PADERBORN UNIVERSITY

Contributions to funnel control of multibody systems

Der Fakultät für Elektrotechnik, Informatik und Mathematik der Universität Paderborn zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften (doctor rerum naturalium) vorgelegte Dissertation von

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Abstract The present thesis contains contributions to output reference tracking of nonlinear multibody systems via feedback control. We focus on *three aspects* of this topic. First, we perform a structural analysis and develop a structurally novel ansatz to decouple and represent the internal dynamics of an input-output control system. It allows to derive the internal dynamics' equations completely algorithmically without the need to compute the Byrnes-Isidori form. Moreover, since the internal dynamics are given in terms of the internal variable and the system's output, we may derive criteria on the system parameters to validate stability of the internal dynamics in advance, i.e., whether a system is minimum phase can be evaluated without decoupling the internal dynamics. The next aspect under consideration concerns a certain feasibility assumption in feedback control. Namely, for systems with relative degree larger than one, it is commonly assumed that the higher output derivatives are available. However, this cannot be guaranteed for general applications. Therefore, we elaborate on the so-called *funnel pre-compensator*, first proposed in [32]. The funnel pre-compensator is a dynamical system, which approximates a given signal arbitrarily good, and the pre-compensator's output and its derivative are explicitly known. We show that the conjunction of a funnel pre-compensator with a minimum phase system with arbitrary relative degree results in an overall system, which is minimum phase as well. As a consequence, we show that, utilizing the funnel pre-compensator, output reference tracking via funnel control is possible with output feedback only, i.e., the derivatives of the system's output need not to be available. The third main contribution consists of two novel feedback control laws, each of which achieves a specific control objective. In the first case, we consider the situation when the system's output is subject to possible measurement losses. We derive a control scheme with an intrinsic availability function, such that in the case of measurement losses, output tracking with predefined accuracy can be achieved, if some conditions on the maximal duration of signal absence and minimal time of signal availability are satisfied. Both conditions are given explicitly. The second control objective under consideration is exact tracking in finite time. We develop a funnel control law, which achieves tracking of a given reference signal such that the error evolves within prescribed bounds, and for a predefined final time, the system's output and its relevant derivatives match the reference exactly.

Zusammenfassung Die vorliegende Dissertation enthält Beiträge zum Thema der Ausgangs-Referenz-Verfolgung von nichtlinearen Mehrkörpersystemen mittels Rückführungs-Regelung (engl. feedback control). Dabei stehen *drei Aspekte* dieses Themenkomplexes im Mittelpunkt. Zunächst wird eine Strukturanalyse durchgeführt und ein strukturell neuer Ansatz zur Entkopplung und Darstellung der internen Dynamik eines Eingangs-Ausgangs-Systems entwickelt. Dieser erlaubt es, die Gleichungen der internen Dynamik vollständig algorithmisch herzuleiten, insbesondere wird die Berechnung der Byrnes-Isidori-Form vermieden. Da die interne Dynamik durch Ausdrücke des internen Zustands und der Systemausgänge dargestellt wird, können Kriterien an die Systemparameter formuliert werden, aus denen die Minimalphasigkeit eines Systems bestimmt werden kann, ohne die interne Dynamik zu entkoppeln und auf Stabilität zu untersuchen. Der nächste untersuchte Aspekt betrifft eine Annahme, welche häufig in der Feedback-Regelung von Systemen mit höherem Relativgrad getroffen wird. Es wird üblicherweise angenommen, dass die höheren Ableitungen des Systemausgangs dem Kontrollschema zugänglich sind. Dies kann jedoch für allgemeine Anwendungen nicht garantiert werden. Aus diesem Grund wird der, in der Arbeit [32] entworfene, *funnel pre-compensator* genauer untersucht. Der *funnel pre-compensator* ist ein dynamisches System, welches ein eingehendes Signal beliebig genau annähert und diese Annäherung ausgibt. Darüber hinaus ist die Ableitung des ausgegebenen Signals explizit bekannt. Es wird gezeigt, dass die Verknüpfung eines minimalphasigen Systems beliebigen Relativgrads mit einem *funnel pre-compensator* wieder ein minimalphasiges System ergibt. Damit lässt sich zeigen, dass unter Zuhilfenahme des *funnel pre-compensators* Ausgangs-Referenz-Verfolgung mittels *funnel control* möglich ist, ohne dass die höheren Ableitungen des Systemausgangs bekannt sein müssen. Der dritte Hauptbeitrag besteht aus zwei neuen Feedback-Regelgesetzen, wobei jedes ein bestimmtes Kontrollziel erreicht. Im ersten Fall werden Systeme betrachtet, deren Ausgänge möglichen Messausfällen unterworfen sind. Es wird ein Regelgesetz entworfen, welches auch in diesem Fall Ausgangs-Referenz-Verfolgung mit vorbestimmtem Fehlerverhalten garantiert, falls bestimmte Bedingungen erfüllt sind. Diese Bedingungen betreffen die maximale Dauer des Signalausfalls und die minimal benötigte Zeit des gesicherten Vorhandenseins des Signals. Beide Bedingungen werden explizit angegeben. Das zweite Kontrollziel ist Ausgangs-Referenz-Verfolgung mit exaktem Wert zu einer vorgegebenen endlichen Zeit. Es wird ein Kontrollgesetz entworfen, welches Ausgangs-Referenz-Verfolgung mit vorgegebenem Fehlerverhalten erreicht, wobei der Systemausgang und seine relevanten Ableitungen für eine vorgegebene Zeit genau mit der Referenz übereinstimmen.

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Contents

1 Introduction

Feedback mechanisms are ubiquitous in every person's everyday life. Being upright, walking, riding a bike, eating, using tools, and even communication is based on processing of impressions. A brief look at the world of plants testifies to the fact that in nature itself feedback mechanisms are developed and utilized, e.g., the turning of the blossom of a sunflower towards the sun; or the flight of a bird between trees. In all these illustrative processes, the actual effects achieved are perceived and compared with the intended effects. Thereupon, the causes of these effects are adjusted according to the registered errors. This defines a *feedback structure*. However, feedback systems not necessarily consist of living matter. In particular, when it comes to *control*, the systems are considered as consisting of inanimate matter. One level of abstraction higher, as regards the mathematical description and the development of control schemes, the objects under consideration are, e.g., functions, operators and trajectories.

To get an impression of the successful history of feedback control, we outline some milestones in its development; for detailed surveys on control see, e.g., [169, 186]. The Greek mathematician and inventor Ktesibios (285 - 222 before our calendar) incorporated a feedback mechanism into a water clock to regulate the flow and so improved its accuracy, cf. [133, 116]. About 1900 years later, the Dutch mathematician, physicist and inventor Christiaan Huygens (1629 - 1695) invented, besides his numerous seminal contributions to science, a centrifugal governor, which regulates the pressure and distance between millstones in windmills, cf. [149]. This control idea was then used by the Scottish inventor and engineer James Watt (1736 - 1819) who developed, in addition to groundbreaking technical achievements, a successor of the centrifugal governor which regulates the admission of steam into the cylinder of his famous steam engine, cf. [141, 183]. Since Watt's centrifugal governor, patented in 1788, combines sensing, actuation and control, it is widely considered as the birth of modern control technology. The theoretical investigation of such regulating systems was started by Maxwell in 1868 [140], where he studied linearized models and performed a stability analysis; moreover, he derived stability conditions of third order closed-loop systems in terms of the roots of algebraic equations. The general case was then treated by Routh [172]. Independently, Vyshnegradskii developed a stability criterion for third order steam engine regulators in 1876 [193]. This article strongly influenced the further developments in various engineering contexts, cf. [4]. In the subsequent years, driven by the Industrial Revolution, different control techniques and feedback strategies were invented in the fields of electricity [184], industrial process control [205], ship steering [144, 177] and flight control [142]. While the invention and development of control systems in the former cases were driven by engineering insights and intuition, in the latter case a solid theoretical basis was extremely important for safety reasons. Unfortunately, as so often in the history of technology and invention, control theory was primarily driven by war. So, since prior to, and during, the Second World War it was realized that science may play an important role in the war, many control systems were developed for military purposes in the first half of the twentieth century, cf. [12]. Linked to this was the further development of methods in control theory, such as the frequency response method [156, 39] and the theory of servomechanisms [79], where in the latter case the transfer function of a system could be determined experimentally and then, involving Nyquist plots, it could be used to design control structures. At that time, the first successes were achieved in replacing mechanical and pneumatic computers with electronic computers, cf. [136, 167]. Now it was possible to simulate large systems and use mathematical models to study the behaviour of diverse systems under different operating conditions. However, since servomechanism theory did not work well for systems with many input and outputs, the construction of models based on the method of frequency response was quite time consuming. In general, performances had to be optimized and new tools and mathematical descriptions had to be developed. At this stage of the new discipline "control theory", Kalman laid the foundations of state-space theory and published a series of seminal works, cf. [102, 34, 103] to name but a few. In order to specify the questions he considered, the fundamental terms of reachability and observability were introduced, and to study questions of stability, the earlier results on stability of differential equations found by Lyapunov [138] were utilized. The new state-space theory strongly influenced the following decades of research and was widely applied in engineering. To give just one example, the so-called "Kalman filter" [101] was of crucial importance in NASA's Apollo program. As mentioned above, feedback techniques were intended to be used in flight control. However, since the dynamic properties of flight vehicles change drastically with flight altitude and speed, ballistic missiles and supersonic flight caused new challenges in control. The existing control systems with constant gain could not be used for the entire flight envelope. For this reason, the research turned towards *adaptive control*, in particular, model reference adaptive control was under consideration, cf. the works [198, 162, 145, 115, 147, 114]. In the subsequent years, more and more techniques and control structures were developed, which achieved tracking of an external (given) trajectory, see, e.g., the articles [96, 109, 86]. This particular control objective, namely *tracking of a given reference trajectory* is the guiding topic of the present thesis and is introduced in the subsequent Section 1.1.

Before we introduce the concepts of *output reference tracking* in detail, we briefly outline the structure of the present thesis, which contains contributions to this particular topic. There are *three main aspects* under consideration. As the *first aspect*, in Chapter 2 we elaborate on the so-called *internal dynamics* of input-output control systems of the form

$$\begin{aligned} \dot{x}(t) &= f(d(t), x(t)) + g(x(t))u(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= h(x(t)), \end{aligned}$$

where $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^n; \mathbb{R}^n)$, $g \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}^{n\times m})$, $h \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^m)$; further, $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is the *output* and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is the *input*. To have a picture in mind, the *internal dynamics* are the dynamics within the system, which are not visible at the output explicitly. The internal dynamics play a crucial role in control theory, since almost all controllers (feedback as well as feedforward) require the internal dynamics to be stable. Hence, in order to verify applicability of a control scheme to a given system, a stability analysis of the internal dynamics has to be performed. To this end, the internal dynamics have to be decoupled. A typical and well-known technique to decouple the internal dynamics is to transform the system into the *Byrnes-Isidori normal form*, cf. [98]. However, finding the respective transformations requires a lot of computational effort, such as solving partial differential equations with feasibility constraints. To avoid this, in Sections 2.1.1 & 2.1.2 we present a structurally new ansatz to decouple the internal dynamics and present a representation of these, which only involves system parameters. The internal dynamics' equations are then given in terms of the system's output and its derivatives and can be derived completely algorithmically. In Section 2.2 we use the novel representation to derive a stability result for the internal dynamics. This allows, for a certain class of systems, to conclude stability of the internal dynamics in advance directly from the system parameters, without the need to decouple the internal dynamics and perform a stability analysis.

The *second aspect* under consideration concerns a certain availability assumption in feedback control of systems with relative degree larger than one. Roughly speaking, the relative degree of a system is the number the system's output has to be differentiated such that the input appears explicitly. Typically, feedback laws for systems with relative degree $r \in \mathbb{N}$ involve the first $r-1$ derivatives of the system's output. However, in general, the derivatives are not known. In this case, the derivatives can be calculated via numerical differentiation, or approximated by high-gain observers, for instance. Both of these techniques have their advantages, however, also particular disadvantages, cf. [190, 76]. To circumvent these, in Chapter 3 we elaborate on the *funnel pre-compensator* first proposed in [32]. This is a dynamical system which takes a signal, e.g., the output of a system, and gives an arbitrary good approximation of the signal as an output. Moreover, the derivative of the compensator's output is known explicitly. To utilize this property for feedback control, we show in Section 3.2 that a minimum phase system

$$y^{(r)}(t) = f(d(t), \mathbf{T}(y, \ldots, y^{(r-1)})(t)) + \Gamma u(t),$$

with arbitrary relative degree $r \in \mathbb{N}$, $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R}^m)$, a causal, bounded-input bounded-output and locally Lipschitz operator $\mathbf{T}$ introduced in Definition 1.4, and symmetric $\Gamma > 0$, in conjunction with a funnel pre-compensator, results in a minimum phase system with the same relative degree. This overall system is then amenable to funnel control, and output reference tracking can be performed with output feedback only, i.e., without involving the output's derivatives.

The *third main contribution* of the present work in Chapter 4 is the introduction of two novel funnel control laws, each of which achieves a certain control objective. In Section 4.1 we consider output reference tracking in the case that the output signal is possibly subject to measurement losses. For linear minimum phase systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned}$$

with matrices $A \in \mathbb{R}^{n\times n}$, $C^\top, B \in \mathbb{R}^{n\times m}$ and arbitrary relative degree $r \in \mathbb{N}$, we derive a funnel feedback law, which guarantees that the tracking error evolves within prescribed bounds, even if the output signal is lost in some intervals. Here, the maximal duration of signal losses and minimal time of signal availability have to satisfy certain conditions, which depend on the system parameters (in particular, these conditions strongly depend on the internal dynamics) and are given explicitly. In Section 4.2 we focus on the long-standing open problem of exact tracking in finite time via feedback control. For nonlinear systems

$$\begin{pmatrix} y_1^{(r_1)}(t) \\ \vdots \\ y_m^{(r_m)}(t) \end{pmatrix} = \begin{pmatrix} f_1(d(t), \mathbf{T}(y_1, \ldots, y_m^{(r_m-1)})(t), u(t)) \\ \vdots \\ f_m(d(t), \mathbf{T}(y_1, \ldots, y_m^{(r_m-1)})(t), u(t)) \end{pmatrix}$$

with vector relative degree $(r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$, $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m; \mathbb{R}^m)$ satisfying the high-gain property in Definition 1.10, and operator $\mathbf{T}$ as above, we present a funnel feedback law, which guarantees that the system's output y and its derivatives have predefined values at a predefined time T, i.e., $y_i^{(j)}(T) = y_{i,\text{ref}}^{(j)}(T)$, $i = 1, \ldots, m$, $j = 0, \ldots, r_i - 1$; and moreover, the tracking error $y - y_{\text{ref}}$ evolves within prescribed bounds.

Although the main content of the present work does not directly concern the *invention and design* of feedback control laws, it is directly related to, in the sense that techniques and tools are provided, which improve and extend the applicability of existing feedback control schemes. More precise, the present work mainly consists of contributions to *output reference tracking via feedback control*, which can be seen as the overarching topic. Therefore, we provide a brief introduction to this topic in the next section.

1.1 Output reference tracking and funnel control

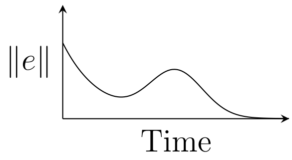
Since almost all results in the present work are derived in the context of *output reference tracking*, we give a brief introduction to this topic. Moreover, since we make extensive use of the idea of *funnel control*, we provide a brief introduction to this topic too.

To begin with, we introduce the problem of output reference tracking. To this end, we consider an input-output control system

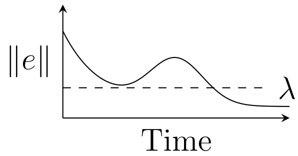
$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(0) = x^0, \\ y(t) &= h(x(t)), \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are to be specified later in the light of the respective contexts. In system (1.1), the function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is called *input* and $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is called *output*. Now, the goal of *output reference tracking* is to find an input signal u such that the output signal y follows a given reference signal y_{ref}. Here "follows" encodes the respective control objective, which in general describes the desired tracking quality in terms of the error $e := y - y_{\text{ref}}$. The control objectives emerged as those that received the most intensive treatment are

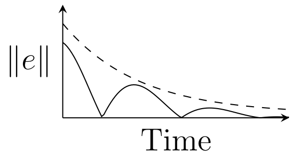
(CO.1) asymptotic tracking, that is, for long times the tracking error approaches zero, i.e., $\lim_{t\to\infty} \|e(t)\| = 0$,



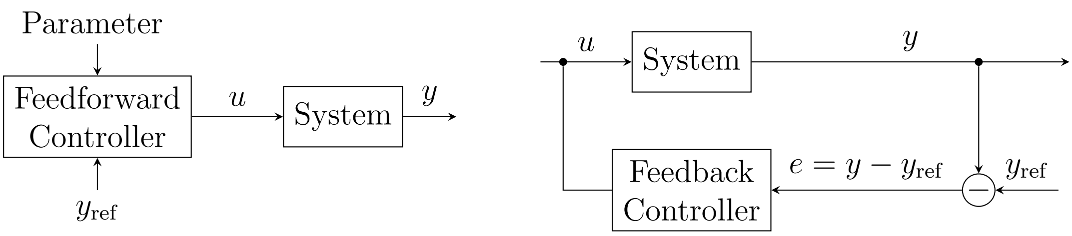
(CO.2) approximate tracking, also known as λ-tracking, where the tracking error becomes smaller than any predefined $\lambda > 0$, i.e., $\limsup_{t\to\infty} \|e(t)\| \leq \lambda$,



(CO.3) prescribed transient behaviour, where the tracking error is guaranteed to evolve within prescribed bounds.



To perform output reference tracking, in general, two main control strategies can be distinguished: *feedforward control* (“open-loop”) and *feedback control* (“closed-loop”), cf. [164, Sec. 9]. The two control strategies are schematically depicted in Figure 1.1.



(a) Feedforward control system.

(b) Feedback control system.

Figure 1.1: Schematic structure of the two control strategies: *feedforward control* and *feedback control*.

Performing output reference tracking via feedforward control means to calculate an input control $u(t)$ based on the system's equations and parameters, the initial data and the reference signal y_{ref}, such that the system's output evolves along the reference trajectory. Therefore, the control signal is based on an inversion of the underlying model - in some sense, not to be specified here. If, however, the system parameters are not known exactly, the application of a feedforward control signal leads to a tracking error, i.e., $y - y_{\text{ref}} \neq 0$. Generally, this tracking error cannot be compensated by the feedforward controller, since it does not incorporate information about the actual output $y(t)$ at time t, see, for instance [183, Sec. 1.4]. Moreover, if the system is subject to unknown disturbances, the tracking may fail. For this reason, often *two degrees of freedom* feedforward controllers are used [83], that is, a feedforward controller combined with a feedback loop, where the feedforward controller is responsible for the tracking of the reference signal and a feedback loop stabilizes the system and rejects disturbances, cf. [70, 58]. Since the present work focuses on *feedback control*, we will not go into detail about feedforward control strategies here, but we refer to the works [55, 70, 37, 71, 199, 97, 158], the survey [129] and the recent work [58] and the references therein, respectively.

In contrast to feedforward control, a *feedback controller* compares the output $y(t)$ at time t with the desired output $y_{\text{ref}}(t)$ and feeds the error $e(t) = y(t) - y_{\text{ref}}(t)$ back to the system, see Figure 1.1b. As in the present thesis we concentrate on *output feedback* control, we exclude the introductory discussion of feedback strategies such as full state feedback, which is the underlying principle in, e.g., pole placement, cf. [164, Sec. 9]. Involving the output tracking error $e(t)$ at time t, a feedback controller calculates a control signal $u(t, e)$ (the dependence of the tracking error is indicated), which affects the system in a regulating manner to achieve that the output follows the reference. Since there are almost countless different feedback control strategies, in this introductory section we outline only two of the most popular and widely used. To begin with, we briefly look at a feedback controller, which inherently relies on a model of the system being controlled, one of the most successful feedback control strategies, so-called *model predictive control*; for a detailed introduction see, e.g., [49, 73]. Such control schemes involve an underlying model, based on which an input control is calculated, which forces the output to follow the given reference and moreover, ensures that the control satisfies certain optimality conditions. One advantage of this strategy is that it makes use of knowledge about the system to reduce the control effort, however, to the prize of high computational costs of solving an optimal control problem. For detailed considerations and applications see, e.g., [146, 170, 49, 73, 2], the survey [122] and the references therein, respectively. In the present work, however, feedback control schemes are of particular interest, which *do not involve any identification of the system* being controlled, but only take the system's output as information. So, in contrast to feedforward control and model predictive control, in what follows we will concentrate on *model free* control strategies which do not involve an underlying model to compute the input control signal. At this point we skip the introduction and discussion of several different methods of designing feedback laws such as for instance sliding mode, cf. [189, 125, 59], H_∞ techniques, cf. [65, 176, 200] or backstepping, cf. [111, 112, 66]. As a prototype of feedback control, we consider a very successful and simple representative of model free feedback controller, the so-called *PID* controller; for an introductory overview see, e.g., [13, 40]. This controller consists of three parts, namely a *proportional* part of the current error $e(t) = y(t) - y_{\text{ref}}(t)$, an *integrated* part, and a *differentiated* part. The control law then reads

$$u(t) = K_p\, e(t) + K_i \int_0^t e(s)\,\mathrm{d}s + K_d\, \dot{e}(t),$$

where the constants $K_p, K_i, K_d \in \mathbb{R}$ are called *gains*, which can be tuned to improve the controller's performance, cf. [161]. Due to its simplicity and success in applications, this controller became the probably most popular and widely used model free feedback control law. For detailed introduction, discussions and applications we refer to [53, 182, 161, 202], the works [126, 5] and the references therein, respectively.

Leading from the general introduction on feedback control to the method of particular interest and to the specific consideration of the control objectives (CO.1) – (CO.3), we exemplary investigate the problem of output stabilization of a linear, single-input single-output system

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t), \quad x(0) = x^0, \\ y(t) &= cx(t), \end{aligned} \tag{1.2}$$

where $a, b, c > 0$. Applying the output feedback law

$$u(t) = -ky(t), \ k > 0,$$

yields the closed-loop system

$$\dot{x}(t) = (a - kcb)\, x(t).$$

If the control gain k is chosen large enough, i.e., if the control input has a *high gain* such that $kcb - a > 0$, then the system stabilizes. More precisely, for k large enough the solution x decays exponentially, and so does the output y. Hence, the goal of output stabilization is achieved, and the control objective (CO.1) is satisfied with reference $y_{\text{ref}}(t) = 0$. We emphasize that for any parameters $a, b, c > 0$ this simple output feedback control law stabilizes the system for $k > 0$ large enough. Now, what if the system parameters a, b, c are unknown? Then it cannot be guaranteed, that the gain $k > 0$ is chosen large enough to stabilize the system, no matter how large k is chosen. In this case, remedy is given by *adaption* of the gain k. The idea is to increase k monotonically, such that after some time it is large enough to stabilize the system. This results in the *adaptive high-gain* feedback law

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = y(t)^2, \ k(0) = 0. \tag{1.3}$$

It was shown in [148, 48, 139] that this adaptive high-gain feedback control law achieves stabilization of any system (1.2). Precisely, it was shown that the feedback law (1.3) applied to a system (1.2) with arbitrary parameters $a, b, c \in \mathbb{R}$, satisfying the relation $cb > 0$, yields a closed-loop initial value problem, which has a unique solution $(x, k) : \mathbb{R}_{\geq 0} \to \mathbb{R} \times \mathbb{R}$ and satisfies $\lim_{t\to\infty} x(t) = 0$. Moreover, the high-gain is finite, i.e., $\lim_{t\to\infty} k(t) = k_\infty < \infty$. Thereby, control objective (CO.1) is achieved with constant reference $y_{\text{ref}}(t) = 0$. A closer investigation of the simple adaptive scheme (1.3) reveals some disadvantages. First, although the monotonically increasing gain k converges to a finite value, it may increase unnecessary strong if perturbations are considered. Moreover, as long as the output y is unequal to zero, the gain grows. Second, this control law is essentially restricted to linear systems. And finally, to achieve tracking of an arbitrary reference and not only stabilization, requires a modification of the adaption scheme, which includes an internal model and thus, the adaption scheme is no longer model free and forfeits its simplicity, cf. [80]. These drawbacks were partially circumvented by the following modification of the adaption scheme and a respective new control law, the so-called λ-tracker

$$u(t) = -k(t)e(t), \quad \dot{k}(t) = \max\left\{|e(t)| - \lambda, 0\right\}, \ k(0) = k^0, \ \lambda > 0, \tag{1.4}$$

where $e = y - y_{\text{ref}}$ is the tracking error. First of all, the feedback law (1.4) is able to perform *tracking* of a given reference signal, not only stabilization. Second, due to the adaption scheme, the gain k only grows if the tracking error is larger than a predefined $\lambda > 0$ and constant otherwise. And finally, the application of this controller is not restricted to linear systems. Feasibility of the feedback law (1.4) for tracking problems was shown in, e.g., [89, 1]. In particular, this controller achieves the control objective (CO.2), i.e., it can be guaranteed that the tracking error converges to a compact interval $[-\lambda, \lambda]$, where $\lambda > 0$ is at the designer's choice. However, similar to (1.3), the adaption scheme (1.4) may lead to unnecessary large gain values k if the system is subject to perturbations. Since the function k is non-decreasing, large values of k remain large and hence perturbations may be amplified with large values. Whilst the adaptive λ-tracker (1.4) improves the adaptive high-gain stabilizer (1.3) in view of tracking, and the latter itself improves the constant high-gain stabilizer, none of the above feedback laws (1.3) and (1.4) may influence how the tracking error (or the stabilization process) evolves over time. In other words, the control objective (CO.3), which concerns the evolution of the tracking error over time, has not been taken into account yet. In [143] a control objective was studied, which in some sense is between (CO.2) and (CO.3). The posed control problem is "of forcing this error [the tracking error between the output and the reference signal] to be less than an (arbitrarily small) prespecified constant after an (arbitrarily short) prespecified period of time, with an (arbitrarily small) prespecified upper bound on the amount of overshoot." [143] The controller proposed there achieves this task for a class of linear time-invariant systems.

In the year 2002, Ilchmann, Ryan and Sangwin introduced in their seminal work "Tracking with prescribed transient behaviour" [91] a control scheme, the so-called *funnel control feedback law*, which in one fell swoop solved many long-standing open problems in feedback control, more precise, in output reference tracking via feedback control. The systems under consideration are of the form

$$\begin{aligned} \dot{y}(t) &= f(d(t), \mathbf{T}(y)(t), u(t)), \\ y|_{[-\sigma,0]} &= y^0 \in \mathcal{C}([-\sigma,0];\mathbb{R}^m), \end{aligned} \tag{1.5}$$

where $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m; \mathbb{R}^m)$ satisfies the so-called *high-gain property* introduced in Definition 1.10, the operator $\mathbf{T}$ belongs to a certain operator class $\mathcal{T}_\sigma^{m,q}$ introduced in Definition 1.4, and $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^p)$ is a bounded disturbance. The breakthrough result in [91] is that the proposed controller achieves output tracking of a given reference, with prescribed transient behaviour of the tracking error. The main ingredient in the new control law is the so-called *funnel function* $\varphi \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, with $\varphi(s) > 0$ for all $s > 0$ and $\liminf_{s\to\infty} \varphi(s) > 0$. It was shown in [91, Thm. 7] that, if the tracking error satisfies the initial constraint $\varphi(0)\|y(0) - y_{\text{ref}}(0)\| < 1$ for some given $y_{\text{ref}} \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, the feedback control law

$$u(t) = -k(t) \cdot \big(y(t) - y_{\text{ref}}(t)\big), \quad k(t) := \frac{1}{1 - \varphi(t)\|y(t) - y_{\text{ref}}(t)\|}, \tag{1.6}$$

achieves output tracking of a given reference signal y_{ref} with prescribed transient behaviour, so it satisfies the control objective (CO.3). The controller (1.6), with this particular gain function k, is a special instance of the control structure proposed in [91] and is particularly well suited to illustrate the fundamental idea of funnel control. By rewriting the gain function k in (1.6) as

$$k(t) = \frac{1}{\varphi(t)} \frac{1}{\psi(t) - \|y(t) - y_{\text{ref}}(t)\|}, \quad \psi(t) := 1/\varphi(t),$$

the intuition behind the funnel controller becomes obvious.

Intuition: Whenever the tracking error is close to zero, the gain is close to one and the resulting input u is close to zero. Whenever the norm of the tracking error

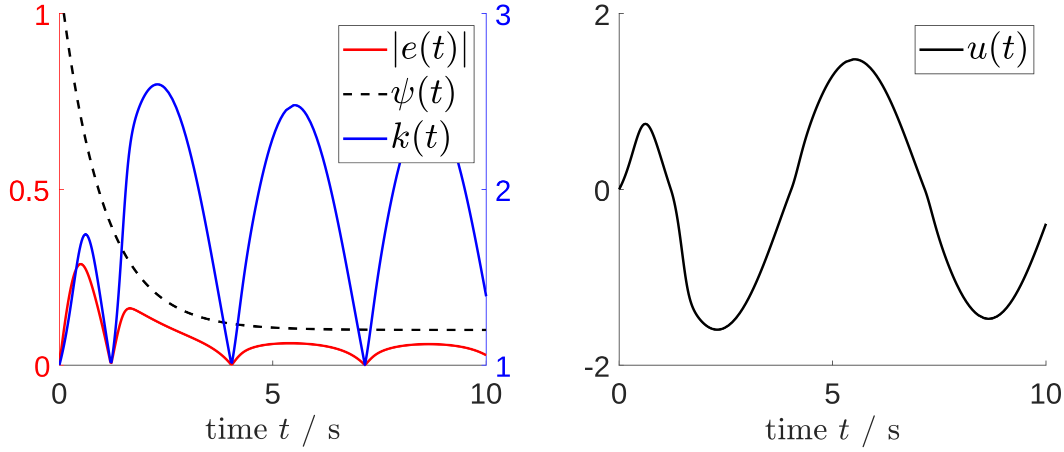
$e = y - y_{\text{ref}}$ is close to the funnel boundary ψ, the gain function k becomes large and the input $u = -k \cdot e$ strongly pushes the output towards the reference, i.e., the error is pushed away from the funnel boundary. In one word, the funnel boundary is repulsive.

Consequence: If the error is initially within the funnel boundaries, it will remain within the funnel boundaries.

We illustrate this by the following simple example. Consider

$$\dot{y}(t) = y(t) + u(t), \quad y(0) = 0,$$

with reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $t \mapsto \sin(t)$. As funnel function we choose $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $t \mapsto (e^{-t} + 0.1)^{-1}$, so the funnel boundary $\psi = 1/\varphi$ is exponentially decaying.



(a) Tracking error e, funnel boundary ψ and corresponding gain function k.

(b) Control input.

Figure 1.2: Exemplary application of the funnel control scheme (1.6).

In Figure 1.2a the absolute value of the tracking error $e = y - y_{\text{ref}}$ and the funnel boundary $\psi = 1/\varphi$, together with the corresponding gain function k are depicted. It can be seen that, if the tracking error is small, the gain function has small values close to one. However, if the tracking error approaches the funnel boundary, the gain function rapidly grows. Note that in particular, the tracking error evolves within the predefined funnel boundary, whereby control objective (CO.3) is satisfied. In Figure 1.2b the control input u as defined in (1.6) is depicted. Here, the behaviour of the tracking error causes the respective input signal, which forces the output y towards the reference signal y_{ref}.

We emphasize some properties of the funnel control strategy.

(i) It is clear from system (1.5) and the control law (1.6), that the funnel controller does not rely on system parameters; only structural assumptions such as the high-gain property of the right-hand side f and the conditions on the operator $\mathbf{T}$ have to be satisfied. So, the funnel controller is model free and hence robust with respect to model uncertainties.

(ii) *One* funnel control scheme is applicable to the entire *class of systems*, which satisfies the structural assumptions.

(iii) The controller is able to compensate any bounded disturbance within the system, so it is robust with respect to noise.

(iv) The controller has a very simple structure, comparable to the PID controller.

(v) The gain function, although time varying, is not a dynamical system, as it is the case for the high-gain stabilizer (1.3) and the λ-tracker (1.4). In particular, the gain function is not monotonically increasing and therefore does not converge to a large value, but becomes large only when needed.

(vi) The funnel controller guarantees that the tracking error evolves within the prescribed error bound $1/\varphi$. Note that the funnel function φ is not necessarily monotonically increasing, i.e., the funnel boundary is not necessarily monotonically decreasing.

Beside the advantages listed above, the control law (1.6) is restricted to systems (1.5), i.e., multi-input multi-output systems with relative degree one. In 2007 a funnel controller for systems with arbitrary relative degree $r \in \mathbb{N}$ was proposed in [93]. However, this controller involves a backstepping procedure, which complicates the controller severely. Since its inception in 2002 it took about eleven years to extend the funnel control scheme in its simple structure to systems of relative degree two, this extension was done in [78]. The new controller does not only involve the tracking error $e = y - y_{\text{ref}}$ but its derivative $\dot{e}$ as well. It was then shown in [78, Thm. 3.1] that the proposed controller achieves the control objective (CO.3) of output reference tracking with prescribed transient behaviour of the tracking error. Another five years of research finally brought the generalization of funnel control schemes to systems of arbitrary high relative degree $r \in \mathbb{N}$ in [24] in the year 2018. The proposed control scheme consists of recursively defined auxiliary error signals involving the tracking error's derivatives up to order $r - 1$, which follows the insights found in [91] and [78]. Up to this stage of the development of funnel control schemes, only the control objective (CO.3) (and hence also (CO.2)) could be achieved. The task of asymptotic tracking (CO.1) remained an open problem and was solved in 2019 for relative degree one systems in [123], where a reformulation of the error variables led to the result [123, Thm. 6] of asymptotic tracking with prescribed transient behaviour. The insights from [123] were used in the recent work [21] to achieve asymptotic tracking for nonlinear systems with arbitrary relative degree $r \in \mathbb{N}$. This controller does not involve the auxiliary error variables as in [24] but straightforwardly uses the derivatives of the tracking error $e, \dot{e}, \ldots, e^{(r-1)}$. The ability to achieve control objectives (CO.1) – (CO.3) was shown in [21, Thm. 1.9].

As it is feasible to achieve control objectives (CO.1) – (CO.3) as well as due to its strikingly simple structure and implementability, the funnel controller proved to be an appropriate tool for tracking problems in various applications such as control of industrial servo-systems [76] and underactuated multibody systems [18, 26], control of electrical circuits [33, 180], control of peak inspiratory pressure [166], adaptive cruise control [28, 29], control of infinite-dimensional systems such as a boundary controlled heat equation [171], a moving water tank [27] and defibrillation processes of the human heart [17], temperature control of chemical reactor models [94], speed control of wind turbine systems [75, 77], DC-link power flow control [180], voltage and current control of electrical circuits [33], and oxygenation control during artificial ventilation therapy [165].

An alternative approach to achieve the control objective (CO.3) is the so-called *prescribed performance controller* first introduced in [11]. Although they follow a similar motivation, funnel control and prescribed performance control are fundamentally different. The latter transforms the system into a new form with new states, where the transformation incorporates a performance function (similar to the funnel function). The control law is then constructed using the new states. This new formulation yields, that boundedness of the new states, which is achieved by the proposed controller, implies the evolution of the tracking error within the prescribed error bounds. The controller proposed in [11] is of rather high complexity, since the unknown nonlinearities from the system are approximated via a neural network. This issue of complexity was addressed in subsequent research and led to the control schemes presented in [10, 8, 56]. The prescribed performance controller was successfully applied in various contexts such as hypersonic flight vehicles [44, 43, 42], spacecraft control [99, 204], control of robotic manipulators [131, 181], transportation systems, e.g., high-speed trains [137] or tractor-trailers [60] and numerous more. For a detailed and comprehensive discussion of applications of the prescribed performance controller see the survey [41]. While both controllers, funnel control and prescribed performance controller, can claim great success in applications, both partly suffer from their feasibility assumptions. For the prescribed performance controller the entire state is required to be available; this was solved in [56], however, via the incorporation of a high-gain observer, which has to be initialized properly, cf. Chapter 3. The funnel controller, on the other hand, involves higher derivatives of the system's output. This issue is addressed in the present work in Section 3.3.

1.2 Preliminary definitions, concepts and results

In this section we provide definitions and introduce concepts which are used throughout the present work. We briefly recall some basic concepts such as relative degree, the representation of a system in Byrnes-Isidori form and the resulting equations for the internal dynamics, which are particularly used in Chapter 2. For $\mathcal{R} \subseteq \mathbb{R}^n$ open and $F : \mathcal{R} \to \mathbb{R}^n$ we consider a dynamical input-output system

$$\begin{aligned} \dot{x}(t) &= F(x(t)) + G(x(t))\, u(t), \quad x(0) = x^0 \in \mathcal{R} \subseteq \mathbb{R}^n, \\ y(t) &= h(x(t)), \end{aligned} \tag{1.7}$$

where the function $h : \mathcal{R} \to \mathbb{R}^m$ is the output measurement, $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is the output, $G : \mathcal{R} \to \mathbb{R}^{n \times m}$ is the input distribution and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is an input affecting the system in an affine form. Note that the input and the output have common dimension $m \in \mathbb{N}$.

Assumption. If not stated otherwise, throughout the present work we assume that the input and the output of an input-output system have the same dimension.

If $m = 1$, a system (1.7) is called *single-input, single-output* (SISO), if $m > 1$, it is called *multi-input, multi-output* (MIMO). In order to introduce the concept of *relative degree*, we recall the following.

Definition 1.1. Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a function which is differentiable on an open set $U \subseteq \mathbb{R}^n$, and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. Then, the *Lie derivative* of h along F is defined by

$$(L_F h)(z) := h'(z) \cdot F(z) \in \mathbb{R}^m, \quad z \in U,$$

where $h' : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is the Jacobian of h. For $z \in U$, $k \in \mathbb{N}$ and $h \in \mathcal{C}^k(U;\mathbb{R}^m)$, we successively may define $(L_F^k h)(z) := (L_F(L_F^{k-1}h))(z)$ with $(L_F^0 h)(z) = h(z)$.

For a system (1.7) we denote the columns of $G(z)$ by $g_i(z)$, $i = 1, ..., m$, $z \in \mathcal{R}$, and may further define the Lie derivative of each component of the output along the input distribution as

$$(L_G h_i)(z) := [(L_{g_1} h_i)(z), ..., (L_{g_m} h_i)(z)] \in \mathbb{R}^{1\times m}, \quad z \in \mathcal{R}.$$

With the aid of Lie derivatives, in accordance with [98, Sec. 5.1] we recall the concept of *vector relative degree* for (SISO and MIMO) systems (1.7).

Definition 1.2. A system (1.7) has vector relative degree $(r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$ at a point $x^0 \in \mathcal{R} \subseteq \mathbb{R}^n$, if there exists an open neighbourhood $U \subseteq \mathcal{R}$ of x^0, such that for all $z \in U$,

i) for all $j \in \{1, \dots, m\}$ and all $k \in \{0, ..., r_j - 2\}$ we have $(L_G L_F^k h_j)(z) = 0$,

ii) and

$$\Gamma(z) := \begin{bmatrix} \Gamma_1(z) \\ \vdots \\ \Gamma_m(z) \end{bmatrix} = \begin{bmatrix} (L_G L_F^{r_1-1} h_1)(z) \\ \vdots \\ (L_G L_F^{r_m-1} h_m)(z) \end{bmatrix} \in \mathbf{Gl}_m(\mathbb{R}),$$

where $\Gamma : U \to \mathbf{Gl}_m(\mathbb{R})$ is called *high-gain matrix*. If $r_1 = \cdots = r_m =: r \in \mathbb{N}$, then we say system (1.7) has *strict relative degree* r.

To have a picture, the strict relative degree is, roughly speaking, the number the output of a system has to be differentiated such that the input occurs explicitly. Consequently, vector relative degree is the collection of numbers of derivatives needed such that the input occurs in the respective output channel. We emphasize that, due to Newton's second law, namely that the rate of change of a body's momentum equals the applied force[1], many physical systems have relative degree two, i.e., the input (the applied force) occurs in the second derivative, i.e., the acceleration, of the measured output (position).

Remark 1.3. Since the class of linear systems $\dot{x}(t) = Ax(t) + Bu(t)$ with output $y(t) = Cx(t)$, $A \in \mathbb{R}^{n\times n}$, input distribution $B = [B_1, \dots, B_m] \in \mathbb{R}^{n\times m}$ and output measurement mapping $C = [C_1^\top, \dots, C_m^\top]^\top \in \mathbb{R}^{m\times n}$, is an important subclass of systems (1.7), we record that, in virtue of Definition 1.2, a linear system has

[1] "Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur" [151], translation "The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed." [152]

(i) strict relative degree $r \in \mathbb{N}$, if

i) for all $k \in \{0, ..., r-2\}$ we have $= CA^kB = 0$,

ii) and the high-gain matrix

$$\Gamma := CA^{r-1}B \in \mathbb{R}^{m \times m}$$

is non-singular,

(ii) vector relative degree $r = (r_1, \ldots, r_m)$, if

i) for all $i, j \in \{1, \ldots, m\}$ and all $k \in \{0, ..., r_j - 2\}$ we have $C_i A^k B_j = 0$,

ii) and the high-gain matrix

$$\Gamma := \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_m \end{bmatrix} = \begin{bmatrix} C_1 A^{r_1-1} B_1 & \cdots & C_1 A^{r_1-1} B_m \\ \vdots & \ddots & \vdots \\ C_m A^{r_m-1} B_1 & \cdots & C_m A^{r_m-1} B_m \end{bmatrix} \in \mathbb{R}^{m \times m},$$

is non-singular.

Next, we introduce a class of operators that play a significant role in the definition of classes of dynamical systems under consideration in Chapters 3 & 4.

Definition 1.4. For $\sigma \geq 0$ and $n, q \in \mathbb{N}$ we define the operator class

$$\mathcal{T}_\sigma^{n,q} := \{\, \mathbf{T} : \mathcal{C}([-\sigma, \infty); \mathbb{R}^n) \to \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^q) \mid \mathbf{T} \text{ satisfies (T.1) } - \text{ (T.3)} \,\},$$

where

(T.1) $\mathbf{T}$ maps bounded trajectories to bounded trajectories, i.e., for all $c_1 > 0$ there exists $c_2 > 0$ such that for all $\xi \in \mathcal{C}([-\sigma, \infty); \mathbb{R}^n)$,

$$\sup_{t \in [-\sigma, \infty)} \|\xi(t)\| \leq c_1 \;\Rightarrow\; \sup_{t \in [0, \infty)} \|\mathbf{T}(\xi)(t)\| \leq c_2,$$

(T.2) $\mathbf{T}$ is causal, i.e., for all $t \geq 0$ and all $\zeta, \xi \in \mathcal{C}([-\sigma, \infty); \mathbb{R}^n)$,

$$\zeta|_{[-\sigma,t)} = \xi|_{[-\sigma,t)} \;\Rightarrow\; \mathbf{T}(\zeta)|_{[0,t)} \overset{a.a.}{=} \mathbf{T}(\xi)|_{[0,t)},$$

(T.3) $\mathbf{T}$ is locally Lipschitz continuous in the following sense: for all $t \geq 0$ and all functions $\xi \in \mathcal{C}([-\sigma, t]; \mathbb{R}^n)$ there exist numbers $\Delta, \delta, c > 0$ such that for all $\zeta_1, \zeta_2 \in \mathcal{C}([-\sigma, \infty); \mathbb{R}^n)$ with $\zeta_1|_{[-\sigma,t]} = \xi$, $\zeta_2|_{[-\sigma,t]} = \xi$ and $\|\zeta_1(s) - \xi(t)\| < \delta$, $\|\zeta_2(s) - \xi(t)\| < \delta$ for all $s \in [t, t+\Delta]$, we have

$$\operatorname{ess\,sup}_{s \in [t, t+\Delta]} \|\mathbf{T}(\zeta_1)(s) - \mathbf{T}(\zeta_2)(s)\| \leq c \, \sup_{\mathrm{s} \in [\mathrm{t}, \mathrm{t}+\Delta]} \|\zeta_1(\mathrm{s}) - \zeta_2(\mathrm{s})\|.$$

Since the definition of the operator class $\mathcal{T}_\sigma^{n,q}$ is rather technical, a commentary seems appropriate. Property (T.1) encodes a stability condition on the internal dynamics of a given system, (T.2) is a causality assumption, which typically arises in physically motivated contexts and is quite naturally satisfied there; this property is important when establishing well-posedness of solutions of respective dynamical systems. The technical property (T.3) comes into play establishing the existence of a solution of closed-loop initial value problems under feedback control.

Remark 1.5. In the context of physically motivated dynamical systems encompassed by the systems under consideration later in Chapters 3 & 4, it is worthwhile to have a closer look on what the operators belonging to $\mathcal{T}_\sigma^{n,q}$ may embrace. We collect a selection of important physical phenomena, which can be modelled by operators belonging to $\mathcal{T}_\sigma^{n,q}$.

(i) *Hysteresis* occurs, if the state of a system depends on the system's history. For *relay hysteresis*, exemplary consider a magnet in a given magnetic field. Then, the magnetic moment of the magnet depends on how the applied field changed in the past. In physical and chemical systems, relay hysteresis effects are often related and associated with irreversible thermodynamic changes and deformations. Moreover, such hysteresis occurs in many artificial systems, such as in Schmitt triggers, which are used in signal processes, for instance to convert analogous signals to digital signals; or heating devices which include thermostats. Formally, let $r_1 < r_2$, and let $\alpha_1 : [-r_1, \infty)$, $\alpha_2 : (-\infty, r_2]$ with $\alpha_1(r_1) = \alpha_2(r_1)$ and $\alpha_1(r_2) = \alpha_2(r_2)$ be globally Lipschitz continuous. Feeding in the continuous input $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ to the hysteresis element, the output $w : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is such that $(v(t), w(t)) \in \operatorname{graph}(\alpha_1) \cup \operatorname{graph}(\alpha_2)$ for all $t \in \mathbb{R}_{\geq 0}$; either $w(t) = \alpha_1(v(t))$ or $w(t) = \alpha_2(v(t))$, depending on the the history of v, namely if the threshold r_1 or r_2 was last attained by v. This situation is schematically depicted in Figure 1.3.

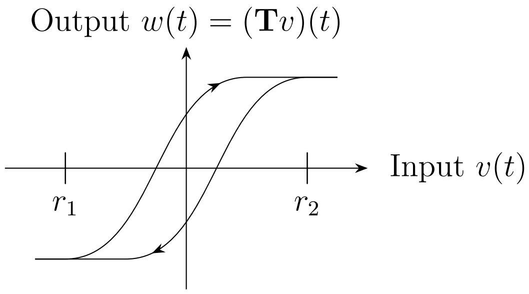


Figure 1.3: Schematic input-output relation of a relay hysteresis element.

A proper initialization then yields that for every input $v \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ the hysteresis element has a unique output $w = \mathbf{T}v \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$, where the operator $\mathbf{T}$ belongs to the class $\mathcal{T}_0^{1,1}$.

A frequently occurring phenomenon in mechanical systems is *backlash hysteresis*, which describes the play between two parts of a solid link. Consider a link consisting of two solid parts P_1 and P_2. For some $b \geq 0$, the displacement of each part at a time instant $t \in \mathbb{R}_{\geq 0}$ is given by $v(t)$ and $w(t)$, respectively, where $|v(t) - w(t)| \leq b$ for all $t \in \mathbb{R}_{\geq 0}$ and $w(0) := v(0) + p$ for some $p \in [-b, b]$. The situation is depicted in Figure 1.4.

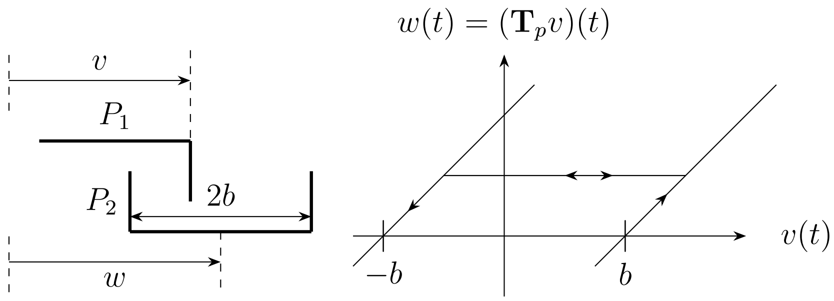


Figure 1.4: Structure of a mechanical backlash hysteresis element and schematic input-output relation.

Now, the position w of part P_2 is constant as long as the displacement v of part P_1 is within the inner part of P_2. Then, for a continuous $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ we have for P_2 that $\dot{w}(t) = 0$ whenever $|v(t) - w(t)| < b$. With this, we may introduce a family of *backlash hysteresis* operators $\mathbf{T}_p$ depending on the parameter p, such that $w = \mathbf{T}_p v$ for a given $v \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$, thus $\mathbf{T}_p \in \mathcal{T}_0^{1,1}$. Further hysteresis effects, which can be modelled by operators belonging to $\mathcal{T}_\sigma^{n,q}$ are *generalized backlash* as introduced in [113], *elastic plastic hysteresis* and general hysteresis effects modelled by so-called *Preisach* operators. For detailed overviews discussing the mathematical properties of the operators briefly discussed above, see, e.g., [113, 134, 90].

(ii) *Delay* is an effect which, due to the finite propagation of information, occurs actually in any real system; see Figure 1.5.

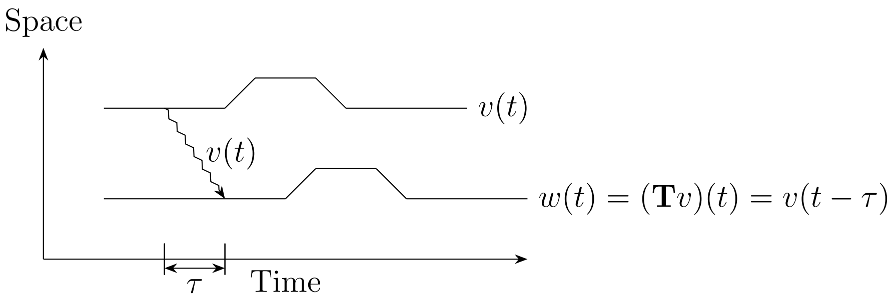


Figure 1.5: Schematic structure of information delay.

In virtue of [173, 90, 21] consider for $k = 0, \ldots, K$ possibly nonlinear functions $\Delta_k : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^q$ such that $\Delta_k(\cdot, z)$ is measurable for each $z \in \mathbb{R}^m$, and for every compact $C \subset \mathbb{R}^m$ there exists $c > 0$ such that

$$\text{for a.a. } t \in \mathbb{R}\ \forall\, z_1, z_2 \in C\ :\ \|\Delta_k(t, z_1) - \Delta_k(t, z_2)\| \leq c\|z_1 - z_2\|.$$

For $k = 0, \ldots, K$ let $\tau_k > 0$ and define $\tau := \max_{k \in \{0,\ldots,K\}} \tau_k$. Then, for $v \in \mathcal{C}([-\tau, \infty), \mathbb{R}^m)$ we may define the following operator,

$$(\mathbf{T}v)(t) := \int_{-\tau_0}^{0} \Delta_0(s, v(t+s))\,\mathrm{d}s + \sum_{k=1}^{K} \Delta_k(t, v(t - \tau_k)),\ t \geq 0,$$

which models point and distributed delays and belongs to the class $\mathcal{T}_\tau^{m,q}$.

(iii) In some contexts, in particular when dealing with partial differential equations, the respective spaces are not finite dimensional any more. For instance in the articles [17, 27] situations are under consideration, where the operator $\mathbf{T}$ represents *infinite dimensional internal dynamics*.

Next, we introduce a class of functions which form the basis to define a concept of a solution of ordinary differential equations. The definition follows [88, App. B].

Definition 1.6. For $n \in \mathbb{N}$ let $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ be a non-empty, connected and relatively open set, i.e., $\mathcal{D}$ is a domain. For $q \in \mathbb{N}$ a function $f : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^n$ is called *Carathéodory function*, if for every $[t_0, t_1] \times K \subset \mathcal{D}$, where $K \subset \mathbb{R}^n$ is a compact set, and every compact set $Q \subset \mathbb{R}^n$

(C.1) $f(t, \cdot, \cdot) : K \times Q \to \mathbb{R}^n$ is continuous for almost all $t \in [t_0, t_1]$,

(C.2) $f(\cdot, x, \mathbf{q}) : [t_0, t_1] \to \mathbb{R}^n$ is measurable for each fixed $(x, \mathbf{q}) \in K \times Q$,

(C.3) there exists an integrable function $\alpha : [t_0, t_1] \to \mathbb{R}_{\geq 0}$ such that for almost all $t \in \mathbb{R}_{\geq 0}$ and all $(x, \mathbf{q}) \in K \times Q$ the estimation $\|f(t, x, \mathbf{q})\| \leq \alpha(t)$ is valid.

Next, we introduce the concept of a *solution* to an initial value problem. For $n, q \in \mathbb{N}$, $\sigma > 0$ and $t_0 \in \mathbb{R}_{\geq 0}$ let $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ be a domain, $F : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^n$ be a *Carathéodory function* as in Definition 1.6, and $\mathbf{T} \in \mathcal{T}_\sigma^{n,q}$ be an operator as in Definition 1.4. We consider the initial value problem

$$\begin{aligned} \dot{x}(t) &= F\big(t, x(t), \mathbf{T}(x)(t)\big), \\ x|_{[-\sigma, t_0]} &= x^0 \in \mathcal{C}([-\sigma, t_0]; \mathbb{R}^n),\ (t_0, x^0(t_0)) \in \mathcal{D}. \end{aligned} \tag{1.8}$$

Then, in virtue of [91, Sec. 5], we have the following definition.

Definition 1.7. A *solution* of (1.8) is a function $x \in \mathcal{C}(I; \mathbb{R}^n)$ on an interval of the form $I = [-\sigma, \bar{\omega}]$, for $t_0 < \bar{\omega} < \infty$, or $I = [-\sigma, \omega)$, for $\omega \leq \infty$, such that, for $J := I \setminus [-\sigma, t_0]$, we have $x|_{[-\sigma, t_0]} = x^0$, $x|_J$ is absolutely continuous with $(t, x(t)) \in \mathcal{D}$ for all $t \in J$, and

$$\forall\, t \in J \;:\; x(t) = x(t_0) + \int_{t_0}^{t} F\big(s, x(s), \mathbf{T}(x)(s)\big) \mathrm{d}s. \tag{1.9}$$

A solution is maximal if it has no proper right extension that is also a solution.

Remark 1.8. As pointed out in [84, p. 234] the interpretation of the “local Lipschitz property” (T.3) in Definition 1.4 and so the interpretation of a solution $x \in \mathcal{C}(I; \mathbb{R}^n)$ of (1.8), where $I = [-\sigma, \tilde{\omega}]$ or $I = [-\sigma, \omega)$ for finite ω, requires special attention since the operator $\mathbf{T}$ has $\mathcal{C}([-\sigma, \infty); \mathbb{R}^n)$ as its domain. Let $J := I \setminus [-\sigma, 0)$ and $x \in \mathcal{C}(I; \mathbb{R}^n)$. For each $\tau \in J$ define the function $x_\tau \in \mathcal{C}([-\sigma, \infty); \mathbb{R}^n)$ by

$$x_\tau(t) = \begin{cases} x(t), & t \in [-\sigma, \tau], \\ x(\tau), & t > \tau. \end{cases}$$

Now, we associate with the operator $\mathbf{T} \in \mathcal{T}_\sigma^{n,q}$ its “localisation”, namely the operator $\tilde{\mathbf{T}} : \mathcal{C}(I; \mathbb{R}^n) \to \mathcal{L}^\infty_{\mathrm{loc}}(J; \mathbb{R}^n)$ defined as

$$\forall\, \tau \in J \;:\; \tilde{\mathbf{T}}(x)|_{[0,\tau]} = \mathbf{T}(x_\tau)|_{[0,\tau]}.$$

Note, that by Definition 1.4 (T.2) the operator $\tilde{\mathbf{T}}$ is well defined. Together, a replacement of $\mathbf{T}$ by $\tilde{\mathbf{T}}$ in (1.9) yields the correct interpretation of a solution; however, throughout the present work we will notationally not distinguish between $\mathbf{T} \in \mathcal{T}_\sigma^{n,q}$ and $\tilde{\mathbf{T}}$.

Since we extensively use the result [91, Thm. 5] concerning the existence of solutions of ordinary differential equations, for reasons of more pleasant readability of the present thesis, we repeat it here.

Proposition 1.9. *[91, Theorem 5]* Consider the initial value problem (1.8) with $(t_0, x^0(t_0)) \in \mathcal{D}$. Then,

(i) the initial value problem (1.8) has a solution $x : [-\sigma, \omega) \to \mathbb{R}^n$, which satisfies $\operatorname{graph}(x|_{[t_0,\omega)}) := \{ (t, x(t)) \mid t \in [t_0, \omega) \} \subset \mathcal{D}$,

(ii) every solution can be extended to a maximal solution,

(iii) if F is locally essentially bounded and $x \in \mathcal{C}([-\sigma, \omega); \mathbb{R}^n)$ is a maximal solution, then the closure of $\operatorname{graph}(x|_{[t_0,\omega)})$ is not a compact subset of $\mathcal{D}$.

We omit the proof here but refer to [91, p. 11].

In order to characterize the class of admissible nonlinearities f appearing in the systems under consideration later, we recall the definition of the "high-gain property" from [21, Sec. 1.2].

Definition 1.10. For $k, q, n \in \mathbb{N}$ a function $f \in \mathcal{C}(\mathbb{R}^k \times \mathbb{R}^q \times \mathbb{R}^n; \mathbb{R}^n)$ satisfies the *high-gain property*, if there exists $v_* \in (0,1)$ such that, for every compact set $K \subset \mathbb{R}^k$ and compact set $Q \subset \mathbb{R}^q$ the continuous function

$$\begin{aligned} &\chi : \mathbb{R} \to \mathbb{R}, \\ &\quad s \mapsto \min \{ \langle v, f(\delta, z, -sv) \rangle \mid (\delta, z) \in K \times Q, \; v \in \mathbb{R}^n, v_* \le \|v\| \le 1 \} \end{aligned}$$

is such that $\sup_{s \in \mathbb{R}} \chi(s) = \infty$.

For a detailed discussion and equivalent conditions of the high-gain property we refer to [21, Rem. 1.3 & 1.4]; we only record the equivalence of f having the high-gain property and the existence of $v_* \in (0,1)$ such that for every compact $K \subset \mathbb{R}^k$, $Q \subset \mathbb{R}^q$ at least one of the following two statements is true

$$\sup_{s>0} \chi(s) = \infty \quad \text{or} \quad \sup_{s<0} \chi(s) = \infty.$$

Later, in Sections 3.2.1 & 4.2.1 we will recall the findings from [21, Sec. 2.1.3] to integrate the high-gain property into the respective context. We complete this introductory section stating the following result about invariant sets of dynamical systems, which will be used frequently throughout the present thesis as it plays a significant role in proving the main results of the present work.

Lemma 1.11. If for $x \in \mathcal{C}^1([\tau, T); \mathbb{R}^m)$ with $\tau < T \in \mathbb{R}$, $m \in \mathbb{N}$, there exists $M \ge \|x(\tau)\| \ge 0$ such that

$$\forall t \in [\tau, T) : \Big(\|x(t)\| \ge M \;\Rightarrow\; \tfrac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2 \le 0 \Big), \tag{1.10}$$

then

$$\forall \; t \in [\tau, T) : \|x(t)\| \le M. \tag{1.11}$$

Proof. Seeking a contradiction we assume that there exists $t_1 \in (\tau, T)$ such that $\|x(t_1)\| > M$. Then, by continuity, there exists $t_0 := \max\{\, t \in [\tau, t_1) \mid \|x(t)\| = M \,\}$, and hence we have $\|x(t)\| \geq M$ for all $t \in [t_0, t_1]$. Then, using (1.10) we obtain

$$\|x(t_1)\|^2 - \|x(t_0)\|^2 = \int_{t_0}^{t_1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{1}{2}\|x(t)\|^2\right) \mathrm{d}t \leq 0,$$

and we find the contradiction

$$M^2 < \|x(t_1)\|^2 \leq \|x(t_0)\|^2 = M^2.$$

Therefore, (1.11) holds for all $t \in [\tau, T)$. □

1.3 Previously published results

As summarized in the following table, parts of the present thesis have been submitted for publication or already have been published. The author of the present thesis made significant contributions to the respective articles.

Section	Contained in	New in the present thesis
Section 2.1	Lanza [118, 120] for the special case $r = 2$	the general case for $r \in \mathbb{N}$, Lemma 2.3, Lemma 2.4, Remark 2.16,
Section 2.1.2	Berger, Drücker, Lanza, Reis and Seifried [18]	Remark 2.23,
Section 2.2	Lanza [118, 120]	Remark 2.31,
Sections 3.2 & 3.3	Lanza [119]	Remark 3.1, Lemma 3.10, *Step four* in the proof of Theorem 3.9 is new as regards properties (T.2) and (T.3), Remark 3.15,
Section 4.1	Berger and Lanza [22]	incorporation of bounded disturbances, Remark 4.2, Corollary 4.12
Section 4.2	Lanza [117] for strict relative degree	the consideration of systems with vector relative degree, Remark 4.16 (ii) & (iv), Remark 4.17, Remark 4.23

2 Internal dynamics

In this chapter we perform a structural analysis of multibody systems; we elaborate on the representation of the so-called *internal dynamics*. Roughly speaking, the internal dynamics of a dynamical system are dynamics within the system, which are not visible at the output explicitly. In particular, for a fixed output possibly there remains motion within the system. Although the internal dynamics do not explicitly affect the output, the system's state is affected by the internal dynamics. This fact is illustrated in Example 2.19. To have a picture in mind, for a linear system $\dot{x} = Ax + Bu$ with linear output $y = Cx$, the internal dynamics are the state's motions within the kernel of C.

In feedforward control as well as in feedback control the internal dynamics play an important role and in particular its stability properties are of high relevance, where *stability* refers to a bounded-input bounded-output, respectively to a bounded-input bounded-state property in most cases. Although there is progress in designing control schemes for systems with unstable internal dynamics, see for instance [16, 23, 18, 178], most control strategies rely on the stability of the internal dynamics, by way of example see [48, 46, 108, 143, 91, 179, 25, 30, 26] and also the survey [87]. Therefore, stability analysis of the internal dynamics of a given system is an important step before applying a certain control scheme to the system. To do so, the internal dynamics have to be decoupled such that a stability analysis can be performed. In [98] a coordinate transformation is introduced such that the system's equations can be represented in terms of *external* and *internal variables*. This idea of representation leads to a normal form for general input-output systems, the so-called *Byrnes-Isidori form*. However, the derivation and explicit computation of this normal form in general requires a lot of effort since typically a set of partial differential equations has to be solved, where additional constraints have to be satisfied; this is illustrated in Example 2.2.

In the present chapter we develop a structurally new ansatz to decouple the internal dynamics and present a set of variables for the internal state. Further, we derive a (local) coordinate transformation, the computation of which, compared to the Byrnes-Isidori normal form, is much less involved and requires less computational effort. In this context, a local coordinate transformation defines a local diffeomorphism between regions of the state space and the space of external and internal states. We will derive a set of feasible coordinates for the internal state in terms of the system's output and the internal variable. This novel representation yields two main results. First, the coordinate transforming diffeomorphism is given explicitly, and moreover, its inverse is given explicitly as well. This allows to reconstruct the system's entire state from the system's output and the internal state.

Second, the internal dynamics are completely decoupled and thus, for instance, are amenable to stability analysis. The new set of variables for the internal state allows to compute the internal dynamics completely algorithmically, and in particular, without the need to compute the Byrnes-Isidori form explicitly.

Although we aim to find a representation of the internal dynamics, which avoids the explicit computation of the Byrnes-Isidori form, we will exploit its structure. To this end, we recall [98, Prop. 5.1.2] which states that for a system (1.7) with well defined (vector) relative degree there exists a state space transformation which diffeomorphically transforms the system into a normal form.

Proposition 2.1. If a system (1.7) has vector relative degree $(r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$ at a point $x^0 \in \mathbb{R}^n$, then, for an open neighbourhood $U \subseteq \mathbb{R}^n$ of x^0, there exists a local diffeomorphism $\Phi : U \to W \subseteq \mathbb{R}^n$, W open, such that

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \Phi(x(t)), \tag{2.1}$$

with $\xi(t) = (\xi_{j,i}(t))_{j=1,\dots,m;i=1,\dots,r_j} \in \mathbb{R}^{\hat{r}}$, $\eta(t) \in \mathbb{R}^{n-\hat{r}}$, where $\hat{r} = \sum_{j=1}^{m} r_j$, transforms system (1.7) (nonlinearly) into *Byrnes-Isidori normal form*

$$\begin{aligned}
y_j(t) &= \xi_{j,1}(t), \\
\dot{\xi}_{j,1}(t) &= \xi_{j,2}(t), \\
&\ \ \vdots \\
\dot{\xi}_{j,r_j-1}(t) &= \xi_{j,r_j}(t), \\
\dot{\xi}_{j,r_j}(t) &= (L_F^{r_j} h_j)\big(\Phi^{-1}(\xi(t),\eta(t))\big) + \Gamma_i\big(\Phi^{-1}(\xi(t),\eta(t))\big)\, u(t), \ \ j = 1, \dots, m, \\
\dot{\eta}(t) &= q(\xi(t),\eta(t)) + p(\xi(t),\eta(t))\, u(t).
\end{aligned} \tag{2.2}$$

We refer to ξ as the *external state* and to η as the *internal state*. Consequently, the last equation in (2.2) represents the *internal dynamics* of system (1.7), which may involve the system's input u exerting force to the internal dynamics.

