Algebraic geometry of the classical Yang-Baxter equation and its generalizations

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Abstract

In this thesis, we use an algebro-geometric approach to study solutions to a generalized version of the classical Yang-Baxter equation (CYBE) for central simple Lie algebras over arbitrary fields of characteristic 0. We assign to these solutions certain geometric data including a cohomology-free sheaf of Lie algebras on a projective curve. The application of geometric methods leads to a new proof of the Belavin-Drinfeld trichotomy, which states that non-degenerate solutions of the CYBE for complex simple Lie algebras are either elliptic, trigonometric, or rational. We give more explicit descriptions of the geometric data as well as the structure theory for solutions from each of these three classes. We also derive a purely geometric version of the Belavin-Drinfeld trichotomy which works over any field of characteristic 0. Moreover, we prove that every non-skew-symmetric solution of the generalized CYBE for central simple Lie algebras over an arbitrary field of characteristic 0 corresponds to a projective curve normalized by $\mathbb{P}^1$ and extends to a rational function by passing to an étale $\mathbb{P}^1$-scheme.

Deutsche Version

In dieser Arbeit verwenden wir einen algebro-geometrischen Zugang zum Studium von Lösungen einer Verallgemeinerung der klassischen Yang-Baxter Gleichung (KYBG) für zentraleinfache Lie Algebren über beliebigen Körpern der Charakteristik 0. Wir ordnen diesen Lösungen bestimmte geometrische Daten zu, unter anderem eine Kohomologie-freie Garbe von Lie Algebren auf einer projektiven Kurve. Unter Verwendung geometrischer Methoden führt dies zu einem neuen Beweis der Belavin-Drinfeld Trichotomie, welche besagt das nicht-entartete Lösungen der KYBG entweder elliptisch, trigonometrisch oder rational sind. Wir geben explizite Beschreibungen der geometrischen Daten und der Strukturtheorie von Lösungen aus jeder einzelnen dieser drei Klassen an. Wir leiten auch eine rein geometrische Version der Belavin-Drinfeld Trichotomie her, die über beliebigen Körpern der Charakteristik 0 gilt. Des weiteren beweisen wir, dass jede nicht schiefsymmetrische Lösung der verallgemeinerten KYBG zu einer von $\mathbb{P}^1$ normalisierten projektiven Kurve gehört und durch den Übergang zu einem étalen $\mathbb{P}^1$-Schema zu einer rationalen Funktion erweitert werden kann.

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Introduction

Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra. The generalized classical Yang-Baxter equation (GCYBE) is the functional equation

$$[r^{12}(x_1,x_2),r^{13}(x_1,x_3)] + [r^{12}(x_1,x_2),r^{23}(x_2,x_3)] + [r^{32}(x_3,x_2),r^{13}(x_1,x_3)] = 0, \quad (0.1)$$

where $r\colon U\times U \to \mathfrak{g}\otimes\mathfrak{g}$ is a meromorphic function for some connected open $U \subseteq \mathbb{C}$. The brackets on the left-hand side of the GCYBE (0.1) are defined in the triple tensor product of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$, where e.g. $r^{13}(x_1,x_3) = (r(x_1,x_3))^{13}$ and $(a\otimes b)^{13} = a\otimes 1\otimes b \in \mathrm{U}(\mathfrak{g})^{\otimes 3}$ for all $a,b\in\mathfrak{g}$. Solutions of the GCYBE (0.1) are called generalized r-matrices and are important in the theory of integrable systems; see e.g. [Mai86; BBT03; Skr06]. For instance, non-degenerate generalized r-matrices can be used to construct Lie algebra splittings, which define particularly well-behaved linear Poisson structures in the Adler-Kostant-Symes scheme; see [Skr13]. More precisely, a non-degenerate solution r of the GCYBE (0.1) defines a Lie subalgebra $\mathfrak{g}(r) \subseteq \mathfrak{g}(\!(z)\!)$, such that $\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(r)$ holds.

In this work, we study non-degenerate generalized r-matrices from an algebro-geometric perspective. To this end, we combine the main result from Ostapenko [Ost92] with well-known projectification schemes (see e.g. [Mum78; Mul90]) to assign a pair $(X,\mathscr{A})$ to any non-degenerate solution r of the GCYBE (0.1), consisting of a coherent sheaf of Lie algebras $\mathscr{A}$ on an irreducible projective curve X; see Subsection 3.1. More precisely:

- X is the completion of $\mathrm{Spec}(O)$ by a smooth point p "at infinity", where

$$O \subseteq \mathrm{Mult}(\mathfrak{g}(r)) := \{\lambda \in \mathbb{C}(\!(z)\!) \mid \lambda\mathfrak{g}(r) \subseteq \mathfrak{g}(r)\} \quad (0.2)$$

  is any unital subalgebra of finite codimension.
- $\mathscr{A}$ is the formal gluing of $\mathfrak{g}(r)$ on $\mathrm{Spec}(O) = X\setminus\{p\}$ with $\mathfrak{g}[\![z]\!]$ on the formal neighbourhood of p.

Note that we can always choose $O = \mathrm{Mult}(\mathfrak{g}(r))$, however in some situations other choices of O are more convenient.

In [Che83b], Cherednik showed that the Szegö kernel of a locally free sheaf of Lie algebras with vanishing cohomology on a smooth algebraic curve solves a geometric version of the GCYBE. This construction of geometric r-matrices was extended by Burban and Galinat in [BG18] to include torsion free sheaves of Lie algebras on singular projective curves. It turns out that the geometric datum $(X,\mathscr{A})$ associated to a non-degenerate solution r of the GCYBE (0.1) satisfies the axioms from [BG18], i.e. $\mathrm{H}^0(\mathscr{A}) = 0 = \mathrm{H}^1(\mathscr{A})$ and $\mathscr{A}|_q \cong \mathfrak{g}$ for all points q in an open neighbourhood of p; see Remark 3.3.1. Hence, it is possible to construct a geometric r-matrix ρ from $(X,\mathscr{A})$. Moreover, r can be recovered by appropriately trivializing ρ; see Theorem 3.3.3. This reveals that r is of geometric nature, which is surprising given its local analytic definition. More precisely, there exists a Riemann surface Y such that r is, up to a natural equivalence transformation, the restriction of a rational map $Y\times Y \to \mathfrak{g}\otimes\mathfrak{g}$; see Corollary 3.3.6.

If a solution r of the GCYBE is skew-symmetric (i.e. $r^{12}(x,y) = -r^{21}(y,x)$), Equation (0.1) reduces to the more commonly known classical Yang-Baxter equation (CYBE)

$$[r^{12}(x_1,x_2),r^{13}(x_1,x_3)] + [r^{12}(x_1,x_2),r^{23}(x_2,x_3)] + [r^{13}(x_1,x_3),r^{23}(x_2,x_3)] = 0. \quad (0.3)$$

Solutions of the CYBE (0.3) are called r-matrices. Additional to the aforementioned linear Poisson structure, an r-matrix defines a quadratic Poisson structure on an appropriate group, which is compatible with the group's multiplication; see [Dri83; FT07; CP95]. For instance, a constant solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ of the CYBE (0.3) with a $\mathfrak{g}$-invariant symmetric part endows the connected, simply-connected Lie group G of $\mathfrak{g}$ with a Poisson bracket such that the multiplication $G \times G \to G$ is a morphism of Poisson manifolds: G is a so-called Poisson-Lie group. The infinitesimal counterpart of such a Poisson bracket on G is a Lie bialgebra structure on $\mathfrak{g}$. Recall that a Lie bialgebra $\mathfrak{a}$ is a Lie algebra equipped with a 1-cocycle $\delta\colon \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$, called Lie bialgebra cobracket, which induces a Lie bracket on $\mathfrak{a}^*$. Lie bialgebra structures can be studied using Manin triples, i.e. triples $(\mathfrak{m}, \mathfrak{m}_+, \mathfrak{m}_-)$ of Lie algebras such that $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$ holds and $\mathfrak{m}_\pm$ are Lagrangian with respect to a non-degenerate, invariant, symmetric bilinear form on $\mathfrak{m}$. We say that a Lie bialgebra cobracket δ on $\mathfrak{m}_+$ is determined by a Manin triple $(\mathfrak{m}, \mathfrak{m}_+, \mathfrak{m}_-)$ if δ is dual to the Lie bracket of $\mathfrak{m}_-$ under the bilinear form on $\mathfrak{m}$. The fundamental example of a Manin triple is the classical double of a Lie bialgebra $\mathfrak{a}$: $\mathfrak{D}(\mathfrak{a}) \coloneqq \mathfrak{a} \oplus \mathfrak{a}^*$, equipped with its canonical bilinear form, possesses a unique Lie algebra structure such that $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, \mathfrak{a}^*)$ is a Manin triple determining the Lie bialgebra cobracket on $\mathfrak{a}$. Non-degenerate r-matrices can be used to construct infinite-dimensional examples of these structures. In fact, for every non-degenerate solution r of the CYBE (0.3), the assignment

$$\delta(a)(x,y) = [a(x) \otimes 1 + 1 \otimes a(y), r(x,y)], \tag{0.4}$$

defines a Lie bialgebra cobracket on $\mathfrak{g}(r)$ and $(\mathfrak{g}(\!(z)\!), \mathfrak{g}(r), \mathfrak{g}[\![z]\!])$ is a Manin triple isomorphic to the classical double of $\mathfrak{g}(r)$ for an appropriate bilinear form on $\mathfrak{g}(\!(z)\!)$; see [ES02, Proposition 6.2].

In [BD83a], Belavin and Drinfeld derived the most important results in the structure theory of non-degenerate solutions r of the CYBE (0.3), under the assumption that r depends on the differences of its variables, i.e. $r(x,y) = \rho(x-y)$ for some meromorphic function $\rho\colon U \to \mathfrak{g} \otimes \mathfrak{g}$. They proved that:

- ρ extends uniquely to a skew-symmetric meromorphic function $\rho\colon \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$ with only simple poles.
- The pole set Λ of ρ is a lattice in $\mathbb{C}$ and they call ρ elliptic (resp. trigonometric, resp. rational) if $\mathrm{rk}(\Lambda) = 2$ (resp. $\mathrm{rk}(\Lambda) = 1$, resp. $\mathrm{rk}(\Lambda) = 0$).
- Elliptic r-matrices are elliptic functions and trigonometric (resp. rational) r-matrices can be transformed into rational functions of exponentials (resp. rational functions) by means of a pole set preserving equivalence transformation.

We refer to this splitting of non-degenerate r-matrices into three classes as Belavin-Drinfeld trichotomy in the following. In [BD83b], Belavin and Drinfeld showed that the assumption $r(x,y) = \rho(x-y)$ can actually be achieved by certain natural equivalence transformations, however, it is unclear whether these transformations respect the classes of the Belavin-Drinfeld trichotomy. Nevertheless, this proves that non-degenerate r-matrices are automatically skew-symmetric. In particular, non-degenerate r-matrices are exactly non-degenerate generalized r-matrices, hence the name.

Recall that there are only three types of connected complex algebraic groups of dimension 1: elliptic curves, the multiplicative group $\mathbb{C}^\times$, and the additive group $\mathbb{C}$. In all three cases the universal covering is $\mathbb{C}$. The Belavin-Drinfeld trichotomy can be reinterpreted as follows: every non-degenerate r-matrix, depending on the difference of variables, is equivalent to the pull-back of a rational function on a one-dimensional, connected, complex algebraic group to the universal covering. This observation gives the Belavin-Drinfeld trichotomy a clear algebro-geometric flavor and is relevant in its proof (where e.g. Chevalley's structure theorem on algebraic groups is used), but a direct interpretation of the trichotomy in algebro-geometric terms remained obscure.

One of the main results of this work is a novel completely algebro-geometric proof of the Belavin-Drinfeld trichotomy, clarifying the geometry behind this splitting. Let r be a non-degenerate solution of the CYBE (0.3). We prove that a particular unital subalgebra $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ exists such that X is of arithmetic genus one, where $(X, \mathscr{A})$ is the geometric datum associated to r; see Theorem 3.2.5. In this case, we call $(X, \mathscr{A})$ the geometric CYBE datum of r. Recall that an irreducible projective curve of arithmetic genus one is a plane cubic curve, i.e. it is determined by an equation $y^2 = x^3 + ax + b$ for some $a, b \in \mathbb{C}$. Furthermore, X is elliptic if $4a^3 \neq -27b^2$ and has a unique singularity s otherwise. In the latter case, s is either nodal (if $4a^3 = -27b^2 \neq 0$) or cuspidal (if $a = 0 = b$) and

$$X \setminus \{s\} \cong \begin{cases} \mathbb{C}^\times & \text{if } s \text{ is nodal,} \\ \mathbb{C} & \text{if } s \text{ is cuspidal.} \end{cases} \tag{0.5}$$

We prove that r is, up to an equivalence transformation preserving X,
- an elliptic function in both variables if and only if X is an elliptic curve,
- a rational function of exponentials if and only if X is a nodal plane cubic curve, and
- a rational function if and only if X is a cuspidal plane cubic curve;

see Theorem 6.2.1. The ambiguity in the categorization of r-matrices that do not depend on the difference of variables is absent in this approach. We point out that the classical double of $\mathfrak{g}(r)$ can be interpreted geometrically. For all $q \in X$ we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathscr{A}) \longrightarrow \Gamma(X \setminus \{q\}, \mathscr{A}) \oplus \widehat{\mathscr{A}_q} \longrightarrow \mathrm{Q}(\widehat{\mathscr{A}_q}) \longrightarrow \mathrm{H}^1(\mathscr{A}) \longrightarrow 0 \tag{0.6}$$

(see [Par01, Proposition 3] or [Gal15, Chapter 3]), so $\mathrm{H}^0(\mathscr{A}) = 0 = \mathrm{H}^1(\mathscr{A})$ implies $\mathrm{Q}(\widehat{\mathscr{A}_q}) = \Gamma(X \setminus \{q\}, \mathscr{A}) \oplus \widehat{\mathscr{A}_q}$ and $(\mathrm{Q}(\widehat{\mathscr{A}_q}), \Gamma(X \setminus \{q\}, \mathscr{A}), \widehat{\mathscr{A}_q})$ becomes a Manin triple, if $\mathrm{Q}(\widehat{\mathscr{A}_q})$ is equipped with a canonical geometric bilinear form. If $q = p$ is the smooth point at infinity, this Manin triple is isomorphic to the classical double $(\mathfrak{g}(\!(z)\!), \mathfrak{g}(r), \mathfrak{g}[\![z]\!])$ of $\mathfrak{g}(r)$.

In [BD83a], Belavin and Drinfeld proved that elliptic r-matrices exist only for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and that all of these were already found by Belavin in [Bel81]. The Lie subalgebras of $\mathfrak{g}(\!(z)\!)$ associated to elliptic r-matrices are precisely the Lie algebras of quasi-periodic functions studied e.g. in [Gol84; RSTS89; Skr12]. Consequently, the geometric CYBE data of elliptic r-matrices have an explicit description. In [BH15], Burban and Heinrich showed that the sheaf of Lie algebras associated to an elliptic r-matrix can be realized as the sheaf of traceless endomorphisms of a simple vector bundle on the underlying elliptic curve.

The classification of rational r-matrices is a representation wild problem, so one cannot expect to obtain a complete list of solutions for arbitrary $\mathfrak{g}$. Nevertheless, Stolin provides reductions of this classification problem to more approachable Lie theoretic problems in [Sto91b; Sto91c] and uses these to calculate all rational r-matrices for certain low-dimensional $\mathfrak{g}$. His methods build on the observation that rational r-matrices

r are in bijection with Lagrangian orders $\mathfrak{W} \subseteq \mathfrak{g}((z^{-1}))$ such that $\mathfrak{g}((z^{-1})) = \mathfrak{g}[z] \oplus \mathfrak{W}$. This is related to the fact that a rational r-matrix defines a Lie bialgebra cobracket on $\mathfrak{g}[z]$ by Formula (0.4). From this perspective, $(\mathfrak{g}((z^{-1})), \mathfrak{g}[z], \mathfrak{W})$ is isomorphic to the associated classical double. Let r be a rational r-matrix. In [BG18], Burban and Galinat used Stolin's theory to show that $\mathbb{C}[z^{-2}, z^{-3}]\mathfrak{g}(r) = \mathfrak{g}(r)$ holds. This results in a more straightforward construction of the geometric datum $(X, \mathscr{A})$ of r. If s denotes the unique cuspidal singularity of X, their construction shows that the geometric Manin triple $(\mathrm{Q}(\widehat{\mathscr{A}_s}), \Gamma(X \setminus \{s\}, \mathscr{A}), \widehat{\mathscr{A}_s})$ is isomorphic to $(\mathfrak{g}((z^{-1})), \mathfrak{g}[z], \mathfrak{W})$.

Trigonometric r-matrices were also classified in [BD83a]. These are parametrized by so-called Belavin-Drinfeld quadruples $Q = ((\Pi_+, \Pi_-, \phi), h)$. More precisely, every trigonometric r-matrix is equivalent to a trigonometric r-matrix r^Q determined by an appropriate Belavin-Drinfeld quadruple Q. Recently, Polishchuck showed that $\mathbb{C}[e^z/(e^z-1)^2, e^z/(e^z-1)^3]\mathfrak{g}(r^Q) = \mathfrak{g}(r^Q)$ and uses this fact to construct the geometric datum of r^Q; see [Pol21].

In this work, we give a different geometrization scheme for trigonometric r-matrices, which yields a more concrete description of the geometric datum to r^Q; see Section 8.1. It is based on the observation that Stolin's theory of rational r-matrices uses Manin triples of the form $(\mathfrak{g}((z^{-1})), \mathfrak{g}[z], \mathfrak{W})$ instead of $(\mathfrak{g}((z)), \mathfrak{W}, \mathfrak{g}[\![z]\!])$, which suggests that there is a more appropriate theory of Manin triples for trigonometric r-matrices as well.

Let $\gamma \in \mathfrak{g} \otimes \mathfrak{g}$ be the Casimir element. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ and note that $\gamma = \gamma_+ + \gamma_\mathfrak{h} + \gamma_-$ for some $\gamma_\mathfrak{h} \in \mathfrak{h} \otimes \mathfrak{h}$ and $\gamma_\pm \in \mathfrak{n}_\pm \otimes \mathfrak{n}_\mp$. The simplest trigonometric r-matrix is given by $\overline{\varrho}^\circ(\exp(x), \exp(y))$, where

$$\overline{\varrho}^\circ(u, v) = \frac{v\gamma}{u - v} + \gamma_- + \gamma_\mathfrak{h}/2. \tag{0.7}$$

Recall that the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[u, u^{-1}] \oplus \mathbb{C}c$ possesses the so-called standard Lie bialgebra structure, which was introduced in [Dri85]; see also [CP95, Example 1.3.8]. It turns out that the Lie bialgebra cobracket on $\mathfrak{g}[u, u^{-1}] \cong \widehat{\mathfrak{g}}/\mathbb{C}c$ induced by the standard structure of $\widehat{\mathfrak{g}}$ is determined by $\overline{\varrho}^\circ$ via Formula (0.4); see [CP95, Example 2.1.10].

We will show that such a connection holds for any affine Kac-Moody algebra $\mathfrak{K}$ and that this yields the desired theory of Manin triples for trigonometric r-matrices; see Subsection 5.4.4. It is well-known (see [Kac90]) that a $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of order $m \in \mathbb{N}$ exists such that $[\mathfrak{K}, \mathfrak{K}]/\mathfrak{c}$ is isomorphic to the twisted loop algebra

$$\mathfrak{L} := \mathfrak{L}(\mathfrak{g}, \sigma) := \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k z^k, \text{ where } \mathfrak{g}_k := \{a \in \mathfrak{g} \mid \sigma(a) = \exp(2\pi i k/m)a\} \tag{0.8}$$

and $\mathfrak{c}$ is the center of $\mathfrak{K}$. The Lie bialgebra cobracket on $\mathfrak{L}$ induced by the standard Lie bialgebra structure of $\mathfrak{K}$ is determined by a σ-twisted version ϱ° of $\overline{\varrho}^\circ$ via Formula (0.4). In fact, any r-matrix $\varrho^t := \varrho^\circ + t$ for some $t \in \mathfrak{L} \otimes \mathfrak{L}$ defines a Lie bialgebra structure on $\mathfrak{L}$ in this way; see Proposition 5.4.8. Furthermore, for every Belavin-Drinfeld quadruple Q, r^Q is of the form $\varrho^t(\exp(x/m), \exp(y/m))$ for an appropriate $t = t_Q \in \wedge^2 \mathfrak{L}$ and $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$; see Lemma 8.2.12. We call the r-matrices of the form ϱ^t, for some $t \in \mathfrak{L} \otimes \mathfrak{L}$, σ-trigonometric r-matrices and argue that these are the appropriate two-parameter versions of trigonometric r-matrices. The standard bialgebra structure of a Kac-Moody algebra is well-known to be determined by a Manin triple; see [CP95, Example 1.3.8]. We will use this fact to deduce a bijection between σ-trigonometric r-matrices and certain subalgebras $\mathfrak{W} \subseteq \mathfrak{L} \times \mathfrak{L}$ such that $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{L}, \mathfrak{W})$ is a Manin triple; see Theorem 5.4.9. Here, $\mathfrak{L} \times \mathfrak{L}$ is equipped with a canonical bilinear form and $\mathfrak{L}$ is identified with its diagonal embedding.

The Belavin-Drinfeld classification of trigonometric r-matrices [BD83a, Theorem 6.1] cannot be transported directly to a classification of the Lie bialgebra structures associated to σ-trigonometric r-matrices. This stems from the fact that holomorphic equivalence transformations are needed to achieve the difference-dependent form and these transformations do not define endomorphisms of twisted loop algebras. We adjust the proof of [BD83a, Theorem 6.1] to deduce that σ-trigonometric r-matrices are determined by Belavin-Drinfeld quadruples, up to equivalence transformations which respect the associated Lie bialgebra structures; see Theorem 8.2.1. Furthermore, we give a concrete description of the Lie algebra $\mathfrak{W}^Q$ associated to r^Q for a Belavin-Drinfeld quadruple Q; see Remark 8.2.7. In our joint work with Maximov [AM21], we use the algebro-geometric theory of the CYBE presented here to derive a direct reduction of Theorem 8.2.1 to [BD83a, Theorem 6.1].

Using results from [KW92], we prove that the Lie algebra $\mathfrak{W} \subseteq \mathfrak{L} \times \mathfrak{L}$ associated to a σ-trigonometric r-matrix ϱ has a natural $\mathbb{C}[u_+, u_-]/(u_+u_-)$-module structure. Therefore, the completed module $\widehat{\mathfrak{W}}$ lives on the formal neighbourhood $\operatorname{Spec}(\mathbb{C}[\![u_+, u_-]\!]/(u_+u_-))$ of the singular point s of a nodal plane cubic curve X. We show that the sheaf $\mathscr{A}$ associated to $\varrho(\exp(x/m), \exp(y/m))$ is given by formally gluing $\mathfrak{L}$ on $X \setminus \{s\} \cong \mathbb{C}^\times$ with $\widehat{\mathfrak{W}}$ on the formal neighbourhood of s; see Lemma 8.1.2 and Theorem 8.1.3. Since the Lie algebra $\mathfrak{W}$ has a concrete description if $\varrho = \varrho^Q$ for some Belavin-Drinfeld quadruple Q, we obtain a tangible description of $\mathscr{A}$ in this case as well. Furthermore, this geometric approach reveals the classical double of the Lie bialgebra structure on $\mathfrak{L}$ defined by ϱ: it is isomorphic to the Manin triple $(\mathrm{Q}(\widehat{\mathscr{A}_s}), \Gamma(X \setminus \{s\}, \mathscr{A}), \widehat{\mathscr{A}_s})$; see Proposition 8.1.4. In particular, we will see that $\mathfrak{D}(\mathfrak{L}) \cong \mathrm{Q}(\widehat{\mathscr{A}_s}) \cong \widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-$, where $\widehat{\mathfrak{L}}^+$ (resp. $\widehat{\mathfrak{L}}^-$) is the completion of $\mathfrak{L}$ in positive (resp. negative) powers of z, i.e. $\widehat{\mathfrak{L}}^\pm = \prod_{\pm j \geqslant 0} \mathfrak{g}_j z^j \oplus \bigoplus_{\pm j < 0} \mathfrak{g}_j z^j$.

To conclude the introduction, let us note that the notion of non-degenerate generalized r-matrices can be generalized to arbitrary fields $\Bbbk$ of characteristic 0 by considering formal power series instead of meromorphic functions. These series will be called formal generalized r-matrices and are especially interesting for $\Bbbk = \mathbb{R}$ from the point of view of integrable systems (see e.g. [BBT03]) and for $\Bbbk = \mathbb{C}(\!(\hbar)\!)$ from the point of view of quantum groups (see e.g. [KPS20]). For $\Bbbk = \mathbb{C}$ the formal and analytic setting lead to the same theory by virtue of Proposition 3.4.5. This work will be formulated in the language of formal generalized r-matrices. We will see that many of the results for generalized r-matrices mentioned above stay valid for formal generalized r-matrices over arbitrary fields of characteristic 0. In particular, we will see that the presented geometrization scheme still works and provides a geometric trichotomy of skew-symmetric formal generalized r-matrices by the three different types of plane cubic curves; see Theorem 3.2.5 and Remark 3.2.6.

Content and structure

In this section, we give a more detailed exposition of the content of this work following precisely the structure of the main body. In the process, we give the precise formulations of our main results. This work is split into two parts.

Part I consists of a detailed look at the universal geometrization procedure for formal generalized r-matrices over a field $\Bbbk$ of characteristic 0 as well as the consequent properties of said objects; it is based on our work in [Abe21]. Chapter 1 is thereby dedicated to the derivation of some necessary results in the theory of sheaves of (not necessarily associative) algebras. In particular, we prove that the results in [Kir78; Kir83], about local triviality of Lie algebra bundles, hold in an algebro-geometric context if the classical topology notion is replaced by the étale topology; see Section 1.2. In Section 1.3 we introduce a special set of subalgebras of $A(\!(z)\!)$, called A-lattices, for any (not necessarily associative) finite-dimensional $\Bbbk$-algebra A. If A is central (see Subsection 1.3.1) and simple, any pair (O, W), consisting of an A-lattice $W \subseteq A(\!(z)\!)$ and a unital subalgebra

$$O \subseteq \mathrm{Mult}(W) := \{\lambda \in \Bbbk(\!(z)\!) \mid \lambda W \subseteq W\} \tag{0.9}$$

of finite codimension, determines a geometric datum $((X, \mathscr{A}), (p, c, \zeta)) := \mathbb{G}(O, W)$. This datum consists of a coherent torsion-free sheaf of algebras $\mathscr{A}$ on an integral projective curve X and a formal trivialization (c, ζ) of $(X, \mathscr{A})$ at a $\Bbbk$-rational smooth point $p \in X$; see Subsection 1.3.5. After introducing appropriate categories of pairs (O, W) and geometric data $((X, \mathscr{A}), (p, c, \zeta))$, $\mathbb{G}$ becomes an equivalence of categories; see Theorem 1.3.6. Let us point out that any finite-dimensional simple algebra over an algebraically closed field is automatically central.

Let $\mathfrak{g}$ be a semi-simple finite-dimensional Lie algebra over $\Bbbk$. In Chapter 2, we introduce formal generalized r-matrices with values in $\mathfrak{g}$. These are certain elements of $(\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$, which solve a formal version of (0.1). We define an appropriate notion of equivalence among them, called formal equivalence. Moreover, we will discuss skew-symmetric formal generalized r-matrices, which are simply called formal r-matrices. We show that a formal generalized r-matrix r determines a Lie algebra $\mathfrak{g}(r) \subseteq \mathfrak{g}(\!(z)\!)$ complementary to $\mathfrak{g}[\![z]\!]$. A central observation in this setting is: if a formal generalized r-matrix is normalized appropriately, the subalgebra $\mathfrak{g}(r) \subseteq \mathfrak{g}(\!(z)\!)$ is Lagrangian if and only if r is skew-symmetric; see Subsection 2.3.2.

In Chapter 3, we use the fact that $\mathfrak{g}(r)$ is a $\mathfrak{g}$-lattice for any formal generalized r-matrix r in the geometrization scheme from Chapter 1 to assign a geometric datum to r and examine the properties of this datum. The main results obtained in Section 3.1 and Section 3.2 can be summarized as follows.

Theorem A.
Let $\mathfrak{g}$ be a finite-dimensional, central, simple Lie algebra over $\Bbbk$, $r \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be any unital subalgebra of finite codimension, and $\mathbb{G}(O, \mathfrak{g}(r)) = ((X, \mathscr{A}), (p, c, \zeta))$.

(1) *The geometric genus of X is at most one, $\mathrm{H}^0(\mathscr{A}) = 0 = \mathrm{H}^1(\mathscr{A})$, and $\mathscr{A}$ is étale $\mathfrak{g}$-locally free at p.*
(2) *If the geometric genus of X is one and $O = \mathrm{Mult}(\mathfrak{g}(r))$, then X is elliptic and r is skew-symmetric up to scaling by an element of $\Bbbk[\![y]\!]^\times$.*
(3) *If r is skew-symmetric and normalized, there is a canonical choice for O such that $\mathrm{h}^1(\mathscr{O}_X) = 1$, $\mathscr{A}$ is étale $\mathfrak{g}$-locally free at the smooth locus C of X, and the Killing form of $\mathscr{A}|_C$ extends to a perfect pairing $\mathscr{A} \times \mathscr{A} \to \mathscr{O}_X$.*

Theorem A.(1) & (2) imply that the normalization of X is $\mathbb{P}^1_\Bbbk$ if r is not equivalent to a formal r-matrix, i.e. the projective curves associated to non-skew-symmetric formal generalized r-matrices are essentially all rational. As for $\Bbbk = \mathbb{C}$, an integral projective curve over $\Bbbk$ of arithmetic genus one with a $\Bbbk$-rational smooth point is either an elliptic curve, a nodal plane cubic curve or a cuspidal plane cubic curve; see Remark 3.2.6. Therefore, Theorem A.(3) can be seen as a splitting of normalized formal r-matrices into three categories by the underlying curves. The three classes of this geometric trichotomy are preserved by arbitrary formal equivalences; see Remark 3.2.7.

In Section 3.3, we derive a scheme that translates properties of a geometric datum $((X, \mathscr{A}), (p, c, \zeta))$ associated to a formal generalized r-matrix r into properties of r. To this end, note that Theorem A.(1) ensures the existence of a smooth open neighbourhood C of p such that $\mathscr{A}|_C$ is étale $\mathfrak{g}$-locally free. We point out that, if $\Bbbk = \mathbb{C}$, this is equivalent to $\mathscr{A}|_q \cong \mathfrak{g}$ for all $q \in C$ closed; see Theorem 1.2.3. We call a quadruple $((X, \mathscr{A}), (C, \eta))$, where η is any non-vanishing 1-form on C, a geometric GCYBE model of r. If r is skew-symmetric and normalized, there is a distinguished choice of GCYBE model called the geometric CYBE model of r (see Subsection 3.3.1), where e.g. O is chosen according to Theorem A.(3). Following [BG18], we construct the geometric r-matrix $\rho \in \Gamma(C \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A})$ (here Δ is the diagonal of C) corresponding to $((X, \mathscr{A}), (C, \eta))$ in Subsection 3.3.2. The section ρ can be seen as a global extension of r and, after étale trivializing $\mathscr{A}$ at p, we obtain a rational extension of r. More precisely, the following results hold; see Theorem 3.3.3 and Theorem 3.3.5.

Theorem B.
Let $\mathfrak{g}$ be a finite-dimensional, central, simple Lie algebra over $\Bbbk$, $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ be a formal generalized r-matrix, $((X, \mathscr{A}), (C, \eta))$ be a geometric GCYBE model of r, and $\rho \in \Gamma(C \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A})$ be the geometric r-matrix of $((X, \mathscr{A}), (C, \eta))$.

(1) *The geometric Taylor expansion of ρ at $C \times \{p\}$ (see Subsection 3.3.4) equals $\lambda(y)r(x, y)$ for some $\lambda \in \Bbbk[\![z]\!]^\times$.*
(2) *If r is skew-symmetric and normalized and $((X, \mathscr{A}), (C, \eta))$ is its geometric CYBE model, we get $\lambda(z) = 1$ in (1).*
(3) *There exists an étale X-scheme Y such that r is equivalent to a Taylor expansion of some rational section of $(\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathscr{O}_{Y \times Y}$.*

Theorem A states that, if r is not equivalent to a formal r-matrix, the normalization of X coincides with $\mathbb{P}^1_\Bbbk$. Therefore, Y can also be thought of as an étale $\mathbb{P}^1_\Bbbk$-scheme in this case. Observe that Y is one-dimensional. If $\Bbbk = \mathbb{C}$, we may assume that Y is a Riemann surface and obtain the extension result mentioned earlier.

We conclude Part I by interpreting formal generalized r-matrices for $\Bbbk = \mathbb{C}$ (resp. $\Bbbk = \mathbb{R}$) as Taylor expansions of non-degenerate solutions of the GCYBE (0.1) (resp. a similar real analytic notion); see Section 3.4. In particular, Proposition 3.4.5 states that in this way the theory of non-degenerate solutions of the GCYBE (0.1) (resp. a similar real analytic notion) coincides with the theory of formal generalized r-matrices over $\mathbb{C}$ (resp. $\mathbb{R}$). Thus, all our established results are applicable to the original analytic context.

Part II consists of an in depth look at the theory of formal r-matrices with values in a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\Bbbk = \mathbb{C}$. We begin by collecting some important results about twisted loop algebras in Chapter 4. Almost all of them are fairly well-known and can be found in [Kac90] or [Hel78, Section X.5]. The only notable exceptions are the results in Subsection 4.1.8 and Subsection 4.2.5 about maximal subalgebras commensurable with a Borel subalgebra by Kac and Wang [KW92].

In Chapter 5, we give an overview of the theory of Lie bialgebras. Most importantly, we discuss their relation with Manin triples (see Section 5.2) and present the procedure of constructing new Lie bialgebras from known ones, called twisting, from [KS02]; see Section 5.3.1. Following [AM21; AB21], we use the relationship between affine Kac-Moody algebras and twisted loop algebras, to define the standard Lie bialgebra structure on $\mathfrak{L} = \mathfrak{L}(\mathfrak{g},\sigma)$ (see (0.8)) for any $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of finite order; see Subsection 5.4.4. Furthermore, we introduce σ-trigonometric r-matrices and show that they define the Lie bialgebra structures on $\mathfrak{L}$ obtained by twisting its standard Lie bialgebra structure. This results in the bijection of σ-trigonometric r-matrices with certain Manin triples $(\mathfrak{L}\times\mathfrak{L},\mathfrak{L},\mathfrak{W})$ in Theorem 5.4.9. We conclude Section 5 with a discussion of rational r-matrices (in the sense of Stolin [Sto91b; Sto91c]) and their relation with Lie bialgebra structures on the polynomial Lie algebra $\mathfrak{g}[z]$; see Subsection 5.4.5.

In Chapter 6, we give the announced new proof of the Belavin-Drinfeld trichotomy, using the theory established in Part I. More precisely, the geometric trichotomy from Theorem A.(3) implies that the sheaf of Lie algebras, appearing in a geometric CYBE model, restricts to an étale $\mathfrak{g}$-locally free sheaf of Lie algebras on a one-dimensional, connected, complex algebraic curve. We classify these sheaves in Subsection 6.1. Combined with Theorem B, this yields the following version of the Belavin-Drinfeld trichotomy; see Theorem 6.2.1.

Theorem C.
Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra, $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a normalized formal r-matrix, and $((X,\mathscr{A}),(C,\eta))$ be the geometric CYBE model of r. Then r is equivalent to the Taylor series in the second variable at 0 of a non-degenerate r-matrix $\mathbb{C}\times\mathbb{C}\to\mathfrak{g}\otimes\mathfrak{g}$, which is
- *elliptic in both variables if and only if X is elliptic,*
- *of the form $\varrho(\exp(x/m),\exp(y/m))$, for some $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of order m and some σ-trigonometric r-matrix ϱ, if and only if X has a nodal singularity, and*
- *a rational r-matrix if and only if X has a cuspidal singularity.*

Note that this approach immediately implies that the three classes of the Belavin-Drinfeld trichotomy are preserved by arbitrary formal equivalences, since this statement was already observed to hold for the geometric trichotomy; see Remark 6.2.6.

To present a complete picture of the situation, we reproduce the explicit geometric construction of elliptic r-matrices from [BH15] as well as their classification from [BD83a]; see Chapter 7. In particular, we see that elliptic r-matrices are parametrized by triples $(\tau,(n,m))$ consisting of a complex number τ with positive imaginary part and a pair $0 < m < n$ of coprime integers. We refine the classification from [BD83a] by proving that the elliptic r-matrices corresponding to the triples $(\tau,(n,m))$ and $(\tau',(n',m'))$ are equivalent if and only if $\mathbb{C}/\langle 1,\tau\rangle_{\mathbb{Z}} \cong \mathbb{C}/\langle 1,\tau'\rangle_{\mathbb{Z}}$, $n = n'$ and $m' \in \{m, n-m\}$; see Proposition 7.2.4.

In Chapter 8, we discuss σ-trigonometric r-matrices in detail beginning with the explicit geometrization procedure from our joint work with Burban [AB21]. More precisely, we combine the theory of Manin triples for σ-trigonometric r-matrices from Subsection 5.4.4 with the results from Subsection 4.2.5 on subalgebras of twisted loop algebras and the construction of torsion-free coherent sheaves on singular curves from [Bod+06] to derive the following results; see Section 8.1.

Theorem D.
Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra, $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ have order $m \in \mathbb{N}$, and $\mathfrak{L} := \mathfrak{L}(\mathfrak{g}, \sigma)$ be the associated twisted loop algebra (0.8). Moreover, let ϱ be a σ-trigonometric r-matrix and $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{L}, \mathfrak{W})$ be the associated Manin triple from Theorem 5.4.9.

(1) *$\mathfrak{W}$ is a torsion-free Lie algebra over $\mathbb{C}[u_+, u_-]/(u_+u_-)$.*
(2) *$\mathfrak{W}$ determines a sheaf of Lie algebras $\mathscr{A}$ on a plane cubic curve X with nodal singularity s such that*

$$\Gamma(X \setminus \{s\}, \mathscr{A}) \cong \mathfrak{L} \text{ and } \widehat{\mathscr{A}_s} \cong \widehat{\mathfrak{W}} := \varprojlim \mathfrak{W}/(u_+, u_-)^k\mathfrak{W}. \tag{0.10}$$

(3) *The geometric CYBE model of the normalized formal r-matrix defined by the Tylor series of $\varrho(\exp(x/m), \exp(y/m))$ in $y = 0$ is $((X, \mathscr{A}), (X \setminus \{s\},\ \mathrm{d}u/u))$, where $\mathrm{d}u/u$ is understood as a rational 1-form on X.*
(4) *The geometric r-matrix of $((X, \mathscr{A}), (X \setminus \{s\},\ \mathrm{d}u/u))$ can be identified with ϱ.*

We present a classification of σ-trigonometric r-matrices, respecting the associated Lie bialgebra structures on $\mathfrak{L}$, in Section 8.2. To this end, we adjust the methods from [BD83a, Section 6] to this more general context. We conclude Chapter 8 by a discussion of equivalences. In particular, we show that the theory from Part I implies that formal equivalences between σ-trigonometric r-matrices are actually regular in nature. This is the key observation used in [AM21] to reduce the aforementioned classification of σ-trigonometric r-matrices to the Belavin-Drinfeld classification of trigonometric r-matrices [BD83a, Theorem 6.1].

We conclude this thesis with a discussion of rational r-matrices in Chapter 9. More precisely, we give an explicit geometrization of rational r-matrices in the vein of [BG18], using the theory of maximal subalgebras commensurable with a Borel subalgebra; see Section 9.2. Moreover, we give a brief overview over Stolin's structure theory of rational r-matrices from [Sto91b; Sto91c]. Finally, we will see in Proposition 9.2.9 that the algebro-geometric theory from Part I implies that formal equivalences between rational r-matrices are actually polynomial in nature.

Conventions in notation and terminology

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the sets of natural, integer, rational, real and complex numbers respectively, where by convention $\mathbb{N}$ excludes $\{0\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this document $\Bbbk$ denotes a field of characteristic 0 with algebraic closure $\overline{\Bbbk}$.

Algebra. All rings are commutative and have a unit and morphisms of rings preserve said unit. For a ring R and R-modules M and N, the space of R-linear maps $M \to N$

(resp. $M \to M$) is denoted by $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{End}_R(M)$), while the tensor product of M and N is written as $M \otimes_R N$. For $R = \Bbbk$ the indices are omitted. The invertible elements of R are denoted by $R^\times$ and $M^* := \mathrm{Hom}_R(M, R)$ is the dual module of M. For any family $\{m_i \in M \mid i \in I\}$, where I is some index set, we define $\langle m_i \mid i \in I\rangle_R := \bigoplus_{i\in I} Rm_i$.

If R is a domain, let $\mathrm{Q}(R) := (R \setminus \{0\})^{-1}R$ denote its quotient field and put $\mathrm{Q}(M) := M \otimes_R \mathrm{Q}(R)$. Let $f: R \to R'$ be a morphism of rings and M' be an R'-module. We say that a map $g: M \to M'$ is f-equivariant if it is a group homomorphism satisfying $g(rm) = f(r)g(m)$ for all $r \in R$, $m \in M$. If R and R' are both domains, the maps $\mathrm{Q}(R) \to \mathrm{Q}(R')$ (resp. $\mathrm{Q}(M) \to \mathrm{Q}(M')$) induced by f (resp. g) is again denoted by f (resp. g).

For any R-bilinear map $B: M \times M \to R$, we write $B^{\mathrm{a}}: M \to M^*$ and $\widetilde{B}: M \otimes_R M \to \mathrm{End}_R(M)$ for the morphisms defined by $a \mapsto B(a, -)$ and $a \otimes b \mapsto B(b, -)a$ respectively. If $R = \Bbbk$, $M, N \subseteq L$ for some vector space L over $\Bbbk$, and $\dim((M + N)/(M \cap N)) < \infty$, we say that M and N are commensurable and write $M \asymp N$.

In this text, an R-algebra A satisfies no additional assumptions, i.e. $A = (A, \mu_A)$ consists of an R-module A equipped with a multiplication map $\mu_A: A \otimes_R A \to A$. In particular, a Lie algebra over R is an R-algebra. For any family $\{a_i \in A \mid i \in I\}$, where I is some index set, $\langle a_i \mid i \in I\rangle_{R\text{-alg}}$ denotes the smallest R-subalgebra of A containing a_i for all $i \in I$, i.e. the R-subalgebra of A generated by $\{a_i \in A \mid i \in I\}$. The group of invertible R-algebra endomorphisms of A, i.e. invertible R-linear maps $f: A \to A$ satisfying $f\mu_A = \mu_A(f \otimes f)$, will be denoted by $\mathrm{Aut}_{R\text{-alg}}(A)$. For another R-algebra A', $A \oplus A'$ will denote the direct sum of R-modules, while $A \times A'$ is used for the direct sum as R-algebras. If A is a Lie algebra, we use expressions of the form $[a \otimes 1, a_1 \otimes a_2] := [a, a_1] \otimes a_2$, $[1 \otimes a, a_1 \otimes a_2] := a_1 \otimes [a, a_2]$, etc. for all $a, a_1, a_2 \in A$.

Algebraic geometry. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on a topological space X and $f: \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. For $U \subseteq X$ open and $p \in X$, we denote by $\mathscr{F}(U) = \Gamma(U, \mathscr{F})$ and $\mathscr{F}_p$ the set of sections of $\mathscr{F}$ over U and the germ of $\mathscr{F}$ in p respectively. Similarly, $f_U = \Gamma(U, f): \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G})$ and $f_p: \mathscr{F}_p \to \mathscr{G}_p$ are the morphisms induced by f on the level of sections and germs, where the indices U and p will be omitted if there is no ambiguity. We write $\mathrm{H}^n(\mathscr{F})$ for the n-th global cohomology group, in particular $\mathrm{H}^0(\mathscr{F}) = \Gamma(X, \mathscr{F}) = \mathscr{F}(X)$. If $\mathrm{H}^n(\mathscr{F})$ is a finite-dimensional $\Bbbk$-vector space, we denote its dimension by $\mathrm{h}^n(\mathscr{F})$.

Assume that $X = (X, \mathscr{O}_X)$ is a ringed space and $\mathscr{F}, \mathscr{G}$ are $\mathscr{O}_X$-modules. For a morphism $f: X \to Y = (Y, \mathscr{O}_Y)$ of ringed spaces, we write $f^\flat: \mathscr{O}_Y \to f_*\mathscr{O}_X$ for the additional structure morphism and let $f^\sharp: f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ be the induced map. The $\mathrm{H}^0(\mathscr{O}_X)$-module of $\mathscr{O}_X$-module homomorphisms $\mathscr{F} \to \mathscr{G}$ (resp. $\mathscr{F} \to \mathscr{F}$) is denoted by $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ (resp. $\mathrm{End}_{\mathscr{O}_X}(\mathscr{F})$), while its sheaf counterpart is denoted by $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ (resp. $\mathscr{E}\!nd_{\mathscr{O}_X}(\mathscr{F})$). The $\mathscr{O}_X$-module tensor product of $\mathscr{F}$ and $\mathscr{G}$ is written as $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$. If X is locally ringed and $p \in X$, the maximal ideal of the local ring $\mathscr{O}_{X,p}$ is denoted by $\mathfrak{m}_p$ and we put $\kappa(p) := \mathscr{O}_{X,p}/\mathfrak{m}_p$ as well as $\mathscr{F}|_p := \mathscr{F}_p/\mathfrak{m}_p\mathscr{F}_p$. The fiber product of two schemes X, Y over a scheme S is denoted by $X \times_S Y$ and the index S will be omitted if $S = \mathrm{Spec}(\Bbbk)$.

Part I

Algebraic geometry of formal generalized r-matrices

1

Some facts about sheaves of algebras

1.1 Basic definitions and properties

In this section, we discuss sheaves of algebras with a special focus on sheaves of Lie algebras. In particular, we define the Killing form of a locally free sheaf of Lie algebras in Subsection 1.1.2, which will turn out to be an important tool while working with the geometric data of formal generalized r-matrices in Chapter 3.

1.1.1 Sheaves of algebras. Let X be a ringed space. A *sheaf of algebras* on X is an $\mathscr{O}_X$-module $\mathscr{A}$ equipped with an $\mathscr{O}_X$-linear morphism $\mu_{\mathscr{A}}\colon \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{A} \to \mathscr{A}$ called *multiplication map*. The *left* (resp. *right*) *multiplication* is the $\mathscr{O}_X$-linear morphism $\ell_{\mathscr{A}}\colon \mathscr{A} \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{A})$ (resp. $r_{\mathscr{A}}\colon \mathscr{A} \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{A})$) given by left (resp. right) multiplication of the $\Gamma(U, \mathscr{O}_X)$-algebra $\Gamma(U, \mathscr{A})$ for each $U \subseteq X$ open. A *morphism of* $\mathscr{O}_X$*-algebras* $f\colon \mathscr{A} \to \mathscr{B}$ is an $\mathscr{O}_X$-linear map such that $\mu_{\mathscr{B}}(f \otimes f) = f\mu_{\mathscr{A}}$.

We call $\mathscr{A}$ a *sheaf of Lie algebras* if for every $U \subseteq X$ open $\mu_{\mathscr{A}}$ defines a Lie bracket on $\Gamma(U, \mathscr{A})$. In this case we write $[,]_{\mathscr{A}} = \mu_{\mathscr{A}}$ and $\mathrm{ad}_{\mathscr{A}} = \ell_{\mathscr{A}} = -r_{\mathscr{A}}$. Morphisms of sheaves of Lie algebras are simply morphisms of sheaves of algebras. Although irrelevant in the following, we note that other important types of algebras such as associative, alternative, Jordan, etc. algebras also generalize to the sheaf setting in the same natural way.

Remark 1.1.1.
Let $f\colon Y \to X$ be a morphism of ringed spaces and $\mathscr{A}$ (resp. $\mathscr{B}$) be a sheaf of algebras on X (resp. Y). The sheaf of algebras $f^*\mathscr{A}$ (resp. $f_*\mathscr{B}$) is naturally a sheaf of algebras on Y (resp. X) with the multiplication defined through

$$f^*\mathscr{A} \otimes_{\mathscr{O}_Y} f^*\mathscr{A} \longrightarrow f^*(\mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{A}) \xrightarrow{f^*\mu_{\mathscr{A}}} \mathscr{A}$$
$$\Big(\text{resp. } f_*\mathscr{B} \otimes_{\mathscr{O}_X} f_*\mathscr{B} \longrightarrow f_*(\mathscr{B} \otimes_{\mathscr{O}_Y} \mathscr{B}) \xrightarrow{f_*\mu_{\mathscr{B}}} \mathscr{B}\Big),$$

where the unlabeled arrow is the canonical one. It is easy to see that $f^*\mathscr{A}$ (resp. $f_*\mathscr{B}$) is a sheaf of Lie algebras, if $\mathscr{A}$ (resp. $\mathscr{B}$) is.
If we assume that X is a locally ringed space and $p \in X$ is an arbitrary point with residue field $\kappa(p)$, the stalk $\mathscr{A}_p$ (resp. the fiber $\mathscr{A}|_p$) is naturally an $\mathscr{O}_{X,p}$-algebra (resp. $\kappa(p)$-algebra), which is a Lie algebra if $\mathscr{A}$ is a sheaf of Lie algebras. Indeed, this can be seen as a special case of the inverse image construction by choosing $Y = (\{p\}, \mathscr{O}_{X,p})$ (resp. $Y = (\{p\}, \kappa(p))$) and considering the inverse image with respect to the canonical morphism $f\colon Y \to X$.

1.1.2 Killing form of a locally free sheaf of Lie algebras. Let X be a ringed space and $\mathscr{A}$ be a sheaf of Lie algebras on X. We call an $\mathscr{O}_X$-bilinear form $B\colon \mathscr{A} \times \mathscr{A} \to \mathscr{O}_X$

invariant (resp. *symmetric, non-degenerate, perfect*), if for all $U \subseteq X$ open, B_U defines an invariant (resp. symmetric, non-degenerate, perfect) bilinear form on $\Gamma(U, \mathscr{A})$. Assume that $\mathscr{A}$ is finite locally free. Then the bilinear form $\mathscr{A} \times \mathscr{A} \to \mathscr{O}_X$ defined by the composition

$$\mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{E}nd(\mathscr{A}) \times \mathscr{E}nd(\mathscr{A}) \longrightarrow \mathscr{E}nd(\mathscr{A}) \longrightarrow \mathscr{O}_X, \tag{1.1}$$

where the first map is $\mathrm{ad}_{\mathscr{A}} \times \mathrm{ad}_{\mathscr{A}}$, the second is given by the sheaf composition and the third is the sheaf trace, is called *Killing form of* $\mathscr{A}$.

Lemma 1.1.2.
Let $\mathscr{A}$ be a finite locally free sheaf of Lie algebras on a ringed space X.
(1) *The Killing form $\mathcal{K}$ of $\mathscr{A}$ is a symmetric invariant bilinear form.*
(2) *For any morphism $f\colon Y \to X$ of ringed spaces, the pairing*

$$f^*\mathcal{K}\colon f^*\mathscr{A} \times f^*\mathscr{A} \to f^*\mathscr{O}_X \cong \mathscr{O}_Y \tag{1.2}$$

can be identified with the Killing form of $f^*\mathscr{A}$.
(3) *$\mathcal{K}_x$ is the Killing form of $\mathscr{A}_x$ as free $\mathscr{O}_{X,x}$-Lie algebra for every $x \in X$ and, if X is a locally ringed space, the fiber $\mathcal{K}|_x$ is the Killing form of $\mathscr{A}|_x$ as a Lie algebra over the residue field $\kappa(x)$ of x.*

Proof. The proof of (1) is clear and (3) follows from (2) and the observation that the functor $(\cdot)_p$ (resp. $(\cdot)|_p$) can be realized by the inverse image via $(\{p\}, \mathscr{O}_{X,p}) \to X$ (resp. $(\{p\}, \kappa(p)) \to X$). Therefore, it remains to prove (2).

The canonical map $\chi\colon f^*\,\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{A}) \to \mathscr{E}nd_{\mathscr{O}_Y}(f^*\mathscr{A})$ coincides with the isomorphism

$$\mathrm{End}_{\mathscr{O}_{X,f(q)}}(\mathscr{A}_{f(q)}) \otimes_{\mathscr{O}_{X,f(q)}} \mathscr{O}_{Y,q} \cong \mathrm{End}_{\mathscr{O}_{Y,q}}(\mathscr{A}_{f(q)} \otimes_{\mathscr{O}_{X,f(q)}} \mathscr{O}_{Y,q}) \tag{1.3}$$

in the stalk in any point $q \in Y$, where we used that $\mathscr{A}$ is finite locally free. This shows that

$$f^*\mathscr{A} \xrightarrow{f^*\mathrm{ad}_{\mathscr{A}}} f^*\,\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{A}) \xrightarrow{\chi} \mathscr{E}nd_{\mathscr{O}_Y}(f^*\mathscr{A})$$

$$\text{and}\quad f^*\,\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{A}) \xrightarrow{\chi} \mathscr{E}nd_{\mathscr{O}_Y}(f^*\mathscr{A}) \xrightarrow{\mathrm{Tr}_{f^*\mathscr{A}}} \mathscr{O}_Y\ ,$$

coincide with $\mathrm{ad}_{f^*\mathscr{A}}$ and $f^*\mathrm{Tr}_{\mathscr{A}}$ respectively and that χ is compatible with the composition of endomorphisms of sheaves. Therefore, applying f^* to (1.1) and using χ implies that $f^*\mathcal{K}_{\mathscr{A}}$ coincides with the Killing form of $f^*\mathscr{A}$. □

1.2 Local triviality of sheaves of algebras

Let $\mathscr{A}$ be a sheaf of algebras on a $\Bbbk$-scheme X. In general, we will see that for our applications we cannot expect $\mathscr{A}$ to be locally free in a Zarsiki sense: the existence of an open subset $U \subseteq X$ and a $\Bbbk$-algebra A such that $\Gamma(U, \mathscr{A}) \cong A \otimes \Gamma(U, \mathscr{O}_X)$ is not guaranteed. However, we will see that local triviality in the less rigid étale topology is ensured in our context. In this section, we explain the formal meaning of étale local triviality and give some important results concerning sheaves of algebras with this property.

1.2.1 Types of local triviality of sheaves of Lie algebras. Let $\mathscr{A}$ be a sheaf of algebras on a $\Bbbk$-scheme X and A be a $\Bbbk$-algebra. There are several different natural notions of A-local triviality for the sheaf of algebras $\mathscr{A}$:

- $\mathscr{A}$ is called *weakly A-locally free* at $x \in X$ if the fiber $\mathscr{A}|_x$ is isomorphic to $A \otimes \kappa(x)$ as a Lie algebra over $\kappa(x)$, where we recall that $\kappa(x)$ is the residue field at x.
- $\mathscr{A}$ is called *formally A-locally free* at $x \in X$ if the completed stalk $\widehat{\mathscr{A}_x}$ is isomorphic to $A \otimes \widehat{\mathscr{O}}_{X,x}$ as Lie algebra over $\widehat{\mathscr{O}}_{X,x}$.
- $\mathscr{A}$ is called *étale A-locally free* at $x \in X$ if there exists a $\Bbbk$-scheme Y and an étale morphism $f\colon Y \to X$ of $\Bbbk$-schemes such that $x \in f(Y)$ and $f^*\mathscr{A}$ is isomorphic to $A \otimes \mathscr{O}_Y$ as sheaf of Lie algebras on Y.
- $\mathscr{A}$ is called *Zariski A-locally free* at $x \in X$ if there exists an open neighbourhood U of x such that $\mathscr{A}|_U$ is isomorphic to $A \otimes \mathscr{O}_U$ as sheaf of Lie algebras on U.
- $\mathscr{A}$ is called weakly (resp. formally, resp. étale, resp. Zariski) A-locally free if $\mathscr{A}$ is weakly (resp. formally, resp. étale, resp. Zariski) A-locally free at all closed points $x \in X$.

Remark 1.2.1.
(1) If $\mathscr{A}$ is étale A-locally free, then $\mathscr{A}$ is locally free as $\mathscr{O}_X$-module; see [GR71, Seconde partie, Théorème 3.1.3].
(2) Let $p \in X$ be an arbitrary point. Obviously, $\mathscr{A}$ is weakly A-locally free in $p \in X$ if it is formally A-locally free in p. Since open immersions are étale, $\mathscr{A}$ is both formally and étale A-locally free in p if it is Zariski A-locally free in p. Furthermore, if $\Bbbk$ is algebraically closed and p is a closed point, $\mathscr{A}$ is formally A-locally free in p if it is étale A-locally free in p.

1.2.2 Étale local triviality of sheaves of algebras. Recall that $\overline{\Bbbk}$ denotes the algebraic closure of $\Bbbk$.

Lemma 1.2.2.
Let $\mathscr{A}$ be a sheaf of algebras on a finite-type $\Bbbk$-scheme X, A be finite-dimensional $\Bbbk$-algebra, and $\pi\colon X \times \operatorname{Spec}(\overline{\Bbbk}) \to X$ be the canonical projection. The sheaf $\mathscr{A}$ is étale A-locally free at a point $p \in X$ if and only if $\pi^*\mathscr{A}$ is étale $(A \otimes \overline{\Bbbk})$-locally free at all $q \in \pi^{-1}(p)$.

Proof. “ $\Longrightarrow$ ” Follows directly from the fact that the property of being étale is stable under base change; see [DG67, Proposition 17.3.3.(iii)].

“$\Longleftarrow$” Étale A-local triviality is a local property and $\mathscr{A}$ is locally free; see Remark 1.2.1.(1). Therefore, we can assume that $M = \Gamma(X, \mathscr{A})$ is a free R-algebra for $X = \operatorname{Spec}(R)$, where $R \coloneqq \Bbbk[x_1, \dots, x_m]/I$ for some ideal $I \subseteq \Bbbk[x_1, \dots, x_m]$. Then $X \times \operatorname{Spec}(\overline{\Bbbk}) \cong \operatorname{Spec}(\overline{R})$ for $\overline{R} \coloneqq \overline{\Bbbk}[x_1, \dots, x_m]/\overline{\Bbbk}I$ and the natural injective morphism $\iota\colon R \to \overline{R}$ induces $\pi\colon X \times \operatorname{Spec}(\overline{\Bbbk}) \to X$. By definition, p is a prime ideal of R. Fix $\overline{q} \in \pi^{-1}(p)$, i.e. $\overline{q} \subseteq \overline{R}$ is a prime ideal such that $\iota^{-1}(\overline{q}) = p$. The étale A-local triviality of $\pi^*\mathscr{A}$ in $\overline{q}$ can now be formulated as: there exists an étale morphism $\overline{f}\colon \overline{R} \to \overline{S}$ of $\overline{\mathbb{k}}$-algebras, such that $\overline{q} = \overline{f}^{-1}(\overline{r})$ for some prime ideal $\overline{r} \subset S$ and there exists an isomorphism

$$\psi\colon M \otimes_R \overline{S} \cong (M \otimes_R \overline{R}) \otimes_{\overline{R}} \overline{S} \to (A \otimes \overline{\mathbb{k}}) \otimes_{\overline{\mathbb{k}}} \overline{S} \cong A \otimes \overline{S}$$

of $\overline{S}$-algebras. We may assume that $\overline{S} = \overline{\mathbb{k}}[x_1, \dots, x_n]/(s_1, \dots, s_k)$ for some $s_1, \dots, s_k \in \overline{\mathbb{k}}[x_1, \dots, x_n]$. The morphism $\overline{f}$ is completely determined by the images $f_i := \overline{f}(x_i + \overline{\mathbb{k}}I) \in \overline{S}$ of $x_i + \overline{\mathbb{k}}I \in \overline{R}$ for $i \in \{1, \dots, m\}$. We can describe ψ by a matrix $[\psi] \in \mathrm{Mat}_{d\times d}(\overline{S})$ after choosing a basis of A. Here $d = \dim(A)$.

Since $\overline{\mathbb{k}}$ is algebraic over $\mathbb{k}$, we can choose a finite field extension $\mathbb{k}'$ of $\mathbb{k}$ such that $s_1, \dots, s_k \in \mathbb{k}'[x_1, \dots, x_n]$, $f_1, \dots, f_m, \det([\psi])^{-1} \in S$ and $[\psi] \in \mathrm{Mat}_{d\times d}(S)$ for

$$R' := \mathbb{k}'[x_1, \dots, x_m]/\mathbb{k}'I \text{ and } S := \mathbb{k}'[x_1, \dots, x_n]/(s_1, \dots, s_k).$$

Here we used that only finitely many elements of $\overline{\mathbb{k}}$ appeared in these constructions. The assignment $x_i + \mathbb{k}'I \mapsto f_i$ for $i \in \{1, \dots, m\}$ defines a $\mathbb{k}$-algebra morphism $f\colon R' \to S$ making the diagram

$$\begin{array}{ccc} R' & \xrightarrow{f} & S \\ {\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \jmath} \\ \overline{R} & \xrightarrow[\overline{f}]{} & \overline{S} \end{array} \tag{1.4}$$

commutative, where the vertical maps are the canonical ones. In particular, it holds that $f^{-1}(r) = q$ for $q := \iota'^{,-1}(\overline{q})$ and $r := \jmath^{-1}(\overline{r})$.

Since $\overline{f}$ can be identified with $f \otimes_{\mathbb{k}'} \mathrm{id}_{\overline{\mathbb{k}}}$, f is étale because $\overline{f}$ is; see [DG67, Proposition 17.7.1.(ii)]. Furthermore, $R \to R'$ is étale since $\mathbb{k} \to \mathbb{k}'$ is finite, the characteristic of $\mathbb{k}$ is 0, and $R' \cong R \otimes \mathbb{k}'$. The composition $g\colon R \to S$ of the canonical morphism $\iota''\colon R \to R'$ with f is étale and satisfies $g^{-1}(r) = \iota''^{,-1}(f^{-1}(r)) = \iota^{-1}(\overline{f}^{-1}(\overline{r})) = p$, where we used $\iota = \iota'\iota''$ and (1.4). The matrix $[\psi]$ defines a morphism $M \otimes_R S \to A \otimes S$ of S-algebras, which is bijective because of $\det([\psi])^{-1} \in S$. It is easy to see now that $\mathscr{A}$ is étale A-locally free in the point p. □

The rest of this section is dedicated to the translation of results by Kiranagi [Kir78; Kir83] about the analytic local triviality of real Lie algebra bundles into the language of algebraic geometry. We start with the algebro-geometric version of the statement from [Kir78] that a Lie algebra bundle with isomorphic fibers is automatically analytically locally free as a Lie algebra bundle.

Theorem 1.2.3.
Let $\mathscr{A}$ be a sheaf of algebras on a reduced finite-type $\mathbb{k}$-scheme X, A be finite-dimensional $\mathbb{k}$-algebra, and $\pi\colon X \times \mathrm{Spec}(\overline{\mathbb{k}}) \to X$ be the canonical projection. The sheaf $\mathscr{A}$ is étale A-locally free if and only if $\pi^*\mathscr{A}$ is weakly $(A \otimes \overline{\mathbb{k}})$-locally free.

Proof. By Lemma 1.2.2 we may assume that $\Bbbk = \overline{\Bbbk}$ and so we have to show that $\mathscr{A}$ is étale A-locally free if and only if $\mathscr{A}|_p \cong A$ for all closed points $p \in X$. The "$\implies$" part was already discussed in Remark 1.2.1. It remains to prove the "$\impliedby$" part. Recall that an algebraic prevariety is a locally ringed space associated to the closed points of a reduced $\Bbbk$-scheme of finite type and that the category of reduced $\Bbbk$-schemes of finite type is equivalent to the category of algebraic prevarieties. As a consequence, we are permitted to work in the latter category.

Étale local triviality is a local property. Therefore, we can assume that X is an affine variety, i.e. the ringed space associated to the closed points of an reduced affine $\Bbbk$-scheme of finite type. Let us identify A with a $\Bbbk$-algebra of the form $(\Bbbk^d, \mu_A)$. The $\Gamma(X, \mathscr{O}_X)$-algebra $\Gamma(X, \mathscr{A})$ can be identified with the $\Gamma(X, \mathscr{O}_X)$-module of all regular maps $X \to \Bbbk^d$ equipped with a multiplication map $\mu_{\mathscr{A}} \colon \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{A} \to \mathscr{A}$. Said multiplication map is determined by a regular map $\theta \colon X \to \mathrm{Alg}_d \coloneqq \mathrm{Hom}(\Bbbk^d \otimes \Bbbk^d, \Bbbk^d)$ via $\mu_{\mathscr{A}}(a \otimes b)(p) = \theta(p)(a(p) \otimes b(p))$ for all $a, b \colon X \to \Bbbk^d$ regular and $p \in X$. By definition, $\mathscr{A}|_p = (\Bbbk^d, \theta(p))$ for all $p \in X$. The group $G = \mathrm{GL}(d, \Bbbk)$ acts on Alg_d by

$$(L \cdot \vartheta)(v \otimes w) = L^{-1}\vartheta(Lv \otimes Lw) \qquad \forall L \in G, \vartheta \in \mathrm{Alg}_d, v, w \in \Bbbk^d. \tag{1.5}$$

The orbit $G \cdot \mu_A$ coincides with the set of multiplications on $\Bbbk^d$ determining an algebra structure isomorphic to A. Therefore, $\theta(X) \subseteq G \cdot \mu_A$ by assumption. Let us split the remainder of the poof into three steps.

Step 1. *The canonical map $o \colon G \to G \cdot \mu_A$ is a surjective smooth morphism of algebraic prevarieties.* Consider the stabiliser H of μ_A in G. The canonical map $o \colon G \to G/H \cong G \cdot \mu_A$ defined by $L \mapsto L \cdot \mu_A$ is a faithfully flat morphism of algebraic prevarieties and the induced morphism $G \times_{G \cdot \mu_A} G \to G \times H$ is an isomorphism; see e.g. [Mil17]. Note that the pull-back diagram

$$\begin{array}{ccc} G \times_{G\cdot\mu_A} G \cong G \times H & \longrightarrow & G \\ \downarrow & & \downarrow \\ G & \xrightarrow{\;o\;} & G \cdot \mu_A \end{array} \quad , \tag{1.6}$$

combined with the fact that H and hence $G \times H \to G$ is smooth and o is flat, implies that o is smooth; see [DG67, Proposition 17.7.4].

Step 2. *For all $p \in X$ there exists an étale morphism $f \colon Y \to X$ and a morphism $s \colon Y \to G$ such that $p \in f(Y)$ and $os = \theta f$.* Consider the pull-back diagram

$$\begin{array}{ccc} G \times_{G\cdot\mu_A} X & \longrightarrow & G \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle o} \\ X & \xrightarrow{\;\theta\;} & G \cdot \mu_A \end{array} \quad , \tag{1.7}$$

Since the morphism o is surjective and smooth, so is g; see e.g. [GW10, Proposition 6.15.(3)]. Let $p \in X$ be an arbitrary point. Using the construction in [DG67, Corollaire 17.16.3], we see that there exists a locally closed affine subvariety $Y \subseteq G \times_{G \cdot \mu_A} X$ such that $f \coloneqq g|_Y$ is étale and $p \in f(Y)$. Let s be the restriction of the canonical projection $G \times_{G \cdot \mu_A} X \to G$ to Y. By construction $os = \theta f$.

Step 3. *s induces an isomorphism $\psi\colon f^*\mathscr{A} \to A \otimes \mathscr{O}_Y$.* We can identify $\Gamma(Y, f^*\mathscr{A})$ with the $\Gamma(Y, \mathscr{O}_Y)$-module of all regular maps $Y \to \Bbbk^d$ equipped with the multiplication $\mu_{f^*\mathscr{A}}$ determined by $\mu_{f^*\mathscr{A}}(a\otimes b)(q) = \theta(f(q))(a(q)\otimes b(q))$ for all $a, b\colon Y \to \Bbbk^d$ regular and $q \in Y$. Evaluating $os = \theta f$ at an arbitrary point $q \in Y$ results in $s(q)\cdot \mu_A = \theta(f(q))$. Therefore,

$$s(q)\theta(f(q))(a(q)\otimes b(q)) = s(q)(s(q)\cdot\mu_A)(a(q)\otimes b(q)) = \mu_A(s(q)a(q)\otimes s(q)b(q))$$

holds for all regular $a, b\colon Y \to \Bbbk^d$ and $q \in Y$. This shows that the $\Gamma(Y, \mathscr{O}_Y)$-linear automorphism ψ of $\{a\colon Y \to \Bbbk^d \mid a \text{ is regular}\}$ defined by $\psi(a)(q) = s(q)a(q)$ for all $a\colon Y \to \Bbbk^d$ regular and $q \in Y$ induces an isomorphism $\psi\colon f^*\mathscr{A} \to A \otimes \mathscr{O}_Y$ of sheaves of algebras. □

Remark 1.2.4.
Note that, under the assumptions of Theorem 1.2.3, $\pi^*\mathscr{A}|_p \cong \mathscr{A}|_{\pi(p)} \otimes_{\kappa(\pi(p))} \overline{\Bbbk}$ for all $p \in X \times \operatorname{Spec}(\overline{\Bbbk})$ since

$$\begin{array}{ccc} (\{p\}, \overline{\Bbbk}) & \longrightarrow & X \times \operatorname{Spec}(\overline{\Bbbk}) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ (\{\pi(p)\}, \kappa(\pi(p))) & \longrightarrow & X \end{array}$$

commutes. Therefore, the statement of Theorem 1.2.3 can be reformulated as: $\mathscr{A}$ is étale A-locally free if and only if $\mathscr{A}_p \otimes_{\kappa(p)} \overline{\Bbbk} \cong A \otimes \overline{\Bbbk}$ for all $p \in X$.

The following result is an algebro-geometric version of [Kir83, Lemma 2.1].

Proposition 1.2.5.
Let $\mathscr{A}$ be a locally free sheaf of Lie algebras on a reduced finite-type $\Bbbk$-scheme X such that $\mathscr{A}|_p$ is semi-simple for some closed point $p \in X$. Then $\mathscr{A}$ is étale $\mathscr{A}|_p$-locally free at $p \in X$.

Proof. Lemma 1.1.2, Remark 1.2.4 and Cartan's criterion for semi-simplicity imply that $\pi^*\mathscr{A}|_q$ is semi-simple for all $q \in \pi^{-1}(p)$ if and only if $\mathscr{A}|_p$ is semi-simple. Therefore, using Lemma 1.2.2, we may assume that $\Bbbk$ is algebraically closed. As in the setting of the proof of Theorem 1.2.3, we can work in the category of algebraic prevarieties, assume that X is an affine variety and that $\mathscr{A}$ is free of rank d. Let $\operatorname{Lie}_d \subseteq \operatorname{Alg}_d = \operatorname{Hom}(\Bbbk^d \otimes \Bbbk^d, \Bbbk^d)$ be the affine subvariety of all of possible Lie brackets on $\Bbbk^d$. Then $\Gamma(X, \mathscr{A})$ can be identified with the $\Gamma(X, \mathscr{O}_X)$-module of all regular maps $Y \to \Bbbk^d$ equipped with a Lie bracket $\mu_{\mathscr{A}}$ defined by a regular map $\theta\colon X \to \operatorname{Lie}_d$ via $\mu_{\mathscr{A}}(a\otimes b)(q) = \theta(q)(a(q)\otimes b(q))$ for all $a, b\colon X \to \Bbbk^d$ regular and $q \in X$.

The action of $G = \operatorname{GL}_n(\Bbbk)$ on Alg_d discussed in the proof of Theorem 1.2.3 restricts to an action on Lie_d. In particular, the orbit $G\cdot\theta(p)$ coincides with the set of Lie brackets on $\Bbbk^d$ which determine a Lie algebra structure isomorphic to $\mathscr{A}|_p = (\Bbbk^d, \theta(p))$. Combining [NR67, Theorem 7.2] with Whitehead's Lemma and the fact that $\mathscr{A}|_p$ is semi-simple, we see that $G\cdot\theta(p) \subseteq \operatorname{Lie}_d$ contains an open neighbourhood V of $\theta(p)$ and as a consequence $U := \theta^{-1}(V)$ is an open neighbourhood of p. This means that for all $q \in U$, we have $\mathscr{A}|_q \cong (\Bbbk^d, \theta(p)) = \mathscr{A}|_p$. Theorem 1.2.3 now asserts that $\mathscr{A}|_U$ is étale A-locally free. □

1.2.3 Interlude: the small étale site. The concept of "étale topology" can be formalized by passing from usual topologies to Grothendieck topologies. Roughly speaking, this means that we replace the set of open subsets with a category of morphisms resembling open immersions in the original context. We will not define Grothendieck topologies here. However, we sketch in the following how in this context topological notions such as sheaves and coverings can be understood for the only example relevant to us: the topology where étale morphisms are treated as open immersions.

Let S be a scheme. The *small étale site* $S_{\text{ét}}$ is the category of étale S-schemes, i.e. the category of schemes S' equipped with an étale structure morphism $f_{S'}\colon S' \to S$ and morphisms of S-schemes as morphisms. Note that all morphisms in $S_{\text{ét}}$ are étale itself (see [DG67, Proposition 17.3.4]) and open subsets $U \subseteq S$ are understood as objects in $S_{\text{ét}}$ via the open immersion $U \to S$. For two morphisms $U, V \in S_{\text{ét}}$ the fiber product $U \times_S V \in S_{\text{ét}}$ takes the role of the intersection of U and V and the natural étale morphisms $U \leftarrow U \times_S V \to V$ can be thought of as the respective inclusion maps. An *étale neighbourhood* of a point $p \in S$ is an object $U \in S_{\text{ét}}$ such that $p \in f_U(U)$. We call a family $\{U_i\}_{i\in I} \subseteq S_{\text{ét}}$ *étale covering* of $U \in S_{\text{ét}}$ if all $\{f_{U_i}\}_{i\in I}$ factor over f_U and $\bigcup_{i\in I} f_{U_i}(U_i) = U$ holds.

A *presheaf* $\mathscr{F}$ on $S_{\text{ét}}$ is a contravariant functor $\mathscr{F}\colon S_{\text{ét}} \to \mathbf{set}$, where **set** is the category of sets. In particular, for every $S' \in S_{\text{ét}}$ we have a set $\Gamma(S', \mathscr{F}) = \mathscr{F}(S')$ of *sections* and for every morphism $S' \to S''$ in $S_{\text{ét}}$ there is a *restriction map*

$$(\cdot)|_{S'}\colon \Gamma(S'', \mathscr{F}) \to \Gamma(S', \mathscr{F}). \tag{1.8}$$

We call $\mathscr{F}$ a sheaf if for every $U \in S_{\text{ét}}$ and every étale covering $\{U_i\}_{i\in I}$ of U the natural sequence

$$\Gamma(U, \mathscr{F}) \longrightarrow \prod_{i\in I} \Gamma(U_i, \mathscr{F}) \rightrightarrows \prod_{i,j\in I} \Gamma(U_i \times_U U_j, \mathscr{F}) \tag{1.9}$$

is an equalizer. In other words, for every family $\{s_i \in \Gamma(U_i, \mathscr{F})\}_{i\in I}$ such that the identities $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ hold for all $i, j \in I$ exists a unique $s \in \Gamma(U, \mathscr{F})$ satisfying $s|_{U_i} = s_i$ for all $i \in I$.

Morphisms of presheaves and sheaves are simply natural transformations as functors. For any $U \in S_{\text{ét}}$, we get a sheaf $\mathscr{F}|_U$ on $U_{\text{ét}}$ via $\Gamma(V, \mathscr{F}|_U) = \Gamma(V, \mathscr{F})$ for all étale morphisms $V \to U$, where on the right-hand side the induced morphism $V \to U \to X$ is used. Notions such as sheaves of (abelian) groups, sheaves of modules, sheaves of algebras, etc. can be naturally generalized to produce subcategories of the category of sheaves on $S_{\text{ét}}$.

The easiest example of a sheaf on $S_{\text{ét}}$ is the sheaf h_Z *represented* by some S-scheme Z, which assigns to each $S' \in S_{\text{ét}}$ the set of morphisms $S' \to Z$ of S-schemes. Clearly, h_G is a sheaf of groups if G is a group scheme over S. For every $\mathscr{O}_S$-module $\mathscr{F}$, the assignment $S_{\text{ét}} \ni U \mapsto \mathrm{H}^0(f_U^*\mathscr{F})$ defines a sheaf $\mathscr{F}_{\text{ét}}$ on $S_{\text{ét}}$ with values in the category of abelian groups. More accurately, $\mathscr{F}_{\text{ét}}$ is an $\mathscr{O}_{S,\acute{e}t}$-module.

We now want to interpret étale local triviality of sheaves of algebras using the small étale site. Let $\mathscr{A}$ be a sheaf of algebras on a $\mathbb{k}$-scheme X and A be a $\mathbb{k}$-algebra. The sheaf $\mathscr{A}$ is étale A-locally free in $p \in X$ if and only if there exists an étale neighbourhood $U \in X_{\text{ét}}$ of p such that $\mathscr{A}_{\text{ét}}|_U \cong A \otimes \mathscr{O}_{U,\text{ét}}$ as $\mathscr{O}_{U,\text{ét}}$-algebras.

1.2.4 Sheaves of algebras and torsors. In this paragraph, we will briefly recall the notion of (étale) torsors and their relation to sheaves of algebras. These objects are algebro-geometric versions of principal fiber bundles. We will restrict ourselves to the definition of torsors using 1-cocycles, which is similar to the definition of principal fiber bundles using 1-cocycles; for a more general approach see e.g. [Mil80, Paragraph III.4]. Our interest in torsors stems from the fact that they are classified by an analog of the first Čech cohomology group. This will be useful in some classification problems concerning sheaves of algebras; see Subsection 6.1.

Let S be a scheme, $\mathscr{G}$ be a group sheaf on $S_{\text{ét}}$ and $\mathbf{U} = \{U_i\}_{i\in I}$ be an étale covering of S. In the following, we use the notation $U_{i_1\ldots i_k} = U_{i_1} \times_S \cdots \times_S U_{i_k}$ for all $i_1, \ldots, i_k \in I$. A set $\{g_{ij} \in \Gamma(U_{ij}, \mathscr{G})\}_{i,j\in I}$ is called *étale 1-cocycle* trivialized at $\mathbf{U}$ if

$$\left(g_{ij}|_{U_{ijk}}\right)\left(g_{jk}|_{U_{ijk}}\right) = g_{ik}|_{U_{ijk}} \tag{1.10}$$

holds for all $i, j, k \in I$. It is called *cohomologous* to another étale 1-cocycle $\{g'_{ij}\}_{i,j\in I}$ if there exists a family $\{h_i \in \Gamma(U_i, \mathscr{G})\}_{i\in I}$ such that $g'_{ij}\left(h_j|_{U_{ij}}\right) = \left(h_i|_{U_{ij}}\right) g_{ij}$ for all $i, j \in I$. The set of cohomology classes of étale 1-cocycles trivialized at $\mathbf{U}$ will be denoted by $\check{\mathrm{H}}^1(\mathbf{U}, \mathscr{G})$. Note that for another étale covering $\mathbf{U}' \coloneqq \{U'_i\}_{i\in I'}$ of S we have an étale covering $\mathbf{U} \times_S \mathbf{U}' \coloneqq \{U_i \times_S U'_j\}_{i\in I, j\in I'}$ and natural maps

$$\check{\mathrm{H}}^1(\mathbf{U}, \mathscr{G}) \longrightarrow \check{\mathrm{H}}^1(\mathbf{U} \times_S \mathbf{U}', \mathscr{G}) \longleftarrow \check{\mathrm{H}}^1(\mathbf{U}', \mathscr{G})$$

making the set of étale covers to a directed set and the étale 1-cocycles to a directed system over this set. An *étale $\mathscr{G}$-torsor* is an element of

$$\check{\mathrm{H}}^1(S_{\text{ét}}, \mathscr{G}) \coloneqq \varinjlim_{\mathbf{U}} \check{\mathrm{H}}^1(\mathbf{U}, G). \tag{1.11}$$

Note that $\check{\mathrm{H}}^1(S_{\text{ét}}, \mathscr{G})$ is a *pointed set*, i.e. it has an distinguished element defined by the neutral element of $\Gamma(S, \mathscr{G})$ interpreted as a 1-cocycle trivialized on the trivial étale covering $S \to S$. If $\mathscr{G}$ is represented by some group scheme G over S, i.e. $\mathscr{G} = \mathrm{h}_G$, we also write $\check{\mathrm{H}}^1(S_{\text{ét}}, G) \coloneqq \check{\mathrm{H}}^1(S_{\text{ét}}, \mathscr{G})$.

Lemma 1.2.6.
Let X be a quasi-compact $\mathbb{k}$-scheme and A be a $\mathbb{k}$-algebra.

(1) *Let $\mathscr{G}$ be the group sheaf on $X_{\text{ét}}$ defined by: $\Gamma(U, \mathscr{G})$ are the sheaf of algebras automorphisms of $A \otimes \mathscr{O}_U$ for all $U \in S_{\text{ét}}$. The isomorphism classes of étale A-locally free sheaves of Lie algebras on X are in bijection with $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathscr{G})$.*

(2) *Assume that X is reduced, A is finite-dimensional, $\mathbb{k}$ is algebraically closed, and $\mathrm{Aut}_{\mathbb{k}\text{-alg}}(A)_X \coloneqq X \times \mathrm{Aut}^{\mathrm{sch}}_{\mathbb{k}\text{-alg}}(A)$, where $\mathrm{Aut}^{\mathrm{sch}}_{\mathbb{k}\text{-alg}}(A)$ is the affine scheme with $\mathbb{k}$-rational points $\mathrm{Aut}_{\mathbb{k}\text{-alg}}(A)$. The isomorphism classes of étale A-locally free sheaves of Lie algebras on X are in bijection with $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\mathbb{k}\text{-alg}}(A)_X)$.*

Proof of (1). Let $\mathscr{A}$ be an étale A-locally free sheaf of algebras on X. Then there exists an étale covering $\{U_i\}_{i\in I}$ of X and isomorphisms $\phi_i\colon f_{U_i}^*\mathscr{A} \to A\otimes\mathscr{O}_{U_i}$ of sheaves of algebras. Let $g_{ij}\in\Gamma(U_{ij},\mathscr{G})$ be defined as the composition

$$A\otimes\mathscr{O}_{U_{ij}} \xrightarrow{\pi_1^*(\phi_i)} \pi_1^* f_{U_i}^*\mathscr{A} \xrightarrow{\cong} \pi_2^* f_{U_j}^*\mathscr{A} \xrightarrow{\pi_2^*(\phi_j)^{-1}} A\otimes\mathscr{O}_{U_{ij}} \tag{1.12}$$

for all $i,j\in I$, where $\pi_1\colon U_{ij}=U_i\times_S U_j\to U_i$ and $\pi_2\colon U_{ij}\to U_j$ are the canonical projections. It is easy to see that $g=\{g_{ij}\}_{i,j\in I}$ is a 1-cocycle and a sheaf of algebras isomorphic to $\mathscr{A}$ defines a 1-cocycle cohomologous to g. Thus, we have defined a mapping from the set of isomorphism classes E of étale A-locally free sheaves of algebras to $\check{\mathrm{H}}^1(X_{\text{ét}},\mathscr{G})$.

We will now construct the inverse of the map $E\to\check{\mathrm{H}}^1(X_{\text{ét}},\mathscr{G})$. Let $g=\{g_{ij}\}_{i,j\in I}$ be an étale 1-cocycle trivialized at the étale covering $\mathbf{U}=\{U_i\}_{i\in I}$ of X. Since X is quasi-compact, we may assume that I is finite. Then $U\coloneqq\coprod_{i\in I}U_i\in X_{\text{ét}}$ and the structure morphism $f_U\colon U\to X$ is faithfully flat since it is étale and surjective. Let $\phi\in\Gamma(U\times_X U,\mathscr{G})$ be the automorphism of $A\otimes\mathscr{O}_U$ defined by $\{g_{ij}\}_{i,j\in I}$. The cocycle condition (1.10) now takes the form $\pi_{31}^*(\phi)=\pi_{32}^*(\phi)\pi_{21}^*(\phi)$, where $\pi_{ij}\colon U\times_X U\times_X U\to U\times_X U$ is the projection $(x_1,x_2,x_3)\mapsto(x_i,x_j)$ for $ij\in\{21,31,32\}$. Faithfully flat descent (see e.g. [Mil80, Proposition 2.22]) provides a coherent sheaf $\mathscr{A}$ on X equipped with a sheaf of algebra structure such that $f_U^*\mathscr{A}\cong A\otimes\mathscr{O}_U$, i.e. $\mathscr{A}$ is an étale A-locally free sheaf of algebras. Indeed, this claim can be reduced to the case that X and U are affine. Then $\mathscr{A}$ coincides with

$$f_{U,*}\left(\operatorname{Ker}(\phi\pi_1^*-\pi_2^*\colon A\otimes\mathscr{O}_U\to A\otimes\mathscr{O}_{U\times U})\right). \tag{1.13}$$

In particular, $\mathscr{A}$ is a subsheaf of algebras of $f_{U,*}(A\otimes\mathscr{O}_U)$, since ϕ is an automorphism of sheaves of algebras. Here, $\pi_1,\pi_2\colon U\times_X U\to U$ are the canonical projections. Furthermore, we can also deduce that an étale 1-cocycle cohomologous to g results in a sheaf of algebras isomorphic to $\mathscr{A}$. Thus, we obtain a map $\check{\mathrm{H}}^1(X_{\text{ét}},\mathscr{G})\to E$ and it is not hard to see that this map is inverse to the map $E\to\check{\mathrm{H}}^1(X_{\text{ét}},\mathscr{G})$ constructed above.

Proof of (2). It suffices to prove that $\mathscr{G}$ is naturally equivalent to the group sheaf on $X_{\text{ét}}$ represented by $\mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X$ as contravariant functor on $X_{\text{ét}}$ with values in the category of groups. For this, note that every automorphism of $A\otimes\mathscr{O}_U$ as sheaves of algebras on a X-scheme U is uniquely determined by a regular map $U(\Bbbk)\to\mathrm{Aut}_{\Bbbk\text{-alg}}(A)$, where $U(\Bbbk)$ are the $\Bbbk$-rational points of U, which coincide with the closed points of U since $\Bbbk$ is algebraically closed. This regular map is uniquely determined by a morphism $U\to\mathrm{Aut}^{\mathrm{sch}}_{\Bbbk\text{-alg}}(A)$, which again defines a morphism $U\to\mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X$ of X-schemes. We obtain an isomorphism $\Gamma(U,\mathscr{G})\cong\mathrm{h}_{\mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X}(U)$. It is straight forward to see that these isomorphisms define the desired natural equivalence. □

1.3 Lattices in current algebras and sheaves of algebras on projective curves

Let A be a finite-dimensional vector space over $\Bbbk$ and W be a subset of the space of A-valued Laurent series $A(\!(z)\!)$ (for an introduction on formal power and Laurent series, see Subsection 2.1.1 below). In the following, we write $W_k := W \cap z^{-k}A(\!(z)\!)$ and note that $\ldots \subseteq W_k \subseteq W_{k+1} \subseteq \ldots$ holds. Moreover, we say that an element $F(z) = z^{-k}a + \ldots \in W_k \setminus W_{k-1}$ is of *order* k and has *main part* $z^{-k}a$.

Let A be a finite-dimensional $\Bbbk$-algebra. A subspace $W \subseteq A(\!(z)\!)$ is called A-*lattice* of *index* (h_0, h_1) if W is a subalgebra and both

$$h_0 := \dim(A[\![z]\!] \cap W) \text{ and } h_1 := \dim(A(\!(z)\!)/(A[\![z]\!] + W)) \tag{1.14}$$

are finite. In other words, $\dim(W_0) < \infty$ and $\ell \in \mathbb{N}_0$ exists such that for all $a \in A$ and $k \geqslant \ell$ an element $\widetilde{a} \in W$ with main part az^{-k} exists.

Remark 1.3.1.
Note that a unital $\Bbbk$-lattice O automatically satisfies $O_0 = O \cap \Bbbk[\![z]\!] = \Bbbk$. Indeed, if there exists $f \in O_0 \setminus \Bbbk$, O_0 would contain the infinite linearly independent set $\{(f - f(0))^k \mid k \in \mathbb{N}\}$, which contradicts the fact that O is a $\Bbbk$-lattice.

