\varepsilon}$ in Beziehung steht. Dies wird für Dimension $d = 3$ und $0 \leq \beta < \frac{1}{3}$ durchgeführt. Die Vorgehensweise ist hauptsächlich analytisch, da wir Konvergenz der Lösungen der nichtlinearen Gleichungen in einem gewichteten Lebesgue-Raum beweisen. Es werden Normabschätzungen für die Resolventen von $-\Delta_\alpha$ und $-\Delta + V_{\alpha,\varepsilon}$ und für die entsprechenden Halbgruppen entwickelt, um die Halbgruppen im gewichteten Raum kontrollieren zu können. Außerdem untersuchen wir grundlegende Eigenschaften der approximierenden Prozesse, wie beispielsweise Pfadregularität.

Contents

1. Introduction

In this work we study a super-Brownian motion with singular mass generation constructed by Fleischmann and Mueller in 2004. We will show that this process can be approximated by a family of processes with regular mass generation.

In the introductory chapter we first explain the construction of superprocesses and first basic properties. Then we introduce the super-Brownian motion with point source. Finally, we will discuss the structure of the work and the main results.

1.1. Superprocesses as scaling limits of branching particle systems

Stochastic processes play a central role in modern mathematics. A classical simple example is the symmetric d-dimensional random walk $(S_n)_{n\in\mathbb{N}_0}$ in the lattice $\mathbb{Z}^d$ given by $S_0 = x \in \mathbb{Z}^d$ and

$$S_n := S_0 + \sum_{j=1}^{n} X_j,$$

where the $(X_j)_{j\in\mathbb{N}}$ are independent random variables with distribution

$$\mathbb{P}(X_j = y | y \in \mathbb{Z}^d, |y| = 1) = \frac{1}{2d}.$$

This can be understood as a particle which moves in one of the $2d$ directions in every step with equal probability. The random walk $(S_n)_{n\in\mathbb{N}_0}$ is discrete in time and space. To obtain a continuous process, we embed the lattice in $\mathbb{R}^d$ and define for $n \in \mathbb{N}$ the rescaled random walk

$$W_t^{(n)} = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}.$$

By increasing n, we decrease the length of the time intervals. The factor $\sqrt{n}^{-1}$ compensates the step length. In 1951, Donsker has shown that for $n \to \infty$ the process $(W_t^{(n)})_{t\geq 0}$ converges in distribution to a standard Brownian motion [13]. This is a famous example on how a continuous stochastic process arises from a discrete particle process as a scaling limit. Note that the Brownian motion has the universal property, that it does not require the approximating random variables to have a particular distribution, they only need to be i.i.d.

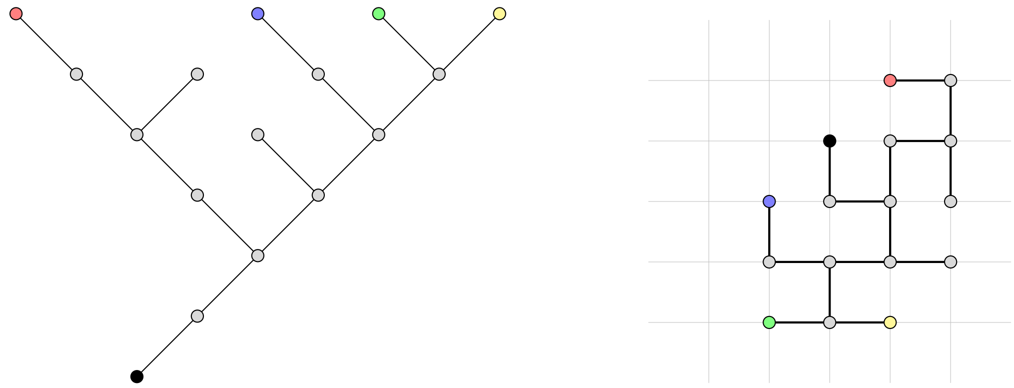

Figure 1.1.: *Example for the construction of the branching particle system. A Galton-Watson tree (left picture) is embedded into the two-dimensional lattice (right picture). The black dot represents the original particle at $t = 0$. The gray particles were alive at times $t = 1$ until $t = 5$. The colored particles are the living descendants at present time $t = 6$.*

Let us consider the discrete model in the two-dimensional lattice. A natural extension is the introduction of particle branching. Imagine we start with a single particle located at $x \in \mathbb{Z}^2$ at time $t = 0$. After one time step, at $t = 1$, the particle dies and gives birth to a random number $\xi \in \{0, 1, 2, \dots\}$ of offsprings. Each of these offsprings spawns at a random lattice point adjacent to x, the points have equal probability $\frac{1}{4}$. In the next step, the offsprings of the first generation die and give birth to a second generation, independent from each other, and so on. The probability distributions of the number of offsprings and the spacial motion remain the same at all times. Note that multiple particles can be present at the same lattice point simultaneously.

There are two sources of randomness in this model: The number of offsprings for each particle and the random spacial motion similar to the simple random walk. We can understand this as follows. In a first random experiment, we fix the genealogy of the population at some time t by choosing a random Galton-Watson tree of depth t. After this step, there is no spacial information yet. In a second random experiment, we embed the Galton-Watson tree into the lattice by placing the root of the tree at $x \in \mathbb{Z}^2$ and incrementally choosing random adjacent lattice points for each new generation as described above. [48, p. 1057-1058] See Figure 1.1 for an example.

Our goal is to scale up this model appropriately in order to pass to a meaningful limit. First we make a restriction on the distribution of the number of offsprings ξ. Let $M = \mathbb{E}(\xi)$. The case $M < 1$ is subcritical, the population will almost surely die out for time t large enough. In the supercritical case $M > 1$ there is a positive probability of survival at all times. Kolmogorov proved in 1938, that for $M = 1$ the probability of survival goes to zero as $t \to \infty$, in fact, this probability is proportional to $t^{-1}\mathrm{Var}(\xi)$. [48, p. 1059] In this case we speak of critical branching.

From now on, assume the critical case $M = 1$. We introduce a scaling variable $n \in \mathbb{N}$. As in the situation of the simple random walk, we scale the space by a factor $\sqrt{n}^{-1}$. Let $m \in \mathbb{N}$ the number of generations in the discrete model. We define scaled time by $t := \frac{m}{n}$. There is no need to restrict ourselves to two dimensions, let $d \in \mathbb{N}$ arbitrary. Now, given a critical Galton-Watson tree T embedded into the scaled grid $\sqrt{n}^{-1}\mathbb{Z}^d$, denote by $R_n^{(m/n)}$ the distribution of particles of the mth generation in the grid. More precisely, $R_n^{(m/n)}$ is the discrete finite measure in $\mathbb{R}^d$, that places mass n^{-1} at each embedded individual of the mth generation, with multiplicity. Since the Galton-Watson tree and the embedding are random, $R_n^{(m/n)}$ is a random measure (but no probability measure). [48, p. 1063].

Because of the critical branching property, the probability that $R_n^{(m/n)}$ is not equal to the zero measure decreases proportional to n^{-1}. To compensate for that, an additional scaling is necessary: We start with n initial particles in generation 0, which evolve independently. Now we obtain a meaningful limit of the mass distribution for $n \to \infty$. The resulting evolving family of random measures $(X_t)_{t\geq 0}$ is a Markov process, taking values in the space of measures $\mathcal{M}(\mathbb{R}^d)$. The initial particle distribution at $t = 0$ also is a finite measure on $\mathbb{R}^d$ in the limit. [48, p. 1065]

We have outlined the construction of the measure-valued super-Brownian motion, starting with a purely discrete model and then passing to the scaling limit with respect to space, time and mass. Now we will introduce another branching particle model, the branching Brownian motion, which also gives rise to a super-Brownian motion. This shows that the super-Brownian motion, similar to ordinary Brownian motion, is a universal object, where different particle models lead to the same limit. The following model is also better suited for a heuristical understanding of the super-Brownian motion as the limit of a population of many Brownian particles. We closely follow the exposition in [18, p. 722].

Again, let $n \in \mathbb{N}$ the scaling variable. Assume we have a number $N_n \in \mathbb{N}$ of particles at positions $x_1^{(n)}, \ldots, x_{N_n}^{(n)} \in \mathbb{R}^d$. This is the situation at time $t = 0$. Now, for $t > 0$, the particles move in space along Brownian paths, independent from each other. Particles have an exponentially distributed lifespan with parameter $cn, c > 0$. When a particle dies, a random number $k \in \{0, 1, 2, \ldots\}$ of offsprings spawn at the same position, which follow Brownian paths independent from each other and from their parent. After i.i.d. exponentially distributed lifetimes, they have offsprings on their own, and so on. In our model, the distribution of the number of offsprings may depend on the current position x of the parent. The probability that a particle dying at point x has exactly k offsprings, is denoted by $p_k^{(n)}(x)$. Assume that the offspring distribution has the expectation

$$e_n(x) := \sum_{k=0}^{\infty} k p_k^{(n)}(x) = 1 + \frac{\gamma(x)}{n} \tag{1.1}$$

and for the variance it holds

$$v_n^2(x) = \sum_{k=0}^{\infty} (k-1)^2 p_k^{(n)}(x) = m(x) + o(1),$$

for $n \to \infty$, uniformly in x. Here $\gamma, m : \mathbb{R}^d \to \mathbb{R}$ are bounded continuous functions with $m > 0$.

For the initial distribution of particles at $t = 0$ we write

$$\mu_n = \frac{1}{n} \sum_{i=1}^{N_n} \delta_{x_i^{(n)}}$$

and assume that the weak limit $\lim_{n\to\infty} \mu_n = \mu \in \mathcal{M}_F(\mathbb{R}^d)$ exists. Let $N_n(t) \in \mathbb{N}$ the number of particles present at time t and $x_1^{(n)}(t), \dots, x_{N_n(t)}^{(n)}(t)$ their positions. Define

$$X_t^{(n)} = \frac{1}{n} \sum_{i=1}^{N_n(t)} \delta_{x_i^{(n)}(t)}.$$

The stochastic process $(X_t^{(n)})_{t\geq 0}$ takes values in the space of all finite measures $\mathcal{M}_F(\mathbb{R}^d)$. Now we scale up $n \to \infty$. The limit

$$X_t = \lim_{n\to\infty} X_t^{(n)} \tag{1.2}$$

in the sense of weak convergence of the induced probability measures exists [18, p. 723]. A first result of this kind was shown by Watanabe in 1968, but only in the sense of convergence of finite-dimensional distributions and with constant $m > 0$ [50, §4]. The resulting measure-valued process $(X_t)_{t\geq 0}$ from (1.2) is related to the solution u of the nonlinear partial differential equation

$$\begin{cases} \partial_t u = (\Delta + V)u - \eta u^2 & \text{on } (0,\infty) \times \mathbb{R}^d, \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d, \end{cases} \tag{1.3}$$

with $V = c\gamma$ and $\eta = \frac{1}{2}cm$ and a bounded and continuous function $f : \mathbb{R}^d \to [0,\infty)$ via the Laplace transition functional

$$\mathbb{E}[e^{-\langle X_t, f\rangle} | X_0 = \mu] = e^{-\langle \mu, u(t,\cdot)\rangle} \tag{1.4}$$

where the measure μ is the weak limit of μ_n for $n \to \infty$. [18, Theorem A2]

Roughly speaking, in this scaling model we increase the number of particles and the branching rate cn, but decrease the weight of each individual particle by a factor $\frac{1}{n}$. Replacing the underlying Brownian motion with a different diffusion process, this concept gives rise to a general class of measure-valued processes, the Dawson-Watanabe superprocesses. The theory was later further developed by Dawson [12] and Dynkin [15] and many others. For a more complete list of references we refer to the monographs of Dynkin [14], Etheridge [19], Li [35] and Perkins [42].

1.2. Super-Brownian motion with point source

As we have seen above, there is a correspondence between superprocesses and certain nonlinear partial differential equations. The Laplace transition formula (1.4) is the pivot at which these two theories are connected with each other. One can imagine this connection as follows: If we want to know the Laplace functional of the superprocess at time $t > 0$ evaluated with the function f, the left side of (1.4), we can take the solution $u(t)$ of (1.3) with initial condition f and compute the right side of (1.4), so we walk time along the solution u instead of the process $(X_t)_{t\geq 0}$.

In the previous section we have constructed the superprocess as the scaling limit of a branching particle system and then we have seen the associated partial differential equation. A natural question is, whether we can go in the other direction: If we start on the analytic side, i.e. with a certain nonlinear PDE, is there a solution of this equation and is there a superprocess corresponding to that solution via the Laplace transition functional?

The equation (1.3) can heuristically be understood as the description of a scaling limit of many particles following Brownian paths, undergoing critical branching everywhere, but with increased mass creation on the support of V. Suppose that the support of V has a certain size, say, $\operatorname{supp}(V) = B_R(0) \subset \mathbb{R}^d$ for a radius $R > 0$. Then the operator $\Delta + V$ in (1.3) is a perturbation of the Laplacian with the property that $(\Delta + V)g = \Delta g$ for all functions satisfying $\operatorname{supp}(g) \subset B_R^c(0)$. In the stochastic interpretation it is intuitive that the Brownian particles enter the ball at some time with a certain probability $p_{R,d} > 0$ and the increased branching rate in this area contributes to the process. But we can also consider the case where $\operatorname{supp}(V)$ becomes very small, which then means that the time a Brownian particle spends in the ball gets shorter. We can try to compensate this effect by increasing the size of the potential V.

In the extreme case, i.e. in the limit $R \to 0$, there is only a single point in $\mathbb{R}^d$ with increased branching rate. To describe this extreme case analytically, we want to replace the perturbed Laplacian $\Delta + V$ in (1.3) by a self-adjoint operator H with the property

$$Hg = \Delta g \text{ for functions } g \in C_0^\infty(\mathbb{R}^d) \text{ with } \operatorname{supp}(g) \subset B_R^c(0) \text{ for } \textit{all } R > 0. \tag{1.5}$$

In other words: Evaluated with functions supported outside an arbitrarily small neighborhood of the origin, the operators H and Δ should conincide. In dimensions $d \geq 4$, this already forces the self-adjoint operator H to be equal to Δ in the Sobolev space $H^{2,2}(\mathbb{R}^d)$ [4, p. 2]. In dimension $d = 1$ there is a 4-parameter family of operators H with property

(1.5). In dimensions $d = 2, 3$ it turns out that there is a one-parameter family of self-adjoint operators H satisfying (1.5), indexed by a renormalized coupling constant $\alpha \in \mathbb{R}$. [4, p. 2-3] We write this family as $(\Delta_\alpha)_{\alpha\in\mathbb{R}}$ and speak of *self-adjoint extensions* of Δ. These operators are of special interest in the field of mathematical physics, describing nonrelativistic quantum mechanical particles interacting via a very short range (in fact zero range) potential with a fixed source [4, p. 1]. The theory goes back to the 1930s, see for example [5]. This model is called *solvable* in the sense, that the resolvents of Δ_α can be given explicitly in terms of α [4, p. 1].

From now on, we focus on dimensions $d = 2, 3$. For every $\alpha \in \mathbb{R}$, the operator Δ_α can roughly be understood as

$$\Delta_\alpha = \Delta + \delta_{0,\alpha},$$

where the scaled Dirac functional $\delta_{0,\alpha}$ describes the point interaction at the origin. Heuristically, Δ_α is the limit of

$$\Delta_\alpha^{(\varepsilon)} := \Delta + h(\alpha, \varepsilon)\mathbf{1}_{B_\varepsilon(0)},$$

as $\varepsilon \downarrow 0$, with a critical rescaling factor h [20, p. 741]. We will give a precise explanation in Chapter 2. Using Δ_α, we can now modify the nonlinear PDE (1.3) and obtain the equation

$$\begin{cases} \partial_t u_\alpha = \Delta_\alpha u_\alpha - \eta u_\alpha^{1+\beta} \ \text{ on } (0,\infty) \times \mathbb{R}^d \setminus \{0\}, \\ u_\alpha(0,\cdot) = f \geq 0 \ \text{ on } \mathbb{R}^d \setminus \{0\}, \end{cases} \tag{1.6}$$

where we replaced the quadratic term with a nonlinearity of order $1 + \beta$ for $0 \leq \beta \leq 1$. The equation (1.6) was investigated by Fleischmann and Mueller 2004 [20]. Existence and uniqueness of the Cauchy problem where shown in weighted L^p spaces for appropriate initial data f. Moreover, it was shown that there is a measure-valued process $X^\alpha := (X_t^\alpha)_{t\geq0}$ such that the solution $u = u_\alpha$ of (1.6) is related to X^α via the Laplace transition functional (1.4). Much more details about these results are given in Chapter 3. The existence of X^α is proven indirectly using mainly analytic methods. In a subsequent work of Fleischmann, Mueller and Vogt, the large-scale behavior of X^α in the three-dimensional case was described [21]. In 2013 Grummt and Kolb extended these results and were also able to prove a version of the strong law of large numbers for X^α in the case $d = 2$, using martingale theory [24].

However, there are still many open questions regarding the properties – e.g. path regularity and long-term behavior – of the superprocess X^α. Moreover, from a stochastic point of view, it is quite surprising that this process even exists, since Brownian particles in $\mathbb{R}^d, d \geq 2$, hit the origin only with probability zero. Consequently, one could intuitively expect that the point source does not contribute to the process and X^α would degenerate to ordinary super-Brownian motion without point source, but this is not the case [20, p. 741]. In his review of [21] Mörters writes

> "This construction is based on the analytical work [...] but remains probabilistically somewhat mysterious [...], as in these dimensions Brownian particles fail to hit single points." *Peter Mörters*

From this perspective, it is desirable to develop methods which allow a further investigation and better understanding of the super-Brownian motion with point source.

In Chapters 3 and 4 we will see that for every $\alpha \in \mathbb{R}$ the transition semigroup $(S_t^\alpha)_{t\geq 0}$ belonging to the process X^α strictly dominates the heat semigroup and it holds

$$\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle] = \langle S_t^\alpha f, \mu\rangle.$$

We want to point out that there are other examples of stochastic processes with these properties, where the behavior at a single point influences the transition semigroup of the process, even though the point is only hit with probability zero. Consider this example from [16, Chapter 4c]: A particle follows a Brownian path in $\mathbb{R}^d$. After a random time, depending on the path, an offspring spawns at the origin and fulfils a Brownian motion on its own. The initial particle moves on undisturbed. After independent random times, both particles produce new offsprings at the origin, and so on. Then the generator of the semigroup $(T_t)_{t\geq 0}$ induced by this process coincides with the Laplacian on $C_0^\infty(\mathbb{R}^d \setminus \{0\})$, because the new particles always spawn at the origin and do not influence the movement of their parents. But because of the mass creation, $(T_t)_{t\geq 0}$ strictly dominates the heat semigroup induced by ordinary Brownian motion. [16, p. 180] However, T_t never coincides with S_t^α, so the models are fundamentally different.

1.3. Structure of the thesis and main results

The main focus of this work lies on the approximation of the superprocess X^α via a family of superprocesses $X^{\alpha,\varepsilon} := (X_t^{\alpha,\varepsilon})_{t\geq 0}$, $\varepsilon > 0$, which are in some sense easier to understand. In order to do so, we start on the analytic side and write the operator Δ_α as the limit for $\varepsilon \to 0$ of a family of operators $H_{\alpha,\varepsilon}$, $\varepsilon > 0$, describing short range interactions. It is the subject of Chapter 2 to construct this family, perform a spectral analysis and prove that the limit – in the norm resolvent sense – is Δ_α, as intended. An important reference in this chapter is the monograph of Albeverio et al. from 1988 [4, Chapter I.1]. As in most parts of the work, we will focus on $d = 3$ here.

In Chapter 3 we give a detailed recapitulation of the Fleischmann-Mueller theory from [20], leading to the well-posedness of the Cauchy problem (1.6) and the existence of the process X^α. Furthermore, using the approximating family $H_{\alpha,\varepsilon}$, $\varepsilon > 0$ and studying the associated operator semigroups, existence of a family of processes $X^{\alpha,\varepsilon}$, $\varepsilon > 0$, is shown. These processes correspond to the solutions of a variant of (1.6), but where Δ_α is replaced by $H_{\alpha,\varepsilon}$. In Chapter 4 we collect properties of the processes, for example explicit moment formulas.

The analytic framework for the theory of Fleischmann and Mueller are weighted L^p spaces. A closer look at these spaces is given in Chapter 5, as well as some important analytic properties of Δ_α and $H_{\alpha,\varepsilon}$ and the associated semigroups $(S_t^\alpha)_{t\geq 0}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ in this context. We want to point out that most results from the literature about Δ_α and $H_{\alpha,\varepsilon}$ and the associated processes are in the setting of unweighted L^p spaces and a main part of the work was to transfer these to the weighted context.

Using the methods obtained in the previous Chapters, weighted L^p–L^q estimates for the resolvents of Δ_α and $H_{\alpha,\varepsilon}$ are developed in Chapter 6. This is done by employing the explicit representations of the resolvents and calculating the involved integrals step by step. It is crucial here, that these estimates are uniform in ε. Moreover, the weighted L^p–L^q estimates can be transferred to the semigroups $(S_t^\alpha)_{t\geq 0}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$, using the representation of analytic semigroups in terms of the resolvents of their generators.

In Chapter 7 we turn our attention to the nonlinear integral equations corresponding to Δ_α and $H_{\alpha,\varepsilon}$ as in (1.3) and the associated solutions u_α and $u_{\alpha,\varepsilon}$. Using the uniform semigroup estimates, we can perform a Picard iteration where the involved Lipschitz constants are independent of ε. The L^p–L^q nature of the estimates helps with controlling the nonlinear terms in the integral equations. Finally, as a main result we can prove that the solutions converge in the weighted space.

Theorem 1.1. *Let $\alpha \geq 0$. Under the conditions of Section 7.1, with the weight $w(x) = |x|^{-1}$ and $p \in (\frac{3}{2}, 2)$, for every $t > 0$ and suitable initial data f it holds*

$$\|u_{\alpha,\varepsilon}(t) - u_\alpha(t)\|_{L^p(w)} \to 0, \quad \textit{for } \varepsilon \to 0.$$

Note that the conditions of this theorem contain some restrictions. In particular we need $\alpha \geq 0$ and for the nonlinear term of the equation it must hold $\beta < \frac{1}{3}$. Refer to Theorem 7.13 for more details. The author conjectures that the restrictions are mainly of technical nature and that the result remains true for a wider range of parameters.

Based on this result, the convergence of the Laplace transforms of the corresponding superprocesses X^α and $X^{\alpha,\varepsilon}$ can directly be obtained. Using methods from the general theory of random measures [29], this gives rise to a mode of convergence of the superprocesses in the space of measures $\mathcal{M}(\mathbb{R}^3)$.

Theorem 1.2. *Under the conditions of Theorem 1.1 let $(X_t^{\alpha,\varepsilon})_{t\geq 0}$ and $(X_t^\alpha)_{t\geq 0}$ the superprocesses associated to $u_{\alpha,\varepsilon}$ and u_α with initial distribution $X_0^{\alpha,\varepsilon} = X_0^\alpha = \mu$. Assume that the measure μ has a density satisfying $\mu(\cdot)|\cdot|^{\frac{1}{p}} \in L^{p'}(\mathbb{R}^3)$. Then it holds for fixed $t \geq 0$*

$$X_t^{\alpha,\varepsilon} \xrightarrow{vd} X_t^\alpha$$

where vd denotes the convergence in distribution with respect to the vague topology on the space of measures $\mathcal{M}(\mathbb{R}^3)$.

More details can be found in Chapter 8. This is a central result, stating that in the three-dimensional case, under some restrictions on the parameters α, β, we can indeed approximate the super-Brownian motion with point source with a family of superprocesses with short-range interaction. This approximating family is more accessible for further investigation of path properties. We show for example, that $X^{\alpha,\varepsilon}$ has càdlàg paths almost surely.

In the last Chapter 9 we give an outlook on open problems and incomplete results. Unfortunately, a convergence result for the 2-dimensional case was outside the scope of this work. Nevertheless, we develop preliminary analytic methods and resolvent estimates for this case as well.

2. Approximation of the Laplacian with point source

The aim of this chapter is to study analytic properties of the self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbb{R}$, of the Laplacian $-\Delta$. In particular, we want to construct a family $(-H_{\alpha,\varepsilon})_{\varepsilon>0}$ of scaled Hamiltonians for each fixed α, that converges towards $-\Delta_\alpha$ for $\varepsilon \downarrow 0$ in an appropriate sense in the Hilbert space $L^2(\mathbb{R}^3)$. The first section focuses on a spectral analysis, in the second section we will obtain the convergence result. A main source is the monograph of Albeverio et al. from 1988 [4]. We will restrict ourselves to the three-dimensional case.

From now on, for $\lambda \in \mathbb{C} \setminus [0,\infty)$, we denote the free resolvent of $-\Delta$ by

$$R_\lambda = (-\Delta - \lambda I)^{-1}$$

with integral kernel

$$R_\lambda(x,y) = \frac{e^{i\sqrt{k}|x-y|}}{4\pi|x-y|}.$$

With a slight abuse of notation we sometimes write $R_\lambda(x-y)$ for $R_\lambda(x,y)$.

2.1. Spectral properties

We start with a spectral analysis of the operator $-\Delta + V$, where V is real-valued, bounded and has compact support. It is well known that the spectrum of the Laplacian $-\Delta$ on $H^2(\mathbb{R}^3)$ is purely absolutely continuous and consists of the nonnegative real axis with no embedded eigenvalues. However, adding a perturbation V to the operator will change the spectrum in general. The particular choice of V plays a central role in the approximation of $-\Delta_\alpha$ in the next section. For convenience of the reader, we also include some known spectral properties, which might still be new for readers with a mainly probabilistic background.

In the case of a real-valued potential V with compact support, the operator $-\Delta + V$ is self-adjoint. This implies that the spectrum is contained on the real axis. More precisely, $\sigma(-\Delta + V)$ consists of the absolutely continuous part $[0,\infty)$ and an at most finite number $N(V)$ of eigenvalues $\lambda_1, \dots, \lambda_{N(V)}$ on the negative real axis. [10, p. 2663]

There are estimates for the number of negative eigenvalues depending on the volume of the potential V. An important example is the following result, which is known as the Cwikel-Lieb-Rozenblum bound.

Lemma 2.1. *[45, Theorem XIII.12]. Let $V : \mathbb{R}^3 \to \mathbb{R}$ bounded with compact support and write $V_- := \min\{V, 0\}$. There is a $C > 0$, independent of V, such that for the number of negative eigenvalues $N(V)$ of the operator $-\Delta + V$ in $L^2(\mathbb{R}^3)$ it holds*

$$N(V) \le C \int_{\mathbb{R}^3} |V_-(x)|^{\frac{3}{2}} \,\mathrm{d}x. \tag{2.1}$$

Note that $V_- \equiv 0$ if V is nonnegative, so in this case the right-hand side of (2.1) vanishes and there are no negative eigenvalues.

In nontrivial cases, we need tools which are finer than the estimate (2.1). The content of the next lemma is the so-called Birman-Schwinger principle, a characterization of the eigenvalues of $-\Delta + V$ in $L^2(\mathbb{R}^3)$. Refer to [44] for a more general treatment.

Lemma 2.2. *Let $V : \mathbb{R}^3 \to \mathbb{R}$ bounded with compact support. Then $\lambda \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of the operator $-\Delta + V$, if and only if -1 is an eigenvalue of the Birman-Schwinger operator $uR_\lambda v$ on $L^2(\mathbb{R}^3)$, where*

$$v(x) = |V(x)|^{1/2}, \quad u(x) = \operatorname{sgn}(V(x))|V(x)|^{1/2}, \quad \text{for } x \in \mathbb{R}^3.$$

Proof. Let $\lambda \in \mathbb{C} \setminus [0, \infty)$. By definition, λ is an eigenvalue of $-\Delta + V$, if and only if there is $\psi \in H^2(\mathbb{R}^3)$ with

$$(-\Delta + V)\psi = \lambda\psi.$$

Clearly, this equation is equivalent to

$$(-\Delta - \lambda I)\psi = -V\psi. \tag{2.2}$$

Since $\sigma(-\Delta) = [0, \infty)$, the resolvent $R_\lambda = (-\Delta - \lambda I)^{-1}$ exists for $\lambda \in \mathbb{C} \setminus [0, \infty)$. By applying R_λ to both sides of (2.2), we get

$$\psi = -R_\lambda V\psi.$$

Multiplying both sides with u gives

$$u\psi = -uR_\lambda V\psi = -uR_\lambda v(u\psi), \tag{2.3}$$

where we have used $V = uv$. Because V is bounded with compact support, $u\psi \in L^2(\mathbb{R}^d)$. Hence, -1 is an eigenvalue of $uR_\lambda v$ with eigenfunction $u\psi$.

Conversely, assume that there is a $\varphi \in L^2(\mathbb{R}^3)$ with

$$uR_\lambda v\varphi = -\varphi. \tag{2.4}$$

Define

$$\psi := R_\lambda v\varphi, \tag{2.5}$$

then $\psi \in H^2(\mathbb{R}^3)$. We can apply $(-\Delta - \lambda I)$ to both sides of (2.5) and obtain

$$(-\Delta - \lambda I)\psi = v\varphi = -vu\psi = -V\psi, \tag{2.6}$$

where we have used $\varphi = -u\psi$, which follows from (2.4). Finally, rearranging (2.6) gives

$$(-\Delta + V)\psi = \lambda\psi,$$

so λ is an eigenvalue of $(-\Delta + V)$ and the proof is complete. □

Remark 2.3. As stated above, if an eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ of $-\Delta + V$ exists, then it is located on the negative real axis. Hence, according to Lemma 2.2, the Birman-Schwinger operator $uR_\lambda v$ can only have -1 as a potential eigenvalue if $\lambda \in (-\infty, 0)$.

For a more detailed analysis, we need to fix a particular V. From now on assume V is the finite spherical square-well potential

$$V = V_R := -\mathbf{1}_{B_R(0)},$$

on $\mathbb{R}^3$, where $R > 0$ is the radius of the potential well. As mentioned above, the operator $-\Delta + V$ has at most finitely many discrete eigenvalues on the negative real axis. The following lemma describes the properties of the lowest eigenvalue, exploiting the radial symmetry of V.

Lemma 2.4. *Suppose that $-\Delta + V$ has an eigenvalue at the bottom of its spectrum*

$$\lambda = \inf \sigma(-\Delta + V).$$

Then λ is nondegenerate, i.e. has multiplicity one. Furthermore, the corresponding eigenfunction is strictly positive and spherically symmetric.

Proof. The potential V is bounded with compact support. From the more general statement [45, Theorem XIII.46] it follows immediately that λ is nondegenerate with strictly positive eigenfunction ψ. We are left to show that ψ is spherically symmetric, this is equivalent to $O\psi = \psi$ for all rotations O of the space $\mathbb{R}^3$. Choose one such O. In particular, O is orthogonal.

Since $-\Delta$ commutes with orthogonal transformations, we have

$$(-\Delta + V)(O\psi) = O((-\Delta + V)\psi) = O(\lambda\psi) = \lambda O\psi, \tag{2.7}$$

so $O\psi$ is an eigenfunction corresponding to λ. But as shown above, λ is nondegenerate, so $O\psi = a\psi$, $a \in \mathbb{C}$. Since O is orthogonal, $|a| = 1$. Furthermore, O is a rotation, so it preserves positivity, this means $a\psi(x) = (O\psi)(x) > 0$ if $\psi(x) > 0$. This implies $a = 1$, because ψ is strictly positive. We have shown $O\psi = \psi$ for an arbitrary rotation O, so ψ is spherically symmetric. □

With Lemma 2.4 we can investigate the existence of negative eigenvalues of the operator $-\Delta + V_R$, depending on the potential radius R.

Theorem 2.5. *For the operator $-\Delta + V_R$ in $L^2(\mathbb{R}^3)$ with domain of definition $\mathcal{D}(-\Delta + V_R) = H^2(\mathbb{R}^3)$ it holds*

(i) For $R < \frac{\pi}{2}$ there are no eigenvalues.

(ii) For $R = \frac{\pi}{2}$ there are no negative eigenvalues. However, $\lambda = 0$ is a simple resonance: There is a $\psi \in L^2_{loc}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$ with $(-\Delta + V_R)\psi = \lambda\psi = 0$. The corresponding eigenspace is one-dimensional.

(iii) For $R > \frac{\pi}{2}$, there are eigenvalues $-1 < \lambda_1 < \ldots < \lambda_{N_R} < 0$. The number of eigenvalues N_R increases with R.

For the proof the following preparatory Lemma is needed.

Lemma 2.6. *Let $k \in \mathbb{R}$. The ordinary second-order differential equation*

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2}{x}\frac{\partial\psi}{\partial x} + k\psi = 0 \tag{2.8}$$

has a two-dimensional solutions space with basis vectors

$$\psi_1(x) = \frac{\sin\sqrt{k}x}{x} \quad \textit{and} \quad \psi_2(x) = \frac{\cos\sqrt{k}x}{x}.$$

Proof. Calculating the derivatives

$$\begin{aligned}\frac{\partial\psi_1}{\partial x} &= \frac{\sqrt{k}x\cos(\sqrt{k}x) - \sin(\sqrt{k}x)}{x^2}, \\ \frac{\partial\psi_2}{\partial x} &= -\frac{\sqrt{k}x\sin(\sqrt{k}x) + \cos(\sqrt{k}x)}{x^2}\end{aligned} \tag{2.9}$$

and the second-order derivatives

$$\frac{\partial^2\psi_1}{\partial x^2} = \frac{2\sin(\sqrt{k}x) - kx^2\sin(\sqrt{k}x) - 2\sqrt{k}x\cos(\sqrt{k}x)}{x^3},$$
$$\frac{\partial^2\psi_2}{\partial x^2} = \frac{2\cos(\sqrt{k}x) - kx^2\cos(\sqrt{k}x) + 2\sqrt{k}x\sin(\sqrt{k}x)}{x^3},$$

it is easy to verify that the functions ψ_1, ψ_2 solve the equation (2.8). Since the solution space of a second-order ordinary differential equation is at most two-dimensional and ψ_1 and ψ_2 are linear independent, they define a basis. □

Proof of Theorem 2.5. Let $\lambda \leq 0$. Assume that

$$(-\Delta + V_R)\psi = \lambda\psi \tag{2.10}$$

for a spherically symmetric function $\psi \in H^2(\mathbb{R}^3)$. We have the following well-known representation of the Laplacian in spherical coordinates

$$\Delta\psi(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}. \tag{2.11}$$

By assumption, ψ is spherically symmetric, so it is constant along any angle. Hence, for the angular derivatives we have

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial\varphi} = 0.$$

In this spherically symmetric situation, the equation (2.11) simplifies to

$$\Delta\psi(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r}.$$

With a slight abuse of notation we can consider ψ as a function of the radius $r = |x|$. Since $V_R = -\mathbf{1}_{B_R(0)}$ is also a radially symmetric function, the equation (2.10) becomes

$$-\frac{\partial^2\psi}{\partial r^2} - \frac{2}{r}\frac{\partial\psi}{\partial r} + V_R\psi = \lambda\psi. \tag{2.12}$$

By definition of V_R, this can be written as

$$\begin{aligned} \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + (1+\lambda)\psi = 0, \quad r \leq R, \\ \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \lambda\psi = 0, \quad r > R. \end{aligned} \tag{2.13}$$

According to Lemma 2.6, the system of ordinary differential equations (2.13) has the solutions

$$\psi(r) = \begin{cases} a_1 \frac{\sin\sqrt{1+\lambda}r}{r} + b_1 \frac{\cos\sqrt{1+\lambda}r}{r}, & r \leq R, \\ a_2 \frac{\sin\sqrt{\lambda}r}{r} + b_2 \frac{\cos\sqrt{\lambda}r}{r}, & r > R. \end{cases} \tag{2.14}$$

with complex-valued coefficients a_1, b_1, a_2, b_2.

First consider the case $r > R$. Since $\lambda \leq 0$, we have $\sqrt{\lambda} = i\sqrt{|\lambda|}$. Using the exponential representations of sin and cos, it holds

$$\begin{aligned} \psi(r) &= a_2 \frac{\sin i\sqrt{|\lambda|}r}{r} + b_2 \frac{\cos i\sqrt{|\lambda|}r}{r} \\ &= ia_2 \frac{e^{\sqrt{|\lambda|}r} - e^{-\sqrt{|\lambda|}r}}{2r} + b_2 \frac{e^{\sqrt{|\lambda|}r} + e^{-\sqrt{|\lambda|}r}}{2r} \\ &= \frac{(ia_2 + b_2)e^{\sqrt{|\lambda|}r} + (b_2 - ia_2)e^{-\sqrt{|\lambda|}r}}{2r} \end{aligned}$$

The coefficient before $e^{\sqrt{|\lambda|}r}$ must vanish, otherwise ψ would not lie in $L^p(\mathbb{R}^3)$ for any p. This leads to the condition $a_2 = ib_2$. Hence, up to linear dependence, the solution for $r > R$ is

$$\psi(r) = \frac{e^{-\sqrt{-\lambda}r}}{r} \tag{2.15}$$

with derivative

$$\frac{\partial\psi}{\partial r} = -\frac{(\sqrt{-\lambda}r + 1)e^{-\sqrt{-\lambda}r}}{r^2}. \tag{2.16}$$

Returning to the system (2.13), we are now focussing on the regularity of the solution at the point $r = R$. For ψ to be in $H^2(\mathbb{R}^3)$, ψ needs to be continuous, so we have the condition

$$a_1 \sin(\sqrt{1+\lambda}R) + b_1 \cos(\sqrt{1+\lambda}R) = e^{-\sqrt{-\lambda}R} \tag{2.17}$$

from (2.14) and (2.15). Furthermore, the first derivative needs to be continuous for the second weak derivative to exist. Using the explicit formulas for the derivatives (2.9) and (2.16), it follows

$$\begin{aligned} &a_1 \frac{\sqrt{k}x\cos(\sqrt{1+\lambda}x) - \sin(\sqrt{1+\lambda}x)}{x^2} - b_1 \frac{\sqrt{1+\lambda}x\sin(\sqrt{1+\lambda}x) + \cos(\sqrt{k}x)}{x^2} \\ &= -\frac{(\sqrt{-\lambda}r + 1)e^{-\sqrt{-\lambda}r}}{r^2}, \end{aligned}$$

which can be simplified to

$$a_1\sqrt{1+\lambda}\cos(\sqrt{1+\lambda}R) - b_1\sqrt{1+\lambda}\sin(\sqrt{1+\lambda}R) = -\sqrt{-\lambda}e^{-\sqrt{-\lambda}R}. \tag{2.18}$$

The equations (2.17) and (2.18) form a system of linear equations in a_1, b_1, which has solutions whenever the matrix

$$\begin{pmatrix} \sin(\sqrt{1+\lambda}R) & \cos(\sqrt{1+\lambda}R) \\ \sqrt{1+\lambda}\cos(\sqrt{1+\lambda}R) & -\sqrt{1+\lambda}\sin(\sqrt{1+\lambda}R) \end{pmatrix}$$

is invertible. This is the case for $\lambda \neq -1$. The solutions are

$$\begin{aligned} a_1 &= -\sin(\sqrt{1+\lambda}R)e^{-\sqrt{-\lambda}R} + \frac{\sqrt{-\lambda}}{\sqrt{1+\lambda}}\cos(\sqrt{1+\lambda}R)e^{-\sqrt{-\lambda}R}, \\ b_1 &= -\cos(\sqrt{1+\lambda}R)e^{-\sqrt{-\lambda}R} - \frac{\sqrt{-\lambda}}{\sqrt{1+\lambda}}\sin(\sqrt{1+\lambda}R)e^{-\sqrt{-\lambda}R}. \end{aligned}$$

Now note that the function

$$r \mapsto \frac{\cos(\sqrt{1+\lambda}r)}{r}$$

is singular at the origin because of $\cos 0 = 1$. Since the eigenfunction needs to be regular, more precisely $\psi \in H^2(\mathbb{R}^3)$, there cannot be a contribution of the singular function for $r \leq R$, because this range includes the origin. This yields the additional condition $b_1 = 0$, which is equivalent to

$$b_R(\lambda) := \cos(\sqrt{1+\lambda}R) + \frac{\sqrt{-\lambda}}{\sqrt{1+\lambda}}\sin(\sqrt{1+\lambda}R) = 0. \tag{2.19}$$

The equation (2.19) cannot have solutions for $\lambda < -1$, where $\sqrt{1+\lambda}$ becomes imaginary and the trigonometric functions turn into hyperblic cosine and hyperbolic sine. The point $\lambda = -1$ has been excluded above. So all potential negative eigenvalues with radial eigenfunctions lie in the range $(-1, 0)$. According to Lemma 2.4, the lowest eigenvalue must have a radial eigenfunction, so all negative eigenvalues lie in this range.

Now assume $\lambda \in (-1, 0)$ and $R < \frac{\pi}{2}$. In this case $\sqrt{1+\lambda}R \in (0, \frac{\pi}{2})$. In this range sine and cosine are strictly positive and the factor $\frac{\sqrt{-\lambda}}{\sqrt{1+\lambda}}$ is positive as well. This implies $b_1 \neq 0$, so there cannot exist any negative eigenvalues with radial eigenfunctions. Moreover, using Lemma 2.4 again, there are no negative eigenvalues whatsoever. This completes the proof of *(i)*.

Let's turn to the case $R = \frac{\pi}{2}$. This case is critical in the sense, that for $\lambda = 0$ we have $b_R(\lambda) = 0$ in equation (2.19) and consequently $b_1 = 0$. So for $r < R$ the regular solution of

(2.14) becomes, up to a constant factor,

$$\psi(r) = \frac{\sin(\sqrt{1+\lambda}r)}{r} = \frac{\sin(r)}{r},$$

where we have used $\lambda = 0$. However, in the range $r > R$, according to (2.15) the solution becomes

$$\psi(r) = \frac{e^{-\sqrt{-\lambda}r}}{r} = \frac{1}{r}. \tag{2.20}$$

The exponential term degenerates to 1, which implies that ψ is not square-integrable, more precisely $\psi \in L^2_{\text{loc}}(\mathbb{R}^3) \smallsetminus L^2(\mathbb{R}^3)$. So ψ is not a proper eigenfunction and $\lambda = 0$ is not an eigenvalue. We call $\lambda = 0$ a resonance and the function

$$\psi(r) = \begin{cases} \frac{\sin(r)}{r}, & r \le \frac{\pi}{2}, \\ \frac{1}{r}, & r > \frac{\pi}{2}, \end{cases} \tag{2.21}$$

the corresponding resonance function. The function ψ is illustrated in Figure 2.1. There is no other linear independent resonance function, because according to [4, p. 19-20], every resonance function is radial if the potential V is radial, which is the case here. This proves *(ii)*.

Finally, for $R > \frac{\pi}{2}$, the function $b_R(\lambda)$ in (2.19) has a growing number of roots $\lambda_j \in (-1, 0)$ with proper square-integrable eigenfunctions. This is illustrated in Figure 2.2. The eigenvalues can be found numerically. This completes the proof. □

Remark 2.7. If $R > \frac{\pi}{2}$ with $N_R \ge 2$, then only the lowest eigenvalue needs to be nondegenerate with spherically symmetric eigenfunction. The eigenvalues $\lambda_2, \dots, \lambda_{N_R}$ can have higher multiplicity and eigenfunctions with angular momentun $l \ge 1$, i.e. functions which are not spherically symmetric. To find these functions, one needs to solve not only the radial, but also the angular part of the partial differential equation (2.10). Solutions of the angular part are related to spherical harmonics, the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^2}$ on the sphere.

We have seen the Birman-Schwinger principle for eigenvalues of $-\Delta + V$ in Lemma 2.2. Now we want to review it in the context of the critical case, where $\lambda = 0$ is a resonance of the operator $-\Delta + V$. Note that the endpoint $\lambda = 0$ was excluded in Lemma 2.2.

Lemma 2.8. *For $V = -\mathbf{1}_{B_{\frac{\pi}{2}}}(0)$ decompose $V = uv$ as in Lemma 2.2. Then -1 is a simple eigenvalue of the Birman-Schwinger operator uR_0v with eigenfunction $\varphi \in L^2(\mathbb{R}^3)$ satisfying*

$$uR_0v\varphi = -\varphi,$$

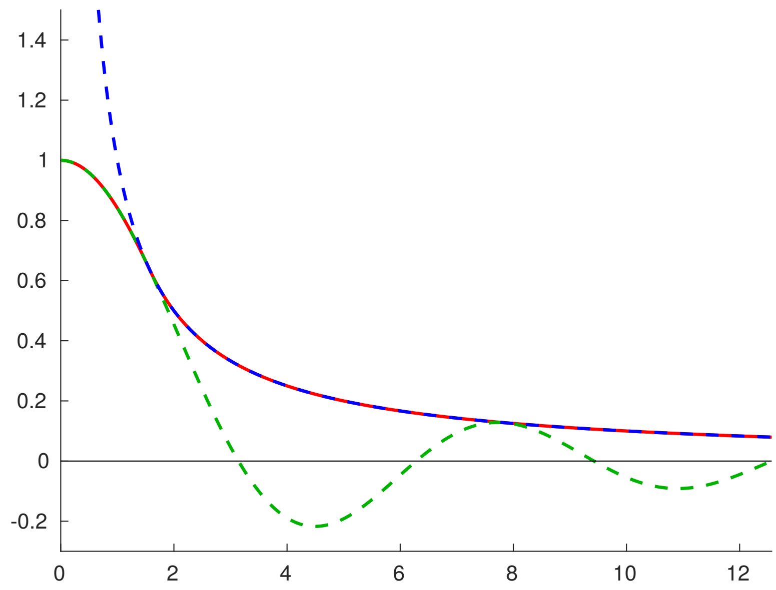

Figure 2.1.: *The function $r \mapsto \frac{1}{r}$ (blue) and the function $r \mapsto \frac{\sin r}{r}$ (green). The red line is the resonance function ψ for $R = \frac{\pi}{2}$ and $\lambda = 0$.*

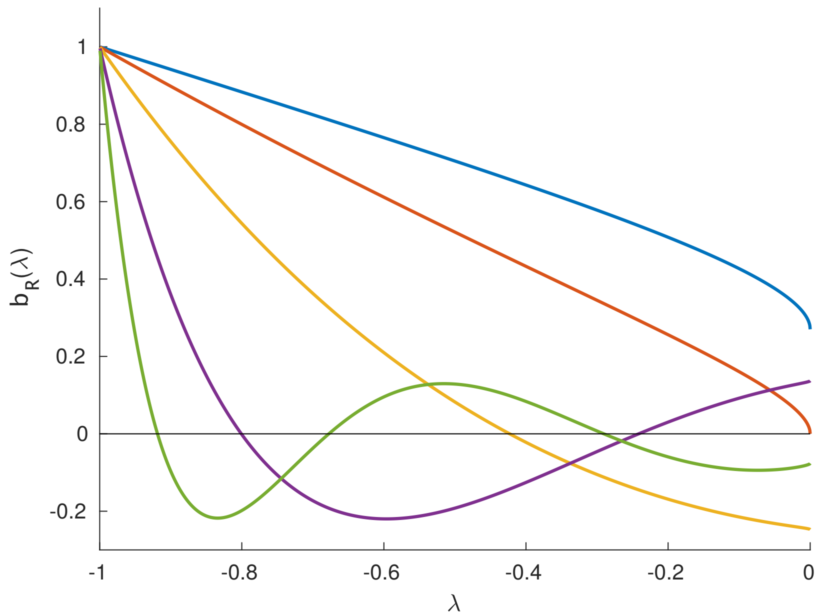

Figure 2.2.: *The function $b_R(\lambda)$ from (2.19) for the subcritical value $R = 1$ (blue), the critical value $R = \frac{\pi}{2}$ (red), and the supercritical values $R = 3$ (yellow), $R = 6$ (purple), $R = 10$ (green). The function graphs are scaled such that $b_R(-1) = 1$. In the supercritical cases, the roots are negative eigenvalues of $-\Delta + V_R$.*

but

$$R_0 v\varphi \in L^2_{loc}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3).$$

Proof. According to Theorem 2.5, the point 0 is a resonance of $-\Delta + V$ whith resonance function $\psi \in L^2_{\text{loc}}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$. This resonance function is unique up to linear dependence. Following the calculations in the first part of the proof of Lemma 2.2, but with $\lambda = 0$ here, the function $\varphi = -u\psi \in L^2(\mathbb{R}^3)$ is an eigenfunction of uR_0v to the eigenvalue -1. Because of the uniqueness of ψ, the point -1 is a simple eigenvalue. It holds

$$R_0 v\varphi = -R_0 vu\psi = -R_0 V\psi = \psi,$$

where we have used equation (2.3) in the last step. Since $\psi \in L^2_{\text{loc}}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$, the proof is complete. □

We now study the scaled potential

$$V_\varepsilon(x) = \varepsilon^{-2} V(x/\varepsilon).$$

for $\varepsilon > 0$. Note that this scaling is not mass preserving, in fact $\|V_\varepsilon\|_{L^1(\mathbb{R}^3)} = o(\varepsilon)$. However, it will turn out that it is the correct scaling to approximate the point source. The next theorem describes the spectral properties of $-\Delta + V_\varepsilon$.