As a special instance, we consider linear systems of the form

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\
y(t) &= Cx(t),
\end{aligned} \tag{2.3}$$

with $A \in \mathbb{R}^{n\times n}$ and $C^\top, B \in \mathbb{R}^{n\times m}$. We assume that this system has strict relative degree $r \in \mathbb{N}$. Note that in this particular case, the relative degree is global, i.e., for all $k < r-1$ we have $CA^kB = 0$ and $CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$ for all $x \in \mathbb{R}^n$. Following the investigations in [95], if system (2.3) has strict relative degree $r \in \mathbb{N}$, then, with

$$\begin{aligned}
\mathcal{B} &:= \begin{bmatrix} B & AB & \dots & A^{r-1}B \end{bmatrix} \in \mathbb{R}^{n\times rm}, \\
\mathcal{C} &:= \begin{bmatrix} C^\top & (CA)^\top & \dots & (CA^{r-1})^\top \end{bmatrix}^\top \in \mathbb{R}^{rm\times n}, \\
V &\in \mathbb{R}^{n\times(n-rm)} \text{ s.t. } \operatorname{im} V = \ker \mathcal{C}, \\
N &:= V^\dagger (I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}) \in \mathbb{R}^{(n-rm)\times n}, \\
U &:= \begin{bmatrix} \mathcal{C} \\ N \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}),
\end{aligned} \tag{2.4}$$

the change of coordinates

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \\ \eta \end{pmatrix} = Ux = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(r-1)} \\ \eta \end{pmatrix}$$

transforms system (2.3) into *Byrnes-Isidori form*

$$\begin{aligned} \dot{\xi}_i(t) &= \xi_{i+1}(t), & \xi_i(0) = \xi_i^0 \in \mathbb{R}^m, \\ \dot{\xi}_r(t) &= \sum_{j=1}^{r} R_j \xi_j(t) + S\eta(t) + \Gamma u(t), & \xi_r(0) = \xi_r^0 \in \mathbb{R}^m, \\ \dot{\eta}(t) &= Q\eta(t) + P\xi_1(t), & \eta(0) = \eta^0 \in \mathbb{R}^{n-rm}, \end{aligned}$$

with output

$$y(t) = \xi_1(t),$$

where

$$\begin{aligned} \begin{bmatrix} R_1 & \dots & R_r & S \end{bmatrix} &:= CA^r U^{-1} \in \mathbb{R}^{m \times (rm+(n-rm))}, \\ P := NA^r B\Gamma^{-1} \in \mathbb{R}^{(n-rm)\times m}&, \ Q := NAV \in \mathbb{R}^{(n-rm)\times(n-rm)}. \end{aligned} \tag{2.5}$$

If $rm < n$, system (2.3) has nontrivial internal dynamics. Moreover, we emphasize, that this particular transformation prevents the occurrence of the input variable u in the internal dynamics, i.e., the internal dynamics are not directly excited by the input signal.

We return to general nonlinear systems (1.7). With the aid of Lie derivatives introduced in Definition 1.1, the diffeomorphism Φ in (2.1) can be represented as

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Phi(x) = \begin{pmatrix} h_1(x) \\ (L_F h_1)(x) \\ \vdots \\ (L_F^{r_1-1} h_1)(x) \\ h_2(x) \\ \vdots \\ (L_F^{r_m-1} h_m)(x) \\ \tilde{\phi}_1(x) \\ \vdots \\ \tilde{\phi}_{n-\hat{r}}(x) \end{pmatrix}, \ x \in U \subseteq \mathbb{R}^n, \tag{2.6}$$

where $\tilde{\phi}_i : U \to \mathbb{R}$, $i = 1, ..., n-\hat{r}$, $\hat{r} = \sum_{j=1}^m r_i$, are such that $\Phi'(x)$ is invertible for all $x \in U$. As in the transformation of linear systems above, we seek for a transformation such that the internal dynamics are not excited by the system's input u. In view of (1.7) and (2.6) this means to find functions $\tilde{\phi}_i$ such that for all $x \in U$

$$\forall\, i = 1, \dots, n-\hat{r} \ : \ \left(L_G \tilde{\phi}_i\right)(x) = \left(\nabla \tilde{\phi}_i(x)\right) \cdot G(x) = 0 \text{ and } \Phi'(x) \in \mathbf{Gl}_n(\mathbb{R}), \tag{2.7}$$

which in fact requires to solve a set of $n - \hat{r}$ nonlinear partial differential equations such that the constraint $\Phi'(x) \in \mathbf{Gl}_n(\mathbb{R})$, $x \in U$, is satisfied. While the result [98, Prop. 5.1.2] yields conditions on the input distribution G such that such functions exist, no strategy is given, how these functions can be constructed. In the present chapter we will focus on the latter aspect.

Example 2.2. We illustrate the aforesaid. Consider a robotic manipulator system as introduced in [179, Sec. 4.2]. The rotational manipulator arm consists of two links with homogeneous mass distribution and mass m with length l. A passive joint consisting of a linear spring-damper combination couples the two links to each other. Passive in this context means, that there is no input force at this point. We stress, that the linearity of the passive joint does not result in linear equations of motion, as we will see in (2.8). As an output, the position of point S on the second link is measured. Using a body fixed coordinate system, the point S on the passive link is described by $0 \le s \le l$. The situation is depicted in Figure 2.1. We

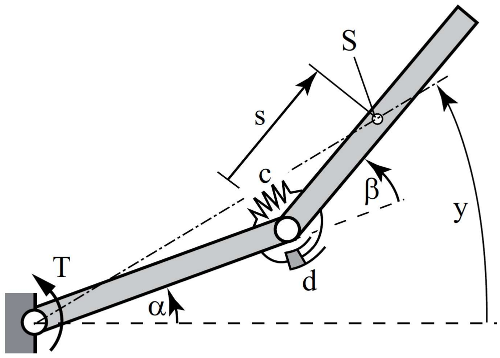


Figure 2.1: Rotational manipulator arm consisting of two links and a passive joint. The figure is taken from from [179].

present the manipulator's equations of motion. As we will see below, it is reasonable to consider the dynamics of the manipulator for $\beta \in \mathcal{B} := \left\{ \beta \in \mathbb{R} \,\middle|\, \cos(\beta) \neq \frac{2l}{3s} \right\}$. Note, that for $s < \frac{2}{3}l$ we have $\mathcal{B} = \mathbb{R}$. Henceforth, we assume $\beta \in \mathcal{B}$ and perform the computations. We define $U_\beta := \mathbb{R} \times \mathcal{B} \times \mathbb{R}^2$, set $x := (x_1, \dots, x_4)^\top$ and

$$
\begin{aligned}
M &: \mathcal{B} \to \mathbb{R}^{2\times 2}, & x_2 &\mapsto l^2 m \begin{bmatrix} \frac{5}{3} + \cos(x_2) & \frac{1}{3} + \frac{1}{2}\cos(x_2) \\ \frac{1}{3} + \frac{1}{2}\cos(x_2) & \frac{1}{3} \end{bmatrix}, \\
f_1 &: U_\beta \to \mathbb{R}, & x &\mapsto \tfrac{1}{2} l^2 m x_4 (2x_3 + x_4) \sin(x_2), \\
f_2 &: U_\beta \to \mathbb{R}, & x &\mapsto -c x_2 - d x_4 - \frac{1}{2} l^2 m x_3^2 \sin(x_2),
\end{aligned}
$$

where $M : \mathcal{B} \to \mathbb{R}^{2\times 2}$ is the mass matrix and (f_1, f_2) encode the internal forces acting on the manipulator arm. With this we obtain the equations of motion

$$
M(\beta(t)) \begin{pmatrix} \ddot{\alpha}(t) \\ \ddot{\beta}(t) \end{pmatrix} = \begin{pmatrix} f_1(\alpha(t), \beta(t), \dot{\alpha}(t), \dot{\beta}(t)) \\ f_2(\alpha(t), \beta(t), \dot{\alpha}(t), \dot{\beta}(t)) \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t). \tag{2.8}
$$

For later use, we compute the inverse of the mass matrix

$$
M(\beta)^{-1} = \frac{36(l^2 m)^{-2}}{16 - 9\cos(\beta)^2} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} - \frac{1}{2}\cos(\beta) \\ -\frac{1}{3} - \frac{1}{2}\cos(\beta) & \frac{5}{3} + \cos(\beta) \end{bmatrix}.
$$

Following the considerations in [179], we take the auxiliary angle

$$y(t) = h(\alpha(t), \beta(t)) = \alpha(t) + \frac{s}{s+l}\beta(t)$$

as output, which approximates the position S on the passive link for small angles α and β. In order to represent the system (2.8) in the form of (1.7) we define for $x = (x_1, \ldots, x_4)^\top$ the functions

$$\begin{aligned}
F &: U_\beta \to \mathbb{R}^4, && x \mapsto \operatorname{diag}(I_2, M(x_2)^{-1}) \begin{bmatrix} x_3 & x_4 & f_1(x) & f_2(x) \end{bmatrix}^\top, \\
G &: U_\beta \to \mathbb{R}^{4\times 1}, && x \mapsto \operatorname{diag}(I_2, M(x_2)^{-1}) \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top, \\
\tilde{h} &: U_\beta \to \mathbb{R}, && x \mapsto \begin{bmatrix} 1 & \frac{s}{s+l} & 0 & 0 \end{bmatrix} x,
\end{aligned}$$

and obtain

$$\begin{aligned}
\dot{x}(t) &= F(x(t)) + G(x(t))u(t), \\
y(t) &= \tilde{h}(x(t)).
\end{aligned}$$

Via a short calculation we obtain

$$\begin{aligned}
\Gamma &: \mathbb{R} \times \mathcal{B} \to \mathbb{R} \\
&\quad x \mapsto (L_G L_F \tilde{h})(x) = \begin{bmatrix} 1 & \frac{s}{s+l} \end{bmatrix} M(x_2)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\qquad = \frac{36(l^2 m)^{-2}}{16 - 9\cos(x_2)^2} \left(\frac{1}{3} - \frac{s}{s+l} \left(\frac{1}{3} + \frac{1}{2}\cos(x_2) \right) \right),
\end{aligned}$$

which is invertible on U_β. Since $\Gamma \neq 0$ for $x \in U_\beta$, according to Definition 1.2, system (2.8) has strict relative degree $r = 2$ on U_β. For $\beta \in \left\{ \beta \in \mathbb{R} \,\middle|\, \cos(\beta) = \frac{2l}{3s} \right\}$ we have $\Gamma = 0$, from which it is clear why to consider the dynamics for $\beta \in \mathcal{B}$ only. With G introduced above the equations (2.7) for $\tilde{\phi}_1, \tilde{\phi}_2$ read

$$\begin{aligned}
\frac{36(l^2 m)^{-1}}{16 - 9\cos(x_2)} \left(\frac{1}{3}\frac{\partial}{\partial x_3}\tilde{\phi}_1(x) - \left(\frac{1}{3} + \frac{1}{2}\cos(x_2) \right) \frac{\partial}{\partial x_4}\tilde{\phi}_1(x) \right) &= 0, \\
\frac{36(l^2 m)^{-1}}{16 - 9\cos(x_2)} \left(\frac{1}{3}\frac{\partial}{\partial x_3}\tilde{\phi}_2(x) - \left(\frac{1}{3} + \frac{1}{2}\cos(x_2) \right) \frac{\partial}{\partial x_4}\tilde{\phi}_2(x) \right) &= 0,
\end{aligned} \tag{2.9a}$$

where $\tilde{\phi}_1, \tilde{\phi}_2$ have to be such that

$$\begin{bmatrix}
1 & \frac{s}{s+l} & 0 & 0 \\
0 & 0 & 1 & \frac{s}{s+l} \\
\frac{\partial}{\partial x_1}\tilde{\phi}_1(x) & \frac{\partial}{\partial x_2}\tilde{\phi}_1(x) & \frac{\partial}{\partial x_3}\tilde{\phi}_1(x) & \frac{\partial}{\partial x_4}\tilde{\phi}_1(x) \\
\frac{\partial}{\partial x_1}\tilde{\phi}_2(x) & \frac{\partial}{\partial x_2}\tilde{\phi}_2(x) & \frac{\partial}{\partial x_3}\tilde{\phi}_2(x) & \frac{\partial}{\partial x_4}\tilde{\phi}_2(x)
\end{bmatrix} \in \mathbf{Gl}_4(\mathbb{R}), \ \ x \in U_\beta. \tag{2.9b}$$

Although (2.9a) is equivalent to

$$\begin{aligned}
\frac{\partial}{\partial x_3}\tilde{\phi}_1(x) &= \left(1 + \frac{3}{2}\cos(x_2) \right) \frac{\partial}{\partial x_4}\tilde{\phi}_1(x), \\
\frac{\partial}{\partial x_3}\tilde{\phi}_2(x) &= \left(1 + \frac{3}{2}\cos(x_2) \right) \frac{\partial}{\partial x_4}\tilde{\phi}_2(x),
\end{aligned}$$

it is far from trivial to find a solution of (2.9a) such that (2.9b) is satisfied. ◇

2.1 Representation of internal dynamics

In this section we introduce a structurally new ansatz to decouple the internal dynamics and present a set of feasible coordinates to represent the internal dynamics of k^{th}-order ODE systems in Section 2.1.1, and multibody systems with kinematic loops and holonomic as well as non-holonomic constraints in Section 2.1.2. Moreover, for a subclass of the system class under consideration, we derive sufficient conditions on the system parameters which allow to verify the stability of the internal dynamics without decoupling these. We show the existence and feasibility of functions forming the novel representation of the internal dynamics, and provide a particular choice simplifying the internal dynamic's structure. We obtain an explicit representation of the coordinate transforming (local) diffeomorphism and its inverse on the one hand, and on the other hand, the internal dynamics can be represented in terms of the system's output and the internal variable without the need to compute the Byrnes-Isidori form explicitly. The representation is such that the internal dynamics are given in the form

$$\dot{\eta}(t) = \Psi(\eta(t), y(t), \dot{y}(t), \dots, y^{(k-1)}(t)),$$

which means that the internal dynamics are completely decoupled and have the system's output and its derivatives as inputs, but no further knowledge of the system's overall (full) state is required. At the end of Section 2.1.1 we revisit Example 2.2 and use the result obtained to present the system's internal dynamics completely determined by the system's parameters, in particular, without the need to solve a set of partial differential equations with constraints. In Section 2.2 we present a stability result which allows to verify the stability of the internal dynamics in advance without the need to decouple the internal dynamics.

Before we start deriving the novel representation of the internal dynamics, we state the following lemma for later use.

Lemma 2.3. For $m, n, p \in \mathbb{N}$ and an open set $U \subseteq \mathbb{R}^n$ let $w : U \to \mathbb{R}^{m\times n}$, $v : U \to \mathbb{R}^{p\times n}$ and $s : U \to \mathbb{R}^{n\times(m+p)}$ be matrix valued functions, where $m + p \le n$. If

$$\forall\, x \in U : \operatorname{rk} w(x) = m,\ \operatorname{rk} v(x) = p,\ \operatorname{rk} s(x) = m + p \text{ and } \operatorname{rk} \begin{bmatrix} v(x) \\ w(x) \end{bmatrix} s(x) = m + p,$$

then, for any $x^0 \in U$ there exist an open neighbourhood $U^0 \subseteq U$ of x^0 and functions $\alpha \in \mathcal{C}^1(U^0; \mathbb{R}^{n-m})$ and $\beta \in \mathcal{C}(U^0; \mathbb{R}^{(n-m-p)\times n})$, such that

$$\begin{bmatrix} w(x) \\ \alpha'(x) \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}),\ \begin{bmatrix} v(x) \\ w(x) \\ \beta(x) \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}),\ \beta(x)s(x) = 0,$$

for all $x \in U^0 \subseteq U$.

Proof. We fix $x^0 \in U$ and make use of [183, Lem. 4.1.5] which states the following. Consider $W \in \mathcal{C}(U; \mathbb{R}^{\omega\times n})$ with $\operatorname{rk} W(x) = \omega$ for all $x \in U$. Then there exist an open neighbourhood $V \subseteq U$ of x^0 and $T \in \mathcal{C}(V; \mathbf{Gl}_n(\mathbb{R}))$ such that

$$\forall\, x \in V : \ W(x)T(x) = \begin{bmatrix} I_\omega & 0 \end{bmatrix}.$$

We use this to show the existence of $\alpha \in \mathcal{C}^1(U^0;\mathbb{R}^{n-m})$. Since by assumption we have $\operatorname{rk} w(x) = m$ for all $x \in U$, there exist an open neighbourhood $V \subseteq U$ of x^0 and $T = [T_1, T_2] \in \mathcal{C}(V; \mathbf{Gl}_n(\mathbb{R}))$ such that

$$\forall\, x \in V: \; w(x)\begin{bmatrix} T_1(x) & T_2(x)\end{bmatrix} = \begin{bmatrix} I_m & 0\end{bmatrix},$$

that is, $\operatorname{im} T_2(x) = \ker w(x)$ and $\operatorname{rk} T_2(x) = n - m$ for all $x \in V$. Let $E = [e_{i_1}^\top, ..., e_{i_{n-m}}^\top]^\top \in \mathbb{R}^{(n-m)\times n}$ with $e_{i_j} \in \mathbb{R}^{1\times n}$ a unit row-vector for $i_j \in \{1, ..., n\}$, $j = 1, \ldots, n-m$. Then,

$$\begin{bmatrix} w(x) \\ E \end{bmatrix}\begin{bmatrix} T_1(x) & T_2(x)\end{bmatrix} = \begin{bmatrix} I_m & 0 \\ * & ET_2(x)\end{bmatrix}.$$

Since $\operatorname{rk} T_2(x^0) = n - m$ it is possible to choose $i_1, \ldots, i_{n-m}$ such that for $x^0 \in V$ we have $ET_2(x^0) \in \mathbf{Gl}_{n-m}(\mathbb{R})$. As $T_2 \in \mathcal{C}(V;\mathbb{R}^{n\times(n-m)})$, the mapping $x \mapsto \det(ET_2(x))$ is continuous on V. Hence, there exists an open neighbourhood $\bar{V} \subseteq V$ of x^0 such that $\det(ET_2(\bar{x})) \neq 0$ for all $\bar{x} \in \bar{V}$. Thus,

$$\forall\, x \in \bar{V}: \; \operatorname{rk}\begin{bmatrix} w(x) \\ E \end{bmatrix} = n.$$

Therefore, with

$$\alpha : \bar{V} \to \mathbb{R}^{n-m}, \quad x \mapsto Ex$$

we have $\alpha \in \mathcal{C}^1(\bar{V};\mathbb{R}^{n-m})$ and $[w(x)^\top, \alpha'(x)^\top] \in \mathbf{Gl}_n(\mathbb{R})$ on $\bar{V}$, via $\alpha'(x) = E$. Next, we show the existence of a function $\beta \in \mathcal{C}(U^0;\mathbb{R}^{(n-m-p)\times n})$ such that $[v(x)^\top, w(x)^\top, \beta(x)^\top] \in \mathbf{Gl}_n(\mathbb{R})$ and $\beta(x)s(x) = 0$ for all $x \in U^0$. We observe that by assumption $\operatorname{rk} s(x) = m + p$ for all $x \in U$. Therefore, again by [183, Lem. 4.1.5], there exist an open neighbourhood $\tilde{V} \subseteq U$ of x^0 and $T = [T_1, T_2] \in \mathcal{C}(\tilde{V}; \mathbf{Gl}_n(\mathbb{R}))$ such that

$$\forall\, x \in \tilde{V}: \; s(x)^\top\begin{bmatrix} T_1(x) & T_2(x)\end{bmatrix} = \begin{bmatrix} I_{m+p} & 0\end{bmatrix},$$

i.e., $\operatorname{im} T_2(x) = \ker s(x)^\top$ and $T_2 \in \mathcal{C}(\tilde{V};\mathbb{R}^{n\times(n-m-p)})$. We observe for $\beta(x) = T_2(x)^\top$

$$\begin{bmatrix} v(x) \\ w(x) \\ \beta(x) \end{bmatrix}\begin{bmatrix} s(x) & \beta(x)^\top\end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v(x) \\ w(x)\end{bmatrix} s(x) & * \\ 0 & T_2(x)^\top T_2(x)\end{bmatrix} \in \mathbb{R}^{n\times n}, \; x \in \tilde{V},$$

which is invertible on $\tilde{V}$ since by assumption $[v(x)^\top, w(x)^\top]^\top s(x)$ is invertible on U, and $\operatorname{rk} T_2(x) = n - m - p$ for all $x \in \tilde{V}$. Therefore, $[v(x)^\top, w(x)^\top, \beta(x)^\top] \in \mathbf{Gl}_n(\mathbb{R})$ on $\tilde{V}$. Moreover, by construction of β we have $\beta(x)s(x) = 0$ for all $x \in \tilde{V}$. We set $U^0 := \bar{V} \cap \tilde{V}$ which completes the proof. □

2.1.1 Systems of ordinary differential equations

We consider systems of ordinary differential equations of order k,

$$\begin{aligned} \zeta^{(k)}(t) &= f(\zeta(t),\dots,\zeta^{(k-1)}(t)) + B(\zeta(t))\,u(t),\\ y(t) &= h(\zeta(t)), \end{aligned} \tag{2.10}$$

with initial values $\zeta^{(i)}(0) = \zeta_i^0 \in \mathbb{R}^n$, $i = 0,\dots,k-1$. In physical contexts, the function $f \in \mathcal{C}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n;\mathbb{R}^n)$ encodes internal and external forces. Further, $B \in \mathcal{C}(\mathbb{R}^n;\mathbb{R}^{n\times m})$ denotes the distribution of the input, and $h \in \mathcal{C}^{k-1}(\mathbb{R}^n;\mathbb{R}^m)$ is the (physical meaningful) measurement. The signal y is called *output*, the signal u is called *input* of system (2.10), respectively. Note that the dimension of the input and output coincide, however, we do not assume colocation, i.e., we allow explicitly for $h'(\zeta) \neq B^\top(\zeta)$.

We aim to rewrite system (2.10) in the form of (1.7). To this end, we set the variables $x_1 := \zeta, x_2 := \dot{\zeta},\dots,x_k := \zeta^{(k-1)}$ and obtain an equivalent representation of (2.10)

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & x_1(0) &= x_1^0 \in \mathbb{R}^n,\\ &\ \ \vdots & &\ \ \vdots\\ \dot{x}_{k-1}(t) &= x_k(t), & x_{k-1}(0) &= x_{k-1}^0 \in \mathbb{R}^n,\\ \dot{x}_k(t) &= f(x_1(t),\dots,x_k(t)) + B(x_1)u(t), & x_k(0) &= x_k^0 \in \mathbb{R}^n. \end{aligned} \tag{2.11}$$

For $N = nk$ we define

$$\begin{aligned} F : \mathbb{R}^n \times \ldots \times \mathbb{R}^n &\to \mathbb{R}^N,\\ (x_1,\dots,x_k) &\mapsto \left(x_2^\top,\dots,x_k^\top, f(x_1,\dots,x_{k-1})^\top\right)^\top,\\ G : \mathbb{R}^n &\to \mathbb{R}^{N\times m},\\ x_1 &\mapsto \left[0,\dots,0,B(x_1)^\top\right]^\top, \end{aligned}$$

and obtain with $x := (x_1^\top,\dots,x_k^\top)^\top \in \mathbb{R}^N$ a system which is equivalent to (2.10)

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} x_2(t)\\ \vdots\\ x_k(t)\\ f(x_1(t),\dots,x_k(t)) \end{pmatrix} + \begin{bmatrix} 0\\ \vdots\\ 0\\ B(x_1) \end{bmatrix} u(t), \quad x(0) = \begin{pmatrix} x_1^0\\ \vdots\\ x_{k-1}^0\\ x_k^0 \end{pmatrix} \in \mathbb{R}^N,\\ &= F(x(t)) + G(x_1(t))u(t),\\ y(t) &= \tilde{h}(x(t)), \end{aligned} \tag{2.12}$$

where $\tilde{h} : \mathbb{R}^N \to \mathbb{R}^m$ with $\tilde{h}(x) = h(x_1)$. Formally, for $L : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ we define recursively for $i \in \mathbb{N}$ the functions $a_0 = 0$ and

$$a_i(x_1,\dots,x_i,L) := \sum_{j=1}^{i-1} \left(\frac{\partial}{\partial x_j}\Big(a_{i-1}(x_1,\dots,x_{i-1}) + L(x_1)\,x_i\Big)\right)\cdot x_{j+1}. \tag{2.13}$$

With this we obtain the following.

Lemma 2.4. For a system (2.12) with $\tilde{h}(x) = h(x_1)$, where $h \in \mathcal{C}^{k-1}(\mathbb{R}^n; \mathbb{R}^m)$ and $h'(x_1) =: H(x_1)$, the i^{th} Lie derivative of the output $\tilde{h}$ along the vector field F is, for $i = 1, \ldots, k-1$, given by

$$\left(L_F^i \tilde{h}\right)(x) = a_i(x_1, \ldots, x_i, H) + H(x_1)\, x_{i+1}, \;\; x \in \mathbb{R}^N,\; x_i \in \mathbb{R}^n, \tag{2.14}$$

where the functions a_i are defined in (2.13).

Proof. We proof (2.14) by induction over i. For $i = 1$ and $x_1 \in \mathbb{R}^n$ we have $a_1(x_1, H) = 0$ and thus, for $x \in \mathbb{R}^N$ with $\tilde{h}(x) = h(x_1)$ and $h'(x_1) = H(x_1)$

$$\left(L_F \tilde{h}\right)(x) = \begin{bmatrix} H(x_1) & 0 & \cdots & 0 \end{bmatrix} F(x) = H(x_1)\, x_2 = a_1(x_1, H) + H(x_1)\, x_2,$$

where $x_1, x_2 \in \mathbb{R}^n$ and $x \in \mathbb{R}^N$. Let (2.14) be true for an $i \in \{1, \ldots, k-2\}$. Then, for $x \in \mathbb{R}^N$ and $x_j \in \mathbb{R}^n$, $j = 1, \ldots, i+1$, we have

$$\begin{aligned}
\left(L_F^{i+1} \tilde{h}\right)(x) &= \left(L_F L_F^i \tilde{h}\right)(x) \\
&= \Big(L_F\big(a_i(x_1, \ldots, x_i, H) + H(x_1)x_{i+1}\big)\Big)(x) \\
&= \begin{bmatrix} \left(\frac{\partial}{\partial x_1}(a_i(x_1, \ldots, x_i, H) + H(x_1)x_{i+1})\right)^\top \\ \vdots \\ \left(\frac{\partial}{\partial x_{i-1}} a_i(x_1, \ldots, x_i, H) + 0\right)^\top \\ \left(\frac{\partial}{\partial x_i} a_i(x_1, \ldots, x_i, H) + 0\right)^\top \\ H(x_1)^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix}^\top F(x) \\
&= \sum_{j=1}^{i} \frac{\partial}{\partial x_j}\Big(a_i(x_1, \ldots, x_i, H) + H(x_1)x_{i+1}\Big)\, x_{j+1} + H(x_1)\, x_{i+1} \\
&= a_{i+1}(x_1, \ldots, x_{i+1}, H) + H(x_1)\, x_{i+1},
\end{aligned}$$

which is the assertion. □

In order to derive a representation of the internal dynamics we make the following assumption.

Assumption 2.5. For $H : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ and $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ there exists some open set $U_1 \subseteq \mathbb{R}^n$ such that

$$\forall x_1 \in U_1 : \;\; \Gamma(x_1) := H(x_1)B(x_1) \in \mathbf{Gl}_m(\mathbb{R}).$$

Note that Assumption 2.5 implies $\operatorname{rk} H(x_1) = \operatorname{rk} B(x_1) = m$ for all $x_1 \in U_1$.

Lemma 2.6. Consider system (2.12) where $\tilde{h}(x) = h(x_1)$ for $x \in \mathbb{R}^N$, $x_1 \in \mathbb{R}^n$, and let Assumption 2.5 be satisfied for $B(z)$ and $H(z) = h'(z)$, $z \in U_1 \subseteq \mathbb{R}^n$, U_1 open as in Assumption 2.5. Then, system (2.12) has relative degree $r = k$ on $U := U_1 \times \mathbb{R}^{(k-1)n}$.

Proof. Using the findings of Lemma 2.4 we may compute for $x_1 \in U_1$, $x_i \in \mathbb{R}^n$ $i = 2, \dots, k-2$, and $x \in U_1 \times \mathbb{R}^{(k-1)n}$

$$
\begin{aligned}
(L_G L_F^i \tilde{h})(x) &= \big[\underbrace{* \;\; \cdots \;\; *}_{i-\text{times}} \;\; H(x_1) \;\; \underbrace{0 \;\; \cdots \;\; 0}_{k-i-1}\big] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B(x_1) \end{bmatrix} \\
&= 0_{m\times m}, \quad i = 1, \dots, k-2, \\
(L_G L_F^{k-1} \tilde{h})(x) &= \big[\underbrace{* \;\; \cdots \;\; *}_{k-1} \;\; H(x_1)\big] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B(x_1) \end{bmatrix} \\
&= H(x_1)B(x_1) =: \Gamma(x_1),
\end{aligned}
$$

where $\Gamma(x_1) = H(x_1)B(x_1)$ is invertible for $x_1 \in U_1$ by Assumption 2.5. Therefore, according to Definition 1.2, system (2.12) has relative degree $r = k$ on $U_1 \times \mathbb{R}^{(k-1)n}$. □

If the system (2.11) has further structure, namely if the input and the output are colocated, we obtain the following.

Corollary 2.7. Consider a system (2.12) and assume colocation of input and output, i.e., $h'(x_1) = B(x_1)^\top$, $x_1 \in \mathbb{R}^n$. Then, system (2.12) has relative degree $r = k$ on $U := U_1 \times \mathbb{R}^{(k-1)n}$, where $U_1 := \{\, x \in \mathbb{R}^n \mid \operatorname{rk} h'(x) = m \,\}$.

Proof. Let $h'(x_1) = B(x_1)^\top$. Then, Assumption 2.5 is satisfied for all $x_1 \in U_1 = \{\, x \in \mathbb{R}^n \mid \operatorname{rk} h'(x) = m \,\}$. Therefore, Lemma 2.6 yields $r = k$ on U. □

Now, we turn towards the representation of the internal dynamics of a system (2.12). In the following we assume that Assumption 2.5 is satisfied on an open set U_1, so system (2.12) has relative degree $r = k$ on $U = U_1 \times \mathbb{R}^{(r-1)n}$. Using (2.6) and the functions a_i defined in (2.13) we obtain with $h'(x_1) = H(x_1)$, $x_1 \in \mathbb{R}^n$, the following candidate for the (local) diffeomorphism Φ

$$
\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Phi(x) = \begin{pmatrix} h(x_1) \\ H(x_1)x_2 \\ a_2(x_1, x_2) + H(x_1)x_3 \\ \vdots \\ a_{r-1}(x_1, \dots, x_{r-1}) + H(x_1)x_r \\ \tilde{\phi}_1(x) \\ \vdots \\ \tilde{\phi}_{n-rm}(x) \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in U \subseteq \mathbb{R}^N, \tag{2.15}
$$

where $N := rn$, and the functions $\tilde{\phi}_i : U \to \mathbb{R}$, $i = 1, ..., N - rm$ are such that $\Phi'(x) \in \mathbf{Gl}_N(\mathbb{R})$ for an open set $U_1 \subseteq \mathbb{R}^n$ and $U = U_1 \times \mathbb{R}^{(r-1)n}$. For the internal state η we make the structural ansatz

$$
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_r \end{pmatrix} = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_1)x_2 \\ \vdots \\ \phi_r(x_1)x_r \end{pmatrix} \in \mathbb{R}^{r(n-m)}, \tag{2.16}
$$

where $\phi_1 \in \mathcal{C}^1(U_1;\mathbb{R}^{n-m})$ and $\phi_i \in \mathcal{C}(U_1;\mathbb{R}^{(n-m)\times n})$, $i = 2,\dots,r$, where we recall $N = rn$. We observe, that the external state ξ

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_r \end{pmatrix} = \begin{pmatrix} h(x_1) \\ H(x_1)x_2 \\ a_2(x_1,x_2) + H(x_1)x_3 \\ \vdots \\ a_{r-1}(x_1,\dots,x_{r-1}) + H(x_1)x_r \end{pmatrix} \tag{2.17}$$

and the internal state η in (2.16) have similar structure. Since Φ in (2.6) is required to be a diffeomorphism from U to $\Phi(U)$, its Jacobian has to be invertible on U. Plugging in the ansatz from (2.16) we obtain

$$\Phi'(x) = \begin{bmatrix} H(x_1) & 0 & \cdots & \cdots & 0 \\ * & H(x_1) & \ddots & & \vdots \\ * & * & H(x_1) & \ddots & \vdots \\ * & * & * & \ddots & 0 \\ * & * & * & * & H(x_1) \\ \phi_1'(x_1) & 0 & \cdots & \cdots & 0 \\ b_2(x_1,x_2) & \phi_2(x_1) & \ddots & & \vdots \\ b_3(x_1,x_3) & 0 & \phi_3(x_1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_r(x_1,x_r) & 0 & \cdots & 0 & \phi_r(x_1) \end{bmatrix}, \tag{2.18}$$

where $*$ is of the form $\frac{\partial}{\partial x_j} a_i(x_1,\dots,x_i)$, $i,j = 1,\dots,r$ appropriate, respectively, and $b_i(x_1,x_i) := \frac{\partial}{\partial x_1}(\phi_i(x_1)\cdot x_i)$. Upon an invertible permutation we observe that $\Phi'(x)$ given in (2.18) being invertible on $U \subseteq \mathbb{R}^N$ is equivalent to

$$\forall\, i = 2,\dots,r \ \forall\, x_1 \in U_1 : \ \begin{bmatrix} H(x_1) \\ \phi_1'(x_1) \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}) \text{ and } \begin{bmatrix} H(x_1) \\ \phi_i(x_1) \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}). \tag{2.19}$$

We aim to investigate the internal dynamics of (2.12) without explicit appearance of the input u. To this end, we seek for functions $\tilde{\phi}_1(x)$, ..., $\tilde{\phi}_{N-rm}(x)$ such that $p(\cdot) = 0$ in equation (2.2), that is

$$\forall\, i = 1,\dots,N-rm \ \forall\, x \in U \ : \ \left(L_G \tilde{\phi}_i\right)(x) = 0.$$

In view of (2.16) this means to find functions $\phi_1,\dots,\phi_r$ such that

$$\forall\, x \in U : \begin{bmatrix} \phi_1'(x_1) & 0 & \cdots & 0 \\ * & \phi_2(x_1) & \ddots & \vdots \\ * & * & \ddots & 0 \\ * & * & * & \phi_r(x_1) \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B(x_1) \end{bmatrix} = 0,$$

which is equivalent to

$$\forall\, x_1 \in U_1 : \ \phi_r(x_1) B(x_1) = 0. \tag{2.20}$$

We show the existence of functions satisfying (2.19) and (2.20) in the following result.

Corollary 2.8. Consider a system (2.12) and let Assumption 2.5 be satisfied. Then, for any $x_1^0 \in U_1$, $U_1 \subseteq \mathbb{R}^n$ open, there exist an open neighbourhood $U_1^0 \subseteq U_1$ of x_1^0 and functions $\phi_1 \in \mathcal{C}^1(U_1^0; \mathbb{R}^{n-m})$, $\phi_i \in \mathcal{C}(U_1^0; \mathbb{R}^{(n-m)\times n})$, $i = 2, \ldots, r$, such that (2.19) and (2.20) hold for all $x_1 \in U_1^0$.

Proof. The assertion is a direct consequence of Lemma 2.3. □

We highlight that Corollary 2.8 justifies the structural ansatz for the internal state η proposed in (2.16).

Assumption. For the remainder of this chapter, let $U_1 = U_1^0 \subseteq \mathbb{R}^n$ with U_1^0 as in Corollary 2.8.

We proceed determining the functions constituting the internal dynamics. While ϕ_1 can basically be chosen freely up to (2.19), we show that the functions ϕ_i, $i = 2, \ldots, r$, are determined by the system parameters up to an invertible left transformation. To find all possible representations, let $P_i : U_1 \to \mathbb{R}^{n\times m}$ and $V_i : U_1 \to \mathbb{R}^{n\times(n-m)}$ be such that

$$\forall\, x_1 \in U_1 : \; [P_i(x_1), V_i(x_1)] \begin{bmatrix} H(x_1) \\ \phi_i(x_1) \end{bmatrix} = I_n, \; i = 2, \ldots, r, \tag{2.21}$$

which exist by (2.19). Then P_i, V_i have pointwise full column rank, by which the pseudoinverse of V_i is given by $V_i(x_1)^\dagger = (V_i(x_1)^\top V_i(x_1))^{-1} V_i(x_1)^\top$, $x_1 \in U_1$. We obtain the following result.

Lemma 2.9. Use the notation and assumptions from Corollary 2.8. Then, the functions $\phi_i : U_1 \to \mathbb{R}^{(n-m)\times n}$, $i = 2, \ldots, r-1$, are determined by (2.19) up to an invertible left transformation. All possible functions are given by

$$\phi_i(x_1) := V_i(x_1)^\dagger \left(I_n - P_i(x_1)H(x_1)\right), \quad x_1 \in U_1, \tag{2.22}$$

for feasible choices of P_i, V_i satisfying (2.21).

Proof. By assumption there exist functions ϕ_i, $i = 2, \ldots, r-1$, such that (2.19) for $x_1 \in U_1$. Then, we have (2.21) for some corresponding P_i and V_i. We multiply (2.21) from the left by $V_i(x_1)^\dagger$ and subtract $V_i(x_1)^\dagger P_i(x_1) H(x_1)$ from both sides. This yields

$$\phi_i(x_1) = V_i(x_1)^\dagger \left(I_n - P_i(x_1)H(x_1)\right), \quad x_1 \in U_1.$$

On the other hand,

$$\begin{bmatrix} H(x_1) \\ \phi_i(x_1) \end{bmatrix} \begin{bmatrix} P_i(x_1) & V_i(x_1) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}.$$

From this we deduce $\phi_i(x_1)V_i(x_1) = I_{n-m}$, $H(x_1)P_i(x_1) = I_m$ and $\operatorname{im} V_i(x_1) = \ker H(x_1)$. Since the letter expressions are independent of i we set $P(x_1) := P_i(x_1)$ and $V(x_1) := V_i(x_1)$ for all $i = 2, \ldots, r-1$. We observe $P(x_1)H(x_1)P(x_1) = P(x_1)$. Therefore, for all $i = 2, \ldots, r-1$ we have

$$\begin{aligned} \phi_i(x_1)P(x_1) &= V(x_1)^\dagger \left(I_n - P(x_1)H(x_1)\right) P(x_1) \\ &= V(x_1)^\dagger \left(P(x_1) - P(x_1)H(x_1)P(x_1)\right) = 0. \end{aligned}$$

Let $\tilde{V}(x_1) := V(x_1)R(x_1)$, $x_1 \in U_1$, for some $R : U_1 \to \mathbf{Gl}_{n-m}$ and consider $\tilde{\phi}_i(x_1) = \tilde{V}(x_1)^\dagger$. A short calculation shows $\tilde{\phi}_i(x_1) = R(x_1)^{-1}\phi_i(x_1)$ for all $x_1 \in U_1$ and all $i = 2, \ldots, r-1$. So the functions $\phi_i(x)$, $i = 2, \ldots, r-1$, are determined by (2.22) up to an invertible left transformation. □

Remark 2.10. A feasible choice for P in (2.22) is $P(x_1) = H(x_1)^\top (H(x_1)H(x_1)^\top)^{-1}$. Then, ϕ_i simplifies to $\phi_i(x_1) = V(x_1)^\dagger$, for all $i = 2, \dots, r-1$, where $\operatorname{im} V(x_1) = \ker H(x_1)$, and ϕ_i is, up to an invertible left transformation, uniquely determined by the system measurement function h.

While Lemma 2.9 leaves some freedom to choose the functions ϕ_i, $i = 2, \dots, r-1$, the remaining function ϕ_r is uniquely determined up to an invertible left transformation. Let $P : U_1 \to \mathbb{R}^{n\times m}$ and $V : U_1 \to \mathbb{R}^{n\times(n-m)}$ be such that (2.21) is satisfied for $i = r$. Similar to (2.22), we define

$$\phi_r(x_1) := V(x_1)^\dagger \Big(I_n - B(x_1)\Gamma(x_1)^{-1}H(x_1) \Big) \tag{2.23}$$

and we find the following result.

Lemma 2.11. We use the notation and assumptions from Corollary 2.8. Then, the function $\phi_r : U_1 \to \mathbb{R}^{(n-m)\times n}$ is uniquely determined by (2.19) and (2.20) up to an invertible left transformation. All possible functions are given by (2.23) for feasible choices of V satisfying (2.21).

Proof. Assume that (2.19) and (2.20) hold. Then, we have (2.21) for some corresponding P and V. We multiply (2.21) from the left by $V(x_1)^\dagger$ and subtract $V(x_1)^\dagger P(x_1)H(x_1)$ from both sides, and obtain

$$\phi_r(x_1) = V(x_1)^\dagger \left(I_n - P(x_1)H(x_1)\right), \quad x_1 \in U_1.$$

Invoking (2.20), we further obtain from (2.21) that

$$P(x_1) = B(x_1)\left(H(x_1)B(x_1)\right)^{-1} = B(x_1)\Gamma(x_1)^{-1},$$

and hence P is uniquely determined by the output measurement h and the input distribution B. Therefore, ϕ_r is given by (2.23). Furthermore, it follows from (2.21) that

$$\begin{bmatrix} H(x_1) \\ \phi_r(x_1) \end{bmatrix} [P(x_1), V(x_1)] = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix},$$

from which we may deduce $\phi_r(x_1)V(x_1) = I_{n-m}$ and in addition $\operatorname{im} V(x_1) = \ker H(x_1)$. Hence, the representation of ϕ_r in (2.23) only depends on the choice of the basis of $\ker H(x_1)$. Lastly, let $\tilde{V}(x_1) := V(x_1)R(x_1)$, $x_1 \in U_1$, for some $R : U_1 \to \mathbf{Gl}_{n-m}$ and consider

$$\tilde{\phi}_r(x_1) = \tilde{V}(x_1)^\dagger \big(I_n - B(x_1)\Gamma(x_1)^{-1}H(x_1)\big).$$

Then, a short calculation shows $\tilde{\phi}_r(x_1) = R(x_1)^{-1}\phi_r(x_1)$ for all $x_1 \in U_1$. □

We highlight that $\operatorname{im} V_i(x_1) = \ker H(x_1)$ for all $i = 2, \dots, r$ and therefore, invoking Remark 2.10, it is possible to have all functions ϕ_i completely determined by the output measurement function h, up to the choice of a basis of $\ker H(x_1)$ and an invertible left transformation.

Corollary 2.12. We use the notation and assumptions from Corollary 2.8. Let ϕ_i be given by (2.22) for $i = 2, \ldots, r-1$, and ϕ_r be given by (2.23). Then,

$$\begin{bmatrix} H(U_1) \\ \phi_i(U_1) \end{bmatrix} =: W_i, \quad \begin{bmatrix} H(U_1) \\ \phi_r(U_1) \end{bmatrix} =: \tilde{W},$$

and for $P_i : U_1 \to \mathbb{R}^{n\times m}$ satisfying $H(x_1)P_i(x_1) = I_m$, and $\operatorname{im} V(x_1) = \ker H(x_1)$, $x_1 \in U_1$, for all $w \in W_i$, $i = 2, \ldots, r-1$, respectively, we have

$$\begin{bmatrix} H(w) \\ \phi_i(w) \end{bmatrix}^{-1} = \begin{bmatrix} P_i(w) & V(w) \end{bmatrix},$$

and for all $\tilde{w} \in \tilde{W}$

$$\begin{bmatrix} H(\tilde{w}) \\ \phi_r(\tilde{w}) \end{bmatrix}^{-1} = \begin{bmatrix} B(\tilde{w})\Gamma(\tilde{w})^{-1} & V(\tilde{w}) \end{bmatrix},$$

where $\Gamma(v) = H(v)B(v)$, $v \in U_1$.

Proof. Clear. □

We continue to derive a representation of the internal dynamics in terms of the system's output and the internal variable. With (2.16) and (2.17) we define

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} h(x_1) \\ \phi_1(x_1) \end{pmatrix} =: \vartheta(x_1), \tag{2.24}$$

which is continuously differentiable for all $x_1 \in U_1$. Moreover for any $x_1^0 \in U_1$ there exists an open neighbourhood U_1^0 of x_1^0 such that the Jacobian $\vartheta'(\tilde{x}_1)$ is invertible for all $\tilde{x}_1 \in U_1^0$. Therefore, ϑ defines a local diffeomorphism on each U_1^0, where U_1^0 is an open neighbourhood of a $x_1^0 \in U_1$, respectively. Hence, any $\tilde{x}_1 \in U_1^0$ can be expressed in terms of $(\xi_1, \eta_1) \in W_1^0 := \vartheta(U_1^0)$, namely

$$\tilde{x}_1 = \vartheta^{-1}(\xi_1, \eta_1), \quad (\xi_1, \eta_1) \in W_1^0.$$

In order to have the coordinate transformation on the entire subset $U_1 \subseteq \mathbb{R}^n$ we make the following assumption.

Assumption 2.13. For $U_1 \subseteq \mathbb{R}^n$ as in Corollary 2.8 and for the function ϑ defined in (2.24), there exist diffeomorphisms $\Delta_1 : \vartheta(U_1) \to \mathbb{R}^n$, $\Delta_2 : U_1 \to \mathbb{R}^n$, and a continuous, non-decreasing function $\delta : [0, \infty) \to (0, \infty)$ with the property $\int_0^\infty \frac{1}{\delta(t)} \mathrm{d}t = \infty$, such that

$$\forall\, x_1 \in U_1 \,:\, \|\Delta_2'(x_1) \cdot (\vartheta'(x_1))^{-1}\| \cdot \|(\Delta_1'(\vartheta(x_1)))^{-1}\| \le \delta(\|\Delta_2(x_1)\|).$$

Then, invoking continuity of the output function h, Lemma 2.3 and equation (2.19), the result [19, Thm. 2.1] yields that ϑ defines a diffeomorphism on U_1 and we can express the system's state x_1 by

$$x_1 = \vartheta^{-1}(\xi_1, \eta_1). \tag{2.25}$$

For the remainder of this section let Assumption 2.13 be true.

If the output of a system (2.10) is linear, we may obtain the following simple representation of ϑ.

Corollary 2.14. Assume there exists $H \in \mathbb{R}^{m\times n}$ such that the output of a system (2.10) is linear with $h(x_1) = Hx_1$, and Assumption 2.5 holds true on some subset $U_1 \subseteq \mathbb{R}^n$. Let $V \in \mathbb{R}^{n\times(n-m)}$ be such that $\operatorname{im} V = \ker H$. Then, for $\vartheta : U_1 \to \mathbb{R}^n$ defined as in (2.24) we find that

$$\vartheta(U_1) = \begin{bmatrix} H \\ V^\top \end{bmatrix} U_1 =: W_1$$

and for all $w_1 \in W_1$ we have

$$\vartheta^{-1}(w_1) = \begin{bmatrix} H \\ V^\top \end{bmatrix}^{-1} w_1 = \begin{bmatrix} H^\top(HH^\top)^{-1} & V(V^\top V)^{-1} \end{bmatrix} w_1.$$

Proof. Clear. □

For later use, we define the following functions as concatenations on $W_1 := \vartheta(U_1)$, where $V \in \mathcal{C}(U_1;\mathbb{R}^{n-m})$ such that $\operatorname{im} V(x_1) = \ker H(x_1)$ for all $x_1 \in U_1$, and $P(x_1)$ satisfies (2.21)

$$\begin{aligned} \phi'_{1,\vartheta}(\cdot) &:= \left(\phi'_1 \circ \vartheta^{-1}\right)(\cdot), & B_\vartheta(\cdot) &:= \left(B \circ \vartheta^{-1}\right)(\cdot), \\ \phi_{i,\vartheta}(\cdot) &:= \left(\phi_i \circ \vartheta^{-1}\right)(\cdot), & H_\vartheta(\cdot) &:= \left(H \circ \vartheta^{-1}\right)(\cdot), \\ V_\vartheta(\cdot) &:= \left(V \circ \vartheta^{-1}\right)(\cdot), & \Gamma_\vartheta(\cdot) &:= \left(\Gamma \circ \vartheta^{-1}\right)(\cdot), \\ P_\vartheta(\cdot) &:= \left(P \circ \vartheta^{-1}\right)(\cdot), \end{aligned}$$

where $i = 2, \dots, r$. In order to represent the internal dynamics in terms of ξ and η, we aim to express the state variables x_i in terms of the external state ξ and the internal state η. With the aid of Lemma 2.4 and equations (2.16), (2.17) and (2.13) we have for $i = 2, \dots, r$

$$\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} a_{i-1}(x_1,\dots,x_{i-1}) + H(x_1)x_i \\ \phi_i(x_1)x_i \end{pmatrix} = \begin{pmatrix} a_{i-1}(x_1,\dots,x_{i-1}) \\ 0 \end{pmatrix} + \begin{bmatrix} H(x_1) \\ \phi_i(x_1) \end{bmatrix} x_i.$$

From this we obtain iteratively for $i = 2, \dots, r-1$, where the matrix valued function P satisfies (2.21), via (2.25) and Corollary 2.12

$$\begin{aligned} x_1 &= \vartheta^{-1}(\xi_1,\eta_1), \\ x_i &= V_\vartheta(\xi_1,\eta_1)\,\eta_i + P_\vartheta(\xi_1,\eta_1)\left(\xi_i - a_{i-1}(\vartheta^{-1}(\xi_1,\eta_1), x_2,\dots,x_{i-1})\right), \\ x_r &= V_\vartheta(\xi_1,\eta_1)\,\eta_r + B_\vartheta(\xi_1,\eta_1)\,\Gamma_\vartheta(\xi_1,\eta_1)^{-1}\left(\xi_r - a_{r-1}(\vartheta^{-1}(\xi_1,\eta_1), x_2,\dots,x_{r-1})\right). \end{aligned} \tag{2.26}$$

Note that $a_i = a_i(x_1,\dots,x_i)$, and thus the representation of x_i in (2.26) depends at most on $x_1,\dots,x_{i-1}$, which are already known from the previous iterations. Therefore, we may successively express the state variables x_i in terms of ξ and the internal variable η, by

$$\begin{aligned} x_1 &= \vartheta^{-1}(\xi_1,\eta_1), \\ x_i &= \ell_i(\xi_1,\dots,\xi_i,\eta_1,\dots,\eta_i), \quad i = 2,\dots,r, \end{aligned} \tag{2.27}$$

for $\ell_i : \mathbb{R}^{im} \times \mathbb{R}^{i(n-m)} \to \mathbb{R}^n$ successively given by (2.26). With the derivations so far, we obtain the following result.

Theorem 2.15. Given a point $x_1^0 \in \mathbb{R}^n$ suppose Assumption 2.5 is true on an open neighbourhood of x_1^0 and for the respective open set $U_1 \subseteq \mathbb{R}^n$ as in Corollary 2.8 let Assumption 2.13 be true as well. Then, the map $\Phi : U = (U_1, \mathbb{R}^{(r-1)n}) \subseteq \mathbb{R}^N \to \mathbb{R}^N$, where $N = rn$, defined in (2.15) with functions ϑ, ϕ_i, $i = 2, \dots, r$ given in (2.22), (2.23) and (2.24), respectively, is a local coordinate transformation, which transforms the state x of a system (2.12) into new variables (ξ, η) defined in (2.16), (2.17), i.e.,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Phi(x), \text{ and } \Phi'(x) \in \mathbf{Gl}_N(\mathbb{R}),\ x \in U \subseteq \mathbb{R}^N,$$

where Φ' denotes the Jacobian of Φ. Moreover, the inverse Φ^{-1} is explicitly given by

$$x = \Phi^{-1}(\xi, \eta) = \begin{pmatrix} \ell_1(\xi_1, \eta_1) \\ \ell_2(\xi_1, \xi_2, \eta_1, \eta_2) \\ \vdots \\ \ell_r(\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_r) \end{pmatrix},$$

where $\ell_1 := \vartheta^{-1}$ from (2.25), and the functions ℓ_i, $i = 2, \dots, r$, from (2.27) are defined via (2.26).

Proof. The theorem is a direct consequence of the derivations so far in this chapter. □

With the results established above, we are in the position to derive equations for the dynamics of the internal state η. To this end, for $j = 1, \dots, r$ we introduce the notation $\bar{\xi}_j := (\xi_1, \dots, \xi_j)$, $\bar{\eta}_j := (\eta_1, \dots, \eta_j)$, and

$$\phi_{j,\vartheta}^{x_1}(\xi_1, \eta_1, \ell_j) := \tfrac{\partial}{\partial x_1}(\phi_i(x_1)\, x_j)|_{x_1 = \vartheta^{-1}(\xi_1, \eta_1), x_j = \ell_j(\bar{\xi}_j, \bar{\eta}_j)}.$$

Then, for $i = 2, \dots, r-1$, using (2.12) and (2.27) the internal dynamics of a system (2.10) are given by

$$\begin{aligned}
\dot{\eta}_1(t) &= \phi_1'(x_1(t)) \cdot \dot{x}_1(t) \\
&= \phi_{1,\vartheta}'\big(\xi_1(t), \eta_1(t)\big) \cdot \ell_2\big(\bar{\xi}_2(t), \bar{\eta}_2(t)\big), \\
\dot{\eta}_i(t) &= \Big\langle \phi_{i,\vartheta}^{x_1}\Big(\xi_1(t), \eta_1(t), \ell_i\big(\bar{\xi}_i(t), \bar{\eta}_i(t)\big)\Big) \cdot \ell_2\big(\bar{\xi}_2(t), \bar{\eta}_2(t)\big), \ell_i\big(\bar{\xi}_i(t), \bar{\eta}_i(t)\big)\Big\rangle \\
&\quad + \phi_{i,\vartheta}\big(\xi_1(t), \eta_1(t)\big) \cdot \ell_{i+1}\big(\bar{\xi}_{i+1}(t), \bar{\eta}_{i+1}(t)\big), \\
\dot{\eta}_r(t) &= \big\langle \phi_{r,\vartheta}^{x_1}\big(\xi_1(t), \eta_1(t), \ell_r(\bar{\xi}_r(t), \bar{\eta}_r(t))\big) \cdot \ell_2\big(\bar{\xi}_2(t), \bar{\eta}_2(t)\big), \ell_r\big(\bar{\xi}_r(t), \bar{\eta}_r(t)\big)\big\rangle \\
&\quad + \phi_{r,\vartheta}(\xi_1(t), \eta_1(t)) \cdot f\Big(\vartheta^{-1}(\xi_1(t), \eta_1(t)), \ell_2(\bar{\xi}_2(t), \bar{\eta}_2(t)), \dots, \ell_r(\bar{\xi}_r(t), \bar{\eta}_r(t))\Big).
\end{aligned}$$

In virtue of Remark 2.10 we may choose $P(x_1) = H(x_1)^\dagger$, whereby $\phi_i(x_1) = V(x_1)^\dagger$, with $\operatorname{im} V(x_1) = \ker H(x_1)$. Then, invoking equations (2.26) we observe

$$\begin{aligned}
\phi_{i,\vartheta}(\xi_1, \eta_1) \cdot \ell_{i+1}(\bar{\xi}_{i+1}, \bar{\eta}_{i+1}) &= V_\vartheta(\xi_1, \eta_1)^\dagger \left(V_\vartheta(\xi_1, \eta_1)\, \eta_{i+1} + P_\vartheta(\xi_1, \eta_1)\big(\xi_{i+1} - a_i\big)\right) \\
&= \eta_{i+1}.
\end{aligned}$$

With this, the internal dynamics of (2.12) simplify to

$$\begin{aligned}
\dot{\eta}_1(t) &= \phi_1'(x_1(t)) \cdot \dot{x}_1(t) \\
&= \phi_{1,\vartheta}'\big(\xi_1(t), \eta_1(t)\big) \cdot \ell_2\big(\bar{\xi}_2(t), \bar{\eta}_2(t)\big), \\
\dot{\eta}_i(t) &= \big\langle \phi_{i,\vartheta}^{x_1}\big(\xi_1(t), \eta_1(t), \ell_i(\bar{\xi}_i(t), \bar{\eta}_i(t))\big) \cdot \ell_2\big(\bar{\xi}_2(t), \bar{\eta}_2(t)\big), \ell_i\big(\bar{\xi}_i(t), \bar{\eta}_i(t)\big)\big\rangle + \eta_{i+1}(t) \\
\dot{\eta}_r(t) &= \big\langle \phi_{r,\vartheta}^{x_1}\big(\xi_1(t), \eta_1(t), \ell_r(\bar{\xi}_r(t), \bar{\eta}_r(t))\big) \cdot \ell_2\big(\bar{\xi}_2(t), \bar{\eta}_2(t)\big), \ell_r\big(\bar{\xi}_r(t), \bar{\eta}_r(t)\big)\big\rangle \\
&\quad + \phi_{r,\vartheta}(\xi_1(t), \eta_1(t)) \cdot f\Big(\vartheta^{-1}(\xi_1(t), \eta_1(t)), \ell_2(\bar{\xi}_2(t), \bar{\eta}_2(t)), \ldots, \ell_r(\bar{\xi}_r(t), \bar{\eta}_r(t))\Big).
\end{aligned} \tag{2.28}$$

Then, identifying $\xi_i = y^{(i-1)}$, the internal dynamics of (2.10) in (2.28) are completely decoupled and are expressed in terms of the system's output and its derivatives, so the internal dynamics are represented in the form

$$\dot{\eta}(t) = \Psi(\eta(t), y(t), \ldots, y^{(r-1)}(t)),$$

and are, for instance, open for stability analysis.

Remark 2.16. If Assumption 2.13 cannot be satisfied (for examples see [19, Sec. 3]) there still exist state space transformations Φ, however, each such transformation is only local for a neighbourhood U_1^0 of any $x_1^0 \in U_1$, where Assumption 2.5 is true on U_1; i.e., for any $x_1^0 \in U_1$ there exists a neighbourhood U_1^0 such that

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Phi(x) \quad \text{and} \quad \Phi'(x) \in \mathbf{Gl}_N(\mathbb{R}), \qquad x \in U_1^0 \times \mathbb{R}^{(r-1)n} \subseteq \mathbb{R}^N,$$

and the internal dynamics are given as in (2.28). This means, since in many applications the focus lies on a certain operation point of a system, if Assumption 2.5 is satisfied in a neighbourhood of such a point, the internal dynamics are given by (2.28), regardless of Assumption 2.13 being satisfied.

Remark 2.17. For systems with certain properties, we may obtain further structure for the internal dynamics (2.28).

(i) We recall the concept of a conservative vector field. A vector field $L : U \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^n$ open, is called conservative, if there exists a scalar field $l : U \to \mathbb{R}$ such that $l'(x) = L(x)^\top$ for all $x \in U$. For $p, q \in \mathbb{N}$ and $i = 1, \ldots, p$ we denote with $A_{i,:}$ the i^{th} row of a matrix $A \in \mathbb{R}^{p \times q}$. If there exist $l_i \in \mathcal{C}^1(U_1 \to \mathbb{R})$ such that $(l_i'(x_1))^\top = V(x_1)_{i,:}^\dagger$ for all $x_1 \in U_1$, $i = 1, \ldots, n-m$, where $\operatorname{im} V(x_1) = \ker H(x_1)$, then, invoking Remark 2.10 and equations (2.26), it is possible to choose $\phi_1 = (\lambda_1 l_1, \ldots, \lambda_{n-m} l_{n-m})^\top$ for some $\lambda_i \in \mathbb{R} \backslash \{0\}$, $i = 1, \ldots, n-m$, and thus $\phi_1'(x_1) = \Lambda V(x_1)^\dagger$ for $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-m})$. Therefore, the dynamics of η_1 in (2.28) reduce to

$$\dot{\eta}_1(t) = \Lambda \eta_2(t),$$

where the entries of Λ can be chosen at will. We will make use of this in Section 2.2.

(ii) If there exists $H \in \mathbb{R}^{m \times n}$ such that the output of system (2.10) is linear with $h(x_1) = Hx_1$, then $V(x_1) = V \in \mathbb{R}^{(n-m) \times n}$ with $\operatorname{im} V = \ker H$ and thus it represents a conservative vector field. Invoking Remark 2.10 we may choose $\phi_i(x_1) = \phi_i = V^\dagger$, $i = 2, \ldots, r-1$, by which $\frac{\partial}{\partial x_1}(\phi_i \cdot x_i) = 0$ for all $i = 2, \ldots, r - r$. If further $\phi_r \in \mathbb{R}^{(n-m)\times n}$ is constant as well, we have $\frac{\partial}{\partial x_1}(\phi_r \cdot x_r) = 0$. Therefore, the equations (2.28) of the internal dynamics of a system (2.10) with linear output simplify to

$$\dot{\eta}(t) = \begin{bmatrix} 0 & \Lambda & 0 & \cdots & 0 \\ 0 & 0 & I_{n-m} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & I_{n-m} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \eta(t) + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \phi_r \cdot f \end{pmatrix},$$

where we omit the arguments of the system's function f.

Example 2.18. In order to demonstrate the decoupling of the internal dynamics using the results from Secion 2.1, we revisit Example 2.2. For the structural ansatz (2.16) we aim to find functions ϕ_1, ϕ_2 satisfying (2.19) and (2.20). We start with the computation of ϕ_1, ϕ_2 satisfying (2.19), (2.20). For the sake of consistent reading we set $\tilde{x}_1 := (x_1, x_2)^\top$ and $\tilde{x}_2 := (x_3, x_4)^\top$. Then we calculate $V : \mathbb{R}\times\mathcal{B} \to \mathbb{R}^2$ such that $\operatorname{im} V(\tilde{x}_1) = \ker h'(\tilde{x}_1)$ and obtain $V = \left[-\frac{s}{s+l} \;\; 1 \right]^\top$. According to Lemma 2.3 we may choose $\phi_1(\tilde{x}_1) = E\tilde{x}_1 := [0, 1] \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \right)$, and using Corollary 2.14 we obtain the expression as in (2.25)

$$\tilde{x}_1 = \vartheta^{-1}(\xi_1, \eta_1) = \begin{bmatrix} 1 & -\frac{s}{s+l} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} \xi_1 - \frac{s}{s+l}\eta_1 \\ \eta_1 \end{pmatrix}. \tag{2.29}$$

Next, we algorithmically compute ϕ_2 according to Lemma 2.11, namely

$$\begin{aligned} \phi_2(\tilde{x}_1) &= \tilde{V}(x_2)^\dagger \left(I_2 - M(x_2)^{-1} B \Gamma(x)^{-1} h'(\tilde{x}_1) \right) \\ &= \tilde{V}(x_2)^\dagger \left(I_2 - M(x_2)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Gamma(x)^{-1} \begin{bmatrix} 1 & \frac{s}{s+l} \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{3} + \frac{1}{2}\cos(x_2) & \frac{1}{3} \end{bmatrix}, \end{aligned}$$

where we chose $\tilde{V}(\tilde{x}_1) = VR(\tilde{x}_1)^{-1}$ with $R(\tilde{x}_1) = \frac{2l - 3s\cos(x_2)}{6(s+l)}$ as a left transformation, and $\tilde{V}^\dagger$ denotes a pseudoinverse of $\tilde{V}$. We stress that $R : \mathbb{R} \times \mathcal{B} \to \mathbb{R}$ is invertible on $\mathbb{R} \times \mathcal{B}$ and we use it only for the sake of better legibility. Thence, we obtain an expression as in (2.26)

$$\tilde{x}_2 = M(\tilde{x}_1)^{-1} B \Gamma(\tilde{x}_1)^{-1} \xi_2 + VR(\tilde{x}_1)^{-1} \eta_2. \tag{2.30}$$

Note that, with the simple calculations above we established the local coordinate transformation $(\xi, \eta) = \Phi(x)$ and its inverse explicitly. Now, we substitute the expressions from (2.29), (2.30) into the respective functions and obtain via (2.28) the internal dynamics of system (2.8). For purposes of legibility we define the functions

$$\begin{aligned} K_1(\eta_1, \eta_2, \xi_2) &= \frac{6(l+s)\eta_2}{2l - 3s\cos(\eta_1)} - \frac{36\xi_2 (3\cos(\eta_1) + 2)(2l - 3s\cos(\eta_1))}{l^4 m^2 (l+s) \left(9\cos(\eta_1)^2 - 16\right)^2}, \\ K_2(\eta_1, \eta_2, \xi_2) &= \frac{6s\eta_2}{2l - 3s\cos(\eta_1)} - \frac{72\xi_2 (2l - 3s\cos(\eta_1))}{l^4 m^2 (l+s) \left(9\cos(\eta_1)^2 - 16\right)^2}. \end{aligned}$$

Then, the internal dynamics of (2.8) are given by

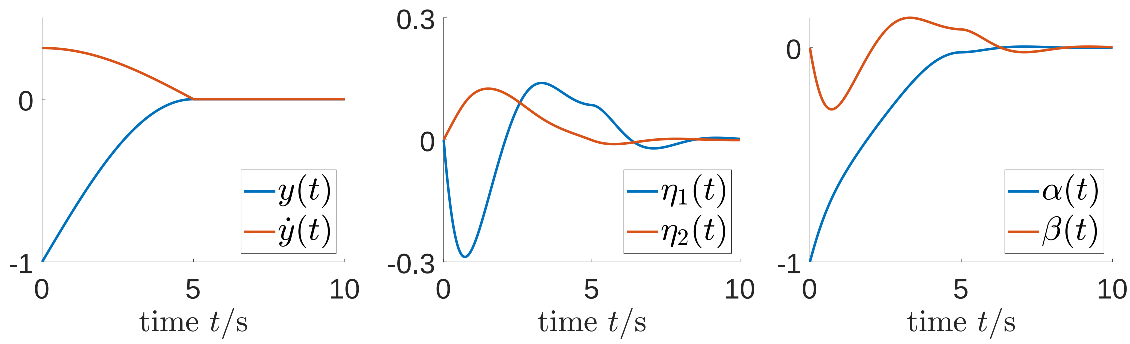
$$
\begin{aligned}
\dot\eta_1(t) &= K_1(\eta_1(t),\eta_2(t),\dot y(t)),\\
\dot\eta_2(t) &= \frac{\sin(\eta_1(t))}{2}K_1(\eta_1(t),\eta_2(t),\dot y(t))\cdot K_2(\eta_1(t),\eta_2(t),\dot y(t))\\
&\quad-\frac{d}{3}K_1(\eta_1(t),\eta_2(t),\dot y(t))-\frac{c}{3}\eta_1(t)\\
&\quad-\frac{l^2 m\sin(\eta_1(t))}{6}K_2(\eta_1(t),\eta_2(t),\dot y(t))^2\\
&\quad-\frac{l^2 m\sin(\eta_1(t))}{2}\Big(2\,K_2(\eta_1(t),\eta_2(t),\dot y(t))-K_1(\eta_1(t),\eta_2(t),\dot y(t))\Big)\cdot\\
&\qquad\cdot\left(\frac{\cos(\eta_1(t))}{2}+\frac{1}{3}\right)K_1(\eta_1(t),\eta_2(t),\dot y(t)).
\end{aligned}
\tag{2.31}
$$

We highlight that the computation of the internal dynamics is completely determined by system parameters and the application of Lemmata 2.3 & 2.11 and Corollary 2.14, and equations (2.22), (2.25) and (2.26). Once chosen ϕ_1, e.g. $\phi_1(x_1) = Ex_1$ as in the proof of Lemma 2.3, the decoupling of the internal dynamics can be performed by an algorithm. We highlight that contrary to equations (2.9a), (2.9b), no partial differential equation has to be solved but - up to the expressions $\phi_2'(x_1)$ and $\phi_2^{x_1}$, for which it is possible to use symbolic differentiation schemes - the internal dynamics are given algebraically. ◇

Example 2.19. In order to illustrate the remaining dynamic of the robotic manipulator for fixed output y we consider equations (2.31) with a given signal $y \in \mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R})$ such that after a time $\tau > 0$ we have $y(t) \equiv 0$ for all $t \geq \tau$ whereby $\dot y(t) \equiv 0$ for all $t \geq \tau$. For illustration purposes we choose

$$
y(t) = \begin{cases} \sin(\frac{\pi t}{10}) - 1, & t \in [0,5) \\ 0, & t \geq 5, \end{cases}
$$

by which $y(t) = \dot y(t) \equiv 0$ for $t \geq 5$. Note, that $y \in \mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R})$. For the simulation we choose $\eta_1(0) = \eta_2(0) = 0$, and the parameters $c = 1$, $d = 0.5$, $l = 1$ and mass $m = 1$.