In this section, we establish a connection between A-lattices, which are modules over $\Bbbk$-lattices, and sheaves of algebras on a projective curve, which are formally A-locally free at a distinguished $\Bbbk$-rational smooth point. More precisely, we introduce appropriate categories of such lattices and geometric data in subsections 1.3.2 and 1.3.3 and construct an equivalence between these categories in the remainder of this section. These constructions are based on the methods used in [Mum78; Mul90]. The existence of a $\Bbbk$-lattice stabilizing an A-lattice, which is necessary for this geometrization procedure, can be settled if A is central simple by a result of Ostapenko; see [Ost92] or Theorem 1.3.3. Let us begin this section by recalling the definition and basic properties of central simple algebras.

1.3.1 Prelude: central simple algebras. Let A be a finite-dimensional $\Bbbk$-algebra with multiplication $\mu_A(a, b) = ab$ for all $a, b \in A$. The *centroid* of A is the $\Bbbk$-subalgebra C of $\operatorname{End}(A)$ consisting of $\Bbbk$-linear maps $\psi\colon A \to A$ such that

$$\psi(ab) = a\psi(b) = \psi(a)b \text{ for all } a, b \in A. \tag{1.15}$$

In other words, C is the centralizer of the subalgebra $\langle r_a, \ell_a \mid a \in A\rangle_{\Bbbk\text{-alg}}$ of $\operatorname{End}(A)$ generated by $\{r_a, \ell_a \mid a \in A\}$, where $r_b(a) := \ell_a(b) := ab$ for all $a, b \in A$. Obviously, $\Bbbk\mathrm{id}_A \subseteq C$ and A is called *central* if $C = \Bbbk\mathrm{id}_A$. Recall that A is called *simple* if $AA \neq \{0\}$ and there is no $\Bbbk$-subspace $I \subseteq A$ such that $IA + AI \subseteq I$ holds except $\{0\}$ and A. The following theorem summarizes the basic properties of central simple $\Bbbk$-algebras, which can be found in e.g. [Jac79, Section X.1].

Theorem 1.3.2.
Let A be a finite-dimensional simple $\Bbbk$-algebra with centroid C. The following results are true.
(1) *C is a field. In particular, $\Bbbk \to C$ is a finite field extension.*
(2) *If A is central and $\Bbbk'$ is any field extension of $\Bbbk$, $A \otimes \Bbbk'$ is again central simple as a $\Bbbk'$-algebra.*
(3) *If A is central, the identity $\operatorname{End}(A) = \langle r_a, \ell_a \mid a \in A\rangle_{\Bbbk\text{-alg}}$ holds.*

1.3.2 The category Lat_A of ringed A-lattices. Let A be a finite-dimensional $\Bbbk$-algebra. A *ringed A-lattice* is a pair (O, W), where O is a unital $\Bbbk$-lattice and W is an A-lattice such that $OW \subseteq W$. A *morphism of ringed A-lattices*

$$(w, \phi)\colon (O_1, W_1) \to (O_2, W_2) \tag{1.16}$$

consists of a series $w \in z\Bbbk[\![z]\!]^\times$ and a map $\phi \in \operatorname{End}_{\Bbbk[\![z]\!]\text{-alg}}(A[\![z]\!]) \subseteq \operatorname{End}(A)[\![z]\!]$ such that $f(z) \mapsto f(w(z))$ maps O_1 to O_2 and $a(z) \mapsto \phi(z)a(w(z))$ maps W_1 to W_2. The class of ringed A-lattices with this notion of morphisms defines the category Lat_A of ringed A-lattices. We have the following important result from Ostapenko [Ost92].

Theorem 1.3.3.
Let A be a finite-dimensional, central, simple $\Bbbk$-algebra and W be some A-lattice. The unital $\Bbbk$-algebra $\operatorname{Mult}(W) := \{f \in \Bbbk(\!(z)\!) \mid fW \subseteq W\}$ is a $\Bbbk$-lattice. In particular, $(O, W) \in \mathrm{Lat}_A$ for all A-lattices W and unital subalgebras $O \subseteq \operatorname{Mult}(W)$ of finite codimension.

Proof. First note that $\operatorname{Mult}(W) \cap \Bbbk[\![z]\!] = \Bbbk$, since otherwise we would get a contradiction with $\dim(W_0) < \infty$ in a similar way as in Remark 1.3.1. Thus, we have to show that $\ell \in \mathbb{N}$ exists such that for all $j \geqslant \ell$ there exists $\lambda \in \operatorname{Mult}(W)$ with main part z^{-j}. Let us denote by r_a (resp. ℓ_a) the right (resp. left) multiplication by an element $a \in A(\!(z)\!)$ considered as an element in $\operatorname{End}_{\Bbbk(\!(z)\!)}(A(\!(z)\!)) \cong \operatorname{End}(A)(\!(z)\!)$, i.e. $\ell_a(b) = ab = r_b(a)$ for all $a, b \in A(\!(z)\!)$. Note that $r_a, \ell_a \in \operatorname{End}(A)$ for $a \in A$. Let $J := \langle r_a, \ell_a \mid a \in W\rangle_{\Bbbk\text{-alg}} \subseteq \operatorname{End}(A)(\!(z)\!)$.

The fact that A is central combined with [Jac79, Chapter X, Theorem 4.] implies that $\operatorname{End}(A) = \langle r_a, \ell_a \mid a \in A\rangle_{\Bbbk\text{-alg}}$. In particular, for every $L \in \operatorname{End}(A)$ exists a non-commutative polynomial $f = f(x_1, \dots, x_q)$ and $a_1, \dots, a_q \in A$ such that $f(m_1, \dots, m_q) = L$, where $m_i \in \{r_{a_i}, \ell_{a_i}\}$ for all $i \in \{1, \dots, q\}$. Let $\{f_i\}_{i=1}^q$ be the unique polynomials defined by the following inductive process: f_q is the sum of all monomials of f depending on x_q and, if $f_{i+1}, \dots, f_q$ are given, f_i is the sum of all monomials of $f - f_{i+1} - \dots - f_q$ which depend on x_i. By construction, f_i depends only on $x_1, \dots, x_i$ and every monomial of f_i contains a factor x_i. Let f_{ij} denote the homogeneous component of f_i of degree j and $g_{ij}(x_1, \dots, x_i; y_i)$ be the polynomial where the leftmost x_i appearing in every monomial of f_{ij} is changed to y_i. Since W is an A-lattice, we can chose $s \in \mathbb{N}$ such that for all $i \in \{1, \dots, q\}$ and $k \in \mathbb{N}_0$ there exists b_i^k in W with main part $z^{-s-k}a_i$. Let $\widetilde{m}_i^k$ be the left (resp. right) multiplication by b_i^k if m_i is the left (resp. right) multiplication by a_i. By construction $g_{ij}\left(\widetilde{m}_1^0,\dots,\widetilde{m}_i^0;\widetilde{m}_i^k\right)$ has main part $z^{-k-sj}f_{ij}(m_1,\dots,m_i)$. Thus,

$$\sum_{i=1}^{q}\sum_{j=1}^{\deg(f_i)} g_{ij}\left(\widetilde{m}_1^0,\dots,\widetilde{m}_i^0;\widetilde{m}_i^{k+s(\deg(f)-j)}\right) \tag{1.17}$$

has main part $z^{-k-s\deg(f)}L$. Putting $r(L) := s\deg(f)$, we see that for all $k\in\mathbb{N}_0$ exists an element of J with main part $z^{-k-r(L)}L$. Let $\{L_i\}_{i=1}^n$ be a basis of $\operatorname{End}(A)$ and set $r := \max\{r(L_i) \mid i\in\{1,\dots,q\}\}$. We have:

$$\text{for all } L\in\operatorname{End}(A), k\in\mathbb{N}_0 \text{ exists } \widetilde{L}^k\in J \text{ with main part } z^{-k-r}L. \tag{1.18}$$

In [Ami76] the author constructs a non-commutative, non-vanishing homogeneous polynomial $P = P(x_1,\dots,x_q)$ in $q := 2\dim(A)^2$-variables which takes values in $\Bbbk\mathrm{id}_A$ when evaluated on $\operatorname{End}(A)^q$. In particular, we may choose $L_1,\dots,L_q\in\operatorname{End}(A)$ such that $P(L_1,\dots,L_q) = \mathrm{id}_A$. Using (1.18) results in

$$P\left(\widetilde{L}_1^0,\dots,\widetilde{L}_{q-1}^0,\widetilde{L}_q^{j-rq}\right) = \lambda_\ell\mathrm{id}_A \tag{1.19}$$

for all $j\geqslant rq$, where $\lambda_j\in\Bbbk(\!(z)\!)$ has main part z^{-j}. Since the left-hand side of (1.19) is an element of J, we can see that $\lambda_j\in\operatorname{Mult}(W)$. □

1.3.3 The category $\mathbf{GeomLat}_A$ geometric A-lattice models. Let A be a finite-dimensional $\Bbbk$-algebra. A *geometric A-lattice model* is a quintuple $((X,\mathscr{A}),(p,c,\zeta))$, where

- X is an integral projective curve over $\Bbbk$, $p\in X$ is a smooth $\Bbbk$-rational point equipped with a $\Bbbk$-algebra isomorphism $c\colon\widehat{\mathscr{O}}_{X,p}\to\Bbbk[\![z]\!]$ and
- $\mathscr{A}$ is a torsion-free coherent $\mathscr{O}_X$-algebra equipped with a c-equivariant algebra isomorphism $\zeta\colon\widehat{\mathscr{A}_p}\to A[\![z]\!]$.

In particular, $\mathscr{A}$ is by definition formally A-locally free in p. Note that c induces an isomorphism $\mathrm{Q}(\widehat{\mathscr{O}}_{X,p})\to\Bbbk(\!(z)\!)$ and ζ induces a c-equivariant algebra isomorphism $\mathrm{Q}(\widehat{\mathscr{A}_p})\to A(\!(z)\!)$ which will again be denoted by c and ζ respectively. Since $\mathscr{A}$ is torsion-free, $\Gamma(U,\mathscr{A})\subseteq\mathrm{Q}(\widehat{\mathscr{A}_p})$ holds for all $U\subseteq X$ open, and ζ is an algebra isomorphism, it can be seen that $\mathscr{A}$ is a sheaf of Lie algebras if A is a Lie algebra. A *morphism of geometric A-lattice models*

$$(f,\phi)\colon((X_1,\mathscr{A}_1),(p_1,c_1,\zeta_1))\to((X_2,\mathscr{A}_2),(p_2,c_2,\zeta_2)) \tag{1.20}$$

consists of

- a birational morphism $f\colon X_2\to X_1$ such that $f(p_2) = p_1$ and

$$\begin{array}{ccc} \widehat{\mathscr{O}}_{X_1,p_1} & \xrightarrow{\ f^\sharp_{p_2}\ } & \widehat{\mathscr{O}}_{X_2,p_2} \\ & {\scriptstyle c_1}\searrow \quad \swarrow{\scriptstyle c_2} & \\ & \Bbbk[\![z]\!] & \end{array}$$

commutes while

- $\phi\colon \mathscr{A}_1 \to f_*\mathscr{A}_2$ is a morphism of $\mathscr{O}_{X_1}$-algebras such that, under consideration of $(f_*\mathscr{A})_{p_1} \cong \mathscr{A}_{p_2}$, the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathscr{A}}_{1,p_1} & \xrightarrow{\widehat{\phi}_{p_1}} & \widehat{\mathscr{A}}_{2,p_2} \\ & {\scriptstyle \zeta_1}\searrow \quad \swarrow{\scriptstyle \zeta_2} & \\ & A[\![z]\!] & \end{array} .$$

1.3.4 The trivialization functor $\mathbb{T}\colon \mathbf{GeomLat}_A \to \mathbf{Lat}_A$. In order to relate the categories $\mathrm{GeomLat}_A$ to Lat_A, for some finite-dimensional $\Bbbk$-algebra A, we note the following general result.

Theorem 1.3.4.
Let X be an integral projective curve over $\Bbbk$ and $p \in X$ be either a smooth closed point or, if X is additionally Gorenstein, an arbitrary closed point. Then we have a canonical exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathscr{F}) \longrightarrow \Gamma(X \setminus \{p\}, \mathscr{F}) \oplus \widehat{\mathscr{F}}_p \longrightarrow \mathrm{Q}(\widehat{\mathscr{F}}_p) \longrightarrow \mathrm{H}^1(\mathscr{F}) \longrightarrow 0$$

for every coherent sheaf $\mathscr{F}$ on X.

The statement concerning a smooth point on an arbitrary integral projective curve can be found in [Par01, Proposition 3], while the Gorenstein singular case is discussed in [Gal15, Chapter 3]. Let $((X, \mathscr{A}), (p, c, \zeta)) \in \mathrm{GeomLat}_A$ for an finite-dimensional $\Bbbk$-algebra A. Then Theorem 1.3.4 implies that

$$(c(\Gamma(X \setminus \{p\}, \mathscr{O}_X)), \zeta(\Gamma(X \setminus \{p\}, \mathscr{A}))) \in \mathrm{Lat}_A\,. \tag{1.21}$$

This assignment is easily seen to define a functor $\mathbb{T}\colon \mathrm{GeomLat}_A \to \mathrm{Lat}_A$, called *trivialization functor*.

1.3.5 The quasi-inverse $\mathbb{G}\colon \mathbf{Lat}_A \to \mathbf{GeomLat}_A$ of $\mathbb{T}$. Let us split the construction of the quasi-inverse of $\mathbb{T}$ into two steps, starting with the observation that any unital $\Bbbk$-lattice defines an integral projective curve over $\Bbbk$.

Proposition 1.3.5.
Let O be a unital $\Bbbk$-lattice of index (h_0, h_1).
(1) $\mathrm{gr}(O) \coloneqq \bigoplus_{k\in\mathbb{N}_0} t^k O_k \subseteq O[t]$ *is a unital graded $\Bbbk$-algebra and $h_0 = 1$. Furthermore, $X \coloneqq \mathrm{Proj}(\mathrm{gr}(O))$ is an integral projective curve over $\Bbbk$ of arithmetic genus h_1.*
(2) *There is a distinguished $\Bbbk$-rational smooth point $p \in X$ equipped with a natural $\Bbbk$-algebra isomorphism $c\colon \widehat{\mathscr{O}}_{X,p} \to \Bbbk[\![z]\!]$ such that the induced map $c\colon \mathrm{Q}(\widehat{\mathscr{O}}_{X,p}) \to \Bbbk(\!(z)\!)$ satisfies $c(\Gamma(X \setminus \{p\}, \mathscr{O}_X)) = O$. In particular, c induces an isomorphism $\mathrm{Spec}(O) \to X \setminus \{p\}$.*

Proof. That $\mathrm{gr}(O)$ is a graded $\Bbbk$-subalgebra of $O[t]$ follows directly from $O_k O_\ell \subseteq O_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$, while $h_0 = 1$ was already shown in Remark 1.3.1. We proceed in five steps.

Step 1. *O has Krull dimension one.* The condition $\dim(\Bbbk(\!(z)\!)/(\Bbbk[\![z]\!]+O)) < \infty$ implies that for a sufficiently large $r \in \mathbb{N}$ there exist elements f and g of O with main parts z^{-r} and z^{-r-1} respectively. The fact $O_0 = \Bbbk$ implies that the canonical projection $O \to \Bbbk[z^{-1}]$ is injective. Consequently, $\Bbbk[f, g] \subseteq O$ is of finite codimension, since for every $\ell_1 \geqslant r$ and $0 \leqslant \ell_2 \leqslant r-1$ the element $f^{\ell_1-\ell_2} g^{\ell_2}$ has main part $z^{-\ell_1 r-\ell_2}$. Therefore, the Krull dimension of O and the Krull dimension of $\Bbbk[f, g]$ coincide. The latter is one, since, if $h_1, \dots, h_k$ is a basis of $\Bbbk[f, g]_{r(r+1)-1}$, we have

$$\Bbbk[f, g]_{r(r+1)-1} \ni f^{r+1} - g^r = c_1 h_1 + \dots + c_k h_k \tag{1.22}$$

for some $c_1, \dots, c_k \in \Bbbk$, which is a polynomial relation of f and g.

Step 2. *Construction of p and c.* By definition of $X = \mathrm{Proj}(\mathrm{gr}(O))$, the homogeneous prime ideal $p := (t) = \bigoplus_{k\in\mathbb{N}_0} t^{k+1} O_k$ generated by $t \in \mathrm{gr}(O)$ is a point of X. Observe that $t^k h$ is an element of the homogeneous elements S of $\mathrm{gr}(O) \setminus (t)$ if and only if h has order k. This shows

$$\mathscr{O}_{X,p} \cong (S^{-1}\mathrm{gr}(O))_0 = \{a/h \mid a, h \in O, a/h \in \Bbbk[\![z]\!]\} = \mathrm{Q}(O) \cap \Bbbk[\![z]\!]. \tag{1.23}$$

Choosing f, g as in Step 2 yields $u := f/g \in \mathrm{Q}(O) \cap z\Bbbk[\![z]\!]^\times$. Therefore, $\Bbbk[u] \subseteq \mathrm{Q}(O) \cap \Bbbk[\![z]\!] \subseteq \Bbbk[\![z]\!]$ and $\Bbbk[\![u]\!] = \Bbbk[\![z]\!]$ results in an isomorphism $c\colon \widehat{\mathscr{O}}_{X,p} \to \Bbbk[\![z]\!]$. We conclude that p is $\Bbbk$-rational and smooth.

Step 3. *We have $c(\Gamma(\mathrm{D}_+(t), \mathscr{O}_X)) = O$, i.e.* $\mathrm{Spec}(O) \cong \mathrm{D}_+(t)$. Since $\Bbbk[\![z]\!]_{(u)} = \Bbbk(\!(z)\!)$ for u from Step 2, we can see from (1.23) that the rational functions on X can be identified with $\mathrm{Q}(O)$ via the isomorphism $c\colon \mathrm{Q}(\widehat{\mathscr{O}}_{X,p}) \to \Bbbk(\!(z)\!)$. More precisely, we can deduce that

$$c(\Gamma(\mathrm{D}_+(t^k h), \mathscr{O}_X)) = \mathrm{gr}(O)[(t^k h)^{-1}]_0 \subseteq \mathrm{Q}(O) \tag{1.24}$$

holds for all $t^k h \in \mathrm{gr}(O)$. More precisely, the formal trivialization c induces the natural isomorphism $\Gamma(\mathrm{D}_+(t^k h), \mathscr{O}_X) \cong \mathrm{gr}(O)[(t^k h)^{-1}]_0$. In particular, we see that

$$c(\Gamma(\mathrm{D}_+(t), \mathscr{O}_X)) = \mathrm{gr}(O)[t^{-1}]_0 = O. \tag{1.25}$$

Therefore, c defines an isomorphism $\mathrm{Spec}(O) \to \mathrm{D}_+(t)$.

Step 4. *X is an integral projective curve over $\Bbbk$.* Using the steps 1,2 and 4, it remains to show that X is a $\Bbbk$-scheme of finite type, since then $\dim(X) = \dim(\mathrm{D}_+(t)) = 1$. Thus, we have to show that $\mathrm{gr}(O)$ is a finitely-generated $\Bbbk$-algebra; see e.g. [GW10, Lemma 13.9.(2) and Proposition 13.12]. We will prove that each basis B of the finite dimensional space $\bigoplus_{k=0}^{r^2} t^k O_k$, containing $t, t^r f$ and $t^{r+1} g$, generates $\mathrm{gr}(O)$, where f, g and r are as in Step 2. Let us write R for the $\Bbbk$-subalgebra of $\mathrm{gr}(O)$ generated by B. We prove by induction on $k \geqslant r^2$ that $t^k O_k \subseteq R$, which is obvious for $k = r^2$. By induction assumption $t^{k-1} O_{k-1} \subseteq R$, so $t^k O_{k-1} \subseteq R$ since $t \in R$. Let $h \in O$ have main part az^{-k} and $\ell_1 \geqslant r, 0 \leqslant \ell_2 \leqslant r-1$ be such that $k = \ell_1 r + \ell_2$. Then

$$t^k h - a(t^r f)^{\ell_1-\ell_2}(t^{r+1} g)^{\ell_2} = t^k h - a t^k f^{\ell_1-\ell_2} g^{\ell_2} \in t^k O_{k-1} \subseteq R,$$

proving $t^k h \in R$ and this gives the induction step.

Step 5. $\mathrm{D}_+(t) = X \setminus \{p\}$ *and* $\mathrm{h}^1(\mathscr{O}_X) = h_1$. Since $O \cap \Bbbk[\![z]\!] = \Bbbk$ by Step 1, the open subscheme $D_+(t) \cup \mathrm{D}_+(t^{r+1}g) = D_+(t) \cup \{p\}$ of X, where g is as in Step 2, is not affine and hence a proper $\Bbbk$-scheme by [Har77, Chapter IV, Exercise 1.4]. Therefore, $\mathrm{D}_+(t) \cup \{p\} \to X$ is both open and closed, so $X = \mathrm{D}_+(t) \cup \{p\}$, since X is integral. Setting $\mathscr{F} = \mathscr{O}_X$ in the exact sequence of Theorem 1.3.4 and applying c results in the desired: $\mathrm{h}^1(\mathscr{O}_X) = \dim(\Bbbk(\!(z)\!)/(\Bbbk[\![z]\!] + O)) = h_1$. □

Theorem 1.3.6.
Let A be a finite-dimensional $\Bbbk$-algebra, $(O, W) \in \mathrm{Lat}_A$ be a ringed A-lattice, (h_0, h_1) be the index of W, and (X, c, p) be the geometric datum associated to O in Proposition 1.3.5.

(1) $\mathrm{gr}(W) := \bigoplus_{k \in \mathbb{Z}} t^k W_k \subseteq W[t, t^{-1}]$ *is a graded* $\mathrm{gr}(O)$*-subalgebra and the associated sheaf of algebras $\mathscr{A}$ on X is coherent, torsion-free and satisfies* $(\mathrm{h}^0(\mathscr{A}), \mathrm{h}^1(\mathscr{A})) = (h_0, h_1)$.

(2) *There is a natural c-equivariant isomorphism* $\zeta\colon \widehat{\mathscr{A}_p} \to A[\![z]\!]$ *such that the induced map* $\zeta\colon \mathrm{Q}(\widehat{\mathscr{A}_p}) \to A(\!(z)\!)$ *satisfies* $\zeta(\Gamma(X \setminus \{p\}, \mathscr{A})) = W$.

(3) $(O, W) \mapsto ((X, \mathscr{A}), (p, c, \zeta))$ *defines a functor* $\mathbb{G}\colon \mathrm{Lat}_A \to \mathrm{GeomLat}_A$ *such that* $\mathbb{T}\mathbb{G} = \mathrm{id}_{\mathrm{Lat}_A}$ *and* $\mathbb{G}\mathbb{T} \cong \mathrm{id}_{\mathrm{GeomLat}_A}$.

Proof. The fact that $O_k W_\ell, \mu_A(W_k \otimes W_\ell) \subseteq W_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$ immediately implies that $\mathrm{gr}(W) \subseteq W[t, t^{-1}]$ is a graded $\mathrm{gr}(O)$-subalgebra. Here,

$$\mu_A\colon A(\!(z)\!) \otimes_{\Bbbk(\!(z)\!)} A(\!(z)\!) \to A(\!(z)\!) \tag{1.26}$$

denotes the multiplication map of $A(\!(z)\!)$, which can be identified with the $\Bbbk(\!(z)\!)$-linear extension of the multiplication map of A. It is obvious that $\mathscr{A}$ is torsion-free since $\mathrm{gr}(W)$ is torsion-free. We split the rest of the proof into several steps.

Step 1. *$\mathscr{A}$ is coherent.* Since X is noetherian, it suffices to prove that $\mathrm{gr}(W)$ is finitely-generated by [Har77, Proposition 5.11.(c)]. Choose $r \in \mathbb{N}$ such that for all $k \geqslant r$ and for all $v \in A$ there exists an element in W with main part vz^{-k} as well as an element in O with main part z^{-k}. Since $W_0 = W \cap A[\![z]\!]$ is finite-dimensional, so is W_k for all $k \in \mathbb{Z}$ and $W_{-k} = \{0\}$ for k sufficiently large. Let B be any basis of the finite-dimensional vector space $\bigoplus_{k=-\infty}^{2r} t^k W_k \subseteq \mathrm{gr}(W)$ and M be the $\mathrm{gr}(O)$-submodule of $\mathrm{gr}(W)$ spanned by B. From $\Bbbk = O_0$ we see that $t^k W_k \subseteq M$ for all $k \leqslant 2r$. We show $\mathrm{gr}(W) = M$ through proving $t^k W_k \subseteq M$ by induction on $k \geqslant 2r$. The base case $k = 2r$ thereby automatically satisfied. By induction assumption $t^{k-1} W_{k-1} \subseteq M$, which immediately implies that $t^k W_{k-1} \subseteq M$ since $t \in \mathrm{gr}(O)$. Let $a \in W$ have main part vz^{-k}. There exists $b \in W$ with main part vz^{-r} and $h \in O$ with main part $z^{-r-\ell}$ for $\ell = k - 2r$. Therefore, $t^k a - t^{r+\ell} h t^r a \in t^k W_{k-1} \subseteq M$. The observations $t^r b \in M$ and $t^{r+\ell} h \in \mathrm{gr}(O)$ show that $t^k a \in M$, which concludes the induction.

Step 2. *Construction of ζ and proof of $\zeta(\Gamma(X \setminus \{p\}, \mathscr{A})) = W$.* The same reasoning as in Step 3 of Proposition 1.3.5 yields $\mathscr{A}_p \cong \mathrm{Q}(W) \cap A[\![z]\!]$. For each $v \in A$ exists $k \in \mathbb{N}$ such that there is some $a \in W$ with main part vz^{-k} and $h \in O$ with main part z^{-k}. Therefore, $a/h \in \mathrm{Q}(W) \cap A[\![z]\!]$ satisfies $a(0) = v$. This shows that $\Bbbk[\![z]\!](Q(W) \cap A[\![z]\!]) = A[\![z]\!]$. Using c we get ζ as the composition

$$\widehat{\mathscr{A}_p} \cong \mathscr{A}_p \otimes_{\mathscr{O}_{X,p}} \widehat{\mathscr{O}}_{X,p} \cong (\mathrm{Q}(W) \cap A[\![z]\!]) \otimes_{\mathrm{Q}(O)\cap\Bbbk[\![z]\!]} \Bbbk[\![z]\!] \cong A[\![z]\!],$$

where the last isomorphism is given by multiplication. The same arguments as in Step 4 of the proof of Proposition 1.3.5 imply that $\zeta(\Gamma(\mathrm{D}_+(t), \mathscr{A})) = \mathrm{gr}(W)[t^{-1}]_0 = W$, so $\mathrm{D}_+(t) = X \setminus \{p\}$ concludes the proof.

Step 3. $\mathrm{h}^0(\mathscr{A}) = h_0, \mathrm{h}^1(\mathscr{A}) = h_1$. Applying ζ to the exact sequence in Theorem 1.3.4 for $\mathscr{F} = \mathscr{A}$ results in $\mathrm{h}^0(\mathscr{A}) = h_0$ and $\mathrm{h}^1(\mathscr{A}) = h_1$.

Step 4. *Proof of* (3). The identity $\mathbb{T}\mathbb{G} = \mathrm{id}_{\mathrm{Lat}_A}$ and the proof of $\mathbb{G}\mathbb{T} \cong \mathrm{id}_{\mathrm{GeomLat}_A}$ are straight forward. It remains to prove the functoriality of $\mathbb{G}$. Let $(w, \phi)\colon (O_1, W_1) \to (O_2, W_2)$ be a morphism in Lat_A and write

$$\mathbb{G}(O_i, W_i) = ((X_i, \mathscr{A}_i), (p_i, c_i, \zeta_i)) \text{ for } i \in \{1, 2\}. \tag{1.27}$$

Then w defines a unital graded morphism $\widetilde{w}\colon \mathrm{gr}(O_1) \to \mathrm{gr}(O_2)$ of $\Bbbk$-algebras via $t^k h(z) \mapsto t^k h(w(z))$. Clearly, $\widetilde{w}(t) = t$ and if $h_1, h_2 \in O_1$ satisfy $h_1/h_2 \in z\Bbbk[\![z]\!]^\times$, we have $\widetilde{w}(h_1)/\widetilde{w}(h_2) \in z\Bbbk[\![z]\!]^\times$. This proves that $X_2 = \mathrm{D}_+(\widetilde{w}(t)) \cup \mathrm{D}_+(\widetilde{w}(t^{r+1}g))$, for g and r as in Step 1 of the proof of Proposition 1.3.5. Thus, $\widetilde{w}$ defines a morphism $f\colon X_2 \to X_1$ by virtue of [GW10, Paragarph 13.2]. Now, $\widetilde{w}^{-1}(t\,\mathrm{gr}(O_2)) = t\,\mathrm{gr}(O_1)$ implies that $f(p_2) = p_1$ and it is easy to see that f is a local isomorphism at p_2. The map $\mathrm{gr}(O_1) \to \mathrm{gr}(O_2)$ defined by $t^k a(w(z)) \mapsto t^k \phi(a)(w(z))$ defines a graded f-equivariant morphism. Therefore, it induces a morphism $\phi\colon \mathscr{A}_1 \to f_*\mathscr{A}_2$ of sheaves of algebras. □

Remark 1.3.7.
Let A be a finite-dimensional $\Bbbk$-algebra, O be an unital $\Bbbk$-lattice, $(O, W) \in \mathrm{Lat}_A$ be a ringed A-lattice, and $\mathbb{G}(O, W) = ((X, \mathscr{A}), (p, c, \zeta))$. By construction, $\mathscr{A}$ is formally A-locally free in p, so $\mathscr{A}|_p \cong A$. Therefore, Theorem 1.2.5 implies that $\mathscr{A}$ is étale A-locally free in p if A is a semi-simple Lie algebra.

1.3.6 Normalization of lattices. In this paragraph, we will see that the integral closure of lattices corresponds to the normalization of lattice models. We fix a finite-dimensional $\Bbbk$-algebra A.

Lemma 1.3.8.
Let $(O, W) \in \mathrm{Lat}_A$, O^ν be the integral closure of O and $W^\nu := O^\nu W$. Then $(O^\nu, W^\nu) \in \mathrm{Lat}_A$ and the natural inclusions $O \subseteq O^\nu, W \subseteq W^\nu$ induce a morphism $(O, W) \to (O^\nu, W^\nu)$. The assignment $(O, W) \mapsto (O^\nu, W^\nu)$ defines an endofunctor of Lat_A.

Proof. The inclusions $O^\nu \subseteq \mathrm{Q}(O^\nu) = \mathrm{Q}(O) \subseteq \Bbbk(\!(z)\!)$ identify O^ν with a unital subalgebra of $\Bbbk(\!(z)\!)$ containing O. It is well-known that $O \subseteq O^\nu$ and $W \subseteq W^\nu$ are of finite codimension. Consequently, $(O^\nu, W^\nu) \in \mathrm{Lat}_A$. The remaining statements are straight forward to prove. □

For a lattice $(O,W) \in \mathrm{Lat}_A$, the ringed lattice (O^ν, W^ν) constructed in Lemma 1.3.8 is called *integral closure* of (O,W).

Lemma 1.3.9.
Let $(O,W) \in \mathrm{Lat}_A$ and $(O^\nu, W^\nu) \in \mathrm{Lat}_A$ be its integral closure and $(\nu,\iota)\colon ((X,\mathscr{A}),(p,c,\zeta)) \to ((X^\nu,\mathscr{A}^\nu),(p^\nu,c^\nu,\zeta^\nu))$ be the image of the canonical morphism $(O,W) \to (O^\nu,W^\nu)$ under $\mathbb{G}$. Then $\nu\colon X^\nu \to X$ is the normalization of X and $\iota\colon \mathscr{A} \to \nu_*\mathscr{A}^\nu$ is injective with a torsion cokernel.

Proof. Since the curve $X^\nu \setminus \{p^\nu\} \cong \mathrm{Spec}(O^\nu)$ is smooth, so is X^ν. Therefore, $\nu\colon X^\nu \to X$ is the normalization, since ν is birational. It is easy to see that ι defines an isomorphism of $\mathscr{A}$ and $\nu_*\mathscr{A}^\nu$ at p, so the kernel and cokernel of ι are torsion sheaves. In particular, the kernel of ι is trivial, since it is a torsion subsheaf of a torsion-free sheaf. □

For a geometric A-lattice model $((X,\mathscr{A}),(p,c,\zeta))$ with trivialization

$$\mathbb{T}((X,\mathscr{A}),(p,c,\zeta)) = (O,W), \tag{1.28}$$

we call $\mathbb{G}(O^\nu,W^\nu) \coloneqq ((X^\nu,\mathscr{A}^\nu),(p^\nu,c^\nu,\zeta^\nu))$ the *normalization* of $((X,\mathscr{A}),(p,c,\zeta))$.

1.3.7 Changing the base field. Let A be a finite-dimenional $\Bbbk$-algebra, $(O,W) \in \mathrm{Lat}_A$, $\Bbbk'$ be an arbitrary field extension of $\Bbbk$, and $A_{\Bbbk'} \coloneqq A \otimes \Bbbk'$. The images $O_{\Bbbk'}$ and $W_{\Bbbk'}$ of $O \otimes \Bbbk'$ and $W \otimes \Bbbk'$ under the multiplication maps $\Bbbk(\!(z)\!) \otimes \Bbbk' \to \Bbbk'(\!(z)\!)$ and $A(\!(z)\!) \otimes \Bbbk' \to A_{\Bbbk'}(\!(z)\!)$ define a ringed $A_{\Bbbk'}$-lattice $(O_{\Bbbk'}, W_{\Bbbk'})$, where $A_{\Bbbk'}$ is considered as $\Bbbk'$-algebra. Then $\mathbb{G}(O,W) = ((X,\mathscr{A}),(p,c,\zeta))$ and the geometric datum $((X_{\Bbbk'},\mathscr{A}_{\Bbbk'}),(p_{\Bbbk'},c_{\Bbbk'},\zeta_{\Bbbk'}))$ constructed from $(O_{\Bbbk'},W_{\Bbbk'})$ in the same vein satisfies the following compatibilities:

- $O_{\Bbbk'} \cong O \otimes \Bbbk'$ induces an isomorphism $X_{\Bbbk'} \cong X \times \mathrm{Spec}(\Bbbk')$ such that the canonical map $O \to O_{\Bbbk'}$ is compatible with the canonical morphism $\pi\colon X_{\Bbbk'} \to X$. Furthermore, $\pi(p_{\Bbbk'}) = p$ and the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathscr{O}}_{X,p} & \xrightarrow{\;c\;} & \Bbbk[\![z]\!] \\ {\scriptstyle \widehat{\pi}_p^\sharp}\downarrow & & \downarrow \\ \widehat{\mathscr{O}}_{X_{\Bbbk'},p_{\Bbbk'}} & \xrightarrow[\;c_{\Bbbk'}\;]{} & \Bbbk'[\![z]\!] \end{array} .$$

- The multiplication map $W \otimes \Bbbk' \cong W_{\Bbbk'}$ induces an isomorphism $\pi^*\mathscr{A} \cong \mathscr{A}_{\Bbbk'}$ of sheaves of Lie algebras such that the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathscr{A}_p} & \xrightarrow{\;\zeta\;} & A[\![z]\!] \\ \downarrow & & \downarrow \\ \widehat{\mathscr{A}_{\Bbbk',p_{\Bbbk'}}} & \xrightarrow[\;\zeta_{\Bbbk'}\;]{} & A_{\Bbbk'}[\![z]\!] \end{array} .$$

In particular, if $\Bbbk \to \Bbbk'$ is Galois with Galois group G, then G acts on $X_{\Bbbk'}$ by automorphisms of X-schemes and on $\mathscr{A}_{\Bbbk'}$ by $\mathscr{O}_X$-linear automorphisms of sheaves of Lie algebras in such a way that X and $\mathscr{A}$ are the respective fixed objects.

2

Formal generalized r-matrices

In this chapter, we use the following notation: $\mathfrak{g}$ is a semi-simple Lie algebra of dimension $d \in \mathbb{N}$ over a field $\Bbbk$ of characteristic 0, K is the Killing form of $\mathfrak{g}$, $\{b_1, \dots, b_d\} \subseteq \mathfrak{g}$ is a basis orthonormal with respect to K, and $\gamma := \sum_{i=1}^d b_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir element.

2.1 Basic definitions and properties

In this section, we introduce the main object of interest: formal generalized r-matrices with values in $\mathfrak{g}$. By definition, these are series in $(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ with the same properties as certain Taylor series of non-degenerate solutions of the GCYBE 0.1, but the ground field is not restricted to be the complex numbers. We begin by recalling some general facts and definitions relating formal power series.

2.1.1 Prelude: formal power series. Let M be a module over a ring R. Then

$$M[\![z]\!] := \left\{ \sum_{k\in\mathbb{N}_0} m_k z^k \,\middle|\, m_k \in M \right\} \tag{2.1}$$

is the module of *M-valued formal power series*. As R-module, it is isomorphic to $\prod_{k\in\mathbb{N}_0} M$, but the representation (2.1) is chosen in order to equip $R[\![z]\!]$ with a ring structure with unit $1 \in R \subseteq R[\![z]\!]$ and multiplication

$$\left(\sum_{k\in\mathbb{N}_0} r_k z^k\right)\left(\sum_{k\in\mathbb{N}_0} s_k z^k\right) = \sum_{k\in\mathbb{N}_0}\sum_{\ell=0}^{k} r_\ell s_{k-\ell} z^k, \text{where } r_k, s_k \in R \text{ for all } k \in \mathbb{N}_0,$$

and $M[\![z]\!]$ with an $R[\![z]\!]$-module structure via

$$\left(\sum_{k\in\mathbb{N}_0} r_k z^k\right)\left(\sum_{k\in\mathbb{N}_0} m_k z^k\right) = \sum_{k\in\mathbb{N}_0}\sum_{\ell=0}^{k} r_\ell m_{k-\ell} z^k, \text{where } r_k \in R, m_k \in M \text{ for all } k \in \mathbb{N}_0.$$

Note that we have a canonical map $M[\![z]\!] \to M$ given by

$$m = \sum_{k\in\mathbb{N}_0} m_k z^k \longmapsto m(0) := m_0. \tag{2.2}$$

It can be easily seen to be equivariant with respect to $R[\![z]\!] \to R$. The $R((z)) := R[\![z]\!][z^{-1}]$-module $M((z)) := M[\![z]\!][z^{-1}]$ is called module of *M-valued formal Laurent series*. Elements p in $M((z))$ (resp. $M((x_1))\dots((x_k))$) will sometimes be denoted with the formal variable (resp. variables) for convenience: $p = p(z)$ (resp. $p = p(x_1, \dots, x_k)$). Moreover, a generic element $m \in M((z))$ is written $m(z) = \sum_{k\in\mathbb{Z}} m_k z^k$, where we keep in mind that if $m \neq 0$ there exists a unique integer $|m|$, called *order of* m, such that $m_{-|m|} \neq 0$ and $m_{-k} = 0$ for all $k \geqslant |m|$. In particular,

$$m(z) \in m_{-|m|}z^{-|m|} + z^{-|m|+1}M[\![z]\!] \tag{2.3}$$

and we call $m_{-|m|}z^{-|m|}$ the *main part* of m. Observe that the module structure of $M(\!(z)\!)$ takes the simple form

$$\left(\sum_{k\in\mathbb{Z}} r_k z^k\right)\left(\sum_{k\in\mathbb{Z}} m_k z^k\right) = \sum_{k,\ell\in\mathbb{Z}} r_\ell m_{k-\ell} z^k, \text{where } r_k \in R, m_k \in M \text{ for all } k \in \mathbb{Z}.$$

Remark 2.1.1.
If M is an R-algebra, $M(\!(z)\!)$ is automatically equipped with the structure of an $R(\!(z)\!)$-algebra via

$$\left(\sum_{k\in\mathbb{Z}} m_k z^k\right)\left(\sum_{k\in\mathbb{Z}} n_k z^k\right) = \sum_{k,\ell\in\mathbb{Z}} m_\ell n_{k-\ell} z^k, \text{where } m_k, n_k \in M \text{ for all } k \in \mathbb{Z}.$$

Clearly, $M[\![z]\!]$ is an $R[\![z]\!]$-subalgebra of $M(\!(z)\!)$ and it is easy to see that if M is a Lie algebra, $M[\![z]\!]$ and $M(\!(z)\!)$ are too.

For an element $m(z) = \sum_{k\in\mathbb{Z}} m_k z^k$, we call the series $m'(z) := \sum_{k\in\mathbb{Z}} k m_k z^{k-1}$ its *formal derivative*. This defines an R-linear derivation of $M(\!(z)\!)$ as $R(\!(z)\!)$-module, since

$$(rm)' = \sum_{k,\ell\in\mathbb{Z}} k r_\ell m_{k-\ell} z^{k-1} = \sum_{k,\ell\in\mathbb{Z}} \left(\ell r_\ell m_{k-\ell} + r_\ell (k-\ell) m_{k-\ell}\right) z^{k-1} = r'm + rm' \tag{2.4}$$

for all $r = \sum_{k\in\mathbb{Z}} r_k z^k \in R(\!(z)\!)$ and $m = \sum_{k\in\mathbb{Z}} m_k z^k \in M(\!(z)\!)$. Henceforth, we write $M[\![x_1, \dots, x_k]\!] := M[\![x_1]\!] \dots [\![x_k]\!]$. Take caution, since $M(\!(x_1, \dots, x_k)\!)$ usually denotes the quotient module of $M[\![x_1, \dots, x_k]\!]$ and does not coincide with $M(\!(x_1)\!) \dots (\!(x_k)\!)$.

Lemma 2.1.2.
Let M be a module over a ring R. The following results are true.
(1) $R[\![z]\!]^\times = \{r \in R[\![z]\!] \mid r(0) \in R^\times\}$ *and for every $u \in zR[\![z]\!]^\times$ exists a unique $w \in zR[\![z]\!]^\times$ such that $u(w(z)) = w(u(z)) = z$. Furthermore, every R-algebra automorphism of $R[\![z]\!]$ is of the form $r(z) \mapsto r(u(z))$ for some $u \in zR[\![z]\!]^\times$.*
(2) *Assume that R has characteristic 0. For any $r \in R[\![z]\!]^\times$ exists a unique $u \in zR[\![z]\!]^\times$ such that $u'(z) = r(u(z))$.*
(3) *Assume that R has characteristic 0. For every $\psi \in \mathrm{End}_R(M)[\![z]\!]$ and $m_0 \in M$ exists a unique $m \in M[\![z]\!]$ such that $m' = \psi m$ and $m(0) = 0$.*
(4) *For every $f \in M[\![x, y]\!]$ with $f(z, z) = 0$ in $M[\![z]\!]$ exists $g \in M[\![x, y]\!]$ such that $f(x, y) = (x - y)g(x, y)$. Furthermore, if $f(x, y) = h(x) - h(y)$ for some $h \in M[\![z]\!]$, the relation $g(z, z) = h'(z)$ holds.*

Proof of (1). It is easy to see that $r(0) \in R^\times$ for all $r \in R[\![z]\!]^\times$. On the other hand, if $r = \sum_{k\in\mathbb{N}_0} r_k z^k \in R[\![z]\!]$ satisfies $r(0) = r_0 \in R^\times$, we can construct $s = \sum_{k\in\mathbb{N}_0} s_k z^k$ inductively by $s_0 = r_0^{-1}$ and $s_{k+1} = -r_0^{-1} \sum_{\ell=0}^{k} s_\ell r_{k+1-\ell}$ and verify that $rs = sr = 1$

holds. For every $r = \sum_{k\in\mathbb{N}_0} r_k z^k \in R[\![z]\!]$ and $u \in \sum_{k\in\mathbb{N}} u_k z^k \in zR[\![z]\!]^\times$ the composition $r(u(z)) = \sum_{k\in\mathbb{N}_0} c_k z^k \in R[\![z]\!]$ exists. Indeed, it can be verified that $c_0 = r_0$ and

$$c_k = \sum_{\ell=0}^{k} \sum_{\substack{(j_1,\dots,j_\ell)\in\mathbb{N}^\ell \\ j_1+\dots+j_\ell=k}} r_\ell u_{j_1}\dots u_{j_\ell}, \tag{2.5}$$

where we emphasize that $\mathbb{N}$ excludes 0. If $r(0) = r_0 = 0$, we can again define inductively $r_1 = u_1^{-1}$ and

$$r_{k+1} = -u_1^{-k-1} \sum_{\ell=0}^{k} \sum_{\substack{(j_1,\dots,j_\ell)\in\mathbb{N}^\ell \\ j_1+\dots+j_\ell=k}} r_\ell u_{j_1}\dots u_{j_\ell} \tag{2.6}$$

to achieve that $c_1 = 1$ and $c_k = 0$ for all $k > 0$. In particular $r(u(z)) = z$ and, since the composition is associative and the above argument shows that r has a left compositional inverse, $u(r(z)) = z$ holds automatically.

Since every R-algebra automorphism of $R[\![z]\!]$ is automatically local, it is continuous in the (z)-adic topology. Therefore, it is completely determined by its image of z, which is necessarily in $zR[\![z]\!]^\times$, concluding the proof.

Proof of (2). From (2.5), we can deduce that for an arbitrary $r = \sum_{k\in\mathbb{N}_0} r_k z^k \in R[\![z]\!]^\times$ the series $u = \sum_{k\in\mathbb{N}} u_k z^k \in z^{-1}\Bbbk[\![z]\!]^\times$, determined inductively by $u_1 = r_0$ and

$$u_{k+1} = \frac{1}{k+1} \sum_{\ell=0}^{k} \sum_{\substack{(j_1,\dots,j_\ell)\in\mathbb{N}^\ell \\ j_1+\dots+j_\ell=k}} r_\ell u_{j_1}\dots u_{j_\ell}, \tag{2.7}$$

satisfies $u'(z) = r(u(z))$.

Proof of (3). Write $\psi = \sum_{k\in\mathbb{N}} \psi_k z^k$. It is easy to verify that defining $m_k \in M$ inductively by $m_{k+1} = \frac{1}{k+1}\sum_{\ell=0}^{k} \psi_\ell(m_{k-\ell})$ yields a unique element $m = \sum_{k\in\mathbb{N}_0} m_k z^k \in M[\![z]\!]$ such that $m' = \psi m$.

Proof of (4). Let $f = \sum_{k\in\mathbb{N}_0} f_k$ be the decomposition of f into homogeneous components, i.e. f_k is the homogeneous component of f with total degree k for all $k \in \mathbb{N}_0$. Then $f(z,z) = 0$ if and only if $f_k(z,z) = 0$ for all $k \in \mathbb{N}_0$, so we may assume that $f = f_k \in M[x,y]$ for some $k \in \mathbb{N}_0$. Since $x - y \in R[x,y] = R[y][x]$ is monic, the polynomial division algorithm provides $g \in M[x,y]$ and $r \in M[y]$ such that

$$f(x,y) = (x-y)g(x,y) + r(y). \tag{2.8}$$

Therefore, $0 = f(z,z) = r(z)$ proves the first part of the statement. The second part can be deduced from the following observation: for $h(z) = h_k z^k$, we have $g(x,y) = h_k \sum_{\ell=0}^{k-1} x^\ell y^{k-1-\ell}$, so $g(z,z) = h_k k z^{k-1}$. □

2.1.2 Series in standard form. Recall that $\mathfrak{g}$ is a semi-simple Lie algebra of dimension $d \in \mathbb{N}$ over $\Bbbk$ with Killing form K, while $\gamma \in \mathfrak{g} \otimes \mathfrak{g}$ denotes the Casimir element of $\mathfrak{g}$. In particular, the identity $\gamma = \sum_{i=1}^{d} b_i \otimes b_i$ holds, where $\{b_1, \dots, b_d\} \subseteq \mathfrak{g}$ is a basis orthonormal with respect to $\mathfrak{g}$. The most important property of γ is its $\mathfrak{g}$-invariance:

$$[a \otimes 1 + 1 \otimes a, \gamma] = 0 \text{ for all } a \in \mathfrak{g}. \tag{2.9}$$

Consider $(x-y)^{-1}$ as the series $\sum_{k=0}^{\infty} x^{-k-1} y^k \in \Bbbk((x))[\![y]\!]$. Then,

$$r_{\text{Yang}}(x,y) := \frac{\gamma}{x-y} = \sum_{k=0}^{\infty} \sum_{i=1}^{d} x^{-k-1} b_i \otimes y^k b_i \tag{2.10}$$

is an element of $(\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!] \cong (\mathfrak{g}((x)) \otimes \mathfrak{g})[\![y]\!] \cong (\mathfrak{g} \otimes \mathfrak{g}) \otimes \Bbbk((x))[\![y]\!]$, which we shall call *Yang's r-matrix*, as is usually done for its analytic counterpart.

A series $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ is said to be in *standard form* if

$$r(x,y) = \frac{\lambda(y)}{x-y}\gamma + r_0(x,y) = \lambda(y) r_{\text{Yang}}(x,y) + r_0(x,y) \tag{2.11}$$

for some $\lambda \in \Bbbk[\![z]\!]^\times$ and $r_0 \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$. In this case

- r is called *normalized* if $\lambda = 1$,
- $\overline{r}(x,y) := \lambda(x) r_{\text{Yang}}(x,y) - \tau(r_0(y,x)) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$, where τ is the $\Bbbk[\![x,y]\!]$-linear extension of the linear automorphism of $\mathfrak{g} \otimes \mathfrak{g}$ defined by $a \otimes b \mapsto b \otimes a$, and
- r is called *skew-symmetric* if $\overline{r} = r$.

Remark 2.1.3.
(1) For a general $r = r(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ there is no appropriate $\overline{r}$ since switching x and y does not define a map of $\Bbbk((x))[\![y]\!]$ into itself. Therefore, a well-defined notion of skew-symmetry for elements of $(\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ needs additional assumptions, e.g. that they are in standard form.
(2) Let $r = r(x,y) = f(x,y) r_{\text{Yang}}(x,y) + r_0(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ for some $f \in \Bbbk[\![x,y]\!]^\times$ and $r_0 \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$. Then

$$r(x,y) = f(y,y) r_{\text{Yang}}(x,y) + \frac{f(x,y) - f(y,y)}{x-y}\gamma + r_0(x,y) \tag{2.12}$$

combined with part (4) of Lemma 2.1.2 shows that r is in standard form. In particular, $\overline{r}$ is also in standard form.

2.1.3 Formal generalized r-matrices. Let $ij \in \{12, 13, 23\}$, $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $R := \Bbbk((x_1))((x_2))[\![x_3]\!]$. The assignments

$$t^{12} := t \otimes 1,\ t^{13} := t_1 \otimes 1 \otimes t_2 \text{ and } t^{23} := 1 \otimes t \text{ for all } t = t_1 \otimes t_2 \in \mathfrak{g} \otimes \mathfrak{g} \tag{2.13}$$

define linear maps $(\cdot)^{ij} \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$. Let s^{ij} be the image of $s \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ via the map

$$(\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!] \cong (\mathfrak{g} \otimes \mathfrak{g})((x_i))[\![x_j]\!] \subseteq (\mathfrak{g} \otimes \mathfrak{g}) \otimes R \xrightarrow{(\cdot)^{ij} \otimes \mathrm{id}_R} (\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})) \otimes R\,.$$

A series $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ is called a *(normalized) formal generalized r-matrix* if it is in (normalized) standard form and solves the *formal generalized classical Yang-Baxter equation* (formal GCYBE)

$$\mathrm{GCYB}(r) = 0, \text{ where } \mathrm{GCYB}(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, \overline{r}^{23}]. \tag{2.14}$$

Here, the brackets in $\mathrm{GCYB}(r)$ are the usual commutators in the associative R-algebra $(\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})) \otimes R$.

Example 2.1.4.
r_{Yang} is a skew-symmetric formal normalized generalized r-matrix since

$$\begin{aligned}(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)&\mathrm{GCYB}(r_{\mathrm{Yang}})(x_1, x_2, x_3) \\ = ((x_2 - x_3) - (x_1 - x_3) + (x_1 - x_2))&[\gamma^{12}, \gamma^{13}] = 0.\end{aligned} \tag{2.15}$$

Here, we used that the fact (2.9) implies that

$$[\gamma^{12}, \gamma^{13}] = -[\gamma^{12}, \gamma^{23}] = [\gamma^{13}, \gamma^{23}]. \tag{2.16}$$

It is easy to see that for any $\lambda \in \Bbbk[\![z]\!]$ and $\widetilde{r}(x, y) := \lambda(y) r_{\mathrm{Yang}}(x, y)$

$$\mathrm{GCYB}(\widetilde{r})(x_1, x_2, x_3) = \lambda(x_2)\lambda(x_3)\mathrm{GCYB}(r_{\mathrm{Yang}})(x_1, x_2, x_3) = 0. \tag{2.17}$$

holds, so $\widetilde{r}$ is a generalized r-matrix. This series is not skew-symmetric if $\lambda \notin \Bbbk^\times$.

We conclude this subsection by summarizing some important observations.

Remark 2.1.5.

(1) Defining $r^{32} := -\overline{r}^{23}$ in (2.14) results in the analog form of the GCYBE used in the introduction (0.1).

(2) Observe that e.g.

$$[(s_1 \otimes s_2)^{13}, (t_1 \otimes t_2)^{23}] = s_1 \otimes t_1 \otimes s_2 t_2 - s_1 \otimes t_1 \otimes t_2 s_2 = s_1 \otimes t_1 \otimes [s_2, t_2]$$

is an element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ for all $s_1, s_2, t_1, t_2 \in \mathfrak{g}$. This and similar calculations show that $\mathrm{GCYB}(r) \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})((x_1))((x_2))[\![x_3]\!]$ for every $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ in standard form. This can be further refined since equation (2.9) implies that $[a \otimes 1 + 1 \otimes a, \gamma] = 0$ for all $a \in \mathfrak{g}[\![z]\!]$. We can use this fact to derive that

$$[r_0^{12}, \gamma^{13}] - [\gamma^{13}, \tau(r_0)^{23}] = [r_0^{12} + \tau(r_0)^{23}, \gamma^{13}] \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_1, x_2, x_3]\!] \tag{2.18}$$

vanishes for $x_1 = x_3$, where r_0 is determined by

$$r(x, y) = \lambda(y) r_{\mathrm{Yang}}(x, y) + r_0(x, y) \text{ for some } \lambda \in \Bbbk[\![z]\!]^\times. \tag{2.19}$$

This and similar identities combined with Lemma 2.1.2.(4) and Example 2.1.4 imply that $\mathrm{GCYB}(r) \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_1, x_2, x_3]\!]$ for all $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ in standard form.

(3) Let $\Bbbk'$ be an arbitrary field extension of $\Bbbk$ and $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ be a formal generalized r-matrix. Then $\mathfrak{g}_{\Bbbk'} := \mathfrak{g} \otimes \Bbbk'$ is a semi-simple Lie algebra over $\Bbbk'$ and the image $r_{\Bbbk'}$ of r under the canonical map

$$(\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!] \to (\mathfrak{g}_{\Bbbk'} \otimes_{\Bbbk'} \mathfrak{g}_{\Bbbk'})((x))[\![y]\!]$$

is again a formal generalized r-matrix, called *extension of r via $\Bbbk \to \Bbbk'$*.

2.1.4 Equivalence of formal r-matrices. For $\varphi \in \mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ the image of φ under

$$\mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!]) \subseteq \mathrm{End}_{\Bbbk[\![z]\!]}(\mathfrak{g}[\![z]\!]) \cong \mathrm{End}(\mathfrak{g})[\![z]\!] \tag{2.20}$$

is again denoted by $\varphi = \varphi(z)$. Furthermore,

$$\varphi \otimes \varphi \in \mathrm{End}(\mathfrak{g})[\![z]\!] \otimes_{\Bbbk[\![z]\!]} \mathrm{End}(\mathfrak{g})[\![z]\!] \cong (\mathrm{End}(\mathfrak{g}) \otimes \mathrm{End}(\mathfrak{g}))[\![z]\!] \tag{2.21}$$

is also denoted by $\varphi(z) \otimes \varphi(z)$ while $\varphi(x) \otimes \varphi(y)$ denotes

$$\varphi \otimes \varphi \in \mathrm{End}(\mathfrak{g})[\![z]\!] \otimes \mathrm{End}(\mathfrak{g})[\![z]\!] \subseteq (\mathrm{End}(\mathfrak{g}) \otimes \mathrm{End}(\mathfrak{g}))[\![x, y]\!]. \tag{2.22}$$

A series $\widetilde{r} \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ is called *equivalent* to $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ if

$$\widetilde{r}(x, y) = \mu(y)(\varphi(x) \otimes \varphi(y))r(w(x), w(y)), \tag{2.23}$$

where the triple (μ, w, φ) is called *equivalence* and consists of a series $\mu \in \Bbbk[\![z]\!]^\times$, called *rescaling*, an invertible series $w \in z\Bbbk[\![z]\!]^\times$, called *coordinate transformation*, and an automorphism $\varphi \in \mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$, called *gauge transformation*. Furthermore, we say that $\widetilde{r}$ is *gauge equivalent* (resp. *coordinate equivalent*) to r if $\mu = 1$ and $w = z$ (resp. $\mu = 1$ and $\varphi = \mathrm{id}_{\mathfrak{g}[\![z]\!]}$). As the name suggests, these transformations define equivalence relations on $(\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$.

Lemma 2.1.6.
The following results are true.
(1) *Equivalences preserve the property of being a formal generalized r-matrix.*
(2) *An equivalence preserves skew-symmetry if and only if it has a constant rescaling part.*
(3) *Every series in standard form is coordinate equivalent to one in normalized standard form.*
(4) *An equivalence (μ, w, φ) of two normalized formal r-matrices automatically satisfies $\mu \in \Bbbk^\times$ and $w(z) = \mu z$.*

Proof. Let $r(x, y) = \lambda(y)r_{\mathrm{Yang}}(x, y) + r_0(x, y)$ for $\mu \in \Bbbk[\![z]\!]^\times, r_0 \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ and $\widetilde{r} \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ be equivalent to r via an equivalence (μ, w, φ). Using of Lemma 2.1.2.(4) and $(\varphi(z) \otimes \varphi(z))\gamma = \gamma$ (see e.g. Remark 2.3.2 below), we can deduce that

$$(\varphi(x) \otimes \varphi(y))r_{\mathrm{Yang}}(w(x), w(y)) - w'(y)^{-1}r_{\mathrm{Yang}}(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!], \tag{2.24}$$

hence $\widetilde{r}$ is in standard form. In particular, if $\mu = 1$, $\varphi = \mathrm{id}_{\mathfrak{g}[\![z]\!]}$ and w is the unique solution of $w'(z) = \lambda(w(z))$ (see Lemma 2.1.2.(2)), $\widetilde{r}(x,y) = r(w(x),w(y))$ is in normalized standard from, so (3) is proven. It is easy to see that $\mathrm{GCYB}(\widetilde{r})(x_1,x_2,x_3)$ equals

$$\mu(x_2)\mu(x_3)(\varphi(x_1)\otimes\varphi(x_2)\otimes\varphi(x_3))\mathrm{GCYB}(r)(w(x_1),w(x_2),w(x_3)),$$

which is zero, since $\mathrm{GCYB}(r) = 0$. This proves (1). It is easy to see that $\widetilde{r}$ is skew-symmetric if r is skew-symmetric and $\mu \in \Bbbk^\times$. Assume that r and $\widetilde{r}$ are skew-symmetric. Then, multiplying $\widetilde{r}(x,y) - \tau(\widetilde{r}(y,x)) = 0$ with $x - y$ and putting $x = y = z$ results in $\mu'(z) = 0$ because of Lemma 2.1.2.(4). Therefore, $\lambda \in \Bbbk^\times$ and (2) is proven. If we now additionally assume that r and $\widetilde{r}$ are normalized, we can deduce from (2.24) that $w(z) = \mu z$. □

Remark 2.1.7.
The study of formal generalized r-matrices will be pursued up to equivalence: equivalent r-matrices will be treated as interchangeable. In light of of Lemma 2.1.6.(2), only equivalences with constant rescaling part will be used if skew-symmetry is relevant to the context. In particular, Lemma 2.1.6.(3) would permit us to restrict our attention to normalized formal generalized r-matrices. Nevertheless, this will be done only if necessary, since non-normalized formal generalized r-matrices appear naturally in the algebro-geometric context; see e.g. Theorem 3.3.3 below.

2.2 Lie subalgebras of $\mathfrak{g}(\!(z)\!)$ complementary to $\mathfrak{g}[\![z]\!]$

In [Che83a], Cherednik assigns a Lie subalgebra of $\mathfrak{g}(\!(z)\!)$ complementary to $\mathfrak{g}[\![z]\!]$ to each non-degenerated solution of (0.3). This was generalized by Skrypnyk to non-degenerate solutions of (0.1); see [Skr13]. In this section, we will discuss the formal analog of that construction.

2.2.1 Lie subalgebras associated to formal generalized r-matrices. For a series

$$s = s(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} s_{k,i}(x)\otimes b_i y^k \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$$

it is always possible to define the vector subspace

$$\mathfrak{g}(s) := \langle s_{k,i}(z) \mid k \in \mathbb{N}_0, i \in \{1,\dots,d\}\rangle_{\Bbbk} \tag{2.25}$$

of $\mathfrak{g}(\!(z)\!)$. It is the smallest subspace of $\mathfrak{g}(\!(z)\!)$ satisfying $s \in (\mathfrak{g}(s)\otimes\mathfrak{g})[\![y]\!]$ and does not depend on the choice of basis $\{b_i\}_{i=1}^d$. The following observation is fundamental to the remainder of this work.

Proposition 2.2.1.
The assignment $r \mapsto \mathfrak{g}(r)$ gives a bijection between normalized formal generalized r-matrices and Lie subalgebras $\mathfrak{W} \subseteq \mathfrak{g}((z))$ satisfying $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{W}$.

Proof. Let $r(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}(x) \otimes y^k b_i$ be a normalized formal generalized r-matrix. The identity $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(r)$ is a direct consequence of $r_{k,i} - z^{-k-1}b_i \in \mathfrak{g}[\![z]\!]$ for all $k \in \mathbb{N}_0, i \in \{1, \dots, d\}$. It remains to show that $\mathfrak{g}(r)$ is a subalgebra of $\mathfrak{g}((z))$. The fact that $[z^k a \otimes 1 + 1 \otimes z^k a, \gamma] = 0$ holds for all $a \in \mathfrak{g}, k \in \mathbb{N}_0$, forces

$$[r^{12} + r^{13}, \gamma^{23}](x_1, x_2, x_3) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}(x_1) \otimes [x_2^k b_i \otimes 1 + 1 \otimes x_3^k b_i, \gamma], \qquad (2.26)$$

which is an element of $(\mathfrak{g}(r) \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_2, x_3]\!]$, to vanish for $x_2 = x_3$. Therefore, Lemma 2.1.2.(4) implies that

$$[r^{12}, r^{23}] + [r^{13}, \overline{r}^{23}] \in (\mathfrak{g}(r) \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_2, x_3]\!]. \qquad (2.27)$$

This, combined with $0 = \mathrm{GCYB}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, \overline{r}^{23}]$, shows that

$$\sum_{k,\ell=0}^{\infty}\sum_{i,j=1}^{d} [r_{k,i}(x_1), r_{\ell,j}(x_1)] \otimes x_2^k b_i \otimes x_3^\ell b_j = [r^{12}, r^{13}](x_1, x_2, x_3) \qquad (2.28)$$

is an element of $(\mathfrak{g}(r) \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_2, x_3]\!]$. We conclude that $[r_{k,i}, r_{\ell,j}] \in \mathfrak{g}(r)$ for all $k, \ell \in \mathbb{N}_0, i, j \in \{1, \dots, d\}$. In particular, $\mathfrak{g}(r)$ is a subalgebra of $\mathfrak{g}((z))$.

Let us consider now a Lie subalgebra $\mathfrak{W} \subset \mathfrak{g}((z))$, satisfying $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{W}$. For every $k \in \mathbb{N}_0, i \in \{1, \dots, d\}$ there is an unique element $r_{k,i}^{\mathfrak{W}} \in \mathfrak{W}$ such that $r_{k,i}^{\mathfrak{W}} - b_i z^{-k-1} \in \mathfrak{g}[\![z]\!]$. By construction,

$$r^{\mathfrak{W}} = r^{\mathfrak{W}}(x,y) := \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}^{\mathfrak{W}}(x) \otimes b_i y^k \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!] \qquad (2.29)$$

is in normalized standard form and satisfies $\mathfrak{g}(r^{\mathfrak{W}}) = \mathfrak{W}$. Furthermore, we can see that $r^{\mathfrak{g}(r)} = r$. Thus, it remains to show that $\mathrm{GCYB}(r^{\mathfrak{W}}) = 0$. In Remark 2.1.5.(2) it was noted that $\mathrm{GCYB}(r^{\mathfrak{W}}) \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_1, x_2, x_3]\!]$. Since $\mathfrak{g}(r^{\mathfrak{W}}) = \mathfrak{W}$ is closed under the Lie bracket, (2.27) and (2.28) show that $\mathrm{GCYB}(r^{\mathfrak{W}}) \in (\mathfrak{W} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_2, x_3]\!]$. Summarized, we obtain

$$\mathrm{GCYB}(r^{\mathfrak{W}}) \in (\mathfrak{W} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_2, x_3]\!] \cap (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_1, x_2, x_3]\!] = \{0\}, \qquad (2.30)$$

since $\mathfrak{g}[\![z]\!] \cap \mathfrak{W} = \{0\}$, concluding the proof. □

Remark 2.2.2.
If one focuses on a certain class of formal generalized r-matrices, there sometimes exist more appropriate ways to assign Lie algebras to said series, which should not be confused with the universal method described here. We will see a examples of this occurrence in the Subsection 5.4.4 and Subsection 5.4.5.

2.2.2 Equivalence of formal generalized r-matrices on the level of subalgebras.

It is always possible to give a finite set of generators for the Lie algebra associated to a formal generalized r-matrix by virtue of the following result.

Lemma 2.2.3.
Let $r = \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}(x)\otimes y^k b_i$ be a normalized formal generalized r-matrix. The identity $\mathfrak{g}(r) = \langle r_{0,i} \mid i \in \{1,\dots,d\}\rangle_{\Bbbk\text{-alg}}$ holds.

Proof. Since $\{r_{0,i}\}_{i=1}^{d}$ is a basis of $\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!]$, it suffices to show that the Lie subalgebra W of $\mathfrak{g}(r)$, generated by $\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!]$, equals $\mathfrak{g}(r)$. Assume that for some $m \in \mathbb{N}$ we have $\mathfrak{g}(r)\cap z^{-m}\mathfrak{g}[\![z]\!] \subseteq W$. For every pair $a_1, a_2 \in \mathfrak{g}$ exist unique $\widetilde{a}_1, \widetilde{a}_2 \in W$ and $s \in \mathfrak{g}(r)$ such that

$$\widetilde{a}_1(z) - a_1 z^{-1}, \widetilde{a}_1(z) - a_2 z^{-m}, s - [a_1, a_2]z^{-m-1} \in \mathfrak{g}[\![z]\!]. \tag{2.31}$$

Since $[\widetilde{a}_1, \widetilde{a}_2] \in W$ and $s - [\widetilde{a}_1, \widetilde{a}_2] \in \mathfrak{g}(r)\cap z^{-m}\mathfrak{g}[\![z]\!] \subseteq W$, we see that $s \in W$. Therefore, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ implies that $\mathfrak{g}(r)\cap z^{-m-1}\mathfrak{g}[\![z]\!] \subseteq W$ and $W = \mathfrak{g}(r)$ is verified by induction on m. □

Lemma 2.2.4.
Let $\widetilde{r} \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be equivalent to a formal generalized r-matrix r via an equivalence (μ, w, φ). Then $\mathfrak{g}(\widetilde{r})$ is the image of $\mathfrak{g}(r)$ under the map $\varphi_w \in$ $\operatorname{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g}(\!(z)\!))$ defined by $a(z) \mapsto \varphi(z)a(w(z))$.

Proof. First note that for any $s(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} s_{k,i}(x)\otimes b_i y^k \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ and $\lambda(z) = \sum_{k=0}^{\infty}\lambda_k z^k \in \Bbbk[\![z]\!]^\times$ we have

$$\widetilde{s}(x,y) := \lambda(y)s(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d}\left(\sum_{\ell=0}^{k}\lambda_\ell s_{k-\ell,i}(x)\right)\otimes b_i y^k$$

and hence $\mathfrak{g}(\widetilde{s}) = \mathfrak{g}(s)$. Therefore, we may assume that r is normalized and $\mu(z) = w'(z)$. Then $\widetilde{r}$ is also a normalized formal generalized r-matrix, as can be seen in the proof of Lemma 2.1.6.

Since $\varphi_w\colon \mathfrak{g}(\!(z)\!) \to \mathfrak{g}(\!(z)\!)$, defined by $a(z) \mapsto \varphi(z)a(w(z))$, is a $\Bbbk$-linear automorphism of Lie algebras and $\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!] = \{(1\otimes\alpha)r(z,0) \mid \alpha \in \mathfrak{g}^*\}$ generates $\mathfrak{g}(r)$ by Lemma 2.2.3, $\varphi_w(\mathfrak{g}(r))$ is generated by $\varphi_w(\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!])$. We have

$$\varphi_w((1\otimes\alpha)r(z,0)) = (\varphi(z)\otimes\alpha)r(w(z),0) = \left(1\otimes\left(\mu(0)^{-1}\alpha\varphi(0)^{-1}\right)\right)\widetilde{r}(z,0),$$

where $w(0) = 0$, $\mu(0) \in \Bbbk^\times$ and $\varphi(0) \in \operatorname{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ was used. Since $\alpha \mapsto \mu(0)^{-1}\alpha\varphi(0)^{-1}$ defines a linear automorphism of $\mathfrak{g}^*$, we see that

$$\varphi_w(\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!]) = \mathfrak{g}(\widetilde{r})\cap z^{-1}\mathfrak{g}[\![z]\!]$$

and hence $\varphi_w(\mathfrak{g}(r)) = \mathfrak{g}(\widetilde{r})$ by applying Lemma 2.2.3. □

2.3 Formal r-matrices

This section is dedicated to the study of skew-symmetric formal generalized r-matrices. Similar to the analytic context, we drop the word "generalized" in this context.

2.3.1 Skew-symmetry and the formal classical Yang-Baxter equation. A series $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ is called a *(normalized) formal r-matrix* if it is in (normalized) standard form and solves the *formal classical Yang-Baxter equation* (formal CYBE)

$$\mathrm{CYB}(r) = 0, \text{ where } \mathrm{CYB}(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]. \tag{2.32}$$

Here, r^{12}, r^{13}, r^{23} are defined in Subsection 2.1.3 and the brackets are commutators in $(\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})) \otimes \Bbbk((x_1))((x_2))[\![x_3]\!]$.

Proposition 2.3.1.
A series $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ is a formal r-matrix if and only if it is a skew-symmetric formal generalized r-matrix.

Proof. It is obvious that a formal generalized r-matrix r solves the formal CYBE (2.32) if $\overline{r} = r$, so we have to prove the contrary, i.e. that each formal r-matrix solves the formal GCYBE (2.14) and satisfies $\overline{r} = r$. The equations (2.27), where the $\overline{r}$ is replaced by r, and (2.28) imply that $\mathfrak{g}(r) \subseteq \mathfrak{g}((z))$ is a Lie subalgebra since $\mathrm{CYB}(r) = 0$. Therefore, Proposition 2.2.1 states that $\mathrm{GCYB}(r) = 0$. In particular, we have:

$$0 = \mathrm{CYB}(r) - \mathrm{GCYB}(r) = [r^{13}, r^{23} - \overline{r}^{23}]. \tag{2.33}$$

Multiplying (2.33) with $x_1 - x_3$, setting $x_1 = x_3$ and subsequently multiplying with an element of $\Bbbk[\![x_3]\!]^\times$ results in $[\gamma^{13}, r^{23} - \overline{r}^{23}] = 0$. Application of the map $a_1 \otimes a_2 \otimes a_3 \longmapsto a_2 \otimes [a_1, a_3]$ gives the desired $\overline{r} = r$. Here we used the following fact:

$$\gamma \mapsto \mathrm{id}_{\mathfrak{g}} \text{ under } \mu\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \text{ defined by } a_1 \otimes a_2 \mapsto \mathrm{ad}(a_1)\mathrm{ad}(a_2). \tag{2.34}$$

Indeed, if $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ is the decomposition of $\mathfrak{g}$ into simple ideals, we have $\gamma = \sum_{i=1}^k \gamma_i$, where γ_i is the Casimir element of $\mathfrak{g}_i$, so we may assume that $\mathfrak{g}$ is simple. Furthermore, an element $f \in \mathrm{End}(\mathfrak{g})$ with the property $f \otimes \mathrm{id}_{\overline{\Bbbk}} = \mathrm{id}_{\mathfrak{g}\otimes\overline{\Bbbk}}$, where $\overline{\Bbbk}$ is the algebraic closure of $\Bbbk$, already satisfies $f = \mathrm{id}_{\mathfrak{g}}$, so we may assume $\Bbbk = \overline{\Bbbk}$. The endomorphism $\mu(\gamma) = \sum_{i=1}^d \mathrm{ad}(b_i)^2$ is the quadratic Casimir operator of the adjoint representation, which is a multiple of the identity because of Schur's Lemma and is equal to the identity since $\mathrm{Tr}(\mathrm{id}_{\mathfrak{g}}) = d = \sum_{i=1}^d K(b_i, b_i) = \mathrm{Tr}(\mu(\gamma))$. □

2.3.2 Skew-symmetry on the level of subalgebras. Let us denote the $\Bbbk((z))$-bilinear extension $\mathfrak{g}((z)) \times \mathfrak{g}((z)) \to \Bbbk((z))$ of the Killing form K of $\mathfrak{g}$ with the same symbol. Then $\mathfrak{g}((z))$ is equipped with the $\Bbbk$-bilinear form K_0 defined by

$$K_{-1}(s,t) := \mathrm{res}_0 K(s,t)\mathrm{d}z = \sum_{k+\ell=-1} K(s_k, t_\ell) \tag{2.35}$$

for all $s = \sum_{k\in\mathbb{Z}} s_k z^k, t = \sum_{k\in\mathbb{Z}} t_k z^k \in \mathfrak{g}((z))$, where $\operatorname{res}_0 \lambda dz = \lambda_{-1}$ for any series $\lambda = \sum_{k\in\mathbb{Z}} \lambda_k z^k \in \Bbbk((z))$.

Remark 2.3.2.
The $\Bbbk((z))$-bilinear extension of K is the Killing form of $\mathfrak{g}((z))$ as a Lie algebra over $\Bbbk((z))$. Therefore, γ can also be understood as the Casimir element of $\mathfrak{g}((z))$. In particular, $[a\otimes 1 + 1\otimes a, \gamma] = 0$ for all $a \in \Bbbk((z))$ and $(\varphi(z)\otimes\varphi(z))\gamma = \gamma$ for all

$$\varphi \in \operatorname{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!]) \subseteq \operatorname{Aut}_{\Bbbk((z))\text{-alg}}(\mathfrak{g}((z))). \tag{2.36}$$

Moreover, K_{-1} is symmetric, non-degenerate and invariant, i.e. $K_{-1}([a,b],c) = K_{-1}(a,[b,c])$ for all $a,b,c \in \mathfrak{g}((z))$.

Using the bilinear form K_{-1} admits us to understand the skew-symmetry of normalized formal generalized r-matrices using the associated Lie subalgebras of $\mathfrak{g}((z))$.

Lemma 2.3.3.
Let r be a normalized formal generalized r-matrix. Then $\mathfrak{g}(r)^\perp = \mathfrak{g}(\overline{r})$ (with respect to the bilinear form (2.35)) is a $\mathfrak{g}(r)$-module satisfying $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(\overline{r})$. In particular, r is skew-symmetric if and only if $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$.

Proof. Let us write

$$r(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}(x)\otimes y^k b_i \text{ and } \overline{r}(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} \overline{r}_{k,i}(x)\otimes y^k b_i \tag{2.37}$$

as well as $r_{k,i}(z) = z^{-k-1}b_i + \sum_{\ell=0}^{\infty}\sum_{j=1}^{d} r_{k,i}^{\ell,j} z^\ell b_j$. Then we have

$$r(x,y) - r_{\text{Yang}}(x,y) = \sum_{k,\ell=0}^{\infty}\sum_{i,j=1}^{d} r_{k,i}^{\ell,j} x^\ell b_j \otimes y^k b_i \tag{2.38}$$

and hence $\overline{r}_{\ell,j}(z) = z^{-\ell-1}b_j - \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}^{\ell,j} z^k b_i$. Therefore, we can deduce

$$K_{-1}(r_{k,i}, \overline{r}_{\ell,j}) = r_{k,i}^{\ell,j} - r_{k,i}^{\ell,j} = 0. \tag{2.39}$$

This implies that $\mathfrak{g}(\overline{r}) \subseteq \mathfrak{g}(r)^\perp$. Now $0 = \mathfrak{g}[\![z]\!] \cap \mathfrak{g}(r)^\perp = (\mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(r))^\perp$ and $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(\overline{r})$, since $\overline{r}$ is in normalized standard form, show that $\mathfrak{g}(r)^\perp = \mathfrak{g}(\overline{r})$. The fact that $[\mathfrak{g}(\overline{r}), \mathfrak{g}(r)] \subseteq \mathfrak{g}(\overline{r})$, i.e. $\mathfrak{g}(\overline{r})$ is a $\mathfrak{g}(r)$-module, is a direct consequence of the inavariance of K_{-1}.

We see that $r = \overline{r}$ implies $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$. On the other hand, since both r and $\overline{r}$ are of normalized standard form, $\mathfrak{g}(r) = \mathfrak{g}(r)^\perp = \mathfrak{g}(\overline{r})$ forces

$$r - \overline{r} \in (\mathfrak{g}(r)\otimes\mathfrak{g})[\![y]\!] \cap (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!] = \{0\}. \tag{2.40}$$

We can conclude that $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$ if and only if $\overline{r} = r$. □

2.3.3 Difference dependence of formal r-matrices. For a normalized formal r-matrix $r(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}(x)\otimes b_i y^k \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ the formal CYBE can be rewritten as

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{i=1}^{d}&[r_{k,i}(x_1)\otimes 1 + 1\otimes r_{k,i}(x_2), r(x_1,x_2)]\otimes b_i x^k\\ &= \sum_{k,\ell=0}^{\infty}\sum_{i,j=1}^{d} r_{k,i}(x_1)\otimes r_{\ell,j}(x_2)\otimes [b_i,b_j]x_3^{k+\ell}.\end{aligned} \tag{2.41}$$

This implies that we can define a linear map $\delta\colon \mathfrak{g}(r)\to\mathfrak{g}(r)\otimes\mathfrak{g}(r)$ by

$$\delta(a)(x,y) := [a(x)\otimes 1 + 1\otimes a(y), r(x,y)],$$

for all $a\in\mathfrak{g}(r)$. This fact can be used to show that formal r-matrices depend on the difference of its formal variables up to equivalence. More precisely we have the following result, which is a variation of the theorem in [BD83b].

Proposition 2.3.4.
Let $r(x,y)$ be a normalized formal r-matrix. There exists $s\in z^{-1}(\mathfrak{g}\otimes\mathfrak{g})[\![z]\!]$ and $\varphi\in \mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ such that $s(x-y) = (\varphi(x)\otimes\varphi(y))r(x,y)$. Furthermore, $\mathfrak{g}(s) := \varphi(\mathfrak{g}(r))$ is closed under the formal derivation $a(z)\mapsto a'(z)$.

Proof. Let $D\colon \mathfrak{g}(r)\to\mathfrak{g}(r)$ be the composition of δ with the Lie bracket $[,]\colon \mathfrak{g}(r)\otimes\mathfrak{g}(r)\to\mathfrak{g}(r)$. Combining the fact (2.34) with $[a\otimes 1 + 1\otimes a,\gamma] = 0$ for all $a\in\mathfrak{g}[\![z]\!]$ and Lemma 2.1.2.(4) results in

$$\left[a(x)\otimes 1 + 1\otimes a(y), \frac{\gamma}{x-y}\right] = \left[\frac{a(x)-a(y)}{x-y}\otimes 1,\gamma\right] \xmapsto{[,]} a'(z). \tag{2.42}$$

If we write $h(x,y)\in\mathfrak{g}[\![x,y]\!]$ for the image of $r(x,y) - r_{\mathrm{Yang}}(x,y)\in(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ under the $\Bbbk[\![x,y]\!]$-linear extension of the Lie bracket $\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ and use the fact that

$$[a\otimes 1 + 1\otimes a, c\otimes d] = [a,c]\otimes d + c\otimes[a,d] \xmapsto{[,]} [[a,c],d] + [c,[a,d]] = [a,[c,d]]$$

holds for all $a,b,c\in\mathfrak{g}$, we can deduce that $D(a)(z) = a'(z) - [h(z,z),a(z)]$.

Let $\psi\in\mathrm{End}(\mathfrak{g})[\![z]\!]$ be the unique solution of $\psi(0) = \mathrm{id}_{\mathfrak{g}}$ and $\psi'(z) = \mathrm{ad}(h(z,z))\psi(z)$ provided by Lemma 2.1.2.(3). For every $a_1,a_2\in\mathfrak{g}$ the series

$$c_1(z) := \psi(z)[a_1,a_2] \text{ and } c_2(z) := [\psi(z)a_1,\psi(z)a_2]\in\mathfrak{g}[\![z]\!] \tag{2.43}$$

satisfy $c_i'(z) = [h(z,z),c_i(z)]$ and $c_i(0) = [a_1,a_2]$ for $i\in\{1,2\}$. Therefore, Lemma 2.1.2.(3) forces $c_1 = c_2$. Thus, ψ defines an element of $\mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$. Let $\varphi(z) := \psi(z)^{-1}\in\mathrm{End}(\mathfrak{g})[\![z]\!]$ and note that $\varphi(z)$ also defines an element of $\mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$. Consider the normalized formal r-matrix $\widetilde{r}(x,y) := (\varphi(x)\otimes\varphi(y))r(x,y)$ and note that $\mathfrak{g}(\widetilde{r}) = \varphi(\mathfrak{g}(r))$ by Lemma 2.2.4. Let $\widetilde{D} := \varphi D\psi\colon \mathfrak{g}(\widetilde{r})\to\mathfrak{g}(\widetilde{r})$ and observe that

$$\widetilde{D}(a)(z) = \varphi(z)D(\psi(a))(z) = \varphi(z)(\psi'(z)a(z) + \psi(z)a'(z) - [h(z,z),\psi(z)a(z)])$$

$$= a'(z) + \varphi(z)([h(z,z),\psi(z)a(z)] - [h(z,z),\psi(z)a(z)]) = a'(z).$$

In particular, $\widetilde{D}\colon \mathfrak{g}(\widetilde{r}) \to \mathfrak{g}(\widetilde{r})$ is the restriction of the formal derivative to $\mathfrak{g}(\widetilde{r})$.

Since $\mathfrak{g}(\widetilde{r})$ is closed under $\widetilde{D}$, which is the restriction of the formal derivative to $\mathfrak{g}(\widetilde{r})$, we have

$$(1 \otimes K(b_i,\cdot))\frac{(-1)^k}{k!}\widetilde{r}^{(k)}(z,0) \in \mathfrak{g}(\widetilde{r}) \cap (b_i z^{-k-1} + \mathfrak{g}[\![z]\!])$$

for all $i \in \{1,\dots,d\}$, $k \in \mathbb{N}_0$, where $\widetilde{r}^{(k)}(z,0) := \widetilde{D}^k\widetilde{r}(z,0)$. The proof of Proposition 2.2.1 and the expansion of $\widetilde{r}(x-y,0)$ as a series imply that

$$\widetilde{r}(x,y) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\widetilde{r}^{(k)}(x,0)y^k = \widetilde{r}(x-y,0)$$

and we can see that putting $s(z) := \widetilde{r}(z,0)$ concludes the proof. □

Remark 2.3.5.
The linear map δ actually defines a so-called *Lie bialgebra structure* on $\mathfrak{g}(r)$. We will discuss these structures in detail in Section 5.1.2 and return to the setting above in Subsection 5.4.2.

3

Algebro-geometric properties of formal generalized r-matrices

Throughout this chapter, $\mathfrak{g}$ is a finite-dimensional, central, simple Lie algebra over a field $\Bbbk$ of characteristic 0 and every formal generalized r-matrix takes values in $\mathfrak{g}$ if not stated otherwise.

3.1 Geometrization of formal generalized r-matrices.

Let $r \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix. The algebra $\mathfrak{g}(r)$ is a $\mathfrak{g}$-lattice of index $(0,0)$ by Proposition 2.2.1, so Theorem 1.3.3 implies that every unital subalgebra $O \subseteq \mathrm{Mult}(\mathfrak{g}(r)) = \{\lambda \in \Bbbk(\!(z)\!) \mid \lambda\mathfrak{g}(r) \subseteq \mathfrak{g}(r)\}$ of finite codimension is a $\Bbbk$-lattice satisfying $O\mathfrak{g}(r) \subseteq \mathfrak{g}(r)$. Therefore, $(O, \mathfrak{g}(r)) \in \mathrm{Lat}_{\mathfrak{g}}$ and Theorem 1.3.6 provides a geometric $\mathfrak{g}$-lattice model $\mathbb{G}(O, \mathfrak{g}(r)) = ((X, \mathscr{A}), (p, c, \zeta))$, where:

- X is an integral projective curve over $\Bbbk$ of arithmetic genus $\dim(\Bbbk(\!(z)\!)/(\Bbbk[\![z]\!] + O))$.
- $p \in X$ is a $\Bbbk$-rational smooth point.
- $c\colon \widehat{\mathscr{O}}_{X,p} \to \Bbbk[\![z]\!]$ is an isomorphism inducing an isomorphism $\mathrm{Spec}(O) \to X \setminus \{p\}$.
- $\mathscr{A}$ is a coherent sheaf of Lie algebras on X which is étale $\mathfrak{g}$-locally free at p (see Remark 1.3.7) and satisfies $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$.
- $\zeta\colon \widehat{\mathscr{A}_p} \to \mathfrak{g}[\![z]\!]$ is a c-equivariant isomorphism such that $\zeta(\Gamma(X \setminus \{p\}, \mathscr{A})) = \mathfrak{g}(r)$.

3.1.1 Geometrization of equivalences. Let $r_1, r_2 \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be formal generalized r-matrices, which are equivalent via some equivalence (μ, w, φ), and $O_1 \subseteq \mathrm{Mult}(\mathfrak{g}(r_1))$ be a unital subalgebra of finite codimension. It is easy to see that the image O_2 of O_1 under the automorphism of $\Bbbk(\!(z)\!)$ defined by $\lambda(z) \mapsto \lambda(w(z))$ is a $\Bbbk$-subalgebra of $\mathrm{Mult}(\mathfrak{g}(r))$ of finite codimension. Let $((X_i, \mathscr{A}_i), (p_i, c_i, \zeta_i)) = \mathbb{G}(O_i, \mathfrak{g}(r_i))$ for $i \in \{1, 2\}$.

Proposition 3.1.1.
The equivalence (μ, w, φ) defines an isomorphism

$$((X_1, \mathscr{A}_1), (p_1, c_1, \zeta_1)) \longrightarrow ((X_2, \mathscr{A}_2), (p_2, c_2, \zeta_2)) \tag{3.1}$$

in the category $\mathrm{GeomLat}_{\mathfrak{g}}$.

Proof. It is easy to see from Lemma 2.2.4 that the equivalence (μ, w, φ) defines an isomorphism $(O_1, W_1) \to (O_2, W_2)$ in Lat_A. Thus, the statement is a direct consequence of the functoriality of $\mathbb{G}$. $\square$

3.1.2 Example: homogeneous formal generalized r-matrices. It is easy to see that $\mathfrak{g}(r_{\mathrm{Yang}}) = z^{-1}\mathfrak{g}[z^{-1}]$. This subalgebra is stable under multiplication by z^{-1}. In general, a subalgebra $W \subseteq \mathfrak{g}(\!(z)\!)$, satisfying $\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!] \oplus W$, is called *homogeneous* if $z^{-1}W \subseteq W$. Such a subalgebra is automatically a deformation of $\mathfrak{g}(r_{\mathrm{Yang}})$ in the sense that there exists $A \in \mathrm{End}(\mathfrak{g})[\![z]\!]$ such that $A(0) = \mathrm{id}_{\mathfrak{g}}$ and

$$W = A\mathfrak{g}(r_{\mathrm{Yang}}) = \langle z^{-k-1}A(z)b_i \mid k \in \mathbb{N}_0, i \in \{1,\dots,d\}\rangle_{\Bbbk}. \tag{3.2}$$

The series A is thereby uniquely determined by $z^{-1}A(z)b_i \in W$ for all $i \in \{1,\dots,d\}$. Recall that two Lie brackets $[,]_1, [,]_2$ on a vector space $\mathfrak{l}$ are called *compatible* if $[,]_1 + [,]_2$ defines a Lie bracket. In this case $\lambda[,]_1 + \mu[,]_2$ is a Lie bracket for all $\lambda, \mu \in \Bbbk$. The condition on A in order for W to be a Lie algebra is examined in [GS02], where the following result is proven.