Theorem 2.9. *For $\varepsilon > 0$ define*

$$V_\varepsilon(x) = \varepsilon^{-2} V(x/\varepsilon).$$

Let $\lambda \in \mathbb{R}$. Then λ is an eigenvalue of $(-\Delta + V)$ with eigenfunction ψ, if and only if $\varepsilon^{-2}\lambda$ is an eigenvalue of $(-\Delta + V_\varepsilon)$ with eigenfunction $\psi_\varepsilon = \psi(\cdot/\varepsilon)$. In particular, if $(-\Delta + V)$ does not have negative eigenvalues, then $(-\Delta + V_\varepsilon)$ does not have negative eigenvalues either.

Proof. First we prove that for $x \in \mathbb{R}^3$ and $f \in H^2(\mathbb{R}^3)$ the identity

$$(-\Delta f(\cdot/\varepsilon))(x) = \varepsilon^{-2}(-\Delta f)(x/\varepsilon) \tag{2.22}$$

holds: For $j \in \{1, 2, 3\}$ consider the partial derivative ∂_j. By chain rule of differentiation it holds

$$\partial_j f(x/\varepsilon) = \varepsilon^{-1}(\partial_j f)(x/\varepsilon)$$

and

$$\partial_j^2 f(\cdot/\varepsilon) = \partial_j(\varepsilon^{-1}(\partial_j f)(x/\varepsilon)) = \varepsilon^{-2}(\partial_j^2 f)(x/\varepsilon).$$

With

$$\Delta f = \sum_{j=1}^{3} \partial_j^2 f$$

identity (2.22) follows.

Now assume

$$(-\Delta + V)\psi = \lambda\psi \tag{2.23}$$

for some $\psi \in H^2(\mathbb{R}^3)$. We can compute

$$\begin{aligned}(-\Delta + V_\varepsilon)\psi_\varepsilon(x) &= (-\Delta\psi(\cdot/\varepsilon))(x) + V_\varepsilon(x)\psi(x/\varepsilon)\\ &= -\varepsilon^{-2}(\Delta\psi)(x/\varepsilon) + \varepsilon^{-2}V(x/\varepsilon)\psi(x/\varepsilon)\\ &= \varepsilon^{-2}(-\Delta + V)\psi(x/\varepsilon)\\ &= \varepsilon^{-2}\lambda\psi_\varepsilon(x),\end{aligned}$$

where we have used (2.22) in the second step and the eigenvalue property (2.23) in the fourth step.

Conversely, performing an analogous calculation with $\varepsilon' := \varepsilon^{-1}$, the other implication follows. Since $\lambda < 0$ if and only if $\varepsilon^{-2}\lambda < 0$, the claim about negative eigenvalues follows immediately. □

2.2. Convergence of resolvents

Consider the self-adjoint extensions $(-\Delta_\alpha)_{\alpha\in\mathbb{R}}$ of the Laplacian in $L^2(\mathbb{R}^3)$ [4, p. 2-3]. As explained in the introduction, these extensions can be understood as Laplacian with point-interaction. In this section we give a summary of the convergence result of an approximating family of operators towards $-\Delta_\alpha$ from [4, p. 19–23]. In order to do so, we introduce the spectrum of the operator $-\Delta_\alpha$, which consists of an absolutely continuous part – the nonnegative real axis – and at most one negative real eigenvalue, depending on the parameter α.

Theorem 2.10. *[4, Theorem 1.1.4]. Let $-\infty < \alpha \le \infty$. Then the essential spectrum of $-\Delta_\alpha$ in $L^2(\mathbb{R}^3)$ is purely absolutely continuous with*

$$\sigma_{ess}(-\Delta_\alpha) = [0, \infty).$$

If $\alpha < 0$, the operator $-\Delta_\alpha$ has precisely one negative simple eigenvalue

$$\lambda_1 = -(4\pi\alpha)^2$$

with the strictly positive eigenfunction

$$\psi_1(x) = \sqrt{-\alpha}\frac{e^{-4\pi\alpha|x|}}{|x|}.$$

If $\alpha \geq 0$*, there are no eigenvalues, i.e.*

$$\sigma(-\Delta_\alpha) = \sigma_{ess}(-\Delta_\alpha) = [0, \infty).$$

For the further analysis we will also need the resolvent of $-\Delta_\alpha$.

Lemma 2.11. *[4, Theorem 1.1.4]. Let* $\alpha \geq 0$*. Then for* $\lambda \in \rho(-\Delta_\alpha)$ *the integral kernel of the resolvent* $R_\lambda^\alpha = (-\Delta_\alpha - \lambda I)^{-1}$ *is given by*

$$R_\lambda^\alpha(x, y) = R_\lambda(x, y) + \overline{R}_\lambda^\alpha(x, y) \tag{2.24}$$

with

$$\overline{R}_\lambda^\alpha(x, y) = \frac{1}{\alpha - \frac{i\sqrt{\lambda}}{4\pi}} \frac{e^{i\sqrt{\lambda}(|x|+|y|)}}{(4\pi)^2|x||y|}.$$

For the construction of the approximating family of operators, fix the critical radius $R = \frac{\pi}{2}$ of the potential from Theorem 2.5.

Definition 2.12. Let $V = -\mathbf{1}_{B_{\frac{\pi}{2}}(0)}$. For $\varepsilon > 0$ define

$$-H_{\alpha,\varepsilon} := -\Delta + V_{\alpha,\varepsilon} = -\Delta + P_\alpha(\varepsilon)\varepsilon^{-2}V\left(\frac{x}{\varepsilon}\right) \tag{2.25}$$

with a polynomial P_α such that $P_\alpha(0) = 1$ and $\alpha = -P_\alpha'(0)|(V, \varphi)|^{-2}$.

Remark 2.13. For example, the polynomial

$$P_\alpha(z) := -|(V, \varphi)|^2\alpha \cdot z + 1, \quad \alpha \in \mathbb{R}, \tag{2.26}$$

satisfies the condition of Definition 2.12.

Now we introduce the resolvents of the family $-H_{\alpha,\varepsilon}$.

Lemma 2.14. *[4, Theorem 1.1.4]. Let* $\varepsilon > 0$ *and* $\alpha \in \mathbb{R}$*. Then for* $\lambda \in \rho(-H_{\alpha,\varepsilon})$ *the resolvent* $R_\lambda^{\alpha,\varepsilon} = (-H_{\alpha,\varepsilon} - \lambda I)^{-1}$ *is given by*

$$\begin{aligned} R_\lambda^{\alpha,\varepsilon} &= R_\lambda + \overline{R}_\lambda^{\alpha,\varepsilon} \\ &:= R_\lambda + P_\alpha(\varepsilon)A_\lambda^\varepsilon\varepsilon[1 + B_\lambda^\varepsilon]^{-1}C_\lambda^\varepsilon \end{aligned} \tag{2.27}$$

with operators given by integral kernels

$$A_\lambda^\varepsilon(x, y) = V(y)R_\lambda(x - \varepsilon y),$$

$$
\begin{aligned}
B^{\varepsilon}_{\lambda}(x,y) &= -V(x)P_{\alpha}(\varepsilon)R_{\lambda\varepsilon^2}(x-y)V(y),\\
C^{\varepsilon}_{\lambda}(x,y) &= V(x)R_{\lambda}(\varepsilon x-y),
\end{aligned}
$$

and a real-analytic function $P_\alpha(\cdot)$ with $P_\alpha(0)=1$.

We are now prepared to prove the convergence of the resolvents in L^2 by applying theory from [4, Chapter 1].

Theorem 2.15. *Let $V=-\mathbf{1}_{B_{\frac{\pi}{2}}(0)}$. Then with $-H_{\alpha,\varepsilon}$ defined as in (2.25), if $\lambda\in\rho(-\Delta_\alpha)$, we have $\lambda\in\rho(-H_{\alpha,\varepsilon})$ for $\varepsilon>0$ small enough, and $-H_{\alpha,\varepsilon}\to-\Delta_\alpha$ in norm resolvent sense for $\varepsilon\to0$. This means*

$$\left\| R^{\alpha,\varepsilon}_{\lambda} - R^{\alpha}_{\lambda}\right\|_{L^2\to L^2} \to 0 \quad as \quad \varepsilon\downarrow 0,$$

with the parameter

$$\alpha = -P'(0)_\alpha |(V,\varphi)|^{-2}, \tag{2.28}$$

where φ is the normalized resonance function.

Proof. We want to apply [4, Theorem 1.2.5], so we need to show that all conditions are fulfilled. Estimating with the Hardy-Littlewood-Sobolev inequality shows that V satisfies

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(x)V(y)|}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \le C\|V\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}. \tag{2.29}$$

and the right side is finite because V is bounded with compact support and lies in any L^p space. The finiteness of the left side is known as the Rollnik condition. Under this condition, the integral operator uG_0v with kernel $u(x)G_0(x,y)v(y)$, where u,v are defined as in Lemma 2.2, is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^3)$ [4, p. 17f]. Clearly it also holds $(1+|\cdot|)V\in L^1(\mathbb{R}^3)$.

According to Theorem 2.5, radius $R=\frac{\pi}{2}$ is the critical case where there is a simple resonance at $\lambda=0$ with resonance function ψ. As shown in Lemma 2.8, this resonance is related to a simple eigenvalue -1 of uR_0v, more precisely it holds

$$
\begin{aligned}
uR_0v\varphi &= -\varphi, \quad \varphi\in L^2(\mathbb{R}^3),\\
\psi &:= R_0v\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)\setminus L^2(\mathbb{R}^3).
\end{aligned} \tag{2.30}
$$

Hence, the conditions of [4, Theorem 1.2.5] are fulfilled. Case II in [4, Formula (1.2.53)] and [4, p. 20] holds true, so the claim follows. □

3. Existence theory for the super-Brownian motion with (approximate) point source

In the year 2004 Fleischmann and Mueller showed the existence of a super-Brownian motion X_t related to the partial differential equation (1.6) in [20]. The existence of this process with point source contradicts intuition, because in three dimensions Brownian motion does not hit a given point with positive probability. The Fleischmann-Mueller theory from [20] is fundamental for our further proceding, and we want to recapitulate it in detail within this chapter. Furthermore, we modify the methods slightly to also analyze the equation

$$\begin{cases} \partial_t u_{\alpha,\varepsilon} = H_{\alpha,\varepsilon} u_{\alpha,\varepsilon} - \eta u_{\alpha,\varepsilon}^{1+\beta} & \text{on } (0,\infty) \times \mathbb{R}^3 \setminus \{0\}, \\ u_{\alpha,\varepsilon}(0,\cdot) = f \geq 0 & \text{on } \mathbb{R}^3 \setminus \{0\}, \end{cases} \tag{3.1}$$

which is the analogue of (1.6) when replacing the point-source operator $-\Delta_\alpha$ by the approximating short-range operators $-H_{\alpha,\varepsilon}$. One method used in this context is the Feynman-Kac formula. Due to the singularities in the integral kernels of the semigroup generated by $-\Delta_\alpha$, the classical solution theory for equations of type (3.1) does not work.

The resolvents and semigroups corresponding to the operators $-\Delta_\alpha$ and $-H_{\alpha,\varepsilon}$ are positivity preserving, i.e.

$$f \geq 0 \Rightarrow S_t^\alpha f \geq 0,$$

so it makes sense to study them in a probabilistic context. Indeed, they can be associated with a family of measure-valued processes $(X_t^{\alpha,\varepsilon})_{t\geq 0}, \varepsilon > 0$. It is important to know that throughout this chapter we consider $-H_{\alpha,\varepsilon}$ for fixed $\varepsilon \in (0,1)$ and that the involved constants generally depend heavily on ε. In later chapters we will develop different methods to acquire results uniformly in ε.

3.1. Preliminaries

In this section we introduce the analytic situation, especially the function spaces, and some important tools. This is adopted from [20, p. 743 f]. However, note that we use a different notation for some important objects compared to the source.

Throughout this whole work we will often deal with inequalities involving some positive constants, which depend on generic parameters such as the dimension of the underlying space or an integrability index. In many cases it is not important to determine the exact value of the constant, because the pure existence of such a constant is sufficient. To avoid keeping track of these constants and for better readability we will often rely on the notation

$$f \lesssim g \text{ for } f \le Cg, C > 0.$$

The particular value of C may change between the different occurrences of the symbol "$\lesssim$".

We introduce the weight function

$$w(x) := |x|^{-1}, \quad x \in \mathbb{R}^3.$$

Now, for every $p \ge 1$, the weighted Lebesgue space $L^p(w)$ is defined as the space of equivalence classes f of measurable functions on $\mathbb{R}^3$, such that

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^3} |f(x)|^p w(x) \, \mathrm{d}x \right)^{1/p} < \infty.$$

Definition 3.1. For fixed $p \ge 1$, we define $\Phi = \Phi^p$ as the set of continuous functions $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ such that $f \in L^p(w)$ and

$$0 \le f \lesssim w, \tag{3.2}$$

with the topology induced by the $\|\cdot\|_{L^p(w)}$ norm.

Note that the space Φ^p is not a Banach space, because the constant in (3.2) is not uniform. For the parameters of the partial differential equation (1.6), we will need the restrictions

$$\alpha \in \mathbb{R}, \eta \ge 0, 0 < \beta < 1$$

and for the exponent p we demand

$$\frac{1}{1 - \frac{\beta}{2}} < p < 2. \tag{3.3}$$

As in Definition 2.12, let

$$-H_{\alpha,\varepsilon} := -\Delta + V_{\alpha,\varepsilon} = -\Delta + P_\alpha(\varepsilon)\varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right)$$

the family of approximating operators with short-range interaction.

Remark 3.2. From Definition 2.12 we know that

$$P'_\alpha(0) = -\alpha|(V, \varphi)|^2 \tag{3.4}$$

Now with the rescaled resonance function $\varphi(x) = \frac{1}{\pi}\frac{\sin|x|}{|x|}$ on the ball $B_{\frac{\pi}{2}}(0)$ from Theorem 2.15 we have in polar coordinates

$$|(V,\varphi)|^2 = \left(4\pi \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \sin r \cdot r \,\mathrm{d}r\right)^2 = 16$$

So we can calculate

$$\begin{aligned} V_{\alpha,\varepsilon} &= P_\alpha(\varepsilon)\varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right) \\ &= [-16\alpha\varepsilon + 1]\varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right) \\ &= [16\alpha\varepsilon^{-1} - \varepsilon^{-2}]\mathbf{1}_{B_{\frac{\varepsilon\pi}{2}}(0)} \end{aligned}$$

Redefining $\varepsilon \mapsto \frac{2}{\pi}\varepsilon$ normalizes the radius of the ball and gives

$$V_{\alpha,\varepsilon} = [16\alpha\frac{\pi}{2}\varepsilon^{-1} - \frac{\pi^2}{4}\varepsilon^{-2}]\mathbf{1}_{B_\varepsilon(0)} = [8\pi\alpha\varepsilon^{-1} - \frac{\pi^2}{4}\varepsilon^{-2}]\mathbf{1}_{B_\varepsilon(0)}$$

which corresponds precisely to the expression in [4, Formula (H.49)] with the parameters $\gamma = \beta = 0$ there.

We introduce the Feynman-Kac formula, which is an important connection between partial differential equations and stochastic processes.

Theorem 3.3. *[33, Proposition 1]. Let $V : \mathbb{R}^3 \to \mathbb{R}$ a bounded piecewise contiunous potential and $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ continuous and of subexponential growth, i.e.* $\lim_{|x|\to\infty}(\ln f(x))|x|^{-1} = 0$. *Then the unique solution of the cauchy problem*

$$\begin{cases} \partial_t u = \Delta u - Vu \ \ on \ \ (0,\infty)\times\mathbb{R}^3, \\ u(0,\cdot) = f, \end{cases}$$

is given by

$$u(t,x) = \mathbb{E}\left[e^{-\int_0^t V(W_s)\,\mathrm{d}s} f(W_t) \mid W_0 = x\right] \tag{3.5}$$

where W_t is a Brownian motion in $\mathbb{R}^3$.

3.2. The heat semigroup with short-range interaction

In this section we give an overview over some properties of the heat flow on the weighted space $L^p(w)$. This is a summary of results from [20, Sections 2.2-2.3]. In addition, we derive results on the action of the semgroup $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ defined below in the weighted setting.

Remember the fundamental solution

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} e^{-|x-y|^2/4t}, \quad t>0, \quad x,y \in \mathbb{R}^3$$

of the classical heat equation

$$\begin{cases} \partial_t u = \Delta u \text{ on } (0,\infty)\times\mathbb{R}^3, \\ u(0,\cdot) = f. \end{cases} \tag{3.6}$$

As an integral kernel, p_t gives rise to the heat semigroup $(S_t)_{t\geq 0}$,

$$S_t f(x) = \int_{\mathbb{R}^3} p_t(x,y) f(y)\,\mathrm{d}y.$$

It is well known that the generator of the semigroup $(S_t)_{t\geq 0}$ is $-\Delta$ and that $u(t,x) = S_t f(x)$ is a solution of the Cauchy problem (3.6). According to Theorem 3.3 with $V=0$ we also have

$$S_t f(x) = \mathbb{E}[f(W_t)|W_0 = x]. \tag{3.7}$$

Now we collect some properties of the heat semigroup. The following lemmas with proofs are from [20, Sec. 2.2], note that we change notation to fit in our setting. We start with a heat flow estimate for the weight w.

Lemma 3.4. *[20, Lemma 2.1] For the weight function $w(x) = |x|^{-1}$ there is some $C>0$ such that*

$$S_t w \leq Cw, \quad t \geq 0$$

Proof. The claim is trivial for $t=0$, so we can assume $t>0$. Let $x \neq 0$. We have to show that

$$\frac{1}{w(x)} S_t w(x)$$

is bounded in $t>0$ and $x\neq 0$. With the transformation $y \mapsto t^{-\frac{1}{2}}(y-x)$ and $z := -t^{-\frac{1}{2}}x$ we calculate

$$\begin{aligned} \frac{1}{w(x)} S_t w(x) &= \frac{1}{w(x)} \int_{\mathbb{R}^3} \frac{1}{(4\pi t)^{3/2}} e^{-|x-y|^2/4t} w(x)\,\mathrm{d}y \\ &\approx \int_{\mathbb{R}^3} w\left(\frac{y-z}{|z|}\right) e^{-\frac{|y|^2}{4}}\,\mathrm{d}y. \end{aligned} \tag{3.8}$$

Now it suffices to show that the remaining integral is bounded in $z\neq 0$. First consider the case $|y| \le \frac{|z|}{2}$. This implies $|y - z| \ge \frac{|z|}{2}$ and consequently

$$w\left(\frac{y-z}{|z|}\right) \le 2.$$

So for the integral in (3.8) restricted to $|y| \le \frac{|z|}{2}$ we have the bound

$$2\int_{\{y:|y|\le|z|/2\}} e^{-\frac{|y|^2}{4}}\,\mathrm{d}y$$

which is of course bounded because of the exponential decay. On the other hand, if we restrict the integral to the subset where $|y| \ge \frac{|z|}{2}$, the exponential expression can be estimated from above by $e^{-\frac{|z|^2}{32}}e^{-\frac{|y|^2}{8}}$. And since $|z|e^{-\frac{|z|^2}{32}}$ is bounded in $\mathbb{R}^3$, there is a $\overline{C} > 0$ such that we have the following bound for the integral in (3.8) restricted to $|y| \ge \frac{|z|}{2}$

$$\overline{C}\int_{\{y:|y|>|z|/2\}} w(y-z)e^{-\frac{|y|^2}{8}}\,\mathrm{d}y.$$

This integral converges because of the exponential decay and the fact that the function w is locally integrable in $\mathbb{R}^3$. This completes the proof. □

Next we cite a maximization result for the heat flow of w.

Lemma 3.5. *[20, Lemma 2.2]. Let $\kappa > 0$. Then*

$$S_t w^\kappa(x) \le S_t w^\kappa(0) \tag{3.9}$$

for $t > 0$ and $x \in \mathbb{R}^3$.

The maximization property (3.9) allows us to obtain the following estimate of the semigroup $(S_t)_{t\ge0}$ in case of an additional singularity.

Lemma 3.6. *[20, Lemma 2.3]. Let $0 \le \beta \le 1$ and p satisfying the condition (3.3). Then it holds for all $f \in L^p(w)$*

$$\left\|S_t(fw^\beta)\right\|_{L^p(w)} \lesssim t^{-\frac{\beta}{2}}\|f\|_{L^p(w)}.$$

Proof. For $t > 0$ and $x \in \mathbb{R}^3$ we define the measure $\mu_{t,x}$ given by the density

$$\mu_{t,x}(y) := t^\kappa p_t(x,y)w^{2\kappa}(y)$$

with $\kappa = \frac{1}{2}\frac{\beta p}{p-1}$. Because of the maximum at the center property (3.9), it holds for the total mass of the measure $\mu_{t,x}$

$$\|\mu_{t,x}\| \le \|\mu_{t,0}\| = \int_{\mathbb{R}^3} p_t(0,y)\,\mathrm{d}y =: C$$

where we used Brownian scaling in the last step. As a consequence, the measures $\mu_{t,x}$ are finite measures with a total mass of at most C, independent of t and x. For every finite measure μ on $\mathbb{R}^3$ and measurable functions f, it holds by Hölder's inequality

$$\Big(\int_{\mathbb{R}^3} |f(y)|\,\mathrm{d}\mu(y)\Big)^p \leq \|\mu\|^{p-1} \int_{\mathbb{R}^3} |f(y)|^p\,\mathrm{d}y.$$

We apply this to the measures $\mu_{t,x}$ and obtain

$$\begin{aligned}
|S_t(fw^\beta)(x)|^p &= \left|\int_{\mathbb{R}^3} f(y)w^\beta(y)p_t(x,y)\,\mathrm{d}y\right| \\
&= t^{-\kappa p}\left|\int_{\mathbb{R}^3} f(y)w^{\beta-2\kappa}(y)\,\mathrm{d}\mu_{t,x}(y)\right| \\
&\leq t^{-\kappa p}\|\mu_{t,x}\|^{p-1}\int_{\mathbb{R}^3} |f(y)|^p w^{p(\beta-2\kappa)}(y)\,\mathrm{d}\mu_{t,x}(y) \\
&\leq t^{-\kappa p+\kappa}C^{p-1}\int_{\mathbb{R}^3} |f(y)|^p w^{p(\beta-2\kappa)+2\kappa}(y)p_t(x,y)\,\mathrm{d}y \\
&= t^{-\kappa(p-1)}C^{p-1}\int_{\mathbb{R}^3} |f(y)|^p p_t(x,y)\,\mathrm{d}y \\
&= t^{-\kappa(p-1)}C^{p-1}S_t(|f|)^p(x)
\end{aligned}$$

where we have used $p(\beta-2\kappa)+2\kappa = 0$. By Lemma 3.4 and Fubini's theorem we have

$$\begin{aligned}
\int_{\mathbb{R}^3} S_t(|f|)^p(x)w(x)\,\mathrm{d}x &= \int_{\mathbb{R}^3} |f(y)|^p S_t w(y)\,\mathrm{d}y \\
&\leq \int_{\mathbb{R}^3} |f(y)|^p \overline{C}w(y)\,\mathrm{d}y \\
&= \overline{C}\|f\|^p_{L^p(w)}
\end{aligned}$$

for a constant $\overline{C} > 0$. Hence,

$$\left\|S_t(fw^\beta)\right\|^p_{L^p(w)} \leq t^{-\kappa(p-1)}C^{p-1}\overline{C}\|f\|^p_{L^p(w)}$$

and the claim follows because of $\kappa(p-1) = \frac{\beta p}{2}$. □

Finally we obtain strong continuity of the heat flow.

Lemma 3.7. *[20, Lemma 3.4]. The semigroup $(S_t)_{t\geq 0}$ acting on $L^p(w)$ is strongly contiuous.*

Outline of proof. The proof uses Lemma 3.6. In a first step we show the claim for bounded functions on compact sets and then later remove this restriction in a second step. Refer to the proof of [20, Lemma 2.4] for full details. □

Now we introduce the semigroup $(S^{\alpha,\varepsilon}_t)_{t\geq 0}$ corresponding to the operator $H_{\alpha,\varepsilon} = -\Delta + V_{\alpha,\varepsilon}$.

Let $\varepsilon \in (0,1)$ and $\alpha \in \mathbb{R}$. For $f \geq 0$ continuous on $\mathbb{R}^3 \setminus 0$, $x \in \mathbb{R}^3$ and $t \geq 0$ define

$$S_t^{\alpha,\varepsilon} f(x) = \mathbb{E}\left[e^{-\int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s} f(W_t) \mid W_0 = x\right], \tag{3.10}$$

where W is a Brownian motion in $\mathbb{R}^3$. According to the Feynman-Kac formula from Theorem 3.3, $u := S_t^{\alpha,\varepsilon} f$ is the unique solution of the cauchy problem

$$\begin{cases} \partial_t u = H_{\alpha,\varepsilon} \text{ on } (0,\infty) \times \mathbb{R}^3, \\ u(0,\cdot) = f. \end{cases} \tag{3.11}$$

The definition (3.10) allows us to easily obtain pointwise estimates of $S_t^{\alpha,\varepsilon} f$ in terms of the heat semigroup, as seen in the next lemma.

Lemma 3.8. *Let $\varepsilon \in (0,1)$, $\alpha \in \mathbb{R}$ and $T > 0$ fixed. Let $f \geq 0$ satisfying the conditions of theorem 3.3. There are constants $c_{T,\alpha,\varepsilon}, C_{T,\alpha,\varepsilon} > 0$ such that for every $t \in [0,T]$ and $x \in \mathbb{R}^3$*

$$c_{T,\alpha,\varepsilon} S_t f(x) \leq S_t^{\alpha,\varepsilon} f(x) \leq C_{T,\alpha,\varepsilon} S_t f(x). \tag{3.12}$$

In particular, the statement is true for all $f \in \Phi^p$.

Proof. We want to estimate the exponential term in (3.10). It holds

$$-T\|V_{\alpha,\varepsilon}\|_\infty \leq -t\|V_{\alpha,\varepsilon}\|_\infty \leq \int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s \leq t\|V_{\alpha,\varepsilon}\|_\infty \leq T\|V_{\alpha,\varepsilon}\|_\infty. \tag{3.13}$$

Consequently

$$e^{-T\|V_{\alpha,\varepsilon}\|_\infty} \leq e^{-\int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s} \leq e^{T\|V_{\alpha,\varepsilon}\|_\infty}.$$

Applied to the semigroup, this gives

$$\begin{aligned} S_t^{\alpha,\varepsilon} f(x) &= \mathbb{E}\left[e^{-\int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s} f(W_t) \mid W_0 = x\right] \\ &\geq \mathbb{E}\left[e^{-T\|V_{\alpha,\varepsilon}\|_\infty} f(W_t) \mid W_0 = x\right] \\ &= e^{-T\|V_{\alpha,\varepsilon}\|_\infty} \mathbb{E}\left[f(W_t) \mid W_0 = x\right] \\ &=: c_{T,\alpha,\varepsilon} \mathbb{E}\left[f(W_t) \mid W_0 = x\right] \end{aligned}$$

and similarly

$$S_t^{\alpha,\varepsilon} f(x) \leq e^{T\|V_{\alpha,\varepsilon}\|_\infty} \mathbb{E}\left[f(W_t) \mid W_0 = x\right] =: C_{T,\alpha,\varepsilon} \mathbb{E}\left[f(W_t) \mid W_0 = x\right].$$

With (3.7) we find that

$$\mathbb{E}\left[f(W_t) \mid W_0 = x\right] = S_t f(x)$$

and the proof is complete. □

Remark 3.9. We want to emphasize here that the constants in Lemma 3.8 depend heavily on α and ε. In particular

$$C_{T,\alpha,\varepsilon} = e^{T\|V_{\alpha,\varepsilon}\|_\infty} = e^{T|P_\alpha(\varepsilon)|\varepsilon^{-2}} = e^{T(|\langle V,\varphi\rangle|^2\alpha\varepsilon^{-1}+\varepsilon^{-2})}$$

by Definition 2.12, and this term clearly tends to infinity for $\varepsilon \downarrow 0$. In this chapter, estimates for fixed ε are sufficient. In later chapters we will develop other tools to obtain estimates that hold uniformly in ε.

Lemma 3.10. *The semigroup $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ is strongly continuous on $L^p(w)$ for $p > \frac{1}{1-\beta/3}$, this means*

$$\|S_t^{\alpha,\varepsilon} f - f\|_{L^p(w)} \to 0 \ \text{ for } t \to 0$$

Proof. Without loss of generality assume $f \geq 0$, otherwise decompose $f = f_+ - f_-$ and use linearity. Let $t > 0$. We expand

$$S_t^{\alpha,\varepsilon} f = S_t f + (S_t^{\alpha,\varepsilon} - S_t) f,$$

so we have

$$\|S_t^{\alpha,\varepsilon} f - f\|_{L^p(w)} \leq \|S_t f - f\|_{L^p(w)} + \|(S_t^{\alpha,\varepsilon} - S_t) f\|_{L^p(w)}. \tag{3.14}$$

The first term tends to zero for $t \to 0$ because of the strong continuity of the heat semigroup in the weighted space, Lemma 3.7. For the expression inside the second norm in (3.14) we have, using (3.10) and (3.7),

$$(S_t^{\alpha,\varepsilon} - S_t) f = \mathbb{E}\left[\left(e^{-\int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s} - 1\right) f(W_t) \mid W_0 = x\right]. \tag{3.15}$$

As in (3.13), it holds

$$-t\|V_{\alpha,\varepsilon}\|_\infty \leq \int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s \leq t\|V_{\alpha,\varepsilon}\|_\infty$$

which leads to the bound

$$\left|e^{-\int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s} - 1\right| \leq e^{t\|V_{\alpha,\varepsilon}\|_\infty} - 1.$$

Applying the norm to (3.15), we obtain

$$\|(S_t^{\alpha,\varepsilon} - S_t) f\|_{L^p(w)} = \left\|\mathbb{E}\left[\left(e^{-\int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s} - 1\right) f(W_t) \mid W_0\right]\right\|_{L^p(w)}$$

$$\leq (e^{t\|V_{\alpha,\varepsilon}\|_\infty} - 1)\left\|\mathbb{E}\left[f(W_t) \mid W_0\right]\right\|_{L^p(w)}$$
$$= (e^{t\|V_{\alpha,\varepsilon}\|_\infty} - 1)\|S_t f\|_{L^p(w)}$$

But because of $\|S_t f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$, which is Lemma 3.6 with $\beta = 0$, and

$$\lim_{t\to 0}(e^{t\|V_{\alpha,\varepsilon}\|_\infty} - 1) = 0$$

we have

$$\|(S_t^{\alpha,\varepsilon} - S_t)f\|_{L^p(w)} \to 0$$

for $t \to 0$ and, returning to (3.14), we have shown strong continuity. □

Corollary 3.11. *The infinitesimal generator of the semigroup $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ is given by the operator $-H_{\alpha,\varepsilon}$. Furthermore, for $f \in \Phi^p$, the map $(t,x) \mapsto S_t^{\alpha,\varepsilon} f(x)$ is continuous on $[0,T] \times \mathbb{R}^3$.*

Proof. Because $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ is strongly continuous and is the unique solution of the abstract Cauchy problem (3.11), the operator $-H_{\alpha,\varepsilon}$ is the infinitesimal generator of $(S_t^{\alpha,\varepsilon})_{t\geq 0}$, see for example [17, Section II.6]. The continuity also follows from the fact that $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ solves the Cauchy problem. □

With Lemma 3.8 we immediately obtain the following two estimates.

Corollary 3.12. *Under the conditions of Lemma 3.8 there is a constant $C > 0$ such that*

$$S_t^{\alpha,\varepsilon} w(x) \leq C w(x).$$

Proof. Because of

$$S_t^{\alpha,\varepsilon} w(x) \leq C_{T,\alpha,\varepsilon} S_t w(x)$$

the claim follows from Lemma 3.4. □

Corollary 3.13. *For $f \in \Phi^p$ and with β, p satisfying* (3.3)*, it holds*

$$\left\|S_t^{\alpha,\varepsilon}(f w^\beta)\right\| \lesssim t^{-\frac{\beta}{2}} \|f\|_{L^p(w)}.$$

Proof. If $f \in \Phi^p$, then $f w^\beta$ satisfies the conditions of Lemma 3.8. This leads to

$$S_t^{\alpha,\varepsilon}(f w^\beta) \leq C_{T,\alpha,\varepsilon} S_t(f w^\beta),$$

and the claim follows from Lemma 3.6 by applying the norm. □

Corollary 3.14. *Under the conditions of Lemma 3.8 and with β, p satisfying* (3.3)*, $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ is a strongly continuous semigroup acting on Φ^p.*

Proof. As shown in Lemma 3.10, the semigroup $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ is strongly continuous on $L^p(w)$ and $\Phi^p \subset L^p(w)$ is a subspace. Hence, it remains to show that for $f \in \Phi^p$ the function $x \mapsto S_t^{\alpha,\varepsilon} f(x)$ is in Φ^p as well. The continuity of this function follows from Corollary 3.11. Furthermore

$$S_t^{\alpha,\varepsilon} f \leq C_{T,\alpha,\varepsilon} S_t f \lesssim S_t w \lesssim w, \tag{3.16}$$

where we have used Lemma 3.8 and Lemma 3.4. This completes the proof. □

3.3. The heat semigroup with point source

In this section we will take a closer look at the solutions of the linear partial differential equation with point interaction, that is

$$\partial_t u = \Delta_\alpha u \ \text{ on } \ (0,\infty) \times \mathbb{R}^3. \tag{3.17}$$

This is a summary of the results of sections 2.4 – 2.7 from [20] and we will omit some details and proofs. First we introduce the fundamental solution of (3.17). Fix $\alpha \in \mathbb{R}$. Define

$$p_{\alpha,t}(x,y) := p_t(x,y) + \frac{2t}{|x||y|} p_t(|x|+|y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty p_t(z+|x|+|y|) e^{-4\pi\alpha z} \,\mathrm{d}z \tag{3.18}$$

where $t > 0$, $x, y \neq 0$. The kernel $p_{\alpha,t}$ is the fundamental solution of (3.17) computed in [2, Formula (3.4)].

Remark 3.15. [20, p. 747]. The last term in (3.18) involving the integral is always finite and disappears for $\alpha = 0$. In the case $\alpha \neq 0$, $p_{\alpha,t}(x,y)$ is continuous and decreasing in α with $p_{\alpha,t} \downarrow p_t$ pointwise as $\alpha \uparrow \infty$ and $p_{\alpha,t} \uparrow \infty$ pointwise as $\alpha \downarrow -\infty$.

Since $-\Delta_\alpha$ is a self-adjoint extension of the Laplacian $-\Delta$ on $\mathbb{R}^3 \setminus \{0\}$ [20, p. 747], we have the following consequence.

Corollary 3.16. *[20, Corollary 2.5]. Let $\alpha \in \mathbb{R}$. Then p_α solves the heat equation. More precisely*

$$\partial_t p_{\alpha,t}(x,y) = \Delta p_{\alpha,t}(x,y) \mathit{on} \ \ (0,\infty) \times \mathbb{R}^3 \setminus \{0\}, \tag{3.19}$$

where the Laplacian acts on x (or y, respectively). In particular, $(t,x,y) \mapsto p_{\alpha,t}(x,y)$ is jointly continuous on $(0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$.

With $(S_t^\alpha)_{t\geq 0}$ we denote the semigroup corresponding to the kernel p_α, that is

$$S_t^\alpha f(x) := \int_{\mathbb{R}^3} p_{\alpha,t}(x,y) f(y) \,\mathrm{d}y, \tag{3.20}$$

for all f such that the right-hand side makes sense.

Lemma 3.17. *[20, Lemma 2.6]. Let $\alpha \in \mathbb{R}$ and $T > 0$. Define*

$$\overline{p}_t(x, y) := t^{-\frac{1}{2}} w(x) w(y) e^{-\frac{|x|^2}{4t}} e^{-\frac{|y|^2}{4t}}. \tag{3.21}$$

Then there is a constant $C(\alpha, T)$ such that

$$p_t(x, y) \le p_{\alpha,t}(x, y) \le p_t(x, y) + C(\alpha, T)\overline{p}_t(x, y) \tag{3.22}$$

for all $t \in (0, T]$ and $x, y \neq 0$.

Outline of proof. According to (3.18) we have to show that

$$\frac{2t}{|x||y|} p_t(|x| + |y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty p_t(z + |x| + |y|) e^{-4\pi\alpha z} \,\mathrm{d}z \le C(\alpha, T)\overline{p}_t(x, y). \tag{3.23}$$

Let $\alpha \ge 0$. Then the right-hand side is bounded by

$$\begin{aligned} \frac{2t}{|x||y|} p_t(|x| + |y|) &= \frac{2t}{|x||y|} (4\pi t)^{-\frac{3}{2}} e^{-(|x|+|y|)^2/4t} \\ &\le \frac{1}{4\pi^{\frac{3}{2}}} t^{-\frac{1}{2}} |x|^{-1} |y|^{-1} e^{-\frac{|x|^2}{4t}} e^{-\frac{|y|^2}{4t}} \\ &= \frac{1}{4\pi^{\frac{3}{2}}} \overline{p}_t(x, y) \end{aligned}$$

where we have used $-(|x| + |y|)^2 \le -|x|^2 - |y|^2$ in the second step. This shows (3.23) in this case.

For $\alpha < 0$ the proof of (3.23) is more difficult. We have to find a constant $C(\alpha, T)$ such that

$$\frac{8\pi\alpha t}{|x||y|} \int_0^\infty p_t(z + |x| + |y|) e^{-4\pi\alpha z} \,\mathrm{d}z \le C(\alpha, T) p_t(|x| + |y|).$$

This is done in [20, p. 748ff], please refer to this source for full details. For the constant it holds $C(\alpha, T) \approx |\alpha| T \exp(8\pi^2 |\alpha|^2 T)$ [20, Expression (2.45)]. Note that this growth of the constant matches with the fact that $p_\alpha \uparrow \infty$ pointwise as $\alpha \downarrow -\infty$. □

Using the kernel $\overline{p}$, we define for $t > 0$ and $x \neq 0$

$$\overline{S}_t f(x) := \int_{\mathbb{R}^3} \overline{p}_t(x, y) f(y) \,\mathrm{d}y, \tag{3.24}$$

as long as the right-hand expression makes sense [20, p. 750]. Using estimate (3.22), it follows for $f \in L^p(w)$

$$\|S_t f\|_{L^p(w)} \le \|S_t^\alpha f\|_{L^p(w)} \le \|S_t f\|_{L^p(w)} + C(\alpha, T)\|\overline{S}_t f\|_{L^p(w)}. \tag{3.25}$$

This allows us to transfer some of our estimates of the heat semigroup to the semigroup $(S_t^\alpha)_{t\ge0}$. We cite these from [20, Subsections 2.6–2.7] without proofs. Refer to the source for detailed proofs. The main strategy is to use the decomposition given by the right-hand side of (3.25), exploit the properties of the heat semigroup and control the residue term $\overline{S}_t f$ appropriately.

Lemma 3.18. *[20, Corollary 2.9]. Let $0 \le \beta \le 1$, $T > 0$ and p satisfying* (3.3). *There is a constant $C(T, \alpha, \beta, p)$ such that*

$$\left\|S_t^\alpha(f w^\beta)\right\| \le C(T, \alpha, \beta, p) t^{-\frac{\beta}{2}} \|f\|_{L^p(w)}.$$

Corollary 3.19. *[20, Corollary 2.11]. Let $p \in (1, 2)$, $T > 0$ and $f \in L^p(w)$ with $0 \le f \lesssim w$. There is a constant $C(T, \alpha, p, f) > 0$ such that*

$$0 \le S_t^\alpha f \le C(T, \alpha, p, f)(1 + t^{\frac{1}{p}-\frac{1}{2}})w$$

for $0 \le t \le T$. In particular, $S_t^\alpha f \in \Phi^p$ for all $t > 0$.

Corollary 3.20. *[20, Corollary 2.10, Corollary 2.12]. Let $p \in (1, 2)$ and $\alpha \in \mathbb{R}$. The semigroup $(S_t^\alpha)_{t\ge0}$ is strongly continuous acting on $L^p(w)$ and strongly continuous acting on Φ^p as well.*

3.4. Solution of the nonlinear integral equation

In this section we prove the existence and uniqueness of solutions of the nonlinear integral equation belonging to the Cauchy problems (1.6) and (3.1) respectively. To avoid redundancies, we introduce the following notation to deal with the semigroups $(S_t^{\alpha,\varepsilon})_{t\ge0}$ and $(S_t^\alpha)_{t\ge0}$ simultaneously.

Assumption 3.21. Let $T > 0, 0 \le \beta < 1$ and $\frac{1}{1-\frac{\beta}{2}} < p < 2$. Assume that $(\tilde{S}_t)_{t\ge0}$ is a semigroup on $L^p(w, \mathbb{R}^3)$ with the following properties.

(i) $(\tilde{S}_t)_{t\ge0}$ is a strongly continuous semigroup acting on Φ^p.

(ii) There are constants $c(T), C(T), \overline{C}(T) \ge 0$ such that for every $0 < t \le T, x \in \mathbb{R}^3$ and $f \in \Phi^p$ it holds

$$c(T) \le \tilde{S}_t f(x) \le C(T) S_t f(x) + \overline{C}(T)\overline{S}_t f(x) \tag{3.26}$$

where $(S_t)_{t\ge0}$ is the heat semigroup and $(\overline{S}_t)_{t\ge0}$ is given by (3.24).

(iii) For $f \in \Phi^p$ it holds

$$\left\|\tilde{S}_t(f w^\beta)\right\|_{L^p(w)} \lesssim t^{-\frac{\beta}{2}} \|f\|_{L^p(w)}. \tag{3.27}$$

(iv) There is a constant $\tilde{C}(T, p, f)$ for every $f \in \Phi^p$ such that for $t \in [0, T]$

$$0 \le \tilde{S}_t f \le \tilde{C}(T, p, f)(1 + t^{\frac{1}{p}-\frac{1}{2}})w. \tag{3.28}$$

Corollary 3.22. *For every fixed $\alpha \in \mathbb{R}$ and $\varepsilon \in (0,1)$, the semigroups $(S_t^{\alpha,\varepsilon})_{t\ge 0}$ and $(S_t^{\alpha})_{t\ge 0}$ satisfy Assumption 3.21.*

Proof. For the semigroup $(S_t^{\alpha,\varepsilon})_{t\ge 0}$ the property (i) follows from Corollary 3.14, (ii) is given by Lemma 3.8 with $\overline{C}(T) = 0$. Property (iii) was shown in Lemma 3.13 and (iv) follows from the stronger estimate (3.16).

As for $(S_t^{\alpha})_{t\ge 0}$, property (i) is stated in Corollary 3.20. Property (ii) follows from Lemma 3.17. Properties (iii) and (iv) are given by Lemma 3.18 and Corollary 3.19 respectively. □

From now on, we will imagine that $\tilde{S}_t$ is either $S_t^{\alpha,\varepsilon}$ or S_t^{α}. Consider the integral equation

$$u(t,x) := \tilde{S}_t f(x) - \eta \int_0^t \tilde{S}_{t-s}(u^{1+\beta}(s))(x)\,\mathrm{d}s \tag{3.29}$$

for $\eta \ge 0$, β, p satisfying (3.3), with $0 \le t \le T$, $x \ne 0$ and $f \in \Phi^p$. We want to show uniqueness and existence of nonnegative solutions for the equation (3.29). Nonnegativity of the nonlinear term also implies the domination

$$0 \le u(t) \le \tilde{S}_t f, \quad t \ge 0. \tag{3.30}$$

The results in this section are based on [20, Subsection 3.1]. We have already studied the properties of the linear term $\tilde{S}_t f$ in (3.29). The following lemma collects some properties of the nonlinear term. We work under the Assumption 3.21.

Lemma 3.23. *[20, Lemma 3.4]. Let $f \in \Phi^p$ and ψ_1, ψ_2 measurable functions on $(0,T] \times \mathbb{R}^3 \setminus \{0\}$ such that*

$$0 \le \psi_1(t,x) \le M(1 + t^{-\kappa})w^\beta(x),$$
$$0 \le \psi_2(t,x) \le \tilde{S}_t f(x)$$

with constants $M = M(T, \psi_1) > 0$ and

$$\kappa = \beta\left(\frac{1}{p} - \frac{1}{2}\right) \in (0,1).$$

There is a constant $C(M,T,\beta,p)$ such that

$$\left\| \int_0^t \tilde{S}_{t-s}(\psi_1(s)\psi_2(s))\,\mathrm{d}s \right\|_{L^p(w)} \le C(M,T,\beta,p)\|f\|_{L^p(w)} I(t), \quad 0 < t \le T, \tag{3.31}$$

where

$$\infty > I(t) := \int_0^t (1+s^{-\kappa})(t-s)^{-\frac{\beta}{2}}\,\mathrm{d}s \downarrow 0 \tag{3.32}$$

as $t \downarrow 0$. Moreover, if for fixed $t \in (0,T]$

$$N_t(x) := \int_0^t \tilde{S}_{t-s}(\psi_1(s)\psi_2(s))(x)\,\mathrm{d}s, \quad x \in \mathbb{R}^3 \setminus \{0\},$$

satisfies

$$N_t(x) \le \tilde{S}_t f(x), \quad x \in \mathbb{R}^3 \setminus \{0\},$$

then $N_t \in \Phi^p$.

Outline of proof. Since $\psi_1(s,x) \le M(1+s^{-\kappa})w^\beta(x)$ and $\psi_2(t,x) \le \tilde{S}_t f(x)$ we can apply the additional singularity estimate (3.27) to obtain the bound (3.31). A computation using $0 < \kappa + \frac{\beta}{2} < 1$ shows that the integral $I(t)$ converges to zero for $t \downarrow 0$. The main work is to show the continuity of the nonlinear term N_t. Here the fact that $(\tilde{S}_t)$ is a semigroup on Φ^p is needed, as well as the estimate (3.28). Refer to [20, p. 754f] for full details. □

In order to prepare for the uniqueness proof, we need the following technical lemma, which is a consequence of the mean value theorem.