(a) Signal y and its derivative $\dot y$. (b) Motion of the internal state $\eta = (\eta_1, \eta_2)$. (c) Motion of the reconstructed state $x_1 = (\alpha, \beta)$.

Figure 2.2: Remaining dynamics of the internal state $\eta = (\eta_1, \eta_2)$ and the system's state $x_1 = (\alpha, \beta)$.

Figure 2.2 shows the motion of the internal state for a given “output signal” y

(as y is the output of the system), and the remaining motion of the system's state after the output is zero. The state has been reconstructed via the formula from the previous Example 2.18. Note that, although the output is zero after $t = 5$, there is still motion within the system, but this motion cannot be observed. The simulation has been performed in MATLAB (solver: `ode45`). ◇

2.1.2 Multibody systems with constraints

In this section we derive a representation of the internal dynamics of multibody systems, which arise in physical contexts and possibly contain kinematic loops and potentially are subject to holonomic and non-holonomic constraints. Kinematic loops arise in multibody systems if the beginning and the end of a kinematic chain are located on the same body, where a kinematic chain is "a set of serially connected rigid bodies." [160] Holonomic constraints are relations between position variables, which do not contain velocities; non-holonomic constraints are relations including velocities and hence also may incorporate changes of the physical units. In what follows, we interpret the constraints as auxiliary inputs and outputs by which the systems under consideration have a vector relative degree. Due to this, the ansatz to decouple the internal dynamics presented in the previous Section 2.1.1 is feasible but needs to be adjusted at some instances.

Since we consider physically meaningful systems, we refer to the dynamics of these systems as *equations of motion*. We investigate multibody systems, the equations of motion of which are of the following form

$$\begin{aligned} \dot q(t) &= v(t), \\ M(q(t))\dot v(t) &= f(q(t), v(t)) + J(q(t))^\top \mu(t) + G(q(t))^\top \lambda(t) + B(q(t))\, u(t), \\ 0 &= J(q(t))v(t) + j(q(t)), \\ 0 &= g(q(t)), \\ y(t) &= h(q(t)), \end{aligned} \tag{2.32}$$

where $q, v : I \to \mathbb{R}^n$ are the generalized coordinates (in the case of no constraints), $I \subseteq \mathbb{R}_{\geq 0}$ an interval; $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the generalized mass matrix; $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are the generalized forces, typically including Coriolis, gyroscopic or centrifugal, or applied forces; $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the distribution of the inputs u, which influence the system in affine form; and $h : \mathbb{R}^n \to \mathbb{R}^m$ are (physical meaningful) output measurements. The holonomic constraints are encoded by $g : \mathbb{R}^n \to \mathbb{R}^\ell$ and $G : \mathbb{R}^n \to \mathbb{R}^{\ell\times n}$ with $g'(w) = G(w)$, $w \in \mathbb{R}^n$, the non-holonomic constraints are encoded by $j : \mathbb{R}^n \to \mathbb{R}^p$ and $J : \mathbb{R}^n \to \mathbb{R}^{p\times n}$, which possibly incorporate a change of the physical units. Accordingly, the functions $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ and $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ are the Lagrange multipliers corresponding to the holonomic and non-holonomic constraints.

In order to decouple the internal dynamics of system (2.32) we interpret the constraints as auxiliary inputs and outputs. We define

$$u_{\text{aux}}(t) = \left(\mu(t)^\top, \lambda(t)^\top, u(t)^\top\right)^\top,$$
$$y_{\text{aux}}(t) = \begin{pmatrix} J(q(t))v(t) + j(q(t)) \\ g(q(t)) \\ h(q(t)) \end{pmatrix} =: \begin{pmatrix} Y_1(q(t), v(t)) \\ Y_2(q(t)) \\ Y_3(q(t)) \end{pmatrix},$$

and set $Y := (Y_1^\top, Y_2^\top, Y_3^\top)^\top \in \mathbb{R}^{p+\ell+m}$. For $x_1 = q \in \mathbb{R}^n$, $x_2 = v \in \mathbb{R}^n$ and $x := (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{2n}$ we define

$$F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \qquad x \mapsto \begin{pmatrix} x_2 \\ M(x_1)^{-1} f(x) \end{pmatrix},$$
$$K : \mathbb{R}^{2n} \to \mathbb{R}^{2n \times (p+\ell+m)}, \quad x \mapsto \operatorname{diag}\left(I_n, M(x_1)^{-1}\right) \begin{bmatrix} 0 & 0 & 0 \\ J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix}.$$

Then, with the expressions above we may rewrite (2.32) equivalently as

$$\begin{aligned} \dot{x}(t) &= F(x(t)) + K(x_1(t))\, u_{\text{aux}}(t), \\ y_{\text{aux}}(t) &= Y(x(t)), \end{aligned}$$

which is of the form (1.7). Following Lemma 2.6 we obtain the following

Corollary 2.20. As before, we denote with $(L_A Z)(z)$ the Lie derivative of Z along the vector field A at z. If there exists $U = (U_1, U_2) \subseteq \mathbb{R}^{2n}$ open such that $(L_K Y_1)(x)$, $(L_K L_F Y_2)(x)$ and $(L_K L_F Y_3)(x)$ exist for all $x \in U$, and if

$$\Gamma(x_1) := \begin{bmatrix} J(x_1) \\ G(x_1) \\ h'(x_1) \end{bmatrix} M(x_1)^{-1} \begin{bmatrix} J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix} \in \mathbf{Gl}_{p+\ell+m}, \tag{2.33}$$

then system (2.32) has vector relative degree $r = (r_1, r_2, r_3) \in \mathbb{N}^{1\times(p+\ell+m)}$ on U with $r_1 = (1, \dots, 1) \in \mathbb{N}^{1\times p}$, $r_2 = (2, \dots, 2) \in \mathbb{N}^{1\times \ell}$, $r_3 = (2, \dots, 2) \in \mathbb{N}^{1\times m}$.

Proof. We calculate for $x \in U$

$$\begin{aligned} (L_K Y_1)(x) &= J(x_1) M(x_1)^{-1} \begin{bmatrix} J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix}, \\ (L_K Y_2)(x) &= \begin{bmatrix} G(x_1) & 0 \end{bmatrix} K(x) = 0, \\ (L_K Y_3)(x) &= \begin{bmatrix} h'(x_1) & 0 \end{bmatrix} K(x) = 0, \\ (L_F Y_2)(x) &= G(x_1) x_2, \\ (L_F Y_3)(x) &= h'(x_1) x_2, \\ (L_K L_F Y_2)(x) &= \begin{bmatrix} \frac{\partial}{\partial x_1}\left(G(x_1)x_2\right) & G(x_1) \end{bmatrix} K(x) \\ &= G(x_1) M(x_1)^{-1} \begin{bmatrix} J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix}, \\ (L_K L_F Y_3)(x) &= \begin{bmatrix} \frac{\partial}{\partial x_1}\left(h'(x_1)x_2\right) & h'(x_1) \end{bmatrix} K(x) \\ &= h'(x_1) M(x_1)^{-1} \begin{bmatrix} J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix}. \end{aligned} \tag{2.34}$$

Since

$$\begin{bmatrix} \Gamma_1(x_1) \\ \Gamma_2(x_1) \\ \Gamma_3(x_1) \end{bmatrix} := \begin{bmatrix} (L_K Y_1)(x) \\ (L_K L_F Y_2)(x) \\ (L_K L_F Y_3(x) \end{bmatrix} = \begin{bmatrix} J(x_1) \\ G(x_1) \\ h'(x_1) \end{bmatrix} M(x_1)^{-1} \begin{bmatrix} J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix}$$

is invertible for $x_1 \in U_1$ by assumption, according to Definition 1.2 system (2.32) has vector relative degree $r = (r_1, r_2, r_3) \in \mathbb{N}^{1\times(p+\ell+m)}$ on U, with $r_1 = (1, \ldots, 1) \in \mathbb{N}^{1\times p}$, $r_2 = (2, \ldots, 2) \in \mathbb{N}^{1\times\ell}$ and $r_3 = (2, \ldots, 2) \in \mathbb{N}^{1\times m}$. $\square$

The third component of the vector relative degree $r_3 = (2, \ldots, 2) \in \mathbb{N}^{1\times m}$ reflects the typical situation in mechanical systems that, following Newton's law, measuring a position, the applied forces appear on acceleration level.

Following the findings in Section 2.1.1 we make the following ansatz for the internal state

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_1)x_2 \end{pmatrix},$$

for some $\phi_1 : U_1 \to \mathbb{R}^{n-\ell-m}$ and $\phi_2 : U_1 \to \mathbb{R}^{(n-\ell-m-p)\times n}$, where $U_1 \subseteq \mathbb{R}^n$ open, and invoking (2.34), with the aid of (2.2) we have for $x \in \mathbb{R}^{2n}$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Phi(x) = \begin{pmatrix} J(x_1)x_2 + j(x_1) \\ g(x_1) \\ G(x_1)x_2 \\ h(x_1) \\ h'(x_1)x_2 \\ \phi_1(x_1) \\ \phi_2(x_1)x_2 \end{pmatrix}, \quad \Phi'(x) = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(J(x_1)x_2\right) + j'(x_1) & J(x_1) \\ G(x_1) & 0 \\ \frac{\partial}{\partial x_1}\left(G(x_1)x_2\right) & G(x_1) \\ h'(x_1) & 0 \\ \frac{\partial}{\partial x_1}\left(h'(x_1)x_2\right) & h'(x_1) \\ \phi_1'(x_1) & 0 \\ \frac{\partial}{\partial x_1}\left(\phi_2(x_1)x_2\right) & \phi_2(x_1) \end{bmatrix}$$

where we used $g'(w) = G(w)$, $w \in \mathbb{R}^n$. Therefore, analogously to the derivations made in the previous section, we have that the Jacobian $\Phi'(x) \in \mathbb{R}^{2n\times 2n}$ is invertible for $x = (x_1, x_2) \in U = (U_1, U_2) \subseteq \mathbb{R}^{2n}$, $U_1, U_2 \subseteq \mathbb{R}^n$, if, and only if,

$$\forall\, x_1 \in U_1 \;:\; \begin{bmatrix} G(x_1) \\ h'(x_1) \\ \phi_1'(x_1) \end{bmatrix} \in \mathbf{Gl}_n \;\text{ and }\; \begin{bmatrix} J(x_1) \\ G(x_1) \\ h'(x_1) \\ \phi_2(x_1) \end{bmatrix} \in \mathbf{Gl}_n\,. \tag{2.35}$$

Via Lemma 2.3 there exist an open set $U_1^0 \subseteq U_1$, a function $\phi_1 \in \mathcal{C}^1(U_1^0; \mathbb{R}^{n-\ell-m})$ and a matrix valued function $\phi_2 \in \mathcal{C}(U_1^0; \mathbb{R}^{(n-\ell-m-p)\times n})$ such that (2.35) is true on U_1^0 and moreover,

$$\forall\, x_1 \in U_1^0 \;:\; (L_K\phi_2)(x) = \phi_2(x_1)M(x_1)^{-1}\begin{bmatrix} J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix} = 0. \tag{2.36}$$

Assumption. With some abuse of notation for the remainder of this section, for U_1^0 just mentioned above we set $U_1 := U_1^0$.

Then, following the derivations made in Section 2.1.1 we have the following result.

Corollary 2.21. Use the notation and assumptions from Corollary 2.20, in particular let (2.33) be true on U_1 defined just above. Then, for $V : U_1 \to \mathbb{R}^{(\ell+m+p)\times n}$ with $\operatorname{im} V(x_1) = \ker[J(x_1)^\top, G(x_1)^\top, h'(x_1)^\top]^\top$ the function $\phi_2 \in \mathcal{C}(U_1; \mathbb{R}^{(n-\ell-m-p)\times n})$ satisfying (2.35) and (2.36) is, up to an invertible left transformation, uniquely given by

$$\phi_2(x_1) = V(x_1)^\dagger\left(I_n - M(x_1)^{-1}[J(x_1)^\top, G(x_1)^\top, B(x_1)]\Gamma(x_1)^{-1}\begin{bmatrix} J(x_1) \\ G(x_1) \\ h'(x_1) \end{bmatrix}\right), \quad x_1 \in U_1.$$

Furthermore, for all $w_1 \in W_1 := [J(U_1)^\top, G(U_1)^\top, h'(U_1)^\top, \phi_2(U_1)^\top]^\top$ we have

$$\begin{bmatrix} J(w_1) \\ G(w_1) \\ h'(w_1) \\ \phi_2(w_1) \end{bmatrix}^{-1} = \begin{bmatrix} M(w_1)^{-1} \begin{bmatrix} J(w_1) \\ G(w_1) \\ B(w_1)^\top \end{bmatrix}^\top \Gamma(w_1)^{-1} & V(w_1) \end{bmatrix}.$$

Proof. Clear. □

We observe that

$$\begin{pmatrix} Y_2 \\ Y_3 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} g(x_1) \\ h(x_1) \\ \phi_1(x_1) \end{pmatrix} =: \vartheta(x_1) \tag{2.37}$$

is continuously differentiable on U_1 and its Jacobian is invertible on U_1 by (2.35). In order to have the coordinate transformation ϑ well defined on the entire subset $U_1 \subseteq \mathbb{R}^n$ we make the following assumption.

Assumption 2.22. For $U_1 \subseteq \mathbb{R}^n$ open as in Corollary 2.21 and ϑ defined in (2.37), let Assumption 2.13 be true.

Then, invoking continuity of g, h, Lemma 2.3 and equation (2.35), [19, Thm. 2.1] yields that ϑ defines a diffeomorphism on U_1, and the state x_1 conversely can be represented in terms of the auxiliary output and the internal variable η_1 by

$$x_1 = \vartheta^{-1}(Y_2, Y_3, \eta_1). \tag{2.38}$$

Accordingly, we seek to find a representation of x_2 in respective terms. To this end, we define

$$\bar{Y} := \begin{bmatrix} G(x_1) \\ h'(x_1) \end{bmatrix} x_2$$

and obtain

$$\begin{pmatrix} Y_1 \\ \bar{Y} \\ \eta_2 \end{pmatrix} = \begin{bmatrix} J(x_1) \\ G(x_1) \\ h'(x_1) \\ \phi_2(x_1) \end{bmatrix} x_2 + \begin{pmatrix} j(x_1) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, with the aid of Corollary 2.21 we may express the state x_2 in terms of the auxiliary output and the internal variable η_2 by

$$x_2 = M(x_1)^{-1} \begin{bmatrix} J(x_1) \\ G(x_1) \\ B(x_1)^\top \end{bmatrix}^\top \Gamma(x_1)^{-1} \begin{pmatrix} Y_1 - j(x_1) \\ \bar{Y} \end{pmatrix} + V(x_1)\,\eta_2. \tag{2.39}$$

Next, we define the following functions as concatenations on $W_1 := \vartheta(U_1)$, where $V \in \mathcal{C}(U_1; \mathbb{R}^{(n-\ell-m-p)\times n})$ be as in Corollary 2.21:

$$\begin{aligned} \phi'_{1,\vartheta}(\cdot) &:= \left(\phi'_1 \circ \vartheta^{-1}\right)(\cdot), & \phi_{2,\vartheta}(\cdot) &:= \left(\phi_i \circ \vartheta^{-1}\right)(\cdot), & B_\vartheta(\cdot) &:= \left(B \circ \vartheta^{-1}\right)(\cdot), \\ H_\vartheta(\cdot) &:= \left(h' \circ \vartheta^{-1}\right)(\cdot), & G_\vartheta(\cdot) &:= \left(G \circ \vartheta^{-1}\right)(\cdot), & J_\vartheta(\cdot) &:= \left(J \circ \vartheta^{-1}\right)(\cdot), \\ j_\vartheta(\cdot) &:= \left(j \circ \vartheta^{-1}\right)(\cdot), & V_\vartheta(\cdot) &:= \left(V \circ \vartheta^{-1}\right)(\cdot), & \Gamma_\vartheta(\cdot) &:= \left(\Gamma \circ \vartheta^{-1}\right)(\cdot). \end{aligned}$$

With this, invoking (2.6) we may write $x_2 = \kappa(\eta, \xi)$ for an appropriate function $\kappa : \mathbb{R}^{2n-2m-2\ell-p} \times \mathbb{R}^{2m+2\ell+p} \to \mathbb{R}^n$. Then, using the notation

$$\phi_{2,\vartheta}^{x_1}(\zeta_1, \kappa(\zeta_2)) := \tfrac{\partial}{\partial x_1}(\phi_2(x_1)x_2)|_{x_1=\vartheta^{-1}(\zeta_1), x_2=\kappa(\zeta_2)}$$

as in the previous section, and invoking the original constraints, namely $Y_1(t) = 0$, $Y_2(t) = 0$, and $\bar{Y}(t) = (\dot{Y}_2(x_1(t))^\top, \dot{Y}_3(x_1(t))^\top)^\top = (0, \dot{y}(t)^\top)^\top$, for $t \geq 0$, the internal dynamics of a multibody system (2.32) are given by

$$\begin{aligned}
\dot{\eta}_1(t) &= \phi_{1,\vartheta}'\big(0, y(t), \eta_1(t)\big) \cdot \Bigg(\eta_2(t) \\
&\quad + M_\vartheta\big(0, y(t), \eta_1(t)\big)^{-1} \begin{bmatrix} J_\vartheta\big(0,y(t),\eta_1(t)\big) \\ G_\vartheta\big(0,y(t),\eta_1(t)\big) \\ B_\vartheta\big(0,y(t),\eta_1(t)\big)^\top \end{bmatrix}^\top \Gamma_\vartheta\big(0, y(t), \eta_1(t)\big)^{-1} \begin{pmatrix} -j_\varphi\big(0,y(t),\eta_1(t)\big) \\ 0 \\ \dot{y}(t) \end{pmatrix}\Bigg) \\
\dot{\eta}_2(t) &= \phi_{2,\vartheta}^{x_1}\big((0, y(t), \eta_1(t)), \kappa(\eta(t), \xi(t))\big) \cdot \kappa(\eta(t), \xi(t)) \\
&\quad + \phi_{2,\varphi}\big(0, y(t), \eta_1(t)\big) M\big(0, y(t), \eta_1(t)\big)^{-1} f\left(\vartheta^{-1}\big(0, y(t), \eta_1(t)\big), \kappa(\eta(t), \xi(t)\right).
\end{aligned} \tag{2.40}$$

As before in Section 2.1.1, the internal dynamics in (2.40) are completely decoupled and given in terms of the system's parameters, its output and the internal variable, where the decoupling process follows step by step an explicit scheme.

Examples. Decoupling the internal dynamics using the ansatz and procedure presented in Sections 2.1.1 & 2.1.2 was successfully applied in the following works. In [23, Sec. 4.2] we performed decoupling of the internal dynamics for the robotic manipulator arm as in Example 2.18; then, we chose the system parameters such that this system was non-minimum phase and, after linearizing the internal dynamics, we performed output reference tracking using the controller from [16]. In [18] we investigated multi-input multi-output systems containing a kinematic loop and being subject to holonomic and non-holonomic constraints. There, the techniques to decouple the internal dynamics were applied in [18, Sec. 3]. The obtained representation was then utilized to design a combined feedforward-feedback controller to perform output reference tracking, where the controller involved the feedforward control schemes from [158, 159] and the feedback controller from [16]. The latter explicitly involves the internal dynamics, which we decoupled using the technique from Section 2.1.2 and linearized it around the operation point. As a real world application we simulated in [18, Sec. 6] output reference tracking of a non-minimum phase robotic manipulator arm, which contained a kinematic loop and was subject to holonomic constraints. ◇

Remark 2.23. The set of variables presented in Sections 2.1.1 & 2.1.2 to decouple the internal dynamics (2.28) of (2.10) and (2.40) of (2.32), respectively, offers an alternative to the Byrnes-Isidori form as in (2.2), whose computation often requires a lot of effort, cf. Example 2.2.

(i) The advantage of the representation of the internal dynamics in (2.28), (2.40), respectively, is, that it involves system parameters only. Lemma 2.11 (Corollary 2.21) shows, that a suitable choice of V is given by $\operatorname{im} V = \ker h'$ ($\operatorname{im} V = \ker(J^\top, G^\top, h')$), and solving (2.25) (or (2.38)) for x_1 only involves h

(or h and g), where the choice $\phi_1(x_1) = Ex_1$ as in the proof of Lemma 2.3 may simplify the computation further. We stress that, once ϕ_1 is chosen, e.g. $\phi_1(x_1) = Ex_1$, the computation of the internal dynamics of a system (2.10) or (2.32), respectively, can be carried out completely algorithmically without the need to choose further functions.

(ii) Equations (2.26) and (2.38), (2.39) provide an explicit representation of the diffeomorphism's inverse, i.e., $x = \Phi^{-1}(\xi, \eta)$, which can be computed algorithmically as well. Since the internal dynamics are given in terms of the system's output, the complete state x can be reconstructed from the output and the solution of the internal dynamics. The situation is depicted in Figure 2.3.

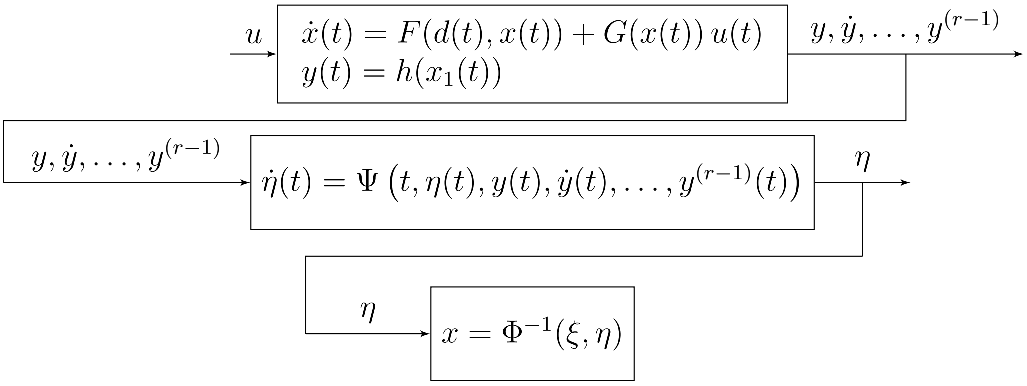


Figure 2.3: Representation of the system's state in terms of the internal variable and the system's output.

2.2 Stability analysis

In this section we utilize the representation of the internal dynamics from Section 2.1 to perform a stability analysis of the internal dynamics of a certain class of multibody systems. We derive sufficient conditions on the system parameters such that the internal dynamics are bounded-input bounded-state stable. These conditions can be verified in advance and hence explicit decoupling and stability analysis of the internal dynamics are not necessary. In particular, it is not necessary to derive (2.28). We consider nonlinear multibody systems without kinematic loops, constant mass matrix $M \in \mathbb{R}^{n \times n}$, constant input distribution $B \in \mathbb{R}^{n \times m}$, and motivated from various applications we assume the output measurement to be linear. The system class under consideration is modelled via generalized coordinates q, v and the equations of motion are of the form

$$\begin{aligned} \dot{q}(t) &= v(t), & q(0) &= q^0 \in \mathbb{R}^n, \\ M\dot{v}(t) &= f(q(t), v(t)) + Bu(t), & v(0) &= v^0 \in \mathbb{R}^n, \\ y(t) &= Hq(t), \end{aligned} \tag{2.41}$$

where $H \in \mathbb{R}^{m \times n}$ is the linear measurement function and we assume the following structure of f

$$f(x_1, x_2) = -K(x_1) - D(x_2) - C(x_1, x_2)x_2, \tag{2.42}$$

where $K \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ may be considered as a nonlinear restoring force, the function $D \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ for example mimics a nonlinear damping or friction and the term

$C \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^{n\times n})$ may take the role of a nonlinear distribution for a position dependent damping or mimic a Coriolis force. As before, the input and output dimensions coincide but we do not assume colocation, i.e., we allow for $H \neq B^\top$. For the sake of notational consistency we introduce the state variables $x_1 := q$, $x_2 := v$, set $x := (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{2n}$, and define the functions

$$F(x) := \begin{pmatrix} x_2 \\ M^{-1}f(x_1, x_2) \end{pmatrix}, \quad G := \begin{pmatrix} 0 \\ M^{-1}B \end{pmatrix}.$$

With this we obtain the following equivalent representation of system (2.41)

$$\begin{aligned} \dot{x}(t) &= F(x(t)) + Gu(t), \\ y(t) &= \tilde{H}x(t), \end{aligned}$$

where $\tilde{H} \in \mathbb{R}^{m\times 2n}$ with $\tilde{H} = [H, 0]$. We assume the high-gain matrix

$$\Gamma := HM^{-1}B \in \mathbf{Gl}_m(\mathbb{R})$$

to be regular, whereby Assumption (2.5) is satisfied on $U_1 = \mathbb{R}^n$ substituting B with $M^{-1}B$. Therefore, via Lemma 2.6, system (2.41) has relative degree $r = 2$ on $\mathbb{R}^{2n}$. Since H, M, B are constant matrices we obtain with the aid of Lemma 2.11

$$\phi_2 = V^\dagger\big(I_n - M^{-1}B\Gamma^{-1}H\big) \in \mathbb{R}^{(n-m)\times n}, \tag{2.43}$$

where $V \in \mathbb{R}^{n\times(n-m)}$ is such that $\operatorname{im} V = \ker H$. Hence, ϕ_2 defines a conservative vector field and thus, according to Remark 2.17, we may choose

$$\phi_1(x_1) = \lambda\phi_2 \cdot x_1, \tag{2.44}$$

for $x_1 \in \mathbb{R}^n$ and some $\lambda \in \mathbb{R} \setminus \{0\}$. Then, via (2.26) we obtain

$$x_1 = \begin{bmatrix} H \\ \lambda\phi_2 \end{bmatrix}^{-1} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = M^{-1}B\Gamma^{-1}\xi_1 + \lambda^{-1}V\eta_1, \tag{2.45}$$

$$x_2 = \begin{bmatrix} H \\ \phi_2 \end{bmatrix}^{-1} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = M^{-1}B\Gamma^{-1}\xi_2 + V\eta_2. \tag{2.46}$$

Defining $\mathcal{M} := M^{-1}B\Gamma^{-1} \in \mathbb{R}^{n\times m}$ and $\Theta := \phi_2 M^{-1} \in \mathbb{R}^{(n-m)\times n}$, via equation (2.28) we obtain the internal dynamics of system (2.41)

$$\begin{aligned} \dot{\eta}_1(t) &= \lambda\eta_2(t), \\ \dot{\eta}_2(t) &= \Theta f\big(\mathcal{M}y(t) + \lambda^{-1}V\eta_1(t), \mathcal{M}\dot{y}(t) + V\eta_2(t)\big), \end{aligned} \tag{2.47}$$

where we identified $\xi_1 = y$, $\xi_2 = \dot{y}$, and the arguments of $f(x_1, x_2)$ with structure as in (2.42) have been substituted via (2.45) and (2.46), respectively. We define the vector field

$$\begin{aligned} &\Psi : \mathbb{R}^{2(n-m)} \times \mathbb{R}^{2m} \to \mathbb{R}^{2(n-m)}, \ (z_1, z_2, v_1, v_2) \mapsto \\ &\begin{pmatrix} \lambda z_2 \\ \Theta\left(-K(\mathcal{M}v_1 + \lambda^{-1}Vz_1) - D(\mathcal{M}v_2 + Vz_2) - C(\mathcal{M}v_1 + \lambda^{-1}Vz_1, \mathcal{M}v_2 + Vz_2)(\mathcal{M}v_2 + Vz_2)\right) \end{pmatrix} \end{aligned} \tag{2.48}$$

with which the internal dynamics of system (2.41) are given by

$$\dot{\eta}(t) = \Psi\big(\eta_1(t), \eta_2(t), y(t), \dot{y}(t)\big).$$

Henceforth let V in (2.43) have orthonormal columns. In order to formulate conditions on f in (2.41) such that the internal dynamics are bounded-input bounded-output stable, we recall the following definition.

Definition 2.24. A function $\alpha : \mathbb{R}^n \to \mathbb{R}$ is called *radially unbounded*, if $\alpha(x) \to \infty$ for $\|x\| \to \infty$.

Next, we define the following set of functions

$$\Sigma_{K,D,C} := \left\{ (K, D, C) \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}^n; \mathbb{R}^{n\times n}) \;\middle|\; (2.49) \text{ are satisfied} \right\},$$

where for $i = 1, 2$ we assume, that there exist a positive constant $z_i^+ > 0$ such that for all $z_i \in Z_i := \{z \in \mathbb{R}^{n-m} \mid \|z\| > z_i^+\}$ and all $v, w \in \mathbb{R}^n$ the functions K, D and C satisfy the following, where $V_K \in \mathcal{C}^1(Z_1; \mathbb{R})$ is a radially unbounded function

$$V_K'(z_1) = (\Theta K(Vz_1))^\top, \tag{2.49a}$$
$$\|K(Vz_1) - K(Vz_1 + w)\| \le g_1(w), \tag{2.49b}$$
$$z_1^\top \Theta K(Vz_1) \ge \kappa \|z_1\|^2, \tag{2.49c}$$

$$\|D(Vz_2) - D(Vz_2 + w)\| \le g_2(w), \tag{2.49d}$$
$$z_2^\top \Theta D(Vz_2) \ge \delta \|z_2\|^2, \tag{2.49e}$$
$$\|D(Vz_2)\| \le d\|z_2\|, \tag{2.49f}$$

$$\|(C(Vz_1, w) - C(Vz_1 + v, w))w\| \le g_3(v)a_3(w), \tag{2.49g}$$
$$z_1^\top \Theta C(Vz_1, w)w \ge \|z_1\|^2 b_3(w), \tag{2.49h}$$
$$\|(C(v, Vz_2) - C(v, Vz_2 + w))(Vz_2 + w)\| \le g_4(w)a_4(v)\|z_2\|, \tag{2.49i}$$
$$z_2^\top \Theta C(v, Vz_2)Vz_2 \ge b_4(v)\|z_2\|\|Vz_2\|^2, \tag{2.49j}$$
$$\|C(v, Vz_2)w\| \le a_4(v)\|z_2\|\|w\|, \tag{2.49k}$$

for suitable functions $a_j, b_j, g_i \in \mathcal{C}(\mathbb{R}^n \to \mathbb{R}_{\ge 0})$, $i = 1, \dots, 4$, $j = 3, 4$, with $a_i(x) \le b_i(x)$ for all $x \in \mathbb{R}^n$, and $\kappa, \delta, d > 0$. A closer inspection yields that conditions (2.49a) – (2.49k) mean that the acting forces are assumed to be basically linear in a certain region, (far) away from the origin. Hence these are merely weak assumptions.

In order to formulate the next result, we set $\tau := \|\Theta\|$ and for some $\rho_1, \rho_2 \ge 0$, $q > 0$ we define the following constants

$$\begin{aligned}
\tilde{K} &:= \max_{z\in B_{\rho_1}(0)} g_1(\mathcal{M}z), & \tilde{D} &:= \max_{z\in B_{\rho_2}(0)} g_2(\mathcal{M}z), \\
\tilde{C}_3 &:= \max_{z\in B_{\rho_1}(0)} g_3(\mathcal{M}z), & \tilde{C}_4 &:= \max_{z\in B_{\rho_2}(0)} g_4(\mathcal{M}z), \\
\varepsilon_1 &:= q(\tfrac{\kappa}{\lambda} - \tfrac{\tau^2}{2}), & \varepsilon_2 &:= \delta - q(\tfrac{d^2}{2} + \lambda), \\
E_1 &:= q\tau(\tilde{K} + d\|\mathcal{M}\|\rho_2), & E_2 &:= \tau(\tilde{K} + \tilde{D}),
\end{aligned} \tag{2.50}$$

which are all nonnegative by the feasible choices $0 < \lambda < 2\kappa/\tau^2$ and $0 < q < 2\delta/(d^2 + 2\lambda)$. Further, we define

$$\gamma_1 := \lambda\tau\tilde{C}_3 \ge 0, \;\; \gamma_2 := \tau(\tilde{C}_4 + \mu\rho_2) \ge 0,$$
$$\tilde{Z}_i := \left\{ z \in \mathbb{R}^{n-m} \;\middle|\; \|z\| > \max\left\{ \frac{E_i}{\varepsilon_i}, \gamma_i \right\} \right\}, \;\; i = 1, 2.$$

We present an explicit Lyapunov function for the internal dynamics (2.47) of a system (2.41) in the following result.

Lemma 2.25. Consider a system (2.41) where f is structured as in (2.42) with $(K, D, C) \in \Sigma_{K,D,C}$. Fix some $\rho_1, \rho_2 \geq 0$ and for $\lambda > 0$ in (2.44) fix $0 < \lambda < 2\kappa/\tau^2$ and $0 < q < 2\delta/(d^2 + 2\lambda)$. Then, for $\mathcal{L} : \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \to \mathbb{R}$ defined by

$$\mathcal{L}(\eta_1, \eta_2) = \frac{1}{2}\|\eta_2\|^2 + q\eta_1^\top \eta_2 + V_K(\lambda^{-1}\eta_1), \tag{2.51}$$

the Lie derivative along the vector field Ψ in (2.48) is nonincreasing for all $y_1 \in B_{\rho_1}(0)$, $y_2 \in B_{\rho_2}(0)$ and all $\eta_i \in Z_i \cap \tilde{Z}_i$, $i = 1, 2$, i.e.,

$$\mathcal{L}'(\eta_1, \eta_2) \cdot \Psi(\eta_1, \eta_2, y_1, y_2) \leq 0. \tag{2.52}$$

Proof. Recall $\Theta = \phi_2 M^{-1} \in \mathbb{R}^{(n-m)\times n}$, and for $\tau := \|\Theta\|$ let $0 < \lambda < 2\kappa/\tau^2$ and $0 < q < 2\delta/(d^2 + 2\lambda)$. For $i = 1, 2$ let $\eta_i \in Z_i$ and $y_i \in B_{\rho_i}(0)$. For the sake of better legibility we set $\bar{\lambda} := \lambda^{-1}$. Then, we calculate the Lie derivative of $\mathcal{L}$ from (2.51) along the vector field Ψ from (2.48).

$$\begin{aligned}
\mathcal{L}'(\eta_1, \eta_2) \cdot \Psi(\eta_1, \eta_2, y_1, y_2) &= K(\bar{\lambda} V \eta_1)^\top \Theta^\top \eta_2 + q\lambda \eta_2^\top \eta_2 \\
&\quad + \eta_2^\top \Theta\big(- K(\mathcal{M}y_1 + \bar{\lambda} V\eta_1) - D(\mathcal{M}y_2 + V\eta_2)\big) \\
&\quad - \eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda} V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2) \\
&\quad + q\eta_1^\top \Theta\big(- K(\mathcal{M}y_1 + \bar{\lambda} V\eta_1) - D(\mathcal{M}y_2 + V\eta_2)\big) \\
&\quad - q\eta_1^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda} V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2) \\
&= \eta_2^\top \Theta\big(K(\bar{\lambda} V\eta_1) - K(\mathcal{M}y_1 + \bar{\lambda} V\eta_1)\big) \\
&\quad + q\lambda\|\eta_2\|^2 - \eta_2^\top \Theta D(\mathcal{M}y_2 + V\eta_2) \\
&\quad - \eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda} V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2) \\
&\quad - q\eta_1^\top \Theta K(\mathcal{M}y_1 + \bar{\lambda} V\eta_1) - q\eta_1^\top \Theta D(\mathcal{M}y_2 + V\eta_2) \\
&\quad - q\eta_1^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda} V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2).
\end{aligned} \tag{2.53}$$

For purpose of better legibility we set $\mu := \|\mathcal{M}\|$. Note, that since V in (2.43) has orthonormal columns we have $\|Vz\| = \|z\|$ for $z \in \mathbb{R}^{n-m}$. We estimate the addends in (2.53) separately for $\eta_i \in Z_i$, respectively.

Step one. We estimate $\eta_2^\top \Theta\big(K(\bar{\lambda} V\eta_1) - K(\mathcal{M}y_1 + \bar{\lambda} V\eta_1)\big)$

$$\begin{aligned}
\eta_2^\top \Theta\big(K(\bar{\lambda} V\eta_1) - K(\mathcal{M}y_1 + \bar{\lambda} V\eta_1)\big) &\leq \tau\|\eta_2\|\|K(\bar{\lambda} V\eta_1) - K(\mathcal{M}y_1 + \bar{\lambda} V\eta_1)\| \\
&\overset{(2.49b)}{\leq} \tau\|\eta_2\| g_1(\mathcal{M}y_1) \leq \tau\tilde{K}\|\eta_2\|.
\end{aligned}$$

Step two. We estimate $-\eta_2^\top \Theta D(\mathcal{M}y_2 + V\eta_2)$

$$\begin{aligned}
-\eta_2^\top \Theta D(\mathcal{M}y_2 + V\eta_2) &= \eta_2^\top \Theta\big(D(V\eta_2) - D(\mathcal{M}y_2 + V\eta_2) - D(V\eta_2)\big) \\
&\leq \tau\|\eta_2\|\|D(V\eta_2) - D(\mathcal{M}y_2 + V\eta_2)\| - \eta_2^\top \Theta D(V\eta_2) \\
&\overset{(2.49d)}{\leq} \tau\|\eta_2\| g_2(\mathcal{M}y_2) - \eta_2^\top \Theta D(V\eta_2) \overset{(2.49e)}{\leq} \tau\tilde{D}\|\eta_2\| - \delta\|\eta_2\|^2.
\end{aligned}$$

Step three. We estimate $-q\eta_1^\top \Theta K(\mathcal{M}y_1 + \bar{\lambda}V\eta_1)$

$$
\begin{aligned}
-q\eta_1^\top \Theta K(\mathcal{M}y_1 + \bar{\lambda}V\eta_1) &\le q\tau\|\eta_1\|\|K(\bar{\lambda}V\eta_1) - K(\mathcal{M}y_1 + \bar{\lambda}V\eta_1)\| \\
&\qquad - q\eta_1^\top \Theta K(\bar{\lambda}V\eta_1) \\
&\overset{(2.49b)}{\le} q\tau\|\eta_1\| g_1(\mathcal{M}y_1) - q\eta_1^\top \Theta K(\bar{\lambda}V\eta_1) \\
&\overset{(2.49c)}{\le} q\tau\tilde{K}\|\eta_1\| - q\kappa\bar{\lambda}\|\eta_1\|^2.
\end{aligned}
$$

Step four. We estimate $-q\eta_1^\top \Theta D(\mathcal{M}y_2 + V\eta_2)$

$$
\begin{aligned}
-q\eta_1^\top \Theta D(\mathcal{M}y_2 + V\eta_2) \le q\tau\|\eta_1\|\|D(\mathcal{M}y_2 + V\eta_2)\| &\overset{(2.49f)}{\le} q\|\eta_1\| d\|\mathcal{M}y_2 + V\eta_2\| \\
&\le q\tau d\mu\rho_2\|\eta_1\| + q\tau d\|\eta_1\|\|\eta_2\| \\
&\le q\tau d\mu\rho_2\|\eta_1\| + q\frac{\tau^2}{2}\|\eta_1\|^2 + q\frac{d^2}{2}\|\eta_2\|^2,
\end{aligned}
$$

where we used $2ab \le a^2 + b^2$ for all $a, b \in \mathbb{R}$ in the last line.
Step five. We estimate $-q\eta_1^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2)$

$$
\begin{aligned}
&-q\eta_1^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2) \\
&\quad = q\eta_1^\top \Theta\Big(C(\bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2) - C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)\Big)(\mathcal{M}y_2 + V\eta_2) \\
&\qquad - q\eta_1^\top \Theta C(\bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2) \\
&\quad \le q\tau\|\eta_1\|\Big\|\Big(C(\bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2) - C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)\Big)(\mathcal{M}y_2 + V\eta_2)\Big\| \\
&\qquad - q\eta_1^\top \Theta C(\bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2) \\
&\quad \overset{(2.49g)}{\le} q\tau\|\eta_1\| g_3(\mathcal{M}y_1) a_3(\mathcal{M}y_2 + V\eta_2) - q\eta_1^\top \Theta C(\bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2) \\
&\quad \overset{(2.49h)}{\le} q\tau\tilde{C}_3\|\eta_1\| a_3(\mathcal{M}y_2 + V\eta_2) - \frac{q}{\lambda}\|\eta_1\|^2\, b_3(\mathcal{M}y_2 + V\eta_2) \\
&\quad \le q\tau\tilde{C}_3\|\eta_1\| a_3(\mathcal{M}y_2 + V\eta_2) - \frac{q}{\lambda}\|\eta_1\|^2\, a_3(\mathcal{M}y_2 + V\eta_2) \\
&\quad = \frac{q}{\lambda}\left(\lambda\tau\tilde{C}_3 - \|\eta_1\|\right)\|\eta_1\| a_3(\mathcal{M}y_2 + V\eta_2).
\end{aligned}
$$

Step six. We estimate $-\eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2)$

$$\begin{aligned}
&-\eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)(\mathcal{M}y_2 + V\eta_2)\\
&\quad = \eta_2^\top \Theta\Big(C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, V\eta_2) - C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)\Big)(\mathcal{M}y_2 + V\eta_2)\\
&\qquad - \eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, V\eta_2)(\mathcal{M}y_2 + V\eta_2)\\
&\quad \le \tau\|\eta_2\|\|\big[C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, V\eta_2) - C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, \mathcal{M}y_2 + V\eta_2)\big](\mathcal{M}y_2 + V\eta_2)\|\\
&\qquad - \eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, V\eta_2)(\mathcal{M}y_2 + V\eta_2)\\
&\quad \overset{(2.49i)}{\le} \tau\|\eta_2\|\, g_4(\mathcal{M}y_2)\, a_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1)\, \|\eta_2\|\eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, V\eta_2)(\mathcal{M}y_2 + V\eta_2)\\
&\quad \overset{(2.49j)}{\le} \tau\|\eta_2\|^2\, g_4(\mathcal{M}y_2)\, a_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1) - b_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1)\|\eta_2\|\|V\eta_2\|^2\\
&\qquad - \eta_2^\top \Theta C(\mathcal{M}y_1 + \bar{\lambda}V\eta_1, V\eta_2)\mathcal{M}y_2\\
&\quad \overset{(2.49k)}{\le} \tau\tilde{C}_4\, a_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1)\|\eta_2\|^2 + \tau\mu\rho_2\, a_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1)\, \|\eta_2\|^2\\
&\qquad - b_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1)\, \|\eta_2\|^3\\
&\quad \le \Big(\tau(\tilde{C}_4 + \mu\rho_2) - \|\eta_2\|\Big)\, \|\eta_2\|^2\, a_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1),
\end{aligned}$$

where we used $\|Vz\| = \|z\|$ for all $z \in \mathbb{R}^{n-m}$ in the second last estimation. We summarize the calculations above to estimate $\mathcal{L}'(\eta_1, \eta_2)\cdot\Psi(\eta_1, \eta_2, y_1, y_2)$ for $\eta_i \in Z_i$ and $y_i \in B_{\rho_i}(0)$, $i = 1, 2$

$$\begin{aligned}
\mathcal{L}'(\eta_1, \eta_2)\cdot\Psi(\eta_1, \eta_2, y_1, y_2) \le\ & q\lambda\|\eta_2\|^2 + \tau\tilde{K}\|\eta_2\| + \tau\tilde{D}\|\eta_2\| - \delta\|\eta_2\|^2 + q\tau\tilde{K}\|\eta_1\|\\
&- q\frac{\kappa}{\lambda}\|\eta_1\|^2 + q\tau d\mu\rho_2\|\eta_1\| + q\frac{\tau^2}{2}\|\eta_1\|^2 + q\frac{d^2}{2}\|\eta_2\|^2\\
&+ \frac{q}{\lambda}\Big(\lambda\tau\tilde{C}_3 - \|\eta_1\|\Big)\, \|\eta_1\|a_3(\mathcal{M}y_2 + V\eta_2)\\
&+ \Big(\tau(\tilde{C}_4 + \mu\rho_2) - \|\eta_2\|\Big)\, \|\eta_2\|^2\, a_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1).
\end{aligned}$$

Sorting these expressions and inserting the constants from (2.50) yields

$$\begin{aligned}
\mathcal{L}'(\eta_1, \eta_2)\cdot\Psi(\eta_1, \eta_2, y_1, y_2) \le & -\varepsilon_1\|\eta_1\|^2 + E_1\|\eta_1\| - \varepsilon_2\|\eta_2\|^2 + E_2\|\eta_2\|\\
&- \frac{q}{\lambda}(\|\eta_1\| - \gamma_1)\|\eta_1\|a_3(\mathcal{M}y_2 + V\eta_2)\\
&- (\|\eta_2\| - \gamma_2)\, \|\eta_2\|^2\, a_4(\mathcal{M}y_1 + \bar{\lambda}V\eta_1),
\end{aligned} \tag{2.54}$$

where $\varepsilon_1, \varepsilon_2 > 0$ and $E_1, E_2 \ge 0$ via the choice of q and λ, and $\gamma_1, \gamma_2 \ge 0$. We consider the function

$$\begin{aligned}
W : \mathbb{R}^{2(n-m)} &\to \mathbb{R}\\
(w_1, w_2) &\mapsto -\varepsilon_1\|w_1\|^2 + E_1\|w_1\| - \varepsilon_2\|w_2\|^2 + E_2\|w_2\|,
\end{aligned} \tag{2.55}$$

with $\varepsilon_i > 0$, $E_i \ge 0$ for $i = 1, 2$ as above. A short calculation yields that for $w_i \in \tilde{Z}_i$ we have $W(w_1, w_2) \le 0$, and $-(\|w_1\| - \gamma_1) < 0$ and $-(\|w_2\| - \gamma_2) < 0$. Comparing (2.54) and (2.55) yields assertion (2.52) via

$$\mathcal{L}'(\eta_1, \eta_2)\cdot\Psi(\eta_1, \eta_2, y_1, y_2) \le W(\eta_1, \eta_2) \le 0,$$

for all $\eta_i \in Z_i \cap \tilde{Z}_i$, $i = 1, 2$, and $y_1 \in B_{\rho_1}(0)$, $y_2 \in B_{\rho_2}(0)$. This completes the proof. □

Seeking a formulation to verify stability of the internal dynamics of a system (2.41) in advance, we formulate an abstract stability result in the spirit of [121, Thm. 4]. Before we do so, we introduce the concept of bounded-input bounded-state stability following the notion in [3].

Definition 2.26. A control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{2.56}$$

where $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, is called *bounded-input bounded-state stable*, if for all $\bar{u} \geq 0$ and all $\bar{x}^0 \geq 0$ there exists $\bar{x} \geq 0$ (depending on $\bar{u}, \bar{x}^0$), such that for all initial conditions $x^0 \in \mathbb{R}^n$ with $\|x^0\| \leq \bar{x}^0$ and all inputs $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ with $\|u\|_\infty \leq \bar{u}$ all solutions $x : I \subseteq \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (2.56), I an interval with $0 \in I$, satisfy

$$\forall\, t \in I \;:\; \|x(t)\| \leq \bar{x}.$$

Lemma 2.27. Consider a control system (2.56), and assume there exist $\bar{u}, c \geq 0$ and a radially unbounded function $\mathcal{L} \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ such that for all $u \in B_{\bar{u}}(0)$ and for all $\zeta \in \{\, z \in \mathbb{R}^n \mid \|z\| > c \,\}$ we have

$$\mathcal{L}'(\zeta) \cdot f(\zeta, u) \leq 0. \tag{2.57}$$

Then, for all $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ with $\|u\|_\infty \leq \bar{u}$ and all solutions $\zeta : I \subseteq \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (2.56) with $\zeta(0) = \zeta^0 \in \mathbb{R}^n$, I an interval with $0 \in I$, there exists $\varepsilon > 0$ such that

$$\forall\, t \in I \;:\; \|\zeta(t)\| \leq \max\{\|\zeta^0\|, c\} + \varepsilon,$$

this means, the system is bounded-input bounded-state stable.

Proof. The following proof is inspired by the proof given in [121, Thm. 4], and some ideas are adopted from the proof in [183, Lem. 5.7.8]. Let $\tilde{c} := \max\{\|\zeta^0\|, c\}$ and $\lambda := \max\{\mathcal{L}(\zeta) \mid \|\zeta\| = \tilde{c}\}$. Since $\mathcal{L}$ is radially unbounded there exists $\varepsilon > 0$ such that $\mathcal{L}(\zeta) > \lambda$ for all $\zeta \in \{\, z \in \mathbb{R}^n \mid \|z\| \geq \tilde{c} + \varepsilon \,\}$. Seeking a contradiction, we suppose the existence of $t_1 \in I$ such that $\|\zeta(t_1)\| > \tilde{c} + \varepsilon$. Let $t_0 := \max\{\, t \in [0, t_1) \mid \|\zeta(t)\| = \tilde{c} + \varepsilon \,\}$. Then, we have $\mathcal{L}(\zeta(t)) > \lambda$ for all $t \in (t_0, t_1]$. Since by (2.57) $\mathcal{L}$ is nonincreasing along solution trajectories of (2.56) for all $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ with $\|u\|_\infty \leq \bar{u}$ and $\|\zeta\| > c$, we have $\lambda < \mathcal{L}(\zeta(t_1)) \leq \mathcal{L}(\zeta(t_0)) \leq \lambda$. A contradiction. Therefore, we conclude $\|\zeta(t)\| \leq \tilde{c} + \varepsilon$ for all $t \in I$. □

Now, we are in the position to formulate a result concerning the stability of the internal dynamics (2.47) of a system (2.41) with f structured as in (2.42).

Theorem 2.28. The internal dynamics (2.47) of a system (2.41), where f is structured as in (2.42) are bounded-input bounded-state stable, if $(K, D, C) \in \Sigma_{K,D,C}$.

Proof. The statement is a direct consequence of Lemmata 2.25 & 2.27. □

Remark 2.29. We emphasize, that conditions (2.49) on f can be verified in advance. Hence, it is not necessary to decouple the internal dynamics, i.e., it is not necessary to derive equations (2.28), or (2.47), respectively.

Example 2.30. We illustrate Theorem 2.28, i.e., we conclude stability of the internal dynamics of a given system (2.41) by verifying that the involved generalized forces f are such that $(K, D, C) \in \Sigma_{K,D,C}$. We consider the mass on a car system presented in [158, Sec. 4.2], which is an extension of the classical mass on a car system under consideration in [179, Sec. 4.1]. The system consists of two cars with mass m_1 (in kg) and m_2 (in kg), respectively. The two cars are coupled via a spring-damper combination with characteristics K_2 (in N/m) and D_2 (in Ns/m), respectively. On the second car a ramp with constant angle $0 < \alpha < \pi/2$ is mounted, on which a third mass m_3 (in kg) is lying and is coupled to the second car via a spring with characteristic K_3 (in N/m), and a damping with characteristic D_3 (in Ns/m). To the first car a force u_1 can be applied, and to the second car individually a force u_2 can be applied. We measure the horizontal position of the first car and the horizontal position of the mass m_3 on the ramp. The situation is depicted in Figure 2.4. For convenience we assume the constant force on m_3 due to gravity,

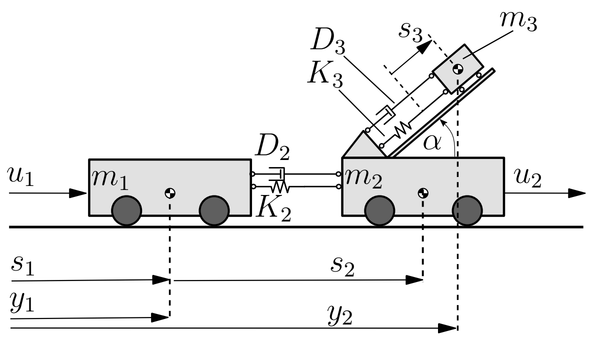


Figure 2.4: Extended mass on a car system. The original figure is taken from [158] and edited for the purpose of the present work.

namely $m_3 g \sin(\alpha)$, where g is the gravitational constant, to be compensated via a linear coordinate transformation, such that $K_3(0) = 0$. Then, according to [158, Sec. 4.2] with $s := (s_1, s_2, s_3)^\top \in \mathbb{R}^3$ the equations of motion for that system are given by

$$
\begin{aligned}
&\underbrace{\begin{bmatrix} m_1+m_2+m_3 & m_2+m_3 & m_3\cos(\alpha) \\ m_2+m_3 & m_2+m_3 & m_3\cos(\alpha) \\ m_3\cos(\alpha) & m_3\cos(\alpha) & m_3 \end{bmatrix}}_{=:M} \begin{pmatrix} \ddot{s}_1(t) \\ \ddot{s}_2(t) \\ \ddot{s}_3(t) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -K_2(s(t)) - D_2(\dot{s}(t)) \\ -K_3(s(t)) - D_3(\dot{s}(t)) \end{pmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=:B} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\
y(t) &= \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} s_1(t) \\ s_1(t) + s_2(t) + \cos(\alpha)s_3(t) \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & \cos(\alpha) \end{bmatrix}}_{=:H} \begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{pmatrix}.
\end{aligned}
\tag{2.58}
$$

In this particular example we have $n = 3$ and $m = 2$, hence we are in the situation of multi-input, multi-output. Note that the input and the output are not colocated, i.e., $H \neq B^\top$. According to the equations given in [158] we assume $K_2(s) = k \cdot s$ and

$D_2(s) = d \cdot s$, where $k, d > 0$. In order to include nonlinear terms, we assume that K_3 and D_3 have the following characteristics, where $\sigma(\cdot)$ denotes the sign function

$$K_3 : \mathbb{R}^3 \to \mathbb{R}, \ q \mapsto \begin{cases} \sigma(q_3)\sqrt{|q_3|}, & |q_3| \leq 1, \\ \frac{1}{2}q_3 + \frac{1}{2}\sigma(q_3), & |q_3| > 1, \end{cases}$$

and

$$D_3 : \mathbb{R}^3 \to \mathbb{R}, \ v \mapsto \begin{cases} \sigma(v_3)v_3^2, & |v_3| \leq 1, \\ 2v_3 - \sigma(v_3) & |v_3| > 1. \end{cases}$$

Note that $K_3, D_3 \in \mathcal{C}(\mathbb{R}^3; \mathbb{R})$. The schematic shapes of K_3 and D_3 are depicted in Figure 2.5.

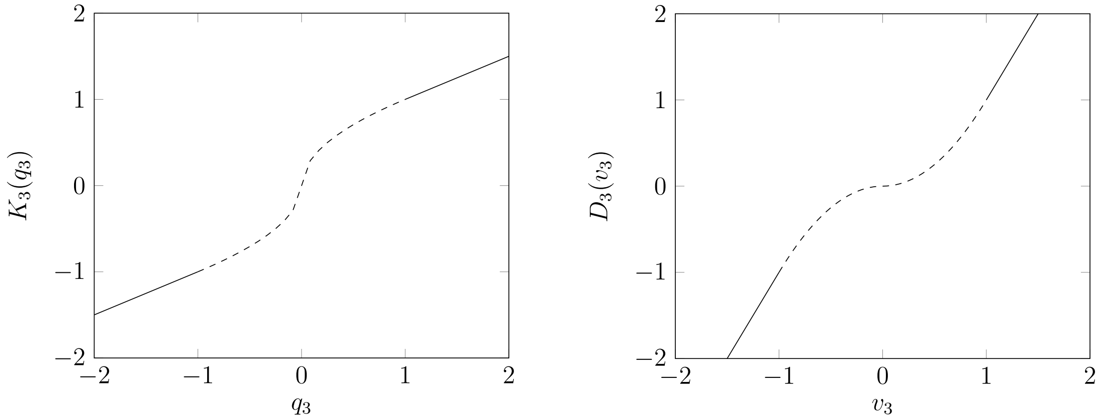


Figure 2.5: Schematic shape of K_3 and D_3, respectively. Solid lines on Z_i, dashed lines on $\mathbb{R} \setminus Z_i$, $i = 1, 2$.

We set $x_1 := (s_1, s_2, s_3)^\top$, $x_2 := \dot{x}_1$, $K := (0, K_2, K_3)^\top$, and $D := (0, D_2, D_3)^\top$, whereby $K, D \in \mathcal{C}(\mathbb{R}^3; \mathbb{R}^3)$, respectively. Further, we define $\tilde{H} := [H, 0]$ and set $\tilde{M} := \operatorname{diag}(I_3, M)$. Then, system (2.58) reads

$$\begin{aligned} \dot{x}(t) &= \tilde{M}^{-1} \begin{pmatrix} x_2(t) \\ -K(x_1(t)) - D(x_2(t)) \end{pmatrix} + \tilde{M}^{-1} \begin{bmatrix} 0 \\ B \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\ y(t) &= \tilde{H}x(t), \end{aligned} \tag{2.59}$$

and (2.59) is of the form (2.41). We set $\mu := m_2 + m_3 \sin(\alpha)^2$ and calculate

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & -\frac{1}{m_1} & 0 \\ -\frac{1}{m_1} & \frac{m_1+\mu}{m_1\mu} & -\frac{\cos(\alpha)}{\mu} \\ 0 & -\frac{\cos(\alpha)}{\mu} & \frac{m_2+m_3}{\mu m_3} \end{bmatrix},$$

and

$$\Gamma = HM^{-1}B = \begin{bmatrix} \frac{1}{m_1} & -\frac{1}{m_1} \\ 0 & \frac{\sin(\alpha)^2}{\mu} \end{bmatrix} \in \mathbf{Gl}_2(\mathbb{R}), \quad \Gamma^{-1} = \begin{bmatrix} m_1 & \frac{\mu}{\sin(\alpha)^2} \\ 0 & \frac{\mu}{\sin(\alpha)^2} \end{bmatrix}.$$

Therefore, assumption 2.5 is satisfied on $\mathbb{R}^3$ (B substituted by $M^{-1}B$) and thus, using Lemma 2.6, system (2.59) has relative degree $r = 2$ on $\mathbb{R}^6$. We calculate

$V = (0, -\cos(\alpha), 1)^\top$ by which $\operatorname{im} V = \ker H$. Then, according to (2.23), ϕ_2 is given by

$$\phi_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} (I_3 - M^{-1} B \Gamma^{-1} H) = \tfrac{1}{\sin(\alpha)^2} \begin{bmatrix} \cos(\alpha) & \cos(\alpha) & 1 \end{bmatrix},$$

and thus

$$\Theta := \phi_2 M^{-1} = \tfrac{1}{m_3 \sin(\alpha)^2} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

For the sets introduced in the definition of $\Sigma_{K,D,C}$ we have $Z_i = \{ z \in \mathbb{R} \mid |z| > 1 \}$, $i = 1, 2$. Now, we validate conditions (2.49a)–(2.49f) step by step. First, consider $V_K : Z_1 \to \mathbb{R}$ defined by

$$V_K : Z_1 \to \mathbb{R}, \; z_1 \mapsto \frac{1}{m_3 \sin(\alpha)^2} (z_1^2 - |z_1|),$$

which is radially unbounded. Note that $Z_1 = \mathbb{R} \setminus [-1, 1]$ and hence $V_K \in \mathcal{C}^1(Z_1; \mathbb{R})$. Then, for $z_1 \in Z_1$ the derivative of V_K is given by

$$\begin{aligned} V_K'(z_1) &= \frac{1}{m_3 \sin(\alpha)^2} (2z_1 - \sigma(z_1)) \\ &= \begin{pmatrix} 0 & -\cos(\alpha)\, k z_1 & 2z_1 - \sigma(z_1) \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_3 \sin(\alpha)^2} \end{bmatrix} = K(V z_1)^\top \Theta^\top, \; z_1 \in Z_1, \end{aligned}$$

thus (2.49a) is satisfied. Furthermore,

$$\|K(Vz_1) - K(Vz_1 + w)\| = \left\| \begin{pmatrix} 0 \\ -\cos(\alpha)\, k z_1 \\ 2z_1 - \sigma(z_1) \end{pmatrix} - \begin{pmatrix} w_1 \\ -\cos(\alpha)\, k z_1 + w_2 \\ 2z_1 - \sigma(z_1) + w_3 \end{pmatrix} \right\| = \|w\|,$$

which proves (2.49b), and

$$\begin{aligned} z_1 \Theta K(V z_1) &= z_1 \frac{1}{m_3 \sin(\alpha)^2} (2z_1 - \sigma(z_1)) \\ &= \frac{1}{m_3 \sin(\alpha)^2} (2z_1^2 - |z_1|) \geq \frac{1}{m_3 \sin(\alpha)^2} z_1^2, \quad z_1 \in Z_1, \end{aligned}$$

which shows (2.49c). Conditions (2.49d)–(2.49e) for D follow analogously for $z_2 \in Z_2 = Z_1$. For (2.49f) consider

$$\begin{aligned} \|D(V z_2)\| &= \left\| \begin{pmatrix} 0 \\ -\cos(\alpha)\, d z_2 \\ 2z_2 - \sigma(z_2) \end{pmatrix} \right\| \\ &\leq d|z_2| + |2z_2 - \sigma(z_2)| \leq (d+2)|z_2| + 1 \leq (d+3)|z_2|, \end{aligned}$$

for $z_2 \in Z_2$, which shows (2.49f). Therefore, via Theorem 2.28 we may deduce stability of the internal dynamics of the extended mass on a car system with equations of motion given in (2.58). Note that we conclude stability of the internal dynamics without having decoupled these explicitly, but just algorithmically calculate the respective functions, in particular, the function ϕ_2, which is given in (2.23). ◇

Remark 2.31. The function f in (2.41) with structure as in (2.42) covers a lot of physical meaningful mechanical systems, however, many other structures and configurations are possible. In particular, when systems of higher order (2.10) are under consideration. Although it is not reasonable to search for conditions similar to (2.49) for arbitrary functions of systems (2.10), Lemma 2.25 in combination with Remark 2.17 gives a good starting point, how potential Lyapunov functions may look like.

3 Output feedback control

In this chapter we focus on the long standing problem that most feedback control laws require derivatives of the output to be available. To this end, we elaborate on the so-called *funnel pre-compensator*, first proposed in [32], which turns out to be an appropriate tool to produce a signal which approximates the original signal and the derivatives of the pre-compensator's signal are available. We show that a conjunction of a minimum phase system of arbitrary relative degree with the funnel pre-compensator results in a minimum phase system of the same relative degree, whereby this overall conjunction is contained in the system class under consideration in [21] and hence, amenable to funnel control. As a consequence, the usage of the funnel pre-compensator enables output reference tracking without requiring to know derivatives of the system's output, that is, to perform output feedback control.