Theorem 3.1.2.
The vector space $W = A\mathfrak{g}(r_{\mathrm{Yang}}) \subseteq \mathfrak{g}(\!(z)\!)$ for $A(z) = \mathrm{id}_{\mathfrak{g}} + zR + \dots \in \mathrm{End}(\mathfrak{g})[\![z]\!]$ is a Lie subalgebra if and only if

$$[A(z)a, A(z)b] = A(z)([a,b] + z[a,b]_R) \tag{3.3}$$

holds for all $a, b \in \mathfrak{g}$, where

$$[a,b]_R := [Ra, b] + [a, Rb] - R[a,b] \tag{3.4}$$

is a Lie bracket of $\mathfrak{g}$, compatible with the original Lie bracket.

We call a formal generalized r-matrix r *homogeneous* if r is normalized and the Lie algebra $\mathfrak{g}(r) = A\mathfrak{g}(r_{\mathrm{Yang}})$ is a homogeneous Lie algebra. Since $\overline{r}$ is in this case also in normalized standard form, we can see that $\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(\overline{r})$, hence there exists a unique invertible series $\overline{A} \in \mathrm{End}(\mathfrak{g})[\![z]\!]$ such that $\overline{A}(0) = \mathrm{id}_{\mathfrak{g}}$ and $\mathfrak{g}(\overline{r}) = \overline{A}\mathfrak{g}(r_{\mathrm{Yang}})$.

Lemma 3.1.3.
Let $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be a homogeneous generalized r-matrix and A, $\overline{A} \in \mathrm{End}(\mathfrak{g})[\![z]\!]$ be given by $\mathfrak{g}(r) = A\mathfrak{g}(r_{\mathrm{Yang}})$, $\mathfrak{g}(\overline{r}) = \overline{A}\mathfrak{g}(r_{\mathrm{Yang}})$. If $A(x) \otimes \overline{A}(y)$ is understood as an element of $(\mathrm{End}(\mathfrak{g}) \otimes \mathrm{End}(\mathfrak{g}))[\![x,y]\!]$,

$$r(x,y) = \frac{A(x) \otimes \overline{A}(y)}{x-y}\gamma \tag{3.5}$$

holds, where $\gamma \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir element. In particular, r is skew-symmetric if and only if $A = \overline{A}$.

Proof. It is easy to see that $\mathfrak{g}(\overline{r}) = \mathfrak{g}(r)^\perp$ implies that

$$\mathrm{res}_0 z^k K(Aa_1, \overline{A}a_2)\mathrm{d}z = \begin{cases} K(a_1,a_2) & k = -1 \\ 0 & k \neq -1 \end{cases} \tag{3.6}$$

for all $a_1, a_2 \in \mathfrak{g}$. From this we can deduce that $K(Aa_1, \overline{A}a_2) = K(a_1, a_2)$ for all

$a_1, a_2 \in \mathfrak{g}(\!(z)\!)$ and as a consequence $(A(z) \otimes \overline{A}(z))\gamma = \gamma$. Therefore,

$$\widetilde{r}(x, y) \coloneqq \frac{A(x) \otimes \overline{A}(y)}{x - y}\gamma \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \tag{3.7}$$

is in normalized standard form by virtue of Lemma 2.1.2.(4). Furthermore, it is straight forward to verify that $\widetilde{r} \in (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!]$ and hence

$$r - \widetilde{r} \in (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!] \cap (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!] = \{0\}, \tag{3.8}$$

where we used that r and $\widetilde{r}$ are both of normalized standard form. □

Geometrically, homogeneous r-matrices are exactly those r-matrices whose "maximal" geometric datum has the underlying curve $\mathbb{P}^1_{\Bbbk}$. More precisely, we have the following statement.

Lemma 3.1.4.
Let r be a normalized formal generalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be a unital subalgebra of finite codimension, and $\mathbb{G}(O, \mathfrak{g}(r)) = ((X, \mathscr{A}), (p, c, \zeta))$.
(1) *r is homogeneous and $O = \mathrm{Mult}(\mathfrak{g}(r)) \implies X = \mathbb{P}^1_{\Bbbk}$.*
(2) *$X \cong \mathbb{P}^1_{\Bbbk} \implies O = \mathrm{Mult}(\mathfrak{g}(r))$ and r is equivalent to a homogeneous formal generalized r-matrix.*

Proof. The proof of (1) is straightforward, so it remains to proof (2). The identity $O = \mathrm{Mult}(\mathfrak{g}(r))$ holds, since O is integrally closed. Furthermore, $\Bbbk(\!(z)\!) = \Bbbk[\![z]\!] + O$ holds. This and $O \cap \Bbbk[\![z]\!] = \Bbbk$ can be used to deduce that $O = \Bbbk[u^{-1}]$ for an arbitrary $u \in z\Bbbk[\![z]\!]^\times$ such that $u^{-1} \in O$. Let $w \in z\Bbbk[\![z]\!]^\times$ be the compositional inverse of u: $w(u(z)) = z$. Then $\widetilde{r}(x, y) \coloneqq r(w(x), w(y))$ satisfies $\mathrm{Mult}(\mathfrak{g}(\widetilde{r})) = \Bbbk[z^{-1}]$. In other words, $\mathfrak{g}(\widetilde{r})$ is homogeneous. After potentially rescaling $\widetilde{r}$, we obtain a homogeneous formal generalized r-matrix equivalent to r. □

3.2 Properties of geometric data associated to formal generalized r-matrices

Let r be a formal generalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be any unital subalgebra of finite codimension, and $((X, \mathscr{A}), (p, c, \zeta)) \coloneqq \mathbb{G}(O, \mathfrak{g}(r))$. In this section, we discuss the properties of $((X, \mathscr{A}), (p, c, \zeta))$. In particular, we will see that there are certain restrictions on X and these lead to splittings of formal generalized r-matrices into distinct categories. For instance, we will see in Subsection 3.2.3 that, if r is normalized and skew-symmetric, X can be chosen to be of arithmetic genus one, resulting in a geometric trichotomy of normalized formal r-matrices. In this case, $\mathscr{A}$ turns out to be étale $\mathfrak{g}$-locally free at the smooth locus of X; see Subsection 3.2.4.

3.2.1 Geometric dichotomy of formal generalized r-matrices. Recall that the *geometric genus* of an algebraic curve is the genus of its normalization.

Proposition 3.2.1.
Let r be a formal generalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be any unital subalgebra of finite codimension, and $((X, \mathscr{A}), (p, c, \zeta)) := \mathbb{G}(O, \mathfrak{g}(r))$ (see Theorem 1.3.6). The geometric genus $\widetilde{g}$ of X is at most one, i.e. $\widetilde{g} \in \{0, 1\}$.

Proof. Let $(\nu, \iota)\colon ((X, \mathscr{A}), (x, c, \varphi)) \to ((X^\nu, \mathscr{A}^\nu), (p^\nu, c^\nu, \zeta^\nu))$ be the image of the integral closure $(O, \mathfrak{g}(r)) \to (O^\nu, \mathfrak{g}(r)^\nu)$ under the geometrization functor $\mathbb{G}$; see Subsection 1.3.6. The cokernel of the injective morphism $\iota\colon \mathscr{A} \to \nu_*\mathscr{A}^\nu$ is a torsion sheaf by Lemma 1.3.9, so $\mathrm{h}^1(\mathrm{Cok}(\iota)) = 0$. The long exact sequence in cohomology of

$$0 \longrightarrow \mathscr{A} \xrightarrow{\iota} \nu_*\mathscr{A}^\nu \longrightarrow \mathrm{Cok}(\iota) \longrightarrow 0 \tag{3.9}$$

combined with $\mathrm{h}^1(\mathscr{A}) = 0 = \mathrm{h}^1(\mathrm{Cok}(\iota))$ implies that $\mathrm{h}^1(\mathscr{A}^\nu) = 0$. Let

$$\mathcal{K}\colon \mathscr{A}^\nu \times \mathscr{A}^\nu \to \mathscr{O}_{X^\nu} \tag{3.10}$$

be the Killing form of the finite locally free sheaf $\mathscr{A}^\nu$ of Lie algebras, and $\mathcal{K}^{\mathrm{a}}\colon \mathscr{A}^\nu \to \mathscr{A}^{\nu,*} = \mathscr{H}om_{\mathscr{O}_{X^\nu}}(\mathscr{A}^\nu, \mathscr{O}_{X^\nu})$ be the natural morphism induced by $\mathcal{K}$. The fiber of $\mathcal{K}$ in p coincides with the Killing form of $\mathscr{A}^\nu|_p \cong \mathscr{A}|_p \cong \mathfrak{g}$ by virtue of Lemma 1.1.2 and is non-degenerate as a consequence. Therefore, $\mathcal{K}^{\mathrm{a}}|_p$ is an isomorphism, so $\mathrm{Ker}(\mathcal{K}^{\mathrm{a}})$ and $\mathrm{Cok}(\mathcal{K}^{\mathrm{a}})$ are torsion sheaves. In particular, $\mathrm{h}^1(\mathrm{Cok}(\mathcal{K}^{\mathrm{a}})) = 0$ and $\mathrm{Ker}(\mathcal{K}^{\mathrm{a}})$ vanishes, since it is a torsion subsheaf of the torsion free sheaf $\mathscr{A}^\nu$. The long exact sequence in cohomology of

$$0 \longrightarrow \mathscr{A}^\nu \xrightarrow{\mathcal{K}^{\mathrm{a}}} \mathscr{A}^{\nu,*} \longrightarrow \mathrm{Cok}(\mathcal{K}^{\mathrm{a}}) \longrightarrow 0 \tag{3.11}$$

combined with $\mathrm{h}^1(\mathscr{A}^\nu) = 0 = \mathrm{h}^1(\mathrm{Cok}(\iota))$ implies that $\mathrm{h}^1(\mathscr{A}^{\nu,*}) = 0$.

The Riemann-Roch theorem for $\mathscr{A}^\nu$ and $\mathscr{A}^{\nu,*}$ (e.g. in the version of [Liu02, Chapter 7, Exercise 3.3]) reads

$$\begin{aligned} 0 &\leqslant \mathrm{h}^0(\mathscr{A}^\nu) - \mathrm{h}^1(\mathscr{A}^\nu) = \deg(\det(\mathscr{A}^\nu)) + (1 - \widetilde{g})\mathrm{rank}(\mathscr{A}^\nu) \\ 0 &\leqslant \mathrm{h}^0(\mathscr{A}^{\nu,*}) - \mathrm{h}^1(\mathscr{A}^{\nu,*}) = -\deg(\det(\mathscr{A}^{\nu,*})) + (1 - \widetilde{g})\mathrm{rank}(\mathscr{A}^\nu), \end{aligned}$$

where we used that $\det(\mathscr{A}^{\nu,*}) = \det(\mathscr{A}^\nu)^*$ implies $\deg(\det(\mathscr{A}^{\nu,*})) = -\deg(\det(\mathscr{A}^\nu))$. We conclude $\widetilde{g} \leqslant 1$. □

Since the geometric genus $\widetilde{g}$ of X is an invariant of the equivalence class of r by Proposition 3.1.1, the Proposition 3.2.1 splits the equivalence classes of formal generalized r-matrices into two types. In order to examine the $\widetilde{g} = 1$ case, we need the following observation.

Remark 3.2.2.
Let ω_X be the dualizing sheaf of X. Since p is smooth, $\omega_{X,p}$ can be identified with the Kähler differentials $\Omega^1_{\mathscr{O}_{X,p}/\Bbbk}$. Therefore, using e.g. [Kun13, Corollary 12.5 and Example 12.7], we obtain a c-equivariant isomorphism $c^*\colon \widehat{\omega}_{X,p} \to \Bbbk[\![z]\!]\mathrm{d}z$. More precisely, the differential $\mathscr{O}_{X,p} \to \Omega^1_{\mathscr{O}_{X,p}/\Bbbk} \cong \omega_{X,p}$ induces a continuous differential

$\mathrm{d}\colon \widehat{\mathscr{O}}_{X,p} \to \widehat{\omega}_{X,p}$, whose image generates $\widehat{\omega}_{X,p}$ and the identity $c^*(\mathrm{d}f) = c(f)'\mathrm{d}z$ holds. The isomorphism c^* respects residues by [Tat68, Theorem 2]: $\mathrm{res}_p\eta = \mathrm{res}_0 c^*\eta$ for all $\eta \in \mathrm{Q}(\widehat{\omega}_{X,p})$. Here we extended c^* to a map $\mathrm{Q}(\widehat{\omega}_{X,p}) \to \Bbbk(\!(z)\!)\mathrm{d}z$.

The following result implies that for $\widetilde{g} = 1$, r is already skew-symmetric up to equivalence. In particular, if r is not equivalent to a formal r-matrix, $X^\nu \cong \mathbb{P}^1_\Bbbk$ holds automatically, i.e. X is rational.

Theorem 3.2.3.
Let r be a formal generalized r-matrix and

$$(\nu,\iota)\colon ((X,\mathscr{A}),(p,c,\varphi)) \longrightarrow ((X^\nu,\mathscr{A}^\nu),(p^\nu,c^\nu,\zeta^\nu)) \tag{3.12}$$

be the image of the integral closure $(\mathrm{Mult}(\mathfrak{g}(r)),\mathfrak{g}(r)) \to (\mathrm{Mult}(\mathfrak{g}(r))^\nu,\mathfrak{g}(r)^\nu)$ *under the functor $\mathbb{G}$ (see Subsection 1.3.6). Assume that the genus of X^ν is one.*

(1) *(ν,ι) is an isomorphism and the Killing form $\mathcal{K}$ of $\mathscr{A}$ is perfect.*

(2) *There exist $\mu \in \Bbbk[\![z]\!]^\times$, $w \in z\Bbbk[\![z]\!]^\times$ such that $\widetilde{r}(x,y) := \mu(y)r(w(x),w(y))$ is normalized and skew-symmetric, i.e. r is equivalent to the normalized formal r-matrix $\widetilde{r}$.*

Proof. Let $\Omega^1_{X^\nu}$ be the sheaf of regular 1-forms on X^ν. We have $\mathrm{H}^1(\Omega^1_{X^\nu}) = \Bbbk\eta$ for some global 1-form η on X^ν and this choice defines an isomorphism $\Omega^1_{X^\nu} \cong \mathscr{O}_{X^\nu}$. Serre duality (e.g. in the version [Liu02, Chapter 6, Remark 4.20 and Theorem 4.32]) and the proof of Proposition 3.2.1 provide $0 = \mathrm{h}^1(\mathscr{A}^{\nu,*}) = \mathrm{h}^0(\mathscr{A}^\nu)$. Combined with $\mathrm{h}^1(\mathscr{A}) = 0$ and the fact that $\mathrm{Cok}(\iota)$ is torsion (see Lemma 1.3.9), the long exact sequence of (3.9) in cohomology implies that $\iota\colon \mathscr{A} \to \nu_*\mathscr{A}^\nu$ is an isomorphism. In particular, $\mathfrak{g}(r)^\nu = \mathfrak{g}(r)$, $\mathrm{Mult}(\mathfrak{g}(r))^\nu = \mathrm{Mult}(\mathfrak{g}(r))$, and $X = X^\nu$. The first part of (1) is proven.

Let $\mathrm{d}u(z) = u'(z)\mathrm{d}z = c^*(\eta)$ (see Remark 3.2.2) and $a,b \in \Gamma(X\setminus\{p\},\mathscr{A})$. If $w \in z\Bbbk[\![z]\!]^\times$ is the series uniquely determined by $w(u(z)) = z$ and $\widetilde{a} := \zeta(a), \widetilde{b} := \zeta(b)$, we may calculate

$$\begin{aligned}K_{-1}\left(\widetilde{a}(w(z)),\widetilde{b}(w(z))\right) &= \mathrm{res}_0 K\left(\widetilde{a}(w(z)),\widetilde{b}(w(z))\right)\mathrm{d}z\\ &= \mathrm{res}_0 K\left(\widetilde{a}(z),\widetilde{b}(z)\right)\mathrm{d}u(z) = \mathrm{res}_p\mathcal{K}(a,b)\eta = 0.\end{aligned}$$

Here, the second to last equality uses the commutative diagram (3.14) below and Remark 3.2.2, while the last equality is due to the residue theorem [Tat68, Corollary of Theorem 3] under consideration of $\mathcal{K}(a,b)\eta \in \Gamma(X\setminus\{p\},\Omega_X)$. Thus, the image $\mathfrak{W}$ of $\mathfrak{g}(r)$ under $a(z) \mapsto a(w(z))$ satisfies $\mathfrak{W}^\perp = \mathfrak{W}$. Lemma 2.2.4 states that $\mathfrak{W} = \mathfrak{g}(\widetilde{r})$ for $\widetilde{r}(x,y) = \mu(y)r(w(x),w(y))$, where $\mu \in \Bbbk[\![z]\!]$ is arbitrary. This shows that $\widetilde{r}$ is a formal r-matrix if we chose μ in such a way that $\widetilde{r}$ is normalized; see Lemma 2.3.3. Thus, (2) is proven.

It remains to prove the second part of (1), i.e. that $\mathcal{K}$ is perfect. This follows from $\mathrm{Cok}(\mathcal{K}^{\mathrm{a}}) = 0$, which is a consequence of using $\mathrm{h}^1(\mathscr{A}) = \mathrm{h}^0(\mathscr{A}^*) = 0$ in the long exact sequence of (3.11) in cohomology. Here, we used the Serre duality again. □

3.2.2 An inclusion result for orthogonal complements. Proposition 3.2.1 can be used to derive the following useful lemma.

Lemma 3.2.4.
Let r be a formal generalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be any unital subalgebra of finite codimension, and $(O^\nu, \mathrm{gr}(r)^\nu)$ be the integral closure of $(O, \mathfrak{g}(r))$. Furthermore, let $\Bbbk(\!(z)\!)$ be equipped with the bilinear form defined by

$$\left(\lambda = \sum_{k\in\mathbb{Z}} \lambda_k z^k, \mu = \sum_{k\in\mathbb{Z}} \mu_k z^k\right) \longmapsto \mathrm{res}_0 \lambda\mu \mathrm{d}z = \sum_{k+\ell=-1} \lambda_k \mu_\ell. \tag{3.13}$$

The chain of inclusions $O^{\nu,\perp}\mathfrak{g}(r) \subseteq O^{\nu,\perp}\mathfrak{g}(r)^\nu \subseteq \mathfrak{g}(r)^{\nu,\perp} \subseteq \mathfrak{g}(r)^\perp$ holds.

Proof. Let $(\nu, \iota)\colon ((X, \mathscr{A}), (x, c, \varphi)) \to ((X^\nu, \mathscr{A}^\nu), (p^\nu, c^\nu, \zeta^\nu))$ be the image of the canonical morphism $(O, \mathfrak{g}(r)) \to (O^\nu, \mathfrak{g}(r)^\nu)$ under $\mathbb{G}$. The diagram

$$\begin{array}{ccc} \Gamma(U, \mathscr{A}^\nu) \times \Gamma(U, \mathscr{A}^\nu) & \xrightarrow{\mathcal{K}} & \Gamma(U, \mathscr{O}_{X^\nu}) \\ \downarrow{\scriptstyle \zeta^\nu \times \zeta^\nu} & & \downarrow{\scriptstyle c^\nu} \\ \mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z)\!) & \xrightarrow[K]{} & \Bbbk(\!(z)\!) \end{array} \tag{3.14}$$

commutes for all affine open $U \subseteq X$ such that $\mathscr{A}^\nu|_U$ is free. Consequently, (3.14) commutes for all $U \subseteq X$ open, by a gluing argument. In particular,

$$K(a, b) \in c^\nu(\Gamma(X^\nu \setminus \{p^\nu\}, \mathscr{O}_{X^\nu})) = O^\nu \tag{3.15}$$

for all $a, b \in \mathfrak{g}(r)^\nu$. Hence, we see that

$$K_{-1}(\lambda a, b) = \mathrm{res}_0 K(\lambda a, b)\mathrm{d}z = \mathrm{res}_0 \lambda K(a, b)\mathrm{d}z = 0$$

for all $\lambda \in O^{\nu,\perp}$. Therefore, $\lambda a \in \mathfrak{g}(r)^{\nu,\perp}$ and we can complete the chain of inclusions by observing that $\mathfrak{g}(r) \subseteq \mathfrak{g}(r)^\nu$ implies $\mathfrak{g}(r)^{\nu,\perp} \subseteq \mathfrak{g}(r)^\perp$. □

3.2.3 Geometric trichotomy of formal r-matrices. We can refine Proposition 3.2.1 for normalized formal r-matrices in the following way.

Theorem 3.2.5.
Let r be a normalized formal r-matrix. There is a canonical unital subalgebra $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ of finite codimension such that X has arithmetic genus one, where $((X, \mathscr{A}), (p, c, \zeta)) := \mathbb{G}(O, \mathfrak{g}(r))$ is the associated geometric $\mathfrak{g}$-lattice model.

Proof. Let $((X^\nu, \mathscr{A}^\nu), (p^\nu, c^\nu, \zeta^\nu)) := \mathbb{G}(O^\nu, \mathfrak{g}(r)^\nu)$, where $(O^\nu, \mathfrak{g}(r)^\nu)$ is the integral closure of $(O, \mathfrak{g}(r))$. If the genus of X^ν is one, the claim is already proven in Theorem 3.2.3, so we may assume that $\mathrm{h}^1(\mathscr{O}_{X^\nu}) = 0$, i.e. $\Bbbk[\![z]\!] + O^\nu = \Bbbk(\!(z)\!)$. Since $O^\nu \cap \Bbbk[\![z]\!] = \Bbbk$, the canonical projection $O \to \Bbbk[z^{-1}]$ is injective and as a consequence $O^\nu = \Bbbk[u]$ for an arbitrary $u \in z^{-1}\Bbbk[\![z]\!]^\times \cap O^\nu$. Lemma 3.2.4 states that

$$O^{\nu,\perp}\mathfrak{g}(r) \subseteq \mathfrak{g}(r)^\perp = \mathfrak{g}(r), \tag{3.16}$$

where $\Bbbk(\!(z)\!)$ is equipped with the bilinear form (3.13) and the last equality follows from Proposition 2.3.1 and Lemma 2.3.3. In particular, $O^{\nu,\perp} \subseteq \mathrm{Mult}(\mathfrak{g}(r)) \subseteq O^\nu$. Since for all $k \in \mathbb{N}_0$

$$\mathrm{res}_0 u^k u' \mathrm{d}z = \mathrm{res}_0 \frac{1}{k+1}\left(u^{k+1}\right)' \mathrm{d}z = 0 \tag{3.17}$$

holds, where $(\cdot)'$ denotes the formal derivative with respect to z, we see that $u'O^\nu \subseteq O^{\nu,\perp} \subseteq O^\nu$. Thus, (3.16) implies that $\Bbbk + u'O^\nu \subseteq \mathrm{Mult}(\mathfrak{g}(r))$. The fact that $u' \in O^\nu = \Bbbk[u]$ has order two can be used to see that $\Bbbk + u'O^\nu = \Bbbk + O^{\nu,\perp} = \Bbbk[u', u'u]$ is an unital $\Bbbk$-subalgebra of $\mathrm{Mult}(\mathfrak{g}(r))$ such that $\dim(\Bbbk(\!(z)\!)/(\Bbbk[\![z]\!] + \Bbbk[u', u'u])) = 1$. Choosing $O = \Bbbk[u', u'u] = \Bbbk + O^{\nu,\perp}$ implies that X has arithmetic genus 1. □

Theorem 3.2.5 can be understood as a geometric trichotomy of formal generalized r-matrices, i.e. a splitting of formal generalized r-matrices into three categories. Indeed, this follows from the following observation.

Remark 3.2.6.
Every integral projective curve Y over $\Bbbk$ of arithmetic genus one with a $\Bbbk$-rational smooth point q is a plane cubic curve, i.e. determined by one cubic equation. This can be seen for example from Theorem 1.3.4: fix an isomorphism $t\colon \widehat{\mathscr{O}}_{Y,q} \to \Bbbk[\![z]\!]$, write $O := t(\Gamma(Y \setminus \{q\}, \mathscr{O}_Y)$, and note that the codimension of $\Bbbk[\![z]\!] + O$ in $\Bbbk(\!(z)\!)$ is one. Then $O \cap \Bbbk[\![z]\!] = t(\Gamma(Y, \mathscr{O}_Y)) = \Bbbk$ implies that $O = \Bbbk[f, g]$, where f has order 2 and g has order 3. After properly adjusting f and g, we get $g^2 = f^3 + af + b$ for some $a, b \in \Bbbk$. This is a minimal polynomial relation between f and g, so $O \cong \Bbbk[x, y]/(y^2 - x^3 - ax - b)$. In other words, Y is a plane cubic curve.
It is easy to see that X is smooth if and only if $4a^3 + 27b^2 \neq 0$, in which case it is elliptic, and has a unique nodal (resp. cuspidal) singularity if $4a^3 = -27b^2 \neq 0$ (resp. $a = b = 0$). In the singular cases Y is rational, i.e. Y has the normalization $\mathbb{P}^1_\Bbbk \to Y$.

Another interesting observation is the fact that the three classes are preserved by formal equivalences.

Remark 3.2.7.
Let $r_1, r_2 \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be two normalized formal r-matrices which are equivalent via the equivalence (μ, w, φ). Furthermore, let $O_i \subseteq \mathrm{Mult}(\mathfrak{g}(r_i))$ be the subalgebra constructed in Theorem 3.2.5 for $r_i = r$, and $((X_i, \mathscr{A}_i), (p_i, c_i, \zeta_i)) = \mathbb{G}(O_i, \mathfrak{g}(r_i))$, where $i \in \{1, 2\}$. By virtue of Lemma 2.1.6.(4), we have $\mu \in \Bbbk^\times$ and $w(z) = \mu z$. Therefore, it is easy to see from the construction of O in the proof of Theorem 3.2.5 that $\lambda(z) \mapsto \lambda(w(z)) = \lambda(\mu z)$ defines an isomorphism $O_1 \to O_2$. We can deduce from Lemma 3.1.1 that $X_1 \cong X_2$. Therefore, X_1 is elliptic (resp. nodal, resp. cuspidal) if and only if X_2 is elliptic (resp. nodal, resp. cuspidal).

3.2.4 Geometric data of normalized formal r-matrices. In this section, we discuss further properties of the geometric datum associated to formal generalized r-matrices.

Proposition 3.2.8.
Let r be a normalized formal r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be chosen according to Theorem 3.2.5, $(\nu,\iota)\colon ((X,\mathscr{A}),(x,c,\varphi)) \to ((X^\nu,\mathscr{A}^\nu),(p^\nu,c^\nu,\zeta^\nu))$ be the image of the integral closure $(O,\mathfrak{g}(r)) \to (O^\nu,\mathrm{gr}(r)^\nu)$ under $\mathbb{G}$, and $\mathcal{K}^\nu$ be the Killing form of $\mathscr{A}^\nu$. There exists a unique perfect pairing $\mathcal{K}\colon \mathscr{A}\times\mathscr{A} \to \mathscr{O}_X$ such that

$$\begin{array}{ccc} \mathscr{A}\times\mathscr{A} & \xrightarrow{\mathcal{K}} & \mathscr{O}_X \\ \downarrow{\scriptstyle \iota\times\iota} & & \downarrow{\scriptstyle \nu^\flat} \\ \nu_*\mathscr{A}^\nu \times \nu_*\mathscr{A}^\nu & \xrightarrow[\nu_*\mathcal{K}^\nu]{} & \nu_*\mathscr{O}_{X^\nu} \end{array} \tag{3.18}$$

commutes.

Proof. The statement is already proven in Theorem 3.2.3 for $\mathrm{h}^1(\mathscr{O}_{X^\nu}) = 1$, so we may assume $\mathrm{h}^1(\mathscr{O}_{X^\nu}) = 0$, i.e. $O^\nu = \Bbbk[u]$ and $O = \Bbbk[u', u'u]$; see the proof of Theorem 3.2.5. Since $X = C \cup (X \setminus \{p\})$, where C is the smooth locus of X, we have to define the paring $\mathcal{K}$ on the affine open set $X \setminus \{p\}$ and show that it is compatible with $\nu_*\mathcal{K}$. The diagram (3.14) implies that $K\colon \mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z)\!) \to \Bbbk(\!(z)\!)$ restricts to a mapping $\mathfrak{g}(r)^\nu \times \mathfrak{g}(r)^\nu \to O^\nu$ and it suffices to show that this pairing restricts further to $\mathfrak{g}(r)\times\mathfrak{g}(r) \to O$. Observe that $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$ implies that K restricts to a bilinear map

$$\mathfrak{g}(r)\times\mathfrak{g}(r) \to P \coloneqq \{\lambda \in O^\nu \mid \mathrm{res}_0 \lambda \mathrm{d}z = 0\}. \tag{3.19}$$

We have $O = \Bbbk + O^{\nu,\perp} \subseteq P$, since $1 \in O^\nu$. The codimension of P in O^ν is one, i.e. O^ν/P is spanned by $u + P$. The same is true for O^ν/O proving $P = O$.

It remains to show that $\mathcal{K}$ is perfect. Similarly to the proof of Proposition 3.2.1, the morphism $\mathcal{K}^{\mathrm{a}}\colon \mathscr{A} \to \mathscr{A}^*$, induced by $\mathcal{K}$, is injective with a torsion cokernel $\mathrm{Cok}(\mathcal{K}^{\mathrm{a}})$. The long exact sequence in cohomology induced by

$$0 \longrightarrow \mathscr{A} \xrightarrow{\mathcal{K}^{\mathrm{a}}} \mathscr{A}^* \longrightarrow \mathrm{Cok}(\mathcal{K}^{\mathrm{a}}) \longrightarrow 0,$$

combined with $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$, implies $\mathrm{h}^0(\mathrm{Cok}(\mathcal{K}^{\mathrm{a}})) = \mathrm{h}^0(\mathscr{A}^*) = \mathrm{h}^1(\mathscr{A}) = 0$. Here, we used the Serre duality (in e.g. the version [Liu02, Chapter 6, Remark 4.20 and Theorem 4.32]). Note that $\mathrm{h}^1(\mathscr{O}_X) = 1$ and the fact that X is locally a complete intersection (see e.g. Remark 3.2.6) thereby implied that the dualizing sheaf of X is trivial. We conclude that $\mathrm{Cok}(\mathcal{K}^{\mathrm{a}})$ vanishes. Thus, $\mathcal{K}^{\mathrm{a}}$ is an isomorphism. This is equivalent to saying that $\mathcal{K}$ is perfect. □

Theorem 3.2.9.
Let r be a normalized formal r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be chosen according to Theorem 3.2.5, and $((X,\mathscr{A}),(p,c,\zeta)) \coloneqq \mathbb{G}(O,\mathfrak{g}(r))$. The sheaf of Lie algebras $\mathscr{A}|_C$ is étale $\mathfrak{g}$-locally free, where C is the smooth locus of X.

Proof. Let $((X_{\overline{\Bbbk}}, \mathscr{A}_{\overline{\Bbbk}}), (p_{\overline{\Bbbk}}, c_{\overline{\Bbbk}}, \zeta_{\overline{\Bbbk}}))$ be the geometric datum associated to $(O_{\overline{\Bbbk}}, W_{\overline{\Bbbk}})$ and $\pi\colon X_{\overline{\Bbbk}} \to X$ be the canonical morphism, where we recall that $\overline{\Bbbk}$ is the algebraic closure of $\Bbbk$ and use the notation from Subsection 1.3.7. Observe that $r_{\overline{\Bbbk}}$ is a normalized formal r-matrix, π factors as $X_{\overline{\Bbbk}} \cong X \times \mathrm{Spec}(\overline{\Bbbk}) \to X$, where the second map is the canonical projection, and $\mathscr{A}_{\overline{\Bbbk}} \cong \pi^*\mathscr{A}$. Lemma 1.2.2 states that $\mathscr{A}|_C$ is étale $\mathfrak{g}$-locally free if $\mathscr{A}_{\overline{\Bbbk}}|_{C_{\overline{\Bbbk}}}$ is étale $(\mathfrak{g} \otimes \overline{\Bbbk})$-locally free, where $\pi^{-1}(C) = C_{\overline{\Bbbk}}$ is the smooth locus of $X_{\overline{\Bbbk}}$. Hence, we may assume that $\Bbbk = \overline{\Bbbk}$.

Using Proposition 3.2.8, Lemma 1.1.2 and Cartan's criterion for semi-simplicity, we see that $\mathscr{A}|_q$ is semi-simple for all closed $q \in C$. Therefore, $\mathscr{A}$ is étale $\mathscr{A}|_q$-locally free in any closed point $q \in C$; see Theorem 1.2.5. Consequently, $\mathscr{A}$ is weakly $\mathscr{A}|_q$-locally free in all closed points in some open neighbourhood of q, since étale maps are open; see also Remark 1.2.1. This forces $\mathscr{A}|_q \cong \mathscr{A}|_p \cong \mathfrak{g}$ for all $q \in C$ closed, since C is connected. Theorem 1.2.3 states that $\mathscr{A}|_C$ is étale $\mathfrak{g}$-locally free. □

Proposition 3.2.10.
Let r be a normalized formal r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be chosen according to Theorem 3.2.5, and $((X, \mathscr{A}), (p, c, \zeta)) := \mathbb{G}(O, \mathfrak{g}(r))$. There exists a unique element $\eta = \eta_r \in \mathrm{H}^0(\omega_X)$ such that $c^*(\eta) = \mathrm{d}z$, where ω_X is the dualizing sheaf of X.

Proof. As in the proof of Theorem 3.2.9, let $((X_{\overline{\Bbbk}}, \mathscr{A}_{\overline{\Bbbk}}), (p_{\overline{\Bbbk}}, c_{\overline{\Bbbk}}, \zeta_{\overline{\Bbbk}}))$ be the geometric datum associated to $(O_{\overline{\Bbbk}}, W_{\overline{\Bbbk}})$ and $\pi\colon X_{\overline{\Bbbk}} \to X$ be the canonical morphism, where the notation from Subsection 1.3.7 is used. Then $r_{\overline{\Bbbk}}$ is a normalized formal r-matrix and π factors as $X_{\overline{\Bbbk}} \cong X \times \mathrm{Spec}(\overline{\Bbbk}) \to X$, where the second map is the canonical projection. In particular, $\pi^*\omega_X \cong \omega_{X_{\overline{\Bbbk}}}$ is the dualizing sheaf of $X_{\overline{\Bbbk}}$ (see e.g. [Con00, Theorem 3.6.1]) and the image of $c^*(\widehat{\eta}_p)$ under $\Bbbk[\![z]\!]\mathrm{d}z \to \overline{\Bbbk}[\![z]\!]\mathrm{d}z$ equals to $c_{\overline{\Bbbk}}^*\big(\widehat{\pi^*\eta}\big)_{p_{\overline{\Bbbk}}}$, for any $\eta \in \mathrm{H}^1(\omega_X)$. Therefore, $c^*(\widehat{\eta}_p) \in \Bbbk\mathrm{d}z$ if and only if $c_{\overline{\Bbbk}}^*\big(\widehat{\pi^*\eta}\big)_{p_{\overline{\Bbbk}}} \in \overline{\Bbbk}\mathrm{d}z$. As a consequence, we may assume that $\Bbbk = \overline{\Bbbk}$.

Let η be a non-zero global section of the dualizing sheaf ω_X of X and $c^*(\widehat{\eta}_p) = \mathrm{d}w(z) = w'(z)\mathrm{d}z$ for some $w(z) \in z\Bbbk[\![z]\!]$. The dualizing sheaf can be identified with the sheaf of Rosenlicht regular 1-forms. More precisely, η is a rational 1-form on the normalization X^ν of X, which is regular on $\nu^{-1}(C)$ (where C is the smooth locus of X) and satisfies

$$\sum_{q\in\nu^{-1}(s)} \mathrm{res}_q f\eta = 0 \text{ for all } s \in X \text{ singular and closed}, f \in \mathscr{O}_{X,s}; \tag{3.20}$$

see e.g. [Con00, Theorem 5.2.3]. The residue theorem on X^ν implies that $\mathrm{res}_p f\eta = 0$ for all $f \in \Gamma(X \setminus \{p\}, \mathscr{O}_X)$. Combining this with Remark 3.2.2 and using diagram (3.14) results in

$$\mathrm{res}_0 K(\zeta(a), \zeta(b))w'\mathrm{d}z = \mathrm{res}_p \mathcal{K}(a, b)\eta = 0 \tag{3.21}$$

for all $a, b \in \Gamma(X \setminus \{p\}, \mathscr{A})$. This implies that $w'\mathfrak{g}(r) \subseteq \mathfrak{g}(r)^\perp = \mathfrak{g}(r)$. In other words, we obtain $w' \in \mathrm{Mult}(\mathfrak{g}(r)) \cap \Bbbk[\![z]\!]^\times = \Bbbk^\times$. We can conclude the proof by replacing η with $(w')^{-1}\eta \in \mathrm{H}^0(\omega_X)$. □

3.3 Global aspects of formal generalized r-matrices

In the last section, we have examined the structure of geometric $\mathfrak{g}$-lattice models associated to formal generalized r-matrices. In this section, we deduce a scheme to transport properties of said models back to properties of formal generalized r-matrices by using the notion of geometric r-matrices (see Subsection 3.3.2) as moderator. The central observation is that formal generalized r-matrices can be recovered as Taylor expansions of geometric r-matrices; see Theorem 3.3.3. As an application, we will see that any formal generalized r-matrix is the Taylor expansion of an appropriate rational map; see Theorem 3.3.5.

3.3.1 Geometric CYBE and GCYBE models. Let r be a formal generalized r-matrix, $O \subseteq \operatorname{Mult}(\mathfrak{g}(r))$ be a unital subalgebra of finite codimension, and $((X, \mathscr{A}), (p, c, \zeta)) := \mathbb{G}(O, \mathfrak{g}(r))$ be the associated geometric $\mathfrak{g}$-lattice model; see Theorem 1.3.6. The sheaf $\mathscr{A}$ is étale $\mathfrak{g}$-locally free in p by Theorem 1.2.5. Since étale morphisms are open, we can chose a smooth open neighbourhood C of p such that $\mathscr{A}|_C$ is étale $\mathfrak{g}$-locally free. After properly shrinking C, there exists a non-vanishing 1-form η on C. We have obtained a geometric datum $((X, \mathscr{A}), (C, \eta))$, where

- X is an integral projective curve over $\Bbbk$.
- $C \subseteq X$ is a non-empty smooth open subset.
- η is a non-vanishing 1-form on C.
- $\mathscr{A}$ is a coherent sheaf of Lie algebras $\mathscr{A}$ on X such that $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$ and $\mathscr{A}|_C$ is étale $\mathfrak{g}$-locally free.

Let us call $((X, \mathscr{A}), (C, \omega))$ a *geometric GCYBE model* of r. If r is normalized and skew-symmetric and O is chosen according to Theorem 3.2.5, we can choose C to be the smooth locus of X (see Theorem 3.2.9) and $\eta = \eta_r$ (see Proposition 3.2.10); in this case $((X, \mathscr{A}), (C, \eta))$ is called the *geometric CYBE model* of r.

Remark 3.3.1.

(1) A geometric GCYBE model of a formal generalized r-matrix is not unique, while the geometric CYBE model of a normalized formal generalized r-matrix is.

(2) For $\Bbbk = \overline{\Bbbk}$ the geometric GCYBE model $((X, \mathscr{A}), (C, \eta))$ satisfies the axioms used in [BG18] to construct a geometric analog of a generalized r-matrix called *geometric r-matrix*. Indeed, in this case Theorem 1.2.3 implies that $\mathscr{A}|_C$ is étale $\mathfrak{g}$-locally free if and only if it is weakly $\mathfrak{g}$-locally free. Therefore, the above conditions can be seen as an appropriate generalization of the axioms used in [BG18] if one works over a non-algebraically closed ground field. In the next subsection, we recall the construction of the geometric r-matrix and observe that it works in our generalized setting.

(3) Assume r is normalized and skew-symmetric, $((X, \mathscr{A}), (C, \eta))$ is the geometric CYBE model of r and $\Bbbk = \overline{\Bbbk}$. Then we can see that $((X, \mathscr{A}), (C, \eta))$ satisfies the geometric axiomatization of skew-symmetry given in [BG18, Theorem 4.3]. Note that the third condition in said theorem can be seen as a consequence of the fact that the Killing form of $\mathscr{A}|_C$ extends to a pairing $\mathscr{A} \times \mathscr{A} \to \mathscr{O}_X$ by Proposition 3.2.8; see [Gal15, Theorem 1.2.(2)]. Hence, we again obtain

a generalization of said axiomatization which works over non-algebraically closed fields.

3.3.2 The geometric r-matrix. Let r be a formal generalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be any unital subalgebra of finite codimension, and $((X,\mathscr{A}),(C,\eta))$ be a geometric GCYBE model of r. Let Δ denote the image of the diagonal embedding $\delta\colon C \to X \times C$. The choice of non-vanishing $\eta \in \Gamma(C,\Omega^1_C)$ induces the so-called *diagonal residue sequence*

$$0 \longrightarrow \mathscr{O}_{X\times C} \longrightarrow \mathscr{O}_{X\times C}(\Delta) \xrightarrow{\mathrm{res}^\eta_\Delta} \delta_*\mathscr{O}_C \longrightarrow 0. \tag{3.22}$$

The map res^η_Δ is thereby determined as follows: for a closed point $q \in C$ with local parameter u, defined on an affine open subset U of C, the sheaves Ω^1_C and $\mathscr{O}_{X\times C}(-\Delta)$ are locally generated by $\mathrm{d}u$ and

$$u - v := u \otimes 1 - 1 \otimes u \in \Gamma(U,\mathscr{O}_X) \otimes \Gamma(U,\mathscr{O}_X) \cong \Gamma(U \times U, \mathscr{O}_{X\times X})$$

around q and (q,q) respectively; res^η_Δ maps $(u-v)^{-1}$ to μ, where μ is defined by $\eta_q = \mu\mathrm{d}u$. Tensoring (3.22) with $\mathscr{A} \boxtimes \mathscr{A}|_C := \mathrm{pr}_1^*\mathscr{A} \otimes_{\mathscr{O}_{X\times C}} \mathrm{pr}_2^*\mathscr{A}|_C$, where $X \xleftarrow{\mathrm{pr}_1} X \times C \xrightarrow{\mathrm{pr}_2} C$ are the canonical projections, gives rise to a short exact sequence

$$0 \longrightarrow \mathscr{A} \boxtimes \mathscr{A}|_C \longrightarrow \mathscr{A} \boxtimes \mathscr{A}|_C(\Delta) \longrightarrow \delta_*(\mathscr{A}|_C \otimes_{\mathscr{O}_C} \mathscr{A}|_C) \longrightarrow 0. \tag{3.23}$$

The Künneth formula implies that

$$\begin{aligned} &\mathrm{H}^0(\mathscr{A} \boxtimes \mathscr{A}|_C) = \mathrm{H}^0(\mathscr{A}) \otimes \mathrm{H}^0(\mathscr{A}|_C) = 0 \text{ and} \\ &\mathrm{H}^1(\mathscr{A} \boxtimes \mathscr{A}|_C) = \big(\mathrm{H}^1(\mathscr{A}) \otimes \mathrm{H}^0(\mathscr{A}|_C)\big) \oplus \big(\mathrm{H}^0(\mathscr{A}) \otimes \mathrm{H}^1(\mathscr{A}|_C)\big) = 0, \end{aligned} \tag{3.24}$$

where we used $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$. The long exact sequence in cohomology, induced by (3.23), yields an isomorphism $R\colon \mathrm{H}^0(\mathscr{A} \boxtimes \mathscr{A}|_C(\Delta)) \to \mathrm{H}^0(\mathscr{A}|_C \otimes \mathscr{A}|_C)$.

For every $q \in C$ exists an étale morphism $f\colon Y \to C$ such that $q \in f(Y)$ and $f^*\mathscr{A}|_C \cong \mathfrak{g} \otimes \mathscr{O}_Y$. Lemma 1.1.2 asserts that the inverse image $f^*\mathcal{K}$ of the Killing form $\mathcal{K}$ of $\mathscr{A}|_C$ can be identified with the Killing form of $\mathfrak{g} \otimes \mathscr{O}_Y$. The pairing $f^*\mathcal{K}$ is prefect, because of the simplicity of $\mathfrak{g}$. Thus, we see that $\mathcal{K}$ is perfect by varying q. This implies that the morphism

$$\widetilde{\mathcal{K}}\colon \mathscr{A}|_C \otimes_{\mathscr{O}_C} \mathscr{A}|_C \to \mathscr{E}nd_{\mathscr{O}_C}(\mathscr{A}|_C), \tag{3.25}$$

defined by $a \otimes b \mapsto \mathcal{K}_U(b,-)a$ for all affine open $U \subseteq C$ and $a, b \in \Gamma(U,\mathscr{A})$, is an isomorphism. Summarized, we obtain an isomorphism

$$\Phi := \widetilde{\mathcal{K}}R\colon \mathrm{H}^0(\mathscr{A} \boxtimes \mathscr{A}|_C(\Delta)) \longrightarrow \mathrm{End}_{\mathscr{O}_C}(\mathscr{A}|_C). \tag{3.26}$$

The section $\rho := \Phi^{-1}(\mathrm{id}_{\mathscr{A}|_C}) \in \mathrm{H}^0(\mathscr{A} \boxtimes \mathscr{A}|_C(\Delta))$ is called *geometric r-matrix* of the geometric GCYBE model $((X,\mathscr{A}),(C,\eta))$ of r.

3.3.3 Equivalence of geometric r-matrices. Let $r_1, r_2 \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be formal generalized r-matrices. which are equivalent via some equivalence (μ, w, φ), let $O_1 \subseteq \operatorname{Mult}(\mathfrak{g}(r_1))$ be a unital subalgebra of finite codimension and let O_2 be the image of O_1 under the automorphism of $\Bbbk(\!(z)\!)$ defined by $\lambda(z) \mapsto \lambda(w(z))$. Moreover, let

$$(f, \varphi)\colon ((X_1, \mathscr{A}_1), (p_1, c_1, \zeta_1)) = \mathbb{G}(O_1, \mathfrak{g}(r_1)) \longrightarrow \mathbb{G}(O_2, \mathfrak{g}(r_2)) = ((X_2, \mathscr{A}_2), (p_2, c_2, \zeta_2))$$

be the isomorphism of geometric $\mathfrak{g}$-lattice models provided by Lemma 3.1.1. In particular, $f\colon X_2 \to X_1$ and $\varphi\colon \mathscr{A}_1 \to f_*\mathscr{A}_2$ are isomorphisms. Chose geometric GCYBE models $((X_i, \mathscr{A}_i), (C_i, \eta_i))$ of r_i for $i \in \{1, 2\}$ in such a way that $f^{-1}(C_2) = C_1$ and $f^*\eta_2 = \eta_1$. Let $\rho_i \in \mathrm{H}^0(\mathscr{A}_i \boxtimes \mathscr{A}_i|_{C_i}(\Delta_i))$ be the geometric r-matrix of $((X_i, \mathscr{A}_i), (C_i, \eta_i))$, where $i \in \{1, 2\}$ and $\Delta_i \subseteq X_i \times C_i$ is the image of the diagonal embedding $\delta_i\colon C_i \to X_i \times C_i$.

Lemma 3.3.2.
The identity $(f^*\varphi \boxtimes f^*\varphi)(f \times f)^*\rho_1 = \rho_2$ holds, where $f^*f_*\mathscr{A}_2 \cong \mathscr{A}_2$ was used implicitly.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\mathscr{A}_1 \boxtimes f^*\mathscr{A}_1|_C & \longrightarrow & f^*\mathscr{A}_1 \boxtimes f^*\mathscr{A}_1|_C(\Delta_2) & \longrightarrow & \delta_{2,*}\mathscr{E}nd_{\mathscr{O}_{C_2}}(f^*\mathscr{A}_1|_{C_1}) & \longrightarrow & 0\,, \\ & & \downarrow{\scriptstyle f^*(\phi)\boxtimes f^*(\phi)} & & \downarrow{\scriptstyle f^*(\phi)\boxtimes f^*(\phi)} & & \downarrow{\scriptstyle f^*(\phi)-f^*(\phi)^{-1}} & & \\ 0 & \longrightarrow & \mathscr{A}_2 \boxtimes \mathscr{A}_2|_C & \longrightarrow & \mathscr{A}_2 \boxtimes \mathscr{A}_2|_C(\Delta_2) & \longrightarrow & \delta_{2,*}\mathscr{E}nd_{\mathscr{O}_{C_2}}(\mathscr{A}_2|_{C_2}) & \longrightarrow & 0 \end{array} \tag{3.27}$$

where the upper row is $(f \times f)^*$ of (3.23) for $\mathscr{A} = \mathscr{A}_1$, the lower row is (3.23) for $\mathscr{A} = \mathscr{A}_2$, and the isomorphisms $\mathscr{A}_i|_{C_i} \otimes \mathscr{A}_i|_{C_i} \to \mathscr{E}nd_{\mathscr{O}_{C_i}}(\mathscr{A}_i|_{C_i})$ for $i \in \{1, 2\}$, constructed analog to (3.25), were used. Here, the commutativity of the right square follows form the following fact: for any free Lie algebra $\mathfrak{l}$ of finite rank over a ring R with Killing form $K_\mathfrak{l}$, the adjoint of $\psi \in \operatorname{Aut}_{R\text{-alg}}(\mathfrak{l})$ with respect to $K_\mathfrak{l}$ coincides with ψ^{-1}, so

$$\widetilde{K}_\mathfrak{l}(\psi(a) \otimes \psi(b)) = \psi \widetilde{K}_\mathfrak{l}(a \otimes b)\psi^{-1} \tag{3.28}$$

holds for $\widetilde{K}_\mathfrak{l}(a \otimes b) := K_\mathfrak{l}(b, \cdot)a \in \operatorname{End}_R(\mathfrak{l})$, where $a, b \in \mathfrak{l}$.

It is straight forward to show that the inverse image along $f \times f$ of the diagonal residue sequence (3.22) for $X = X_1$ and $\eta = \eta_1$ coincides with the diagonal residue sequence for $X = X_2$ and $\eta = f^*\eta_1 = \eta_2$. This implies that $(f \times f)^*\rho_1$ is mapped to $\mathrm{id}_{f^*\mathscr{A}_1|_{C_1}}$ under $\mathrm{H}^0(f^*\mathscr{A}_1 \boxtimes f^*\mathscr{A}_1|_{C_1}(\Delta_2)) \to \operatorname{End}_{\mathscr{O}_{C_2}}(f^*\mathscr{A}_1|_{C_1})$. Thus, the commutativity of (3.27) implies that

$$(f^*(\phi) \boxtimes f^*(\phi))(f \times f)^*\rho_1 \longmapsto \mathrm{id}_{\mathscr{A}_2|_{C_2}} \tag{3.29}$$

under $\mathrm{H}^0(\mathscr{A}_2 \boxtimes \mathscr{A}_2|_{C_2}(\Delta_2)) \to \operatorname{End}_{\mathscr{O}_{C_2}}(\mathscr{A}_2|_{C_2})$, so $\rho_2 = (f^*(\phi) \boxtimes f^*(\phi))(f \times f)^*\rho_1$ holds. □

3.3.4 Interlude: geometric Taylor series. Let S be a scheme and $T \subseteq S$ be a closed subscheme, defined by a sheaf of ideals $\mathscr{I} \subseteq \mathscr{O}_S$. The *formal completion* of S along T is the locally ringed space $\mathfrak{S}$ whose topological space coincides with the topological space of T and $\mathscr{O}_{\mathfrak{S}} = \varprojlim(\mathscr{O}_S/\mathscr{I}^n)|_T$. Here, we recall that for any inverse system $\{\mathscr{F}\}_{n\in\mathbb{N}}$ of sheaves of abelian groups on S, we have $\Gamma(U, \varprojlim \mathscr{F}_n) = \varprojlim \Gamma(U, \mathscr{F}_n)$ for all $U \subseteq S$ open. There is a canonical morphism $\jmath\colon \mathfrak{S} \to S$, which is simply the canonical inclusion of T into S on the level of topological spaces and $\jmath^\flat\colon \mathscr{O}_S \to \jmath_*\mathscr{O}_{\mathfrak{S}}$ is the canonical morphism induced by the completion on the level of local sections. If S is integral and noetherian, the canonical morphism $\jmath^*\colon \mathscr{F} \to \jmath_*\jmath^*\mathscr{F}$ is injective for any coherent sheaf $\mathscr{F}$ on S. For any open subset $U \subseteq T$ and $a \in \Gamma(U, \mathscr{F})$, we call the section $\jmath^* a \in \Gamma(\jmath^{-1}(U), \jmath^*\mathscr{F})$ *Taylor expansion of a in T.*

In the following, we will use a concrete special case of this construction. Assume that $S = T \times T$ for some one dimensional finite type $\Bbbk$-scheme T and let $t \in T$ be a smooth closed point. We can fix an isomorphism $c_t\colon \widehat{\mathscr{O}}_{T,t} \to \kappa(t)[\![z]\!]$ by virtue of the Cohen structure theorem; see e.g. [DG64, p. 19.6.4]. Let $\mathfrak{S}$ be the completion of S along $T \times \{t\}$, where the underlying topological space is identified with T and the underlying sheaf of rings is identified with the sheaf of rings $U \mapsto c_t(\Gamma(U, \mathscr{O}_T))[\![y]\!] \subseteq \kappa(t)(\!(x)\!)[\![y]\!]$ on T. Here, recall that we also use c_t to denote the isomorphism $\mathrm{Q}(\widehat{\mathscr{O}}_{T,t}) \to \kappa(t)(\!(z)\!)$ induced by c_t. Let $\mathscr{B}$ be a sheaf of algebras on T which is formally A-locally free at t for some finite-dimensional $\Bbbk$-algebra A. We can identify $\jmath^*(\mathscr{B} \boxtimes \mathscr{B})$ with the coherent sheaf

$$U \longmapsto (\zeta_t(\Gamma(U, \mathscr{B})) \otimes_{\kappa(t)} A_{\kappa(t)})[\![y]\!] \subseteq (A_{\kappa(t)} \otimes_{\kappa(t)} A_{\kappa(t)})(\!(x)\!)[\![y]\!] \tag{3.30}$$

on $\mathfrak{S}$, where $A_{\kappa(t)} := A \otimes \kappa(t)$ and $\zeta_t\colon \widehat{\mathscr{B}}_t \to A \otimes \widehat{\mathscr{O}}_{X,t} \cong A_{\kappa(t)}[\![z]\!]$ is some fixed isomorphism. In particular, for every $V \subseteq T \times T$ open with non-empty intersection with $T \times \{p\}$, the Taylor expansion in $T \times \{t\}$ takes the form

$$\Gamma(V, \mathscr{B} \boxtimes \mathscr{B}) \xrightarrow{\jmath^*} (\zeta_t(\Gamma(\jmath^{-1}(V), \mathscr{B})) \otimes_{\kappa(t)} A_{\kappa(t)})[\![y]\!] \subseteq (A_{\kappa(t)} \otimes_{\kappa(t)} A_{\kappa(t)})(\!(x)\!)[\![y]\!]. \tag{3.31}$$

3.3.5 Taylor series of the geometric r-matrix. The following statement can be seen as a generalization of [BG18, Theorem 6.4].

Theorem 3.3.3.
Let r be a formal generalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be any unital subalgebra of finite codimension, and $((X, \mathscr{A}), (p, c, \zeta)) := \mathbb{G}(O, \mathfrak{g}(r))$. Furthermore, choose a geometric GCYBE model $((X, \mathscr{A}), (C, \eta))$ of r and let

$$\rho \in \mathrm{H}^0(\mathscr{A} \boxtimes \mathscr{A}|_C(\Delta)) \subseteq \Gamma(X \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A}) \tag{3.32}$$

be the geometric r-matrix of $((X, \mathscr{A}), (C, \eta))$.

(1) *There exists a $\lambda \in \Bbbk[\![z]\!]^\times$ such that $\jmath^*(\rho) = \lambda(y) r(x, y)$, where*

$$\jmath^*\colon \Gamma(X \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A}) \longrightarrow (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!] \subseteq (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \tag{3.33}$$

is the Taylor expansion in $X \times \{p\}$ given by (3.31) for the datum $T = X$, $\mathscr{B} = \mathscr{A}$, $t = p$, $A = \mathfrak{g}$, $\zeta_t = \zeta$, $c_t = c$ and $V = X \times C \setminus \Delta$.

(2) *If $((X, \mathscr{A}), (C, \eta))$ is the geometric CYBE model of r, $\jmath^*(\rho) = r$.*

3.3.6 Proof of Theorem 3.3.3 Let $u \in \Gamma(V, \mathscr{O}_X)$ be any local parameter of p, where $V \subseteq C$ is an affine open neighbourhood of p. Chose U to be some affine open subset of the intersection of

- the projection of the open set $\{(q,q') \in V \times V \mid (u-v)(q,q') \neq 0\}$ to the first component with
- an open neighbourhood of p where $\mathrm{d}u$ generates Ω^1_C.

It is straight forward to show that $\mathrm{d}u$ generates $\Gamma(U, \Omega^1_C)$ and $u - v$ generates the space $\Gamma(U \times U, \mathscr{O}_{X \times C}(-\Delta))$. By definition $\mathrm{res}^\eta_\Delta\left(\frac{1\otimes\mu}{u-v}\right) = 1$ if $\eta|_U = \mu^{-1}\mathrm{d}u$. Let χ be a preimage of $\mathrm{id}_{\mathscr{A}|_U}$ under

$$\Gamma(U \times U, \mathscr{A} \boxtimes \mathscr{A}) \longrightarrow \Gamma(U, \mathscr{A} \otimes \mathscr{A}) \cong \mathrm{End}_{\mathscr{O}_U}(\mathscr{A}|_U). \tag{3.34}$$

Then both $\rho|_{U\times U}$ and $\frac{1\otimes\mu}{u-v}\chi$ map to $\mathrm{id}_{\mathscr{A}_U}$ under the canonical map

$$\Gamma(U \times U, \mathscr{A} \boxtimes \mathscr{A}|_C(\Delta)) \longrightarrow \mathrm{End}_{\mathscr{O}_U}(\mathscr{A}|_U). \tag{3.35}$$

Since (3.23) and $\widetilde{\mathcal{K}}$ induce a short exact sequence

$$0 \longrightarrow \mathscr{A} \boxtimes \mathscr{A}|_C \longrightarrow \mathscr{A} \boxtimes \mathscr{A}|_C(\Delta) \longrightarrow \delta_* \mathrm{End}_{\mathscr{O}_C}(\mathscr{A}|_C) \longrightarrow 0, \tag{3.36}$$

and U is affine, we obtain $\rho_0 := \rho|_{U\times U} - \frac{1\otimes\mu}{u-v}\chi \in \Gamma(U \times U, \mathscr{A} \boxtimes \mathscr{A})$, i.e.

$$\rho|_{U\times U} = \frac{1\otimes\mu}{u-v}\chi + \rho_0. \tag{3.37}$$

Write $\widetilde{\mu} := c(\mu), \widetilde{u} := c(u) \in \Bbbk[\![z]\!]$. Then

$$\jmath^\flat(1\otimes\mu) = \widetilde{\mu}(y), \jmath^\flat(u-v) = \widetilde{u}(x) - \widetilde{u}(y) \in \Bbbk[\![x,y]\!]. \tag{3.38}$$

Similarly, let $\widetilde{\rho}_0 := \jmath^*(\rho_0), \widetilde{\chi} := \jmath^*(\chi) \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$. The diagram

$$\begin{array}{ccccc}
\Gamma(X \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A}) & \xrightarrow{\jmath^*} & (\mathfrak{g}(r)\otimes\mathfrak{g})[\![y]\!] & \xrightarrow{\subseteq} & (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] \\
\downarrow & & & \nearrow{\scriptstyle\subseteq} & \\
\Gamma(U \times U \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A}) & \xrightarrow[\jmath^*]{} & (\zeta(\Gamma(U\setminus\{p\}, \mathscr{A}))\otimes\mathfrak{g})[\![y]\!] & &
\end{array} \tag{3.39}$$

commutes. Therefore, (3.37) implies that the image of ρ under (3.33) is of the form

$$\widetilde{r}(x,y) = \frac{\widetilde{\mu}(y)}{\widetilde{u}(x) - \widetilde{u}(y)}\widetilde{\chi}(x,y) + \widetilde{\rho}_0(x,y). \tag{3.40}$$

Lemma 2.1.2.(4) can be used to see that

$$(\widetilde{u}(x) - \widetilde{u}(y))^{-1} - (\widetilde{u}'(y)(x-y))^{-1} \in \Bbbk[\![x,y]\!] \tag{3.41}$$

and $\widetilde{\chi}(x,y) - \gamma \in (x-y)(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$. Summarized, for $\lambda_1 := \widetilde{\mu}/\widetilde{u}'$, we obtain

$$\widetilde{r}(x,y) = \frac{\lambda_1(y)\gamma}{x-y} + \widetilde{r}_0(x,y) \tag{3.42}$$

for some $\widetilde{r}_0 \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$.

Let $\lambda_2 \in \Bbbk[\![z]\!]$ satisfy $r(x,y) - \lambda_2(y)\gamma/(x-y) \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and put $\lambda = \lambda_1/\lambda_2$. The image $\widetilde{r}$ of ρ under (3.33) is by definition in $(\mathfrak{g}(r) \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$. But so is $\lambda(y)r(x,y) \in \lambda_1(y)r_{\mathrm{Yang}}(x,y) + (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and

$$\widetilde{r}(x,y) - \lambda(y)r(x,y) \in (\mathfrak{g}(r)\otimes\mathfrak{g})[\![y]\!] \cap (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!] = \{0\}$$

concludes the proof of Theorem 3.3.3.(1).

For (2), observe that if r is skew-symmetric and normalized (i.e. $\lambda_2 = 1$), and $((X,\mathscr{A}),(C,\eta))$ is the geometric CYBE datum of r, $\lambda_1 = 1$ by virtue of Proposition 3.2.10. Indeed, the above construction of λ_1 implies that $\lambda_1^{-1}\mathrm{d}z = c^*\eta$ holds. Hence, $\lambda = 1$ in this case.

3.3.7 The geometric GCYBE. Let r be a formal generalized r-matrix, $((X,\mathscr{A}),(C,\eta))$ be a geometric GCYBE model of r, and ρ be the geometric r-matrix of $((X,\mathscr{A}),(C,\eta))$. Assume that C is affine and let $\mathscr{U}$ be the quasi-coherent sheaf on C associated to the universal enveloping sheaf of $\mathrm{H}^0(\mathscr{A}|_C)$ as $\mathrm{H}^0(\mathscr{O}_C)$-Lie algebra and $\iota\colon \mathrm{H}^0(\mathscr{A}|_C) \to \mathrm{H}^0(\mathscr{U})$ be the canonical map. It can be shown that ι is injective; see [Gal15, Lemma 1.6]. For $ij \in \{12,13,23\}$, let $\pi_{ij}\colon C\times C\times C \to C\times C$ denote the natural projections, defined through $(x_1,x_2,x_3) \mapsto (x_i,x_j)$. Moreover, note that there are natural maps

$$(\cdot)^{ij}\colon \mathscr{A}|_C \boxtimes \mathscr{A}|_C \to \pi_{ij,*}(\mathscr{U}\boxtimes\mathscr{U}\boxtimes\mathscr{U}), \tag{3.43}$$

defined, under consideration of the Künneth formulas

$$\mathrm{H}^0(\mathscr{A}|_C \boxtimes \mathscr{A}|_C) \cong \mathrm{H}^0(\mathscr{A}|_C)\otimes \mathrm{H}^0(\mathscr{A}|_C)$$
$$\mathrm{H}^0(\mathscr{U}\boxtimes\mathscr{U}\boxtimes\mathscr{U}) \cong \mathrm{H}^0(\mathscr{U})\otimes\mathrm{H}^0(\mathscr{U})\otimes\mathrm{H}^0(\mathscr{U}),$$

by $t^{12} = \iota(a)\otimes\iota(b)\otimes 1, t^{13} = \iota(a)\otimes 1\otimes\iota(b)$ and $t^{23} = 1\otimes\iota(a)\otimes\iota(b)$ for $t = a\otimes b \in \mathrm{H}^0(\mathscr{A}|_C)\otimes\mathrm{H}^0(\mathscr{A}|_C)$. Furthermore, if $\sigma\colon C\times C \to C\times C$ denotes the map $(x,y)\mapsto(y,x)$, let $\overline{(\cdot)}\colon \mathscr{A}|_C\boxtimes\mathscr{A}|_C \to \sigma_*(\mathscr{A}|_C\boxtimes\mathscr{A}|_C)$ be the morphism defined on global sections by the σ-equivariant automorphism $a\otimes b\mapsto -b\otimes a$. The following result is a version of [BG18, Theorem 3.11 & Theorem 4.3] in our setting.

Theorem 3.3.4.
In the notation of this subsection, the geometric r-matrix ρ, treated as an element of $\Gamma(C\times C\setminus\Delta,\mathscr{A}\boxtimes\mathscr{A})$, solves the geometric GCYBE

$$[\rho^{12},\rho^{13}] + [\rho^{12},\rho^{23}] + [\rho^{13},\overline{\rho}^{23}] = 0, \tag{3.44}$$

where the left-hand side is defined in $\Gamma(C\times C\times C\setminus\Sigma,\mathscr{U}\boxtimes\mathscr{U}\boxtimes\mathscr{U})$ for $\Sigma = \{(x_1,x_2,x_3)\in C\times C\times C \mid x_i\neq x_j, i\neq j\}$. Furthermore, if r is normalized and skew-symmetric and $((X,\mathscr{A}),(C,\eta))$ is its geometric CYBE model, $\overline{\rho} = \rho$.

Proof. Similar calculations as in Remark 2.1.5 show that the left-hand side of the geometric GCYBE (3.44) is actually contained in $\Gamma(C\times C\times C\setminus\Sigma,\mathscr{A}\boxtimes\mathscr{A}\boxtimes\mathscr{A})$.

Let $\mathfrak{X}_2$ be the formal completion of $X\times X\times X$ along $X\times X\times\{p\}$ (here p is the point at infinity of X) and write $\jmath_2\colon \mathfrak{X}_2 \to X\times X\times X$ for the canonical morphism.

We obtain an injective morphism

$$\jmath_2^*\colon \Gamma(C\times C\times C\setminus\Sigma,\mathscr{A}\boxtimes\mathscr{A}\boxtimes\mathscr{A})\longrightarrow(\Gamma((C\setminus\{p\})\times(C\setminus\{p\})\setminus\Delta,\mathscr{A}\boxtimes\mathscr{A})\otimes\mathfrak{g})[\![x_3]\!],$$

where ζ was used. Since $\jmath^*\colon\Gamma(C\times C\setminus\Delta,\mathscr{A}\boxtimes\mathscr{A})\to(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ from (3.33) extends to an injective morphism

$$\Gamma((C\setminus\{p\})\times(C\setminus\{p\})\setminus\Delta,\mathscr{A}\boxtimes\mathscr{A})\longrightarrow(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)(\!(y)\!),\tag{3.45}$$

we obtain an injective morphism

$$\Gamma(C\times C\times C\setminus\Sigma,\mathscr{A}\boxtimes\mathscr{A}\boxtimes\mathscr{A})\longrightarrow(\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})(\!(x_1)\!)(\!(x_2)\!)[\![x_3]\!].\tag{3.46}$$

Using Theorem 3.3.3, it can be shown that the map (3.46) sends the left-hand side of (3.44) to $\mathrm{GCYB}(\widetilde{r})$, where $\widetilde{r}(x,y)=\lambda(y)r(x,y)$. This is 0, since r is a formal generalized r-matrix. The injectivity of (3.46) implies that ρ solves (3.44).

If r is normalized and skew-symmetric and $((X,\mathscr{A}),(C,\eta))$ is its geometric CYBE model, ρ is mapped to r via (3.33). This implies that $\overline{\rho}$ is mapped to $\overline{r}$, so $r=\overline{r}$ implies $\rho=\overline{\rho}$, since $\jmath^*$ is injective. □

3.3.8 Rational extension of a formal generalized r-matrix. Since the sheaf of Lie algebras associated to a formal generalized r-matrix is étale trivial around the point at infinity, we can trivialize the associated geometric r-matrix to obtain a proper rational map. Theorem 3.3.3 then implies that the Taylor expansion of said map is equivalent to r, up to passing to a finite field extension. More precisely, we have the following result.

Theorem 3.3.5.
Let $r\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix and $X,\mathscr{A},p$, and ρ be the same datum assigned to r as in Theorem 3.3.3. Furthermore, let $f\colon Y\to X$ be an étale morphism such that $p\in f(Y)$ and an isomorphism $\psi\colon f^*\mathscr{A}\to\mathfrak{g}\otimes\mathscr{O}_Y$ of sheaves of Lie algebras exists. Fix an isomorphism $c_q\colon\widehat{\mathscr{O}}_{Y,p}\to\kappa(q)[\![z]\!]$ for some $q\in f^{-1}(p)$ and let

$$\jmath_q^*\colon(\mathfrak{g}\otimes\mathfrak{g})\otimes\Gamma(Y\times Y\setminus\Delta_f,\mathscr{O}_{Y\times Y})\longrightarrow(\mathfrak{g}_{\kappa(q)}\otimes_{\kappa(q)}\mathfrak{g}_{\kappa(q)})(\!(x)\!)[\![y]\!]\tag{3.47}$$

be the Taylor expansion (3.31) for the datum $T=Y$, $\mathscr{B}=\mathfrak{g}\otimes\mathscr{O}_Y$, $t=q$, $A=\mathfrak{g}$, $\zeta_t=\mathrm{id}_\mathfrak{g}\otimes c_q$, $c_t=c_q$ and $V=Y\times Y\setminus\Delta$, where $\Delta_f:=(f\times f)^{-1}\Delta$. Then

$$\varrho:=(\psi\times\psi)(f\times f)^*\rho\in(\mathfrak{g}\otimes\mathfrak{g})\otimes\Gamma(Y\times Y\setminus\Delta_f,\mathscr{O}_{Y\times Y})\tag{3.48}$$

satisfies: $\jmath_q^*\varrho$ is equivalent to the extension $r_{\kappa(q)}$ of r via $\Bbbk\to\kappa(q)$ (see Remark 2.1.5.(3)).

Proof. We proceed using the notation from Theorem 3.3.3 and Remark 2.1.5.(3). We see that the image of ρ under

$$\Gamma(X\times C\setminus\Delta,\mathscr{A}\boxtimes\mathscr{A})\xrightarrow{\jmath^*}(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]\longrightarrow(\mathfrak{g}_{\kappa(q)}\otimes_{\kappa(q)}\mathfrak{g}_{\kappa(q)})(\!(x)\!)[\![y]\!]$$

is equivalent to $r_{\kappa(q)}$ by a rescaling $\lambda \in \Bbbk[\![z]\!]^\times \subseteq \kappa(q)[\![z]\!]^\times$. The image of z under

$$\Bbbk[\![z]\!] \xrightarrow{c^{-1}} \widehat{\mathscr{O}}_{X,p} \xrightarrow{\widehat{f}_p^\sharp} \widehat{\mathscr{O}}_{Y,q} \xrightarrow{c_q} \kappa(q)[\![z]\!]$$

defines a coordinate transform $w \in z\kappa(q)[\![z]\!]^\times$. Similarly, the chain of maps

$$\mathfrak{g}[\![z]\!] \xrightarrow{\zeta^{-1}} \widehat{\mathscr{A}_p} \longrightarrow \widehat{f^*\mathscr{A}}_q \xrightarrow{\widehat{\psi}_q} \mathfrak{g} \otimes \widehat{\mathscr{O}}_{Y,q} \xrightarrow{\mathrm{id}_\mathfrak{g} \otimes c_q} \mathfrak{g}_{\kappa(q)}[\![z]\!] \tag{3.49}$$

induces a gauge transformation $\varphi \in \mathrm{Aut}_{\kappa(q)\text{-alg}}(\mathfrak{g}_{\kappa(q)}[\![z]\!])$. Here, the middle arrow of (3.49) is the composition of $\widehat{f}_p^*$ with the completion of the canonical map $(f_*f^*\mathscr{A})_p \to f^*\mathscr{A}_q$. It is straight forward to show from the construction of $\jmath$ and $\jmath_q$ that the diagram

$$\begin{array}{ccccc} \Gamma(X \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A}) & \xrightarrow{\jmath^*} & (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] & \longrightarrow & (\mathfrak{g}_{\kappa(q)} \otimes_{\kappa(q)} \mathfrak{g}_{\kappa(q)})(\!(x)\!)[\![y]\!] \\ \downarrow{\scriptstyle (f\times f)^*} & & & \nearrow{\scriptstyle \jmath_q^*} & \\ \Gamma(Y \times Y, f^*\mathscr{A} \boxtimes f^*\mathscr{A}) & \xrightarrow[\psi\boxtimes\psi]{} & (\mathfrak{g} \otimes \mathfrak{g}) \otimes \Gamma(Y \times Y \setminus \Delta_f, \mathscr{O}_{Y\times Y}) & & \end{array}$$

commutes. Here, the arrow in the upper right is defined by

$$s(x,y) \longmapsto (\varphi(x) \otimes \varphi(y))s(w(x), w(y)) \tag{3.50}$$

for all $s \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$. In particular, this implies that the series $\jmath_q^*\varrho$ is equivalent to $r_{\kappa(q)}$ via the equivalence (λ, w, φ). □

The section ϱ from Theorem 3.3.5 is the solution of yet another version of the GCYBE. For $ij \in \{12, 13, 23\}$, let $\pi_{ij}\colon Y \times Y \times Y \to Y \times Y$ be the canonical projections $(y_1, y_2, y_3) \to (y_i, y_j)$ and ϱ^{ij} be the image of ϱ under

$$(\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathscr{O}_{Y\times Y} \xrightarrow{\pi_{ij}^*} (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathscr{O}_{Y\times Y\times Y} \xrightarrow{(\cdot)^{ij} \otimes \mathrm{id}_{\mathscr{O}_{Y\times Y\times Y}}} (\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})) \otimes \pi_{ij,*}\mathscr{O}_{Y\times Y\times Y}\,,$$

where $(\cdot)^{ij}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$ has the same meaning as in the definition of the formal GCYBE (2.14). Furthermore, let $\overline{\varrho} = -\tau\sigma^*\varrho$, where $\sigma\colon Y \times Y \to Y \times Y$ is given by $\sigma(x,y) = (y,x)$ and $\tau\colon (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathscr{O}_{Y\times Y} \to (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathscr{O}_{Y\times Y}$ is the $\mathscr{O}_{Y\times Y}$-linear extension of $a \otimes b \mapsto b \otimes a$ for $a, b \in \mathfrak{g}$. Then we have

$$[\varrho^{12}, \varrho^{13}] + [\varrho^{12}, \varrho^{23}] + [\varrho^{13}, \overline{\varrho}^{23}] = 0. \tag{3.51}$$

Moreover, if r is normalized and skew-symmetric and $((X, \mathscr{A}), (C, \eta))$ is its geometric CYBE datum, the identity $\varrho = \overline{\varrho}$ holds. Indeed, these statements follow by applying

$$(\psi \boxtimes \psi \boxtimes \psi)(f \times f \times f)^* \tag{3.52}$$

to (3.44), using $\overline{\varrho} = (\psi \times \psi)(f \times f)^*\overline{\rho}$, and Theorem 3.3.4.

Since any smooth projective curve over the complex numbers is a compact Riemann surface, a consequence of Theorem 3.3.5 in the case of $\Bbbk = \mathbb{C}$ can be formulated as follows.

Corollary 3.3.6.
Let $\Bbbk = \mathbb{C}$, X_g be the compact Riemann surface of genus $g \in \mathbb{N}_0$, and $\mathbb{K}_g$ be the field of rational functions on $X_g \times X_g$. Then every formal generalized r-matrix is equivalent to the Taylor series of an element of $(\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathbb{K}_g$ for an appropriate $g \in \mathbb{N}_0$.

3.4 Analytic generalized r-matrices

In this section, $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$ is equipped with the standard euclidean topology and $\mathrm{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Recall that $\mathfrak{g}$ is central and simple over $\Bbbk$.

3.4.1 The (generalized) classical Yang-Baxter equation. For two real vector spaces W, V of finite dimension and any open subset $U \subseteq W$, we say that $U \to V$ is a *(real) meromorphic map* if it is a map defined on a dense open subset of U with values in V that coincides with the restriction of some meromorphic map $\widetilde{U} \to V \otimes_{\mathbb{R}} \mathbb{C}$, where $\widetilde{U}$ is an open neighbourhood of U in $W \otimes_{\mathbb{R}} \mathbb{C}$.

Recall that for $ij \in \{12, 13, 23\}$ the linear maps $(\cdot)^{ij} \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$ are defined by (2.13) and let us write $(\cdot)^{ji} := (\cdot)^{ij}\tau$, where $\tau \in \mathrm{End}(\mathfrak{g} \otimes \mathfrak{g})$ is defined by $\tau(a \otimes b) = b \otimes a$ for all $a, b \in \mathfrak{g}$. Let $U \subset \Bbbk$ be a connected open subset and $r \colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ be a meromorphic map. Then r is called a *generalized r-matrix* if it solves the *generalized classical Yang-Baxter equation* (GCYBE)

$$[r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{32}(x_3, x_2), r^{13}(x_1, x_3)] = 0, \tag{3.53}$$

and it is called *r-matrix* if it solves the *classical Yang-Baxter equation* (CYBE)

$$[r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), r^{23}(x_1, x_3)] = 0, \tag{3.54}$$

for all $x_1, x_2, x_3 \in \Bbbk$ for which these equations are defined respectively. Here, for example $r^{12}(x_1, x_2) = r(x_1, x_2)^{12}$ and the brackets on the left-hand side of both equations are understood as the usual commutators in the associative $\Bbbk$-algebra $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$. If we want to emphasize that $\Bbbk = \mathbb{R}$ (resp. $\Bbbk = \mathbb{C}$) we call r *real* (resp. *complex*) and if we want to distinguish this notion of (generalized) r-matrix from its formal counterpart, we will refer to it as *analytic*. Observe that calculations similar to Remark 2.1.5 show that the left-hand side of (3.53) and (3.54) are meromorphic functions $U \times U \times U \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

3.4.2 Non-degeneracy and skew-symmetry Let $r \colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ be a meromorphic map for some connected open $U \subseteq \mathbb{C}$, K be the Killing form of $\mathfrak{g}$, and $\widetilde{K} \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the isomorphism defined by $a \otimes b \mapsto K(b, -)a$. We call r

- *non-degenerate* if for some (x, y) in the domain of definition of r, $\widetilde{K}(r(x, y))$ is an isomorphism and
- *skew-symmetric* if $r(x, y) = -\tau(r(y, x))$ for all (x, y) in the domain of definition of r.

Non-degeneracy is a generic property: if r is non-degenerate, $\widetilde{K}(r(x,y))$ is an isomorphism for all (x,y) in a dense open subset of the domain of definition of r.

Let $\gamma \in \mathfrak{g} \otimes \mathfrak{g}$ be the Casimir element of $\mathfrak{g}$, i.e. $\widetilde{K}(\gamma) = \mathrm{id}_{\mathfrak{g}}$. There is a particular class of non-degenerate meromorphic functions which will be important in the following. We say that r is in *standard form*, if $0 \in U$ and

$$r(x,y) = \frac{\lambda(y)}{x-y}\gamma + r_0(x,y) \tag{3.55}$$

for some analytic maps $\lambda\colon U \to \Bbbk^\times$ and $r_0\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$. Indeed,

$$(x-y)\widetilde{K}(r(x,y))|_{x=y} = \lambda(y)\,\mathrm{id}_{\mathfrak{g}} \tag{3.56}$$

implies that $\det(\widetilde{K}(r(x,y)))$ is non-vanishing in an open neighbourhood of the diagonal $\{(x,y) \in U \times U \mid x = y\}$. The importance of the standard form is provided by the following result, which is a generalization of an observation from [BD83b]. We point out that the proof in the real case relies on the centrality of $\mathfrak{g}$.

Proposition 3.4.1.
Every non-degenerate generalized r-matrix $r\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ is in standard form (3.55), after properly shrinking U and relocating the origin.

Proof. It suffices to prove that r has the form (3.55) locally around some $(p,p) \in U \times U$, since we can then relocate the origin to put $p = 0$. We split the proof into four steps.

Step 1. *Translating the CYBE and GCYBE into operator language.* We will apply the coordinate-free methods presented in [KK11]. Consider the isomorphisms $\widetilde{K}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ and $\widetilde{K}^{(3)}\colon \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ defined by

$$\widetilde{K}(a_1 \otimes a_2)(b_1) = K(a_2,b_1)a_1 \text{ and } \widetilde{K}^{(3)}(a_1 \otimes a_2 \otimes a_3)(b_1 \otimes b_2) = K(b_2,a_3)K(b_1,a_2)a_1$$

for all $a_1,a_2,a_3,b_1,b_2 \in \mathfrak{g}$. Assume first that r is a generalized r-matrix. Then, applying $\widetilde{K}^{(3)}$ to the GCYBE (3.53) and evaluating in $t_1 \otimes t_2 \in \mathfrak{g} \otimes \mathfrak{g}$ yields

$$\begin{aligned}[\widetilde{K}(r(x_1,x_2))t_1, \widetilde{K}(r(x_1,x_3))t_2] &= \widetilde{K}(r(x_1,x_2))[t_1, \widetilde{K}(r(x_2,x_3))t_2] \\ &\quad + \widetilde{K}(r(x_1,x_3))[\widetilde{K}(r(x_3,x_2))t_1, t_2],\end{aligned} \tag{3.57}$$

where it was used that e.g. for any $a,b,c,d \in \mathfrak{g}$ we have:

$$\begin{aligned}&\widetilde{K}^{(3)}([(a \otimes b)^{32}, (c \otimes d)^{13}])(t_1 \otimes t_2) = \widetilde{K}^{(3)}(c \otimes b \otimes [a,d])(t_1 \otimes t_2) = K([a,d],t_2)K(b,t_1)c \\ &= K([K(b,t_1)a,d],t_2)c = -K([K(b,t_1)a,t_2],d)c = -\widetilde{K}(c \otimes d)[\widetilde{K}(a \otimes b)t_1, t_2].\end{aligned}$$

The other used identities can be derived similarly; see [KK11, Proposition 2.14]. If r is a solution of the CYBE (3.54), we find

$$\begin{aligned}[\widetilde{K}(r(x_1,x_2))t_1, \widetilde{K}(r(x_1,x_3))t_2] &= \widetilde{K}(r(x_1,x_2))[t_1, \widetilde{K}(r(x_2,x_3))t_2] \\ &\quad - \widetilde{K}(r(x_1,x_3))[\widetilde{K}(r(x_2,x_3))^* t_1, t_2]\end{aligned} \tag{3.58}$$

by applying $\widetilde{K}^{(3)}$. Here, $(\cdot)^*$ denotes the adjoint with respect to K.

Step 2. *r has poles along the diagonal.* Since non-degeneracy is a generic property, we can choose a point (x_0, y_0) in the domain of r such that $\widetilde{K}(r(x_0, y_0))$ is an isomorphism and $z \mapsto T_z := \widetilde{K}(r(z, y_0))$ is analytic along the line, connecting x_0 and y_0, but excluding y_0. We prove by contradiction that r has a pole at (y_0, y_0). Assume that r is analytic in (y_0, y_0), i.e. $z \mapsto T_z$ is analytic in y_0. The equations (3.57) and (3.58) reduce to

$$[T_z t_1, T_z t_2] = T_z([t_1, T_{y_0} t_2] + [T_{y_0} t_1, t_2]) \tag{3.59}$$

$$[T_z t_1, T_z t_2] = T_z([t_1, T_{y_0} t_2] - [T_{y_0}^* t_1, t_2]) \tag{3.60}$$

respectively, by setting $x_1 = z$ and $x_2, x_3 = y_0$. Applying $\psi_z := T_z \circ T_{y_0}^{-1}$ to (3.59) and (3.60) evaluated at $z = x_0$ results in

$$\psi_z[T_{x_0} t_1, T_{x_0} t_2] = T_z([t_1, T_{y_0} t_2] + [T_{y_0} t_1, t_2]), \tag{3.61}$$

$$\psi_z[T_{x_0} t_1, T_{x_0} t_2] = T_z([t_1, T_{y_0} t_2] - [T_{y_0}^* t_1, t_2]). \tag{3.62}$$

Comparing these equations with (3.59) and (3.60), evaluated at $z = x_0$, and using the fact that T_{x_0} is bijective, we see that ψ_z is a Lie algebra homomorphism in both cases. Therefore, the fact that ψ_z is orthogonal with respect to K if it is invertible, implies that $\det(\psi_u) \in \{0, \pm 1\}$; see e.g. [KK11, Lemma 2.3.] for details. A continuity argument and $\psi_{x_0} = \mathrm{id}_\mathfrak{g}$ force ψ_{y_0} and consequently $T_{y_0} = \psi_{y_0} \circ T_{x_0}$ to be an isomorphism.

Setting $z = y_0$ in equation (3.59), we see that $T_{y_0}^{-1}$ is an invertible derivation of $\mathfrak{g}$, contradicting the simplicity of $\mathfrak{g}$. Setting $z = y_0$ in equation (3.60) leads to the same contradiction, considering the fact that

$$\det(T_{y_0}) \neq 0 \implies T_{y_0}^* = -T_{y_0}. \tag{3.63}$$

The proof of (3.63) can be found in [KK11, Lemma 3.2 and Lemma 3.4] for $\Bbbk = \mathbb{C}$ and uses Schur's Lemma as well as the fact that every automorphism of $\mathfrak{g}$ has a fixed vector. Since $\mathfrak{g}$ is assumed to be central, Schur's Lemma applies for $\Bbbk = \mathbb{R}$ and an automorphism of $\mathfrak{g}$ without fixed vector defines one on the simple complex Lie algebra $\mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ by extension of scalars. Thus, the proof in [KK11, Lemma 3.2 and Lemma 3.4] also applies to the case $\Bbbk = \mathbb{R}$. Summarized, the assumption that r is a solution of either the CYBE or GCYBE without a pole along the diagonal leads to a contradiction. We have shown that r has a pole along the diagonal.