Lemma 3.24. *[20, Lemma 3.6]. Let $\beta > 0$ and $a, b \in \mathbb{R}$. Then*

$$|a(a \vee 0)^\beta - b(b \vee 0)^\beta| \le (1+\beta)(|a|+|b|)^\beta |a-b|. \tag{3.33}$$

Theorem 3.25 (Uniqueness). *[20, Lemma 3.7]. Impose Assumption 3.21 and* (3.30). *Fix $f \in \Phi^p$. Suppose that u, v are Φ^p-valued solutions of* (3.29). *Then $u = v$.*

Proof. We proceed as in the proof of [20, Lemma 3.7]. For $0 \le t \le T$ and $x \in \mathbb{R}^3 \setminus \{0\}$ define the difference

$$D(t,x) := u(t,x) - v(t,x).$$

Using (3.30) and estimate (3.27) with $\beta = 0$, we obtain

$$\|D(t)\|_{L^p(w)} \le \|u(t)\|_{L^p(w)} + \|v(t)\|_{L^p(w)} \le 2\left\|\tilde{S}_t f\right\|_{L^p(w)} \lesssim 2\|f\|_{L^p(w)}. \tag{3.34}$$

With the inequality (3.33) and the fact u and v are nonnegative, we get

$$\begin{aligned}|D(t,x)| &= \eta \left| \int_0^t \tilde{S}_{t-s}(u^{1+\beta}(s) - v^{1+\beta}(s))(x)\,\mathrm{d}s \right| \\ &\leq \eta \int_0^t \tilde{S}_{t-s} \left|u^{1+\beta}(s) - v^{1+\beta}(s)\right| (x)\,\mathrm{d}s \\ &\leq (1+\beta)\eta \int_0^t \tilde{S}_{t-s}[(u(s)+v(s))^\beta |u(s)-v(s)|](x)\,\mathrm{d}s \\ &\leq 2\eta \int_0^t \tilde{S}_{t-s}[(u^\beta(s)+v^\beta(s))|D(s)|](x)\,\mathrm{d}s.\end{aligned} \tag{3.35}$$

Using estimate (3.28), we have in the remaining integral

$$(u^\beta(s) + v^\beta(s)) \leq 2\tilde{S}_s f \lesssim (1+s^\kappa)w$$

with $\kappa = \beta(\frac{1}{p} - \frac{1}{2})$ and an implicit constant depending on T and f. Inserting in (3.35) and applying the norm gives

$$\begin{aligned}\|D(t)\|_{L^p(w)} &\lesssim 4\eta \int_0^t (1+s^\kappa) \left\|\tilde{S}_{t-s}(|D(s)|w^\beta)\right\|_{L^p(w)} \mathrm{d}s \\ &\lesssim 4\eta \int_0^t (1+s^\kappa)(t-s)^{-\frac{\beta}{2}} \|D(s)\|_{L^p(w)}\,\mathrm{d}s \\ &\leq 4 \sup_{0<s\leq t} \|D(s)\|_{L^p(w)} \eta \int_0^t (1+s^\kappa)(t-s)^{-\frac{\beta}{2}}\,\mathrm{d}s \\ &= 4\eta \sup_{0<s\leq t} \|D(s)\|_{L^p(w)} I(t)\end{aligned} \tag{3.36}$$

where property (3.27) of the semigroup was used in the second step and with the notation $I(t)$ from (3.32). Applying the supremum to the first and last expression in (3.36) we obtain

$$\sup_{0<s\leq t} \|D(s)\|_{L^p(w)} \lesssim \sup_{0<s\leq t} \|D(s)\|_{L^p(w)} \sup_{0<s\leq t} I(s) = \sup_{0<s\leq t} \|D(s)\|_{L^p(w)} I(t), \tag{3.37}$$

note that $I(t)$ is increasing in t. The implicit constant depends on T and f. Because the term $\sup_{0<s\leq t} \|D(s)\|_{L^p(w)}$ is finite due to (3.34) and the integral $I(t)$ tends to zero as shown in Lemma 3.23, it follows $\sup_{0<s\leq t} \|D(s)\|_{L^p(w)} = 0$ on a subinterval $[0, t_0]$ for $0 < t_0 < T$ sufficiently small. This is equivalent to $u = v$ on this interval, note that by assumption u and v are continuous. Since the model is time-homogeneous, we can repeat the argument finitely often to extend to the whole interval $[0, T]$ [20, p. 757]. □

Theorem 3.26 (Well-posedness)**.** *[20, Theorem 3.3]. Under the conditions of Assumption 3.21, if $f \in \Phi^p$, then the integral equation* (3.29) *has a unique Φ^p-valued solution $u = u_f$ satisfying* (3.30). *Moreover, the semigroup $(\mathcal{U}_t)$ given by $\mathcal{U}_t f(x) = u_f(t,x)$ is a nonlinear strongly continuous semigroup acting on Φ^p.*

Outline of proof. We collect references for the missing steps. For the proof of existence of solutions, one can construct a sequence $(u_n)_{n\in\mathbb{N}}$ of functions inductively defined by

$$\begin{aligned} u_0(t,x) &:= \tilde{S}_t f(x), \\ u_n(t,x) &:= \tilde{S}_t f(x) - \eta \int_0^t \tilde{S}_{t-s}(u_{n-1}(s)\psi_1(s))(x)\,\mathrm{d}s, \quad n \in \mathbb{N}, \end{aligned} \tag{3.38}$$

with ψ_1 as in Lemma 3.23 [20, p. 758f] and then pass to the limit $n \to \infty$ via a fixed-point iteration argument. Lemma 3.23 is also the main tool for dealing with the integral term in the linearized equation (3.38). Details are given in [20, Lemma 3.10 and Lemma 3.11]. For the proof of the nonnegativity and domination

$$0 \le u(t) \le \tilde{S}_t f, \quad t \ge 0,$$

please refer to [20, Lemma 3.8] or Subsection 7.2 below. Uniqueness was shown in Theorem 3.25, and the semigroup property of $(\mathcal{U}_t)$ follows from the uniqueness of solutions as well [20, p. 762].

As for strong continuity of $(\mathcal{U}_t)$, note that

$$|\mathcal{U}_t f - f| \le |\mathcal{U}_t f - \tilde{S} f| + |\tilde{S} f - f|.$$

Because the semigroup $(\tilde{S}_t)_{t\ge 0}$ is strongly continuous, it suffices to consider the first term on the right-hand side. It holds

$$|\mathcal{U}_t f - \tilde{S}_t f| = \eta \left| \int_0^t \tilde{S}_{t-s}(v^{1+\beta}(s))(x)\,\mathrm{d}s \right|$$

and this remaining integral can be controlled using property (3.28) of the semigroup $(\tilde{S}_t)_{t\ge 0}$ and Lemma 3.23. Also refer to [20, Lemma 3.5]. □

3.5. Construction of the measure-valued processes

We remain in the situation of the previous section. In particular, we continue to impose Assumption 3.21 on the semigroup $(\tilde{S}_t)$. From now on, let u the solution of the nonlinear integral equation (3.29) with initial value $u(0,\cdot) = f$ given by Theorem 3.26. In this section, we describe the construction of the measure-valued process X related to u via the Laplace transition functional [20, Formula 1.6]

$$\mathbb{E}[e^{-\langle X_t, f\rangle} | X_0 = \mu] =: \mathbb{E}_\mu e^{-\langle X_t, f\rangle} = e^{-\langle \mu, u(t)\rangle},$$

where we use the notation for evaluating a measure μ with a nonnegative measurable function g

$$\langle \mu, g \rangle := \int_{\mathbb{R}^3} g(x) \, \mathrm{d}\mu(x).$$

The strategy to construct X will be a Trotter product approach: We introduce an approximating integral equation with solutions $\overline{u}_n$, by separating critical continuous-state branching with index $1+\beta$ and mass flow according to $(\tilde{S}_t)_{t\geq 0}$ on alternate time intervals of length $\frac{1}{n}$ [20, p. 762]. We start with an inductive construction of the functions $\overline{u}_n$ on $[0,T] \times \mathbb{R}^3 \setminus \{0\}$ following closely [20, Subsection 4.1]. Under Assumption 3.21 fix $n \in \mathbb{N}$ and $f \in \Phi^p$. Define

$$\overline{u}_n(0,x) := \tilde{S}_{\frac{1}{n}} f(x). \tag{3.39}$$

This means we let evolve the mass flow until time $\frac{1}{n}$. Now assume that $\overline{u}_n(\frac{k}{n})$ is defined for some $k \in \mathbb{N}$. For the time interval $\frac{k}{n} \leq t < \frac{k+1}{n}$ and $x \neq 0$ set

$$\overline{u}_n(t,x) := \frac{\overline{u}_n(\frac{k}{n},x)}{\left[1+\eta\beta\overline{u}_n^{\beta}(\frac{k}{n},x)(t-\frac{k}{n})\right]^{\frac{1}{\beta}}}. \tag{3.40}$$

By computing the derivative

$$\begin{aligned}
\partial_t \overline{u}_n(t,x) &= -\frac{1}{\beta}\frac{\overline{u}_n(\frac{k}{n},x)}{\left[1+\eta\beta\overline{u}_n^{\beta}(\frac{k}{n},x)(t-\frac{k}{n})\right]^{1+\frac{1}{\beta}}}\eta\beta\overline{u}_n^{\beta}(\tfrac{k}{n},x) \\
&= -\eta\frac{\overline{u}_n^{1+\beta}(\frac{k}{n},x)}{\left[1+\eta\beta\overline{u}_n^{\beta}(\frac{k}{n},x)(t-\frac{k}{n})\right]^{\frac{1}{\beta}(1+\beta)}} \\
&= -\eta\overline{u}_n^{1+\beta}(\tfrac{k}{n},x),
\end{aligned} \tag{3.41}$$

so $\overline{u}_n(t,x)$ is the Laplace transition function of a critical continuous-state branching process with index $1+\beta$ on the time interval $(\frac{k}{n},\frac{k+1}{n})$ [20, p. 762]. Roughly speaking, $\overline{u}_n$ solves the cauchy problem (3.1) on this time interval, but without the linear term. Also note that by construction for the left-hand limit we have $\overline{u}_n(\frac{k}{n}+,x) = \overline{u}_n(\frac{k}{n},x)$ and the right-hand limit $\overline{u}_n(\frac{k}{n}-,x)$ exists as well. For $x \neq 0$ put

$$\overline{u}_n\left(\frac{k+1}{n},x\right) := \tilde{S}_{\frac{1}{n}}\overline{u}_n\left(\frac{k+1}{n}-,\cdot\right)(x), \tag{3.42}$$

which completes the inductive definition. The function $\overline{u}_n$ is nonnegative by constructiuon.

Lemma 3.27. *[20, Lemma 4.1] For every fixed $n \in \mathbb{N}$ the function $\overline{u}_n$ defined above satisfies*

$$\overline{u}_n(t,x) = \tilde{S}_{(1+\lfloor tn \rfloor)/n} f(x) - \eta \int_0^t \tilde{S}_{(\lfloor tn \rfloor - \lfloor sn \rfloor)/n} (\overline{u}_n^{1+\beta}(s))(x)\, \mathrm{d}s \tag{3.43}$$

on $[0,T] \times \mathbb{R}^3 \setminus \{0\}$.

Proof. First assume $t \neq \frac{k}{n}, k \in \mathbb{N}$. Differentiating the equation (3.43) gives the true statement (3.41), because

$$\partial_t \tilde{S}_{(1+\lfloor tn \rfloor)/n} f(x) = 0$$

for $t \neq \frac{k}{n}$. As for the case $t = \frac{k}{n}, k \in \mathbb{N}$, the right-hand side of (3.43) becomes

$$\tilde{S}_{(1+k)/n} f(x) - \eta \sum_{i=0}^{k} \left[\tilde{S}_{(k-(i-1))/n} \left(\int_{(i-1)/n}^{i/n} \overline{u}_n^{1+\beta}(s)\, \mathrm{d}s \right)(x) \right],$$

since $\lfloor sn \rfloor / n \equiv (i-1)/n$ for $s \in (\frac{i-1}{n}, \frac{i}{n})$. Using (3.41) and the fundamental theorem of calculus, this expression equals $\overline{u}_n(\frac{k}{n}, x)$, completing the proof [20, p. 763]. □

The following theorem states that the functions $\overline{u}_n$ converge towards the solution v of Theorem 3.26.

Theorem 3.28. *[20, Proposition 4.3]. Fix $f \in \Phi^p$. Define $\overline{u}_n$ as in* (3.39)–(3.42). *Let v the unique Φ^p-valued solution of the integral equation* (3.29) *given by Theorem 3.26. Then, for each $t \in [0,T]$,*

$$\lim_{n \to \infty} \| u(t) - \overline{u}_n(t) \|_{L^p(w)} = 0.$$

Outline of proof. For $n \in \mathbb{N}$ we obtain by subtracting (3.43) from (3.29) and decomposing the integral range

$$\begin{aligned}
\| u(t) - \overline{u}_n(t) \|_{L^p(w)} \leq{} & \left\| \tilde{S}_t f - \tilde{S}_{(1+\lfloor tn \rfloor)/n} f \right\|_{L^p(w)} \\
& + \eta \int_0^{\lfloor tn \rfloor / n} \left\| \tilde{S}_{t-s} u^{1+\beta}(s) - \tilde{S}_{(\lfloor tn \rfloor - \lfloor sn \rfloor)/n} u^{1+\beta}(s) \right\|_{L^p(w)} \mathrm{d}s \\
& + \eta \int_0^{\lfloor tn \rfloor / n} \left\| \tilde{S}_{(\lfloor tn \rfloor - \lfloor sn \rfloor)/n} | u^{1+\beta}(s) - \overline{u}_n^{1+\beta}(s) | \right\|_{L^p(w)} \mathrm{d}s \\
& + \eta \left\| \int_{\lfloor tn \rfloor / n}^t \tilde{S}_{t-s} u^{1+\beta}(s)\, \mathrm{d}s \right\|_{L^p(w)} + \eta \left\| \int_{\lfloor tn \rfloor / n}^t \tilde{S}_{(\lfloor tn \rfloor - \lfloor sn \rfloor)/n} \overline{u}_n^{1+\beta}(s)\, \mathrm{d}s \right\|_{L^p(w)}.
\end{aligned}$$

These terms can be estimated separately using the properties of the semigroup $(\overline{S}_t)$ from Assumption 3.21. Refer to [20, p. 764ff] for a detailed proof. □

With the convergence of the approximating solution $\overline{u}_n$, Theorem 3.28, in place, we are able to make the transition to the setting of measure-valued processes.

Theorem 3.29. *[20, Theorem 4.4] Let $\mu \in \mathcal{M}(\mathbb{R}^3 \setminus \{0\}) =: \mathcal{M}$. Under Assumption 3.21, there is a unique in law nondegenerate $\mathcal{M}$-valued time-homogeneous Markov process $X = (X_t)_{t\geq 0}$ with Laplace transition functional*

$$\mathbb{E}_\mu e^{-\langle X_t, f\rangle} = e^{-\langle \mu, u(t)\rangle} \tag{3.44}$$

using test functions $f \in \Phi^p$ and where $u = \mathcal{U}f$ is the unique Φ^p-valued solution of (3.29) *from Theorem 3.26.*

Proof. First we prove that the process X, if it exists, is nondegenerate, i.e. not equal to its expectation everywhere. It holds

$$\mathbb{E}_\mu[\langle X_t, f\rangle] = \langle \mu, \tilde{S}_t f\rangle$$

for every $\mu \in \mathcal{M}, f \in \Phi^p, t \geq 0$, as shown in Theorem 4.1 below. But if $f \neq 0$ and $t > 0$, the integral term in (3.29) does not vanish and hence $u(t) \neq \tilde{S}_t f$. Applying (3.44), we obtain

$$\mathbb{E}_\mu e^{-\langle X_t, f\rangle} = e^{-\langle \mu, u(t)\rangle} \neq e^{\langle \mu, \tilde{S}_t f\rangle}.$$

Thus, $\langle X_t, f\rangle \not\equiv \langle \mu, \tilde{S}_t f\rangle$, which means that X is nondegenerate.

Now we turn to the existence proof following closely the proof of [20, Theorem 4.4]. Fix $\mu \in \mathcal{M}$ and $n \in \mathbb{N}$. Using the approximating solutions $\overline{u}_n$ defined in (3.39)–(3.42), we want to construct a random measure $X_t^n \in \mathcal{M}$, $t \geq 0$ fixed, satisfying

$$\mathbb{E}_\mu e^{-\langle X_t^n, f\rangle} = e^{-\langle \mu, \overline{u}_n(t)\rangle}, \quad f \in \Phi^p. \tag{3.45}$$

Then we later let $n \uparrow \infty$ to obtain a random measure $X_t \in \mathcal{M}$, $t \geq 0$ fixed, satisfying (3.44). This will give us a probability kernel Q_t. Because of the semigroup property of $\mathcal{U}_t f$, the family $(Q_t)_{t\geq 0}$ satisfies the conditions of the Chapman-Kolmogorov theorem. Consequently, $(Q_t)_{t\geq 0}$ is then the transition kernel of a time-homogeneous Markov process $(X_t)_{t\geq 0}$. [20, p. 769, 771]

Now we begin with the construction of $(X_t^n)_{t\geq 0}$ for fixed $n \in \mathbb{N}$. This is done again by alternating operations of continuous-state branching and mass flow according to the semigroup $(\tilde{S}_t)_{t\geq 0}$ on time intervals of length $\frac{1}{n}$. Because these two alternating operations do not commute, we must apply them in reversed order on the dual level of measures compared to the construction of $\overline{u}_n$.

Consider the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = -\eta g^{1+\beta}(t) \quad \text{on } [0,\infty) \text{ with } g(0) = \theta, \tag{3.46}$$

where $\theta \geq 0$. The unique solution g of (3.46) yields the Laplace transition functional of a critical continuous-state branching process $(y_t)_{t\geq 0}$ with index $1+\beta$ via

$$\mathbb{E}[e^{-y_t\theta}|y_0 = a] = ag(t).$$

This was shown in [34], see also [20, p. 769]. In the case $\beta = 1$, The process $(y_t)_{t\geq 0}$ is the critical Feller branching diffusion.

To add the spacial component, we let the process $(y_t)_{t\geq 0}$ evolve independently at each point $x \neq 0$ by replacing the starting point θ with the initial data $f(x)$ in (3.46). For each fixed x, this leads to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t,x) = -\eta g^{1+\beta}(t,x) \quad \text{on } [0,\infty) \text{ with } g(0,x) = f(x). \tag{3.47}$$

The resulting $\mathcal{M}$-valued Markov process $(Y_t)_{t\geq 0}$ with càdlàg paths has the Laplace transition functional

$$\mathbb{E}_\mu e^{-\langle Y_t, f\rangle} = e^{-\langle \mu, G_t f\rangle} \tag{3.48}$$

where

$$G_t f(x) := g(t,x)$$

for $t \geq 0$, $f \in \Phi^p$ and $\mu \in \mathcal{M}$. [20, p. 770]. Now we want to apply the mass flow operation and inductively define the random measures X_t^n satisfying (3.45). We introduce the following notation of smearing out a measure μ according to the flow of $\tilde{S}_t$:

$$\langle \tilde{S}_t\mu, f\rangle := \langle \mu, \tilde{S}_t f\rangle. \tag{3.49}$$

For fixed $\mu \in \mathcal{M}$, $n \in \mathbb{N}$ and $t \in [0, \frac{1}{n})$ define

$$X_t^n := \tilde{S}_{1/n} Y_t, \quad Y_0 := \mu.$$

With (3.48) and (3.49) it follows

$$\mathbb{E}_\mu e^{-\langle X_t^n, f\rangle} = \mathbb{E}_\mu e^{-\langle \tilde{S}_{1/n} Y_t, f\rangle} = \mathbb{E}_\mu e^{-\langle Y_t, \tilde{S}_{1/n} f\rangle} = e^{-\langle \mu, G_t \tilde{S}_{1/n} f\rangle}.$$

But because of the uniqueness of solutions to (3.46) and by (3.41) with $k = 0$, we have that $G_t\tilde{S}_{1/n}f = \overline{u}_n(t)$. Consequently, $e^{-\langle \mu, G_t\tilde{S}_{1/n}f\rangle} = e^{-\langle \mu, \overline{u}_n(t)\rangle}$ and we have shown (3.45) for $0 \leq t < \frac{1}{n}$. [20, p. 770]. To proceed with the induction, assume that for $k \in \mathbb{N}$ the random measures X_t^n are defined for $\frac{k}{n} \leq t < \frac{k+1}{n}$ and satisfy (3.45). Then, fix $t \in [\frac{k+1}{n}, \frac{k+2}{n})$ and set

$$X_t^n := \tilde{S}_{1/n} Y_{1/n}, \quad Y_0 := X_{t-\frac{1}{n}}.$$

Replacing μ with $X_{t-\frac{1}{n}}$ in equation (3.48) implies

$$\mathbb{E}_\mu e^{-\langle X_t^n, f\rangle} = \mathbb{E}\big[e^{-\langle Y_{1/n}, \tilde{S}_{1/n} f\rangle} \big| Y_0 := X_{t-\frac{1}{n}}\big] = e^{-\langle X_{t-\frac{1}{n}}, G_t \tilde{S}_{1/n} f\rangle} = e^{-\langle \mu, \overline{u}_n(1-\frac{1}{n})\rangle},$$

using the induction hypothesis in the last step, but with $\overline{u}_n(0) = \tilde{S}_{1/n} G_{1/n} \tilde{S}_{1/n} f$ instead of $\tilde{S}_{1/n} f$. But since this new $\overline{u}_n(0)$ differs from the original $\overline{u}_n(0)$ by critical branching and mass flow of length $\frac{1}{n}$, we obtain from the construction (3.39)–(3.42) that the new $\overline{u}_n(t-\frac{1}{n})$ coincides with the original $\overline{u}_n(t)$. This yields (3.45) on the time interval $[\frac{k+1}{n}, \frac{k+2}{n})$. By induction, we have shown (3.45) for all $t \geq 0$. [20, p. 770]

Now we pass to the limit $n \uparrow \infty$. According to Theorem 3.28, $\overline{u}_n(t) \to u(t)$ as $n \uparrow \infty$ for $t \geq 0$ fixed. This implies

$$e^{-\langle \mu, \overline{u}_n(t)\rangle} \to e^{-\langle \mu, u(t)\rangle}, \quad n \uparrow \infty, t \geq 0.$$

Therefore, the Laplace transforms at the left-hand side of (3.45) converge to $\langle \mu, u(t)\rangle$, too. From the integral representation (3.29) we get $\langle \mu, u(t)\rangle \downarrow 0$ as $f \downarrow 0$. Hence, the limit of the Laplace transforms in (3.45) is again a Laplace transform of a random measure in $\mathcal{M}$ [Dynkin 94]. Denote this random measure by X_t. Consequently, for t fixed, $X_t^n \to X_t$ in distribution as $n \uparrow \infty$. Since the map $\mu \mapsto \langle \mu, \mathcal{U}_t f\rangle$ is measurable, via $\mu \to X_t$ we get a probability kernel Q_t in $\mathcal{M}$ for fixed t. As mentioned at the beginning of the proof, using Chapman-Kolmogorov, this implies that $(Q_t)_{t\geq 0}$ is the transition kernel of a time-homogeneous Markov process in $\mathcal{M}$, which is the desired superprocess $(X_t)_{t\geq 0}$ [20, p. 771]. In fact

$$\int_{\mathcal{M}} e^{-\langle \lambda, f\rangle} Q_t(\mu, \mathrm{d}\lambda) = e^{-\langle \mu, u(t)\rangle}$$

for $\lambda, \mu \in \mathcal{M}$ [35, p. 42]. The kernel Q_t satisfies the branching property

$$Q_t(\mu_1 + \mu_2, \cdot) = Q_t(\mu_1, \cdot) * Q_t(\mu_2, \cdot)$$

where $\mu_1, \mu_2 \in \mathcal{M}$ [35, Formula 2.1]. □

We summarize the results by returning to the semigroups $(S_t^\alpha)_{t\geq 0}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$.

Corollary 3.30. *Let $\mu \in \mathcal{M}, 0 \leq \beta < 1$ and $\frac{1}{1-\beta/2} < p < 2$. Fix $\alpha \in \mathbb{R}$ and $\varepsilon \in (0,1)$. There are unique in law nondegenerate $\mathcal{M}$-valued time-homogeneous Markov processes X^α and $X^{\alpha,\varepsilon}$ with Laplace transition functionals*

$$\mathbb{E}_\mu e^{-\langle X_t^\alpha, f\rangle} = e^{-\langle \mu, u_\alpha(t)\rangle} \quad \textit{and} \tag{3.50}$$

$$\mathbb{E}_\mu e^{-\langle X_t^{\alpha,\varepsilon}, f\rangle} = e^{-\langle \mu, u_{\alpha,\varepsilon}(t)\rangle} \tag{3.51}$$

respectively, using test functions $f \in \Phi^p$ and where $u_\alpha, u_{\alpha,\varepsilon}$ are the unique Φ^p-valued solutions *of*

$$u_\alpha(t,x) = S_t^\alpha f(x) - \eta \int_0^t S_{t-s}^\alpha(u_\alpha^{1+\beta}(s))(x)\,\mathrm{d}s \quad \textit{and}$$
$$u_{\alpha,\varepsilon}(t,x) = S_t^{\alpha,\varepsilon} f(x) - \eta \int_0^t S_{t-s}^{\alpha,\varepsilon}(u_{\alpha,\varepsilon}^{1+\beta}(s))(x)\,\mathrm{d}s$$

respectively, with initial data $u_\alpha(0) = u_{\alpha,\varepsilon}(0) = f$.

Proof. The semigroups $(S_t^\alpha)_{t\geq 0}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ satisfy Assumption 3.21 as stated in Corollary 3.22. Hence, the claim follows immediately from the previous Theorem 3.29 □

Remark 3.31. The existence of the process X^α is the subject of the work from Fleischmann-Mueller 2004 [20], which was also our main source. Concerning the process $X^{\alpha,\varepsilon}$, there are some previous results for similar processes in special cases, for example the case $\beta = 1$ in [18]. Also, the existence of the process $X^{\alpha,\varepsilon}$ follows from the more general theory of [35], as discussed in Chapter 4. However, to the knowledge of the author, the approach to this process that we discussed above, is new.

4. Basic properties of the measure-valued processes

In the previous chapter we have shown the existence of the measure-valued processes X^α and $X^{\alpha,\varepsilon}$. Next up we want to study first properties of these processes, including moment formulas and an analysis of the dependence on the parameter η. For the process $X^{\alpha,\varepsilon}$, a kind of path regularity is also shown.

4.1. Moment formulas

The following theorem gives an explicit formula for the expectation of X_t^α. The main idea in the computation is to write the first moment as a derivative of the Laplace transform of the process. We continue to impose Assumption 3.21, recall that $0 \le \beta < 1$ and $\frac{1}{1-\frac{\beta}{2}} < p < 2$.

Theorem 4.1. *Under Assumption 3.21, let initial data $f \in \Phi^p$ and u_f the corresponding solution of the integral equation*

$$u_f(t,x) = S_t^\alpha f(x) - \eta \int_0^t S_{t-s}^\alpha(u_f^{1+\beta}(s))(x)\,\mathrm{d}s, \tag{4.1}$$
$$u_f(0,\cdot) = f,$$

given by Theorem 3.26. Let $\alpha \ge 0$ and $(X_t^\alpha)_{t\ge 0}$ the associated superprocess given by Corollary 3.30 and μ a measure with density $\mu(\cdot) \in L^1(w)$. Then it holds

$$\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle] = \langle \mu, S_t^\alpha f\rangle < \infty. \tag{4.2}$$

Note that we wrote the solution of (4.1) with index f to emphasize the dependency on the initial data. This is because we will need to vary the initial data in the following proof. We follow the usual strategy, the proofs are included because we have not found reference covering the situation where a singularity is present. Throughout this section we frequently use the fact that $f \ge 0$ and that cosenquently the expression $S_t^\alpha f$ and the solution $u_f(t)$, $t \ge 0$, of (4.1) are nonnegative. See Section 7.2 for more details about this nonnegativity.

Proof of Theorem 4.1. Let $r \geq 0$. We have

$$\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle] = -\mathbb{E}_\mu\left[\left.\frac{\partial}{\partial r}\right|_{r=0} \exp(-r\langle X_t^\alpha, f\rangle)\right]. \tag{4.3}$$

Our aim is to interchange derivative and expectation to compute the right-hand side of (4.3). In order to do this, consider first $r > 0$. It holds

$$\left|\frac{\partial}{\partial r}\exp(-r\langle X_t^\alpha, f\rangle)\right| = \langle X_t^\alpha, f\rangle \exp(-r\langle X_t^\alpha, f\rangle) \leq \frac{1}{er}$$

for every fixed $r > 0$, because the function $[0, \infty) \ni x \mapsto xe^{-rx}$ takes its maximum $\frac{1}{er}$ at the point $x_0 = \frac{1}{r}$. It follows

$$\mathbb{E}_\mu\left[\left|\frac{\partial}{\partial r}\exp(-r\langle X_t^\alpha, f\rangle)\right|\right] \leq \frac{1}{er} < \infty.$$

Hence, the dominated convergence theorem can be applied and yields

$$\mathbb{E}_\mu\left[\frac{\partial}{\partial r}\exp(-r\langle X_t^\alpha, f\rangle)\right] = \frac{\partial}{\partial r}\mathbb{E}_\mu\left[\exp(-r\langle X_t^\alpha, f\rangle)\right] \tag{4.4}$$

for every $r > 0$. Furthermore, we have the monotone convergence

$$\begin{aligned} -\frac{\partial}{\partial r}\exp(-r\langle X_t^\alpha, f\rangle) &= \langle X_t^\alpha, f\rangle \exp(-r\langle X_t^\alpha, f\rangle) \\ &\uparrow \langle X_t^\alpha, f\rangle \\ &= -\left.\frac{\partial}{\partial r}\right|_{r=0} \exp(-r\langle X_t^\alpha, f\rangle) \end{aligned}$$

for $r \downarrow 0$. Thus, we can apply the monotone convergence theorem to the expectation on the right-hand side of (4.3) and obtain with (4.4)

$$\begin{aligned} \mathbb{E}_\mu\left[\left.\frac{\partial}{\partial r}\right|_{r=0} \exp(-r\langle X_t^\alpha, f\rangle)\right] &= \mathbb{E}_\mu\left[\lim_{r\downarrow 0}\frac{\partial}{\partial r}\exp(-r\langle X_t^\alpha, f\rangle)\right] \\ &= \lim_{r\downarrow 0}\mathbb{E}_\mu\left[\frac{\partial}{\partial r}\exp(-r\langle X_t^\alpha, f\rangle)\right] \\ &= \lim_{r\downarrow 0}\frac{\partial}{\partial r}\mathbb{E}_\mu\left[\exp(-r\langle X_t^\alpha, f\rangle)\right] \\ &= \left.\frac{\partial}{\partial r}\right|_{r=0} \mathbb{E}_\mu\left[\exp(-r\langle X_t^\alpha, f\rangle)\right]. \end{aligned} \tag{4.5}$$

For $r \geq 0$ we write u_{rf} for the solution of the integral equation (4.1) corresponding to initial data rf. Via the Laplace transition functional

$$\mathbb{E}_\mu[\exp(-r\langle X_t^\alpha, f\rangle)] = \mathbb{E}_\mu[\exp(-\langle X_t^\alpha, rf\rangle)] = \exp(-\langle \mu, u_{rf}(t)\rangle), \tag{4.6}$$

refer to (3.50), it follows with the identity (4.5) and the chain rule

$$\begin{aligned}
\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle] &= -\mathbb{E}_\mu\left[\frac{\partial}{\partial r}\bigg|_{r=0}\exp(-r\langle X_t^\alpha, f\rangle)\right]\\
&= -\frac{\partial}{\partial r}\bigg|_{r=0}\mathbb{E}_\mu[\exp(-r\langle X_t^\alpha, f\rangle)]\\
&= -\frac{\partial}{\partial r}\bigg|_{r=0}\exp(-\langle \mu, u_{rf}(t)\rangle)\\
&= \left[\exp(-\langle \mu, u_{rf}(t)\rangle)\right]_{r=0}\frac{\partial}{\partial r}\bigg|_{r=0}\langle \mu, u_{rf}(t)\rangle\\
&= \frac{\partial}{\partial r}\bigg|_{r=0}\langle \mu, u_{rf}(t)\rangle.
\end{aligned}$$

Now let's assume for the moment that we can show

$$\frac{\partial}{\partial r}\bigg|_{r=0}\langle \mu, u_{rf}(t)\rangle = \int_{\mathbb{R}^d}\frac{\partial}{\partial r}\bigg|_{r=0}u_{rf}(t)\,\mathrm{d}\mu = \langle \mu, S_t^\alpha f\rangle, \tag{4.7}$$

then the proof would be finished.

In order to do this, we calculate $\frac{\partial}{\partial r}u_{rf}$ using the Picard iteration

$$\begin{aligned}
u_{0,rf}(t,x) &:= S_t^\alpha f(x),\\
u_{n,rf}(t,x) &:= S_t^\alpha f(x) - \eta\int_0^t S_{t-s}^\alpha(u_{n-1,rf}^{1+\beta}(s))(x)\,\mathrm{d}s, \quad n \in \mathbb{N},
\end{aligned} \tag{4.8}$$

as in the proof of Theorem 3.26 Note that the $L^p(w)$-convergence of this iteration towards u_{rf} is uniform in r for fixed f and $r \le C < \infty$. This follows for example from a calculation in a later part of the work: The Lipschitz constant from (7.19) for the function Cf also holds for all functions $rf, r \le C$. This implies

$$\frac{\partial}{\partial r}u_{rf}(t) = \lim_{n\to\infty}\frac{\partial}{\partial r}u_{n,rf}(t), \tag{4.9}$$

where the limit is taken in $L^p(w)$.

Now we are left to calculate $\frac{\partial}{\partial r}u_{n,rf}$ at $r = 0$. For $n = 0$ we have

$$u_{0,rf}(t) = S_t^\alpha(rf) = rS_t^\alpha(f),$$

and hence

$$\left.\frac{\partial}{\partial r}\right|_{r=0} u_{0,rf}(t) = \frac{\partial}{\partial r} u_{0,rf}(t) = S_t^\alpha f.$$

In the next step

$$\begin{aligned}\frac{\partial}{\partial r} u_{1,rf}(t) &= \frac{\partial}{\partial r} S_t^\alpha(rf) - \eta \frac{\partial}{\partial r} \int_0^t S_{t-s}^\alpha(S_s^\alpha(rf))^{1+\beta}\,\mathrm{d}s \\ &= S_t^\alpha f - \eta \frac{\partial}{\partial r} r^{1+\beta} \int_0^t S_{t-s}^\alpha(S_s^\alpha f)^{1+\beta}\,\mathrm{d}s \\ &= S_t^\alpha f - \eta(1+\beta) r^\beta \int_0^t S_{t-s}^\alpha(S_s^\alpha f)^{1+\beta}\,\mathrm{d}s.\end{aligned} \tag{4.10}$$

This again leads to

$$\left.\frac{\partial}{\partial r}\right|_{r=0} u_{1,rf}(t) = S_t^\alpha f.$$

Furthermore, the representation (4.10) and Lemma 3.23 imply $\frac{\partial}{\partial r} u_{1,rf} \in \Phi^p$.

Now let's assume as our induction hypothesis that

$$\left.\frac{\partial}{\partial r}\right|_{r=0} u_{n-1,rf}(t) = S_t^\alpha f$$

and $\frac{\partial}{\partial r} u_{n-1,rf}(t) \in \Phi^p$ for some $n \in \mathbb{N}$ and all $t > 0$. The latter fact yields an integrable majorant for $\frac{\partial}{\partial r} u_{n-1,rf}$ and allows interchanging derivative and integral in the following calculation.

$$\begin{aligned}\frac{\partial}{\partial r} u_{n,rf}(t) &= S_t^\alpha f - \eta \frac{\partial}{\partial r} \int_0^t S_{t-s}^\alpha(u_{n-1,rf}^{1+\beta}(s))\,\mathrm{d}s \\ &= S_t^\alpha f - \eta(1+\beta) \int_0^t S_{t-s}^\alpha \left(u_{n-1,rf}^\beta(s) \frac{\partial}{\partial r} u_{n-1,rf}(s)\right) \mathrm{d}s \\ &= S_t^\alpha f - \eta(1+\beta) \int_0^t S_{t-s}^\alpha \left(u_{n-1,rf}^\beta(s) S_s^\alpha f\right) \mathrm{d}s\end{aligned}$$

It follows again $\frac{\partial}{\partial r} u_{n,rf}(t) \in \Phi^p$. Furthermore

$$\left.\frac{\partial}{\partial r}\right|_{r=0} u_{n,rf}(t) = S_t^\alpha f - \eta(1+\beta) \int_0^t S_{t-s}^\alpha \left(u_{n-1,0}^\beta(s) S_s^\alpha f\right) \mathrm{d}s = S_t^\alpha f,$$

because $u_{n-1,0} = 0$. This completes the induction. Using (4.9), we infer

$$\left.\frac{\partial}{\partial r}\right|_{r=0} u_{rf}(t) = S_t^\alpha f$$

and $\frac{\partial}{\partial r} u_{rf}(t) \in \Phi^p$. This allows us to calculate

$$\frac{\partial}{\partial r}\bigg|_{r=0} \langle \mu, u_{rf}(t) \rangle = \int_{\mathbb{R}^d} \frac{\partial}{\partial r}\bigg|_{r=0} u_{rf}(t)\, \mathrm{d}\mu = \int_{\mathbb{R}^d} S_t^\alpha f\, \mathrm{d}\mu < \infty,$$

because $\mu(\cdot) \in L^1(w)$. Hence we have shown (4.7) and the proof is complete. □

In complete analogy we obtain the corresponding result for the process $X^{\alpha,\varepsilon}$ by replacing the semigroup $(S_t^\alpha)_{t\geq 0}$ with $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ in the proof of Theorem 4.1. This result can be deduced from the usual literature.

Corollary 4.2. *Under Assumption 3.21 let initial data $f \in \Phi^p$ and u_f the corresponding solution of the integral equation*

$$\begin{aligned} u_f(t,x) &= S_t^{\alpha,\varepsilon} f(x) - \eta \int_0^t S_{t-s}^{\alpha,\varepsilon}(u_f^{1+\beta}(s))(x)\, \mathrm{d}s, \\ u_f(0,\cdot) &= f, \end{aligned}$$

given by Theorem 3.26. Let $\alpha \geq 0$, $\varepsilon \in (0,1)$ and $(X_t^{\alpha,\varepsilon})_{t\geq 0}$ the associated superprocess given by Corollary 3.30 and μ a measure with density $\mu(\cdot) \in L^1(w)$. Then it holds

$$\mathbb{E}_\mu[\langle X_t^{\alpha,\varepsilon}, f\rangle] = \langle \mu, S_t^{\alpha,\varepsilon} f\rangle < \infty. \tag{4.11}$$

We have shown that the first moment of X_t^α exists and can be explicitly calculated by formula 4.2. This raises the natural question, whether higher moments exist. Our main focus is the three-dimensional case. However, in this case we are restricted to $\beta < 1$, refer to Assumption 3.21. In [20] the two-dimensional case was also investigated and in this situation $\beta = 1$ is admissible. The following result shows that for $\beta = 1$ the second moment exists.

Theorem 4.3. *Let $d = 2$ and initial data $f \in \Phi^p$ and u_f the corresponding solution of the integral equation* (4.1) *in the case $\beta = 1$. Let $(X_t^\alpha)_{t\geq 0}$ the associated superprocess and μ a measure with density $\mu(\cdot) \in L^1(w)$. Then it holds*

$$\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle^2] = \langle \mu, S_t^\alpha f\rangle^2 + 2\eta \left\langle \mu, \int_0^t S_{t-s}^\alpha (S_s^\alpha f)^2\, \mathrm{d}s \right\rangle < \infty.$$

Proof. First of all, the result of Theorem 4.1 is also valid in the two-dimensional case, since all calculations in the proof are precisely the same, even for $\beta = 1$. We proceed as in that proof, but use the second derivative instead of the first. This gives us

$$\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle^2] = \frac{\partial^2}{\partial r^2}\bigg|_{r=0} \mathbb{E}_\mu[\exp(-r\langle X_t^\alpha, f\rangle)].$$

Again, we use the Laplace transition functional and calculate

$$\begin{aligned}
&\frac{\partial^2}{\partial r^2}\bigg|_{r=0} \mathbb{E}_\mu[\exp(-r\langle X_t^\alpha, f\rangle)] \\
&\qquad = \frac{\partial^2}{\partial r^2}\bigg|_{r=0} \exp(-\langle \mu, u_{rf}(t)\rangle) \\
&\qquad = -\frac{\partial}{\partial r}\bigg|_{r=0} \left[\exp(-\langle \mu, u_{rf}(t)\rangle)\frac{\partial}{\partial r}\langle \mu, u_{rf}(t)\rangle\right] \\
&\qquad = \left[\frac{\partial}{\partial r}\langle \mu, u_{rf}(t)\rangle\right]^2\bigg|_{r=0} - \frac{\partial^2}{\partial r^2}\bigg|_{r=0}\langle \mu, u_{rf}(t)\rangle \\
&\qquad = \langle \mu, S_t^\alpha f\rangle^2 - \frac{\partial^2}{\partial r^2}\bigg|_{r=0}\langle \mu, u_{rf}(t)\rangle,
\end{aligned} \tag{4.12}$$

where we have used (4.7) in the last step. So it suffices to show that

$$-\frac{\partial^2}{\partial r^2}\bigg|_{r=0}\langle \mu, u_{rf}(t)\rangle = -\int_{\mathbb{R}^2}\frac{\partial^2}{\partial r^2}\bigg|_{r=0} u_{rf}(t)\,\mathrm{d}\mu = 2\eta\left\langle \mu, \int_0^t S_{t-s}^\alpha (S_s^\alpha f)^2\,\mathrm{d}s\right\rangle. \tag{4.13}$$

To do so, we once again employ the Picard iteration as in (4.8). As seen in [20, Subsections 3.4–3.7] this can be done in the two-dimensional case as well. Inserting $\beta = 1$, we obtain

$$\begin{aligned}
-\frac{\partial^2}{\partial r^2} u_{1,rf}(t) &= -\frac{\partial^2}{\partial r^2} rS_t^\alpha(f) + \eta\frac{\partial^2}{\partial r^2}\int_0^t S_{t-s}^\alpha(S_s^\alpha(rf))^2\,\mathrm{d}s \\
&= \eta\frac{\partial^2}{\partial r^2} r^2\int_0^t S_{t-s}^\alpha(S_s^\alpha f)^2\,\mathrm{d}s \\
&= 2\eta\int_0^t S_{t-s}^\alpha(S_s^\alpha f)^2\,\mathrm{d}s.
\end{aligned} \tag{4.14}$$

The last representation implies $-\frac{\partial^2}{\partial r^2}u_{1,rf}(t) = |\frac{\partial^2}{\partial r^2}u_{1,rf}(t)| \in \Phi^p$ according to Lemma 3.23. We proceed with induction over n. As the induction hypothesis, assume

$$-\frac{\partial^2}{\partial r^2}\bigg|_{r=0} u_{n-1,rf}(t) = 2\eta\int_0^t S_{t-s}^\alpha(S_s^\alpha f)^2\,\mathrm{d}s$$

and $|\frac{\partial^2}{\partial r^2}u_{n-1,rf}(t)| \in \Phi^p$ for some $n \in \mathbb{N}$ and all $t > 0$. The latter fact gives an integrable majorant and allows to interchange derivative and integral in the next iteration step. We compute

$$\begin{aligned}
-\frac{\partial^2}{\partial r^2} u_{n,rf}(t) &= -\frac{\partial^2}{\partial r^2} rS_t^\alpha(f) + \eta\frac{\partial^2}{\partial r^2}\int_0^t S_{t-s}^\alpha(u_{n-1,rf}^2(s))\,\mathrm{d}s \\
&= 2\eta\int_0^t \frac{\partial}{\partial r}S_{t-s}^\alpha\left(u_{n-1,rf}(s)\frac{\partial}{\partial r}u_{n-1,rf}(s)\right)\,\mathrm{d}s
\end{aligned}$$

$$= 2\eta \int_0^t S_{t-s}^\alpha \left(\left[\frac{\partial}{\partial r} u_{n-1,rf}(s) \right]^2 + u_{n-1,rf}(s) \frac{\partial^2}{\partial r^2} u_{n-1,rf}(s) \right) \mathrm{d}s.$$

We already know from the calculations in the proof of the previous theorem, that

$$\frac{\partial}{\partial r} u_{n-1,rf}(s) \Big|_{r=0} = S_t^\alpha f.$$

Furthermore, $u_{n-1,rf} = 0$ for $r = 0$. Inserting these two expressions implies

$$-\frac{\partial^2}{\partial r^2} u_{n,rf}(t) \Big|_{r=0} = 2\eta \int_0^t S_{t-s}^\alpha (S_s^\alpha f)^2 \, \mathrm{d}s$$

and the identity

$$\left| \frac{\partial^2}{\partial r^2} u_{n,rf}(t) \right| = 2\eta \int_0^t S_{t-s}^\alpha \left(\left[\frac{\partial}{\partial r} u_{n-1,rf}(s) \right]^2 + u_{n-1,rf}(s) \left| \frac{\partial^2}{\partial r^2} u_{n-1,rf}(s) \right| \right) \mathrm{d}s$$

implies $|\frac{\partial^2}{\partial r^2} u_{n,rf}(t)| \in \Phi^p$ because of the induction hypothesis and the fact that $\frac{\partial}{\partial r} u_{n-1,rf}(s) \in \Phi^p$.

Taking limits as in (4.9) gives us

$$-\frac{\partial^2}{\partial r^2} u_{rf}(t) \Big|_{r=0} = 2\eta \int_0^t S_{t-s}^\alpha (S_s^\alpha f)^2 \, \mathrm{d}s$$

and $|\frac{\partial^2}{\partial r^2} u_{rf}(t)| \in \Phi^p$. We can infer (4.13), which completes the proof. □

Remark 4.4. In the case $\beta < 1$ the second moment cannot be computed as in Theorem 4.3. In fact, calculating as in (4.14), we obtain

$$\begin{aligned} -\frac{\partial^2}{\partial r^2} u_{1,rf}(t) &= \eta \frac{\partial^2}{\partial r^2} r^{1+\beta} \int_0^t S_{t-s}^\alpha (S_s^\alpha f)^{1+\beta} \, \mathrm{d}s \\ &= \eta(1+\beta)\beta r^{\beta-1} \int_0^t S_{t-s}^\alpha (S_s^\alpha f)^2 \, \mathrm{d}s \end{aligned}$$

and the last term becomes singular for $r \downarrow 0$ because of $\beta - 1 < 0$. So it seems likely that the second moment and higher moments do not exist in this case, which means that the process has infinite variance.

Now we show existence of fractional moments, up to but not including the moment of order $1 + \beta$. We closely follow the strategy in [40, Lemma 2.1], where a corresponding existence result is given in the context of Dawson-Watanabe superprocesses with $(1 + \beta)$-stable branching mechanisms. More details about general branching mechanisms can be found in the next section.

Theorem 4.5. *Let $0 < \zeta < \beta$. Then it holds*

$$\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle^{1+\zeta}] \leq 1 + c(\zeta,\beta) \left[\langle \mu, S_t^\alpha f\rangle^{1+\beta} + \langle \mu, \int_0^t S_{t-s}^\alpha (S_s^\alpha f)^{1+\beta}\, \mathrm{d}s\rangle\right]$$

for a constant $c(\zeta,\beta) > 0$.

For the proof we will need the following technical Lemma.