The funnel pre-compensator is a simple adaptive dynamical system of high-gain type which receives signals from a certain class of signals specified later, and has an output which approximates the input signal in the sense that the error between the input signal and the pre-compensator's output evolves within a prescribed performance funnel. Moreover, the derivatives of the pre-compensator's output are known explicitly. Comparing the preprint [31] to the work [32], it is clear that the funnel pre-compensator was inspired by the concept of high-gain observers, mainly inspired by the adaptive high-gain observer proposed in [45]. For detailed literature on high-gain observers we refer to [61, 108, 175, 187] and the survey [107] as well as the references therein, respectively. Nevertheless, we explicitly highlight two important properties of high-gain observers. First, high-gain observers can be used to estimate the system's state without the exact knowledge of the system parameters but only some structural assumptions, as for instance the relative degree of the system, are required. Second, high-gain observers are robust. However, in most cases the value of the high-gain parameter has to be evaluated via offline simulations. If it is chosen unnecessary large, the high-gain observer is very sensitive to high-frequent measurement noise. To overcome this drawback, in [45] the constant high-gain parameter has been replaced by a dynamical parameter function, which is determined by a differential equation involving the observation error. This adaption scheme produces a monotonically increasing parameter function as long as the error is larger than a predefined value; once reached this value the function stops increasing. However, the nondecreasing parameter function leads to high sensitivity with respect to strongly increasing signals or perturbations. Although convergence to a prescribed error is guaranteed the transient behaviour of the error cannot be prescribed. In contrast, the funnel pre-compensator introduced in [32] has an inherent different structure of the output, compared to high-gain observers; namely, it does not approximate the derivatives of the input signal but produces a "synthetic output", the derivatives of which are known exactly and this output evolves within a prescribed performance funnel around the input signal, that is, transient behaviour of the error can be prescribed. Although it is quite different, the funnel pre-compensator resembles a high-gain observer and adopts some of its benefits. Namely, the estimation of the signal requires only structural assumptions, such as knowledge of the relative degree of the underlying system. Furthermore, it is robust with respect to signal noise, cf. [32, Rem. 1]. An immensely important aspect of the funnel pre-compensator's simple adaption scheme, compared to, e.g., backstepping schemes, is the fact that no high powers of the gain function are involved. Therefore, issues in numerical implementation are circumvented without any estimations of the underlying model. For detailed discussion of this aspect in the context of high-gain observers see, e.g., the works [6, 105].

There is plenty of properly working high-gain based feedback controllers guaranteeing prescribed error performance. For *funnel control* schemes see, e.g., [91, 25], the recent work [21] or the construction of a bang-bang funnel controller cf. [127]. For *prescribed performance controller* see [9, 8]. However, all suffer from the problem that the output signal's derivatives (funnel control) or the full state (prescribed performance controller) have to be available to the control scheme. For funnel control, this means that if the output's derivatives are not available from measurement, the output measurement has to be differentiated, which is an ill-posed problem as pointed out in [74, Sec. 1.4.4]. A widespread idea in the literature to handle this topic is the so-called *backstepping* procedure, see for instance [114, 106]. This technique was studied in conjunction with an input filter, e.g., in the works [92, 93]. However, the backstepping procedure typically involves high powers of a "large-valued" gain function, which causes numerical issues and leads to impractical performances, see [74, Sec. 4.4.3]. Another approach to solve an *arbitrary good transient and steady-state response problem* for linear minimum phase systems with arbitrary relative degree is presented in [143]. The proposed controller involves an internal compensator scheme of LTI type, which allows to achieve an arbitrarily small error within an arbitrarily short time, while only receiving the system's output and the reference signal. Although this control scheme has a number of advantageous features, such as noise tolerance and applicability to systems with unknown relative degree to name but two (see also the survey [85]), it is an adaptive scheme with a monotonically non-decreasing gain-function and involves a (piecewise constant) switching function, where the switching times are determined in a two phase scheme of rather high complexity. In the works [50, 56] approaches to realize output tracking with prescribed error behaviour via output feedback only are presented. In [50] single-input single-output systems of known arbitrary relative degree with bounded input bounded state stable internal dynamics are under consideration. The control scheme involves higher derivatives of the output which are approximated by a high-gain observer. With this, tracking via output feedback can be realized. However, in this setting knowledge of the control coefficient is required and hence the particular control scheme is, in contrast to standard funnel control schemes, not model free. In [56] an extension of the prescribed performance controller [8] is used to achieve output tracking with prescribed error performance of unknown nonlinear multi-input multi-output systems with known vector relative degree. A high-gain observer scheme is used to make the required derivatives available. Since the control schemes in [50, 56] involve high-gain observers, both suffer from the problem of proper initializing, i.e., the high-gain parameters are to be predetermined appropriately. However, it is not clear how to choose these parameters appropriately in advance. In [130] an output feedback funnel control scheme is developed, which achieves output tracking with prescribed transient behaviour for a class of nonlinear single-input single-output systems, where the nonlinearity is a function of time and the output variable only. In particular, the problem of choosing parameters appropriately in advance is circumvented.

As mentioned above the derivatives of the pre-compensator's output are known explicitly, and hence the aforesaid gives rise to the idea that the funnel pre-compensator scheme proposed in [32] can help resolving the long-standing problem of adaptive feedback control with prescribed error performance of nonlinear systems with relative degree higher than one with unknown output derivatives. In order to resolve this problem, we show that the application of a cascade of funnel pre-compensators to a minimum phase system of arbitrary relative degree yields a system of the same relative degree, which is minimum phase as well. In particular, the derivatives of the pre-compensator's output are known explicitly. Therefore, output reference tracking with prescribed transient behaviour using well known funnel control schemes for systems of arbitrary (possibly high) relative degree, as for instance from [25] or the recent work [21], is possible without knowledge of the system's output derivatives. In particular, the tracking error between the original system's output and the desired reference trajectory evolves within a prescribed performance funnel. For systems of relative degree two this was shown in [32] and this result was used for funnel control in [30], but for arbitrary relative degree $r \in \mathbb{N}$ this remained an open problem which we solve in the present chapter.

3.1 The funnel pre-compensator

The *funnel pre-compensator*, first introduced in [32], is a dynamical input-output system of high-gain type in the spirit of funnel control. For details concerning the idea of funnel control we refer to Section 1.1, and the classical works [91, 87], the recent works [25, 21], and the references therein, respectively. The funnel pre-compensator receives signals u and w belonging to a certain set of signals $\mathcal{P}_r$ introduced below, and gives a signal z as an output, the derivative of which is explicitly given by the pre-compensator's equations; and z tracks the signal w with prescribed error.

Before we recall and investigate the funnel pre-compensator introduced in [32] we highlight that, contrary to most approaches, the funnel pre-compensator does not necessarily receive signals u and w which are input and output of a dynamical system or a corresponding plant, but we consider signals u and w belonging to the large set

$$\mathcal{P}_r := \left\{ (u,w) \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{R}^m) \times \mathcal{W}^{r,\infty}_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{R}^m) \,\middle|\, \begin{array}{l} \exists\, \Gamma \in \mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R}^{m\times m}) : \\ \Gamma w^{(r-1)} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^m), \\ \frac{\mathrm{d}}{\mathrm{d}t}\big(\Gamma w^{(r-1)}\big) - u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^m) \end{array} \right\}.$$

We emphasize that it is not assumed to know the matrix valued function Γ. Only knowledge of the signals u and w and the number $r \in \mathbb{N}$ is assumed. It is self evident that the signals u and w can be the input and output of a corresponding plant. For example, for a suitable continuous function f, a bounded disturbance d and an input distribution matrix $B \in \mathbf{Gl}_m(\mathbb{R})$, let $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ be the input and $y \in \mathcal{C}^r(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ the output of a system

$$y^{(r)}(t) = f\left(d(t), y(t), \ldots, y^{(r-1)}(t)\right) + Bu(t).$$

Then, if $y, \dot{y}, \ldots, y^{(r-1)}$ are bounded we have $(u, y) \in \mathcal{P}_r$. The signal set $\mathcal{P}_r$, however, allows for a much larger class of dynamical systems, cf. [25] and the works [20, 91].

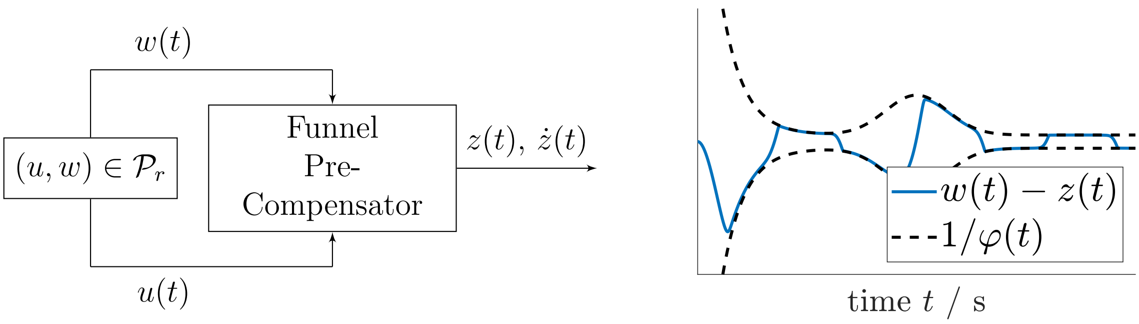
As mentioned above, the output z of the funnel pre-compensator approximates the input signal w with prescribed accuracy. To precise this property, we denote the error between the signals w and z with $e := w - z$. Then the error evolves within a performance funnel

$$\mathcal{F}_\varphi := \{\, (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \,\},$$

the shape of which is prescribed by a funnel function φ, which, for $r \in \mathbb{N}$, belongs to the set

$$\Phi_r := \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0}; \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded}, \\ \varphi(s) > 0 \text{ for all } s > 0, \liminf_{s \to \infty} \varphi(s) > 0 \end{array} \right\}.$$

The boundary of a performance funnel is given by the reciprocal of the associated funnel function, i.e., the boundary is given by $1/\varphi$, see Figure 3.1b. The schematic structure of the funnel pre-compensator is depicted in Figure 3.1a. We highlight



(a) Schematic funnel pre-compensator.

(b) Approximation error $e = w - z$ and funnel boundary $1/\varphi$.

Figure 3.1: Schematic structure of an application of the funnel pre-compensator (3.1) to signals $(u, w) \in \mathcal{P}_r$, and its error. The figures are based on the respective figures in [32].

two important properties of funnel functions $\varphi \in \Phi_r$, cf. [32, Sec. 2].

(i) Functions with $\varphi(0) = 0$ are explicitly allowed. This means that the boundary $1/\varphi$ has a pole at $t = 0$. This property is important in the context of funnel based feedback control, where initial conditions of the form $\varphi(0)\|e(0)\| < 1$ occur, which are satisfied trivially for $\varphi(0) = 0$.

(ii) It is not required that the funnel functions increase monotonically, see Figure 3.1b. Although in most situations a monotonically decreasing funnel boundary is desirable, there may be situations where widening the funnel boundary over some time interval is beneficial, e.g., if the signal w is changing strongly or in the presence of (periodic) disturbances.

Now, after we established the pre-compensator's general shape and functionality, we recall its mathematical formulation as proposed in [32]. The funnel pre-compensator

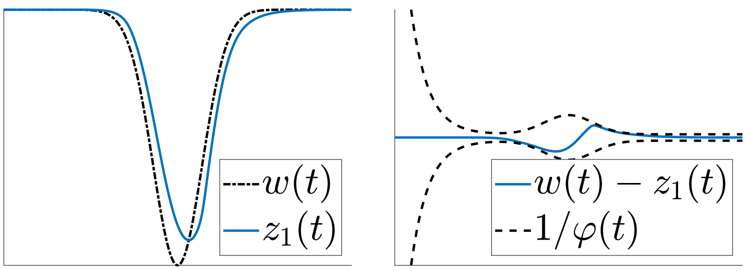
$$FP(a, p, \tilde{\Gamma}, \varphi) : (u, w) \mapsto (u, z_1)$$

is in general defined for signals $(u, w) \in \mathcal{P}_r$ and $\varphi \in \Phi_r$ via the following system of ordinary differential equations

$$\begin{aligned}
\dot{z}_1(t) &= \big(a_1 + p_1 h(t)\big)\big(w(t) - z_1(t)\big) + z_2(t), & z_1(0) &= z_1^0 \in \mathbb{R}^m,\\
\dot{z}_2(t) &= \big(a_2 + p_2 h(t)\big)\big(w(t) - z_1(t)\big) + z_3(t), & z_2(0) &= z_2^0 \in \mathbb{R}^m,\\
&\;\;\vdots & &\;\;\vdots\\
\dot{z}_{r-1}(t) &= \big(a_{r-1} + p_{r-1} h(t)\big)\big(w(t) - z_1(t)\big) + z_r(t), & z_{r-1}(0) &= z_{r-1}^0 \in \mathbb{R}^m,\\
\dot{z}_r(t) &= \big(a_r \quad + p_r \quad h(t)\big)\big(w(t) - z_1(t)\big) + \tilde{\Gamma} u(t), & z_r(0) &= z_r^0 \in \mathbb{R}^m,
\end{aligned} \tag{3.1}$$

$$h(t) := \frac{1}{1 - \varphi(t)^2 \|w(t) - z_1(t)\|^2},$$

where $\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ is the so-called high-gain matrix, which satisfies certain properties, see Definition 3.2. Further, the constants $a := (a_1, \ldots, a_r)$, $p := (p_1, \ldots, p_r)$ and the function φ are design parameters to be introduced and determined later in Definition 3.2. The pre-compensator being of funnel type shows up in the last line in (3.1), where the gain function h is introduced. If the error between the signals w and z_1 is small, the gain is close or equal to one; if the error is near the funnel boundary, i.e., if $\varphi \|w - z_1\|$ is close to one, the gain growths rapidly and forces z_1 into the direction of w. The situation is illustrated in Figure 3.2.



(a) Signal w and pre-compensator state z_1.

(b) Error $w - z_1$ and funnel boundary $1/\varphi$.

(c) Gain funtion h.

Figure 3.2: Exemplary application of the funnel pre-compensator to a signal w, the error $w - z_1$ and the shape of the gain function h determined by $\varphi \in \Phi_r$ and $w - z_1$.

The first result in [32] concerns feasibility of the funnel pre-compensator applied to given signals, namely [32, Prop. 1] guarantees prescribed performance of the error between the signal w and the pre-compensator's output $z := z_1$ as shown in Figure 3.2b. Moreover, the derivative $\dot{z}$ is known explicitly. However, although z approximates the input signal w arbitrarily good (as good as prescribed by the chosen funnel function), the higher derivatives $\dot{w}, \dots, w^{(r-1)}$, are not approximated. Neither in the sense that the higher derivatives $\dot{z}, \dots, z^{(r-1)}$ approximate the higher derivatives of w, nor that the states z_i, $i = 1, \dots, r$, approximate the higher derivatives of w in the sense of prescribed performance of the respective errors. Having in mind that we are searching for a signal, the higher derivatives of which are known, the previous observation motivates a successive application of the funnel pre-compensator, resulting in a *cascade of funnel pre-compensators* as introduced in [32, Sec. 2]. This means, to apply funnel pre-compensators in a row to the preceding system, which is already a funnel pre-compensator, i.e., for $i, n \in \mathbb{N}$ we have $FP : \mathcal{P}_n \to \mathcal{P}_n, (u, z_{i-1,1}) \mapsto (u, z_{i,1})$. Note that applying pre-compensators in a row requires an adaptation of notation. We introduce double indices to indicate the states, i.e., $z_{i,j}$ denotes the j^{th} state of the i^{th} pre-compensator, and the gain function $h_i := 1/(1 - \varphi^2 \|z_{i-1,1} - z_{i,1}\|^2)$ is indexed respectively. Applying $r - 1$ pre-compensators in a row we obtain for $\varphi_1, \varphi \in \Phi_r$ the following structure

$$\begin{aligned} FP(a, p, \tilde{\Gamma}, \varphi) \circ \dots \circ FP(a, p, \tilde{\Gamma}, \varphi) \circ FP(a, p, \tilde{\Gamma}, \varphi_1) : \mathcal{P}_r \to \mathcal{P}_r, \\ (u, w) \mapsto (u, z), \end{aligned} \tag{3.2}$$

where, except of the first, all pre-compensators in the cascade have the same funnel function $\varphi \in \Phi_r$ and the same high-gain matrix $\tilde{\Gamma}$. Further, as in the case of only one pre-compensator, the parameters $a, p > 0$ (componentwise) are introduced and discussed in Definition 3.2. The situation is depicted in Figure 3.3. For such a

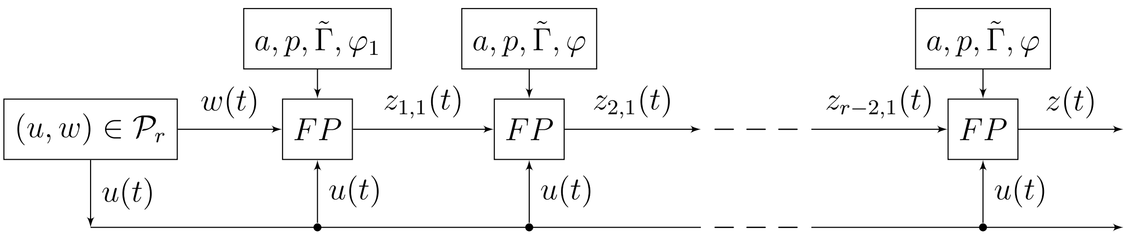


Figure 3.3: Cascade of $r-1$ funnel pre-compensators given in (3.2) applied to signals $(u, w) \in \mathcal{P}_r$. The figure is based on the respective figure in [32].

cascade of funnel pre-compensators (3.2) the result [32, Thm. 1] states that applied to signals $(u, w) \in \mathcal{P}_r$, the cascade yields a system with output $z := z_{r-1,1}$ such that the error $e = w - z$ evolves within a prescribed performance funnel, and moreover, all derivatives $\dot{z}, \dots, z^{(r-1)}$ are known and given explicitly via the intermediate states $z_{i,j}$, $i = 1, \dots, r - 1$, $j = 1, \dots, r$.

Remark 3.1. Consider a cascade of $r - 1$ funnel pre-compensators (3.2). We discuss and present an explicit representation of higher derivatives of the funnel pre-compensator's output $z = z_{r-1,1}$. For $i \geq 2$ we perform the following calculation involving equations (3.1)

$$\begin{aligned} \dot{z}_{i,1}(t) &= z_{i,2}(t) + (a_1 + p_1 h_i(t))(z_{i-1,1}(t) - z_{i,1}(t)), \\ \ddot{z}_{i,1}(t) &= z_{i,3}(t) + (a_2 + p_2 h_i(t))(z_{i-1,1}(t) - z_{i,1}(t)) \\ &\quad + p_1 \dot{h}_i(t)(z_{i-1,1}(t) - z_{i,1}(t)) + (a_1 + p_1 h_i(t))(\dot{z}_{i-1,1}(t) - \dot{z}_{i,1}(t)), \end{aligned}$$

with the gain functions h_i given in (3.1), the derivative of which are given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}h_i(t) = 2h_i(t)^2\Big(\varphi(t)\dot{\varphi}(t)\|z_{i-1,1}(t)-z_{i,1}(t)\|^2 \\ + \varphi(t)^2\langle z_{i-1,1}(t)-z_{i,1}(t), \dot{z}_{i-1,1}(t)-\dot{z}_{i,1}(t)\rangle\Big).$$

This motivates to introduce, in reference to [32, Rem. 3], the following functions. Let $h_k, \varphi_i \in \mathbb{R}$ and $\xi_i \in \mathbb{R}^m$. Then, for $k, l \in \{1,\dots,r-1\}$ and $i \geq 2$, we define recursively

$$\tilde{L}_0^{k,l}(h_k,\varphi_0,\xi_0) := (a_l + p_l h_k)\xi_0,$$
$$\tilde{L}_{i+1}^{k,l}(h_k,\varphi_0,\dots,\varphi_{i+1},\xi_0,\dots,\xi_{i+1}) := \frac{\partial \tilde{L}_i^{k,l}}{\partial h_k}\cdot\left(\varphi_0\varphi_1\|\xi_0\|^2 + \varphi_0^2\langle \xi_0,\xi_1\rangle\right) \\ + \sum_{j=0}^{i}\frac{\partial \tilde{L}_i^{k,l}}{\partial \varphi_j}\cdot\varphi_{j+1} + \sum_{j=0}^{i}\frac{\partial \tilde{L}_i^{k,l}}{\partial \xi_j}\cdot\xi_{j+1}.$$

With this, defining $v_i := z_{i-1,1} - z_{i,1}$ and invoking equations (3.1), we obtain for $i \geq 2$ and $j < \min\{k,i\}$

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^j\Big((a_{k-j}+p_{k-j}h_i(t))v_i(t)\Big) = \tilde{L}_j^{i,k-j}\big(h_i(t),\varphi(t),\dots,\varphi^{(j)}(t),v_i(t),\dots,v_i^{(j)}(t)\big).$$

For $(u,w) \in \mathcal{P}_r$ and $i = 2,\dots,r-1$, $j = 0,\dots,r-1$ we define

$$L_j^1(t) := \sum_{s=0}^{j-1}\tilde{L}_s^{1,j-1}\Big(h_1(t),\varphi_1(t),\dots,\varphi_1^{(s)}(t),w(t)-z_{1,1}(t),\dots,w^{(s)}(t)-z_{1,1}^{(s)}(t)\Big),$$
$$L_j^i(t) := \sum_{s=0}^{j-1}\tilde{L}_s^{i,j-1}\Big(h_i(t),\varphi(t),\dots,\varphi^{(s)}(t),v_i(t),\dots,v_i^{(s)}(t)\Big).$$

Then, invoking equations (3.1), (3.2) we have

$$\begin{aligned} z_{1,1}^{(j)}(t) &= z_{1,j+1} + L_j^1(t), \\ z_{i,1}^{(j)}(t) &= z_{i,j+1} + L_j^i(t). \end{aligned} \tag{3.3}$$

In particular, for the funnel pre-compensator's output $z := z_{r-1,1}$ we make the following observation. The function L_j^{r-1} depends on the first $j-1$ derivatives of z and on the same number of derivatives of $z_{r-2,1}$. Whilst the dependence on the derivatives of z successively leads to a dependency of $z^{(j)}$ on $z_{r-1,1},\dots,z_{r-1,j+1}$, the recursive reapplication of (3.3) to $z_{r-2,1},\dots,z_{r-2,1}^{(j-1)}$ leads to an explicit dependence of $z^{(j)}$ on the intermediate pre-compensator states $z_{r-1,1},\dots,z_{r-1,j+1},\dots,z_{r-j,1},z_{r-j,2},z_{r-j-1,1}$. Figure 3.4 gives a clarifying picture of the aforesaid. Observing that for $j = r-1$ the derivative $z^{(r-1)}$ depends on $z_{r-j-1,1} = z_{0,1} = w$ it is clear that the "length" $r-1$ of the cascade corresponds to the number of required derivatives of $z = z_{r-1,1}$. We exemplify formula (3.3) for $z^{(j)}$ for the case $r = 3$ and $j = 2$. We set $v_0 := w - z_{1,1}$ and $v_1 := z_{1,1} - z_{2,1} = z_{1,1} - z$. Then for $(u,w) \in \mathcal{P}_3$ we obtain

$$\begin{aligned}\ddot{z}(t) &= z_{2,3}(t) + (a_2 + p_2h_2(t))v_1(t) \\ &\quad + (a_1+p_1h_2(t))\Big(z_{1,2}(t) + (a_1+p_1h_1(t))v_0(t) - \big(z_{2,2}(t) + (a_1+p_1h_2(t))v_1(t)\big)\Big) \\ &\quad + 2p_1h_2(t)^2\varphi(t)\dot{\varphi}(t)\|v_1(t)\|^2 \\ &\quad + 2p_1h_2(t)^2\varphi(t)^2\Big\langle v_1(t), z_{1,2}(t) + (a_1+p_1h_1(t))v_0(t) - \big(z_{2,2}(t)+(a_1+p_1h_2(t))v_1(t)\big)\Big\rangle.\end{aligned}$$

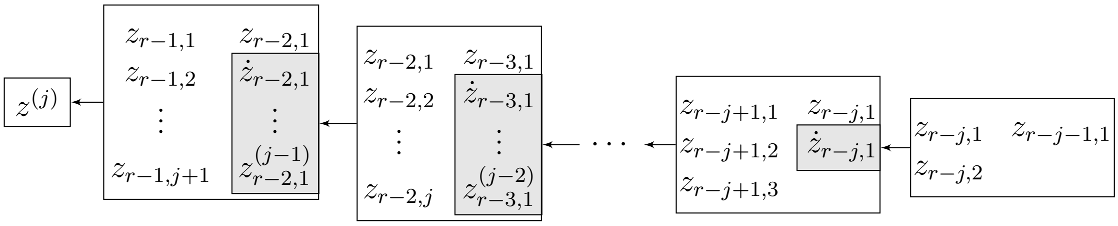


Figure 3.4: Schematic dependence of the derivatives $z^{(j)}$ on the intermediate pre-compensator states $z_{i,j}$. The figure is based on the respective figure in [32].

At the first glance, the expression for $\ddot z$ looks lengthy and awkward to handle. However, we stress that with the aid of formula (3.3) the computation of the derivatives of z can be performed completely algorithmically, involving symbolic differentiation and matrix vector multiplication.

Next, we introduce the set of feasible design parameters for the funnel pre-compensator.

Definition 3.2. We define the set

$$\Sigma := \left\{ (a, p, \varphi, \varphi_1, \rho, \Theta, \tilde\Gamma) \in \mathbb{R}^r \times \mathbb{R}^r \times \Phi_r \times \Phi_r \times \mathbb{R} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times m} \,\middle|\, \text{(A.1)–(A.4) hold} \right\},$$

where (A.1) – (A.4) denote the following properties.

(A.1) The numbers a_i are such that $a_i > 0$ for all $i = 1, \dots, r$, and

$$A := \begin{bmatrix} -a_1 & 1 & & \\ \vdots & & \ddots & \\ -a_{r-1} & & & 1 \\ -a_r & & & 0 \end{bmatrix} \in \mathbb{R}^{r\times r}$$

is Hurwitz, i.e., $\sigma(A) \subseteq \mathbb{C}_-$. Furthermore, let $P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_4 \end{bmatrix} > 0$, with $P_1 \in \mathbb{R}$, $P_2 \in \mathbb{R}^{1\times(r-1)}$, $P_4 \in \mathbb{R}^{(r-1)\times(r-1)}$ be the solution of

$$A^\top P + PA + Q = 0$$

for some symmetric $Q \in \mathbb{R}^{r\times r}$ with $Q > 0$; then p is defined as

$$\begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} := P^{-1} \begin{pmatrix} P_1 - P_2 P_4^{-1} P_2^\top \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -P_4^{-1} P_2^\top \end{pmatrix}.$$

(A.2) The funnel functions $\varphi_1, \varphi \in \Phi_r$ from (3.2) satisfy

$$\exists\, \rho > 1\ \forall\, t \ge 0 : \ \varphi(t) = \rho\, \varphi_1(t).$$

(A.3) The matrix $\tilde\Gamma$ from (3.2) is symmetric and definite (w.l.o.g. we may assume $\tilde\Gamma > 0$) and moreover, for a given symmetric $0 < \Theta \in \mathbb{R}^{m\times m}$ we have

$$\Theta \tilde\Gamma^{-1} = \left(\Theta \tilde\Gamma^{-1}\right)^\top > 0.$$

(A.4) For the matrix $\Theta \in \mathbb{R}^{m\times m}$ from (A.3), $\tilde{\Gamma}$ from (3.2) and ρ from (A.2) the matrix $G := I_m - \Theta\tilde{\Gamma}^{-1}$ satisfies

$$\|G\| < \min\left\{\frac{\rho-1}{r-2}, \frac{\rho}{4\rho^2(\rho+1)^{r-2}-1}\right\}.$$

If conditions (A.1) – (A.4) hold we write $(a, p, \varphi, \varphi_1, \rho, \Theta, \tilde{\Gamma}) \in \Sigma$.

Remark 3.3. We comment on the conditions (A.1) - (A.4).

(i) At the first glance, there are r parameters to be chosen appropriately satisfying (A.1). However, considering the polynomial $(s + s_0)^r$, which has all its roots in $\mathbb{C}_-$ for $s_0 > 0$, it is clear, with

$$(s+s_0)^r = s^r + \sum_{i=1}^{r} a_i s^{r-i},$$

that

$$A := \begin{bmatrix} -a_1 & 1 & & \\ \vdots & & \ddots & \\ -a_{r-1} & & & 1 \\ -a_r & & & 0 \end{bmatrix} \in \mathbb{R}^{r\times r}, \quad \sigma(A) \subseteq \mathbb{C}_-,$$

that is, the matrix A is Hurwitz. Moreover, the simple choice $Q = I_m$ is always feasible; and the matrix P is completely determined by the choice of A and Q. With this, since $p_1, \ldots, p_r$ are given by the Lyapunov matrix P, all parameters required to satisfy (A.1) can be determined by choosing one real number $s_0 > 0$.

(ii) Condition (A.2) means that the first funnel, which limits the error $w - z_{1,1}$, is somewhat wider than the others. As we will see later in Section 3.3 this is relevant in the case of output tracking.

(iii) Property (A.3) in particular asks for regularity of the matrix product $\Theta\tilde{\Gamma}^{-1}$, and (A.4) ensures that the matrix $\tilde{\Gamma}$ is "not too different" from matrix Θ. We highlight that, although the matrix Θ is not assumed to be known, (A.3) & (A.4) are feasible. In the special case if Θ is known, (A.3) & (A.4) can be trivially satisfied via the choice $\tilde{\Gamma} = \Theta$.

(iv) If the funnel pre-compensator is applied to signals $(u, w) \in \mathcal{P}_r$, then conditions (A.3) & (A.4) are asked to be satisfied for $\Theta = \Gamma$.

Example 3.4. We illustrate an application of a cascade of funnel pre-compensators to given signals $(u, w) \in \mathcal{P}_r$. Moreover, we qualitatively compare how the parameters influence its performance. We choose the signals

$$w(t) = e^{-(t-5)^2} - 0.5, \quad u(t) = \sin(t),$$

and, since $(u, w) \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$, we choose to simulate the application of a cascade consisting of two funnel pre-compensators (3.2), i.e., we consider the case $r = 3$. Initially we choose a Hurwitz polynomial, i.e., a polynomial whose roots have strict negative real part, to determine the matrix $A \in \mathbb{R}^{3\times 3}$ from (A.1) via the polynomial's coefficients, cf. Remark 3.3. During various simulations with different parameters it turned out that the parameters a_i influence the pre-compensator's performance the most. So we compare three sets of these parameters. To this end, we choose the Hurwitz polynomials

$$\begin{aligned}(\alpha+1)^3 &= \alpha^3 + 3\alpha^2 + 3\alpha + 1,\\ (\beta+3)^3 &= \beta^3 + 9\beta^2 + 27\beta + 27,\\ (\gamma+5)^3 &= \gamma^3 + 15\gamma^2 + 75\gamma + 125,\end{aligned}$$

which determine corresponding matrices

$$A = \begin{bmatrix} -3 & 1 & 0\\ -3 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}, \qquad B = \begin{bmatrix} -9 & 1 & 0\\ -27 & 0 & 1\\ -27 & 0 & 0\end{bmatrix}, \qquad C = \begin{bmatrix} -15 & 1 & 0\\ -75 & 0 & 1\\ -125 & 0 & 0\end{bmatrix},$$

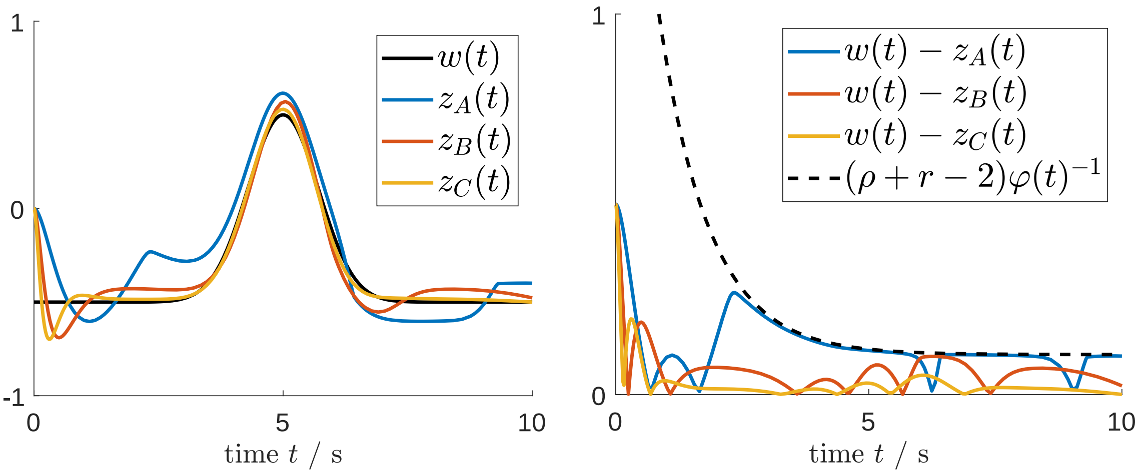
so $a_1 = 3$, $a_2 = 3$, $a_3 = 1$, $b_1 = 9$, $b_2 = 27$, $b_3 = 27$, and $c_1 = 15$, $c_2 = 75$, $c_3 = 125$. Choosing $Q := I_3$ the respective Lyapunov matrix P_i, $i \in \{A, B, C\}$, is given as

$$P_A = \begin{bmatrix} 1 & -\frac{1}{2} & -1\\ -\frac{1}{2} & 1 & -\frac{1}{2}\\ -1 & -\frac{1}{2} & 4\end{bmatrix}, \quad P_B = \begin{bmatrix} 4 & -\frac{1}{2} & -\frac{22}{27}\\ -\frac{1}{2} & \frac{22}{27} & -\frac{1}{2}\\ -\frac{22}{27} & -\frac{1}{2} & \frac{61}{81}\end{bmatrix}, \quad P_C = \begin{bmatrix} \frac{58}{5} & -\frac{1}{2} & -\frac{136}{125}\\ -\frac{1}{2} & \frac{136}{125} & -\frac{1}{2}\\ -\frac{136}{125} & -\frac{1}{2} & \frac{1333}{3125}\end{bmatrix},$$

and so

$$p_a = \begin{pmatrix} 1\\ \frac{2}{3}\\ \frac{1}{3}\end{pmatrix}, \quad p_b = \begin{pmatrix} 1\\ \frac{1037}{481}\\ \frac{1787}{711}\end{pmatrix}, \quad p_c = \begin{pmatrix} 1\\ \frac{1383}{391}\\ \frac{2230}{333}\end{pmatrix}.$$

Next, we choose the funnel function $\varphi(t) = (e^{-t} + 0.05)^{-1}$, and $\varphi_1(t) = \varphi(t)/\rho$ for $\rho = 1.1$. As initial values we set $z_{i,j}(0) = 0$ for $i = 1, 2$, $j = 1, 2, 3$ satisfying the conditions in [32, Thm. 1]. We run the simulation over the time interval $0 - 10$ seconds. The outcome is depicted in Figure 3.5. In Figure 3.5a the signal w and the



(a) Signal w and the pre-compensator's output for different choices of parameters, respectively.

(b) Errors between the signal w and the pre-compensator's output for different choices of parameters, respectively.

Figure 3.5: Illustration of functionality of the funnel pre-compensator applied to given signals $(u, w) \in \mathcal{P}_3$.

pre-compensator's respective output z_i, $i \in \{A, B, C\}$, is shown. We observe that the "approximation quality" strongly depends on the parameters a_i satisfying (A.1). While in the first case the signals w and z_A differ quite much, the second case shows a much better approximation; and in the third case the signal w and the output z_C are almost identical. In Figure 3.5b the respective error between the signal w and the pre-compensator's output z is depicted. Here the difference of approximation quality mentioned before crystallises from the viewpoint of errors, which are quite different. Although the signals z_A, z_B, z_C approximate the signal w quite differently, we emphasize that in all three cases the error evolves within the prescribed funnel boundaries. This means all approximations are at least as good as a "predetermined quality". The simulations have been performed in MATLAB (solver: `ode23tb`, default tolerances). ◇

3.2 Preservation of minimum phase

As discussed in Chapter 2, most high-gain feedback control schemes require the internal dynamics to be stable, i.e., the application of such control laws is restricted to minimum phase systems. Therefore, in order to utilize the funnel pre-compensator to enable high-gain feedback control with unknown output derivatives, it is decisive that the conjunction of a minimum phase system with a funnel pre-compensator results in a minimum phase system. For this reason, in Section 3.2.2 we focus on the question formulated in [32, Rem. 4], namely if the interconnection of a minimum phase system with a cascade of funnel pre-compensators yields a minimum phase system for relative degree larger than three. In fact, we aim to answer this question for the case of relative degree larger than two, since a careful inspection reveals that the proof of [32, Thm. 2], where this question seems to be answered for relative degree two and three, is incomplete as regards the boundedness of the gain functions h_1, h_2 in the case $r = 3$ on the one hand and on the other hand, the property (T.3) of the involved operator $\tilde{\mathbf{T}}$. In this section we show that for arbitrary $r \in \mathbb{N}$ the interconnection of a cascade of $r-1$ funnel pre-compensators with a minimum phase system with relative degree r yields a system of the same relative degree which is minimum phase as well.

3.2.1 System class

We introduce the system class under consideration in this chapter. Recalling the operator class $\mathcal{T}_\sigma^{n,q}$ from Definition 1.4, we introduce a system class $\mathcal{L}^{m,r}$, which is the same class of systems under consideration in [32], i.e., multi-input multi-output minimum phase systems, where the input and the output have common dimension.

Definition 3.5. If for $m, p, q \in \mathbb{N}$ we have $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R}^m)$, $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^p)$, for some $\tau > 0$ the operator $\mathbf{T}$ belongs to the class $\mathcal{T}_\tau^{rm,q}$, and the matrix $\Gamma \in \mathbb{R}^{m \times m}$ is symmetric and definite (w.l.o.g. we assume $\Gamma > 0$), then for $R_i \in \mathbb{R}^{m \times m}$, $i = 1, \ldots, r$, we say a system

$$y^{(r)}(t) = \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\left(d(t), \mathbf{T}\big(y, \dot{y}, \ldots, y^{(r-1)}\big)(t)\right) + \Gamma\, u(t), \tag{3.4}$$

with given initial trajectory $y|_{[-\tau,0]} = y^0 \in \mathcal{W}^{r-1,\infty}([-\tau,0];\mathbb{R}^m)$, belongs to the class $\mathcal{L}^{m,r}$; with the auxiliary tupel $R := (R_1,\dots,R_r) \in \mathbb{R}^{m\times m} \times \dots \times \mathbb{R}^{m\times m}$ we write

$$(d, f, \mathbf{T}, \Gamma, R) \in \mathcal{L}^{m,r}.$$

In Definition 3.5 of the system class $\mathcal{L}^{m,r}$ the number $r \in \mathbb{N}$ is the *relative degree* and $\tau > 0$ is the *memory* of the system. The function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is called *input*, the function $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ *output* of system (3.4), respectively. From the viewpoint of applications, the function f in (3.4) encodes internal and external forces such as restoring forces, friction and Coriolis forces, to name but a few. Moreover, the operator $\mathbf{T}$ allows to model systems with hysteresis effects, nonlinear delay elements or systems with infinite dimensional linear internal dynamics, cf. Remark 1.5, for detailed applications see, e.g., [28, 18, 17, 27]. In the context of systems (3.4), condition (T.1) in Definition 1.4 resembles a minimum phase property or, more precisely, an input to state stability of the internal dynamics, where from the viewpoint of the internal dynamics the system's output and its derivatives act as inputs, cf. Section 2.1.

In what follows, we investigate the conjunction of a minimum phase system with a cascade of funnel pre-compensators. We aim to show that this conjunction results in a minimum phase system. The assumption that the only information available from the system is its output y motivates the following definition of a subclass of $\mathcal{T}^{n,q}_{\sigma}$, which then allows to introduce a subclass of the system class $\mathcal{L}^{m,r}$.

Definition 3.6. For $r, m \in \mathbb{N}$, $n = rm$ and $1 \leq k \leq r$ we define the operator class $\mathcal{T}^{n,q}_{\sigma,k} := \left\{ \mathbf{T} \in \mathcal{T}^{n,q}_{\sigma} \,\middle|\, \mathbf{T} \text{ satisfies } (\mathrm{T}^{\mathrm{k}}.1) \right\} \subseteq \mathcal{T}^{n,q}_{\sigma}$ (equality if $k = r$), where

(T^{k}.1) for all $c_1 > 0$ there exists $c_2 > 0$ such that for all $\xi_1,\dots,\xi_r \in \mathcal{C}([-\sigma,\infty);\mathbb{R}^m)$

$$\sup_{t\in[-\tau,\infty)} \left\| \left(\xi_1(t)^\top,\dots,\xi_k(t)^\top\right)^\top \right\| \leq c_1 \;\Rightarrow\; \sup_{t\in[0,\infty)} \|\mathbf{T}(\xi_1,\dots,\xi_r)(t)\| \leq c_2.$$

Condition (T^{k}.1) means boundedness of $\mathbf{T}(\xi_1,\dots,\xi_r)$ whenever its first $k \leq r$ input arguments are bounded, which is, for $k < r$, a stronger version of condition (T.1) in Definition 1.4. With this, we may introduce a subclass of $\mathcal{L}^{m,r}$, namely systems of the form (3.4), where the operator satisfies (T^{k}.1), i.e., $\mathbf{T} \in \mathcal{T}^{n,q}_{\sigma,k}$. We set

$$\mathcal{L}^{m,r}_k := \left\{ (d, f, \mathbf{T}, \Gamma, R) \in \mathcal{L}^{m,r} \,\middle|\, \mathbf{T} \in \mathcal{T}^{n,q}_{\sigma,k} \right\} \subseteq \mathcal{L}^{m,r} \text{ (equality if } k = r).$$

Remark 3.7. The somewhat arcane condition $\mathbf{T} \in \mathcal{T}^{rm,q}_{\sigma,1}$ reflects the intuition that in order to have the conjunction of the system with the funnel pre-compensator being a minimum phase system, we must be able to conclude from the available information (only the output y) that the internal dynamics stay bounded. Note that this, however, *does not* mean that the operator $\mathbf{T}$ does not act on the output's derivatives, see Example 3.17.

Remark 3.8. We highlight an important subclass of (3.4), namely linear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned} \tag{3.5}$$

where $A \in \mathbb{R}^{n\times n}$ is the system matrix , $B \in \mathbb{R}^{n\times m}$ is the input distribution matrix, and $C : \mathbb{R}^n \to \mathbb{R}^m$ the linear output measurement, i.e., $C \in \mathbb{R}^{m\times n}$ for $m \le n$. Furthermore, $\operatorname{rk} C = \operatorname{rk} B = m$, whereby it is clear that the dimensions of the input and the output coincide. We define the set of functions

$$\mathcal{D}(\mathbb{R}_{\ge 0};\mathbb{R}^n) := \left\{ d \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0};\mathbb{R}^n) \;\middle|\; \begin{array}{l} \forall\, j = 0,\dots,r-1 : \\ CA^j d \in \mathcal{W}^{r-1-j,\infty}(\mathbb{R}_{\ge 0};\mathbb{R}^n) \end{array} \right\}.$$

We stress that $\mathcal{W}^{r-1,\infty}(\mathbb{R}_{\ge 0};\mathbb{R}^n) \subset \mathcal{D}(\mathbb{R}_{\ge 0};\mathbb{R}^n)$. If

$$\forall\, k \in \{0,\dots,r-2\} : CA^k B = 0,\ \Gamma := CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R}),\ \text{and } d \in \mathcal{D}(\mathbb{R}_{\ge 0};\mathbb{R}^n), \tag{3.6}$$

then, following the derivations and calculations in [95, Thm. 3] and involving the transformation U given by (2.4) in Section 2.1, with the operator

$$\begin{aligned} &L : \mathcal{D}(\mathbb{R}_{\ge 0};\mathbb{R}^n) \to \mathcal{L}^\infty(\mathbb{R}_{\ge 0};\mathbb{R}^n), \\ &\qquad d(\cdot) \mapsto \left(t \mapsto \begin{pmatrix} l_1(t) \\ \vdots \\ l_r(t) \\ 0_{n-rm} \end{pmatrix} \right),\ l_i(t) = \sum_{j=0}^{i-2} CA^j d^{(i-2-j)}(t),\ i = 1,\dots,r, \end{aligned}$$

the change of coordinates

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \\ \eta \end{pmatrix} = Ux + L(d) = \begin{pmatrix} y \\ \vdots \\ y^{(r-1)} \\ \eta \end{pmatrix}$$

transforms system (3.5) into *Byrnes-Isidori form*

$$\begin{aligned} \dot\xi_i(t) &= \xi_{i+1}(t), & \xi_i(0) &= \xi_i^0 \in \mathbb{R}^m, \\ \dot\xi_r(t) &= \sum_{j=1}^{r} R_j \xi_j(t) + S\eta(t) + \Gamma u(t) + d_r(t), & \xi_r(0) &= \xi_r^0 \in \mathbb{R}^m, \\ \dot\eta(t) &= Q\eta(t) + P\xi_1(t) + d_\eta(t), & \eta(0) &= \eta^0 \in \mathbb{R}^{n-rm}, \end{aligned} \tag{3.7a}$$

where $i = 1,\dots,r-1$, with output

$$y(t) = \xi_1(t). \tag{3.7b}$$

The matrices $R_1,\dots,R_r,S,P,Q$ are given in (2.5), and

$$\begin{aligned} d_r(t) &:= \sum_{j=0}^{r-1} \left(CA^j d^{(r-1-j)}(t) - R_{j+1} l_{j+1}(t) \right) \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to R^m), \\ d_\eta(t) &:= N\left(d(t) - AU^{-1}L(d)(t) \right) \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^{n-rm}), \end{aligned} \tag{3.8}$$

and the high-gain matrix Γ is given in (3.6). As we have already seen in Chapter 2, the last differential equation in (3.7a) describes the internal dynamics of system (3.5). Associating the linear integral operator

$$J : y(\cdot) \mapsto \left(t \mapsto \int_0^t e^{Q(t-s)} P y(s)\, \mathrm{d}s \right)$$

with the internal dynamics given in (3.7a) we obtain for $H(t) := e^{Qt}[0, I_{n-rm}]Ux^0$ and $D(t) := e^{Qt}\left(d_\eta(0) + \int_0^t e^{-Qs} d_\eta(s)\,\mathrm{d}s\right)$ the internal state

$$\eta(t) = D(t) + H(t) + J(y)(t).$$

Then, invoking (3.7) we find that the linear system (3.5) is equivalent to the functional differential equation

$$y^{(r)}(t) = \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\big(S(D(t) + H(t)), SJ(y)(t)\big) + \Gamma u(t) + d_r(t),$$

where $f(v, w) = v + w$ for $v, w \in \mathbb{R}^m$, and the operator J introduced above satisfies conditions (T.2) and (T.3) in Definition 1.4 specifying the operator class. The minimum phase property (condition (T.1)) for linear systems has various equivalent conditions, which have been studied extensively, see for instance [48, 15, 188]. In the present work, we restrict ourself mentioning the equivalence between system (3.5) being minimum phase, i.e., $\sigma(Q) \subseteq \mathbb{C}_-$, and having asymptotically stable zero dynamics, cf. [95], where referring exemplarily to [93, 98] the latter means

$$\forall\, \lambda \in \mathbb{C}_- \;:\; \operatorname{rk} \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} = n + m. \tag{3.9}$$

We highlight that for $\sigma(Q) \subseteq \mathbb{C}_-$ we have $D \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. Therefore, if system (3.5) has relative degree $r \in \mathbb{N}$ as in (3.6) and satisfies (3.9), the class of linear minimum phase systems is contained in the system class $\mathcal{L}_1^{m,r}$. If the disturbance does not affect the integrator chain but enters the system on the same level as the input does, i.e.,

$$\forall\, k = 0, \dots, r-2 \;:\; CA^k d(\cdot) = 0,$$

which is an often used assumption, cf. [16, 14], then $\mathcal{D}(\mathbb{R}_{\geq 0}; \mathbb{R}^n) = \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$. In this case the coordinate transformation simplifies to $(\xi^\top, \eta^\top)^\top = Ux$ and for the disturbance $(d_r^\top, d_\eta^\top)^\top = [(CA^{r-1})^\top, N^\top]^\top d$.

Moreover, $\mathcal{L}_1^{m,r}$ encompasses systems of the following form

$$\begin{aligned} y(t) &= \xi_1(t), \\ \dot{\xi}_i(t) &= \xi_{i+1}(t), \quad i = 1, \dots, r-1, \\ \dot{\xi}_r(t) &= f(t, \xi(t), \eta(t)) + \Gamma u(t), \\ \dot{\eta}(t) &= g(\eta(t), \xi_1(t)), \end{aligned} \tag{3.10}$$

where for $rm \leq n \in \mathbb{N}$ the function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm} \times \mathbb{R}^{n-rm} \to \mathbb{R}^m$ is locally Lipschitz in $(\xi, \eta) \in \mathbb{R}^{rm} \times \mathbb{R}^{n-rm}$, and piecewise continuous and bounded in t; $g : \mathbb{R}^{n-rm} \times \mathbb{R}^m; \mathbb{R}^{n-rm}$ is such that for $\xi_1 \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ the corresponding differential equation has a bounded solution; and $\Gamma \in \mathbf{Gl}_m(\mathbb{R})$ is symmetric and definite. Therefore, with constant input parameters, a subclass of the system class under consideration in [50] is contained in $\mathcal{L}_1^{1,r} \subset \mathcal{L}_1^{m,r}$. Moreover, the class $\mathcal{L}_1^{m,r}$ encompasses the system class under consideration in [93]. In [47, Cor. 5.7] explicit criteria on the parameters a, b of nonlinear systems of the form $\dot{x}(t) = a(x(t)) + b(x(t))u(t)$ are given such that it can be transformed into a system (3.10).

We conclude the introduction of the system class under consideration with the preceding remark and turn towards the conjunction of a minimum phase system with a cascade of funnel pre-compensators and its preservation of minimum phase.

3.2.2 A cascade of funnel pre-compensators in conjunction with a minimum phase system

In this section we present and prove the extension of [32, Thm. 2] for arbitrary relative degree $r \in \mathbb{N}$, that is, the conjunction of a minimum phase system belonging to the class $\mathcal{L}_1^{m,r}$ with a cascade of $r-1$ funnel pre-compensators results in a system of the same relative degree which is minimum phase as well.

Theorem 3.9. Consider a system (3.4) with $(d, f, \mathbf{T}, \Gamma, R) \in \mathcal{L}_1^{m,r}$ and given initial data $y^0 \in \mathcal{W}^{r-1,\infty}([-\tau, 0]; \mathbb{R}^m)$ (we emphasize $\mathbf{T} \in \mathcal{T}_{\tau,1}^{rm,q}$, $q \in \mathbb{N}$). Further, consider the cascade of funnel pre-compensators defined by (3.2) with $(a, p, \varphi, \varphi_1, \rho, \Gamma, \tilde{\Gamma}) \in \Sigma$ as in Definition 3.2 (note that (A.3) & (A.4) are asked to hold for $\Theta = \Gamma$). Assume the initial conditions

$$\varphi_1(0)\|y(0) - z_{1,1}^0\| < 1, \quad \varphi(0)\|z_{i-1,1}^0 - z_{i,1}^0\| < 1, \quad i = 2, \ldots, r-1, \tag{3.11}$$

are satisfied. Then, for $\bar{q} = rm(r-1) + r$, there exist $\tilde{d} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^r)$, a function $\tilde{F} \in \mathcal{C}(\mathbb{R}^r \times \mathbb{R}^{\bar{q}}; \mathbb{R}^m)$ and an operator $\tilde{\mathbf{T}} : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{rm}) \to \mathcal{L}_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{\bar{q}})$ with

$$(\tilde{d}, \tilde{F}, \tilde{\mathbf{T}}, \tilde{\Gamma}, 0_{rm\times m}) \in \mathcal{L}^{m,r},$$

such that the conjunction of (3.2) and (3.4) with input u and output $z := z_{r-1,1}$ can be equivalently written as

$$z^{(r)}(t) = \tilde{F}\big(\tilde{d}(t), \tilde{\mathbf{T}}(z, \dot{z}, \ldots, z^{(r-1)})(t)\big) + \tilde{\Gamma}u(t), \tag{3.12}$$

with respective initial conditions.

Sketch of proof. Before we prove Theorem 3.9 rigorously, we present a sketch of the proof, such that the reader has a guidance leading through the partly technical and quite long proof; the proof itself follows hereafter and is subdivided in four main steps. In the *Step one* we recall the transformations given in [32, pp. 4759-4760] which allow to analyse the error dynamics of two successive pre-compensators. *Step two* is the main part of the proof consisting of preparatory work to show that there exists an operator $\tilde{\mathbf{T}} \in \mathcal{T}_0^{rm,\bar{q}}$ such that the conjunction of a minimum phase system (3.4) with a cascade of funnel pre-compensators (3.2) can be written as in (3.12); the functions $\tilde{d}$ and $\tilde{F}$ are then given naturally. We define an operator $\tilde{\mathbf{T}}$ mapping the pre-compensator's output z and its derivatives to the state of an overall auxiliary error system (3.14) and the respective gain functions. In order to show that $\tilde{\mathbf{T}}$ satisfies condition (T.1) in Definition 1.4 we establish boundedness of the solution of the auxiliary system (3.14) and the respective gain functions. Here, *Step two* splits into two parts. First, using that $\mathbf{T}(y, \ldots, y^{(r-1)})$ is bounded whenever y is bounded (by $\mathbf{T} \in \mathcal{T}_{\tau,1}^{rm,q}$), we may define an overall system of errors of two successive pre-compensators, namely sytem (3.18) for $i = 3, \dots, r-1$,

$$\begin{aligned}
\dot{w}_1(t) &= \hat{A}w_1(t) - h_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}\bar{w}(t) + B_1(t), \\
\dot{w}_2(t) &= \hat{A}w_2(t) - h_2(t)\bar{P}w_{2,1}(t) + h_1(t)\bar{P}\bar{w}(t) + B_2(t), \\
\dot{w}_i(t) &= \hat{A}w_i(t) - h_i(t)\bar{P}w_{i,1}(t) + h_{i-1}(t)\bar{P}w_{i-1,1}(t) + B_i(t),
\end{aligned}$$

where the error-states $w_{i,j}$ stem from the transformations in *Step one*, and the compactly written states $w_i, \bar{w}$ are defined at the beginning of *Step two*; B_1, B_2, B_i are bounded functions. For this overall error system (3.18) we find a Lyapunov function, which in combination with Grönwall's lemma allows us to deduce boundedness of the error states w_{ij} for all $i = 1, \dots, r-1$ and $j = 1, \dots, r$. In the second part of *Step two*, which is the most technical part of the proof, we show that the gain functions h_i are bounded. This demands particular accuracy since each of the functions h_i may introduce a singularity. Due to the shape of the gain functions, namely $h(t) = \left(1 - \varphi(t)^2\|x(t)\|^2\right)^{-1}$, boundedness is equivalent to the existence of $\nu > 0$ such that $\|x(t)\| \le \varphi(t)^{-1} - \nu$, which is commonly utilized in standard funnel control proofs, cf. [25, pp. 350-351]. However, unlike the standard funnel case, the auxiliary error dynamics involve the respective previous gain function, and the first equation involves the last gain function. To see this, we anticipate the set of equations (3.26), which is for $i = 2, \dots, r-1$ given by

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_1(t)\|^2 &= -h_1(t)\|x_1(t)\|^2 + \langle x_1(t), h_{r-1}(t)\, Gx_{r-1}(t) + b_1(t)\rangle, \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_i(t)\|^2 &= -h_i(t)\|x_i(t)\|^2 + \langle x_i(t), h_{i-1}(t)x_{i-1}(t) + b_i(t)\rangle,
\end{aligned}$$

where x_i are auxiliary states defined in *Step two*; b_1, b_i are bounded functions. It turns out that this loop structure demands some technical derivations and requires accurate estimations of the involved expressions. Exploiting properties (A.1) – (A.4) of the design parameters we can show by contradiction that there exist $\kappa_i > 0$ such that $\|x_i(t)\| \le \varphi(t)^{-1} - \kappa_i$ for all $i = 2, \dots, r-1$, and $\|x(t)_1\| \le \varphi_1(t)^{-1} - \kappa_1$, respectively, which then implies boundedness of all gain functions h_i. In *Step three* we summarize the previously established results to deduce that the solution of the auxiliary error system (3.14) is globally defined. In *Step four* we show that the operator $\tilde{\mathbf{T}}$ belongs to the operator class $\mathcal{T}_0^{rm,\bar{q}}$. $\tilde{\mathbf{T}}$ satisfying conditions (T.1) & (T.2) in Definition 1.4 is immediate from the previous steps; so it remains to show that $\tilde{\mathbf{T}}$ satisfies the technical condition (T.3). We do so by interpreting the arguments of the operator as inputs for the error system (3.14) and show that the right-hand side of (3.14) satisfies a Lipschitz estimation. Then, with the aid of Grönwall's lemma we deduce $\tilde{\mathbf{T}} \in \mathcal{T}_0^{rm,\bar{q}}$. Finally, the functions $\tilde{d}$ and $\tilde{F}$ arise naturally in equation (3.60). Together, we may conclude that a minimum phase system (3.4) with $(d, f, \mathbf{T}, \Gamma, R) \in \mathcal{L}_1^{m,r}$, in conjunction with a cascade of funnel pre-compensators (3.2) with $(a, p, \varphi, \varphi_1, \rho, \tilde{\Gamma}) \in \Sigma$ as in Definition 3.2, can be equivalently written as a minimum phase system (3.12) belonging to $\mathcal{L}^{m,r}$.

Before we present the proof of Theorem 3.9, we establish the following statement about the Kronecker product of two matrices, which will be used within the proof. For two matrices $L \in \mathbb{R}^{l \times m}$ and $K = (k_{ij})_{i=1,\dots,k; j=1,\dots,n} \in \mathbb{R}^{k \times n}$ the Kronecker product is defined by

$$K \otimes L := \begin{bmatrix} k_{11}L & \cdots & k_{1n}L \\ \vdots & \ddots & \vdots \\ k_{k1}L & \cdots & k_{kn}L \end{bmatrix} \in \mathbb{R}^{kl \times mn}.$$

With this we state the following result from linear algebra.

Lemma 3.10. Let $A \in \mathbb{R}^{n \times n}$. Then,

(i) for the spectrum we have $\sigma(A \otimes I_n) = \sigma(A)$,

(ii) if $P = P^\top \in \mathbb{R}^{n \times n}$ solves $A^\top P + PA + Q = 0$ for some $Q = Q^\top \in \mathbb{R}^{n \times n}$, then

$$(A \otimes I_n)^\top (P \otimes I_n) + (P \otimes I_n)^\top (A \otimes I_n) + (Q \otimes I_n) = 0.$$

Proof. Assertion (i) is a direct consequence of [82, Thm. 4.2.12]. Invoking the rules of calculation for the Kronecker product, specifically $(A \otimes I_n)(P \otimes I_n) = AP \otimes I_n$, assertion (ii) is immediate. □

With the sketch above and Lemma 3.10 at hand, we may now present the proof of Theorem 3.9.

Proof of Theorem 3.9. In the following we assume $r \geq 3$, since the case $r = 2$ was already proven in [32, Thm. 2]. The proof is subdivided in four main steps.
Step one. We present the transformations performed in [32, p. 4758-4760] to study the error dynamics of two successive funnel pre-compensators. We define the error $e_{1,j} := y^{(j-1)} - z_{1,j}$ for $j = 1, \ldots, r-1$, and $e_{1,r} := y^{(r-1)} - \Gamma\tilde{\Gamma}^{-1} z_{1,r}$. Then, we obtain

$$\begin{aligned}
\dot{e}_{1,1}(t) &= e_{1,2}(t) - (a_1 + p_1 h_1(t)) e_{1,1}(t), \\
&\ \ \vdots \\
\dot{e}_{1,r-2}(t) &= e_{1,r-1}(t) - (a_{r-2} + p_{r-2} h_1(t)) e_{1,1}(t), \\
\dot{e}_{1,r-1}(t) &= e_{1,r}(t) - (a_{r-1} + p_{r-1} h_1(t)) e_{1,1}(t) + (\Gamma\tilde{\Gamma}^{-1} - I_m) z_{1,r}(t), \\
\dot{e}_{1,r}(t) &= -\Gamma\tilde{\Gamma}^{-1}(a_r + p_r h_1(t)) e_{1,1}(t) \\
&\quad + \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\big(d(t), \mathbf{T}(y, \dot{y}, \ldots, y^{(r-1)})(t)\big).
\end{aligned}$$

We set $v_{1,1} := e_{1,1}$ and $\tilde{v} := \sum_{i=1}^{r-1} v_{i,1}$, and for $j = 2, \ldots, r$ we define the auxiliary error $v_{1,j} := e_{1,j} - \sum_{k=1}^{j-1} R_{r-j+k+1} \tilde{v}^{(k-1)}$. With this we obtain

$$\begin{aligned}
\dot{v}_{1,1}(t) &= v_{1,2}(t) - (a_1 + p_1 h_1(t)) v_{1,1}(t) + R_r \tilde{v}(t), \\
&\ \ \vdots \\
\dot{v}_{1,r-2}(t) &= v_{1,r-1}(t) - (a_{r-2} + p_{r-2} h_1(t)) v_{1,1}(t) + R_3 \tilde{v}(t), \\
\dot{v}_{1,r-1}(t) &= v_{1,r}(t) - (a_{r-1} + p_{r-1} h_1(t)) v_{1,1}(t) + R_2 \tilde{v}(t) + (\Gamma\tilde{\Gamma}^{-1} - I_m) z_{1,r}(t), \\
\dot{v}_{1,r}(t) &= -\Gamma\tilde{\Gamma}^{-1}(a_r + p_r h_1(t)) v_{1,1}(t) + R_1 \tilde{v}(t) \\
&\quad + \sum_{i=1}^{r} R_i \big(y^{(i-1)}(t) - \tilde{v}^{(i-1)}(t)\big) + f\big(d(t), \mathbf{T}(y, \dot{y}, \ldots, y^{(r-1)})(t)\big).
\end{aligned}$$

Next, for $i = 2, \ldots, r-1$ and $j = 1, \ldots, r$ we set $v_{i,j} := z_{i-1,j} - z_{i,j}$. Then,

$$\begin{aligned}
\dot{v}_{i,1}(t) &= v_{i,2}(t) - (a_1 + p_1 h_i(t))v_{i,1}(t) + (a_1 + p_1 h_{i-1}(t))v_{i-1,1}(t),\\
&\ \ \vdots\\
\dot{v}_{i,r-1}(t) &= v_{i,r}(t) - (a_{r-1} + p_{r-1} h_i(t))v_{i,1}(t) + (a_{r-1} + p_{r-1} h_{i-1}(t))v_{i-1,1}(t),\\
\dot{v}_{i,r}(t) &= -(a_r + p_r h_i(t))v_{i,1}(t) + (a_r + p_r h_{i-1}(t))v_{i-1,1}(t).
\end{aligned}$$

We record the following observations. For $z := z_{r-1,1}$ we have

$$\begin{aligned}
y(t) - \tilde{v}(t) &= y(t) - \sum_{i=1}^{r-1} v_{i,1}(t)\\
&= y(t) - (y(t) - z_{2,1}(t)) - \ldots - (z_{r-2,1}(t) - z_{r-1,1}(t))\\
&= z_{r-1,1}(t) = z(t),
\end{aligned} \tag{3.13a}$$

the following relation for $z_{1,r}$

$$z_{1,r}(t) = z_{1,1}^{(r-1)}(t) - \sum_{k=0}^{r-2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \Big((a_{r-k-1} + p_{r-k-1} h_1(t))v_{1,1}(t)\Big), \tag{3.13b}$$

and for $z_{1,1}$

$$z_{1,1}(t) = y(t) - v_{1,1}(t) = z(t) + \tilde{v}(t) - v_{1,1}(t) = z(t) + \sum_{i=2}^{r-1} v_{i,1}(t). \tag{3.13c}$$

Then with (3.13) we have

$$z_{1,r}(t) = z^{(r-1)}(t) + \sum_{i=2}^{r-1} v_{i,1}^{(r-1)}(t) - \sum_{k=0}^{r-2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \Big((a_{r-k-1} + p_{r-k-1} h_1(t))v_{1,1}(t)\Big).$$

We complete *Step one* by introducing a further transformation and consider the dynamics of the resulting error variables. For $j = 1, \ldots, r$ and $i = 2, \ldots, r-1$ we define $w_{i,j} := v_{i,j}$. Further, for $j = 1, \ldots, r-1$ we define with $G := I - \Gamma\tilde{\Gamma}^{-1}$

$$\begin{aligned}
w_{1,r-j}(t) := v_{1,r-j}(t) + G \sum_{k=2}^{r-1} v_{k,1}^{(r-1-j)}(t)&\\
- G \sum_{k=j}^{r-2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-j} &\Big((a_{r-k-1} + p_{r-k-1} h_1(t))v_{1,1}(t)\Big),
\end{aligned}$$

and $w_{1,r} := v_{1,r}$. With these definitions we investigate the dynamics of the auxiliary error variable $w_{i,j}$. We set $\tilde{w} := \sum_{i=2}^{r-1} w_{i,1}$ and obtain for $i = 1$

$$
\begin{aligned}
\dot{w}_{1,1}(t) &= w_{1,2}(t) - \Gamma\tilde{\Gamma}^{-1}(a_1 + p_1 h_1(t))(w_{1,1}(t) - G\tilde{w}(t)) \\
&\qquad + R_r(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)), \\
\dot{w}_{1,2}(t) &= w_{1,3}(t) - \Gamma\tilde{\Gamma}^{-1}(a_2 + p_2 h_1(t))(w_{1,1}(t) - G\tilde{w}(t)) \\
&\qquad + R_{r-1}(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)), \\
&\ \vdots \\
\dot{w}_{1,r-2}(t) &= w_{1,r-1}(t) - \Gamma\tilde{\Gamma}^{-1}(a_{r-2} + p_{r-2} h_1(t))(w_{1,1}(t) - G\tilde{w}(t)) \\
&\qquad + R_3(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)), \\
\dot{w}_{1,r-1}(t) &= w_{1,r}(t) - \Gamma\tilde{\Gamma}^{-1}(a_{r-1} + p_{r-1} h_1(t))(w_{1,1}(t) - G\tilde{w}(t)) \\
&\qquad + R_2(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)) - G z^{(r-1)}(t), \\
\dot{w}_{1,r}(t) &= -\Gamma\tilde{\Gamma}^{-1}(a_r + p_r h_1(t))(w_{1,1}(t) - G\tilde{w}(t)) + R_1(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)) \\
&\qquad + \sum_{i=1}^{r} R_i z^{(i-1)}(t) + f\big(d(t), \mathbf{T}(y, \dot{y}, \ldots, y^{(r-1)})(t)\big), \\
h_1(t) &= \frac{1}{1 - \varphi_1(t)^2 \|w_{1,1}(t) - G\tilde{w}(t)\|^2}
\end{aligned}
\tag{3.14a}
$$

for $i = 2$

$$
\begin{aligned}
\dot{w}_{2,1}(t) &= -(a_1 + p_1 h_2(t)) w_{2,1}(t) + (a_1 + p_1 h_1(t))(w_{1,1}(t) - G\tilde{w}(t)) + w_{2,2}(t), \\
&\ \vdots \\
\dot{w}_{2,r-1}(t) &= -(a_{r-1} + p_{r-1} h_2(t)) w_{2,1}(t) + (a_{r-1} + p_{r-1} h_1(t))(w_{1,1}(t) - G\tilde{w}(t)) + w_{2,r}(t), \\
\dot{w}_{2,r}(t) &= -(a_r + p_r h_2(t)) w_{2,1}(t) + (a_r + p_r h_1(t))(w_{1,1}(t) - G\tilde{w}(t)), \\
h_2(t) &= \frac{1}{1 - \varphi(t)^2 \|w_{2,1}(t)\|^2},
\end{aligned}
\tag{3.14b}
$$

and for $i = 3, \ldots, r-1$ we find

$$
\begin{aligned}
\dot{w}_{i,1}(t) &= -(a_1 + p_1 h_i(t)) w_{i,1}(t) + (a_1 + p_1 h_{i-1}(t)) w_{i-1,1}(t) + w_{i,2}(t), \\
&\ \vdots \\
\dot{w}_{i,r-1}(t) &= -(a_{r-1} + p_{r-1} h_i(t)) w_{i,1}(t) + (a_{r-1} + p_{r-1} h_{i-1}(t)) w_{i-1,1}(t) + w_{i,r}(t), \\
\dot{w}_{i,r}(t) &= -(a_r \quad + p_r \quad h_i(t)) w_{i,1}(t) + (a_r \quad + p_r \quad h_{i-1}(t)) w_{i-1,1}(t), \\
h_i(t) &= \frac{1}{1 - \varphi(t)^2 \|w_{i,1}(t)\|^2}.
\end{aligned}
\tag{3.14c}
$$

Step two. We aim to show the existence of $\tilde{\mathbf{T}} \in \mathcal{T}_0^{rm,\bar{q}}$ as claimed in (3.12). First, we define an operator and prove that it satisfies condition (T.1) in Definition 1.4. This step consists of two parts. In the first part we show $w_{i,j} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$, $i = 1, \ldots, r-1$ and $j = 1, \ldots, r$; in the second part we show $h_i \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$, $i = 1, \ldots, r-1$. For $\bar{q} = rm(r-1) + r$ let $\tilde{\mathbf{T}} : \mathcal{C}([0,\infty);\mathbb{R}^{rm}) \to \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0};\mathbb{R}^{\bar{q}})$ be the solution operator of (3.14) in the following sense: for $\xi_1, \ldots, \xi_r \in \mathcal{C}([0,\infty);\mathbb{R}^m)$

let $w_{i,j} : [0,\omega) \to \mathbb{R}^m$, $\omega \in (0,\infty]$, be the unique maximal solution of (3.14), which exists thanks to Proposition 1.9; with $z = \xi_1, \dot{z} = \xi_2, \ldots, z^{(r-1)} = \xi_r$, and with suitable initial values $w_{i,j}(0)$ according to the transformations. Then, we define for $t \in [0,\omega)$

$$\tilde{\mathbf{T}}(\xi_1,\ldots,\xi_r)(t) := \left(w_{1,1}(t),\ldots,w_{1,r}(t),w_{2,1}(t),\ldots,w_{r-1,r}(t),h_1(t),\ldots,h_{r-1}(t)\right)^\top.$$

Note, that in (3.14a) the expressions $y,\dot{y},\ldots,y^{(r-1)}$ can be expressed in terms of $w_{i,j}$ and $z,\dot{z},\ldots,z^{(r-1)}$ using $y^{(i)} = z^{(i)} + w_{1,1}^{(i)} + \Gamma\tilde{\Gamma}^{-1}\tilde{w}^{(i)}$ and equations (3.14). Defining the set

$$\mathcal{D} := \left\{ (t,\zeta_{1,1},\ldots,\zeta_{r-1,r}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm(r-1)} \;\middle|\; \begin{array}{l} \varphi_1(t)\|\zeta_{1,1}(t) - G\tilde{\zeta}(t)\| < 1, \\ \varphi(t)\|\zeta_{i,1}(t)\| < 1,\ i = 2,\ldots,r-1 \end{array} \right\},$$

where $\tilde{\zeta} := \sum_{i=2}^{r-1}\zeta_{i,1}$, we have $(t,w_{1,1}(t),\ldots,w_{r-1,r}(t)) \in \mathcal{D}$ for all $t \in [0,\omega)$. Furthermore, by Proposition 1.9 (iii) the closure of the graph of the solution $(w_{1,1},\ldots,w_{r-1,r})$ of (3.14) is not a compact subset of $\mathcal{D}$.