Step 3. *After shrinking U, $r(x, y) = \frac{\lambda(y)\gamma}{(x-y)^k} + \frac{f(x,y)}{(x-y)^{k-1}}$.* Using Lemma 2.1.2.(4) for $M = \mathfrak{g} \otimes \mathfrak{g}$ as well as $\bigcap_{k=0}^\infty (x-y)^k M[\![x, y]\!] = \{0\}$, we can find a $k \in \mathbb{N}_0$ and shrink U in such a way that $s(x, y) = (x-y)^k r(x, y)$ is analytic on $U \times U$ and $h(z) := s(z, z)$ is an analytic function on U which is not identically 0. After properly shrinking U further, we may assume that h is non-vanishing and $s(x, y) - h(y) = (x - y)f(x, y)$ for an analytic function $f \colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$. Multiplying either (3.59) or (3.60) with $(x_1 - x_2)^k$ and setting $x_1 = x_2$ results in

$$[\widetilde{K}(h(x_2))t_1, \widetilde{K}(r(x_2, x_3))t_2] = \widetilde{K}(h(x_2))[t_1, \widetilde{K}(r(x_2, x_3))t_2] \tag{3.64}$$

in both cases. Choosing x_3 in such a way that $\widetilde{K}(r(x_2, x_3))$ is an isomorphism, we see that $\widetilde{K}(h(x_2))$ is an equivariant endomorphism of $\mathfrak{g}$ with respect to the adjoint representation. In other words, $\widetilde{K}(h(x_2))$ is in the centroid of $\mathfrak{g}$ (see Subsection 1.3.1). Hence, $\widetilde{K}(h(x_2)) = \lambda(x_2)\mathrm{id}_\mathfrak{g}$ since $\mathfrak{g}$ is central, where $\lambda\colon U \to \Bbbk^\times$ is an analytic function. This implies that $h(x_2) = \lambda(x_2)\gamma$. Summarized, we obtain

$$r(x,y) = \frac{\lambda(y)\gamma}{(x-y)^k} + \frac{f(x,y)}{(x-y)^{k-1}}.$$

Step 4. $k = 1$. Assume that $k > 1$. Then

$$(x_1-x_2)^{k-1}[r^{32}(x_3,x_2), r^{13}(x_1,x_3)] \text{ and } (x_1-x_2)^{k-1}[r^{13}(x_1,x_3), r^{23}(x_2,x_3)]$$

vanish for $x_1 = x_2$. Therefore, multiplying the CYBE or the GCYBE with $(x_1-x_2)^{k-1}$, using

$$[\gamma^{12}, r(x_2,y_3)^{23}] = -[\gamma^{12}, r(x_2,y_3)^{13}],$$

and taking the limit $x_1 \to x_2$ results in

$$\begin{aligned} 0 &= [h(x_2)^{12}, \partial_{x_2} r(x_2,x_3)^{13}] + [f(x_2,x_2)^{12}, r(x_2,x_3)^{13} + r(x_2,x_3)^{23}] \\ &= [h(x_2)^{12}, (x_2-x_3)^{-k}\partial_{x_2} s(x_2,x_3)^{13} - k(x_2-x_3)^{-k-1}s(x_2,x_3)^{13}] \\ &\quad + [f(x_2,x_2)^{12}, r(x_2,x_3)^{13} + r(x_2,x_3)^{23}], \end{aligned}$$

where h, s and f are defined in Step 3. and the limit definition of ∂_{x_2} was used. Multiplying this with $(x_2-x_3)^{k+1}$ and taking the limit $x_1 \to x_3$, under consideration of $h(z) = \lambda(z)\gamma$, yields $-k\lambda(x_3)^2[\gamma^{12},\gamma^{13}] = 0$. This contradicts the fact that $[\gamma^{12},\gamma^{13}] \neq 0$. Indeed, $[\gamma^{12},\gamma^{13}]$ maps to γ with respect to the linear map defined by $a \otimes b \otimes c \mapsto [b,a] \otimes c$; see (2.34). Therefore, the assumption $k > 1$ leads to a contradiction and we can conclude that $k = 1$. □

As an immediate consequence of this result, we can relate non-degenerate (generalized) r-matrices to formal (generalized) r-matrices.

Corollary 3.4.2.
Let $r\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ be a non-degenerate (generalized) r-matrix. Then, after properly shrinking U and relocating the origin, the Taylor series of r in $y = 0$ is a formal (generalized) r-matrix.

In particular, Proposition 2.3.1 combined with the identity theorem (see e.g. [GR65, Chapter I.A, Theorem 6]) provides the following result.

Corollary 3.4.3.
A meromorphic function $r\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ for some connected open $U \subseteq \Bbbk$ is a non-degenerate solution of the CYBE (3.54) if and only if r is a non-degenerate skew-symmetric solution of the GCYBE (3.53).

In the following, we restrict our attention to analytic generalized r-matrices in standard form.

3.4.3 Analytic equivalence. Let $U_i \subseteq \Bbbk$ be a connected open subset and $r_i\colon U_i \times U_i \to \mathfrak{g} \otimes \mathfrak{g}$ be a meromorphic map in standard form for $i \in \{1, 2\}$. We call r_2 *analytically equivalent* to r_1 via the *analytic equivalence* (μ, w, φ) if

$$r_2(x, y) = \mu(y)(\varphi(x) \otimes \varphi(y))r_1(w(x), w(y)), \tag{3.65}$$

where (μ, w, φ) consists of

- a non-zero analytic $\mu\colon W \to \Bbbk^\times$, called *rescaling*, for some connected open neighbourhood $W \subseteq U_2$ of 0,
- an analytic embedding $w\colon W \to U_1$ such that $w(0) = 0$, called *coordinate transformation*, and
- an analytic map $\varphi\colon W \to \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$, called *gauge transformation*.

A similar result to Lemma 2.1.6 holds for analytic equivalences. More precisely, analytic equivalences define an equivalence transformation on the set of generalized r-matrices in standard form and analytic equivalences with constant rescaling part preserve skew-symmetry and thus the property of solving the CYBE (3.54). The proof of these statements uses a reduction to the complex case and the identity theorem (see e.g. [GR65, Chapter I.A, Theorem 6]), under consideration that the domain of definition of a complex meromorphic function on a connected open set is connected.

3.4.4 Comparison of analytic and formal generalized *r*-matrices. Let us denote the Taylor series of any meromorphic map r in standard form in $y = 0$ by Θr, i.e.

$$\Theta r(x, y) \coloneqq \sum_{j=0}^{\infty} \frac{1}{k!} \left(\partial_y^k r(x, y)\right)\Big|_{y=0} y^k \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]. \tag{3.66}$$

By virtue of Proposition 3.4.1, any non-degenerate generalized r-matrix r may be assumed to be in standard form and Θr is a formal generalized r-matrix in standard form. Furthermore, the identity theorem (see e.g. [GR65, Chapter I.A, Theorem 6]) implies that r is uniquely determined by Θr, so Θ defines an embedding of non-degenerate generalized r-matrices (up to relocating the origin) into the set of formal generalized r-matrices. However, there might be formal generalized r-matrices which do not converge to analytic generalized r-matrices. Nevertheless, we can see that non-degenerate analytic r-matrices actually provide representatives for any equivalence class of formal generalized r-matrices.

> **Proposition 3.4.4.**
> *Every formal generalized r-matrix is equivalent to a series of the form Θr (see (3.66)) for some analytic generalized r-matrix r in standard form.*

Proof. Let $r \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix, $((X, \mathscr{A}), (p, c, \zeta)) \coloneqq \mathbb{G}(\mathrm{Mult}(\mathfrak{g}(r)), \mathfrak{g}(r))$, chose a geometric GCYBE model $((X, \mathscr{A}), (C, \eta))$ of r and let ρ be the associated geometric r-matrix.

The analytic manifold $C^{\mathrm{an}} = (C^{\mathrm{an}}, \mathscr{O}_C^{\mathrm{an}})$, defined by the $\Bbbk$-rational points of C, is 1-dimensional. In the real case this may be seen through the implicit function theorem and the fact that $p \in C^{\mathrm{an}}$. Let $U \to C^{\mathrm{an}}$ be an analytic parameterization around p, where $U = (U, \mathscr{O}_U^{\mathrm{an}})$ is the locally ringed space associated to an open disc (resp. an open interval) if $\Bbbk = \mathbb{C}$ (resp. $\Bbbk = \mathbb{R}$). We write $\iota\colon U \to C^{\text{an}} \to X$ for the resulting morphism of locally ringed spaces and we may assume $0 \in U$ and $\iota(0) = p$. The sheaf of Lie algebras $\iota^*\mathscr{A}$ can be identified with an analytic fiber bundle on U with fiber $\mathfrak{g}$ and structure group $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$. Indeed, for $\Bbbk = \mathbb{C}$ this follows from Theorem 1.2.3 and the observation that étale $\mathfrak{g}$-local triviality implies local triviality in the complex topology, while for $\Bbbk = \mathbb{R}$ this is due to [Kir83, Lemma 2.1]. These fiber bundles are always trivial since U is contractible; see [Gra58, Satz 6] for the complex and [GMT86, Chapter VIII, Propositions 1.10 & 1.19] for the real case. Thus, there exists an isomorphism $\psi\colon \iota^*\mathscr{A} \to \mathfrak{g} \otimes \mathscr{O}_U^{\text{an}}$ of sheaves of Lie algebras and $\psi \boxtimes \psi$ defines an isomorphism $\iota^*\mathscr{A} \boxtimes \iota^*\mathscr{A} \to (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathscr{O}_{U\times U}^{\text{an}}$.

Consider the meromorphic map

$$\varrho := (\psi \boxtimes \psi)(\iota \times \iota)^*\rho\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}.$$

The Taylor series of ϱ in the second variable in the preimage of p under ι is equivalent to r. This can be deduced with an argument similar to the proof of Theorem 3.3.5. □

Proposition 3.4.5.
Two analytic generalized r-matrices in standard form are analytically equivalent if and only if their Taylor series in $y = 0$ are formally equivalent. In particular, Θ from (3.66) defines a bijection between

- *the set of analytic equivalence classes of real (resp. complex) analytic generalized r-matrices in standard form and*
- *the set of equivalence classes of formal generalized r-matrices over $\Bbbk = \mathbb{R}$ (resp. $\Bbbk = \mathbb{C}$).*

Proof. Let $r_1, r_2 \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be equivalent formal generalized r-matrices and

$$((X_1, \mathscr{A}_1), (p_1, c_1, \zeta_1)) := \mathbb{G}(\mathrm{Mult}(\mathfrak{g}(r_1)), \mathfrak{g}(r_1)) \xrightarrow{(f,\varphi)} \mathbb{G}(\mathrm{Mult}(\mathfrak{g}(r_2)), \mathfrak{g}(r_2)) =: ((X_2, \mathscr{A}_2), (p_2, c_2, \zeta_2))$$

be the isomorphism of their respective geometric $\mathfrak{g}$-lattice models provided by Lemma 3.1.1. In particular, $f\colon X_2 \to X_1$ is an isomorphism such that $f(p_2) = p_1$ and $\varphi\colon \mathscr{A}_1 \to f_*\mathscr{A}_2$ is an isomorphism of sheaves of Lie algebras. Chose geometric GCYBE models $((X_i, \mathscr{A}_i), (C_i, \eta_i))$ of r_i for $i \in \{1, 2\}$ in such a way that $f^{-1}(C_2) = C_1$ and $f^*\eta_2 = \eta_1$. For $i \in \{1, 2\}$, let ρ_i be the geometric r-matrix of $((X_i, \mathscr{A}_i), (C_i, \eta_i))$, let C_i^{an} be the analytic manifold defined by the $\Bbbk$-rational points of C_i and let $\iota_i\colon U \to C_i^{\text{an}} \to X$ be a local parametrization around p_i, where U_i is an open interval (resp. open disc) around the origin if $\Bbbk = \mathbb{R}$ (resp. $\Bbbk = \mathbb{C}$). As explained in the proof of Proposition 3.4.4, there exist isomorphisms $\psi_i\colon \iota_i^*\mathscr{A}_i \to \mathfrak{g} \otimes \mathscr{O}_{U_i}$ such that

$$\varrho_i := (\psi_i \boxtimes \psi_i)(\iota_i \times \iota_i)^*\rho_i\colon U_i \times U_i \to \mathfrak{g} \otimes \mathfrak{g} \tag{3.67}$$

is a meromorphic map whose Taylor series is equivalent to r_i. Thus, it remains to show that ϱ_1 and ϱ_2 are analytically equivalent.

By virtue of Lemma 3.3.2, we have $(f^*\varphi \boxtimes f^*\varphi)(f \times f)^*\rho_1 = \rho_2$. Application of $(\psi \boxtimes \psi)(\iota_2 \times \iota_2)^*$ results in

$$\Big(\psi_2\big((f\iota_2)^*\varphi\big) \boxtimes \psi_2\big((f\iota_2)^*\varphi\big)\Big)(f\iota_2 \times f\iota_2)^*\rho_1 = \varrho_2. \tag{3.68}$$

After properly shrinking U_2, there exists an analytic embedding $w\colon U_2 \to U_1$ such that $\iota_1 w = f\iota_2$ and $w(0) = 0$, since f is an isomorphism that maps p_2 to p_1. We can rewrite (3.68) as $(\varphi(x)\otimes\varphi(y))\varrho_1(w(x), w(y)) = \varrho_2(x, y)$, where $\varphi\colon U_2 \to \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ is the analytic map induced by the chain

$$\mathfrak{g} \otimes \mathscr{O}_{U_2} \xrightarrow{w^*\left(\psi_1^{-1}\right)} w^*\iota_1^*\mathscr{A}_1 = (f\iota_2)^*\mathscr{A}_1 \xrightarrow{(f\iota_2)^*\phi} \iota_2^*\mathscr{A}_2 \xrightarrow{\psi_2} \mathfrak{g} \otimes \mathscr{O}_{U_2} \tag{3.69}$$

of isomorphisms of sheaves of Lie algebras. □

Part II

Formal r-matrices over $\mathbb{C}$

4 Twisted loop algebras

4.1 Kac-Moody algebras

This section provides a survey of the theory of Kac-Moody algebras, with the purpose of applying said theory to twisted loop algebras in the next section. All statements will be presented without proof; we refer to [Kac90] for the statements from subsections 4.1.1–4.1.6 and to [KW92] for the results in the remaining subsections 4.1.7 and 4.1.8. As usual, $\Bbbk$ denotes an arbitrary field of characteristic 0.

4.1.1 Generalized Cartan matrices. A *generalized Cartan matrix* A is an element $A = (a_{ij})_{i,j=1}^q \in \mathbb{Z}^{q\times q}$ such that $a_{ii} = 2$, $a_{ij} \leqslant 0$ and $a_{ij} = 0$ implies $a_{ji} = 0$ for all $i, j \in \{1, \dots, q\}$. The matrix A is said to be *symmetrizable*, if there exists a diagonal matrix D such that DA is symmetric. Another generalized Cartan matrix $B = (b_{ij})_{i,j=1}^q$ is called *equivalent* to A if there exists a permutation σ of $\{1, \dots, q\}$ such that $b_{ij} = a_{\sigma(i)\sigma(j)}$ for all $i, j \in \{1, \dots, q\}$. Furthermore, A is called *decomposable* if it is equivalent to a generalized Cartan matrix in block diagonal form with at least two blocks and *indecomposable* otherwise. Finally, an indecomposable generalized Cartan matrix is said to be of *finite type* if $\mathrm{rk}(A) = q$, of *affine type* if $\mathrm{rk}(A) = q - 1$ and of *indefinite type* if $\mathrm{rk}(A) < q - 1$.

4.1.2 Realizations of generalized Cartan matrices and Kac-Moody algebras. A quadruple $(\mathfrak{K}, \mathfrak{h}, \Pi, \Pi^\vee)$ is called *realization* of a generalized Cartan matrix $A = (a_{ij})_{i,j=1}^q$ if $\mathfrak{h}$ is a vector space of dimension $2q - \mathrm{rk}(A)$, $\Pi = \{\alpha_1, \dots, \alpha_q\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_q^\vee\} \subset \mathfrak{h}$ are linearly independent subsets satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$ and $\mathfrak{K}$ is the $\Bbbk$-Lie algebra generated by $\mathfrak{h}$ and symbols $\left\{e_i^+, e_i^- \,\middle|\, i \in \{1, \dots, q\}\right\}$, called *Chevalley generators* of $\mathfrak{K}$, with relations

$$[\mathfrak{h}, \mathfrak{h}] = 0; \qquad [h, e_i^\pm] = \pm\alpha_i(h)e_i^\pm;$$
$$[e_i^+, e_j^-] = \delta_{ij}h_i; \qquad \mathrm{ad}(e_i^\pm)^{1-a_{ij}}e_j^\pm = 0,$$

where $h \in \mathfrak{h}, i, j \in \{1, \dots, q\}$ and $i \neq j$ in the last relation. A Lie algebra is called *Kac-Moody algebra* with Cartan matrix A if it is isomorphic to $\mathfrak{K}$ for some generalized Cartan matrix A with realization $(\mathfrak{K}, \mathfrak{h}, \Pi, \Pi^\vee)$.

Lemma 4.1.1.
Let A be a generalized Cartan matrix. The following results are true.

(1) *There exists a realization $(\mathfrak{K}, \mathfrak{h}, \Pi, \Pi^\vee)$ of A.*

(2) *For any realization $(\widetilde{\mathfrak{K}}, \widetilde{\mathfrak{h}}, \widetilde{\Pi}, \widetilde{\Pi}^\vee)$ of a generalized Cartan matrix B equivalent to A exists an isomorphism $\mathfrak{h} \to \widetilde{\mathfrak{h}}$ which identifies Π (resp. $\Pi^\vee$) with $\widetilde{\Pi}$ (resp. $\widetilde{\Pi}^\vee$) and extends to an isomorphism of Lie algebras $\mathfrak{K} \to \widetilde{\mathfrak{K}}$ by mapping the respective Chevalley generators onto each other.*

(3) $[\mathfrak{K},\mathfrak{K}]$ *is generated by the Chevalley generators of* $\mathfrak{K}$. *Furthermore, the center of* $\mathfrak{K}$ *is contained in* $\mathfrak{h}' := \mathfrak{h} \cap [\mathfrak{K},\mathfrak{K}] = \langle \Pi^\vee \rangle_{\Bbbk}$.
(4) *If* A *is indecomposable, every ideal of* $\mathfrak{K}$ *either contains* $[\mathfrak{K},\mathfrak{K}]$ *or is contained in the center of* $\mathfrak{K}$.

4.1.3 Weyl group and root structure of Kac-Moody algebras. Let us fix a realization

$$\left(\mathfrak{K}, \mathfrak{h}, \Pi = \{\alpha_1, \dots, \alpha_q\}, \Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_q^\vee\}\right) \tag{4.1}$$

of a generalized Cartan matrix A with Chevalley generators $\left\{e_i^+, e_i^- \,\middle|\, i \in \{1,\dots,q\}\right\}$ for the rest of this section. For $\alpha \in \mathfrak{h}^*$, we write

$$\mathfrak{K}_\alpha := \{x \in \mathfrak{K} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{H}\} \tag{4.2}$$

and call α *root* with *root space* $\mathfrak{K}_\alpha$ if $\alpha \neq 0$ and $\mathfrak{K}_\alpha \neq 0$. Let us denote the set of roots by Φ. For any $i \in \{1,\dots,q\}$, let r_i be the $\Bbbk$-linear automorphism of $\mathfrak{h}$ defined by $h \mapsto h - \alpha_i(h)\alpha_i^\vee$, which is called *i-th fundamental reflection.* The group W of $\Bbbk$-linear automorphisms generated by $\{r_1,\dots,r_q\}$ is called *Weyl group.* Note that W naturally acts on $\mathfrak{h}^*$ by $w \cdot \lambda = \lambda w$ for all $\lambda \in \mathfrak{h}^*$ and $w \in W$.

Lemma 4.1.2.
The following results are true.
(1) $\mathfrak{K} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{K}_\alpha$ *is a direct sum of* $\mathfrak{h}$*-modules.*
(2) $\Phi \subseteq \langle \alpha_1,\dots,\alpha_q\rangle_{\mathbb{Z}}$ *and* $\Phi = \Phi_+ \sqcup \Phi_-$*, where*

$$\Phi_+ := \{\alpha = a_1\alpha_1 + \dots a_q\alpha_q \mid (a_0,\dots,a_q) \in \mathbb{N}_0^q \setminus \{0\}, \alpha \in \Phi\} \tag{4.3}$$

and $\Phi_- := -\Phi_+$.
(3) *The action of* W *on* $\mathfrak{h}^*$ *restricts to an action on* Φ*, i.e.* $W\Phi = \Phi$.
(4) *For* $\alpha \in W\Pi \subseteq \Phi$ *it holds that* $\dim(\mathfrak{K}_\alpha) = 1$ *and* $\Phi \cap \mathbb{Z}\alpha = \{\alpha, -\alpha\}$.

A root α is called *positive* (resp. *negative*), if $\alpha \in \Phi_+$ (resp. $\alpha \in \Phi_-$). Furthermore, α is called *real* (resp. *imaginary*) if $\alpha \in \Phi^{\mathrm{re}} := W\Pi$ (resp. $\alpha \in \Phi^{\mathrm{im}} := \Phi \setminus \Phi^{\mathrm{re}}$).

Lemma 4.1.3.
Assume that A *is indecomposable.*
(1) *If* A *is of finite type,* $\Phi = \Phi^{\mathrm{re}}$ *is finite and there exist unique* $k_1,\dots,k_q \in \mathbb{N}$ *such that* $\alpha_0 = \sum_{i=1}^q k_i\alpha_i \in \Phi^+$ *and* $\sum_{i=1}^q k_i$ *is maximal.*
(2) *If* A *is of affine type, there are unique relatively prime* $k_1,\dots,k_q \in \mathbb{N}$ *such that* $\sum_{i=1}^q k_i a_{ij} = 0$ *and in this case* $\Phi^{\mathrm{im}} = \langle\delta\rangle_{\mathbb{Z}}$ *for* $\delta = \sum_{i=1}^q k_i\alpha_i$. *In particular,* $\delta|_{\langle\Pi^\vee\rangle_{\Bbbk}} = 0$.

4.1.4 $\mathbb{Z}$-grading of Kac-Moody algebras. Recall that for an abelian group A (written additively with neutral element 0) and a Lie algebra $\mathfrak{A}$ an *A-grading* of $\mathfrak{A}$ is a vector space decomposition $\mathfrak{A} = \bigoplus_{a \in A} \mathfrak{A}_a$ such that $[\mathfrak{A}_a, \mathfrak{A}_b] \subseteq \mathfrak{A}_{a+b}$. Note that $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is automatically a subalgebra and $\mathfrak{A} = \bigoplus_{a \in A} \mathfrak{A}_a$ is a direct sum of $\mathfrak{A}_0$-modules. For another A-graded Lie algebra $\widetilde{\mathfrak{A}}$, a Lie algebra morphism $\varphi \colon \mathfrak{A} \to \widetilde{\mathfrak{A}}$ is called A-graded if $\varphi(\mathfrak{A}_a) \subseteq \widetilde{\mathfrak{A}}_a$ for all $a \in A$.

For any $\alpha \in \Phi$ and $s = (s_1, \ldots, s_q) \in \mathbb{N}_0^q \setminus \{0\}$, the *s-root height* is defined by $\mathrm{ht}_s(\alpha) := \sum_{j=1}^q a_j s_j$, where $a_1, \ldots, a_q \in \mathbb{Z}$ are uniquely given by $\alpha = \sum_{j=1}^q a_j \alpha_j$. Let $\mathfrak{K}_j^s$ be the direct sum of the vector spaces $\mathfrak{K}_\alpha$, where α runs over all roots satisfying $\mathrm{ht}_s(\alpha) = j$.

Lemma 4.1.4.
$\mathfrak{K} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{K}_j^s$ *is a* $\mathbb{Z}$*-grading as Lie algebra, called* grading of type s.

4.1.5 Invariant bilinear forms. The discussion of invariant bilinear forms in the theory of Kac-Moody algebras is summarized in the following lemma.

Lemma 4.1.5.
There exists a non-degenerate, invariant, symmetric, bilinear map $B \colon \mathfrak{K} \times \mathfrak{K} \to \Bbbk$ *if and only if* A *is symmetrizable and in this case* B *has the following properties:*
(1) *For* $\alpha, \beta \in \mathfrak{h}^*$ *we have* $B(\mathfrak{K}_\alpha, \mathfrak{K}_\beta) \neq 0$ *if and only if* $\alpha = -\beta$ *and in this case* $B|_{\mathfrak{K}_\alpha \times \mathfrak{K}_\beta}$ *is non-degenerate.*
(2) *For* $\alpha \in \mathfrak{h}^*$, *let* χ_α *be the preimage of the identity under the isomorphism* $\mathfrak{K}_\alpha \otimes \mathfrak{K}_{-\alpha} \to \mathrm{End}(\mathfrak{K}_\alpha)$ *defined by* $a \otimes b \mapsto B(b, -)a$. *Then*
$$[a \otimes 1, \chi_\beta] + [1 \otimes a, \chi_{\alpha+\beta}] = 0 \tag{4.4}$$
for all $\alpha, \beta \in \mathfrak{h}^*$ *and* $a \in \mathfrak{K}_\alpha$.
(3) *The kernel of* B *restricted to* $[\mathfrak{K}, \mathfrak{K}]$ *is the center of* $\mathfrak{K}$.
(4) *If* A *is indecomposable,* B *is unique up to scaling and the choice of a vector subspace of* $\mathfrak{h}$ *complementary to* $\mathfrak{h} \cap [\mathfrak{K}, \mathfrak{K}]$.

4.1.6 Dynkin diagrams Assume that A is indecomposable. The *Dynkin diagram* $D(A)$ of A is the graph consisting of q vertices and the vertices $i, j \in D(A)$ are connected with respect to the following rules:
- If $a_{ij}a_{ji} \leqslant 4$ and $|a_{ij}| \geqslant |a_{ji}|$ there are $|a_{ij}|$ lines connecting i and j as well as an arrow pointing towards i if $|a_{ij}| > 1$.
- If $a_{ij}a_{ji} > 4$ the vertices i and j are connected by a bold-faced line equipped with the orderd pair of integeres $|a_{ij}|, |a_{ji}|$ as integers.

Theorem 4.1.6.
The following results are true.
(1) A *can be uniquely reconstructed from* $D(A)$.
(2) *If* A *is of finite type,* $D(A)$ *is one of the diagrams shown in Figure 4.1. In particular* $\mathfrak{K}$ *is a finite dimensional split simple Lie algebra. All such Lie algebras arise in such a way from appropriate generalized Cartan matrices of*

finite type.

(3) *If A is of affine type, $D(A)$ is one of the diagrams shown in Figure 4.2.*

We see that the theory of Kac-Moody algebras for generalized Cartan matrices of finite type is essentially the theory of finite-dimensional split simple Lie algebras. The next section is completely dedicated to a detailed discussion of Kac-Moody algebras arising for generalized Cartan matrices of affine type over an algebraically closed field.

Figure 4.1: If A is of finite type, $D(A)$ is one of the following diagrams. Here, $n = q$ and we can choose a labeling of the simple roots of $\mathfrak{K}$ such that the vertex with label i represents the simple root α_i and the grey subscript of said vertex is k_i from Lemma 4.1.3.(1).

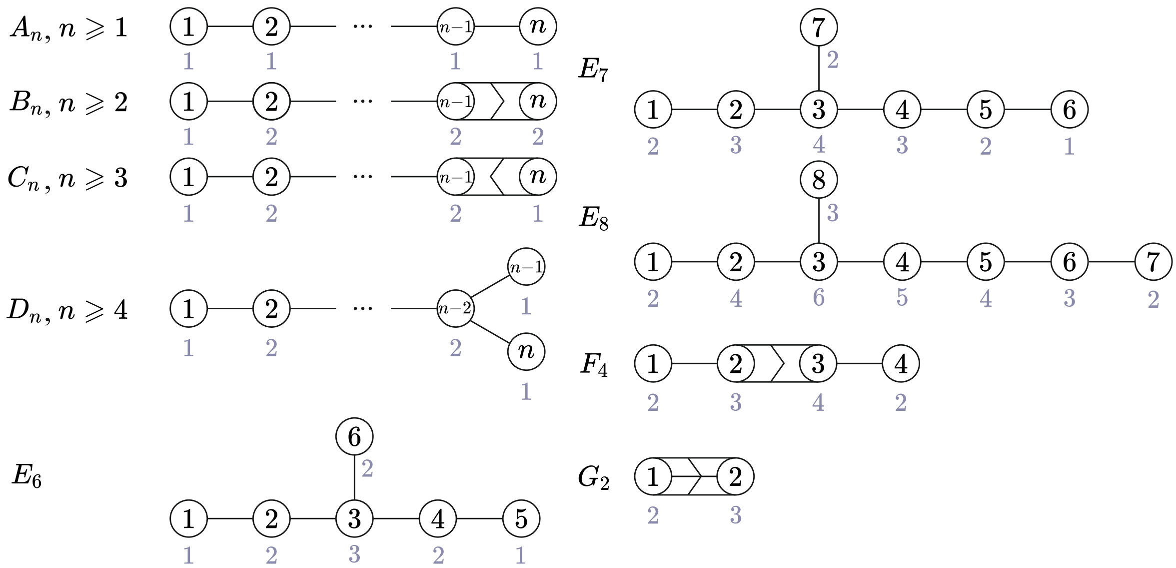
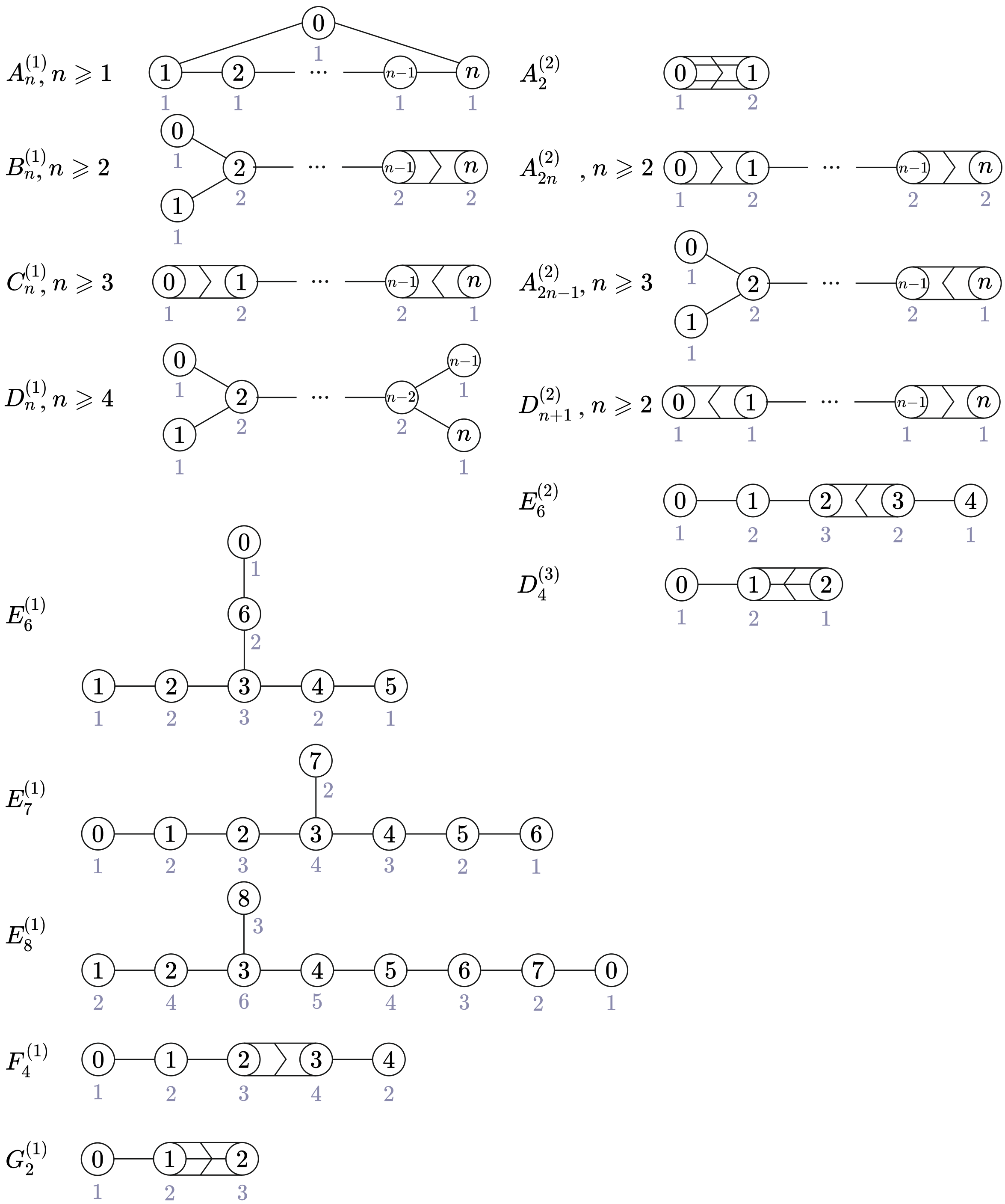


Figure 4.2: If A is of affine type, $D(A)$ is one of the following diagrams. Here, $n = q - 1$ and we can choose a labeling of the simple roots of $\mathfrak{K}$ such that the vertex with label i represents the simple root α_{i+1} and the grey subscript of said vertex is k_{i+1} from Lemma 4.1.3.(2).



4.1.7 Automorphisms of a derived Kac-Moody algebra modulo its center. The goal of this subsection is to describe the set of automorphisms of $\mathfrak{K}' := [\mathfrak{K}, \mathfrak{K}]$ modulo its center $\mathfrak{c}$. For brevity let us assume that A is indecomposable, however, note that all notions introduced in the following can be generalized to the decomposable case. First, we have the following result.

Lemma 4.1.7.
For every real root α and $x \in \mathfrak{K}_\alpha$ the endomorphism $\operatorname{ad}(x)$ of $\mathfrak{K}$ is locally nilpotent: for all $y \in \mathfrak{K}$ exists a $k \in \mathbb{N}$ such that $\operatorname{ad}(x)^k y = 0$.

Let $\operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$ (resp. $\operatorname{Inn}^{\pm}_{\mathrm{ad}}(\mathfrak{K})$) $\subseteq \operatorname{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K})$ be the subgroup generated by

$$\{\exp(\operatorname{ad}(x)) \mid \alpha \in \Phi^{\mathrm{re}}, x \in \mathfrak{K}_\alpha\} \ \ (\text{resp. } \{\exp(\operatorname{ad}(x)) \mid \alpha \in \Phi^{\mathrm{re}} \cap \Phi^{\pm}, x \in \mathfrak{K}_\alpha\}). \tag{4.5}$$

An automorphism of $\mathfrak{K}$ is said to be *inner of adjoint type* if it is contained in $\operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$.

Remark 4.1.8.
The group $\operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$ is actually the image of the so called *Kac-Moody group* of $\mathfrak{K}'$ under its natural action on $\mathfrak{K}$. We will skip a detailed introduction of this group, since we do not need any of its representation theoretical properties besides its action on $\mathfrak{K}$.

Let $i \in \{1, \ldots, q\}$. A straight forward calculation shows that the restriction of $\exp(\operatorname{ad}(e_i^-))\exp(-\operatorname{ad}(e_i^+))\exp(\operatorname{ad}(e_i^-)) \in \operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$ to $\mathfrak{h}$ coincides with the i-th fundamental reflection $r_i \in W$. Therefore, the Weyl group W can be understood as a subgroup of $\operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$. Moreover, there exists a natural homomorphism $\psi_i \colon \mathrm{SL}_2(\Bbbk) \to \operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$ such that

$$\psi_i \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \exp(\operatorname{ad}(\lambda e_i^+)) \text{ and } \psi_i \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \exp(\operatorname{ad}(\lambda e_i^-)) \text{ for all } \lambda \in \Bbbk. \tag{4.6}$$

Let $H_i := \psi_i(\{\operatorname{diag}(\lambda, \lambda^{-1}) \mid \lambda \in \Bbbk^\times\}) \subseteq \operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$ and H be the subgroup of $\operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K})$ generated by $H_1, \ldots, H_q$. The group H is stable under conjugation by elements of W and the normalizer of H can be identified with $W \ltimes H$.

Proposition 4.1.9.
It holds that $\operatorname{Inn}_{\mathrm{ad}}(\mathfrak{K}) = \sqcup_{w \in W} B_+ w B_+ = \sqcup_{w \in W} B_+ w B_-$, for $B_\pm := H \operatorname{Inn}^{\pm}_{\mathrm{ad}}(\mathfrak{K})$.

The group $\widetilde{H} := \operatorname{Hom}_{\mathbb{Z}}(\langle \alpha_1, \ldots, \alpha_q \rangle_{\mathbb{Z}}, \Bbbk^\times)$ acts on $\mathfrak{K}$ by

$$\lambda \cdot x := \lambda(\alpha) x \text{ for all } \lambda \in \widetilde{H}, x \in \mathfrak{K}_\alpha \text{ and } \alpha \in \Phi. \tag{4.7}$$

The automorphisms in the image $\operatorname{Inn}_{\mathrm{sc}}(\mathfrak{K})$ of this action in $\operatorname{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K})$ are called *inner automorphisms of scaling type*. Write $\operatorname{Aut}(A)$ for all permutations σ of $\{1, \ldots, q\}$ satisfying $a_{\sigma(i)\sigma(j)} = a_{ij}$, where $A = (a_{ij})_{i,j=1}^q$ and let $\{e_i^+, e_i^-\}$ be Chevalley generators of $\mathfrak{K}$.

Lemma 4.1.10.
The following results are true.
(1) $\mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K})$ *is stable under the conjugation of elements of* $\mathrm{Inn}_{\mathrm{sc}}(\mathfrak{K})$.
(2) *For every* $\sigma \in \mathrm{Aut}(A)$ *the assignment* $\widetilde{\sigma}(e_i^{\pm}) := e_{\sigma(i)}^{\pm}$ *determines a unique* $\widetilde{\sigma} \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K}')$.
(3) *The assignment* $\omega(e_i^{\pm}) = -e_i^{\mp}$ *for all* $i \in \{1, \dots, q\}$ *and* $\omega(h) = -h$ *for all* $h \in \mathfrak{h}$ *determines a unique* $\omega \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K})$ *called* Chevalley involution.
(4) $\mathrm{Inn}_{\mathrm{sc}}(\mathfrak{K}) \cap \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K}) \neq \{\mathrm{id}_{\mathfrak{K}}\}$ *and* $\omega \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K})$ *if and only if* A *is of finite type in which case* $\mathrm{Inn}_{\mathrm{sc}}(\mathfrak{K}) \subseteq \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K})$.

Let $\mathfrak{c}$ be the center of $\mathfrak{K}$ and recall that $\mathfrak{c} \subseteq \mathfrak{K}'$. Define the group $\mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K}'/\mathfrak{c})$ (resp. $\mathrm{Inn}_{\mathrm{sc}}(\mathfrak{K}'/\mathfrak{c})$) as the image of $\mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K})$ (resp. $\mathrm{Inn}_{\mathrm{sc}}(\mathfrak{K})$) under the canonical map $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K}) \to \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K}'/\mathfrak{c})$. We write

$$\mathrm{Inn}(\mathfrak{K}'/\mathfrak{c}) = \begin{cases} \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K}'/\mathfrak{c}) & \text{if } A \text{ is of finite type} \\ \mathrm{Inn}_{\mathrm{sc}}(\mathfrak{K}'/\mathfrak{c}) \ltimes \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K}'/\mathfrak{c}) & \text{otherwise} \end{cases} \tag{4.8}$$

for the group of inner automorphisms of $\mathfrak{K}'/\mathfrak{c}$. Similarly, let the subgroup $\mathrm{Out}(\mathfrak{K}'/\mathfrak{c}) \subseteq \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K}'/\mathfrak{c})$ of *outer automorphisms* of $\mathfrak{K}'/\mathfrak{c}$ consist of all elements of the form $\omega^i\widetilde{\sigma}$, where $\sigma \in \mathrm{Aut}(A)$ and $i = 0$ if A is of finite type while $i \in \{0, 1\}$ otherwise.

Theorem 4.1.11.
The identity $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K}'/\mathfrak{c}) = \mathrm{Out}(\mathfrak{K}'/\mathfrak{c}) \ltimes \mathrm{Inn}(\mathfrak{K}'/\mathfrak{c})$ *holds. In particular, for any symmetric, non-degenerate, invariant bilinear form* B *of* $\mathfrak{K}$, *the induced non-degenerate invariant bilinearform* $\overline{B}$ *on* $\mathfrak{K}'/\mathfrak{c}$ *is stable under* $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K}'/\mathfrak{c})$*:*

$$\overline{B}(\varphi(a), \varphi(b)) = \overline{B}(a, b) \text{ for all } a, b \in \mathfrak{K}'/\mathfrak{c}, \varphi \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{K}'/\mathfrak{c}). \tag{4.9}$$

4.1.8 Subalgebras of Kac-Moody algebras. For any subset $S \subseteq \{1, \dots, q\}$, let $\mathfrak{n}_{\pm}^S$ denote the subalgebras of $\mathfrak{K}$ generated by $\{e_i^{\pm} \mid i \in S\}$ and omit the superscript if $S = \{1, \dots, q\}$. The algebras $\mathfrak{b}_{\pm} := \mathfrak{h} \oplus \mathfrak{n}_{\pm}$ are called *standard Borel subalgebras*. For any proper subset $S \subsetneq \{1, \dots, q\}$ the subalgebras $\mathfrak{p}_{\pm}^S := \mathfrak{b}_{\pm} \oplus \mathfrak{n}_{\mp}^S$ of $\mathfrak{K}$ are said to be *standard parabolic*.

Proposition 4.1.12.
The following statements are true.
(1) *For any proper subalgebra* $\mathfrak{p}$ *of* $\mathfrak{K}$ *containing* $\mathfrak{b}_{\pm}$ *exists* $S \subseteq \{1, \dots, q\}$ *such that* $\mathfrak{p} = \mathfrak{p}_{\pm}^S$.
(2) *Let* $S \subseteq \{1, \dots, q\}$ *and* $\Phi_{\pm}^S := \Phi_{\pm} \cap \langle \alpha_i \mid i \in S\rangle_{\mathbb{Z}}$. *Then* $\mathfrak{n}_{\pm}^S = \bigoplus_{\alpha \in \Phi_{\pm}^S} \mathfrak{K}_{\alpha}$.
(3) *Let* $\mathfrak{a}_1 \subseteq \mathfrak{K}$ *be a subalgebra and* $\mathfrak{a}_2 \subseteq \mathfrak{n}_{\pm}$ *be a subalgebra of finite codimension such that* $\mathfrak{a}_2$ *is also an ideal in* $\mathfrak{a}_1$ *and* $\mathfrak{a}_1/\mathfrak{a}_2$ *is a finite-dimensional solvable Lie algebra. Then there exists* $\psi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K})$ *such that* $\psi(\mathfrak{a}_1) \subseteq \mathfrak{b}_{\pm}$.
(4) *Let* $\mathfrak{a}_1 \subseteq \mathfrak{K}$ *be a subalgebra and* $\mathfrak{a}_2 \subseteq \mathfrak{n}_{\pm}$ *be a subalgebra of finite codimension such that* $[\mathfrak{a}_1, \mathfrak{a}_2] \subseteq \mathfrak{a}_1$. *Then there exists* $S \subseteq \{1, \dots, q\}$ *and* $\psi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{K})$ *such that* $[\psi(\mathfrak{a}_1), \mathfrak{p}_{\pm}^S] \subseteq \mathfrak{p}_{\pm}^S$.

4.2 Twisted Loop algebras and affine Kac-Moody algebras

In this section, we will discuss the theory of twisted loop algebras. As we will see, this is closely related to the theory of Kac-Moody algebras arising from generalized Cartan matrices of affine type or affine Kac-Moody algebras for short. The main references for this section are [Kac90, Chapter 6-8] and [Hel78, Chapter X.5]. All statements presented without proof or explicit reference can be found there. Throughout this section, the field $\Bbbk$, which is as usual of characteristic 0, is additionally assumed to be algebraically closed.

4.2.1 Definition and generalities. Let $\mathfrak{g}$ be a simple Lie algebra over $\Bbbk$, $\sigma \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ have order $m \in \mathbb{N}$ and fix an m-th primitive root of unity $\varepsilon \in \Bbbk$. The *loop algebra* of $\mathfrak{g}$ is the free $\Bbbk[\widetilde{u}, \widetilde{u}^{-1}]$-Lie algebra $\mathfrak{L}(\mathfrak{g}) := \mathfrak{g}[\widetilde{u}, \widetilde{u}^{-1}] = \mathfrak{g} \otimes \Bbbk[\widetilde{u}, \widetilde{u}^{-1}]$ with Lie bracket determined by

$$[a\widetilde{u}^k, b\widetilde{u}^\ell] = [a,b]\widetilde{u}^{k+\ell} \text{ for all } a, b \in \mathfrak{g}, k, \ell \in \mathbb{Z}. \tag{4.10}$$

It can be identified with the Lie algebra of regular functions $\Bbbk^\times \to \mathfrak{g}$ of affine varieties over $\Bbbk$. The *twisted loop algebra* $\mathfrak{L}(\mathfrak{g}, \sigma)$ of the pair $(\mathfrak{g}, \sigma)$ is the fixed point subalgebra of $\mathfrak{L}(\mathfrak{g})$ with respect to the action $a(\widetilde{u}) \mapsto \sigma(a(\varepsilon^{-1}\widetilde{u}))$, where σ is applied coefficientwise. This means

$$\mathfrak{L}(\mathfrak{g}, \sigma) := \{a \in \mathfrak{L}(\mathfrak{g}) \mid a(\varepsilon\widetilde{u}) = \sigma(a(\widetilde{u}))\} \subseteq \mathfrak{L}(\mathfrak{g}). \tag{4.11}$$

Observe that $\mathfrak{L}(\mathfrak{g}, \mathrm{id}_\mathfrak{g}) = \mathfrak{L}(\mathfrak{g})$. Let $\mathfrak{g}_j^\sigma$ be the eigenspace of σ to the eigenvalue ε^j and write $\mathfrak{L}(\mathfrak{g}, \sigma)_j := \widetilde{u}^j \mathfrak{g}_j^\sigma$. Clearly, $\mathfrak{g}_{j+km}^\sigma = \mathfrak{g}_j^\sigma$ for all $k \in \mathbb{Z}$, so $\mathfrak{L}(\mathfrak{g}, \sigma)$ is stable under the multiplication by $\Bbbk[u, u^{-1}] \subseteq \Bbbk[\widetilde{u}, \widetilde{u}^{-1}]$, where $u = \widetilde{u}^m$.

Lemma 4.2.1.
The following results are true.
(1) $\mathfrak{L}(\mathfrak{g}, \sigma) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{L}(\mathfrak{g}, \sigma)_j$ *is a $\mathbb{Z}$-grading of a Lie algebra.*
(2) $\mathfrak{L}(\mathfrak{g}, \sigma)$ *is a free* $\Bbbk[u, u^{-1}]$-submodule *of* $\mathfrak{L}(\mathfrak{g})$ *and*

$$\mathfrak{L}(\mathfrak{g}) \cong \mathfrak{L}(\mathfrak{g}, \sigma) \otimes_{\Bbbk[u,u^{-1}]} \Bbbk[\widetilde{u}, \widetilde{u}^{-1}] \tag{4.12}$$

as $\Bbbk[\widetilde{u}, \widetilde{u}^{-1}]$*-Lie algebras.*
(3) *For every* $\rho \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ *there is a canonical* $\Bbbk[\widetilde{u}, \widetilde{u}^{-1}]$*-linear isomorphism* $\mathfrak{L}(\mathfrak{g}, \sigma) \to \mathfrak{L}(\mathfrak{g}, \rho\sigma\rho^{-1})$ *of $\mathbb{Z}$-graded Lie algebras.*
(4) *Let K be the Killing form of $\mathfrak{g}$. Then the map* $K_k \colon \mathfrak{L}(\mathfrak{g}, \sigma) \times \mathfrak{L}(\mathfrak{g}, \sigma) \to \Bbbk$ *defined by*

$$K_k(a, b) := \mathrm{res}_0 K(a(\widetilde{u}), b(\widetilde{u}))\widetilde{u}^{-mk-1} d\widetilde{u} \text{ for all } a, b \in \mathfrak{L}(\mathfrak{g}, \sigma) \tag{4.13}$$

is non-degenerate, invariant, symmetric, and bilinear for every $k \in \mathbb{Z}$.

Proof. Observe that $\mathfrak{L}(\mathfrak{g}, \sigma) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{L}(\mathfrak{g}, \sigma)_j$ holds as vector spaces and since σ is an automorphism of $\mathfrak{g}$ as a Lie algebra, it is easy to see that $[\mathfrak{g}_j^\sigma, \mathfrak{g}_k^\sigma] \subseteq \mathfrak{g}_{k+j}^\sigma$.

This implies $[\mathfrak{L}(\mathfrak{g},\sigma)_j, \mathfrak{L}(\mathfrak{g},\sigma)_k] \subseteq \mathfrak{L}(\mathfrak{g},\sigma)_{k+j}$, proving (1). For (2), note that $\mathfrak{L}(\mathfrak{g},\sigma)$ is torsion free, hence free, as a $\Bbbk[u,u^{-1}]$-submodule of $\mathfrak{L}(\mathfrak{g})$. Therefore, the multiplication map $\mathfrak{L}(\mathfrak{g},\sigma) \otimes_{\Bbbk[u,u^{-1}]} \Bbbk[\tilde{u},\tilde{u}^{-1}] \to \mathfrak{L}(\mathfrak{g})$, given by $a \otimes \lambda \mapsto \lambda a$, is injective. It is also surjective, since $\mathfrak{g} = \bigoplus_{j=0}^{m-1} \mathfrak{g}_j^\sigma$. Now $\tilde{u}^k a \mapsto \tilde{u}^k \rho(a)$, where $a \in \mathfrak{g}_k^\sigma$, defines an isomorphism $\mathfrak{L}(\mathfrak{g},\sigma) \to \mathfrak{L}(\mathfrak{g},\rho\sigma\rho^{-1})$ with the properties stated in (3) and (4) is a simple straight forward verification. □

Remark 4.2.2.
(1) The map $\mathrm{Spec}(\Bbbk[\tilde{u},\tilde{u}^{-1}]) \to \mathrm{Spec}(\Bbbk[u,u^{-1}])$ is étale induced by $\Bbbk[u,u^{-1}] \subseteq \Bbbk[\tilde{u},\tilde{u}^{-1}]$. Therefore, Lemma 4.2.1.(2) implies that the sheaf of Lie algebras on $\mathrm{Spec}(\Bbbk[u,u^{-1}])$ defined by $\mathfrak{L}(\mathfrak{g},\sigma)$ is étale $\mathfrak{g}$-locally free in the sense of Subsection 1.2.1.
(2) Let $d \in \mathbb{N}$ satisfy $\gcd(m,d) = 1$. There exists $\tilde{\sigma} \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ of order m such that $\tilde{\sigma}^d = \sigma$, so $\mathfrak{L}(\mathfrak{g},\tilde{\sigma}) = \{a \colon \Bbbk^\times \to \mathfrak{g} \text{ regular} \mid a(\varepsilon^d z) = \sigma a(z)\}$. Indeed, choose $k,\ell \in \mathbb{Z}$ such that $kd + \ell m = 1$, then $\sigma = \sigma^{kd+m\ell} = \sigma^{kd}$ and we can take $\tilde{\sigma} = \sigma^k$. Since $\gcd(n,d) = \gcd(k,m) = 1$, it is easy to see that σ^k has order m and $a \in \mathfrak{L}(\mathfrak{g},\sigma^k)$ if and only if $\sigma(\varepsilon^d z) = \sigma^{kd} a(z) = \sigma a(z)$. We can conclude that choosing a primitive root of unity different from ε does not lead to new twisted loop algebras.

4.2.2 Connection to affine Kac-Moody algebras. Henceforth, we write $\mathfrak{L} = \mathfrak{L}(\mathfrak{g},\sigma)$ and $L = \Bbbk[u,u^{-1}]$. As already mentioned, twisted loop algebras are closely related to affine Kac-Moody algebras. In fact, they are derived algebras of affine Kac-Moody algebras modulo the respective center. To understand this, we recall the construction from [Kac90, Section 8.2]. There exists a Lie algebra structure on the vector space

$$\widehat{\mathfrak{L}} := \mathfrak{L} \oplus \Bbbk c \oplus \Bbbk d \tag{4.14}$$

with the following properties.
(1) The Lie bracket is uniquely determined by the following rules
- $[\tilde{u}^k a, \tilde{u}^\ell b] = \tilde{u}^{k+\ell}[a,b] + kK_0(\tilde{u}^k a, \tilde{u}^\ell b)c$ for all $\tilde{u}^k a, \tilde{u}^\ell b \in \mathfrak{L}$.
- c is the center of $\widehat{\mathfrak{L}}$ and $[d, \tilde{u}^k a] = k\tilde{u}^k a$ for all $\tilde{u}^k a \in \mathfrak{L}$.

(2) $\widehat{\mathfrak{L}} = \bigoplus_{j\in\mathbb{Z}} \widehat{\mathfrak{L}}_j$ is a $\mathbb{Z}$-graded Lie algebra, where $\widehat{\mathfrak{L}}_0 = \mathfrak{L}_0 \dotplus \Bbbk c \dotplus \Bbbk d$ and $\widehat{\mathfrak{L}}_j = \mathfrak{L}_j$ for all $j \in \mathbb{Z} \setminus \{0\}$.
(3) K_0 extends to a non-degenerate, symmetric, invariant bilinear form $\widehat{K_0}$ on $\widehat{\mathfrak{L}}$ by putting $\widehat{K_0}(c,d) = 1$ and

$$\widehat{K_0}(a,c) = \widehat{K_0}(a,d) = \widehat{K_0}(c,c) = \widehat{K_0}(d,d) = 0 \text{ for all } a \in \mathfrak{L}. \tag{4.15}$$

In particular, $\widehat{K_0}|_{\mathfrak{H}\times\mathfrak{H}} = K_0|_{\mathfrak{H}\times\mathfrak{H}}$ is non-degenerate.
(4) $[\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}]/\Bbbk c \cong \mathfrak{L}$ as $\mathbb{Z}$-graded Lie algebras.

The connection between affine Kac-Moody algebras and twisted loop algebras is now summarized in the following theorem.

Theorem 4.2.3.
The Lie algebra $\mathfrak{L}_0 = \mathfrak{g}_0^\sigma$ *is reductive. Let* $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$ *be a triangular decomposition and set* $\widehat{\mathfrak{H}} := \mathfrak{H} \oplus \Bbbk c \oplus \Bbbk d \subseteq \widehat{\mathfrak{L}}_0$*. There exist uniquely determined* $\Pi \subseteq (\mathfrak{H} \oplus \Bbbk d)^* \subseteq \widehat{\mathfrak{H}}^*$ *and* $\Pi^\vee \subseteq \mathfrak{H} \oplus \Bbbk c \subseteq \widehat{\mathfrak{H}}$ *as well as a generalized Cartan matrix* A *of affine type such that* $(\widehat{\mathfrak{L}}, \widehat{\mathfrak{H}}, \Pi, \Pi^\vee)$ *is a realization of* A *with the following properties.*

(1) *There exists a tuple* $s \in \mathbb{N}_0^{\dim(\mathfrak{H})+1} \setminus \{0\}$ *such that the standard* $\mathbb{Z}$*-grading of* $\widehat{\mathfrak{L}}$ *coincides with the grading of type* s *as a Kac-Moody algebra.*

(2) *Let* $\mathfrak{g}_0^\sigma = \mathfrak{n}'_+ \dotplus \mathfrak{H}' \dotplus \mathfrak{n}'_-$ *be another triangular decomposition and* $(\widehat{\mathfrak{L}}, \widehat{\mathfrak{H}}', \Pi', \Pi'^\vee)$ *be the associated realization of some generalized Cartan matrix* A' *of affine type. Then* A *and* A' *are equivalent and the automorphism of* $\widehat{\mathfrak{L}}$ *identifying the respective realizations is graded and identifies the respective triangular decompositions of* $\mathfrak{g}_0^\sigma$*.*

Proof. Everything except (2) is explicitly stated in [Kac90, Chapter 8]. For (2), note that the triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \dotplus \mathfrak{H} \dotplus \mathfrak{n}_-$ can be transformed into $\mathfrak{g}_0^\sigma = \mathfrak{n}'_+ \dotplus \mathfrak{H}' \dotplus \mathfrak{n}'_-$ by using an automorphism of $\mathfrak{g}_0^\sigma$ of the form $\varphi = \exp(\mathrm{ad}(a_1)) \dots \exp(\mathrm{ad}(a_k))$ for $a_1, \dots, a_k \in \mathfrak{g}_0^\sigma$. Indeed, this can be seen e.g. by translating the description of inner automorphisms from Subsection 4.1.7 to this context and using the well-known fact that Borel subalgebras of reductive Lie algebras can be transformed into each other by means of inner automorphisms. The automorphism φ extends to a graded automorphism of $\widehat{\mathfrak{L}}$ and is seen to satisfy the desired properties, after recalling the constructions in [Kac90, Chapter 8]. □

The rest of this section is dedicated to a detailed discussion of the consequences of this theorem.

4.2.3 Root structure of loop algebras. As in Theorem 4.2.3, let $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \dotplus \mathfrak{H} \dotplus \mathfrak{n}_-$ be a fixed triangular decomposition and

$$\left(\widehat{\mathfrak{L}}, \mathfrak{h} := \widehat{\mathfrak{H}} = \mathfrak{H} \dotplus \Bbbk c \dotplus \Bbbk d, \Pi = \{\alpha_0, \dots, \alpha_n\}, \Pi^\vee = \{\alpha_0^\vee, \dots, \alpha_n^\vee\}\right)$$

be the associated realization of $A = (a_{ij})_{i,j=0}^n$, where $n := \dim(\mathfrak{H})$. Write

$$\Phi = \Phi_+ \sqcup \Phi_- = \Phi^{\mathrm{re}} \sqcup \Phi^{\mathrm{im}} \tag{4.16}$$

for the associated polarized root system, $\left\{e_i^+, e_i^- \,\middle|\, i \in \{0, \dots, n\}\right\} \subseteq \mathfrak{L}$ be a set of Chevalley generators and let $s = (s_0, \dots, s_n) \in \mathbb{N}_0^{n+1} \setminus \{0\}$ be the tuple such that the natural $\mathbb{Z}$-grading of $\widehat{\mathfrak{L}}$ coincides with the grading of type s. Then for every $\alpha \in \Phi$ it holds that $\alpha(d) = \mathrm{ht}_s(\alpha) = \deg(a) \in \mathbb{Z}$ for any $a \in \widehat{\mathfrak{L}}_\alpha$ and $\alpha(c) = 0$. Note that

$$\mathfrak{L}_\alpha := \widehat{\mathfrak{L}}_\alpha = \left\{a \in \mathfrak{L}_{\alpha(d)} \,\middle|\, [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{H}\right\} \subseteq \mathfrak{L} \tag{4.17}$$

In particular, $\mathfrak{H}$ is called *Cartan subalgebra* of $\mathfrak{L}$ and

$$\mathfrak{B}_\pm := \mathfrak{H} \oplus \mathfrak{N}_\pm \subseteq \mathfrak{L}, \text{ where } \mathfrak{N}_\pm := \langle e_0^\pm, \dots, e_n^\pm \rangle_{\Bbbk\text{-alg}} \subseteq \mathfrak{L}, \tag{4.18}$$

are its *standard Borel subalgebras*.

Lemma 4.2.4.
The following results are true.
(1) $\alpha \in \Phi^{\mathrm{re}}$ *if and only if* $\alpha|_{\mathfrak{H}} \neq 0$.
(2) *There are unique relatively prime* $k_0, \ldots, k_n \in \mathbb{N}$ *such that*

$$\Phi^{\mathrm{im}} = \{\alpha \in \langle \alpha_0, \ldots, \alpha_n \rangle_{\mathbb{Z}} \mid \alpha|_{\mathfrak{H}} = 0\} = \langle \delta \rangle_{\mathbb{Z}}. \tag{4.19}$$

(3) K_0 *defines a positive definite bilinear form on* $\mathfrak{H}_{\mathbb{Q}} \coloneqq \sum_{\alpha \in \Phi} \mathbb{Q}\alpha|_{\mathfrak{H}} \subset \mathfrak{H}^*$, *where we consider* $\mathbb{Q}$ *as a subfield of* $\mathbb{k}$.

Proof. Part (1) follows from (2) since Φ is the disjoint union of Φ^{re} and Φ^{im}. By virtue of Lemma 4.1.3, there exist unique $k_0, \ldots, k_n \in \mathbb{N}$ such that $\Phi^{\mathrm{im}} = \langle \delta \rangle_{\mathbb{Z}}$ for $\delta \coloneqq \sum_{i=0}^{n} k_i \alpha_i$. It holds that $\delta|_{\mathfrak{H}} = 0$, since $\delta|_{\langle \Pi^\vee \rangle_{\mathbb{k}}} = 0$ and $\mathfrak{H} \subseteq \langle \Pi^\vee \rangle_{\mathbb{k}}$. The fact that $\alpha_0, \ldots, \alpha_n \in \mathfrak{h}^*$ are linearly independent and satisfy $\alpha_0(c) = \cdots = \alpha_n(c) = 0$ implies that $\delta|_{\mathfrak{H}} = 0$ is, up to scalar multiple, the only linear relation between $\alpha_0|_{\mathfrak{H}}, \ldots, \alpha_n|_{\mathfrak{H}}$. Let $\alpha = \sum_{i=0}^{n} k'_i \alpha_i$ satisfy $\alpha|_{\mathfrak{H}} = 0$ for some $k'_0, \ldots, k'_n \in \mathbb{Z}$. Then there exists $\lambda \in \mathbb{k}$ such that $k'_i = \lambda k_i$ for all $i \in \{0, \ldots, n\}$. Since $k'_0, \ldots, k'_n$ are integers and $k_0, \ldots, k_n$ are relatively prime positive integers, we can conclude that λ is an integer. Therefore, $\alpha \in \langle \delta \rangle_{\mathbb{Z}} = \Phi^{\mathrm{im}}$ proves Part *2*. For the third part of the assertion, we can observe that

$$K(h, h') = \mathrm{Tr}(\mathrm{ad}(h)\,\mathrm{ad}(h')) = \sum_{\alpha \in \Phi} \alpha(h)\alpha(h') \text{ holds for all } h, h' \in \mathfrak{H}. \tag{4.20}$$

Here, the second equality uses the fact that the trace of an operator coincides with the sum of its eigenvalues. This implies that $K_0(\lambda, \lambda) = \sum_{\alpha \in \Phi} K_0(\lambda, \alpha|_{\mathfrak{H}})^2$ for all $\lambda \in \sum_{\alpha \in \Phi} \mathbb{Q}\alpha|_{\mathfrak{H}}$, so it remains to argue that $K_0(\alpha|_{\mathfrak{H}}, \beta|_{\mathfrak{H}}) = \widehat{K}_0(\alpha, \beta) \in \mathbb{Q}$, for all $\alpha, \beta \in \Phi$. This is a consequence of [Kac90, Proposition 5.1]. □

Remark 4.2.5.
It is common to identify roots of $\mathfrak{L}$ with pairs $\mathfrak{H} \times \mathbb{Z}$ by applying $\widehat{\mathfrak{H}} \to \mathfrak{H} \times \mathbb{Z}$ defined by $\alpha \mapsto (\alpha|_{\mathfrak{H}}, \alpha(d))$. This results in a notion of roots that is intrinsic to $\mathfrak{L}$: it can be used to omit the reference to the affine Kac-Moody algebra $\widehat{\mathfrak{L}}$. However, we will not do so in order to make the application of the theory established in Section 4.1 more unambiguous.

4.2.4 Automorphisms of Loop algebras. For all $\lambda \in \mathbb{k}^\times$ the assignment $\mu_\lambda(a)(\tilde{u}) = a(\lambda\tilde{u})$ defines an element $\mu_\lambda \in \mathrm{Aut}_{\mathbb{k}\text{-alg}}(\mathfrak{L})$, which is not L-linear if $\lambda^m \neq 1$.

Lemma 4.2.6.
The following results are true.
(1) *The assignment* $\varphi \mapsto \tilde{\varphi}$, *where* $\varphi(a)(\tilde{u}) = \tilde{\varphi}(\tilde{u})a(\tilde{u})$ *for all* $a \in \mathfrak{L}$, *defines a bijection between* $\mathrm{Aut}_{L\text{-alg}}(\mathfrak{L})$ *and the set of invertible elements* $\psi(\tilde{u}) \in \mathrm{End}(\mathfrak{g})[\tilde{u}, \tilde{u}^{-1}]$ *with the properties* $\psi(\varepsilon\tilde{u}) = \sigma\psi(\tilde{u})\sigma^{-1}$ *and*

$$\psi(\tilde{u})[a(\tilde{u}), b(\tilde{u})] = [\psi(\tilde{u})a(\tilde{u}), \psi(\tilde{u})b(\tilde{u})] \text{ for all } a, b \in \mathfrak{L}. \tag{4.21}$$

(2) $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{L}) = \{\mu_\lambda, \omega\mu_\lambda \mid \lambda \in \Bbbk^\times\} \ltimes \mathrm{Aut}_{L\text{-alg}}(\mathfrak{L})$.
(3) *A mapping* $\varphi \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{L})$ *fixing* $\mathfrak{B}_+$ *or* $\mathfrak{B}_-$ *(see (4.18)) induces a unique* $\nu \in \mathrm{Aut}(A)$ *such that* $\varphi(\mathfrak{L}_\alpha) = \tilde{\nu}(\mathfrak{L}_\alpha)$ *for all* $\alpha \in \Phi$.

Proof. The proof of (1) is straight forward. Using the statements in Subsection 4.1.7 it can be seen that $\mathrm{Inn}(\mathfrak{L}) \subseteq \mathrm{Aut}_{L\text{-alg}}(\mathfrak{L})$. Furthermore, for any $\nu \in \mathrm{Aut}(A)$ we have $\tilde{\nu}((u-1)\mathfrak{L}) = (\lambda u - 1)\mathfrak{L}$ for some $\lambda \in \Bbbk^\times$; see 4.2.8.(3) below. This implies that $\mu_{\lambda^{-1}}\tilde{\nu} \in \mathrm{Aut}_{L\text{-alg}}(\mathfrak{L})$, proving (2). For (3), we may assume that $\varphi(\mathfrak{B}_+) = \mathfrak{B}_+$. Since $\mathfrak{N}_+ = [\mathfrak{B}_+, \mathfrak{B}_+]$, we see that φ also fixes $\mathfrak{N}_+$ and $\mathfrak{H}$. By [KW92, Lemma 1.29], there exists a linearly independent set $\{\beta_0, \dots, \beta_n\} \subseteq \Phi$ such that $\varphi(\mathfrak{L}_{\alpha_i}) = \mathfrak{L}_{\beta_i}$ for $i \in \{0, \dots, n\}$ and for all $\alpha \in \Phi^+$ exist $a_0, \dots, a_n \in \mathbb{N}_0$ such that $\alpha = \sum_{i=0}^n a_i\beta_i$ or $\alpha = -\sum_{i=0}^n a_i\beta_i$. The roots $\beta_0, \dots, \beta_n$ are positive, since φ fixes $\mathfrak{N}_+$. Thus, $\{\beta_0, \dots, \beta_n\} = \Pi$ has to hold and $\beta_i = \alpha_{\nu(i)}$ for $i \in \{0, \dots, n\}$ defines $\nu \in \mathrm{Aut}(A)$ with the desired properties. □

The following result is more specific to our applications.

Lemma 4.2.7.
Let $\mathfrak{A}_\pm \subseteq \mathfrak{L}$ *be subspaces such that* $\mathfrak{L} = \mathfrak{A}_+ + \mathfrak{A}_-$ *and there exists* $\varphi_\pm \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L})$ *with* $\varphi_\pm(\mathfrak{A}_\pm) \subseteq \mathfrak{B}_\pm$. *Then there exists* $\varphi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L})$ *such that* $\varphi(\mathfrak{A}_\pm) \subseteq \mathfrak{B}_\pm$.

Proof. Let $\varphi_- = b_- w b_+$ for $b_\pm \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L})$ satisfying $b_\pm(\mathfrak{B}_\pm) = \mathfrak{B}_\pm$ and $w(\mathfrak{H}) = \mathfrak{H}$; see Proposition 4.1.9. Then $\mathfrak{B}_+ + \varphi_-^{-1}(\mathfrak{B}_-) \supseteq \mathfrak{A}_+ + \mathfrak{A}_-$ implies $\mathfrak{B}_+ + w^{-1}(\mathfrak{B}_-) = \mathfrak{L}$. In particular, there exist $\{\beta_0, \dots, \beta_n\} \subseteq \Phi_+$ such that $w(\mathfrak{L}_{-\alpha_i}) = \mathfrak{L}_{-\beta_i}$ for all $i \in \{0, \dots, n\}$. By [KW92, Lemma 1.29], $\{\beta_0, \dots, \beta_n\} \subseteq \Phi_+$ are linearly independent and for all $\alpha \in \Phi^+$ we have $\alpha \in \sum_{i=0}^n \mathbb{N}_0\beta_i$, so $\Pi = \{\beta_0, \dots, \beta_n\}$. This implies that $w(\mathfrak{B}_-) = \mathfrak{B}_-$. Therefore, $\varphi := b_+\varphi_+$ has the desired property $\varphi(\mathfrak{A}_\pm) \subseteq \mathfrak{B}_\pm$. □

4.2.5 Some facts about subalgebras of Loop algebras. For any $S \subsetneq \{0, \dots, n\}$, the canonical projection $\mathfrak{P}_\pm^S$ of the standard parabolic $\mathfrak{p}_\pm^S \subseteq \widehat{\mathfrak{L}}$ to $\mathfrak{L}$ is called *standard parabolic subalgebra* of $\mathfrak{L}$ with respect to S.

Lemma 4.2.8.
Let $\mathfrak{A} \subseteq \mathfrak{L}$ *be a subalgebra.*
(1) *If* $\mathfrak{A}$ *contains a subspace* $\mathfrak{f}$ *of* $\mathfrak{H}$ *satisfying* $\mathfrak{f}^\perp \subseteq \mathfrak{f}$, *it holds that* $[\mathfrak{H}, \mathfrak{A}] \subseteq \mathfrak{A}$.
(2) *If* $\mathfrak{B}_\pm \subseteq \mathfrak{A}$, *there exists* $S \subsetneq \{0, \dots, n\}$ *such that* $\mathfrak{A} = \mathfrak{P}_\pm^S$.
(3) *If* $\mathfrak{A}$ *is an ideal,* $\mathfrak{A} = f\mathfrak{L}$ *for some* $f \in \mathbb{C}[u]$. *In particular, all maximal proper Lie ideals of* $\mathfrak{L}$ *are of the form* $(\lambda u - 1)\mathfrak{L}$ *for some* $\lambda \in \Bbbk^\times$.

Proof of (1). Assume there are distinct $\alpha_1, \alpha_2 \in \Phi \cup \{0\}$ such that $(\alpha_1 - \alpha_2)|_\mathfrak{f} = 0$, i.e. $(\alpha_1 - \alpha_2)|_\mathfrak{H}$ is in the kernel of $\mathfrak{h}^* \to \mathfrak{f}^*$, which coincides with the image of $\mathfrak{f}^\perp$ under the isomorphism $\mathfrak{H} \to \mathfrak{H}^*$ defined by K_0. Since $\mathfrak{f}$ is coisotropic, this results in $K_0((\alpha_1 - \alpha_2)|_\mathfrak{H}, (\alpha_1 - \alpha_2)|_\mathfrak{H}) = 0$. Lemma 4.2.4.(3) implies that $\alpha_1|_\mathfrak{H} = \alpha_2|_\mathfrak{H}$, so we can conclude that

$$\alpha_1|_\mathfrak{f} = \alpha_2|_\mathfrak{f} \implies \alpha_1|_\mathfrak{H} = \alpha_2|_\mathfrak{H} \text{ for all } \alpha_1, \alpha_2 \in \Phi \cup \{0\}. \tag{4.22}$$

For a subspace $\mathfrak{i} \subseteq \mathfrak{h} = \widehat{\mathfrak{H}}$ and $\lambda \in \mathfrak{i}^*$ we write

$$\mathfrak{L}_\lambda := \{a \in \mathfrak{L} \mid [h, a] = \lambda(h)a \text{ for all } h \in \mathfrak{i}\} = \bigoplus_{\substack{\alpha \in \Phi \cup \{0\} \\ \alpha|_{\mathfrak{i}} = \lambda}} \mathfrak{L}_\alpha. \tag{4.23}$$

Using [Kac90, Proposition 1.5] and $[\mathfrak{f}, \mathfrak{A}] \subseteq \mathfrak{A}$ we can write

$$\mathfrak{A} = \bigoplus_{\lambda \in \mathfrak{f}^*} (\mathfrak{L}_\lambda \cap \mathfrak{A}). \tag{4.24}$$

Therefore, (4.22) and (4.23) imply that for every $\lambda \in \mathfrak{f}^*$ such that $\mathfrak{L}_\lambda \neq \{0\}$ exists $\alpha \in \Phi \cup \{0\}$ such that $\mathfrak{L}_\lambda = \mathfrak{L}_{(\alpha|_{\mathfrak{H}})}$, where by definition (4.23) and under consideration of Lemma 4.2.4

$$\mathfrak{L}_{(\alpha|_{\mathfrak{H}})} = \bigoplus_{\alpha' \in \alpha + \Phi^{\mathrm{im}}} \mathfrak{L}_{\alpha'}. \tag{4.25}$$

This and (4.24) implies that $[\mathfrak{H}, \mathfrak{A}] \subseteq \mathfrak{A}$.

Proof of (2). Without loss of generality assume that $\mathfrak{A}$ contains $\mathfrak{B}_+$. The inclusion $\mathfrak{H} \subseteq \mathfrak{A}$ and [Kac90, Proposition 1.5] imply that $\mathfrak{A} = \bigoplus_{\alpha \in \Phi \cup \{0\}} \mathfrak{A} \cap \mathfrak{L}_{(\alpha|_{\mathfrak{H}})}$, where we use the notation from (4.25). Take $a \in \mathfrak{A} \cap \mathfrak{L}_{(\alpha|_{\mathfrak{H}})} \cap \mathfrak{N}_-$ for some $\alpha \in \Phi^{\mathrm{re}}$ and assume $\alpha \in \Phi$ was chosen in such a way that the $\mathfrak{L}_\alpha$-component of a is non-zero and $\alpha(d)$ is minimal with this property. Using $\mathfrak{L}_\delta \subseteq \mathfrak{B}_+ \subseteq \mathfrak{A}$ for δ from Lemma 4.2.4, $\dim(\mathfrak{L}_{\alpha'}) = 1$ for all $\alpha' \in \Phi^{\mathrm{re}}$ and applying [Hel78, Lemma X.5.5'.(iii)] iteratively we see that

$$\mathfrak{L}_{\alpha'} \subseteq \mathfrak{A} \cap \mathfrak{L}_{(\alpha|_{\mathfrak{H}})} \text{ for all } \alpha' \in \Phi \text{ such that } \alpha'|_{\mathfrak{H}} = \alpha|_{\mathfrak{H}} \text{ and } \alpha'(d) \geqslant \alpha(d). \tag{4.26}$$

Following the proof of [KW92, Lemma 1.5] we show that

$$\mathfrak{A} = \mathfrak{P}_S^+, \text{ where } S = \{\alpha \in \Pi \mid \mathfrak{L}_{-\alpha} \subseteq \mathfrak{A}\}. \tag{4.27}$$

Assume the contrary. Let $\gamma \notin \langle S \rangle_{\mathbb{Z}}$ be a negative root of minimal height such that there exists an element $a_1 \in \mathfrak{A}_{(\gamma|_{\mathfrak{H}})} \cap \mathfrak{N}_-$ with a non-zero $\mathfrak{L}_\gamma$-component $\overline{a}_1$. Then there exists $\alpha_j \in \Pi \setminus S$ such that $[e_j^+, \overline{a}_1] \neq 0$ and $\gamma + \alpha_j \in \langle S \rangle_{\mathbb{Z}}$; see [Hel78, Lemma X.5.5'.(iii)]. Note that equation (4.26) implies $\gamma \neq \alpha_j$. We can find $a_2 \in \mathfrak{L}_{-\gamma-\alpha_j} \subseteq \mathfrak{B}_+ \subseteq \mathfrak{A}$ such that

$$B([e_j^+, \overline{a}_1], a_2) \neq 0. \tag{4.28}$$

The invariance of the form B then gives $0 \neq [\overline{a}_1, a_2] \in \mathfrak{L}_{-\alpha_j}$. Applying (4.26) to $a := [a_1, a_2] \in \mathfrak{A}$ results in $\alpha_j \in S$ contradicting our choice of α_j.

Proof of Part (3). This is [Kac90, Lemma 8.6]. □

Recall that two vector subspaces V_1, V_2 of a vector space V are said to be *commensurable*, in symbols $V_1 \asymp V_2$, if $\dim((V_1 + V_2)/(V_1 \cap V_2)) < \infty$.

Proposition 4.2.9.
Let $\mathfrak{A} \subseteq \mathfrak{L}$ be a subalgebra commensurable to $\mathfrak{B}_\pm$.

(1) *There exists* $i \in \{0, \dots, n\}$ *and* $\varphi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L}) \subseteq \mathrm{Aut}_{L\text{-alg}}(\mathfrak{L})$ *such that* $\varphi(\mathfrak{A}) \subseteq \mathfrak{P}_{\pm}^{(i)}$*, where* $\mathfrak{P}_{\pm}^{(i)}$ *is the standard parabolic subalgebra of* $\mathfrak{L}$ *to* $\{0, \dots, n\} \setminus \{i\}$.
(2) *If* $\mathfrak{A}^{\perp} \subseteq \mathfrak{A}$ *it automatically holds that* $u^{\pm 1}\mathfrak{A} \subseteq \mathfrak{A}^{\perp} \subseteq \mathfrak{A}$*. In particular,* $\mathfrak{A}$ *is a free Lie algebra over* $\Bbbk[u^{\pm 1}]$.
(3) *If* $\mathfrak{A}^{\perp} \subseteq \mathfrak{L}$ *is a subalgebra containing* $\mathfrak{A}$*, there exists* $\varphi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L}) \subseteq \mathrm{Aut}_{L\text{-alg}}(\mathfrak{L})$ *such that* $\varphi(\mathfrak{A}) \subseteq \mathfrak{B}_{\pm}$*. Furthermore, if* $\mathfrak{A} \subseteq \mathfrak{A}^{\perp}$ *is an ideal and* $\mathfrak{A}^{\perp}/\mathfrak{A}$ *is solvable, we have* $\varphi(\mathfrak{A}^{\perp}) \subseteq \mathfrak{B}_{\pm}$.

Proof of (1). We may assume without loss of generality that $\mathfrak{A}$ is commensurable with $\mathfrak{B}_+$. Let $k \in \mathbb{N}$ be such that $\bigoplus_{j \geqslant mk} \mathfrak{L}_j \subseteq \mathfrak{A} \subseteq \bigoplus_{j \geqslant -mk} \mathfrak{L}_j$ and $\mathfrak{I} := u^{2k+1}\mathfrak{A}$. Obviously, $\mathfrak{I}$ is an ideal in $\mathfrak{A}$ and

$$\bigoplus_{j=(3k+1)m} \mathfrak{L}_j \subseteq \mathfrak{I} \subseteq \bigoplus_{j \geqslant (k+1)m} \mathfrak{L}_j. \tag{4.29}$$

The subspace $\widetilde{\mathfrak{A}} := \mathfrak{A} \dotplus \Bbbk c$ of $\widehat{\mathfrak{L}}$ is a subalgebra. Since $\mathfrak{I} \subseteq \bigoplus_{j \geqslant (k+1)m} \mathfrak{L}_j$ and $\mathfrak{A} \subseteq \bigoplus_{j \geqslant -km} \mathfrak{L}_j$, the construction of the commutator of $\widehat{\mathfrak{L}}$ in Subsection 4.2.2 implies that $[a, b]_{\mathfrak{L}} = [a, b]_{\widehat{\mathfrak{L}}}$ for all $a \in \mathfrak{I}$ and $b \in \mathfrak{A}$. Hence, $\mathfrak{I} \subset \widetilde{\mathfrak{A}}$ is an ideal with respect to the Lie bracket of $\widehat{\mathfrak{L}}$. By (4) of Proposition 4.1.12, there exists an inner automorphism ψ of $\widehat{\mathfrak{L}}$ and $i \in \{0, \dots, n\}$ such that $[\mathfrak{p}_+^{(i)}, \psi(\mathfrak{A})] \subseteq \widetilde{\mathfrak{p}}_+^{(i)}$, where $\mathfrak{p}_+^{(i)} := \mathfrak{P}_+^{(i)} \oplus \Bbbk c \oplus \Bbbk d$ denotes the positive standard parabolic subalgebra of $\widehat{\mathfrak{L}}$ to $\{0, \dots, n\} \setminus \{i\}$. Since the only non-trivial ideals of $\widehat{\mathfrak{L}}$ are $\mathfrak{L} \oplus \Bbbk c$ and $\Bbbk c$ (see Lemma 4.1.1), we can deduce that

$$\left\{ a \in \widehat{\mathfrak{L}} \,\middle|\, [a, \mathfrak{p}_+^{(i)}] \subseteq \mathfrak{p}_+^{(i)} \right\} = \mathfrak{p}_+^{(i)}. \tag{4.30}$$

It follows that $\psi(\widetilde{\mathfrak{A}}) \subseteq \mathfrak{p}_+^{(i)}$. Consider $\varphi \colon \mathfrak{L} \to \mathfrak{L}$ induced by ψ. The map φ is L-linear, since it is inner; see Lemma 4.2.6. Applying the canonical projection $\widehat{\mathfrak{L}} \to \mathfrak{L}$ to $\psi(\widetilde{\mathfrak{A}}) \subseteq \mathfrak{p}_+^{(i)}$, we end up with an inclusion $\varphi(\mathfrak{A}) \subseteq \mathfrak{P}_i^+$.

Proof of (2) We may assume without loss of generality that $\mathfrak{A}$ is commensurable with $\mathfrak{B}_+$. We know from (1) that there exists an $i \in \{0, \dots, n\}$ and $\varphi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L})$ such that $\varphi(\mathfrak{A}) \subseteq \mathfrak{P}_+^{(i)}$. Let $\Phi_i \subseteq \Phi$ be the subset such that $\mathfrak{P}_+^{(i)} = \bigoplus_{\alpha \in \Phi_i \cup \{0\}} \mathfrak{L}_\alpha$ and note that $\Phi_i \cap \Phi^- \subseteq \Phi^{\mathrm{re}}$ by Lemma 4.2.4. Let γ be the imaginary root such that $\mathfrak{L}_\gamma = u\mathfrak{H}$, then $u\mathfrak{P}_i^+ = \bigoplus_{\alpha \in \Phi_i} \mathfrak{L}_{\alpha+\gamma}$. We claim that

$$u\mathfrak{P}_+^{(i)} \subseteq \mathfrak{P}_+^{(i),\perp}, \text{ in other words } K_0\left(u\mathfrak{P}_+^{(i)}, \mathfrak{P}_+^{(i)}\right) = \{0\}. \tag{4.31}$$

Since $B(\mathfrak{L}_\alpha, \mathfrak{L}_{\beta+\gamma}) \neq \{0\}$ if and only if $\alpha = -\beta - \gamma$, it suffices to show that $\alpha \in \Phi_i$ implies $-\alpha - \gamma \notin \Phi_i$. We have two cases to consider:

- If $\alpha \in \Phi_i \cap \Phi^{\mathrm{im}}$ or $\alpha = 0$, then $-\alpha - \gamma$ is imaginary and $\Phi_i \cap \Phi^+ \subseteq \Phi^{\mathrm{re}}$ implies that $-\alpha - \gamma \notin \Phi_i$.
- If $\alpha \in \Phi_i \cap \Phi^{\mathrm{re}}$ and we assume that $-\alpha - \gamma \in \Phi_i$, then there exist $a \in \mathfrak{L}_\alpha, b \in \mathfrak{L}_{-\alpha-\gamma}$ such that $0 \neq [a, b] \in \mathfrak{L}_{-\gamma} \cap \mathfrak{P}_i^+$ (see [Hel78, Chapter X, Lemma 5.5']), so $-\gamma \in \Phi_i$. This contradicts $\Phi_i \cap \Phi^+ \subseteq \Phi^{\mathrm{re}}$ and we can conclude that $-\alpha - \gamma \notin \Phi_i$.

Thus, the claim (4.31) is proven. Using $\varphi(\mathfrak{A}) \subseteq \mathfrak{P}_i^+$ and the fact that $(\cdot)^{\perp}$ is inclusion reversing yields

$$u\mathfrak{A} \subseteq u\varphi^{-1}\left(\mathfrak{P}_+^{(i)}\right) \subseteq \varphi^{-1}\left(\mathfrak{P}_+^{(i),\perp}\right) \subseteq \mathfrak{A}^{\perp} \subseteq \mathfrak{A}. \tag{4.32}$$

Here, we used that the L-linearity of φ^{-1} implies

$$u\varphi^{-1}\left(\mathfrak{P}_+^{(i)}\right) = \varphi^{-1}\left(u\mathfrak{P}_+^{(i)}\right) \subseteq \varphi^{-1}\left(\mathfrak{P}_+^{(i),\perp}\right), \tag{4.33}$$

and φ^{-1} respects K_0 (see Theorem 4.1.11), so $\varphi^{-1}\left(\mathfrak{P}_+^{(i),\perp}\right) = \varphi^{-1}\left(\mathfrak{P}_+^{(i)}\right)^\perp \subseteq \mathfrak{A}^\perp$. That $\mathfrak{A}$ is free is a direct consequence of the fact that it is torsion-free.

Proof of (3) We may assume without loss of generality that $\mathfrak{A}$ is commensurable with $\mathfrak{B}^+$. Therefore, there exists $k \in \mathbb{N}$ such that $\bigoplus_{j\geqslant k}\mathfrak{L}_j \subseteq \mathfrak{A} \subseteq \bigoplus_{j\geqslant -k}\mathfrak{L}_j$, so $\bigoplus_{j\geqslant k+1}\mathfrak{L}_j \subseteq \mathfrak{A}^\perp \subseteq \bigoplus_{j\geqslant -k-1}\mathfrak{L}_j$ holds and we see that $\mathfrak{A}^\perp$ is also commensurable with $\mathfrak{B}_+$. Part *1.* states that there exists $\psi_1 \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L})$ and $i \in \{0,\dots,n\}$ such that $\psi_1(\mathfrak{A}) \subseteq \psi_1(\mathfrak{A}^\perp) = \varphi(\mathfrak{A})^\perp \subseteq \mathfrak{P}_+^{(i)}$, where we used that ψ_1 respects K_0 by virtue of Theorem 4.1.11. Put $\mathfrak{I} := \mathfrak{P}_+^{(i),\perp} \cap \psi_1(\mathfrak{A})$ and observe that

$$\psi_1(\mathfrak{A})/\mathfrak{I} \cong (\psi_1(\mathfrak{A}) + \mathfrak{P}_+^{(i),\perp})/\mathfrak{P}_+^{(i),\perp} \subseteq \psi_1(\mathfrak{A})^\perp/\mathfrak{P}_+^{(i),\perp} \subseteq \mathfrak{s}_i := \mathfrak{P}_+^{(i)}/\mathfrak{P}_+^{(i),\perp}, \tag{4.34}$$

where $\mathfrak{s}_i$ is the finite-dimensional semi-simple Lie algebra with Chevalley generators $\{e_j^+, e_j^- \mid j \in \{0,\dots,r\} \setminus \{i\}\}$. It can be seen that the bilinear form on $\mathfrak{s}_i$ induced be K_0 coincides with the Killing form of $\mathfrak{s}_i$. Therefore,

$$K_0\left(\psi_1(\mathfrak{A}) + \mathfrak{P}_+^{(i),\perp}, \psi_1(\mathfrak{A}) + \mathfrak{P}_+^{(i),\perp}\right) = \{0\} \tag{4.35}$$

and Cartan's criterion for solvability implies that

$$\psi_1(\mathfrak{A})/\mathfrak{I} \cong \left(\psi_1(\mathfrak{A}) + \mathfrak{P}_+^{(i),\perp}\right)/\mathfrak{P}_+^{(i),\perp} \text{ is solvable.} \tag{4.36}$$

Note that $\widetilde{\mathfrak{A}} := \psi_1(\mathfrak{A}) + \Bbbk c \subseteq \widehat{\mathfrak{L}}$ is a subalgebra, $\widetilde{\mathfrak{A}} \cap \mathfrak{P}_+^{(i),\perp} = \mathfrak{I} \subseteq \mathfrak{N}^+$ is a subalgebra of finite codimension and an ideal in $\widetilde{\mathfrak{A}}$ and (4.36) implies that $\widetilde{\mathfrak{A}}/\mathfrak{I}$ is solvable. Since $\mathfrak{I} \subseteq \mathfrak{N}_+$ is a subalgebra of finite codimension, [KW92, Lemma 2.4] and [PV83, Theorem 3] implies that there exists $\widetilde{\psi}_2 \in \mathrm{Inn}_{\mathrm{ad}}(\widehat{\mathfrak{L}})$ (in the notation of Subsection 4.1.7) such that $\widetilde{\psi}_2(\widetilde{\mathfrak{A}}) \subseteq \mathfrak{b}_+ := \mathfrak{B}_+ \oplus \Bbbk c \oplus \Bbbk d$. The automorphism $\widetilde{\psi}_2$ induces $\psi_2 \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L})$ such that $\psi_2\psi_1(\mathfrak{A}) \subseteq \mathfrak{B}_+$. Putting $\varphi := \psi_2\psi_1$ concludes the first part of the proof.