Lemma 4.6. *[12, Lemma 5.5.2, (c)–(e)]. Let $0 < \zeta, \beta \leq 1$. There are constants $c_1(\zeta), c_2(\beta) > 0$, such that for every nonnegative random variable Y it holds*

$$\begin{aligned} &\text{(i)} \quad \mathbb{E}[Y^{1+\zeta}] \leq 1 + c_1(\zeta) \int_1^\infty z^{1+\zeta} \int_0^{2/z} \mathbb{E}\left[e^{-rY} - 1 + rY\right] \mathrm{d}r\, \mathrm{d}z, \\ &\text{(ii)} \quad Y - 1 \leq c_2(\beta) Y^{1+\beta} - e^{-Y}. \end{aligned}$$

Proof of Theorem 4.5. We proceed as in the proof of [40, Lemma 2.1]. Using Lemma 4.6(i) it holds

$$\begin{aligned} \mathbb{E}_\mu[\langle X_t^\alpha, f\rangle^{1+\zeta}] &\leq 1 + c_1(\zeta) \int_1^\infty z^{1+\zeta} \int_0^{2/z} \mathbb{E}_\mu\left[e^{-\langle X_t^\alpha, rf\rangle} - 1 + \langle X_t^\alpha, rf\rangle\right] \mathrm{d}r\, \mathrm{d}z \\ &\leq 1 + c_1(\zeta) \int_1^\infty z^{1+\zeta} \int_0^{2/z} e^{-\langle \mu, u_{rf}(t)\rangle} - 1 + \langle \mu, S_t^\alpha(rf)\rangle\, \mathrm{d}r\, \mathrm{d}z, \end{aligned} \tag{4.15}$$

where we used the Laplace transition functional (4.6) and the formula (4.2) for the first moment of X_t^α in the second step. Remember here that u_{rf} is the solution of the nonlinear integral equation (4.1) with initial data rf. Let us turn our attention to the inner integrand in (4.15). It is bounded by

$$\begin{aligned} &c_2(\beta)\langle \mu, S_t^\alpha(rf)\rangle^{1+\beta} + \left|e^{-\langle \mu, u_{rf}(t)\rangle} - e^{-\langle \mu, S_t^\alpha(rf)\rangle}\right| \\ &\quad \leq c_2(\beta)\langle \mu, S_t^\alpha(rf)\rangle^{1+\beta} + \left|\langle \mu, u_{rf}(t)\rangle - \langle \mu, S_t^\alpha(rf)\rangle\right| \end{aligned} \tag{4.16}$$

for $c_2(\beta) > 0$, where we used Lemma 4.6(ii) to estimate the term $\langle \mu, S_t^\alpha(rf)\rangle - 1$ and then applied the mean value theorem to the function $[0,\infty) \ni y \mapsto e^{-y}$. The function $u_{rf}(t)$ solves the integral equation

$$u_{rf}(t) - S_t^\alpha(rf) = \int_0^t S_{t-s}^\alpha (u_{rf}(s))^{1+\beta}\, \mathrm{d}s,$$

which yields the bound

$$\begin{aligned} &c_2(\beta)\langle \mu, S_t^\alpha(rf)\rangle^{1+\beta} + \langle \mu, \int_0^t S_{t-s}^\alpha (u_{rf}(s))^{1+\beta}\, \mathrm{d}s\rangle \\ &\leq (1 + c_2(\beta)) r^{1+\beta} \left[\langle \mu, S_t^\alpha(f)\rangle^{1+\beta} + \langle \mu, \int_0^t S_{t-s}^\alpha (S_s^\alpha(f))^{1+\beta}\, \mathrm{d}s\rangle\right] \end{aligned}$$

for (4.16). We have used the domination $u_{rf}(s) \leq S_s^\alpha(rf)$ from (3.30) here. Returning to (4.15) results in

$$\mathbb{E}_\mu[\langle X_t^\alpha, f\rangle^{1+\zeta}] \leq 1 + c_1(\zeta)(1+c_2(\beta)) \int_1^\infty z^{1+\zeta} \int_0^{2/z} r^{1+\beta}\,\mathrm{d}r\,\mathrm{d}z \\ \cdot \left[\langle \mu, S_t^\alpha(f)\rangle^{1+\beta} + \langle \mu, \int_0^t S_{t-s}^\alpha(S_s^\alpha(f))^{1+\beta}\,\mathrm{d}s\rangle\right].$$

It remains to show that the constant

$$c(\zeta,\beta) := c_1(\zeta)(1+c_2(\beta)) \int_1^\infty z^{1+\zeta} \int_0^{2/z} r^{1+\beta}\,\mathrm{d}r\,\mathrm{d}z$$

is finite. A direct computation shows that

$$\int_1^\infty z^{1+\zeta} \int_0^{2/z} r^{1+\beta}\,\mathrm{d}r\,\mathrm{d}z = \frac{2^{2+\beta}}{(2+\beta)(\zeta-\beta)} \int_1^\infty z^{\zeta-\beta-1}\,\mathrm{d}z = \frac{2^{2+\beta}}{(2+\beta)(\beta-\zeta)} < \infty,$$

using $\zeta - \beta < 0$. This completes the proof. □

Again, we can repeat the previous proof step by step for the process $X^{\alpha,\varepsilon}$ and the associated semigroup and obtain

Corollary 4.7. *Let $0 < \zeta < \beta$ and $\varepsilon \in (0,1)$. Then it holds*

$$\mathbb{E}_\mu[\langle X_t^{\alpha,\varepsilon}, f\rangle^{1+\zeta}] \leq 1 + c(\zeta,\beta)\left[\langle \mu, S_t^{\alpha,\varepsilon} f\rangle^{1+\beta} + \langle \mu, \int_0^t S_{t-s}^{\alpha,\varepsilon}(S_s^{\alpha,\varepsilon} f)^{1+\beta}\,\mathrm{d}s\rangle\right]$$

for a constant $c(\zeta,\beta) > 0$.

4.2. Pathwise properties of the superprocess with approximate point source

In Chapter 3 we have adopted the methods from Fleischmann-Mueller [20], to prove the existence of the measure-valued process $X^{\alpha,\varepsilon}$ related to the partial differential equation

$$u(t,x) := S_t^{\alpha,\varepsilon} f(x) - \eta \int_0^t S_{t-s}^{\alpha,\varepsilon}(u^{1+\beta}(s))(x)\,\mathrm{d}s, \quad u(0,\cdot) = f,$$

where the semigroup $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ is given by the Feynman-Kac formula

$$S_t^{\alpha,\varepsilon} f(x) = \mathbb{E}\left[e^{-\int_0^t V_{\alpha,\varepsilon}(W_s)\,\mathrm{d}s} f(W_t) \mid W_0 = x\right].$$

It turns out that this situation is a special case of a more general theory of measure-valued branching processes. Given a bounded function V on $\mathbb{R}^3$, the expression $\int_0^t V(W_s)\,\mathrm{d}s$ is an additive functional of a brownian motion. This gives a transition semigroup (P_t^V) via

$$P_t^V f(x) = \mathbb{E}\left[e^{-\int_0^t V(W_s)\,\mathrm{d}s} f(W_t) \mid W_0 = x\right]. \tag{4.17}$$

This leads to the question, whether a superprocess X exists, satisfying the transition formula

$$\mathbb{E}[e^{-X_t f} | \xi_0 = \mu] =: = e^{-\langle \mu, u(t) \rangle},$$

where u is the unique solution of the integral equation

$$u(t,x) := P_t^V f(x) + \int_0^t P_{t-s}^V \psi(x, u(s))(x)\,\mathrm{d}s, \quad u(0,\cdot) = f.$$

The function ψ is called the branching mechanism. In our special case the branching mechanism is given by $\psi_{\eta,\beta}(x,z) := -\eta z^{1+\beta}$. In [35] the existence of superprocesses is shown for branching mechanisms ψ of the form

$$\psi(x,z) := -a(x)z - b(x)z^2 + \int_0^\infty (1 - e^{-zv} - zv) k(x, \mathrm{d}v) \tag{4.18}$$

such that $(v \wedge v^2)k(x, \mathrm{d}v)$ is a bounded positive integral kernel and a, b are bounded functions with $b \geq 0$. In fact, this is a special case of the abstract setting [35, Formula 2.26], described in [35, Example 2.4] and [6].

To apply this theory, we need to show that the function $\psi_{\eta,\beta}$ is of the shape (4.18): Choose $a = b \equiv 0$ and the kernel $k(x, \mathrm{d}v) := \eta \frac{\beta(\beta+1)}{\Gamma(1-\beta)} v^{-2-\beta}\,\mathrm{d}v$ where Γ denotes the Gamma function $\Gamma(y) = \int_0^\infty s^{1-y} e^{-s}\,\mathrm{d}s$, $y \geq 0$. With partial integration we calculate

$$z^\beta = \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty \frac{1 - e^{-zv}}{v^{1+\beta}}\,\mathrm{d}v \tag{4.19}$$

and obtain the identity

$$-z^{1+\beta} = \frac{\beta(\beta+1)}{\Gamma(1-\beta)} \int_0^\infty \frac{1 - e^{-zv} - zv}{v^{2+\beta}}\,\mathrm{d}v, \tag{4.20}$$

since differentiating both sides of (4.20) with respect to z gives the true statement (4.19). Also refer to [6, Subsection 3.2].

Comparing (4.20) and (4.18) shows that the process $X^{\alpha,\varepsilon}$, $\alpha \in \mathbb{R}$, $\varepsilon \in (0,1)$, with branching mechanism $\psi_{\eta,\beta}$ satisfies the conditions in [35, Subsection 2.3] with $a = b = 0$. Note that the function $V_{\alpha,\varepsilon}$ defined in (2.25) is bounded for every fixed α, ε. Hence, $X^{\alpha,\varepsilon}$ is a Dawson-Watanabe superprocess with spatially constant branching mechanism, i.e. $\psi(x,z)$ in (4.18) does not depend on x [35, p.42].

This allows us to deduce some properties of the superprocess $X^{\alpha,\varepsilon}$. Applying [35, Theorem

2.22], the paths of $X^{\alpha,\varepsilon}$ are right-continuous in probability, that is

$$\lim_{t\to r+} \mathbb{P}[\rho(X_t^{\alpha,\varepsilon}, X_r^{\alpha,\varepsilon}) > \varepsilon] = 0$$

for every $r > 0$, using the metric ρ on $\mathcal{M}(\mathbb{R}^3)$ defined in [35, Formula (1.3)]. The following Theorem gives a stronger càdlàg property of the paths.

Theorem 4.8. *There is a unique measure $\overline{\mathrm{Q}}_\mu$, satisfying $\overline{\mathrm{Q}}_\mu[X_0^{\alpha,\varepsilon} = \mu] = 1$ and $X^{\alpha,\varepsilon}$ is a Markov process on $\mathcal{M}$ relative to $\overline{\mathrm{Q}}_\mu$ with transition semigroup $(Q_t)_{t\geq 0}$. Then, $X^{\alpha,\varepsilon}$ has $\overline{\mathrm{Q}}_\mu$-a.s. càdlàg paths in $\mathcal{M}$.*

Proof. The existence of the unique probability measure $\overline{\mathrm{Q}}_\mu$ is shown in [35, Theorem 5.1]. According to [35, Theorem 5.7], the paths of the process $X^{\alpha,\varepsilon}$ are $\overline{\mathrm{Q}}_\mu$-a.s. right-continuous. Now note that the underlying spacial motion of $X^{\alpha,\varepsilon}$ in the sense of [35, p. 42] is the Brownian motion $(W_t)_{t\geq 0}$. The Brownian motion has continuous paths. In particular, W is a Hunt process, i.e. it is right-continuous, has the strong Markov property and is quasi-left continuous (refer to the definition in [9, p. 74]). Using [35, Theorem 5.11], we have that $X^{\alpha,\varepsilon}$ is a Hunt process in $\mathcal{M}(\mathbb{R}^3)$ as well. According to [9, Theorem 3.1.1] this implies that the paths of $X^{\alpha,\varepsilon}$ have left limits almost surely. Together with the aforementioned right-continuity we have the desired càdlàg property. □

Another useful property is the following martingale representation from [35, Theorem 7.26]: There is a martingale measure M_t, $t \geq 0$, such that

$$\langle X_t^{\alpha,\varepsilon}, f\rangle = \langle X_0^{\alpha,\varepsilon}, u(t)\rangle + \int_0^t \int_{\mathbb{R}^3} u(t-s,x) M(\,\mathrm{d}s,\,\mathrm{d}x)$$

and u is the unique solution of the integral equation

$$u(t,x) := S_t^{\alpha,\varepsilon} f(x) - \eta \int_0^t S_{t-s}^{\alpha,\varepsilon}(u(s))(x)\,\mathrm{d}s,$$

where f is bounded.

Remark 4.9. We want to emphasize that the results in this section do not transfer directly to the measure-valued process with point interaction X^α. The transition semigroup of this process cannot be described via perturbation of the heat semigroup $(S_t)_{t\geq 0}$ with a bounded potential V as in (4.17), hence, the theory of [35, Chapter 2] is not applicable. However, we will show the convergence $X_t^{\alpha,\varepsilon} \to X_t^\alpha$, $\varepsilon \downarrow 0$, in an appropriate sense in a later chapter. So it seems likely that some kind of regularity of the paths of X^α can be proven by using the properties of $X^{\alpha,\varepsilon}$ and passing to the limit.

4.3. Dependence on the parameter of the nonlinear term

In the nonlinear term of the partial differential equation (1.6) we have a multiplicative parameter $\eta > 0$. Intuitively, one would expect that there is a kind of monotonic dependence of the solution on η. The following theorem shows that this is indeed the case. We will also analyze how this influences the corresponding superprocesses.

Lemma 4.10. *Under Assumption 3.21, let initial data $f \in \Phi^p$ and $\alpha \in \mathbb{R}$. Let $0 < \eta_1 \le \eta_2$. For the solutions u_1, u_2 of*

$$\begin{aligned} \partial_t u_1 &= \Delta_\alpha u_1 - \eta_1 u_1^{1+\beta} \quad \text{on } (0,\infty) \times \mathbb{R}^d \setminus \{0\}, \\ \partial_t u_2 &= \Delta_\alpha u_2 - \eta_2 u_2^{1+\beta} \quad \text{on } (0,\infty) \times \mathbb{R}^d \setminus \{0\}, \\ u_1(0,\cdot) = u_2(0,\cdot) &= f \quad \text{on } \mathbb{R}^d \setminus \{0\}, \end{aligned} \tag{4.21}$$

it holds $u_1(t,x) \ge u_2(t,x)$ for fixed $t \ge 0$ and $x \in \mathbb{R}^3$.

Proof. As in Theorem 3.26, we have the following iteration converging to the solutions u_1 and u_2 respectively

$$\begin{aligned} u_{1,0} &:= u_{2,0} := S_t^\alpha f, \\ u_{j,m}(t,x) &= S_t^\alpha f(x) - \eta_j \int_0^t S_{t-s}^\alpha (u_{j,m-1}^{1+\beta}(s))(x)\, \mathrm{d}s \end{aligned}$$

for $m \in \mathbb{N}$ and $j = 1,2$. Our goal is to show the assertion $u_{1,m}(t,x) \ge u_{2,m}(t,x)$ for all m. For $m = 0$ we have equality by construction. For $m \ge 1$ it holds

$$u_{1,m} - u_{2,m} = \eta_2 \int_0^t S_{t-s}^\alpha (u_{2,m-1}^{1+\beta})\, \mathrm{d}s - \eta_1 \int_0^t S_{t-s}^\alpha (u_{1,m-1}^{1+\beta})\, \mathrm{d}s. \tag{4.22}$$

In the case $m = 1$ the right-hand side simplifies to

$$(\eta_2 - \eta_1) \int_0^t S_{t-s}^\alpha ((S_s^\alpha f)^{1+\beta})\, \mathrm{d}s,$$

so the claim is true here because of $\eta_2 \ge \eta_1$ and the nonnegativity of solutions, Lemma 7.5. We proceed by induction over $m \ge 2$. Assume the assertion holds for $m-1$ and $m-2$. We expand the right-hand side of (4.22) and obtain

$$u_{1,m} - u_{2,m} = (\eta_2 - \eta_1) \int_0^t S_{t-s}^\alpha (u_{1,m-1}^{1+\beta})\, \mathrm{d}s - \eta_2 \int_0^t S_{t-s}^\alpha (u_{1,m-1}^{1+\beta} - u_{2,m-1}^{1+\beta})\, \mathrm{d}s.$$

For this expression to be nonnegative, it is sufficient to show that

$$(\eta_2 - \eta_1) u_{1,m-1}^{1+\beta} \ge \eta_2 (u_{1,m-1}^{1+\beta} - u_{2,m-1}^{1+\beta}) \tag{4.23}$$

holds pointwise in t, x. Because of the induction assumption $u_{1,m-1} \geq u_{2,m-1}$ and the elementary estimate (7.6) we have

$$\eta_2(u_{1,m-1}^{1+\beta} - u_{2,m-1}^{1+\beta}) \leq \eta_2(1+\beta)u_{1,m-1}^{\beta}(u_{1,m-1} - u_{2,m-1}).$$

Hence, the statement

$$(\eta_2 - \eta_1)u_{1,m-1} \geq \eta_2(1+\beta)(u_{1,m-1} - u_{2,m-1})$$

is sufficient for (4.23). Note that we divided both sides by $u_{1,m-1}^{\beta}$ here. We rewrite the last inequality

$$u_{1,m-1} - \frac{\eta_2(1+\beta)}{(\eta_2 - \eta_1)}(u_{1,m-1} - u_{2,m-1}) \geq 0.$$

This is equivalent to

$$S_t^\alpha f - \left(\eta_1 - \frac{\eta_1\eta_2(1+\beta)}{\eta_2 - \eta_1}\right)\int_0^t S_{t-s}^\alpha(u_{1,m-2}^{1+\beta})\,\mathrm{d}s - \frac{\eta_2^2(1+\beta)}{\eta_2 - \eta_1}\int_0^t S_{t-s}^\alpha(u_{2,m-2}^{1+\beta})\,\mathrm{d}s \geq 0.$$

Using $u_{1,m-2} \geq u_{2,m-2}$, we obtain the following lower bound for the left-hand side

$$\begin{aligned}
S_t^\alpha f &- \left(\eta_1 - \frac{\eta_1\eta_2(1+\beta)}{\eta_2 - \eta_1} + \frac{\eta_2^2(1+\beta)}{\eta_2 - \eta_1}\right)\int_0^t S_{t-s}^\alpha(u_{1,m-2}^{1+\beta})\,\mathrm{d}s \\
&\geq S_t^\alpha f - (\eta_2(1+\beta) + \eta_1)\int_0^t S_{t-s}^\alpha(u_{1,m-2}^{1+\beta})\,\mathrm{d}s \\
&\geq S_t^\alpha f - (\eta_2(1+\beta) + \eta_1)\int_0^t S_{t-s}^\alpha((S_s^\alpha f)^{1+\beta})\,\mathrm{d}s.
\end{aligned}$$

Because of the nonnegativity of solutions, more precisely case $n = 1$ in Theorem 7.4, the last expression is nonnegative. This completes the proof. □

Now we want to use this result to compare the Laplace transforms of the corresponding processes evaluated with fixed f. This can be done directly via the Laplace transition functional (1.4). In order to do this in an appropriate sense, the following definition from the theory of stochastic orders in [46, p. 233] is needed.

Definition 4.11. Let Y, Z two real-valued nonnegative random variables such that

$$\mathbb{E}[\exp(-sY)] \geq \mathbb{E}[\exp(-sY)] \quad \text{for all } s > 0.$$

Then Y is said to be smaller than Z in Laplace transform order, denoted by $Y \leq_{\mathrm{Lt}} Z$.

There are alternative characterizations of the Laplace transform order. For instance, $Y \leq_{\mathrm{Lt}} Z$ if and only if $\mathbb{E}[\varphi(Y)] \geq \mathbb{E}[\varphi(Z)]$ for all completely monotone functions $\varphi : [0, \infty) \to \mathbb{R}$

[46, Theorem 5.A.3]. A function φ is said to be completely monotone, if all derivatives $\varphi^{(n)}, n \in \mathbb{N}$ exist and $(-1)^n\varphi^{(n)} \geq 0$. This characterization gives rise to several results about Laplace transform ordered random variables, see [46, Chapter 5.A]. Now we will show that the result for the solutions from Lemma 4.12 translates to the superprocesses in terms of Laplace transform order.

Lemma 4.12. *Under the conditions of Lemma 4.10, let $X^{\alpha,1}$ and $X^{\alpha,2}$ the superprocessses corresponding to the solutions u_1 and u_2 of (4.21) with $0 < \eta_1 \leq \eta_2$, satisfying $X_0^{\alpha,1} = X_0^{\alpha,2} = \mu$. Then it holds for fixed $t \geq 0$*

$$\langle X_t^{\alpha,2}, f\rangle \leq_{Lt} \langle X_t^{\alpha,1}, f\rangle.$$

Proof. Let $s > 0$. Let $u_{sf,1}$ and $u_{sf,2}$ the solutions of (4.21), but with initial data sf instead of f. This notation was also used in Section 4.1. We have from Lemma 4.10 the pointwise order

$$u_{sf,1}(t,x) \geq u_{sf,2}(t,x).$$

With the monotonicity of the exponential function and $X_0^{\alpha,1} = X_0^{\alpha,2}$ we obtain

$$\exp(-\langle X_0^{\alpha,1}, u_{sf,1}(t)\rangle) \leq \exp(-\langle X_0^{\alpha,2}, u_{sf,2}(t)\rangle). \tag{4.24}$$

Using the Laplace transition functional (1.4), it holds for $j = 1, 2$

$$\exp(-\langle X_0^{\alpha,j}, u_{sf,j}(t)\rangle) = \mathbb{E}_\mu[\exp(-\langle X_t^{\alpha,j}, sf\rangle)] = \mathbb{E}_\mu[\exp(-s\langle X_t^{\alpha,j}, f\rangle)]. \tag{4.25}$$

Combining (4.24) and (4.25) yields

$$\mathbb{E}_\mu[\exp(-s\langle X_t^{\alpha,1}, f\rangle)] \leq \mathbb{E}_\mu[\exp(-s\langle X_t^{\alpha,2}, f\rangle)]$$

and the proof is complete by Definition 4.11. Note that $\langle X_t^{\alpha,j}, f\rangle$, $j = 1, 2$, are nonnegative random variables for every fixed t, f. □

In complete analogy, the results of this section can be transferred to the approximating superprocess $X^{\alpha,\varepsilon}$.

Corollary 4.13. *Under Assumption 3.21 let initial data $f \in \Phi^p$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Let $0 < \eta_1 \leq \eta_2$. For the solutions $u_{\alpha,\varepsilon,1}, u_{\alpha,\varepsilon,2}$ of*

$$\begin{aligned} \partial_t u_{\alpha,\varepsilon,1} &= H_{\alpha,\varepsilon} u_{\alpha,\varepsilon,1} - \eta_1 u_{\alpha,\varepsilon,1}^{1+\beta} \quad \textit{on } (0,\infty) \times \mathbb{R}^d \setminus \{0\}, \\ \partial_t u_{\alpha,\varepsilon,2} &= H_{\alpha,\varepsilon} u_{\alpha,\varepsilon,2} - \eta_2 u_{\alpha,\varepsilon,2}^{1+\beta} \quad \textit{on } (0,\infty) \times \mathbb{R}^d \setminus \{0\}, \\ u_{\alpha,\varepsilon,1}(0,\cdot) &= u_{\alpha,\varepsilon,2}(0,\cdot) = f \quad \textit{on } \mathbb{R}^d \setminus \{0\}, \end{aligned} \tag{4.26}$$

it holds $u_{\alpha,\varepsilon,1}(t,x) \geq u_{\alpha,\varepsilon,2}(t,x)$ *for fixed* $t \geq 0$ *and* $x \in \mathbb{R}^3$*. Furthermore, for the associated superprocesses satisfying* $X_0^{\alpha,\varepsilon,1} = X_0^{\alpha,\varepsilon,2} = \mu$ *we have*

$$\langle X_t^{\alpha,\varepsilon,2}, f\rangle \leq_{Lt} \langle X_t^{\alpha,\varepsilon,1}, f\rangle. \tag{4.27}$$

Proof. We can repeat the proof of Lemma 4.10 step by step, replacing S_t^α with $S_t^{\alpha,\varepsilon}$ everywhere and obtain (4.26). Then, (4.27) follows from (4.26) as described in Lemma 4.12. □

As a conclusion of this chapter, we summarize that there are multiple similarities between the super-Brownian motion with point source and the approximating processes. Concerning the path regularity properties of $X^{\alpha,\varepsilon}$ from Section 4.2, it seems likely that a similar property holds for X^α as well.

5. Sectoriality of Laplacians with (approximate) point source

In Chapter 3, we have investigated solutions of the partial differential equations in the context of Lebesgue spaces with an additional weight function $|x|^{-1}$, following closely the work of Fleischmann and Mueller from [20]. Weighted spaces will play a central role further on. In this chapter we want to take a closer look at these spaces, introduce them from a more general perspective and collect some helpful properties. In the second part of the chapter we establish the class of sectorial operators in these weighted spaces. This allows us to obtain existence of the semigroups $(S_t^{\alpha})_{t\geq 0}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ in the weighted space and useful representations of these semigroups via the resolvents of the generators.

5.1. Weighted Lebesgue-Spaces

We now want to study the behavior of the resolvents of $-\Delta$, $-\Delta_\alpha$ and $-H_{\alpha,\varepsilon}$ in weighted spaces, i.e. Lebesgue spaces with an additional weight function $w : \mathbb{R}^d \to [0,\infty)$. For these spaces we use the notation $L^p(w) := L^p(w, \mathbb{R}^d)$. They belong to the larger class of weighted *Muckenhoupt spaces*.

Definition 5.1. [49, p. 194]. A weight function w is an A_p weight or Muckenhoupt weight, if there is a $C > 0$ such that for all balls $B \in \mathbb{R}^d$

$$\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x \cdot \left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}\,\mathrm{d}x\right)^{\frac{p}{p'}} \leq C < \infty.$$

We will focus again on the case $d = 3$ now. In particular, let us consider the weight $w(x) := |x|^{-1}$ in this context.

Lemma 5.2. *The weight function $w : \mathbb{R}^3 \to [0,\infty), x \mapsto |x|^{-1}$, is in A_p for every $p \geq 1$.*

Proof. We have this characterization of A_p weights of type $w(x) = |x|^a$, $a \in \mathbb{R}$ from [49, Paragraph V.6.4, p. 218]: The function $x \mapsto |x|^a$ is in A_p if and only if

$$-d < a < d(p-1).$$

Clearly for $a = -1$ it holds $-d = -3 < -1 = a$ and $a = -1 < 0 \leq d(p-1)$ for every $p \geq 1$, so the claim follows immediately. $\square$

A nontrivial property of A_p weights is the following: w is in A_p, if and only if the Hardy-Littlewood operator defined by

$$(Mf)(x) = \sup_{r>0} \frac{c_d}{r^d} \int_{|y|\le r} |f(x-y)| \, \mathrm{d}y$$

is bounded in $L^p(w)$, this means

$$\int_{\mathbb{R}^d} (Mf(x))^p w(x) \, \mathrm{d}x \lesssim \int_{\mathbb{R}^d} |f(x)|^p w(x) \, \mathrm{d}x.$$

This characterization is given in [49, p. 193f]. As a consequence, A_p weights have very good properties regarding singular integral operators. This allows us to develop weighted integral inequalities for the resolvents in later chapters.

The following theorem gives a criterion for a convolution operator to be bounded in $L^p(w)$.

Theorem 5.3. *[49, Section V.4.2, Theorem 2 and Remark 4.5.2]. Let $T : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ a convolution operator given by $Tf = f * K$. Assume*

(i) the operator T is bounded on $L^2(\mathbb{R}^3)$

$$\|Tf\|_{L^2} \lesssim A\|f\|_{L^2},$$

(ii) for the derivatives of the kernel K it holds for $x \neq 0$ and multi-indices α with $|\alpha| \le 1$

$$|\partial_x^\alpha K(x)| \lesssim |x|^{-3-|\alpha|}.$$

Furthermore, for $1 < p < \infty$ let $w \in A_p$. Then we have for $f \in L^p(w)$

$$\int_{\mathbb{R}^3} |Tf(x)|^p w(x) \, \mathrm{d}x \lesssim \int_{\mathbb{R}^3} |f(x)|^p w(x) \, \mathrm{d}x,$$

thus T is a bounded linear operator on $L^p(w)$

In the unweighted L^p spaces, it is well known that the resolvent R_λ of the Laplacian is bounded. We now apply the previous theorem to show that for $\lambda \in \rho(-\Delta)$ the resolvent R_λ is bounded in the weighted space as well.

Lemma 5.4. *Let $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $R_\lambda = (-\Delta - \lambda I)^{-1}$ the resolvent to λ. Let $w \in A_p$. For every $p > 1$, R_λ is a bounded linear operator from $L^p(w)$ to $L^p(w)$. In particular, $\sigma(-\Delta) \subseteq [0, \infty)$ in $L^p(w)$.*

Proof. The convolution kernel of R_λ is

$$R_\lambda(x) = \frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|}.$$

To apply Theorem 5.3 we state first that R_λ is bounded as an operator in $L^2(\mathbb{R}^3)$. Now we have to show that $R_\lambda(x)$ decays like $|x|^{-3}$, this is the case $\alpha = 0$. Write $\sqrt{\lambda} = a + ib$. Then

$$\left|e^{i\sqrt{\lambda}|x|}\right| = \left|e^{ia|x|}\right|\left|e^{-b|x|}\right| = \left|e^{-b|x|}\right|.$$

By assumption $\lambda \notin [0,\infty)$, thus $b = \operatorname{Im}\sqrt{\lambda} > 0$. Now consider two cases.

Case 1: $|x| \leq 1$. In this case we immediately get

$$4\pi|R_\lambda(x)| = \frac{e^{-b|x|}}{|x|} \leq \frac{1}{|x|^3}.$$

Case 2: $|x| > 1$. Since $e^{-b|x|}$ decays faster than any polynomial in $|x|$, there is a constant $C_b > 0$, such that $e^{-b|x|} \leq C_b|x|^{-2}$ for every $|x| \geq 1$. This implies

$$\frac{e^{-b|x|}}{|x|} \lesssim \frac{1}{|x|^3}$$

in this case, but note that the constant depends on $|\sqrt{\lambda}|$.

Now we study the first order derivatives of $R_\lambda(x)$, so $\alpha = 1$. We have for $j \in \{1,2,3\}$

$$\begin{aligned} 4\pi\left|\partial_{x_i} R_\lambda(x)\right| &= \left|\frac{e^{i\sqrt{\lambda}|x|}x_i}{|x|^2}(i\sqrt{\lambda} - |x|^{-1})\right| \\ &\leq \left|\frac{e^{-b|x|}}{|x|}(i\sqrt{\lambda} - |x|^{-1})\right| \\ &\leq \frac{|\sqrt{\lambda}|e^{-b|x|}}{|x|} + \frac{e^{-b|x|}}{|x|^2}. \end{aligned}$$

The second term decays like $|x|^{-4}$ by directly applying the first part of the proof. For the first term we follow the same argumentation as above, but with the higher exponent 4:

Case 1: $|x| \leq 1$. In this case we get

$$\frac{|\sqrt{\lambda}|e^{-b|x|}}{|x|} \lesssim \frac{1}{|x|^4}$$

with the implicit constant $|\sqrt{\lambda}|$.

Case 2: $|x| > 1$. Since $e^{-b|x|}$ decays faster than any polynomial in $|x|$, there is a constant $C_b' > 0$, such that $e^{-b|x|} \leq C_b'|x|^{-3}$ for every $|x| \geq 1$. This implies

$$\frac{e^{-b|x|}}{|x|} \lesssim \frac{1}{|x|^4}$$

in this case, with a constant depending on $|\sqrt{\lambda}|$. $\square$

Now let's turn our attention to the resolvents of the Laplacian with point interaction $-\Delta_\alpha$, where $\alpha \geq 0$. We do not only prove $L^p(w) \to L^p(w)$ boundedness of $\overline{R}_\lambda^\alpha$, but a more general $L^p(w) \to L^q(w)$ result. Due to the simple structure of $\overline{R}_\lambda^\alpha$, this can be shown with elementary calculations. Similar regularizing effects are known for the heat semigroup in the unweighted Lebesgue spaces, refer for example to [11, Chapter 3]. Weighted estimates of this kind will play an important role in the semigroup estimates of the following chapters.

Lemma 5.5. *Let $\alpha \geq 0, \lambda \in \mathbb{C} \setminus [0,\infty)$ and $R_\lambda^\alpha = (-\Delta_\alpha - \lambda I)^{-1}$ the resolvent to λ. Let $w = |x|^{-1}$. For every $p > 1$ and $1 < q < 2$, the residue term $\overline{R}_\lambda^\alpha$ as defined in (2.24) is a bounded linear operator from $L^p(w)$ to $L^q(w)$.*

Proof. Let $f \in L^p(w)$. We have

$$\overline{R}_\lambda^\alpha f = \int_{\mathbb{R}^3} \overline{R}_\lambda^\alpha(x,y) f(y)\,\mathrm{d}y = \frac{1}{\alpha - \frac{i\sqrt{\lambda}}{4\pi}} \frac{e^{i\sqrt{\lambda}|x|}}{|x|} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|y|}}{|y|} f\,\mathrm{d}y.$$

From this it follows

$$\begin{aligned}\left\|\overline{R}_\lambda^\alpha f\right\|_{L^q(w)} &= \left\| \frac{1}{\alpha - \frac{i\sqrt{\lambda}}{4\pi}} \frac{e^{i\sqrt{\lambda}|\cdot|}}{|\cdot|} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|y|}}{|y|} f(y)\,\mathrm{d}y \right\|_{L^q(w)} \\ &= \left| \frac{1}{\alpha - \frac{i\sqrt{\lambda}}{4\pi}} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|y|}}{|y|} f(y)\,\mathrm{d}y \right| \cdot \left\| \frac{e^{i\sqrt{\lambda}|\cdot|}}{|\cdot|} \right\|_{L^q(w)}. \end{aligned} \tag{5.1}$$

First of all, it holds $|\alpha - \frac{i\sqrt{\lambda}}{4\pi}| > 0$ because of $\alpha \geq 0$ and $\operatorname{Im}\sqrt{\lambda} > 0$. Next we show that the function $g(x) = \frac{e^{i\sqrt{\lambda}|x|}}{|x|}$ is in $L^q(w)$ and hence the last factor in (5.1) is finite. With $b = \operatorname{Im}\sqrt{\lambda} > 0$ we write

$$\begin{aligned}\|g\|_{L^q(w)}^q &= \int_{\mathbb{R}^3} \left| \frac{e^{i\sqrt{\lambda}|x|}}{|x|} \right|^q w(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} |e^{i\sqrt{\lambda}|x|}|^q |x|^{-q-1}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} e^{-bq|x|} |x|^{-q-1}\,\mathrm{d}x \\ &\approx \int_0^\infty e^{-bqr} r^{1-q}\,\mathrm{d}r.\end{aligned}$$

Now decompose the integration range $[0,\infty) = [0,1) \cup [1,\infty)$. For the second integral with $r \geq 1$ we have

$$\int_1^\infty e^{-bqr} r^{1-q}\,\mathrm{d}r \leq \int_1^\infty e^{-bqr}\,\mathrm{d}r < \infty,$$

since the exponential function e^{-t} is integrable. In the range of the first integral we have

$$\int_0^1 e^{-bqr} r^{1-q} \,\mathrm{d}r < \int_0^1 r^{1-q} \,\mathrm{d}r < \infty,$$

because with $1 - q > -1$ for $q < 2$ the singularity of the one-dimensional integral is mild enough. Now it remains to estimate the first integral in (5.1). We have with Hölder's inequality

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|y|}}{|y|} f(y) \,\mathrm{d}y \right| &= \left| \int_{\mathbb{R}^3} e^{i\sqrt{\lambda}|y|} f(y) w(y) \,\mathrm{d}y \right| \\
&\le \int_{\mathbb{R}^3} |e^{-b|y|}| w(y)^{1/p'} |f(y)| w(y)^{1/p} \,\mathrm{d}y \\
&\le \left(\int_{\mathbb{R}^3} |e^{-b|y|}|^{p'} w(y) \,\mathrm{d}y \right)^{1/p'} \|f\|_{L^p(w)},
\end{aligned}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ and again $b = \operatorname{Im} \sqrt{\lambda}$. The remaining y integral converges since

$$\int_{\mathbb{R}^3} |e^{-b|y|}|^{p'} w(y) \,\mathrm{d}y \approx \int_0^\infty e^{-bp'r} r \,\mathrm{d}r = -\frac{1}{bp'} [e^{-bp'r}]_0^\infty = \frac{1}{bp'}.$$

By recollecting the terms in (5.1) we get the desired boundedness of $\overline{R}_\lambda^\alpha$. □

Now one easily obtains $L^p(w)$ boundedness of R_λ^α.

Corollary 5.6. *Let $\alpha \ge 0, \lambda \in \mathbb{C} \setminus [0, \infty)$ and $R_\lambda^\alpha = (-\Delta_\alpha - \lambda I)^{-1}$ the resolvent to λ. Let $w = |x|^{-1}$. For every $1 < p < 2$, R_λ^α is a bounded linear operator from $L^p(w)$ to $L^p(w)$. In particular, $\sigma(-\Delta_\alpha) \subseteq [0, \infty)$ in $L^p(w)$, and equation (2.24) is valid in the $L^p(w)$ sense, i.e. as equality of bounded linear operators in $L^p(w)$.*

Proof. This follows immediately by applying Lemma 5.4 and Lemma 5.5 to the right-hand side of (2.24) with $q = p$ respectively. □

Remark 5.7. The $L^p(w)$-boundedness of the resolvent $R_\lambda^{\alpha,\varepsilon}$ is difficult to prove with elementary calculations because of the less explicit structure of the resolvent (2.27). It will follow from more abstract results in the next chapter. $L^p(w)$–$L^q(w)$ estimates will then also be derived in this context.

In the following lemma we formulate the well-known Hölder's inequality for the space $L^p(w)$ in dimension $d = 3$, as well as a convolution inequality. These will be refered to in later parts of the work.

Lemma 5.8. *Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ measurable functions.*

(i) *Let* $p, p' > 1$ *such that* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then it holds*

$$\|fg\|_{L^r(w)} \leq \|f\|_{L^p(w)} \|g\|_{L^{p'}(w)}.$$

(ii) *Let* $p, q, r \geq 1$ *with* $\frac{1}{p} + \frac{1}{r} = 1 + \frac{2}{3q}$. *Then it holds*

$$\|f * g\|_{L^q(w)} \lesssim \|f\|_{L^p(w)} \left\|g w^{-\frac{1}{p}}\right\|_{L^r}$$

with a constant depending on p, q, r.

Proof. Hölder's inequality holds for arbitrary measure spaces, so in particular for the measure $w\,\mathrm{d}x$. The inequality in (ii) is a special case of the weighted convolution inequality from [30, p. 208]. □

5.2. Analytic semigroups and sectoriality

The following definition introduces a class of operators with certain spectral properties. They allow us to represent the semigroups $(S_t^\alpha)_{t\geq 0}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ in terms of the resolvents of the corresponding generator. Semigroups with this property are called *analytic in* $L^p(w)$.

It is known that the operator $-\Delta_\alpha$ generates an analytic semigroup in $L^2(\mathbb{R}^3)$, just by the fact that $-\Delta_\alpha$ is self-adjoint and semibounded from below, see [4, Chapter I]. In 1993, by Caspers and Clément this was also shown in the setting of unweighted L^p spaces for $\frac{3}{2} < p < 3$ using the Sobolev embedding theorem [8]. However, these results do not transfer trivially to the weighted space $L^p(w)$, $w(x) = |x|^{-1}$, since this space has a different structure due to the singular nature of the weight. In particular, for any $p > 1$ the space $L^p(w)$ is not contained in L^p and vice versa. To the knowledge of the author, there is no literature yet about the analycity of the semigroups $(S_t^\alpha)_{t\geq 0}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ in $L^p(w)$.

The restrictions on p differ slightly compared to the unweighted case. Most importantly, the upper bound is 2 instead of 3, because the weight introduces an extra singularity of order 1, see the proof of Lemma 5.17 below.

Definition 5.9. [41, p. 30]. Let X a Banach space and $A : X \to X$ a linear operator on X. A is called *sectorial*, if

(i) there is an angle $\varphi \in (0, \frac{\pi}{2})$, such that

$$\rho(A) \supset \Sigma_\varphi = \{\lambda \in \mathbb{C} : \varphi < \arg\lambda < 2\pi - \varphi\},$$

(ii) there is a constant $M > 0$, such that for every $\lambda \in \Sigma_\varphi$

$$\|R_\lambda A\|_{X\to X} \leq \frac{M}{|\lambda|}.$$

Remark 5.10. Note that the situation in this definition of sectoriality is reflected at the imaginary axis in comparison with the literature [41] or [25]. This is due to the fact that we are interested in operators with spectrum in or around $[0, \infty)$. The transition between these notations can be performed by using the identity $-R_\lambda(A) = R_{-\lambda}(-A)$.

The following representation of a semigroup via the resolvents of its generator is an important application of sectoriality.

Theorem 5.11. *[41, Theorem 1.7.7]. Let $A : X \to X$ be a sectorial operator. Then, A is the infinitesimal generator of a C_0 semigroup $(T_t)_{t\geq 0}$ satisfying $\|T_t\|_X \leq C$ for some $C \geq 0$. Moreover,*

$$T_t = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} R_\lambda(A)\, \mathrm{d}\lambda, \tag{5.2}$$

where Γ is a smooth unbounded curve in Σ_φ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\varphi < \theta < \frac{\pi}{2}$. A semigroup that satisfies these properties is called a bounded analytic semigroup.

The curve integral in (5.2) is sometimes referred to as the *Dunford integral* [38]. The curve Γ can be chosen to be uniformly away from the origin, i.e. for any $c > 0$ we can choose a curve Γ satisfying $|\lambda| > c > 0$ for all $\lambda \in \Gamma$ [38, Section 1.3]. We introduce a class of curves depending only on a radius (the distance from the origin) and an opening angle in the following definition. See also Figure 5.1 below.

Definition 5.12. From now on, let $\Gamma = \Gamma(r, \theta) = \Gamma_1(r, \theta) \cup \Gamma_2(r, \theta) \cup \Gamma_3(r, \theta)$ with

$$\begin{aligned}
\Gamma_1 &= \Gamma_1(r, \theta) = \{se^{-i\theta} \in \mathbb{C} : s \geq r\}, \\
\Gamma_2 &= \Gamma_2(r, \theta) = \{re^{-i\eta} \in \mathbb{C} : \eta \in [\theta, 2\pi - \theta]\}, \\
\Gamma_3 &= \Gamma_3(r, \theta) = \{se^{i\theta} \in \mathbb{C} : s \geq r\},
\end{aligned}$$

where $r = r(\Gamma) > 0$ describes the radius of the curve and $\theta = \theta(\Gamma) \in (0, \frac{\pi}{2})$ the opening angle.

In the following chapters we will often deal with square roots of elements of Γ, because the resolvent kernels contain a factor of the shape $e^{i\sqrt{\lambda}|x|}$, $\lambda \in \Gamma$. The next lemma will be particularly useful for different kinds of estimates in these situations. Note that we are always using the principal branch of the complex square root with $\sqrt{z} := \sqrt{|z|} e^{i\frac{\arg z}{2}}$.

Lemma 5.13. *Let Γ as in Defintion 5.12 and define*

$$\sqrt{\Gamma} := \{\sqrt{\gamma} : \gamma \in \Gamma\}.$$

It holds for all $\sqrt{\gamma} \in \sqrt{\Gamma}$

$$\operatorname{Im} \sqrt{\gamma} \geq \sqrt{r} \sin \frac{\theta}{2} > 0.$$

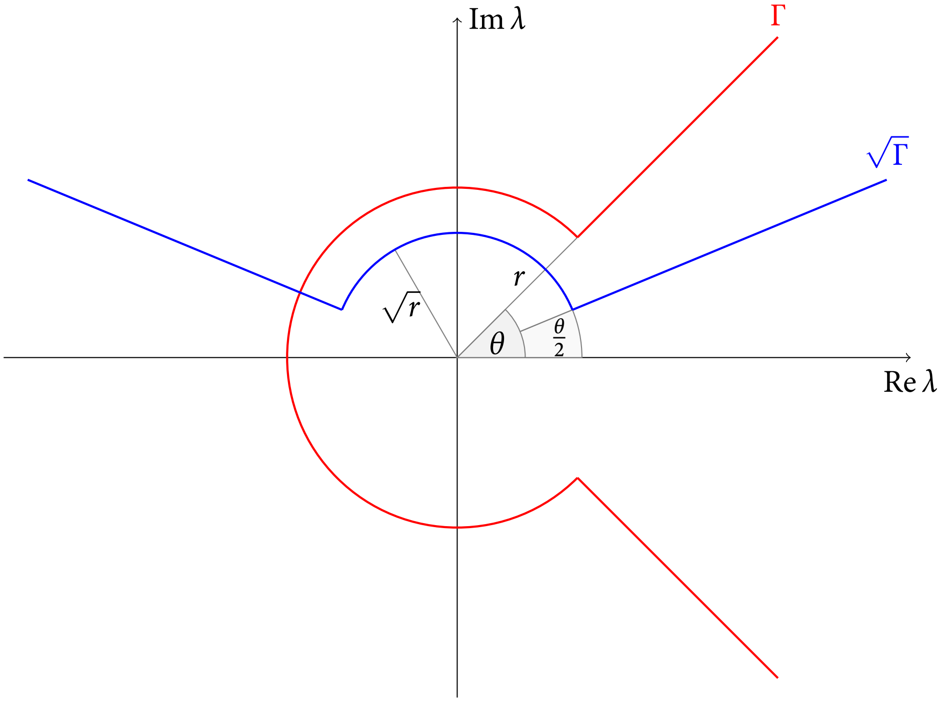

Figure 5.1.: *Illustration of the path Γ and $\sqrt{\Gamma}$. The elements of $\sqrt{\Gamma}$ have positive imaginary part and $\sqrt{\Gamma}$ is uniformly away from the real axis.*

Proof. First let $\gamma \in \Gamma_1$, so $\gamma = se^{i\theta}$ for some $s \geq r$. We calculate

$$\sqrt{\gamma} = \sqrt{se^{i\theta}} = \sqrt{s}e^{i\frac{\theta}{2}}.$$

Now, using Euler's identity,

$$\begin{aligned}\operatorname{Im}\sqrt{\gamma} &= \operatorname{Im}(\sqrt{s}e^{i\frac{\theta}{2}}) = \sqrt{s}\operatorname{Im}(e^{i\frac{\theta}{2}}) \\ &= \sqrt{s}\operatorname{Im}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) = \sqrt{s}\sin\frac{\theta}{2} \geq \sqrt{r}\sin\frac{\theta}{2}.\end{aligned}$$

In complete analogy, for $\gamma \in \Gamma_3$ it holds for some $s \geq r$

$$\operatorname{Im}\sqrt{\gamma} = \operatorname{Im}(\sqrt{s}e^{-i\frac{\theta}{2}}) \geq \sqrt{r}\sin\left(-\frac{\theta}{2}\right) = \sqrt{r}\sin\frac{\theta}{2}.$$

It remains to consider $\gamma \in \Gamma_2$. This means

$$\sqrt{\gamma} = \sqrt{r}e^{-i\eta}$$

for some $\eta \in [\frac{\theta}{2}, \pi - \frac{\theta}{2}]$. Thus,

$$\operatorname{Im} \sqrt{\gamma} = \sqrt{r} \sin \eta,$$

but $\sin \eta \geq \sin \frac{\theta}{2}$ for $\eta \in [\frac{\theta}{2}, \pi - \frac{\theta}{2}]$ and $\theta < \frac{\pi}{2}$. □

It is important to note that the value of the curve integral in (5.2) does not depend on the radius of the curve $r(\Gamma)$, as shown next.