We show $w_{i,j} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$ for $i = 1,\ldots,r-1$ and $j = 1,\ldots,r$. We compactly write $w_i := (w_{i,1}^\top,\ldots,w_{i,r}^\top)^\top \in \mathbb{R}^{rm}$ for $i = 1,\ldots,r-1$, and set $\bar{w} := w_{1,1} - G\tilde{w}$. For corresponding matrices A,Q,P satisfying (A.1), using the Kronecker matrix product, we define

$$\hat{A} := A \otimes I_m \in \mathbb{R}^{rm\times rm}, \quad \hat{P} := P \otimes I_m \in \mathbb{R}^{rm\times rm}, \quad \hat{Q} := Q \otimes I_m \in \mathbb{R}^{rm\times rm}. \tag{3.15}$$

Then, using Lemma 3.10 we have for the spectra of $\hat{A},\hat{P},\hat{Q}$ that $\sigma(\hat{A}) = \sigma(A)$, $\sigma(\hat{P}) = \sigma(P)$ and $\sigma(\hat{Q}) = \sigma(Q)$, and

$$\hat{A}^\top\hat{P} + \hat{P}\hat{A} + \hat{Q} = 0. \tag{3.16}$$

Furthermore, for $p_1,\ldots,p_r$ from (A.1), setting $\bar{P} := (p_1,\ldots,p_r)^\top \otimes I_m$ we have

$$\hat{P}\bar{P} = \left[\tilde{p}I_m,0,\ldots,0\right]^\top \in \mathbb{R}^{rm\times m}, \tag{3.17}$$

where $\tilde{p} := P_1 - P_2P_4^{-1}P_2^\top > 0$. Aiming to show the bounded input to bounded output property (T.1), we assume that $z,\dot{z},\ldots,z^{(r-1)}$ are bounded on $[0,\omega)$. As the solution evolves in $\mathcal{D}$, $w_{1,1} - G\tilde{w}, w_{2,1},\ldots,w_{r-1,1}$ are bounded on $[0,\omega)$. Thus, $y = z + w_{1,1} + \Gamma\tilde{\Gamma}^{-1}\tilde{w}$ is bounded and hence $\mathbf{T}\left(y,\dot{y},\ldots,y^{(r-1)}\right)$ is bounded via $\mathbf{T} \in \mathcal{T}_{\tau,1}^{rm,q}$, and therefore $f\left(d,\mathbf{T}\left(y,\dot{y},\ldots,y^{(r-1)}\right)\right)$ is bounded on $[0,\omega)$. Hence, invoking (3.15), we may rewrite (3.14) as

$$\begin{aligned} \dot{w}_1(t) &= \hat{A}w_1(t) - h_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}\bar{w}(t) + B_1(t), \\ \dot{w}_2(t) &= \hat{A}w_2(t) - h_2(t)\bar{P}w_{2,1}(t) + h_1(t)\bar{P}\bar{w}(t) + B_2(t), \\ \dot{w}_i(t) &= \hat{A}w_i(t) - h_i(t)\bar{P}w_{i,1}(t) + h_{i-1}(t)\bar{P}w_{i-1,1}(t) + B_i(t), \end{aligned} \tag{3.18}$$

for $i = 3,\ldots,r-1$ and suitable bounded functions $B_1,B_2,B_i \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^{rm})$. Seeking a suitable Lyapunov function for the overall system (3.18) we define with $\hat{P}$ from (3.15) the matrix

$$\hat{P}_1 := \left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right)\hat{P}\left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right),$$

which is possible since by (A.3) the matrix product $\Gamma\tilde{\Gamma}^{-1}$ is regular. Moreover, $\hat{P}_1$ is symmetric and we observe $\hat{P}_1 > 0$. Thus, with $\hat{P}\bar{P} = [\tilde{p}I_m, 0, \ldots, 0]^\top \in \mathbb{R}^{rm\times m}$ via (3.17), we obtain

$$\hat{P}_1\bar{P} = [\tilde{p}(\Gamma\tilde{\Gamma}^{-1})^{-1}, 0, \ldots, 0]^\top \in \mathbb{R}^{rm\times m}. \tag{3.19}$$

Since for all $M \in \mathbb{R}^{m\times m}$ we have $\hat{A}(I_r \otimes M) = (I_r \otimes M)\hat{A}$, $\hat{A}^\top(I_r \otimes M) = (I_r \otimes M)\hat{A}^\top$ we obtain

$$\begin{aligned} \hat{A}^\top\hat{P}_1 &= \left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right)\hat{A}^\top\hat{P}\left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right), \\ \hat{P}_1\hat{A} &= \left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right)\hat{P}\hat{A}\left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{A}^\top\hat{P}_1 + \hat{P}_1\hat{A} &= \left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right)\left(\hat{A}^\top\hat{P} + \hat{P}\hat{A}\right)\left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right) \\ &\overset{(3.16)}{=} -\left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right)\hat{Q}\left(I_r \otimes (\Gamma\tilde{\Gamma}^{-1})^{-\frac{1}{2}}\right) =: -\hat{Q}_1, \end{aligned} \tag{3.20}$$

where $\hat{Q}_1 = \hat{Q}_1^\top$ by (A.3), and $\hat{Q}_1 > 0$ via (3.16) and (A.3). We define

$$0 < \mathcal{P} := \begin{bmatrix} \hat{P}_1 & 0 & \ldots & 0 \\ 0 & \hat{P} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \hat{P} \end{bmatrix} = \mathcal{P}^\top \in \mathbb{R}^{rm(r-1)\times rm(r-1)}, \tag{3.21}$$

and set $w := (w_1^\top, \ldots, w_{r-1}^\top)^\top \in \mathbb{R}^{mr(r-1)}$. With the definitions above, we consider the Lyapunov function candidate

$$\begin{aligned} V &: \mathbb{R}^{rm} \times \cdots \times \mathbb{R}^{rm} \to \mathbb{R}, \\ &(w_1, \ldots, w_{r-1}) \mapsto \langle w, \mathcal{P}w\rangle = \langle w_1, \hat{P}_1 w_1\rangle + \sum_{i=2}^{r-1}\langle w_i, \hat{P}w_i\rangle, \end{aligned}$$

and study its evolution along the solution trajectories of the respective differential equations (3.18). We fix $\theta \in (0, \omega)$ and note that $w_i \in \mathcal{L}^\infty([0,\theta); \mathbb{R}^{rm})$ for all $i = 1, \ldots, r-1$. Using (3.19) and (3.20) we obtain for $t \in [\theta, \omega)$

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\langle w(t), \mathcal{P}w(t)\rangle &= \langle w_1(t), \hat{P}_1\hat{A}w_1(t)\rangle + \langle \hat{P}_1\hat{A}w_1(t), w_1(t)\rangle \\ &\quad + \sum_{i=2}^{r-1}\langle w_i(t), \hat{P}\hat{A}w_i(t)\rangle + \langle \hat{P}\hat{A}w_i(t), w_i(t)\rangle \\ &\quad + 2\langle w_1(t), \hat{P}_1 B_1(t)\rangle + 2\sum_{i=2}^{r-1}\langle w_i(t), \hat{P}B_i(t)\rangle \\ &\quad + 2h_1(t)\langle w_2(t), \hat{P}\bar{P}\bar{w}(t)\rangle - 2h_1(t)\langle w_1(t), \hat{P}_1\bar{P}\Gamma\tilde{\Gamma}^{-1}\bar{w}(t)\rangle \\ &\quad + 2\sum_{i=2}^{r-2} h_i(t)\langle w_{i+1}(t), \hat{P}\bar{P}w_{i,1}(t)\rangle - 2\sum_{i=2}^{r-1} h_i(t)\langle w_i(t), \hat{P}\bar{P}w_{i,1}(t)\rangle. \end{aligned} \tag{3.22}$$

We estimate the addends separately. With the aid of (3.15) and (3.20) we have

$$
\begin{aligned}
&\langle w_1(t), \hat{P}_1\hat{A}w_1(t)\rangle + \langle \hat{P}_1\hat{A}w_1(t), w_1(t)\rangle \\
&\qquad + \sum_{i=2}^{r-1}\langle w_i(t), \hat{P}\hat{A}w_i(t)\rangle + \langle \hat{P}\hat{A}w_i(t), w_i(t)\rangle \\
&\qquad\qquad \leq -\lambda_{\min}(\hat{Q}_1)\|w_1(t)\|^2 - \sum_{i=2}^{r-1}\lambda_{\min}(\hat{Q})\|w_i(t)\|^2,
\end{aligned}
\tag{3.23a}
$$

where $0 < \lambda_{\min}(\hat{Q}_1), \lambda_{\min}(\hat{Q})$ denotes the smallest eigenvalue of $\hat{Q}_1, \hat{Q}$, respectively. Boundedness of the functions B_i, $i = 1, \ldots, r-1$, introduced in (3.18) gives

$$
\begin{aligned}
&\langle w_1(t), \hat{P}_1 B_1(t)\rangle + \sum_{i=2}^{r-1}\langle w_i(t), \hat{P}B_i(t)\rangle \\
&\qquad \leq \|\hat{P}_1\|\|B_1\|_\infty\|w_1(t)\| + \|\hat{P}\|\sum_{i=2}^{r-1}\|B_i\|_\infty\|w_i(t)\|.
\end{aligned}
\tag{3.23b}
$$

Invoking (3.17) and the definition of h_i we obtain

$$
\begin{aligned}
&\sum_{i=2}^{r-2} h_i(t)\langle w_{i+1}(t), \hat{P}\bar{P}w_{i,1}(t)\rangle - \sum_{i=2}^{r-1} h_i(t)\langle w_i(t), \hat{P}\bar{P}w_{i,1}(t)\rangle \\
&\qquad = \tilde{p}\sum_{i=2}^{r-2} h_i(t)\langle w_{i+1,1}(t) - w_{i,1}(t), w_{i,1}(t)\rangle - \tilde{p}h_{r-1}(t)\|w_{r-1,1}(t)\|^2 \\
&\qquad \leq \tilde{p}\sum_{i=2}^{r-2} h_i(t)\langle w_{i+1,1}(t) - w_{i,1}(t), w_{i,1}(t)\rangle,
\end{aligned}
\tag{3.23c}
$$

and with the aid of (A.2) we observe for $i = 2, \ldots, r-2$

$$
\begin{aligned}
&h_i(t)\langle w_{i+1,1}(t) - w_{i,1}(t), w_{i,1}(t)\rangle \\
&= -h_i(t)\|w_{i,1}(t)\|^2 + h_i(t)\langle w_{i,1}(t), w_{i+1,1}(t)\rangle \\
&\leq -h_i(t)\|w_{i,1}(t)\|^2 + h_i(t)\|w_{i,1}(t)\|\|w_{i+1,1}(t)\| \\
&< -h_i(t)\|w_{i,1}(t)\|^2 + h_i(t)\|w_{i,1}(t)\|\frac{1}{\varphi(t)} \\
&= -h_i(t)\|w_{i,1}(t)\|\left(\|w_{i,1}(t)\| - \frac{1}{\varphi(t)}\right) \\
&= -\|w_{i,1}(t)\|\frac{1}{(1+\varphi(t)\|w_{i,1}(t)\|)(1-\varphi(t)\|w_{i,1}(t)\|)}\left(\|w_{i,1}(t)\| - \frac{1}{\varphi(t)}\right) \\
&= -\|w_{i,1}(t)\|\frac{\varphi(t)\|w_{i,1}(t)\| - 1}{(1+\varphi(t)\|w_{i,1}(t)\|)(1-\varphi(t)\|w_{i,1}(t)\|)}\frac{1}{\varphi(t)} \\
&= \;\|w_{i,1}(t)\|\frac{1}{1+\varphi(t)\|w_{i,1}(t)\|}\frac{1}{\varphi(t)} \\
&\leq \;\|w_{i,1}(t)\|\frac{1}{\varphi(t)} \leq \|w_{i,1}(t)\|\sup_{s\geq\theta}\frac{1}{\varphi(s)} \leq \|w_i(t)\|\sup_{s\geq\theta}\frac{1}{\varphi(s)}.
\end{aligned}
\tag{3.23d}
$$

We recall $\bar{w} = w_{1,1} - G\tilde{w}$ and therefore, for $t \in [\theta, \omega)$ we observe $\|w_{2,1}(t)\| < \varphi(t)^{-1}$ and $\|\tilde{w}(t)\| < \sum_{i=2}^{r-1} \varphi(t)^{-1} = (r-2)\varphi(t)^{-1}$. Hence, we obtain for $t \in [\theta, \omega)$

$$
\begin{aligned}
&-h_1(t)\langle w_{1,1}(t), \bar{w}(t)\rangle + h_1(t)\langle w_{2,1}(t), \bar{w}(t)\rangle \\
&\quad = -h_1(t)\langle \bar{w}(t), \bar{w}(t)\rangle - h_1(t)\langle G\tilde{w}(t), \bar{w}(t)\rangle + h_1(t)\langle w_{2,1}(t), \bar{w}(t)\rangle \\
&\quad \le -h_1(t)\|\bar{w}(t)\|^2 + h_1(t)\big(\|G\|\|\tilde{w}(t)\| + \|w_{2,1}(t)\|\big)\|\bar{w}(t)\| \\
&\quad \le -h_1(t)\|\bar{w}(t)\| \left(\|\bar{w}(t)\| - \frac{\|G\|(r-2)+1}{\varphi(t)}\right) \\
&\quad \overset{\text{(A.4)}}{<} -h_1(t)\|\bar{w}(t)\| \left(\|\bar{w}(t)\| - \frac{\frac{\rho-1}{r-2}(r-2)+1}{\varphi(t)}\right) \\
&\quad \overset{\text{(A.2)}}{=} -h_1(t)\|\bar{w}(t)\| \left(\|\bar{w}(t)\| - \frac{1}{\varphi_1(t)}\right) \\
&\quad \le \|\bar{w}(t)\| \sup_{s\ge\theta} \frac{1}{\varphi_1(s)} \\
&\quad \le \sup_{s\ge\theta} \frac{1}{\varphi_1(s)} \big(\|w_{1,1}(t) - G\tilde{w}(t)\|\big) \\
&\quad \le \sup_{s\ge\theta} \frac{1}{\varphi_1(s)} \big(\|w_{1,1}(t)\| + \|G\|\|\tilde{w}(t)\|\big) \\
&\quad \le \sup_{s\ge\theta} \frac{1}{\varphi_1(s)} \left(\|w_1(t)\| + \|G\| \sum_{i=2}^{r-1} \|w_i(t)\|\right).
\end{aligned}
\tag{3.23e}
$$

For better readability we define the constants

$$
\begin{aligned}
M_1 &:= \|\hat{P}_1\|\|B_1\|_\infty + \tilde{p} \sup_{s\ge\theta} \varphi_1(s)^{-1}, \\
M_i &:= \|\hat{P}\|\|B_i\|_\infty + \tilde{p} \sup_{s\ge\theta} \varphi(s)^{-1} + \tilde{p}\|G\| \sup_{s\ge\theta} \varphi_1(s)^{-1}, \ i = 2, \dots, r-1, \\
N &:= \frac{2M_1^2}{\lambda_{\min}(\hat{Q}_1)} + \sum_{i=2}^{r-1} \frac{2M_i^2}{\lambda_{\min}(\hat{Q})}.
\end{aligned}
$$

Then, using $2ab \le 2a^2 + \frac{1}{2}b^2$ for $a, b \in \mathbb{R}$, we may estimate (3.22) with the aid of (3.23) for $t \in [\theta, \omega)$

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\langle w(t), \mathcal{P}w(t)\rangle &\le -\lambda_{\min}(\hat{Q}_1)\|w_1(t)\|^2 - \lambda_{\min}(\hat{Q}) \sum_{i=2}^{r-1} \|w_i(t)\|^2 + \sum_{i=1}^{r-1} 2M_i\|w_i(t)\| \\
&= -\lambda_{\min}(\hat{Q}_1)\|w_1(t)\|^2 - \lambda_{\min}(\hat{Q}) \sum_{i=2}^{r-1} \|w_i(t)\|^2 \\
&\quad + \frac{2M_1}{\sqrt{\lambda_{\min}(\hat{Q}_1)}}\|w_1(t)\|\sqrt{\lambda_{\min}(\hat{Q}_1)} + \sum_{i=2}^{r-1} \frac{2M_i}{\sqrt{\lambda_{\min}(\hat{Q})}}\|w_i(t)\|\sqrt{\lambda_{\min}(\hat{Q})} \\
&\le -\frac{\lambda_{\min}(\hat{Q}_1)}{2}\|w_1(t)\|^2 - \frac{\lambda_{\min}(\hat{Q})}{2} \sum_{i=2}^{r-1} \|w_i(t)\|^2 + N \\
&\le -\frac{\mu}{2} w(t)^\top \mathcal{P} w(t) + N,
\end{aligned}
$$

where we set $\mu := \frac{\min\{\lambda_{\min}(\hat{Q}),\lambda_{\min}(\hat{Q}_1)\}}{\max\{\lambda_{\max}(\hat{P}),\lambda_{\max}(\hat{P}_1)\}} > 0$. Then, with the aid of Grönwall's lemma [72] we obtain

$$\langle w(t), \mathcal{P}w(t)\rangle \le \langle w(\theta), \mathcal{P}w(\theta)\rangle e^{-\frac{\mu}{2}(t-\theta)} + \frac{2N}{\mu},$$

from which we obtain, since $\lambda_{\max}(\mathcal{P}), \lambda_{\min}(\mathcal{P}) > 0$ by (3.21), the following estimation for $t \in [\theta, \omega)$

$$\|w(t)\|^2 \le \frac{\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})}\|w(\theta)\|^2 e^{-\frac{\mu}{2}(t-\theta)} + \frac{2N}{\mu\lambda_{\min}(\mathcal{P})}. \tag{3.24}$$

Clearly, inequality (3.24) implies $w \in \mathcal{L}^\infty\left([\theta,\omega);\mathbb{R}^{rm(r-1)}\right)$. Therefore, recalling $w \in \mathcal{L}^\infty\left([0,\theta);\mathbb{R}^{rm(r-1)}\right)$, we have $w_i \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^{rm})$ for all $i = 1,\dots,r-1$. In particular, we obtain $\bar{w} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$. This completes part one of *Step two.*

In the second part we show $h_i \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$ for all $i = 1,\dots,r-1$. For the sake of consistency, we introduce the variables $x_1 := \bar{w}$ and $x_i := w_{i,1}$ for $i = 2,\dots,r-1$. Observing $-\Gamma\tilde{\Gamma}^{-1} - G = -\Gamma\tilde{\Gamma}^{-1} - (I - \Gamma\tilde{\Gamma}^{-1}) = -I_m$ and setting $\tilde{w}_2 := \sum_{i=2}^{r-1} w_{i,2}$, we obtain via (3.14)

$$\begin{aligned}
\dot{x}_1(t) &= \dot{w}_{1,1}(t) - G\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{w}(t)\\
&= w_{1,2}(t) - \Gamma\tilde{\Gamma}^{-1}(a_1 + p_1 h_1(t))\underbrace{(w_{1,1}(t) - G\tilde{w}(t))}_{=x_1(t)} + R_r(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t))\\
&\qquad - G\Big(\tilde{w}_2(t) - (a_1 + p_1 h_{r-1}(t))w_{r-1,1}(t) + (a_1 + p_1 h_1(t))x_1(t)\Big)\\
&= w_{1,2}(t) - G\tilde{w}_2(t) + G(a_1 + p_1 h_{r-1}(t))w_{r-1,1}(t) + R_r(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t))\\
&\qquad - \Gamma\tilde{\Gamma}^{-1}(a_1 + p_1 h_1(t))x_1(t) - G(a_1 + p_1 h_1(t))x_1(t)\\
&= w_{1,2}(t) - G\tilde{w}_2(t) + \underbrace{(-\Gamma\tilde{\Gamma}^{-1} - G)}_{=-I_m}(a_1 + p_1 h_1(t))x_1(t)\\
&\qquad + G(a_1 + p_1 h_{r-1}(t))w_{r-1,1}(t) + R_r(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t))\\
&= -(a_1 + p_1 h_1(t))x_1(t) + G(a_1 + p_1 h_{r-1}(t))w_{r-1,1}(t)\\
&\qquad + R_r(w_{1,1}(t) + \Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)) + w_{1,2}(t) - G\tilde{w}_2(t).
\end{aligned} \tag{3.25}$$

Since $p_1 = 1$, using (3.14) and (3.25) we have for $t \in [0,\omega)$ and $i = 2,\dots,r-1$

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_1(t)\|^2 &= -h_1(t)\|x_1(t)\|^2 + h_{r-1}(t)\langle x_1(t), Gx_{r-1}(t)\rangle + \langle x_1(t), b_1(t)\rangle,\\
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_i(t)\|^2 &= -h_i(t)\|x_i(t)\|^2 + h_{i-1}(t)\langle x_i(t), x_{i-1}(t)\rangle + \langle x_i(t), b_i(t)\rangle,
\end{aligned} \tag{3.26}$$

for suitable functions $b_i \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$, $i = 1,\dots,r-1$. We observe that in (3.26) for all $i = 2,\dots,r-1$ the i^{th} equation depends on the preceding gain function h_{i-1}, respectively, and the first equation depends on the last gain function h_{r-1}. Therefore, we cannot apply standard funnel arguments to show boundedness of the gain functions, cf. [91, p. 484-485], [84, p. 241-242] or [25, p. 350-351]. However, on closer examination of the respective proofs in the aforementioned references and due to the shape of the gain functions h_i we may retain

$$\begin{aligned}
h_1 \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0};\mathbb{R}) &\iff \exists\, \nu_1 > 0\, \forall\, t \ge 0 \,:\, \varphi_1(t)^{-1} - \|x_1(t)\| \ge \nu_1,\\
h_i \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0};\mathbb{R}) &\iff \exists\, \nu_i > 0\, \forall\, t \ge 0 \,:\, \varphi(t)^{-1} - \|x_i(t)\| \ge \nu_i,\ i = 2,\dots,r-1.
\end{aligned}$$

Moreover, due to the loop structure in (3.26) it suffices to show boundedness of one gain function, which implies boundedness of all remaining gain functions by means of Lemma 1.11. In order to properly proceed with the proof, we provide some preparatory definitions. Let $\psi_1 := 1/\varphi_1$, $\psi := 1/\varphi$, set $\lambda_1 := \inf_{s\in(0,\omega)} \psi_1(s) > 0$ and $\lambda := \inf_{s\in(0,\omega)} \psi(s) > 0$. We fix $\beta \in (0,\omega)$. Since $\liminf_{s\to\infty} \varphi(s) > 0$ and $\dot{\varphi}$ is bounded, we have that $\frac{\mathrm{d}}{\mathrm{d}t}\psi|_{[\beta,\infty)}$ is bounded, and respective for $\frac{\mathrm{d}}{\mathrm{d}t}\psi_1|_{[\beta,\infty)}$. Thus, there exists a Lipschitz bound $L > 0$ of $\psi|_{[\beta,\infty)}$ and respectively a Lipschitz bound $L_1 > 0$ of $\psi_1|_{[\beta,\infty)}$. For $\rho > 1$ as in (A.2) we fix $\delta > 0$ as

$$\frac{1}{\rho+1} < \delta < \frac{1}{2}, \tag{3.27}$$

and define

$$\Delta_1 := \rho - \|G\| \left(4\rho^2(\rho+1)^{r-2} - 1\right) \overset{(A.4)}{>} 0, \quad \Delta := 1 - 2\delta \overset{(3.27)}{>} 0. \tag{3.28}$$

With δ in (3.27) and Δ_1, Δ given in (3.28) we choose the number $\kappa > 0$ such that

$$\begin{aligned} 0 < \kappa < \min\Bigg\{ & \frac{\lambda_1}{1+\rho}, \frac{\lambda}{2}, \inf_{s\in(0,\beta]} \left(\psi_1(s) - \|x_1(s)\|\right), \\ & \min_{i\in\{2,\dots,r-1\}} \left\{ \inf_{s\in(0,\beta]} \left(\psi(s) - \|x_i(s)\|\right) \right\} \Bigg\} \end{aligned} \tag{3.29}$$

and small enough such that for Δ_1, $\Delta > 0$ from (3.28)

$$0 < L_1 \le \min\Bigg\{ \frac{\lambda_1^2}{4\kappa}\frac{\rho - \|G\|}{\rho} - \sup_{s\in[0,\omega)} \|b_1(s)\|, \tag{3.30a}$$
$$\frac{\Delta_1\lambda_1^2}{4\rho\kappa} - 2\|G\| \sup_{s\in[\beta,\omega)} \psi_1(s)\rho(\rho+1)^{r-2} - \sup_{s\in[0,\omega)} \|b_1(s)\| \Bigg\},$$

$$0 < L \le \min\Bigg\{ \frac{\rho^2\lambda^2}{2\kappa} - \sup_{s\in[0,\omega)} \|b_2(s)\|, \tag{3.30b}$$
$$\min_{i\in\{3,\dots,r-1\}} \left\{ 2^{i-1}\Delta\,\frac{\rho^2\lambda^2}{\kappa} - \sup_{s\in[0,\omega)} \|b_i(s)\| \right\} \Bigg\}.$$

We emphasize that, since L_1, L, Δ_1, $\Delta > 0$ and $\rho > \|G\|$, the number $\kappa > 0$ is well defined. Further, since $\rho > 1$ we choose $\hat{\delta} > 0$ satisfying

$$\frac{1}{\rho} < \hat{\delta} \le 1. \tag{3.31}$$

With $\hat{\delta}$ from (3.31), $\delta > 0$ from (3.27) and $\kappa > 0$ given in (3.29) we define

$$\begin{aligned} \kappa_1 &:= \kappa, \\ \kappa_i &:= \frac{\delta^{i-1}}{2\rho^2}\kappa, \quad i = 2,\dots,r-2, \\ \kappa_{r-1} &:= \frac{\delta^{r-2}\hat{\delta}}{2\rho^2}\kappa. \end{aligned} \tag{3.32}$$

From this we have κ_i ordered as $\kappa_{r-1} < \kappa_{r-2} < \cdots < \kappa_2 < \kappa_1$. We aim to show

$$\forall\, t \in [0,\omega) \,:\, \psi_1(t) - \|x_1(t)\| \geq \kappa_1 \ \wedge\ \psi(t) - \|x_i(t)\| \geq \kappa_i,\ i = 2,\dots,r-1. \tag{3.33}$$

Note that this is true on $(0,\beta]$ by definition of κ in (3.29). We conclude the preparatory definitions by setting

$$\begin{aligned} t_0^1 &:= \inf\{\, t \in (\beta,\omega) \mid \psi_1(t) - \|x_1(t)\| < \kappa_1 \,\}\,, \\ t_0^i &:= \inf\{\, t \in (\beta,\omega) \mid \psi(t) - \|x_i(t)\| < \kappa_i \,\}\,,\ i = 2,\dots,r-1, \end{aligned}$$

where the infimum of the empty set is infinity as usual. Seeking a contradiction we suppose $t_0^\ell < \infty$ for some $\ell \in \{1,\dots,r-1\}$. Then either

$$\big(\exists\, \ell \in \{2,\dots,r-1\} \,:\, t_0^\ell < t_0^{\ell-1}\big) \ \vee\ t_0^1 < t_0^{r-1}, \tag{3.34a}$$

or

$$t_0^1 = \cdots = t_0^{r-1}. \tag{3.34b}$$

If (3.34b) is true, we decrease $\hat{\delta}$ from (3.31) by which we decrease κ_{r-1} in (3.32) and therefore $t_0^1 < t_0^{r-1}$; hence, without loss of generality we may assume that (3.34a) is true. We set $t_0 := t_0^\ell$, i.e., t_0 denotes the very first moment when $\psi(t) - \|x_\ell(t)\|$ exceeds the threshold κ_ℓ. As will become clear from the subsequent steps, we must distinguish three cases, namely either $\ell = 1$, or $\ell = 2$, or $3 \leq \ell \leq r-1$.

If $\ell = 1$ there may occur two possible cases, namely either

$$\psi_1(t_0) - \|x_1(t_0)\| \leq \psi(t_0) - \|x_{r-1}(t_0)\|, \tag{3.35a}$$

or

$$\psi_1(t_0) - \|x_1(t_0)\| > \psi(t_0) - \|x_{r-1}(t_0)\|. \tag{3.35b}$$

First, we draw our attention to case (3.35a) and observe

$$\begin{aligned} \psi_1(t_0) - \|x_1(t_0)\| &\leq \psi(t_0) - \|x_{r-1}(t_0)\| \\ \iff\ \|x_1(t_0)\| &\geq \psi_1(t_0) - \psi(t_0) + \|x_{r-1}(t_0)\| \\ &= \frac{\varphi(t_0) - \varphi_1(t_0)}{\varphi_1(t_0)\varphi(t_0)} + \|x_{r-1}(t_0)\|, \end{aligned}$$

and thus, invoking (A.2), we have

$$\begin{aligned} \|x_1(t_0)\| &\geq \frac{\varphi(t_0) - \varphi_1(t_0)}{\varphi_1(t_0)\varphi(t_0)} + \|x_{r-1}(t_0)\| = \frac{\rho - 1}{\varphi(t_0)} + \|x_{r-1}(t_0)\| \\ &> (\rho - 1)\|x_{r-1}(t_0)\| + \|x_{r-1}(t_0)\| = \rho\|x_{r-1}(t_0)\|. \end{aligned}$$

Due to the definition of t_0, there exists $t_1 \in (t_0,\omega)$ such that $\psi_1(t_1) - \|x_1(t_1)\| < \kappa_1$, and therefore, we have

$$\forall\, t \in [t_0,t_1] \,:\ \|x_1(t)\| > \rho\|x_{r-1}(t)\|. \tag{3.36}$$

Thus, for $t \in [t_0,t_1]$ we readily deduce the following relations

$$\begin{aligned} \psi_1(t) - \|x_1(t)\| &\leq \kappa_1, \\ \|x_1(t)\| &\geq \psi_1(t) - \kappa_1 \geq \frac{\lambda_1}{2}, \\ h_1(t) = \frac{1}{1 - \varphi_1(t)^2\|x_1(t)\|^2} &\geq \frac{\lambda_1}{2\kappa_1}. \end{aligned} \tag{3.37}$$

For $t \in [t_0, t_1]$ we consider the first equation in (3.26)

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_1(t)\|^2 &= -h_1(t)\|x_1(t)\|^2 + \langle x_1(t), h_{r-1}(t)Gx_{r-1}(t) + b_1(t)\rangle \\
&\le -h_1(t)\|x_1(t)\|^2 + h_{r-1}(t)\|x_1(t)\|\|G\|\|x_{r-1}(t)\| + \|x_1(t)\|\|b_1(t)\| \\
&\overset{(3.36)}{<} \big(-h_1(t)\rho + h_{r-1}(t)\|G\|\big)\frac{\|x_1(t)\|^2}{\rho} + \|x_1(t)\|\|b_1(t)\|.
\end{aligned}$$

We estimate the first term. By (A.2) and relation (3.36) we obtain for $t \in [t_0, t_1]$

$$\begin{aligned}
h_1(t)\rho - h_{r-1}(t)\|G\| &= h_1(t)h_{r-1}(t)\Big(\rho - \rho\varphi(t)^2\|x_{r-1}(t)\|^2 \\
&\qquad - \|G\| + \|G\|\varphi_1(t)^2\|x_1(t)\|^2\Big) \\
&\overset{(3.36)}{>} h_1(t)h_{r-1}(t)\Big(\rho - \rho\varphi(t)^2\|x_{r-1}(t)\|^2 \\
&\qquad - \|G\| + \|G\|\varphi_1(t)^2\rho^2\|x_{r-1}(t)\|^2\Big) \\
&\overset{(A.2)}{=} h_1(t)h_{r-1}(t)(\rho - \|G\|)\big(1 - \varphi(t)^2\|x_{r-1}(t)\|^2\big) \\
&= h_1(t)(\rho - \|G\|) \overset{(3.37)}{\ge} \frac{\lambda_1}{2\kappa_1}(\rho - \|G\|).
\end{aligned}$$

Hence, using the estimation above and the relations from (3.37) we estimate the first equation in (3.26) for $t \in [t_0, t_1]$

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_1(t)\|^2 &< -\frac{\lambda_1}{2\kappa_1}(\rho - \|G\|)\frac{\|x_1(t)\|^2}{\rho} + \|x_1(t)\|\|b_1(t)\| \\
&\le \left(-\frac{\lambda_1^2}{4\kappa_1}\frac{\rho - \|G\|}{\rho} + \sup_{s\in[0,\omega)}\|b_1(s)\|\right)\|x_1(t)\| \\
&\overset{(3.32)}{=} \left(-\frac{\lambda_1^2}{4\kappa}\frac{\rho - \|G\|}{\rho} + \sup_{s\in[0,\omega)}\|b_1(s)\|\right)\|x_1(t)\| \overset{(3.30a)}{\le} -L_1\|x_1(t)\|.
\end{aligned}$$

From this we calculate

$$\begin{aligned}
\|x_1(t_1)\| - \|x_1(t_0)\| &= \int_{t_0}^{t_1} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|x_1(t)\|^2\right)}{\|x_1(t)\|}\,\mathrm{d}t \\
&\le \int_{t_0}^{t_1} -L_1\,\mathrm{d}t = -L_1(t_1 - t_0) \le \psi_1(t_1) - \psi_1(t_0)
\end{aligned} \tag{3.38a}$$

and therefore,

$$\kappa_1 = \psi_1(t_0) - \|x_1(t_0)\| \le \psi_1(t_1) - \|x_1(t_1)\| < \kappa_1, \tag{3.38b}$$

a contradiction. Next, we consider the case (3.35b) where by $t_0 = t_0^1 < t_0^{r-1}$ we have

$$\kappa_{r-1} \le \psi(t_0) - \|x_{r-1}(t_0)\| < \psi_1(t_0) - \|x_1(t_0)\| = \kappa_1.$$

This, together with the definition of t_0 implies the existence of $t_1 \in (t_0, t_0^{r-1})$ such that

$$\forall\, t \in [t_0, t_1] \,:\; \kappa_{r-1} \le \psi(t) - \|x_{r-1}(t)\| < \psi_1(t) - \|x_1(t)\| \le \kappa_1, \tag{3.39}$$

from which we obtain for $t \in [t_0, t_1]$

$$\begin{aligned} \|x_{r-1}(t)\| \le \psi(t) - \kappa_{r-1} &\overset{(A.2)}{=} \frac{1}{\rho}\left(\frac{1}{\varphi_1(t)} - \rho\kappa_{r-1}\right) \\ &= \frac{1}{\rho}\left(\psi_1(t) - \kappa_1 + \kappa_1 - \rho\kappa_{r-1}\right) \\ &\le \frac{\|x_1(t)\| + \kappa_1 - \rho\kappa_{r-1}}{\rho} \end{aligned} \tag{3.40a}$$

and therefore,

$$\forall\, t \in [t_0, t_1] \;:\; \|x_1(t)\| \ge \rho\|x_{r-1}(t)\| - (\kappa_1 - \rho\kappa_{r-1}) > 0. \tag{3.40b}$$

Note that the last inequality holds due to

$$\begin{aligned} \rho\|x_{r-1}(t)\| - (\kappa_1 - \rho\kappa_{r-1}) &\overset{(A.2),\,(3.39)}{\ge} \frac{\rho}{\rho\varphi_1(t)} - \rho\kappa_1 - (\kappa_1 - \rho\kappa_{r-1}) \\ &\ge \lambda_1 - (\rho+1)\kappa_1 + \rho\kappa_{r-1} > \rho\kappa_{r-1}, \end{aligned}$$

where we used $\kappa_1 < \frac{\lambda_1}{\rho+1}$ in the last step. Furthermore, recalling h_{r-1} and using the definition of κ_1 and κ_{r-1} we deduce for $t \in [t_0, t_1]$

$$\begin{aligned} \frac{1}{h_{r-1}(t)} &= 1 - \varphi(t)^2\|x_{r-1}(t)\|^2 \\ &= (1 + \varphi(t)\|x_{r-1}(t)\|)(1 - \varphi(t)\|x_{r-1}(t)\|) \\ &= \varphi(t)(1 + \varphi(t)\|x_{r-1}(t)\|)(\psi(t) - \|x_{r-1}(t)\|) \\ &\ge \varphi(t)(1 + \varphi(t)\|x_{r-1}(t)\|)\kappa_{r-1} \ge \varphi(t)\kappa_{r-1} = \rho\varphi_1(t)\kappa_{r-1}, \end{aligned}$$

and hence we obtain for $t \in [t_0, t_1]$ the following estimation

$$h_{r-1}(t) = \frac{1}{1 - \varphi(t)^2\|x_{r-1}(t)\|^2} \le \frac{1}{\varphi(t)\kappa_{r-1}} \overset{(A.2)}{=} \frac{1}{\rho\varphi_1(t)\kappa_{r-1}}. \tag{3.41}$$

Again, we consider the first equation in (3.26) for $t \in [t_0, t_1]$ and obtain

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_1(t)\|^2 &= -h_1(t)\|x_1(t)\|^2 + \langle x_1(t), h_{r-1}(t)Gx_{r-1}(t) + b_1(t)\rangle \\ &\le -h_1(t)\|x_1(t)\|^2 + h_{r-1}(t)\|x_1(t)\|\|G\|\|x_{r-1}(t)\| + \|x_1(t)\|\|b_1(t)\| \\ &\overset{(3.40a)}{\le} -h_1(t)\|x_1(t)\|^2 + \|G\|h_{r-1}(t)\|x_1(t)\|\frac{\|x_1(t)\| + (\kappa_1 - \rho\kappa_{r-1})}{\rho} \\ &\qquad + \|x_1(t)\|\|b_1(t)\| \\ &= \Big(-\rho h_1(t) + \|G\|h_{r-1}(t)\Big)\frac{\|x_1(t)\|^2}{\rho} \\ &\qquad + \|G\|\frac{\kappa_1 - \rho\kappa_{r-1}}{\rho}h_{r-1}(t)\|x_1(t)\| + \|x_1(t)\|\|b_1(t)\|. \end{aligned}$$

Noting that by definition of κ_1, κ_{r-1} we have $\kappa_1 - \rho\kappa_{r-1} = \kappa\left(1 - \frac{\delta^{r-2}\hat{\delta}}{2\rho}\right) > 0$, we estimate the first term in the above expression for $t \in [t_0, t_1]$

$$
\begin{aligned}
&\rho h_1(t) - \|G\| h_{r-1}(t) \\
&\quad = h_1(t) h_{r-1}(t)\big(\rho - \rho\varphi(t)^2\|x_{r-1}(t)\|^2 - \|G\| + \|G\|\varphi_1(t)^2\|x_1(t)\|^2\big) \\
&\quad \overset{(3.40b)}{\geq} h_1(t) h_{r-1}(t)\Big(\rho - \rho\varphi(t)^2\|x_{r-1}(t)\|^2 \\
&\qquad\quad - \|G\| + \|G\|\varphi_1(t)^2\big(\rho\|x_{r-1}(t)\| - (\kappa_1 - \rho\kappa_{r-1})\big)^2\Big) \\
&\quad = h_1(t) h_{r-1}(t)\Big(\rho - \rho\varphi(t)^2\|x_{r-1}(t)\|^2 \\
&\qquad\quad - \|G\| + \|G\|\varphi_1(t)^2\Big(\rho^2\|x_{r-1}(t)\|^2 - 2\rho\|x_{r-1}(t)\|(\kappa_1 - \rho\kappa_{r-1}) + (\kappa_1 - \rho\kappa_{r-1})^2\Big)\Big) \\
&\quad \overset{(A.2)}{=} h_1(t) h_{r-1}(t)(\rho - \|G\|)\big(1 - \varphi(t)^2\|x_{r-1}(t)\|^2\big) \\
&\qquad\quad + h_1(t) h_{r-1}(t)\|G\|\varphi_1(t)^2\big(-2\rho\|x_{r-1}(t)\|(\kappa_1 - \rho\kappa_{r-1}) + (\kappa_1 - \rho\kappa_{r-1})^2\big) \\
&\quad > h_1(t) h_{r-1}(t)(\rho - \|G\|)\big(1 - \varphi(t)^2\|x_{r-1}(t)\|^2\big) \\
&\qquad\quad + h_1(t) h_{r-1}(t)\|G\|\varphi_1(t)^2\left(-2\rho\frac{1}{\varphi(t)}(\kappa_1 - \rho\kappa_{r-1}) + (\kappa_1 - \rho\kappa_{r-1})^2\right) \\
&\quad \overset{(A.2)}{=} h_1(t) h_{r-1}(t)(\rho - \|G\|)\big(1 - \varphi(t)^2\|x_{r-1}(t)\|^2\big) \\
&\qquad\quad + h_1(t) h_{r-1}(t)\|G\|\varphi_1(t)^2\left(-\frac{2}{\varphi_1(t)}(\kappa_1 - \rho\kappa_{r-1}) + (\kappa_1 - \rho\kappa_{r-1})^2\right) \\
&\quad = h_1(t)(\rho - \|G\|) + h_1(t) h_{r-1}(t)\|G\|\varphi_1(t)\big(-2(\kappa_1 - \rho\kappa_{r-1}) + \varphi_1(t)(\kappa_1 - \rho\kappa_{r-1})^2\big) \\
&\quad > h_1(t)(\rho - \|G\|) - 2h_1(t) h_{r-1}(t)\|G\|\varphi_1(t)(\kappa_1 - \rho\kappa_{r-1}) \\
&\quad = h_1(t)\Big(\rho - \|G\| - 2\varphi_1(t)\|G\| h_{r-1}(t)(\kappa_1 - \rho\kappa_{r-1})\Big) \\
&\quad \overset{(3.41)}{\geq} h_1(t)\left(\rho - \|G\| - 2\varphi_1(t)\|G\|\frac{1}{\rho\varphi_1(t)\kappa_{r-1}}(\kappa_1 - \rho\kappa_{r-1})\right) \\
&\quad = h_1(t)\left(\rho + \|G\| - \frac{2\|G\|}{\rho}\frac{\kappa_1}{\kappa_{r-1}}\right) \\
&\quad \overset{(3.32)}{=} h_1(t)\left(\rho + \|G\| - \frac{4\rho^2\|G\|}{\rho}\frac{\kappa}{\delta^{r-2}\hat{\delta}\kappa}\right) \\
&\quad \overset{(3.27),(3.31)}{>} h_1(t)\Big(\rho + \|G\| - 4\|G\|\rho^2(\rho+1)^{r-2}\Big) \\
&\quad \overset{(3.28)}{=} h_1(t)\Delta_1.
\end{aligned}
$$

Condensing the estimations above, we find for $t \in [t_0, t_1]$

$$
\rho h_1(t) - \|G\| h_{r-1}(t) > h_1(t)\Delta_1 \overset{(3.37)}{\geq} \frac{\lambda_1\Delta_1}{2\kappa} > 0. \tag{3.42}
$$

To sum up, with (3.37), (3.41) and (3.42), we obtain for $t \in [t_0, t_1]$

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_1(t)\|^2 &\overset{(3.42)}{<} -\frac{\Delta_1\lambda_1}{2\kappa}\frac{\|x_1(t)\|^2}{\rho} + \left(\frac{\|G\|(\kappa_1-\rho\kappa_{r-1})}{\rho}h_{r-1}(t) + \|b_1(t)\|\right)\|x_1(t)\| \\
&\overset{(3.41)}{\leq} -\frac{\Delta_1\lambda_1}{2\rho\kappa}\|x_1(t)\|^2 + \frac{\|G\|}{\rho^2\varphi_1(t)}\frac{\kappa_1}{\kappa_{r-1}}\|x_1(t)\| + \|x_1(t)\|\|b_1(t)\| \\
&\overset{(3.32)}{=} -\frac{\Delta_1\lambda_1}{2\rho\kappa}\|x_1(t)\|^2 + \frac{2\|G\|}{\varphi_1(t)}\frac{1}{\delta^{r-2}\hat{\delta}}\|x_1(t)\| + \|x_1(t)\|\|b_1(t)\| \\
&\overset{(3.27)}{\leq} -\frac{\Delta_1\lambda_1}{2\rho\kappa}\|x_1(t)\|^2 + \left(2\|G\| \sup_{s\in[\beta,\omega)}\psi_1(s)\rho(\rho+1)^{r-2} + \|b_1(t)\|\right)\|x_1(t)\| \\
&\overset{(3.37)}{\leq} \left(-\frac{\Delta_1\lambda_1^2}{4\rho\kappa} + 2\|G\| \sup_{s\in[\beta,\omega)}\psi_1(s)\rho(\rho+1)^{r-2} + \sup_{s\in[0,\omega)}\|b_1(s)\|\right)\|x_1(t)\|.
\end{aligned}$$

Then, similar to (3.38) we deduce a contradiction via Lemma 1.11. Therefore, in both cases (3.35a) and (3.35b) we have for $t \in [\beta, \omega)$ that $\psi_1(t) - \|x_1(t)\| \geq \kappa_1$. Moreover, for $t \in [0, \beta)$ we have $\psi_1(t) - \|x_1(t)\| \geq \kappa_1$ by definition of κ. Therefore, $h_1 \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$. Then, successively we obtain $h_i \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$ for all remaining $i \in \{2,\ldots,r-1\}$.

If $\ell = 2$ we have, because $t_0 = t_0^2 < t_0^1$ by (3.34a),

$$\psi(t_0) - \|x_2(t_0)\| = \kappa_2 \overset{(3.32)}{=} \frac{\delta\kappa}{2\rho^2} < \frac{\kappa_1}{\rho} \leq \frac{\psi_1(t_0)}{\rho} - \frac{\|x_1(t_0)\|}{\rho}.$$

Invoking the definition of $t_0 = t_0^2$ there exists $t_1 \in (t_0, t_0^1)$ such that for $t \in [t_0, t_1]$

$$\begin{aligned}
&\psi(t) - \|x_2(t)\| \leq \kappa_2 < \frac{\kappa_1}{\rho} \leq \psi(t) - \frac{\|x_1(t)\|}{\rho} \\
\iff & -\|x_2(t)\| \leq \kappa_2 - \psi(t) < \frac{\kappa_1}{\rho} - \psi(t) \leq -\frac{\|x_1(t)\|}{\rho} \\
\iff & 0 \leq \kappa_2 - \psi(t) + \|x_2(t)\| < \frac{\kappa_1}{\rho} - \psi(t) + \|x_2(t)\| \leq \|x_2(t)\| - \frac{\|x_1(t)\|}{\rho}.
\end{aligned} \tag{3.43}$$

So we readily conclude for $t \in [t_0, t_1]$ the relations

$$\begin{aligned}
\psi(t) - \|x_2(t)\| &\leq \kappa_2, \\
\|x_2(t)\| &\geq \psi(t) - \kappa_2 \geq \frac{\lambda}{2}, \\
h_2(t) = \frac{1}{1-\varphi(t)^2\|x_2(t)\|^2} &\geq \frac{1}{2\varphi(t)\kappa_2} \geq \frac{\lambda}{2\kappa_2},
\end{aligned} \tag{3.44a}$$

and analogously to (3.41) we find

$$h_1(t) = \frac{1}{1-\varphi_1(t)^2\|x_1(t)\|^2} \leq \frac{1}{\varphi_1(t)\kappa_1}. \tag{3.44b}$$

Furthermore, using (3.43) and estimation (3.44a), we have for $t \in [t_0, t_1]$

$$\begin{aligned}
\|x_2(t)\| - \frac{\|x_1(t)\|}{\rho} &\geq \|x_2(t)\| - \psi(t) + \frac{\kappa_1}{\rho} \\
&\geq \psi(t) - \kappa_2 - \psi(t) + \frac{\kappa_1}{\rho} \overset{(3.32)}{=} \frac{\kappa}{\rho} - \frac{\delta\kappa}{2\rho^2} \overset{(3.27)}{>} 0.
\end{aligned} \tag{3.45}$$

We observe that (3.45) implies $\|x_1(t)\| < \rho\|x_2(t)\|$ for $t \in [t_0, t_1]$. Considering equation (3.26) for $i = 2$ and $t \in [t_0, t_1]$ we obtain

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_2(t)\|^2 &= -h_2(t)\|x_2(t)\|^2 + \langle x_2(t), h_1(t)x_1(t) + b_2(t)\rangle \\ &\le -h_2(t)\|x_2(t)\|^2 + h_1(t)\|x_2(t)\|\|x_1(t)\| + \|x_2(t)\|\|b_2(t)\| \\ &\overset{(3.45)}{<} -h_2(t)\|x_2(t)\|^2 + h_1(t)\rho\|x_2(t)\|^2 + \|x_2(t)\|\|b_2(t)\| \\ &= \Big(-h_2(t) + \rho h_1(t)\Big)\|x_2(t)\|^2 + \|x_2(t)\|\|b_2(t)\|. \end{aligned} \tag{3.46}$$

Thanks to $0 < \delta < 1$ in (3.27) and $\kappa > 0$ given in (3.32) we have

$$\kappa_1 - 2\rho^2\kappa_2 \overset{(3.32)}{=} \kappa - 2\rho^2\frac{\delta\kappa}{2\rho^2} = \kappa(1-\delta) \overset{(3.27)}{>} 0. \tag{3.47}$$

Hence, using property (A.2) and the relations from (3.44) we obtain for $t \in [t_0, t_1]$

$$\begin{aligned} h_2(t) - \rho h_1(t) &\overset{(3.44)}{\ge} \frac{1}{2\varphi(t)\kappa_2} - \frac{\rho}{\varphi_1(t)\kappa_1} \overset{(A.2)}{=} \frac{\kappa_1 - 2\rho^2\kappa_2}{2\varphi(t)\kappa_1\kappa_2} \\ &\overset{(3.32),(3.47)}{=} \frac{2\rho^2(1-\delta)\kappa}{2\varphi(t)\delta\kappa^2} \overset{(3.44a)}{\ge} \frac{1-\delta}{\delta}\frac{\rho^2\lambda}{\kappa} \overset{(3.27)}{>} \frac{\rho^2\lambda}{\kappa}. \end{aligned}$$

With this, using (3.44a) we estimate (3.46) for $t \in [t_0, t_1]$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_2(t)\|^2 \overset{(3.44a)}{<} \left(-\frac{\rho^2\lambda^2}{2\kappa} + \sup_{s\in[0,\omega)}\|b_2(s)\|\right)\|x_2(t)\| \overset{(3.30b)}{\le} -L\|x_2(t)\|.$$

Then, similar to (3.38) a contradiction arises. Therefore, for $t \in [\beta, \omega)$ we have $\psi(t) - \|x_2(t)\| \ge \kappa_2$. Moreover, for $t \in [0, \beta)$ we have $\psi(t) - \|x_2(t)\| \ge \kappa_2$ by definition of κ. Hence, $h_2 \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$. Then successively we obtain $h_i \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$ for all remaining $i \in \{1, \ldots, r-1\} \setminus \{2\}$.

If $3 \le \ell \le r-1$ we have, because $t_0 = t_0^\ell < t_0^{\ell-1}$ by (3.34a),

$$\psi(t_0) - \|x_\ell(t_0)\| = \kappa_\ell < \kappa_{\ell-1} \le \psi(t_0) - \|x_{\ell-1}(t_0)\|.$$

Then, by invoking the definition of $t_0 = t_0^\ell$ there exists $t_1 \in (t_0, t_0^{\ell-1})$ such that

$$\forall\, t \in [t_0, t_1] \;:\; \psi(t) - \|x_\ell(t)\| \le \kappa_\ell < \kappa_{\ell-1} \le \psi(t) - \|x_{\ell-1}(t)\|. \tag{3.48}$$

As before, we deduce for $t \in [t_0, t_1]$

$$\begin{aligned} \psi(t) - \|x_\ell(t)\| &\le \kappa_\ell, \\ \|x_\ell(t)\| &\ge \psi(t) - \kappa_\ell \ge \frac{\lambda}{2}, \\ h_\ell(t) = \frac{1}{1 - \varphi(t)^2\|x_\ell(t)\|^2} &\ge \frac{1}{2\varphi(t)\kappa_\ell}, \end{aligned} \tag{3.49a}$$

similar to (3.41) we obtain

$$h_{\ell-1}(t) = \frac{1}{1 - \varphi(t)^2\|x_{\ell-1}(t)\|^2} \le \frac{1}{\varphi(t)\kappa_{\ell-1}}, \tag{3.49b}$$

and, using (3.48) and (3.49a), we have for $t \in [t_0, t_1]$

$$\begin{aligned} \|x_\ell(t)\| - \|x_{\ell-1}(t)\| &\geq \|x_\ell(t)\| - \psi(t) + \kappa_{\ell-1} \\ &\geq \psi(t) - \kappa_\ell - \psi(t) + \kappa_{\ell-1} \overset{(3.32)}{=} \frac{\delta^{\ell-2}}{2\rho^2}(1-\delta)\,\kappa > 0. \end{aligned} \tag{3.50}$$

We observe that (3.50) implies $\|x_\ell(t)\| > \|x_{\ell-1}(t)\|$ for $t \in [t_0, t_1]$. Considering equation (3.26) for $i = \ell$ and $t \in [t_0, t_1]$ we obtain

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_\ell(t)\|^2 &= -h_\ell(t)\|x_\ell(t)\|^2 + \langle x_\ell(t), h_{\ell-1}(t)x_{\ell-1}(t) + b_\ell(t)\rangle \\ &\leq -h_\ell(t)\|x_\ell(t)\|^2 + h_{\ell-1}(t)\|x_\ell(t)\|\|x_{\ell-1}(t)\| + \|x_\ell(t)\|\|b_\ell(t)\| \\ &\overset{(3.50)}{<} \Big(-h_\ell(t) + h_{\ell-1}(t)\Big)\|x_\ell(t)\|^2 + \|x_\ell(t)\|\|b_\ell(t)\|. \end{aligned} \tag{3.51}$$

Noting that

$$\kappa_{\ell-1} - 2\kappa_\ell \overset{(3.32)}{=} \frac{\delta^{\ell-2}\kappa}{2\rho^2}(1-2\delta) \overset{(3.28)}{=} \frac{\delta^{\ell-2}\Delta}{2\rho^2}\kappa > 0, \tag{3.52}$$

and using (3.49) we estimate

$$\begin{aligned} h_\ell(t) - h_{\ell-1}(t) &\overset{(3.49)}{\geq} \frac{1}{2\varphi(t)\kappa_\ell} - \frac{1}{\varphi(t)\kappa_{\ell-1}} = \frac{\kappa_{\ell-1} - 2\kappa_\ell}{2\varphi(t)\kappa_{\ell-1}\kappa_\ell} \\ &\overset{(3.32),(3.52)}{=} \frac{1}{2\varphi(t)}\,\frac{\delta^{\ell-2}\Delta\kappa}{2\rho^2}\,\frac{4\rho^4}{\delta^{\ell-2}\delta^{\ell-1}\kappa^2} \overset{(3.27),(3.49a)}{>} 2^{\ell-1}\Delta\,\frac{\rho^2\lambda}{\kappa}. \end{aligned}$$

With this and using (3.49a) we estimate (3.51) for $t \in [t_0, t_1]$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|x_\ell(t)\|^2 < \left(-2^{\ell-1}\Delta\frac{\rho^2\lambda^2}{\kappa} + \sup_{s\in[0,\omega)}\|b_\ell(s)\|\right)\|x_\ell(t)\| \overset{(3.30b)}{\leq} -L\|x_\ell(t)\|.$$

As before, a contradiction arises from analogous calculations as in (3.38). Hence, for $t \in [\beta, \omega)$ we have $\psi(t) - \|x_\ell(t)\| \geq \kappa_\ell$. Moreover, for $t \in [0, \beta)$ we have $\psi(t) - \|x_\ell(t)\| \geq \kappa_\ell$ by definition of κ. Therefore, $h_\ell \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$. Then, successively we obtain $h_i \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$ for all remaining $i \in \{1,\dots,r-1\}\setminus\{\ell\}$. This completes *Step two*.

Step three. We show $\omega = \infty$. Seeking a contradiction let $\omega < \infty$. Then, since h_i and $w_{i,j}$ for $i = 1,\dots,r-1$ and $j = 1,\dots,r$ are bounded via *Step two*, it follows that the graph of the solution of (3.14) is a compact subset of $\mathcal{D}$, which contradicts the findings in *Step one*. Thus, $\omega = \infty$.