If $\mathfrak{A}^\perp/\mathfrak{A}$ is solvable, $\psi_1(\mathfrak{A}^\perp)/\mathfrak{P}_+^{(i),\perp}$ is too, since $(\psi_1(\mathfrak{A}) + \mathfrak{P}_+^{(i),\perp})/\mathfrak{P}_+^{(i),\perp}$ is solvable and $\mathfrak{P}_+^{(i),\perp} \subseteq \psi_1(\mathfrak{A}^\perp)$. Repeating the argument from (4.36) onward for $\psi_1(\mathfrak{A})$ replaced by $\psi_1(\mathfrak{A}^\perp)$ concludes the proof. □

4.2.6 The case that σ is an outer automorphism.

Let the Cartan matrix of $\mathfrak{g}$ have a Dynkin diagram of type X_n from Figure 4.1. Henceforth, fix a triangular decomposition $\mathfrak{g} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$, $\nu \in \mathrm{Aut}(X_n)$ and assume that that $\sigma = \tilde{\nu} \in \mathrm{Out}(\mathfrak{g})$ in the notation of Subsection 4.1.7.

Proposition 4.2.10.
The fixed point algebra $\mathfrak{g}_0^\sigma$ of $\mathfrak{g}$ is simple, $\mathfrak{g}_0^\sigma = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+$ is a triangular decomposition, where $\mathfrak{H} := \tilde{\mathfrak{h}} \cap \mathfrak{g}_0^\sigma, \mathfrak{n}_\pm = \mathfrak{g}_0^\sigma \cap \tilde{\mathfrak{n}}_\pm$, and $\mathfrak{g}_1^\sigma$ is an irreducible $\mathfrak{g}_0^\sigma$-module isomorphic to $\mathfrak{g}_{-1}^\sigma$.

Let $\mathring{\Pi} = \left\{\mathring{\alpha}_1, \dots, \mathring{\alpha}_n\right\}$ be the set of simple roots of $\mathfrak{g}_0^\sigma$ with respect to the triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+$, $\left\{\mathring{e}_j^+, \mathring{e}_j^- \mid j \in \{1, \ldots, n\}\right\}$ be the associated Chevalley generators, $\mathfrak{h} = \widehat{\mathfrak{H}} = \mathfrak{H} \oplus \Bbbk c \oplus \Bbbk d \subseteq \widehat{\mathfrak{L}}$ and $\theta = \sum_{i=1}^n \ell_i \, \mathring{\alpha}_i$ be the highest weight of $\mathfrak{L}_1 \cong \mathfrak{g}_1^\sigma \cong \mathfrak{g}_{-1}^\sigma \cong \mathfrak{L}_{-1}$. Define $\alpha_0, \ldots, \alpha_n \in \mathfrak{h}^*$ by

$$\alpha_0|_{\mathfrak{H}} = -\theta, \alpha_i|_{\mathfrak{H}} = \mathring{\alpha}_i, \alpha_0(c) = 0 = \alpha_i(c) \text{ and } \alpha_0(d) = 1, \alpha_i(d) = 0 \tag{4.37}$$

for all $i \in \{1, \ldots, n\}$. Chose $\mathring{e}_j^\pm \in \mathfrak{L}_{\pm\alpha_0}$ in such a way that $\alpha_0\left(\left[\mathring{e}_j^+, \mathring{e}_j^-\right]\right) = 2$ and write $\alpha_i^\vee := \left[\mathring{e}_j^+, \mathring{e}_j^-\right]_{\widehat{\mathfrak{L}}} \in \mathfrak{h}$.

Proposition 4.2.11.
The quadruple $(\widehat{\mathfrak{L}}, \mathfrak{h}, \Pi := \{\alpha_0, \ldots, \alpha_n\}, \Pi^\vee := \{\alpha_0^\vee, \ldots, \alpha_n^\vee\})$ is the realization of the Cartan matrix with Dynkin diagram of type $X_n^{(m)}$ mentioned in Theorem 4.2.3 with Chevalley generators defined by $e_0^\pm = \tilde{u}^{\pm 1} \, \mathring{e}_i^\pm$ and $e_i^\pm = \mathring{e}_i^\pm$ for $i \in \{1, \ldots, n\}$. Furthermore, $k_0 = 1, k_1 = \ell_1, \ldots, k_n = \ell_n$ coincide with the integers given in Lemma 4.2.4.(2) and $s = (1, 0, \ldots, 0)$.

4.2.7 Classification of finite order automorphisms The structure theory of twisted loop algebras inherited by the theory of Kac-Moody algebras can be used to classify the finite order automorphisms up to conjugacy; see [Kac90, Section 8.6].

Theorem 4.2.12.
The following results are true.
(1) $\left\{\mathring{e}_0^+, \ldots, \mathring{e}_n^+\right\} \subseteq \mathfrak{g}$ *generates* $\mathfrak{g}$ *as a Lie algebra.*
(2) *For all $s' = (s_0', \ldots, s_n') \in \mathbb{N}_0^{n+1}$ there exists a unique $\sigma_{(s';m)} \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ of order $m' = m \sum_{i=0}^n k_i s_i'$, called* automorphism of type $(s'; m)$, *determined by $\sigma'(e_i^+(1)) = \varepsilon^{s_i'} e_i^+(1)$. Furthermore, the image of $\sigma_{(s';m)}$ under the projection* $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g}) \to \mathrm{Out}(\mathfrak{g})$ *is σ.*
(3) *The mapping $s' \mapsto \sigma_{(s';m)}$ defines a bijection between $\mathbb{N}_0^{n+1} \setminus \{0\}$ and the conjugacy classes of finite order automorphisms of $\mathfrak{g}$ with representative of order m in* $\mathrm{Out}(\mathfrak{g})$.

Note that the given definition of an automorphism of type (s', m) depends only on $\nu \in \mathrm{Aut}(X_n)$ and the triangular decomposition of $\mathfrak{g}$.

4.2.8 Regrading. Let $\sigma' \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ have finite order m' and a representative of order m in $\mathrm{Out}(\mathfrak{g})$. Then σ' is conjugate to $\sigma_{(s',m)}$ for some $s' \in \mathbb{N}_0^{r+1} \setminus \{0\}$, so $\mathfrak{L}(\mathfrak{g}, \sigma')$ is canonically isomorphic to $\mathfrak{L}^{s'} := \mathfrak{L}(\mathfrak{g}, \sigma_{(s',m)})$ as $\mathbb{Z}$-graded Lie algebras. This Lie algebra can be related back to $\mathfrak{L}$ via a process which we call *regrading.*

Lemma 4.2.13.
The mapping $\tilde{u}^{\pm s_i} e_i^\pm \mapsto \tilde{u}^{\pm s_i'} e_i^\pm$ defines an Lie algebra isomorphism $\mathfrak{L} \to \mathfrak{L}^{s'}$.

From Theorem 4.2.3 we now that $\widehat{\mathfrak{L}}$ (resp. $\widehat{\mathfrak{L}}^{s'}$) is the Kac-Moody algebra of the generalized Cartan matrix of type $X_n^{(m)}$ equipped with the $\mathbb{Z}$-grading of type s (resp

s'). Thus, the isomorphism from Lemma 4.2.13 can simply be interpreted as equipping the Kac-Moody algebra of $X_n^{(m)}$ with another $\mathbb{Z}$-grading. This motivates us to call said isomorphism *regrading of type* s'.

5

Lie bialgebras

As usual, $\Bbbk$ is a field of characteristic 0 throughout this chapter.

5.1 Basic definitions and properties

In this chapter, we give a brief overview of the theory of Lie bialgebras, which is fundamentally linked to the notion of r-matrices and especially their skew-symmetry property. We begin by introducing Lie coalgebras, i.e. algebraic objects dual to Lie algebras, in Subsection 5.1.1. A Lie bialgebra consists of a vector space equipped with compatible Lie algebra and Lie coalgebra structures; see Subsection 5.1.2.

5.1.1 The category of Lie coalgebras. For a vector space V over $\Bbbk$ the linear maps $\tau_V : V \otimes V \to V \otimes V$ and $\mathrm{Alt}_V : V \otimes V \otimes V \to V \otimes V \otimes V$ are defined by $\tau_V(v_1 \otimes v_2) \coloneqq v_2 \otimes v_1$ and

$$\mathrm{Alt}_V(v_1 \otimes v_2 \otimes v_3) \coloneqq v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2 \tag{5.1}$$

for all $v_1, v_2, v_3 \in V$. A vector space $\mathfrak{c}$ over $\Bbbk$ equipped with a linear map $\delta_{\mathfrak{c}} : \mathfrak{c} \to \mathfrak{c} \otimes \mathfrak{c}$ is called *Lie coalgebra* if said map is skew-symmetric, i.e. $\delta_{\mathfrak{c}}(a) + \tau_{\mathfrak{c}}\delta_{\mathfrak{c}}(a) = 0$ and satisfies the *co-Jacobi identity* $\mathrm{Alt}_{\mathfrak{c}}((\delta_{\mathfrak{c}} \otimes \mathrm{id}_{\mathfrak{c}})\delta_{\mathfrak{c}}(a)) = 0$ for all $a \in \mathfrak{c}$. If there is no ambiguity, the index $\mathfrak{c}$ in $\delta_{\mathfrak{c}}$ is dropped. A *Lie coalgebra morphism* $\varphi : \mathfrak{c} \to \mathfrak{c}'$ is a linear map satisfying $(\varphi \otimes \varphi)\delta_{\mathfrak{c}} = \delta_{\mathfrak{c}'}\varphi$. We have defined the *category of Lie coalgebras*.

Lemma 5.1.1.
The following results are true.

(1) *For a Lie coalgebra $\mathfrak{c}$ the restriction of $\delta_{\mathfrak{c}}^* : (\mathfrak{c} \otimes \mathfrak{c})^* \to \mathfrak{c}^*$ to $\mathfrak{c}^* \otimes \mathfrak{c}^* \subseteq (\mathfrak{c} \otimes \mathfrak{c})^*$ defines a Lie algebra structure on $\mathfrak{c}^*$. This procedure defines a contravariant functor from the category of Lie coalgebras to the category of Lie algebras. This functor restricts to an equivalence of categories on the respective subcategories of finite-dimensional objects.*

(2) *For a Lie algebra $\mathfrak{a}$ the linear map $\delta_{\mathfrak{a}^*} \coloneqq [,]_{\mathfrak{a}}^* : \mathfrak{a}^* \to (\mathfrak{a} \otimes \mathfrak{a})^*$ defines a Lie coalgebra on $\mathfrak{a}^*$ if and only if this map takes values in $\mathfrak{a}^* \otimes \mathfrak{a}^* \subseteq (\mathfrak{a} \otimes \mathfrak{a})^*$.*

Proof of (1). By definition $[\lambda_1, \lambda_2]_{\mathfrak{c}^*}(a) \coloneqq \delta_{\mathfrak{c}}^*(\lambda_1 \otimes \lambda_2)(a) = (\lambda_1 \otimes \lambda_2)\delta_{\mathfrak{c}}(a)$, for all $\lambda_1, \lambda_2, \lambda_3 \in \mathfrak{c}^*$ and $a \in \mathfrak{c}$. Therefore, we can deduce that

$$[\lambda_1, \lambda_2]_{\mathfrak{c}^*} + [\lambda_2, \lambda_2]_{\mathfrak{c}^*} = (\lambda_1 \otimes \lambda_2)(\delta_{\mathfrak{c}}(a) + \tau_{\mathfrak{c}}\delta_{\mathfrak{c}}(a)) = 0. \tag{5.2}$$

It holds that

$$[[\lambda_1, \lambda_2]_{\mathfrak{c}^*}, \lambda_3]_{\mathfrak{c}^*}(a) = [(\lambda_1 \otimes \lambda_2)\delta_{\mathfrak{c}}, \lambda_3]_{\mathfrak{c}^*}(a) = (\lambda_1 \otimes \lambda_2 \otimes \lambda_3)(\delta_{\mathfrak{c}} \otimes \mathrm{id}_{\mathfrak{g}})\delta_{\mathfrak{c}}(a), \tag{5.3}$$

so the Jacobi identity in $\lambda_1, \lambda_2, \lambda_3$ equals $(\lambda_1 \otimes \lambda_2 \otimes \lambda_3)\mathrm{Alt}_{\mathfrak{c}}((\delta_{\mathfrak{c}} \otimes \mathrm{id}_{\mathfrak{c}})\delta_{\mathfrak{c}}(a)) = 0$ by permuting $\lambda_1, \lambda_2, \lambda_3$ appropriately in each term. The fact that $\mathfrak{c} \mapsto \mathfrak{c}^*$ is functorial is clear, since $(\cdot)^*$ is an contravariant endofunctor of the category of $\Bbbk$-vector spaces. The assertion about finite dimensional coalgebras follows from Part *2.* and the facts that the restriction of $(\cdot)^*$ to the category of finite dimensional vector spaces is its own quasi-inverse and commutes with tensor products.

Proof of (2) The " $\implies$ " direction is clear from the definition of a Lie coalgebra, so it remains to prove " $\impliedby$ ". Let $\lambda \in \mathfrak{a}^*$ and $a_1, a_2, a_3 \in \mathfrak{a}$. By definition, $\delta_{\mathfrak{a}^*} = [,]_{\mathfrak{a}}^*$ satisfies $\delta_{\mathfrak{a}^*}(\lambda)(a_1 \otimes a_2) = \lambda([a_1, a_2]_{\mathfrak{a}})$ and the assumption that $\delta_{\mathfrak{a}^*}$ takes values in $\mathfrak{a}^* \otimes \mathfrak{a}^*$ means that $\delta_{\mathfrak{a}^*}(\lambda) = \sum_{j=1}^k \lambda_j^{(1)} \otimes \lambda_j^{(2)}$ for some $\lambda_j^{(i)} \in \mathfrak{a}^*$, where $i \in \{1, 2\}, j \in \{1, \dots, k\}$, so $\lambda([a_1, a_2]_{\mathfrak{a}}) = \sum_{j=1}^k \lambda_j^{(1)}(a_1)\lambda_j^{(2)}(a_2)$. Combined we see that

$$\begin{aligned}(\delta_{\mathfrak{a}^*}(\lambda) + \tau_{\mathfrak{a}^*}\delta_{\mathfrak{a}^*}(\lambda))(a_1 \otimes a_2) &= \sum_{j=1}^k (\lambda_j^{(1)}(a_1)\lambda_j^{(2)}(a_2) + \lambda_j^{(1)}(a_2)\lambda_j^{(2)}(a_1)) \\ &= \lambda([a_1, a_2]_{\mathfrak{a}^*} + [a_2, a_1]_{\mathfrak{a}^*}) = 0\end{aligned} \tag{5.4}$$

holds. Furthermore, we can calculate

$$\begin{aligned}(\delta_{\mathfrak{a}^*} \otimes \mathrm{id}_{\mathfrak{a}*})\delta_{\mathfrak{a}^*}(\lambda)(a_1 \otimes a_2 \otimes a_3) &= \sum_{j=1}^k \delta_{\mathfrak{a}^*}(\lambda_j^{(1)})(a_1 \otimes a_2)\lambda_j^{(2)}(a_3) \\ &= \sum_{j=1}^k \lambda_j^{(1)}([a_1, a_2]_{\mathfrak{a}})\lambda_j^{(2)}(a_3) = \lambda([[a_1, a_2]_{\mathfrak{a}}, a_3]_{\mathfrak{a}}).\end{aligned}$$

Therefore, we can deduce that

$$\begin{aligned}&\mathrm{Alt}_{\mathfrak{a}^*}((\delta_{\mathfrak{a}^*} \otimes 1)\delta_{\mathfrak{a}^*}(\lambda))(a_1 \otimes a_2 \otimes a_3) = (\delta_{\mathfrak{a}^*} \otimes 1)\delta_{\mathfrak{a}^*}(\lambda)\mathrm{Alt}(a_1 \otimes a_2 \otimes a_3) \\ &= \lambda([[a_1, a_2], a_3] + [[a_2, a_3], a_1] + [[a_3, a_1], a_2]) = 0\end{aligned} \tag{5.5}$$

holds. This concludes the proof, since λ, a_1, a_2 and a_3 where chosen arbitrarily. □

5.1.2 The category of Lie bialgebras. For a Lie algebra $\mathfrak{a}$ over $\Bbbk$ the space $\mathfrak{a}^{\otimes n}$ is an $\mathfrak{a}$-module, where the action is defined by

$$a \cdot (a_1 \otimes \cdots \otimes a_n) \coloneqq \sum_{j=1}^n a_1 \otimes \cdots \otimes a_{j-1} \otimes [a, a_j] \otimes a_{j+1} \otimes \cdots \otimes a_n, \tag{5.6}$$

for all $a, a_1, \dots, a_n \in \mathfrak{a}$. In particular, we have a linear map $\partial \colon \mathfrak{a}^{\otimes n} \to \mathrm{Hom}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$ defined by $\partial t(x) = x \cdot t$ for all $x \in \mathfrak{a}$ and $t \in \mathfrak{a}^{\otimes n}$. Elements in the image of ∂ are called 1-*coboundaries* and elements in the kernel of ∂ are called $\mathfrak{a}$-*invariant*. Observe that

$$[a \otimes 1 + 1 \otimes a, t] \coloneqq a \cdot t = \partial t(a) \tag{5.7}$$

hold for all $a \in \mathfrak{a}, t \in \mathfrak{a} \otimes \mathfrak{a}$.

Remark 5.1.2.
Let $t \in \mathfrak{a}^{\otimes(n+1)}$ be $\mathfrak{a}$-invariant and write $t = \sum_{i_1,\dots,i_n \in I} t_{i_1,\dots,i_n} \otimes b_{i_1} \otimes \cdots \otimes b_{i_n}$ for some basis $\{b_i\}_{i \in I}$, where I is some (possibly infinite) index set. Note that the sum is finite. Then $\mathfrak{i} := \langle t_{i_1,\dots,i_n} \mid i_1, \dots, i_n \in I \rangle_{\Bbbk} \subseteq \mathfrak{a}$ is a finite-dimensional subspace. The identity $b_k \cdot t = 0$ implies that

$$[b_k, t_{i_1,\dots,i_n}] = \sum_{\ell=1}^{n} \sum_{j \in I} C^{i_\ell}_{jk} t_{i_1,\dots,i_{\ell-1},j,i_{\ell+1},\dots,i_n} \in \mathfrak{i}, \tag{5.8}$$

where $\{C^i_{jk}\}_{i,j,k \in I} \subseteq \Bbbk$ is determined by $[b_j, b_k] = \sum_{i \in I} C^i_{jk} b_i$. Observe that all sums here are finite. This shows that $\mathfrak{i} \subseteq \mathfrak{a}$ is a finite-dimensional ideal.

A Lie algebra $\mathfrak{a}$ is called *Lie bialgebra* if it is also a Lie coalgebra and the Lie cobracket $\delta_{\mathfrak{a}}$ is a *1-cocycle*, i.e. $\delta_{\mathfrak{a}}([a_1, a_2]_{\mathfrak{a}}) = a_1 \cdot \delta_{\mathfrak{a}}(a_2) - a_2 \cdot \delta_{\mathfrak{a}}(a_1)$ holds for all $a_1, a_2 \in \mathfrak{a}$. In this case, $\delta_{\mathfrak{a}}$ is called *Lie bialgebra cobracket*. A linear map between two Lie bialgebras is a *Lie bialgebra morphism* if it is a morphism of both Lie algebra and Lie coalgebra structures. We have defined the *category of Lie bialgebras* over $\Bbbk$.

Remark 5.1.3.
Under consideration of Lemma 5.1.1, the functor $\mathfrak{a} \mapsto \mathfrak{a}^*$ defines a contravariant autoequivalence of the category of finite-dimensional Lie bialgebras over $\Bbbk$.

5.2 Manin triples

In this section, we will discuss how Lie bialgebras can be encoded in terms of Lie algebras using the notion of Manin triples. Roughly speaking, the Lie algebra and Lie coalgebra structure of some Lie bialgebra can be described by two separate Lie algebras and the compatibility between them can be understood using an enveloping Lie algebra equipped with an invariant, non-degenerate, symmetric bilinear form.

5.2.1 Definition. Recall that a subspace W of a vector space V over $\Bbbk$ equipped with a symmetric bilinear form B is called *isotropic* (resp. *coisotropic*, resp. *Lagrangian*) if $W^\perp := \{w \in W \mid B(v, w) = 0 \text{ for all } v \in V\}$ satisfies $W \subseteq W^\perp$ (resp. $W^\perp \subseteq W$, resp. $W^\perp = W$).

A *Manin triple* $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$ consists of a Lie algebra $\mathfrak{m}$ equipped with an invariant, non-degenerate, symmetric bilinear form $B\colon \mathfrak{m} \times \mathfrak{m} \to \Bbbk$ and isotropic subalgebras $\mathfrak{m}_\pm \subseteq \mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$. If the choice of bilinear form on $\mathfrak{m}$ is clear from the context, we omit B in the datum of the Manin triple, i.e. we write $(\mathfrak{m}, \mathfrak{m}_+, \mathfrak{m}_-)$ instead of $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$.

Lemma 5.2.1.
Let $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$ be a Manin triple.
(1) *Then $\mathfrak{m}_\pm = \mathfrak{m}_\pm^\perp$, i.e. $\mathfrak{m}_\pm$ are Lagrangian.*
(2) *The canonical injection $B^{\mathrm{a}}\colon \mathfrak{m} \to \mathfrak{m}^*$ induced by B restricts to injective maps*

$B^{\mathrm{a}}_{\pm} \colon \mathfrak{m}_{\pm} \to \mathfrak{m}^*_{\mp}$.

Proof. Let $a = a_+ + a_- \in \mathfrak{m}_+^{\perp} \supseteq \mathfrak{m}_+$ be an arbitrary element, where $a_\pm \in \mathfrak{m}_\pm$ are uniquely determined. By definition, $0 = B(a,b) = B(a_-,b)$ for all $b \in \mathfrak{m}_+$ and $B(a_-,b) = 0$ for all $b \in \mathfrak{m}_- \subseteq \mathfrak{m}_-^{\perp}$. We conclude $a_- = 0$, since B is non-degenerate, so $a = a_+ \in \mathfrak{m}_+$. Since $\mathfrak{m}_+$ and $\mathfrak{m}_-$ play symmetric roles, this proves (1). For (2), notice that the kernel of the composition B^{a} with the restriction $\mathfrak{m}^* \to \mathfrak{m}^*_{\pm}$ is easily seen to be $\mathfrak{m}_{\pm}^{\perp} = \mathfrak{m}_{\pm}$, hence the isomorphism theorem and $\mathfrak{m}/\mathfrak{m}_{\pm} = \mathfrak{m}_{\mp}$ conclude the proof. □

5.2.2 The classical double. Historically, Manin triples originated in [Dri83] as a construction associated with Lie bialgebras: the so-called classical double. To understand said construction, we first recall the following easy construction from linear algebra. Let V, W be two vector space over $\Bbbk$ and $B \colon V \times W \to \Bbbk$ be bilinear. Then there is an induced bilinear pairing $B^{\otimes k} \colon V^{\otimes k} \times W^{\otimes k} \to \Bbbk$ defined by

$$B^{\otimes k}(v_1 \otimes \dots \otimes v_k, w_1 \otimes \dots \otimes w_k) \coloneqq B(v_1, w_1) \dots B(v_k, w_k), \tag{5.9}$$

for all $v_1, \dots, v_k \in V, w_1, \dots w_k \in W$. Assume now that B is non-degenerate, i.e. the canonical maps $V \to W^*$ and $W \to V^*$ induced by B are injective. Then $B^{\otimes k}$ is non-degenerate for all $k \in \mathbb{N}$.

Proposition 5.2.2.
Let $\mathfrak{a}$ be a Lie bialgebra over $\Bbbk$. There exists a unique Lie algebra structure on $\mathfrak{D}(\mathfrak{a}) \coloneqq \mathfrak{a} \oplus \mathfrak{a}^*$ such that both $\mathfrak{a}$ and $\mathfrak{a}^*$ are subalgebras of $\mathfrak{D}(\mathfrak{a})$ and the canonical pairing

$$B(a_1 + \lambda_1, a_2 + \lambda_2) = \lambda_1(a_1) + \lambda_2(a_2) \qquad \forall a_1, a_2 \in \mathfrak{a}, \lambda_1, \lambda_2 \in \mathfrak{a}^* \tag{5.10}$$

is invariant. In this case, $\mathfrak{D}(\mathfrak{a})$ is called the classical double *of $\mathfrak{a}$ and $((\mathfrak{D}(\mathfrak{a}), B), \mathfrak{a}, \mathfrak{a}^*)$ is a Manin triple satisfying $B^{\otimes 2}(\delta_{\mathfrak{a}}(a), \lambda_1 \otimes \lambda_2) = B(a, [\lambda_1, \lambda_2]_{\mathfrak{a}*})$ for all $a \in \mathfrak{a}$ and $\lambda_1, \lambda_2 \in \mathfrak{a}^*$.*

Proof. Note that for all $a \in \mathfrak{a}$ and $\lambda_1, \lambda_2 \in \mathfrak{a}^*$

$$B(\delta_{\mathfrak{a}}(a), \lambda_1 \otimes \lambda_2) = (\lambda_1 \otimes \lambda_2)\delta(a) = [\lambda_1, \lambda_2]_{\mathfrak{a}^*}(a) = B(a, [\lambda_1, \lambda_2]_{\mathfrak{a}^*}) \tag{5.11}$$

holds by definition. Furthermore, $\mathfrak{a}^{\perp} = \mathfrak{a}$ and $\mathfrak{a}^{*,\perp} = \mathfrak{a}^*$ is also obvious. Let $[,] \coloneqq [,]_{\mathfrak{D}(\mathfrak{a})} \colon \mathfrak{D}(\mathfrak{a}) \times \mathfrak{D}(\mathfrak{a}) \to \mathfrak{D}(\mathfrak{a})$ be the unique skew-symmetric map such that

$$[,]|_{\mathfrak{a}\times\mathfrak{a}} = [,]_{\mathfrak{a}}, [,]|_{\mathfrak{a}^*\times\mathfrak{a}^*} = [,]_{\mathfrak{a}^*} \text{ and } [a, \lambda] = -\lambda\,\mathrm{ad}(a) + (\lambda \otimes \mathrm{id}_{\mathfrak{a}})\delta_{\mathfrak{a}}(a) \tag{5.12}$$

for all $a \in \mathfrak{a}, \lambda \in \mathfrak{a}^*$. It is easy to see that $[,]$ is the only map satisfying $B([d_1, d_2], d_3) = B(d_1, [d_2, d_3])$ for all $d_1, d_2, d_3 \in \mathfrak{D}(\mathfrak{a})$.

It remains to prove that $[,]$ satisfies the Jacobi-identity. In the following $\delta = \delta_{\mathfrak{a}}$, $a_1, a_2 \in \mathfrak{a}$ and $\lambda_1, \lambda_2 \in \mathfrak{a}^*$. It suffices to show that the Jacobi identity in

a_1, a_2, λ_1 (resp. $a_2, \lambda_1, \lambda_2$) is satisfied, which will be denoted by $\mathrm{Jac}(a_1, a_2, \lambda_1)$ (resp. $\mathrm{Jac}(a_2, \lambda_1, \lambda_2)$). We have

$$\begin{aligned} &B([a_1,[a_2,\lambda_1]],\lambda_2) = B(-[a_1,\lambda_1 \operatorname{ad}(a_2)],\lambda_2) + B((\mathrm{id}_{\mathfrak{a}} \otimes \operatorname{ad}(a_1))\delta(a_2),\lambda_1 \otimes \lambda_2) \\ &= B(a_1,-[\lambda_1 \operatorname{ad}(a_2),\lambda_2]) + B((\mathrm{id}_{\mathfrak{a}} \otimes \operatorname{ad}(a_1))\delta(a_2),\lambda_1 \otimes \lambda_2) \\ &= -B((\operatorname{ad}(a_2) \otimes \mathrm{id}_{\mathfrak{g}})\delta(a_1),\lambda_1 \otimes \lambda_2) + B((\mathrm{id}_{\mathfrak{a}} \otimes \operatorname{ad}(a_1))\delta(a_2),\lambda_1 \otimes \lambda_2). \end{aligned} \tag{5.13}$$

Similarly, one shows that

$$B([[a_1,\lambda_1],a_2],\lambda_2) = -B((\mathrm{id}_{\mathfrak{a}} \otimes \operatorname{ad}(a_2))\delta(a_1),\lambda_1 \otimes \lambda_2) + B((\operatorname{ad}(a_1) \otimes \mathrm{id}_{\mathfrak{a}})\delta(a_2),\lambda_1 \otimes \lambda_2)$$

and combined this results in

$$B^{\otimes 2}(a_1 \cdot \delta(a_2) - a_2 \cdot \delta(a_1), \lambda_1 \otimes \lambda_2) = B([a_1,[a_2,\lambda_1]] + [[a_1,\lambda_1],a_2],\lambda_2). \tag{5.14}$$

Furthermore, a similar argument shows

$$B^{\otimes 2}(a_1 \cdot \delta(a_2) - a_2 \cdot \delta(a_1), \lambda_1 \otimes \lambda_2) = B(a_1,[[a_2,\lambda_1],\lambda_2] + [\lambda_1,[a_2,\lambda_2]]). \tag{5.15}$$

Comparing these expressions to

$$B^{\otimes 2}(\delta([a_1,a_2]),\lambda_1 \otimes \lambda_2) = B([a_1,a_2],[\lambda_1,\lambda_2]) = B([[a_1,a_2],\lambda_1],\lambda_2) = B(a_1,[a_2,[\lambda_1,\lambda_2]]),$$

we can conclude

$$\begin{aligned} &B^{\otimes 2}(\delta([a_1,a_2]) - a_1 \cdot \delta(a_2) + a_2 \cdot \delta(a_1), \lambda_1 \otimes \lambda_2) \\ &= B(\mathrm{Jac}(a_1,a_2,\lambda_1),\lambda_2) = B(a_1,\mathrm{Jac}(a_2,\lambda_1,\lambda_2)). \end{aligned} \tag{5.16}$$

The left-hand side of this equation vanishes since δ is a 1-cocycle and the non-degeneracy of B concludes the proof. □

5.2.3 Manin triples determining Lie bialgebra structures. The duality of Lie cobracket and Lie bracket under the bilinear form of a Manin triple can be used to generalize the notion of classical double and provides the possibility to construct Lie bialgebra structures from Manin triples. This will be used in Section 5.4.

Proposition 5.2.3.
Let $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$ be a Manin triple, $S \subseteq \mathfrak{m}_+$ generate $\mathfrak{m}_+$ as Lie algebra and assume there exists a linear map $\delta \colon \mathfrak{m}_+ \to \mathfrak{m}_+ \otimes \mathfrak{m}_+$ such that

$$B^{\otimes 2}(\delta(a), w_1 \otimes w_2) = B(a,[w_1,w_2]_{\mathfrak{m}}) \text{ for all } a \in S, w_1, w_2 \in \mathfrak{m}_-. \tag{5.17}$$

Then δ is a 1-cocycle if and only if (5.17) holds for all $a \in \mathfrak{m}_+$ and in this case δ is unique with this property and defines a Lie bialgebra structure on $\mathfrak{m}_+$.

Proof. Let $\pi_-\colon \mathfrak{m} \to \mathfrak{m}_-$ be the canonical projection. Repeating the derivation of (5.14) in the proof of Proposition 5.2.2, where $\lambda \mapsto \lambda\,\mathrm{ad}(a)$ is replaced by $w \mapsto \pi_-[a,w]$, results in

$$\begin{aligned}&B^{\otimes 2}(a_1\cdot\delta(a_2) - a_2\cdot\delta(a_1), w_1\otimes w_2) = B([a_1,[a_2,w_1]] + [[a_1,w_1],a_2],w_2)\\ &= B([[a_1,a_2],w_1],w_2) = B([a_1,a_2],[w_1,w_2])\end{aligned} \tag{5.18}$$

for all $a_1,a_2 \in M \coloneqq \{a \in \mathfrak{m} \mid (5.17) \text{ holds for } a\}$ and $w_1,w_2 \in \mathfrak{m}_+$. Note that if

$$\delta \text{ is a 1-cocycle} \iff (5.17) \text{ holds for all } a \in S \tag{5.19}$$

is true and these equivalent conditions are satisfied, then δ is uniquely determined by (5.17) since B is non-degenerate. Furthermore, in this case δ defines a Lie bialgebra structure since the skew-symmetry and co-Jacobi identity can be checked using (5.17) and the invariance of B. It remains to prove (5.19).

" $\implies$ " If δ is a 1-cocycle, (5.18) takes the form

$$\begin{aligned}B^{\otimes 2}(\delta([a_1,a_2]),w_1\otimes w_2) &= B^{\otimes 2}(a_1\cdot\delta(a_2) - a_2\cdot\delta(a_1), w_1\otimes w_2)\\ &= B([a_1,a_2],[w_1,w_2]),\end{aligned} \tag{5.20}$$

for all $a_1,a_2\in M$. In particular, M is a Lie subalgebra of $\mathfrak{m}_+$ containing S. Since S generates $\mathfrak{m}_+$, we have $M = \mathfrak{m}_+$.

" $\impliedby$ " If (5.17) is satisfied for all $a \in \mathfrak{m}_+$, we can combine

$$B^{\otimes 2}(\delta([a_1,a_2]),w_1\otimes w_2) = B([a_1,a_2],[w_1,w_2]) \tag{5.21}$$

with the equality (5.18) and the Jacobi identity in $\mathfrak{m}$ to obtain

$$B^{\otimes 2}(\delta([a_1,a_2]) - a_1\cdot\delta(a_2) + a_2\cdot\delta(a_1), w_1\otimes w_2) = 0, \tag{5.22}$$

for all $a_1,a_2\in\mathfrak{m}_+$ and $w_1,w_2\in\mathfrak{m}_-$. Therefore, δ is an 1-cocycle since B is non-degenerate. □

We say that a Lie bialgebra cobracket $\delta_{\mathfrak{m}_+}$ on $\mathfrak{m}_+$ is *determined* by a Manin triple $((\mathfrak{m},B),\mathfrak{m}_+,\mathfrak{m}_-)$ if

$$B^{\otimes 2}(\delta_{\mathfrak{m}_+}(a),w_1\otimes w_2) = B(a,[w_1,w_2]_{\mathfrak{m}_-}) \text{ for all } a\in\mathfrak{m}_+, w_1,w_2\in\mathfrak{m}_- \tag{5.23}$$

holds. The classical double construction shows that any Lie bialgebra is determined by some Manin triple.

5.2.4 Manin triples and isomorphisms of Lie bialgebras. It is possible to construct isomorphisms between Lie bialgebra structures by relating Manin triples defining said structures.

Lemma 5.2.4.
The following statements are true.
(1) *Let* $\mathfrak{m}_+$ *and* $\mathfrak{m}'_+$ *be two Lie bialgebras determined by Manin triples* $(\mathfrak{m},\mathfrak{m}_+,\mathfrak{m}_-)$

and $(\mathfrak{m}', \mathfrak{m}'_+, \mathfrak{m}'_-)$ *respectively. If there exists an orthogonal Lie algebra isomorphism* $\varphi\colon \mathfrak{m} \to \mathfrak{m}'$ *satisfying* $\varphi(\mathfrak{m}_\pm) \subseteq \mathfrak{m}'_\pm$*, then* $\varphi|_{\mathfrak{m}_+}\colon \mathfrak{m}_+ \to \mathfrak{m}'_+$ *is an isomorphism of Lie bialgebras and we say that the Manin triples* $(\mathfrak{m}, \mathfrak{m}_+, \mathfrak{m}_-)$ *and* $(\mathfrak{m}', \mathfrak{m}'_+, \mathfrak{m}'_-)$ *are* isomorphic*.*

(2) *Every Manin triple consisting of finite dimensional algebras arises up to isomorphism as the classical double of a Lie bialgebra.*

Proof. The proof of Part (1) is straight forward. For (2), note that for any Manin triple $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$ we have the canonical isomorphism $B^{\mathrm{a}}_-\colon \mathfrak{m}_- \to \mathfrak{m}_+^*$. This identifies $[,]^*_{\mathfrak{m}_-}$ with a Lie bialgebra cobracket on $\mathfrak{m}_+$ which is determined by $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$. It is now easy to see that $\varphi\colon \mathfrak{m} \to \mathfrak{D}(\mathfrak{a})$ defined by $\varphi|_{\mathfrak{m}_+} = \mathrm{id}_{\mathfrak{m}_+}$ and $\varphi|_{\mathfrak{m}_-} = \mathrm{B}^{\mathrm{a}}_-$ results in the desired isomorphism of Manin triples. □

5.3 Twisting Lie bialgebra structures

Following [KS02], we examine in Subsection 5.3.1 the condition for which a 1-cocycle defined by adding a 1-coboundary to a given Lie bialgebra cobracket defines a new Lie bialgebra structure. In Subsection 5.3.2, we will see that this procedure can be described using Manin triples if the original Lie bialgebra structure is determined by a Manin triple.

5.3.1 Classical Twists. For a Lie algebra $\mathfrak{a}$ we denote by $\mathrm{U}(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$ and by $(\cdot)^{ij}\colon \mathfrak{a} \otimes \mathfrak{a} \to \mathrm{U}(\mathfrak{a}) \otimes \mathrm{U}(\mathfrak{a}) \otimes \mathrm{U}(\mathfrak{a})$ for $ij \in \{12, 13, 23\}$ the linear maps defined by $t^{12} = t \otimes 1$, $t^{13} = a_1 \otimes 1 \otimes a_2$ and $t^{23} = 1 \otimes t$, where $t = a_1 \otimes a_2 \in \mathfrak{a} \otimes \mathfrak{a}$. We can calculate e.g.

$$[(a_1 \otimes a_2)^{13}, (b_1 \otimes b_2)^{23}] = a_1 \otimes b_1 \otimes [a_2, b_2] \in \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a}, \tag{5.24}$$

for all $a_1, a_2, b_1, b_2 \in \mathfrak{a}$, where the brackets denote the commutator of the unital associative $\Bbbk$-algebra $\mathrm{U}(\mathfrak{a}) \otimes \mathrm{U}(\mathfrak{a}) \otimes \mathrm{U}(\mathfrak{a})$. This shows that for all $t \in \mathfrak{a} \otimes \mathfrak{a}$

$$\mathrm{CYB}(t) := [t^{12}, t^{13}] + [t^{12}, t^{23}] + [t^{13}, t^{23}] \tag{5.25}$$

defines a quadratic map $\mathrm{CYB}\colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a}$.

Proposition 5.3.1.
Let $\mathfrak{a}$ *be a Lie bialgebra with Lie bialgebra cobracket* $\delta = \delta_{\mathfrak{a}}$*. Then* $\delta^t := \delta + \partial t$ *defines a Lie bialgebra structure on* $\mathfrak{a}$ *for some* $t \in \mathfrak{a} \otimes \mathfrak{a}$ *if and only if both* $t + \tau(t) \in \mathfrak{a} \otimes \mathfrak{a}$ *and* $\mathrm{CYB}(t) - \mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})t) \in \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a}$ *are* $\mathfrak{a}$*-invariant.*

Proof. Clearly, δ^t is an 1-cocycle for all $t \in \mathfrak{a} \otimes \mathfrak{a}$ and it is skew-symmetric if and only if $a \cdot (t + \tau(t)) = 0$ for all $a \in \mathfrak{a}$. Hence, it remains to be show that $\mathrm{Alt}\left((\delta^t \otimes \mathrm{id}_{\mathfrak{a}}) \circ \delta^t\right)(a) = 0$ for all $a \in \mathfrak{a}$ if and only if $\mathrm{CYB}(t) - \mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})t) \in \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a}$ is $\mathfrak{a}$-invariant. Since $\mathfrak{a}$ equipped with δ is a Lie bialgebra, we have $\mathrm{Alt}\left((\delta \otimes 1) \circ \delta\right) = 0$. In the following we write $t = \sum_{i=1}^k a_i \otimes b_i$, fix an element $a \in \mathfrak{a}$ and write $\delta(a) = \sum_{i=1}^n x_i \otimes y_i$. We proceed in three steps.

Step 1. $\mathrm{Alt}((\partial t \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a)) = -a \cdot \mathrm{CYB}(t)$. A direct computation shows that $a \cdot [t^{12}, t^{13}]$ equals

$$\sum_{i,j=1}^{k} ([a,[a_i,a_j]] \otimes b_i \otimes b_j + [a_i,a_j] \otimes [a,b_i] \otimes b_j + [a_i,a_j] \otimes b_i \otimes [a,b_j])$$
$$= \sum_{i,j=1}^{k} ([[a,a_i],a_j] \otimes b_i \otimes b_j + [a_i,[a,a_j]] \otimes b_i \otimes b_j + [a_i,a_j] \otimes [a,b_i] \otimes b_j$$
$$+[a_i,a_j] \otimes b_i \otimes [a,b_j]) = [(a\cdot t)^{12}, t^{13}] + [t^{12}, (a \cdot t)^{13}].$$

Similarly, we can derive the identities

$$a \cdot [t^{12}, t^{23}] = [(a\cdot t)^{12}, t^{23}] + [t^{12}, (a\cdot t)^{23}] \text{ and } a \cdot [t^{13}, t^{23}] = [(a\cdot t)^{13}, t^{23}] + [t^{13}, (a\cdot t)^{23}].$$

By definition

$$(\partial t \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a) = \sum_{i,j=1}^{k} ([[a,a_i],a_j] \otimes b_j \otimes b_i + a_j \otimes [[a,a_i],b_j] \otimes b_i$$
$$+[a_i,a_j] \otimes b_j \otimes [a,b_i] + a_j \otimes [a_i,b_j] \otimes [a,b_i]) = [(a\cdot t)^{13}, t^{12}] + [(a\cdot t)^{23}, t^{12}].$$

Write $(x_1 \otimes x_2 \otimes x_3)^{ijk} = x_i \otimes x_j \otimes x_k$ for $\{i,j,k\} = \{1,2,3\}$ and $x_1, x_2, x_3 \in \mathfrak{a}$. Then it is easy to see that for skew-symmetric $s_1, s_2 \in \mathfrak{a} \otimes \mathfrak{a}$ we have $[s_1^{12}, s_2^{13}]^{231} = [s_1^{13}, s_2^{23}]$. This and similar identities combined with the above equalities shows

$$\mathrm{Alt}((\partial t \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a)) = -a \cdot \mathrm{CYB}(t). \tag{5.26}$$

Step 2. $\mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a) + (\partial t \otimes \mathrm{id}_{\mathfrak{a}})\delta(a)) = a \cdot \mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})t)$. We can calculate

$$(\delta \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a) = \sum_{i=1}^{k}\Big(\delta([a,a_i]) \otimes b_i + \delta(a_i) \otimes [a,b_i]\Big) = \sum_{i=1}^{k}\Big((a \circ \delta(a_i) - a_i \cdot \delta(a)) \otimes b_i$$
$$+ \delta(a_i) \otimes [a,b_i]\Big) = (\delta \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a) = a \cdot (\delta \otimes \mathrm{id}_{\mathfrak{a}})(t) - \sum_{i=1}^{k}(a_i \cdot \delta(a)) \otimes b_i.$$

Then we have the following identities

$$\begin{aligned}(\partial t \otimes \mathrm{id}_{\mathfrak{a}})\delta(a) &= \sum_{j=1}^{\ell}\sum_{i=1}^{n}\Big([x_j,a_i] \otimes b_i \otimes y_j + a_i \otimes [x_j,b_i] \otimes y_j\Big) \\ \sum_{i=1}^{n}\Big(a_i \circ \delta(a)\Big) \otimes b_i &= \sum_{j=1}^{m}\sum_{i=1}^{n}\Big([a_i,x_j] \otimes y_j \otimes b_i + x_j \otimes [a_i,y_j] \otimes b_i\Big).\end{aligned} \tag{5.27}$$

Since t is skew-symmetric, i.e. $t = -\sum_{i=1}^{k} b_i \otimes a_i$, we have

$$\sum_{j=1}^{\ell}\sum_{i=1}^{k}[a_i,x_j] \otimes y_j \otimes b_i = \sum_{j=1}^{\ell}\sum_{i=1}^{k}[x_j,b_i] \otimes y_j \otimes a_i.$$

and consequently $\mathrm{Alt}\Big(\sum_{j=1}^{\ell}\sum_{i=1}^{k} a_i \otimes [x_j,b_i] \otimes y_j - [a_i,x_j] \otimes y_j \otimes b_i\Big) = 0$. Similarly, the skew-symmetry of $\delta(a)$ yields $\mathrm{Alt}\left(\sum_{j=1}^{m}\sum_{i=1}^{n}[x_j,a_i] \otimes b_i \otimes y_j - x_j \otimes [a_i,y_j] \otimes b_i\right) = 0$. Summarized, we arrive at $\mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a) + (\partial t \otimes \mathrm{id}_{\mathfrak{a}})\delta(a)) = a \cdot \mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})t)$.

Step 3. *Conclusion.* Combining steps 1 and 2, we see that $\mathrm{Alt}((\delta^t \otimes 1)\delta^t(a))$ equals

$$\begin{aligned}&\mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})\delta(a)) + \mathrm{Alt}((\delta \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a) + (\delta_t \otimes \mathrm{id}_{\mathfrak{a}})\delta(a)) + \mathrm{Alt}((\partial t \otimes \mathrm{id}_{\mathfrak{a}})\partial t(a)) \\ &= 0 + a \cdot \mathrm{Alt}(\delta \otimes \mathrm{id}_{\mathfrak{a}}(t)) - a \cdot \mathrm{CYB}(t) = a \cdot (\mathrm{Alt}(\delta \otimes \mathrm{id}_{\mathfrak{a}}(t)) - \mathrm{CYB}(t)).\end{aligned}$$

This concludes the proof. □

For a Lie cobracket δ of a Lie bialgebra $\mathfrak{a}$, a tensor $t \in \mathfrak{a} \otimes \mathfrak{a}$ is called *classical twist* of δ if it is skew-symmetric, in symbols $t + \tau(t) = 0$, and $\mathrm{Alt}(\delta \otimes \mathrm{id}_{\mathfrak{a}}(t)) = \mathrm{CYB}(t)$ holds. In particular, the Lie bialgebra cobrackets of the form $\delta^t = \delta + \partial t$ for a classical twist t of δ are called *twists of* δ.

5.3.2 Twisting Manin triples determining Lie bialgebra structures. The procedure of twisting translates nicely to Manin triples. Before stating the formal result, recall that two subspaces V_1, V_2 of a vector space V over $\Bbbk$ are called commensurable, written $V_1 \asymp V_2$, if $\dim((V_1 + V_2)/(V_1 \cap V_2)) < \infty$. Furthermore, recall that for any Manin triple $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$, the bilinear form B induces injective linear maps $\widetilde{B}_\pm \colon \mathfrak{m}_\pm \otimes \mathfrak{m}_\pm \to \mathrm{Hom}(\mathfrak{m}_\mp, \mathfrak{m}_\pm)$ by $a \otimes b \mapsto B(b, -)a$. The following result from [AM21] is a generalization of the methods from [Sto91b; Sto91c].

Theorem 5.3.2.
Let δ be a Lie bialgebra cobracket on a Lie algebra $\mathfrak{m}_+$, determined by a Manin triple $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$. Then

$$t \longmapsto \mathfrak{m}_-^t := \left\{ \widetilde{B}_+(t)(w) - w \,\middle|\, w \in \mathfrak{m}_- \right\} \tag{5.28}$$

defines a bijection of classical twists $t \in \mathfrak{m}_+ \otimes \mathfrak{m}_+$ of δ and subalgebras $\mathfrak{w} \subseteq \mathfrak{m}$ such that $\mathfrak{w} \asymp \mathfrak{m}_-$ and $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{w})$ is a Manin triple. Moreover, for any classical twist t of $\mathfrak{m}_+$, the Manin triple $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-^t)$ determines the Lie bialgebra cobracket δ^t.

Proof. Let $t = \sum_{i=1}^k a_i \otimes b_i \in \mathfrak{m}_+ \otimes \mathfrak{m}_+$ be an arbitrary tensor, write $T := \widetilde{B}_+(t)$, and $\mathfrak{m}_-^t := \{Tw - w \mid w \in \mathfrak{m}_-\} \subseteq \mathfrak{m}$. The proof is conducted in five steps.

Step 1. $\mathfrak{m}_-^t \asymp \mathfrak{m}_-$. The definition of T implies that $\dim(\mathrm{Im}(T)) < \infty$, so $\dim(\mathfrak{m}_-/\operatorname{Ker}(T)) < \infty$. Combining this with $\operatorname{Ker}(T) = \mathfrak{m}_- \cap \mathfrak{m}_-^t$ and $\mathfrak{m}_- + \mathfrak{m}_-^t \subseteq \mathfrak{m}_- + \mathrm{Im}(T)$ yields $\dim((\mathfrak{w} + \mathfrak{m}_-)/(\mathfrak{w} \cap \mathfrak{m}_-)) < \infty$.

Step 2. $\mathfrak{m}_-^t$ *is isotropic if and only if* $t + \tau_{\mathfrak{m}}(t) = 0$. For all $w_1, w_2 \in \mathfrak{m}_-$ we have

$$\begin{aligned} B(Tw_1 - w_1, Tw_2 - w_2) &= -B(Tw_1, w_2) - B(w_1, Tw_2) \\ &= -\left(\sum_{i=1}^k B(b_i, w_1) B(a_i, w_2) + B(b_i, w_2) B(a_i, w_1) \right). \end{aligned} \tag{5.29}$$

Therefore, $t = -\tau_{\mathfrak{m}}(t)$ is equivalent to the fact that $\mathfrak{m}_-^t$ is isotropic.

Step 3. $\mathfrak{m}_-^t$ *is an isotropic subalgebra if and only if t is a classical twist of δ.*

For every $w_1, w_2, w_3 \in \mathfrak{m}_-$ we have

$$\begin{aligned} &B^{\otimes 3}(w_1 \otimes w_2 \otimes w_3, [t^{12}, t^{13}]) = \sum_{i,j=1}^{k} B(w_1, [a_i, a_j])B(w_2, b_i)B(w_3, b_j) \\ &= \sum_{i,j=1}^{k} B(w_1, [B(w_2, b_i)a_i, B(w_3, b_j)a_j]) = B(w_1, [Tw_2, Tw_3]), \end{aligned} \tag{5.30}$$

and similarly we obtain

$$\begin{aligned} B^{\otimes 3}(w_1 \otimes w_2 \otimes w_3, [t^{12}, t^{23}]) &= B(w_2, [Tw_3, Tw_1]) \text{ and} \\ B^{\otimes 3}(w_1 \otimes w_2 \otimes w_3, [t^{13}, t^{23}]) &= B(w_3, [Tw_1, Tw_2]). \end{aligned} \tag{5.31}$$

Since $\mathfrak{m}_+$ is determined by $(\mathfrak{m}, \mathfrak{m}_+, \mathfrak{m}_-)$, we see for $i, j, k \in \{1, 2, 3\}$

$$B(w_i \otimes w_j \otimes w_k, (\delta \otimes \mathrm{id}_{\mathfrak{m}_+})t) = B(Tw_k, [w_i, w_j]). \tag{5.32}$$

Combined, this shows

$$B(w_1 \otimes w_2 \otimes w_3, \mathrm{Alt}((\delta \otimes 1)t) - \mathrm{CYB}(t)) = B([Tw_1 - w_1, Tw_2 - w_2], Tw_3 - w_3).$$

The non-degeneracy of B implies that $\mathfrak{m}_-^t$ is an isotropic subalgebra of $\mathfrak{m}$ if and only if t is a classical twist of $\delta := \delta_{\mathfrak{m}_+}$.

Step 4. *For an isotropic subalgebra $\mathfrak{w} \subseteq \mathfrak{m}$ such that $\mathfrak{w} \asymp \mathfrak{m}_-$ and $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{w})$ is a Manin triple, there exists a classical twist t of $\mathfrak{m}_+$ such that $\mathfrak{w} = \mathfrak{m}_-^t$.* For any $w \in \mathfrak{m}_-$ we have a unique splitting $w_+ + v$, for some $w_+ \in \mathfrak{m}_+$ and $v \in \mathfrak{w}$. Let $T\colon \mathfrak{m}_- \to \mathfrak{m}_+$ be defined by $T(w) := w_+$. Then by construction $\mathfrak{w} = \{Tw - w \mid w \in L_-\}$ and $\dim((\mathfrak{w} + \mathfrak{m}_-)/(\mathfrak{w} \cap \mathfrak{m}_-)) < \infty$ implies that $\dim(\mathrm{Im}(T)) < \infty$. Since $\mathfrak{w}$ is isotropic, we have $B(Tw_1, w_2) = -B(w_1, Tw_2)$ for all $w_1, w_2 \in \mathfrak{m}_-$. Note that B gives a non-degenerate pairing between the finite-dimensional spaces $\mathfrak{m}_-/\operatorname{Ker}(T)$ and $\mathrm{Im}(T)$. Let $\{Tw_i\}_{i=1}^n$ be a basis for $\mathrm{Im}(T)$ and $\{v_i + \operatorname{Ker}(T)\}_{i=1}^n$ be its dual basis for $\mathfrak{m}_-/\ker(T)$. Then

$$\sum_{i=1}^{n} B(w_k, -Tv_i)Tw_i = \sum_{i=1}^{n} B(Tw_k, v_i)Tw_i = Tw_k, \tag{5.33}$$

for all $k \in \{1, \dots, n\}$. Since T is completely determined by its action on $\{w_i\}_{i=1}^n$, we have the equality $T = -\sum_{i=1}^n B(Tv_i, -)Tw_i$. Setting $t := -\sum_{i=1}^n Tw_i \otimes Tv_i$, we see from steps 2 and 3 that t is a classical twist of δ such that $\mathfrak{w} = \mathfrak{m}_-^t$.

Step 5. *The Manin triple $(\mathfrak{m}, \mathfrak{m}_+, \mathfrak{m}_-^t)$ determines δ^t.* For all $w_1, w_2 \in \mathfrak{m}_-$ and $a \in \mathfrak{m}_+$ we have

$$\begin{aligned} &B^{\otimes 2}(\delta^t(a), (Tw_1 - w_1) \otimes (Tw_2 - w_2)) = B^{\otimes 2}(\delta(a), w_1 \otimes w_2) + B^{\otimes 2}(a \cdot t, w_1 \otimes w_2) \\ &= B(a, [w_1, w_2]) + \sum_{i=1}^{k} (B([a, a_i], w_1)B(b_i, w_2) + B(a_i, w_1)B([a, b_i], w_2)) \\ &= B(a, [w_1, w_2]) + B([a, Tw_2], w_1) - B([a, Tw_1], w_2) = B(a, [Tw_1 - w_1, Tw_2 - w_2]), \end{aligned}$$

where we used the fact that $\mathfrak{m}_+$ is Lagrangian in the first and last equality. □

5.4 Examples of Lie bialgebras

We now look at some examples of Lie bialgebras. First, we discuss the process of defining Lie bialgebra cobrackets using structure constants, which we call constant r-matrices in Subsection 5.4.1. Then, in Subsection 5.4.2, we explain how formal r-matrices can be viewed as an infinite dimensional counterpart to this approach. Finally, in subsections 5.4.3-5.4.5, we present certain infinite-dimensional Lie bialgebras which will be central to the remainder of this work.

5.4.1 Quasitriangular Lie bialgebras and constant r-matrices. A basic but important example that reveals the effectiveness of the twisting method is the case of the trivial Lie bialgebra cobracket, i.e. $\delta = 0$. We have the following special case of Proposition 5.3.1.

Lemma 5.4.1.
For a Lie algebra $\mathfrak{a}$ and a tensor $t \in \mathfrak{a} \otimes \mathfrak{a}$ the linear map $\delta := \partial t$ defines a Lie bialgebra structure on $\mathfrak{a}$ if and only if both $t + \tau(t)$ and $\mathrm{CYB}(t)$ are $\mathfrak{a}$-invariant.

Let $\delta = \partial r$ be a Lie bialgebra cobracket on a Lie algebra $\mathfrak{a}$ for some $r \in \mathfrak{a} \otimes \mathfrak{a}$. We call δ *quasitriangular* if r solves the *constant classical Yang-Baxter equation* $\mathrm{CYB}(r) = 0$ and *triangular* if additionally $r + \tau(r) = 0$. In this case, r is called *constant r-matrix*. Examples of quasitriangular Lie bialgebras structures can be constructed from finite-dimensional Manin triples.

Lemma 5.4.2.
Let $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$ be a Manin triple such that $\dim(\mathfrak{m}) = 2d < \infty$ and let $\{b_i^\pm\}_{i=1}^d \subseteq \mathfrak{m}_\pm$ be bases such that $B(b_i^+, b_j^-) = \delta_{ij}$. Then $r = \sum_{i=1}^d b_i^+ \otimes b_i^-$ is a constant r-matrix and $\delta = \partial r$ defines a Lie bialgebra structure on $\mathfrak{m}$ restricting to a Lie bialgebra structure on $\mathfrak{m}_+$ determined by $((\mathfrak{m}, B), \mathfrak{m}_+, \mathfrak{m}_-)$.

Proof. Let $[b_i^\pm, b_j^\pm] = \sum_{k=1}^d C_{ijk}^\pm b_k$. Then $B([b_i^+, b_j^-], b_k^+) = B(b_j^-, [b_k^+, b_i^+]) = C_{kij}^+$ and $B([b_i^+, b_j^-], b_k^-) = B(b_i^+, [b_j^-, b_k^-]) = C_{jki}^-$ implies that

$$[b_i^+, b_j^-] = \sum_{k=1}^d (C_{kij}^+ b_k^- + C_{jki}^- b_k^+). \tag{5.34}$$

Therefore, we see that

$$\mathrm{CYB}(r) = \sum_{i,j,k=1}^d \left(C_{ijk}^+ b_k^+ \otimes b_i^- \otimes b_j^- - b_i^+ \otimes (C_{kji}^+ b_k^- + C_{ikj}^- b_k^+) \otimes b_j^- + b_i^+ \otimes b_j^+ \otimes C_{ijk}^- b_k^- \right).$$

This expression vanishes since $C_{ijk}^\pm = -C_{jik}^\pm$ and relabeling indices yields

$$\begin{aligned} \sum_{i,j,k=1}^d b_i^+ \otimes C_{kji}^+ b_k^- \otimes b_j^- &= \sum_{i,j,k=1}^d C_{ijk}^+ b_k^+ \otimes b_i^- \otimes b_j^- \text{ and} \\ \sum_{i,j,k=1}^d b_i^+ \otimes C_{ikj}^- b_k^+ \otimes b_j^- &= \sum_{i,j,k=1}^d b_i^+ \otimes b_j^+ \otimes C_{ijk}^- b_k^-. \end{aligned} \tag{5.35}$$

It is easy to see that the isomorphism $\mathfrak{m} \otimes \mathfrak{m} \to \mathrm{End}(\mathfrak{m})$ defined by $a_1 \otimes a_2 \mapsto B(a_2, -)a_1$ maps $r + \tau(r)$ to $\mathrm{id}_{\mathfrak{m}}$. This implies that $r + \tau(r)$ is $\mathfrak{m}$-invariant, so r is a constant r-matrix and ∂r defines a Lie bialgebra structure on $\mathfrak{m}$.

Observe that (5.34) implies

$$\begin{aligned}\partial r(b_i^+) &= \sum_{j,k=1}^{d} \left(C_{ijk}^+ b_k^+ \otimes b_j^- + b_j^+ \otimes (C_{kij}^+ b_k^- + C_{jki}^- b_k^+)\right)\\ &= \sum_{j,k=1}^{d} C_{ijk}^+ b_k^+ \otimes b_j^- - \sum_{j,k=1}^{d} C_{ikj}^+ b_j^+ \otimes b_k^- + \sum_{j,k=1}^{d} C_{jki}^+ b_j^+ \otimes b_k^+ = \sum_{j,k=1}^{d} C_{jki}^- b_j^+ \otimes b_k^+,\end{aligned}$$

so ∂r restricts to a Lie bialgebra cobracket on $\mathfrak{m}_+$ and

$$B^{\otimes 2}(\delta(b_i^+), b_j^- \otimes b_k^-) = C_{jki}^- = B(b_i^+, [b_j^-, b_k^-]) \tag{5.36}$$

concludes the proof. □

Lemma 5.4.3.
Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over an algebraically closed field $\Bbbk$ of characteristic 0 with Casimir element $\gamma \in \mathfrak{g} \otimes \mathfrak{g}$. Every Lie bialgebra cobracket on $\mathfrak{g}$ is of the form ∂r for an $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying $r + \tau(r) = \lambda\gamma$ and $\mathrm{CYB}_{\mathfrak{g}}(r) = 0$.

Proof. By Whitehead's Lemma, every 1-cocycle, in particular every Lie bialgebra cobracket, is of the form ∂r for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. Lemma 5.4.1 states that $r + \tau(r)$ and $\mathrm{CYB}(r)$ are $\mathfrak{g}$-invariant. The space of $\mathfrak{g}$-invariant elements of $\mathfrak{g} \otimes \mathfrak{g}$ is spanned by γ, i.e. $r + \tau(r) = \lambda\gamma$ for some $\lambda \in \Bbbk$. It is not hard to see that this implies $\mathrm{CYB}(r) \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. The space of $\mathfrak{g}$-invariant elements of $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is spanned by $\mathrm{CYB}(\gamma)$ (see [Kos50, Theorem 11.2]), i.e. $\mathrm{CYB}(r) = \mu\mathrm{CYB}(\gamma)$ for some $\mu \in \Bbbk$. Let $\alpha \in \Bbbk$ satisfy $\alpha^2 = -\mu$. Then $\partial(r + \alpha\gamma) = \partial r$ and $\mathrm{CYB}(r + \alpha\gamma) = \mathrm{CYB}(r) + \alpha^2\mathrm{CYB}(\gamma) = 0$ to conclude the proof. □

Constant r-matrices are, as the name suggests, closely related to formal r-matrices. The following lemma depicts examples of this. The proof is straightforward.

Lemma 5.4.4.
Let $\mathfrak{g}$ be a semi-simple finite-dimensional Lie algebra over $\Bbbk$ with Casimir element $\gamma \in \mathfrak{g} \otimes \mathfrak{g}$. For a $t \in \mathfrak{g} \otimes \mathfrak{g}$, the following results are true:

(1) *$t + \tau(t) = 0$ and $\mathrm{CYB}(t) = 0$ if and only if*

$$r(x, y) := \frac{\gamma}{x - y} + t \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!] \tag{5.37}$$

is a formal r-matrix.

(2) *$t + \tau(t) = \gamma$ and $\mathrm{CYB}(t) = 0$ if and only if*

$$r(x, y) := \frac{\gamma}{\exp(x - y) - 1} + t \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!] \tag{5.38}$$

is a formal r-matrix.

5.4.2 The Lie bialgebra structure associated to a formal r-matrix. Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\Bbbk$ and $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a normalized formal r-matrix. In Subsection 2.3.3, it is shown that

$$\delta(a)(x,y) := [a(x)\otimes 1 + 1\otimes a(y), r(x,y)] \tag{5.39}$$

defines a linear map $\delta\colon \mathfrak{g}(r) \to \mathfrak{g}(r)\otimes\mathfrak{g}(r)$. This map defines a Lie bialgebra structure on $\mathfrak{g}(r)$. More precisely, we have the following result.

Proposition 5.4.5.
Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\Bbbk$ and $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a normalized formal r-matrix. The linear map δ defined by (5.39) is a Lie bialgebra cobracket on $\mathfrak{g}(r)$ and $(\mathfrak{g}((z)), \mathfrak{g}(r), \mathfrak{g}[\![z]\!])$ is a Manin triple isomorphic to the classical double of $\mathfrak{g}(r)$.

Proof. It is easy to see that δ is a 1-cocycle and $(\mathfrak{g}((z)), \mathfrak{g}(r), \mathfrak{g}[\![z]\!])$ is a Manin triple; see Proposition 2.2.1 and Lemma 2.3.3. Let $r(x,y) = \sum_{k\in\mathbb{N}_0}\sum_{i=1}^d r_{k,i}(x)\otimes y^k b_i$ for some orthonormal basis $\{b_i\}_{i=1}^d$ of $\mathfrak{g}$ with respect to the Killing form K of $\mathfrak{g}$. Equation (2.41) reads

$$\sum_{k\in\mathbb{N}_0}\sum_{i=1}^d \delta(r_{k,i})\otimes y^k b_i = \sum_{k,\ell\in\mathbb{N}_0}\sum_{i,j=1}^d r_{k,i}(x_1)\otimes r_{\ell,j}(x_2)\otimes [b_i,b_j]x_3^{k+\ell}. \tag{5.40}$$

Combining $K_{-1}(r_{k,i}(z), b_j z^\ell) = \delta_{ij}\delta_{k\ell}$ for all $i,j\in\{1,\dots,d\}, k,\ell\in\mathbb{N}_0$ with the fact that $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!]\oplus\mathfrak{g}(r)$ shows that $a\mapsto K_{-1}(a,-)$ defines an isomorphism $K_{-1}^{\mathrm{a}}\colon \mathfrak{g}[\![z]\!]\to\mathfrak{g}(r)^*$. Applying $K_{-1}^{\otimes 3}(-, b_i x_1^k\otimes b_j x_2^\ell\otimes r_{m,n}(x_3))$ to (5.40) yields

$$K_{-1}^{\otimes 2}(\delta(r_{m,n})(x,y), b_i x^k\otimes b_j y^\ell) = K_{-1}(r_{m,n}(z), [b_i z^k, b_j z^\ell]). \tag{5.41}$$

This implies that δ defines a Lie bialgebra structure on $\mathfrak{g}(r)$ and $K_{-1}^{\mathrm{a}}\colon \mathfrak{g}[\![z]\!]\to\mathfrak{g}(r)^*$ is an isomorphism of Lie algebras, so $(\mathfrak{g}((z)), \mathfrak{g}(r), \mathfrak{g}[\![z]\!])$ is isomorphic to the classical double of $\mathfrak{g}(r)$. □

It is easy to see that the Lie bialgebras $\mathfrak{g}(r)$ and $\mathfrak{g}(\widetilde{r})$ are isomorphic if and only if $\widetilde{r}\in(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ is gauge equivalent to r. In this language, Theorem 2.3.4 can be understood as: the canonical derivation of $\mathfrak{g}(r)$ is, up to equivalence in r, given by the restriction of the formal derivative $\mathrm{d}/\mathrm{d}z$ to $\mathfrak{g}(r)$.

5.4.3 The standard bialgebra structure on Kac-Moody algebras. Let

$$\mathbf{R} := \left(\mathfrak{K}, \mathfrak{h}, \Pi := \{\alpha_1,\dots,\alpha_q\}, \Pi^\vee := \left\{\alpha_1^\vee,\dots,\alpha_q^\vee\right\}\right)$$

be the realization of a symmetrizable generalized Cartan matrix A,

$$\{e_i^+, e_i^- \mid i\in\{1,\dots,q\}\} \tag{5.42}$$

be a choice of Chevalley generators, $\Phi = \Phi_+\sqcup\Phi_-$ be the associated polarized root system and B be a fixed non-degenerate invariant bilinear form. We can extend the action $\mathfrak{K}$ on

$\mathfrak{K}\otimes\mathfrak{K}$ to $M := \prod_{\alpha,\beta\in\Phi\cup\{0\}} \mathfrak{K}_\alpha\otimes\mathfrak{K}_\beta$ to turn this vector space into a $\mathfrak{K}$-module. The bilinear form B induces an isomorphism $\mathfrak{K}_\alpha \otimes \mathfrak{K}_{-\alpha} \to \mathrm{End}(\mathfrak{K}_\alpha)$ by $a \otimes b \mapsto B(b,-)a$. Let χ_α be the preimage of the identity under this map and define $r_{(\mathbf{R},B)} := \chi_0/2 + \sum_{\alpha\in\Phi_-} \chi_\alpha \in M$. Then $a \mapsto a \cdot r_{(\mathbf{R},B)}$ defines a linear map $\delta_{(\mathbf{R},B)} \colon \mathfrak{K} \to M$. As the notation suggests, $r_{(\mathbf{R},B)}$ and $\delta_{(\mathbf{R},B)}$ are completely determined by the choice of realization $\mathbf{R}$ and bilinear form B. From now on we write r and δ instead of $r_{(\mathbf{R},B)}$ and $\delta_{(\mathbf{R},B)}$.

Lemma 5.4.6.
For δ defined as above, the identities $\delta(h) = 0$ for all $h \in \mathfrak{h}$ and $\delta(e_i^\pm) = 1/2\alpha_i^* \wedge e_i^\pm$ for all $i \in \{1,\dots,q\}$ hold, where $\alpha_i^* \in \mathfrak{h}$ is uniquely determined by $B(\alpha_i^*,\cdot) = \alpha_i$. In particular, $\delta\colon \mathfrak{K} \to \mathfrak{K}\otimes\mathfrak{K}$ is a 1-cocycle.

Proof. Part *2.* of Lemma 4.1.5 states that $[a \otimes 1, \chi_\beta] + [1 \otimes a, \chi_{\alpha+\beta}] = 0$ holds for all $\alpha,\beta \in \Phi\cup\{0\}$ and $a \in \mathfrak{K}_\alpha$. For $h \in \mathfrak{K}_0 = \mathfrak{h}$ this immediately implies $\delta(h) = h \cdot r = 0$ and we can see that

$$\begin{aligned}\delta(e_i^+) &= \frac{1}{2}e_i^+ \cdot \chi_0 + \sum_{\alpha\in\Phi_-}\Big([e_i^+\otimes 1, \chi_{\alpha-\alpha_i}] + [1\otimes e_i^+, \chi_\alpha]\Big)\\ &= \frac{1}{2}e_i^+ \cdot \chi_0 + [e_i^+\otimes 1, \chi_{-\alpha_i}] + \sum_{\alpha\in\Phi_-}\Big([e_i^+\otimes 1, \chi_{\alpha-\alpha_i}] + [1\otimes e_i^+, \chi_\alpha]\Big) \qquad (5.43)\\ &= \frac{1}{2}e_i^+ \cdot \chi_0 + [e_i^+\otimes 1, \chi_{-\alpha_i}] = \frac{1}{2}[e_i^+\otimes 1 - 1\otimes e_i^+, \chi_0].\end{aligned}$$

Here, we used the fact that for any $\alpha \in \Phi\cup\{0\}$, $[e_i^+ \otimes 1, \chi_\alpha] \neq 0$ if and only if $\alpha + \alpha_i \in \Phi\cup\{0\}$.

Let $\{h_j\}_{j=1}^n \subset \mathfrak{h}$ be a basis orthonormal with respect to B, where $n := \dim(\mathfrak{h})$. Then $\chi_0 = \sum_{j=1}^n h_j \otimes h_j$ and

$$[e_i^+ \otimes 1, \chi_0] = -\sum_{j=1}^n \alpha_i(h_j)e_i^+ \otimes h_j = -\sum_{j=1}^n e_i^+ \otimes \alpha_i(h_j)h_j = -e_i^+ \otimes \alpha_i^*, \qquad (5.44)$$

where $\sum_{j=1}^n \alpha_i(h_j)h_j = \sum_{j=1}^n B(\alpha_i^*, h_j)h_j = \alpha_i^*$ was used. In a similar fashion, we see $[1\otimes e_i^+, \chi_0] = -\alpha_i^* \otimes e_i^+$. Combined we conclude $\delta(e_i^+) = 1/2\alpha_i^* \wedge e_i^+$. A similar calculation yields $\delta(e_i^-) = 1/2\alpha_i^* \wedge e_i^-$. Since $\{e_i^+, e_i^-\} \cup \mathfrak{h}$ generates $\mathfrak{K}$, δ takes values in any submodule of M containing $\delta(e_i^\pm)$, in particular $\mathfrak{K}\otimes\mathfrak{K}$. □

Our next goal is to show that δ defines a Lie bialgebra structure determined by a Manin triple. Therefore, let $\mathfrak{b}_\pm := \mathfrak{h} \oplus \mathfrak{n}_\pm$ be the standard Borel subalgebras of $\mathfrak{K}$ with respect to the realization $\mathbf{R}$. Note that

$$B^{(2)}((a_+,a_-),(b_+,b_-)) := B(a_+,b_+) - B(a_-,b_-) \qquad a_\pm, b_\pm \in \mathfrak{K}$$

defines a symmetric, non-degenerate, invariant bilinear form on $\mathfrak{K}\times\mathfrak{K}$.

Proposition 5.4.7.
Let $\iota\colon \mathfrak{K} \to \mathfrak{K}\times\mathfrak{K}$ be the morphism defined by $a \mapsto (a,a)$, $\mathfrak{d} := \mathrm{Im}(\iota)$ and $\mathfrak{w} := \{(a_+,a_-) \in \mathfrak{b}_+ \times \mathfrak{b}_- \mid a_+ + a_- \in \mathfrak{n}_+ + \mathfrak{n}_-\}$. Then δ defines a Lie bialgebra

structure on $\mathfrak{K}$ and $(\mathfrak{K} \times \mathfrak{K}, \mathfrak{d}, \mathfrak{w})$ is a Manin triple determining the Lie bialgebra cobracket $\mathfrak{d} \to \mathfrak{d} \otimes \mathfrak{d}$ defined by $(a, a) \mapsto (\iota \otimes \iota)\delta(a)$.

Proof. The fact that $\mathfrak{K} \times \mathfrak{K} = \mathfrak{d} \oplus \mathfrak{w}$ and $\mathfrak{d} \subseteq \mathfrak{d}^\perp$ holds can be directly verified. For any $(a_+, a_-), (b_+, b_-) \in \mathfrak{w}$, let $a_{\pm,\mathfrak{h}}$ and $b_{\pm,\mathfrak{h}}$ be the $\mathfrak{h}$-component of $a_\pm$ and $b_\pm$ respectively. Then $a_{+,\mathfrak{h}} = -a_{-,\mathfrak{h}}$ and $b_{+,\mathfrak{h}} = -b_{-,\mathfrak{h}}$ implies

$$B^{(2)}((a_+, a_-), (b_+, b_-)) = B(a_{+,\mathfrak{h}}, b_{+,\mathfrak{h}}) - B(a_{-,\mathfrak{h}}, b_{-,\mathfrak{h}}) = 0, \tag{5.45}$$

so $(\mathfrak{K} \times \mathfrak{K}, \mathfrak{d}, \mathfrak{w})$ is a Manin triple. By virtue of Proposition 5.2.3 and the symmetry of $\mathfrak{b}_+$ and $\mathfrak{b}_-$ it remains to prove for all $(a_+, a_-), (b_+, b_-) \in \mathfrak{w}$ that

$$\begin{aligned} &B^{\otimes 2}(\delta(e_i^+), (a_+ - a_-) \otimes (b_+ - b_-)) = B^{(2),\otimes 2}((\iota \otimes \iota)\delta(e_i^+), (a_+, a_-) \otimes (b_+, b_-)) \\ &= B^{(2)}((e_i^+, e_i^+), [(a_+, a_-), (b_+, b_-)]) = B(e_i^+, [a_+, b_+] - [a_-, b_-]) \end{aligned}$$

holds. Both sides of this equation are non-zero if and only if $a_+ - a_-, b_+ - b_- \in \mathfrak{h} + \Bbbk e_i^-$. Since δ is skew-symmetric, B is bilinear and the map $(a_+, a_-) \mapsto a_+ - a_-$ defines an linear isomorphism $\mathfrak{w} \to \mathfrak{K}$, we may assume that $a/2 = a_+ = -a_- \in \mathfrak{h}, b_+ = 0$ and $b_- = e_i^-$. Then

$$\begin{aligned} &B(\delta(e_i^+), (a_+ - a_-) \otimes (-e_i^-)) = -\frac{1}{2}B(\alpha_i^*, a)B(e_i^+, e_i^-) \\ &= -\frac{1}{2}B(e_i^+, \alpha_i(a)e_i^-) = B(e_i^+, -[a_-, e_i^-]). \end{aligned}$$

concludes the proof. □

The Lie bialgebra structure δ is called the *standard Lie bialgebra structure* of $\mathfrak{K}$ with respect to $((\mathfrak{K}, \mathfrak{h}, \Pi, \Pi^\vee), B)$.

5.4.4 The standard bialgebra structure on twisted loop algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\Bbbk = \mathbb{C}$ and $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ have order $m \in \mathbb{N}$. Furthermore, let $\mathfrak{L} = \mathfrak{L}(\mathfrak{g}, \sigma) \subseteq \mathfrak{g}[\widetilde{u}, \widetilde{u}^{-1}]$ be the associated loop algebra and K_0 be the bilinear form of $\mathfrak{L}$ described in Lemma 4.2.1. Fix a triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+$ and let $(\widehat{\mathfrak{L}}, \widehat{\mathfrak{H}}, \Pi, \Pi^\vee)$ be the associated realization due to Theorem 4.2.3. Recall that K_0 induces a non-degenerate, invariant, symmetric bilinear form $\widehat{K_0}$ on $\widehat{\mathfrak{L}}$.

Let δ be the standard bialgebra structure of $\widehat{\mathfrak{L}}$ with respect to $((\widehat{\mathfrak{L}}, \widehat{\mathfrak{H}}, \Pi, \Pi^\vee), \widehat{K_0})$ (see Subsection 5.4.3). Since $\delta([\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}]) \subseteq [\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}] \otimes [\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}]$ and $\delta(c) = 0$, we can see that δ induces a bialgebra structure δ° on $\mathfrak{L} \cong [\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}]/\mathbb{C}c = \mathfrak{L}$ by

$$\delta^\circ(a + \mathbb{C}c) = \delta(a) + (\mathbb{C}c \otimes [\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}] + [\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}] \otimes \mathbb{C}c) \tag{5.46}$$

for all $a \in [\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}]$. We call the Lie bialgebra cobracket δ° the *standard Lie bialgebra structure of* $\mathfrak{L} = \mathfrak{L}(\mathfrak{g}, \sigma)$ with respect to triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$.

Let us choose Chevalley generators $\{e_i^+, e_i^- \mid i \in \{0, \dots, n\}\} \subseteq \mathfrak{L}$. The definition of δ immediately implies that

$$\delta^\circ(e_i^\pm) = 1/2\alpha_i^* \wedge e_i^\pm \tag{5.47}$$

holds for all $i \in \{0,\dots,n\}$, where $\alpha_i^* \in \mathfrak{H}$ is uniquely determined by $K_0(\alpha_i^*,\cdot) = \alpha_i|_{\mathfrak{H}}$.

Let $\mathfrak{B}_\pm = \mathfrak{H} \oplus \mathfrak{N}_\pm \subseteq \mathfrak{L}$ be the standard Borel subalgebras and $\iota\colon \mathfrak{L} \to \mathfrak{L} \times \mathfrak{L}$ be the embedding $a \mapsto (a,a)$. Then the Lie bialgebra cobracket defined by $(a,a) \mapsto (\iota\otimes\iota)\delta^\circ(a)$ is determined by the Manin triple $(\mathfrak{L}\times\mathfrak{L}, \mathfrak{D}, \mathfrak{W}^\circ)$, where $\mathfrak{D} := \operatorname{Im}(\iota)$,

$$\mathfrak{W}^\circ := \{(a_+,a_-) \in \mathfrak{B}_+ \times \mathfrak{B}_- \mid a_+ + a_- \in \mathfrak{N}_+ + \mathfrak{N}_-\}, \tag{5.48}$$

and $\mathfrak{L}\times\mathfrak{L}$ is equipped with the bilinear form

$$K_0^{(2)}((a_+,a_-),(b_+,b_-)) = K_0(a_+,b_+) - K_0(a_-,b_-) \text{ for all } a_\pm, b_\pm \in \mathfrak{L}. \tag{5.49}$$

Indeed, this can be seen using the same argument as in Proposition 5.4.7.

The next goal of this subsection is to see that δ° and its twisted versions are closely related to a certain class of r-matrices. Since $(\sigma\otimes\sigma)\gamma = \gamma$, we have $\gamma \in \bigoplus_{j=0}^{m-1}(\mathfrak{g}_j^\sigma \otimes \mathfrak{g}_{-j}^\sigma)$. Let γ_j be the $\mathfrak{g}_j^\sigma \otimes \mathfrak{g}_{-j}^\sigma$-component of γ with respect to this decomposition. Furthermore, let $\gamma_0 = \gamma_0^+ + \gamma_{\mathfrak{H}} + \gamma_0^-$ be the unique decomposition of γ_0 such that $\gamma_{\mathfrak{H}} \in \mathfrak{H}\otimes\mathfrak{H}$ and $\gamma_0^\pm \in \mathfrak{n}_\pm \otimes \mathfrak{n}_\mp$. Consider the rational map $\varrho^\circ\colon \mathbb{C}\times\mathbb{C} \to \mathfrak{g}\otimes\mathfrak{g}$ defined by

$$\varrho^\circ(\tilde{u},\tilde{v}) := \frac{1}{(\tilde{u}/\tilde{v})^m - 1}\sum_{j=0}^{m-1}\left(\frac{\tilde{u}}{\tilde{v}}\right)^j \gamma_j + \gamma_0^- + \gamma_{\mathfrak{H}}/2 \tag{5.50}$$

as an element of the $\mathfrak{L}$-module $(\mathfrak{L}\otimes\mathfrak{L})[((\tilde{u}/\tilde{v})^m - 1)^{-1}]$.

Proposition 5.4.8.
Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra, $\sigma \in \operatorname{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ have order $m \in \mathbb{N}$, and fix a triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$. Furthermore, let δ° be the standard Lie bialgebra structure on $\mathfrak{L} := \mathfrak{L}(\mathfrak{g},\sigma)$ with respect to $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$ and ϱ° be given by (5.50).

(1) *$\delta^t := \delta^\circ + \partial t$ defines a Lie bialgebra structure on $\mathfrak{L}$ for some tensor $t \in \mathfrak{L}\otimes\mathfrak{L}$ if and only if t is a classical twist of δ° if and only if $\varrho^t := \varrho^\circ + t$ solves the CYBE (3.54).*

(2) *$\delta^t(a)(\tilde{u},\tilde{v}) = [a(\tilde{u})\otimes 1 + 1\otimes a(\tilde{v}), \varrho^t(\tilde{u},\tilde{v})]$ for all $t \in \mathfrak{L}\otimes\mathfrak{L}, a \in \mathfrak{L}$.*

Proof. Note that the inclusions

$$(\mathfrak{L}\otimes\mathfrak{L})[((\tilde{u}/\tilde{v})^m - 1)^{-1}] \subseteq (\mathfrak{g}\otimes\mathfrak{g})[\tilde{u},\tilde{u}^{-1},\tilde{v},\tilde{v}^{-1},((\tilde{u}/\tilde{v})^m - 1)^{-1}] \subseteq (\mathfrak{g}\otimes\mathfrak{g})(\!(\tilde{u})\!)(\!(\tilde{v})\!)$$

defined by the Laurent expansion in $\tilde{v} = 0$ are compatible with the respective $\mathfrak{L}$-module structures. Here,

$$\frac{1}{(\tilde{u}/\tilde{v})^m - 1}\sum_{j=0}^{m-1}\left(\frac{\tilde{u}}{\tilde{v}}\right)^j \gamma_j = \sum_{j=1}^{\infty}\left(\frac{\tilde{v}}{\tilde{u}}\right)^{mj}\sum_{j=0}^{m-1}\left(\frac{\tilde{v}}{\tilde{u}}\right)^{j-m}\gamma_j = \sum_{j=1}^{\infty}\left(\frac{\tilde{v}}{\tilde{u}}\right)^j \gamma_{-j} \tag{5.51}$$

in $(\mathfrak{g}\otimes\mathfrak{g})(\!(\tilde{u})\!)(\!(\tilde{v})\!)$, where we wrote $\gamma_{j+km} = \gamma_j$ for all $k \in \mathbb{N}$, $j \in \{0,\dots,m-1\}$.

Let $\Phi = \Phi_+ \sqcup \Phi_-$ be the polarized root system of $\widehat{\mathfrak{L}}$ associated to $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$ and let $s \in \mathbb{N}^{n+1} \setminus \{0\}$ be chosen such that the natural grading of $\widehat{\mathfrak{L}}$ coincides with the $\mathbb{Z}$-grading of type s as Kac-Moody algebra (see Subsection 4.1.4). For any $j \in \mathbb{Z} \setminus \{0\}$

$$\left(\frac{\widetilde{v}}{\widetilde{u}}\right)^j \gamma_{-j} = \sum_{\substack{\alpha \in \Phi \\ \mathrm{ht}_s(\alpha) = -j}} \chi_\alpha, \tag{5.52}$$

holds. Here, for all $\alpha \in \Phi$, χ_α is the preimage of the identity under the canonical isomorphism $\mathfrak{L}_\alpha \otimes \mathfrak{L}_{-\alpha} \to \mathrm{End}(\mathfrak{L}_\alpha)$ defined by $a \otimes b \mapsto B(b, \cdot)a$. We obtain

$$\varrho^\circ = \gamma_{\mathfrak{H}}/2 + \sum_{\alpha \in \Phi^-} \chi_\alpha \tag{5.53}$$

and the identity $\delta^\circ(a)(\widetilde{u}, \widetilde{v}) = [a(\widetilde{u}) \otimes 1 + 1 \otimes a(\widetilde{v}), \varrho^\circ(\widetilde{u}, \widetilde{v})]$ for all $a \in \mathfrak{L}$ is now a direct consequence of the definition of the standard structure of $\widehat{\mathfrak{L}}$ in Subsection 5.4.3. Since

$$\partial t(a)(\widetilde{u}, \widetilde{v}) = [a(\widetilde{u}) \otimes 1 + 1 \otimes a(\widetilde{v}), t(\widetilde{u}, \widetilde{v})] \tag{5.54}$$

holds for all $a \in \mathfrak{L} \subseteq \mathfrak{g}[\widetilde{u}, \widetilde{v}]$ and $t \in \mathfrak{L} \otimes \mathfrak{L} \subseteq (\mathfrak{g} \otimes \mathfrak{g})[\widetilde{u}, \widetilde{v}, \widetilde{u}^{-1}, \widetilde{v}^{-1}]$, this concludes the proof of (2).

The map $\delta^t := \delta^\circ + \partial t$ defines a Lie bialgebra structure on $\mathfrak{L}$ if and only if $t + \tau_{\mathfrak{L}}(t)$ and $\mathrm{Alt}_{\mathfrak{L}}((\delta^\circ \otimes 1)t) - \mathrm{CYB}(t)$ are $\mathfrak{L}$-invariant elements; see Proposition 5.3.1. But Remark 5.1.2 implies that

$$\mathfrak{L}^{\otimes n} \text{ has no non-zero } \mathfrak{L}\text{-invariant elements} \tag{5.55}$$

since $\mathfrak{L}$ has no finite-dimensional ideals (see e.g. Lemma 4.2.8.(3)), so t is a classical twist. Observe that, under consideration of $\mathfrak{L} \otimes \mathfrak{L} \subseteq (\mathfrak{g} \otimes \mathfrak{g})[\widetilde{u}, \widetilde{u}^{-1}, \widetilde{v}, \widetilde{v}^{-1}]$,

$$T \in \mathfrak{L} \otimes \mathfrak{L}, T(\widetilde{u}, \widetilde{u}) = 0 \implies T = ((\widetilde{u}/\widetilde{v})^m - 1)\widetilde{T} \text{ for some } \widetilde{T} \in \mathfrak{L} \otimes \mathfrak{L}, \tag{5.56}$$

holds. This can be used to see that the left-hand side of the CYBE (3.54) in ϱ^t is an element $\mathrm{CYB}(\varrho^t)$ of $\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}$. A similar calculation as in Proposition 5.3.1 shows that $\mathrm{Alt}_{\mathfrak{L}}((\delta^t \otimes \mathrm{id}_{\mathfrak{L}})\delta^t(a)) = 0$ is equivalent to $a \cdot \mathrm{CYB}(\varrho^t) = 0$ for all $a \in \mathfrak{L}$. In particular, $\mathrm{Alt}_{\mathfrak{L}}((\delta^t \otimes \mathrm{id}_{\mathfrak{L}})\delta^t(a)) = 0$ is equivalent to $\mathrm{CYB}(\varrho^t) = 0$ by virtue of (5.55), which concludes the proof of (1). □

Elements of the form $\varrho^t := \varrho^\circ + t$ for some classical twist $t \in \mathfrak{L} \otimes \mathfrak{L}$ are called *σ-trigonometric r-matrices* and we call ϱ° the *standard σ-trigonometric r-matrix* with respect to the triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+$. Furthermore, the Lie bialgebra cobrackets of $\mathfrak{L}$ of the form $\delta^t := \delta^\circ + \partial t$ for $t \in \mathfrak{L} \otimes \mathfrak{L}$ are called *twisted standard Lie bialgebra structures of* $\mathfrak{L}$. Combining Theorem 5.4.8 with Theorem 5.3.2 and the fact that δ° is determined by the Manin triple $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{D}, \mathfrak{W}^\circ)$ provides the following theory of Manin triples for σ-trigonometric r-matrices.