Lemma 5.14. *Let Γ, Γ' two paths as in Definition 5.12 with $\theta(\Gamma) = \theta(\Gamma')$ but $r(\Gamma) > r(\Gamma')$. In the situation of Theorem 5.11 it holds*

$$\int_\Gamma e^{-\lambda t} R_\lambda(A) \,\mathrm{d}\lambda = \int_{\Gamma'} e^{-\lambda t} R_\lambda(A) \,\mathrm{d}\lambda.$$

Proof. Because of $\theta(\Gamma) = \theta(\Gamma')$ and $r(\Gamma) > r(\Gamma')$, both curves Γ and Γ' contain the points $r(\Gamma)e^{i\theta}$ and $r(\Gamma)e^{-i\theta}$. Furthermore, by Definition 5.12, Γ and Γ' coincide on the piece connecting $\infty e^{-i\theta}$ and $r(\Gamma)e^{-i\theta}$ and on the piece connecting $r(\Gamma)e^{i\theta}$ and $\infty e^{i\theta}$. Hence, it suffices to show

$$\int_{\overline{\Gamma}} e^{-\lambda t} R_\lambda(A) \,\mathrm{d}\lambda = \int_{\overline{\Gamma}'} e^{-\lambda t} R_\lambda(A) \,\mathrm{d}\lambda, \tag{5.3}$$

where $\overline{\Gamma} = \{\lambda \in \Gamma : |\lambda| = r(\Gamma)\}$ and $\overline{\Gamma}' = \{\lambda \in \Gamma' : |\lambda| \leq r(\Gamma)\}$ are the two different paths connecting $r(\Gamma)e^{-i\theta}$ and $r(\Gamma)e^{i\theta}$. This is illustrated in Figure 5.2.

We know that

$$R_\lambda(A) = (A - \lambda I)^{-1}$$

exists for $\lambda \in \rho(A)$. According to the Analytic Fredholm Theorem [45, Theorem VI.14], this implies that the function $\lambda \mapsto R_\lambda(A)$ is a holomorphic operator-valued function in $\rho(A)$. Of course, then for every $t \geq 0$ the function $\lambda \mapsto e^{-\lambda t} R_\lambda(A)$ is holomorphic as well.

Remember that by Definition 5.9

$$\rho(A) \supset \Sigma_\theta = \{\lambda \in \mathbb{C} : \pi < \arg \lambda < 2\pi - \theta\},$$

because A is a sectorial operator. Note that the curves $\overline{\Gamma}$ and $\overline{\Gamma}'$ are homotopic to each other in the simply connected region Σ_θ. Now we are able to employ Cauchy's integral theorem: Because $\overline{\Gamma}$ and $\overline{\Gamma}'$ are homotopic curves connecting $r(\Gamma)e^{-i\theta}$ and $r(\Gamma)e^{i\theta}$ and the integrand is holomorphic, we obtain (5.3) and the proof is complete. □

Depending on the structure of the generator and the resolvents, it is sometimes quite difficult to prove sectoriality directly with Definition 5.9. In these situations the following characterization of sectorial operators is useful. In particular, the resolvent structure of $-\Delta_\alpha$

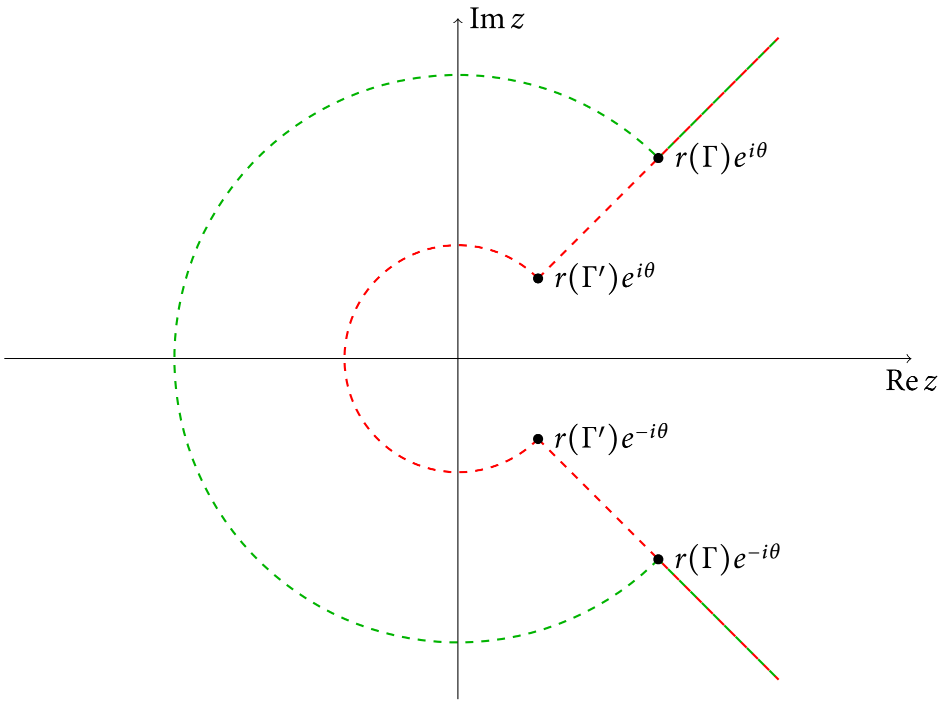

Figure 5.2.: *Illustration of the curves Γ (green) and Γ' (red) in the complex plane. The curves form two homotopic paths connecting $r(\Gamma)e^{-i\theta}$ and $r(\Gamma)e^{i\theta}$.*

allows us to show a decay in the imaginary part of the parameter λ. By part *(iii)* of the next theorem, this is sufficient for sectoriality in our situation, see Lemma 5.17 for further details. On the other hand, the resolvents of $R_\lambda^{\alpha,\varepsilon}$ are difficult to estimate with direct calculations, but the generator $-H_{\alpha,\varepsilon}$ is a bounded perturbation of $-\Delta$, so we can apply part *(iv)* of the next theorem, see Corollary 5.18.

Theorem 5.15. *[17]. Let X a Banach space and $A : X \to X$ a densely defined operator with generated semigroup $(T_t)_{t\geq 0}$. The following statements are equivalent.*

(i) *A is sectorial.*

(ii) *$(T_t)_{t\geq 0}$ is a bounded analytic semigroup.*

(iii) *$(T_t)_{t\geq 0}$ is a bounded strongly continuous semigroup and there is $C > 0$ such that*

$$\|R_\lambda(A)\|_{X\to X} \leq \frac{C}{(\operatorname{Im}\sqrt{\lambda})^2}$$

for $\lambda \in \Sigma_\varphi, \varphi \in (0, \frac{\pi}{2})$.

(iv) *$(T_t)_{t\geq 0}$ is bounded and $A = B + P$ where B is a sectorial operator and $P : X \to X$ is densely defined and bounded.*

Proof. The equivalence of *(i),(ii)* and *(iii)* is the subject of [17, Theorem II.4.6]. The equivalence of *(ii)* and *(iv)* follows from the bounded perturbation theorem in [17, Proposition III.1.12]. □

We investigate the sectorial property on the weighted spaces $L^p(w)$.

Lemma 5.16. *The positive definite Laplacian $-\Delta$ is sectorial on $L^p(w)$ for $w(x) = |x|^{-1}$, $p \in (1,2)$.*

Proof. First of all, $-\Delta$ is a linear operator on $L^p(w)$ and $\sigma(-\Delta) \subseteq [0,\infty)$ as shown in Lemma 5.4. For $\lambda \in [0,\infty)$ we have $\arg\lambda = 0$, hence the condition (i) of Definition 5.9 is trivially fulfilled for any angle in $(0, \frac{\pi}{2})$. With this property in place, the estimate in condition (ii) follows from the more general result for elliptic differential operators [25, Corollary 6.8]. □

Now we prove sectoriality of the operator $-\Delta_\alpha$. A similar result was shown by Caspers and Clément in 1993, but in the context of unweighted spaces $L^p(\mathbb{R}^3)$, $\frac{3}{2} < p < 3$ [8, Theorem 4.4].

Lemma 5.17. *For $\alpha \geq 0$, the operator $-\Delta_\alpha$ is sectorial on $L^p(w)$ for $w(x) = |x|^{-1}$, $p \in (1,2)$.*

Proof. We already know that $\sigma(-\Delta_\alpha) \subseteq [0,\infty)$ for $\alpha \geq 0$, cf. Lemma 5.6. For the resolvent estimate, note that

$$R_\lambda^\alpha = R_\lambda + \overline{R}_\lambda^\alpha.$$

Because of the previous Lemma 5.16, we only have to consider the residue term $\overline{R}_\lambda^\alpha$ for $\lambda \in \mathbb{C} \setminus [0,\infty)$. Remember

$$\overline{R}_\lambda^\alpha(x,y) = \frac{1}{\alpha - \frac{i\sqrt{\lambda}}{4\pi}} \frac{e^{i\sqrt{\lambda}(|x|+|y|)}}{(4\pi)^2|x||y|}.$$

Since $\alpha \geq 0$ and $b = \operatorname{Im}\lambda > 0$, the absolute value of the denominator of the first fraction is

$$\left|\alpha - \frac{i\sqrt{\lambda}}{4\pi}\right| = \left|\alpha + \frac{b - i\operatorname{Re}\sqrt{\lambda}}{4\pi}\right| \geq \frac{b}{4\pi} > 0.$$

So, for fixed α we have the decay

$$\left|\frac{1}{\alpha - \frac{i\sqrt{\lambda}}{4\pi}}\right| \leq \frac{C(\alpha)}{b},$$

where $C(\alpha) > 0$. Proceding as in the proof of Lemma 5.5 around formula (5.1), we have

$$\|\overline{R}_\lambda^\alpha\|_{L^p(w)\to L^p(w)} \leq \frac{C(\alpha)}{b}\|e^{-b|\cdot|}\|_{L^{p'}(w)}\left\|\frac{e^{i\sqrt{\lambda}|\cdot|}}{|\cdot|}\right\|_{L^p(w)}.$$

For the last factor it follows by transforming $r \to \frac{r}{b}$

$$\left\| \frac{e^{i\sqrt{\lambda}|\cdot|}}{|\cdot|} \right\|_{L^p(w)} \lesssim \left(\int_0^\infty e^{-bpr} r^{1-p} \,\mathrm{d}r \right)^{1/p}$$
$$= b^{1-2/p} \left(\int_0^\infty e^{-pr} r^{1-p} \,\mathrm{d}r \right)^{1/p}$$

and the remaining integral converges for $p < 2$. Analogously it follows for the p' norm

$$\left\| e^{-b|\cdot|} \right\|_{L^{p'}(w)} \lesssim b^{-2/p'} \left(\int_0^\infty e^{-p'r} r \,\mathrm{d}r \right)^{1/p'} .$$

Since

$$1 - \frac{2}{p} - \frac{2}{p'} = -1,$$

we have in summary a decay of

$$\left\| \overline{R}_\lambda^\alpha \right\|_{L^p(w) \to L^p(w)} \le \frac{C(\alpha)}{(\operatorname{Im} \sqrt{\lambda})^2}. \tag{5.4}$$

According to Corollary 3.20, $(S_t^\alpha)_{t\ge 0}$ is a bounded strongly continuous semigroup on $L^p(w)$. Together with the estimate for the imaginary part (5.4) and Theorem 5.15, this is sufficient for sectoriality, so the proof is complete. □

Finally we also have

Corollary 5.18. *For $\varepsilon > 0$ and $\alpha \ge 0$, the operator $-H_{\alpha,\varepsilon}$ is sectorial on $L^p(w)$ for $w(x) = |x|^{-1}$, $p \in (1,2)$.*

Proof. Clearly the multiplication operator $f \mapsto V_{\alpha,\varepsilon} f$ is bounded in $L^p(w)$ for fixed α, ε. Hence, the operator $-H_{\alpha,\varepsilon} = -\Delta + V_{\alpha,\varepsilon}$ is a bounded perturbation of $-\Delta$. Since $-\Delta$ is sectorial and the semigroup $(S_t^{\alpha,\varepsilon})_{t\ge 0}$ is bounded, we also have by Theorem 5.15 that $-H_{\alpha,\varepsilon}$ is sectorial. □

The sectoriality of the operators $-\Delta, -\Delta_\alpha$ and $-H_\varepsilon$ allows the following representation of the corresponding semigroups.

Theorem 5.19. *Let $\alpha \ge 0$ and $\varepsilon > 0$. Let $p \in (1,2)$ and $f \in L^p(w)$. For the semigroups $(S_t)_{t\ge 0}$, $(S_t^\alpha)_{t\ge 0}$ and $(S_t^{\alpha,\varepsilon})_{t\ge 0}$ with generators $-\Delta, -\Delta_\alpha, -H_{\alpha,\varepsilon}$ respectively, it holds*

(i) $$S_t f = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} R_\lambda f \,\mathrm{d}\lambda,$$

(ii) $$S_t^\alpha f = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} R_\lambda^\alpha f \,\mathrm{d}\lambda,$$

$$\textit{(iii)} \quad S_t^{\alpha,\varepsilon} f = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} R_\lambda^{\alpha,\varepsilon} f \, \mathrm{d}\lambda,$$

with Γ *as in Defintion 5.12.*

Proof. As shown in Lemma 5.16, Lemma 5.17 and Corollary 5.18, the operators are sectorial in $L^p(w)$. Thus, the claim follows directly from Theorem 5.11. □

6. Semigroup estimates and convergence properties

In this chapter we develop analytic tools, which are crucial to obtain convergence of solutions of the nonlinear integral equations later on. In particular, time-dependent resolvent estimates are proven, which then lead to $L^p(w)$–$L^q(w)$ semigroup estimates using the representations from Theorem 5.19. It is of special importance to ensure that the estimates for $R_\lambda^{\alpha,\varepsilon}$ and $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ are uniform in $\varepsilon \in (0,1)$. Furthermore, we will prove convergence of $S_t^{\alpha,\varepsilon}$ towards S_t^α for $\varepsilon \to 0$ in the weighted space. Results of this type have been derived only in unweighted Lebesgue spaces so far.

Because this chapter will become quite technical, we want to give a brief heuristic motivation for the following steps. For full details refer to Chapter 7. Remember that one of our main goals is to prove convergence for $\varepsilon \to 0$ of $u_{\alpha,\varepsilon}$ towards u_α, which are the solutions of the nonlinear integral equation (3.29) with semigroups $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ and $(S_t^\alpha)_{t\geq 0}$ respectively. Let's assume for the moment that for fixed $\alpha, \varepsilon > 0$ we have sequences of functions $(u_\alpha^n)_{n\in\mathbb{N}}$ and $(u_{\alpha,\varepsilon}^{(n)})_{n\in\mathbb{N}}$, which are the results of a fixed-point interation of the nonlinear integral equation, such that for $t \in [0,T]$

$$
\begin{aligned}
u_\alpha^{(n)}(t) &\to u_\alpha(t), \quad n \to \infty, \\
u_{\alpha,\varepsilon}^{(n)}(t) &\to u_{\alpha,\varepsilon}(t), \quad n \to \infty,
\end{aligned}
$$

where the limit is taken in the weighted space $L^p(w)$ for a suitable p. Assume further that we can write the limits in $L^p(w)$ as telescopic sums

$$
\begin{aligned}
u_\alpha &= u_\alpha^{(0)} + \sum_{n=0}^{\infty} u_\alpha^{(n+1)} - u_\alpha^{(n)}, \\
u_{\alpha,\varepsilon} &= u_{\alpha,\varepsilon}^{(0)} + \sum_{n=0}^{\infty} u_{\alpha,\varepsilon}^{(n+1)} - u_{\alpha,\varepsilon}^{(n)}.
\end{aligned}
$$

Consequently it holds for each $N \in \mathbb{N}$

$$
\begin{aligned}
\|u_\alpha(t) - u_{\alpha,\varepsilon}(t)\|_{L^p(w)} \leq& \left\| u_\alpha^{(N)}(t) - u_{\alpha,\varepsilon}^{(N)}(t) \right\|_{L^p(w)} \\
&+ \left\| \sum_{n=N}^{\infty} u_\alpha^{(n+1)}(t) - u_\alpha^{(n)}(t) \right\|_{L^p(w)} + \left\| \sum_{n=N}^{\infty} u_{\alpha,\varepsilon}^{(n+1)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^p(w)}
\end{aligned}
$$

$$:= \left\| u_\alpha^{(N)}(t) - u_{\alpha,\varepsilon}^{(N)}(t) \right\|_{L^p(w)} + \delta_{\alpha,\varepsilon}(N,t) \tag{6.1}$$

We could now show the convergence

$$\|u_\alpha(t) - u_{\alpha,\varepsilon}(t)\|_{L^p(w)} \to 0, \quad \varepsilon \to 0,$$

if we would have the following two properties in place: On the one hand, for fixed N the first term in (6.1) must tend to zero for $\varepsilon \to 0$. On the other hand, the term $\delta_{\alpha,\varepsilon}(N,t)$ must become arbitrarily small *uniformly in* ε when choosing N large enough.

In this chapter we develop the tools to prove the mentioned two properties. Resolvent and semigroup norm estimates are proven which are uniform in ε. Furthermore, convergence of the resolvents and semigroups, i.e. the solutions of the linear equations, is shown in the weighted space. In the next chapter this can then be extended to solutions of the nonlinear equations.

6.1. Resolvent estimates for the approximate point source

In order to further analyze the semigroups $(S_t^{\alpha,\varepsilon})_{t\geq 0}$ and $(S_t^\alpha)_{t\geq 0}$ via the representations from Theorem 5.19, we need to develop some tools to control the involved resolvent terms R_λ^α and $R_\lambda^{\alpha,\varepsilon}$ for $\lambda \in \Gamma$, where Γ is the complex curve from Definition 5.12. In this section we focus on the resolvent $R_\lambda^{\alpha,\varepsilon}$ corresponding to the approximate point source. Remember that the resolvents of $-H_{\alpha,\varepsilon}$ are given by

$$R_\lambda^{\alpha,\varepsilon} = R_\lambda + \overline{R}_\lambda^{\alpha,\varepsilon} = R_\lambda + P_\alpha(\varepsilon) A_\lambda^\varepsilon \varepsilon [1 + B_\lambda^\varepsilon]^{-1} C_\lambda^\varepsilon$$

in the L^2 sense, see (2.27).

For the analysis of the resolvents $R_\lambda^{\alpha,\varepsilon}$ we need the following variant of [4, Lemma 1.2.4], a convergence result in the unweighted L^2 space. As in Lemma 2.2 write for $V : \mathbb{R}^3 \to \mathbb{R}$

$$v(x) = |V(x)|^{1/2}, \quad u(x) = \operatorname{sgn}(V(x))|V(x)|^{1/2}, \quad \text{for } x \in \mathbb{R}^3.$$

Lemma 6.1. *For $V = -\mathbf{1}_{B_{\frac{\pi}{2}}(0)}(x)$, $\lambda \in \mathbb{C} \setminus [0,\infty)$ and B_λ^ε as in Lemma 2.14 it holds*

$$\varepsilon[1 + B_\lambda^\varepsilon]^{-1} \longrightarrow - \left[\frac{i\sqrt{\lambda}|(v,\varphi)|^2}{4\pi} + P'_\alpha(0) \right]^{-1} (\varphi, \cdot)\varphi$$

where the limit is taken for $\varepsilon \to 0$ in L^2 norm and $(\cdot,\cdot)$ denotes the scalar product of Hilbert space L^2 and φ is the eigenfunction from Lemma 2.8. The convergence is uniform in λ if $|\lambda| \leq c$ for a fixed $c > 0$.

Proof. We follow the proof of [4, Lemma 1.2.4]. First, without loss of generality, we can assume that φ is scaled such that $(\varphi,\varphi) = \|\varphi\|_{L^2}^2 = 1$. Then $(\varphi,\cdot)\varphi =: \Pi$ is the orthogonal projection on the eigenspace of $B_0 = uR_0v$ to the eigenvalue -1 which is seen as follows: For $f \in L^2$ it holds

$$\Pi^2 f = (\varphi,(\varphi,f)\varphi)\varphi = (\varphi,\varphi)(\varphi,f)\varphi = (\varphi,f)\varphi = \Pi f,$$

so Π is a projection, and on the other hand

$$uR_0v\Pi f = uR_0v(\varphi,f)\varphi = (\varphi,f)uR_0v\varphi = -(\varphi,f)\varphi = -\Pi f.$$

Now define

$$B_\lambda^1 := P_\alpha'(0)uR_0v + \frac{i\sqrt{\lambda}}{4\pi}(v,\cdot)u.$$

We are going to show that

$$B_\lambda^\varepsilon = B_0 + \varepsilon B_\lambda^1 + o(\varepsilon) \tag{6.2}$$

with respect to the Hilbert-Schmidt norm. As in formula (1.2.43) in the proof of [4, Lemma 1.2.4], the mean value theorem implies

$$P(\varepsilon)R_{\varepsilon^2\lambda}(x,y) = R_0(x,y) - \varepsilon P_\alpha'(\varepsilon\widetilde{\theta}(\varepsilon))R_0(x,y) + \frac{\varepsilon i\sqrt{\lambda}}{4\pi}e^{i\varepsilon\theta(\varepsilon)\sqrt{\lambda}|x-y|},$$

for functions $0 \le \theta(\varepsilon), \widetilde{\theta}(\varepsilon) \le 1$. This again implies

$$\begin{aligned}
&B_\lambda^\varepsilon(x,y)\\
&\quad = P_\alpha(\varepsilon)u(x)R_{\varepsilon^2\lambda}(x,y)v(y)\\
&\quad = u(x)R_0(x,y)v(y) - \varepsilon P_\alpha'(\varepsilon\widetilde{\theta}(\varepsilon))u(x)R_0(x,y)v(y) + \frac{\varepsilon i\sqrt{\lambda}}{4\pi}u(x)e^{i\varepsilon\theta(\varepsilon)\sqrt{\lambda}|x-y|}v(y).
\end{aligned}$$

Now compute

$$\begin{aligned}
\left\|B_\lambda^\varepsilon - B_0 - \varepsilon B_\lambda^1\right\|_2^2 \le& \left\|B_\lambda^\varepsilon - uR_0v - \varepsilon P'(0)uR_0v - \varepsilon\frac{i\sqrt{\lambda}}{4\pi}(v,\cdot)u\right\|_2^2\\
\le& 2\varepsilon^2|P_\alpha'(\varepsilon\widetilde{\theta}(\varepsilon)) - P_\alpha'(0)|\|uR_0v\|_2^2\\
&+ 2\left(\frac{\varepsilon|\sqrt{\lambda}|}{4\pi}\right)^2\left\|u|e^{i\varepsilon\theta(\varepsilon)\sqrt{\lambda}|\cdot|} - 1|(v,\cdot)\right\|_2^2,
\end{aligned} \tag{6.3}$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. To conclude (6.2), it suffices to show that the last norm is uniformly bounded in ε. This is the case because

$$\left\| u\left|e^{i\varepsilon\theta(\varepsilon)\sqrt{\lambda}|\cdot|}-1\right|(v,\cdot)\right\|_2^2 \le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u(x)|^2|v(y)|^2\left|e^{i\varepsilon\theta(\varepsilon)\sqrt{\lambda}|x-y|}-1\right|^2 \mathrm{d}y\ \mathrm{d}x$$
$$\lesssim \left(\int_{\mathbb{R}^3}|V(x)|\,\mathrm{d}x\right)^2,$$

which is clearly finite since $V \in L^1(\mathbb{R}^3)$, and this completes the proof of (6.2). Now, applying (6.2) yields

$$\begin{aligned}
\varepsilon[1+B_\lambda^\varepsilon]^{-1} &= \varepsilon[1+B_0+\varepsilon B_\lambda^1+o(\varepsilon)]^{-1}\\
&= \varepsilon\left[1+\varepsilon+B_0+\varepsilon(B_1-1+o(\varepsilon))\right]^{-1}\\
&= \left[1+\varepsilon(1+\varepsilon+B_0)^{-1}(B_1-1+o(\varepsilon))\right]^{-1}\varepsilon(1+\varepsilon+B_0)^{-1}\\
&= \left[1+\Pi(B_1-1)+o(\varepsilon)\right]^{-1}\left[\Pi+o(\varepsilon)\right]
\end{aligned}$$

in the L^2 sense. The last step needs further explanation: As in [4, formula 1.2.35], we have the series expansion

$$(1+\varepsilon+B_0)^{-1} = \varepsilon^{-1}\Pi + \sum_{m=0}^{\infty}(-\varepsilon)^m T^{m+1},$$

where

$$T = \lim_{\varepsilon\to 0}(1+\varepsilon+B_0)^{-1}(1-\Pi) = o(1),$$

so it follows

$$\varepsilon(1+\varepsilon+B_0)^{-1} = \Pi + o(\varepsilon).$$

From our calculation we conclude the L^2 convergence

$$\lim_{\varepsilon\to 0}\varepsilon[1+B_\lambda^\varepsilon]^{-1} = \left[1+\Pi(B_1-1)\right]^{-1}\Pi. \tag{6.4}$$

For the first factor it holds

$$\begin{aligned}
\left[1+\Pi(B_1-1)\right]^{-1} &= 1-\frac{\Pi(B_1-1)}{(\varphi,B_1\varphi)}\\
&= 1-\frac{\left[1+P_\alpha'(0)\right]\Pi+\frac{i\lambda}{4\pi}(\varphi,v)(v,\cdot)\varphi}{\frac{i\sqrt{\lambda}|(v,\varphi)|^2}{4\pi}+P_\alpha'(0)}.
\end{aligned}$$

Now multiply Π from the right side

$$
\begin{aligned}
\Pi - &\frac{\left[1 + P_\alpha'(0)\right]\Pi + \frac{i\sqrt{\lambda}}{4\pi}(\varphi,\nu)(\nu,\cdot)\varphi}{\frac{i\sqrt{\lambda}|(\nu,\varphi)|^2}{4\pi} + P_\alpha'(0)}\Pi \\
&= -\frac{1}{\frac{i\sqrt{\lambda}|(\nu,\varphi)|^2}{4\pi} + P_\alpha'(0)}\Pi + \frac{\frac{i\sqrt{\lambda}}{4\pi}(\varphi,\nu)(\nu,\varphi)(\varphi,\cdot)\varphi + P_\alpha'(0)\Pi}{\frac{i\sqrt{\lambda}|(\nu,\varphi)|^2}{4\pi} + P_\alpha'(0)} + \Pi \\
&= -\frac{1}{\frac{i\sqrt{\lambda}|(\nu,\varphi)|^2}{4\pi} + P_\alpha'(0)}\Pi + \frac{\frac{i\sqrt{\lambda}}{4\pi}|(\nu,\varphi)|^2\Pi + P_\alpha'(0)\Pi}{\frac{i\sqrt{\lambda}|(\nu,\varphi)|^2}{4\pi} + P_\alpha'(0)} + \Pi \\
&= -\frac{1}{\frac{i\sqrt{\lambda}|(\nu,\varphi)|^2}{4\pi} + P_\alpha'(0)}\Pi.
\end{aligned}
$$

Inserting in (6.4), the proof is almost complete: The additional claim about uniformity follows from the fact, that the first factor in (6.3) decays uniformly in $|\lambda|$ for $\varepsilon \to 0$ if we have a bound on λ. □

With this convergence result in place, we are able to prove the following time-dependent estimate for the term $\left[1 + B^\varepsilon_{t^{-1}\lambda}\right]^{-1}$.

Lemma 6.2. *Let $\alpha \geq 0, \varepsilon \in (0,1)$ and $t \in [0,T]$ for some $T > 0$. Furthermore, let $\lambda \in \Gamma$, where Γ is the path from Definition 5.12. There is a $\delta > 0$ such that*

(i) for $|\varepsilon\sqrt{t^{-1}\lambda}| < \delta$ it holds

$$
\varepsilon\left\| \left[1 + B^\varepsilon_{t^{-1}\lambda}\right]^{-1} \right\|_{L^2\to L^2} \lesssim \sqrt{t} \tag{6.5}
$$

with a constant independent of ε and λ.

(ii) for $|\varepsilon\sqrt{t^{-1}\lambda}| \geq \delta$ it holds

$$
\left\| \left[1 + B^\varepsilon_{t^{-1}\lambda}\right]^{-1} \right\|_{L^2\to L^2} \lesssim 1 \tag{6.6}
$$

with a constant independent of ε, t, λ.

Proof. In case *(i)* we make use of Lemma 6.1. This convergence result implies that for every $\gamma > 0$ there is a $\delta > 0$ such that

$$
\left\| \frac{\varepsilon}{\sqrt{t}} \left[1 + B_\lambda^{\frac{\varepsilon}{\sqrt{t}}}\right]^{-1} \right\|_{L^2\to L^2} \leq (1+\gamma)\|T_\lambda\|_{L^2\to L^2}
$$

for $\frac{\varepsilon|\sqrt{\lambda}|}{\sqrt{t}} \leq \delta$, where

$$T_\lambda = -\left[\frac{i\sqrt{\lambda}|(\nu,\varphi)|^2}{4\pi} + P'(0)\right]^{-1}(\varphi,\cdot)\varphi.$$

It follows

$$\begin{aligned}\left\|\varepsilon\left[1+B^{\varepsilon}_{t^{-1}\lambda}\right]^{-1}\right\|_{L^2\to L^2} &= \sqrt{t}\left\|\frac{\varepsilon}{\sqrt{t}}\left[1+B^{\frac{\varepsilon}{\sqrt{t}}}_{\lambda}\right]^{-1}\right\|_{L^2\to L^2}\\ &\leq \sqrt{t}c(\lambda)\|(\varphi,\cdot)\varphi\|_{L^2\to L^2},\end{aligned}$$

with

$$c(\lambda) = o((1+\sqrt{\lambda})^{-1}).$$

for some δ small enough. Fix this δ for the whole proof. So we can conclude

$$\left\|\varepsilon\left[1+B^{\varepsilon}_{t^{-1}\lambda}\right]^{-1}\right\|_{L^2\to L^2} \lesssim \sqrt{t}o((1+\sqrt{\lambda})^{-1}) \tag{6.7}$$

in this case. Because of

$$|\sqrt{\lambda}| > \sqrt{r(\Gamma)}\sin\frac{\theta}{2} > 0, \tag{6.8}$$

the expression $(1+\sqrt{\lambda})^{-1}$ is uniformly bounded for $\lambda \in \Gamma$.

To prove *(ii)*, we distinguish two subregions, which are illustrated in Figure 6.1. First, assume that $|\varepsilon\sqrt{t^{-1}\lambda}| > \frac{\sqrt{2}}{\sin\frac{\theta}{2}}$. Write $b := \operatorname{Im}\sqrt{\lambda}$. Then, with Young's convolution inequality, we can estimate the operator norm of $\left[1+B^{\varepsilon}_{t^{-1}\lambda}\right]^{-1}$ as follows.

$$\begin{aligned}(P_\alpha(\varepsilon))^{-1}\left\|B^{\varepsilon}_{t^{-1}\lambda}\right\|_{L^2\to L^2} &= \sup_{\|g\|_{L^2}=1}\|V(R_{\varepsilon^2 t^{-1}\lambda}(\cdot) * Vg)\|_{L^2}\\ &\leq \sup_{\|g\|_{L^2}=1}\|R_{\varepsilon^2 t^{-1}\lambda}(\cdot) * Vg\|_{L^2}\\ &\leq \sup_{\|g\|_{L^2}=1}\|R_{\varepsilon^2 t^{-1}\lambda}(\cdot)\|_{L^1}\|g\|_{L^2}\\ &\leq \|R_{\varepsilon^2 t^{-1}\lambda}(\cdot)\|_{L^1}\\ &\leq \int_{\mathbb{R}^3}\frac{e^{-\varepsilon b\sqrt{t^{-1}}|y|}}{4\pi|y|}\,\mathrm{d}y\\ &= \frac{t}{\varepsilon^2 b^2}\int_{\mathbb{R}^3}\frac{e^{-|y|}}{4\pi|y|}\,\mathrm{d}y\end{aligned}$$

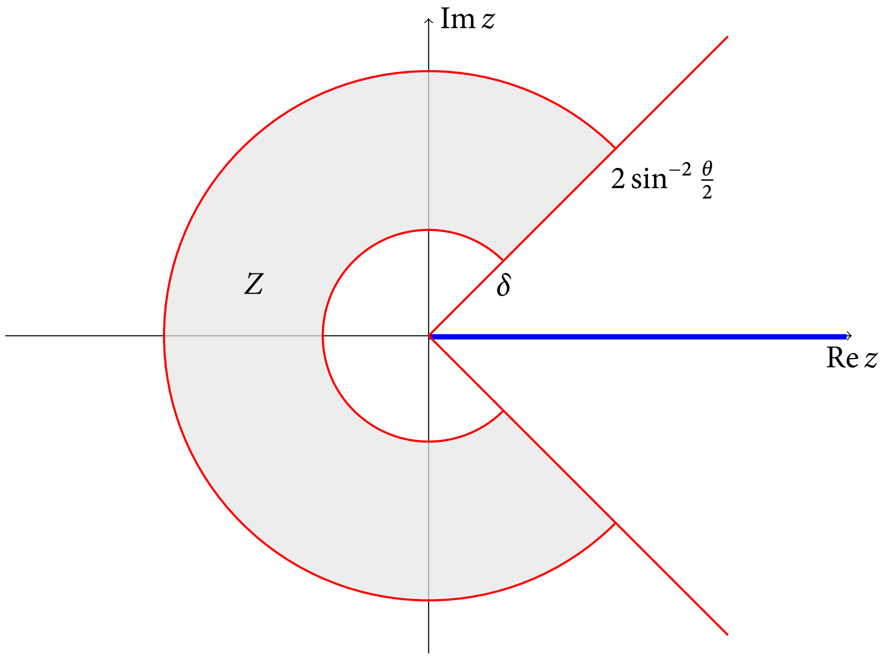

Figure 6.1.: *Illustration of the cases in the proof in the complex plane. In case* (i) *the point z lies in the area surrounded by the inner arc of radius δ and the diagonals of angle θ. In the first part of case* (ii) *the point z lies outside the outer arc of radius $2\sin^{-2}\frac{\theta}{2}$. In the last subcase z is contained in the area between inner and outer arc and the diagonals. This subset $Z \in \mathbb{C}$ (gray background) is compact.*

and the remaining integral is equal to 1. We have $b \geq |\sqrt{\lambda}| \sin\frac{\theta}{2}$. So it follows

$$\frac{t}{\varepsilon^2 b^2} \leq \frac{t}{\varepsilon^2 |\sqrt{\lambda}|^2 \sin^2\frac{\theta}{2}} \leq \frac{\sin^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{1}{2}. \tag{6.9}$$

Consequently

$$\left\| \left[1 + B^{\varepsilon}_{t^{-1}\lambda}\right]^{-1} \right\|_{L^2\to L^2} \leq \frac{1}{1 - \left\| B^{\varepsilon}_{t^{-1}\lambda} \right\|_{L^2\to L^2}} \leq \frac{1}{1 - \frac{P_\alpha(\varepsilon)}{2}} \leq 2. \tag{6.10}$$

So we have a uniform bound in this situation.

It remains to investigate the case $\delta \leq |\varepsilon\sqrt{t^{-1}\lambda}| \leq \frac{\sqrt{2}}{\sin\frac{\theta}{2}}$. Define $z := \varepsilon^2 t^{-1}\lambda$ and consider the operator-valued function

$$\mathbb{C} \ni z \mapsto F(z) := [1 + B_z]^{-1} = \left[1 + B_{\varepsilon^2 t^{-1}\lambda}\right]^{-1} = \left[1 + B^{\varepsilon}_{t^{-1}\lambda}\right]^{-1}.$$

We already know that $F(z)$ exists for $|z| > \frac{\sqrt{2}}{\sin\frac{\theta}{2}}$. Then, according to the analytic fredholm theorem [45, Theorem VI.14], the function F is meromorphic in $\mathbb{C} \setminus [0,\infty)$, i.e. it has an at most discrete set of poles. Assume $z_0 \in \mathbb{C} \setminus [0,\infty)$ is one such pole. Then the kernel of the operator $1 + B_{z_0}$ is nonzero, so there is a $g \in L^2(\mathbb{R}^3)$ with

$$B_{z_0} g = -g.$$

Hence, -1 is an eigenvalue of B_{z_0} with eigenfunction g. Now recall the Birman-Schwinger principle from Lemma 2.2: If -1 is an eigenvalue of the operator $B_{z_0} = P_\alpha(\varepsilon) u R_{z_0} v$, then z_0 is an eigenvalue of $-\Delta + P_\alpha(\varepsilon) V$. However, the latter operator has no eigenvalues in $\mathbb{C} \setminus [0,\infty)$, according to Theorem 2.5 and 2.9, because for $\alpha \geq 0$ we have $P_\alpha(\varepsilon) \leq 1$, cf. Definition 2.12 and (2.26). Thus, there is no pole z_0 of $F(z)$. This implies that F is holomorphic on $\mathbb{C} \setminus [0,\infty)$.

Because of $\theta \leq \arg \lambda \leq 2\pi - \theta$ and $\delta \leq |\varepsilon \sqrt{t^{-1}\lambda}| \leq \frac{\sqrt{2}}{\sin \frac{\theta}{2}}$, it follows that z lies in the compact set

$$Z := \{ z \in \mathbb{C} : \delta^2 \leq |z| \leq \frac{2}{\sin^2 \frac{\theta}{2}}, \theta \leq \arg z \leq 2\pi - \theta \},$$

which is uniformly away from the nonnegative real axis. The holomorphic operator-valued function F is bounded on this compact set. Hence,

$$\left\| [1 + B_z]^{-1} \right\|_{L^2 \to L^2} \leq c$$

for $z \in Z$ and some $c > 0$, which is a uniform bound and implies *(ii)*. □

Remark 6.3. In the proof of Lemma 6.2 the condition $\alpha \geq 0$ is needed to ensure that the function $z \mapsto [1 + B_z]^{-1}$ is holomorphic in the set $Z \in \mathbb{C}$. For $\alpha < 0$, we have that $P_\alpha(\varepsilon) > 0$ for small ε according to (2.26). This again raises the problem that $-\Delta + P_\alpha(\varepsilon) V$ has at least one negative eigenvalue, because V was chosen critical in Chapter 2, so multiplying with a factor larger than one leads into the supercritical case, where negative eigenvalues exist. Hence, the function $z \mapsto [1 + B_z]^{-1}$ has at least one pole in the set Z. Also refer to Section 9.2 for more details, where we deal with negative eigenvalues in the two-dimensional case.

Now we are prepared to prove $L^p(w)$–$L^q(w)$ boundedness of the operator

$$\overline{R}_\lambda^{\alpha,\varepsilon} = P_\alpha(\varepsilon) A_\lambda^\varepsilon \varepsilon [1 + B_\lambda^\varepsilon]^{-1} C_\lambda^\varepsilon \tag{6.11}$$

from Lemma 2.14 for suitable exponents p, q. Moreover, in this context we will investigate the dependency of $\overline{R}_{t^{-1}\lambda}^{\alpha,\varepsilon}$ on time $t \in [0, T]$. This will play a central role in the semigroup estimates following afterwards.

Theorem 6.4. *Let $\alpha \geq 0$ and $\varepsilon \in (0,1)$. Let $\lambda \in \Gamma$, where Γ is the path from Definition 5.12. Let $\overline{R}_\lambda^{\alpha,\varepsilon}$ given by* (6.11) *the residue of the resolvent of $-H_{\alpha,\varepsilon}$ as in Lemma 2.14. Furthermore, let*

$p > \frac{3}{2}$ and $1 < q < 2$ and $f \in L^p(w)$ and fix $T > 0$. We then have for time $t \in [0, T]$

$$\left\|\overline{R}^{\alpha,\varepsilon}_{t^{-1}\lambda} f(x)\right\|_{L^q(w)} \lesssim \max\{t^{\frac{3}{2}-\frac{3}{2p}}, t^{\frac{5}{4}-\frac{3}{2p}}, t^{\frac{1}{q}+\frac{1}{2}-\frac{3}{2p}}\}\|f\|_{L^p(w)}, \tag{6.12}$$

where the implicit constant depends only on p, q, T.

Proof. Using (2.27), it holds

$$\left\|\overline{R}^{\alpha,\varepsilon}_{t^{-1}\lambda} f\right\|_{L^q(w)} \le |P_\alpha(\varepsilon)|\left\|A^\varepsilon_{t^{-1}\lambda}\right\|_{L^2\to L^q(w)}\left\|\varepsilon[1 + B^\varepsilon_{t^{-1}\lambda}]^{-1}\right\|_{L^2\to L^2}\left\|C^\varepsilon_{t^{-1}\lambda}\right\|_{L^p(w)\to L^2}\|f\|_{L^p(w)}.$$

Now the operator norms can be dealt with separately. In the following two steps we develop estimates of the $C^\varepsilon_{t^{-1}\lambda}$ and the $A^\varepsilon_{t^{-1}\lambda}$ term. This is done by explicitly calculating and estimating the corresponding integrals. This part is quite technical. In the final step of the proof we will collect the calculated bounds and employ Lemma 6.2 to estimate the norm of $\varepsilon[1 + B^\varepsilon_{t^{-1}\lambda}]^{-1}$.

Step 1. By definition of the operator norm it holds

$$\left\|C^\varepsilon_{t^{-1}\lambda}\right\|_{L^p(w)\to L^2} = \sup_{\|g\|_{L^p(w)}=1}\left\|C^\varepsilon_{t^{-1}\lambda} g\right\|_{L^2}.$$

Now

$$\begin{aligned}
\left\|C^\varepsilon_{t^{-1}\lambda} g\right\|^2_{L^2} &= \int_{\mathbb{R}^3} V(x)\left(\int_{\mathbb{R}^3}\frac{e^{i\sqrt{t^{-1}\lambda}|\varepsilon x-y|}}{4\pi|\varepsilon x - y|} g(y)\,\mathrm{d}y\right)^2 \mathrm{d}x \\
&\le \|g\|^2_{L^p(w)}\int_{\mathbb{R}^3} V(x)\left(\int_{\mathbb{R}^3}\frac{e^{-t^{-\frac{1}{2}}bp'|\varepsilon x-y|}}{(4\pi|\varepsilon x - y|)^{p'}}|y|^{\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{2}{p'}} \mathrm{d}x \\
&= \|g\|^2_{L^p(w)}\int_{\mathbb{R}^3} V(x)\left(\int_{\mathbb{R}^3}\frac{e^{-t^{-\frac{1}{2}}bp'|y|}}{(4\pi|y|)^{p'}}|y + \varepsilon x|^{\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{2}{p'}} \mathrm{d}x \\
&= t^{\frac{3}{p'}-1}\|g\|^2_{L^p(w)}\int_{\mathbb{R}^3} V(x)\left(\int_{\mathbb{R}^3}\frac{e^{-bp'|y|}}{(4\pi|y|)^{p'}}|\sqrt{t}y + \varepsilon x|^{\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{2}{p'}} \mathrm{d}x.
\end{aligned}$$

Because of $\sqrt{t} \le \sqrt{T}$ and $\varepsilon \le 1$ it holds

$$|\sqrt{t}y + \varepsilon x| \lesssim \max\{\sqrt{T}|y|, |x|\}.$$

So, for $|\sqrt{t}y + \varepsilon x| \lesssim \sqrt{T}|y|$, using $\frac{p'}{p} - p' = -1$, one obtains the bound

$$\left\|C^\varepsilon_{t^{-1}\lambda}\right\|^2_{L^p(w)\to L^2} \lesssim t^{\frac{3}{p'}-1}T^{\frac{1}{p}}\int_{\mathbb{R}^3} V(x)\left(\int_{\mathbb{R}^3}\frac{e^{-bp'|y|}}{(4\pi|y|)}\,\mathrm{d}y\right)^{\frac{2}{p'}} \mathrm{d}x \lesssim t^{\frac{3}{p'}-1}T^{\frac{1}{p}},$$

whereas for $|\sqrt{t}y + \varepsilon x| \lesssim |x|$ we get

$$\left\|C^{\varepsilon}_{t^{-1}\lambda}\right\|^{2}_{L^p(w)\to L^2} \lesssim t^{\frac{3}{p'}-1} \int_{\mathbb{R}^3} V(x)|x|^{\frac{2}{p}} \left(\int_{\mathbb{R}^3} \frac{e^{-bp'|y|}}{(4\pi|y|)^{p'}}\,\mathrm{d}y\right)^{\frac{2}{p'}} \mathrm{d}x \lesssim t^{\frac{3}{p'}-1},$$

note that because of

$$b = \operatorname{Im}\sqrt{\lambda} > \sqrt{r(\Gamma)}\sin\frac{\theta}{2} > 0 \tag{6.13}$$

the exponential decay of $e^{-bp'|\cdot|}$ does not vanish and the remaining y integral is uniformly bounded in b and consequently also in λ. Concerning the outer integral, $|x|^{\frac{2}{p}}$ is clearly integrable on a compact set. In summary

$$\left\|C^{\varepsilon}_{t^{-1}\lambda}\right\|_{L^p(w)\to L^2} \lesssim t^{\frac{1}{2}(\frac{3}{p'}-1)}(1+\sqrt{T}^{\frac{1}{p}}) = t^{1-\frac{3}{2p}}(1+\sqrt{T}^{\frac{1}{p}}). \tag{6.14}$$

Step 2. By definition

$$\left\|A^{\varepsilon}_{t^{-1}\lambda}\right\|_{L^2\to L^q(w)} = \sup_{\|g\|_{L^2}=1}\left\|A^{\varepsilon}_{t^{-1}\lambda}g\right\|_{L^q(w)}.$$

Using Hölder's inequality we can extract $\|g\|_{L^2}$ and obtain, in analogy to Step 1,

$$\left\|A^{\varepsilon}_{t^{-1}\lambda}\right\|^{q}_{L^2\to L^q(w)} \le \int_{\mathbb{R}^3} |x|^{-1}\left(\int_{\mathbb{R}^3} \frac{e^{-2t^{-\frac{1}{2}}b|x-\varepsilon y|}}{(4\pi|x-\varepsilon y|)^2} V(y)\,\mathrm{d}y\right)^{\frac{q}{2}} \mathrm{d}x \tag{6.15}$$

First we restrict the outer integral to the complement of a centered ball, i.e. we consider the case where $(2\varepsilon)^{-1}x \in B_{\frac{\pi}{2}}(0)^c$. This implies $|x| \ge 2\varepsilon|y|$ in the inner integral, so $|x-\varepsilon y| \ge \frac{1}{2}|x|$. We then have, because of $\varepsilon \le 1$, the bound

$$\frac{e^{-t^{-\frac{1}{2}}b|x|}}{(2\pi|x|)^2}\int_{\mathbb{R}^3} V(y)\,\mathrm{d}y = \frac{\pi^4}{6}\frac{e^{-t^{-\frac{1}{2}}b|x|}}{|x|^2}$$

for the inner integral, which leads to the estimate

$$\left\|A^{\varepsilon}_{t^{-1}\lambda}\right\|^{q}_{L^2\to L^q(w)} \lesssim \int_{\mathbb{R}^3} \frac{e^{-t^{-\frac{1}{2}}\frac{q}{2}b|x|}}{|x|^{q+1}}\,\mathrm{d}x = t^{1-\frac{q}{2}}\int_{\mathbb{R}^3} \frac{e^{-\frac{q}{2}b|x|}}{|x|^{q+1}}\,\mathrm{d}x,$$

and the remaining integral exists because of $q+1<3$ and is uniformly bounded in $\lambda \in \Gamma$ because of (6.13). So we have

$$\left\|A^{\varepsilon}_{t^{-1}\lambda}\right\|_{L^2\to L^q(w)} \lesssim t^{\frac{1}{q}-\frac{1}{2}}. \tag{6.16}$$

We still have to deal with the case where $(2\varepsilon)^{-1}x$ lies in the ball $B_{\frac{\pi}{2}}(0)$. In this situation we distinguish again subcases concerning the ratio $|\varepsilon\sqrt{t^{-1}\lambda}|$. Let δ as in Lemma 6.2.