Step four. We show $\tilde{\mathbf{T}} \in \mathcal{T}_0^{rm,\bar{q}}$. First, we observe that by *Step three* the operator $\tilde{\mathbf{T}}$ is well defined, and satisfies condition (T.1) of Definition 1.4 by *Step two*. Moreover, since $\tilde{\mathbf{T}}$ is defined via the solution of (3.14), property (T.2) follows from uniqueness of the maximal solution. It remains to show that the operator $\tilde{\mathbf{T}}$ satisfies property (T.3) in Definition 1.4. To this end, we consider the auxiliary error system (3.14) and interpret $z, \dot{z}, \dots, z^{(r-1)}$ as inputs for this system. We show that for all $i = 1,\dots,r-1$ the respective right-hand side of (3.14) is Lipschitz continuous with respect to the state and the variables which are interpreted as inputs. From this we then conclude that the operator $\tilde{\mathbf{T}}$ satisfies (T.3) with "memory" $\sigma = 0$ in the present context. Before we establish the estimations to conclude that $\tilde{\mathbf{T}}$ satisfies (T.3), we record the following observations concerning the gain functions h. For a strictly positive $\gamma > 0$ and $\varphi \in \Phi_r$ we define the set

$$\mathcal{F}_{\varphi,\gamma} := \{\, (s,\eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid 1 - \varphi(s)\|\eta\| \geq \gamma \,\}. \tag{3.53}$$

Further, for $(s,\eta) \in \mathcal{F}_{\varphi,\gamma}$ we define the function $h(s,\eta) := 1/(1-\varphi(s)^2\|\eta\|^2)$. Then, for $(s,\eta) \in \mathcal{F}_{\varphi,\gamma}$ we observe

$$\frac{1}{h(s,\eta)} = (1+\varphi(s)\|\eta\|)(1-\varphi(s)\|\eta\|) \geq \gamma,$$

which implies

$$\forall\,(s,\eta) \in \mathcal{F}_{\varphi,\gamma} \;:\; h(s,\eta) \leq \frac{1}{\gamma}, \tag{3.54}$$

so $h(s,\eta)$ is uniformly bounded for $(s,\eta) \in \mathcal{F}_{\varphi,\gamma}$. Using this, for $(s,\eta),(s,\hat{\eta}) \in \mathcal{F}_{\varphi,\gamma}$ we obtain

$$\begin{aligned}
\|\eta h(s,\eta) - \hat{\eta} h(s,\hat{\eta})\| &= \left\| \frac{\eta}{1-\varphi(s)^2\|\eta\|^2} - \frac{\hat{\eta}}{1-\varphi(s)^2\|\hat{\eta}\|^2} \right\| \\
&\leq h(s,\eta)h(s,\hat{\eta})\|\eta-\hat{\eta}\| + h(s,\eta)h(s,\hat{\eta})\varphi(s)^2 \left\| \eta\|\hat{\eta}\|^2 - \hat{\eta}\|\eta\|^2 \right\| \\
&\overset{(3.54)}{\leq} \frac{1}{\gamma^2}\|\eta-\hat{\eta}\| + \frac{\varphi(s)^2}{\gamma^2} \left\| \eta\|\hat{\eta}\|^2 - \eta\|\eta\|^2 + \eta\|\eta\|^2 - \hat{\eta}\|\eta\|^2 \right\| \\
&\leq \frac{1}{\gamma^2}\|\eta-\hat{\eta}\| + \frac{\varphi(s)^2}{\gamma^2}\|\eta\|^2\|\eta-\hat{\eta}\| + \frac{\varphi(s)^2}{\gamma^2}\|\eta\| \left| \|\hat{\eta}\|^2 - \|\eta\|^2 \right| \\
&< \frac{1}{\gamma^2}\|\eta-\hat{\eta}\| + \frac{1}{\gamma^2}\|\eta-\hat{\eta}\| + \frac{\varphi(s)^2}{\gamma^2}\|\eta\| \left(\|\hat{\eta}\| + \|\eta\|\right) \left| \|\hat{\eta}\| - \|\eta\| \right| \\
&< \frac{4}{\gamma^2}\|\eta-\hat{\eta}\|,
\end{aligned} \tag{3.55a}$$

where we used $\varphi(s)\|\eta\| < 1$, $\varphi(s)\|\hat{\eta}\| < 1$, and the reversed triangular inequality in the last estimation. Further, for $(s,\eta),(s,\hat{\eta}) \in \mathcal{F}_{\varphi,\gamma}$ we obtain

$$\begin{aligned}
|h(s,\eta) - h(s,\hat{\eta})| &= \left| \frac{1}{1-\varphi(s)^2\|\eta\|^2} - \frac{1}{1-\varphi(s)^2\|\hat{\eta}\|^2} \right| \\
&\leq h(s,\eta)h(s,\hat{\eta})\varphi(s)^2 \left| \|\eta\|^2 - \|\hat{\eta}\|^2 \right| \\
&\overset{(3.54)}{\leq} \frac{\varphi(s)^2}{\gamma^2}(\|\eta\| + \|\hat{\eta}\|) \left| \|\eta\| - \|\hat{\eta}\| \right| \\
&< \frac{2}{\gamma^2} \sup_{\sigma \in \mathbb{R}_{\geq 0}} \varphi(\sigma)\|\eta-\hat{\eta}\|,
\end{aligned} \tag{3.55b}$$

where we used $\varphi(s)\|\eta\| < 1$, $\varphi(s)\|\hat{\eta}\| < 1$, and the reversed triangular inequality in the last estimation. Respective estimates hold true for $(s,\eta),(s,\hat{\eta}) \in \mathcal{F}_{\varphi_1,\gamma}$. With this preparatory observations at hand, we return to the task of establishing estimations, which allow us to conclude that the operator $\tilde{\mathbf{T}}$ satisfies (T.3). To this end, for $t \geq 0$, some $\Delta, \delta > 0$ (with some abuse of notation Δ, δ are not the same as in *Step two*), and $\xi \in \mathcal{C}([0,t];\mathbb{R}^{rm})$ we define the set

$$Z := \left\{ \zeta \in \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^{rm}) \;\middle|\; \begin{array}{l} \zeta|_{[0,t]} = \xi, \\ \forall\, s \in [t,t+\Delta] \;:\; \|\zeta(s) - \xi(t)\| < \delta, \\ \forall\, s \geq t+\Delta \;:\; \zeta(s) = \zeta(t+\Delta) \end{array} \right\}.$$

According to condition (T.3) we seek to show that there exists $C_0 > 0$ such that

$$\zeta, \hat{\zeta} \in Z \;\Rightarrow\; \operatorname{ess\,sup}_{s \in [t,t+\Delta]} \|\tilde{\mathbf{T}}(\zeta)(s) - \tilde{\mathbf{T}}(\hat{\zeta})(s)\| \leq C_0 \sup_{s \in [t,t+\Delta]} \|\zeta(s) - \hat{\zeta}(s)\|. \tag{3.56}$$

First, we observe that for $\zeta \in Z$ we have

$$\forall\, s \in [t, t+\Delta] \; : \; \|\zeta(s)\| = \|\zeta(s) - \xi(t) + \xi(t)\| < \delta + \|\xi\|_\infty.$$

Thus, $\zeta := (z, \dots, z^{(r-1)}) : \mathbb{R}_{\ge 0} \to \mathbb{R}^{rm}$ to be considered as input in (3.14) is uniformly bounded on the interval $[t, t+\Delta]$. Therefore, the results established in *Step two* are valid and can be used for the subsequent reasoning. Inspecting the definition of κ in (3.29) and κ_i, $i = 1, \dots, r-1$, in (3.32), we see that these numbers depend on ζ. We define

$$\begin{aligned}
\varepsilon_1 &:= \inf_{\sigma \in \mathbb{R}_{\ge 0}} \varphi_1(\sigma) \inf_{\zeta \in Z} \kappa_1 > 0, \\
\varepsilon_j &:= \inf_{\sigma \in \mathbb{R}_{\ge 0}} \varphi(\sigma) \inf_{\zeta \in Z} \kappa_j > 0, \quad j = 2, \dots, r-1, \\
\varepsilon &:= \min_{i \in \{1, \dots, r-1\}} \varepsilon_i > 0.
\end{aligned}$$

Clearly, we have $\varepsilon \le \kappa_i$ for all $i = 1, \dots, r-1$. Therefore, by (3.33) in *Step two* we have for all $i = 2, \dots, r-1$

$$\forall\, s \in [t, t+\Delta] \; : \; \Big(1 - \varphi_1(s)\|w_{1,1}(s) - G\tilde{w}(s)\| \ge \varepsilon \; \wedge \; 1 - \varphi(s)\|w_{i,1}(s)\| \ge \varepsilon\Big),$$

that is, for $\mathcal{F}_{\varphi,\varepsilon}$ defined in (3.53) we have $(s, w_{1,1}(s) - G\tilde{w}(s)) \in \mathcal{F}_{\varphi_1,\varepsilon}$ and $(s, w_{i,1}(s)) \in \mathcal{F}_{\varphi,\varepsilon}$ for all $i = 2, \dots, r-1$ and all $s \in [t, t+\Delta]$. Next, we consider (3.14) separately for $i = 1$, $i = 2$ and $i = 3, \dots, r-1$. Recalling $\bar{P} = (p_1, \dots, p_r)^\top \otimes I_m$, we refine the compactly written dynamics (3.18) with $\zeta = (\zeta_1, \dots, \zeta_r) = (z, \dots, z^{(r-1)})$ and obtain

$$\begin{aligned}
\dot{w}_1(t) = {}& \underbrace{\begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}}_{=: M_1} w_1(t) + \underbrace{\begin{bmatrix} R_r - a_1 \Gamma \tilde{\Gamma}^{-1} \\ \vdots \\ R_2 - a_{r-1} \Gamma \tilde{\Gamma}^{-1} \\ R_1 - a_r \Gamma \tilde{\Gamma}^{-1} \end{bmatrix}}_{=: M_2} w_{1,1}(t) \\
& + \underbrace{\left(\begin{bmatrix} R_r \\ \vdots \\ R_1 \end{bmatrix} - \begin{bmatrix} R_r - a_1 \Gamma \tilde{\Gamma}^{-1} \\ \vdots \\ R_1 - a_r \Gamma \tilde{\Gamma}^{-1} \end{bmatrix} G \right)}_{=: M_3} \tilde{w}(t) - \bar{P} \Gamma \tilde{\Gamma}^{-1} \frac{\bar{w}(t)}{1 - \varphi_1(t)^2 \|\bar{w}(t)\|^2} \\
& + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ -G\zeta_r(t) \\ \sum_{i=1}^r R_i \zeta_i(t) \end{pmatrix}}_{=: M_4 \zeta(t)} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ f(d(t), \mathbf{T}(y, \dot{y}, \dots, y^{(r-1)})(t)) \end{pmatrix},
\end{aligned}$$

where $\bar{w} = w_{1,1} - G\tilde{w}$. Explicitly indicating the dependence of the “input” ζ, let $W(\cdot, \zeta)$ denote the solution of the overall differential equation (3.14), and $w_i(\cdot, \zeta)$ denotes the respective solution of the subsystems in (3.14a) – (3.14c) for $i = 1, \dots, r-1$, respectively. Further, let $\tilde{w}(\cdot, \zeta), \bar{w}(\cdot, \zeta)$ be the corresponding expressions to $\tilde{w}, \bar{w}$. By the observations above, $\tilde{w}(\cdot, \zeta), \bar{w}(\cdot, \zeta)$ are uniformly bounded on the interval

$[t, t+\Delta]$, in particular, $(s, \bar{w}(s,\zeta)), (s, \bar{w}(s,\hat{\zeta})) \in \mathcal{F}_{\varphi_1,\varepsilon}$ for $s \in [t, t+\Delta]$. We set $\mu_j := \|M_j\|$ for $j = 1,2,3$ and $c_0 := \|M_4\|$. With this, invoking (3.55a) we may infer the existence of $C_1 > 0$ such that for $s \in [t, t+\Delta]$ and $\zeta, \hat{\zeta} \in Z$ we have

$$\begin{aligned}
\|\dot{w}_1(s,\zeta) - \dot{w}_1(s,\hat{\zeta})\| &\le \mu_1 \|w_1(s,\zeta) - w_1(s,\hat{\zeta})\| + \mu_2 \|w_{1,1}(s,\zeta) - w_{1,1}(s,\hat{\zeta})\| \\
&\quad + \mu_3 \|\tilde{w}(s,\zeta) - \tilde{w}(s,\hat{\zeta})\| + c_0 \|\zeta(s) - \hat{\zeta}(s)\| \\
&\quad + \|\bar{P}\Gamma\tilde{\Gamma}^{-1}\| \left\| \frac{\bar{w}(s,\zeta)}{1-\varphi_1(s)^2\|\bar{w}(s,\zeta)\|^2} - \frac{\bar{w}(s,\hat{\zeta})}{1-\varphi_1(s)^2\|\bar{w}(s,\hat{\zeta})\|^2} \right\| \\
&\overset{(3.55a)}{\le} \mu_1 \|w_1(s,\zeta) - w_1(s,\hat{\zeta})\| + \mu_2 \|w_{1,1}(s,\zeta) - w_{1,1}(s,\hat{\zeta})\| \\
&\quad + \mu_3 \|\tilde{w}(s,\zeta) - \tilde{w}(s,\hat{\zeta})\| + c_0 \|\zeta(s) - \hat{\zeta}(s)\| \\
&\quad + \frac{4\|\bar{P}\Gamma\tilde{\Gamma}^{-1}\|}{\varepsilon^2} \|\bar{w}(s,\zeta) - \bar{w}(s,\hat{\zeta})\| \\
&\le C_1 \|W(s,\zeta) - W(s,\hat{\zeta})\| + c_0 \|\zeta(s) - \hat{\zeta}(s)\|,
\end{aligned}$$

so, compactly written, for $s \in [t, t+\Delta]$ we have

$$\|\dot{w}_1(s,\zeta) - \dot{w}_1(s,\hat{\zeta})\| \le C_1 \|W(s,\zeta) - W(s,\hat{\zeta})\| + c_0 \|\zeta(s) - \hat{\zeta}(s)\|, \tag{3.57a}$$

where in particular, the term $f(d(t), \mathbf{T}(y, \ldots, y^{(r-1)}(t))$ does not occur since it is not affected by different "inputs" $\zeta, \hat{\zeta}$. Next, for $i = 2$ we consider (3.14b), and obtain in accordance with (3.18)

$$\dot{w}_2(s) = \hat{A} w_2(s) - h_2(s) \bar{P} w_{2,1}(s) + \big(\bar{A} + \bar{P} h_1(s)\big)\, \bar{w}(s)$$

where $\hat{A} = A \otimes I_m$ and $\bar{A} := (a_1, \ldots, a_r)^\top \otimes I_m$. Then, invoking the findings from above and (3.55a), we may infer the existence of $C_2 > 0$ such that for $s \in [t, t+\Delta]$ we have

$$\begin{aligned}
\|\dot{w}_2(s,\zeta) - \dot{w}_2(s,\hat{\zeta})\| &\le \|\hat{A}\| \|w_2(s,\zeta) - w_2(s,\hat{\zeta})\| \\
&\quad + \frac{4\|\bar{P}\|}{\varepsilon^2} \|w_{2,1}(s,\zeta) - w_{2,1}(s,\hat{\zeta})\| \\
&\quad + \left(\|\bar{A}\| + \frac{4}{\varepsilon^2} \|\bar{P}\| \right) \|\bar{w}(s,\zeta) - \bar{w}(s,\hat{\zeta})\| \\
&\le C_2 \|W(s,\zeta) - W(s,\hat{\zeta})\|.
\end{aligned} \tag{3.57b}$$

Accordingly, for $i = 3, \ldots, r-1$ in (3.14c) we have

$$\dot{w}_i(s) = \hat{A} w_i(s) - h_i(s) \bar{P} w_{i,1}(s) + \big(\bar{A} + \bar{P} h_{i-1}(s)\big)\, w_{i-1,1}(s).$$

Then, invoking the previously established inequalities, we infer the existence of $C_i > 0$ such that for $s \in [t, t+\Delta]$ and $i = 3, \ldots, r-1$ we have

$$\|\dot{w}_i(s,\zeta) - \dot{w}_i(s,\hat{\zeta})\| \le C_i \|W(s,\zeta) - W(s,\hat{\zeta})\|. \tag{3.57c}$$

We set $C := 2\sum_{i=1}^{r-1} C_i + 1$. With this, invoking estimations (3.57a) – (3.57c), we calculate

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}s}\|W(s,\zeta)-W(s,\hat\zeta)\|^2 &\le 2\|W(s,\zeta)-W(s,\hat\zeta)\|\sum_{i=1}^{r-1}\|\dot w_i(s,\zeta)-\dot w_i(s,\hat\zeta)\| \\
&\overset{(3.57)}{\le} 2\sum_{i=1}^{r-1}C_i\|W(s,\zeta)-W(s,\hat\zeta)\|^2 \\
&\quad + 2\|W(s,\zeta)-W(s,\hat\zeta)\|c_0\|\zeta(s)-\hat\zeta(s)\| \\
&\le 2\sum_{i=1}^{r-1}C_i\|W(s,\zeta)-W(s,\hat\zeta)\|^2 \\
&\quad + 2\|W(s,\zeta)-W(s,\hat\zeta)\|c_0\sup_{\sigma\in[t,t+\Delta]}\|\zeta(\sigma)-\hat\zeta(\sigma)\| \\
&\le 2\sum_{i=1}^{r-1}C_i\|W(s,\zeta)-W(s,\hat\zeta)\|^2 \\
&\quad + \|W(s,\zeta)-W(s,\hat\zeta)\|^2 + \Big(c_0\sup_{\sigma\in[t,t+\Delta]}\|\zeta(\sigma)-\hat\zeta(\sigma)\|\Big)^2 \\
&\le C\|W(s,\zeta)-W(s,\hat\zeta)\|^2 + \Big(c_0\sup_{\sigma\in[t,t+\Delta]}\|\zeta(\sigma)-\hat\zeta(\sigma)\|\Big)^2.
\end{aligned}$$

We note that for $\zeta,\hat\zeta\in Z$ we have $W(0,\zeta)=W(0,\hat\zeta)$. Thus, for $s\in[t,t+\Delta]$ we obtain upon integration

$$\begin{aligned}
\|W(s,\zeta)-W(s,\hat\zeta)\|^2 &\le \int_t^s C\|W(\sigma,\zeta)-W(\sigma,\hat\zeta)\|^2\mathrm{d}\sigma \\
&\quad + \Big(c_0\sup_{\sigma\in[t,t+\Delta]}\|\zeta(\sigma)-\hat\zeta(\sigma)\|\Big)^2\int_t^{t+\Delta}\mathrm{d}\sigma.
\end{aligned}$$

We set $C_0:=\sqrt{\Delta}\,e^{\frac12 C\Delta}c_0$. Then for $s\in[t,t+\Delta]$ Grönwall's lemma yields

$$\|W(s,\zeta)-W(s,\hat\zeta)\|\le C_0\sup_{\sigma\in[t,t+\Delta]}\|\zeta(\sigma)-\hat\zeta(\sigma)\|, \tag{3.58}$$

where the right-hand side is independent of s. Next, we invoke the previous observation that $(s,w_{i,1}(s,\zeta))\in\mathcal{F}_{\varphi,\varepsilon}$ for $\zeta\in Z$ and $s\in[t,t+\Delta]$. We consider the gain functions h_i, where we denote $h_i(\cdot,\zeta):=1/(1-\varphi(\cdot)^2\|w_{1,i}(\cdot,\zeta)\|^2)$, $i=2,\dots,r-1$. Then, for $s\in[t,t+\Delta]$ and using (3.55b), we have

$$\begin{aligned}
\Big|h_i(s,\zeta)-h_i(s,\hat\zeta)\Big| &\le \frac{2}{\varepsilon^2}\sup_{\sigma\in\mathbb{R}_{\ge0}}\varphi(\sigma)\|w_{i,1}(s,\zeta)-w_i(s,\hat\zeta)\| \\
&\le \frac{2}{\varepsilon^2}\sup_{\sigma\in\mathbb{R}_{\ge0}}\varphi(\sigma)\|W(s,\zeta)-W(s,\hat\zeta)\| \qquad (3.59a)\\
&\overset{(3.58)}{\le} \frac{2}{\varepsilon^2}\sup_{\sigma\in\mathbb{R}_{\ge0}}\varphi(\sigma)C_0\sup_{\sigma\in[t,t+\Delta]}\|\zeta(\sigma)-\hat\zeta(\sigma)\|.
\end{aligned}$$

Using $\varphi_1=\varphi/\rho$ and $(s,\bar w(s,\zeta))\in\mathcal{F}_{\varphi_1,\varepsilon}$, for $h_1(\cdot,\zeta):=1/(1-\varphi_1(\cdot)^2\|\bar w(\cdot,\zeta)\|^2)$ we have for $s\in[t,t+\Delta]$

$$|h_1(s,\zeta)-h_1(s,\hat\zeta)|\le\frac{2}{\varepsilon^2}\sup_{\sigma\in\mathbb{R}_{\ge0}}\varphi(\sigma)C_0\sup_{\sigma\in[t,t+\Delta]}\|\zeta(\sigma)-\hat\zeta(\sigma)\|. \tag{3.59b}$$

Note that the right-hand side of (3.59) is independent of s. Together, estimations (3.58) and (3.59) imply (3.56), and so the operator $\tilde{\mathbf{T}}$ satisfies (T.3) in Definition 1.4. Therefore, we may conclude $\tilde{\mathbf{T}} \in \mathcal{T}_0^{rm,\bar{q}}$.

Lastly, we observe that the higher derivatives of z can be calculated via successive application of the cascade's equations (3.2) and result in

$$z^{(j)}(t) = z_{r-1,j+1} + \sum_{k=0}^{j-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k \big((a_{r-k} + p_{r-k}h_{r-1}(t))w_{r-1,1}(t)\big),$$

where $w_{r-1,1} = z_{r-2,1} - z_{r-1,1}$. Then, with $z_{r-1,r+1} := \tilde{\Gamma}u$ the results above allow us to write the conjunction of (3.2) and (3.4) with input u and output $z = z_{r-1,1}$ as

$$\begin{aligned} z^{(r)}(t) &= \sum_{k=0}^{r-1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \Big((a_{r-k} + p_{r-k}h_{r-1}(t))w_{r-1,1}(t)\Big) + \tilde{\Gamma}u(t) \\ &=: \tilde{F}\left(\tilde{d}(t), \tilde{\mathbf{T}}(z, \dot{z}, \ldots, z^{(r-1)})(t)\right) + \tilde{\Gamma}u(t), \end{aligned} \tag{3.60}$$

where $\tilde{F} \in \mathcal{C}(\mathbb{R}^r \times \mathbb{R}^{\bar{q}}; \mathbb{R}^m)$, $\tilde{d}(t) := (\varphi(t), \dot{\varphi}(t), \ldots, \varphi^{(r-1)}(t))^\top \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^r)$ and $\tilde{\mathbf{T}} \in \mathcal{T}_0^{rm,\bar{q}}$. Therefore, $(\tilde{d}, \tilde{F}, \tilde{\mathbf{T}}, \tilde{\Gamma}, 0_{rm\times m}) \in \mathcal{L}^{m,r}$, which completes the proof. □

As a direct consequence of the pre-compensator's design, namely to be of funnel type, we have the following result.

Corollary 3.11. We use the notation and assumptions from Theorem 3.9. Then, for any $u \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and any solution of (3.2), (3.4) with initial conditions (3.11) we have

$$\exists\, \varepsilon > 0\ \forall\, t > 0 : \ \|y(t) - z(t)\| < (\rho + r - 2)\varphi(t)^{-1} - \varepsilon. \tag{3.61}$$

Proof. The prescribed transient behaviour (3.61) follows directly from an iterative application of [32, Prop. 1]. □

Remark 3.12. A careful inspection of the proof of Theorem 3.9 reveals that conditions (A.2) – (A.4) on the design parameters are sufficient but we cannot claim necessity. Condition (A.4) on the norm of the matrix $G = I_m - \Gamma\tilde{\Gamma}^{-1}$ can be interpreted as a "small gain condition" as conjectured in [32, Rem. 4]. Roughly speaking it means "choose the matrix $\tilde{\Gamma}$ close enough to the matrix Γ". Examining the proof shows that this condition plays a crucial role in the estimations (3.23e) and (3.42), however, it allows for various reformulations and small changes which still are sufficient to prove the theorem. Especially, the condition $\|G\| < \rho/(4\rho^2(\rho+1)^{r-2} - 1)$ has many varieties - we cannot claim having found the weakest. If Γ is known, the simple choice $\tilde{\Gamma} = \Gamma$ is feasible and the proof simplifies significantly. Moreover, in this case (A.3) & (A.4) are satisfied at once. However, in general the matrix Γ is not (or only partially) known and hence verification of conditions (A.3) & (A.4) causes problems. In such general cases methods for parameter identification can be useful. For linear systems of type (3.5) there is plenty of literature on system identification, see for instance [168, 36], and the recent work [194] where under the assumptions of controllability and persistently exciting inputs system identification is performed; in [203] the estimated parameters result from a least square problem. Note that although the system identification in [194, 203] is developed for time-discrete linear systems the results can be applied to time-continuous linear systems to some extend, see e.g. [36, Sec. 1]. In [64] parameter identification for nonlinear systems is studied, where under an identifiability condition and with the aid of a high-gain observer system parameters are identified. In [63] an extended high-gain observer is introduced to identify the state and the unknown parameters dynamically; and in the recent (rather technical) work [104] parameter identification via an adaption scheme for nonlinear systems is proposed and an error bound between the nominal and the estimated parameter is given. However, the approaches [64, 63, 104] involve the system equations and hence the parameter identification is not model free. Nevertheless, if a model is available an extension of [104, Prop. 2.1] to matrix valued parameters may yield error bounds such that the stronger version of (A.4)

$$\|\tilde{\Gamma}^{-1}\|\|\tilde{\Gamma} - \Gamma\| < \min\left\{\frac{\rho - 1}{r - 2}, \frac{\rho}{4\rho^2(\rho + 1)^{r-2} - 1}\right\}$$

can ensured to be satisfied.

Remark 3.13. If the first $k \leq r - 1$ derivatives of the output signal y are known, the funnel pre-compensator can be applied to $y^{(k)}$. Then, the condition $\mathbf{T} \in \mathcal{T}^{rm,q}_{\sigma,1}$ in Theorem 3.9 becomes the relaxed condition $\mathbf{T} \in \mathcal{T}^{rm,q}_{\sigma,k}$, with $\mathcal{T}^{rm,q}_{\sigma,k}$ as in Definition 3.6. Moreover, the error bound tightens and we have

$$\forall\, t \geq 0 \;:\; \|y(t) - z(t)\| < (\rho + r - 2 - k)\varphi(t)^{-1} - \varepsilon.$$

Remark 3.14. Although the funnel pre-compensator introduced in [32] may take signals y and u with different dimensions, the system class $\mathcal{L}^{m,r}$ under consideration is restricted to systems where the input and output have the same dimension. This comes into play when applying control schemes to the conjunction of a minimum phase system with a cascade of funnel pre-compensators, see Section 3.3. However, a careful inspection of the proof of Theorem 3.9 yields that an extension of Theorem 3.9 to systems with different input ($u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$) and output ($y : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$) dimensions is possible, if $m > p$. In the case $m < p$ (fewer inputs than outputs) with $\operatorname{rk}\Gamma = \operatorname{rk}\tilde{\Gamma} = m$ one would require the matrix product $\Gamma\tilde{\Gamma}^\dagger \in \mathbb{R}^{p \times p}$ ($\tilde{\Gamma}^\dagger$ denotes a pseudoinverse of $\tilde{\Gamma}$) to be strictly positive definite. However, since $\operatorname{rk}(\Gamma\tilde{\Gamma}^\dagger) \leq \min\{m, p\} < p$ only positive semi-definiteness can be demanded, which is not sufficient. If $m > p$ (more inputs than outputs) the proof of Theorem 3.9 can be adapted such that the statement is still true in the following two cases:

(i) Known $m - p$ entries of the input are set to zero, w.l.o.g. the last $m - p$. Therefore, $\Gamma u = \Gamma_p(u_1, \ldots, u_p)^\top$, where $\Gamma_p = \Gamma_p^\top \in \mathbb{R}^{p \times p}$ and definiteness is required. Then conditions (A.3) & (A.4) have to be satisfied for Γ_p and $\tilde{\Gamma}_p$.

(ii) The system itself ignores known $m - p$ entries of the input (w.l.o.g. the last $m - p$), i.e., $\Gamma = [\Gamma_p, 0]$, where $\Gamma_p = \Gamma_p^\top \in \mathbb{R}^{p \times p}$ and definiteness is required. Then, $\Gamma u = \Gamma_p(u_1, \ldots, u_p)^\top$ and conditions (A.3) & (A.4) have to be satisfied for Γ_p and a symmetric $0 < \tilde{\Gamma}_p \in \mathbb{R}^{p \times p}$.

In both cases the respective transformations in *Step one* of the proof are feasible and the proof of Theorem 3.9 can be done with corresponding matrices Γ_p and $\tilde{\Gamma}_p$.

3.3 An application of the funnel pre-compensator: output feedback control

In this section we discuss the combination of the funnel pre-compensator with feedback control schemes. Theorem 3.9 yields that the conjunction of a minimum phase system (3.4) with a cascade of funnel pre-compensators (3.2) is again a minimum phase system and hence amenable to funnel control; for detailed results on funnel control for systems with higher relative degree see the works [93, 25] and the recent work [21]. We show that the combination of the funnel pre-compensator with a funnel control scheme achieves output tracking with prescribed transient behaviour of the tracking error via output feedback only. This resolves the long-standing problem that for the application of a funnel controller to a system with higher relative degree the derivatives of the system's output are required to be known. In Figure 3.6 the idea is depicted, how the funnel pre-compensator enters the tracking problem.

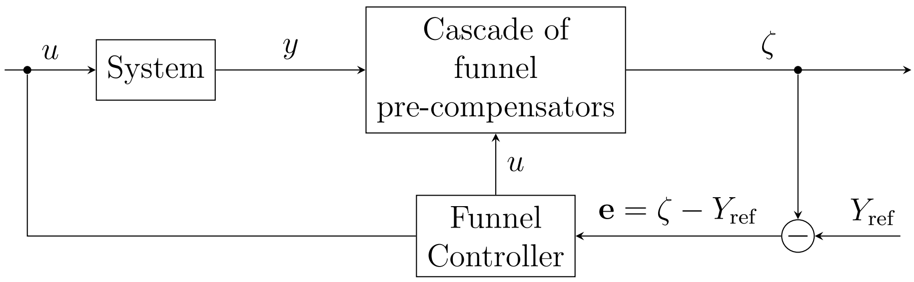


Figure 3.6: Utilizing the funnel pre-compensator for output feedback tracking. The derivatives required by the controller are given via the pre-compensator's output $\zeta = (z, \ldots, z^{(r-1)})$. Here, we set $Y_{\text{ref}} := (y_{\text{ref}}, \ldots, y_{\text{ref}}^{(r-1)})$.

We recall the funnel control scheme from [21]. For $e := z - y_{\text{ref}} \in \mathbb{R}^m$ define the instantaneous available error vector $\mathbf{e}(t) := (e^{(0)}(t)^\top, \ldots, e^{(r-1)}(t)^\top)^\top \in \mathbb{R}^{rm}$, that is, the error vector between the signal z and the reference signal y_{ref} and their derivatives, respectively. Next, the control parameters are chosen. The funnel function ϕ belongs to the set

$$\Phi_{\text{FC}} := \left\{ \phi \in AC_{\text{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}) \;\middle|\; \begin{array}{l} \forall\, s > 0 \;:\; \phi(s) > 0,\ \liminf_{s\to\infty} \phi(s) > 0, \\ \exists\, c > 0 \;:\; |\dot\phi(s)| \leq c\,(1 + \phi(s)) \text{ for a. a. } s \geq 0 \end{array} \right\}.$$

Let $N \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ be a surjection, $\alpha \in \mathcal{C}^1([0,1); [1,\infty))$ be a bijection, and for $\mathcal{B} := \{\, w \in \mathbb{R}^m \mid \|w\| < 1 \,\}$ define the map $\gamma : \mathcal{B} \to \mathbb{R}^m$, $w \mapsto \alpha(\|w\|^2)w$. With this, the maps $\rho_k : \mathcal{D}_k \to \mathcal{B}$ are defined recursively for $k = 1, \ldots, r$ as follows

$$\begin{aligned} &\mathcal{D}_1 := \mathcal{B},\ \rho_1 : \mathcal{D}_1 \to \mathcal{B},\ \eta_1 \mapsto \eta_1, \\ &\mathcal{D}_k := \left\{ (\eta_1^\top, \ldots, \eta_k^\top)^\top \in \mathbb{R}^{km} \;\middle|\; \begin{array}{l} (\eta_1^\top, \ldots, \eta_{k-1}^\top)^\top \in \mathcal{D}_{k-1}, \\ \eta_k + \gamma\left(\rho_{k-1}(\eta_1^\top, \ldots, \eta_{k-1}^\top)^\top\right) \in \mathcal{B} \end{array} \right\}, \\ &\rho_k : \mathcal{D}_k \to \mathcal{B},\ (\eta_1^\top, \ldots, \eta_k^\top)^\top \mapsto \eta_k + \gamma\left(\rho_{k-1}(\eta_1^\top, \ldots, \eta_{k-1}^\top)^\top\right). \end{aligned}$$

Then the funnel control scheme from [21, Sec. 1.4, Eq. (9)] is given by

$$u(t) = (N \circ \alpha)(\|w(t)\|^2)\, w(t),\ \ w(t) := \rho_r(\varphi(t)\,\mathbf{e}(t)). \tag{3.62}$$

Remark 3.15. Before we establish the next result, namely the application of the feedback law (3.62) to the conjunction of a minimum phase system with a cascade of funnel pre-compensators to achieve output reference tracking, we comment on this particular application.

(i) We recall the findings from [21] concerning the high-gain property. Namely, a system

$$\dot{x}(t) = F(d(t), \mathbf{T}(x)(t)) + \Theta\, u(t), \tag{3.63}$$

where $F \in \mathcal{C}(\mathbb{R}^k \times \mathbb{R}^q; \mathbb{R}^n)$, $\mathbf{T} \in \mathcal{T}_\tau^{n,q}$ and $\Theta \in \mathbb{R}^{n\times n}$, has the high-gain property from Definition 1.10 if, and only if, the matrix Θ is strictly sign definite, i.e., for all $v \in \mathbb{R}^m \setminus \{0\}$ we have $\langle v, \Theta v\rangle \neq 0$. Along the lines of [21, Sec. 2.1.3] this can be seen as follows. For $v_* \in (0,1)$ define the set $V := \{\, v \in \mathbb{R}^n \mid v_* \le \|v\| \le 1 \,\}$. First, assume that (3.63) has the high-gain property. Seeking a contradiction, suppose that there exists $v \neq 0$ such that $\langle v, \Theta v\rangle = 0$. Then, there exists $\bar{v} \in V$ such that $\langle \bar{v}, \Theta \bar{v}\rangle = 0$. Choose the compact sets $K = \{0\} \subset \mathbb{R}^k$ and $Q = \{0\} \subset \mathbb{R}^q$. Then, for all $s \in \mathbb{R}$ we have

$$\chi(s) = \min_{v\in V} \left(\langle -sv, \Theta v\rangle\right) \le -s\langle \bar{v}, \Theta\bar{v}\rangle = 0.$$

This, however, contradicts the high-gain property. The remaining implication can be seen as follows. Assume that the matrix Θ is sign definite. Then, there exists $\delta \in \{-1, 1\}$ such that $\delta\Theta$ is positive definite. This implies that the smallest eigenvalue λ^- of the symmetric matrix $\frac{\delta}{2}(\Theta + \Theta^\top)$ is strict positive. So it follows that for all $v \in V$ we have $\langle v, (\Theta + \Theta^\top)v\rangle \ge \lambda^- v_*^2$. Choosing compact sets $K \subset \mathbb{R}^k$, $Q \subset \mathbb{R}^q$ we may define $C := \min\{\, \langle v, \kappa + \mathbf{q}\rangle \mid (\kappa, \mathbf{q}) \in K \times Q, v \in V \,\}$. With this we calculate

$$\forall\, s \in \mathbb{R} : \chi(s) - C \ge \min_{v\in V} \left(\langle -sv, \Theta v\rangle\right) = \min_{v\in V} \left(-\frac{\delta s}{2} \langle v, (\Theta + \Theta^\top) v\rangle \right).$$

Then, for a real sequence (s_n) with $\delta s_n < 0$ for all $n \in \mathbb{N}$ and $\delta s_n \to -\infty$ for $n \to \infty$ it directly follows

$$\forall\, n \in \mathbb{N} : \chi(s_n) \ge C - \delta s_n \frac{v_*^2 \lambda^-}{2},$$

by which $\chi(s_n) \to \infty$ as $n \to \mathbb{N}$. Therefore, since without loss of generality we assumed $\Gamma, \tilde{\Gamma} > 0$, the system class $\mathcal{L}^{m,r}$ under consideration in the present chapter has the high-gain property. Hence, it is meaningful to apply the control scheme (3.62) proposed in [21] to the conjunction of a $\mathcal{L}_1^{m,r}$ system with a cascade of funnel pre-compensators (3.2).

(ii) Since by assumption we have $\Gamma > 0$, and by (A.3) we have $\tilde{\Gamma} > 0$, according to [21, Rem. 1.8.(b)] the surjection $N : \mathbb{R}_{\ge 0} \to \mathbb{R}$ in control scheme (3.62) can be chosen as $N(s) = -s$. Moreover, similar to the gain functions h_i in (3.1) we may choose $\alpha(s) = 1/(1-s)$ by which the control scheme (3.62) is given by

$$u(t) = -\frac{w(t)}{1 - \|w(t)\|^2}, \quad w(t) := \rho_r(\varphi(t)\,\mathbf{e}(t)). \tag{3.64}$$

Now, if the reference trajectory satisfies

$$y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m),$$

we have the following result.

Corollary 3.16. Consider a system (3.4) with $(d, f, \mathbf{T}, \Gamma, R) \in \mathcal{L}_1^{m,r}$ and initial trajectory $y^0 \in \mathcal{W}^{r-1,\infty}([-\tau, 0]; \mathbb{R}^m)$ in conjunction with a cascade of funnel pre-compensators (3.2) with $(a, p, \varphi, \varphi_1, \rho, \Gamma, \tilde{\Gamma}) \in \Sigma$. Furthermore, assume that the initial conditions (3.11) in Theorem 3.9 are satisfied. Moreover, let $\phi \in \Phi_{\text{FC}}$ and assume that for the pre-compensator's output $z := z_{r-1,1}$ the funnel control initial value constraint

$$\phi(0)\,\mathbf{e}(0) \in \mathcal{D}_r,$$

is satisfied, where $e := z - y_{\text{ref}}$ and $\mathbf{e} := (e, \dot{e}, \ldots, e^{(r-1)})$. Then, for any reference $y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, the funnel controller (3.62), with $z = z_{r-1,1}$ from (3.2), applied to system (3.4) yields an initial value problem, which has a solution, and every solution can be extended to a maximal solution $(y, \zeta) : [-\tau, \omega) \to \mathbb{R}^m \times \mathbb{R}^{rm(r-1)}$, where $\omega \in (0, \infty]$, and $\zeta = (z_{1,1}^\top, \ldots, z_{r-1,r}^\top)^\top \in \mathbb{R}^{m(r-1)r}$, and the maximal solution has the properties

(i) the solution is global, i.e., $\omega = \infty$,

(ii) the input u, the compensator states ζ, the pre-compensator gain functions $h_1, \ldots, h_{r-1}$, and the original system's output and its derivatives $y, \dot{y}, \ldots, y^{(r-1)}$ are bounded, that is, for all $i = 1, \ldots, r-1$ we have $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, $\zeta \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{m(r-1)r})$, $h_i \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$, $y \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$,

(iii) the errors evolve in their respective performance funnels, that is

$$\begin{aligned} &\exists\, \varepsilon_1 > 0 \quad \forall\, t \geq 0 : \|y(t) - z_{1,1}(t)\| < \varphi_1(t)^{-1} - \varepsilon_1, \\ \forall\, i = 2, \ldots, r-1\ &\exists\, \varepsilon_i > 0 \quad \forall\, t \geq 0 : \|z_{i-1,1}(t) - z_{i,1}(t)\| < \varphi(t)^{-1} - \varepsilon_i, \\ &\exists\, \beta > 0 \quad \forall\, t \geq 0 : \|z(t) - y_{\text{ref}}(t)\| < \phi(t)^{-1} - \beta. \end{aligned}$$

In particular, with $\varepsilon := \sum_{i=1}^{r-1} \varepsilon_i$, the tracking error $y - y_{\text{ref}}$ evolves within a prescribed funnel, i.e.,

$$\forall\, t \geq 0 : \ \|y(t) - y_{\text{ref}}(t)\| < (\rho + r - 2)\varphi(t)^{-1} + \phi(t)^{-1} - (\varepsilon + \beta).$$

Proof. Theorem 3.9 yields that a system (3.4) with $(d, f, \mathbf{T}, \Gamma, R) \in \mathcal{L}_1^{m,r}$ in conjunction with a cascade of funnel pre-compensators (3.2) with $(a, p, \varphi, \varphi_1, \rho, \Gamma, \tilde{\Gamma}) \in \Sigma$ results in a minimum phase system with $(\tilde{d}, \tilde{F}, \tilde{\mathbf{T}}, \tilde{\Gamma}, 0_{rm \times m}) \in \mathcal{L}^{m,r}$. Furthermore, the respective aspects of assertions (ii) & (iii), namely concerning ζ, h_i, are true. Therefore, invoking Remark 3.15, the conjunction belongs to the system class under consideration in [21]. Then [21, Thm. 1.9] is applicable and yields the remaining aspects of assertions (i) – (iii). Furthermore, the transient behaviour of the tracking error $y - y_{\text{ref}}$ is a direct consequence of (iii). □

The remainder of this chapter consists of some numerical simulations to illustrate the results developed for the funnel pre-compensator.

Example 3.17. We apply the funnel control scheme (3.64) to the conjunction of a cascade of funnel pre-compensators (3.2) with a minimum phase system (3.4) to achieve output tracking with prescribed transient behaviour of the tracking error via output feedback only, i.e., we illustrate an application of Corollary 3.16. We emphasize that the funnel pre-compensator receives only the measurement of the output signal y. Then, the applied controller (3.64) takes the pre-compensator's output z and its derivatives which are known explicitly, cf. Figure 3.6. Since the application of the funnel pre-compensator to the standard illustrative example *mass on a car system* from [179] was already discussed in detail in [32, Sec. 5.1], and the application of the controller (3.62) to this particular example was elaborated in [21, Sec. 3.1], we consider the following artificial, in particular, nonlinear multi-input multi-output ODE of relative degree $r = 3$ with $m = 2$ and initial conditions $y|_{[-\tau,0]} \equiv 0 \in \mathbb{R}^2$ for some $\tau > 0$,

$$y^{(3)}(t) = R_1 y(t) + R_2 \dot{y}(t) + R_3 \ddot{y}(t) + f\left(d(t), \mathbf{T}(y, \dot{y}, \ddot{y})(t)\right) + \Gamma u(t), \tag{3.65}$$

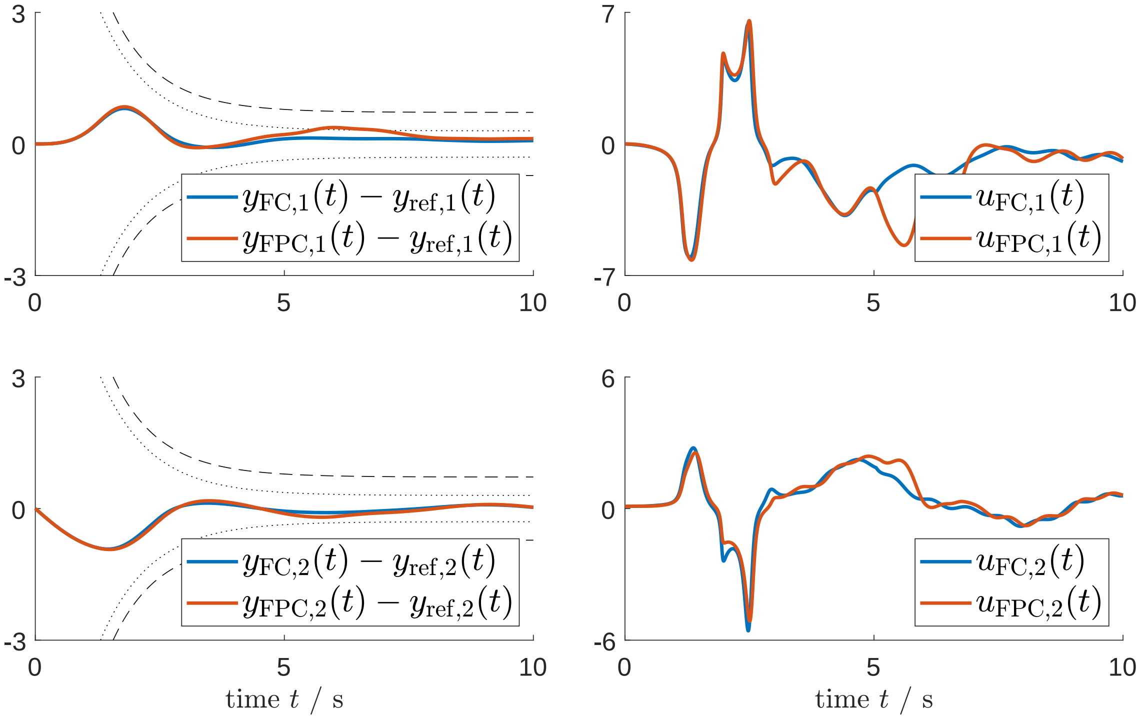
where

$$R_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 2 & 0.2 \\ 0.2 & 2 \end{bmatrix} = \Gamma^\top > 0,$$

and for $d = (d_1, d_2)^\top$, $\xi_i = (\xi_{i,1}, \xi_{i,2})^\top$, $i = 1, 2, 3$

$$\begin{aligned} \mathbf{T} : \mathcal{C}([-\tau, \infty); \mathbb{R}^6) &\to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^3), \\ (\xi_1(\cdot), \xi_2(\cdot), \xi_3(\cdot)) &\mapsto \left(t \mapsto \begin{pmatrix} \xi_{1,1}(t)^2 + e^{\xi_{1,1}(t) - |\xi_{2,1}(t)|} \\ \xi_{1,2}(t)^3 - \sin(\xi_{2,2}(t)) \\ \int_0^t e^{-(t-s)} \|\xi_1(t)\|^2 \tanh(\|\xi_3(t)\|^2) \, \mathrm{d}s \end{pmatrix} \right), \\ f : \mathbb{R}^2 \times \mathbb{R}^3 &\to \mathbb{R}^2, \\ (d_1, d_2, \zeta_1, \zeta_2, \eta) &\mapsto \begin{pmatrix} d_1 + \zeta_1 + \eta^3 \\ d_2 + \zeta_2 - \eta \end{pmatrix}, \end{aligned}$$

whereby the internal dynamics are bounded-input bounded-state stable and the associated operator $\mathbf{T}$ belongs to the class $\mathcal{T}^{6,3}_{\tau,1}$. The disturbance is chosen as $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, $t \mapsto (0.2\sin(5t) + 0.2\cos(7t), 0.25\sin(9t) + 0.2\cos(3t))^\top$ by which $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$ and hence $(d, f, \mathbf{T}, \Gamma) \in \mathcal{L}^{2,3}_1$. For the funnel pre-compensator we choose the Hurwitz polynomial $(s + s_0)^3$ with $s_0 = 4$ and $Q = I_3$ to determine matrices A and P satisfying (A.1) and obtain the respective parameters $a_1 = 12$, $a_2 = 48$, $a_3 = 64$ and $p_1 = 1$, $p_2 = 1563/548$, $p_3 = 1589/365$. We choose the pre-compensator's funnel function $\varphi(t) = (2\,e^{-2t} + 0.2)^{-1}$, and for $\tilde{\Gamma} = 2 \cdot I_2$ with $\rho = 1.1$ conditions (A.3) & (A.4) are satisfied. We stress that Γ and $\tilde{\Gamma}$ are quite different; while Γ distributes both input signals (u_1, u_2) to both output directions (y_1, y_2), $\tilde{\Gamma}$ only allocates the input signal u_i to output direction y_i, $i = 1, 2$. Next, we choose the controller's funnel function $\phi_{\mathrm{FC}}(t) = (10\,e^{-t} + 0.3)^{-1}$. Then, with $z_{i,j}(0) = 0$, $i = 1, 2$, $j = 1, 2, 3$, the assumptions on the initial values from Corollary 3.16 are satisfied. Further, we choose the reference trajectory as $y_{\mathrm{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, $t \mapsto (e^{-(t-5)^2}, \sin(t))^\top$. We simulate the output tracking over a time interval $0 - 10$ seconds. In order to illustrate the funnel pre-compensator's contribution we compare the two cases, first, if the derivatives of the output of system (3.65) are available to the controller and second, if not. The outcomes of the simulations are depicted in Figure 3.7, where the subscript FC (y_{FC}) denotes the case when the derivatives of the system are available, i.e., the funnel pre-compensator is not necessary and hence not present; and the subscript FPC (y_{FPC}) indicates the situation when the system's output is approximated by the pre-compensator and the derivatives of the latter are handed over to the controller.



(a) Tracking error $y - y_{\mathrm{ref}}$. The dashed line $- - -$ represents the funnel boundary given by $(\rho + r - 2)/\varphi + 1/\phi_{\mathrm{FC}}$, the dotted line $\cdots$ represents the funnel boundary given by $1/\phi_{\mathrm{FC}}$.

(b) Input signals u generated by the funnel control scheme (3.64).

Figure 3.7: Output reference tracking of the nonlinear multi-input multi-output minimum phase system (3.65) via output feedback.

Figure 3.7a shows the tracking error between the system's output y_i and the reference trajectory $y_{\mathrm{ref,i}}$ in both cases, for $i = 1, 2$. Note that in the case when the derivatives of the system's output are available the error evolves within the funnel boundaries defined by $1/\phi_{FC}$ as it can be expected from the results in [21, Thm. 1.9]. We emphasize that in the second case (the derivatives of the system's output are not available), although the error leaves the funnel $1/\phi_{\mathrm{FC}}$, the tracking error can be guaranteed to evolve within the wider boundaries given by $(\rho + r - 2)/\varphi + 1/\phi_{\mathrm{FC}} = (\rho + 1)/\varphi + 1/\phi_{\mathrm{FC}}$, where $r = 3$ and $\rho = 1.1$ in this particular example. Figure 3.7b shows the control input u_i, $i = 1, 2$, generated by (3.64). Both signals show oscillations which arise from the influence of the disturbance d, i.e., the controller compensates the disturbance's effect to the system. Analogously to the approximation performance discussed in the previous Example 3.4, the tracking performance (and so the input signal u) strongly depends on the choice of the pre-compensator's parameters, i.e., on the choice of A (for fixed $Q = I_m$) and can be improved with larger values of s_0. The simulations have been performed in MATLAB (solver: `ode15s`, rel. tol: 10^{-6}, abs. tol: 10^{-6}). ◇

Remark 3.18. Recalling Remark 3.8, we highlight that Corollary 3.16 achieves output tracking with prescribed transient behaviour of the tracking error via output feedback only for the large class of linear minimum phase systems. Note that this result applies to single-input, single-output systems as well as to multi-input, multi-output systems. The controller proposed in [56] achieves output tracking with prescribed performance of the error via output feedback for minimum phase multi-input, multi-output systems of arbitrary relative degree as well. In particular, this controller as well as the control scheme (3.62) is applicable to linear minimum phase systems. However, as it involves a high-gain observer structure, the controller from [56] suffers from the problem of proper initializing, i.e., some parameters have to be chosen large enough in advance, however, it is not clear how large. Contrary, conditions (A.1) – (A.4) explicitly determine the set Σ of feasible design parameters of the funnel pre-compensator. This resolves a long-standing problem in the field of high-gain based output feedback control with prescribed transient behaviour.

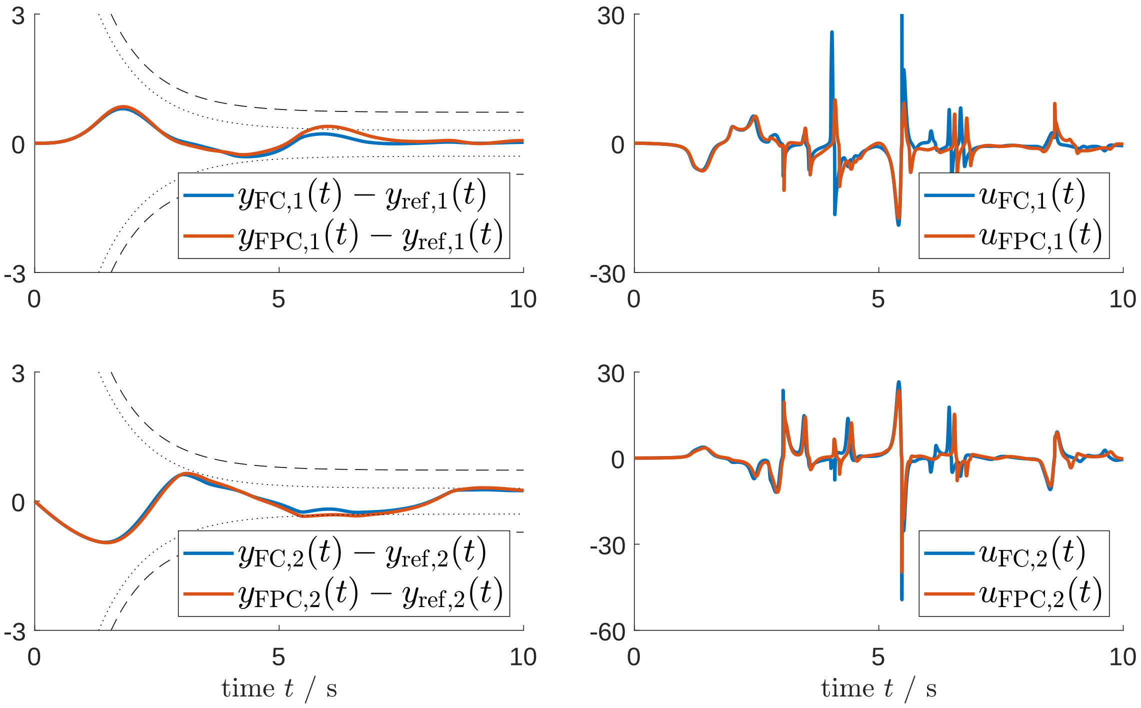
Remark 3.19. We comment on three aspects of the application of Corollary 3.16.

(i) According to [21, Sec. 1.4], in particular [21, Rem. 1.7 (c)], tracking of a given reference is also possible if the number of the available derivatives of the reference signal is smaller than the relative degree. This means the following. Let $\hat{r}$ be the number of derivatives of the reference signal y_{ref} available to the feedback controller. Then the control scheme (3.62) achieves output tracking with prescribed performance of the tracking error in the case $\hat{r} < r$. This means for instance, target tracking of a given “smooth” trajectory is possible, where the derivatives of the reference are unknown, as are the derivatives of the system's output. This situation is illustrated in Example 3.20.

(ii) In the case $\hat{r} = r$ exact asymptotic tracking can be achieved, see [21, Rem. 1.7 (f)]. In the present context this means $\lim_{t\to\infty}(z(t) - y_{\text{ref}}(t)) = 0$. However, $\lim_{t\to\infty}(y(t) - z(t)) = 0$ cannot be guaranteed since $\varphi \in \Phi_r$. Moreover, anticipating the results presented in Section 4.2, exact output reference tracking in finite time $T > 0$ is possible as well, however, suffering from the same limitations, namely, while $z(T) - y_{\text{ref}}(T) = 0$ can be achieved, the error between the system's output y and the reference y_{ref} cannot be guaranteed to be zero.

(iii) An application of the funnel pre-compensator to linear non-minimum phase system under consideration in [16] allows output feedback tracking for a certain class of linear non-minimum phase systems with the controller scheme proposed in [16, Sec. 3]. However, in combination with the funnel pre-compensator the bound of the tracking error discussed in [16, Sec. 4] is not valid any more. For deeper insights regarding output tracking of linear non-minimum phase systems see the article [16].

Example 3.20. We illustrate Remark 3.19 (i). To this end, we revisit Example 3.17 and assume that the reference signal is only available without information of its derivatives, i.e., we perform output reference tracking, where the derivatives of the output of the system are as unavailable as the derivatives of the reference signal. We choose the same parameters as in Example 3.17 and simulate tracking over the time interval $0 - 10$ seconds.



(a) Tracking error $y - y_{\text{ref}}$. The dashed line $---$ represents the funnel boundary given by $(\rho + r - 2)/\varphi + 1/\phi_{\text{FC}}$, the dotted line $\cdots$ represents the funnel boundary given by $1/\phi_{\text{FC}}$.

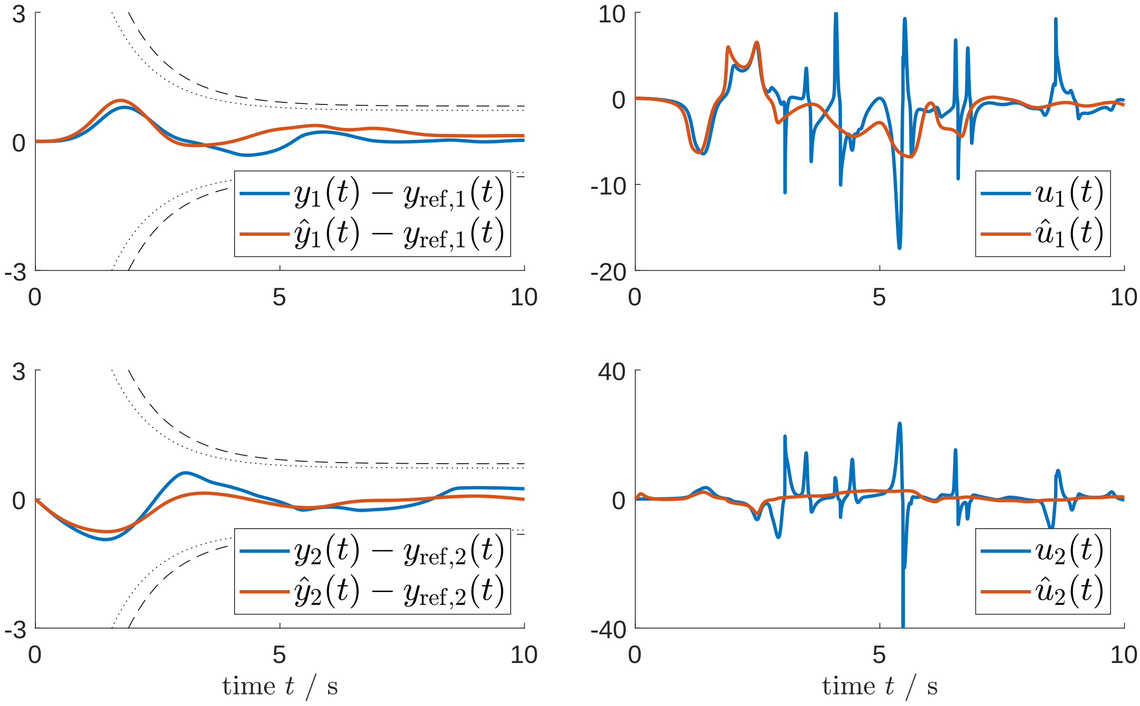
(b) Input signals u generated by the funnel control scheme (3.64).

Figure 3.8: Output reference tracking of system (3.65) via output feedback, where the derivatives of the reference signal are unknown.

Figure 3.8a shows the tracking error. The subscript FC refers to the situation where the system's output derivatives are available; the subscript FPC refers to the situation where the system's output derivatives are not available and the funnel pre-compensator is applied. Note that the error $y_{\text{FPC}} - y_{\text{ref}}$ crosses the dotted funnel boundary line; however, as expectable from Corollary 3.16, the error strictly evolves within the wider funnel boundary $(\rho+1)/\varphi + 1/\phi_{\text{FC}}$. Figure 3.8b shows the control input. Compared to the control input in Example 3.17, the control effort here is much higher. This observation supports the conjecture formulated in [21, Sec. 1.1] that the more information about the reference signal is available to the controller, the better is the control performance. The simulations have been performed in MATLAB (solver: `ode15s`, rel. tol: 10^{-6}, abs. tol: 10^{-6}). ◇

The observations in the previous example regarding the control input in the situation where the derivatives of the reference signal are not available lead to the following reasoning. If only the reference signal y_{ref} is available, and its derivatives are not, an application of the funnel pre-compensator to the reference signal produces a signal, the derivatives of which are available. This may improve the controller's performance. We will illustrate this thought in the next Example 3.21.

Example 3.21. We revisit Example 3.20 and perform output reference tracking in the case when the derivatives of the reference are unavailable, as are the derivatives of the system's output. Then we apply a cascade of funnel pre-compensators to the reference signal to obtain a signal, the derivatives of which are available. We compare this with the situation in Example 3.20 where the derivatives of the reference are not available and the system's output is approximated by a cascade of funnel pre-compensators. For the pre-compensator applied to the system's output and for the control, we choose the same parameters as in Example 3.20 and the same funnel function $\varphi_{\mathrm{FPC,sig}} = \varphi$. For the pre-compensator applied to the reference signal, we take the same parameters a, p and choose the funnel function $\varphi_{\mathrm{FPC,ref}} = (2\,e^{-t} + 0.05)^{-1}$. We simulate tracking over the time interval $0 - 10$ seconds.



(a) Tracking error $y - y_{\mathrm{ref}}$. The dashed line $---$ represents the funnel boundary given by $(\rho+1)/\varphi_{\mathrm{FPC,sig}} + 1/\varphi_{\mathrm{FPC,ref}} + 1/\phi_{\mathrm{FC}}$, the dotted line $\cdots$ represents the funnel boundary given by $(\rho + 1)/\varphi_{\mathrm{FPC,sig}} + 1/\phi_{\mathrm{FC}}$.

(b) Controls generated by (3.64). Inputs u_1, u_2 are the same as in Example 3.20 (red signals in Fig. 3.8b); inputs $\hat{u}_1, \hat{u}_2$ are those, where a funnel pre-compensator is applied to the reference.

Figure 3.9: Output reference tracking of system (3.65) via output feedback, where the reference signal is approximated by a cascade of funnel pre-compensators.

Figure 3.9 shows the results of the simulation. Expressions with a hat, i.e., terms of the form $\hat{z}$, refer to the situation when the funnel pre-compensator is applied to the reference signal. Figure 3.9a shows the tracking error, which is comparable in both situations. In Figure 3.9b the input signal is depicted. It can be seen that using the approximation of the reference signal and the approximation's derivatives reduces the control effort significantly. The simulations have been performed in MATLAB (solver: `ode15s`, rel. tol: 10^{-6}, abs. tol: 10^{-6}). ◇

We conclude this section by an illustration of the stability of the conjunction of a $\mathcal{L}_1^{m,r}$ system with a cascade of funnel pre-compensators. We consider a simple integrator system of third order and simulate tracking of a chaotic reference.

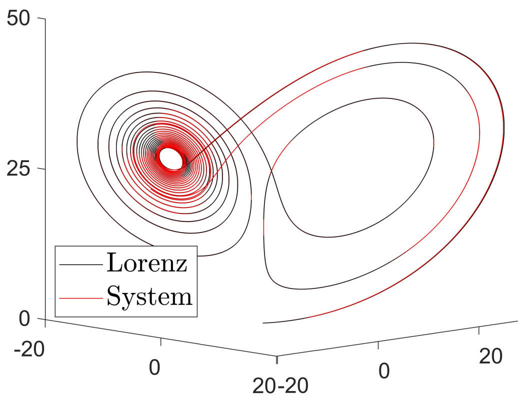
Example 3.22. In order to demonstrate the funnel pre-compensator's capability, we simulate tracking of a trajectory generated by the *Lorenz equations* for parameters which cause chaotic behaviour, i.e., the reference trajectory $y_{\text{ref}} = (y_{\text{ref},1}, y_{\text{ref},2}, y_{\text{ref},3})$ is given by the solution of

$$\begin{aligned}
\dot y_{\text{ref},1}(t) &= \alpha(y_{\text{ref},2}(t) - y_{\text{ref},1}(t)), & y_{\text{ref},1}(0) &= y^0_{\text{ref},1} \in \mathbb{R},\\
\dot y_{\text{ref},2}(t) &= y_{\text{ref},1}(t)(\beta - y_{\text{ref},2}(t)) - y_{\text{ref},2}(t), & y_{\text{ref},2}(0) &= y^0_{\text{ref},2} \in \mathbb{R},\\
\dot y_{\text{ref},3}(t) &= y_{\text{ref},1}(t)y_{\text{ref},2}(t) - \gamma y_{\text{ref},3}(t), & y_{\text{ref},3}(0) &= y^0_{\text{ref},3} \in \mathbb{R},
\end{aligned}$$

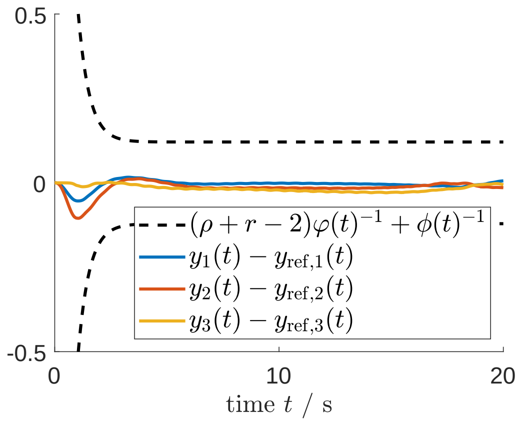
where $\alpha = 10, \beta = 28, \gamma = 8/3$ as initially introduced in [135]. We consider the third order integrator system

$$y^{(3)}(t) = u(t),$$

with $y^{(i)}(0) = y^{(i)}_{\text{ref}}(0)$, $i = 0, 1, 2$, and initialize the funnel pre-compensator likewise. For the pre-compensator we choose the funnel function $\varphi(t) = (e^{-2t} + 0.01)^{-1}$, so for $\rho = 1.1$ the function φ_1 is given according to (A.2). As in the previous examples, we choose a polynomial $(s + s_0)^r$ to generate the parameters a_i, p_i satisfying (A.1). In this case we choose $s_0 = 15$ and $Q = I_3$ and obtain $a_1 = 45, a_2 = 675, a_3 = 3375$, and correspondingly $p_1 = 1$, $p_2 = 9586/927$, $p_3 = 6053/104$. For the funnel controller (3.64) we choose $\phi(t) = (e^{-2t} + 0.1)^{-1}$.



(a) Chaotic Lorenz attractor and the system's output chasing the chaos.



(b) Tracking error.

Figure 3.10: Output reference tracking via output feedback. The reference is generated by the Lorenz equations.

This example highlights two important aspects. First, the funnel control scheme proposed in [21] achieves output reference tracking with guaranteed error bounds for quite challenging references; and second, the funnel pre-compensator's output approximates the system's output sufficiently good such that the tracking of a chaotic reference is possible without the availability of the system's output derivatives. The simulations have been performed in MATLAB (solver: `ode15s`, rel. tol: 10^{-12}, abs. tol: 10^{-12}). ◇

4 Funnel control

The results presented in Chapters 2 & 3 are related to, but not strictly restricted to *funnel control*. The findings there are practical tools to address certain aspects of well known problems in high-gain feedback control, but neither novel tracking objectives were under consideration nor new control schemes were developed. In the present chapter, we present two recent results, each addressing new aspects of funnel control. The first result is related to system properties, namely reliability of the output measurement. The second result deals with a specific control objective, namely exact tracking. First, in Section 4.1 we investigate the situation of output reference tracking when the output measurement is subject to losses, i.e., within some time intervals no information from the system is available. Second, in Section 4.2 we focus on the long standing open problem of exact tracking in finite time via feedback control. We develop a novel funnel control law, which forces the output of a system to have exact specified values at a predefined finite time while before that time the tracking error evolves within prescribed bounds.

4.1 Tracking under output measurement losses

In this section we consider output tracking for linear minimum phase systems with arbitrary relative degree, which may be subject to possible output measurement losses. The study of such situations is of high practical importance, since in the presence of output measurement losses the performance of closed-loop control strategies can deteriorate significantly and even lead to instability. Measurement losses can occur whenever signals are transmitted analogously over large distances or via digital communication networks and may hence be prone to signal losses or package dropouts. In this section we present a control strategy for linear systems, which guarantees that the tracking error evolves within prescribed error bounds whenever the output signal is available. Moreover, involving a time-varying error bound, after output measurement losses the controller is able to recapture the error within this error bound by appropriately shifting it.

A typical framework to study systems which may be subject to output measurement losses is the concept of networked control systems, see for instance the works [68, 197, 52, 150]. Within the framework of networked control systems, so-called *event-triggered* controllers have been developed to achieve global asymptotic stability, for linear systems see [124, 38, 128], and for nonlinear systems we refer to [196, 57]. Further, so-called H_∞ control approaches were studied in [67, 185], and in [163, 132] *model predictive control* strategies were exploited to deal with output measurement losses. To the best of the author's knowledge, output tracking control with prescribed performance of the error, where the output is subject to measurement losses, has not been considered yet. In this section, we present a novel funnel control law to achieve this. This particular control strategy relies on an intrinsic "availability function", which, as a binary value, encodes whether the output measurement is available or not. As a consequence, no *a priori* information about the time instants where the measurement is lost or recaptured is necessary. Involving the availability function, the basic idea for the controller is the following. Whenever the signal is available, we apply a classical funnel controller [21] such as already used in Section 3.3; whenever the output signal is lost, we set the input to zero and restart the controller when the output measurement signal is available again.

4.1.1 System class, control objective, feedback law

We consider linear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned} \tag{4.1}$$

with matrices $A \in \mathbb{R}^{n\times n}$ and $B, C^\top \in \mathbb{R}^{n\times m}$. Note that the dimensions of the input u and the output y coincide. Further, $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$ is a bounded disturbance. In virtue of Definition 1.2 we assume that system (4.1) has strict relative degree $r \in \mathbb{N}$, and we define $\Gamma := CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$. Then, as pointed out in Remark 3.8, a straightforward generalization of [95, Thm. 3] yields that there exist $R_i \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, r$, $S, P^\top \in \mathbb{R}^{m\times(n-rm)}$ and $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$ such that system (4.1) is equivalent to

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t) + d_r(t), \\ \dot{\eta}(t) &= Q\eta(t) + Py(t) + d_\eta(t), \end{aligned} \tag{4.2}$$

with initial conditions

$$(y(0), \ldots, y^{(r-1)}(0)) = (y_0^0, \ldots, y_{r-1}^0) \in \mathbb{R}^{rm}, \quad \eta(0) = \eta^0 \in \mathbb{R}^{n-rm},$$

where d_r and d_η are given via the transformation presented in Remark 3.8. With this, we introduce the system class under consideration in the present section.