Theorem 5.4.9.
Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra, $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ have order $m \in \mathbb{N}$, and fix a triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$. Furthermore,

let δ° be the standard Lie bialgebra structure on $\mathfrak{L} := \mathfrak{L}(\mathfrak{g}, \sigma)$ with respect to $\mathfrak{g}_0^\sigma = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$ and ϱ° (resp. $\mathfrak{W}^\circ$) be given by (5.50) (resp. (5.48)).

(1) *$\varrho^t = \varrho^\circ + t \mapsto \mathfrak{W}^t$ defines a bijection between σ-trigonometric r-matrices and subalgebras $\mathfrak{W} \subseteq \mathfrak{L} \times \mathfrak{L}$ such that $((\mathfrak{L} \times \mathfrak{L}, K_0^{(2)}), \mathfrak{D}, \mathfrak{W})$ is a Manin triple and $\mathfrak{W} \asymp \mathfrak{W}^\circ$. Here, $\mathfrak{W}^t := \mathfrak{W}^{\circ,t}$ is defined in Theorem 5.3.2 and $K_0^{(2)}$ is given by formula (5.49).*

(2) *Let $t \in \mathfrak{L} \otimes \mathfrak{L}$ be a classical twist of δ°. The Manin triple $((\mathfrak{L} \times \mathfrak{L}, K_0^{(2)}), \mathfrak{D}, \mathfrak{W}^t)$ determines (up to $\mathfrak{L} \cong \mathfrak{D}$) the Lie bialgebra cobracket $\delta^t := \delta^\circ + \partial t \colon \mathfrak{L} \to \mathfrak{L} \otimes \mathfrak{L}$, given by the formula*

$$\delta^t(a)(\widetilde{u}, \widetilde{v}) = [a(\widetilde{u}) \otimes 1 + 1 \otimes a(\widetilde{v}), \varrho^t(\widetilde{u}, \widetilde{v})] \text{ for all } a \in \mathfrak{L}. \tag{5.57}$$

The notation was purposefully chosen in such a way that $\varrho^0 = \varrho^\circ, \delta^0 = \delta^\circ$ and $\mathfrak{W}^0 = \mathfrak{W}^\circ$. Let us conclude this subsection with an investigation, under which circumstances two different twisted standard Lie bialgebra structure of $\mathfrak{L}$ are isomorphic.

Lemma 5.4.10.
Let $\varphi \in \mathrm{Aut}_{\mathbb{C}[u,u^{-1}]\text{-alg}}(\mathfrak{L})$, $\widetilde{\varphi} \in \mathrm{End}(\mathfrak{g})[\widetilde{u}, \widetilde{u}^{-1}]$ be the associated Laurent polynomial and t_1, t_2 be two classical twists of δ. Then

$$(\varphi \otimes \varphi)\delta^{t_1} = \delta^{t_2}\varphi \iff (\widetilde{\varphi} \otimes \widetilde{\varphi})\varrho^{t_1} = \varrho^{t_2}. \tag{5.58}$$

Proof. Combining $(\widetilde{\varphi}(\widetilde{u}) \otimes \widetilde{\varphi}(\widetilde{u}))\gamma = \gamma$ with (5.56) implies that

$$(\widetilde{\varphi}(\widetilde{u}) \otimes \widetilde{\varphi}(\widetilde{v}))\varrho^{t_1}(\widetilde{u}, \widetilde{v}) - \varrho^{t_2}(\widetilde{u}, \widetilde{v}) \in \mathfrak{L} \otimes \mathfrak{L}. \tag{5.59}$$

For all $a \in \mathfrak{L}$, $(\varphi \otimes \varphi)\delta^{t_1}\varphi^{-1}(a) = \delta^{t_2}$ is equivalent to

$$[a(\widetilde{u}) \otimes 1 + 1 \otimes a(\widetilde{v}), (\widetilde{\varphi}(\widetilde{u}) \otimes \widetilde{\varphi}(\widetilde{v}))\varrho^{t_1}(\widetilde{u}, \widetilde{v}) - \varrho^{t_2}(\widetilde{u}, \widetilde{v})] = 0. \tag{5.60}$$

In particular, $(\widetilde{\varphi}(\widetilde{u}) \otimes \widetilde{\varphi}(\widetilde{v}))\varrho^{t_1}(\widetilde{u}, \widetilde{v}) - \varrho^{t_2}(\widetilde{u}, \widetilde{v}) \in \mathfrak{L} \otimes \mathfrak{L}$ is $\mathfrak{L}$-invariant. Therefore, this is equivalent to $(\widetilde{\varphi}(\widetilde{u}) \otimes \widetilde{\varphi}(\widetilde{v}))\varrho^{t_1}(\widetilde{u}, \widetilde{v}) = \varrho^{t_2}(\widetilde{u}, \widetilde{v})$ by virtue of (5.55). □

We say that two classical twists t_1 and t_2 of δ° (resp. two σ-trigonometric r-matrices ϱ^{t_1} and ϱ^{t_2}) are *regularly equivalent* if there exists $\varphi \in \mathrm{Aut}_{\mathbb{C}[u,u^{-1}]\text{-alg}}(\mathfrak{L})$ satisfying (5.58). We will study the structural and geometric theory of σ-trigonometric r-matrices up to regular equivalence in detail in Chapter 8.

Remark 5.4.11.
The results and definitions in this paragraph can be repeated ad verbatim if $\mathbb{C}$ is replaced by any algebraically closed field $\Bbbk$ of characteristic 0.

5.4.5 The standard Lie bialgebra structure on polynomial Lie algebras. Let us consider a finite dimensional semi-simple Lie algebra $\mathfrak{g}$ over $\Bbbk$ with Killing form K and Casimir element γ. If we equip $\mathfrak{g}[z, z^{-1}]$ with the non-degenerate, symmetric, invariant bilinear form K_{-1}, defined by

$$K_{-1}(a, b) = \mathrm{res}_0 K(a(z), b(z))\mathrm{d}z \text{ for all } a, b \in \mathfrak{g}[z, z^{-1}],$$

then $(\mathfrak{g}[z,z^{-1}],\mathfrak{g}[z],z^{-1}\mathfrak{g}[z^{-1}])$ becomes a Manin triple. Consider the $\mathfrak{g}[z]$-module $M = (\mathfrak{g}\otimes\mathfrak{g})[x,y,(x-y)^{-1}]$ and observe that $\mathfrak{g}[z]\otimes\mathfrak{g}[z]\cong(\mathfrak{g}\otimes\mathfrak{g})[x,y]$ is a $\mathfrak{g}[z]$-submodule of M and

$$r_{\mathrm{Yang}}(x,y) := \frac{\gamma}{x-y} \in M. \tag{5.61}$$

Therefore, $a(z)\mapsto[a(x)\otimes 1+1\otimes a(y), r_{\mathrm{Yang}}(x,y)]$ defines a linear map $\partial r_{\mathrm{Yang}}\colon \mathfrak{g}[z]\to M$.

Since $[a\otimes 1+1\otimes a,\gamma]=0$ for all $a\in\mathfrak{g}$, we can see that $\delta(a)=0$ and

$$\delta(za) = [(x-y)a\otimes 1, r_{\mathrm{Yang}}(x,y)] + y[a\otimes 1+1\otimes a, r_{\mathrm{Yang}}(x,y)] = [a\otimes 1,\gamma].$$

Therefore, $\partial r_{\mathrm{Yang}}$ takes values in $(\mathfrak{g}\otimes\mathfrak{g})[x,y]$, since $\mathfrak{g}[z]$ is generated by $\mathfrak{g}\sqcup z\mathfrak{g}$. Furthermore, it is not hard to see that

$$\begin{aligned} &K_{-1}^{\otimes 2}(\delta(za), z^{-1}w_1\otimes z^{-1}w_2) = K^{\otimes 2}([a\otimes 1,\gamma], w_1\otimes w_2)\\ &= K([a,w_1],w_2) = K(a,[w_1,w_2]) = K_{-1}(za, z^{-2}[w_1,w_2]) \end{aligned} \tag{5.62}$$

holds for all $w_1,w_2\in\mathfrak{g}$ and consequently for all $w_1,w_2\in\mathfrak{g}[z^{-1}]$. Thus, Proposition 5.2.3 implies that $\partial r_{\mathrm{Yang}}$ defines a Lie bialgebra cobracket on $\mathfrak{g}[z]$ determined by the Manin triple $(\mathfrak{g}[z,z^{-1}],\mathfrak{g}[z],z^{-1}\mathfrak{g}[z^{-1}])$. We call $\partial r_{\mathrm{Yang}}$ the *standard Lie bialgebra structure* on $\mathfrak{g}[z]$. Following Stolin [Sto91b; Sto91c], we call the formal r-matrices of the form $r^t := r_{\mathrm{Yang}} + t$ for $t\in(\mathfrak{g}\otimes\mathfrak{g})[x,y]$ *rational r-matrices.*

Theorem 5.4.12.
Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ over a field $\Bbbk$ of characteristic 0 with Killing form K and Casimir element γ. Let $r_{\mathrm{Yang}}(x,y) := \gamma/(x-y)$ and define $\partial r_{\mathrm{Yang}}\colon \mathfrak{g}[z]\to(\mathfrak{g}\otimes\mathfrak{g})[x,y]$ by $a(z)\mapsto[a(x)\otimes 1+1\otimes a(y), r_{\mathrm{Yang}}(x,y)]$.
(1) *$\partial r^t := \partial r_{\mathrm{Yang}} + \partial t$ defines a Lie bialgebra structure on $\mathfrak{g}[z]$ for some tensor $t\in(\mathfrak{g}\otimes\mathfrak{g})[x,y]$ if and only if t is a classical twist of $\partial r_{\mathrm{Yang}}$ if and only if $r^t = r_{\mathrm{Yang}} + t$ is a rational r-matrix.*
(2) *$r^t \mapsto \mathfrak{W}^t := (z^{-1}\mathfrak{g}[z^{-1}])^t$ (see Theorem 5.3.2) defines a bijection between rational r-matrices and subalgebras $\mathfrak{W}\subseteq\mathfrak{g}[z,z^{-1}]$ such that $\mathfrak{W}\asymp z^{-1}\mathfrak{g}[z^{-1}]$ and $((\mathfrak{g}[z,z^{-1}],K_{-1}),\mathfrak{g}[z],\mathfrak{W})$ is a Manin triple.*
(3) *For any rational r-matrix r^t, the Lie bialgebra corbacket ∂r^t is determined by $((\mathfrak{g}[z,z^{-1}],K_{-1}),\mathfrak{g}[z],\mathfrak{W}^t)$.*

Proof. Note that (2) and (3) follow from (1) and Theorem 5.3.2, so we only need to prove (1). The map $\partial r_{\mathrm{Yang}} + \partial t$ defines a Lie bialgebra structure on $\mathfrak{g}[z]$ if and only if $t+\tau_{\mathfrak{g}[z]}(t)$ and $\mathrm{Alt}_{\mathfrak{g}[z]}((\partial r_{\mathrm{Yang}}\otimes 1)t) - \mathrm{CYB}(t)$ are $\mathfrak{g}[z]$-invariant elements; see Proposition 5.3.1. But Remark 5.1.2 implies that

$$\mathfrak{g}[z]^{\otimes n} \text{ has no non-zero } \mathfrak{g}[z]\text{-invariant elements} \tag{5.63}$$

since it has no finite-dimensional ideals. Indeed, any non-zero ideal $\mathfrak{i}\subseteq\mathfrak{g}[z]$ contains the torsion-free $\Bbbk[z]$-module $[\mathfrak{i},\mathfrak{g}[z]]$, which is non-zero due to $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ and infinite-dimensional as a consequence. Therefore, t is a classical twist of $\partial r_{\mathrm{Yang}}$.

The argument in Example 2.1.4 shows that r_{Yang} is a solution of the formal CYBE (2.32). Therefore, the CYBE for $r^t = r_{\mathrm{Yang}} + t$ is easily seen to coincide with

$$\mathrm{CYB}(r_{\mathrm{Yang}}) - \mathrm{Alt}_{\mathfrak{g}[z]}((\partial r_{\mathrm{Yang}} \otimes 1)t) + \mathrm{CYB}(t), \tag{5.64}$$

which is zero if and only if t is a classical twist. □

Remark 5.4.13.
The bilinear form K_{-1} has a natural extension to $\mathfrak{g}((z^{-1}))$. It is easy to see that the Manin triples $(\mathfrak{g}[z,z^{-1}], \mathfrak{g}[z], \mathfrak{W})$ such that $\mathfrak{W} \asymp z^{-1}\mathfrak{g}[z^{-1}]$ and $(\mathfrak{g}((z^{-1})), \mathfrak{g}[z], \widehat{\mathfrak{W}})$ such that $\widehat{\mathfrak{W}} \asymp z^{-1}\mathfrak{g}[\![z^{-1}]\!]$ are in bijection; see e.g. [Sto95]. Therefore, the theory of Manin triples for rational r-matrices presented in Theorem 5.4.12 is equivalent to the one used by Stolin [Sto91b; Sto91c].

Let us conclude this section by discussing the relation of equivalence of r-matrices and isomorphism of the associated bialgebra structures.

Lemma 5.4.14.
Let $\varphi \in \mathrm{Aut}_{\Bbbk[z]\text{-alg}}(\mathfrak{g}[z])$, $\widetilde{\varphi} \in \mathrm{End}(\mathfrak{g})[z]$ be the associated polynomial and t_1, t_2 be two classical twists of δ. Then

$$(\varphi \otimes \varphi)\delta^{t_1} = \delta^{t_2}\varphi \iff (\widetilde{\varphi} \otimes \widetilde{\varphi})r^{t_1} = r^{t_2}. \tag{5.65}$$

Proof. Combining $\widetilde{\varphi}(z) \otimes \widetilde{\varphi}(z)(\gamma) = \gamma$ with of Lemma 2.1.2.(4) implies that

$$(\widetilde{\varphi}(x) \otimes \widetilde{\varphi}(y))r^{t_1}(x,y) - r_{\mathrm{Yang}}(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})[x,y]. \tag{5.66}$$

For all $a \in \mathfrak{g}[z]$, $(\varphi \otimes \varphi)\partial r^{t_1}\varphi^{-1}(a) = \partial r^{t_2}$ is equivalent to

$$[a(x) \otimes 1 + 1 \otimes a(y), (\widetilde{\varphi}(x) \otimes \widetilde{\varphi}(y))r^{t_1}(x,y) - r^{t_2}(x,y)] = 0. \tag{5.67}$$

In particular, $(\widetilde{\varphi}(x) \otimes \widetilde{\varphi}(y))r^{t_1}(x,y) - r^{t_2}(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})[x,y]$ is $\mathfrak{g}[z]$-invariant. Therefore, this is equivalent to $(\widetilde{\varphi}(x) \otimes \widetilde{\varphi}(y))r^{t_1}(x,y) = r^{t_2}(x,y)$ by virtue of (5.63). □

We say that two classical twists t_1 and t_2 of $\partial r_{\mathrm{Yang}}$ (resp. two rational r-matrices r^{t_1} and r^{t_2}) are *polynomially equivalent* if there exists $\varphi \in \mathrm{Aut}_{\Bbbk[z]\text{-alg}}(\mathfrak{g}[z])$ satisfying (5.65). We will study the structure and geometry of rational r-matrices up to polynomial equivalence in detail in Section 9.

6

The Belavin-Drinfeld trichotomy

6.1 Sheaves of algebras on algebraic groups of dimension one

It is well-known that any connected complex algebraic group of dimension one is either isomorphic to an elliptic curve or affine, in which case it is either the additive group $\mathrm{Spec}(\mathbb{C}[z])$ or the multiplicative group $\mathrm{Spec}(\mathbb{C}[u, u^{-1}])$. In this section, we will derive some classification results for sheaves of algebras on these schemes. The affine case is considered in Subsection 6.1.1 and is based on the relation of sheaves of algebras and torsors discussed in Subsection 1.2.4 and the results of Pianzola on the latter from [Pia05]. The elliptic case will be presented in Subsection 6.1.3 and is based on the well-known relation between analytic and algebraic geometry (see Subsection 6.1.2, [Ser56]) and the description of vector bundles on elliptic curves using factors of automorphy; see e.g. [Ien11].

6.1.1 Affine case. The classification of weakly trivial sheaves of algebras over one-dimensional, connected, affine algebraic groups over an algebraically closed field of characteristic 0 is provided by the following statement.

Theorem 6.1.1.
Let $\Bbbk$ be an algebraically closed field of characteristic 0, A be a finite-dimensional $\Bbbk$-algebra, and $\mathscr{A}$ be a weakly A-locally free sheaf of algebras on a $\Bbbk$-scheme X.

(1) *If $X = \mathrm{Spec}(\Bbbk[u, u^{-1}])$, there exists $\sigma \in \mathrm{Aut}_{\Bbbk\text{-alg}}(A)$ of order $m \in \mathbb{N}$ and a primitive m-th root of unity $\varepsilon \in \Bbbk$ such that $\mathscr{A}$ is isomorphic to the sheaf of algebras associated to*

$$\mathrm{L}(A, \sigma) := \{a(\tilde{u}) \in A[\tilde{u}, \tilde{u}^{-1}] \mid a(\varepsilon\tilde{u}) = \sigma(f(\tilde{u}))\}$$

on X, where the module structure of $\mathrm{L}(A, \sigma)$ is defined by $u = \tilde{u}^m$.

(2) *If $X = \mathrm{Spec}(\Bbbk[z])$, $\mathscr{A}$ is isomorphic to the sheaf of algebras associated to $A[z]$ on X.*

Proof. In both cases, Theorem 1.2.3 states that $\mathscr{A}$ is automatically étale A-locally free on X and for this reason, up to isomorphism, determined by an element of $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X)$; see Lemma 1.2.6. The arguments in [Pia05] imply that there is a canonical injection $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X) \to \check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Out}(A)_X)$, where $\mathrm{Out}(A)$ is the group of connected components of $\mathrm{Aut}_{\Bbbk\text{-alg}}(A)$ and $\mathrm{Out}(A)_X := X \times \mathrm{Out}(A)$. The case of $X = \mathrm{Spec}(\Bbbk[u, u^{-1}])$ is thereby considered explicitly in [Pia05] while the case of $X = \mathrm{Spec}(\Bbbk[u])$ works analogous. Since $\mathrm{Out}(A)$ is finite, we have a bijection of $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Out}(A)_X)$ and the non-abelian continuous cohomology group $\mathrm{H}^1(\pi_1(X, x), \mathrm{Out}(A))$, where $x \in X$ is a closed point, $\pi_1(X, x)$ is the associated étale fundamental group and the action of $\pi_1(X,x)$ on $\mathrm{Out}(A)$ is trivial.

Proof of (1). If $X = \mathrm{Spec}(\Bbbk[u,u^{-1}])$, we can choose $x = (u-1)$ and then it is explained in [Pia05] that $\mathrm{H}^1(\pi_1(X,x), \mathrm{Out}(A))$ is in bijection with the conjugacy classes in $\mathrm{Out}(A)$ and the $\mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X$-torsor of $\mathrm{L}(A,\sigma)$ is mapped to the conjugacy class of the class of σ^{-1} in $\mathrm{Out}(A)$. In particular, every $\mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X$-torsor is represented by some $\mathrm{L}(A,\sigma)$ for an appropriate σ.

Proof of (2). If $X = \mathrm{Spec}(\Bbbk[z])$ and $x = (z)$, the group $\pi_1(X,x)$ is trivial. Therefore, $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X)$ has only one element, consisting of the trivial $\mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X$-torsor on X, i.e. the one represented by $A[z]$. □

Corollary 6.1.2.
Every weakly locally trivial sheaf of Lie algebras on $\mathrm{Spec}(\Bbbk[u,u^{-1}])$ *is associated to a twisted loop algebra. Every weakly locally free sheaf of Lie algebras on* $\mathrm{Spec}(\Bbbk[z])$ *is trivial, i.e. associated to a polynomial Lie algebra.*

6.1.2 Interlude: analytification. A ringed space $(X, \mathscr{O}_X)$ is called $\mathbb{C}$-*space*, if $\mathscr{O}_X$ is a sheaf of $\mathbb{C}$-algebras. A locally ringed $\mathbb{C}$-space $(V, \mathscr{O}_V)$ is called *local model space* if there exists an open subset $U \subseteq \mathbb{C}^n$ (in the complex analytic topology) and holomorphic functions $f_1, \dots, f_k$ on U, such that

$$V = \{z \in \mathbb{C}^n \mid f_1(z) = \dots = f_k(z) = 0\} \tag{6.1}$$

is the *vanishing locus* of $f_1, \dots, f_k$ and $\mathscr{O}_V = \mathscr{H}_U/(f_1, \dots, f_k)$ are the *holomorphic functions on* V, where $\mathscr{H}_U$ is the sheaf of holomorphic functions on U. A *complex analytic space* is a locally ringed $\mathbb{C}$-space which is locally isomorphic to a local model space.

For every reduced and separated $\mathbb{C}$-scheme X of finite type, it is clear that the closed points of X can be equipped with the structure of a complex analytic space, which will be denoted by $(X^{\mathrm{an}}, \mathscr{O}_X^{\mathrm{an}})$ and is called *analytification of* X. There is a natural morphism $\iota\colon X^{\mathrm{an}} \to X$ of locally ringed spaces, which identifies points of X^{an} as closed points of X and regular functions of X as holomorphic functions on X^{an}. The following results can be found in [Ser56].

Theorem 6.1.3.
Let X be a projective $\mathbb{C}$-scheme.
(1) *The assignment $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}} := \iota^*\mathscr{F}$ defines an equivalence of the category of coherent sheaves on X and coherent sheaves on X^{an}.*
(2) *For any coherent sheaf $\mathscr{F}$ on X and any $n \in \mathbb{N}_0$ there is a natural isomorphism* $\mathrm{H}^n(X, \mathscr{F}) \to \mathrm{H}^n(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$.
(3) *For any coherent sheaf $\mathscr{F}$ on X and any $q \in X^{\mathrm{an}}$, the canonical morphism* $\widehat{\mathscr{F}}_{\iota(q)} \to \widehat{\mathscr{F}}^{\mathrm{an}}_q$ *is bijective.*

6.1.3 Elliptic case. Let X be an elliptic curve over $\Bbbk = \mathbb{C}$. There exists an biholomorphic map $\nu\colon X^{\mathrm{an}} \to \mathbb{C}/\Lambda$ for some lattice $\Lambda = \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}} \subset \mathbb{C}$ of rank two. Let

$\mathrm{pr}\colon \mathbb{C} \to \mathbb{C}/\Lambda$ be the canonical map, $\mathscr{A}$ be a locally free sheaf of rank d on X and $E \to \mathbb{C}/\Lambda$ be the vector bundle with sheaf of holomorphic sections $\nu_*\mathscr{A}^{\mathrm{an}}$. In e.g. [Ien11] it is explained that, since pr^*E is trivial as a vector bundle, E is determined by some holomorphic map $\phi\colon \Lambda \times \mathbb{C} \to \mathrm{GL}(d, \mathbb{C})$ satisfying

$$\Phi(\lambda + \lambda', z) = \Phi(\lambda, z + \lambda')\Phi(\lambda', z) \text{ for all } \lambda, \lambda' \in \Lambda \text{ and } z \in \mathbb{C}, \tag{6.2}$$

called *factor of automorphy*, in the sense that

$$E = \mathbb{C} \times \mathbb{C}^d/\sim, \text{ where } (z, a) \sim (z + \lambda, \Phi(\lambda, z)a) \text{ for all } \lambda \in \Lambda. \tag{6.3}$$

Assume that $\mathscr{A}$ is an étale $\mathfrak{g}$-locally free sheaf of Lie algebras for some finite-dimensional complex Lie algebra $\mathfrak{g}$. Then it is easy to see that E is a holomorphic fiber bundle with fiber $\mathfrak{g}$ and structure group $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$. Therefore, [Gra58, Satz 6] implies that $\pi^*E \cong \mathbb{C} \times \mathfrak{g}$ as holomorphic fiber bundles. This implies that Φ takes values in $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$.

Theorem 6.1.4.
Let $\mathfrak{g}$ be a simple, finite-dimensional, complex Lie algebra, $\mathscr{A}$ be a weakly $\mathfrak{g}$-locally free acyclic (i.e. $\mathrm{h}^1(\mathscr{A}) = 0$) sheaf of Lie algebras on an elliptic curve X, and $\nu\colon X^{\mathrm{an}} \to \mathbb{C}/\Lambda$ be a biholomorphic map for some $\Lambda := \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}} \subseteq \mathbb{C}$ of rank two. There exist commuting $\phi_1, \phi_2 \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of finite order without common non-zero fixed vector such that $\nu_*\mathscr{A}^{\mathrm{an}}$ is isomorphic to the sheaf of holomorphic sections of

$$\mathbb{C} \times \mathfrak{g}/\sim \qquad (z, a) \sim (z + \lambda_1, \phi_1(a)) \sim (z + \lambda_2, \phi_2(a)). \tag{6.4}$$

Proof. Let $\pi\colon \mathbb{C} \to X^{\mathrm{an}}$ be the map induced by $\mathrm{pr}\colon \mathbb{C} \to \mathbb{C}/\Lambda$ and $\nu\colon X^{\mathrm{an}} \to \mathbb{C}/\Lambda$. We split the proof into four steps.

Step 1. $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$. The morphism $\mathscr{A} \to \mathscr{A}^*$ induced by the Killing form of $\mathscr{A}$ is an isomorphism, since $\mathscr{A}|_p \cong \mathfrak{g}$ is simple for all $p \in X$ closed; see Lemma 1.1.2. Therefore, the Serre duality implies $\mathrm{h}^0(\mathscr{A}) = \mathrm{h}^1(\mathscr{A}^*) = \mathrm{h}^1(\mathscr{A}) = 0$.

Step 2. *Description of $\Gamma(X \setminus \{p\}, \mathscr{A})$, where $p := \iota\pi(0)$.* Theorem 1.2.3 states that $\mathscr{A}$ is étale $\mathfrak{g}$-locally trivial. As argued above, this implies that the holomorphic vector bundle $E \to \mathbb{C}/\Lambda$ with holomorphic sheaf of sections $\nu_*\mathscr{A}^{\mathrm{an}}$ is determined by a factor of automorphy $\Phi\colon \mathbb{C} \times \Lambda \to \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$. Using the first step and Theorem 1.3.4 results in $\Gamma(X \setminus \{p\}, \mathscr{A}) \oplus \widehat{\mathscr{A}_p} = \mathrm{Q}(\widehat{\mathscr{A}_p})$. Let $\mathfrak{W}$ be the algebra of global meromorphic sections of $\nu_*\mathscr{A}^{\mathrm{an}}$ which are holomorphic except in $\mathrm{pr}(0) \in \mathbb{C}/\Lambda$, i.e.

$$\mathfrak{W} = \left\{ a\colon \mathbb{C} \to \mathfrak{g} \text{ meromorphic} \;\middle|\; \begin{array}{c} a \text{ is holomorphic on } \mathbb{C} \setminus \Lambda \\ a(z + \lambda) = \Phi(\lambda, z)a(z), \forall \lambda \in \Lambda, z \in \mathbb{C} \setminus \Lambda \end{array} \right\}.$$

The Laurent expansion of local sections of $\nu_*\mathscr{A}^{\mathrm{an}}$ in $\pi(0)$ with respect to some holomorphic coordinate z of $\mathbb{C}$ in 0 combined with $\mathrm{h}^i(\mathscr{A}^{\mathrm{an}}) = \mathrm{h}^i(\mathscr{A}) = 0$ for $i \in \{0, 1\}$ induces the identification $\mathfrak{W} \oplus \mathfrak{g}[\![z]\!] = \mathfrak{g}(\!(z)\!)$. The canonical isomorphism

$\zeta\colon \widehat{\mathscr{A}_p} \to \widehat{\mathscr{A}^{\mathrm{an}}_{\pi(0)}} \cong \mathfrak{g}[\![z]\!]$ induces a commutative diagram

$$\begin{array}{ccc} \Gamma(X\setminus\{p\},\mathscr{A})\oplus\widehat{\mathscr{A}_p} & \xrightarrow{\cong} & \mathrm{Q}(\widehat{\mathscr{A}_p}) \\ \downarrow & & \downarrow\cong \\ \mathfrak{W}\oplus\mathfrak{g}[\![z]\!] & \xrightarrow[\cong]{} & \mathfrak{g}(\!(z)\!) \end{array} , \tag{6.5}$$

so $\zeta(\Gamma(X\setminus\{p\},\mathscr{A})) = \mathfrak{W}$.

Step 3. *Φ is locally constant up to isomorphism.* Proposition 2.2.1 and $\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!]\oplus\mathfrak{W}$ implies that there exists a normalized formal generalized r-matrix r such that $\mathfrak{g}(r) = \mathfrak{W} = \zeta(\Gamma(X\setminus\{p\},\mathscr{A}))$. We can chose a global 1-form η on X such that $\pi^*\iota^*\eta = \mathrm{d}z$ as a holomorphic 1-form on $\mathbb{C}$. The residue theorem forces

$$\mathrm{res}_0 K(\zeta(a),\zeta(b))\mathrm{d}z = \mathrm{res}_p\mathcal{K}(a,b)\eta = 0 \tag{6.6}$$

for all $a,b\in\Gamma(X\setminus\{p\},\mathscr{A})$, where $\mathcal{K}$ is the Killing form of $\mathscr{A}$. Therefore, $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$ and Proposition 2.3.1 forces r to be skew-symmetric. Combining Lemma 3.1.1 and Proposition 2.3.4, we may assume that $\mathfrak{g}(r)$ is closed under the derivation with respect to z, after probably replacing $\mathscr{A}$ with an isomorphic sheaf of Lie algebras. In particular, we have

$$\Phi(\lambda,z)\frac{da}{dz}(z) = \frac{da}{dz}(z+\lambda) = \frac{\partial\Phi}{\partial z}(\lambda,z)a(z) + \Phi(\lambda,z)\frac{da}{dz}(z). \tag{6.7}$$

for every $a\in\mathfrak{W}$. Therefore, $\frac{\partial\Phi}{\partial z}(\lambda,z)a(z) = 0$ for all $z\in\mathbb{C}\setminus\Lambda$, $a\in\mathfrak{W}$. Since $\mathfrak{W}\otimes\mathbb{C}(\!(z)\!)\cong\mathfrak{g}(\!(z)\!)$, we see that $\frac{\partial\Phi}{\partial z}(\lambda,z) = 0$, and thus $\Phi_\lambda := \Phi(\lambda,z)\in\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ is independent of z.

Step 4. *$P := \{\Phi_\lambda\}_{\lambda\in\Lambda}$ is a finite abelian group.* Equation (6.2) implies that $\Phi_\lambda\Phi_{\lambda'} = \Phi_{\lambda+\lambda'}$ for all $\lambda,\lambda'\in\Lambda$, so $P := \{\Phi_\lambda\}_{\lambda\in\Lambda}$ is a commutative subgroup of $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ generated by $\phi_1 := \Phi_{\lambda_1}, \phi_2 := \Phi_{\lambda_2}$. A non-zero element in $\mathfrak{g}$ which is fixed by all elements in P would define a global section of $\mathscr{A}$. Hence, such an element does not exist by Step 1. Assume that P has infinite order and let $\mathfrak{s}$ be the Lie algebra of the smallest algebraic subgroup S of $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ containing P. Since P is infinite, $\mathfrak{s}$ can be identified with a non-zero subalgebra of $\mathfrak{g}$. Since P is abelian and dense (with respect to the Zariski topology) in S, it can be shown that S is abelian. Therefore, the action of S on $\mathfrak{s}$ is trivial and each non-zero element of $\mathfrak{s}$ is fixed by all elements in P. This is a contradiction. We can conclude that P has finite order. □

6.2 The trichotomy theorem

The results of Subsection 6.1 and the geometric trichotomy presented in Theorem 3.2.5 and Remark 3.2.6 can now be combined with the notion of geometric r-matrix to derive a new proof of the Belavin-Drinfeld trichotomy. In particular, in this section we proof the following result:

Theorem 6.2.1.
Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra, $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a normalized formal r-matrix, and $((X,\mathscr{A}),(C,\eta))$ be the geometric CYBE datum of r. The following results are true.
(1) *X is an elliptic curve if and only if r is equivalent to the Taylor series of an analytic r-matrix $\widetilde{r}\colon \mathbb{C}\times\mathbb{C}\to\mathfrak{g}\otimes\mathfrak{g}$ in $y=0$, such that $\widetilde{r}(x+\lambda,y+\lambda')=\widetilde{r}(x,y)$ for all λ, λ' in some rank-two lattice in $\mathbb{C}$.*
(2) *X is a nodal plane cubic curve if and only if there exist some $\sigma\in\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of order $m\in\mathbb{N}$ and σ-trigonometric r-matrix ϱ (see Subsection 5.4.4) such that r is equivalent to the Taylor series of $\widetilde{r}(x,y)=\varrho(\exp(x/m),\exp(y/m))$ in $y=0$.*
(3) *X is a cuspidal plane cubic curve if and only if r is equivalent to the Taylor series of a rational r-matrix $\widetilde{r}$ (see Subsection 5.4.5) in $y=0$.*

6.2.1 Proof of Theorem 6.2.1. Let $O\subseteq\mathrm{Mult}(\mathfrak{g}(r))$ be chosen as in Theorem 3.2.5, $((X,\mathscr{A}),(p,c,\zeta)) := \mathbb{G}(O,\mathfrak{g}(r))$, and ρ be the geometric r-matrix of $((X,\mathscr{A}),(C,\eta))$. Let $X^{\mathrm{an}}=(X^{\mathrm{an}},\mathscr{O}_X^{\mathrm{an}})$ denote the complex analytic space associated to X and $\iota\colon X^{\mathrm{an}}\to X$ be the canonical morphism of locally ringed spaces. Recall that C is the smooth locus of X and write $C^{\mathrm{an}} := \iota^{-1}(C)\subseteq X^{\mathrm{an}}$ for the respective smooth locus of X^{an}.

If $X=C$ is smooth, it is a complex elliptic curve, so there exists a lattice $\Lambda\subseteq\mathbb{C}$ of rank two as well as a biholomorphic map $\widetilde{\nu}\colon\mathbb{C}/\Lambda\to X^{\mathrm{an}}$ such that $\widetilde{\nu}(\Lambda)=p$. Otherwise, X is a plane cubic curve with a unique singular closed point s and normalization $\nu\colon\mathbb{P}^1_{\mathbb{C}}\to X$, where one of the following cases occurs:

- s is nodal and we can choose coordinates $(u_0:u_1)$ on $\mathbb{P}^1_{\mathbb{C}}$ such that
$$\nu^{-1}(s)=\{(1:0),(0:1)\} \text{ and } \nu(1:1)=p. \tag{6.8}$$
In particular, ν restricts to an isomorphism $\mathrm{Spec}(\mathbb{C}[u,u^{-1}])\to C$ for $u=u_1/u_0$ and induces a biholomorphic map $\widetilde{\nu}\colon\mathbb{C}^\times\to C^{\mathrm{an}}$.
- s is cuspidal and we can choose coordinates $(z_0:z_1)$ on $\mathbb{P}^1_{\mathbb{C}}$ such that
$$\nu^{-1}(s)=\{(0:1)\} \text{ and } \nu(1:0)=p. \tag{6.9}$$
In particular, ν restricts to an isomorphism $\mathrm{Spec}(\mathbb{C}[z])\to C$ for $z=z_1/z_0$ and induces a biholomorphic map $\widetilde{\nu}\colon\mathbb{C}\to C^{\mathrm{an}}$.

Summarized, we have a holomorphic covering $\widetilde{\pi}\colon\mathbb{C}=(\mathbb{C},\mathscr{O}_{\mathbb{C}}^{\mathrm{an}})\to C^{\mathrm{an}}$ satisfying $\iota\widetilde{\pi}(0)=p$, defined by
$$\widetilde{\pi}(\widetilde{z})=\begin{cases}\widetilde{\nu}(\widetilde{z}+\Lambda) & \text{if } X \text{ is elliptic}\\ \widetilde{\nu}\left(\exp\left(\widetilde{z}\right)\right) & \text{if } X \text{ is nodal}\\ \widetilde{\nu}(\widetilde{z}) & \text{if } X \text{ is cuspidal}\end{cases} \tag{6.10}$$
in some holomorphic coordinates $\widetilde{z}$ on $\mathbb{C}$. Let $\pi := \iota\widetilde{\pi}\colon\mathbb{C}\to X$. The invertible sheaf $\Omega_C^{\mathrm{an}}=\iota^*\Omega_C$ can be identified with the sheaf of holomorphic 1-forms on C^{an}, so $\iota^*\eta$ can be viewed as holomorphic 1-form. We can assume that $\pi^*\eta=\mathrm{d}\widetilde{z}$. Indeed, it is well-known that there exists a $\lambda\in\mathbb{C}^\times$ such that $\pi^*\eta=\lambda\mathrm{d}\widetilde{z}$ if X is elliptic and
$$\nu^*(\eta)=\begin{cases}\lambda\mathrm{d}u/u & \text{if } X \text{ is nodal}\\ \lambda\mathrm{d}z & \text{if } X \text{ is cuspidal}\end{cases}. \tag{6.11}$$

Hence, we can achieve that $\lambda = 1$ by replacing r with $\lambda r(\lambda x, \lambda y)$.

Lemma 6.2.2.
Let $\theta\colon \widehat{\mathscr{O}}^{\mathrm{an}}_{\mathbb{C},0} \to \mathbb{C}[\![\tilde{z}]\!]$ be the isomorphism defined by the Taylor series in 0. Then the diagram

$$\begin{array}{ccc} \widehat{\mathscr{O}}_{X,p} & \xrightarrow{\widehat{\pi}_0^\sharp} & \widehat{\mathscr{O}}^{\mathrm{an}}_{\mathbb{C},0} \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathbb{C}[\![z]\!] & \xrightarrow[z\mapsto\tilde{z}]{} & \mathbb{C}[\![\tilde{z}]\!] \end{array} \tag{6.12}$$

commutes. Furthermore, for any rational function f on X, $c(f) \in \mathbb{C}(\!(z)\!)$ coincides with the Laurent series of a meromorphic function on $\mathbb{C}$ in 0 which is
(1) *elliptic if and only if X is an elliptic curve,*
(2) *a rational function of exponentials if and only if X is a nodal plane cubic curve, and*
(3) *rational if and only if X is a cuspidal plane cubic curve.*

Proof. As in Remark 3.2.2, the canonical derivations $\mathscr{O}_{X,p} \to \omega_{X,p}$ and $\mathscr{O}^{\mathrm{an}}_{\mathbb{C},0} \to \Omega^{\mathrm{an}}_{\mathbb{C},0}$ induce continuous derivations $\widehat{\mathscr{O}}_{X,p} \to \widehat{\omega}_{X,p}$ and $\widehat{\mathscr{O}}^{\mathrm{an}}_{\mathbb{C},0} \to \widehat{\Omega}^{\mathrm{an}}_{\mathbb{C},0}$ whose images generate the respective modules. These derivations will both be denoted by d, since it will be clear from the context which one is in use. The completion $\widehat{\omega}_{X,p} \to \widehat{\Omega}^{\mathrm{an}}_{\mathbb{C},0}$ of

$$\omega_{X,p} \xrightarrow{\pi_p^*} (\pi_*\pi^*\Omega^{\mathrm{an}}_{\mathbb{C}})_p \longrightarrow \Omega^{\mathrm{an}}_{\mathbb{C},0} \tag{6.13}$$

is described by $\mathrm{d}f \mapsto \mathrm{d}\widehat{\pi}_0^\sharp(f)$ for all $f \in \widehat{\mathscr{O}}_{X,p}$. The identity $c^*(\widehat{\eta}_p) = \mathrm{d}z$ implies that $\widehat{\eta}_p = \mathrm{d}c^{-1}(z)$ (see Remark 3.2.2) and $\pi^*\eta = \mathrm{d}\tilde{z}$ implies that $\mathrm{d}\widehat{\pi}_0^\sharp(c^{-1}(z)) = \mathrm{d}\tilde{z}$. This yields $\widehat{\pi}_0^\sharp(c^{-1}(z)) = \tilde{z}$, i.e. (6.12) is commutative, since $\theta(\tilde{z}) = \tilde{z}$.

Let f be a rational function of X. Then $\pi^\flat(f)(\tilde{z}) = f(\pi(\tilde{z}))$ is a meromorphic function on $\mathbb{C}$ and its Laurent series in 0 coincides with its image of the extension of θ to the respective quotient fields. The commutativity of (6.12) implies that this Laurent series evaluated in z coincides with $c(f) \in \mathbb{C}(\!(z)\!)$. Looking at (6.10), we can see that $f(\pi(\tilde{z}))$ is elliptic if and only if X is elliptic, a rational function of exponentials if and only if X is nodal, and a rational function if and only if X is cuspidal. Here, we used $f\iota\tilde{\nu} = f\nu\iota$ and the fact that $f\nu$ is a rational function on $\mathbb{P}^1_{\mathbb{C}}$ if X is singular, i.e. simply a quotient of two polynomials. □

Lemma 6.2.3.
The "$\Longleftarrow$" directions hold in Theorem 6.2.1.(1)-(3).

Proof. Assume that $\tilde{r}$ is of the form stated in Theorem 6.2.1. By Lemma 2.2.3, $\mathfrak{g}(\tilde{r})$ is generated by $\{(1 \otimes \alpha)\tilde{r}(z,0) \mid \alpha \in \mathfrak{g}^*\}$, so it can be identified with a subalgebra of meromorphic maps $\mathbb{C} \to \mathfrak{g}$, which are elliptic in case (1), rational functions of $\exp(z/m)$ in case (2), and rational in case (3). Since r and $\tilde{r}$ are equivalent r-matrices in normalized standard form, $\mathrm{Mult}(\mathfrak{g}(r))$ coincides with

$\mathrm{Mult}(\mathfrak{g}(\widetilde{r}))$, up to $z \mapsto \lambda z$ for some $\lambda \in \Bbbk^\times$; see Lemma 2.1.6.(4). We have seen in the proof of Theorem 1.3.3 that there exists a subalgebra $O \subseteq \mathrm{Mult}(\mathfrak{g}(\widetilde{r}))$ of finite codimension with the property: for every $f \in O$ exists a non-commutative polynomial $P = P(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n \in \mathfrak{g}(\widetilde{r})$ satisfying

$$f\mathrm{id}_{\mathfrak{g}((z))} = P(\mathrm{ad}(a_1), \dots, \mathrm{ad}(a_n)). \tag{6.14}$$

In particular, O consists of elliptic functions in case (1), rational functions of exponentials in case (2), and rational functions in case (3). Thus, the same is true for the image $\widetilde{O}$ of O in $\mathrm{Mult}(\mathfrak{g}(r))$. Since the quotient field of $\widetilde{O}$ coincides with the rational functions on X, this observation combined with Lemma 6.2.2 proves all "$\Longleftarrow$" directions. □

Lemma 6.2.4.
There exists an analytic r-matrix $\widetilde{r}\colon \mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$ of the form

$$\widetilde{r} = (\psi \boxtimes \psi)(\pi \times \pi)^* \rho|_{C \times C} \tag{6.15}$$

for an isomorphism $\psi\colon \pi^*\mathscr{A} \to \mathfrak{g} \otimes \mathscr{O}_\mathbb{C}$ such that:
(1) *If X is an elliptic curve, $\widetilde{r}(x + \lambda, y + \lambda') = \widetilde{r}(x, y)$ for all $\lambda, \lambda' \in n\Lambda$, where n is some natural number.*
(2) *If X is a nodal plane cubic curve, there exist some $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of order m and σ-trigonometric r-matrix ϱ such that $\widetilde{r}(x, y) = \varrho(\exp(x/m), \exp(y/m))$.*
(3) *If X is a cuspidal plane cubic curve, $\widetilde{r}$ is a rational r-matrix.*

Proof of (1). Since $\mathscr{A}$ is weakly $\mathfrak{g}$-locally free on the elliptic curve X, Theorem 6.1.4 provides an isomorphism $\psi_1\colon \widetilde{\nu}^*\iota^*\mathscr{A} \to \mathscr{S}$, where $\mathscr{S}$ is the sheaf on $\mathbb{C}/\Lambda$ of holomorphic sections of

$$\mathbb{C} \times \mathfrak{g}/\sim, \text{ where } (z, a) \sim (z + \lambda_1, \phi_1(a)) \sim (z + \lambda_2, \phi_2(a))$$

for $\phi_1, \phi_2 \in \mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}(\mathfrak{g})$ of finite order. Choose $n \in \mathbb{N}$ such that $\phi_1^n = \mathrm{id}_\mathfrak{g} = \phi_2^n$. Let $\psi_2\colon \mathrm{pr}^*\mathscr{S} \to \mathfrak{g} \otimes \mathscr{O}_\mathbb{C}^{\mathrm{an}}$ denote the canonical isomorphism, where $\mathrm{pr}\colon \mathbb{C} \to \mathbb{C}/\Lambda$ is the canonical projection. Then $\psi_2\mathrm{pr}^*a\colon \mathbb{C} \to \mathfrak{g}$ is an $n\Lambda$-periodic meromorphic function for any rational section a of $\mathscr{S}$. Therefore,

$$\widetilde{r} \coloneqq (\psi \boxtimes \psi)(\pi \times \pi)^*\rho\colon \mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g} \tag{6.16}$$

is an analytic r-matrix (see Theorem 3.3.4) satisfying the desired periodicity, where $\psi \coloneqq \psi_2(\mathrm{pr}^*\psi_1)$ and $\pi = \iota\widetilde{\nu}\mathrm{pr}$ was used.

Proof of (2). Recall that $\nu\colon \mathrm{Spec}(\mathbb{C}[u, u^{-1}]) \to C$ is an isomorphism (since X is nodal) and $\mathscr{A}|_C$ is weakly $\mathfrak{g}$-locally free. Theorem 6.1.1.(1) provides an isomorphism $\psi_1\colon \nu^*\mathscr{A}|_C \to \mathscr{L}$, where $\mathscr{L}$ is the sheaf associated to $\mathfrak{L}(\mathfrak{g}, \sigma) \subseteq \mathfrak{g}[\tilde{u}, \tilde{u}^{-1}]$ on $\mathrm{Spec}(\mathbb{C}[u, u^{-1}]) = \nu^{-1}(C)$, for some $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of finite order m and $u = \tilde{u}^m$. Recall that $\gamma = \sum_{j=0}^{m-1} \gamma_j$ for $\gamma_j \in \mathfrak{g}_j^\sigma \otimes \mathfrak{g}_{-j}^\sigma$, since $(\sigma \otimes \sigma)\gamma = \gamma$. Choosing the local parameter $\nu^{\flat,-1}(u-1)$ of p and

$$\chi = (\psi_1 \boxtimes \psi_1)(\nu^{-1} \times \nu^{-1})^*\widetilde{\chi}, \text{ where } \widetilde{\chi} \coloneqq \sum_{j=0}^{m-1} (\tilde{u}/\tilde{v})^j \gamma_j \in \mathfrak{L}(\mathfrak{g}, \sigma) \otimes \mathfrak{L}(\mathfrak{g}, \sigma),$$

in (3.37) yields

$$\varrho \coloneqq (\psi_1 \boxtimes \psi_1)(\nu \times \nu)^* \rho|_{C\times C} = \frac{v\widetilde{\chi}}{u-v} + \varrho_0, \tag{6.17}$$

for some $\varrho_0 \in \mathfrak{L}(\mathfrak{g},\sigma) \otimes \mathfrak{L}(\mathfrak{g},\sigma)$. In particular, ϱ is a σ-trigonometric r-matrix by virtue of Theorem 3.3.4. Here, we used $\tilde{u}^m = u$, $\eta = \mathrm{d}u/u$, and the fact that $\gamma = \sum_{j=0}^{m-1}\gamma_j$ implies that χ is indeed a preimage of $\mathrm{id}_{\mathscr{A}|_C}$ under (3.34). The mapping $a(\tilde{u}) \mapsto a(\exp(z/m)))$ induces an isomorphism $\psi_2\colon \exp^*\iota^*\mathscr{L} \to \mathfrak{g} \otimes \mathscr{O}_{\mathbb{C}}^{\mathrm{an}}$ such that

$$\widetilde{r} \coloneqq (\psi_2 \boxtimes \psi_2)(\iota\exp \times \iota\exp)^*\varrho|_{C\times C} = (\psi \boxtimes \psi)(\pi \times \pi)^*\rho|_{C\times C}\colon \mathbb{C}\times\mathbb{C} \to \mathfrak{g}\otimes\mathfrak{g}$$

is of the desired form, where $\psi \coloneqq \psi_2\,(\exp^*\iota^*\psi_1)$ and $\nu\iota\exp = \iota\widetilde{\nu}\exp = \pi$ was used.

Proof of (3). If X is cuspidal, the construction of $\widetilde{r}$ can be carried out similar to the nodal case: using Theorem 6.1.1.(2), the local parameter $\nu^{\flat,-1}(z)$ of p, an appropriate χ that is constructed from γ, and (3.37) gives $\widetilde{r}$ in the desired form, under consideration of $\eta = \mathrm{d}u$. □

Lemma 6.2.5.
The formal r-matrix obtained from the Taylor expansion of $\widetilde{r}$ in $y = 0$ is gauge equivalent to r.

Proof. By construction in Lemma 6.2.4, $\widetilde{r} = (\psi \boxtimes \psi)(\pi\times\pi)^*\rho|_{C\times C}$ for an isomorphism $\psi\colon \pi^*\mathscr{A} \to \mathfrak{g}\otimes\mathscr{O}_{\mathbb{C}}$. Using Lemma 6.2.2, we can see that the composition

$$\mathfrak{g}[\![z]\!] \xrightarrow{\zeta^{-1}} \widehat{\mathscr{A}_p} \longrightarrow \widehat{\pi^*\mathscr{A}}_0 \xrightarrow{\widehat{\psi}_0} \mathfrak{g}\otimes\widehat{\mathscr{O}}^{\mathrm{an}}_{\mathbb{C},0} \xrightarrow{\mathrm{id}_{\mathfrak{g}}\otimes\theta} \mathfrak{g}[\![z]\!] \tag{6.18}$$

defines a $\mathbb{C}[\![z]\!]$-linear Lie algebra automorphism φ of $\mathfrak{g}[\![z]\!]$. Here, the second arrow is the isomorphism obtained by completing $\mathscr{A}_p \to (\pi_*\pi^*\mathscr{A})_p \to (\pi^*\mathscr{A})_0$ and θ is the map defined by the Taylor expansion in 0. It is straight forward to show that the diagram

$$\begin{array}{ccccc}
\Gamma(C\times C\setminus\Delta, \mathscr{A}\boxtimes\mathscr{A}) & \xrightarrow{\jmath^*} & (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] & \xrightarrow{\varphi(x)\otimes\varphi(y)} & (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] \\
\Big\downarrow{\scriptstyle(\pi\times\pi)^*} & & & \nearrow & \\
\Gamma(\mathbb{C}\times\mathbb{C}\setminus\Delta_\pi, \pi^*\mathscr{A}\boxtimes\pi^*\mathscr{A}) & \xrightarrow[\psi\boxtimes\psi]{} & (\mathfrak{g}\otimes\mathfrak{g})\otimes\Gamma(\mathbb{C}\times\mathbb{C}\setminus\Delta_\pi, \mathscr{O}^{\mathrm{an}}_{\mathbb{C}\times\mathbb{C}}) & &
\end{array}$$

commutes, where $\Delta_\pi \coloneqq \{(x,y)\in\mathbb{C}\times\mathbb{C} \mid \pi(x) = \pi(y)\}$, $\jmath^*$ is given in (3.33), and the unlabeled arrow is given by Taylor expansion in $y = 0$. The upper row maps $\rho|_{C\times C}$ to $(\varphi(x)\otimes\varphi(y))r(x,y)$ by virtue of Theorem 3.3.3. This concludes the proof. □

Remark 6.2.6.
Note that the three classes from Theorem 6.2.1 are preserved by arbitrary formal equivalences because of Remark 3.2.7. This fact remained unclear in the classical

approach from [BD83a; BD83b].

7 Elliptic r-matrices

Throughout this chapter, $\mathfrak{g}$ is a finite-dimensional, simple, complex Lie algebra.

7.1 Acyclic weakly $\mathfrak{g}$-locally free sheaves of algebras on elliptic curves.

In this section, we present a classification of acyclic weakly $\mathfrak{g}$-locally free sheaves of algebras on elliptic curves, refining Theorem 6.1.4 based on the results of [BD83a, Section 5]. Translating said reference to the algebro-geometric language will result in the classification of elliptic r-matrices in the next section.

7.1.1 Prelude: pairs of automorphisms of $\mathfrak{g}$ without common fixed vector. The classification of pairs of automorphisms of $\mathfrak{g}$ without a common eigenvector of eigenvalue 1 relies on the following solvability criterion for finite-dimensional Lie algebras from [BD84, Section 9].

Proposition 7.1.1.
Let $\mathfrak{l}$ be a finite-dimensional Lie algebra and assume $\phi \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{l})$ exists such that $\det(\phi - \mathrm{id}_{\mathfrak{l}}) \neq 0$. Then $\mathfrak{l}$ is solvable.

Proof. If ϕ has finite order and $\mathfrak{l}$ is simple, we can deduce from Theorem 4.2.3 that $\mathfrak{l}_0^\phi := \{a \in \mathfrak{l} \mid \phi(a) = a\} \neq \{0\}$, so $\det(\phi - \mathrm{id}_{\mathfrak{l}}) = 0$. If ϕ has infinite order and $\mathfrak{l}$ is simple, we can repeat the argument in Step 4. of the proof of Theorem 6.1.4 for the group P generated by ϕ to see that again a fixed vector of ϕ necessarily exists, so $\det(\phi - \mathrm{id}_{\mathfrak{l}}) = 0$.

Assume now that $\mathfrak{l}$ is semi-simple and let $\mathfrak{i} \subseteq \mathfrak{l}$ be a simple ideal. Since $\mathfrak{l}$ is a finite direct sum of its simple ideals, there exists $n \in \mathbb{N}$ such that $\phi^n(\mathfrak{i}) = \mathfrak{i}$. Therefore, as argued above, there exists $a \in \mathfrak{i}$ such that $\phi^n(a) = a$. Choosing n to be minimal satisfying $\phi^n(\mathfrak{i}) = \mathfrak{i}$, we see that $a, \phi(a), \dots, \phi^{n-1}(a) \in \mathfrak{l}$ are linearly independent, so $0 \neq \widetilde{a} := \sum_{k=0}^{n-1} \phi^k(a)$ satisfies $\phi(\widetilde{a}) = \widetilde{a}$. In particular, we conclude $\det(\phi - \mathrm{id}_{\mathfrak{l}}) = 0$.

Finally, let $\mathfrak{l}$ be an arbitrary finite-dimensional Lie algebra and $\phi \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{l})$ satisfy $\det(\phi - \mathrm{id}_{\mathfrak{l}}) \neq 0$. The automorphism ϕ respects the radical $\mathrm{rad}(\mathfrak{l})$ of $\mathfrak{l}$. Therefore, ϕ induces an automorphism of the semi-simple Lie algebra $\mathfrak{l}/\mathrm{rad}(\mathfrak{l})$ without fixed point. This forces $\mathfrak{l}/\mathrm{rad}(\mathfrak{l}) = \{0\}$, so $\mathfrak{l} = \mathrm{rad}(\mathfrak{l})$ is solvable. □

The following result can be found in [BD83a, Proposition 5.2]. For sake of completeness, we will give the proof presented there.

Theorem 7.1.2.
Let $\phi_1, \phi_2 \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ *be of finite order such that* $\phi_1\phi_2 = \phi_2\phi_1$ *and*

$$\{a \in \mathfrak{g} \mid \phi_1(a) = a = \phi_2(a)\} = \{0\}. \tag{7.1}$$

There exist coprime integers $0 < m < n$ *and an isomorphism* $\psi\colon \mathfrak{g} \to \mathfrak{sl}_n(\mathbb{C})$ *of complex Lie algebras such that* $\psi\phi_1\psi^{-1} = \mathrm{Ad}_{T_1^m}$ *and* $\psi\phi_2\psi^{-1} = \mathrm{Ad}_{T_2}$ *for*

$$T_1 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{pmatrix}, T_2 := \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \text{ where } \varepsilon := \exp\left(\frac{2\pi i}{n}\right). \tag{7.2}$$

Proof. We will split the proof into several parts.

Step 1. ϕ_2 *induces an automorphism* ν *of the Dynkin diagram of* $\mathfrak{L}(\mathfrak{g}, \phi_1)$. The Lie algebra $\mathfrak{g}_0 := \{a \in \mathfrak{g} \mid \phi_1(a) = a\}$ is non-zero and reductive; see Theorem 4.2.3. Furthermore, ϕ_2 defines an Lie algebra automorphism on $\mathfrak{g}_0$ without fixed vector, so $\mathfrak{g}_0$ is solvable by virtue of Proposition 7.1.1. Combined, we see that $\mathfrak{g}_0$ is abelian. In particular, $\mathfrak{g}_0 = \mathfrak{H}$ is the only triangular decomposition of $\mathfrak{g}_0$. Let $(\widehat{\mathfrak{L}}(\mathfrak{g}, \phi_1), \widehat{\mathfrak{H}}, \Pi, \Pi^\vee)$ be the associated realization from Theorem 4.2.3. Since ϕ_1 and ϕ_2 commute, ϕ_2 defines an automorphism of $\mathfrak{L}(\mathfrak{g}, \phi_1)$ which fixes the standard Borel subalgebras induced by the realization $(\widehat{\mathfrak{L}}(\mathfrak{g}, \phi_1), \widehat{\mathfrak{H}}, \Pi, \Pi^\vee)$. Therefore, ϕ_2 induces an automorphism of the Dynkin Diagram of $\mathfrak{L}(\mathfrak{g}, \phi_1)$; see Lemma 4.2.6.

Step 2. *There exists an isomorphism* $\widetilde{\psi}\colon \mathfrak{g} \to \mathfrak{sl}_n(\mathbb{C})$. We have seen that $\phi_2(\alpha_i^*) = \alpha_{\nu(i)}^*$ for some automorphism ν of the Dynkin diagram of $\mathfrak{L}(\mathfrak{g}, \phi_1)$, where $\Pi = \{\alpha_0, \dots, \alpha_\ell\}$ and $\alpha_i^* \in \mathfrak{H}$ is determined by $\alpha_i|_{\mathfrak{H}} = K(\alpha_i^*, -)$ for the Killing form K of $\mathfrak{g}$ and $i \in \{0, \dots, \ell\}$. Assume that $S \subseteq \{0, \dots, \ell\}$ is stable under ν. Then $0 \neq a := \sum_{i \in S} \alpha_i^* \in \mathfrak{g}_0$ satisfies $\phi_2(a) = a$, so $a = 0$ by assumption. Since Π has exactly one linear relation among its elements, we can deduce that $S = \{0, \dots, \ell\}$, so $\{\nu^j \mid j \in \mathbb{N}_0\}$ operates transitively on Π. This forces the Dynkin diagram of $\mathfrak{L}(\mathfrak{g}, \phi_1)$ to be of type $A_\ell^{(1)}$; see Figure 4.2. Therefore, ϕ_1 is inner and there exists an isomorphism $\widetilde{\psi}\colon \mathfrak{g} \to \mathfrak{sl}_n(\mathbb{C})$ for $n = \ell + 1$.

Step 3. ϕ_1, ϕ_2 *define a homomorphism* $\pi\colon \mathbb{Z}_N^2 \to \mathrm{PGL}_n(\mathbb{C})$ *for some* $N \in \mathbb{N}$. Repeating the first two steps for ϕ_1 and ϕ_2 switched implies that ϕ_2 is also inner. Therefore, there exists $A_1, A_2 \in \mathrm{SL}_n(\mathbb{C})$ such that $\widetilde{\psi}\phi_i\widetilde{\psi}^{-1} = \mathrm{Ad}_{A_i}$ for $i \in \{1, 2\}$, where $\mathrm{Ad}_A(a) = AaA^{-1}$ for all $A \in \mathrm{SL}_n(\mathbb{C})$ and $a \in \mathfrak{sl}_n(\mathbb{C})$. The facts $\phi_1^N = \mathrm{id}_{\mathfrak{g}} = \phi_2^N$ and $\phi_1\phi_2 = \phi_2\phi_1$ imply

$$A_1A_2A_1^{-1}A_2^{-1}, A_1^N, A_2^N \in \mathrm{Ker}(\mathrm{Ad}) = \mathbb{C}^\times \mathrm{id}_{\mathbb{C}^n}. \tag{7.3}$$

Therefore, $\pi\colon \mathbb{Z}_N^2 \to \mathrm{PGL}_n(\mathbb{C}) := \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times \mathrm{id}_{\mathbb{C}^n}$ defined by $(k, \ell) \mapsto A_1^k A_2^\ell$ is a homomorphism.

Step 4. π *is irreducible understood as projective representation.* Since π is a projective representation of a finite group, it is irreducible if and only if it is indecomposable; see [Kar85, Chapter 3, theorems 2.5 & 2.10]. Assume π is decomposable: there exists a non-trivial decomposition $\mathbb{C}^n = V_1 \oplus V_2$ such that $A_iV_j = V_j$ for $i, j \in \{1, 2\}$. Let $\mathrm{pr}\colon \mathbb{C}^n \to \mathbb{C}^n$ be the canonical projection onto V_1.

It is easy to see that $A_i \mathrm{pr} A_i^{-1} = \mathrm{pr}$ for $i \in \{1,2\}$. Then $a := \mathrm{pr} - \frac{\dim(V_1)}{n}\mathrm{id}_{\mathbb{C}^n}$ can be interpreted as a non-zero element of $\mathrm{sl}_n(\mathbb{C})$ such that $\phi_i(\widetilde{\psi}(a)) = \widetilde{\psi} A_i a A_i^{-1} = \widetilde{\psi}(a)$ for $i \in \{1,2\}$. This is a contradiction. We deduce that π is irreducible.

Step 5. *Concluding the proof.* Consider the generators $g_1 = (0,1)$ and $g_2 = (1,1)$ of $\mathbb{Z}_N^2$ and note that $\mathbb{Z}_N^2$ is the product of the cyclic groups generated by g_1 and g_2. By virtue of [Kar85, Chapter 3, Theorem 7.1], we have $N = n$ and π is projectively equivalent to the projective representation π' determined by $g_1 \mapsto T_2$ and $g_2 \mapsto T_2 T_1^m$, where T_1 and T_2 are given in (7.2). Note that $T_1 T_2 T_1^{-1} T_2^{-1} \in \mathbb{C}\,\mathrm{id}_{\mathbb{C}^n}$, so $\pi' \colon \mathbb{Z}_n^2 \mapsto \mathrm{PGL}_n(\mathbb{C})$ is given by $(k,\ell) \mapsto T_1^{km} T_2^{\ell}$. The fact that π and π' are projectively equivalent can be formulated as: there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $PA_1P^{-1} = \mu_1 T_1^m$ and $PA_2P^{-1} = \mu_2 T_2$ for some $\mu_1, \mu_2 \in \mathbb{C}^\times$. In particular, $\mathrm{Ad}_P \mathrm{Ad}_{A_1} \mathrm{Ad}_P = \mathrm{Ad}_{T_1^m}$ and $\mathrm{Ad}_P \mathrm{Ad}_{A_2} \mathrm{Ad}_P = \mathrm{Ad}_{T_2}$. Therefore, the isomorphism $\psi := \mathrm{Ad}_P \widetilde{\psi} \colon \mathfrak{g} \to \mathfrak{sl}_n(\mathbb{C})$ satisfies $\psi \phi_1 \psi^{-1} = \mathrm{Ad}_{T_1^m}$ and $\psi \phi_2 \psi^{-1} = \mathrm{Ad}_{T_2}$. □

7.1.2 Refinement of Theorem 6.1.4. Recall that for every complex elliptic curve X exists an isomorphism $X^{\mathrm{an}} \cong \mathbb{C}/\langle 1, \tau \rangle_{\mathbb{Z}}$, for an appropriate

$$\tau \in \mathbb{H} := \{ z \in \mathbb{C} \mid z \text{ has positive imaginary part} \}. \tag{7.4}$$

Combining Theorem 7.1.2 and Theorem 6.1.4 immediately results in the following refinement of the classification of acyclic weakly $\mathfrak{g}$-locally free sheaves of Lie algebras on an elliptic curve.

Theorem 7.1.3.
Let $\mathscr{A}$ be an acyclic weakly $\mathfrak{g}$-locally free sheaf of Lie algebras on an elliptic curve X and $\nu \colon X^{\mathrm{an}} \to \mathbb{C}/\langle 1, \tau \rangle_{\mathbb{C}}$ be a biholomorphic map for an appropriate $\tau \in \mathbb{H}$. There exist coprime integers $0 < m < n$ such that $\nu_* \mathscr{A}^{\mathrm{an}}$ is isomorphic to the sheaf $\mathscr{Q}_{(\tau,(n,m))}$ of holomorphic sections of

$$\mathbb{C} \times \mathfrak{sl}_n(\mathbb{C}) / \sim \qquad (z, a) \sim (z+1, T_1^m a T_1^{-m}) \sim (z+\tau, T_2 a T_2^{-1}) \tag{7.5}$$

as sheaves of Lie algebras, where T_1 and T_2 are defined in (7.2).

7.1.3 Description via simple vector bundles. The following theorem from [BH15] gives an intrinsic algebro-geometric description of the sheaves appearing in Theorem 7.1.3.

Theorem 7.1.4.
Let $\mathscr{A}$ be an acyclic weakly $\mathfrak{g}$-locally free sheaf of Lie algebras on an elliptic curve X, $\nu \colon X^{\mathrm{an}} \to \mathbb{C}/\langle 1, \tau \rangle_{\mathbb{C}}$ be a biholomorphic map for an appropriate $\tau \in \mathbb{H}$, and $0 < m < n$ be coprime integers. For any simple locally free sheaf $\mathscr{S}$ (i.e. $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{S}) \cong \mathscr{O}_X$) of rank n and degree m, the sheaf $\mathscr{A}$ defined by the short exact sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{S}) \xrightarrow{\mathrm{Tr}_{\mathscr{S}}} \mathscr{O}_X \longrightarrow 0. \tag{7.6}$$

satisfies $\nu_*\mathscr{A}^{\mathrm{an}} \cong \mathscr{Q}_{(\tau,(n,m))}$.

Proof. If $\mathscr{S}$ and $\mathscr{S}'$ are simple locally free sheaves of Lie algebras with the same rank and degree, there is a canonical isomorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{S}) \cong \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{S}')$ respecting the trace; see [BH15, Proposition 2.14] for details. Therefore, it suffices to prove the assertion for some simple vector bundle $\mathscr{S}$ of rank n and degree m. We split the proof in three steps.

Step 1. *Setup.* It is easy to see that $\mathscr{A}^{\mathrm{an}}$ is the kernel of $\mathrm{Tr}_{\mathscr{S}^{\mathrm{an}}}$. By virtue of [Ien11, Theorem 5.24], we can chose $\mathscr{S}$ in such a way that $\mathscr{S}^{\mathrm{an}}$ is isomorphic to the sheaf of sections of $\mathbb{C} \times \mathbb{C}^n/\sim_1$, where

$$(z,v) \sim_1 (z+1,v) \sim_1 (z+\tau,\Phi(z)v) \text{ for } \Phi(z) = \begin{pmatrix} 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\\ \phi(z)^n & 0 & \dots & 0 \end{pmatrix} \tag{7.7}$$

and $\phi(z) = \exp(-\pi i m\tau - 2\pi i m z/n)$. Consider the sheaf $\widetilde{\mathscr{S}}$ of sections of $\mathbb{C} \times \mathbb{C}^n/\sim_2$, where

$$(z,v) \sim_2 (z+1,T_1'v) \sim_2 (z+\tau,T_2'v) \text{ for } T_1' = \varepsilon^m T_1^m, T_2' = c\phi(z)T_2. \tag{7.8}$$

Here, $\eta \coloneqq \exp(-2\pi i m\tau/n)$, $c \in \mathbb{C}^\times$ is chosen in such a way that $c^n = \eta^{n(n-1)/2}$ and T_1, T_2, ε are given in (7.2).

Step 2. $\mathscr{S}^{\mathrm{an}} \cong \widetilde{\mathscr{S}}$. Consider $A\colon \Lambda \times \mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ defined inductively by

$$A(0,z) \coloneqq \mathrm{id}_{\mathbb{C}^n}, A(k\tau+\ell,z) \coloneqq A(k\tau,z) \coloneqq \Phi(z+(k-1)\tau)A((k-1)\tau,z) \tag{7.9}$$

for $k,\ell \in \mathbb{Z}, k > 0$ and by $A(k\tau+\ell,z) \coloneqq A(-k\tau+\ell,z-k\tau)^{-1}$ for $k,\ell \in \mathbb{Z}, k < 0$. Furthermore, let $B(\lambda,z) \coloneqq D(z+\lambda)A(\lambda,z)D(z)^{-1}$ for $z \in \mathbb{C}, \lambda \in \Lambda$, where

$$D(z) \coloneqq \begin{pmatrix} \phi(z)^{n-1}a_1 & 0 & \dots & 0\\ 0 & \phi(z)^{n-2}a_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_n \end{pmatrix} \text{ for some } a_1,\dots,a_n \in \mathbb{C}^\times. \tag{7.10}$$

Since $\phi(z+1) = \phi(z)\varepsilon^{-m}$, the identity $B(1,z) = D(z+1)D(z)^{-1} = \varepsilon^m T_1^m = T_1'$ holds for any choice of $a_1,\dots,a_n$. Therefore, [Ien11, Theorem 2.6] provides the desired isomorphism $\mathscr{S}^{\mathrm{an}} \cong \widetilde{\mathscr{S}}$ if $a_1,\dots,a_n$ are chosen in such a way that $T_2' = B(\tau,z)$ holds. Using $\phi(z+\tau) = \phi(z)\eta$ and choosing $a_1,\dots a_n \in \mathbb{C}$ such that

$$a_n a_1^{-1} = \eta^{n-k}a_k a_{k+1}^{-1} = c \text{ for all } 1 \leqslant k \leqslant n-1 \tag{7.11}$$

implies $B(\tau,z) = D(z+\tau)\Phi(z)D(z) = c\phi(z)T_2 = T_2'$. The system of equations (7.11) has a solution for $a_1,\dots,a_n$. Indeed, putting $a_n = 1$, we can see inductively that $a_{n-k} = c^k\eta^{-1-2-\dots-k} = c^k\eta^{-k(k-1)/2}$ has to hold for all $k \in \{1,\dots,n-1\}$ and (7.11) is consistent since $c = a_n a_1^{-1} = a_1^{-1} = c^{-n+1}\eta^{n(n-1)/2}$ holds due to $c^n = \eta^{n(n-1)/2}$. In particular, we have shown $\mathscr{S}^{\mathrm{an}} \cong \widetilde{\mathscr{S}}$.

Step 3. $\mathscr{A} \cong \mathscr{Q}_{(n,m)}$. It is straight-forward to see that the kernel of the sheaf trace $\mathrm{Tr}_{\widetilde{\mathscr{S}}}\colon \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{\mathscr{S}}) \to \mathscr{O}_X$ is given by

$$\mathbb{C} \times \mathfrak{sl}_n(\mathbb{C})/\sim_3 \text{ for } (z, A) \sim_3 (z+1, \mathrm{Ad}_{T_1'}(A)) \sim_3 (z+\tau, \mathrm{Ad}_{T_2'}(A)). \tag{7.12}$$

Now the identities $\mathrm{Ad}_{T_1'} = \mathrm{Ad}_{T_1^m}$ and $\mathrm{Ad}_{T_2'} = \mathrm{Ad}_{T_2}$ conclude the proof. □

7.2 Classification of elliptic r-matrices.

In this section, we derive the well-known explicit formulas for elliptic r-matrices from [Bel81], using Theorem 7.1.3.

7.2.1 Preparation. Let us define $Z_{k,\ell} = T_1^k T_2^{-\ell}$ for $k, \ell \in \mathbb{Z}_n$, where T_1, T_2 were defined in (7.2). Observe that the labeling is well-defined since $T_1^n = \mathrm{id}_{\mathbb{C}^n} = T_2^n$. The following lemma summarizes the basic properties of these elements.

Lemma 7.2.1.
Let $\varepsilon = \exp(2\pi i/n)$. The following results are true.
(1) $\{Z_{k_1,k_2} \mid (k_1,k_2) \in \mathbb{Z}_n^2 \setminus \{0\}\} \subseteq \mathfrak{sl}_n(\mathbb{C})$ *is a basis.*
(2) $T_i Z_{k_1,k_2} T_i^{-1} = \varepsilon^{k_i} Z_{k_1,k_2}$ *for all* $i \in \{1,2\}, (k_1,k_2) \in \mathbb{Z}_n^2 \setminus \{0\}$.
(3) $\mathrm{Tr}(Z_{k_1,k_2} Z_{\ell_1,\ell_2}) = n\varepsilon^{k_1k_2}\delta_{k_1,-\ell_1}\delta_{k_2,-\ell_2}$ *for all* $(k_1,k_2),(\ell_1,\ell_2) \in \mathbb{Z}_n^2 \setminus \{0\}$ *and*

$$\sum_{k,\ell \in \mathbb{Z}_n^2 \setminus \{0\}} \frac{\varepsilon^{k\ell}}{2n^2} Z_{k,\ell} \otimes Z_{-k,-\ell} \in \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C}) \tag{7.13}$$

is the Casimir element.

Proof. It is easy to see that the identities

$$T_1 T_2^{-1} = \varepsilon T_2^{-1} T_1 \text{ and } T_2 T_1^{-1} = \varepsilon^{-1} T_1^{-1} T_2 \tag{7.14}$$

hold. Therefore, $\mathrm{Tr}(Z_{k,\ell}) = \varepsilon^k \mathrm{Tr}(T_2^{-1} T_1^k T_2^{-\ell+1}) = \varepsilon^k \mathrm{Tr}(Z_{k,\ell})$ implies that $\mathrm{Tr}(Z_{k,\ell})$ vanishes if $k \neq 0$ in $\mathbb{Z}_n$. A similar argument shows $\mathrm{Tr}(Z_{k,\ell}) = 0$ if $\ell \neq 0$ in $\mathbb{Z}_n$, so $Z_{k,\ell} \in \mathfrak{sl}_n(\mathbb{C})$ for $(k,\ell) \in \mathbb{Z}_n^2 \setminus \{0\}$. The fact (7.14) also implies

$$\mathrm{Tr}(Z_{k_1,k_2} Z_{\ell_1,\ell_2}) = \varepsilon^{-k_2\ell_1} \mathrm{Tr}(Z_{k_1+\ell_1,k_2+\ell_2}) = \begin{cases} n\varepsilon^{k_1k_2} & (k_1,k_2) = -(\ell_1,\ell_2) \in \mathbb{Z}_n \\ 0 & \text{otherwise} \end{cases},$$

where $\mathrm{Tr}(Z_{0,0}) = \mathrm{Tr}(\mathrm{id}_{\mathbb{C}^n}) = n$ was used. From this we deduce that

$$\left\{Z_{k_1,k_2} \mid (k_1,k_2) \in \mathbb{Z}_n^2 \setminus \{0\}\right\} \subseteq \mathfrak{sl}_n(\mathbb{C}) \tag{7.15}$$

is linearly independent, hence (1) is proven. Part (2) is a direct consequence of (7.14) while the remaining assertion in (3) follows from the fact that the Killing form of $\mathfrak{sl}_n(\mathbb{C})$ is given by $(a,b) \mapsto 2n\mathrm{Tr}(ab)$. □

7.2.2 The classification theorem. For any $(k,\ell) \in \mathbb{Z}_n^2 \setminus \{0\}$ and $\tau \in \mathbb{H}$, a unique meromorphic function $z \mapsto \lambda_{k,\ell}(z;\tau)$ on $\mathbb{C}$ with pole set $\langle 1,\tau\rangle_{\mathbb{Z}}$ exists such that

- $\lim_{z\to 0} z\lambda_{k,\ell}(z;\tau) = 1$ and
- $\lambda_{k,\ell}(z+1;\tau) = \varepsilon^k\lambda_{k,\ell}(z;\tau)$ and $\lambda_{k,\ell}(z+\tau;\tau) = \varepsilon^\ell\lambda_{k,\ell}(z;\tau)$.

Existence and uniqueness of these functions can be seen by using the Mittag-Lefler theorem or the following observation; see [Skr12].

Remark 7.2.2.
For every $k,\ell \in \mathbb{Z}$ with image $\overline{k},\overline{\ell} \in \mathbb{Z}_n$, the quasi-periodic functions $\lambda_{k,\ell}$ can be described using the theta function

$$\vartheta_{(a_1,a_2)}(z;\tau) := \sum_{k\in\mathbb{Z}} \exp\Big(\pi i(k+a_1)^2\tau + 2\pi i(k+a_1)(z+a_2)\Big) \tag{7.16}$$

with characteristic $(a_1,a_2) \in \mathbb{C}^2$ via the formula

$$\lambda_{\overline{k},\overline{\ell}}(z;\tau) = \frac{\vartheta'_{(1/2,1/2)}(0;\tau)\vartheta_{(k/n+1/2,-\ell/n+1/2)}(z;\tau)}{\vartheta_{(1/2,1/2)}(z;\tau)\vartheta_{(k/n+1/2,-\ell/n+1/2)}(0;\tau)}. \tag{7.17}$$

The following theorem settles the classification of elliptic r-matrices.

Theorem 7.2.3.
Let $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a normalized r-matrix, $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be chosen according to Theorem 3.2.5, and $\mathbb{G}(O,\mathfrak{g}(r)) = ((X,\mathscr{A}),(p,c,\zeta))$. Assume that X is elliptic. There exist $\tau \in \mathbb{H}$ and coprime integers $0 < m < n$ such that r is gauge equivalent to the Taylor series of $r_{(\tau,(n,m))}$ in $y = 0$, where

$$r_{(\tau,(n,m))}(x,y) := \sum_{k,\ell\in\mathbb{Z}_n^2\setminus\{0\}} \frac{\varepsilon^{k\ell}}{2n^2}\lambda_{mk,\ell}(x-y;\tau)Z_{k,\ell}\otimes Z_{-k,-\ell}. \tag{7.18}$$

Proof. Choose $(\tau,(n,m))$ such that there exist isomorphisms $\nu\colon X^{\mathrm{an}} \to \mathbb{C}/\langle 1,\tau\rangle_{\mathbb{Z}}$ and $\psi\colon \mu^*\mathscr{A} \to \mathscr{Q}_{(\tau,(n,m))}$, where $\mu := \iota\nu^{-1}\colon \mathbb{C}/\Lambda \to X$ for the canonical morphism $\iota\colon X^{\mathrm{an}} \to X$. The latter is possible because of Theorem 7.1.3.

Let $\widetilde{r} \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be the Talyor series of $r_{(\tau,(n,m))}(x,y)$ at $y = 0$. Lemma 7.2.1 implies that $\widetilde{r}$ is in normalized standard form. The j-th derivative $\lambda^{(j)}_{mk,\ell}(z;\tau)Z_{k,\ell}$ in z of $\lambda_{mk,\ell}(z;\tau)Z_{k,\ell}$ is an element of the subspace $\mathfrak{W}$ of meromorphic sections in $\Gamma(U,\mathscr{Q}_{(\tau,(n,m))})$, where $U := (\mathbb{C}/\Lambda)\setminus\{\Lambda\}$. Indeed, this follows from the j-th derivative in z of the identities

$$\begin{aligned} &\mathrm{Ad}_{T_1^m}(\lambda_{mk,\ell}(z;\tau)Z_{k,\ell}) = \varepsilon^{mk}\lambda_{mk,\ell}(z;\tau)Z_{k,\ell} = \lambda_{mk,\ell}(z+1;\tau)Z_{k,\ell} \text{ and} \\ &\mathrm{Ad}_{T_2}(\lambda_{mk,\ell}(z;\tau)Z_{k,\ell}) = \varepsilon^{\ell}\lambda_{mk,\ell}(z;\tau)Z_{k,\ell} = \lambda_{mk,\ell}(z+\tau;\tau)Z_{k,\ell}. \end{aligned} \tag{7.19}$$

Since global periodic functions are constant due to Lioville's theorem and $\mathrm{Ad}_{T_1^m}, \mathrm{Ad}_{T_2}$ have no common fixed vector, we can see that

$$L_0(\mathfrak{W}) = \mathfrak{g}(\widetilde{r}) = \langle L_0(\lambda^{(j)}_{mk,\ell}(z;\tau)Z_{k,\ell}) \mid k,\ell \in \mathbb{Z}_n^2\setminus\{0\}, j \in \mathbb{N}_0\rangle_{\mathbb{C}}, \tag{7.20}$$

where L_0 is the Laurent expansion in $z = 0$. Moreover, $\zeta(\Gamma(X \setminus \{p\}, \mathscr{A})) = \mathfrak{g}(r)$ and the chain

$$\Gamma(X \setminus \{p\}, \mathscr{A}) \xrightarrow{\mu^*} \Gamma(U, \mu^*\mathscr{A}) \xrightarrow{\psi} \Gamma(U, \mathscr{Q}_{(\tau,(n,m))}) \tag{7.21}$$

results in an isomorphism $\Gamma(X \setminus \{p\}, \mathscr{A}) \cong \mathfrak{W}$. Therefore, $\varphi(\mathfrak{g}(r)) = \mathfrak{g}(\widetilde{r})$ holds, where $\varphi \in \mathrm{Aut}_{\mathbb{C}[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ is defined by the chain

$$\mathfrak{g}[\![z]\!] \xrightarrow{\zeta^{-1}} \widehat{\mathscr{A}_p} \longrightarrow \widehat{\mu^*\mathscr{A}}_\Lambda \xrightarrow{\widehat{\psi}_\Lambda} \widehat{\mathscr{Q}}_{(\tau,(n.m)),\Lambda} \xrightarrow{L_0} \mathfrak{g}[\![z]\!]. \tag{7.22}$$

Here, the second arrow is the completion of $\mathscr{A}_p \to (\mu_*\mu^*\mathscr{A})_p \to \mu^*\mathscr{A}_\Lambda$ and the $\mathbb{C}[\![z]\!]$-linearity of φ is a consequence of Lemma 6.2.2. Application of Lemma 2.2.4 concludes the proof. □

7.2.3 Equivalences of elliptic r-matrices. To complete the classification of elliptic r-matrices, it remains to investigate which elliptic r-matrices of the form (7.18) are equivalent.

Proposition 7.2.4.
Let $\tau, \tau' \in \mathbb{H}$ and $0 < m < n$, $0 < m' < n'$ be two pairs of coprime integers. The elliptic r-matrices $r_{(\tau,(n,m))}$ and $r_{(\tau',(n',m'))}$ are formally equivalent if and only if $\mathbb{C}/\langle 1,\tau\rangle_\mathbb{Z} \cong \mathbb{C}/\langle 1,\tau'\rangle_\mathbb{Z}, n = n'$ and $m' \in \{m, n-m\}$.

Proof. Using the construction in Section 7.2 and Lemma 3.1.1, we can deduce that $r_{(\tau,(n,m))}$ and $r_{(\tau',(n',m'))}$ are equivalent if and only if there exist isomorphisms $f\colon \mathbb{C}/\langle 1,\tau'\rangle_\mathbb{Z} \to \mathbb{C}/\langle 1,\tau\rangle_\mathbb{Z}$ and $\psi\colon \mathscr{Q}_{(\tau,(n,m))} \to \mathscr{Q}_{(\tau,(n',m'))}$, where we used $f_*\mathscr{Q}_{(\tau',(n',m'))} \cong \mathscr{Q}_{(\tau,(n',m'))}$. Obviously, $n = n'$ has to hold. The isomorphism ψ is determined by a holomorphic map $\mathbb{C} \to \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$, which we will also denote by ψ, satisfying

$$\psi(z+1)\,\mathrm{Ad}_{T_1^m} = \mathrm{Ad}_{T_1^{m'}}\,\psi(z) \text{ and } \psi(z+\tau)\,\mathrm{Ad}_{T_2} = \mathrm{Ad}_{T_2}\,\psi(z); \tag{7.23}$$

see e.g. [Ien11]. We see that ψ is bounded, so constant by virtue of Liouville's theorem. There are two possible cases.

Case 1. *ψ is inner.* Let $A \in \mathrm{SL}_n(\mathbb{C})$ satisfy $\psi = \mathrm{Ad}_A$. The identities (7.23) are equivalent to

$$T_1^{m'}AT_1^{-m} = \varepsilon^{k_1}A \text{ and } T_2^{-1}AT_2 = \varepsilon^{k_2}A \tag{7.24}$$

for some $k_1, k_2 \in \{0,\dots,n-1\}$. Let $A = (a_{ij})_{i,j=1}^n$ and chose $i,j \in \{1,\dots,n\}$ such that $a_{ij} \neq 0$. Then $T_2AT_2^{-1} = \varepsilon^{k_2}A$ implies $a_{i'j'} \neq 0$ for all $i',j' \in \{1,\dots,n\}$ such that $i'-j' \equiv i-j \pmod n$ and the identity $T_1^{m'}AT_1^{-m} = \varepsilon^{k_1}A$ states $\varepsilon^{(i'-1)m'-(j'-1)m}a_{i'j'} = \varepsilon^{k_1}a_{i'j'}$. Chose i' and j' in such a way that $i' \equiv i+1 \pmod n$ and $j' \equiv j+1 \pmod n$. Then

$$\varepsilon^{im'-jm}a_{i'j'} = \varepsilon^{k_1}a_{i'j'} \text{ and } \varepsilon^{(i-1)m'-(j-1)m}a_{ij} = \varepsilon^{k_1}a_{ij} \tag{7.25}$$

imply that $im' - jm \equiv k_1 \equiv (i-1)m' - (j-1)m \pmod{n}$. Therefore, $m - m' = 0 \pmod{n}$ and $0 < m, m' < n$ implies $m = m'$.

Case 2. *ψ is outer.* Let $A \in \mathrm{SL}_n(\mathbb{C})$ satisfy $\psi = \mathrm{Ad}_A \sigma$, where $\sigma(a) = -a^T$ for all $a \in \mathfrak{sl}_n(\mathbb{C})$ represents the only non-trivial outer automorphism of $\mathfrak{sl}_n(\mathbb{C})$. Since $T_1^T = T_1$ and $T_2^T = T_2^{-1}$, we have $\sigma\,\mathrm{Ad}_{T_1} = \mathrm{Ad}_{T_1^{-1}} \sigma$ and $\sigma\,\mathrm{Ad}_{T_2} = \mathrm{Ad}_{T_2} \sigma$. Therefore, we can see that (7.23) reads

$$\mathrm{Ad}_A \mathrm{Ad}_{T_1^{-m}} \sigma = \mathrm{Ad}_{T_1^{m'}} \mathrm{Ad}_A \sigma \text{ and } \mathrm{Ad}_A \mathrm{Ad}_{T_2} \sigma = \mathrm{Ad}_{T_2} \mathrm{Ad}_A \sigma. \tag{7.26}$$

Thus, there exist $k_1, k_2 \in \{0, \dots, n-1\}$ such that

$$T_1^{m'} A T_1^m = \varepsilon^{k_1} A \text{ and } T_2^{-1} A T_2 = \varepsilon^{k_2} A. \tag{7.27}$$

Using $T_1^m = T_1^{-(n-m)}$ and repeating the proof of Case 1. yields $m' = n - m$. □

7.2.4 Example: elliptic r-matrices over $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Let $\tau \in \mathbb{H}$, $\vartheta_{(a_1,a_2)}$ be the theta function of characteristic $(a_1, a_2) \in \mathbb{C}^2$ (see (7.16)), and write

$$\alpha = \frac{\vartheta'_{(1/2,1/2)}(0;\tau)\vartheta_{(1,1)}(0;\tau)}{\vartheta_{(1,1/2)}(0;\tau)\vartheta_{(1/2,1)}(0;\tau)}, \text{ and } k = \sqrt{1 - \left(\frac{\vartheta_{(1,1/2)}(0;\tau)}{\vartheta_{(1,1)}(0;\tau)}\right)^2}. \tag{7.28}$$

Then the quasi-periodic functions $\lambda_{1,1}, \lambda_{1,0}$ and $\lambda_{0,1}$ can be expressed using the Jacobi elliptic functions sn, cn and dn via

$$\lambda_{1,1}(z;\tau) = \alpha\frac{\mathrm{dn}(\alpha z;k)}{\mathrm{sn}(\alpha z;k)}, \lambda_{0,1}(z;\tau) = \alpha\frac{\mathrm{cn}(\alpha z;k)}{\mathrm{sn}(\alpha z;k)} \text{ and } \lambda_{1,0}(z;\tau) = \frac{\alpha}{\mathrm{sn}(\alpha z;k)};$$

see [Skr12]. Let us put

$$P_3 := \frac{i}{2}Z_{1,0} = \frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, P_2 := \frac{i}{2}Z_{0,1} = \frac{i}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, P_1 := -\frac{1}{2}Z_{1,1} = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

then $[P_i, P_j] = -\epsilon_{ijk} P_k$ holds for all $\{i,j,k\} = \{1,2,3\}$, where ϵ_{ijk} is the Levi-Civita tensor. Moreover, $K(P_i, P_j) = 4\mathrm{Tr}(P_i P_j) = -2\delta_{ij}$ for all $i, j \in \{1,2,3\}$. Then

$$r_{(\tau,(2,1))}(x,y) = -\frac{\alpha}{2}\left(\frac{\mathrm{dn}(\alpha z;k)}{\mathrm{sn}(\alpha z;k)} P_1 \otimes P_1 + \frac{\mathrm{cn}(\alpha z;k)}{\mathrm{sn}(\alpha z;k)} P_2 \otimes P_2 + \frac{1}{\mathrm{sn}(\alpha z;k)} P_3 \otimes P_3\right),$$

where $z = x - y$, is essentially the only elliptic r-matrix for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ associated to $\tau \in \mathbb{H}$.