Case 1: $|\varepsilon\sqrt{t^{-1}\lambda}| < \delta$. Here we estimate the inner integral in (6.15) as follows

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{e^{-2t^{-\frac{1}{2}}b\varepsilon|y|}}{(4\pi\varepsilon|y|)^2} V(\varepsilon^{-1}x+y)\,\mathrm{d}y \\
&\quad \le \varepsilon^{-2}\int_{\mathbb{R}^3} \frac{1}{(4\pi|y|)^2} V(\varepsilon^{-1}x+y)\,\mathrm{d}y \\
&\quad \le \varepsilon^{-2}\frac{3\pi}{2}\int_{B(\frac{3\pi}{2})} \frac{1}{(4\pi|y|)^2}\,\mathrm{d}y.
\end{aligned}$$

Note that $|\varepsilon^{-1}x+y| \le \frac{\pi}{2}$ can only hold for $|y| \le \frac{3\pi}{2}$ because $\varepsilon^{-1}|x| \le \pi$. Thus, for the outer integral in (6.15) one obtains the bound

$$\|A^{\varepsilon}_{t^{-1}\lambda}\|^q_{L^2(w)\to L^q(w)} \lesssim \varepsilon^{-q}\int_{\mathbb{R}^3}\frac{V((2\varepsilon)^{-1}x)}{|x|}\,\mathrm{d}x \le \varepsilon^{2-q}\int_{\mathbb{R}^3}\frac{V(\frac{x}{2})}{|x|}\,\mathrm{d}x \lesssim 1,$$

using the condition $(2\varepsilon)^{-1}x \in B_{\frac{\pi}{2}}(0)$ and the fact $2-q>0$.

Case 2: $|\varepsilon\sqrt{t^{-1}\lambda}| \ge \delta$. Here we choose a slightly different approach to estimate the inner term in (6.15) in order to gain a decay in t. It holds

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{e^{-2t^{-\frac{1}{2}}b\varepsilon|y|}}{(4\pi\varepsilon|y|)^2} V(\varepsilon^{-1}x+y)\,\mathrm{d}y \\
&\quad \le \frac{\sqrt{t}}{\varepsilon^3}\int_{\mathbb{R}^3}\frac{e^{-2b|y|}}{(4\pi|y|)^2}\,\mathrm{d}y \\
&\quad \lesssim \frac{\sqrt{t}}{\varepsilon^3},
\end{aligned}$$

where we have used (6.13). Returning to the outer integral of (6.15) with $(2\varepsilon)^{-1}x \in B_{\frac{\pi}{2}}(0)$ we obtain

$$\|A^{\varepsilon}_{t^{-1}\lambda}\|^q_{L^2\to L^q(w)} \lesssim \varepsilon^{-\frac{3}{2}q}t^{\frac{q}{4}}\int_{\mathbb{R}^3}\frac{V((2\varepsilon)^{-1}x)}{|x|}\,\mathrm{d}x \le \varepsilon^{2-\frac{3}{2}q}t^{\frac{q}{4}}\int_{\mathbb{R}^3}\frac{V(\frac{x}{2})}{|x|}\,\mathrm{d}x \lesssim \varepsilon^{2-\frac{3}{2}q}t^{\frac{q}{4}}$$

and finally

$$\|A^{\varepsilon}_{t^{-1}\lambda}\|_{L^2\to L^q(w)} \lesssim \varepsilon^{\frac{2}{q}-\frac{3}{2}}t^{\frac{1}{4}}. \tag{6.17}$$

Step 3: Conclusion. Now it remains to collect terms in the different cases.

Case 1: $|\varepsilon\sqrt{t^{-1}\lambda}| < \delta$. It holds

$$\begin{aligned}
\|\overline{R}^{\alpha,\varepsilon}_{t^{-1}\lambda}\|_{L^p(w)\to L^q(w)} &\le P_\alpha(\varepsilon)\|A^{\varepsilon}_{t^{-1}\lambda}\|_{L^2\to L^q(w)}\|\varepsilon[1+B^{\varepsilon}_{t^{-1}\lambda}]^{-1}\|_{L^2\to L^2}\|C^{\varepsilon}_{t^{-1}\lambda}\|_{L^p(w)\to L^2}. \\
&\lesssim t^{\frac{1}{q}-\frac{1}{2}}t^{\frac{1}{2}}t^{1-\frac{3}{2p}}
\end{aligned}$$

$$= t^{1+\frac{1}{q}-\frac{3}{2p}}$$

using (6.16), (6.5) and (6.14).

Case 2: $|\varepsilon\sqrt{t^{-1}\lambda}| \geq \delta$. We have

$$\begin{aligned}
\|\overline{R}^{\alpha,\varepsilon}_{t^{-1}\lambda}\|_{L^p(w)\to L^q(w)} &\leq P_\alpha(\varepsilon)\|A^\varepsilon_{t^{-1}\lambda}\|_{L^2\to L^q(w)}\|\varepsilon[1+B^\varepsilon_{t^{-1}\lambda}]^{-1}\|_{L^2\to L^2}\|C^\varepsilon_{t^{-1}\lambda}\|_{L^p(w)\to L^2}. \\
&\lesssim \max\{t^{\frac{1}{q}-\frac{1}{2}}, \varepsilon^{\frac{2}{q}-\frac{3}{2}}t^{\frac{1}{4}}\}\varepsilon t^{1-\frac{3}{2p}} \\
&\leq \max\{t^{\frac{1}{q}-\frac{1}{2}}, t^{\frac{1}{4}}\}t^{1-\frac{3}{2p}},
\end{aligned}$$

using (6.16), (6.17), (6.6) and (6.14). Note that $\frac{2}{q}-\frac{3}{2} > -1$ implies $\varepsilon^{\frac{2}{q}-\frac{3}{2}}\varepsilon \leq 1$.

Concluding the proof, by summing up both cases we have

$$\|\overline{R}^{\alpha,\varepsilon}_{t^{-1}\lambda}f(x)\|_{L^q(w)} \leq C(p,q,T)\max\{t^{1+\frac{1}{q}-\frac{3}{2p}}, t^{\frac{1}{q}+\frac{1}{2}-\frac{3}{2p}}, t^{\frac{5}{4}-\frac{3}{2p}}\}\|f\|_{L^p(w)},$$

where $C(p,q,T) = o(1+T^{\frac{1}{2p}})$ for fixed p, q. □

From this theorem, we immediately obtain the following consequence by setting $t \equiv 1$.

Corollary 6.5. *Under the conditions of Theorem 6.4, the resolvent residue $\overline{R}^{\alpha,\varepsilon}_\lambda$ is a bounded operator from $L^p(w)$ to $L^q(w)$. This holds uniformly in $\varepsilon > 0$.*

Now we prepare ourselves to transfer the previous results for the resolvents to the corresponding semigroup. Under the conditions of Theorem 6.4 we introduce for $t \geq 0$ and $f \in L^p(w)$

$$\overline{S}^{\alpha,\varepsilon}_t f = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}\overline{R}^{\alpha,\varepsilon}_\lambda f \,\mathrm{d}\lambda. \tag{6.18}$$

This is well-defined, because $-\Delta$ and $-H_{\alpha,\varepsilon}$ are sectorial in $L^p(w)$ and hence

$$\|\overline{R}^{\alpha,\varepsilon}_\lambda\|_{L^p(w)\to L^p(w)} \leq \|R_\lambda\|_{L^p(w)\to L^p(w)} + \|R^{\alpha,\varepsilon}_\lambda\|_{L^p(w)\to L^p(w)} \lesssim \frac{1}{|\lambda|}$$

using (2.27), with an implicit constant depending on α and ε. It follows

$$\begin{aligned}
S_t f + \overline{S}^{\alpha,\varepsilon}_t f &= \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}(R_\lambda + \overline{R}^{\alpha,\varepsilon}_\lambda) f \,\mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}R^{\alpha,\varepsilon}_\lambda f \,\mathrm{d}\lambda \\
&= S^{\alpha,\varepsilon}_t f.
\end{aligned}$$

Theorem 6.6. *Let $p > \frac{3}{2}$ and $1 < q < 2$ and $f \in L^p(w)$. We then have for time $0 < t \le T$*

$$\left\| \overline{S}_t^{\alpha,\varepsilon} f(x) \right\|_{L^q(w)} \lesssim \max\{t^{\frac{1}{2}-\frac{3}{2p}}, t^{\frac{1}{4}-\frac{3}{2p}}, t^{\frac{1}{q}-\frac{1}{2}-\frac{3}{2p}}\} \|f\|_{L^p(w)}, \tag{6.19}$$

where the implicit constant depends only on p, q, T.

Proof. We immediately have

$$\begin{aligned} \overline{S}_t^{\alpha,\varepsilon} f &= \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} \overline{R}_\lambda^{\alpha,\varepsilon} f \, \mathrm{d}\lambda \\ &= \frac{1}{2\pi} \left| \int_\Gamma e^{-\lambda t} \overline{R}_\lambda^{\alpha,\varepsilon} f \, \mathrm{d}\lambda \right|, \end{aligned}$$

where we used (6.18). For the path integral it follows by substitution $\lambda \to t^{-1}\lambda$

$$\begin{aligned} \int_\Gamma e^{-\lambda t} \overline{R}_\lambda^{\alpha,\varepsilon} f \, \mathrm{d}\lambda &= \int_{t^{-1}\Gamma} e^{-\lambda t} \overline{R}_\lambda^{\alpha,\varepsilon} f \, \mathrm{d}\lambda \\ &= t^{-1} \int_\Gamma e^{-\lambda} \overline{R}_{t^{-1}\lambda}^{\alpha,\varepsilon} f \, \mathrm{d}\lambda, \end{aligned}$$

because rescaling of the curve Γ doesn't change the value of the integral, as seen in Lemma 5.14. Hence

$$\left\| \overline{S}_t^{\alpha,\varepsilon} f(x) \right\|_{L^q(w)} \le \frac{1}{2\pi t} \int_\Gamma e^{-\operatorname{Re}\lambda} \left\| \overline{R}_{t^{-1}\lambda}^{\alpha,\varepsilon} f \right\|_{L^q(w)} \mathrm{d}\lambda.$$

By applying Theorem 6.4 one obtains

$$\begin{aligned} \left\| \overline{S}_t^{\alpha,\varepsilon} f(x) \right\|_{L^q(w)} &\le \frac{1}{2\pi t} \int_\Gamma e^{-\operatorname{Re}\lambda} \max\{t^{\frac{3}{2}-\frac{3}{2p}}, t^{\frac{5}{4}-\frac{3}{2p}}, t^{\frac{1}{q}+\frac{1}{2}-\frac{3}{2p}}\} \|f\|_{L^p(w)} \, \mathrm{d}\lambda \\ &\lesssim \max\{t^{\frac{1}{2}-\frac{3}{2p}}, t^{\frac{1}{4}-\frac{3}{2p}}, t^{\frac{1}{q}-\frac{1}{2}-\frac{3}{2p}}\} \|f\|_{L^p(w)} \end{aligned}$$

and the proof is complete. □

6.2. Resolvent estimates for the point source

We turn our attention to the resolvents of $-\Delta_\alpha$ and the associated semigroup. Similar to (6.18) we introduce for $t \ge 0$ and $f \in L^p(w)$

$$\overline{S}_t^{\alpha} f = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} \overline{R}_\lambda^{\alpha} f \, \mathrm{d}\lambda \tag{6.20}$$

which is well-defined because of the sectoriality of $-\Delta$ and $-\Delta_\alpha$ and the decomposition (2.24). The path Γ is given by Definition 5.12. Again, for $t \geq 0$ it holds

$$S_t^\alpha = S_t + \overline{S}_t^\alpha .$$

In this section, we can proceed more directly, because the resolvent kernel is explicitly given. We start with a pointwise estimate.

Lemma 6.7. *For $\lambda \in \mathbb{C} \setminus [0, \infty)$ let $\overline{R}_\lambda^\alpha$ the residue of the resolvent of $-\Delta_\alpha$ as introduced in Lemma 2.11, with integral kernel*

$$\overline{R}_\lambda^\alpha(x, y) = \frac{1}{\alpha - \frac{i\sqrt{\lambda}}{4\pi}} \frac{e^{i\sqrt{\lambda}(|x|+|y|)}}{(4\pi)^2|x||y|}.$$

Let $f \in L^p(w)$, $p \geq 1$. We then have

$$\left|\overline{R}_{t^{-1}\lambda}^\alpha f(x)\right| \lesssim \frac{e^{-\sqrt{t^{-1}}\,\mathrm{Im}\sqrt{\lambda}|x|}}{4\pi|x|} t^{\frac{3}{2}-\frac{1}{p}} \|f\|_{L^p(w)} \tag{6.21}$$

for $\lambda \in \Gamma$, and

$$0 \leq \overline{S}_t^\alpha f \lesssim t^{\frac{1}{2}-\frac{1}{p}} w \|f\|_{L^p(w)}, \tag{6.22}$$

where $\overline{S}_t^\alpha f$ given by (6.20) *for $t > 0$ and the implicit constants depend on p.*

Proof. With (6.20) and the nonnegativity of f we obtain

$$\begin{aligned}(0, \infty) \ni \overline{S}_t^\alpha f &= \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} \overline{R}_\lambda^\alpha f \,\mathrm{d}\lambda \\ &= \frac{1}{2\pi} \left| \int_\Gamma e^{-\lambda t} \overline{R}_\lambda^\alpha f \,\mathrm{d}\lambda \right|.\end{aligned}$$

Now, for the path integral it follows

$$\begin{aligned}\int_\Gamma e^{-\lambda t} \overline{R}_\lambda^\alpha f \,\mathrm{d}\lambda &= t^{-1} \int_{t\Gamma} e^{-\lambda} \overline{R}_{t^{-1}\lambda}^\alpha f \,\mathrm{d}\lambda \\ &= t^{-1} \int_\Gamma e^{-\lambda} \overline{R}_{t^{-1}\lambda}^\alpha f \,\mathrm{d}\lambda,\end{aligned}$$

because rescaling of the curve Γ doesn't change the value of the integral, which has been shown in Lemma 5.14. Hence

$$\overline{S}_t^\alpha f \leq \frac{1}{2\pi t} \int_\Gamma e^{-\mathrm{Re}\,\lambda} \left|\overline{R}_{t^{-1}\lambda}^\alpha f\right| \mathrm{d}\lambda. \tag{6.23}$$

We calculate with $b = \operatorname{Im}\sqrt{\lambda}$

$$\left|\overline{R}^{\alpha}_{t^{-1}\lambda} f(x)\right| \le \frac{1}{\left|\alpha - \frac{i\sqrt{t^{-1}\lambda}}{4\pi}\right|} \frac{e^{-\sqrt{t^{-1}}b|x|}}{4\pi|x|} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{t^{-1}}b|y|}}{4\pi|y|} |f(y)| \,\mathrm{d}y. \tag{6.24}$$

The remaining integral is, using Hölder's inequality in the first step and then switching to polar coordinates and transforming $r \to t^{-\frac{1}{2}} r$,

$$\begin{aligned} \int_{\mathbb{R}^3} e^{-\sqrt{t^{-1}}b|y|} |f(y)||w(y)| \,\mathrm{d}y &\le \left(\int_{\mathbb{R}^3} e^{-\sqrt{t^{-1}}bp'|y|} w(y) \,\mathrm{d}y \right)^{\frac{1}{p'}} \|f\|_{L^p(w)} \\ &= \left(4\pi \int_0^\infty e^{-bp'r} r t \,\mathrm{d}r \right)^{\frac{1}{p'}} \|f\|_{L^p(w)} \\ &\lesssim t^{\frac{1}{p'}} \|f\|_{L^p(w)}. \end{aligned} \tag{6.25}$$

Because of $\alpha \in [0, \infty)$, which is uniformly away from the curve Γ, we have

$$\frac{1}{\left|\alpha - \frac{i\sqrt{t^{-1}\lambda}}{4\pi}\right|} \lesssim \sqrt{t}|\sqrt{\lambda}|^{-1} \le \sqrt{t}\sqrt{r(\Gamma)}^{-1}. \tag{6.26}$$

Collecting terms from (6.25) and (6.26) in (6.24) completes the proof of (6.21), because $\sqrt{t} t^{\frac{1}{p'}} = t^{\frac{3}{2} - \frac{1}{p}}$. On the other hand, we can up to a constant estimate the right-hand side of (6.24) from above by

$$w(x) t^{\frac{3}{2} - \frac{1}{p}} \|f\|_{L^p(w)}.$$

Collecting terms in (6.23) we get

$$\begin{aligned} \overline{S}^{\alpha}_t f(x) &\lesssim t^{-1} t^{\frac{3}{2} - \frac{1}{p}} \sqrt{t} w(x) \|f\|_{L^p(w)} \int_\Gamma e^{-\operatorname{Re}\lambda} \,\mathrm{d}\lambda \\ &\lesssim t^{\frac{1}{2} - \frac{1}{p}} w(x) \|f\|_{L^p(w)} \end{aligned}$$

because of $\frac{1}{p'} = 1 - \frac{1}{p}$, which completes the proof of (6.22). □

Corollary 6.8. *Under the conditions of Lemma 6.7 it holds for $f \in \Phi^p(w)$*

$$0 \le S^{\alpha}_t f \lesssim (1 + \|f\|_{L^p(w)} t^{\frac{1}{2} - \frac{1}{p}}) w. \tag{6.27}$$

Proof. Since $S^{\alpha}_t = S_t + \overline{S}^{\alpha}_t$, the claim follows from the previous Lemma and $S_t f \lesssim S_t(w) \lesssim w$, cf. estimate 3.9 □

Remark 6.9. The exponent $\frac{1}{2} - \frac{1}{p}$ of t in Corollary 6.8 corresponds precisely to the $d = 3$ case of estimate (2.76) in [20] with exponent

$$-\frac{1}{2}+\frac{(d+1)(p-1)}{4p}=-\frac{1}{2}+\frac{p-1}{p}=\frac{1}{2}-\frac{1}{p}.$$

Also refer to Corollary 3.19 above. The estimate there was obtained with different methods, estimating the kernel $p_{\alpha,t}$ directly.

Corollary 6.10. *Let $f \in L^p(w)$, $1 < q < 2$ and $p \geq 1$. We then have*

$$\left\|\overline{R}^{\alpha}_{t^{-1}\lambda} f\right\|_{L^q(w)} \lesssim t^{1+\frac{1}{q}-\frac{1}{p}}\|f\|_{L^p(w)}. \tag{6.28}$$

for $\lambda \in \Gamma$, and

$$\left\|\overline{S}^{\alpha}_t f\right\|_{L^q(w)} \lesssim t^{\frac{1}{q}-\frac{1}{p}}\|f\|_{L^p(w)}. \tag{6.29}$$

Proof. We apply the $L^q(w)$ norm to formula (6.21)

$$\left\|\overline{R}^{\alpha}_{t^{-1}\lambda} f\right\|_{L^q(w)} \lesssim \left\|\frac{e^{-\sqrt{t^{-1}}b|\cdot|}}{4\pi|\cdot|}\right\|_{L^q(w)} t^{\frac{3}{2}-\frac{1}{p}}\|f\|_{L^p(w)}. \tag{6.30}$$

Now an easy calculation shows that

$$\left\|\frac{e^{-\sqrt{t^{-1}}b|\cdot|}}{4\pi|\cdot|}\right\|_{L^q(w)} \lesssim t^{\frac{1}{q}-\frac{1}{2}}$$

for $q < 2$, so we have shown (6.28). Consequently, applying the norm to (6.23), following the steps in the proof of Lemma 6.7, yields (6.29) and the proof is complete. □

6.3. Summary of semigroup estimates

In the norm estimates of the previous chapter, most importantly (6.12) and (6.29), different potencies of t occured on the right-hand side. In order to unify the estimates and obtain more practicable semigroup bounds for further calculations in the next chapter about the nonlinear equations, we introduce the following technical lemma.

Lemma 6.11. *Let $N \in \mathbb{N}$, $T > 0$ and $t \in [0, T]$. Let $z_1, \ldots, z_N \in \mathbb{R}$. It holds*

$$\max_j t^{z_j} \leq \sum_{i=1}^{N} t^{z_i} \leq NC(T)t^{\min_j z_j} \tag{6.31}$$

with

$$C(T) = (1 + T^{\max_{i,j}(z_i - z_j)}).$$

Proof. The first estimate in (6.31) is trivial. It remains to show the second part. Let $i \in \{1, \dots, N\}$. We have

$$t^{z_i} = t^{\min_j z_j} t^{z_i - \min_j z_j} \leq t^{\min_j z_j} \cdot \begin{cases} T^{z_i - \min_j z_j}, & T \geq 1, \\ 1, & T \leq 1, \end{cases}$$

where the last estimate follows from $t \leq T$ and $z_i - \min_j z_j \geq 0$. Summing up

$$\begin{aligned} \sum_{i=1}^{N} t^{z_i} &\leq t^{\min_j z_j} \sum_{i=1}^{N} \max\{1, T^{z_i - \min_j z_j}\} \\ &\leq t^{\min_j z_j} \max\{N, \sum_{i=1}^{N} T^{z_i - \min_j z_j}\} \\ &\leq t^{\min_j z_j} N(1 + T^{\max_{i,j}(z_i - z_j)}) \end{aligned}$$

completes the proof. □

Note that Lemma 6.11 will only be applied for $N \leq 4$, which is the number of different t-potencies we obtain in the semigroup estimates.

Theorem 6.12. *Let $p > \frac{3}{2}$ and $1 < q < 2$ and $f \in L^p(w)$ and $0 < t \leq T$ for $T > 0$. There is an exponent $z(p,q) > -1$ and a constant $C(p,q,T) \in o(1+T)$ such that*

$$\|S_t^\alpha f\|_{L^q(w)} \leq C(p,q,T) t^{z(p,q)} \|f\|_{L^p(w)}, \tag{6.32}$$

and, uniformly in ε,

$$\|S_t^{\alpha,\varepsilon} f\|_{L^q(w)} \leq C(p,q,T) t^{z(p,q)} \|f\|_{L^p(w)}. \tag{6.33}$$

Furthermore, for p large enough, we have $z(p,q) \geq 0$.

Proof. By definition of the semigroup $(S_t^\alpha)_{t \geq 0}$ we have the decomposition

$$\|S_t^\alpha f\|_{L^q(w)} \leq \|S_t f\|_{L^q(w)} + \|\overline{S}_t^\alpha f\|_{L^q(w)}. \tag{6.34}$$

For the second term we have the estimate (6.29)

$$\left\|\overline{S}_t^\alpha f\right\|_{L^q(w)} \lesssim t^{\frac{1}{q} - \frac{1}{p}} \|f\|_{L^p(w)}.$$

Now let's turn to the heat semigroup in the first term. We compute directly, using the heat kernel p_t,

$$\begin{aligned}\|S_t f\|_{L^q(w)} &= \|p_t * f\|_{L^q(w)}\\ &\lesssim \left\|p_t w^{-\frac{1}{p}}\right\|_{L^r} \|f\|_{L^p(w)},\end{aligned}$$

using the inequality from Lemma 5.8(ii), with $\frac{1}{r} = 1 + \frac{2}{3q} - \frac{1}{p}$. For the remaining L^r norm involving the heat kernel, it follows with a transformation $x \mapsto \sqrt{t}x$

$$\left\|p_t w^{-\frac{1}{p}}\right\|_{L^r} = \frac{1}{4\pi t^{\frac{3}{2}}} \left(\int_{\mathbb{R}^3} e^{-\frac{r|x|^2}{4t}} |x|^{\frac{r}{p}} \,\mathrm{d}x\right)^{\frac{1}{r}} = t^{\frac{3}{2r}+\frac{1}{2p}-\frac{3}{2}} \frac{1}{4\pi} \left(\int_{\mathbb{R}^3} e^{-\frac{r|x|^2}{4}} |x|^{\frac{r}{p}} \,\mathrm{d}x\right)^{\frac{1}{r}}.$$

This exponent of t is

$$\frac{3}{2r} + \frac{1}{2p} - \frac{3}{2} = \frac{3}{2}\left(1 + \frac{2}{3q} - \frac{1}{p}\right) + \frac{1}{2p} - \frac{3}{2} = \frac{1}{q} - \frac{1}{p}.$$

The remaining integral is a constant depending only on p, q. So it holds

$$\|S_t f\|_{L^q(w)} \lesssim t^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(w)}. \tag{6.35}$$

Returning with (6.29) and (6.35) to (6.34), we obtain

$$\|S_t^\alpha f\|_{L^q(w)} \lesssim t^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(w)}.$$

Since $p, q > 1$, clearly the exponent $z_1(p,q) := \frac{1}{q} - \frac{1}{p}$ is greater than -1 and nonnegative for $p \geq q$. So $z_1(p,q)$ satisfies the desired properties.

Similarly, we can decompose

$$\|S_t^\varepsilon f\|_{L^q(w)} \leq \|S_t f\|_{L^q(w)} + \|\overline{S}_t^{\alpha,\varepsilon} f\|_{L^q(w)}. \tag{6.36}$$

With estimate (6.19) and (6.35) this leads to

$$\|S_t^{\alpha,\varepsilon} f(x)\|_{L^q(w)} \lesssim \left(t^{\frac{1}{q}-\frac{1}{p}} + \max\{t^{\frac{1}{2}-\frac{3}{2p}}, t^{\frac{1}{4}-\frac{3}{2p}}, t^{\frac{1}{q}-\frac{1}{2}-\frac{3}{2p}}\}\right) \|f\|_{L^p(w)}, \tag{6.37}$$

with an implicit constant depending on p, q, T. Now define

$$z_2(p,q) = \min\left\{\frac{1}{q} - \frac{1}{p}, \frac{1}{2} - \frac{3}{2p}, \frac{1}{4} - \frac{3}{2p}, \frac{1}{q} - \frac{1}{2} - \frac{3}{2p}\right\} \geq 0.$$

Using the inequality (6.31), this implies

$$\|S_t^{\alpha,\varepsilon} f(x)\|_{L^q(w)} \lesssim t^{z_2(p,q)} \|f\|_{L^p(w)}.$$

Because of $p, q > 1$ and $q < 2$, all four numbers

$$\frac{1}{q} - \frac{1}{p}, \frac{1}{2} - \frac{3}{2p}, \frac{1}{4} - \frac{3}{2p}, \frac{1}{q} - \frac{1}{2} - \frac{3}{2p}$$

are greater than -1 and nonnegative for p large enough. Clearly this means that $z_2(p,q)$ satisfies these conditions too. Defining $z(p,q) := \min\{z_1(p,q), z_2(p,q)\}$ and using inequality (6.31) again completes the proof. □

6.4. Convergence of the resolvents in the weighted space

With our resolvent estimates in place, we can now show convergence of the resolvents in the weighted space $L^q(w, \mathbb{R}^3)$. This is a variant of the result from [4, Theorem 1.2.5] in the L^2 space. Note that similar convergence results for unweighted L^p spaces can be found in the literature, for example [3, Theorem 3.1] in the case $p \in (\frac{3}{2}, 3)$.

Theorem 6.13. *Let $\frac{3}{2} < p, q < 2$. Then for every $\lambda \in \rho(-\Delta_\alpha)$ it holds*

$$\left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha) f\right\|_{L^q(w)} \to 0 \quad as \quad \varepsilon \downarrow 0$$

for $f \in L^2(\mathbb{R}^3) \cap L^p(w, \mathbb{R}^3)$.

Proof. For the whole proof fix $f \in L^2(\mathbb{R}^3) \cap L^p(w, \mathbb{R}^3)$ and $\lambda \in \rho(-\Delta_\alpha)$. First we restrict our attention to the area away from the origin, more precisely $B_1(0)^c$. With Hölder's inequality we get

$$\begin{aligned} \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha) f\right\|_{L^q(w, B_1(0)^c)} &= \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha) f \cdot w^{1/q}\right\|_{L^q(B_1(0)^c)} \\ &\lesssim \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha) f\right\|_{L^2(B_1(0)^c)} \left\|w^{1/q}\right\|_{L^{q'}(B_1(0)^c)}, \end{aligned} \tag{6.38}$$

where $\frac{1}{q'} + \frac{1}{2} = \frac{1}{q}$. The first factor converges to zero for $\varepsilon \to 0$ according to Theorem 2.15 because

$$\left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha) f\right\|_{L^2(B_1(0)^c)} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)\right\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)}, \tag{6.39}$$

and $f \in L^2(\mathbb{R}^3)$ by assumption. For the second factor of (6.38) we have, independent of ε and f,

$$\left\|w^{1/q}\right\|_{L^{q'}(B_1(0)^c)} = \left\||\cdot|^{-q'/q}\right\|_{L^1(B_1(0)^c)}^{1/q'} < \infty,$$

since the exponent satisfies $\frac{q'}{q} > 4$, which implies integrability in any region away from the origin.

Inside the unit ball, for $\delta \in (0,1)$ we further decompose $B_1(0) = U_\delta \cup V_\delta$, where $U_\delta = B_\delta(0)$ and $V_\delta = B_1(0) \setminus U_\delta$. Of course, for every δ in the range it holds

$$\left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f\right\|_{L^q(w,B_1(0))} \le \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f\right\|_{L^q(w,U_\delta)} + \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f\right\|_{L^q(w,V_\delta)} \tag{6.40}$$

On the annulus V_δ we have $L^2(V_\delta) \subseteq L^q(V_\delta)$, so with Hölder's inequality it follows

$$\begin{aligned}\left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f\right\|_{L^q(w,V_\delta)} &\lesssim \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f \cdot w^{1/q}\right\|_{L^2(V_\delta)} \\ &\le \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f\right\|_{L^2(V_\delta)} \left\|w^{1/q}\right\|_{L^\infty(V_\delta)},\end{aligned}$$

and again it follows convergence to zero of the first factor as in (6.39). We have $\|w^{1/q}\|_{L^\infty(V_\delta)} \approx \delta^{-1/q} < \infty$. So for each fixed δ the second term in (6.40) converges to zero for $\varepsilon \to 0$.

Now we consider the first term in (6.40). First we show that for each $\eta > 0$ there is a $\delta_0 > 0$ such that

$$\limsup_{\varepsilon \to 0} \left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f\right\|_{L^q(w,U_\delta)} \le c_0 \eta \|f\|_{L^p(w)}^2 \tag{6.41}$$

for all $\delta \le \delta_0$ and some $c_0 > 0$.

We have

$$\left\|(R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha)f\right\|_{L^q(w,U_\delta)} \le \left\|\overline{R}_\lambda^\alpha f\right\|_{L^q(w,U_\delta)} + \left\|\overline{R}_\lambda^\varepsilon f\right\|_{L^q(w,U_\delta)}.$$

As in the proof of Lemma 5.5 around formula (5.1),

$$\begin{aligned}\left\|\overline{R}_\lambda^\alpha f\right\|_{L^q(w,U_\delta)} &\lesssim \left\|\frac{e^{i\sqrt{\lambda}|\cdot|}}{|\cdot|}\int_{\mathbb{R}^3}\frac{e^{i\sqrt{\lambda}|y|}}{|y|}f(y)\,\mathrm{d}y\right\|_{L^q(w,U_\delta)} \\ &= \left|\int_{\mathbb{R}^3}\frac{e^{i\sqrt{\lambda}|y|}}{|y|}f(y)\,\mathrm{d}y\right| \left\|\frac{e^{i\sqrt{\lambda}|\cdot|}}{|\cdot|}\right\|_{L^q(w,U_\delta)} \\ &= \left|\int_{\mathbb{R}^3}\frac{e^{i\sqrt{\lambda}|y|}}{|y|}f(y)\,\mathrm{d}y\right| \|R_\lambda(\cdot)\|_{L^q(w,U_\delta)} \\ &\le \left(\int_{\mathbb{R}^3}|e^{-b|y|}|^{p'}w(y)\,\mathrm{d}y\right)^{1/p'} \|f\|_{L^p(w)}\|R_\lambda(\cdot)\|_{L^q(w,U_\delta)} \\ &\lesssim \|f\|_{L^p(w)}\|R_\lambda(\cdot)\|_{L^q(w,U_\delta)},\end{aligned}$$

using Hölder's inequality. Now

$$\begin{aligned}\|R_\lambda(\cdot)\|_{L^q(U_\delta,w)}^q &= \int_{U_\delta}\frac{e^{-bq|x|}}{|x|^{q+1}}\,\mathrm{d}x \\ &= 4\pi\int_0^\delta e^{-bqr}r^{1-q}\,\mathrm{d}r \\ &\le 4\pi\int_0^\delta r^{1-q}\,\mathrm{d}r\end{aligned}$$

$$= \frac{4\pi}{2-q}\delta^{2-q}$$

So choose

$$\delta_1 = \delta_1(\eta) = \eta^{\frac{q}{2-q}} \left(\frac{2-q}{4\pi}\right)^{\frac{1}{2-q}}$$

to obtain

$$\left\|\overline{R}_\lambda^\alpha f\right\|_{L^q(w,U_\delta)} \leq c_1 \eta \|f\|^2_{L^p(w)} \tag{6.42}$$

for $\delta \leq \delta_1$ and some $c_1 > 0$.

Concerning the residue term $\overline{R}_\lambda^{\alpha,\varepsilon}$, remember that

$$\overline{R}_\lambda^{\alpha,\varepsilon} = P_\alpha(\varepsilon) A_\lambda^\varepsilon \varepsilon [1 + B_\lambda^\varepsilon]^{-1} C_\lambda^\varepsilon.$$

According to Lemma 6.2 and Theorem 6.4 and the calculations in its proof, we have that the operator $P_\alpha(\varepsilon) A_\lambda^\varepsilon \varepsilon [1 + B_\lambda^\varepsilon]^{-1}$ is bounded from L^2 to $L^q(w)$, uniformly in ε. Hence, it suffices to show that $\left\|C_\lambda^\varepsilon f\right\|_{L^2(U_\delta)}$ becomes sufficiently small for decreasing δ. The arguments in step 1 of the proof of Theorem 6.4 imply

$$\begin{aligned}
\|C_\lambda^\varepsilon f\|^2_{L^2(U_\delta)} &\lesssim \|f\|^2_{L^p(w)} \int_{U_\delta} V(x) \left(\int_{\mathbb{R}^3} \frac{e^{-bp'|y|}}{(4\pi|y|)^{p'}}\, \mathrm{d}y\right)^{\frac{2}{p'}} \mathrm{d}x \\
&\lesssim \|f\|^2_{L^p(w)} \int_{U_\delta} 1\, \mathrm{d}x \\
&= \|f\|^2_{L^p(w)} \frac{4}{3}\pi\delta^3.
\end{aligned}$$

Thus, we choose

$$\delta_2 = \delta_2(\eta) = \eta^{\frac{3}{2}}$$

to obtain

$$\left\|\overline{R}_\lambda^{\alpha,\varepsilon} f\right\|_{L^q(w,U_\delta)} \leq c_2 \eta \|f\|^2_{L^p(w)} \tag{6.43}$$

for $\delta \leq \delta_2$ and some $c_2 > 0$ up to a generic constant depending only on p. Finally choose

$$\delta_0 := \min\{\delta_1, \delta_2\}, \quad c_0 := c_1 + c_2,$$

which yields (6.41). By nonnegativity of the norm it follows

$$0 \leq \lim_{\varepsilon\to 0} \left\| (R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha) f \right\|_{L^q(w,U_\delta)} \leq \limsup_{\varepsilon\to 0} \left\| (R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha) f \right\|_{L^q(w,U_\delta)} = 0,$$

so the proof is complete. □

6.5. Convergence of semigroups in weighted spaces

With the convergence of resolvents from Theorem 6.13 in place, it is now possible to prove the convergence of the semigroups for fixed t. This is done using the representation of semigroups with path integrals.

Theorem 6.14. *Let $p, q \in (\frac{3}{2}, 2)$ and $t \in [0, T]$. For $f \in L^2 \cap L^p(w)$ it holds for $\varepsilon \to 0$*

$$\|(S_t^\alpha - S_t^{\alpha,\varepsilon}) f\|_{L^q(w)} \to 0.$$

Proof. By the representation of sectorial semigroups (5.2) we have

$$\begin{aligned} \|(S_t^\alpha - S_t^{\alpha,\varepsilon}) f\|_{L^q(w)} &= \frac{1}{2\pi} \left\| \int_\Gamma e^{-\lambda t} (R_\lambda^\alpha - R_\lambda^{\alpha,\varepsilon}) f \,\mathrm{d}\lambda \right\|_{L^q(w)} \\ &\leq \frac{1}{2\pi} \int_\Gamma e^{-t \operatorname{Re}\lambda} \left\| (\overline{R}_\lambda^\alpha - \overline{R}_\lambda^{\alpha,\varepsilon}) f \right\|_{L^q(w)} \mathrm{d}\lambda \end{aligned} \tag{6.44}$$

with the path Γ as in Definition 5.12. By Theorem 6.4 with $t = 1$ it follows uniformly in ε on Γ

$$\left\| \overline{R}_\lambda^{\alpha,\varepsilon} f(x) \right\|_{L^q(w)} \lesssim \|f\|_{L^p(w)}.$$

A similar estimate holds for $\overline{R}_\lambda^\alpha$: By setting $t = 1$ in (6.28) we obtain

$$\left\| \overline{R}_\lambda^\alpha f(x) \right\|_{L^q(w)} \lesssim |\lambda|^{-1} \|f\|_{L^p(w)}.$$

Using the structure of the path Γ, these estimates imply

$$e^{-t \operatorname{Re}\lambda} \left\| (\overline{R}_\lambda^\alpha - \overline{R}_\lambda^{\alpha,\varepsilon}) f \right\|_{L^q(w)} \lesssim e^{-t \operatorname{Re}\lambda} (|\lambda|^{-1} + 1) \|f\|_{L^p(w)} \leq e^{-t|\lambda| \cos\theta} (|\lambda|^{-1} + 1) \|f\|_{L^p(w)},$$

which is an integrable majorant for the integrand in (6.44) because of the exponential decay, note that for the opening angle θ of the curve Γ it holds $\cos\theta > 0$, because $\theta \in (0, \frac{\pi}{2})$. Now, using Lebesgue's theorem,

$$\lim_{\varepsilon\to 0} \|(S_t^\alpha - S_t^{\alpha,\varepsilon}) f\|_{L^q(w)} \lesssim \int_\Gamma e^{-t \operatorname{Re}\lambda} \lim_{\varepsilon\to 0} \left\| (\overline{R}_\lambda^\alpha - \overline{R}_\lambda^{\alpha,\varepsilon}) f \right\|_{L^q(w)} \mathrm{d}\lambda = 0$$

by Theorem 6.13. □

7. Convergence of solutions of the nonlinear equations

In this chapter we will prove the main convergence result for the solutions of the nonlinear integral equations

$$u_{\alpha,\varepsilon}(t,x) = S_t^{\alpha,\varepsilon} f(x) - \eta \int_0^t S_{t-s}^{\alpha,\varepsilon}(u_{\alpha,\varepsilon}^{1+\beta}(s))(x)\,\mathrm{d}s, \quad \varepsilon \in (0,1), \tag{7.1}$$

towards

$$u_\alpha(t,x) = S_t^\alpha f(x) - \eta \int_0^t S_{t-s}^\alpha(u_\alpha^{1+\beta}(s))(x)\,\mathrm{d}s. \tag{7.2}$$

for $\varepsilon \to 0$, where $t \geq 0$ is fixed and f is a function with sufficiently small norm in the weighted space, satisfying the conditions of Section 7.1. In order to do so, using the analytic tools from the previous chapters, we establish a Picard iteration

$$\begin{aligned} u_{\alpha,\varepsilon}^{(0)}(t,x) &:= S_t^{\alpha,\varepsilon} f(x) \\ u_{\alpha,\varepsilon}^{(n)}(t,x) &:= S_t^{\alpha,\varepsilon} f(x) - \eta \int_0^t S_{t-s}^{\alpha,\varepsilon}((u_{\alpha,\varepsilon}^{(n-1)})^{1+\beta}(s))(x)\,\mathrm{d}s, \quad n \in \mathbb{N}. \end{aligned} \tag{7.3}$$

Similarly, we define for the point interaction the iteration

$$\begin{aligned} u_\alpha^{(0)}(t,x) &:= S_t^\alpha f(x) \\ u_\alpha^{(n)}(t,x) &:= S_t^\alpha f(x) - \eta \int_0^t S_{t-s}^\alpha((u_\alpha^{(n-1)})^{1+\beta}(s))(x)\,\mathrm{d}s, \quad n \in \mathbb{N}. \end{aligned} \tag{7.4}$$

We start with the technical framework and some important pointwise properties of the solutions, before we prove the main convergence result in Section 7.3.

7.1. Technical settings

We briefly introduce the analytic framework for the whole chapter. First we choose the following set of parameters:

$$0 \leq \beta < \frac{1}{3}, \quad p, q \in \left(\frac{3}{2}, 2\right), \quad q = (1+\beta)p. \tag{7.5}$$

Furthermore, let $\kappa > \frac{3}{2}$ large enough such that for the corresponding exponent in Theorem 6.12 it holds $z(\kappa, q) \geq 0$.

From now on we assume that $f : \mathbb{R}^3 \to [0, \infty)$ is a function satisfying

$$f \in \Phi^p \cap L^2 \cap L^\kappa(w)$$

with Φ^p as in Definition 3.1. For example, every continuous nonnegative bounded function with bounded support satisfies these requirements.

We will also need the following elementary estimate, introduced in [20, Lemma 3.6].

Lemma 7.1. *Let $a, b \geq 0$ and $0 < \beta \leq 1$. It holds*

$$|a^{1+\beta} - b^{1+\beta}| \leq (1+\beta) \max\{a, b\}^\beta |a - b|. \tag{7.6}$$

Proof. The estimate follows directly from the mean value theorem. □

7.2. Nonnegativity of solutions

In this section we want to collect some pointwise properties, especially nonnegativity, of the iterated solutions $u^{(n)}_{\alpha,\varepsilon}, u^{(n)}_\alpha$, $n \in \mathbb{N}$, defined in (7.3) and (7.4). The main technique here are pointwise estimates of the semigroups $(S^\alpha_t)_{t\geq 0}$ and $(S^{\alpha,\varepsilon}_t)_{t\geq 0}$ for fixed $\varepsilon \in (0, 1]$.

We follow the argumentation in [20, Section 3.2 and 3.4]. Fix a time $T > 0$ and a measurable function $\psi : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ with

$$0 \leq \psi(t, x) \leq M(1 + t^{-\kappa}) w^\beta(x). \tag{7.7}$$

For a suitable $M = M(T) > 0$ and $\kappa = \frac{\beta}{2} - \frac{\beta(d+1)(p-1)}{4p}$ it holds

$$(S^\alpha_t f(x))^\beta \leq \psi(t, x), \tag{7.8}$$

this follows from [20, Corollary 2.11]. Now define for $N \in \mathbb{N}$, $N \geq 2$,

$$\psi_N := \psi \wedge \eta^{-1} N.$$

The next lemma is an adaption of [20, Lemma 3.8].

Lemma 7.2. *Let* $f \in \Phi^p$*,* $T > 0$ *and* $v_{N,n}, N, n \in \mathbb{N}, N \geq 2$*, given by*

$$\begin{aligned} v_{N,0}(t,x) &= S_t^\alpha f(x), \\ v_{N,n}(t,x) &= S_t^\alpha f(x) - \eta \int_0^t S_{t-s}^\alpha(\psi_N(s) v_{N,n-1}(s))(x)\,\mathrm{d}s. \end{aligned} \tag{7.9}$$

Then it holds

$$0 \leq v_{N,n}(t,x) \leq S_t^\alpha f(x)$$

for $0 \leq t \leq T$.

Proof. We want to fix N and proceed by induction over n. The claim is trivially true for $n = 0$. Assume as induction hypothesis, that $0 \leq v_{N,n-1} \leq S_t^\alpha f(x)$. We have

$$S_{t-s}^\alpha(\psi_N(s) v_{N,n-1}(s)) \leq S_{t-s}^\alpha(\eta^{-1} N S_s^\alpha f) \leq \eta^{-1} N S_t^\alpha f. \tag{7.10}$$

For now we restrict our attention to the time interval $0 \leq t \leq N^{-1}$. Here it holds because of (7.10)

$$S_t^\alpha f \geq v_{N,n}(t) \geq S_t^\alpha f - S_t^\alpha f \eta \int_0^{N^{-1}} \eta^{-1} N\,\mathrm{d}s = 0.$$

The upper bound follows from the nonnegativity of the integral in (7.9). This bound also implies $v_{N,n}(N^{-1}) \in \Phi^p$.

In the next step we want to use induction over time intervals of length N^{-1}. We have shown that $v_{N,n}$ is nonnegative on $[0, N^{-1}]$ and that $v_{N,n}(N^{-1}) \in \Phi^p$. To proceed with our induction, assume that $v_{N,n}$ is nonnegative on $[(k-1)N^{-1}, kN^{-1}]$ for some $0 \leq k \leq NT - 1$ and that $v_{N,n}(kN^{-1}) \in \Phi^p$. We now want to shift time and start the iteration with $v_{N,n}(kN^{-1})$ instead of f. In order to do so, define

$$\begin{aligned} v_{N,n}^{(k)}(t) &:= v_{N,n}(t + kN^{-1}), \\ f_{N,n}^{(k)} &:= v_{N,n}(N^{-1}), \\ \psi_N^{(k)}(t) &:= \psi_N(t + kN^{-1}). \end{aligned} \tag{7.11}$$

To apply our result from the previous step, we need to show that

$$v_{N,n}^{(k)}(t) = S_t^\alpha f_{N,n}^{(k)} - \eta \int_0^t S_{t-s}^\alpha(\psi_N^{(k)}(s) v_{N,n-1}^{(k)}(s))\,\mathrm{d}s \tag{7.12}$$

holds for for $0 \leq t \leq N^{-1}$. By applying S_t^α to both sides of definition (7.9), it holds for a time

$r > 0$

$$S_t^\alpha v_{N,n}(r) = S_{t+r}^\alpha f - \eta \int_0^r S_{t+r-s}^\alpha (\psi_N(s) v_{N,n-1}(s)\, \mathrm{d}s.$$

Fix $r = kN^{-1}$ and $0 \le t \le N^{-1}$. Then, using definition (7.11),

$$S_t^\alpha v_{N,n}(kN^{-1}) = S_t^\alpha f_{N,n}^{(k)} = S_{t+kN^{-1}}^\alpha f - \eta \int_0^{kN^{-1}} S_{t+kN^{-1}-s}^\alpha (\psi_N(s) v_{N,n-1}(s))\, \mathrm{d}s. \tag{7.13}$$

With the change of variables $s \mapsto s - kN^{-1}$ it also holds

$$\begin{aligned} \eta \int_0^t S_{t-s}^\alpha (\psi_N^{(k)}(s) v_{N,n-1}^{(k)}(s))\, \mathrm{d}s &= \eta \int_0^t S_{t-s}^\alpha (\psi_N(s+kN^{-1}) v_{N,n-1}(s+kN^{-1}))\, \mathrm{d}s \\ &= \eta \int_{kN^{-1}}^{t+kN^{-1}} S_{t+kN^{-1}-s}^\alpha (\psi_N(s) v_{N,n-1}(s))\, \mathrm{d}s. \end{aligned} \tag{7.14}$$

Inserting (7.13) and (7.14) into the rigt side of (7.12), it follows

$$\begin{aligned} &S_t^\alpha f_{N,n}^{(k)} - \eta \int_0^t S_{t-s}^\alpha (\psi_N^{(k)}(s) v_{N,n-1}^{(k)}(s))\, \mathrm{d}s \\ &\quad = S_{t+kN^{-1}}^\alpha f - \eta \int_0^{t+kN^{-1}} S_{t+kN^{-1}-s}^\alpha (\psi_N(s) v_{N,n-1}(s))\, \mathrm{d}s \\ &\quad = v_{N,n}(t+kN^{-1}) = v_{N,n}^{(k)}(t) \end{aligned}$$

by applying definitions (7.9) and (7.11) in the last two steps. So we have verified (7.12). The first step of the proof implies that $0 \le v_{N,n}^{(k)}(t) \le S_t^\alpha f$ for $0 \le t \le N^{-1}$. But this is equivalent to $0 \le v_{N,n}(t) \le S_t^\alpha f$ for $kN^{-1} \le t \le (k+1)N^{-1}$. Thus, the claim follows on the whole interval $[0, T]$ and the proof is complete. □

Lemma 7.3. *Let $f \in \Phi^p$, $T > 0$ and v_n, $n \in \mathbb{N}$, given by*

$$\begin{aligned} v_0(t,x) &= S_t^\alpha f(x), \\ v_n(t,x) &= S_t^\alpha f(x) - \eta \int_0^t S_{t-s}^\alpha (\psi(s) v_{n-1}(s))(x)\, \mathrm{d}s. \end{aligned}$$

Then it holds

$$0 \le v_n(t,x) \le S_t^\alpha f(x) \tag{7.15}$$

for $0 \le t \le T$.