Definition 4.1. For $m, r \in \mathbb{N}$ a system (4.2) belongs to the system class $\Sigma_{m,r}$, if

(i) the high-gain matrix $\Gamma = CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$ is sign definite, w.l.o.g. we assume $\Gamma + \Gamma^\top > 0$,

(ii) the system is minimum phase, i.e., $\sigma(Q) \subseteq \mathbb{C}_-$,

(iii) the disturbance $(d_r, d_\eta) : \mathbb{R}_{\geq 0} \to \mathbb{R}^m \times \mathbb{R}^{n-rm}$ is bounded.

Then, we write $(A, B, C) \in \Sigma_{m,r}$.

Note that the system class in Definition 4.1 is a subclass of $\mathcal{L}^{m,r}$ defined in Section 3.2.1 and coincides with the class of linear systems discussed in Remark 3.8.

Remark 4.2. We emphasize that the restriction to linear systems is a consequence of the assumption of no *a priori* knowledge of the time instances where the measurement is lost. To see this, consider the nonlinear control system $\dot{y}(t) = y(t)^2 + u(t)$ with $y(0) = y^0$. Now, assume $y(t_1) >$ for some $t_1 > 0$ and the signal is lost at $t = t_1$, and the input is set to zero. Then, the solution for $t \geq t_1$ is

$$y(t) = \frac{y(t_1)}{1 - y(t_1)(t - t_1)},$$

which has a finite escape time, i.e., for $t \to (1 + t_1\, y(t_1))/y(t_1)$ the solution blows up. In particular, the interval $[t_1, \frac{1+t_1\, y(t_1)}{y(t_1)})$ where the solution is bounded depends on the time instance t_1 and on the state $y(t_1)$. So no assumption on the maximal allowable duration of measurement losses is possible in advance. Contrary, a linear system (4.2) with $(A, B, C) \in \Sigma_{m,r}$ does not have a finite escape time, and so for $u \equiv 0$ the solution is bounded on any compact interval.

In order to introduce the assumptions on the system parameters and the maximal allowable duration of signal absence properly, we state the following result, the proof of which is straightforward, cf. [69].

Lemma 4.3. Let $L \in \mathbb{R}^{p\times p}$ with $\sigma(L) \subseteq \mathbb{C}_-$. Then there exists $0 < K = K^\top$ such that $KL + L^\top K + I_p = 0$ and moreover,

$$\forall\, t \geq 0 \,:\, \|e^{Lt}\| \leq \sqrt{\|K^{-1}\|\|K\|}\, e^{-\frac{1}{2\|K\|}t}.$$

For the matrix Q from (4.2) we define, in virtue of Lemma 4.3, the following constants

$$M := \sqrt{\|K^{-1}\|\|K\|}, \quad \mu := \frac{1}{2\|K\|}, \tag{4.3}$$

where the matrix K is such that $KQ + Q^\top K + I_{n-rm} = 0$. If $n - rm = 0$, i.e., if system (4.2) has trivial internal dynamics, we set $M := 0$ and $\mu := 1$. Utilizing Lemma 4.3 we record for later use that, for $t \geq t_0 \geq 0$, we have

$$\int_{t_0}^{t} \|e^{Q(s-t_0)}\|\mathrm{d}s \leq \frac{M}{\mu}(1 - e^{-\mu(t-t_0)}) \leq \frac{M}{\mu}, \tag{4.4a}$$

$$\int_{t_0}^{t} \|e^{Q(s-t_0)}\|\mathrm{d}s \leq M \int_{t_0}^{t} |e^{-\mu(s-t_0)}|\mathrm{d}s \leq M(t - t_0). \tag{4.4b}$$

Further, thanks to the well known variation of constants formula, cf. [195, § 2], we record that the second equation in (4.2) has the solution

$$\eta(t) = e^{Q(t-t_0)}\eta(t_0) + \int_{t_0}^{t} e^{Q(t-s)}(Py(s) + d_\eta(s))\,\mathrm{d}s. \tag{4.5}$$

Thus, for any signals $y \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and $d_\eta \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{n-rm})$ with $\|d_\eta\|_\infty =: \hat{d}_\eta$ we may estimate

$$\|\eta(t)\| \leq Me^{-\mu(t-t_0)}\|\eta(t_0)\| + M(\|P\|\, \|y|_{[t_0,t]}\|_\infty + \hat{d}_\eta) \int_{t_0}^{t} e^{-\mu(s-t_0)}\,\mathrm{d}s. \tag{4.6}$$

Next, we propose assumptions relating the maximal duration of measurement losses and minimal time of measurement availability. As mentioned above, we do not assume the measurement losses to happen in previously known intervals, but we only assume that it is possible to determine, at every time instant t, whether the measurement of $y(t)$ is available or not. If the availability is uncertain, then it should be rendered "signal unavailable". This also encompasses the situation that, after a disconnection, the availability of the measurement is only determined with some delay. Based on the aforesaid we define the following "availability function"

$$a : \mathbb{R}_{\geq 0} \to \{0,1\},\qquad t \mapsto \begin{cases} 1, & \text{measurement of } y(t) \text{ available,} \\ 0, & \text{measurement of } y(t) \text{ not available.} \end{cases} \tag{4.7}$$

In order to introduce the assumptions on the maximal duration of measurement losses and the minimal time of measurement availability involving the function (4.7), we make the following feasibility assumption.

Assumption 1. The availability function (4.7) is left-continuous and has only finitely many jumps in each compact interval. Then, we define the strictly increasing sequences (t_k^-), (t_k^+), with $t_k^- < t_k^+ < t_{k+1}^- < t_{k+1}^+$ such that

$$\begin{aligned} \{\, t \geq 0 \mid a(t) = 1 \,\} &= \bigcup_{k \in \mathbb{N}} (t_k^+, t_{k+1}^-], \\ \{\, t \geq 0 \mid a(t) = 0 \,\} &= \bigcup_{k \in \mathbb{N}} (t_k^-, t_k^+], \end{aligned} \tag{4.8}$$

that is, on the interval $(t_k^+, t_{k+1}^-]$ the signal is available, and on the interval $(t_k^-, t_k^+]$ the signal is not available.

We note that it is possible that both sequences contain only finitely many points. In this situation we have either $a(t) = 1$ for $t > t_N^+$, or $a(t) = 0$ for $t > t_N^-$ for some $N \in \mathbb{N}$. With the definitions above we are in the position to introduce the assumptions relating the maximal duration of measurement losses and the minimal time of measurement availability.

Assumption 2. For $S^\top, P \in \mathbb{R}^{(n-rm)\times m}, Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$ and $R_i \in \mathbb{R}^{m\times m}$, $i = 1, \dots, r$, from (4.2), let $p := \|P\|$, $s := \|S\|$ and $\beta := 1 + \frac{spM}{\mu} + \sum_{i=1}^r \|R_i\|$, with M, μ given in (4.3). Further, let q, A_r be the constants introduced below in (4.9). The signal is lost for at most $\Delta > 0$, i.e., for $t_k^\pm$ as in (4.8) we have $|t_k^- - t_k^+| \leq \Delta$ for all $k \in \mathbb{N}$, such that for some $\kappa \geq 2$ and some $\theta > s$ we have that Δ satisfies

$$spM\Delta^2 e^{\beta\Delta} \leq 1, \tag{Δ_1}$$

$$pM^2\Delta e^{\beta\Delta} \leq \frac{q}{A_r}\frac{\mu(\kappa-1)}{2\kappa\theta}. \tag{Δ_2}$$

Assumption 3. The signal is available for at least $\delta > 0$, i.e., for $t_k^\pm$ as in (4.8) we have $|t_k^+ - t_{k+1}^-| \geq \delta$ for all $k \in \mathbb{N}$, such that for Δ, β, κ, θ from Assumption 2 and M, μ from (4.3) we have that δ satisfies

$$e^{\mu\delta} \geq 2\kappa M \left(\Delta p e^{\beta\Delta} + 2(1 + spM\Delta^2 e^{\beta\Delta})\right), \tag{δ_1}$$

$$e^{\mu\delta} \geq \frac{\kappa(3 + 2sM^2)}{\theta}. \tag{δ_2}$$

Remark 4.4. For systems with trivial internal dynamics, i.e., if the second equation in (4.2) is not present, Assumptions 2 & 3 can be formulated much weaker. To be precise, in the case of trivial internal dynamics we have $p = 0$, $s = 0$ and $M = 0$ with which the inequalities (Δ_1), (Δ_2) and (δ_1), (δ_2) are always satisfied (for $\theta = 3\kappa$). This means, that arbitrary large durations $\Delta > 0$ of signal losses, and arbitrary small intervals of length $\delta > 0$ where the signal is available, are possible, so that $|t_k^- - t_k^+| \leq \Delta$ and $|t_k^+ - t_{k+1}^-| \geq \delta$ for all $k \in \mathbb{N}$. Here the only implicit assumption is boundedness of the sequence $(|t_k^- - t_k^+|)_k$, $k \in \mathbb{N}$.

We develop a control scheme which achieves output tracking of a given reference trajectory for systems where the output measurement is subject to outages. The control objective is that the tracking error evolves within prescribed bounds in the following sense. For a system (4.2) with $(A, B, C) \in \Sigma_{m,r}$ and a given reference signal $y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ the output y tracks the reference in the sense that, whenever the measurement of y is available to the controller, the error $e := y - y_{\text{ref}}$ evolves within a prescribed *performance funnel*

$$\mathcal{F}_\varphi := \{\, (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \,\},$$

where φ belongs to the following set of monotonically increasing functions

$$\Phi := \left\{ \phi \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \;\middle|\; \begin{array}{l} \forall\, t_2 \geq t_1 \geq 0 : \; 0 < \phi(t_1) \leq \phi(t_2), \\ \exists\, d > 0 \;\forall\, t \geq 0 : \; |\dot{\phi}(t)| \leq d(1 + \phi(t)) \end{array} \right\}.$$

The performance funnel's boundary is given by the reciprocal of the funnel function φ, see Figure 4.2. We emphasize that the function φ may be unbounded. In this case, and if no measurement losses occur for $t \geq T$ for some $T > 0$, asymptotic tracking may be achieved, i.e., $\lim_{t\to\infty} e(t) = 0$, cf. Remark 3.19. Next, we introduce the controller's design parameters $\eta^* \in \mathbb{R}$ and $\varphi_0 \in \Phi$, the control law itself is introduced hereinafter in equation (4.13). In Figure 4.1 the five steps towards the choice of the design parameters are depicted.

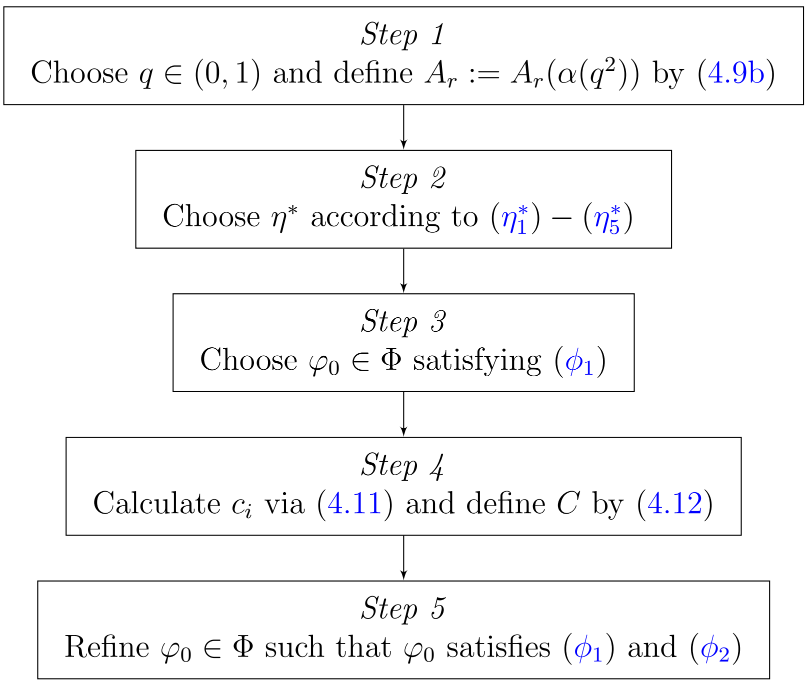


Figure 4.1: Flowchart for the choice of the controller's design parameters.

As depicted in the flowchart 4.1, we choose the design parameters η^* and φ_0 via the following five steps.

Step 1 Choose $q \in (0,1)$ and define the bijection

$$\alpha : [0,1) \to [1,\infty), \quad s \mapsto \frac{1}{1-s}. \tag{4.9a}$$

Further, for $k \geq 0$ define the function

$$A_k : \mathbb{R} \to \mathbb{R}, \quad s \mapsto \sum_{j=0}^{k} s^j, \tag{4.9b}$$

and set $A_r := A_r(\alpha(q^2))$.

Step 2 For the disturbances in (4.2) we set $\hat{d}_r := \|d_r\|_\infty$ and $\hat{d}_\eta := \|d_\eta\|_\infty$. For $\Delta, \delta, p, s, \beta, \kappa, \theta$ from Assumptions 2 & 3, and for M, μ from (4.3), setting $x_{\text{ref}} := (y_{\text{ref}}, \dot{y}_{\text{ref}}, \ldots, y_{\text{ref}}^{(r-1)})$, we choose

$$\eta^* \in \{\, \nu \in \mathbb{R}_{\geq 0} \mid \nu \text{ satisfies } (\eta_1^*) - (\eta_5^*) \,\},$$

where

$$\eta^* \geq \frac{p}{\mu}\|y_{\text{ref}}\|_\infty e^{\mu\delta}, \tag{η_1^*}$$

$$\eta^* \geq \|x_{\text{ref}}\|_\infty e^{\mu\delta}, \tag{η_2^*}$$

$$\eta^* \geq \frac{\|x_{\text{ref}}\|_\infty (1+e^{-\beta\Delta})e^{\mu\delta}}{\Delta}, \tag{η_3^*}$$

$$\eta^* \geq \left(\hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta\right) e^{\mu\delta}, \tag{η_4^*}$$

$$\eta^* \geq 2\kappa\left(\Delta p M\left(\|x_{\text{ref}}\|_\infty + \Delta\hat{d}_r + \Delta\frac{sM}{\mu}\hat{d}_\eta\right)e^{\beta\Delta} + \Delta M\hat{d}_\eta\right), \tag{η_5^*}$$

and set $E := \theta\Delta e^{\beta\Delta}\eta^* > 0$.

Step 3 Let $\varphi_0 \in \Phi$ such that

$$\varphi_{0,\min} := \frac{2\kappa p M^2}{\mu(\kappa-1)\eta^*} \leq \varphi_0(0) \leq \frac{q}{A_r E} =: \varphi_{0,\max}, \tag{ϕ_1}$$

which is possible by (Δ_2).

Step 4 In this step we introduce some constants which are necessary to exploit the estimations given in [21, Cor. 1.10]. Let $\hat{\alpha}^\dagger(z) = z/(1+z)$ which obviously yields $\hat{\alpha}^\dagger(s\alpha(s)) = s$, and define $\tilde{\alpha}(s) = 2s\alpha'(s) + \alpha(s) = (1+s)/(1-s)^2$. Further, invoking properties of $\varphi_0 \in \Phi$, namely that $|\dot{\varphi}_0(t)| \leq d(1+\varphi_0(t))$ for all $t \geq 0$, we set $\mu_0 := \frac{d(1+\varphi_0(0))}{\varphi_0(0)}$, and observe $\operatorname{ess\,sup}_{t\geq 0}(|\dot{\varphi}_0(t)|/\varphi_0(t)) \leq \mu_0$. Note that for slowly increasing funnel functions $\varphi_0 \in \Phi$ we might have $\operatorname{ess\,sup}_{t\geq 0}(|\dot{\varphi}_0(t)|/\varphi_0(t)) < \mu_0$ and so the constant μ_0 is possibly chosen too large. However, we use this possibly larger μ_0 to ensure that it only depends on the initialization $\varphi_0(0)$. Then, in virtue of [21, Eq. (12)], for $k = 1, \ldots, r-1$ we recursively define the constants $c_0 = 0$ and

$$\begin{aligned} e_1^0 &:= \varphi_0(0)e(0), \\ c_1 &:= \max\{\|e_1^0\|^2, \hat{\alpha}^\dagger(1+\mu_0), q^2\}^{1/2} < 1, \\ \mu_k &:= 1 + \mu_0\big(1 + c_{k-1}\alpha(c_{k-1}^2)\big) + \tilde{\alpha}(c_{k-1}^2)\big(\mu_{k-1} + c_{k-1}\alpha(c_{k-1}^2)\big), \\ e_k^0 &:= \varphi_0(0)e^{(k-1)}(0) + \alpha(\|e_{k-1}^0\|^2)e_{k-1}^0, \\ c_k &:= \max\{\|e_k^0\|^2, \hat{\alpha}^\dagger(\mu_k), q^2\}^{1/2} < 1, \end{aligned} \tag{4.11}$$

where $e^{(i)}(0) = y^{(i)}(0) - y_{\text{ref}}^{(i)}(0)$ for $i = 0, \ldots, r-1$, and we set

$$\begin{aligned} e_r^0 &:= \varphi_0(0)e^{(r-1)}(0) + \alpha(\|e_{r-1}^0\|^2)e_{r-1}^0, \\ C &:= \sum_{i=1}^{r-1} c_i + c_{i-1}\alpha(c_{i-1}^2) + (1 + c_{r-1}\alpha(c_{r-1}^2)). \end{aligned} \tag{4.12}$$

Step 5 We refine the function $\varphi_0 \in \Phi$ satisfying (ϕ_1) such that for an intermediate $\rho \in (0, \delta)$ the estimation

$$\varphi_0(\rho) \geq \max\left\{\frac{Ce^{\mu\delta}}{\eta^*}, \frac{Ce^{\mu\delta}}{\Delta\eta^*}\right\} \tag{ϕ_2}$$

is satisfied.

Remark 4.5. We comment on the constants and design parameters introduced in the five-steps process above.

(i) The purpose of the constant $q \in (0,1)$ chosen in *Step 1* is to determine the minimal initial width of the performance funnel, described by the upper bound $\varphi_{0,\max}$ given by (ϕ_1) in *Step 3*. Then, condition (ϕ_2) in *Step 5* ensures that the width of the funnel, and so the tracking error, is not too large before the signal possibly vanishes the next time.

(ii) The parameter η^* chosen in *Step 2*, as we will see later, describes the invariant set in which the internal dynamics evolve when no measurement is available and hence no control is applied.

(iii) The purpose of the constants defined in *Step 4* is to make use of [21, Cor. 1.10], which we will recapitulate in Lemma 4.10.

With the constants introduced above at hand, we introduce the feedback law. The idea for the controller design is to choose a funnel function $\varphi_0 \in \Phi$, which is reset whenever no output measurement is available, i.e., when $a(t) = 0$. Then, as soon as the output measurement is available again, i.e., when we have $a(t^*) = 1$ for some $t^* \geq 0$, the funnel controller is restarted with $\varphi(t) = \varphi_0(t - t^*)$ so that $\varphi(t^*) > 0$ and the performance funnel is sufficiently large at t^* to ensure applicability of the funnel controller result [21, Thm. 1.9]. With this, and recalling the bijection

$\alpha(s) = 1/(1-s)$ from (4.9a), we introduce the following control law for systems (4.2) under possible output measurement losses:

$$\tau(t) = \begin{cases} t, & a(t) = 0, \\ \tau(t-), & a(t) = 1, \end{cases} \quad \varphi(t) = \begin{cases} \varphi_0(0), & a(t) = 0, \\ \varphi_0(t-\tau(t)), & a(t) = 1, \end{cases}$$

$$\begin{aligned} e_1(t) &= \varphi(t)\, e(t) = \varphi(t)\big(y(t) - y_{\text{ref}}(t)\big), \\ e_{i+1}(t) &= \varphi(t)\, e^{(i)}(t) + \alpha(\|e_i(t)\|^2)\, e_i(t), \ i = 1, \dots, r-1, \\ u(t) &= -a(t)\, \alpha(\|e_r(t)\|^2)\, e_r(t). \end{aligned} \tag{4.13}$$

Remark 4.6. We comment on some aspects of the presented control law.

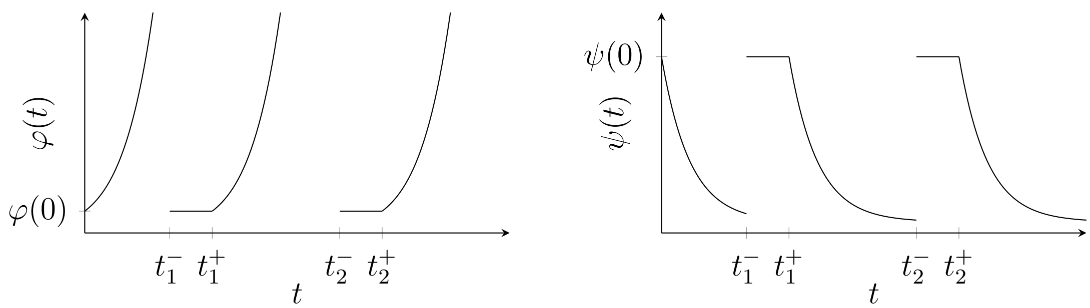
(i) In Definition 4.1 of the system class we assumed $\Gamma + \Gamma^\top > 0$. If $\Gamma + \Gamma^\top < 0$ the control would read $u(t) = a(t)\alpha(\|e_r(t)\|^2)e_r(t)$.

(ii) If the output measurement is always available, i.e., $a(t) = 1$ for all $t \geq 0$, then the controller (4.13) coincides with that proposed in [21] and the existence of a global solution of the closed-loop system follows from the results presented there.

(iii) Since it is not known *a priori* when output measurement losses occur, the funnel function φ cannot be globally defined in advance. Therefore, φ is defined online as part of the control law (4.13); it is equal to a shifted version of the funnel function φ_0 whenever measurements are available, and constantly $\varphi_0(0)$ otherwise.

(iv) Contrary to standard funnel control laws, the loss of the system's output signal possibly introduces a discontinuity in the control signal.

A typical choice for a funnel function is $\varphi_0(t) = (ae^{-bt} + c)^{-1}$ with $a, b, c > 0$, which is depicted in Figure 4.2.



(a) Shape of a funnel function φ for a typical $\varphi_0 \in \Phi$.

(b) Corresponding funnel boundary function $\psi = 1/\varphi$.

Figure 4.2: Schematic shape of a typical funnel function and corresponding boundary, respectively with shifts.

4.1.2 Tracking under output measurement losses via funnel control

In this section we show that the application of the funnel controller (4.13) to a system (4.2) under possible output measurement losses leads to a closed-loop initial value problem which has a global solution in the sense of Definition 1.7.

Theorem 4.7. Consider a system (4.2) with $(A, B, C) \in \Sigma_{m,r}$ and initial values $(y_0^0, \dots, y_{r-1}^0) \in \mathbb{R}^{rm}$ and $\eta^0 \in \mathbb{R}^{n-rm}$. Let $y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, $a(\cdot)$ be the availability function given in (4.7) satisfying Assumption 1. Choose design the parameter η^* satisfying (η_1^*) – (η_5^*), and let $\varphi_0 \in \Phi$ such that (ϕ_1), (ϕ_2) are valid. If the initial conditions

$$\forall\, i = 1, \dots, r \;:\; \|e_i(0)\| < 1, \tag{4.14a}$$

$$\|\eta^0\| \leq \eta^* \tag{4.14b}$$

are satisfied, then the application of the control scheme (4.13) to a system (4.2) yields an initial value problem which has a solution, every solution can be extended to a maximal solution and every maximal solution $(y, \eta) : [0, \omega) \to \mathbb{R}^m \times \mathbb{R}^{n-rm}$ has the following properties:

(i) the solution is global, i.e., $\omega = \infty$,

(ii) the tracking error $e = y - y_{\text{ref}}$ evolves within the funnel boundaries, i.e.,

$$\forall\, t \geq 0 \;:\; \varphi(t)\|e(t)\| < 1,$$

(iii) the control signal is globally bounded, i.e., $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, and for the output signal we have $y \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

Sketch of proof. Since the proof of Theorem 4.7 is quite technical in some respects, we briefly outline the main ideas here; the proof itself follows after establishing some technical results. The proof consists of four steps. In *Step one* we establish the existence of a local solution of the closed-loop initial value problem (4.2), (4.13), (4.14). In *Step two* we show that the tracking error evolves within the prescribed funnel boundary. Since the system's output strongly depends on the internal dynamics, it is essential to find an invariant set of the internal dynamics such that, after the reappearance of the output signal, the funnel controller can be applied with a shifted funnel, where the funnel has to be sufficiently wide. We utilize the constraints on the maximal duration of signal losses and minimal time of signal availability, and the result [21, Cor. 1.10] to derive respective estimations. In *Step three* we show global boundedness of the input signal; similar to standard funnel control proofs, we show this by contradiction. Lastly, in *Step four* we use the findings from *Step one* to conclude that the solution of the closed-loop system is globally defined.

Before we present the proof of Theorem 4.7, we provide some technical results to be used. First, we derive an explicit exponential bound for the solution of (4.2) whenever no measurement is available.

Lemma 4.8. Consider a linear system (4.2) with $(A, B, C) \in \Sigma_{m,r}$. Then, for M, μ from (4.3), β given in Assumption 2, $s = \|S\|$ and $\hat{d}_r := \|d_r\|_\infty$, $\hat{d}_\eta := \|d_\eta\|_\infty$, we have that for all solutions $(y, \eta) \in \mathcal{C}^{r-1}([0, \omega); \mathbb{R}^m) \times \mathcal{C}([0, \omega); \mathbb{R}^{n-rm})$, $\omega \in (0, \infty]$, of (4.2) with $u|_{(t_0,t_1)} = 0$ for $0 \le t_0 < t_1 \le \omega$ and with $x = (y^\top, \dot{y}^\top, \dots, (y^{(r-1)})^\top)^\top$ that for all $t \in [t_0, t_1)$

$$\|x|_{[t_0,t]}\|_\infty \le \left(\|x(t_0)\| + \int_{t_0}^t \left(sM\|\eta(t_0)\|e^{-\mu(\tau-t_0)} + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta \right) \mathrm{d}\tau \right) e^{\beta(t-t_0)}.$$

Proof. Let $x = (x_1^\top, \dots, x_r^\top)^\top$. Then, we have that

$$\dot{x}(t) = \begin{pmatrix} x_2(t) \\ \vdots \\ x_r(t) \\ \sum_{i=1}^r R_i x_i(t) + S\eta(t) + d_r(t) \end{pmatrix}$$

for almost all $t \in [t_0, t_1]$, and upon integration we obtain

$$\|x(t)\| \le \|x(t_0)\| + \int_{t_0}^t \left(\|x(\tau)\| + \sum_{i=1}^r \|R_i\|\|x_i(\tau)\| + s\|\eta(\tau)\| + \|d_r(\tau)\| \right) \mathrm{d}\tau.$$

For $t \in [t_0, \omega)$ we define $w(t) := \|x|_{[t_0,t]}\|_\infty$. Then, utilizing (4.4a) and (4.5) we have

$$\begin{aligned} w(t) \le \|x(t_0)\| + \sup_{\sigma\in[t_0,t]} \int_{t_0}^\sigma \Bigg[& w(\tau) + s\|e^{Q(\tau-t_0)}\eta(t_0)\| + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta \\ & + \sum_{i=2}^r \|R_i\|\|x_i|_{[t_0,\tau]}\|_\infty + \left(\|R_1\| + \frac{spM}{\mu} \right) \|x_1|_{[t_0,\tau]}\|_\infty \Bigg] \mathrm{d}\tau \\ \le \|x(t_0)\| + & \int_{t_0}^t \underbrace{\left[1 + \left(\sum_{i=1}^r \|R_i\| + \frac{spM}{\mu} \right) \right]}_{=\beta} w(\tau)\,\mathrm{d}\tau \\ & + \int_{t_0}^t \left(sM\|\eta(t_0)\|e^{-\mu(\tau-t_0)} + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta \right) \mathrm{d}\tau. \end{aligned}$$

Then, an application of Grönwall's lemma yields the assertion. □

The second technical lemma provides an estimate involving the function A_k defined in (4.9b).

Lemma 4.9. For $k = 0, \dots, r$, $r \in \mathbb{N}$, let A_k be given as in (4.9b). Further, let $\alpha : [0, 1) \to [1, \infty)$ be a bijection, $q \in (0, 1)$ and $\lambda, E \ge 0$ with

$$\lambda \le \frac{q}{A_r(\alpha(q^2))E}. \tag{4.15}$$

Let $\xi_0, \dots, \xi_{r-1} \in \mathbb{R}^n$ with

$$\forall\, k \in \{0, \dots, r-1\} : \ \|\xi_k\| \le E, \tag{4.16}$$

and define $\zeta_0 := 0$ and $\zeta_{k+1} \in \mathbb{R}^n$ for $k = 0, \dots, r-1$ by

$$\zeta_{k+1} := \lambda\xi_k + \alpha(\|\zeta_k\|^2)\zeta_k. \tag{4.17}$$

Then,

$$\forall\, k \in \{1, \dots, r\} : \ \|\zeta_k\| \le \lambda E A_{k-1}(\alpha(q^2)) \le q.$$

Proof. First observe that for $s \geq 0$ we have

$$\forall\, k \in \mathbb{N} : A_k(s) \leq A_k(s) + s^{k+1} = A_{k+1}(s),$$

and furthermore, for $\tilde{A}_k := A_k\big(\alpha(q^2)\big)$ we have that

$$\lambda E \tilde{A}_k \leq \lambda E A_r(\alpha(q^2)) \overset{(4.15)}{\leq} q.$$

We show that

$$\forall\, k \in \{1, \dots, r\} : \ \|\zeta_k\| \leq \lambda E \tilde{A}_{k-1} \tag{4.18}$$

by induction over k. For $k = 1$ we have

$$\|\zeta_1\| \overset{(4.17)}{\leq} \lambda \|\xi_0\| \overset{(4.16)}{\leq} \lambda E.$$

Let (4.18) be true for some $k \in \{1, \dots, r-1\}$. Then, using monotonicity of the bijection α, we obtain

$$\begin{aligned}
\|\zeta_{k+1}\| &\overset{(4.17)}{\leq} \lambda \|\xi_k\| + \alpha(\|\zeta_k\|^2)\|\zeta_k\| \\
&\overset{(4.16),(4.18)}{\leq} \lambda E + \alpha\big((\lambda E \tilde{A}_{k-1})^2\big)\lambda E \tilde{A}_{k-1} \\
&\leq \lambda E\big(1 + \alpha(q^2)\tilde{A}_{k-1}\big) \\
&= \lambda E\big(1 + \alpha(q^2) A_{k-1}(\alpha(q^2))\big) = \lambda E A_k(\alpha(q^2)),
\end{aligned}$$

where we have used that $1 + sA_{k-1}(s) = A_k(s)$. This proves (4.18). □

As the third statement, to improve readability of the proof of Theorem 4.7, we recapitulate a special case of [21, Cor. 1.10], adapted to the present context, which will be given without a proof. For details we refer to [21, p. 166 & pp. 190–191].

Lemma 4.10. *[21, Cor. 1.10]* Assume all hypotheses of Theorem 4.7 are satisfied. Then, for every maximal solution (y, η) of (4.2), (4.13) the tracking error $e = y - y_{\text{ref}}$ satisfies for the constants c_i defined in (4.11)

$$\forall\, k = 0, \dots, r-2 \,\forall\, t \in [0, t_1^-) \ : \ \varphi(t)\, \|e^{(k)}(t)\| \leq c_{k+1} + \alpha(c_k^2)c_k,$$

where t_1^- is as in (4.8) and the bijection α is given in (4.9a).

With the sketch and the three lemmata presented above at hand, we give a proof of Theorem 4.7.

Proof of Theorem 4.7. The proof consists of four steps.
Step one. We establish the existence of a solution of the initial value closed-loop system (4.2), (4.13). Following *Step 1* in the proof of [21, Thm. 1.9], we introduce the set $\mathcal{B} = \{\, w \in \mathbb{R}^m \mid \|w\| < 1 \,\}$ and for $\alpha(s) = 1/(1-s)$ the map

$$\gamma : \mathcal{B} \to \mathbb{R}^m, \ w \mapsto \alpha(\|w\|^2)w.$$

Further, we define the sets $\mathcal{D}_k$ and maps $\sigma_k : \mathcal{D}_k \to \mathcal{B}$, $k = 1, \dots, r$ recursively as follows:

$$\begin{aligned}
&\mathcal{D}_1 := \mathcal{B}, \ \sigma_1 : \mathcal{D}_1 \to \mathcal{B}, \ \zeta_1 \mapsto \zeta_1, \\
&\mathcal{D}_k := \left\{ (\zeta_1, \dots, \zeta_k) \in \mathbb{R}^{km} \ \middle| \ \begin{array}{l} Z := (\zeta_1, \dots, \zeta_{k-1}) \in \mathcal{D}_{k-1}, \\ \zeta_k + \gamma(\sigma_{k-1}(Z)) \in \mathcal{B} \end{array} \right\}, \\
&\sigma_k : \mathcal{D}_k \to \mathcal{B}, \ (\zeta_1, \dots, \zeta_k) \mapsto \zeta_k + \gamma(\sigma_{k-1}(\zeta_1, \dots, \zeta_{k-1})).
\end{aligned}$$

With this, and $x_{\text{ref}} := (y_{\text{ref}}^\top, \dot{y}_{\text{ref}}^\top, \ldots, (y_{\text{ref}}^{(r-1)})^\top)^\top$, we define the set

$$\mathcal{D} := \{ (t,\xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm} \mid \varphi(t)\|\xi - x_{\text{ref}}(t)\| \in \mathcal{D}_r \},$$

and $\sigma : \mathcal{D} \to \mathcal{B}$, $(t,\xi) \mapsto \sigma_r\big(\varphi(t)\big(\xi - x_{\text{ref}}(t)\big)\big)$. Note that since the availability function $a(\cdot)$ is left-continuous, the set $\mathcal{D}$ is relatively open. With the definitions so far, u in (4.13) can be written as

$$u(t) = -a(t) \cdot \alpha(\|\sigma(t,x(t))\|^2) \cdot \sigma(t,x(t)).$$

To proceed, we formally define the function

$$F : \mathcal{D} \times \mathbb{R}^{n-rm} \to \mathbb{R}^n,$$
$$(t,\xi_1,\ldots,\xi_r,\eta) \mapsto \begin{pmatrix} \xi_2 \\ \vdots \\ \xi_r \\ \sum_{i=1}^r R_i\xi_i + S\eta - a(t)\alpha(\|\sigma(t,\xi)\|^2)\sigma(t,\xi) + d_r(t) \\ Q\eta + P\xi_1 + d_\eta(t) \end{pmatrix}.$$

With this we obtain, setting $x := (y^\top, \dot{y}^\top, \ldots, (y^{(r-1)})^\top)^\top$, an initial value problem

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{\eta}(t) \end{pmatrix} &= F\left(t, x(t), \eta(t)\right), \\ x(0) &= \big(y_0^0, \ldots, y_{r-1}^0\big),\ \eta(0) = \eta^0, \end{aligned} \tag{4.19}$$

which is equivalent to (4.2), (4.13). Note that F is continuous in $(\xi_1,\ldots,\xi_r,\eta)$ and locally essentially bounded and, in particular, measurable in the variable t regardless of the possible discontinuities introduced by the availability function $a(\cdot)$. Therefore, since we have $(0,x(0)) \in \mathcal{D}$, Proposition 1.9 yields the existence of a maximal solution $(x,\eta) : [0,\omega) \to \mathbb{R}^n$ of (4.19), where $\omega \in (0,\infty]$. Moreover, the closure of the graph of the solution of (4.19) is not a compact subset of $\mathcal{D} \times \mathbb{R}^{n-rm}$.

Step two. In this step we establish (ii) on $[0,\omega)$. To this end, let (t_k^-), (t_k^+) be sequences as defined in (4.8). It is also possible that both sequences contain only finitely many points, then either $a(t) = 1$ for $t \geq t_N^+$, or $a(t) = 0$ for $t \geq t_N^-$ for some $N \in \mathbb{N}$. The following arguments apply, mutatis mutandis, in both cases. To proceed, let $\mathbf{e} := x - x_{\text{ref}}$. Since we consider a subclass of the system class under consideration in [21], and since by (4.14a) we have $\varphi(0)\mathbf{e}(0) \in \mathcal{D}_r$, the feasibility result [21, Thm. 1.9] restricted to the interval $[0,t_1^-]$ is applicable and ensures assertion (ii) for $t \in [0,t_1^-] \subseteq [0,\omega)$. The inclusion $[0,t_1^-] \subseteq [0,\omega)$ since without measurement losses, [21, Thm. 1.9] yields $\omega = \infty$. In order to reapply [21, Thm. 1.9] at $t = t_1^+$, we establish that the initial conditions (4.14) are satisfied for $t = t_1^+$. First, we show (4.14a) at t_1^+. We define the function $\psi := 1/\varphi_0$. Then by (4.5) and (4.6), the initial condition (4.14b) and using (η_1^*) we have

$$\begin{aligned} \|\eta(t_1^-)\| &\overset{(4.4a)}{\leq} Me^{-\mu\delta}\eta^* + \frac{M}{\mu}\left(p\psi(0) + p\|y_{\text{ref}}\|_\infty + \hat{d}_\eta\right) \\ &\overset{(\eta_1^*)}{\leq} 2Me^{-\mu\delta}\eta^* + \frac{pM}{\mu}\psi(0). \end{aligned} \tag{4.20}$$

Invoking the constants defined in (4.11) via the result [21, Cor. 1.10] we have that

$$\forall\, i = 0,\ldots,r-2\ \forall\, t \in [0,t_1^-) : \ \|e^{(i)}(t)\| \leq \psi(t)(c_{i+1} + c_i\alpha(c_i^2)), \tag{4.21}$$

and moreover, since $\|e_r(t)\| \le 1$ for $t \in [0, t_1^-)$, we have

$$\|e^{(r-1)}(t)\| \le \psi(t)(1 + c_{r-1}\alpha(c_{r-1}^2)).$$

Thus, for C defined in (4.12) we have $\|\mathbf{e}(t)\| \le C\psi(t)$, and since ψ is monotonically decreasing by properties of Φ, for some $\rho < \delta \le t_1^-$ as in (ϕ_2) we have in particular

$$\|\mathbf{e}(t_1^-)\| \le C\psi(t_1^-) \le C\psi(\rho). \tag{4.22}$$

With this estimation at hand, invoking Lemma 4.8 we obtain

$$\begin{aligned}
\|x|_{[t_1^-,t_1^+]}\|_\infty &\le \left(\|x(t_1^-)\| + \int_{t_1^-}^{t_1^+} \left(sM\|\eta(t_1^-)\|e^{-\mu(s-t_1^-)} + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta\right) \mathrm{d}s\right) e^{\beta(t_1^+-t_1^-)} \\
&\overset{(4.4b)}{\le} \left(\|\mathbf{e}(t_1^-)\| + \|x_{\mathrm{ref}}\|_\infty + \left(sM\|\eta(t_1^-)\| + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta\right)\Delta\right) e^{\beta(t_1^+-t_1^-)} \\
&\overset{(4.22)}{\le} C\psi(\rho)e^{\beta\Delta} + \|x_{\mathrm{ref}}\|_\infty e^{\beta\Delta} + \left(sM\|\eta(t_1^-)\| + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta\right)\Delta e^{\beta\Delta},
\end{aligned} \tag{4.23}$$

and therefore, for the error $\mathbf{e}(t)$ at $t = t_1^+$ we have

$$\begin{aligned}
\|\mathbf{e}(t_1^+)\| &\le \|x_{\mathrm{ref}}(t_1^+)\| + \|x(t_1^+)\| \le \|x_{\mathrm{ref}}\|_\infty + \|x|_{[t_1^-,t_1^+]}\|_\infty \\
&\overset{(4.23)}{\le} \|x_{\mathrm{ref}}\|_\infty\left(1 + e^{\beta\Delta}\right) + C\psi(\rho)e^{\beta\Delta} + \left(sM\|\eta(t_1^-)\| + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta\right)\Delta e^{\beta\Delta} \\
&\overset{(\eta_3^*),(\eta_4^*),(\phi_2)}{\le} \Delta e^{\beta\Delta}\eta^* e^{-\mu\delta} + \Delta e^{\beta\Delta}\eta^* e^{-\mu\delta} + \Delta e^{\beta\Delta}\eta^* e^{-\mu\delta} + sM\Delta e^{\beta\Delta}\|\eta(t_1^-)\| \\
&\overset{(4.20)}{\le} \Delta\eta^* e^{\beta\Delta-\mu\delta}\left(3 + 2sM^2\right) + \Delta e^{\beta\Delta}\frac{spM^2}{\mu}\psi(0) \\
&\overset{(\phi_2)}{\le} \Delta\eta^* e^{\beta\Delta-\mu\delta}\left(3 + 2sM^2\right) + s\Delta e^{\beta\Delta}\frac{\kappa-1}{\kappa}\eta^* \\
&\overset{(\delta_2)}{\le} \Delta e^{\beta\Delta}\frac{\theta}{\kappa}\eta^* + s\Delta e^{\beta\Delta}\frac{\kappa-1}{\kappa}\eta^* \\
&\overset{s<\theta}{<} \theta\Delta e^{\beta\Delta}\eta^* = E.
\end{aligned} \tag{4.24}$$

In particular, estimation (4.24) yields

$$\forall\, i = 0, \ldots, r-1 \;:\; \|e^{(i)}(t_1^+)\| < E.$$

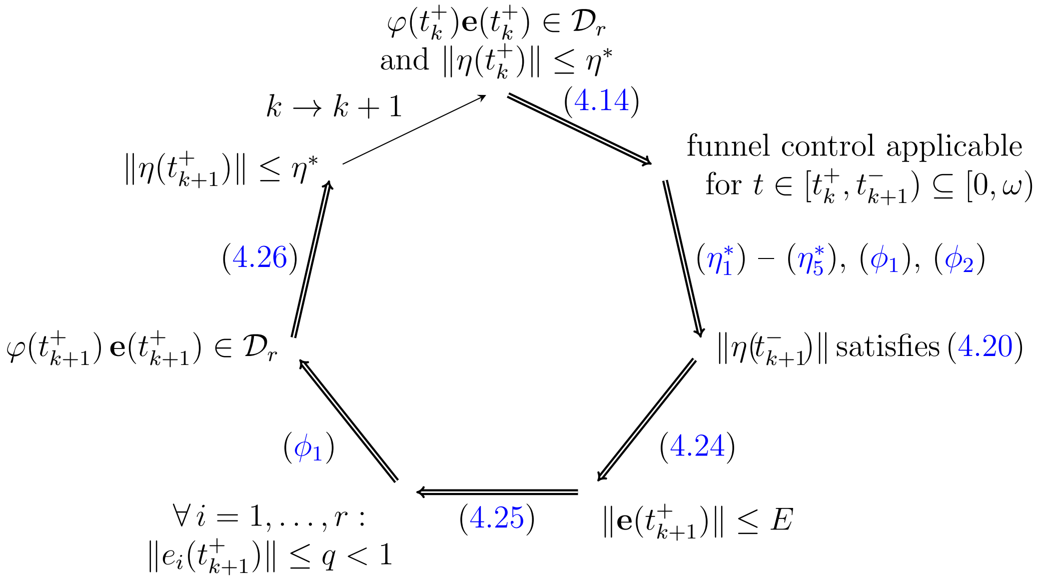
Invoking (ϕ_1), an application of Lemma 4.9 (applied with $\lambda = \varphi(t_1^+) = \varphi_0(0)$) yields

$$\begin{aligned}
&\|e_i(t_1^+)\| \le q \le c_i < 1, \; i = 1, \ldots, r-1, \\
&\|e_r(t_1^+)\| \le q,
\end{aligned} \tag{4.25}$$

whereby $\varphi(t_1^+)\,\mathbf{e}(t_1^+) \in \mathcal{D}_r$. Moreover, since by construction we have $u|_{[t_1^-,t_1^+)} \equiv 0$, assertion (ii) is true for $t \in [t_1^-, t_1^+) \subseteq [0, \omega)$ by the previous estimations (4.24) & (4.25). The inclusion of the interval via standard theory of linear differential equations. Furthermore, invoking equations (4.20) & (4.22) and using Lemma 4.8, we obtain with similar estimates as above

$$
\begin{aligned}
\|\eta(t_1^+)\| &\overset{(4.6),(4.4b)}{\leq} M\|\eta(t_1^-)\| + \Delta p M \|y|_{[t_1^-,t_1^+]}\|_\infty + \Delta M \hat{d}_\eta \\
&\overset{(4.23)}{\leq} M\|\eta(t_1^-)\| + \Delta M \hat{d}_\eta \\
&\quad + \Delta p M \left(C\psi(\rho) + \|x_{\mathrm{ref}}\|_\infty \right) e^{\beta\Delta} \\
&\quad + pM \left(sM\|\eta(t_1^-)\| + \hat{d}_r + \frac{sM}{\mu}\hat{d}_\eta \right) \Delta^2 e^{\beta\Delta} \\
&= \Delta p M \left(\|x_{\mathrm{ref}}\|_\infty + \Delta \hat{d}_r + \Delta \frac{sM}{\mu}\hat{d}_\eta \right) e^{\beta\Delta} + \Delta M \hat{d}_\eta \\
&\quad + M\|\eta(t_1^-)\| \left(1 + spM\Delta^2 e^{\beta\Delta}\right) + \Delta p M C e^{\beta\Delta}\psi(\rho) \\
&\overset{(\eta_5^*),(\phi_2)}{\leq} \frac{\eta^*}{2\kappa} + \Delta p M \eta^* e^{\beta\Delta-\mu\delta} + M\|\eta(t_1^-)\| \left(1 + spM\Delta^2 e^{\beta\Delta}\right) \\
&\overset{(4.20)}{\leq} \frac{\eta^*}{2\kappa} + \Delta p M \eta^* e^{\beta\Delta-\mu\delta} \\
&\quad + \left(1 + spM\Delta^2 e^{\beta\Delta}\right) \left(2Me^{-\mu\delta}\eta^* + \frac{pM}{\mu}\psi(0) \right) \\
&\overset{(\Delta_1)}{\leq} \frac{\eta^*}{2\kappa} + \eta^* \Big(\Delta p M e^{\beta\Delta} + 2M\left(1 + spM\Delta^2 e^{\beta\Delta}\right) \Big) e^{-\mu\delta} + \frac{2pM}{\mu}\psi(0) \\
&\overset{(\delta_1),(\phi_1)}{\leq} \frac{\eta^*}{2\kappa} + \frac{\eta^*}{2\kappa} + \frac{\kappa-1}{\kappa}\eta^* = \eta^*.
\end{aligned}
\tag{4.26}
$$

Therefore, the initial conditions (4.14) are satisfied at $t = t_1^+$ and [21, Thm. 1.9] is applicable for $t \geq t_1^+$. Moreover, invoking (4.25), the estimates (4.20), (4.24) and (4.26) are valid for $t = t_2^-$ and $t = t_2^+$, respectively, since $\|\eta(t_1^+)\| \leq \eta^*$ and $[t_1^+, t_2^+] \subseteq [0, \omega)$ via the same arguments as above. Therefore, we obtain the following chain of inductive implications



This means, the funnel control can be reapplied at $t = t_k^+$ for all $k \in \mathbb{N}$ with $[t_k^+, t_{k+1}^-) \subseteq [0, \omega)$. This yields assertion (ii) on $[0, \omega)$.

Step three. We show $y \in \mathcal{W}^{r,\infty}([0,\omega);\mathbb{R}^m)$ and $u \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$. Invoking (4.21) and (4.23), we obtain $y \in \mathcal{W}^{r-1,\infty}([0,\omega);\mathbb{R}^m)$. We aim to obtain a global bound for u and $y^{(r)}$. We set $Y_{\max} := \max_{i=0,\dots,r} \|y_{\text{ref}}^{(i)}\|_\infty$ and $\lambda := \inf_{s\geq 0} \psi(s)$; further, let $\gamma > 0$ be such that $\frac{1}{2}\langle v, (\Gamma + \Gamma^\top)v\rangle \geq \gamma\|v\|^2$ for all $v \in \mathbb{R}^m$, which is possible by assumption. Additionally, we define the constant

$$\bar{\eta} := \max\left\{\eta^*, M\eta^* + \frac{M}{\mu}\left(p\psi(0) + pY_{\max} + \hat{d}_\eta\right)\right\}.$$

With this we observe $\|\eta(t)\| \leq \bar{\eta}$ for all $t \in [t_k^+, t_{k+1}^-]$ by a similar estimate as in (4.20), and that $\|\eta(t)\| \leq \eta^* \leq \bar{\eta}$ by a similar estimate as in (4.26) for $t \in [t_k^-, t_k^+]$. Finally, recalling the function $\tilde{\alpha}(s) = (1+s)/(1-s)^2$ defined in *Step 4* of the parameter selection process with c_i from (4.11), we define the constant

$$\begin{aligned}\tilde{C} := \mu_0\left(1 + \frac{c_{r-1}}{1-c_{r-1}^2}\right) + \tilde{\alpha}(c_{r-1}^2)\left(\mu_{r-1} + \frac{c_{r-1}}{1-c_{r-1}^2}\right)\\ + \sum_{i=1}^{r-1}\|R_i\|\left(c_i + \frac{c_{i-1}}{1-c_{i-1}^2} + \frac{Y_{\max}}{\lambda}\right) + 1 + \frac{c_{r-1}}{1-c_{r-1}^2} + \frac{2Y_{\max} + s\bar{\eta} + \hat{d}_r}{\lambda}.\end{aligned}$$

Let $\varepsilon \in (0,1)$ be the unique point such that

$$\frac{\tilde{C}}{\gamma\varphi_0(0)} = \frac{\varepsilon}{1-\varepsilon^2},$$

and set

$$c_r := \max\left\{\|e_r^0\|^2, \varepsilon, q^2\right\}^{1/2} < 1.$$

We show that $\|e_r(t)\| \leq c_r$ for all $t \in [0, t_1^-)$ (or on any interval of existence, respectively). We prove this by contradiction. To this end, suppose there exists $t_1 \in [0, t_1^-)$ such that $c_r < \|e_r(t_1)\| < 1$ and define

$$t_0 := \max\left\{\, t \in [0, t_1] \mid \|e_r(t)\| = c_r \,\right\},$$

which is well-defined since $\|e_r(0)\| \leq c_r$. Now, observe that, by the same calculations as in the proof of [21, Cor. 1.10], we have for $\gamma_{r-1}(t) := \alpha(\|e_{r-1}(t)\|^2)e_{r-1}(t)$ that

$$\begin{aligned}\|\dot{\gamma}_{r-1}(t)\| &\leq \tilde{\alpha}(c_{r-1}^2)\left(\mu_{r-1} + \alpha(c_{r-1}^2)c_{r-1}\right),\\ \|e^{(r-1)}(t)\| &\leq 1 + \frac{c_{r-1}}{1-c_{r-1}},\end{aligned}$$

for $t \in [t_0, t_1]$, where μ_{r-1} is given by (4.11). Furthermore, since $\|e_r(t)\| \geq c_r$ for all $t \in [t_0, t_1]$ we have $\alpha(\|e_r(t)\|^2) \geq 1/(1-c_r^2)$ and hence, invoking Lemma 4.10, we may calculate

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_r(t)\|^2 &= \langle e_r(t), \big(\dot\varphi(t)e^{(r-1)}(t) + \varphi(t)e^{(r)}(t) + \dot\gamma_{r-1}(t)\big)\rangle \\
&\le \|e_r(t)\| \left(\mu_0\varphi(t)\|e^{(r-1)}(t)\| + \tilde\alpha(c_{r-1}^2)\left(\mu_{r-1} + \frac{c_{r-1}}{1-c_{r-1}^2}\right)\right) \\
&\quad + \varphi(t)\|e_r(t)\| \left(Y_{\max} + \sum_{i=1}^{r} \|R_i\|\|y^{(i-1)}(t)\| + s\bar\eta + \hat d_r\right) \\
&\quad - \frac{1}{2}\varphi(t)\alpha(\|e_r(t)\|^2)\langle e_r(t), (\Gamma+\Gamma^\top)e_r(t)\rangle \\
&\le \|e_r(t)\| \left(\mu_0\varphi(t)\|e^{(r-1)}(t)\| + \tilde\alpha(c_{r-1}^2)\left(\mu_{r-1} + \frac{c_{r-1}}{1-c_{r-1}^2}\right)\right) \\
&\quad + \left(\sum_{i=1}^{r-1}\|R_i\|\left(c_i + \frac{c_{i-1}}{1-c_{i-1}^2} + \frac{Y_{\max}}{\lambda}\right) + 1 + \frac{c_{r-1}}{1-c_{r-1}^2}\right)\|e_r(t)\| \\
&\quad + \frac{2Y_{\max} + s\bar\eta + \hat d_r}{\lambda}\|e_r(t)\| - \frac{\gamma\varphi(0)}{1-c_r^2}\|e_r(t)\|^2 \\
&\le \left(\tilde C - \gamma\varphi(0)\frac{c_r}{1-c_r^2}\right)\|e_r(t)\| \le 0,
\end{aligned}$$

by which $c_r < \|e_r(t_1)\| \le \|e_r(t_0)\| = c_r$, a contradiction. By (4.25) we have that $\|e_r(t_k^+)\| \le q \le c_r$ for all $k \in \mathbb{N}$ with $t_k^+ \in [0,\omega)$. Therefore, the arguments above can be reapplied on any interval $[t_k^+, t_{k+1}^-) \subseteq [0,\omega)$ to obtain $\|e_r(t)\| \le c_r$ for all $t \in [t_k^+, t_{k+1}^-)$. Then, invoking $u|_{[t_k^-, t_k^+)} = 0$, it follows from (4.13) that for the input we have $\|u(t)\| \le c_r/(1-c_r^2)$ for all $t \in [0,\omega)$, whereby $u \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$. As a consequence, it follows from (4.2) that $y^{(r)} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$.

Step four. Finally, we show that the solution is global. Suppose the opposite, i.e., assume $\omega < \infty$. Then, since $\|\eta(t)\| \le \bar\eta$ and for all $i = 1,\dots,r$ we have for the error variable $\|e_i(t)\| \le c_i < 1$ for $t \in [t_k^+, t_{k+1}^-)$ by [21, Cor. 1.10] and *Step three*, and extending the estimate (4.24) straightforwardly to $t \in [t_k^-, t_k^+)$ to obtain $\|e_i(t)\| \le q \le c_i$ for $t \in [t_k^-, t_k^+)$ by (4.25), it follows that the closure of the graph of the solution of (4.19) is a compact subset of $\mathcal{D} \times \mathbb{R}^{n-rm}$, which contradicts the findings of *Step one*. This yields assertion (i) and consequently assertions (ii) & (iii) follow. This completes the proof. □

Remark 4.11. We comment on some aspects of the proof of Theorem 4.7.

(i) The proof is constructive and we provide explicit bounds of the tracking error $e = y - y_{\mathrm{ref}}$, as well as for the control input u. We emphasize that these bounds only depend on the system parameters, the initial values of the tracking error, the funnel function and the design parameters. Therefore, with these values given, the maximal control effort can be calculated in advance.

(ii) The crucial obstacle in the feasibility proof of the feedback control law in Theorem 4.7 is to show that the resulting control input in the closed-loop system is globally bounded. To this end, we require appropriate assumptions on the maximal duration of measurement losses and the minimal time of measurement availability. The bounds for these durations essentially depend on the internal dynamics of the system. If the internal dynamics are absent, no restrictions must be made. However, if they are present, a decisive task is to find an invariant set for the internal dynamics and to choose the initial width of the performance funnel large enough.

We collect the observations about systems with trivial internal dynamics made in Remark 4.4 and Remark 4.11 (ii) to formulate the following result.

Corollary 4.12. Consider a system (4.2) with $(A, B, C) \in \Sigma_{m,r}$ such that $n = rm$, i.e., the system has trivial internal dynamics. Let this system have initial values $(y_0^0, \dots, y_{r-1}^0) \in \mathbb{R}^{rm}$. Let $y_{\mathrm{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, and $a(\cdot)$ be the availability function given by (4.7) satisfying Assumption 1. Let $\Delta > 0$ be an arbitrary long duration of possible signal losses, and $\delta > 0$ be an arbitrary short duration of guaranteed signal availability. Choose the design parameter η^* satisfying (η_1^*) – (η_5^*), where $p = M = s = 0$ in Assumptions 2 & 3. Further, let $\varphi_0 \in \Phi$ such that (ϕ_1), (ϕ_2) are valid. If the initial conditions

$$\forall\, i = 1, \dots, r \,:\; \|e_i(0)\| < 1,$$

are satisfied, then the application of the control scheme (4.13) to a system (4.2) yields an initial value problem which has a solution, every solution can be extended to a maximal solution and every maximal solution $y : [0, \omega) \to \mathbb{R}^m$ has the following properties:

(i) the solution is global, i.e., $\omega = \infty$,

(ii) the tracking error $e = y - y_{\mathrm{ref}}$ evolves within the funnel boundaries, i.e.,

$$\forall\, t \geq 0 \,:\; \varphi(t)\|e(t)\| < 1,$$

(iii) the control signal is globally bounded, i.e., $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, and for the output signal we have $y \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

Proof. Clear. □

We conclude this section with a numerical example.

Example 4.13. To illustrate Theorem 4.7, we numerically simulate an application of the funnel control scheme (4.13) to a system (4.2). We consider a smaller version of Example 2.30, namely the *mass-on-car* system introduced in [179], where on a car with mass m_1 (in kg) a ramp is mounted on which a mass m_2 (in kg), coupled to the car by a spring-damper-component with spring constant $k > 0$ (in N/m) and damping $\sigma > 0$ (in Ns/m), passively moves. A control force $F = u$ (in N) can be applied to the car. The situation is depicted in Figure 4.3.

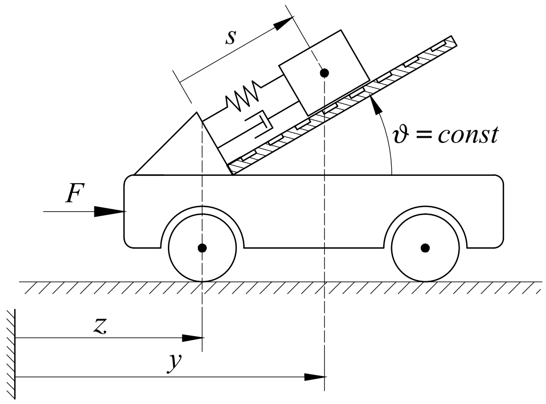


Figure 4.3: Mass-on-car system. The figure is taken from [21], which itself is based on the figure in [179].

We assume some bounded disturbance $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$ acting along the input direction. Then, the equations of motion for the system read

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos(\vartheta) \\ m_2 \cos(\vartheta) & m_2 \end{bmatrix} \begin{pmatrix} \ddot{z}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + \sigma \dot{s}(t) \end{pmatrix} + \begin{pmatrix} d(t) \\ 0 \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}, \tag{4.27a}$$

with the horizontal position of the second mass m_2 as output

$$y(t) = z(t) + \cos(\vartheta) s(t). \tag{4.27b}$$

To have this system in the form (4.1), we set

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m_1 + m_2 & m_2 \cos(\vartheta) \\ 0 & 0 & m_2 \cos(\vartheta) & m_2 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \ A := M^{-1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -k & 0 & 0 \\ 0 & 0 & 0 & -\sigma \end{bmatrix} \in \mathbb{R}^{4 \times 4},$$

and $B := M^{-1}[0, 0, 1, 0]^\top \in \mathbb{R}^{4 \times 1}$, $C := [1, \cos(\vartheta), 0, 0] \in \mathbb{R}^{1 \times 4}$. So, in this example we have $n = 4$ and $m = 1$. Then, setting $x_1(t) := (z(t), s(t))^\top$, $x_2(t) := \dot{x}_1(t)$ and $x(t) := (x_1(t)^\top, x_2(t)^\top)^\top$ we obtain

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \tilde{d}(t), \\ y(t) &= Cx(t), \end{aligned}$$

where $\tilde{d}(t) := -d(t) M^{-1}[0, 0, 1, 0]^\top$. For the simulation we choose the system parameters $m_1 = 4$, $m_2 = 1$, $k = 2$, $\sigma = 1$, $\vartheta = \pi/4$, and the initial values $z(0) = s(0) = \dot{z}(0) = \dot{s}(0) = 0$. As a reference signal we choose $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $t \mapsto \cos(t)$, by which $\|y_{\text{ref}}\|_\infty = \|(y_{\text{ref}}, \dot{y}_{\text{ref}})\|_\infty = 1$. As a disturbance we insert $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $t \mapsto 0.1 \sin(11t)$. For the above parameters a brief calculation yields $CB = 0$ and $CAB = 1/9$. Hence, system (4.27) has relative degree two with respect to the output (4.27b), and so we have $(A, B, C) \in \Sigma_{1,2}$. We define

$$\mathcal{B} := \begin{bmatrix} B & AB \end{bmatrix}, \ \mathcal{C} := \begin{bmatrix} C \\ CA \end{bmatrix}, \ V \in \mathbb{R}^{4 \times 2} \text{ such that } \operatorname{im} V = \ker \mathcal{C},$$

and $N := V^\dagger (I_4 - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}) \in \mathbb{R}^{2 \times 4}$. Thus, following [95], via the transformations given in (2.4) and (3.8), system (4.27) can equivalently be written in the form (4.2) with

$$R_1 = 0,\ R_2 = \frac{8}{9},\ S = \frac{-4\sqrt{2}}{9}\begin{bmatrix}2 & 1\end{bmatrix},\ \Gamma = \frac{1}{9},\ Q = \begin{bmatrix}0 & 1\\ -4 & -2\end{bmatrix},\ P = 2\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix},$$

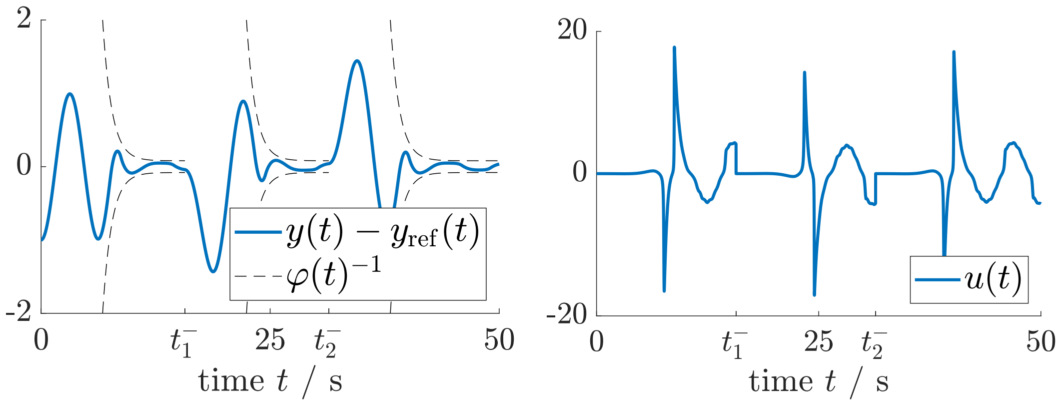
and, since $C\tilde{d}(t) = 0$, the respective components (d_r, d_η) of the disturbance are given by the transformation formula (3.8) in Remark 3.8 as

$$\begin{pmatrix} d_r(t)\\ d_\eta(t)\end{pmatrix} = \begin{bmatrix} CA\\ N\end{bmatrix}\tilde{d}(t) = \begin{pmatrix} d(t)\\ 0\\ 0\end{pmatrix}.$$

Next, we choose the controller's design parameters. According to Assumptions 2 & 3 with $q = 0.95$, $\kappa = 15$, $\theta = (1 + 0.01)\|S\|$, $\mu = 0.3305$, $M = 2.2477$ we assume $\Delta \le 2.4 \cdot 10^{-3}$ and $\delta \ge 15.7$. Conditions $(\eta_1^*) - (\eta_5^*)$ are satisfied with $\eta^* = 146\,527$. We choose the funnel function $\varphi_0(t) = (ae^{-bt} + c)^{-1}$. According to (ϕ_1) the funnel function has to satisfy

$$\varphi_{0,\min} = 6.3236 \cdot 10^{-4} \le \varphi_0(0) \le 6.3236 \cdot 10^{-4} = \varphi_{0,\max},$$

and we choose $c = 0.08$, $a = 1/\varphi_{0,\min} - c$ and $b = 1$. Then, the constant from (4.12) is given as $C = 21.4683$, and condition (ϕ_2) is satisfied with $\varphi(\rho) = 12$, where $\rho = 0.99\,\delta$. We simulate output tracking over the interval $0 - 50$ seconds. For illustration purposes we consider two losses and reappearances of the output signal. The results of the simulation are shown in Figure 4.4.



(a) Error between the output y and the reference signal y_{ref}, and funnel boundary $1/\varphi$.

(b) Control input u.

Figure 4.4: Tracking error $y - y_{\mathrm{ref}}$, funnel boundary $1/\varphi$, and control input u.