8 Trigonometric r-matrices

Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra. We have seen in Theorem 6.2.1 that a normalized formal r-matrix whose associated curve is a nodal Weierstraß cubic is equivalent to a certain Taylor series of a σ-trigonometric r-matrix for an appropriate $\sigma \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of finite order. In this chapter, we investigate these r-matrices in detail. For this purpose, we use the notation from Subsection 5.4.4:

- σ is a Lie algebra automorphism of $\mathfrak{g}$ of order $m \in \mathbb{N}$, γ is the Casimir element of $\mathfrak{g}$ and K is the Killing form of $\mathfrak{g}$;
- $\mathfrak{g}_0^\sigma = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+$ is a fixed triangular decomposition and $\gamma = \sum_{j=0}^{m-1} \gamma_j \in \bigoplus_{j=0}^{m-1} \mathfrak{g}_j^\sigma \otimes \mathfrak{g}_{-j}^\sigma$ (resp. $\gamma_0 = \gamma_0^+ + \gamma_\mathfrak{h} + \gamma_0^- \in (\mathfrak{n}_+ \otimes \mathfrak{n}_-) \oplus (\mathfrak{H} \otimes \mathfrak{H}) \oplus (\mathfrak{n}_- \otimes \mathfrak{n}_+)$) is the natural decomposition;
- $\mathfrak{L} := \mathfrak{L}(\mathfrak{g}, \sigma) \subseteq \mathfrak{g}[\widetilde{u}, \widetilde{u}^{-1}]$, $L := \mathbb{C}[u, u^{-1}]$ for $u = \widetilde{u}^m$ and K_0 is the bilinear form of $\mathfrak{L}$ defined in Lemma 4.2.1;
- $(\widehat{\mathfrak{L}}, \mathfrak{h} := \widehat{\mathfrak{H}}, \Pi := \{\alpha_0, \dots, \alpha_n\}, \Pi^\vee)$ is the realization provided by Theorem 4.2.3 to the given triangular decomposition of $\mathfrak{g}_0^\sigma$, $\left\{e_j^+, e_j^- \,\middle|\, j \in \{0, \dots, n\}\right\}$ is a set of Chevalley generators of said realization and $\Phi = \Phi^+ \sqcup \Phi^- = \Phi^{\mathrm{re}} \sqcup \Phi^{\mathrm{im}}$ is the associated root system;
- $\mathfrak{B}_\pm = \mathfrak{H} \oplus \mathfrak{N}_\pm$ are the standard Borel subalgebras, i.e.

$$\mathfrak{N}_\pm := \mathfrak{n}_\pm \oplus \bigoplus_{\pm j > 0} z^j \mathfrak{L}_j = \langle e_j^\pm \mid j \in \{0, \dots, n\} \rangle_{\mathbb{C}\text{-alg}}; \tag{8.1}$$

- $\delta^\circ = \partial \varrho^\circ$ is the standard Lie bialgebra structure of $\mathfrak{L}$ to $\mathfrak{g}_0^\sigma = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+$, where

$$\varrho^\circ(\widetilde{u}, \widetilde{v}) = \frac{1}{(u/v) - 1} \sum_{j=0}^{m-1} \left(\frac{\widetilde{u}}{\widetilde{v}}\right)^j \gamma_j + \gamma_0^- + \gamma_\mathfrak{h}/2 \tag{8.2}$$

  is the standard σ-trigonometric r-matrix and we write $\varrho^t := \varrho^\circ + t, \delta^t := \delta^\circ + \partial t$ for all $t \in \mathfrak{L} \otimes \mathfrak{L}$. The tensor t is a classical twist of δ° if and only if δ^t is a Lie bialgebra cobracket if and only if ϱ^t is a σ-trigonometric r-matrix;
- $((\mathfrak{L} \times \mathfrak{L}, K_0^{(2)}), \mathfrak{D}, \mathfrak{W}^\circ)$ is the Manin triple determining δ° (up to $\mathfrak{L} \cong \mathfrak{D}$), where $\mathfrak{W}^\circ$ is given in (5.48), and $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{D}, \mathfrak{W}^t)$ is the consequent Manin triple determining δ^t (up to $\mathfrak{L} \cong \mathfrak{D}$) for any classical twist t of δ°; see Theorem 5.4.9.

8.1 Explicit geometrization

In this section, we give an explicit description of the geometric data associated to σ-trigonometric r-matrices. More precisely, we assign a geometric $\mathfrak{g}$-lattice model to any classical twists t of δ° in Subsection 8.1.1 and prove that this construction yields the geometric datum of ϱ^t in Theorem 8.1.3. The sheaf of Lie algebras in this model is constructed from $\mathfrak{W}^t$ following the theory of torsion-free sheaves from [Bod+06]. We will see in Proposition 8.1.4 that the classical double of δ^t appears naturally in this geometric picture.

8.1.1 Construction of geometric $\mathfrak{g}$-lattice models from classical twists of δ°. Let t be a classical twist of δ° and write $\mathfrak{W} := \mathfrak{W}^t$. Recall that $\mathfrak{D} = \operatorname{Im}(\iota)$ for the injective Lie algebra morphism $\iota\colon \mathfrak{L} \to \mathfrak{L} \times \mathfrak{L}$ defined by $a \mapsto (a, a)$ and

$$\mathfrak{W}^\circ := \{(a_+, a_-) \in \mathfrak{B}_+ \times \mathfrak{B}_- \mid a_+ + a_- \in \mathfrak{N}_+ + \mathfrak{N}_-\}. \tag{8.3}$$

Theorem 5.4.9 implies that the Manin triple $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{D}, \mathfrak{W})$ determines the Lie bialgebra cobracket $\mathfrak{D} \to \mathfrak{D} \otimes \mathfrak{D}$ given by $(a, a) \mapsto (\iota \otimes \iota)\delta^t(a)$ and $\mathfrak{W} \asymp \mathfrak{W}^\circ$ holds. Let $\operatorname{pr}_\pm\colon \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ be the projections defined by $\operatorname{pr}_\pm(a_+, a_-) = a_\pm$. Then the subalgebras $\operatorname{pr}_\pm(\mathfrak{W}) = \mathfrak{W}_\pm \subseteq \mathfrak{L}$ satisfy

$$\mathfrak{W}_+^\perp \times \mathfrak{W}_-^\perp = (\mathfrak{W}_+ \times \mathfrak{W}_-)^\perp \subseteq \mathfrak{W}^\perp = \mathfrak{W} \subseteq \mathfrak{W}_+ \times \mathfrak{W}_-. \tag{8.4}$$

Summarized, $\mathfrak{W}_\pm \subseteq \mathfrak{L}$ are coisotropic subalgebras of $\mathfrak{L}$ commensurable with $\mathfrak{B}_\pm$, i.e. $\mathfrak{W}_\pm^\perp \subseteq \mathfrak{W}_\pm$ and $\mathfrak{W}_\pm^\perp \asymp \mathfrak{B}_\pm$ in symbols.

Proposition 4.2.9 implies that $u^{\pm 1}\mathfrak{W}_\pm \subseteq \mathfrak{W}_\pm^\perp \subseteq \mathfrak{W}_\pm$, so

$$\mathfrak{W}_\pm \text{ is a free Lie algebra over } L_\pm := \mathbb{C}[u^{\pm 1}] \subseteq L. \tag{8.5}$$

Combining this with (8.4) implies that

$$\mathfrak{W} \text{ is a torsion-free Lie algebra over } \mathbb{C}[u_+, u_-]/(u_+u_-), \tag{8.6}$$

where the multiplication is defined by $u_+(a_+, a_-) = (ua_+, 0)$ and $u_-(a_+, a_-) = (0, u^{-1}a_-)$ for all $(a_+, a_-) \in \mathfrak{W}$.

Let us write $U_\pm := \operatorname{Spec}(L_\pm)$ and $U := \operatorname{Spec}(L)$ as well as $\mathscr{L}$, $\mathscr{W}_\pm$ for the sheaves of Lie algebras on $U, U_\pm$ defined by $\mathfrak{L}, \mathfrak{W}_\pm$ respectively. We identify $\mathbb{P}^1_\mathbb{C}$ with the gluing of U_+ and U_- along U. Since $\mathfrak{W}_\pm \asymp \mathfrak{B}_\pm$, the multiplication map gives rise to isomorphisms $\mathfrak{W}_\pm \otimes_{L_\pm} L \cong \mathfrak{L}$. Therefore, we can glue $\mathscr{W}_+$ and $\mathscr{W}_-$ along $\mathscr{L}$ to produce a sheaf of Lie algebras $\mathscr{B}$ on $\mathbb{P}^1_\mathbb{C}$. In particular, we have $\mathbb{P}^1_\mathbb{C} = U_+ \cup U_-$, $U = U_+ \cap U_-$ and there are canonical isomorphisms $\mathscr{B}|_{U_\pm} \cong \mathscr{W}_\pm$, $\mathscr{B}|_U \cong \mathscr{L}$.

The Mayer-Vietoris sequence combined with the fact that $\mathfrak{L} \times \mathfrak{L} = \mathfrak{D} \oplus \mathfrak{W}$ gives $\mathfrak{W}_+ + \mathfrak{W}_- = \mathfrak{L}$ implies that $\mathrm{H}^0(\mathscr{B}) \cong \mathfrak{W}_- \cap \mathfrak{W}_+$ and $\mathrm{h}^1(\mathscr{B}) = 0$. Note that

$$\mathscr{B}|_0 \cong \mathfrak{W}_+/u\mathfrak{W}_+ =: \overline{\mathfrak{W}}_+ \text{ and } \mathscr{B}|_\infty \cong \mathfrak{W}_-/u^{-1}\mathfrak{W}_- =: \overline{\mathfrak{W}}_-, \tag{8.7}$$

where $0 := (u) \in U_+$ and $\infty := (u^{-1}) \in U_-$.

Lemma 8.1.1.
The following results are true.
(1) *The Lie algebra $\overline{\mathfrak{W}} := \mathfrak{W}/(u_+, u_-)\mathfrak{W}$ admits a linear isomorphism*

$$\overline{\mathfrak{W}} \times (\mathfrak{W}_+ \cap \mathfrak{W}_-) \longrightarrow \overline{\mathfrak{W}}_+ \times \overline{\mathfrak{W}}_-, \quad (\overline{(w_+, w_-)}, a) \longmapsto (\overline{w_+ - a}, \overline{w_- - a}).$$

(2) $\mathfrak{W} \neq \mathfrak{W}_+ \times \mathfrak{W}_-$.

Proof of (1). The identity $\mathfrak{L} \times \mathfrak{L} = \mathfrak{D} \oplus \mathfrak{W}$ yields

$$\mathfrak{W}_+ \times \mathfrak{W}_- = \iota(\mathfrak{W}_+ \cap \mathfrak{W}_-) \oplus \mathfrak{W}. \tag{8.8}$$

Combining this with $(u_+, u_-)\mathfrak{W} = u\mathfrak{W}_+ \times u^{-1}\mathfrak{W}_-$ immediately implies the statement.

Proof of (2). Assume $\mathfrak{W} = \mathfrak{W}_+ \times \mathfrak{W}_-$. Then (8.4) implies $\mathfrak{W}_\pm = \mathfrak{W}_\pm^\perp$ and (8.8) implies $\mathfrak{L} = \mathfrak{W}_+ \oplus \mathfrak{W}_-$. Combining Proposition 4.2.9.(3) with Lemma 4.2.7 provides $\varphi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{L})$ such that $\varphi(\mathfrak{W}_\pm) \subseteq \mathfrak{B}_\pm$. Thus,

$$\mathfrak{N}_\pm \subseteq \varphi(\mathfrak{W}_\pm)^\perp = \varphi(\mathfrak{W}_\pm^\perp) = \varphi(\mathfrak{W}_\pm) \subseteq \mathfrak{B}_\pm, \tag{8.9}$$

so $\varphi(\mathfrak{W}_\pm) = \mathfrak{H}_\pm \oplus \mathfrak{N}_\pm$ for some $\mathfrak{H}_\pm \subseteq \mathfrak{H}$. The identities $\varphi(\mathfrak{W}_\pm)^\perp = \varphi(\mathfrak{W}_\pm)$ and $\varphi(\mathfrak{W}_+) \oplus \varphi(\mathfrak{W}_-) = \mathfrak{L}$ are equivalent to $\mathfrak{H}_\pm^\perp = \mathfrak{H}_\pm$ and $\mathfrak{H}_+ \oplus \mathfrak{H}_- = \mathfrak{H}$.

The proof of Lemma 4.2.4.(3) can also be used to deduce that K_0 defines a positive-definite bilinear form on $\mathfrak{H}_\mathbb{R} := \mathfrak{H}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$. Clearly, $\mathfrak{H} = \mathfrak{H}_\mathbb{R} \oplus i\mathfrak{H}_\mathbb{R}$ and the projection $\pi\colon \mathfrak{H} \to \mathfrak{H}_\mathbb{R}$ restricted to $\mathfrak{H}_\pm$ has kernel $\mathfrak{H}_\pm \cap i\mathfrak{H}_\mathbb{R}$. The identities $\mathfrak{H}_\pm^\perp = \mathfrak{H}_\pm$ and the fact that K_0 is positive-definite on $\mathfrak{H}_\mathbb{R}$ imply $\mathfrak{H}_\pm \cap i\mathfrak{H}_\mathbb{R} = \{0\}$, so $\pi|_{\mathfrak{H}_\pm}$ are isomorphisms. This implies that $\mathfrak{H}_\pm = \{a + iA_\pm a \mid a \in \mathfrak{H}_\mathbb{R}\}$ for some $A_\pm \in \mathrm{End}_\mathbb{R}(\mathfrak{H}_\mathbb{R})$. But this contradicts $\mathfrak{H}_+ \oplus \mathfrak{H}_- = \mathfrak{H}$. We conclude that $\mathfrak{W} = \mathfrak{W}_+ \times \mathfrak{W}_-$ is impossible. □

Let us define X by the push-out diagram

$$\begin{array}{ccc} \{0,\infty\} & \longrightarrow & \{s\} \\ \downarrow & & \downarrow \\ \mathbb{P}^1_\mathbb{C} & \xrightarrow{\ \nu\ } & X \end{array} \tag{8.10}$$

Then X is a curve of arithmetic genus one with nodal singularity $s \in X$, smooth locus $C := X \setminus \{s\}$, and normalization ν. Consider the sheaf of Lie algebras $\mathscr{A}$ defined by the pull-back diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ \iota\ } & \nu_*\mathscr{B} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{W}} & \xrightarrow{\ \subseteq\ } & \overline{\mathfrak{W}}_+ \times \overline{\mathfrak{W}}_- \end{array} \tag{8.11}$$

where $\overline{\mathfrak{W}}$ and $\overline{\mathfrak{W}}_+ \times \overline{\mathfrak{W}}_-$ are understood as skyscraper sheaves at s. Note that the canonical projections $\overline{\mathfrak{W}} \to \overline{\mathfrak{W}}_\pm$ are surjective, so they induce a surjective morphism $\theta\colon \overline{\mathfrak{W}} \to \overline{\mathfrak{W}}_+ \times \overline{\mathfrak{W}}_-$. The sheaf $\mathscr{A}$ is the torsion-free sheaf on X associated to the triple $(\mathscr{B}, \theta, \overline{\mathfrak{W}})$ in [Bod+06, Theorem 16].

Lemma 8.1.2.
We have $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$, $\Gamma(C, \mathscr{A}) \cong \mathfrak{L}$, *and* $\widehat{\mathscr{A}_s} \cong \widehat{\mathfrak{W}} := \varprojlim \mathfrak{W}/(u_+, u_-)^k\mathfrak{W}$.

Proof. By construction $\Gamma(C, \mathscr{A}) = \Gamma(U, \mathscr{B}) = \mathfrak{L}$. The short exact sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow \overline{\mathfrak{W}} \times \nu_*\mathscr{B} \longrightarrow \overline{\mathfrak{W}}_+ \times \overline{\mathfrak{W}}_- \longrightarrow 0, \tag{8.12}$$

$\mathrm{H}^0(\mathscr{B}) \cong \mathfrak{W}_+ \cap \mathfrak{W}_-$, and $\mathrm{h}^1(\mathscr{B}) = 0$ induce the exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathscr{A}) \longrightarrow \overline{\mathfrak{W}} \times (\mathfrak{W}_+ \cap \mathfrak{W}_-) \longrightarrow \overline{\mathfrak{W}}_+ \times \overline{\mathfrak{W}}_- \longrightarrow \mathrm{H}^1(\mathscr{A}) \longrightarrow 0 \tag{8.13}$$

in cohomology. The middle arrow is the isomorphism from Lemma 8.1.1, so $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$. Observe that

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{\subseteq} & \mathfrak{W}_+ \times \mathfrak{W}_- \\ \downarrow & & \downarrow \\ \overline{\mathfrak{W}} & \xrightarrow[\subseteq]{} & \overline{\mathfrak{W}}_- \times \overline{\mathfrak{W}}_+ \end{array} \tag{8.14}$$

is a pull-back diagram, since $(u_+, u_-)\mathfrak{W} = u\mathfrak{W}_+ \times u^{-1}\mathfrak{W}_-$. Similarly, the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{W}} & \xrightarrow{\subseteq} & \widehat{\mathfrak{W}}_+ \times \widehat{\mathfrak{W}}_- \\ \downarrow & & \downarrow \\ \overline{\mathfrak{W}} & \xrightarrow[\subseteq]{} & \overline{\mathfrak{W}}_- \times \overline{\mathfrak{W}}_+ \end{array} \tag{8.15}$$

is also a pull-back diagram. Here, we wrote $\widehat{\mathfrak{W}}_\pm := \varprojlim \mathfrak{W}_\pm / u^{\pm k}\mathfrak{W}_\pm$. Note that the completion of $(\nu_*\mathscr{B})_s$ is isomorphic to $\widehat{\mathfrak{W}}_+ \times \widehat{\mathfrak{W}}_-$ as $\mathbb{C}[\![u_+, u_-]\!]/(u_+u_-)$-Lie algebras. Thus, applying the completion functor $\widehat{(\cdot)}_s$ to the pull-back diagram (8.11) yields $\widehat{\mathscr{A}_s} \cong \widehat{\mathfrak{W}}$. □

Let $p := \nu(u-1) \in X$ and $c\colon \widehat{\mathscr{O}}_{X,p} \to \mathbb{C}[\![z]\!]$ be the isomorphism obtained by the completing $L = \mathbb{C}[u, u^{-1}] \to \mathbb{C}[\![z]\!]$ defined by $h(u) \mapsto h(\exp(z))$. We have an c-equivariant isomorphism $\zeta\colon \widehat{\mathscr{A}_p} \to \mathfrak{g}[\![z]\!]$ of Lie algebras induced by the completion of the morphism $\mathfrak{L} \to \mathfrak{g}[\![z]\!]$ defined by $a(u) \mapsto a(\exp(z/m))$. Summarized, we have constructed a geometric $\mathfrak{g}$-lattice model $((X, \mathscr{A}), (p, c, \zeta))$.

8.1.2 The comparison theorem. The following theorem verifies that the geometric datum constructed from a classical twist of δ° in the Subsection 8.1.1 is indeed the geometric datum of the formal r-matrix associated to said classical twist. It is a version of one of the main results from [AB21].

Theorem 8.1.3.
Let $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be the Taylor series of $\varrho^t(\exp(x/m), \exp(y/m))$ in $y = 0$ for some classical twist t of δ° and $((X, \mathscr{A}), (p, c, \zeta))$ be the geometric $\mathfrak{g}$-lattice model associated to t in Subsection 8.1.1.
(1) $\mathbb{T}((X, \mathscr{A}), (p, c, \zeta)) = (\mathrm{Mult}(\mathfrak{g}(r)), \mathfrak{g}(r))$.
(2) *The geometric CYBE model of r is $((X, \mathscr{A}), (X \setminus \{s\}, \mathrm{d}u/u))$, where $\mathrm{d}u/u$ is understood as Rosenlicht regular 1-form on X.*
(3) *The associated geometric r-matrix ρ of $((X, \mathscr{A}), (X \setminus \{s\}, \mathrm{d}u/u))$ satisfies $(\nu \times \nu)^*\rho = \varrho^t \in (\mathfrak{L} \otimes \mathfrak{L})[((u/v) - 1)^{-1}]$.*

Proof. First, note that $\zeta(\Gamma(X \setminus \{p\}, \mathscr{A})) = \mathfrak{g}(r)$ would be a consequence of the identity $(\nu \times \nu)^*\rho = \varrho^t$ and Theorem 3.3.3. Since $\mathbb{C}[\![z]\!] + c(\Gamma(X \setminus \{p\}, \mathscr{O}_X))$ has codimension one in $\mathbb{C}(\!(z)\!)$, we can see that either $\mathrm{Mult}(\mathfrak{g}(r)) = c(\Gamma(X \setminus \{p\}, \mathscr{A}))$ or $\mathbb{C}[\![z]\!] + \mathrm{Mult}(\mathfrak{g}(r)) = \mathbb{C}(\!(z)\!)$. The latter would imply that $\iota\colon \mathscr{A} \to \mathscr{B}$ is an isomorphism, so $\mathfrak{W} = \mathfrak{W}_+ \times \mathfrak{W}_-$. This does not occur by virtue of Lemma 8.1.1. Therefore, it remains to proof $(\nu \times \nu)^*\rho = \varrho^t$. We split the derivation of this assertion into several steps.

Step 1. *Some preliminary notations.* For any $k \in \mathbb{N}$ let

- $P^{(k)} := \mathbb{C}[u_+, u_-]/(u_-, u_+)^k$ and $\widetilde{P}^{(k)}_{\pm} := \mathbb{C}[u^{\pm 1}]/(u^{\pm k})$ as well as
- $Y^{(k)} := \operatorname{Spec}(P^{(k)}) \times U$, $\widetilde{Y}^{(k)}_{\pm} := \operatorname{Spec}(\widetilde{P}^{(k)}_{\pm}) \times U$ and $\widetilde{Y}^{(k)} := \widetilde{Y}^{(k)}_{+} \sqcup \widetilde{Y}^{(k)}_{-}$.

Then we set: $Y := \varinjlim Y^{(k)}$, $\widetilde{Y}_{\pm} := \varinjlim \widetilde{Y}^{(k)}_{\pm}$, $\widetilde{Y} = \widetilde{Y}_+ \times \widetilde{Y}_-$. Note that $Y = \operatorname{Spec}(P)$ and $\widetilde{Y}_{\pm} = \operatorname{Spec}(\widetilde{P}_{\pm})$, where $P := \mathbb{C}[v, v^{-1}][\![u_+, u_-]\!]/(u_+u_-)$ and $\widetilde{P}_{\pm} := \mathbb{C}[v, v^{-1}][\![u^{\pm 1}]\!]$. Finally, let $S_{\pm} := \mathbb{C}[v, v^{-1}](\!(u^{\pm 1})\!)$, $Z_{\pm} := \operatorname{Spec}(S_{\pm})$, $S := S_+ \times S_-$ and $Z := Z_+ \sqcup Z_-$. The canonical embedding $\mathbb{C}[u, u^{-1}, v, v^{-1}] \subseteq S_{\pm}$ induce an embedding $\psi \colon \mathbb{C}[u, u^{-1}, v, v^{-1}] \to S$. The identities

$$-(u - v)\sum_{k=0}^{\infty} v^{-k-1}u^k = 1 = (u - v)\sum_{k=0}^{\infty} v^k u^{-k-1} \tag{8.16}$$

imply that $\psi(u - v)$ is a unit in S. As a consequence, ψ can be extended uniquely to an algebra homomorphism $\widetilde{\psi} \colon \mathbb{C}[u, u^{-1}, v, v^{-1}, (u - v)^{-1}] \to S$, where

$$\widetilde{\psi}\left(\frac{v}{u - v}\right) = \left(-\sum_{k=0}^{\infty} v^{-k}u^k, \sum_{k=1}^{\infty} v^k u^{-k}\right). \tag{8.17}$$

Thus, considering $U = \operatorname{Spec}(\mathbb{C}[u, u^{-1}]) = \nu^{-1}(C)$, we obtain a morphism

$$\jmath \colon Z \to \operatorname{Spec}(\mathbb{C}[u, u^{-1}, v, v^{-1}, (u - v)^{-1}]) = U \times U \setminus \widetilde{\Delta}, \tag{8.18}$$

where $\widetilde{\Delta} = (\nu \times \nu)^{-1}\Delta$ and Δ is the diagonal of $C \times C$. Next, we have a family of morphisms of schemes a

$$\left(\varepsilon_k \colon Y^{(k)} \to X \times C \setminus \Delta\right)_{k \in \mathbb{N}}, \tag{8.19}$$

defined as the product of the morphism $\operatorname{Spec}(P^{(k)}) \to X$ mapping the unique closed point to s and $\nu \colon U \to C$. Taking the corresponding direct limit, we get a morphism $\varepsilon \colon Y \to X \times C \setminus \Delta$. In a similar way, we have a family of morphisms $\left(\widetilde{\varepsilon}_k \colon \widetilde{Y}^{(k)} \to \mathbb{P}^1_{\mathbb{C}} \times U \setminus \widetilde{\Delta}\right)_{k \in \mathbb{N}}$ as well as the corresponding direct limit $\widetilde{\varepsilon} \colon \widetilde{Y} \to \mathbb{P}^1_{\mathbb{C}} \times U \setminus \widetilde{\Delta}$. Summing up, we get the following commutative diagram in the category of schemes:

$$\begin{array}{ccccc}
X \times C \setminus \Delta & \xleftarrow{\ \nu \times \nu\ } & \mathbb{P}^1 \times U \setminus \widetilde{\Delta} & \xhookleftarrow{\ \imath\ } & U \times U \setminus \widetilde{\Delta} \\
\uparrow{\scriptstyle \varepsilon_k} & & \uparrow{\scriptstyle \widetilde{\varepsilon}_k} & & \\
Y^{(k)} & \xleftarrow{\ \mu_k\ } & \widetilde{Y}^{(k)} & & \uparrow{\scriptstyle \jmath} \\
\downarrow & & \downarrow & & \\
Y & \xleftarrow{\ \mu\ } & \widetilde{Y} & \xleftarrow{\ \eta\ } & Z
\end{array}
\qquad (\varepsilon \colon Y \to X \times C \setminus \Delta,\ \widetilde{\varepsilon} \colon \widetilde{Y} \to \mathbb{P}^1 \times U \setminus \widetilde{\Delta}) \tag{8.20}$$

where $\widetilde{\nu}, \widetilde{\nu}_k$ and η are the morphism induced by the canonical inclusions $P^{(k)} \subseteq \widetilde{P}^{(k)}_{-} \times \widetilde{P}^{(k)}_{+}$ and $\mathbb{C}[v, v^{-1}][\![u^{\pm 1}]\!] \subseteq S_{\pm}$.

Step 2. *Formal series expression of ρ at s.* Since $\mathscr{A}$ is torsion free, we get an injective map

$$\Gamma(X \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A}) \xrightarrow{\varepsilon^*} \widehat{\mathfrak{W}}\widehat{\otimes}\mathfrak{L} := \varprojlim \left(\mathfrak{W}/(u_+,u_-)^k\mathfrak{W} \otimes \mathfrak{L}\right)$$

Let $\Upsilon := \{(k,i) \in \mathbb{Z}^2 \mid 1 \leqslant i \leqslant \dim(\mathfrak{g}_i^\sigma)\}$ and $\{b_{(k,i)}\}_{i=1}^{\dim(\mathfrak{g}_i)}$ be bases for all $k \in \mathbb{Z}$, which are chosen in such a way that $K(b_{(k,i)}, b_{(\ell,j)}) = \delta_{k,-\ell}\delta_{i,j}$. Then $\left\{a_{(k,i)} := b_{(k,i)}\tilde{u}^k\right\}_{(k,i)\in\Upsilon}$ is a basis of $\mathfrak{L}$ with the property $K_0(a_{(k,i)}, a_{(\ell,j)}) = \delta_{k,-\ell}\delta_{i,j}$. There exists a uniquely determined family $(c_{(k,i)})_{(k,i)\in\Upsilon}$ of elements of $\widehat{\mathfrak{W}}$ such that for any $k \in \mathbb{N}$ there exists a finite subset $\Upsilon_k \subset \Upsilon$ satisfying the following properties:

- The class $c_{(k,i)}^{(k)}$ of $c_{(k,i)}$ in $\mathfrak{W}/(u_-,u_+)^k\mathfrak{W}$ is zero for all $(k,i) \notin \Upsilon_k$ and
- $\varepsilon_k^*(\rho) = \sum_{l\in\Upsilon_k} c_{(k,i)}^{(k)} \otimes a_{(k,i)} \in \mathfrak{W}/(u_+,u_-)^k\mathfrak{W} \otimes \mathfrak{L}$.

Recall that $\rho \in \Gamma(X \times C, \mathscr{A} \boxtimes \mathscr{A}|_C(\Delta)) \subseteq \Gamma(X \times C \setminus \Delta, \mathscr{A} \boxtimes \mathscr{A})$. In these terms we may write

$$\varepsilon^*(\rho) = \sum_{(k,i)\in\Upsilon} c_{(k,i)} \otimes a_{(k,i)}. \tag{8.21}$$

Step 3. *An expression for ρ on the smooth locus.* Choosing the local parameter $\nu^{\flat,-1}(u-1)$ of p and

$$\chi := \sum_{j=0}^{m-1} (\tilde{u}/\tilde{v})^j \gamma_j \in \mathfrak{L} \otimes \mathfrak{L}, \text{ where } \gamma = \sum_{j=0}^{m-1} \gamma_j \in \bigoplus_{j=0}^{m-1} (\mathfrak{g}_j^\sigma \otimes \mathfrak{g}_{-j}^\sigma).$$

in (3.37) yields

$$(\nu \times \nu)^*\rho|_{C\times C} = \frac{v\chi}{u-v} + h, \tag{8.22}$$

for some $h \in \mathfrak{L} \otimes \mathfrak{L}$. We have to show that $h = t + \gamma_{0,\mathfrak{n}_-} + \gamma_{0,\mathfrak{h}}/2$ to conclude the proof.

Step 4. *Comparing the expressions for ρ.* It follows from (8.17) and (8.22) that

$$\jmath^*(\nu \times \nu)^*\rho = \sum_{(k,i)\in\Upsilon} (w_{(k,i)} + (h_{(k,i)}, h_{(k,i)})) \otimes a_{(k,i)} \in (\widehat{\mathfrak{W}}\widehat{\otimes}\mathfrak{L}) \otimes_P S, \tag{8.23}$$

where the $(h_{(k,i)}, h_{(k,i)}) \in \mathfrak{D} \subset \widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-$ are determined by the expression $h = \sum_{(k,i)\in\Upsilon} h_{(k,i)} \otimes a_{(k,i)}$ (which is a finite sum in $\mathfrak{L} \otimes \mathfrak{L}$) and

$$\widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^- \ni w_{(k,i)} = \begin{cases} \left(0, a_{(-k,i)}\right) & \text{if} \quad k \geqslant 1 \\ \left(-a_{(-k,i)}, 0\right) & \text{if} \quad k \leqslant 0. \end{cases} \tag{8.24}$$

Here, $\widehat{\mathfrak{L}}^\pm := \mathfrak{L} \otimes \mathbb{C}(\!(u^{\pm 1})\!)$ and we note that

$$\mathrm{Q}(\widehat{\mathfrak{W}}) = \widehat{\mathfrak{W}} \otimes_P (\mathbb{C}(\!(u_+)\!) \times \mathbb{C}(\!(u_-)\!)) = \widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-. \tag{8.25}$$

It follows from (8.20) that $(\eta\mu)^*\varepsilon^*\rho = \jmath^*(\nu \times \nu)^*\iota^*\rho$. Hence, for any $(k,i) \in \Upsilon$ we have:

$$\widehat{\mathfrak{W}} \ni c_{(k,i)} = w_{(k,i)} + (h_{(k,i)}, h_{(k,i)}) \in \widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-. \tag{8.26}$$

It follows that $c_{(k,i)} = w_{(k,i)}$ for all but finitely many $(k,i) \in \Upsilon$, since all $h_{(k,i)}$ but finitely many are zero. As $\mathfrak{D}$ is an isotropic subalgebra of $\widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-$, we deduce from (8.24) the following orthogonality relation:

$$B_2\Big(c_{(k',i')}, (a_{(k'',i'')}, a_{(k'',i'')})\Big) = B_2\Big(w_{(k',i')}, (a_{(k'',i'')}, a_{(k'',i'')})\Big) = -\delta_{k'k''}\delta_{i'i''}. \tag{8.27}$$

Note that (8.26) actually completely determines the $h_{(k,i)}$, since $\widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^- = \mathfrak{D} \oplus \widehat{\mathfrak{W}}$.

Step 5. $h = \gamma_{0,\mathfrak{n}_-} + \gamma_{0,\mathfrak{h}}/2$ *for* $t = 0$. We first assume that $t = 0$, i.e. $\widehat{\mathfrak{W}} = \widehat{\mathfrak{W}}^\circ$, and write $c_{(k,i)} := c^\circ_{(k,i)}$ for every $(k,i) \in \Upsilon$. Then $w_{(k,i)} \in \widehat{\mathfrak{W}}^\circ$ for all $(k,i) \in \Upsilon$, $k \neq 0$, so $c^\circ_{(k,i)} = w_{(k,i)}$ in this case. Furthermore, we have

$$(\mathrm{pr}_\pm \otimes \mathrm{id}_\mathfrak{g})\left(\sum_{i=1}^{\dim(\mathfrak{g}_0)} w_{(0,i)} \otimes a_{(0,i)}\right) = \begin{cases} 0 & \text{if } - \\ -\gamma_0 & \text{if } + \end{cases} \tag{8.28}$$

Consider $h = \gamma_0^- + \gamma_\mathfrak{h}/2$. Then

$$(\mathrm{pr}_\pm \otimes \mathrm{id}_\mathfrak{g})\left(\sum_{i=1}^{\dim(\mathfrak{g}_0)} c^\circ_{(0,i)} \otimes a_{(0,i)}\right) = \mp\gamma_0^\pm \mp \gamma_\mathfrak{h}/2, \tag{8.29}$$

from which we can deduce $\sum_{i=1}^{\dim(\mathfrak{g}_0)} c^\circ_{(0,i)} \otimes a_{(0,i)} \in \widehat{\mathfrak{W}}^\circ \otimes \mathfrak{L}$ or more specifically $c_{(0,i)} \in \widehat{\mathfrak{W}}^\circ$.

Step 6. *In general* $h = \gamma_{0,\mathfrak{n}_-} + \gamma_{0,\mathfrak{h}}/2 + t$. Recalling the definition of $\mathfrak{W}$ via an endomorphism T constructed from t in Theorem 5.3.2, we may see that $c_{(k,i)} = c^\circ_{(k,i)} - Tc^\circ_{(k,i)} \in \mathfrak{W} \subseteq \widehat{\mathfrak{W}}$ is negatively dual to $(a_{(k,i)}, a_{(k,i)})$. Furthermore, by definition of T, the identity

$$(\mathrm{pr}_\pm \otimes \mathrm{id}_\mathfrak{L})\left(\sum_{(k,i)\in\Upsilon} T(-c^\circ_{(k,i)}) \otimes a_{(k,i)}\right) = t \tag{8.30}$$

holds. Combined, we can conclude that $h = \gamma_{0,\mathfrak{n}_-} + \gamma_{0,\mathfrak{h}}/2 + t$. □

8.1.3 The classical double of δ^t. Let t be a classical twist of δ° and write $\mathfrak{W} := \mathfrak{W}^t$. Although δ^t is determined by the Manin triple $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{D}, \mathfrak{W})$, this is not the classical double, since $\mathfrak{W} \to (\mathfrak{L} \times \mathfrak{L})^*$ is injective but not surjective. However, the classical double emerges naturally from the geometric CYBE datum $((X, \mathscr{A}), (C, \eta))$ in Theorem 8.1.3 in the form of

$$\mathrm{Q}(\widehat{\mathscr{A}_s}) \cong \mathrm{Q}(\widehat{\mathscr{B}_0}) \times \mathrm{Q}(\widehat{\mathscr{B}_\infty}) \cong \widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-, \tag{8.31}$$

where $\widehat{\mathfrak{L}}^{\pm} := \mathfrak{L} \otimes_L \mathbb{C}((u^{\pm 1}))$ and $\mathscr{B}$ was defined in Subsection 8.1.1. The bilinear form can thereby be given using the pairing $\mathcal{K}$ of $\mathscr{A}$ (see Proposition 3.2.8) via the formula

$$\begin{aligned} \widehat{K}_0^{(2)}(a,b) &:= \sum_{q \in \nu^{-1}(s)} \mathrm{res}_q \widehat{\mathcal{K}}(a_q, b_q)\widehat{\eta}_q = \mathrm{res}_0 \widehat{\mathcal{K}}(a_0, b_0)\widehat{\eta}_0 + \mathrm{res}_\infty \widehat{\mathcal{K}}(a_\infty, b_\infty)\widehat{\eta}_\infty \\ &= \sum_{j+k=0} (K(a_{+,j}, b_{+,k}) - K(a_{-,j}, b_{-,k})) \end{aligned}$$

for $a, b \in \mathrm{Q}(\widehat{\mathscr{A}_s})$, where a_q, b_q are the $\mathrm{Q}(\widehat{\mathscr{B}_q})$-components of a, b for $q \in \nu^{-1}(s) = \{0, \infty\}$ and $a_\pm = \sum_{j \in \mathbb{Z}} a_{\pm,j}\widetilde{u}^j, b_\pm = \sum_{j\in\mathbb{Z}} b_{\pm,j}\widetilde{u}^j \in \widehat{\mathfrak{L}}^{\pm} \subseteq \mathfrak{g}((\widetilde{u}^{\pm 1}))$ are the respective images under (8.31).

Proposition 8.1.4.
Let $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ be the Taylor series of $\varrho^t(\exp(x/m), \exp(y/m))$ in $y = 0$ for some classical twist t of δ°, $((X, \mathscr{A}), (C, \eta))$ be the associated geometric CYBE datum used in Theorem 8.1.3, and $\mathfrak{W} := \mathfrak{W}^t$. Then

$$(\mathrm{Q}(\widehat{\mathscr{A}_s}), \Gamma(C, \mathscr{A}), \widehat{\mathscr{A}_s}) \cong (\widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-, \mathfrak{D}, \widehat{\mathfrak{W}}) \tag{8.32}$$

is a Manin triple determining δ^t (up to $\mathfrak{L} \cong \mathfrak{D}$), identifying $\widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-$ with the classical double of δ^t.

Proof. First, recall that $\widehat{\mathscr{A}_s} = \widehat{\mathfrak{W}}$; see Lemma 8.1.2. The fact that $(\widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^-, \mathfrak{D}, \widehat{\mathfrak{W}})$ is a Manin triple that determines δ^t up to the identification $\mathfrak{L} \cong \mathfrak{D}$ can be derived from the fact that $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{D}, \mathfrak{W})$ is a Manin triple determining δ^t up to the same identification; see Theorem 5.4.9. Therefore, it remains to prove that $\widehat{K}_0^{(2)}$ induces an linear isomorphism $\widehat{\mathfrak{W}} \to \mathfrak{D}^* \cong \mathfrak{L}^*$, because of the uniqueness condition of the Lie algebra structure on the classical double from Proposition 5.2.2.

First, observe that the canonical projection $\widehat{\mathfrak{W}} \subseteq \widehat{\mathfrak{L}}^+ \times \widehat{\mathfrak{L}}^- = \mathfrak{D} \oplus \widehat{\mathfrak{W}}^\circ \to \widehat{\mathfrak{W}}^\circ$ defines an linear isomorphism $\widehat{\mathfrak{W}} \cong \widehat{\mathfrak{W}}^\circ$ compatible with the respective maps $\widehat{\mathfrak{W}}^\circ \to \mathfrak{D}^* \leftarrow \widehat{\mathfrak{W}}$ induced by $\widehat{K}_0^{(2)}$. Thus, it remains to prove that $K_0^{(2)}$ defines an isomorphism $\widehat{\mathfrak{W}}_0 \to \mathfrak{D}^*$. Clearly, $\widehat{K}_0^{(2)}$ induces linear isomorphisms

$$\prod_{j\in\mathbb{N}} (\mathfrak{L}_j \times \{0\}) \longrightarrow \left(\bigoplus_{j\in\mathbb{N}} \mathfrak{L}_{-j}\right)^* \text{ and } \prod_{j\in\mathbb{N}} (\{0\} \times \mathfrak{L}_{-j}) \longrightarrow \left(\bigoplus_{j\in\mathbb{N}} \mathfrak{L}_{j}\right)^*. \tag{8.33}$$

Furthermore, $\widehat{K}_0^{(2)}$ also induces an linear isomorphism

$$\mathfrak{w} := \{(w_+, w_-) \in (\mathfrak{n}_+ \oplus \mathfrak{H}) \times (\mathfrak{H} \oplus \mathfrak{n}_-) \mid w_+ + w_- \in \mathfrak{n}_+ + \mathfrak{n}_-\} \longrightarrow \mathfrak{L}_0^*. \tag{8.34}$$

Combined, we deduce that the map

$$\widehat{\mathfrak{W}}^\circ = \prod_{j\in\mathbb{N}} (\mathfrak{L}_j \times \{0\}) \oplus \mathfrak{w} \oplus \prod_{j\in\mathbb{N}} (\{0\} \times \mathfrak{L}_{-j}) \longrightarrow \mathfrak{D}^* \tag{8.35}$$

induced by $\widehat{K}_0^{(2)}$ is an isomorphism, concluding the proof. ∎

8.2 Classification

In this section, we translate the classification of trigonometric solutions of the CYBE with one spectral parameter from [BD83a] to a classification of σ-trigonometric r-matrices. The latter is equivalent to the classification of classical twists of δ°.

8.2.1 The classification theorem. A *Belavin-Drinfeld triple* (Π_+, Π_-, ϕ) consists of two proper (possibly empty) subsets $\Pi_\pm \subsetneq \Pi$ and a bijection $\phi\colon \Pi_+ \to \Pi_-$, satisfying

- $K_0(\phi(\alpha)|_{\mathfrak{H}}, \phi(\beta)|_{\mathfrak{H}}) = K_0(\alpha|_{\mathfrak{H}}, \beta|_{\mathfrak{H}})$ for all $\alpha, \beta \in \Pi_+$ and
- for any $\alpha \in \Pi_+$ exists an integer $\ell = \ell(\alpha) \in \mathbb{N}$ such that

$$\phi(\alpha), \dots, \phi^{\ell-1}(\alpha) \in \Pi_+ \text{ but } \phi^\ell(\alpha) \notin \Pi_+. \tag{8.36}$$

A *Belavin-Drinfeld quadruple* $((\Pi_+, \Pi_-, \phi), h)$ consists of a Belavin-Drinfeld triple (Π_+, Π_-, ϕ) and a tensor $h \in \mathfrak{H} \wedge \mathfrak{H}$ such that

$$(\phi(\alpha) \otimes 1 + 1 \otimes \alpha)(h + \gamma_{\mathfrak{H}}/2) = 0 \tag{8.37}$$

holds for all $\alpha \in \Pi_+$.

Let $\mathfrak{s}^S := \langle e_i^+, e_i^- \mid \alpha_i \in S\rangle_{\mathbb{C}\text{-alg}} \subseteq \mathfrak{L}$ for any $S \subseteq \Pi$. Note that if $S \neq \Pi$, $\mathfrak{s}^S$ is a finite-dimensional semi-simple Lie algebra with Chevalley generators $\{e_i^+, e_i^- \mid \alpha_i \in S\}$, which induce a triangular decomposition $\mathfrak{s}^S = \mathfrak{N}_+^S \oplus \mathfrak{H}^S \oplus \mathfrak{N}_-^S$, where $\mathfrak{N}_\pm^S = \bigoplus_{\alpha \in \Phi_\pm^S} \mathfrak{L}_\alpha$ for some subsets $\Phi_\pm^S \subseteq \Phi_\pm$.

For a Belavin-Drinfeld triple (Π_+, Π_-, ϕ), let $\theta^\phi\colon \mathfrak{s}_{\Pi_+} \to \mathfrak{s}_{\Pi_-}$ be the unique Lie algebra isomorphism defined by

$$\theta^\phi(e_i^\pm) = e_{\phi(i)}^\pm \text{ for all } \alpha_i \in \Pi_+, \text{ where by abuse of notation } \phi(\alpha_i) = \alpha_{\phi(i)}. \tag{8.38}$$

Moreover, let $\theta_\pm^\phi\colon \mathfrak{L} \to \mathfrak{L}$ be the unique extension of $\theta^\phi|_{\mathfrak{N}_\pm^S}$ by 0 with respect to the decomposition $\mathfrak{L} = (\mathfrak{H} \oplus \bigoplus_{\alpha \in \Phi \setminus \Phi_\pm^S} \mathfrak{L}_\alpha) \oplus \mathfrak{N}_\pm^S$. Note that $\theta_\pm^\phi$ are both nilpotent because of (8.36).

Consider a Belavin-Drinfeld quadruple $Q = ((\Pi_+, \Pi_-, \phi), h)$, let us choose root vectors $\{b_\alpha \in \mathfrak{L}_\alpha \mid \alpha \in \Phi_-^{\Pi_+} \sqcup \Phi_+^{\Pi_+}\}$ such that $B(b_\alpha, b_{-\alpha}) = 1$ for all $\alpha \in \Phi_+^{\Pi_+}$ and write

$$t_Q := h + \sum_{\alpha \in \Phi_+^{\Pi_+}} \sum_{j=1}^{\infty} b_{-\alpha} \wedge \theta_+^{\phi,j}(b_\alpha) \in \mathfrak{L} \otimes \mathfrak{L}. \tag{8.39}$$

Recall the notion of regular equivalence between classical twists of δ° (resp. σ-trigonometric r-matrices) introduced in Subsection 5.4.4. The subsections 8.2.2-8.2.6 below are dedicated to the proof of the following theorem.

Theorem 8.2.1.
The tensor t_Q defined in (8.39) is a classical twist of δ° for any Belavin-Drinfeld quadruple Q and any classical twist t of δ° is regularly equivalent to t_Q for an appropriate Belavin-Drinfeld quadruple Q. In particular, any σ-trigonometric r-matrix ϱ^t is regularly equivalent to $\varrho^Q := \varrho^{t_Q}$ for an appropriate Belavin-Drinfeld

quadruple Q.

Theorem 8.2.1 can be seen as a generalization of [BD83a, Theorem 6.1]. It can in fact be reduced to [BD83a, Theorem 6.1], by using the relation of formal and regular equivalences discussed in Subsection 8.3.1 below, which relies on the explicit geometrization of σ-trigonometric r-matrices from Theorem 8.1.3 and the geometry of equivalences from Lemma 3.3.2; see [AM21]. In this work, we will proof Theorem 8.2.1 by adjusting the proof of [BD83a, Theorem 6.1] using the structure theory of loop algebras established in Section 4.2.

8.2.2 Interlude: the Cayley transform. Let V be a $\Bbbk$-vector space equipped with a non-degenerate symmetric bilinear form B and $A \in \operatorname{End}(V)$ possess an adjoint $A^* \in \operatorname{End}(V)$ such that $A = \mathrm{id}_V - A^*$, or, equivalently,

$$B(Av, w) + B(v, Aw) = B(v, w) \text{ for all } v, w \in V. \tag{8.40}$$

Clearly, $\operatorname{Im}(A - \mathrm{id}_V)^\perp = \operatorname{Ker}((A - \mathrm{id}_V)^*) = \operatorname{Ker}(-A) = \operatorname{Ker}(A)$ and similarly the identity $\operatorname{Im}(A)^\perp = \operatorname{Ker}(A - \mathrm{id}_V)$ holds. Note that for all $v_1 \in \operatorname{Ker}(A)$ and $v_2 \in \operatorname{Ker}(A - \mathrm{id}_V)$

$$v_1 = (A - \mathrm{id}_V)(-v_1) \in \operatorname{Im}(A - \mathrm{id}_V) \text{ and } v_2 = Av_2 \in \operatorname{Im}(A) \tag{8.41}$$

hold. Combined, we see that

$$\operatorname{Ker}(A) = \operatorname{Im}(A - \mathrm{id}_V)^\perp \subseteq \operatorname{Im}(A - \mathrm{id}_V) \text{ and } \operatorname{Ker}(A - \mathrm{id}_V) = \operatorname{Im}(A)^\perp \subseteq \operatorname{Im}(A).$$

Therefore, the bilinear form B induces non-degenerate bilinear forms on the quotient spaces $\operatorname{Im}(A - \mathrm{id}_V)/\operatorname{Ker}(A)$ and $\operatorname{Im}(A)/\operatorname{Ker}(A - \mathrm{id}_V)$ and it is easy to see that the map $\theta\colon \operatorname{Im}(A - \mathrm{id}_V)/\operatorname{Ker}(A) \to \operatorname{Im}(A)/\operatorname{Ker}(A - \mathrm{id}_V)$ given by

$$(A - \mathrm{id}_V)v + \operatorname{Ker}(A) \longmapsto Av + \operatorname{Ker}(A - \mathrm{id}_V) \tag{8.42}$$

is a well-defined orthogonal linear isomorphism. The triple

$$\operatorname{CT}(A) := (\operatorname{Im}(A - \mathrm{id}_V), \operatorname{Im}(A), \theta) \tag{8.43}$$

is called *Cayley transform* of A.

Remark 8.2.2.
Classically, the Cayley transform of a real skew-symmetric $k \times k$-matrix M is (up to sign convention) the orthogonal matrix $(M + \mathrm{id}_{\mathbb{R}^k})(M - \mathrm{id}_{\mathbb{R}^k})^{-1}$; see e.g. [GVL96, Problem 2.5.1]. Assume that A and $A - \mathrm{id}_V$ are invertible and consider $S := 2A - \mathrm{id}_V$. Then $V = \operatorname{Im}(A - \mathrm{id}_V) = \operatorname{Im}(A)$, $S^* = 2(\mathrm{id}_V - A) - \mathrm{id}_V = -S$ and

$$\theta = A(A - \mathrm{id}_V)^{-1} = 2A(2A - 2\,\mathrm{id}_V)^{-1} = (S + \mathrm{id}_V)(S - \mathrm{id}_V)^{-1} \tag{8.44}$$

satisfies $\theta^* = \theta^{-1}$. Thus, the notion of Cayley transform defined here is a generalization of the aforementioned classical notion from linear algebra.

8.2.3 Proof of Theorem 8.2.1 I: Cayley transform and classical twists of δ°. Let $t = \sum_{i=1}^k a_i \otimes b_i \in \mathfrak{L} \otimes \mathfrak{L}$ be an arbitrary skew-symmetric tensor and recall that $((\mathfrak{L} \times \mathfrak{L}, K_0^{(2)}), \mathfrak{D}, \mathfrak{W}^\circ)$ denotes the Manin triple determining δ° from Subsection 5.4.4. The map $\mu\colon \mathfrak{W}^\circ \to \mathfrak{L}$ defined by $\mu(a_+, a_-) = a_+ - a_-$ is a linear isomorphism with inverse

$$\mu^{-1}(a) = ((\pi_+ + \pi_{\mathfrak{H}}/2)a, -(\pi_- + \pi_{\mathfrak{H}}/2)a) \text{ for all } a \in \mathfrak{L}, \tag{8.45}$$

where $\pi_\pm, \pi_{\mathfrak{H}}\colon \mathfrak{L} \to \mathfrak{L}$ are the canonical projections of $\mathfrak{N}_\pm$ and $\mathfrak{H}$ respectively. Consider

$$R^t \coloneqq \mathrm{pr}_-(T\mu^{-1} - \mu^{-1}) = \pi_- + \pi_{\mathfrak{H}}/2 + \widetilde{K}_0(t) \in \mathrm{End}(\mathfrak{L}) \tag{8.46}$$

where, for $a, a_\pm \in \mathfrak{L}$ and $\iota(a) = (a, a)$, $T \in \mathrm{End}(\mathfrak{L} \times \mathfrak{L})$ is defined by

$$T(a_+, a_-) = \sum_{i=1}^k K_0^{(2)}((b_i, b_i), (a_+, a_-))(a_i, a_i) = \iota\widetilde{K}_0(t)(a_+ - a_-) \tag{8.47}$$

and $\widetilde{K}_0\colon \mathfrak{L} \otimes \mathfrak{L} \to \mathrm{End}(\mathfrak{L})$ is given by $a \otimes b \mapsto K_0(b, -)a$. Then

$$\begin{aligned} R^t - \mathrm{id}_{\mathfrak{L}} &= -\pi_+ - \pi_{\mathfrak{H}}/2 + \widetilde{K}_0(t) = \mathrm{pr}_+(T\mu^{-1} - \mu^{-1}) \text{ and} \\ \mathfrak{W}^t &\coloneqq \{Tw - w \mid w \in \mathfrak{W}^\circ\} = \{((R^t - \mathrm{id}_{\mathfrak{L}})a, R^t a) \mid a \in \mathfrak{L}\} \end{aligned} \tag{8.48}$$

hold. Moreover, $\pi_\pm^* = \pi_\mp$, $\pi_{\mathfrak{H}}^* = \pi_{\mathfrak{H}}$ and $\widetilde{K}_0(t)^* = \widetilde{K}_0(\tau_{\mathfrak{L}}(t)) = -\widetilde{K}_0(t)$ implies that $R^{t,*} = \mathrm{id}_{\mathfrak{L}} - R^t$.

Lemma 8.2.3.
Let $t \in \wedge^2\mathfrak{L}$, $R \coloneqq R^t$ be given by (8.46), and $\mathrm{CT}(R) \coloneqq (\mathfrak{W}_+, \mathfrak{W}_-, \theta)$ be the Cayley transform of R. Then $\mathfrak{W} \coloneqq \mathfrak{W}^t$ defined in (8.48) satisfies

$$\mathfrak{W} = \{(w_+, w_-) \in \mathfrak{W}_+ \times \mathfrak{W}_- \mid \theta(w_+ + \mathfrak{W}_+^\perp) = w_- + \mathfrak{W}_-^\perp\}. \tag{8.49}$$

Furthermore, t is a classical twist of δ° if and only if
- *$\mathfrak{W}_\pm \subseteq \mathfrak{L}$ are subalgebras (then $\mathfrak{W}_\pm^\perp \subseteq \mathfrak{W}_\pm$ are automatically ideals) and*
- *$\theta\colon \mathfrak{W}_+/\mathfrak{W}_+^\perp \to \mathfrak{W}_-/\mathfrak{W}_-^\perp$ is a Lie algebra homomorphism.*

In this case, the notation $\mathfrak{W} = \mathfrak{W}^t$ is consistent with Theorem 5.4.9.

Proof. The definition of the Cayley transform immediately implies that

$$\begin{aligned} \mathfrak{W} &= \{((R - \mathrm{id}_{\mathfrak{L}})a, Ra) \mid a \in \mathfrak{L}\} \\ &\subseteq \{(w_+, w_-) \in \mathfrak{W}_+ \times \mathfrak{W}_- \mid \theta(a + \mathfrak{W}_+^\perp) = b + \mathfrak{W}_-^\perp\} \\ &= \{((R - \mathrm{id}_{\mathfrak{L}})a, Rb) \mid R(a - b) \in \mathrm{Ker}(R - \mathrm{id}_{\mathfrak{L}}), a, b \in \mathfrak{L}\}. \end{aligned} \tag{8.50}$$

For every $a, b \in \mathfrak{L}$ such that $(R - \mathrm{id}_{\mathfrak{L}})a = Rb$ and $R(a - b) \in \mathrm{Ker}(R - \mathrm{id}_{\mathfrak{L}})$, we can deduce that $a = R(a - b) \in \mathrm{Ker}(R - \mathrm{id}_{\mathfrak{L}})$, so $0 = (R - \mathrm{id}_{\mathfrak{L}})a = Rb$ implies

$$\{((R - \mathrm{id}_{\mathfrak{L}})a, Rb) \mid R(a - b) \in \mathrm{Ker}(R - \mathrm{id}_{\mathfrak{L}}), a, b \in \mathfrak{L}\} \cap \mathfrak{D} = \{0\}. \tag{8.51}$$

Combining this identity with $\mathfrak{L} \times \mathfrak{L} = \mathfrak{D} \oplus \mathfrak{W}$, forces the inclusion in the second line of (8.50) to be an equality, so the formula from the first part of the assertion is proven.

The fact that, if $\mathfrak{W}_\pm \subseteq \mathfrak{L}$ is a subalgebra, $\mathfrak{W}_\pm^\perp \subseteq \mathfrak{W}_\pm$ is an ideal, is a direct consequence of the invariance of K_0. By virtue of Theorem 5.3.2, t is a classical twist of δ° if and only if $\mathfrak{W} \subseteq \mathfrak{L} \times \mathfrak{L}$ is a subalgebra. Using the formula from the first part of the assertion, it is easy to see that $\mathfrak{W} \subseteq \mathfrak{L} \times \mathfrak{L}$ is a subalgebra if and only if $\mathfrak{W}_\pm \subseteq \mathfrak{L}$ are subalgebras and θ is a Lie algebra homomorphism. □

Lemma 8.2.4.
Let $t \in \wedge^2\mathfrak{L}$ and $R := R^t$ be defined as in (8.46). The tensor t is a classical twist of δ° if and only if R satisfies

$$[Ra, Rb] = R([Ra, b] + [a, Rb] - [a, b]) \tag{8.52}$$

for all $a, b \in \mathfrak{L}$.

Proof. Let $\mathrm{CT}(R) := (\mathfrak{W}_+, \mathfrak{W}_-, \theta)$ be the Cayley transform of R. By virtue of Lemma 8.2.3, we have to prove that $\mathfrak{W}_\pm \subseteq \mathfrak{L}$ are subalgebras and θ is a Lie algebra homomorphism if and only if (8.52) holds for all $a, b \in \mathfrak{L}$.

" $\Longrightarrow$ " Since $\mathfrak{W}_+ = \mathrm{Im}(R - \mathrm{id}_\mathfrak{L}) \subseteq \mathfrak{L}$ is a subalgebra, there exists some $c \in \mathfrak{L}$ such that $[(R - \mathrm{id}_\mathfrak{L})a, (R - \mathrm{id}_\mathfrak{L})b] = (R - \mathrm{id}_\mathfrak{L})c$. Applying the Lie algebra morphism θ yields $[R_-a, R_-b] = R_-c + d$ for some $d \in \mathfrak{W}_-^\perp = \mathrm{Ker}(R - \mathrm{id}_\mathfrak{L})$. Subtracting $[(R - \mathrm{id}_\mathfrak{L})a, (R - \mathrm{id}_\mathfrak{L})b] = (R - \mathrm{id}_\mathfrak{L})c$ from this equation results in $[Ra, b] + [a, Rb] - [a, b] = c + d$. Finally, applying $R - \mathrm{id}_\mathfrak{L}$ leaves us with

$$\begin{aligned} &(R - \mathrm{id}_\mathfrak{L})([Ra, b] + [a, Rb] - [a, b]) \\ &= (R - \mathrm{id}_\mathfrak{L})c = [(R - \mathrm{id}_\mathfrak{L})a, (R - \mathrm{id}_\mathfrak{L})b]. \end{aligned} \tag{8.53}$$

This equation is easily seen to be equivalent to (8.52).

" $\Longleftarrow$ " The equations (8.52) and (8.53) imply that $\mathfrak{W}_\pm \subseteq \mathfrak{L}$ are subalgebras and θ is a Lie algebra homomorphism. □

Let us conclude this subsection by investigating the compatibility of the objects discussed here with the notion of regular equivalence introduced in Subsection 5.4.4.

Lemma 8.2.5.
Let $t_1, t_2 \in \mathfrak{L} \otimes \mathfrak{L}$ be classical twists of δ° and $\varphi \in \mathrm{Aut}_{L\text{-alg}}(\mathfrak{L})$. Then, φ defines a regular equivalence of t_1 and t_2 if and only if $(\varphi \times \varphi)\mathfrak{W}^{t_1} = \mathfrak{W}^{t_2}$ if and only if $\varphi R^{t_1}\varphi^{-1} = R^{t_2}$ (see (8.46)).

Proof. Let us write $\widetilde{K} \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ for the isomorphism defined by the assignment $a \otimes b \mapsto K(b, -)a$. Observe that

$$\widetilde{K}_0(a \otimes b)(c)(\widetilde{u}) = \mathrm{res}_0 \left(\widetilde{K}(b(\widetilde{v}), c(\widetilde{v}))a(\widetilde{u}) \frac{\mathrm{d}\widetilde{v}}{\widetilde{v}} \right) = \mathrm{res}_0 \left(\widetilde{K}(a(\widetilde{u}) \otimes b(\widetilde{v}))c(\widetilde{v}) \frac{\mathrm{d}\widetilde{v}}{\widetilde{v}} \right)$$

holds for all $a, b, c \in \mathfrak{L} \subseteq \mathfrak{g}[\widetilde{u}, \widetilde{u}^{-1}]$, where we recall $\widetilde{u}^m = u, \widetilde{v}^m = v$. Equation (5.53) implies that

$$\left(\left(\pi_- + \frac{\pi_\mathfrak{H}}{2} \right) a \right)(\widetilde{u}) = \mathrm{res}_0 \left(\widetilde{K}(\varrho(\widetilde{u}, \widetilde{v}))a(\widetilde{v}) \frac{\mathrm{d}\widetilde{v}}{\widetilde{v}} \right) \tag{8.54}$$

holds for all $a \in \mathfrak{L}$. Therefore, we can deduce that

$$R^{t_i}(a)(\tilde{u}) = \operatorname{res}_0\left(\widetilde{K}(\varrho^{t_i}(\tilde{u},\tilde{v}))a(\tilde{v})\frac{\mathrm{d}v}{v}\right) \tag{8.55}$$

holds for any $a \in \mathfrak{L}$ and $i \in \{1,2\}$. From this we can see that

$$(\tilde{\varphi}(\tilde{u}) \otimes \tilde{\varphi}(\tilde{v}))\varrho^{t_1}(\tilde{u},\tilde{v}) = \varrho^{t_2}(\tilde{u},\tilde{v}) \tag{8.56}$$

implies the identity $\varphi R^{t_1}\varphi^{-1} = R^{t_2}$. Here, $\tilde{\varphi} \in \operatorname{End}(\mathfrak{g})[\tilde{u},\tilde{u}^{-1}]$ is defined by $\varphi(a)(\tilde{u}) = \tilde{\varphi}(\tilde{u})a(\tilde{u})$ for all $a \in \mathfrak{L}$ and we use that the adjoint of φ with respect to the Killing form $\mathfrak{L} \times \mathfrak{L} \to L$ is φ^{-1}. It is clear that $\varphi R^{t_1}\varphi^{-1} = R^{t_2}$ implies $\varphi(R^{t_1} - \mathrm{id}_{\mathfrak{L}})\varphi^{-1} = (R^{t_2} - \mathrm{id}_{\mathfrak{L}})$, so $(\varphi \times \varphi)\mathfrak{W}^{t_1} = \mathfrak{W}^{t_2}$. Finally, assume that $(\varphi \times \varphi)\mathfrak{W}^{t_1} = \mathfrak{W}^{t_2}$ holds. Then $(\varphi \otimes \varphi)\delta^{t_1}\varphi^{-1}$ and δ^{t_2} are both determined by $(\mathfrak{L} \times \mathfrak{L}, \mathfrak{D}, \mathfrak{W}^{t_2})$. Therefore, $(\varphi \otimes \varphi)\delta^{t_1}\varphi^{-1} = \delta^{t_2}$, i.e. t_1 and t_2 are regularly equivalent. □

8.2.4 Proof of Theorem 8.2.1 II: Equation (8.39) defines a classical twist of δ°.

Let us begin this paragraph with the following observation.

Lemma 8.2.6.
Let (Π_+,Π_-,ϕ) be a Belavin-Drinfeld triple, $h \in \mathfrak{H} \wedge \mathfrak{H}$, and $\mathrm{CT}(H) = (\mathfrak{H}_+,\mathfrak{H}_-,\theta^h)$ be the Cayley transform of $H := \widetilde{K}_0(h) + \mathrm{id}_{\mathfrak{H}}/2 \in \operatorname{End}(\mathfrak{H})$. Then $((\Pi_+,\Pi_-,\phi),h)$ is a Belavin-Drinfeld quadruple if and only if $\mathfrak{H}_\pm = \mathfrak{H}^{\Pi_\pm} \oplus (\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp})$ and $\theta^h|_{\mathfrak{H}^{\Pi_+}} = \theta^\phi|_{\mathfrak{H}^{\Pi_+}}$, where $\mathfrak{H}_\pm^\perp \subseteq \mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp}$ was used to identify $\mathfrak{H}^{\Pi_\pm}$ as a subspace of $\mathfrak{H}_\pm/\mathfrak{H}_\pm^\perp$.

Proof. It is easy to see that (8.37) is equivalent to

$$(H - \mathrm{id}_{\mathfrak{H}})\phi(\alpha)^* = (\widetilde{K}_0(h) - \mathrm{id}_{\mathfrak{H}}/2)\phi(\alpha)^* = (\widetilde{K}_0(h) + \mathrm{id}_{\mathfrak{H}}/2)\alpha^* = H\alpha^* \tag{8.57}$$

for all $\alpha \in \Gamma_+$. Here, for any $\alpha \in \Phi$, $\alpha^* \in \mathfrak{H}$ denotes the element uniquely determined by $\alpha|_{\mathfrak{H}} = K_0(\alpha^*,\cdot) \in \mathfrak{H}^*$.

" $\Longrightarrow$ " For every $\alpha \in \Pi_+$, (8.57) implies that

$$\alpha^* = H\alpha^* - (H - \mathrm{id}_{\mathfrak{H}})\alpha^* = (H - \mathrm{id}_{\mathfrak{H}})(\phi(\alpha)^* - \alpha^*) \in \mathfrak{H}_+ \tag{8.58}$$

holds. Furthermore, we can deduce

$$\phi(\alpha)^* = H(\phi(\alpha)^* - \alpha^*) \in \mathfrak{H}_- \tag{8.59}$$

in a similar fashion. In particular, $\mathfrak{H}^{\Pi_\pm} \subseteq \mathfrak{H}_\pm$ holds. Lemma 4.2.4.(4) implies that $\mathfrak{H} = \mathfrak{H}^{\Pi_\pm} \oplus \mathfrak{H}^{\Pi_\pm,\perp}$. Therefore, $\mathfrak{H}_\pm \supseteq \mathfrak{H}^{\Pi_\pm} \oplus (\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp})$ and $\mathfrak{H}_\pm^\perp \subseteq \mathfrak{H}^{\Pi_\pm,\perp}$ implies $\mathfrak{H}_\pm^\perp \subseteq \mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp}$. Note that $\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp} = (\mathfrak{H}^{\Pi_\pm} \oplus \mathfrak{H}_\pm^\perp)^\perp$ implies

$$\begin{aligned}\dim(\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp}) &= n - \dim(\mathfrak{H}^{\Pi_\pm} \oplus \mathfrak{H}_\pm^\perp)\\ &= n - (\dim(\mathfrak{H}^{\Pi_\pm}) + (n - \dim(\mathfrak{H}_\pm)) = \dim(\mathfrak{H}_\pm) - \dim(\mathfrak{H}^{\Pi_\pm})\end{aligned} \tag{8.60}$$

so $\mathfrak{H}_\pm = \mathfrak{H}^{\Pi_\pm} \oplus (\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp})$. The identities (8.58) & (8.59) provide

$$\theta^h(\alpha^*) = \theta^h((H - \mathrm{id}_{\mathfrak{H}})(\phi(\alpha)^* - \alpha^*) = \theta^h(H(\phi(\alpha)^* - \alpha^*)) = \phi(\alpha)^* = \theta^\phi(\alpha^*),$$

concluding the proof.

" $\Longleftarrow$ " For every $\alpha \in \Pi_+$ exists an $a \in \mathfrak{H}$ such that $\alpha^* = (H - \mathrm{id}_{\mathfrak{H}})a$ and $\phi(\alpha)^* = Ha$. Subtracting the first from the second equation gives $a = \phi(\alpha)^* - \alpha^*$. Therefore,

$$(H - \mathrm{id}_{\mathfrak{H}})(\phi(\alpha)^* - \alpha^*) = \alpha^* = H\alpha^* - (H - \mathrm{id}_{\mathfrak{H}})\alpha^*$$

implies (8.57), proving the assertion. □

Let $Q = ((\Pi_+, \Pi_-, \phi), h)$ be a Belavin Drinfeld quadruple. It is easy to see from the defining formula (8.39), that

$$\widetilde{K}_0(t_Q) = \frac{\theta_-^{\phi,*}}{\pi_- - \theta_-^{\phi,*}} + \widetilde{K}_0(h) - \frac{\theta_+^\phi}{\pi_+ - \theta_+^\phi}, \tag{8.61}$$

where we used $\theta_+^\phi/(\pi_+ - \theta_+^\phi) = \sum_{j=1}^\infty \theta_+^{\phi,j}$, $(\theta_+^\phi/(\pi_+ - \theta_+^\phi))^* = \theta_+^{\phi,*}/(\pi_- - \theta_+^{\phi,*})$ and $\widetilde{K}_0(t_Q)|_{\mathfrak{N}_-} = -(\widetilde{K}_0(t_Q)|_{\mathfrak{N}_+})^*$. This implies that

$$R := R^{t_Q} = \frac{\pi_-}{\pi_- - \theta_-^{\phi,*}} + \left(\widetilde{K}_0(h) + \frac{\mathrm{id}_{\mathfrak{H}}}{2}\right) - \frac{\theta_+^\phi}{\pi_+ - \theta_+^\phi} \tag{8.62}$$

holds. Let $(\mathfrak{W}_+, \mathfrak{W}_-, \theta)$ be the Cayley transform of R and $(\mathfrak{H}_+, \mathfrak{H}_-, \theta^h)$ be the Cayley transform of $\widetilde{K}_0(h) + \mathrm{id}_{\mathfrak{H}}/2$. Then the identities

$$\mathfrak{W}_\pm = \mathfrak{N}_\mp^{\Pi_\pm} \oplus \mathfrak{H}_\pm \oplus \mathfrak{N}_\pm. \tag{8.63}$$

hold. Therefore, Lemma 8.2.6 implies that $\mathfrak{W}_\pm \subseteq \mathfrak{L}$ are subalgebras and we have canonical isomorphisms $\mathfrak{W}_\pm/\mathfrak{W}_\pm^\perp \cong \mathfrak{s}^{\Pi_\pm} \oplus \mathfrak{f}_\pm$, identifying the center of $\mathfrak{W}_\pm/\mathfrak{W}_\pm^\perp$ with $\mathfrak{f}_\pm := (\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp})/\mathfrak{H}_\pm^\perp$. Note that $\theta|_{\mathfrak{H}^{\Pi_\pm}\oplus\mathfrak{f}_\pm} = \theta^h$ holds by definition and $\theta(\mathfrak{f}_+) = \mathfrak{f}_-$, since θ is orthogonal. To conclude that t_Q is a classical twist of δ° (or, equivalently, that $\varrho^Q = \varrho^\circ + t_Q$ is a σ-trigonometric r-matrix), it remains to show that $\theta|_{\mathfrak{s}^{\Pi_+}} : \mathfrak{s}^{\Pi_+} \to \mathfrak{s}^{\Pi_-}$ is a homomorphism of Lie algebras; see Lemma 8.2.3.

Every element $b \in \mathfrak{N}_+^{\Pi_+}$ can be written as

$$b = (R - \mathrm{id}_{\mathfrak{L}})a = -\left(\frac{\pi_+}{\pi_+ - \theta_+^\phi}\right)a \tag{8.64}$$

for some $a \in \mathfrak{N}_+$. Therefore,

$$\theta(b) = -\left(\frac{\theta_+^\phi}{\pi_+ - \theta_+^\phi}\right)a = \theta_+^\phi(b) \implies \theta|_{\mathfrak{N}_\pm^{\Pi_\pm}} = \theta^\phi|_{\mathfrak{N}_\pm^{\Pi_\pm}}. \tag{8.65}$$

This, Lemma 8.2.6, and $\theta^* = \theta^{-1}$ implies that $\theta|_{\mathfrak{s}^{\Pi_+}} = \theta^\phi$. In particular, θ is a homomorphism of Lie algebras, so t_Q is a classical twist of δ.

Remark 8.2.7.
Let us clarify the explicit form of the Lie algebra $\mathfrak{W}^Q := \mathfrak{W}^{t_Q}$ associated to $\varrho^Q = \varrho^\circ + t_Q$ for the Belavin-Drinfeld quadruple $Q = ((\Pi_+, \Pi_-, \phi), h)$. Let $(\mathfrak{H}_+, \mathfrak{H}_-, \theta^h)$ be the Cayley transform of $\widetilde{K}_0(h) + \mathrm{id}_{\mathfrak{H}}/2$. Then $\mathfrak{W}^Q = \mathfrak{W}^{(\Pi_+,\Pi_-,\phi)} \oplus \mathfrak{W}^h$, where

$$\mathfrak{W}^{(\Pi_+,\Pi_-,\phi)} := \left\{(a,b) \in (\mathfrak{s}^{\Pi_+} + \mathfrak{N}_+) \times (\mathfrak{s}^{\Pi_-} + \mathfrak{N}_-) \mid \theta^\phi(a|_{\Pi_+}) = b|_{\Pi_-}\right\}$$
$$\mathfrak{W}^h := \left\{(a,b) \in \left(\mathfrak{H}_+ \cap \mathfrak{H}^{\Pi_+,\perp}\right) \times \left(\mathfrak{H}_- \cap \mathfrak{H}^{\Pi_-,\perp}\right) \middle| \, \theta^h\left(a + \mathfrak{H}_+^\perp\right) = b + \mathfrak{H}_-^\perp\right\}.$$

Here, θ^ϕ was defined in (8.38) and $a|_{\Pi_\pm}$ denotes the image of $a \in \mathfrak{s}^{\Pi_\pm} + \mathfrak{N}_\pm$ under the canonical projection

$$\mathfrak{s}^{\Pi_\pm} + \mathfrak{N}_\pm = \mathfrak{s}^{\Pi_\pm} \oplus \bigoplus_{\alpha \in \Phi_\pm \setminus \Phi_\pm^{\Pi_\pm}} \mathfrak{L}_\alpha \longrightarrow \mathfrak{s}^{\Pi_\pm}. \tag{8.66}$$

8.2.5 Proof of Theorem 8.2.1 III: Spectral properties of R^t. In order to prove Theorem 8.2.1, it remains to show that any classical twist of δ° is regularly equivalent to t_Q for an appropriate Belavin-Drinfeld quadruple Q. Fix a classical twist t of δ° and put $R := R^t$; see (8.46). Let $\mathfrak{L}^\lambda := \bigcup_{j=1}^\infty \mathrm{Ker}((R - \lambda\,\mathrm{id}_{\mathfrak{L}})^j)$ be the generalized Eigenspace of R to $\lambda \in \mathbb{C}$. Observe that there exists a vector space splitting $\mathfrak{L} = V_- \oplus V_0 \oplus V_+$ such that

$$R(V_0) \subseteq V_0,\ R|_{V_-} = \mathrm{id}_{V_-},\ R|_{V_+} = 0, \text{ and } \dim(V_0) < \infty,\ V_\pm \asymp \mathfrak{B}_\pm \tag{8.67}$$

since $R = \pi_- + \pi_{\mathfrak{H}}/2 + \widetilde{K}_0(t)$. Therefore, $\mathfrak{L}^0$ (resp. $\mathfrak{L}^1$) is commensurable to $\mathfrak{B}_+$ (resp. $\mathfrak{B}_-$) and we can immediately deduce the following facts by reduction to the finite-dimensional case: $\mathfrak{L} = \bigoplus_{\lambda\in\mathbb{C}} \mathfrak{L}^\lambda$ holds and for every $\lambda \in \mathbb{C}$ exists $k = k(\lambda) \in \mathbb{N}$ such that

$$\mathfrak{L}^\lambda = \mathrm{Ker}((R - \lambda\,\mathrm{id}_{\mathfrak{L}})^k) \text{ and } \mathrm{Im}((R - \lambda\,\mathrm{id}_{\mathfrak{L}})^k) = \bigoplus_{\lambda' \in \mathbb{C}\setminus\{\lambda\}} \mathfrak{L}^{\lambda'}. \tag{8.68}$$

Lemma 8.2.8.
For $\lambda_1, \lambda_2 \in \mathbb{C}$, the following results are true:
(1) $K_0(\mathfrak{L}^{\lambda_1}, \mathfrak{L}^{\lambda_2}) \neq \{0\}$ *if and only if* $\lambda_2 = 1 - \lambda_1$.
(2) *If* $\lambda_1 + \lambda_2 \neq 1$ *we have* $[\mathfrak{L}^{\lambda_1}, \mathfrak{L}^{\lambda_2}] \subseteq \mathfrak{L}^{\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2-1}}$.
(3) *If* $\lambda_1 \notin \{0,1\}$ *we have* $[\mathfrak{L}^{\lambda_1}, \mathfrak{L}^{1-\lambda_1}] = \{0\}$.

Proof. Part (1) is a direct consequence of $(R - \lambda\,\mathrm{id}_{\mathfrak{L}})^{k,*} = (-1)^k(R - (1-\lambda)\,\mathrm{id}_{\mathfrak{L}})^k$, $\mathrm{Im}((R - \lambda\,\mathrm{id}_{\mathfrak{L}})^k)^\perp = \mathrm{Ker}((R - \lambda\,\mathrm{id}_{\mathfrak{L}})^{k,*})$ and $\mathrm{Im}((R - \lambda\,\mathrm{id}_{\mathfrak{L}})^k) = \bigoplus_{\lambda'\in\mathbb{C}\setminus\{\lambda\}} \mathfrak{L}^{\lambda'}$, for any $\lambda \in \mathbb{C}$ and $k = k(\lambda) \in \mathbb{N}$.

Equation (8.52) implies that

$$\begin{aligned}&[(R - \lambda_1\mathrm{id}_{\mathfrak{L}})a, (R - \lambda_2\mathrm{id}_{\mathfrak{L}})b] = (R - \lambda_1\,\mathrm{id}_{\mathfrak{L}})[a, (R - \lambda_2\,\mathrm{id}_{\mathfrak{L}})b] \\ &+ (R - \lambda_2\,\mathrm{id}_{\mathfrak{L}})[(R - \lambda_1\,\mathrm{id}_{\mathfrak{L}})a, b] + ((\lambda_1 + \lambda_2 - 1)R - \lambda_1\lambda_2\,\mathrm{id}_{\mathfrak{L}})[a,b]\end{aligned} \tag{8.69}$$

holds for all $a, b \in \mathfrak{L}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Let $\lambda_1, \lambda_2 \in \mathbb{C}, a_0, b_0 \in \mathfrak{L}$ and write

$$a_k := (R - \lambda_1 \operatorname{id}_{\mathfrak{L}})^k a_0 \text{ and } b_k = (R - \lambda_2 \operatorname{id}_{\mathfrak{L}})^k b_0 \tag{8.70}$$

for any $k \in \mathbb{N}$. We can see by induction on $n = k + \ell$ that $a_k = b_\ell = 0$ implies $((\lambda_1+\lambda_2-1)R-\lambda_1\lambda_2 \operatorname{id}_{\mathfrak{L}})^n[a_0, b_0] = 0$. Indeed, combining the induction assumption with (8.69), results in the desired

$$\begin{aligned} 0 &= ((\lambda_1 + \lambda_2 - 1)R - \lambda_1\lambda_2 \operatorname{id}_{\mathfrak{L}})^{n-1}([a_1, b_1] - (R - \lambda_1 \operatorname{id}_{\mathfrak{L}})[a_0, b_1] - (R - \lambda_2 \operatorname{id}_{\mathfrak{L}})[a_1, b_0]) \\ &= ((\lambda_1 + \lambda_2 - 1)R - \lambda_1\lambda_2 \operatorname{id}_{\mathfrak{L}})^n[a_0, b_0]. \end{aligned}$$

The identity $((\lambda_1 + \lambda_2 - 1)R - \lambda_1\lambda_2)^n[a_0, b_0] = 0$ for $n = k + \ell$ if $a_k = b_\ell = 0$ concludes the proof of (2) and (3). □

Lemma 8.2.8 implies that $\mathfrak{L}' := \bigoplus_{\lambda\in\mathbb{C}\setminus\{0,1\}} \mathfrak{L}^\lambda$ is a finite-dimensional subalgebra of $\mathfrak{L}$ and $\mathfrak{L}^0$ (resp. $\mathfrak{L}^1$) is a subalgebra of $\mathfrak{L}$ commensurable with $\mathfrak{B}_+$ (resp. $\mathfrak{B}_-$). Moreover, we see that $\mathfrak{L}^{0,\perp} = \mathfrak{L}^0 \oplus \mathfrak{L}'$ (resp. $\mathfrak{L}^{1,\perp} = \mathfrak{L}^1 \oplus \mathfrak{L}'$) is a subalgebra of $\mathfrak{L}$ containing $\mathfrak{L}^0$ (resp. $\mathfrak{L}^1$) as an ideal. Note that $R' := R|_{\mathfrak{L}'}$ satisfies $\det(R') \neq 0 \neq \det(R' - \operatorname{id}_{\mathfrak{L}})$ by construction. Therefore, $\mathfrak{L}' \subseteq \operatorname{Im}(R - \operatorname{id}_{\mathfrak{L}}) \cap \operatorname{Im}(R)$ combined with (8.52) and (8.53) implies that $R'/(R' - \operatorname{id}_{\mathfrak{L}'})$ defines an automorphism of $\mathfrak{L}'$ without fixed vector. In particular, $\mathfrak{L}'$ is solvable by virtue of Proposition 7.1.1.

Proposition 8.2.9.
Replacing t by a regularly equivalent classical twist of δ°, the normalizer of $\mathfrak{L}^0$ (resp. $\mathfrak{L}^1$) is equal to $\mathfrak{B}_+$ (resp. $\mathfrak{B}_-$). Furthermore, $R(\mathfrak{N}_+) \subseteq \mathfrak{N}_+$, $R(\mathfrak{H}) \subseteq \mathfrak{H}$ and $R(\mathfrak{N}_-) \subseteq \mathfrak{N}_-$.

Proof. Since for $i \in \{0, 1\}$ the subalgebra $\mathfrak{L}^i \oplus \mathfrak{L}' \subseteq \mathfrak{L}$ contains $\mathfrak{L}^i = (\mathfrak{L}^i \oplus \mathfrak{L}')^\perp$ as ideal and $\mathfrak{L}'$ is finite-dimensional and solvable, there exists $\varphi, \psi \in \operatorname{Aut}_{L\text{-alg}}(\mathfrak{L})$ such that $\varphi(\mathfrak{L}^0 \oplus \mathfrak{L}') \subseteq \mathfrak{B}_+$ and $\psi(\mathfrak{L}^1 \oplus \mathfrak{L}') \subseteq \mathfrak{B}_-$; see Proposition 4.2.9.(3). Since $\mathfrak{L}^0 \oplus \mathfrak{L}' \oplus \mathfrak{L}^1 = \mathfrak{L}$, we can assume $\varphi = \psi$ by virtue of Lemma 4.2.7. Replacing R with $\varphi R \varphi^{-1}$ amounts to replacing t with an regularly equivalent classical twist of δ° (see Lemma 8.2.5) and $\mathfrak{L}^\lambda$ by $\varphi(\mathfrak{L}^\lambda)$ for all $\lambda \in \mathbb{C}$. Thus, we may assume $\mathfrak{L}^0 \oplus \mathfrak{L}' \subseteq \mathfrak{B}_+$ and $\mathfrak{L}^1 \oplus \mathfrak{L}' \subseteq \mathfrak{B}_-$. Therefore,

$$\mathfrak{N}_+ = \mathfrak{B}_+^\perp \subseteq (\mathfrak{L}^0 \oplus \mathfrak{L}')^\perp = \mathfrak{L}^0 \subseteq \mathfrak{B}_+ \text{ and } \mathfrak{N}_- = \mathfrak{B}_-^\perp \subseteq (\mathfrak{L}^1 \oplus \mathfrak{L}')^\perp = \mathfrak{L}^1 \subseteq \mathfrak{B}_-.$$

The identity $\mathfrak{N}_\pm = [\mathfrak{B}_\pm, \mathfrak{B}_\pm]$ implies that $\widetilde{\mathfrak{L}}^0 = \mathfrak{B}_+$ and $\widetilde{\mathfrak{L}}^1 = \mathfrak{B}_-$, where

$$\widetilde{\mathfrak{L}}^\lambda := \{a \in \mathfrak{L} \mid [a, \mathfrak{L}^\lambda] \subseteq \mathfrak{L}^\lambda\} \text{ for all } \lambda \in \mathbb{C}. \tag{8.71}$$

Therefore, considering $\mathfrak{B}_+ \cap \mathfrak{B}_- = \mathfrak{H}$, it remains to prove

$$R(\widetilde{\mathfrak{L}}^\lambda) \subseteq \widetilde{\mathfrak{L}}^\lambda \text{ for all } \lambda \in \mathbb{C}. \tag{8.72}$$

These inclusions are a consequence of: for any $\lambda \in \mathbb{C}$, $a \in \widetilde{\mathfrak{L}}^\lambda$ and $b \in \mathfrak{L}$ such that $(R - \lambda \operatorname{id}_{\mathfrak{L}})^k b = 0$, $[Ra, b] \in \mathfrak{L}^\lambda$ holds by induction on $k \in \mathbb{N}$. Indeed, this is equivalent to $(R - \lambda \operatorname{id}_{\mathfrak{L}})[Ra, b] \in \mathfrak{L}^\lambda$ and combining (8.52), the induction assumption, and $Rb \in \mathfrak{L}^\lambda$ implies that

$$(R - \lambda \operatorname{id}_{\mathfrak{L}})[Ra, b] = [Ra, (R - \lambda \operatorname{id}_{\mathfrak{L}})b] - R[a, Rb] \tag{8.73}$$

is an element of $\mathfrak{L}^\lambda$. □

8.2.6 Proof of Theorem 8.2.1 IV: Conclusion. Let t be a classical twist of δ°, $R := R^t$ be the associated endomorphism from (8.46), and $\mathrm{CT}(R) := (\mathfrak{W}_+, \mathfrak{W}_-, \theta)$ be the Cayley transform of R. Proposition 8.2.9 implies that we may assume $\widetilde{K}_0(t)(\mathfrak{N}_\pm) \subseteq \mathfrak{N}_\pm$ and $\widetilde{K}_0(t)(\mathfrak{H}) \subseteq \mathfrak{H}$, which yields

$$t = t_+ + h + t_- \tag{8.74}$$

for some $t_\pm \in \mathfrak{N}_\pm \otimes \mathfrak{N}_\mp$ and $h \in \mathfrak{H} \wedge \mathfrak{H}$. Therefore, if $(\mathfrak{H}_+, \mathfrak{H}_-, \theta^h)$ denotes the Cayley transform of $\widetilde{K}_0(h) + \mathrm{id}_{\mathfrak{H}}/2 \in \mathrm{End}(\mathfrak{H})$, this implies that

$$\mathfrak{W}_\pm = \mathfrak{N}_\pm \oplus \mathfrak{H}_\pm \oplus (\mathfrak{W}_\pm \cap \mathfrak{N}_\mp). \tag{8.75}$$

Then, $\mathfrak{W}_\pm^\perp \subseteq \mathfrak{W}_\pm$ forces $\mathfrak{H}_\pm^\perp \subseteq \mathfrak{H}_\pm$, so $[\mathfrak{H}, \mathfrak{W}_\pm] \subseteq \mathfrak{W}_\pm$ by virtue of Lemma 4.2.8.(1). Consequently, $\mathfrak{H} + \mathfrak{W}_\pm$ is a subalgebra of $\mathfrak{L}$ containing $\mathfrak{B}_\pm$ and Lemma 4.2.8.(2) gives $\mathfrak{H} + \mathfrak{W}_\pm = \mathfrak{P}_\pm^{\Pi_\pm}$ for some proper subsets $\Pi_\pm \subsetneq \Pi$. In particular, this shows $\mathfrak{H}^{\Pi_\pm} \subseteq \mathfrak{H}_\pm$ and $\mathfrak{W}_\pm \cap \mathfrak{N}_\mp = \mathfrak{N}_\mp^{\Pi_\pm}$.