Proof. The claim follows with an induction over n from Lemma 7.2 and the fact that

$$\lim_{N \uparrow \infty} |v_{N,n}(t) - v_n(t)| = 0 \tag{7.16}$$

for each n. For a proof of this convergence and more details refer to [20, Lemma 3.9]. $\square$

Theorem 7.4. *Let $f \in \Phi^p$, $T > 0$ and $u_\alpha^{(n)}$, $n \in \mathbb{N}$, given by* (7.4) *Then it holds*

$$0 \le u_\alpha^{(n)}(t,x) \le S_t^\alpha f(x) \tag{7.17}$$

for $0 \le t \le T$

Proof. We proceed by induction over n. The claim is trivially true for $n = 0$. Now let $n \ge 1$. Assume as induction hypothesis, that

$$0 \le u_\alpha^{(n)}(t,x) \le S_t^\alpha f(x)$$

holds. Then the integral in (7.3) is nonnegative and the upper bound in (7.17) follows. Furthermore

$$\begin{aligned} \eta \int_0^t S_{t-s}^\alpha (u_\alpha^{(n)}(s)^{1+\beta})\,\mathrm{d}s &\le \eta \int_0^t S_{t-s}^\alpha ((S_s^\alpha f)^{1+\beta})\,\mathrm{d}s \\ &\le \eta \int_0^t S_{t-s}^\alpha (\psi(s) S_s^\alpha f)\,\mathrm{d}s. \end{aligned}$$

Using Lemma 7.3, this implies the nonnegativity

$$u_\alpha^{(n)}(t) \ge S_t^\alpha f - \eta \int_0^t S_{t-s}^\alpha (\psi(s) S_s^\alpha f)\,\mathrm{d}s = v_1(t) \ge 0.$$

This implies the lower bound in (7.17). $\square$

Lemma 7.5. *Under the conditions of Section 7.1 it holds for all $n \in \mathbb{N}$ and every $\varepsilon > 0$*

$$\begin{aligned} 0 &\le u_\alpha^{(n)}(t,x) \le S_t^\alpha f(x), \\ 0 &\le u_{\alpha,\varepsilon}^{(n)}(t,x) \le S_t^{\alpha,\varepsilon} f(x). \end{aligned}$$

Proof. The estimates for $u_\alpha^{(n)}$ where subject of the previous Theorem 7.4. Concerning the solutions $u_{\alpha,\varepsilon}^{(n)}$, $n \in \mathbb{N}$, note the following: We can repeat the proof of Theorem 7.4 using Lemma 7.2 and Lemma 7.3 and substitute $S_t^{\alpha,\varepsilon}$ for S_t^α everywhere. Instead of estimate (7.8) we have for every fixed ε

$$(S_t^{\alpha,\varepsilon} f(x))^\beta \le C(\varepsilon) w^\beta(x), \tag{7.18}$$

using (3.16). Hence, we can conclude the claimed estimates for $u_{\alpha,\varepsilon}^{(n)}$. $\square$

Corollary 7.6. *For each $t \in [0,T]$ and $\varepsilon > 0$ it holds*

$$(S_t^\alpha f)^{1+\beta}, (S_t^{\alpha,\varepsilon} f)^{1+\beta} \in L^p(w).$$

Moreover, for every $n \in \mathbb{N}$ it holds

$$u_\alpha^{(n)}(t,\cdot)^{1+\beta}, u_{\alpha,\varepsilon}^{(n)}(t,\cdot)^{1+\beta} \in L^p(w).$$

Proof. Using Theorem 6.12 and (7.5), it follows

$$\left\|(S_t^\alpha f)^{1+\beta}\right\|_{L^p(w)} = \|S_t^\alpha f\|_{L^q(w)}^{1+\beta} \le C(\kappa,q,T)^{1+\beta} t^{z(\kappa,q)(1+\beta)} \|f\|_{L^\kappa(w)}^{1+\beta},$$

and we have the same estimate for $\|(S_t^{\alpha,\varepsilon} f)^{1+\beta}\|_{L^p(w)}$. The claim for $u_\alpha^{(n)}(t,\cdot)^{1+\beta}, u_{\alpha,\varepsilon}^{(n)}(t,\cdot)^{1+\beta}$ follows with Lemma 7.5. □

7.3. Convergence of solutions in the weighted space

No we are able to prove the main convergence result stated at the beginning of the chapter. The strategy will be to prove a cauchy property and then convergence of the functions $u_{\alpha,\varepsilon}^{(n)}(t)$ to $u_{\alpha,\varepsilon}(t)$ for $n \to \infty$, but uniformly in ε. This uniformity can then be exploited to write for each $\delta > 0$ the function $u_{\alpha,\varepsilon}(t)$ as a finite sum consisting of $N \in \mathbb{N}$ terms involving $(u_{\alpha,\varepsilon}^{(n)}(t))_{n \le \mathbb{N}}$ plus an $o(\delta)$ term. Because N, δ are uniform in ε, it then suffices to prove convergence for $\varepsilon \to 0$ of the first N summands using the convergence of semigroups from the previous chapter, Theorem 6.14. Also refer to the motivation given at the beginning of Chapter 6.

Uniform Cauchy property

We need to show that the sequence $(u_{\alpha,\varepsilon}^{(n)})_{n\in\mathbb{N}}$ is a cauchy sequence in the weighted space, uniformly in ε. In the following lemma we deal with the first iteration step.

Lemma 7.7. *Under the conditions of Section 7.1, for every $T > 0$ there is a constant $C_T > 0$ such that for all $t \in [0,T]$*

$$\sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(1)}(t,\cdot) - u_{\alpha,\varepsilon}^{(0)}(t,\cdot) \right\|_{L^q(w)} \le 1$$

if $\|f\|_{L^\kappa(w)} \le C_T$.

Proof. We have by definition of (7.3)

$$\begin{aligned} \left\| u_{\alpha,\varepsilon}^{(1)}(t,\cdot) - u_{\alpha,\varepsilon}^{(0)}(t,\cdot) \right\|_{L^q(w)} &= \eta \left\| \int_0^t S_{t-s}^{\alpha,\varepsilon}(S_s^{\alpha,\varepsilon} f)^{1+\beta}\, \mathrm{d}s \right\|_{L^q(w)} \\ &\le \eta \int_0^t \left\| S_{t-s}^{\alpha,\varepsilon}(S_s^{\alpha,\varepsilon} f)^{1+\beta} \right\|_{L^q(w)} \mathrm{d}s. \end{aligned}$$

Using Corollary 7.6, it follows $(S_s^{\alpha,\varepsilon} f)^{1+\beta} \in L^p(w)$. This allows us to apply Theorem 6.12 again, to obtain the bound

$$\begin{aligned}\left\| S_{t-s}^{\alpha,\varepsilon}(S_s^{\alpha,\varepsilon} f)^{1+\beta} \right\|_{L^q(w)} &\le C(p,q,T)(t-s)^{z(p.q)} \left\| (S_s^{\alpha,\varepsilon} f)^{1+\beta} \right\|_{L^p(w)} \\ &= C(p,q,T)(t-s)^{z(p,q)} \| S_s^{\alpha,\varepsilon} f \|_{L^q(w)}^{1+\beta} \\ &\le C(p,q,T) C(\kappa,q,T)^{1+\beta} (t-s)^{z(p,q)} s^{z(\kappa,q)(1+\beta)} \| f \|_{L^\kappa(w)}^{1+\beta}.\end{aligned}$$

By the choice of parameters it holds $z(\kappa,q)(1+\beta) \ge 0$. This leads to

$$\begin{aligned}\left\| u_{\alpha,\varepsilon}^{(1)}(t,\cdot) - u_{\alpha,\varepsilon}^{(0)}(t,\cdot) \right\|_{L^q(w)} &\le \eta C(p,q,T) C(\kappa,q,T)^{1+\beta} \| f \|_{L^\kappa(w)}^{1+\beta} t^{z(\kappa,q)(1+\beta)} \int_0^t s^{z(p,q)} \, \mathrm{d}s \\ &= \frac{\eta C(p,q,T) C(\kappa,q,T)^{1+\beta}}{1+z(p,q)} \| f \|_{L^\kappa(w)}^{1+\beta} t^{z(\kappa,q)(1+\beta)+z(p,q)+1}.\end{aligned}$$

Because of $z(p,q) > -1$ the exponent of t is positive and we can apply the supremum

$$\sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(1)}(t,\cdot) - u_{\alpha,\varepsilon}^{(0)}(t,\cdot) \right\|_{L^q(w)} \le \frac{\eta C(p,q,T) C(\kappa,q,T)^{1+\beta}}{1+z(p,q)} \| f \|_{L^\kappa(w)}^{1+\beta} T^{z(\kappa,q)(1+\beta)+z(p,q)+1}$$

and choose

$$\| f \|_{L^\kappa(w)} \le C_T := \left(\frac{\eta C(p,q,T) C(\kappa,q,T)^{1+\beta}}{1+z(p,q)} T^{z(\kappa,q)(1+\beta)+z(p,q)+1} \right)^{-\frac{1}{1+\beta}},$$

which completes the proof. □

With this initial case in place, we can proceed with the induction.

Lemma 7.8. *Under the conditions of Lemma 7.7, for every $\varepsilon > 0$ the sequence $(u_{\alpha,\varepsilon}^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence with convergence rate uniformly in ε with respect to the norm $\sup_{t\in[0,T]} \|\cdot\|_{L^q(w)}$. More precisely, for every $\delta > 0$ there is a $N_\delta \in \mathbb{N}$ such that*

$$\sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n+1)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} < \delta$$

for all $n \ge N_\delta$, independent of ε.

Proof. Using the conditions on the parameters (7.5) and the norm estimate from Theorem 6.12 we have

$$\begin{aligned}&\left\| u_{\alpha,\varepsilon}^{(n+1)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \\ &\le \eta \int_0^t \left\| S_{t-s}^{\alpha,\varepsilon} n((u_{\alpha,\varepsilon}^{(n)})^{1+\beta}(s) - (u_{\alpha,\varepsilon}^{(n-1)})^{1+\beta}(s)) \right\|_{L^q(w)} \mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leq \eta \int_0^t (t-s)^{z(p,q)} C(p,q,T) \left\| (u_{\alpha,\varepsilon}^{(n)})^{1+\beta}(s) - (u_{\alpha,\varepsilon}^{(n-1)})^{1+\beta}(s) \right\|_{L^p(w)} \mathrm{d}s \\
&\leq \eta(1+\beta) C(p,q,T) \\
&\quad \cdot \int_0^t (t-s)^{z(p,q)} \left\| \max\{u_{\alpha,\varepsilon}^{(n)}(s), u_{\alpha,\varepsilon}^{(n-1)}(s)\}^\beta \cdot (u_{\alpha,\varepsilon}^{(n)}(s) - u_{\alpha,\varepsilon}^{(n-1)}(s)) \right\|_{L^p(w)} \mathrm{d}s.
\end{aligned}
$$

Here we have used estimate (7.6). Now with Hölder's inequality in the weighted space (Lemma 5.8(i)) and $\frac{1}{p} = \frac{\beta}{q} + \frac{1}{q}$ and again Theorem 6.12, the remaining time integral is bounded by

$$
\begin{aligned}
&\int_0^t (t-s)^{z(p,q)} \left\| \max\{u_{\alpha,\varepsilon}^{(n)}(s), u_{\alpha,\varepsilon}^{(n-1)}(s)\}^\beta \right\|_{L^{\frac{q}{\beta}}(w)} \left\| u_{\alpha,\varepsilon}^{(n)}(s) - u_{\alpha,\varepsilon}^{(n-1)}(s) \right\|_{L^q(w)} \mathrm{d}s \\
&\leq \int_0^t (t-s)^{z(p,q)} \| S_s^{\alpha,\varepsilon} f \|_{L^q(w)}^\beta \left\| u_{\alpha,\varepsilon}^{(n)}(s) - u_{\alpha,\varepsilon}^{(n-1)}(s) \right\|_{L^q(w)} \mathrm{d}s \\
&\leq C(\kappa,q,T)^\beta \|f\|_{L^\kappa(w)}^\beta \int_0^t (t-s)^{z(p,q)} s^{\beta z(\kappa,q)} \left\| u_{\alpha,\varepsilon}^{(n)}(s) - u_{\alpha,\varepsilon}^{(n-1)}(s) \right\|_{L^q(w)} \mathrm{d}s \\
&\leq C(\kappa,q,T)^\beta \|f\|_{L^\kappa(w)}^\beta \sup_{s\in[0,t]} \left\| u_{\alpha,\varepsilon}^{(n)}(s) - u_{\alpha,\varepsilon}^{(n-1)}(s) \right\|_{L^q(w)} \int_0^t (t-s)^{z(p,q)} s^{\beta z(\kappa,q)} \mathrm{d}s.
\end{aligned}
$$

Note that we have used $0 \leq u_m^{\alpha,\varepsilon} \leq S_t^{\alpha,\varepsilon} f$, $m \in \mathbb{N}$, from Lemma 7.5 in the second step.

This implies for $T > 0$ that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n+1)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \\
&\leq \eta(1+\beta) C(p,q,T) C(\kappa,q,T)^\beta \|f\|_{L^\kappa(w)}^\beta \\
&\qquad \cdot \sup_{t\in[0,T]} \left[\left(\sup_{s\in[0,t]} \left\| u_{\alpha,\varepsilon}^{(n)}(s) - u_{\alpha,\varepsilon}^{(n-1)}(s) \right\|_{L^q(w)} \right) \int_0^t (t-s)^{z(p,q)} s^{\beta z(\kappa,q)} \mathrm{d}s \right] \\
&\leq \eta(1+\beta) C(p,q,T) C(\kappa,q,T)^\beta \|f\|_{L^\kappa(w)}^\beta T^{\beta z(\kappa,q)} \int_0^T s^{z(p,q)} \mathrm{d}s \\
&\qquad \cdot \sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n)}(t) - u_{\alpha,\varepsilon}^{(n-1)}(t) \right\|_{L^q(w)} \\
&\leq \frac{\eta(1+\beta) C(p,q,T) C(\kappa,q,T)^\beta}{1+z(p,q)} \|f\|_{L^\kappa(w)}^\beta T^{\beta z(\kappa,q)+z(p,q)+1} \sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n)}(t) - u_{\alpha,\varepsilon}^{(n-1)}(t) \right\|_{L^q(w)}.
\end{aligned}
$$

We have $z(\kappa,q) \geq 0$ and $z(p,q) > -1$, hence the denominator of the constant doesn't vanish and the exponent of T is strictly positive.

Now choose $\|f\|_{L^\kappa(w)}$ small enough, such that

$$
L_{p,q,\eta,\beta}(f,T) := \frac{\eta(1+\beta) C(p,q,T) C(\kappa,q,T)^\beta}{1+z(p,q)} \|f\|_{L^\kappa(w)}^\beta T^{\beta z(\kappa,q)+z(p,q)+1} < 1. \tag{7.19}
$$

So we have

$$\sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n+1)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \leq L_{p,q,\eta,\beta}(f,T) \sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n)}(t) - u_{\alpha,\varepsilon}^{(n-1)}(t) \right\|_{L^q(w)}$$

for all $n \geq 1$ and it follows

$$\sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n+1)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \downarrow 0$$

for $n \to \infty$ uniformly in ε, so we have the desired Cauchy property. □

Corollary 7.9. *Under the conditions of Lemma 7.7, for every $\varepsilon > 0$ the sequence $(u_{\alpha,\varepsilon}^{(n)})_{n\in\mathbb{N}}$ converges with ε-independent rate with respect to the norm $\sup_{t\in[0,T]} \|\cdot\|_{L^q(w)}$. More precisely, for every $\varepsilon > 0$ there is a function $u_{\alpha,\varepsilon}$, such that for every $\delta > 0$ there exists an $N_\delta \in \mathbb{N}$ with*

$$\sup_{t\in[0,T]} \left\| u_{\alpha,\varepsilon}^{(n)}(t) - u_{\alpha,\varepsilon}(t) \right\|_{L^q(w)} < \delta$$

for all $n \geq N_\delta$, uniformly in ε. The function $u_{\alpha,\varepsilon}$ is the unique solution of the integral equation (7.1).

Proof. With the previous Lemma 7.8 and the completeness of $L^q(w)$, for every $\varepsilon > 0$ there is a function $u_{\alpha,\varepsilon} \in L^q(w)$ such that for every $\delta > 0$ there exists an $N_\delta \in \mathbb{N}$ with

$$\left\| u_{\alpha,\varepsilon}^{(n)}(t) - u_{\alpha,\varepsilon}(t) \right\|_{L^q(w)} < \delta$$

for all $n \geq N_\delta$, uniformly in ε and every fixed $t \in [0, T]$. But the uniform cauchy property from Lemma 7.8 implies that this δ is also uniformly in t. With the uniqueness of solutions from Theorem 3.25 the claim follows. □

With the notation

$$d_{\alpha,\varepsilon}^{(n)} := u_{\alpha,\varepsilon}^{(n+1)} - u_{\alpha,\varepsilon}^{(n)}, \quad n \in \mathbb{N}_0,$$

we have the series representation

$$u_{\alpha,\varepsilon} := u_{\alpha,\varepsilon}^{(0)} + \sum_{n=0}^{\infty} d_{\alpha,\varepsilon}^{(n)}.$$

The following corollary shows that this is well-defined. As an immediate consequence of Lemma 7.8, the convergence speed of the series is uniform in ε.

Corollary 7.10. *For every $\delta > 0$ there is a $N_\delta \in \mathbb{N}$ such that uniformly in ε*

$$\sum_{n=N_\delta}^{\infty} \sup_{t\in[0,T]} \left\| d_{\alpha,\varepsilon}^{(n)} \right\|_{L^q(w)} < \delta.$$

Proof. Let $\delta > 0$. By Lemma 7.8 we have for $\tilde{\delta} := \delta \left(\sum_{n=0}^{\infty} L^n\right)^{-1}$, where $L < 1$ is the Lipschitz constant in the proof,

$$\sup_{t\in[0,T]} \left\| d_{\alpha,\varepsilon}^{(n)} \right\|_{L^q(w)} < \tilde{\delta}$$

for all n greater than some $N_{\tilde{\delta}}$. Now choose $N_\delta = N_{\tilde{\delta}}$. Because of

$$\sup_{t\in[0,T]} \left\| d_{\alpha,\varepsilon}^{(n+1)} \right\|_{L^q(w)} \leq L \cdot \sup_{t\in[0,T]} \left\| d_{\alpha,\varepsilon}^{(n)} \right\|_{L^q(w)}$$

for all $n \in \mathbb{N}$ we can conclude

$$\sum_{n=N_\delta}^{\infty} \sup_{t\in[0,T]} \left\| d_{\alpha,\varepsilon}^{(n)} \right\|_{L^q(w)} < \tilde{\delta} \sum_{n=0}^{\infty} L^n = \delta,$$

and the proof is complete. □

Repeating the previous proofs of Lemma 7.7, Lemma 7.8, Corollary 7.9 and Corollary 7.10 step by step, but for the point interaction semigroup $(S_t^\alpha)_{t\geq 0}$, we obtain the analogous result

Corollary 7.11. *Under the conditions of Section 7.1, the sequence $(u_\alpha^{(n)})_{n\in\mathbb{N}}$ converges with respect to the norm $\sup_{t\in[0,T]} \|\cdot\|_{L^q(w)}$. More precisely, there is a function u_α, such that for every $\delta > 0$ there exists an $N_\delta \in \mathbb{N}$ with*

$$\sup_{t\in[0,T]} \left\| u_\alpha^{(n)}(t) - u_\alpha(t) \right\|_{L^q(w)} < \delta$$

for all $n \geq N_\delta$. The function u_α is the unique solution of the nonlinear integral equation (7.2) *and has the series representation*

$$u_\alpha = u_\alpha^{(0)} + \sum_{n=0}^{\infty} d_\alpha^{(n)}$$

with

$$d_\alpha^{(n)} := u_\alpha^{(n+1)} - u_\alpha^{(n)}, \quad n \in \mathbb{N}_0.$$

The main convergence result

We are now prepared to show the convergence of $u_{\alpha,\varepsilon}$ towards u_α for $\varepsilon \to 0$ in the weighted space. The following Lemma will be needed in the proof to apply the convergence theorem for semigroups, Theorem 6.14.

Lemma 7.12. *It holds for every fixed $t \in (0, T]$*

$$(S_t^\alpha f)^{1+\beta} \in L^2 \cap L^p(w).$$

Proof. First we estimate the L^2 norm

$$\begin{aligned}
\left\|(S_t^\alpha f)^{1+\beta}\right\|_{L^2}^2 &= \int_{\mathbb{R}^3} |S_t^\alpha f(x)|^{2+2\beta}\,\mathrm{d}x \\
&\lesssim (1 + \|f\|_{L^p(w)} t^{\frac{1}{2}-\frac{1}{p}}) \int_{\mathbb{R}^3} |S_t^\alpha f(x)|^{1+2\beta} w(x)\,\mathrm{d}x \\
&= (1 + \|f\|_{L^p(w)} t^{\frac{1}{2}-\frac{1}{p}}) \|S_t^\alpha f\|_{L^{1+2\beta}(w)}^{1+2\beta} \\
&\le C(\kappa, 1+2\beta, T)^{1+2\beta} (1 + \|f\|_{L^p(w)} t^{\frac{1}{2}-\frac{1}{p}}) t^{z(\kappa,1+2\beta)(1+2\beta)} \|f\|_{L^\kappa(w)}^{1+2\beta}.
\end{aligned}$$

Here we have used $S_t^\alpha f \lesssim (1 + \|f\|_{L^p(w)} t^{\frac{1}{2}-\frac{1}{p}}) w$ from Corollary 6.8 in the second and Theorem 6.12 in the fourth step. Since $f \in L^\kappa(w)$, we have found an L^2-bound for every $t > 0$.

Concerning the $L^p(w)$-bound, it holds, using again Theorem 6.12,

$$\left\|(S_t^\alpha f)^{1+\beta}\right\|_{L^p(w)} = \|S_t^\alpha f\|_{L^q(w)}^{1+\beta} \le C(\kappa, q, T)^{1+\beta} t^{z(\kappa,q)(1+\beta)} \|f\|_{L^\kappa(w)}^{1+\beta},$$

this completes the proof. □

It follows the main convergence result for the solutions of the nonlinear integral equations.

Theorem 7.13. *Under the conditions of Section 7.1, for every $T > 0$ there is a constant $C_T > 0$ such that for all $t \in [0, T]$ and initial data $f \in \Phi^p \cap L^\kappa(w)$ satisfying $\|f\|_{L^\kappa(w)} \le C_T$ it holds*

$$\|u_\alpha(t) - u_{\alpha,\varepsilon}(t)\|_{L^q(w)} \to 0.$$

Proof. Let $\delta > 0$. By Corollaries 7.9 and 7.3 there exists an $N_\delta \in \mathbb{N}$ such that

$$\sum_{n=N_\delta}^{\infty} \sup_{t\in[0,T]} \left\|d_\alpha^{(n)}\right\|_{L^q(w)} + \sum_{n=N_\delta}^{\infty} \sup_{t\in[0,T]} \left\|d_{\alpha,\varepsilon}^{(n)}\right\|_{L^q(w)} < \delta.$$

uniformly in ε, under the condition that for the initial data f it holds $\|f\|_{L^\kappa(w)} \le C_T$. Using this, we estimate

$$\|u_\alpha(t) - u_{\alpha,\varepsilon}(t)\|_{L^q(w)}$$

$$
\begin{aligned}
&= \left\| u_\alpha^{(0)}(t) + \sum_{n=0}^{\infty} d_\alpha^{(n)}(t) - u_{\alpha,\varepsilon}^{(0)}(t) - \sum_{n=0}^{\infty} d_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \\
&\leq \left\| u_\alpha^{(0)}(t) + \sum_{n=0}^{N_\delta - 1} d_\alpha^{(n)}(t) - u_{\alpha,\varepsilon}^{(0)}(t) - \sum_{n=0}^{N_\delta - 1} d_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \\
&\qquad + \left\| \sum_{n=N_\delta}^{\infty} d_\alpha^{(n)}(t) \right\|_{L^q(w)} + \left\| \sum_{n=N_\delta}^{\infty} d_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \\
&\leq \left\| u_\alpha^{(N_\delta)}(t) - u_{\alpha,\varepsilon}^{(N_\delta)}(t) \right\|_{L^q(w)} + \delta.
\end{aligned}
$$

So we are left to show that

$$
\left\| u_\alpha^{(N_\delta)}(t) - u_{\alpha,\varepsilon}^{(N_\delta)}(t) \right\|_{L^q(w)} \to 0
$$

for $\varepsilon \to 0$. In order to do this, we are showing via an induction argument that in fact for all $n \in \mathbb{N}$

$$
\left\| u_\alpha^{(n)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \to 0, \quad \varepsilon \to 0.
$$

From Theorem 6.14 we know for each fixed $t \in [0, T]$

$$
\left\| u_\alpha^{(0)}(t) - u_{\alpha,\varepsilon}^{(0)}(t) \right\|_{L^q(w)} = \| (S_t^\alpha - S_t^{\alpha,\varepsilon}) f \|_{L^q(w)} \to 0
$$

for $\varepsilon \to 0$. Now for $n \in \mathbb{N}$, as our induction hypothesis we assume that we have shown the claim for $n-1$, more precisely

$$
\lim_{\varepsilon \to 0} \left\| u_{\alpha,\varepsilon}^{(n-1)}(t) - u_\alpha^{(n-1)}(t) \right\|_{L^q(w)} = 0
$$

for all $t \in [0, T]$. For the transition $n - 1 \to n$ we calculate

$$
\begin{aligned}
&\left\| u_\alpha^{(n)}(t) - u_{\alpha,\varepsilon}^{(n)}(t) \right\|_{L^q(w)} \\
&= \left\| u_\alpha^{(0)}(t) - u_{\alpha,\varepsilon}^{(0)}(t) + \eta \int_0^t S_{t-s}^{\alpha,\varepsilon} (u_{\alpha,\varepsilon}^{(n-1)}(s))^{1+\beta} \, \mathrm{d}s - \eta \int_0^t S_{t-s}^{\alpha} (u_{\alpha}^{(n-1)}(s))^{1+\beta} \, \mathrm{d}s \right\|_{L^q(w)} \\
&\leq \left\| u_\alpha^{(0)}(t) - u_{\alpha,\varepsilon}^{(0)}(t) \right\|_{L^q(w)} + \eta \left\| \int_0^t S_{t-s}^{\alpha,\varepsilon} (u_{\alpha,\varepsilon}^{(n-1)}(s))^{1+\beta} \, \mathrm{d}s - \int_0^t S_{t-s}^{\alpha} (u_{\alpha}^{(n-1)}(s))^{1+\beta} \, \mathrm{d}s \right\|_{L^q(w)}.
\end{aligned}
$$

The first term goes to zero as mentioned above. For the second term we have

$$
\eta \left\| \int_0^t S_{t-s}^{\alpha,\varepsilon} (u_{\alpha,\varepsilon}^{(n-1)}(s))^{1+\beta} \, \mathrm{d}s - \int_0^t S_{t-s}^{\alpha} (u_{\alpha}^{(n-1)}(s))^{1+\beta} \, \mathrm{d}s \right\|_{L^q(w)}
$$

$$= \eta \left\| \int_0^t (S_{t-s}^{\alpha,\varepsilon} - S_{t-s}^{\alpha})(u_\alpha^{(n-1)}(s))^{1+\beta}\,\mathrm{d}s + \int_0^t S_{t-s}^{\alpha,\varepsilon}((u_{\alpha,\varepsilon}^{(n-1)}(s))^{1+\beta} - (u_\alpha^{(n-1)}(s))^{1+\beta})\,\mathrm{d}s \right\|_{L^q(w)}$$
$$=: \eta \| I_{1,n}(t,\varepsilon) + I_{2,n}(t,\varepsilon)\|_{L^q(w)}.$$

It holds

$$\|I_{1,n}(t,\varepsilon)\|_{L^q(w)} \leq \int_0^t \left\| (S_{t-s}^{\alpha,\varepsilon} - S_{t-s}^{\alpha})(u_\alpha^{(n-1)}(s))^{1+\beta}\right\|_{L^q(w)} \mathrm{d}s.$$

Because of Lemma 7.12 and 7.5 we know that $(u_\alpha^{(n-1)}(s))^{1+\beta} \in L^2 \cap L^p(w)$. So by Theorem 6.14 and dominated convergence the integral converges to zero if there is an integrable majorant that holds for all ε. This is the fact, as follows.

$$\begin{aligned}
&\left\| (S_{t-s}^{\alpha,\varepsilon} - S_{t-s}^{\alpha})(u_\alpha^{(n-1)}(s))^{1+\beta}\right\|_{L^q(w)} \\
&\leq \|S_{t-s}^{\alpha,\varepsilon} - S_{t-s}^{\alpha}\|_{L^p(w)\to L^q(w)} \left\| (S_s^\alpha f)^{1+\beta}\right\|_{L^p(w)} \\
&= \|S_{t-s}^{\alpha,\varepsilon} - S_{t-s}^{\alpha}\|_{L^p(w)\to L^q(w)} \|S_s^\alpha f\|_{L^q(w)}^{1+\beta} \\
&\leq C(\kappa,q,T)^{1+\beta} \|S_{t-s}^{\alpha,\varepsilon} - S_{t-s}^{\alpha}\|_{L^p(w)\to L^q(w)} \|f\|_{L^\kappa(w)}^{1+\beta} s^{z(\kappa,q)(1+\beta)} \\
&\leq C(\kappa,q,T)^{1+\beta} \left(\|S_{t-s}^{\alpha,\varepsilon}\|_{L^p(w)\to L^q(w)} + \|S_{t-s}^{\alpha}\|_{L^p(w)\to L^q(w)} \right) \|f\|_{L^\kappa(w)}^{1+\beta} s^{z(\kappa,q)(1+\beta)} \\
&\leq 2C(p,q,T) C(\kappa,q,T)^{1+\beta} \|f\|_{L^\kappa(w)}^{1+\beta} (t-s)^{z(p,q)} s^{z(\kappa,q)(1+\beta)},
\end{aligned}$$

where we have used $p(1+\beta) = q$ Lemma 7.5 and Theorem 6.12. Integrability in s is given because of $z(p,q) > -1$ and $z(\kappa,q) \geq 0$.

One obtains

$$\lim_{\varepsilon\to 0} \|I_{1,n}(t,\varepsilon)\|_{L^q(w)} = \int_0^t \lim_{\varepsilon\to 0} \left\| (S_{t-s}^{\alpha,\varepsilon} - S_{t-s}^{\alpha})(u_\alpha^{(n-1)}(s))^{1+\beta}\right\|_{L^q(w)} \mathrm{d}s = 0.$$

We are left to deal with $\|I_{2,n}(t,\varepsilon)\|_{L^q(w)}$. In order to do this, we use nonnegativity of semigroups and estimate (7.6) to obtain the upper bound

$$\begin{aligned}
&\int_0^t \|S_{t-s}^{\alpha,\varepsilon}\|_{L^p(w)\to L^q(w)} \left\| (u_{\alpha,\varepsilon}^{(n-1)}(s))^{1+\beta} - (u_\alpha^{(n-1)}(s))^{1+\beta}\right\|_{L^p(w)} \mathrm{d}s \\
&\quad \leq (1+\beta) \int_0^t \|S_{t-s}^{\alpha,\varepsilon}\|_{L^p(w)\to L^q(w)} \\
&\qquad \cdot \left\| \max\{u_{\alpha,\varepsilon}^{(n-1)}(s), u_\alpha^{(n-1)}(s)\}^\beta \left(u_{\alpha,\varepsilon}^{(n-1)}(s) - u_\alpha^{(n-1)}(s)\right)\right\|_{L^p(w)} \mathrm{d}s \\
&\quad \leq (1+\beta) \int_0^t \|S_{t-s}^{\alpha,\varepsilon}\|_{L^p(w)\to L^q(w)} \left\| \max\{S_s^{\alpha,\varepsilon} f, S_s^\alpha f\}^\beta \right\|_{L^{\frac{q}{\beta}}(w)} \left\| u_{\alpha,\varepsilon}^{(n-1)}(s) - u_\alpha^{(n-1)}(s)\right\|_{L^q(w)} \mathrm{d}s \\
&\quad \leq (1+\beta) C(p,q,T) \\
&\qquad \cdot \int_0^t (t-s)^{z(p,q)} \|\max\{S_s^{\alpha,\varepsilon} f, S_s^\alpha f\}\|_{L^q(w)}^\beta \left\| u_{\alpha,\varepsilon}^{(n-1)}(s) - u_\alpha^{(n-1)}(s)\right\|_{L^q(w)} \mathrm{d}s
\end{aligned}$$

$$\leq (1+\beta)C(p,q,T)C(\kappa,q,T)^{\beta} \int_0^t (t-s)^{z(p,q)} s^{\beta z(\kappa,q)} \left\| u_{\alpha,\varepsilon}^{(n-1)}(s) - u_{\alpha}^{(n-1)}(s) \right\|_{L^q(w)} \mathrm{d}s.$$

Here we have used Theorem 6.12 again and Hölder's inequality, note that $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{q}$. Also remember that $(u_{\alpha,\varepsilon}^{(n-1)}(s))^{1+\beta}, (u_{\alpha}^{(n-1)}(s))^{1+\beta} \in L^p(w)$ according to Corollary 7.6. As in the previous investigation of I_1, we need to find an integrable majorant. Because of $f \in L^{\kappa}(w)$ and

$$\left\| u_{\alpha,\varepsilon}^{(n-1)}(s) - u_{\alpha}^{(n-1)}(s) \right\|_{L^q(w)} \leq \|S_s^{\alpha,\varepsilon} f\|_{L^q(w)} + \|S_s^{\alpha} f\|_{L^q(w)}$$

due to Lemma 7.5, this majorant is given by

$$2C(\kappa,q,T)\|f\|_{L^{\kappa}(w)} (t-s)^{z(p,q)} s^{z(\kappa,q)(1+\beta)},$$

which is again integrable because of the fact that $z(p,q) > -1$ and $z(\kappa,q) \geq 0$.

Interchanging limit and integral again, we have shown

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \|I_{2,n}(t,\varepsilon)\|_{L^q(w)} \\
&\leq (1+\beta)C(p,q,T)C(\kappa,q,T)^{\beta} \int_0^t (t-s)^{z(p,q)} s^{\beta z(\kappa,q)} \lim_{\varepsilon\to 0} \left\| u_{\alpha,\varepsilon}^{(n-1)}(s) - u_{\alpha}^{(n-1)}(s) \right\|_{L^q(w)} \mathrm{d}s \\
&= 0
\end{aligned}$$

because of the induction hypothesis. This completes the proof. □

8. Approximation of the measure-valued process

In Theorem 7.13 we have shown the convergence of solutions of the nonlinear integral equations (7.3) and (7.4)

$$\|u_\alpha(t) - u_{\alpha,\varepsilon}(t)\|_{L^p(w)} \to 0$$

in the weighted space for fixed $t \in [0, T]$ and initial data $f \in \Phi^p \cap L^\kappa(w)$ satisfying $\|f\|_{L^\kappa(w)} \leq C_T$. Now we are able to prove a convergence result for the associated superprocesses $X^{\alpha,\varepsilon}$ and X^α. In a first step, the convergence of the Laplace transforms will be shown. With this in place, in the second section a result from the theory of random measures will be exploited to obtain the convergence of the processes in the sense of vague convergence in distribution for fixed $t \geq 0$.

8.1. Convergence of the Laplace transforms

By the Laplace transition functional, the Laplace transforms of the associated stochastic processes are given by

$$\begin{aligned} \mathbb{E}_\mu e^{-\langle X_t^\alpha, f\rangle} &= e^{-\langle \mu, u_\alpha(t)\rangle}, \\ \mathbb{E}_\mu e^{-\langle X_t^{\alpha,\varepsilon}, f\rangle} &= e^{-\langle \mu, u_{\alpha,\varepsilon}(t)\rangle}, \end{aligned} \tag{8.1}$$

where $\mu = X_0^\alpha = X_0^{\alpha,\varepsilon}$, refer to (1.4). The notation $\langle \mu, f\rangle = \int f \,\mathrm{d}\mu$ for a measure μ and a function f will be used from now on. Also for a process $(X_t)_{t\geq 0}$ we write $\mathbb{E}_\mu(\cdot) := \mathbb{E}(\,\cdot\,|X_0 = \mu)$.

Our aim is to show the convergence of the Laplace transforms $\mathbb{E}_\mu e^{-\langle X_t^{\alpha,\varepsilon}, f\rangle}$.

Lemma 8.1. *Assume the conditions of Theorem 7.13. Furthermore, let μ a measure with density $\mu(\cdot)$ such that $\mu(\cdot)|\cdot|^{\frac{1}{p}} \in L^{p'}(\mathbb{R}^3)$. Then it holds*

$$e^{-\langle \mu, u_{\alpha,\varepsilon}(t)\rangle} \to e^{-\langle \mu, u_\alpha(t)\rangle}$$

for $\varepsilon \to 0$ and every $t \in [0, T]$, provided $\|f\|_{L^\kappa(w)} \leq C_T$.

Proof. We estimate with Hölder's inequality

$$\begin{aligned}
|\langle \mu, u_{\alpha,\varepsilon}(t)\rangle - \langle \mu, u_\alpha(t)\rangle| &= \left| \int_{\mathbb{R}^3} u_{\alpha,\varepsilon}(t,x)\mu(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} u_\alpha(t,x)\mu(x)\,\mathrm{d}x \right| \\
&\le \int_{\mathbb{R}^3} |u_{\alpha,\varepsilon}(t,x) - u_\alpha(t,x)|\mu(x)\,\mathrm{d}x \\
&\le \left(\int_{\mathbb{R}^3} |u_{\alpha,\varepsilon}(t,x) - u_\alpha(t,x)|^p w(x)\,\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^3} \mu(x)^{p'} |x|^{\frac{p'}{p}}\,\mathrm{d}x \right)^{\frac{1}{p'}} \\
&= \|u_\alpha(t) - u_{\alpha,\varepsilon}(t)\|_{L^p(w)} \left\| \mu(\cdot)|\cdot|^{\frac{1}{p}} \right\|_{L^{p'}}.
\end{aligned}$$

Due to the assumptions, the last expression tends to zero for $\varepsilon \to 0$. So we have

$$\langle \mu, u_{\alpha,\varepsilon}(t)\rangle \to \langle \mu, u_\alpha(t)\rangle$$

and by continuity

$$e^{-\langle \mu, u_{\alpha,\varepsilon}(t)\rangle} \to e^{-\langle \mu, u_\alpha(t)\rangle},$$

so the proof is complete. □

Corollary 8.2. *For $\alpha \ge 0$ and $\varepsilon \in (0,1)$ let X^α and $X^{\alpha,\varepsilon}$ the measure-valued processes from Corollary 3.30, with $X_0^\alpha = X_0^{\alpha,\varepsilon} =: \mu$, where μ is a measure with density $\mu(\cdot)$ satisfying $\mu(\cdot)|\cdot|^{\frac{1}{p}} \in L^{p'}(\mathbb{R}^3)$. Under the conditions of Theorem 7.13 it holds*

$$\mathbb{E}_\mu e^{-\langle X_t^{\alpha,\varepsilon}, f\rangle} \to \mathbb{E}_\mu e^{-\langle X_t^\alpha, f\rangle}$$

for $\varepsilon \to 0$ and every $t \in [0,T]$.

Proof. By assumption, the conditions of Lemma 8.1 are fulfilled. Using the Laplace transition formula (8.1) we obtain directly

$$\mathbb{E}_\mu^{-\langle X_t^{\alpha,\varepsilon}, f\rangle} = e^{-\langle \mu, u_{\alpha,\varepsilon}(t)\rangle} \to e^{-\langle \mu, u_\alpha(t)\rangle} = \mathbb{E}_\mu e^{-\langle X_t^\alpha, f\rangle}$$

for $\varepsilon \to 0$ and every fixed t. □

This completes the analysis of the Laplace transforms. We will use this result to obtain a mode of distributional convergence of the measure-valued processes in the next section.

8.2. Convergence of random measures

Our aim is to deduce a meaningful mode of convergence for the measure-valued processes $X_t^{\alpha,\varepsilon} \to X_t^\alpha$, $\varepsilon \to 0$. For fixed $t \ge 0$, we can consider $X_t^{\alpha,\varepsilon}$ and X_t^α as random measures in the space $\mathcal{M} := \mathcal{M}(\mathbb{R}^3)$ of all measures on $\mathbb{R}^3$. The existence theory of the measure-values processes of Chapter 3 was formulated in the underlying space $\mathbb{R}^3 \setminus \{0\}$, but we can continuously embed this space into $\mathbb{R}^3$.

We introduce the *vague convergence in distribution* on $\mathcal{M}$: For a sequence $\xi, \xi_1, \xi_2, \ldots \in \mathcal{M}$ we write

$$\xi_n \xrightarrow{vd} \xi$$

if $\mathbb{E}f(\xi_n) \to \mathbb{E}f(\xi)$ for all bounded functions f on $\mathcal{M}$ which are continuous with respect to the *vague topology*. This topology is generated by the integration maps $\pi_g : \mu \mapsto \langle \mu, g \rangle = \int g \, \mathrm{d}\mu$ for bounded continuous functions with bounded support g. [29, p. 109] The following theorem describes the connection between the convergence of Laplace transforms and the vd-convergence.

Theorem 8.3. *[29, Theorem 4.11]. Let $\xi, \xi_1, \xi_2, \ldots \in \mathcal{M}$ a sequence of random measures. Suppose that for all continuous functions $f : \mathbb{R}^3 \to [0,1]$ with bounded support it holds*

$$\mathbb{E}e^{-\langle \xi_n, f \rangle} \to \mathbb{E}e^{-\langle \xi, f \rangle}, \quad n \to \infty. \tag{8.2}$$

Then the sequence $(\xi_n)_{n \in \mathbb{N}}$ convergences in the sense of vague convergence in distribution, i.e.

$$\xi_n \xrightarrow{vd} \xi, \quad n \to \infty.$$

Concerning the measures $X_t^{\alpha,\varepsilon}$ and X_t^α for fixed $t \geq 0$, remember that we already showed the convergence of Laplace transforms in Corollary 8.2, so it seems that with Theorem 8.3 we can almost immediately deduce vd-convergence. But some care is needed because of the norm bound on f in Lemma 8.1, which *a priori* does not admit the whole class of functions required in the condition of Theorem 8.3. So we will first prove a localized convergence result and then later extend it to the whole space. This requires to choose a more abstract approach before we can return to $X_t^{\alpha,\varepsilon}$ and X_t^α in Theorem (8.9).

The following definition introduces a class of set systems in the general context of metric spaces that will be needed to apply the theory of random measures from [29, Chapter 4].

Definition 8.4. Let $\mathcal{S}$ a separable and complete metric space.

(i) A *semi-ring* over $\mathcal{S}$ is a class $\mathcal{I}$ of subsets of $\mathcal{S}$ that is closed under finite intersections and such that every proper difference in $\mathcal{I}$ is a finite disjoint union of $\mathcal{I}$-sets. [29, p. 16]

(ii) A semi-ring $\mathcal{I}$ is called *dissecting*, if every open set $U \in \mathcal{S}$ is a countable union of sets in I and every bounded set $B \in \mathcal{S}$ is covered by finitely many sets in $\mathcal{I}$. [29, p. 24]

We now introduce a set system in $\mathcal{B}(\mathbb{R}^3)$, consisting of sets which are small enough in an appropriate sense. This system will turn out to be a dissecting semiring over $\mathbb{R}^3$. It will help us to deal with the $L^\kappa(w)$ norm bound on the initial data f of the nonlinear equations.

Definition 8.5. Let $C > 0$.

(i) With $\mathcal{I}_C$ we denote the class of all $I \in \mathcal{B}(\mathbb{R}^3)$, such that there is a continuous cutoff function $\chi_I : \mathbb{R}^3 \to [0,1]$ satisfying $\chi_I = 1$ on I, $\chi_I = 0$ outside a compact set containing I and

$$\|\chi_I\|_{L^\kappa(w,\mathbb{R}^3)} \leq C. \tag{8.3}$$

(ii) For a cutoff function χ_I corresponding to a set $I \in \mathcal{I}_C$ define the localized measure

$$\mu^{\chi_I}(M) := \int_{\mathbb{R}^3} \mathbf{1}_M \chi_I \, \mathrm{d}\mu.$$

By Urysohn's Lemma, $\mathcal{I}_C$ contains for example all open balls $B \subset \mathbb{R}^3$ with radius small enough. Consequently we have

Corollary 8.6. *Let $C > 0$. The class $\mathcal{I}_C$ defined above is a dissecting semi-ring over $\mathbb{R}^3$.*

Now we show that under the additional constraint of the norm bound (8.3), the statement of Theorem 8.3 remains valid, at least for suitable restrictions of the random measures.

Lemma 8.7. *Let $\xi, \xi_1, \xi_2, \ldots \in \mathcal{M}$ a sequence of random measures. Suppose that*

$$\mathbb{E}e^{-\langle \xi_n, f\rangle} \to \mathbb{E}e^{-\langle \xi, f\rangle}, \quad n \to \infty,$$

for all continuous functions $0 \leq f \leq 1$ satisfying the norm bound (8.3) with a constant $C > 0$. Then it holds for every $I \in \mathcal{I}_C$ with cutoff function χ_I as in Definition 8.5(i)

$$\xi_n^{\chi_I} \xrightarrow{vd} \xi^{\chi_I},$$

here we used the notation for restricted measures from Definition 8.5(ii).