We have seen in the proof of Lemma 8.2.6 that $\mathfrak{H} = \mathfrak{H}^{\Pi_\pm} \oplus \mathfrak{H}^{\Pi_\pm,\perp}$ implies the identity $\mathfrak{H}_\pm = \mathfrak{H}^{\Pi_\pm} \oplus (\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp})$. Therefore,

$$\mathfrak{W}_\pm/\mathfrak{W}_\pm^\perp \cong \mathfrak{s}^{\Pi_\pm} \oplus \mathfrak{f}_\pm \text{ for } \mathfrak{f}_\pm := (\mathfrak{H}_\pm \cap \mathfrak{H}^{\Pi_\pm,\perp})/\mathfrak{H}_\pm^\perp. \tag{8.76}$$

The orthogonal isomorphism θ of Lie algebras satisfies $\theta(\mathfrak{f}_+) = \mathfrak{f}_-$ and $\theta(\mathfrak{s}^{\Pi_+}) = \mathfrak{s}^{\Pi_-}$. In particular, θ defines an isomorphism $\mathfrak{s}^{\Pi_+} \to \mathfrak{s}^{\Pi_-}$ preserving the respective natural triangular decompositions induced by $\mathfrak{L} = \mathfrak{N}_+ \oplus \mathfrak{H} \oplus \mathfrak{N}_-$. It is a standard routine to see that there exist $\{\lambda_i \in \mathbb{C}^\times \mid \alpha_i \in \Pi_+\}$ such that

$$\theta(e_i^\pm) = \lambda_i^{\pm 1} e_{\phi(i)}^\pm \text{ for all } \alpha_i \in \Pi_+, \tag{8.77}$$

where $\phi\colon \Pi_+ \to \Pi_-$ is a bijection satisfying $K_0(\phi(\alpha)|_{\mathfrak{H}}, \phi(\beta)|_{\mathfrak{H}}) = K_0(\alpha|_{\mathfrak{H}}, \beta|_{\mathfrak{H}})$ for all $\alpha, \beta \in \Pi_+$.

The restriction $R_+ := R|_{\mathfrak{N}_+}$ is nilpotent, since $\mathfrak{N}_+ \subseteq \mathfrak{L}^0$. Therefore, the extension of $\theta|_{\mathfrak{N}_+^{\Pi_+}}$ to an element of $\mathrm{End}(\mathfrak{N}_+)$ by zero can be identified with $R_+/(R_+ - \mathrm{id}_{\mathfrak{N}_+})$. In particular, said extension is nilpotent too, so ϕ satisfies (8.36). This implies that (Π_+, Π_-, ϕ) is a Belavin-Drinfeld triple. Lemma 8.2.6 implies that $Q := ((\Pi_+, \Pi_-, \phi), h)$ is a Belavin-Drinfeld quadruple.

Since $\mathfrak{W}$, and consequently $(\mathfrak{W}_+, \mathfrak{W}_-, \theta)$, determines t uniquely, we can see from Subsection 8.2.4 that $t = t_Q$ if $\lambda_i = 1$ for all $\alpha_i \in \Pi_+$, where λ_i are given in (8.77). In general, this can be achieved up to regular equivalence: let $\{\widetilde{\lambda}_i \in \mathbb{C} \mid i \in \{0, \dots, n\}\}$ be determined by $\widetilde{\lambda}_i = 0$ for $\alpha_i \notin \Pi_+$ and $\exp(\widetilde{\lambda}_{\phi(i)})\exp(-\widetilde{\lambda}_i) = \lambda_i^{-1}$ for all $\alpha_i \in \Pi_+$. These are seen to exist because ϕ satisfies (8.36). There exists a unique $a \in \mathfrak{H}^{\Pi_+}$ such that $\alpha_i(a) = \widetilde{\lambda}_i$ for all $i \in \{0, \dots, n\}$ and it is easy to see that $\psi := \exp(\mathrm{ad}(a))$ defines a regular equivalence of t to t_Q. Summarized, we have seen that $t = t_Q$ up to regular equivalence, which concludes the proof of Theorem 8.2.1.

8.2.7 Existence of Belavin-Drinfeld quadruple. In this section, we prove that for every Belavin-Drinfeld triple (Π_+, Π_-, ϕ) exists a solution of equation (8.37). More precisely, it is easy to see that the space of solutions of (8.37) is affine with underlying vector space $\mathfrak{H}_0 \wedge \mathfrak{H}_0$, where $\mathfrak{H}_0 = \{a \in \mathfrak{H} \mid (\phi(\alpha) - \alpha)(a) = 0, \alpha \in \Pi_+\}$. Therefore, we will prove that this affine space is non-empty and determine that $\dim(\mathfrak{H}_0)$ coincides with the order of the set $\Pi \setminus \Pi_+$ to obtain the following result.

Lemma 8.2.10.
Let (Π_+, Π_-, ϕ) be a Belavin-Drinfeld triple and assume that $\Pi \backslash \Pi_+$ has k elements. Then the $h \in \mathfrak{H} \wedge \mathfrak{H}$, such that $((\Pi_+, \Pi_-, \phi), h)$ is a Belavin-Drinfeld quadruple, form an affine space of dimension $(k-1)(k-2)/2$.

Proof. The assignment $h \mapsto ((\alpha_i \otimes \alpha_j)h)_{i,j=0}^n$ defines an injective map

$$m\colon \mathfrak{H} \otimes \mathfrak{H} \longrightarrow \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{C}), \tag{8.78}$$

since $\{\alpha_0|_{\mathfrak{H}}, \dots, \alpha_n|_{\mathfrak{H}}\}$ spans $\mathfrak{H}^*$. The image of m consists of all $(a_{ij})_{i,j=0}^n$ such that

$$\sum_{i=0}^n k_i a_{ij} = 0 = \sum_{i=0}^n k_i a_{ji} \tag{8.79}$$

for all $j \in \{0, \dots, n\}$, where $k_0, \dots, k_n \in \mathbb{N}$ are the unique coprime numbers such that $\sum_{i=0}^n k_i \alpha_i = 0$; see Lemma 4.2.4.

A matrix $(a_{ij})_{i,j=0}^n$ defines a tensor $h \in \mathfrak{H} \otimes \mathfrak{H}$ such that $((\Pi_+, \Pi_-, \phi), h)$ is a Belavin-Drinfeld quadruple if and only if $a_{ij} + a_{ji} = 0$ for all $i, j \in \{0, \dots, n\}$, $\sum_{i=0}^n k_i a_{ij} = 0$ for all $j \in \{0, \dots, n\}$, and

$$a_{\phi(i)j} - a_{ij} + K((\phi(\alpha_i) + \alpha_i)|_{\mathfrak{H}}, \alpha_j|_{\mathfrak{H}}) = 0 \tag{8.80}$$

for all $j \in \{0, \dots, n\}$, $\alpha_i \in \Pi_+$. Using (8.80) iteratively combined with (8.36) implies that $(a_{ij})_{i,j=0}^n$ is completely determined by $\{a_{ij} \mid \alpha_i, \alpha_j \in \Pi_+\}$, so the remaining conditions $a_{ij} + a_{ji} = 0$ for all $i, j \in \{0, \dots, n\}$ imply that $(a_{ij})_{i,j=0}^n$ is completely determined by

$$\{a_{ij} \mid \alpha_i, \alpha_j \in \Pi \setminus \Pi_+, i < j\}. \tag{8.81}$$

Therefore, $\sum_{i=0}^n k_i a_{ij} = 0$ defines exactly one linear relation between the elements in (8.81) for $k \geqslant 2$, proving the fact that the affine space of solutions to (8.37) has dimension $(k-1)(k-2)/2$ in this case.

It remains to prove the statement for $k = 1$, i.e. we have to show that if $\Pi \setminus \Pi_+ = \{\alpha_i\}$ for some $i \in \{0, \dots, n\}$, there exists exactly one solution of (8.37). Note that $\{\phi(\alpha) - \alpha \mid \alpha \in \Pi_+\}$ is a linearly independent set of $n = \dim(\mathfrak{H})$ elements, i.e.

$$\{\phi(\alpha) - \alpha \mid \alpha \in \Pi_+\} \text{ is a basis of } \mathfrak{H}^* \text{ if } \Pi \setminus \Pi_+ = \{\alpha_i\}. \tag{8.82}$$

Indeed, $\sum_{\alpha \in \Pi_+} \lambda_\alpha (\phi(\alpha) - \alpha) = 0$ can be rewritten as $\sum_{j=0}^n \widetilde{\lambda}_j \alpha_j = 0$ for some $\widetilde{\lambda}_0, \dots, \widetilde{\lambda}_n \in \mathbb{C}$ satisfying $\sum_{j=0}^n \widetilde{\lambda}_j = 0$ and Lemma 4.2.4 states that $\widetilde{\lambda}_j = \lambda k_j$ for $j \in \{0, \dots, n\}$. This forces $\widetilde{\lambda}_0 = \dots = \widetilde{\lambda}_n = 0$ and so $\phi(S) = S$ holds for $S := \{\alpha \in \Pi_+ \mid \lambda_\alpha \neq 0\}$. But (8.36) forces $S = \emptyset$, proving (8.82). In particular, $\dim(\mathfrak{H}_0) = k - 1 = 0$.

Let $\{b_\alpha\}_{\alpha \in \Pi_+} \subseteq \mathfrak{H}$ be the unique elements such that $(\phi(\alpha) - \alpha)(b_\beta) = \delta_{\alpha\beta}$. It can be easily verified that $h = \sum_{\alpha,\beta \in \Pi_+} h_{\alpha\beta} b_\alpha \otimes b_\beta \in \mathfrak{H} \otimes \mathfrak{H}$, where

$$h_{\alpha\beta} = -K((\phi(\alpha) + \alpha)|_{\mathfrak{H}}, (\phi(\beta) - \beta)|_{\mathfrak{H}})/2 = (K(\phi(\alpha)|_{\mathfrak{H}}, \beta|_{\mathfrak{H}}) - K(\alpha|_{\mathfrak{H}}, \phi(\beta)|_{\mathfrak{H}}))/2,$$

is a skew-symmetric tensor solving (8.37). Here, we used $K(\phi(\alpha)|_{\mathfrak{H}}, \phi(\beta)|_{\mathfrak{H}}) = K(\alpha|_{\mathfrak{H}}, \beta|_{\mathfrak{H}})$ for $\alpha, \beta \in \Pi_+$. Summarized, we see that h is the only skew-symmtric solution of (8.37), concluding the proof. □

Remark 8.2.11.
In [CP95, Section 3.2.C] the authors present the results of [BD83a] with a slight flaw: they implicitly state that $k(k-1)/2$ is the dimension of the affine space of solutions of (8.37), in the notation of Lemma 8.2.10.

8.2.8 The classification theorem in the Coxeter twisting. In case the automorphism σ is Coxeter, it is possible to describe Belavin-Drinfeld quadruples and the associated σ-trigonometric r-matrices intrinsically in $\mathfrak{g}$. This is the language chosen in [BD83a]. In this subsection, we explain the relation of this description to the one presented here.

Let $\mathfrak{g} = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{H}} \oplus \widetilde{\mathfrak{n}}_-$ be a triangular decomposition of $\mathfrak{g}$, ν be an automorphism of the Dynkin diagram of $\mathfrak{g}$ of order k and $\widetilde{\nu}$ be the associated outer automorphism of $\mathfrak{g}$ with respect to the chosen triangular decomposition. We say that σ is a *Coxeter automorphism* of type k if σ is conjugate to $\widetilde{\nu}_{((1,\dots,1);k)}$ in the notation of Subsection 4.2.7. Equivalently, σ is an automorphism of $\mathfrak{g}$ of minimal order with the properties: the representative of σ in $\mathrm{Out}(\mathfrak{g})$ is of order k and $\mathfrak{g}_0^\sigma$ is abelian.

Let $\mathfrak{g}_k := \mathfrak{g}_k^\sigma$ for $1 \leqslant k \leqslant m-1$, write

$$\mathfrak{g}_k^\alpha := \{a \in \mathfrak{g}_k \mid [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{H}\} \tag{8.83}$$

for any $\alpha \in \mathfrak{H}^*$ as well as

$$\Lambda_k := \{\alpha \in \mathfrak{H}^* \mid \mathfrak{g}_k^\alpha \neq \{0\}\} \text{ and } \Xi := \{(\alpha, k) \in \mathfrak{H}^* \times \{1, \dots, m-1\} \mid \alpha \in \Lambda_k\}. \tag{8.84}$$

Observe that

$$\mathfrak{g} = \mathfrak{H} \oplus \bigoplus_{k=1}^{m-1} \bigoplus_{\alpha \in \Lambda_k} \mathfrak{g}_k^\alpha \tag{8.85}$$

and $\Lambda_1 = \{\alpha^\circ := \alpha|_{\mathfrak{H}} \mid \alpha \in \Pi\}$ completely determines Π, since $\alpha(d) = 1$ for all $\alpha \in \Pi$. Therefore, a Belavin-Drinfeld triple is determined by a triple $(\Gamma_+, \Gamma_-, \tau)$, where $\Gamma_\pm \subsetneq \Lambda_1$ and $\tau \colon \Gamma_+ \to \Gamma_-$ is a bijection satisfying the following conditions:

- $K(\tau(\alpha), \tau(\beta)) = K(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma_+$ and
- for any $\alpha \in \Gamma_+$ there exists $\ell = \ell(\alpha) \in \mathbb{N}$ such that

$$\alpha, \dots, \tau^{\ell-1}(\alpha) \in \Gamma_+ \text{ and } \tau^\ell(\alpha) \notin \Gamma_+. \tag{8.86}$$

Let $\mathfrak{g}^{\Gamma_\pm}$ be the Lie subalgebras of $\mathfrak{g}$ generated by the vector subspace $\bigoplus_{\alpha \in \Gamma_\pm} \mathfrak{g}_1^\alpha$. Then $\mathfrak{g}^{\Gamma_\pm}$ is isomorphic to the positive part of the semi-simple Lie algebra defined by the Dynkin diagram associated to $\Gamma_\pm$ and we have a direct sum decomposition

$$\mathfrak{g}^{\Gamma_\pm} = \bigoplus_{(\alpha,k) \in \Xi_\pm} \mathfrak{g}_k^\alpha \tag{8.87}$$

for appropriate subsets $\Xi_\pm \subseteq \Xi$. Fixing non-zero elements in $\mathfrak{g}_1^\alpha$ for all $\alpha \in \Lambda_1$, one can extend the bijection τ to an isomorphism of Lie algebras $\widetilde{\tau} \colon \mathfrak{g}^{\Gamma_+} \to \mathfrak{g}^{\Gamma_-}$.

Let $\vartheta \colon \mathfrak{g} \to \mathfrak{g}$ be a linear map defined as the composition of the

$$\mathfrak{g} \xrightarrow{\mathrm{pr}} \mathfrak{g}^{\Gamma_+} \xrightarrow{\widetilde{\tau}} \mathfrak{g}^{\Gamma_-} \xrightarrow{\subseteq} \mathfrak{g}, \tag{8.88}$$

where pr is the canonical projection with respect to the direct sum decompositions (8.85) and (8.87). Then ϑ is nilpotent and $\vartheta(\mathfrak{g}_k) \subseteq \mathfrak{g}_k$ for all $k \in \{1, \dots, m-1\}$. Let $\psi = \frac{\vartheta}{\mathrm{id}_\mathfrak{g} - \vartheta} = \sum_{j=1}^\infty \vartheta^j$. It follows that $\psi(\mathfrak{g}_k) \subseteq \mathfrak{g}_k$ for all $k \in \{1, \dots, m-1\}$ as well.

For the triple $(\Gamma_+, \Gamma_-, \tau)$, we can chose a tensor $h \in \mathfrak{H} \wedge \mathfrak{H}$

$$\Big(\tau(\alpha) \otimes \mathrm{id}_\mathfrak{H} + \mathrm{id}_\mathfrak{H} \otimes \alpha\Big)\Big(h + \gamma_0/2\Big) = 0 \text{ for all } \alpha \in \Gamma_1 \tag{8.89}$$

Then $((\Gamma_+, \Gamma_-, \tau), h)$ determines a Belavin-Dinfeld quadruple Q in the obvious way. In [BD83b, Theorem 6.1], Belavin and Drinfeld consider the trigonometric r-matrix

$$X^Q(\widetilde{u}, \widetilde{v}) := \varrho^\circ(\widetilde{u}, \widetilde{v}) + h + \sum_{j=1}^{m-1} \left(\left(\frac{\widetilde{v}}{\widetilde{u}}\right)^j (\mathrm{id}_\mathfrak{g} \otimes \psi)\gamma_{-j} - \left(\frac{\widetilde{u}}{\widetilde{v}}\right)^j (\psi \otimes \mathrm{id}_\mathfrak{g})\gamma_j \right). \tag{8.90}$$

Lemma 8.2.12.
Equation (8.90) defines a σ-trigonometric r-matrix which is regularly equivalent to ϱ^Q.

Proof. Let $Q = ((\Pi_+, \Pi_-, \phi), h)$. By construction, there exists $\{\lambda_i \in \mathbb{C}^\times \mid \alpha_i \in \Pi_+\}$ such that $\vartheta(e_i^+(1)) = \lambda_i e_i^+(1)$ for all $\alpha_i \in \Pi_+$. Therefore, the same argument as in the end of Subsection 8.2.6 shows that we may assume $\lambda_i = 1$ for all $\alpha_i \in \Pi_+$ up to regular equivalence. In particular, $\theta_+^\phi(e_i^+)(1) = \vartheta(e_i^+(1))$. This and the observation

$$\sum_{j=0}^{m-1} \left(\left(\frac{\widetilde{v}}{\widetilde{u}}\right)^j (\mathrm{id}_\mathfrak{g} \otimes \psi)\gamma_{-j} - \left(\frac{\widetilde{u}}{\widetilde{v}}\right)^j (\psi \otimes \mathrm{id}_\mathfrak{g})\gamma_j \right) = \sum_{\alpha \in \Phi_+^{\Pi_+}} \sum_{j=1}^\infty b_{-\alpha} \wedge \theta_+^{\phi, j}(b_\alpha) \tag{8.91}$$

for any $\{b_\alpha \in \mathfrak{L}_\alpha \mid \alpha \in \Phi\}$, satisfying $B(b_\alpha, b_{-\alpha}) = 1$ for all $\alpha \in \Phi_+^{\Pi_+}$, concludes the proof. Here, we used that, for d in $\widehat{\mathfrak{L}} = \mathfrak{L} \oplus \mathbb{C}c \oplus \mathbb{C}d$ (see Subsection 4.2.2), we have:

- $\alpha(d) < m = \sum_{i=0}^n k_i$ for all $\alpha \in \Phi_+^{\Pi_+}$, where $k_0, \dots, k_n$ are the unique integers defined in Lemma 4.2.4,
- $\sum_{\alpha(d)=j} b_\alpha \otimes b_{-\alpha} = (\widetilde{u}/\widetilde{v})^j \gamma_j$ for all $j \in \mathbb{Z} \setminus \{0\}$ and
- $\psi(b_\alpha) = 0$ for all $\alpha \notin \Phi_+^{\Pi_+}$.

The first fact is thereby a consequence of $\Phi_+^{\Pi_+} \subseteq \Phi^{\mathrm{re}}$. □

8.3 Equivalences

In this section, we take a closer look at equivalences between σ-trigonometric r-matrices. More precisely, we investigate the difference between formal and regular equivalences in Subsection 8.3.1 and the compatibility of the latter with Belavin-Drinfeld quadruples in Subsection 8.3.2. Furthermore, we discuss the relationship between the regrading of loop algebras described in 4.2.8 and the classical twists of the standard structure resp. σ-trigonometric r-matrices in Subsection 8.3.3.

8.3.1 Regular and formal equivalence. We can consider σ-trigonometric r-matrices either up to regular equivalence or up to formal equivalence in the exponential language. At first glance, the former seems stricter than the latter. However, the geometric theory implies that this is indeed not the case.

Proposition 8.3.1.
Two σ-trigonometric r-matrices ϱ_1 and ϱ_2 are regularly equivalent if and only if the Taylor series of $\varrho_1(\exp(x/m), \exp(y/m))$ and $\varrho_2(\exp(x/m), \exp(y/m))$ in $y = 0$ are formally equivalent.

Proof. " $\Longrightarrow$ " is clear, so we have to prove "$\Longleftarrow$". Let (λ, w, φ) be a formal equivalence between the Taylor series r_1 and r_2 of $\varrho_1(\exp(x/m), \exp(y/m))$ and $\varrho_1(\exp(x/m), \exp(y/m))$ in $y = 0$. Since r_1 and r_2 are normalized and skew-symmetric, $\lambda = w' \in \mathbb{C}^\times$ holds; see Lemma (2.1.6). For $i \in \{1, 2\}$, let $((X, \mathscr{A}_i), (p, c, \zeta))$ be the geometric datum of r_i constructed in Section 8.1. Lemma 3.1.1 states that we have an automorphism f of X fixing p and an isomorphism $\psi\colon \mathscr{A}_1 \to f_*\mathscr{A}_2$. The automorphism f defines an automorphism of $\mathbb{P}^1_\mathbb{C}$, i.e. a Möbius transformation, which restricts to an automorphism of $\operatorname{Spec}(\mathbb{C}[u, u^{-1}])$ fixing $(u-1)$. Thus, f is the identity. In particular, this means that $\lambda = 1$ and $w(z) = z$. Let C be the smooth locus of X. By construction, $\Gamma(C, \mathscr{A}_1) \cong \mathfrak{L} \cong \Gamma(C, \mathscr{A}_2)$, so ψ induces a $\mathbb{C}[u, u^{-1}]$-linear automorphism of $\mathfrak{L}$. Application of Lemma 3.3.2 and Theorem 8.1.3.(3) now concludes the proof. □

Remark 8.3.2.
Proposition 8.3.1 can be used to reduce the classification of classical twists of δ° presented in Section 8.2 directly to the classification of trigonometric r-matrices in [BD83a]; see [AM21].

8.3.2 Equivalence and Belavin-Drinfeld quadruple. In the context of the classification presented in Subsection 8.2, it is natural to ask under which circumstances the classical twists (and so the σ-trigonometric r-matrices defined by said twists) determined by Belavin-Drinfeld quadruples are equivalent. This question is settled by the following statement, which was already remarked after [BD83b, Theorem 6.1] without proof and was proven in [AM21].

Proposition 8.3.3.
Let $Q^{(i)} = ((\Pi_+^{(i)}, \Pi_-^{(i)}, \phi^{(i)}), h^{(i)})$ be a Belavin-Drinfeld quadruple for $i \in \{1, 2\}$. Then $t_{Q^{(1)}}$ and $t_{Q^{(2)}}$ are regularly equivalent if and only if there exists $\nu \in \operatorname{Aut}(A)$ such that $\widetilde{\nu}(\Pi_\pm^{(1)}) = \Pi_\pm^{(2)}$, $\widetilde{\nu}\phi^{(1)}\widetilde{\nu}^{-1} = \phi^{(2)}$, and $\widetilde{\nu}(h^{(1)}) = h^{(2)}$ hold (where $\widetilde{\nu} \in \operatorname{Out}(\mathfrak{L})$ is defined in Lemma 4.1.10.(2)).

Proof. Let us write $t_1 := t_{Q^{(1)}}$ and $t_2 := t_{Q^{(2)}}$.
" $\Longrightarrow$ " Lemma 8.2.5 states that $\varphi R^{t_1}\varphi^{-1} = R^{t_2}$ for some $\varphi \in \operatorname{Aut}_{L\text{-alg}}(\mathfrak{L})$. If $\mathfrak{L}_1^0$ and $\mathfrak{L}_2^0$ are the generalized eigenspaces of R^{t_1} and R^{t_2} to 0 respectively, $\varphi(\mathfrak{L}_1^0) = \mathfrak{L}_2^0$

holds. We have seen in the proof of Proposition 8.2.9 that the normalizer of $\mathfrak{L}_i^0$ is $\mathfrak{B}_+$ for $i \in \{1,2\}$. Since an automorphism of Lie algebras respects the process of normalizing, φ is an automorphism of $\mathfrak{L}$ fixing the Borel subalgebra $\mathfrak{B}_+$. Lemma 4.2.8 states that φ induces an automorphism ν of the Dynkin diagram of $\mathfrak{L}$. The identity $(\varphi \otimes \varphi)\mathfrak{W}^{t_1} = \mathfrak{W}^{t_2}$, (8.63) combined with Lemma 8.2.3.(4) and Lemma 8.2.5 now imply that $\widetilde{\nu}(\Pi_\pm^{(1)}) = \Pi_\pm^{(2)}$, $\widetilde{\nu}\phi^{(1)}\widetilde{\nu}^{-1} = \phi^{(2)}$, and $\widetilde{\nu}(h^{(1)}) = h^{(2)}$.

" $\Longleftarrow$ " Let ν be a Dynkin diagram automorphism, such that $\widetilde{\nu}(\Pi_\pm^{(1)}) = \Pi_\pm^{(2)}$, $\widetilde{\nu}\phi^{(1)}\widetilde{\nu}^{-1} = \phi^{(2)}$, and $\widetilde{\nu}(h^{(1)}) = h^{(2)}$ hold. Using Lemma 4.2.6, $\widetilde{\nu}$ defines a $L = \mathbb{C}[u,u^{-1}]$-linear automorphism φ of $\mathfrak{L}$ after probably precomposing an scaling automorphism. It is easy to see from (8.63) that $(\varphi \otimes \varphi)\mathfrak{W}^{t_1} = \mathfrak{W}^{t_2}$ holds, so φ is the desired regular equivalence by virtue of Lemma 8.2.5. □

Remark 8.3.4.
Combining the propositions 8.3.3 and 8.3.1 we obtain the following statement: for $i \in \{1,2\}$ the Taylor series of $\varrho^{Q^{(i)}}(\exp(x/m), \exp(y/m))$ in $y = 0$ for Belavin-Drinfeld quadruples $Q^{(i)} = ((\Pi_+^{(i)}, \Pi_-^{(i)}, \phi^{(i)}), h^{(i)})$ are formally equivalent if and only if $\widetilde{\nu}(\Pi_\pm^{(1)}) = \Pi_\pm^{(2)}$, $\widetilde{\nu}\phi^{(1)}\widetilde{\nu}^{-1} = \phi^{(2)}$, and $\widetilde{\nu}(h^{(1)}) = h^{(2)}$ hold.

8.3.3 Changing the twisting automorphism. Let us fix a triangular decomposition $\mathfrak{g} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{H}} \oplus \widetilde{\mathfrak{n}}_+$ and assume that $\sigma = \widetilde{\nu}$ for some automormophism ν of the Dynkin diagram of $\mathfrak{g}$, while $\mathfrak{H} = \widetilde{\mathfrak{H}} \cap \mathfrak{g}_0^\sigma, \mathfrak{n}_\pm = \widetilde{\mathfrak{n}}_\pm \cap \mathfrak{g}_0^\sigma$. Consider two Lie algebra automomrphisms σ_1 and σ_2 of finite order m_1 and m_2 respectively and assume that $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{g}, \sigma_1) \cong \mathfrak{L}(\mathfrak{g}, \sigma_2)$ as Lie algebras over $\mathbb{C}$, i.e. the class of both σ_1 and σ_2 in $\mathrm{Out}(\mathfrak{g})$ has order m; see Subsection 4.2.6. Then, by virtue of Subsection 4.2.7, there exist $\upsilon_1, \upsilon_2 \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ and

$$s_1 = (s_{10}, \dots, s_{1n}), s_2 = (s_{20}, \dots, s_{2n}) \in \mathbb{N}_0^{n+1} \setminus \{0\} \tag{8.92}$$

such that $\sigma_1 = \upsilon_1 \sigma_{(s_1;m)} \upsilon_1^{-1}$ and $\sigma_2 = \upsilon_2 \sigma_{(s_2;m)} \upsilon_2^{-1}$. We have a canonical isomorphism $\psi \colon \mathfrak{L}(\mathfrak{g}, \sigma_1) \to \mathfrak{L}(\mathfrak{g}, \sigma_2)$, which factors as

$$\mathfrak{L}(\mathfrak{g}, \sigma_1) \xrightarrow{\upsilon_1^{-1}} \mathfrak{L}^{s_1} \xrightarrow{\text{regrading}} \mathfrak{L}^{s_2} \xrightarrow{\upsilon_2} \mathfrak{L}(\mathfrak{g}, \sigma_2), \tag{8.93}$$

where the middle arrow is defined by $\mathfrak{L}_\alpha^{s_1} \ni X \mapsto X(1) z^{\mathrm{ht}_{s_2}(\alpha)} \in \mathfrak{L}^{s_2}$ for all $\alpha \in \Phi$; see Subsection 4.2.8.

Let $i \in \{1,2\}$. Note that $\mathfrak{g}_0^{\sigma(s_i,m)} = \mathfrak{n}_-^{s_i} \oplus \mathfrak{H} \oplus \mathfrak{n}_+^{s_i}$, where

$$\mathfrak{n}_\pm^{s_i} = \langle e_j^\pm(1) \mid j \in \{0, \dots, n\} \text{ such that } s_j^{(i)} = 0 \rangle_{\mathbb{C}\text{-alg}} = \bigoplus_{\substack{\alpha \in \Phi \\ \mathrm{ht}_{s_i}(\alpha) = 0}} \mathrm{ev}_1(\mathfrak{L}_\alpha). \tag{8.94}$$

Let δ_i° and ϱ_i° be the standard Lie bialgebra structure of $\mathfrak{L}(\mathfrak{g}, \sigma_i)$ with respect to $\mathfrak{g}_0^{\sigma_i} = \upsilon_i(\mathfrak{n}_+^{s_i}) \oplus \upsilon_i(\mathfrak{H}) \oplus \upsilon_i(\mathfrak{n}_-^{s_i})$ for $i \in \{1,2\}$. It is easy to see that for any classical twist t_1 of δ_1° the tensor $t_2 := (\psi \otimes \psi) t_1$ is a classical twist of δ_2° and $\psi \colon (\mathfrak{L}(\mathfrak{g}, \sigma_1), \delta_1^{t_1}) \to (\mathfrak{L}(\mathfrak{g}, \sigma_1), \delta_2^{t_2})$ is an isomorphism of Lie bialgebra structures. The following result settles the connection between the respective trigonometric r-matrices $\varrho_1^{t_1} = \varrho_1^\circ + t_1$ and $\varrho_2^{t_2} = \varrho_1^\circ + t_2$, which can be seen as a generalization of [BD83b, Lemma 6.22]; see also [AM21, Lemma 3.2].

Lemma 8.3.5.
Let $a_0 \in \mathfrak{H}$ be the unique element such that $\alpha_j(a_0) = s_{2j}/m_2 - s_{1j}/m_1$ for all $j \in \{0, \dots, n\}$. Then we have

$$\begin{aligned}&(\upsilon_2 \exp(x\,\mathrm{ad}(a_0))\upsilon_1^{-1} \otimes \upsilon_2 \exp(y\,\mathrm{ad}(a_0))\upsilon_1^{-1})\varrho_1^{t_1}\left(\exp\left(x/m_1\right), \exp\left(y/m_1\right)\right)\\ &= \varrho_2^{t_2}\left(\exp\left(x/m_2\right), \exp\left(y/m_2\right)\right)\end{aligned}$$

Proof. First of all, a_0 is well-defined, since $m_i = m\sum_{j=0}^n k_j s_{ij}$ for $i \in \{1,2\}$, so $\sum_{j=0}^n k_j\alpha_j(a_0) = 0$, where $k_0, \dots, k_n$ is given in Lemma 4.2.4. We can assume that $\upsilon_1 = \mathrm{id}_{\mathfrak{g}} = \upsilon_2$, since these simply define regular equivalences; see Lemma 5.4.10. Notice that for $i \in \{1,2\}$ the Lie algebra $\mathfrak{L}(\mathfrak{g}, \sigma_i)$ is generated by

$$\{e_j^+(1)\widetilde{u}^{s_{ij}}, e_i^-(1)\widetilde{u}^{-s_{ij}} \mid j \in \{0, \dots, n\}\} \tag{8.95}$$

and it is easy to see that $\psi(e_j^\pm(1)\widetilde{u}^{\pm s_{1j}})$ evaluated at $\exp(z/m_2)$ coincides with $e_j^\pm \exp(s_{2j}z/m_2) = \exp(z\,\mathrm{ad}(a_0))e_j^\pm(1)\exp(s_{1j}z/m_1)$. This proves that

$$\psi(a)(\exp(z/m_2)) = \exp(z\,\mathrm{ad}(a))a(\exp(z/m_1)) \text{ for all } a \in \mathfrak{L}^{s_1}. \tag{8.96}$$

In particular, we may assume that $t_1 = 0 = t_2$. Using (5.53) for $\varrho = \varrho_1$ and $\varrho = \varrho_2$ results in

$$\begin{aligned}&(\exp(x\,\mathrm{ad}(a_0)) \otimes \exp(y\,\mathrm{ad}(a_0)))\varrho_1\left(\exp\left(x/m_1\right), \exp\left(y/m_1\right)\right)\\ &= \gamma_{\mathfrak{H}}/2 + \sum_{\alpha\in\Phi_-}\left((\psi\otimes\psi)\chi_\alpha^{s_1}\right)\left(\exp\left(x/m_2\right), \exp\left(y/m_2\right)\right)\\ &= \gamma_{\mathfrak{H}}/2 + \sum_{\alpha\in\Phi_-}\chi_\alpha^{s_2}\left(\exp\left(x/m_2\right), \exp\left(y/m_2\right)\right)\\ &= \varrho_2\left(\exp\left(x/m_2\right), \exp\left(y/m_2\right)\right),\end{aligned} \tag{8.97}$$

where $\chi_\alpha^{s_i}$ is the image of χ_α under $\mathfrak{L}_\alpha \otimes \mathfrak{L}_\alpha \cong \mathfrak{L}_\alpha^{s_i} \otimes \mathfrak{L}_\alpha^{s_i}$ for $i \in \{1,2\}$ and $\alpha \in \Phi$. □

8.3.4 Example: σ-trignometric r-matrices over $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Let us assume that $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_-$ is the standard decomposition into traceless diagonal, uppter trigangular and lower triangular matrices. Since $\mathfrak{sl}_2(\mathbb{C})$ has no non-trivial automorphism of the Dynkin diagram, we can assume that $\sigma = \mathrm{id}_{\mathfrak{g}}$ by virtue of Subsection 8.3.3. Then $\mathfrak{L}(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}}) = \mathfrak{L}(\mathfrak{g}) = \mathfrak{sl}_2(\mathbb{C}[u, u^{-1}])$ has two simple roots α_0, α_1 with

$$K_0(\alpha_0|_{\mathfrak{H}}, \alpha_0|_{\mathfrak{H}}) = K_0(\alpha_1|_{\mathfrak{H}}, \alpha_1|_{\mathfrak{H}}), \tag{8.98}$$

so by Proposition 8.3.3 there are essentially two Belavin-Drinfeld triple (Π_+, Π_-, ϕ):

$$\Pi_+ = \Pi_- = \emptyset \text{ and } \Pi_+ = \{\alpha_1\}, \Pi_- = \{\alpha_0\}, \phi(\alpha_0) = \alpha_1. \tag{8.99}$$

Since $\mathfrak{H} \wedge \mathfrak{H} = \{0\}$, these uniquely determine Belavin-Drinfeld quadruple and hence $\mathrm{id}_{\mathfrak{sl}_2}$-trigonometric r-matrices.

In the first case, we obtain the standard $\mathrm{id}_{\mathfrak{sl}_2}$-trigonometric r-matrix

$$\begin{aligned}\varrho^\circ(u,v) =& \frac{v}{u-v}\left(\frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\\ &+ \frac{1}{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \frac{1}{8}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.\end{aligned}$$

In the second case, $\theta_+^\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = u \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\theta_+^{\phi,2} = 0$, so the associated r-matrix is given by the formula

$$\varrho^\circ(u,v) + (v-u) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{8.100}$$

Remark 8.3.6.
In [BD83a], the authors present all Belavin-Drinfeld quadruple and the associated r-matrices for $\mathfrak{g} \in \{\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_3(\mathbb{C})\}$ in the language of Subsection 8.2.8.

8.4 Restrictions of the standard structure

It is easy to see that δ° restricts to a Lie bialgebra cobracket on both the semi-simple subalgebra $\mathfrak{s}^S$ and the standard parabolic subalgebra $\mathfrak{P}_\pm^S$ of $\mathfrak{L}$ for all $S \subseteq \Pi$. This section is dedicated to a study of these Lie bialgebra structures.

8.4.1 Restricted standard structures on parabolic subalgebras. Note that t_Q is an element of $\mathfrak{P}_+^S \otimes \mathfrak{P}_+^S$ for some Belavin-Drinfeld quadruple $Q = ((\Pi_+, \Pi_-, \phi), h)$ if and only if $\Pi_+ \subseteq S$. The following theorem from [AM21] gives a classification of classical twists of the restricted Lie bialgebra structure $\delta^\circ|_{\mathfrak{P}_+^S}$ or, equivalently, classical twists of δ° contained in $\mathfrak{P}_+^S \otimes \mathfrak{P}_+^S$.

Theorem 8.4.1.
For any $S \subseteq \Pi$ and classical twist $t \in \mathfrak{P}_+^S \otimes \mathfrak{P}_+^S$ of δ° exists a Belavin-Drinfeld quadruple $Q = ((\Pi_+, \Pi_-, \phi), h)$ such that $\Pi_+ \subseteq S$ and t is regularly equivalent to t_Q via a regular equivalence that restricts to an automorphism of $\mathfrak{P}_+^S$.

Proof. By Theorem 8.2.1, there is a regular equivalence φ_1 and a Belavin-Drinfeld quadruple $Q' = ((\Pi'_+, \Pi'_-, \phi'), h')$ such that $(\varphi_1 \times \varphi_1)\mathfrak{W}^t = \mathfrak{W}^{Q'}$. Since $t \in \mathfrak{P}_+^S \otimes \mathfrak{P}_+^S$ we have $\mathfrak{W}^t \subseteq \mathfrak{P}_+^S \times \mathfrak{L}$. Let $\mathfrak{H}'_\pm \subseteq \mathfrak{H}$ be the image of $\widetilde{K}_0(h') \mp \mathrm{id}_{\mathfrak{H}}/2$. Since h' is skew-symmetric, this is easily seen to be a coisotropic subspace of $\mathfrak{H}$. Then

$$\mathfrak{W}_+^Q = \mathfrak{N}_+ \oplus \mathfrak{H}'_+ \oplus \mathfrak{N}_-^{\Pi_+} \subseteq \varphi_1(\mathfrak{P}_+^S) \tag{8.101}$$

and, in particular, we have the inclusion $\mathfrak{H}'_+ \subseteq \varphi_1(\mathfrak{P}_+^S)$. By the first part of Lemma 4.2.8 we have

$$[\mathfrak{H}, \varphi_1(\mathfrak{P}_+^S)] \subseteq \varphi_1(\mathfrak{P}_+^S). \tag{8.102}$$

Since $\mathfrak{P}_+^S$ is self-normalizing, $\varphi_1(\mathfrak{P}_+^S)$ is self-normalizing as well. Therefore, we get $\mathfrak{H} \subseteq \varphi_1(\mathfrak{P}_+^S)$ and consequently $\mathfrak{B}_+ \subseteq \varphi_1(\mathfrak{P}_+^S)$. Then Lemma 4.2.8 shows that $\varphi_1(\mathfrak{P}_+^S) = \mathfrak{P}_+^{S'}$ for some $S' \subsetneq \Pi$. The inclusion (8.101) implies that $\Pi'_+ \subseteq S'$.

Define $\mathfrak{B}' := \varphi_1^{-1}(\mathfrak{B}_+)$. The subalgebra $\mathfrak{B}'/\mathfrak{P}_+^{S,\perp}$, being the preimage of the Borel subalgebra $\mathfrak{B}_+/\mathfrak{P}_+^{S',\perp}$ of $\mathfrak{s}^{S'} + \mathfrak{H} = \mathfrak{P}_+^{S'}/\mathfrak{P}_+^{S',\perp}$ under φ_1, is a Borel subalgebra of $\mathfrak{s}^S + \mathfrak{H} = \mathfrak{P}_+^S/\mathfrak{P}_+^{S,\perp}$. Therefore, by the conjugacy theorem for Borel subalgebras, there exists an inner automorphism φ_2 of $\mathfrak{s}^S + \mathfrak{H}$ mapping $\mathfrak{B}'/\mathfrak{P}_+^{S,\perp}$ to $\mathfrak{B}_+/\mathfrak{P}_+^{S,\perp}$. Combining this result with the fact that $\mathrm{Inn}(\mathfrak{s}^S + \mathfrak{H})$ is generated by

$$\{\exp(\mathrm{ad}(a)) \mid a \in \mathfrak{L}_\alpha, \alpha \in \Phi^S\} \tag{8.103}$$

(see e.g. [Bou05, §3.2]), we can view φ_2 as a regular equivalence on $\mathfrak{L}$ that restricts to an automorphism of $\mathfrak{P}_+^S$ and maps $\mathfrak{B}'$ to $\mathfrak{B}_+$. The composition $\varphi_2\varphi_1^{-1}$ is then an automorphism of $\mathfrak{L}$ mapping $\mathfrak{P}_+^{S'}$ to $\mathfrak{P}_+^S$ and fixing the Borel subalgebra $\mathfrak{B}_+$. Lemma 4.2.8 implies that $\varphi_2\varphi_1^{-1}$ induces $\nu \in \mathrm{Out}(\mathfrak{L})$ such that $\nu(S') = S$. Using Lemma 4.2.6, $\widetilde{\nu}$ defines a regular equivalence φ_3 such that $(\varphi_3 \times \varphi_3)\mathfrak{W}^{Q'} = \mathfrak{W}^Q$, where $Q = ((\nu(\Pi'_+), \nu(\Pi'_-), \nu\phi'\nu^{-1}), \nu(h'))$. Since $\nu(\Pi'_+) \subseteq \nu(S') = S$, $\varphi := \varphi_3\varphi_1$ is the desired regular equivalence. □

Remark 8.4.2.
Let $Q^{(i)} = ((\Pi_+^{(i)}, \Pi_-^{(i)}, \phi^{(i)}), h^{(i)})$ be a Belavin-Drinfeld quadruple such that $\Pi_+^{(i)} \subseteq S$ for $i \in \{1, 2\}$. Then Theorem 8.3.3 immediately implies that $t_{Q^{(1)}}$ and $t_{Q^{(2)}}$ are regularly equivalent via a regular equivalence fixing $\mathfrak{P}_+^S$ if and only if there exists an automorphism ν of the Dynkin diagram of $\mathfrak{L}$ such that ν fixes the subdiagram defined by S, $\widetilde{\nu}(\Pi_\pm^{(1)}) = \Pi_\pm^{(2)}$, $\widetilde{\nu}\phi^{(1)}\widetilde{\nu}^{-1} = \phi^{(2)}$, and $\widetilde{\nu}(h^{(1)}) = h^{(2)}$. We point out that the set of elements of $\mathrm{Aut}_{\mathbb{C}[u,u^{-1}]\text{-alg}}(\mathfrak{L})$ fixing $\mathfrak{P}_+^S$ can be identified with $\mathrm{Aut}_{\mathbb{C}[u]\text{-alg}}(\mathfrak{P}_+^S)$ by restricting to $\mathfrak{P}_+^S$.

8.4.2 Quasi-trigonometric r-matrices. In [Kho+08; PS08] the authors study r-matrices of the form

$$\frac{v\gamma}{u - v} + p(u, v) \text{ for some } p \in (\mathfrak{g} \otimes \mathfrak{g})[u, v], \tag{8.104}$$

which they refer to as *quasi-trigonometric r-matrices*. In our approach, these are exactly σ-trigonometric r-matrices ϱ^t for a classical twists t of $\delta^\circ|_{\mathfrak{P}_+^S}$, where $\sigma = \mathrm{id}_\mathfrak{g}$ and $S = \{\alpha_1, \dots, \alpha_n\}$ in the notation of Subsection 4.2.6. In particular, Theorem 8.4.1 and Remark 8.4.2 give a classification of said r-matrices, up to $r(u, v) \sim (\widetilde{\varphi}(u) \otimes \widetilde{\varphi}(v))r(u, v)$, where $\varphi \in \mathrm{Aut}_{\mathbb{C}[u]\text{-alg}}(\mathfrak{g}[u])$ and $\widetilde{\varphi} \in \mathrm{End}(\mathfrak{g})[u]$ is defined by $\varphi(a)(u) = \widetilde{\varphi}(u)a(u)$ for all $a \in \mathfrak{g}[u]$. This classification was achieved using different methods in [PS08].

We note that e.g. for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ it can be shown that all $\mathrm{id}_\mathfrak{g}$-trigonometric r-matrices are quasi-trigonometric up to regular equivalence, by using Proposition 8.3.3 or using the theory of maximal orders from [Sto91c]; see [AM21, Subsection 4.3]. However, it is noted in [AM21, Subsection 4.2] that there are $\mathrm{id}_\mathfrak{g}$-trigonometric r-matrices which are not regularly equivalent to quasi-trigonometric ones.

8.4.3 Lie bialgebra structures on $\mathfrak{g}$ and constant r-matrices. Note that t_Q is an element of $\mathfrak{s}^S \otimes \mathfrak{s}^S$ for some Belavin-Drinfeld quadruple $Q = ((\Pi_+, \Pi_-, \phi), h)$ if and only if $\Pi_+, \Pi_- \subseteq S$. The following theorem gives a classification of classical twists of the restricted Lie bialgebra structure $\delta^\circ|_{\mathfrak{s}^S}$ or, equivalently, classical twists of δ° contained in $\mathfrak{s}_+^S \otimes \mathfrak{s}_+^S$.

Theorem 8.4.3.
For any $S \subseteq \Pi$ and classical twist $t \in \mathfrak{s}^S \otimes \mathfrak{s}^S$ of δ° exists a Belavin-Drinfeld quadruple $Q = ((\Pi_+, \Pi_-, \phi), h)$ such that $\Pi_+, \Pi_- \subseteq S$ and t is regularly equivalent to t_Q via a regular equivalence that restricts to an automorphism of $\mathfrak{s}^S$.

Proof. This follows by applying Theorem 8.4.1 for both $\delta^\circ|_{\mathfrak{P}_+^S}$ and $-\delta^\circ|_{\mathfrak{P}_-^S}$ under consideration of the Cartan involution ω of $\mathfrak{L}$. □

Remark 8.4.4.
Let $Q^{(i)} = ((\Pi_+^{(i)}, \Pi_-^{(i)}, \phi^{(i)}), h^{(i)})$ be a Belavin-Drinfeld quadruple such that $\Pi_+^{(i)}, \Pi_-^{(i)} \subseteq S$ for $i \in \{1, 2\}$. Then Proposition 8.3.3 immediately implies that $t_{Q^{(1)}}$ and $t_{Q^{(2)}}$ are regularly equivalent via a regular equivalence fixing $\mathfrak{s}^S$ if and only if there exists an automorphism ν of the Dynkin diagram of $\mathfrak{L}$ such that $\widetilde{\nu}(S) = S$, $\widetilde{\nu}(\Pi_\pm^{(1)}) = \Pi_\pm^{(2)}$, $\widetilde{\nu}\phi^{(1)}\widetilde{\nu}^{-1} = \phi^{(2)}$, and $\widetilde{\nu}(h^{(1)}) = h^{(2)}$. We point out that all inner automorphisms of $\mathfrak{s}^S$ are restrictions to $\mathfrak{s}^S$ of elements of $\mathrm{Aut}_{\mathbb{C}[u,u^{-1}]\text{-alg}}(\mathfrak{L})$ fixing $\mathfrak{s}^S$, but, in general, not every outer automorphism of $\mathfrak{s}^S$ is of this form. Nevertheless, if σ and S are chosen as in Subsection 8.4.4 below, every automorphism of $\mathfrak{s}^S$ arises as the restriction of an element of $\mathrm{Aut}_{\mathbb{C}[u,u^{-1}]\text{-alg}}(\mathfrak{L})$ fixing $\mathfrak{s}^S$.

8.4.4 Constant quasi-triangular r-matrices. Assume that $\sigma = \mathrm{id}_\mathfrak{g}$ and note that in this case $\mathfrak{g} = \mathfrak{s}^S$ for $S = \{\alpha_1, \dots, \alpha_n\}$ in the notation of Subsection 4.2.6. Lemma 5.4.4 states that $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies $r + \tau_\mathfrak{g}(r) = \gamma$ and $\mathrm{CYB}(r) = 0$ if and only if

$$\frac{v\gamma}{u - v} + r \tag{8.105}$$

is an $\mathrm{id}_\mathfrak{g}$-trigonometric r-matrix. Indeed, put $u = \exp(x)$, $v = \exp(y)$ and use the second part of said Lemma. In other words, r is a quasi-triangular constant r-matrix satisfying $r + \tau_\mathfrak{g}(r) = \gamma$ if and only if $t := r - \gamma_0^+ - \gamma_\mathfrak{h}/2$ is a classical twist of $\delta^\circ|_\mathfrak{g}$. Therefore, they are classified, up to $r \sim (\varphi \otimes \varphi)r$ for $\varphi \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$, by Theorem 8.4.3 and Remark 8.4.4.

9 Rational r-matrices

Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra with Killing form K and Casimir element γ. We have seen in Theorem 6.2.1 that a normalized formal r-matrix, whose associated curve is a cuspidal Weierstraß cubic, is gauge equivalent to a rational r-matrix, i.e. an r-matrix of the form

$$r^t := r_{\mathrm{Yang}} + t \text{ for } r_{\mathrm{Yang}} = \frac{\gamma}{x-y} \text{ and } t \in (\mathfrak{g} \otimes \mathfrak{g})[x,y]. \tag{9.1}$$

In Subsection 5.4.5, we have seen that these r-matrices correspond to classical twists of the standard Lie bialgebra cobracket $\partial r_{\mathrm{Yang}}$ of $\mathfrak{g}[z]$ in the sense that r^t is a rational r-matrix if and only if $\delta^t := \partial r^t$ defines a Lie bialgebra cobracket on $\mathfrak{g}[z]$.

This chapter is dedicated to the study of rational r-matrices. For this purpose, we fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{H} \oplus \mathfrak{n}_+$, let $(\widehat{\mathfrak{L}}(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}}), \widehat{\mathfrak{H}}, \Pi = \{\alpha_0, \dots, \alpha_n\}, \Pi^\vee)$ be the realization constructed in Subsection 4.2.6 and $((\mathfrak{g}[z,z^{-1}], K_{-1}), \mathfrak{g}[z], \mathfrak{W}^t)$ be the Manin triple associated to a classical twist t of δ in Theorem 5.4.12.

9.1 Explicit geometrization

We begin by presenting an explicit construction of the geometric data of a rational r-matrix. Our approach differs slightly from the one in [BG18], since we use the theory of maximal subalgebras of $\mathfrak{L}(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}}) = \mathfrak{g}[z,z^{-1}]$ commensurable with a Boral subalgebra from Subsection 4.2.5 instead of the theory of maximal orders from [Sto91b; Sto91c].

9.1.1 Construction of the geometric data associated to rational r-matrices. Let $r = r^t$ be a rational r-matrix and write $\mathfrak{W} = \mathfrak{W}^t$. The subalgebra $\mathfrak{W} \subseteq \mathfrak{g}[z,z^{-1}]$ is commensurable with $\mathfrak{W}^0 = z^{-1}\mathfrak{g}[z^{-1}]$, so $\mathfrak{W}$ is also commensurable with the standard Borel subalgebra $\mathfrak{B}_- := \mathfrak{H} \oplus \mathfrak{n}_- \oplus z^{-1}\mathfrak{g}[z^{-1}]$ of $\mathfrak{g}[z,z^{-1}]$. Therefore, Proposition 4.2.9 implies that $\varphi(\mathfrak{W}) \subseteq \mathfrak{P}_-^{(i)}$ for some $i \in \{0,\dots,n\}$, where $\mathfrak{P}_-^{(i)} \subseteq \mathfrak{g}[z,z^{-1}]$ is the standard parabolic subalgebra to $\{0,\dots,n\} \setminus \{i\}$ and $\psi \in \mathrm{Aut}_{\mathbb{C}[z,z^{-1}]\text{-alg}}(\mathfrak{g}[z,z^{-1}])$. Note that the orthogonal complement of a linear subspace $V \subseteq \mathfrak{L}$ with respect to K_0 coincides with $zV^\perp$, where here and henceforth $(\cdot)^\perp$ denotes the orthogonal complement with respect to K_{-1}; see Lemma 4.2.1. Using Proposition 4.2.9.(2) yields

$$z^{-2}\mathbb{C}[z^{-1}]\mathfrak{W} \subseteq \mathfrak{W}^\perp = \mathfrak{W}. \tag{9.2}$$

In particular, $\mathfrak{W}$ is a torsion-free Lie algebra over $\mathbb{C}[z^{-2}, z^{-3}]$.

Let X be defined as the gluing of $C := \mathrm{Spec}(\mathbb{C}[z])$ and $U := \mathrm{Spec}(\mathbb{C}[z^{-2},z^{-3}])$ along $\mathrm{Spec}(\mathbb{C}[z,z^{-1}])$. Then X is a projective curve of arithmetic genus one with smooth locus C and cuspidal singular closed point $s := (z^{-2}, z^{-3}) \in U \subseteq X$.

Let $\mathscr{W}$ be the sheaf of Lie algebras on U defined by $\mathfrak{W}$. Since the multiplication $\mathfrak{W} \otimes_{\mathbb{C}[z]} \mathbb{C}[z,z^{-1}] \to \mathfrak{g}[z,z^{-1}]$ is an isomorphism, we can glue $\mathfrak{g} \otimes \mathscr{O}_C$ and $\mathscr{W}$ along

$U \cap V_- = \operatorname{Spec}(\mathbb{C}[z,z^{-1}])$ to obtain a coherent sheaf of Lie algebras $\mathscr{A}$ on X. In particular, $\mathscr{A}|_C \cong \mathfrak{g} \otimes \mathscr{O}_C$ and $\mathscr{A}|_U \cong \mathscr{W}$. The Mayer-Vietoris sequence and $\mathfrak{g}[z,z^{-1}] = \mathfrak{g}[z] \oplus \mathfrak{W}$ implies that $\mathrm{h}^0(\mathscr{A}) = 0 = \mathrm{h}^1(\mathscr{A})$. Put $p := (z) \in C$ and let $c\colon \widehat{\mathscr{O}}_{X,p} \to \mathbb{C}[\![z]\!]$ and $\zeta\colon \widehat{\mathscr{A}_p} \to \mathfrak{g}[\![z]\!]$ be the isomorphisms induced by $\Gamma(C,\mathscr{O}_X) \cong \mathbb{C}[z] \subseteq \mathbb{C}[\![z]\!]$ and $\Gamma(C,\mathscr{A}) \cong \mathfrak{g}[z] \subseteq \mathfrak{g}[\![z]\!]$ respectively. We have constructed a geometric $\mathfrak{g}$-lattice model $((X,\mathscr{A}),(p,c,\zeta))$.

9.1.2 The comparison theorem. The following theorem verifies that the geometric datum constructed from a rational r-matrix in the last subsection is indeed the geometric datum of said r-matrix. It is a variation of [BG18, Theorem 5.3].

Theorem 9.1.1.
Let $((X,\mathscr{A}),(p,c,\zeta))$ be the geometric datum constructed from the rational r-matrix $r = r^t$ in Subsection 9.1.1. Then:
(1) $\mathbb{T}((X,\mathscr{A}),(p,c,\zeta)) = (\mathbb{C}[z^{-2},z^{-3}],\mathfrak{g}(r))$.
(2) *$((X,\mathscr{A}),(C,\mathrm{d}z))$ is the geometric CYBE model of r, where $\mathrm{d}z$ is understood as global Rosentlicht regular 1-form on X.*
(3) *The geometric r-matrix ρ of $((X,\mathscr{A}),(C,\mathrm{d}z))$ satisfies $\rho|_{C\times C} = r$.*

Proof. It is easy to see that $\mathbb{T}((X,\mathscr{A}),(p,c,\zeta)) = (\mathbb{C}[z^{-2},z^{-3}],\mathfrak{W})$ and the identity $\mathfrak{W} = \mathfrak{g}(r)$ actually holds almost by construction, proving the first part of the claim. The remainder is now a consequence of Theorem 3.3.3. □

9.2 Structure theory

It was already noticed in [BD83a] that the classification of rational r-matrices for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ includes the problem of classifying commuting matrices up to simultaneous similarity. This problem is known to be representation wild and as such hopeless to solve in an appropriate sense. However, in this section we present structure theoretical results about ration r-matrices based on the work of Stolin in [Sto91b; Sto91c; Sto95].

9.2.1 Prelude: maximal bounded subalgebras up to polynomial equivalence. Let us begin by deriving the theory of maximal bounded subalgebras of $\mathfrak{g}[z,z^{-1}]$ used in [Sto95] from the results in Subsection 4.2.5. It is easy to see that a subalgebra $\mathfrak{A} \subseteq \mathfrak{g}[z,z^{-1}]$ is commensurable with

$$\mathfrak{B}_+ := \mathfrak{H} \oplus \mathfrak{n}_+ \oplus z\mathfrak{g}[z] \text{ (resp. } \mathfrak{B}_- := \mathfrak{H} \oplus \mathfrak{n}_- \oplus z^{-1}\mathfrak{g}[z^{-1}]) \tag{9.3}$$

if and only if it is *positively* (resp. *negatively*) *bounded*, that is $z^k\mathfrak{g}[z] \subseteq \mathfrak{A} \subseteq z^{-k}\mathfrak{g}[z]$ (resp. $z^{-k}\mathfrak{g}[z^{-1}] \subseteq \mathfrak{A} \subseteq z^k\mathfrak{g}[z^{-1}]$) for some $k \in \mathbb{N}_0$. In particular, Theorem 4.2.9 states that for a negatively bounded $\mathfrak{A} \subseteq \mathfrak{g}[z,z^{-1}]$ exists $\varphi \in \mathrm{Inn}_{\mathrm{ad}}(\mathfrak{g}[z,z^{-1}])$ such that $\mathfrak{A} \subseteq \varphi(\mathfrak{P}_-^{(i)})$ for some $i \in \{0,\dots,n\}$. Using Theorem 4.1.9, we can write $\varphi = b_+wb_-$, where w is an element of the Weyl group W associated to $\mathfrak{g}[z,z^{-1}]$ and $b_+(\mathfrak{g}[z]) = \mathfrak{g}[z]$ as well as $b_-(\mathfrak{P}_-^{(i)}) = \mathfrak{P}_-^{(i)}$. In particular, $b_+^{-1}(\mathfrak{A}) \subseteq w(\mathfrak{P}_-^{(i)})$ and we obtain the following result.

Lemma 9.2.1.
For any negatively bounded subalgebra $\mathfrak{A} \subseteq \mathfrak{g}[z,z^{-1}]$ *exist* $\varphi \in \mathrm{Aut}_{\mathbb{C}[z]\text{-alg}}(\mathfrak{g}[z])$, $i \in \{0,\dots,n\}$, *and* $w \in W$ *such that* $\varphi(\mathfrak{A}) \subseteq w(\mathfrak{P}_-^{(i)})$.

Let Φ be the root system of $(\widehat{\mathfrak{L}}(\mathfrak{g},\mathrm{id}_\mathfrak{g}),\widehat{\mathfrak{H}},\Pi,\Pi^\vee)$ and

$$\mathfrak{H}_\mathbb{R} := \{h \in \mathfrak{H} \mid \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in \Phi\}. \tag{9.4}$$

For any $h \in \mathfrak{H}_\mathbb{R} \oplus \mathbb{C}c \oplus \mathbb{R}d \subseteq \widehat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$ (see Subsection 4.2.2), we can consider the subalgebra

$$\mathfrak{P}_-^h := \bigoplus_{\substack{\alpha\in\Phi\\ \alpha(h)\leqslant 0}} \mathfrak{g}[z,z^{-1}]_\alpha \subseteq \mathfrak{g}[z,z^{-1}]. \tag{9.5}$$

Then, $w(\mathfrak{g}[z,z^{-1}]_\alpha) = \mathfrak{g}[z,z^{-1}]_{w(\alpha)}$ for all $\alpha \in \Phi$ implies $w(\mathfrak{P}_-^{w(h)}) = \mathfrak{P}_-^h$. Furthermore, $\mathfrak{P}_-^{(i)} = \mathfrak{P}_-^{h_i}$ holds, where $h_0,\dots,h_n \in \mathfrak{H}_\mathbb{R} \oplus \mathbb{R}d$ are uniquely determined by the conditions $\alpha_j(h_k) = \delta_{jk}/k_j$ for $j,k \in \{0,\dots,n\}$ and $k_0,\dots,k_n$ are defined in Lemma 4.2.4. In particular, $h_0 = d$. The set $\{h_0,\dots,h_n\}$ consists exactly of the vertices of the simplex

$$\Sigma := \{h \in \mathfrak{H}_\mathbb{R} \oplus \mathbb{R}d \mid \alpha_i(h) \in \mathbb{R}_{\geqslant 0}, i \in \{0,\dots,n\}\}. \tag{9.6}$$

Lemma 9.2.2 states that for any negatively bounded subalgebra $\mathfrak{A} \subseteq \mathfrak{g}[z,z^{-1}]$ exists $\varphi \in \mathrm{Aut}_{\mathbb{C}[z]\text{-alg}}(\mathfrak{g}[z])$, $i \in \{0,\dots,n\}$, and $w \in W$ such that $\varphi(\mathfrak{A}) \subseteq \mathfrak{P}_-^{w(h_i)}$.

Assume that $\mathfrak{g}[z]+\mathfrak{A} = \mathfrak{g}[z,z^{-1}]$. Then $\mathfrak{g}[z]+\mathfrak{P}_-^{w(h_i)} = \mathfrak{g}[z,z^{-1}]$ forces $-\alpha_0(w(h_i)) \leqslant 0$. There exists an element w' in the Weyl group of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{H} \subseteq \mathfrak{g}$ such that $\alpha_i(w'w(h_i)) \geqslant 0$ for all $i \in \{1,\dots,n\}$, where we consider w' as an element of the Weyl group of $\mathfrak{g}[z,z^{-1}]$. Using $w(d) - d \in \mathfrak{H} \oplus \Bbbk c$ and $\alpha_0 w' = \alpha_0$, we see that $\alpha_i(h_j) \geqslant 0$ for all $i \in \{0,\dots,n\}$. In particular, the image of h_j under the canonical projection $\widehat{\mathfrak{H}} \to \mathfrak{H} \oplus \mathbb{C}d$ is a vertex of the simplex Σ. Therefore, $w'w(h_j) \in \{h_0,\dots,h_n\} + \mathbb{C}c$ for all $j \in \{0,\dots,n\}$. Since $\mathfrak{P}_-^{h+\lambda c} = \mathfrak{P}_-^h$ for all $h \in \mathfrak{H}_\mathbb{R} \oplus \mathbb{R}d$ and $\lambda \in \mathbb{C}$, we obtain the following result.

Lemma 9.2.2.
For all negatively bounded subalgebras $\mathfrak{A} \subseteq \mathfrak{g}[z,z^{-1}]$ *such that* $\mathfrak{g}[z] + \mathfrak{A} = \mathfrak{g}[z,z^{-1}]$ *exist* $\varphi \in \mathrm{Aut}_{\mathbb{C}[z]\text{-alg}}(\mathfrak{g}[z])$ *and* $i \in \{0,\dots,n\}$ *such that* $\varphi(\mathfrak{A}) \subseteq \mathfrak{P}_-^{(i)}$.

It was observed in [Sto95] that Lemma 9.2.2 admits a geometric counterpart. Let us identify $\mathbb{P}^1_\mathbb{C}$ as the gluing of $U = \mathrm{Spec}(\mathbb{C}[z])$ and $V = \mathrm{Spec}(\mathbb{C}[z^{-1}])$ along $\mathrm{Spec}(\mathbb{C}[z,z^{-1}])$ and $\mathscr{P}_-^h$ be the sheaf of Lie algebras on $\mathbb{P}^1_\mathbb{C}$ defined by gluing the free $\mathbb{C}[z]$-Lie algebra $\mathfrak{g}[z]$ with the free $\mathbb{C}[z^{-1}]$-Lie algebra $\mathfrak{P}_-^h$ along $\mathfrak{g}[z,z^{-1}]$ for any $h \in \mathfrak{H}_\mathbb{R} \oplus \mathbb{C}c \oplus \mathbb{R}d$. Moreover, we write $\mathscr{P}_-^{(i)} := \mathscr{P}_-^{h_i}$ for any $i \in \{0,\dots,n\}$.

Let $\mathscr{L}$ be any locally free sheaf of Lie algebras on $\mathbb{P}^1_\mathbb{C}$, which is weakly $\mathfrak{g}$-locally free on U. Then, $\Gamma(U,\mathscr{L}) = \mathfrak{g}[z]$ up to isomorphism of sheaves of Lie algebras by virtue of Theorem 6.1.1. Hence, $\mathrm{h}^0(\mathscr{L}),\mathrm{h}^1(\mathscr{L}) < \infty$ implies that $\Gamma(V,\mathscr{L}) \subseteq \mathfrak{g}[z,z^{-1}]$ is a negatively bounded subalgebra. Application of Lemma 9.2.1 and Lemma 9.2.2 yields the following result.

Lemma 9.2.3.
Any locally free sheaf $\mathscr{L}$ of Lie algebras on $\mathbb{P}^1_\mathbb{C}$, which is weakly $\mathfrak{g}$-locally free on U, is isomorphic to a subsheaf of Lie algebras of $\mathscr{P}^h_-$ for some $h \in \mathfrak{H}_\mathbb{R} \oplus \mathbb{R}d$. Furthermore, if $\mathrm{h}^1(\mathscr{L}) = 0$, $\mathscr{L}$ is isomorphic to a subsheaf of Lie algebras of $\mathscr{P}^{(i)}_-$ for some $i \in \{0, \dots, n\}$.

9.2.2 Proof of Drinfeld's conjecture about rational r-matrices. Lemma 9.2.2 states that for any classical twist t of $\partial r_{\mathrm{Yang}}$, up to polynomial equivalence, $\mathfrak{W}^t \subseteq \mathfrak{P}^{(i)}_-$ for some $i \in \{0, \dots, n\}$. Therefore, the process of reconstructing the rational r-matrix r^t from $\mathfrak{W}^t$ (as in the proof of Theorem 5.3.2) combined with $\mathfrak{W}^t \subseteq \mathfrak{P}^{(i)}_- \subseteq z\mathfrak{g}[z^{-1}]$ implies that the total degree of t is at most one. This is Stolin's proof from [Sto91b; Sto91c] of the following conjecture by Drinfeld.

Theorem 9.2.4.
Let $t \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ be a classical twist of $\partial r_{\mathrm{Yang}}$, i.e. r^t is a rational r-matrix. Then t is polynomially equivalent to a classical twist of $\partial r_{\mathrm{Yang}}$ of total degree at most one.

9.2.3 An algebro-geometric reduction. Let $((X, \mathscr{A}), (p, c, \zeta))$ be the geometric CYBE datum of a rational r-matrix $r = r^t$ from Theorem 9.1.1. The inclusion $\Gamma(X \setminus \{p\}, \mathscr{A}) \cong \mathfrak{W}^t \subseteq \mathfrak{P}^{(i)}_-$ is equivalent to $\mathscr{A} \subseteq \nu_* \mathscr{P}^{(i)}_-$, where $\nu\colon \mathbb{P}^1_\mathbb{C} \to X$ is the canonical map. Note that $\mathfrak{g}[z, z^{-1}] = \mathfrak{g}[z] \oplus \mathfrak{W}^t$ and $\mathfrak{W}^t \subseteq \mathfrak{P}^{(i)}_-$ is equivalent to the identity $\mathfrak{P}^{(i)}_- = (\mathfrak{g}[z] \cap \mathfrak{P}^{(i)}_-) \oplus \mathfrak{W}$. Using $\mathrm{H}^0(\mathscr{P}^{(i)}_-) = \mathfrak{g}[z] \cap \mathfrak{P}^{(i)}_-$, we arrive at

$$\Gamma(U_-, \mathscr{A}) \oplus \mathrm{H}^0(\mathscr{P}^{(i)}_-) = \mathfrak{W}^t \oplus \mathrm{H}^0(\mathscr{P}^{(i)}_-) = \mathfrak{P}^{(i)}_- = \Gamma(U_-, \mathscr{P}^{(i)}_-). \tag{9.7}$$

This gives a defining algebro-geometric condition for rational r-matrices, which was originally found in [Sto95].

9.2.4 Reduction to a finite-dimensional problem. Let r^t be a rational r-matrix such that $\mathfrak{W}^t \subseteq \mathfrak{P}^{(i)}_-$ for some $i \in \{0, \dots, n\}$. This inclusion can be used to reduce the problem of classifying rational r-matrices to a finite-dimensional problem. The bilinear form K_{-1} (see Lemma 4.2.1) induces a non-degenerate invariant bilinear form on the finite-dimensional reductive Lie algebra $\mathfrak{p}^{(i)} := \mathfrak{P}^{(i)}_- / \mathfrak{P}^{(i),\perp}_-$ and $(\mathfrak{p}^{(i)}, \mathfrak{a}^{(i)}, \mathfrak{w}^t)$ is a Manin triple, where $\mathfrak{a}^{(i)}$ is the image of $\mathfrak{g}[z] \cap \mathfrak{P}^{(i)}_-$ under the canonical projection $\mathfrak{P}^{(i)}_- \to \mathfrak{p}^{(i)}$ and $\mathfrak{w}^t := \mathfrak{W}^t / \mathfrak{P}^{(i),\perp}_-$. Here, we used $\mathfrak{P}^{(i),\perp}_- \subseteq \mathfrak{W}^{t,\perp} = \mathfrak{W}^t \subseteq \mathfrak{P}^{(i)}_-$ and (9.7).

Proposition 9.2.5.
The map $r^t \mapsto \mathfrak{w}^t := \mathfrak{W}^t / \mathfrak{P}^{(i),\perp}_-$ defines a bijection between rational r-matrices r^t such that $\mathfrak{W}^t \subseteq \mathfrak{P}^{(i)}_-$ and subalgebras $\mathfrak{w} \subseteq \mathfrak{p}^{(i)} := \mathfrak{P}^{(i)}_- / \mathfrak{P}^{(i),\perp}_-$ such that $(\mathfrak{p}^{(i)}, \mathfrak{a}^{(i)}, \mathfrak{w})$ is a Manin triple, where $\mathfrak{a}^{(i)}$ is the image of $\mathfrak{g}[z] \cap \mathfrak{P}^{(i)}_-$ in $\mathfrak{p}^{(i)}$.

Proof. We have already discussed above that the assignment is well-defined, so it remains to provide an inverse. Let $(\mathfrak{p}^{(i)}, \mathfrak{a}^{(i)}, \mathfrak{w})$ be any Manin triple. Since $\mathfrak{P}_-^{(i)}$ can be identified with $\mathfrak{p}^{(i)} \oplus \mathfrak{P}_-^{(i),\perp}$ as a vector space, we can identify $\mathfrak{A} := \mathfrak{w} \oplus \mathfrak{P}_-^{(i),\perp}$ with a subalgebra of $\mathfrak{P}_-^{(i)}$. It is straight forward to show that $(\mathfrak{g}[z, z^{-1}], \mathfrak{g}[z], \mathfrak{A})$ is a Manin triple and $\mathfrak{P}_-^{(i),\perp} \subseteq \mathfrak{A}$ immediately implies that $\mathfrak{A}$ is commensurable with $z^{-1}\mathfrak{g}[z^{-1}]$, so $\mathfrak{A} = \mathfrak{W}^t$ for some rational r-matrix r^t by virtue of Theorem 5.4.12. It is easy to see that this assignment is inverse to $r^t \mapsto \mathfrak{w}^t$, concluding the proof. □

9.2.5 Stolin pairs. Let $\mathfrak{p}_+^{(i)} \subseteq \mathfrak{g}$ be the standard parabolic Lie algebra of $\mathfrak{g}$ to $\{1, \dots, n\} \setminus \{i\}$ for any $i \in \{0, \dots, n\}$. In particular, $\mathfrak{p}_+^{(0)} = \mathfrak{g}$. A *Stolin pair* $(\mathfrak{l}, B)$ *of index* $i \in \{0, \dots, n\}$ consists of a Lie subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$, satisfying $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}_+^{(i)}$, equipped with a skew-symmetric bilinear form $B\colon \mathfrak{l} \times \mathfrak{l} \to \mathbb{C}$ such that

$$B([a, b], c) + B([c, a], b) + B([b, c], a) = 0 \tag{9.8}$$

for all $a, b, c \in \mathfrak{l}$ and B defines a non-degenerated bilinear form on $\mathfrak{l} \cap \mathfrak{p}_+^{(i)}$. We point out that, in the language of Lie algebra cohomology, (9.8) means that B is a 2-cocycle on $\mathfrak{l}$.

We say that $\alpha_i \in \Pi$ is *singular* if there exists an automorphism ν of the Dynkin diagram of $\mathfrak{g}[z, z^{-1}]$ which maps α_i to α_0. This is equivalent to $k_i = 1$ for $k_0, \dots, k_n$ as given in Lemma 4.2.4; see Figure 4.2. For instance, every root in Π is singular for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. For such a simple root, $\widetilde{\nu}$ induces an isomorphism

$$\mathfrak{p}^{(i)} := \mathfrak{P}_-^{(i)}/\mathfrak{P}_-^{(i),\perp} \cong \mathfrak{g}[\epsilon]/\epsilon^2\mathfrak{g}[\epsilon] \cong \mathfrak{g} \oplus \epsilon\mathfrak{g}. \tag{9.9}$$

This isomorphism identifies $\mathfrak{a}^{(i)}$ with $\mathfrak{p}_+^{(j)} \oplus \epsilon\mathfrak{p}_+^{(j),\perp}$, where $\nu(\alpha_0) = \alpha_j$, and equips $\mathfrak{g} \oplus \epsilon\mathfrak{g}$ with the non-degenerate invariant bilinear form

$$K_\epsilon(a_0 + \epsilon a_\epsilon, b_0 + \epsilon b_\epsilon) := K(a_0, b_\epsilon) + K(b_0, a_\epsilon) \text{ for all } a_0, a_\epsilon, b_0, b_\epsilon \in \mathfrak{g}. \tag{9.10}$$

Therefore, the assignment $t \mapsto \mathfrak{w}^t$ defined in Proposition 9.2.5 yields a bijection between rational r-matrices r^t such that $\mathfrak{W}^t \subseteq \mathfrak{P}_-^{(i)}$ and Manin triples $(\mathfrak{g} \oplus \epsilon\mathfrak{g}, \mathfrak{p}_+^{(j)} \oplus \epsilon\mathfrak{p}_+^{(j),\perp}, \mathfrak{w})$.

Let $(\mathfrak{g} \oplus \epsilon\mathfrak{g}, \mathfrak{p}_+^{(j)} \oplus \epsilon\mathfrak{p}_+^{(j),\perp}, \mathfrak{w})$ be any Manin triple and $\mathfrak{w}_0 := \mathrm{pr}_0(\mathfrak{w})$ for $\mathrm{pr}_0\colon \mathfrak{g} \oplus \epsilon\mathfrak{g} \to \mathfrak{g}$ defined by $a_0 + \epsilon a_\varepsilon \mapsto a_0$. Note that $\mathfrak{w} \subseteq \mathfrak{w}_0 \oplus \epsilon\mathfrak{g}$ implies $\mathfrak{w} = \mathfrak{w}^\perp \supseteq (\mathfrak{w}_0 \oplus \epsilon\mathfrak{g})^\perp = \epsilon\mathfrak{w}_0^\perp$. Hence, $\dim(\mathfrak{w}_0) + \dim(\mathfrak{w}_0^\perp) = \dim(\mathfrak{g}) = \dim(\mathfrak{w})$ implies that $\mathrm{Ker}(\mathrm{pr}_0|_\mathfrak{w}) = \mathfrak{w} \cap \varepsilon\mathfrak{g} = \varepsilon\mathfrak{w}_0^\perp$, so there exists a unique $f_\mathfrak{w}\colon \mathfrak{w}_0 \to \mathfrak{g}/\mathfrak{w}_0^\perp$ such that

$$\mathfrak{w}/\epsilon\mathfrak{w}_0^\perp = \{a + \epsilon f_\mathfrak{w}(a) \mid a \in \mathfrak{w}_0\} \subseteq \mathfrak{w}_0 \oplus \epsilon(\mathfrak{g}/\mathfrak{w}_0^\perp). \tag{9.11}$$

We write $B_\mathfrak{w}(a, b) := K(f_\mathfrak{w}(a), b)$ for all $a, b \in \mathfrak{w}_0$. The following result allows the calculation of all rational r-matrices in low dimensions; see [Sto91b; Sto91c] for details.

Proposition 9.2.6.
Let $i \in \{0, \dots, n\}$, ν be an automorphism of the Dynkin diagram which maps α_i to α_0, and put $\alpha_j := \nu(\alpha_0)$. The assignment $r^t \mapsto (\mathfrak{w}_0^t, B_{\mathfrak{w}^t})$ defines a bijection between rational r-matrices r^t satisfying $\mathfrak{W}^t \subseteq \mathfrak{P}_-^{(i)}$ and Stolin pairs of index j.

Proof. If suffices to prove that

$$\left(\mathfrak{g} \oplus \epsilon\mathfrak{g}, \mathfrak{p}_+^{(j)} \oplus \epsilon\mathfrak{p}_+^{(j),\perp}, \mathfrak{w}\right) \longmapsto (\mathfrak{w}_0, B_\mathfrak{w}) \tag{9.12}$$

defines a bijection of Manin triples $\left(\mathfrak{g} \oplus \epsilon\mathfrak{g}, \mathfrak{p}_+^{(j)} \oplus \epsilon\mathfrak{p}_+^{(j),\perp}, \mathfrak{w}\right)$ and Stolin pairs $(\mathfrak{l}, B)$ of index j. We first prove that this assignment is well-defined, i.e. that $(\mathfrak{w}_0, B_\mathfrak{w})$ is indeed a Stolin pair of index j. Since $\mathfrak{w}$ is Lagrangian, $\mathfrak{w}/\epsilon\mathfrak{w}_0^\perp$ is isotropic, so

$$0 = K_\epsilon(a + \epsilon f_\mathfrak{w}(a), b + \epsilon f_\mathfrak{w}(b)) = B(a,b) + B(b,a) \text{ for all } a, b \in \mathfrak{w}_0. \tag{9.13}$$

In particular, B is skew-symmetric. The fact that $\mathfrak{w}$ is a Lie algebra and $\epsilon\mathfrak{w}_0^\perp \subseteq \mathfrak{w}$ is an ideal implies that

$$[a,b] + \epsilon f_\mathfrak{w}([a,b]) = [a + \epsilon f_\mathfrak{w}(a), b + \epsilon f_\mathfrak{w}(b)] = [a,b] + \epsilon([f_\mathfrak{w}(a), b] + [a, f_\mathfrak{w}(b)]),$$

for all $a, b \in \mathfrak{w}_0$. In particular, $f_\mathfrak{w}([a,b]) = [f_\mathfrak{w}(a), b] + [a, f_\mathfrak{w}(b)]$ holds. Combining this with the invariance of K shows

$$B_\mathfrak{w}([a,b],c) = K(f_\mathfrak{w}([a,b]),c) = K([f_\mathfrak{w}(a),b] + [a, f_\mathfrak{w}(b)], c) = B_\mathfrak{w}(a,[b,c]) + B_\mathfrak{w}(b,[c,a]).$$

To conclude that $(\mathfrak{w}_0, B_\mathfrak{w})$ is a Stolin pair of index j, it remains to show that $B_\mathfrak{w}$ is non-degenerate if restricted to $\mathfrak{w}_0 \cap \mathfrak{p}_+^{(j)}$, since $\mathfrak{w}_0 + \mathfrak{p}_+^{(j)} = \mathfrak{g}$ is immediately implied by $\mathfrak{g} \oplus \epsilon\mathfrak{g} = \left(\mathfrak{p}_+^{(j)} \oplus \epsilon\mathfrak{p}_+^{(j),\perp}\right) \oplus \mathfrak{w}$. Let $a \in \mathfrak{w}_0 \cap \mathfrak{p}_+^{(j)}$ satisfy $B_\mathfrak{w}\left(a, \mathfrak{w}_0 \cap \mathfrak{p}_+^{(j)}\right) = \{0\}$. This is equivalent to

$$f_\mathfrak{w}(a) \in \left(\mathfrak{w}_0 \cap \mathfrak{p}_+^{(j)}\right)^\perp / \mathfrak{w}_0^\perp = \left(\mathfrak{w}_0^\perp + \mathfrak{p}_+^{(j),\perp}\right)/\mathfrak{w}_0^\perp = \mathfrak{p}_+^{(j),\perp} / \left(\mathfrak{w}_0^\perp \cap \mathfrak{p}_+^{(j),\perp}\right) = \mathfrak{p}_+^{(j),\perp},$$

where we used $\mathfrak{w}_0^\perp \cap \mathfrak{p}_+^{(j),\perp} = \left(\mathfrak{w}_0 + \mathfrak{p}_+^{(j)}\right)^\perp = \mathfrak{g}^\perp = \{0\}$. Since $a + \epsilon f_\mathfrak{w}(a) \in \mathfrak{w}/\epsilon\mathfrak{w}_0^\perp$, the identity $\mathfrak{w} \cap \left(\mathfrak{p}_+^{(i)} \oplus \epsilon\mathfrak{p}_+^{(i),\perp}\right) = \{0\}$ implies $a = 0$.

Now we have to construct the inverse assignment. Let $P := (\mathfrak{l}, B)$ be a Stolin pair of index j and $f_P := \mathfrak{l} \to \mathfrak{g}/\mathfrak{l}^\perp$ defined by B. Then

$$\mathfrak{w}_P := \{a + \epsilon f_P(a) \mid a \in \mathfrak{l}\} \oplus \epsilon\mathfrak{l}^\perp \subseteq \mathfrak{l} \oplus \epsilon\mathfrak{g} \tag{9.14}$$

defines a Lie subalgebra of $\mathfrak{g} \oplus \epsilon\mathfrak{g}$. Similar considerations as above show that $\left(\mathfrak{g} \oplus \epsilon\mathfrak{g}, \mathfrak{p}_+^{(j)} \oplus \epsilon\mathfrak{p}_+^{(j),\perp}, \mathfrak{w}_P\right)$ is a Manin triple and this defines the inverse of (9.12). □

Remark 9.2.7.

(1) There are versions of Stolin pairs admitting a reduction in the vein of Proposition 9.2.6 for non-singular roots in Π; see [Sto91c]. However, we will not go into this depth here.

(2) In some cases it is possible to assess that rational r-matrices are polynomially equivalent by studying relations between the finite-dimensional data determining them in the sense of subsections 9.2.3 or 9.2.6. For instance, in the case of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, two rational r-matrices r_1 and r_2 determined by Stolin pairs

$(\mathfrak{l}_1, B_1)$ and $(\mathfrak{l}_2, B_2)$ of the same class j are polynomial equivalent if there exists $g \in \mathrm{SL}_n(\mathbb{C})$ such that $\mathrm{Ad}_g\left(\mathfrak{p}_+^{(j)}\right) = \mathfrak{p}_+^{(j)}$, $\mathrm{Ad}_g(\mathfrak{l}_1) = \mathfrak{l}_2$ and

$$B_2(\mathrm{Ad}_g(a), \mathrm{Ad}_g(b)) - B_1(a,b) = f([a,b]) \text{ for all } a, b \in \mathfrak{l}_1; \tag{9.15}$$

see [Sto91b]. The Equation (9.15) can be reformulated as: $B_2 \circ (\mathrm{Ad}_g \times \mathrm{Ad}_g)$ and B_1 define the same class in the Lie algebra cohomology group $\mathrm{H}^2(\mathfrak{l}_1, \mathbb{C})$.

9.2.6 Constant triangular r-matrices. Lemma 5.4.4 states that if r^t is a rational r-matrix for $t \in \mathfrak{g} \otimes \mathfrak{g}$, t is a skew-symmetric constant r-matrix, i.e. $t + \tau_{\mathfrak{g}}(t) = 0$ and $\mathrm{CYB}(t) = 0$. Furthermore, $\mathfrak{W}^t \subseteq \mathfrak{g}[z^{-1}]$ in this case. Thus, Proposition 9.2.6 states that skew-symmetric constant r-matrices are in bijection with Stolin pairs of index 0. Note that a Stolin pair of index 0 is exactly a *quasi-Frobenius Lie subalgebra* of $\mathfrak{g}$. In particular, classifying quasi-Frobenius Lie subalgebras of $\mathfrak{g}$ yields a classification of skew-symmetric constant r-matrices.

9.2.7 Example: rational r-matrices over $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Let us list all rational r-matrices for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ up to polynomial equivalence. First of all, we have the trivial rational r-matrix: Yang's r-matrix

$$r_{\mathrm{Yang}}(x,y) = \frac{1}{x-y}\left(\frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right).$$

Since two-dimensional Lie algebras are automatically solvable, there is only one two-dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ up to conjugation, namely the standard Borel subalgebra $\mathfrak{b}_+$ of traceless upper triangular matrices. Quasi-Frobenius Lie algebras are automatically even dimensional and $\mathrm{H}^2(\mathfrak{b}_+, \mathbb{C}) = 0$, so there is at most one non-trivial rational r-matrix of class 0 up to polynomial equivalence; see part Remark 9.2.7.(2). It is given by the formula

$$r(x,y) = r_{\mathrm{Yang}} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \tag{9.16}$$

It can be shown that a Stolin pair $(\mathfrak{l}, B)$, where $\mathfrak{l}$ is solvable, determines a solution polynomially equivalent to a constant one; see the Proposition after [Sto91b, Lemma 4.4]. Therefore, a non-trivial Stolin pair $(\mathfrak{l}, B)$ of index 1 automatically satisfies $\mathfrak{l} = \mathfrak{sl}_2(\mathbb{C})$. Since Whitehead's lemma implies $\mathrm{H}^2(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C}) = 0$, we again have at most one rational r-matrix of this class. It is given by the formula

$$r(x,y) = r_{\mathrm{Yang}} + \begin{pmatrix}0 & -1/2\\ 0 & 0\end{pmatrix} x \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}0 & -1/2\\ 0 & 0\end{pmatrix} y. \tag{9.17}$$

Remark 9.2.8.
A list of all rational r-matrices for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ can be found in [Sto91a; Sto91b].

9.2.8 Comparison of formal and polynomial equivalence. As for σ-trigonometric r-matrices, we have introduced two different notions of equivalence for rational r-matrices: formal and polynomial equivalence. At first glance, polynomial equivalence looks stronger than the formal one, but the geometric theory implies that this is not the case.

Proposition 9.2.9.
Two rational r-matrices r_1 and r_2 are formally equivalent if and only if there exists $\lambda \in \mathbb{C}^\times$ such that $r_1(x,y)$ and $\lambda r_2(\lambda x, \lambda y)$ are polynomially equivalent.

Proof. Let (λ, w, φ) be a formal equivalence of r_1 and r_2. Since both r_1 and r_2 are normalized and skew-symmetric, $\lambda = w' \in \mathbb{C}^\times$; see Lemma 2.1.6.(4). For $i \in \{1,2\}$, let $((X, \mathscr{A}_i), (p, c, \zeta))$ be the geometric datum of r_i constructed in Section 9.2. Lemma 3.1.1 states that w defines an automorphism f of X fixing p and φ defines an isomorphism $\psi\colon \mathscr{A}_1 \to f_*\mathscr{A}_2$. Let C be the smooth locus of X. By construction, $\Gamma(C, \mathscr{A}_1) \cong \mathfrak{g}[z] \cong \Gamma(C, \mathscr{A}_2)$, so ψ induces an $\mathbb{C}[z]$-linear automorphism of $\mathfrak{g}[z]$. Application of Lemma 3.3.2 combined with Theorem 9.1.1.(3) now concludes the proof. □

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