Proof. Let $\widehat{C}$ the set of all continuous functions from $\mathbb{R}^d$ to $[0,1]$ and $g \in \widehat{C}$ arbitrary. For $I \in \mathcal{I}_C$ the function $g\chi_I$ is continuous and $[0,1]$-valued. It also holds $g\chi_I \leq \chi_I$, so $g\chi_I$ satisfies the norm bound (8.3). Consequently, $f := g\chi_I$ fulfills the assumption and thus we have

$$\mathbb{E}e^{-\langle \xi_n, g\chi_I\rangle} \to \mathbb{E}e^{-\langle \xi, g\chi_I\rangle}.$$

But for every measure μ

$$\mathbb{E}e^{-\langle \mu, g\chi_I\rangle} = \mathbb{E}\left[-\exp\int_{\mathbb{R}^3} g\chi_I \, \mathrm{d}\mu\right] = \mathbb{E}\left[-\exp\int_{\mathbb{R}^3} g \, \mathrm{d}\mu^{\chi_I}\right] = \mathbb{E}e^{-\langle \mu^{\chi_I}, g\rangle}.$$

It follows

$$\mathbb{E}e^{-\langle \xi_n^{\chi_I}, g\rangle} \to \mathbb{E}e^{-\langle \xi^{\chi_I}, g\rangle}.$$

Since $g \in \widehat{C}$ was chosen arbitrarily, this implies

$$\xi_n^{\chi_I} \xrightarrow{vd} \xi^{\chi_I}$$

by Theorem 8.3. □

Due to the dissecting property of $\mathcal{I}_C$, it is possible to extend this convergence to the whole space.

Lemma 8.8. *Under the conditions of Lemma 8.7, it holds*

$$\xi_n \xrightarrow{vd} \xi, n \to \infty.$$

Proof. According to the definition of vague convergence in distribution, from

$$\xi_n^{\chi_I} \xrightarrow{vd} \xi^{\chi_I}$$

it follows that $\mathbb{E}f(\xi_n^{\chi_I}) \to \mathbb{E}f(\xi^{\chi_I})$ for all bounded and vaguely continuous functions f on $\mathcal{M}$. In particular, this holds for the projections $\pi_g : \mu \mapsto \langle \mu, g\rangle$, where $g : \mathbb{R}^3 \to [0, \infty)$ is a bounded continuous function with bounded support. Denote this class of functions by C_0^b and fix one $g \in C_0^b$. Since

$$\pi_g(\mu^{\chi_I}) = \int_{\mathbb{R}^3} g \, \mathrm{d}\mu^{\chi_I} = \int_{\mathbb{R}^3} g\chi_I \, \mathrm{d}\mu = \pi_{g\chi_I}(\mu)$$

for every measure $\mu \in \mathcal{M}$, it follows

$$\mathbb{E}\pi_{g\chi_I}(\xi_n) \to \mathbb{E}\pi_{g\chi_I}(\xi).$$

Now we make use of the dissecting property of $\mathcal{I}_C$. Because g has bounded support, we can find a finite number of cutoff functions $\chi_1, \ldots, \chi_N$, $N \in \mathbb{N}$, satisfying the norm bound (8.3), such that

$$\operatorname{supp} g \subset \bigcup_{j=1}^{n} \operatorname{supp} \chi_j.$$

Without loss of generality, we can assume that the χ_j form a partition of unity on $\operatorname{supp} g$, i.e. $\chi_1 + \ldots + \chi_N \equiv 1$ on $\operatorname{supp} g$. Consequently

$$g = \sum_{j=1}^{n} g\chi_j.$$

We obtain the convergence

$$\mathbb{E}\pi_g(\xi_n) = \sum_{j=1}^{n} \mathbb{E}\pi_{g\chi_j}(\xi_n) \to \sum_{j=1}^{n} \mathbb{E}\pi_{g\chi_j}(\xi) = \mathbb{E}\pi_g(\xi), \quad n \to \infty.$$

Now we proceed as in [29, Theorem 2.2]. The set $\mathcal{F}$ of all functions, such that $\mathbb{E}f(\xi_n) \to \mathbb{E}f(\xi)$, is a monotone class, this means $f, h \in \mathcal{F}$, $a \in \mathbb{R}$, implies $af + h \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $f_n \uparrow f$, implies $f \in \mathcal{F}$. The latter statement follows from the monotone convergence theorem.

As we have shown, $\pi_g \in \mathcal{F}$ for all $g \in C_0^b$. Using the monotone class theorem [29, Lemma 1.2], we have $\xi_n \xrightarrow{d} \xi$ on the σ-field Σ generated by the projections $\pi_g, g \in C_0^b$. But according to [29, Theorem 4.7], Σ coincides with the Borel σ-field corresponding to the vague topology. So we conclude

$$\xi_n \xrightarrow{vd} \xi$$

and the proof is complete. □

With the results of Lemma 8.7 and Lemma 8.7 for a general sequence of random measures in $\mathcal{M}(\mathbb{R}^3)$ in place, we are now able to apply these tools to our superprocesses $X^{\alpha,\varepsilon}$ and X^{α}. This will yield the desired vd-convergence for fixed t.

Theorem 8.9. *For $T > 0$ let $C_T > 0$ as in Theorem 7.13. Let p, β satisfying* (7.5) *and $f \in \Phi^p \cap L^\kappa(w)$ with κ satisfying $z(\kappa, q) > 0$ and $\|f\|_{L^\kappa(w)} \le C_T$. Let $u_{\alpha,\varepsilon}, u_\alpha$ the solutions of the nonlinear problems* (7.3) *and* (7.4) *with $u_{\alpha,\varepsilon}(0,\cdot) = u_\alpha(0,\cdot) = f$ and the corresponding stochastic processes $X_t^{\alpha,\varepsilon}, X_t^\alpha$ given by Corollary 3.30. Furthermore, let μ a measure satisfying the conditions of Lemma 8.1. Then it holds for every $t \in [0,T]$ and $\varepsilon \to 0$*

$$X_t^{\alpha,\varepsilon_n} \xrightarrow{vd} X_t^\alpha,$$

where $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ is an arbitrary zero sequence.

Proof. According to Corollary 8.2 and Theorem 7.13, choosing a zero sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$, we have the convergence of Laplace transforms

$$\mathbb{E}_\mu e^{-\langle X_t^{\alpha,\varepsilon_n}, f\rangle} \to \mathbb{E}_\mu e^{-\langle X_t^{\alpha}, f\rangle}, \tag{8.4}$$

for all $f \in \Phi^p \cap L^\kappa(w)$ with $\|f\|_{L^\kappa(w)} \le C_T$. Remember that the space Φ^p consists of $L^p(w)$ functions which are continuous on $\mathbb{R}^3 \setminus \{0\}$. So in particular, (8.4) holds for all continuous functions $f : \mathbb{R}^3 \to [0,1]$ with $\operatorname{supp} f \in \mathcal{I}_C$ for a sufficiently small C depending on T. Thus, using Lemmas 8.7 and 8.8, the assertion follows. □

9. Outlook

We have shown that the super-Brownian motion with point source $(X_t^\alpha)_{t\geq 0}$ related to the partial differential equation (1.6) can be approximated by a family of processes with short-range interaction in a certain sense. This is the central result of this work. In this last chapter, we discuss open problems and perspectives for further research. One important aspect are possible stronger modes of convergence. We will also take a look at the restrictions we had to impose on the dimension and the parameters α and β and explain the problems that arise in other cases.

9.1. Stronger modes of convergence

In Chapter 8, we were able to employ the convergence of solutions of the nonlinear equations to show convergence of the Laplace transforms of $X_t^{\alpha,\varepsilon}$ towards the Laplace transform of X_t^α. This again allowed us to deduce vague convergence in distribution

$$X_t^{\alpha,\varepsilon} \xrightarrow{vd} X_t^\alpha, \quad \varepsilon \to 0,$$

for fixed $t \in [0, T]$. Ideally we would have a kind of uniformity in time t, to be able to transfer properties like the path regularity of $X^{\alpha,\varepsilon}$, which was shown in Section 4.2, to the limit X^α. We briefly outline a strategy how this could be achieved. This is smilar to the proceding in [27, p. 314].

Based on our distributional convergence result for a fixed $t \in [0, T]$, in a subsequent step we would need to prove vague convergence of the finite-dimensional distributions, i.e.

$$(X_{t_0}^{\alpha,\varepsilon}, \dots, X_{t_n}^{\alpha,\varepsilon}) \xrightarrow{vd} (X_{t_0}^\alpha, \dots, X_{t_n}^\alpha), \quad \varepsilon \to 0,$$

for all $(t_0, \dots, t_n) \in [0, T]$, $n \in \mathbb{N}$. Now assume we could show $((X_t^{\alpha,\varepsilon})_{t\in[0,T]})_{\varepsilon\in(0,1)}$ is relatively compact with respect to the Skorokhod topology. Then this sequence has cluster points in the Skorokhod space and beacuse of the convergence of finite-dimensional distributions the cluster point is unique. This is equivalent with convergence in the Skorokhod topology and would allow us to prove path properties for the limit process X^α.

9.2. The two-dimensional situation

Until now we have restricted ourselves to the three-dimensional case. However, as stated in the introduction, the one-parameter family of self-adjoint extensions $(-\Delta_\alpha)_{\alpha\in\mathbb{R}}$ exists in the two-dimensional case as well [4, p. 2-3]. In the work of Fleischmann and Mueller [20], well-posedness of equation (1.6) is also shown in the space $L^p(w,\mathbb{R}^2)$ note that here we have the weight

$$w(x) = |x|^{-\frac{d-1}{2}} = |x|^{-\frac{1}{2}}$$

and p is restricted to $(1-\frac{\beta}{3})^{-1} < p < 3$ in this case. This condition is weaker than in the case $d = 3$. This even allows to admit $\beta = 1$ here, in contrary to the three-dimensional situation, where $\beta < 1$ is a necessary condition. [20, p. 753] The existence of the corresponding super-Brownian motion $(X_t^\alpha)_{t\geq 0}$ with point source is also shown [20, section 4]. In complete analogy to our proceding in Chapter 3, for fixed $\varepsilon > 0$ we can obtain well-posedness of the equation 3.1 with

$$-H_{\alpha,\varepsilon} = -\Delta + V_{\alpha,\varepsilon}$$

where $V_{\alpha,\varepsilon}$ is a suitably scaled indicator function of a centered ball as in (2.25). Then the existence of the corresponding superprocess $(X_t^{\alpha,\varepsilon})_{t\geq 0}$ can be proven as in Section 3.5.

But when we look at the spectral properties, we encounter an important difference between the cases. Remember that for $d = 3$ and $\alpha \geq 0$ the spectrum of $-\Delta_\alpha$ is purely absolutely continuous on the nonnegative real axis. In particular, there are no negative eigenvalues. The same holds for $-H_{\alpha,\varepsilon}$ if the function $V_{\alpha,\varepsilon}$ is scaled appropriately, as shown in Theorem 2.5. This is fundamentally different in the case $d = 2$: For every $\alpha \in \mathbb{R}$ there is precisely one negative simple eigenvalue of $-\Delta_\alpha$ [4, Theorem 5.4]. The operator $-H_{\alpha,\varepsilon}$ also has at least one negative eigenvalue, which does not vanish even when the scaling of $V_{\alpha,\varepsilon}$ is chosen arbitrarily small. In fact, for every function $V \geq 0$ with compact support and positive $L^1(\mathbb{R}^2)$-norm, the two-dimensional operator $-\Delta - V$ has one or more negative eigenvalues [32, Theorem 2.22]. However, as in the three-dimensional case the spectrum is contained on the real axis.

The presence of negative eigenvalues leads to difficulties when dealing with resolvent estimates. Due to this issue, the proof of two-dimensional $L^p(w)$–$L^q(w)$ estimates needed for the uniform convergence of solutions could not be completed in this work. This also affects the case $d = 3$, $\alpha < 0$, where we have a negative eigenvalue as well. In the appendix we give first estimates for the resolvents of $-H_{\alpha,\varepsilon}$, but they are incomplete because the case where the negative eigenvalue has to be considered is not covered there. These preliminary estimates may be helpful as a starting point for further research.

Let us discuss how the negative eigenvalue disturbs the analysis of the resolvents. Similar to the three-dimensional case, in the representation of the resolvent $R_\lambda^{\alpha,\varepsilon}$ the expression

$$A_\lambda^\varepsilon [1 + B_\lambda^\varepsilon]^{-1} C_\lambda^\varepsilon$$

occurs, see Lemma A.6 in the Appendix. We can prove norm estimates for the operators $A_{t^{-1}\lambda}^\varepsilon$ and $C_{t^{-1}\lambda}^\varepsilon$ (Lemma A.12). Moreover, L^2 estimates are given for $[1 + B_{t^{-1}\lambda}^\varepsilon]^{-1}$ in the case where either $|\varepsilon\sqrt{t^{-1}\lambda}| < \delta_1$ or $|\varepsilon\sqrt{t^{-1}\lambda}| > \delta_2$, with suitable $0 < \delta_1 < \delta_2$, as shown after Lemma A.13. So it remains to deal with the intermediate case $\delta_1 \le |\varepsilon\sqrt{t^{-1}\lambda}| \le \delta_2$ in (A.9). However, here we encounter the fundamental difference between the two-dimensional and three-dimensional situation: The presence of negative eigenvalues of $-H_{\alpha,\varepsilon}$. Now the details follow.

Define $z = \varepsilon\sqrt{t^{-1}\lambda}$. Remember that in the three-dimensional situation we dealt with the case $\delta_1 \le |z| \le \delta_2$ by employing the Birman-Schwinger principle, Lemma 2.2: The operator B_z has -1 as an eigenvalue, if and only if z is an eigenvalue of $-\Delta + P_\alpha(\varepsilon)V$. Consequently, the absence of such eigenvalues away from the nonnegative real axis allowed us to conclude that the operator-valued function $z \mapsto [1 + B_z]^{-1}$ is holomorphic on the concerning compact region in $\mathbb{C}$. This implied the desired boundedness.

In two dimensions, the operator $-\Delta + P_\alpha(\varepsilon)V$ has at least one negative eigenvalue, as explained above. Again, by the Birman-Schwinger principle, the function $z \mapsto [1 + B_z]^{-1}$ has at least one pole z_0 on the negative real axis. By choosing δ_1 and δ_2 appropriately, we were able avoid this singularity in our estimates above for sufficiently small and large z, which implies $-\delta_2 \le z_0 \le -\delta_1$. Consequently, at least one singularity is located in this intermediate region, and in contrary to the three-dimensional case, $[1 + B_z]^{-1}$ is not holomorphic there. Thus, using this approach, we cannot conclude that the norm of $[1 + B_z]^{-1}$ is uniformly bounded.

This raises the problem, that the curve Γ in (A.7) cannot simply be rescaled by t^{-1} to obtain (A.8). This is due to the fact, that Cauchy's integral theorem is only applicable if the integrand is holomorphic in the region enclosed by the curve. As we have just seen, this is not the case here.

It is left for further research to find a solution for this problem. A thorough spectral analysis of the operator $-H_{\alpha,\varepsilon}$ seems necessary to determine the number and positions of negative eigenvalues. Then a method must be found to deal with the poles in the curve integral, using for example the residue theorem. A similar problem occurs in the case $d = 3$, $\alpha < 0$, as stated in Remark 6.3.

9.3. A time-independent equation

The starting point for the construction of the measure-valued process in Chapter 3 was the nonlinear partial differential equation

$$-\Delta_\alpha u + \eta u^{1+\beta} = -\partial_t u \tag{9.1}$$

for $\alpha \in \mathbb{R}$. In the three-dimensional case, it is known that the operator $-\Delta_\alpha$ has contiunous spectrum $[0, \infty)$ and has a single negative eigenvalue $\lambda_0 := -(4\pi\alpha)^2$ if and only if $\alpha < 0$, refer to Theorem 2.10. Now we want to compare this equation to a related differential equation, where the time-derivative is replaced by a linear term λu for $\lambda \in \mathbb{R}$.

Consider

$$-\Delta_\alpha u + u^{1+\beta} = \lambda u \tag{9.2}$$

in $L^2(\mathbb{R}^3)$ with $0 < \beta < \frac{1}{2}$. This equation has been studied by Caspers and Clément [7]. To connect the theories behind the equations (9.1) and (9.2) with each other, in future research one could choose the following approach: Define

$$u(t,x) := e^{\lambda t} f(x).$$

Then it holds for every t

$$\partial_t u(t,x) = \lambda e^{\lambda t} f(x) = \lambda u(t,x).$$

Under this condition, the solution of (9.2) from [7] could be studied in the context of this work, which may lead to interesting results.

A. Preliminary resolvent estimates in two dimensions

As announced in Section 9.2, analytic properties of the resolvents and semigroups as well as preliminary resolvent estimates for the two-dimensional case are collected in this appendix. The resolvent estimates are incomplete, but may be a starting point for further research.

A.1. Elemental properties and sectoriality

We want to introduce the resolvents of the operators $-\Delta, -\Delta_\alpha$ and $-H_{\alpha,\varepsilon}, \varepsilon > 0$ for $d = 2$. As in the three-dimensional case, there are explicit representations for the kernels. They have a different structure here, given in terms of linear combinations of Bessel functions, known as Hankel functions. We start with the resolvents of the Laplacian.

Lemma A.1. *[4, p. 99] Let $\lambda \in \mathbb{C} \setminus [0, \infty) = \rho(-\Delta)$. Then the resolvent $R_\lambda = (-\Delta - \lambda I)^{-1}$ as an operator in $L^2(\mathbb{R}^2)$ has integral kernel*

$$R_\lambda(x, y) = \frac{i}{4} H_0^{(1)}(\sqrt{\lambda}|x - y|), \tag{A.1}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and order zero, given by $H_0^{(1)}(\sqrt{\lambda}x) := J_0(\sqrt{\lambda}x) + iY_0(\sqrt{\lambda}x)$, with the bessel functions

$$\begin{aligned} J_0(z) &= \frac{1}{\pi} \int_0^\pi \cos(z \cos r)\,\mathrm{d}r, \\ Y_0(z) &= \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} \cos(z \cos r) \ln(2z \sin^2 r)\,\mathrm{d}r. \end{aligned}$$

[1, 9.1.3, 9.1.18, 9.1.19]

For further analysis and estimates we need to understand how the Hankel function behaves asymptotically for small and large $x \in \mathbb{R}^2$ respectively. Heuristically, there is a logarithmic singularity at the origin, but exponential decay for $|x| \to \infty$. This is the subject of the next lemma.

Lemma A.2 (Properties of the Hankel function)**.** *Let $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $y \in \mathbb{R}$.*

(i) *For* $y \to 0$ *it holds*

$$-iH_0^{(1)}(\sqrt{\lambda}y) \approx \frac{2}{\pi}\ln(\sqrt{\lambda}y).$$

(ii) *For* $|y| \to \infty$ *it holds*

$$H_0^{(1)}(\sqrt{\lambda}y) \approx \sqrt{\frac{2}{\pi\sqrt{\lambda}y}}e^{i(\sqrt{\lambda}y-\frac{\pi}{4})}.$$

Proof. Statement (i) corresponds to [1, 9.1.8] and (ii) is [1, 9.2.3]. □

With these first estimates in place, we can prove $L^p(w)$-boundedness of the kernel $R_\lambda(\cdot)$. In particular, this implies that Lemma A.1 also holds in the $L^p(w)$-sense. Remember that $w(x) = |x|^{-\frac{1}{2}}$ in the two-dimensional case.

Lemma A.3. *For fixed* $\lambda \in \mathbb{C} \setminus [0,\infty) = \rho(-\Delta)$ *and* $p \geq 1$ *it holds*

$$R_\lambda(\cdot) \in L^p(w).$$

Proof. We have

$$\|R_\lambda(\cdot)\|^p_{L^p(w)} = 4^{-p}\int_{\mathbb{R}^2}\left|H_0^{(1)}(\sqrt{\lambda}|x|)\right|^p |x|^{-\frac{1}{2}}\,\mathrm{d}x.$$

Because of the asymptotic property around the origin from Lemma A.2(i), there is a radius $\rho(\lambda, p) = \rho > 0$ such that

$$\int_{B_\rho(0)}\left|H_0^{(1)}(\sqrt{\lambda}|x|)\right|^p |x|^{-\frac{1}{2}}\,\mathrm{d}x \approx \int_{B_\rho(0)}\left|\ln(\sqrt{\lambda}|x|)\right|^p |x|^{-\frac{1}{2}}\,\mathrm{d}x \quad = 2\pi\int_0^\rho\left|\ln(\sqrt{\lambda}r)\right|^p |r|^{\frac{1}{2}}\,\mathrm{d}r$$

which is finite because of $|\sqrt{\lambda}| > 0$.

On the complement $B_\rho(0)^c$ the function $x \mapsto H_0^{(1)}(\sqrt{\lambda}|x|)$ is bounded, this follows from the structure of J_0 and Z_0 given in Lemma A.1. Furthermore, we have exponential decay for $|x| \to \infty$ from Lemma A.2(ii) because of $\operatorname{Im}\sqrt{\lambda} > 0$. This together implies

$$\int_{B_\rho(0)^c}\left|H_0^{(1)}(\sqrt{\lambda}|x|)\right|^p |x|^{-\frac{1}{2}}\,\mathrm{d}x < \infty$$

and finishes the proof. □

Now we give a precise definition of the operator $-H_{\alpha,\varepsilon}$ in a way that allows convergence to $-\Delta_\alpha$ in the norm resolvent sense. The following definition is a summary of the situation in [4, p. 103] and [4, Theorem 5.5].

Definition A.4. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be measurable such that for a $\delta > 0$

$$\int_{\mathbb{R}^2} (1 + |x|^{2+\delta}) V(x) \, \mathrm{d}x < \infty, \quad \int_{\mathbb{R}^2} |V(x)|^{1+\delta} \, \mathrm{d}x < \infty.$$

Let

$$v(x) = |V(x)|^{1/2}, \quad u(x) = \operatorname{sgn}(x)|V(x)|^{1/2}, \quad \text{for } x \in \mathbb{R}^2$$

and let D a Hilbert-Schmidt operator in $L^2(\mathbb{R}^2)$ with integral kernel

$$D(x, y) = u(x) \ln |x - y| v(y), \quad x \neq y.$$

For $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ define

$$-H_{\alpha,\varepsilon} := -\Delta + V_{\alpha,\varepsilon} = -\Delta + P_\alpha((\ln \varepsilon)^{-1})\varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right) \tag{A.2}$$

where the polynomial $P_\alpha(z) = \mu_1 z + \mu_2(\alpha) z^2$ is given by the coefficients

$$\begin{aligned} \mu_1 &= \frac{2\pi}{(v, u)} \\ \mu_2(\alpha) &= \frac{\alpha(2\pi)^2}{(v, u)} - \frac{2\pi(v, Du)}{(v, u)^3}. \end{aligned} \tag{A.3}$$

Theorem A.5. *Let $\alpha \in \mathbb{R}$. For $\varepsilon > 0$ let $-H_{\alpha,\varepsilon}$ as in in* (A.2)*. If $\lambda \in \rho(-\Delta_\alpha)$, we have $\lambda \in \rho(-H_{\alpha,\varepsilon})$ for $\varepsilon > 0$ small enough, and $-H_\varepsilon \to -\Delta_\alpha$ in norm resolvent sense for $\varepsilon \to 0$. This means*

$$\left\| R_\lambda^{\alpha,\varepsilon} - R_\lambda^\alpha \right\|_{L^2 \to L^2} \to 0 \quad as \quad \varepsilon \downarrow 0.$$

Lemma A.6. *[4, Formulas (5.49)–(5.53), p. 103] For $\lambda \in \rho(-H_{\alpha,\varepsilon})$ the resolvent $R_\lambda^{\alpha,\varepsilon} = (-H_{\alpha,\varepsilon} - \lambda I)^{-1}$ is given by*

$$\begin{aligned} R_\lambda^{\alpha,\varepsilon} &= R_\lambda + \overline{R}_\lambda^{\alpha,\varepsilon} \\ &:= R_\lambda - P_\alpha((\ln \varepsilon)^{-1}) A_\lambda^\varepsilon [1 + B_\lambda^\varepsilon]^{-1} C_\lambda^\varepsilon \end{aligned}$$

with operators given by integral kernels

$$\begin{aligned} A_\lambda^\varepsilon(x, y) &= V(y) R_\lambda(x - \varepsilon y), \\ B_\lambda^\varepsilon(x, y) &= V(x) V(y) P_\alpha((\ln \varepsilon)^{-1}) R_{\lambda\varepsilon^2}(x - y), \\ C_\lambda^\varepsilon(x, y) &= V(x) R_\lambda(\varepsilon x - y), \end{aligned}$$

and the polynomial P_α as in Definition A.4.

Lemma A.7. *Let $1 < p < \infty$.*

(i) The Laplacian $-\Delta$ is a sectorial operator on $L^p(w)$.

(ii) For every fixed $\varepsilon > 0$, the operator $-H_{\alpha,\varepsilon}$ generates an analytic semigroup in $L^p(w)$.

Proof. This follows exactly as in the three-dimensional situation in Lemma 5.16 and Corollary 5.18 from the more general results cited there. □

This implies that the operator semigroups generated by $-\Delta$ and $-H_{\alpha,\varepsilon}$ can be represented via a curve integral as in (5.2). Note that the presence of negative eigenvalues of $-H_{\alpha,\varepsilon}$ makes it necessary to choose the curve radius $r(\Gamma) > |\lambda_0|$, where $\lambda_0 < 0$ is the smallest eigenvalue. Since the positions of the eigenvalues depend on α and ε, the same is true for the restriction on $r(\Gamma)$.

Next, we study the operator $-\Delta_\alpha$ in two dimensions and its spectrum and resolvents.

Lemma A.8. *[4, Theorem I.5.2/I.5.4]. Let $\alpha \in \mathbb{R}$. Then $\sigma(-\Delta_\alpha) = [0, \infty) \cup \{-4e^{2(-2\pi\alpha+\Psi(1))}\}$ and for $\lambda \in \rho(-\Delta_\alpha)$ the integral kernel of the resolvent $R_\lambda^\alpha = (-\Delta_\alpha - \lambda I)^{-1}$ is given by*

$$R_\lambda^\alpha(x, y) = R_\lambda(x, y) + \overline{R}_\lambda^\alpha(x, y) \tag{A.4}$$

with

$$\overline{R}_\lambda^\alpha(x, y) = -\frac{\pi/8}{2\pi\alpha - \Psi(1) + \ln(\sqrt{\lambda}/2i)} H_0^{(1)}(\sqrt{\lambda}|x|) H_0^{(1)}(\sqrt{\lambda}|y|)$$

where $\Psi(1) \in \mathbb{C}$ is a constant.

Again, $-\Delta_\alpha$ has a single negative eigenvalue λ_0, while the continuous spectrum consists of the nonnegative real axis. However, if we can verify the resolvent estimate from 5.9 for $\lambda \in \rho(-\Delta_\alpha)$ large enough, it follows that $-\Delta_\alpha$ generates an analytic semigroup and thus can be represented by the Dunford integral (5.2) [17, 4.14(6)]. So we prove

Lemma A.9. *For $p > 1$ and $\lambda \in \rho(-\Delta_\alpha)$ with $|\lambda|$ large enough and $|arg\lambda| > \theta$ for some $\theta \in (0, \frac{\pi}{2})$ it holds for all $f \in L^p(w)$*

$$\|R_\lambda^\alpha f\|_{L^p(w)} \lesssim |\lambda|^{-1} \|f\|_{L^p(w)}.$$

Proof. Due to the sectoriality of $-\Delta$ and (A.4), we only need to show the estimate for $\overline{R}_\lambda^\alpha$. So let $f \in L^p(w)$.

According to Lemma A.8

$$\|\overline{R}_\lambda^\alpha\|_{L^p(w)} = -C(\alpha, \lambda) \left| \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x|) f(x)\,\mathrm{d}x \right| \left\| H_0^{(1)}(\sqrt{\lambda}|\cdot|) \right\|_{L^p(w)} \tag{A.5}$$

Clearly the factor

$$C(\alpha, \lambda) = \frac{\pi/8}{2\pi\alpha - \Psi(1) + \ln(\sqrt{\lambda}/2i)}$$

is uniformly bounded in the range $|\lambda| > \delta$ for some δ large enough. Concerning the second factor in (A.5) we have with Hölder's inequality and the transformation $x \mapsto |\sqrt{\lambda}|^{-1}x$

$$\begin{aligned}\left|\int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x|)f(x)\,\mathrm{d}x\right| &\leq \|f\|_{L^p(w)}\left(\int_{\mathbb{R}^2}|H_0^{(1)}(\sqrt{\lambda}|x|)|^{p'}|x|^{\frac{p'}{2p}}\,\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\leq |\lambda|^{-\frac{1}{p'}-\frac{1}{4p}}\|f\|_{L^p(w)}\left(\int_{\mathbb{R}^2}|H_0^{(1)}(\sqrt{\lambda}|\sqrt{\lambda}|^{-1}|x|)|^{p'}|x|^{\frac{p'}{2p}}\,\mathrm{d}x\right)^{\frac{1}{p'}}\end{aligned}$$

The remaining integral is bounded because of the properties of the Hankel function from Lemma A.2. Furthermore, there is a bound uniformly in λ, because $\sqrt{\lambda}|\sqrt{\lambda}|^{-1}$ lies in the compact subset of the unit circle $\{e^{i\varphi} \in \mathbb{C} : \frac{\theta}{2} \leq \varphi \leq \frac{2\pi-\theta}{2}\}$.

Concerning the last factor in (A.5) we calculate with the same transformation

$$\left(\int_{\mathbb{R}^2}|H_0^{(1)}(\sqrt{\lambda}|x|)|^{p}|x|^{-\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{1}{p}} = |\lambda|^{-\frac{3}{4p}}\left(\int_{\mathbb{R}^2}|H_0^{(1)}(\sqrt{\lambda}|\sqrt{\lambda}|^{-1}|x|)|^{p}|x|^{-\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{1}{p}}$$

and again the remaining integral is bounded as described above. Summing up we have

$$\|\overline{R}_\lambda^\alpha f\|_{L^p(w)} \lesssim |\lambda|^{-\frac{1}{p'}-\frac{1}{4p}-\frac{3}{4p}}\|f\|_{L^p(w)}$$

and

$$-\frac{1}{p'} - \frac{1}{4p} - \frac{3}{4p} = -\frac{1}{p'} - \frac{1}{p} = -1$$

which concludes the proof. □

Note that we didn't need to impose an upper bound on p in Lemma A.9. Technically this is due to the fact that the logarithmic singularity at the origin remains integrable for any $p < \infty$.

We can summarize our results about the semigroup representations in analogy to Theorem 5.19.

Theorem A.10. *Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Let $1 < p < \infty$ and $f \in L^p(w)$. For the semigroups $(S_t), (S_t^\alpha)$ and $(S_t^{\alpha,\varepsilon})$ with generators $-\Delta, -\Delta_\alpha, -H_{\alpha,\varepsilon}$ respectively, it holds*

$$\begin{aligned}&\textit{(i)} \quad S_t f = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}R_\lambda f\,\mathrm{d}\lambda,\\ &\textit{(ii)} \quad S_t^\alpha f = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}R_\lambda^\alpha f\,\mathrm{d}\lambda,\\ &\textit{(iii)} \quad S_t^{\alpha,\varepsilon} f = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}R_\lambda^{\alpha,\varepsilon} f\,\mathrm{d}\lambda,\end{aligned}$$

with Γ as in Defintion 5.12, provided $r(\Gamma)$ large enough depending on α, ε.

A.2. First weighted norm estimates

We have seen that the semigroups generated by $-\Delta_\alpha$ and $-H_{\alpha,\varepsilon}$ are analytic in $L^p(w,\mathbb{R}^2)$. As a consequence, the next step to obtain convergence results as in the three-dimensional case would be to establish $L^p(w)$–$L^q(w)$ resolvent estimates, which could then be transferred to the semigroups using the representations from Theorem A.10. The estimates for $(S_t^{\alpha,\varepsilon})$ are required to be uniform in ε.

First we consider the residue term $(\overline{S}_t^\alpha)_{t\geq 0}$ given by

$$\overline{S}_t^\alpha f = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}\overline{R}_\lambda^\alpha f \,\mathrm{d}\lambda.$$

Lemma A.11. *Let $p,q>1$ and $t\in(0,T)$ for $T\geq 0$. It holds for $f\in L^p(w)$*

$$\left\|\overline{S}_t^\alpha f\right\|_{L^q(w)} \lesssim t^{\frac{3}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^p(w)}$$

with an implicit constant only depending on p, q and α.

Proof. Let $f\in L^p(w)$. By transformation of the path integral we have as in the three-dimensional case

$$\left\|\overline{S}_t^\alpha f\right\|_{L^q(w)} \leq \frac{1}{2\pi t}\int_\Gamma e^{-\lambda}\|\overline{R}_{t^{-1}\lambda}^\alpha f\|_{L^q(w)}\,\mathrm{d}\lambda.$$

Similar to (A.5) it holds

$$\|\overline{R}_{t^{-1}\lambda}^\alpha\|_{L^q(w)} = C(\alpha,\lambda,t)\left|\int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{t^{-1}\lambda}|x|)f(x)\,\mathrm{d}x\right|\left\|H_0^{(1)}(\sqrt{t^{-1}\lambda}|\cdot|)\right\|_{L^q(w)} \tag{A.6}$$

with

$$C(\alpha,\lambda,t) = \frac{\pi/8}{|2\pi\alpha-\Psi(1)+\ln(\sqrt{t^{-1}\lambda}/2i)|}.$$

Since $t^{-1}>T^{-1}$, the factor $C(\alpha,\lambda,t)$ is uniformly bounded for $r(\Gamma)$ chosen large enough as in the proof of Lemma A.9. So let's turn to the second factor in (A.6). It holds

$$\begin{aligned}\left|\int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x|)f(x)\,\mathrm{d}x\right| &\leq \|f\|_{L^p(w)}\left(\int_{\mathbb{R}^2}|H_0^{(1)}(\sqrt{\lambda}|x|)|^{p'}|x|^{\frac{p'}{2p}}\,\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\leq t^{\frac{1}{p'}+\frac{1}{4p}}\|f\|_{L^p(w)}\left(\int_{\mathbb{R}^2}|H_0^{(1)}(\sqrt{\lambda}|x|)|^{p'}|x|^{\frac{p'}{2p}}\,\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\approx t^{1-\frac{3}{4p}}\|f\|_{L^p(w)}.\end{aligned}$$

The explanation why the remaining integral is bounded can also be found in the proof of

Lemma A.9, note that $|\lambda|^{-1} \leq r(\Gamma)^{-1}$. Concerning the last factor in (A.6) we calculate

$$\left(\int_{\mathbb{R}^2} |H_0^{(1)}(\sqrt{t^{-1}\lambda}|x|)|^q |x|^{-\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{1}{q}} = t^{\frac{3}{4q}} \left(\int_{\mathbb{R}^2} |H_0^{(1)}(\sqrt{\lambda}|x|)|^q |x|^{-\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{1}{q}} \approx t^{\frac{3}{4q}}.$$

Summarizing all factors, we have

$$\left\|\overline{S}_t^{\alpha} f\right\|_{L^q(w)} \lesssim \frac{1}{2\pi t} t^{\frac{3}{4q}} t^{1-\frac{3}{4p}} \|f\|_{L^p(w)} \int_\Gamma e^{-\lambda}\,\mathrm{d}\lambda$$

and the proof is finished. $\square$

We turn our attention to $\overline{R}_\lambda^{\alpha,\varepsilon}$ and the corresponding residue term

$$\overline{S}_t^{\alpha,\varepsilon} f = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} \overline{R}_\lambda^{\alpha,\varepsilon} f\,\mathrm{d}\lambda \tag{A.7}$$

for $t \geq 0$. If we could prove a $L^p(w)$–$L^q(w)$ estimate for $\overline{R}_{t^{-1}\lambda}^{\alpha,\varepsilon}$, uniform in ε, we would obtain

$$\left\|\overline{S}_t^{\alpha,\varepsilon} f\right\|_{L^q(w)} \leq \frac{1}{2\pi t} \int_\Gamma e^{-\lambda} \left\|\overline{R}_{t^{-1}\lambda}^{\alpha,\varepsilon} f\right\|_{L^q(w)} \mathrm{d}\lambda. \tag{A.8}$$

This would give rise to a uniform norm estimate for $\overline{S}_t^{\alpha,\varepsilon} f$. Using the resolvent formula from Lemma A.6 leads to the estimate

$$\begin{aligned}\left\|\overline{R}_{t^{-1}\lambda}^{\alpha,\varepsilon} f\right\|_{L^q(w)} \leq & P_\alpha((\ln\varepsilon)^{-1}) \left\|A_{t^{-1}\lambda}^{\varepsilon}\right\|_{L^2\to L^q(w)} \left\|[1 + B_{t^{-1}\lambda}^{\varepsilon}]^{-1}\right\|_{L^2\to L^2} \\ & \cdot \left\|C_{t^{-1}\lambda}^{\varepsilon}\right\|_{L^p(w)\to L^2} \|f\|_{L^p(w)}.\end{aligned} \tag{A.9}$$

Now the operator norms in (A.9) can be dealt with separately. We have the following estimates for the terms involving the operators $A_{t^{-1}\lambda}^{\varepsilon}$ and $C_{t^{-1}\lambda}^{\varepsilon}$

Lemma A.12. *Let $p > 1$ and $1 < q \leq 2$ and $t \in (0, T)$ for $T \geq 0$. It holds uniformly in $\varepsilon > 0$*

$$\left\|A_{t^{-1}\lambda}^{\varepsilon}\right\|_{L^2\to L^q(w)} \lesssim t^{\frac{1}{2}-\frac{1}{4q}}, \tag{A.10}$$

$$\left\|C_{t^{-1}\lambda}^{\varepsilon}\right\|_{L^p(w)\to L^2} \lesssim t^{1-\frac{1}{p}}, \tag{A.11}$$

with implicit constants only depending on p, q and T.

Proof. Step 1: Let $g \in L^2$. We compute

$$\begin{aligned}\left\|A_{t^{-1}\lambda}^{\varepsilon}\right\|_{L^q(w)}^q &= \int_{\mathbb{R}^2} |x|^{-\frac{1}{2}} \left|\int_{\mathbb{R}^2} V(y) H_0^{(1)}(\sqrt{t^{-1}\lambda}|x - \varepsilon y|) g(y)\,\mathrm{d}y\right|^q \mathrm{d}x \\ &\leq \int_{\mathbb{R}^2} |x|^{-\frac{1}{2}} \left(\int_{\mathbb{R}^2} V(y) \left|H_0^{(1)}(\sqrt{t^{-1}\lambda}|x - \varepsilon y|)\right| |g(y)|\,\mathrm{d}y\right)^q \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le \|g\|_{L^2}^q \int_{\mathbb{R}^2} |x|^{-\frac{1}{2}} \left(\int_{\mathbb{R}^2} V(y) \left| H_0^{(1)}(\sqrt{t^{-1}\lambda}|x-\varepsilon y|) \right|^2 \mathrm{d}y \right)^{\frac{q}{2}} \mathrm{d}x \\
&\lesssim \|g\|_{L^2}^q \left(\int_{\mathbb{R}^2} V(y) \int_{\mathbb{R}^2} \left| H_0^{(1)}(\sqrt{t^{-1}\lambda}|x-\varepsilon y|) \right|^2 |x|^{-\frac{1}{q}} \mathrm{d}x \, \mathrm{d}y \right)^{\frac{q}{2}},
\end{aligned}
$$

where we have used the fact $\frac{q}{2} \le 1$ in the last step. Now let's focus on the inner integral. We have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left| H_0^{(1)}(\sqrt{t^{-1}\lambda}|x-\varepsilon y|) \right|^2 |x|^{-\frac{1}{q}} \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^2} \left| H_0^{(1)}(\sqrt{t^{-1}\lambda}|x|) \right|^2 |x+\varepsilon y|^{-\frac{1}{q}} \mathrm{d}x \\
&\quad = t \int_{\mathbb{R}^2} \left| H_0^{(1)}(\sqrt{\lambda}|x|) \right|^2 |\sqrt{t}x+\varepsilon y|^{-\frac{1}{q}} \mathrm{d}x \\
&\quad = t^{1-\frac{1}{2q}} \int_{\mathbb{R}^2} \left| H_0^{(1)}(\sqrt{\lambda}|x|) \right|^2 \left| |x + \frac{\varepsilon}{\sqrt{t}} y \right|^{-\frac{1}{q}} \mathrm{d}x.
\end{aligned}
$$

Because of the Hardy-Littlewood rearrangement inequality [37, Theorem 3.4], the remaining integral in the last line is bounded by

$$
\int_{\mathbb{R}^2} \left| H_0^{(1)}(\sqrt{\lambda}|x|) \right|^2 ||x|^{-\frac{1}{q}} \mathrm{d}x,
$$

which is, as in previous calculations, finite uniformly in λ because of the asymptotic properties of the Hankel function. We conclude

$$
\begin{aligned}
\left\| A_{t^{-1}\lambda}^{\varepsilon} \right\|_{L^q(w)} &\lesssim \|g\|_{L^2} \left(t^{1-\frac{1}{2q}} \int_{\mathbb{R}^2} V(y) |y|^{\frac{1}{2}} \mathrm{d}y \right)^{\frac{1}{2}} \\
&\lesssim \|g\|_{L^2} t^{\frac{1}{2}-\frac{1}{4q}}.
\end{aligned}
$$

This implies

$$
\left\| A_{t^{-1}\lambda}^{\varepsilon} \right\|_{L^2 \to L^q(w)} \lesssim t^{\frac{1}{2}-\frac{1}{4q}},
$$

so we have proven (A.10).

Step 2. In analogy to the first step, for $g \in L^p(w)$ we have

$$
C_\lambda^\varepsilon g(x) = V(x) \int_{\mathbb{R}^2} R_\lambda(\varepsilon x - y) f(y) \, \mathrm{d}y.
$$

We calculate

$$
\left\| C_{t^{-1}\lambda}^{\varepsilon} g \right\|_{L^2}^2 = \int_{\mathbb{R}^2} V(x) \left| \int_{\mathbb{R}^2} R_{t^{-1}\lambda}(\varepsilon x - y) g(y) \, \mathrm{d}y \right|^2 \mathrm{d}x
$$

$$\approx \int_{\mathbb{R}^2} V(x) \left| \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{t^{-1}\lambda}|\varepsilon x - y|)|y|^{-\frac{1}{2p}}|y|^{\frac{1}{2p}} g(y)\,\mathrm{d}y \right|^2 \mathrm{d}x$$
$$\leq \|g\|^2_{L^p(w)} \int_{\mathbb{R}^2} V(x) \left(\int_{\mathbb{R}^2} |H_0^{(1)}(\sqrt{t^{-1}\lambda}|\varepsilon x - y|)|^{p'} |y|^{\frac{p'}{2p}} \,\mathrm{d}y \right)^{\frac{2}{p'}} \mathrm{d}x. \tag{A.12}$$

The inner integral in (A.12) coincides with

$$\int_{\mathbb{R}^2} |H_0^{(1)}(\sqrt{t^{-1}\lambda}|y|)|^{p'} |y + \varepsilon x|^{\frac{p'}{2p}} \,\mathrm{d}y = t \int_{\mathbb{R}^2} |H_0^{(1)}(\sqrt{\lambda}|y|)|^{p'} |\sqrt{t}y + \varepsilon x|^{\frac{p'}{2p}} \,\mathrm{d}y.$$

Because of

$$|\sqrt{t}y + \varepsilon x| \lesssim \max\{2\pi, \sqrt{T}|y|\},$$

the properties of the path Γ and the exponential decay of the Hankel function, the integral on the right side is bounded uniformly in $\varepsilon \leq 1$, $\lambda \in \Gamma$. Clearly the outer integral in (A.12) is bounded because V has compact support. It follows

$$\left\| C^{\varepsilon}_{t^{-1}\lambda} f \right\|^2_{L^2} \lesssim t^{\frac{2}{p'}} \|g\|^2_{L^p(w)}$$

and because of $\frac{1}{p'} = 1 - \frac{1}{p}$ we have

$$\left\| C^{\varepsilon}_{t^{-1}\lambda} \right\|_{L^p(w)\to L^2} \lesssim t^{1-\frac{1}{p}},$$

which completes the proof of (A.11). □

To control the right-hand side of (A.9), we still need a uniform bound for the $L^2(\mathbb{R}^2)$ operator norm of $[1 + B^{\varepsilon}_{t^{-1}\lambda}]^{-1}$. Remember that in the three-dimensional case this is done by distinguishing the three cases

$$|\varepsilon\sqrt{t^{-1}\lambda}| \leq \delta_1,$$
$$|\varepsilon\sqrt{t^{-1}\lambda}| \geq \delta_2,$$
$$\delta_1 \leq |\varepsilon\sqrt{t^{-1}\lambda}| \leq \delta_2$$

for suitable $0 < \delta_1 < \delta_2$ in the proof of Lemma 6.2. Concerning the case where $|\varepsilon\sqrt{t^{-1}\lambda}|$ is small enough, we have in similarity to Lemma 6.1 the convergence

Lemma A.13. *[4, p. 103f, Case (d)] Let the operators $B^{\varepsilon}_{\lambda}$ and D as in Lemma A.6. Let the polynomial P_α as in* (A.3) *and* $\Psi(1)$ *as in Lemma A.8. It holds*

$$[1 + B^{\varepsilon}_{\lambda}]^{-1} = -2\pi(\ln\varepsilon)\Big[2\pi(v,u)(-\Psi(1) + \ln(\sqrt{\lambda}/2i)) + \mu_2(\alpha)(v,u)^2$$
$$+(2\pi(v,Du)/(v,u))\Big](v,\cdot)u + O(1)$$

for $\varepsilon \to 0$ *in* $L^2(\mathbb{R}^2)$.

This allows to control the operator norm of $[1 + B^{\varepsilon}_{t^{-1}\lambda}]^{-1}$ in this case, proceding as in the first part of the proof of Lemma 6.2 on page 81.

In the case $|\varepsilon\sqrt{t^{-1}\lambda}| > \delta_2$ we can estimate the operator norm of $[1 + B^{\varepsilon}_{t^{-1}\lambda}]^{-1}$, similar to the calculations leading to estimate (6.10). For $0 < \varepsilon < 1$ it holds

$$\begin{aligned}
\left\|B^{\varepsilon}_{t^{-1}\lambda}\right\|_{L^2\to L^2} &= \sup_{\|f\|_{L^2}=1} \left\|V(B^{\varepsilon}_{t^{-1}\lambda} * Vf)\right\|_{L^2} \\
&\le \sup_{\|f\|_{L^2}=1} \left\|B^{\varepsilon}_{t^{-1}\lambda} * Vf\right\|_{L^2} \\
&\le \sup_{\|f\|_{L^2}=1} \left\|B^{\varepsilon}_{t^{-1}\lambda}\right\|_{L^1} \|f\|_{L^2} \\
&= \left\|B^{\varepsilon}_{t^{-1}\lambda}(\cdot)\right\|_{L^1} \\
&\lesssim P_\alpha((\ln\varepsilon)^{-1}) \int_{\mathbb{R}^2} \left|H_0^{(1)}(\varepsilon\sqrt{t^{-1}\lambda}|y|)\right| \,\mathrm{d}y \\
&\approx \left(\frac{\sqrt{t}}{\varepsilon|\sqrt{\lambda}|}\right) \int_{\mathbb{R}^2} \left|H_0^{(1)}(\sqrt{\lambda}|\sqrt{\lambda}|^{-1}|y|)\right| \,\mathrm{d}y,
\end{aligned}$$

where we have used the transformation $y \mapsto \frac{\sqrt{t}}{\varepsilon|\sqrt{\lambda}|}$ in the last step. The remaining integral is uniformly bounded in λ as explained in the proof of Lemma A.9. We obtain

$$\left\|B^{\varepsilon}_{t^{-1}\lambda}\right\|_{L^2\to L^2} \lesssim \left(\frac{\sqrt{t}}{\varepsilon|\sqrt{\lambda}|}\right) < \delta_2^{-1}.$$

Choosing δ_2 large enough yields a uniform bound for

$$\left\|\left[1 + B^{\varepsilon}_{t^{-1}\lambda}\right]^{-1}\right\|_{L^2\to L^2}$$

in this case. The remaining case $\delta_1 \le |\varepsilon\sqrt{t^{-1}\lambda}| \le \delta_2$ can not be covered, as discussed in Section 9.2.

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