Figure 4.4a shows the error $e = y - y_{\mathrm{ref}}$ between the system's output and the reference signal. As expected, the error evolves within the prescribed funnel boundaries whenever the output signal is available, and remains bounded whenever the signal is not available. In Figure 4.4b the control input is depicted. It can be seen that on large time intervals, especially after t_1^- and t_2^-, the input signal is approximately zero. Only when the performance funnel gets tighter again a large control action is necessary, which induces some small peaks in the input when a small tracking error is enforced. Moreover, the controller compensates the disturbances. The simulation has been performed in MATLAB (solver: `ode15s`, default tolerances). ◇

4.2 Exact tracking in finite time

In this section we focus on the long standing problem of exact tracking in finite time via feedback control. Since rendezvous of spacecraft has been a hot topic from the early sixties to the present, a brief motivation from the engineering point of view seems adequate. The first successful automated docking manoeuvre was carried out in 1967 between *Cosmos 186* and *Cosmos 188* in the context of the Soviet space program. In the subsequent years more and more such docking manoeuvres became relevant as for bringing crew members to a space station (successful in 1971 for the first time), supplying a space station as well as for service missions for, e.g., the Hubble Space Telescope. In view of the recent space mission to Mars (rover "Perseverance" and helicopter "Ingenuity"), due to the finite speed of information travelling through space-time, the farer away from the earth spacecraft are, the more and more *feedback control schemes* become relevant in launching, landing and docking manoeuvres. However, back to earth, in daily life contexts there is use of docking manoeuvres as well, for instance, in automatic production processes, linking up two parts of a train or charging of autonomous vehicles to name but a few possible fields of application. Reviewing the current literature on feedback based output reference tracking, one gets the impression that *exact tracking* (see Section 1.1 (CO.1)) is an old problem which has only recently been addressed, and *exact tracking in finite time* represents a chapter that has hardly been written yet. Referring to the results in [48], in [91] it is shown that the proposed funnel control scheme can achieve global asymptotic stabilization for a class of linear MIMO systems of relative degree one. A generalization to a class of nonlinear relative degree one MIMO systems is proposed in [174]. In [154] an extended sliding mode controller is proposed which achieves asymptotic tracking of linear SISO systems. This controller is extended to linear MIMO systems in [153]. In [54] backstepping is combined with feedback linearization techniques and higher order sliding modes to design a controller which achieves exponential accurate tracking. In [157] a high-gain based sliding mode controller is introduced, where the peaking related to the high-gain observer is obviated introducing a dwell-time activation scheme. This controller achieves asymptotic tracking for a class of nonlinear SISO systems of arbitrary relative degree, where the reference signal is generated by a reference model. At the price of a discontinuous control, asymptotic tracking for nonlinear MIMO systems is achieved in [191, 192]. In [123] a funnel control scheme is proposed, which achieves asymptotic tracking for a class of nonlinear relative degree one MIMO systems. Using the proof techniques and the idea of rewriting the error variable as in [123], this result was extended in the recent work [21], where it is shown that the proposed funnel controller achieves asymptotic tracking of nonlinear MIMO systems with arbitrary relative degree whereas the tracking error has prescribed transient behaviour. Now, we turn from asymptotic tracking towards exact tracking in finite time. In [7] sliding mode control concepts and results from [35] are used to establish control schemes which achieve finite time stabilization for linear (SISO & MIMO) systems. In [62] backstepping and higher order sliding mode control are combined to construct a control scheme which achieves exact output tracking in finite time for nonlinear MIMO systems in nonlinear block controllable form. Similar to the prescribed performance controller in [56] this controller suffers from the proper initialization problem, where it is not clear how large to choose the involved parameters. The controller in [62], along with limiting conditions on the system class, presumes knowledge of the system's functions and explicitly involves inverses of some. In [201] a controller is introduced which achieves exact tracking in finite time for a class of nonlinear SISO systems satisfying a certain homogeneity assumption. This controller relies on estimating techniques of the external disturbances, where the problem of proper initialization is avoided by assuming explicit knowledge of the bounds of the disturbances and the reference. The control scheme explicitly involves (parts of) the system's right-hand side and is of relatively high complexity. As far as we understand, in the control schemes for exact tracking in finite time discussed above, the final time, this is the time when the desired reference should be matched, cannot be prescribed; only the existence of such a finite time is ensured. Contrary, in [100] a control scheme is introduced which solves a *predefined-time exact tracking problem* for the class of fully actuated mechanical (relative degree two) systems. The controller relies on a backstepping procedure and consists of a *predefined-time stabilization function* and involves the system's equations explicitly. The controller introduced in [123] achieves asymptotic tracking as well as convergence to zero of the tracking error in finite time for a class of relative degree one MIMO systems.

Circumventing the drawbacks mentioned above, in the present section we introduce a new funnel control scheme which achieves *exact tracking in predefined finite time*. Since the control scheme is of funnel type, it inherits the advantages of robustness with respect to noise and being model free. Moreover, the tracking error evolves within prescribed bounds.

4.2.1 System class, control objective, feedback law

We consider nonlinear systems of the following type

$$\begin{pmatrix} y_1^{(r_1)}(t) \\ \vdots \\ y_m^{(r_m)}(t) \end{pmatrix} = \begin{pmatrix} f_1\left(d(t), \mathbf{T}(y_1, \dot y_1, \ldots, y_1^{(r_1-1)}, y_2, \ldots, y_m^{(r_m-1)})(t), u(t)\right) \\ \vdots \\ f_m\left(d(t), \mathbf{T}(y_1, \dot y_1, \ldots, y_1^{(r_1-1)}, y_2, \ldots, y_m^{(r_m-1)})(t), u(t)\right) \end{pmatrix}, \tag{4.28}$$
$$y_j|_{[-\sigma,0]} = y_j^0 \in \mathcal{C}^{r_j-1}([-\sigma,0];\mathbb{R}), \quad j = 1,\ldots,m,$$

where $f_j \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m;\mathbb{R})$ for $j = 1,\ldots,m$, $d \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0};\mathbb{R}^p)$ is a bounded disturbance, $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ is an input, the real number $\sigma > 0$ is the "memory" of the system and the parameters $r_j \in \mathbb{N}$ are related to the concept of vector relative degree. Referring to the system above, we introduce the system class $\mathcal{N}^{m,\bar r}$ under consideration in this section.

Definition 4.14. For $m, p, q, r_1, \ldots, r_m \in \mathbb{N}$ and $\bar r := (r_1,\ldots,r_m) \in \mathbb{N}^{1\times m}$ a system (4.28) is said to belong to the system class $\mathcal{N}^{m,\bar r}$, if the operator $\mathbf{T}$ satisfies Definition 1.4, i.e., $\mathbf{T} \in \mathcal{T}_\sigma^{n,q}$, the function

$$F(d,\eta,u) := (f_1(d,\eta,u),\ldots,f_m(d,\eta,u))^\top \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m;\mathbb{R}^m)$$

satisfies the high-gain property from Definition 1.10, and the dimension of the input u coincides with the number m of output channels.

In this section we aim to find a control structure which achieves exact output reference tracking in the following sense. For every system (4.28) belonging to $\mathcal{N}^{m,\bar{r}}$ and any suitable reference trajectory $y_{\text{ref}} = (y_{1,\text{ref}}, \dots, y_{m,\text{ref}})^\top : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ with $y_{j,\text{ref}} \in \mathcal{W}^{r_j,\infty}([0,T);\mathbb{R})$, where $j = 1, \dots, m$ and $T > 0$, the output approaches the reference within the interval $[0,T)$, whereas for a given function φ introduced next, the componentwise tracking error $e_j := y_j - y_{j,\text{ref}}$, evolves within the performance funnel defined by

$$\mathcal{F}^m_{\varphi_j} := \left\{ (t,e) \in [0,T) \times \mathbb{R} \;\middle|\; \varphi_j(t)|e| < \frac{1}{\sqrt{m}} \right\},$$

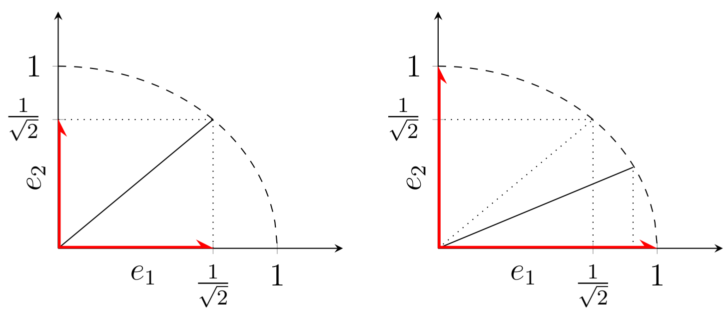
and the control achieves exact tracking in finite time. That is, we aim to achieve the following control objective

$$\forall\, j = 1, \dots, m \;\forall\, t \in [0,T) \;:\; (t, e_j(t)) \in \mathcal{F}^m_{\varphi_j}, \tag{4.29a}$$

$$\forall\, j = 1, \dots, m \;\forall\, k = 0, \dots, r_j - 1 \;:\; \lim_{t \to T} \|e_j^{(k)}(t)\| = 0, \tag{4.29b}$$

where $e_j^{(k)}$ denotes the k^{th} time derivative of e_j.

Remark 4.15. The parametrization of the performance funnel $\mathcal{F}^m_{\varphi_j}$ by the number of input/output channels m is due to the fact that each error variable $e_{j,k}$ (introduced below in (4.31)) has to evolve independently within its individual funnel boundary. Since $\|(e_{1,r_1}, \dots, e_{m,r_m})\| < 1$ is required due to the construction of the control law, the respective components have to be bounded by the root of the number of inputs. The situation is geometrically illustrated in Figure 4.5.



(a) $|e_1| < \frac{1}{\sqrt{2}}$ and $|e_2| < \frac{1}{\sqrt{2}}$. (b) $\|(e_1, e_2)\| < 1$.

Figure 4.5: Geometric illustration of the fact that, if $|e_1| < 1/\sqrt{2}$ and $|e_2| < 1/\sqrt{2}$, then $\|(e_1, e_2)\| < 1$; the converse, however is not true.

The parametrization could also be shifted to the control function β, however, this would require an adaption of the formulation of the high-gain property. Thus, for the sake of better legibility, we will use the parametrization of the performance funnel $\mathcal{F}^m_{\varphi_j}$.

In order to establish the control structure which achieves the control objective (4.29), we introduce the following funnel control parameters. Choose the final time $T > 0$ and for $j = 1, \dots, m$ choose $c_j > 0$. Then, we define the funnel function

$$\varphi_j(t) = \frac{1}{c_j} \frac{1}{T - t}, \qquad t \in [0,T), \tag{4.30a}$$

where we highlight $\lim_{t\to T}\varphi_j(t)=\infty$, further choose

$$N \in \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}) \text{ a surjection,} \tag{4.30b}$$

and for $c_j > 0$ from (4.30a) choose

$$\alpha_j \in \mathcal{C}^{r_j-1}([0,1/m);[c_j(r_j+1),\infty)) \text{ a bijection,} \tag{4.30c}$$

and

$$\beta \in \mathcal{C}([0,1);[1,\infty)) \text{ a bijection.} \tag{4.30d}$$

We will comment on the functions N, α_j and β below in Remark 4.17.

Remark 4.16. Compared to the funnel control schemes proposed in the common literature, see e.g. [91, 84, 25, 123, 21], the explicit choice of φ_j in (4.30a) seems to be quite restrictive. For this reason, we give a brief comment on that.

(i) Anticipating the initial condition (4.34) in Theorem 4.20, the choice of the funnel function φ in (4.30a) reflects the intuition that the shorter the final time T is chosen, the better the initial guess has to be. Further, the tuning parameter c_j in (4.30a) link the funnel function to the respective gain function α_j in (4.30c). If the final time T is short, large c_j can ensure the initial condition (4.34) to be satisfied. This in turn causes a larger lower bound of the gain function α_j which means that small tracking errors cause high input values.

(ii) If, however, the initial error is completely unknown, a combination of well known funnel control schemes and the proposed feedback law achieves the control objective (4.29) as follows. For some $\tau \in (0,T)$ apply for $t \in [0,\tau)$ a standard funnel controller, for instance the control scheme from [21], to force the tracking error to a certain value, such that with a suitable choice of c_j the initial conditions (4.34) are satisfied at $t=\tau$. Then apply the feedback law (4.32) for $t \in [\tau,T)$. Note that this may introduce a discontinuity in the control at the "switching" time $t=\tau$.

(iii) A careful inspection of the proof of [21, Thm. 1.9] yields that an essential property of the feasible funnel functions is a growth condition, namely

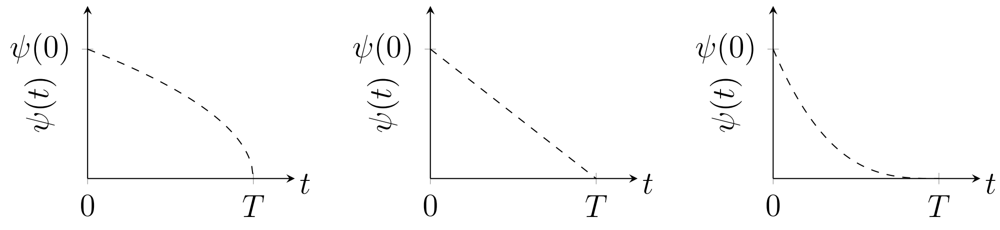
$$\exists\, d>0 \;:\; |\dot\varphi(t)| \leq d\big(1+\varphi(t)\big) \text{ for almost all } t \geq 0.$$

This condition prevents a "blow up" in finite time, i.e., the funnel function φ is bounded on $[0,\infty)$. With this, however, exact tracking in finite time via funnel control is impossible. Contrary, the funnel functions φ_j defined in (4.30a) do not satisfy this growth condition. Therefore, in order to achieve exact tracking in finite time via funnel control, we propose the modified feedback law (4.31), (4.32).

(iv) Anticipating Theorem 4.20 and its proof, we note that it is possible to choose the funnel functions φ_j from (4.30a) as

$$\varphi_j(t) = \frac{1}{c_j}\frac{1}{(T-t)^{\rho_j}}, \quad \rho_j>0, \quad j=1,\dots,r-1.$$

Different choices of ρ are depicted in Figure 4.6.



(a) $\rho = 0.5$. (b) $\rho = 1$. (c) $\rho = 2$.

Figure 4.6: Shapes of the funnel boundary $\psi = 1/\varphi$ for different $\rho > 0$.

However, for the sake of readability of the proof, this additional option is not taken into account in Theorem 4.20.

We introduce the control law which achieves the previously formulated control objective (4.29). To this end, for $j = 1, \ldots, m$, and $i = 0, \ldots, r_j - 1$ we set $e_j^{(i)} := y_j^{(i)} - y_{j,\mathrm{ref}}^{(i)}$ and recursively define for $k = 1, \ldots, r_j$ with α_j from (4.30c) and $\gamma_{j,0} \equiv 0$ the functions

$$
\begin{aligned}
e_{j,k}(t) &:= \varphi_j(t) e_j^{(k-1)}(t) + \varphi_j(t) \sum_{i=1}^{k-1} \gamma_{j,i}^{(k-1-i)}(t), && \text{(4.31a)}\\
\alpha_{j,k}(t) &:= \alpha_j(e_{j,k}(t)^2), && \text{(4.31b)}\\
\gamma_{j,k}(t) &:= \alpha_{j,k}(t) e_{j,k}(t). && \text{(4.31c)}
\end{aligned}
$$

Then, setting $e_r := (e_{1,r_1}, \ldots, e_{m,r_m})^\top$, with the functions introduced in (4.30), (4.31) we define the feedback law $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ as

$$
u(t) := (N \circ \beta)(\|e_r(t)\|^2)\, e_r(t). \tag{4.32}
$$

Remark 4.17. Anticipating the results of Theorem 4.20, we discuss purposes and possible choices of the surjection N introduced in (4.30b), and illustrate this with an example. Further, we briefly consider possible choices of the functions α_j and β introduced in (4.30c), (4.30d).

(i) As pointed out in [21, Rem. 1.4 & 1.8, Sec. 2.2], the surjection N can be interpreted as a "testing function", best illustrated by a linear system. For a linear system (4.1) the matrix $\sigma\Gamma = CA^{r-1}B$ is positive definite for a $\sigma \in \{-1, +1\}$. Then σ is the "control direction". If the control direction is known, the high-gain adaptive stabilizer (1.3) with $u(t) = -\sigma k(t) y(t)$ stabilizes the system, and the gain k is bounded. If the control direction of a system is not known, then the control input possibly steers the system into the wrong direction. Morse conjectured in [148], that there does not exist an adaptive controller, which smoothly stabilizes every linear SISO system (4.1), where $\Gamma \neq 0$. Nussbaum proved the falsity of this conjecture in [155], where he showed that the incorporation of a "sign-testing" function solves the problem. The incorporation of a continuous surjection in [21] generalizes the idea of Nussbaum and allows for a larger class of "testing functions". The idea is, if the control direction is unknown and the control steers the system into the wrong direction, the input becomes larger and larger as the controller aims to satisfy the control objective. Then, since $N : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defines a continuous surjection, as the error grows it will change its sign at some point and the control input is directed into the right direction.

(ii) As elaborated in [21, Rem. 1.4(c)], if a system has a "negative high-gain property", i.e., if for χ given in Definition 1.10 we have $\sup_{s>0} \chi(s) = \infty$, then the surjection $N : \mathbb{R}_{\geq 0} \to \mathbb{R}$ can be replaced by a continuous surjection $[0, \infty) \to [0, \infty)$, e.g., the map $s \mapsto s$; in the case $\sup_{s<0} \chi(s) = \infty$ it can be replaced by a continuous surjection $[0, \infty) \to (-\infty, 0]$, e.g., the map $s \mapsto -s$.

(iii) If the control direction is unknown, suitable choices are, e.g., $N : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $s \mapsto s\cos(s)$, or $s \mapsto s\sin(s)$.

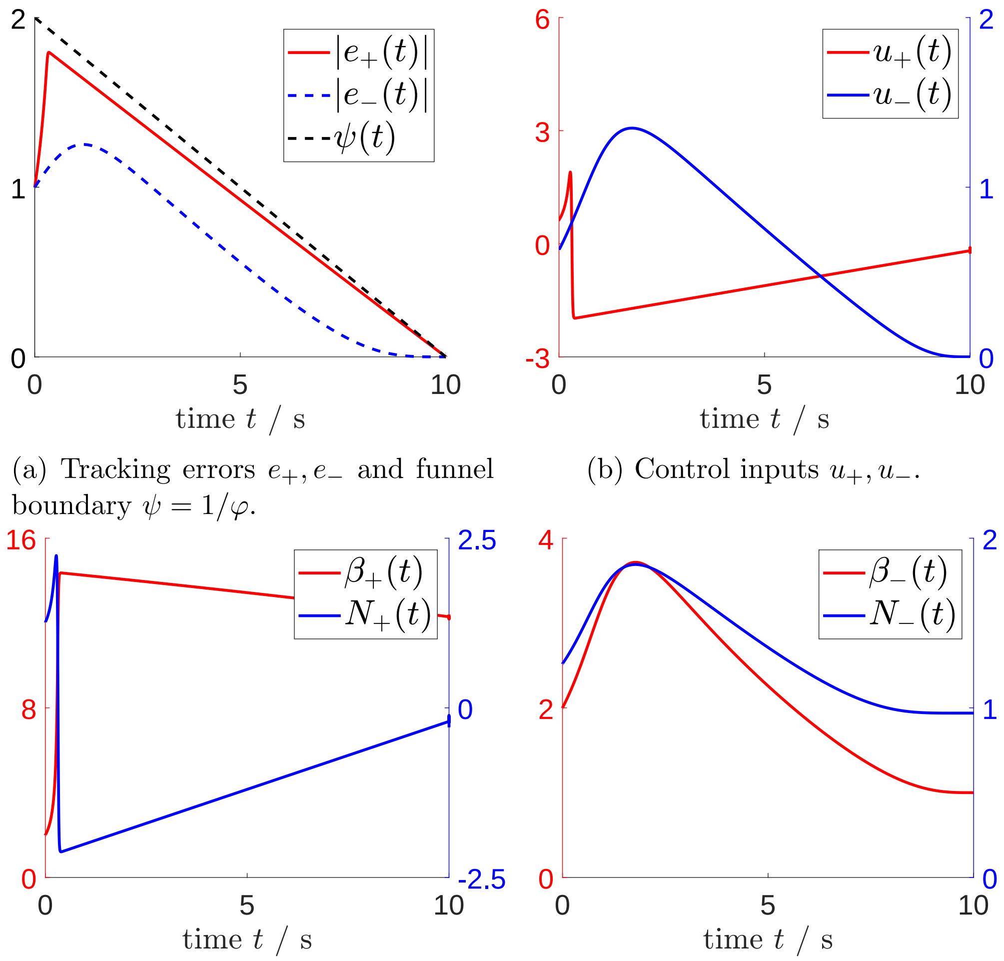
(iv) We illustrate the purpose of the surjection N and its "testing" with the simple example

$$\dot{y}(t) = y(t) + \sigma u(t), \quad y(0) = 1,$$

where $\sigma \in \{-1, +1\}$ is unknown, and so the control direction is unknown. According to Remark 3.15 this system has the high-gain property. If, for a reference $y_{\text{ref}} \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$, the control from (4.32) with $e := y - y_{\text{ref}}$

$$u(t) = N\left(\beta\big((\varphi(t)e(t))^2\big)\right)\varphi(t)e(t)$$

is applied, then two cases may occur. Either the control input steers the system into the right direction, or the control steers the system into the wrong direction. The latter case means that the tracking error $e = y - y_{\text{ref}}$ is pushed towards the funnel boundary. This results in a larger value of $\beta(\|\varphi(t)e(t)\|^2)$. In this case, the surjection changes its sign at some point and the control acts into the right direction. This is illustrated in the following simulation, where we choose $\varphi(t) = 5/(10 - t)$ and $y_{\text{ref}}(t) = 0$, the bijection $\beta(s) = 1/(1 - s)$ and as surjection we choose $N : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $s \mapsto s\cos(0.25s)$.



(a) Tracking errors e_+, e_- and funnel boundary $\psi = 1/\varphi$.

(b) Control inputs u_+, u_-.

(c) Gain function β_+ and corresponding surjection N_+.

(d) Gain function β_- and corresponding surjection N_-.

Figure 4.7: Illustration of the influence of the surjection N.

Figure 4.7 shows the influence of the surjection N, where the subscript indicates the value of $\sigma \in \{-1, +1\}$, i.e., e_+ is the tracking error in the case $\sigma = +1$; further, we set $\beta_+(t) := \beta((\varphi(t)e_+(t))^2)$ and $N_+(t) := N(\beta_+(t))$, and respectively for $\sigma = -1$. In Figure 4.7a the tracking errors and the funnel boundary are depicted. It can be seen, that for $\sigma = +1$ in the first seconds the input causes a larger tracking error, which in turn causes a larger value of $\beta_+(t)$. Then, the surjection changes its sign and the control directs into the right direction. For $\sigma = -1$ the control points into the right direction. Figure 4.7b shows the respective control signals $u_\pm$. In Figures 4.7c & 4.7d the aforesaid crystallizes from the perspective of the gain functions $\beta_\pm$ and the surjections $N_\pm$.

(v) Regardless of the control direction, suitable choices for the bijections in (4.31b), (4.30d) are, for instance,

$$\alpha_j(s) = \frac{c_j(r_j + 1)}{(1 - sm)^{\rho_j}}, \quad \beta(s) = \frac{1}{(1 - s^\kappa)^\rho}, \quad \rho_j, \rho, \kappa > 0,$$

see also the funnel functions in Remark 4.16 (iv), and the pre-compensator's gain functions in Section 3.1.

Remark 4.18. We comment on the computation of the expressions recursively given in (4.31). We define the set

$$\mathcal{D}_0 := \left\{ \zeta \in \mathbb{R} \;\middle|\; |\zeta| < 1/\sqrt{m} \right\},$$

and, for α_j from (4.30c), the function $\Gamma_{j,0} : \mathbb{R}_{\geq 0} \times \mathcal{D}_0 \to \mathbb{R}$ by

$$\Gamma_{j,0}(t,\zeta) := \alpha_j(\zeta^2)\,\zeta.$$

For $j = 1,\dots,m$ and $k = 1,\dots,r_j - 1$ we recursively define the sets $\mathcal{D}_k$ and the functions $\Gamma_{j,k} : [0,T) \times \mathcal{D}_k \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{D}_k &:= \underbrace{\mathcal{D}_0 \times \cdots \times \mathcal{D}_0}_{k-\text{times}} \times \mathbb{R},\\ \Gamma_{j,k}(t,\zeta_1,\dots,\zeta_{k+1}) &:= \frac{\partial \Gamma_{j,k-1}(t,\zeta_1,\dots,\zeta_k)}{\partial t}\\ &\quad + \sum_{i=1}^{k} \frac{\partial \Gamma_{j,k-1}(t,\zeta_1,\dots,\zeta_k)}{\partial \zeta_i}\Big(\varphi_j(t)\left(c_j\zeta_i - \Gamma_{j,0}(\zeta_i)\right) + \zeta_{i+1}\Big).\end{aligned} \tag{4.33}$$

Then with $e_{j,k}$ from (4.31a) and $\gamma_{j,k}$ from (4.31c) we obtain

$$\gamma_{j,k}^{(q)}(t) = \Gamma_{j,q}\big(t, e_{j,k}(t),\dots,e_{j,k+q}(t)\big),\ \ 0 \leq q \leq r_j - k,$$

which can be seen via a brief induction over q using equations (4.31).

Remark 4.19. Comparing the control scheme (4.32) with the common literature, we make the following two observations.

(i) Due to the recursive structure, implementation of the control scheme (4.32) is not as simple as for the control scheme in [21, Eq. (9)]. However, with the explicit recursion (4.33) given in Remark 4.18 calculation of the required expressions can be done completely algorithmically.

(ii) Comparing the control scheme (4.32) with the control scheme in [25, Eq. (5)] a certain similarity is recognizable. Both control schemes involve recursively defined signals from which the input is constructed. In view of the findings in the recent work [21] this similarity gives reason to hope that for the control objective formulated in (4.29) a non-recursive control scheme can be found in future research.

4.2.2 Exact tracking in finite time via funnel control

In this section we show that the application of the control scheme (4.32) to a system (4.28) belonging to $\mathcal{N}^{m,\bar{r}}$ yields an initial value problem that has a solution, the input and output signals are bounded and in particular, we show that the introduced control scheme achieves exact output tracking in finite time while the tracking error evolves within prescribed bounds. As highlighted in earlier works, see, e.g., [84, 25, 21], some care is required when showing boundedness of the involved signals since the bijections α_j and β may introduce a singularity. Moreover, in the present context expressions involving the unbounded funnel functions φ_j demand particularly high attention.

Theorem 4.20. For $\bar{r} := (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$, $m \in \mathbb{N}$ and $j = 1, \ldots, m$ consider a system (4.28) belonging to $\mathcal{N}^{m,\bar{r}}$, with initial data $y_j^0 \in \mathcal{C}^{r_j-1}([-\sigma_j, 0]; \mathbb{R})$. Let $T > 0$ and $y_{j,\text{ref}} \in \mathcal{W}^{r_j,\infty}([0, T); \mathbb{R})$ be a reference trajectory. Assume that for the control design parameters introduced in (4.30) and $e_{j,k}$ defined in (4.31) the initial conditions

$$\forall\, j = 1 \ldots, m \;\forall\, k = 1, \ldots, r_j \;:\; |e_{j,k}(0)| < \frac{1}{\sqrt{m}} \tag{4.34}$$

are satisfied. Then, the application of the funnel control scheme (4.32) to a system (4.28) yields an initial value problem which has a solution and every maximal solution $y : [-\sigma, \omega) \to \mathbb{R}^m$ has the following properties

(i) $\omega = T$,

(ii) $u \in \mathcal{L}^\infty([0, T); \mathbb{R}^m)$, and for all $j = 1, \ldots, m$ we have $y_j \in \mathcal{W}^{r_j,\infty}([-\sigma, T); \mathbb{R})$,

(iii) the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ evolves within predefined margins; precisely, for $e_r := (e_{1,r_1}, \ldots, e_{m,r_m})^\top$ we have

$$\forall\, t \in [0, T) \;:\; \|e_r(t)\| < 1,$$

and for the componentwise tracking error $e_j(t) = y_j(t) - y_{j,\text{ref}}(t)$ we have

$$\forall\, j \in \{\, i \in \{1, \ldots, m\} \mid r_i > 1\,\} \;\forall\, t \in [0, T) \;:\; |e_{j,1}(t)| = \varphi_j(t)|e_j(t)| < \frac{1}{\sqrt{m}},$$

(iv) the tracking of the reference and its derivatives is exact at $t = T$, that is,

$$\forall\, j = 1, \ldots, m \;\forall\, k = 0, \ldots, r_j - 1 \;:\; \lim_{t\to T} |e_j^{(k)}(t)| = 0.$$

Sketch of proof. Since the proof of Theorem 4.20 goes beyond standard funnel control techniques, we briefly outline the main steps. In *Step one* we establish the existence of a local solution of the closed-loop initial value problem (4.28), (4.32), (4.34). In *Step two*, we show by contradiction boundedness of the auxiliary variables $e_{j,k}$ introduced in (4.31). The next step is preparatory work to show boundedness of the higher derivatives of the tracking error. To this end, we show boundedness of the higher derivatives of the auxiliary error variables via similar arguments as in *Step two* and using Lemma 1.11. Both of these steps require particular attention, since the unbounded funnel functions φ_j are involved with higher powers. In *Step four* we use the previously established estimations to conclude boundedness of the local solution of the closed-loop system. In *Step five* we show boundedness of the input signal u. This is done mainly by invoking the high-gain property of the system's right-hand side. From the findings established so far, we conclude in *Step six* that the solution is defined on the entire interval $[0, T)$. The remaining two steps, namely to establish that the errors evolve within their respective funnel boundaries and vanish for $t \to T$, are straightforward.

Proof of Theorem 4.20. The proof is subdivided in eight steps.
Step one. We show existence of a solution of (4.28), (4.32). To this end, we aim to reformulate (4.28), (4.32) as an initial value problem of the form

$$\begin{aligned} \dot{x}(t) &= F\left(t, x(t), \mathbf{T}(x)(t)\right), \\ x(0) &= \left(y_1^0(0), \dot{y}_1^0(0), \ldots, (\tfrac{\mathrm{d}}{\mathrm{d}t})^{r_m-1} y_m^0(0)\right), \end{aligned} \tag{4.35}$$

where we set $n := \sum_{j=1}^m r_j$, and

$$x := \left(y_1, \dot{y}_1, \ldots, y_1^{(r_1-1)}, y_2, \ldots, y_m^{(r_m-1)}\right).$$

Setting $\mathcal{D}_0 := \{\, v \in \mathbb{R} \mid |v| < 1/\sqrt{m} \,\}$ we choose some interval $I \subseteq [0,T)$ with $0 \in I$ such that $(e_{1,1}, \ldots, e_{1,r_1}, e_{2,1}, \ldots, e_{m,r_m}) : I \to \mathbb{R}^n$ satisfy the relations in (4.31) and such that for all $t \in I$ we have $e_{1,1}(t), \ldots, e_{m,r_m-1}(t) \in \mathcal{D}_0$, which is possible via initial conditions (4.34). With the aid of (4.33) for all $j = 1\ldots,m$ and $k = 1, \ldots, r_j - 1$ the higher derivatives of $\gamma_{j,k}$ are given by

$$\gamma_{j,k}^{(q)}(t) = \Gamma_{j,q}\big(t, e_{j,k}(t), \ldots, e_{j,k+q}(t)\big),\ \ 0 \le q \le r_j - k,\ \ t \in I.$$

Next, for $j = 1\ldots,m$ we define the functions

$$\begin{aligned} \tilde{e}_{j,1} : [0,T) \times \mathbb{R} &\to \mathbb{R}, \\ (t, \xi_{j,0}) &\mapsto \varphi_j(t)\left(\xi_{j,0} - y_{j,\mathrm{ref}}(t)\right), \end{aligned}$$

and the set

$$\tilde{\mathcal{D}}_{j,1} := \{\, (t, \xi_{j,0}) \in [0,T) \times \mathbb{R} \mid \tilde{e}_{j,1}(t, \xi_{j,0}) \in \mathcal{D}_0 \,\}.$$

With this, again for $j = 1\ldots,m$ we recursively define for $k = 2, \ldots, r_j$ the functions

$$\begin{aligned} &\tilde{e}_{j,k} : \tilde{\mathcal{D}}_{j,k-1} \times \mathbb{R} \to \mathbb{R}, \\ &(t, \xi_{j,0}, \ldots, \xi_{j,k-1}) \mapsto \varphi_j(t)\left(\xi_{j,k-1} - y_{j,\mathrm{ref}}^{(k-1)}(t)\right) + \varphi_j(t) \sum_{i=1}^{k-1} \Gamma_{j,k-1-i}\big(t, \tilde{e}_{j,i}, \ldots, \tilde{e}_{j,k-1}\big), \end{aligned}$$

where for the sake of better legibility we omit the arguments of $\tilde{e}_{j,q}$, $q = 1, \ldots, k-1$. Further, we define the sets

$$\tilde{\mathcal{D}}_{j,k} := \left\{ (t, \xi_{j,0}, \ldots, \xi_{j,k-1}) \in \tilde{\mathcal{D}}_{j,k-1} \times \mathbb{R} \,\middle|\, \tilde{e}_{j,i}(t, \xi_{j,0}, \ldots, \xi_{j,k-1}) \in \mathcal{D}_0, i = 1, \ldots, k \right\}.$$

Setting $\tilde{e}_r := (\tilde{e}_{1,r_1}, \ldots, \tilde{e}_{m,r_m})^\top$ we observe $\|\tilde{e}_r(t)\| < 1$ for $t \in I$, and with α_j, β, N given in (4.30) we define for $t \in I$

$$N_r(t) := (N \circ \beta)\left(\|\tilde{e}_r\big(t, y_1(t), \dot{y}_1(t), \ldots, y_1^{(r_1-1)}(t), y_2(t), \ldots, y_m^{(r_m-1)}(t)\big)\|^2\right).$$

Then, the control u defined in (4.32) reads

$$u(t) = N_r(t) \cdot \tilde{e}_r\big(t, y_1(t), \ldots, y_1^{(r_1-1)}(t), y_2(t), \ldots, y_m^{(r_m-1)}(t)\big),\ \ t \in I.$$

Lastly, setting $\tilde{\mathcal{D}}_{r-1} := \tilde{\mathcal{D}}_{1,1} \times \cdots \times \tilde{\mathcal{D}}_{m,r_m-1}$ and for $\nu \in \mathbb{N}$

$$f(d, \eta, u) := (f_1(d, \eta, u), \ldots, f_m(d, \eta, u))^\top \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^\nu \times \mathbb{R}^m; \mathbb{R}^m), \tag{4.36}$$

we define the function

$$F : \tilde{\mathcal{D}}_{r-1} \times \mathbb{R}^\nu \to \mathbb{R}^n$$
$$(t, \xi_{1,0}, \dots, \xi_{1,r_1-1}, \xi_{2,0}, \dots, \xi_{m,r_m-1}, \eta) \mapsto \begin{pmatrix} \xi_{1,1} \\ \vdots \\ \xi_{m,r_m-1} \\ f\big(d(t), \eta, N_r(t) \cdot \tilde{e}_r\big) \end{pmatrix},$$

where in $\tilde{e}_r = \tilde{e}_r(t, \xi_{1,0}, \dots, \xi_{1,r_1-1}, \xi_{2,0}, \dots, \xi_{m,r_m-1})$ we omit the arguments for the sake of better legibility. Together, the initial value problem (4.28), (4.32) is equivalent to (4.35). In particular, we have $(0, x(0)) \in \tilde{\mathcal{D}}_{r-1}$, the function F is measurable in the variable t, continuous in $(\xi_{1,0}, \dots, \xi_{m,r_m-1}, \eta)$ and locally essentially bounded. Hence, invoking Remark 1.8, Proposition 1.9 yields the existence of a maximal solution $x : [-\sigma, \omega) \to \mathbb{R}^n$ of (4.35), $0 < \omega \le T$. In particular, the graph of the solution of (4.35) is not a compact subset of $\tilde{\mathcal{D}}_{r-1}$.

Step two. For $j = 1, \dots, m$ we show for the functions $e_{j,k}$ introduced in (4.31a) that for all $k = 1, \dots, r_j - 1$ there exists $\varepsilon_{j,k} \in (0, 1/\sqrt{m})$ such that $|e_{j,k}(t)| \le \varepsilon_{j,k}$ for all $t \in [0, \omega)$. We observe that for $t \in [0, \omega)$ and $k = 1, \dots, r_j$ we have

$$e_{j,k}(t) - \varphi_j(t) \sum_{i=1}^{k-1} \gamma_{j,i}^{(k-1-i)}(t) = \varphi_j(t) e_j^{(k-1)}(t).$$

With this, and using $\dot{\varphi}_j(t) = c_j \varphi_j(t)^2$, we calculate for $k = 1, \dots, r_j - 1$

$$\begin{aligned} \dot{e}_{j,k}(t) &= \dot{\varphi}_j(t) e_j^{(k-1)}(t) + \varphi_j(t) e_j^{(k)}(t) + \dot{\varphi}_j(t) \sum_{i=1}^{k-1} \gamma_{j,i}^{(k-1-i)}(t) + \varphi_j(t) \sum_{i=1}^{k-1} \gamma_{j,i}^{(k-i)}(t) \\ &= \frac{\dot{\varphi}_j(t)}{\varphi_j(t)} \Big(e_{j,k}(t) - \varphi_j(t) \sum_{i=1}^{k-1} \gamma_{j,i}^{(k-1-i)}(t) \Big) + \Big(e_{j,k+1}(t) - \varphi_j(t) \sum_{i=1}^{k} \gamma_{j,i}^{(k-i)}(t) \Big) \\ &\quad + \dot{\varphi}_j(t) \sum_{i=1}^{k-1} \gamma_{j,i}^{(k-1-i)}(t) + \varphi_j(t) \sum_{i=1}^{k-1} \gamma_{j,i}^{(k-i)}(t) \\ &= (c_j - \alpha_{j,k}(t)) \varphi_j(t) e_{j,k}(t) + e_{j,k+1}(t), \\ \dot{e}_{j,r_j}(t) &= c_j \varphi_j(t) e_{j,r_j}(t) + \varphi_j(t) e^{(r_j)}(t) + \varphi_j(t) \sum_{i=1}^{r-1} \gamma_{j,i}^{(r-i)}(t). \end{aligned} \tag{4.37}$$

Further, using the definitions of $\alpha_{j,k}$ and $\gamma_{j,k}$, we record for later use

$$\begin{aligned} \dot{\gamma}_{j,k}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t} (\alpha_{j,k}(t) e_{j,k}(t)) \\ &= 2\alpha_j'(e_{j,k}(t)^2)\, e_{j,k}(t)^2\, \dot{e}_{j,k}(t) + \alpha_{j,k}(t) \dot{e}_{j,k}(t). \end{aligned} \tag{4.38}$$

We observe $e_{j,k} = \tilde{e}_{j,k}(y_j, \dots, y_j^{(k-1)})$. Therefore, since $\tilde{e}_{j,k}(t) \in \mathcal{D}_0$ for $t \in [0, \omega)$ due to the initial conditions (4.34), we have

$$\forall\, j = 1, \dots, m \ \forall\, k = 1, \dots, r_j \ \forall\, t \in [0, \omega) \ : \ |e_{j,k}(t)| < \frac{1}{\sqrt{m}}.$$

For $j = 1, \dots, m$ we set

$$\hat{\varepsilon}_{j,k} := |e_{j,k}(0)|^2 < \frac{1}{m} \quad \text{and} \quad \lambda_j := \inf_{s \in [0,T)} \varphi_j(s) > 0.$$

Let ε_j be the unique point in $(0, \frac{1}{m})$ such that $\alpha_j(\varepsilon_j)\varepsilon_j = (1 + c_j\lambda_j)/\lambda_j$ and define $\varepsilon_{j,k} := \max\{\varepsilon_j, \hat{\varepsilon}_{j,k}\} < \frac{1}{m}$. We show that

$$\forall\, j = 1, \dots, m \;\forall\, k = 1, \dots, r_j - 1 \;\forall\, t \in [0, \omega) \;:\; e_{j,k}(t)^2 \le \varepsilon_{j,k}. \tag{4.39}$$

Seeking a contradiction, we suppose (4.39) is false for at least one $\hat{\imath} \in \{1, \dots, m\}$ and at least one $\ell \in \{1, \dots, r_{\hat{\imath}} - 1\}$. Then $e_{\hat{\imath},\ell}(t_1)^2 > \varepsilon_{\hat{\imath},\ell}$ for some $t_1 \in (0, \omega)$ and we define

$$t_0 := \max \left\{ t \in [0, t_1) \;\middle|\; e_{\hat{\imath},\ell}(t)^2 = \varepsilon_{\hat{\imath},\ell} \right\}.$$

With this we have

$$\forall\, t \in [t_0, t_1] \;:\; \varepsilon_{\hat{\imath}} \le \varepsilon_{\hat{\imath},\ell} \le e_{\hat{\imath},\ell}(t)^2,$$

which gives, invoking monotonicity of the bijection $\alpha_{\hat{\imath}}$, the following relation

$$\forall\, t \in [t_0, t_1] \;:\; \alpha_{\hat{\imath}}(\varepsilon_{\hat{\imath}}) \le \alpha_{\hat{\imath}}(e_{\hat{\imath},\ell}(t)^2) = \alpha_{\hat{\imath},\ell}(t).$$

Hence,

$$\forall\, t \in [t_0, t_1] \;:\; \alpha_{\hat{\imath},\ell}(t) e_{\hat{\imath},\ell}(t)^2 \ge \alpha_{\hat{\imath}}(\varepsilon_{\hat{\imath}})\varepsilon_{\hat{\imath}} = \frac{1 + c_{\hat{\imath}}\lambda_{\hat{\imath}}}{\lambda_{\hat{\imath}}}.$$

Using $\alpha_{\hat{\imath},\ell} \ge c_{\hat{\imath}}$ via (4.30c), and the relations in (4.37), we calculate for $t \in [t_0, t_1]$

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2} e_{\hat{\imath},\ell}(t)^2 &= e_{\hat{\imath},\ell}(t)\Big((c_{\hat{\imath}} - \alpha_{\hat{\imath},\ell}(t))\varphi_{\hat{\imath}}(t) e_{\hat{\imath},\ell}(t) + e_{\hat{\imath},\ell+1}(t) \Big) \\
&= -\varphi_{\hat{\imath}}(t)(\alpha_{\hat{\imath},\ell}(t) - c_{\hat{\imath}}) e_{\hat{\imath},\ell}(t)^2 + e_{\hat{\imath},\ell}(t) e_{\hat{\imath},\ell+1}(t) \\
&< -\varphi_{\hat{\imath}}(t)(\alpha_{\hat{\imath},\ell}(t) - c_{\hat{\imath}}) e_{\hat{\imath},\ell}(t)^2 + 1 \\
&\le -\lambda_{\hat{\imath}}(\alpha_{\hat{\imath},\ell}(t) - c_{\hat{\imath}}) e_{\hat{\imath},\ell}(t)^2 + 1 \\
&< -\lambda_{\hat{\imath}}\alpha_{\hat{\imath},\ell}(t) e_{\hat{\imath},\ell}(t)^2 + c_{\hat{\imath}}\lambda_{\hat{\imath}} + 1 \le 0,
\end{aligned}$$

which implies the contradiction

$$\varepsilon_{\hat{\imath},\ell} < e_{\hat{\imath},\ell}(t_1)^2 < e_{\hat{\imath},\ell}(t_0)^2 = \varepsilon_{\hat{\imath},\ell}.$$

Therefore, (4.39) is true. This implies boundedness of $\alpha_{j,k}$ (bounded by $\alpha(\varepsilon_{j,k})$) and boundedness of $\gamma_{j,k}$ (bounded by $\alpha(\varepsilon_{j,k})\sqrt{\varepsilon_{j,k}}$) for all $j = 1, \dots, m$ and all $k = 1, \dots, r_j - 1$.

Step three. In this step we derive some preparatory estimations to conclude boundedness of the solution x of (4.35) in the next step. Since the functions $e_{j,k}$ defined in (4.31a) involve higher derivatives of the functions $\gamma_{j,k}$ we aim to show boundedness of the latter. In order to do so, recalling the definition of $\gamma_{j,k}$ we establish boundedness of higher derivatives of $\alpha_{j,k}$ on $[0, \omega)$, which in turn involve higher derivatives of $e_{j,k}$. For this reason, we show boundedness of higher the derivatives of $e_{j,k}$ on $[0, \omega)$; more precisely, we show boundedness of $e_{j,k}^{(r_j - k)}$ on $[0, \omega)$ for $k = 1, \dots, r_j - 1$, where $j = 1, \dots, m$ as before. Recalling the definition of the funnel functions φ_j in (4.30a) we have the relation $\varphi_j^{(q)}(t) = c_j^q q! \varphi_j(t)^{q+1}$ for $q \in \mathbb{N}$. Using the generalized Leibniz rule, we obtain via (4.37) for $j = 1, \dots, m$, $k = 1, \dots, r_j - 1$ and $1 \le q \le r_j - k$ the formula

$$\begin{aligned}
e_{j,k}^{(q)}(t) &= \big((c_j - \alpha_{j,k}(t))\varphi_j(t) e_{j,k}(t) \big)^{(q-1)} + e_{j,k+1}^{(q-1)}(t) \\
&= \sum_{q_1+q_2+q_3=q-1} \frac{(q-1)!}{q_1! q_2! q_3!} (c_j - \alpha_{j,k}(t))^{(q_1)} \varphi_j^{(q_2)}(t) e_{j,k}^{(q_3)}(t) + e_{j,k+1}^{(q-1)}(t) \\
&= \sum_{q_1+q_2+q_3=q-1} \frac{(q-1)!}{q_1! q_3!} (c_j - \alpha_{j,k}(t))^{(q_1)} c_j^{q_2} \varphi_j(t)^{q_2+1} e_{j,k}^{(q_3)}(t) + e_{j,k+1}^{(q-1)}(t),
\end{aligned} \tag{4.40}$$

which gives the derivatives recursively. To have an example, we calculate the expression above for $q = 2 \le r_j - k$, where we use the expressions from (4.37)

$$\begin{aligned}\ddot{e}_{j,k}(t) = \big(&-2\alpha_j'(e_{j,k}(t)^2)e_{j,k}(t)\dot{e}_{j,k}(t)\big)\varphi_j(t)e_{j,k}(t)\\ &+ (c_j - \alpha_{j,k}(t))c_j\varphi_j^2 e_{j,k}(t)\\ &+ (c_j - \alpha_{j,k}(t))\varphi_j(t)\Big((c_j - \alpha_{j,k}(t))\varphi_j(t)e_{j,k}(t) + e_{j,k+1}(t)\Big)\\ &+ (c_j - \alpha_{j,k+1}(t))\varphi_j(t)e_{j,k+1}(t) + e_{j,k+2}(t).\end{aligned}$$

The recursion (4.40) successively leads to the following observations. Since the numbers q_1, q_2, q_3 satisfy the relation $q_1 + q_2 + q_3 = q - 1$ we have

(i) for $q_1 = 0$ the expression $\varphi_j^{q_2+1}e_{j,k}^{(q_3)}$ involves at most the $q-1^{\text{st}}$ derivative of $e_{j,k}$, and at most the q^{th} power of φ_j; the other terms involve (at most) derivatives and powers of the form $\varphi_j^{q_2+1}e_{j,k}^{(q-1-q_2)}$ for $q \le r_j - k$,

(ii) $e_{j,k}^{(q)}$ involves $e_{j,k+1}^{(q-1)}$ which itself involves $e_{j,k+2}^{(q-2)}$ and so forth; therefore the expression $e_{j,k}^{(q)}$ involves the term $e_{j,k+q}$,

(iii) the highest derivative of the bijection $\alpha_{j,k}$ appearing in $e_{j,k}^{(q)}$ is $\alpha_{j,k}^{(q-1)}$, which itself involves at most the $q - 1^{\text{st}}$ derivative of $e_{j,k}$.

These observations together with the fact that

$$\forall\, q \in \mathbb{N}:\ \varphi_j^q e_{j,k} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}) \Rightarrow \varphi_j^{q-1}e_{j,k} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}) \tag{4.41}$$

yield that boundedness of $e_{j,k}^{(q)}$ on $[0,\omega)$, $q \le r_j - k$, can be established by showing boundedness of $\varphi_j^{r_j-k}e_{j,k}$ for $j = 1,\dots,m$ and all $k = 1,\dots,r_j-1$. In order to show this, we initially establish the following: for all $k = 1,\dots,r_j - 1$ we have

$$\varphi_j^{r_j-k-1}e_{j,k+1} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}) \Rightarrow \varphi_j^{r_j-k}e_{j,k} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}). \tag{4.42}$$

This can be seen as follows. We assume $\varphi_j^{r_j-k-1}e_{j,k+1} \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$ and define the non-negative finite constant $M_{j,k+1} := \sup_{s\in[0,\omega)} |\varphi_j(s)^{r_j-k-1}e_{j,k+1}(s)| < \infty$. Then, invoking (4.30a), (4.30c) and (4.37), for $t \in [0,\omega)$ we calculate

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\big(\varphi_j(t)^{r_j-k}e_{j,k}(t)\big)^2 &= \varphi_j(t)^{r_j-k}e_{j,k}(t)\varphi_j(t)^{r_j-k}\dot{e}_{j,k}(t)\\ &\qquad + \varphi_j(t)^{r_j-k}e_{j,k}(t)c_j(r_j-k)\varphi_j(t)^{r_j-k+1}e_{j,k}(t)\\ &= \varphi_j(t)^{r_j-k}e_{j,k}(t)\varphi_j(t)^{r_j-k}\Big(\big(c_j-\alpha_{j,k}(t)\big)\varphi_j(t)e_{j,k}(t)+e_{j,k+1}(t)\Big)\\ &\qquad + \varphi_j(t)^{r_j-k}e_{j,k}(t)c_j(r_j-k)\varphi_j(t)^{r_j-k+1}e_{j,k}(t)\\ &\le -\varphi_j(t)\big(c_j(r_j+1) - c_j(r_j-k+1)\big)\Big(\varphi_j(t)^{r_j-k}e_{j,k}(t)\Big)^2\\ &\qquad + \varphi_j(t)M_{j,k+1}\,|\varphi_j(t)^{r_j-k}e_{j,k}(t)|\\ = -\varphi_j(t)\Big(c_jk\,|\varphi_j(t)^{r_j-k}e_{j,k}(t)| - M_{j,k+1}\Big)&|\varphi_j(t)^{r_j-k}e_{j,k}(t)|\end{aligned} \tag{4.43}$$

which is non-positive for $|\varphi_j(t)^{r_j-k} e_{j,k}(t)| \geq \frac{M_{j,k}}{c_j k} \geq 0$ and hence Lemma 1.11 yields boundedness of $\varphi_j^{r_j-k+1} e_{j,k-1}$ on $[0,\omega)$ as claimed in (4.42). Then, invoking (4.39), a successive application of (4.42) implies

$$\forall\, j = 1,\dots,m \;\forall\, k = 1,\dots,r_j - 1 \,:\; \varphi_j^{r_j-k} e_{j,k} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}). \tag{4.44}$$

In particular, $\varphi_j e_{j,k} \in \mathcal{L}^\infty([0,\omega);\mathbb{R})$ for all $j = 1,\dots,m$ and all $k = 1,\dots,r_j - 1$. Then, boundedness of $\dot e_{j,k}$ follows, which in turn implies boundedness of $\dot\alpha_{j,k}$ and $\dot\gamma_{j,k}$ on $[0,\omega)$. Now, via (4.40), (4.41) and (4.42) boundedness of $e_{j,k}^{(q)}$ successively follows for all $q \leq r_j - k$, from which we may deduce boundedness of $\alpha_{j,k}^{(q)}$ and $\gamma_{j,k}^{(q)}$ for $q \leq r_j - k$. Therefore, for $j = 1,\dots,m$, all $k = 1,\dots,r_j - 1$ and $q \leq r_j - k$ there exists

$$\bar\gamma_{j,k}^q := \sup_{s\in[0,\omega)} \gamma_{j,k}^{(q)}(s) < \infty.$$

Step four. We show boundedness of the solution x of (4.35) on $[0,\omega)$. Recalling the definition of $e_{j,k}$ we see that for $j = 1,\dots,m$ and all $k = 1,\dots,r_j$ we have via the previous steps

$$\begin{aligned}\forall\, t \in [0,\omega) \,:\; |e_j^{(k-1)}(t)| &\leq \left|\frac{e_{j,k}(t)}{\varphi_j(t)}\right| + \left|\sum_{i=1}^{k-1} \gamma_i^{(k-1-i)}(t)\right| \\ &\leq \frac{1}{\lambda_j} + \sum_{i=1}^{k-1} \bar\gamma_{j,i}^{k-1-i} < \infty.\end{aligned} \tag{4.45}$$

We recall

$$\begin{aligned} x(t)^\top &= (y_1(t), \dot y_1(t), \dots, y_1^{(r_1-1)}(t), y_2(t), \dots, y_m^{(r_m-1)}(t)) \\ &= \left(e_1(t) + y_{1,\mathrm{ref}}(t), \dot e_1(t) + \dot y_{1,\mathrm{ref}}(t), \dots, e_m^{(r_m-1)}(t) + y_{m,\mathrm{ref}}(t)\right), \end{aligned}$$

where by assumption $y_{j,\mathrm{ref}} \in \mathcal{W}^{r_j,\infty}([0,T);\mathbb{R})$. Therefore, we have $x \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^n)$.

Step five. We recall $e_r = (e_{1,r_1},\dots,e_{m,r_m})^\top : [0,\omega) \to \mathbb{R}^m$ and show boundedness of $\beta_r(t) := \beta(\|e_r(t)\|^2)$ for $t \in [0,\omega)$. Invoking the previous steps, in particular boundedness of x on $[0,\omega)$, and the properties of the operator class $\mathcal{T}_\sigma^{n,q}$ we deduce the existence of a compact $K_q \subset \mathbb{R}^q$ such that $\mathbf{T}(x)(t) \in K_q$ for $t \in [0,\omega)$. Furthermore, since $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^p)$ there exists a compact $K_p \subset \mathbb{R}^p$ such that $d(t) \in K_p$ for $t \in [0,\omega)$. Since the function f defined in (4.36) satisfies the high-gain property, there exists $v_* \in (0,1)$ such that for the compact set $V := \{\, v \in \mathbb{R}^m \mid v_* \leq \|v\| \leq 1 \,\}$ the continuous function

$$\begin{aligned} \chi : \mathbb{R} &\to \mathbb{R}, \\ s &\mapsto \min\{\, \langle v, f(\delta,\eta,-sv)\rangle \mid (\delta,\eta,v) \in K_p \times K_q \times V \,\} \end{aligned}$$

is unbounded from above. We show boundedness of β_r by contradiction. Since $N : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is surjective, the set $\{\, \kappa > \rho_0 \mid N(\kappa) = \rho_1 \,\}$ is non-empty for every $\rho_0 \in \mathbb{R}_{\geq 0}$ and every $\rho_1 \in \mathbb{R}$. Following the proof in [21, pp. 188-190], we choose a real sequence (s_i) such that the corresponding sequence $(\chi(s_i))$ is positive, strictly increasing and in particular unbounded. We initialize a sequence (κ_i) by choosing $\kappa_1 > \beta(v_*^2) + \beta_r(0)$ such that $N(\kappa_1) = s_1$, and hereinafter define the strictly increasing sequence (κ_i) by

$$\kappa_{i+1} := \inf\{\, \kappa > \kappa_i \mid N(\kappa) = s_{i+1} \,\},$$

which obviously yields that $\lim_{i\to\infty} \chi(N(\kappa_i)) = \lim_{i\to\infty} \chi(s_i) = \infty$. Now, since we assumed β_r to be unbounded and we have $\kappa_{i+1} > \kappa_1 > \beta_r(0)$ for all $i \in \mathbb{N}$, we may define the sequence

$$\tau_i := \inf \{ t \in [0, \omega) \mid \beta_r(t) = \kappa_{i+1} \}, \ i \in \mathbb{N}_0,$$

which lies within $(0, \omega)$. Note that (τ_i) is strictly increasing. Inserting the relations from above, we have $N(\beta_r(\tau_i)) = N(\kappa_{i+1}) = s_{i+1}$ for each $i \in \mathbb{N}_0$. We define a second sequence in $(0, \omega)$ by

$$\sigma_i = \sup \{ t \in [\tau_{i-1}, \tau_i] \mid \chi(N(\beta_r(t))) = \chi(s_i) \}, \ i \in \mathbb{N}.$$

With this, since the sequence $(\chi(s_i))$ is strictly increasing, we obtain for all $i \in \mathbb{N}$

$$\chi(N(\beta_r(\sigma_i))) = \chi(s_i) < \chi(s_{i+1}) = \chi(N(\beta_r(\tau_i))),$$

and therefore,

$$\forall\, i \in \mathbb{N} \, : \, \sigma_i < \tau_i, \tag{4.46a}$$

as well as

$$\forall\, i \in \mathbb{N} \, \forall\, t \in (\sigma_i, \tau_i] \, : \, \chi(N(\beta_r(\sigma_i))) = \chi(s_i) < \chi(N(\beta_r(t))). \tag{4.46b}$$

As an auxiliary intermediate result, we show by contradiction

$$\forall\, i \in \mathbb{N} \, \forall\, t \in [\sigma_i, \tau_i] \, : \, e_r(t) \in V.$$

To this end, we first show

$$\forall\, i \in \mathbb{N} \, \forall\, t \in [\sigma_i, \tau_i] \, : \, \beta_r(t) \geq \kappa_i,$$

by contradiction as well. Suppose that $\beta_r(t) < \kappa_i$ for some $t \in [\sigma_i, \tau_i]$. Then, by $\beta_r(\tau_i) = \kappa_{i+1} > \kappa_i$ and by continuity of β_r there exists $\tilde{t} \in (\sigma_i, \tau_i)$ such that $\beta_r(\tilde{t}) = \kappa_i$. Hence, we find $\chi(N(\beta_r(\tilde{t}))) = \chi(N(\kappa_i)) = \chi(s_i)$, which contradicts the definition of σ_i. Therefore, $\beta_r(t) \geq \kappa_i$ for all $t \in [\sigma_i, \tau_i]$. Now, suppose $e_r(t) \notin V$. Since for all $t \in [0, \omega)$ we have $\|e_r(t)\| < 1$, this means to suppose $\|e_r(t)\| < v_*$ for some $[\sigma_i, \tau_i]$. This, together with $\beta_r(t) \geq \kappa_i$, leads to the contradiction

$$\forall\, i \in \mathbb{N} \, : \, \beta(v_*^2) < \kappa_1 \leq \kappa_i \leq \beta_r(t) = \beta(\|e_r(t)\|^2) < \beta(v_*^2).$$

Hence, we deduce

$$\forall\, i \in \mathbb{N} \, \forall\, t \in [\sigma_i, \tau_i] \, : \, e_r(t) \in V.$$

Since $d(t) \in K_p$ and $\mathbf{T}(x)(t) \in K_q$ for $t \in [0, \omega)$ we obtain, using (4.46), for all $i \in \mathbb{N}$ and $t \in [\sigma_i, \tau_i]$ the following estimation

$$\begin{aligned} \langle e_r(t), f\big(d(t), \mathbf{T}(x)(t), u(t)\big)\rangle &= -\langle -e_r(t), f\big(d(t), \mathbf{T}(x)(t), -N(\beta_r(t))(-e_r(t))\rangle \\ &\leq -\min \left\{ \langle v, f(\delta, z, -N(\beta_r(t))v\rangle \;\middle|\; \begin{array}{l} (\delta, z, v) \in \\ K_p \times K_q \times V \end{array} \right\} \\ &= -\chi(N(\beta_r(t))) \\ &\leq -\chi(s_i). \end{aligned} \tag{4.47}$$

Since for $j = 1, \ldots, m$ we have $y_{j,\mathrm{ref}} \in \mathcal{W}^{r_j,\infty}([0,T);\mathbb{R})$ we may define the bounded constant $c_{\mathrm{ref}} := \max_{j\in\{1,\ldots,m\}} \sup_{s\geq 0} \|y_{\mathrm{ref}}^{(r_j)}(s)\| < \infty$, and we recall $\sum_{i=1}^{r_j-1} \bar{\gamma}_{j,i}^{r_j-i} < \infty$ from the previous steps. Furthermore, we observe $\sigma_1 > 0$ and therefore, by properties of the funnel functions φ_j we may define $0 < \min_{j\in\{1,\ldots,m\}} \inf_{s\in[\sigma_1,T)} \varphi_j(s) =: c_\varphi$. We set $\Phi(t) = \mathrm{diag}(\varphi_1(t), \ldots, \varphi_m(t))$ and thus by (4.30a) we have $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = C\Phi(t)^2$, where $C = \mathrm{diag}(c_1, \ldots, c_m)$. Then, with the aid of (4.37) and (4.47), we obtain for all $i \in \mathbb{N}$ and $t \in [\sigma_i, \tau_i]$

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|e_r(t)\|^2 &= \langle e_r(t), C\Phi(t)e_r(t)\rangle + \Big\langle e_r(t), \Phi(t)\big(f(d(t),\mathbf{T}(x)(t),u(t)) - y_{\mathrm{ref}}^{(r)}(t)\big)\Big\rangle \\ &\quad + \sum_{j=1}^{m}\left(e_{j,r}(t)\varphi_j(t)\sum_{\ell=1}^{r_j-1}\gamma_{j,\ell}^{(r_j-\ell)}(t)\right) \\ &< \min_{j\in\{1,\ldots,m\}}\varphi_j(t)\Big(\min_{j\in\{1,\ldots,m\}} c_j + c_{\mathrm{ref}} + \sum_{j=1}^{m}\sum_{i=\ell}^{r_j-1}\bar{\gamma}_{j,\ell}^{r_j-i} - \chi(s_i)\Big).\end{aligned}$$

Therefore, still seeking a contradiction, we may choose $J \in \mathbb{N}$ large enough such that for $t \in [\sigma_J, \tau_J]$ we have

$$\begin{aligned}&\min_{j\in\{1,\ldots,m\}}\varphi_j(t)\Big(\min_{j\in\{1,\ldots,m\}} c_j + c_{\mathrm{ref}} + \sum_{j=1}^{m}\sum_{i=1}^{r_j-1}\bar{\gamma}_{j,i}^{r_j-i} - \chi(s_J)\Big) \\ &\qquad \leq -c_\varphi\Big(\chi(s_J) - \Big(\min_{j\in\{1,\ldots,m\}} c_j + c_{\mathrm{ref}} + \sum_{j=1}^{m}\sum_{i=1}^{r_j-1}\bar{\gamma}_{j,i}^{r_j-i}\Big)\Big) < 0,\end{aligned}$$

which yields $\|e_r(\tau_J)\|^2 < \|e_r(\sigma_J)\|^2$, which in turn gives for $t \in [\sigma_J, \tau_J]$

$$\beta_r(\tau_J) = \beta(\|e_r(\tau_J)\|^2) < \beta(\|e_r(\sigma_J)\|^2) = \beta_r(\sigma_J).$$

This, however, contradicts the definition of τ_J, by which we have $\beta_r(t) < \beta_r(\tau_J)$ for all $t \in [0, \tau_J)$. Therefore, the assumption of an unbounded β_r cannot be true. As a direct consequence thereof, we may infer the existence of $\varepsilon_r \in (0,1)$ such that

$$\forall\, t \in [0,\omega) \,:\, \|e_r(t)\|^2 \leq \varepsilon_r.$$

Step six. We show $\omega = T$. Via the previous steps we have for all $j = 1, \ldots, m$ and $k = 1, \ldots, r_j$

$$\forall\, t \in [0,\omega) \,:\, |e_{j,k}(t)| \leq \varepsilon^* := \sqrt{\max\{\varepsilon_{1,1}, \ldots, \varepsilon_{m,r_m}\}} < \frac{1}{\sqrt{m}},$$

and

$$\forall\, t \in [0,\omega) \,:\, \|e_r(t)\| \leq \varepsilon_r,$$

by which the set

$$\hat{\mathcal{D}} := \left\{ (\zeta_{1,1}, \ldots, \zeta_{m,r_m}) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} j = 1, \ldots, m, i = 1, \ldots, r_j - 1, \\ |\zeta_{j,i}| \leq \varepsilon^*, \zeta_r := (\zeta_{1,r_1}, \ldots, \zeta_{m,r_m}),\ \|\zeta_r\| \leq \varepsilon_r, \end{array} \right\}$$

is a compact subset of $\tilde{\mathcal{D}}_{r-1}$. Assume $\omega < T$. Then,

$$\forall\, t \in [0,\omega) \,:\, x(t) \in \hat{\mathcal{D}} \subset \tilde{\mathcal{D}}_{r-1}.$$

So, by compactness of $\hat{\mathcal{D}}$ the closure of the graph of the solution x of (4.35) on $[0,\omega)$ is a compact subset of $\tilde{\mathcal{D}}_{r-1}$ which contradicts the findings of *Step one*. Thus, $\omega = T$.

Step seven. We show assertions (ii) & (iii). Assertion (ii) follows immediately from *Step four* and *Step six*, and assertion (iii) is a direct consequence of *Step two* and *Step six*.

Step eight. We show that the tracking error $e = y - y_{\text{ref}}$ and its derivatives tend to zero as $t \to T$, that is, we show

$$\forall\, j = 1,\ldots,m\ \forall\, k = 1,\ldots,r_j\ :\ \lim_{t\to T} |e_j^{(k-1)}(t)| = 0. \tag{4.48}$$

We note that the estimation in (4.45) is too rough to show (4.48). For $j = 1,\ldots,m$ recalling the definition (4.31c) of $\gamma_{j,k} = \alpha_{j,k} e_{j,k}$ and exemplary its derivative (4.38), we see that by *Step three* not only $\gamma_{j,k}^{(q)}$ is bounded on $[0,\omega)$ for $q \le r_j - k - 1$ but with the aid of (4.44) even the product $\varphi_j\, \gamma_{j,k}^{(q)}$ is bounded on $[0,\omega)$, i.e., for all $\ell = 1\ldots, r_j - 1$ there exists $\hat{\gamma}_{j,\ell}^{q} := \sup_{s\in[0,\omega)} \varphi_j(s)\gamma_{j,\ell}^{(q)}(s) < \infty$ for $0 \le q \le r_j - \ell - 1$. Invoking *Step two* and *Step five* we may improve estimation (4.45) for $j = 1,\ldots,m$ and $k = 1,\ldots,r_j$ for $t \in [0,\omega)$ as follows

$$\begin{aligned}
|e_j^{(k-1)}(t)| &\le \frac{|e_{j,k}(t)|}{\varphi_j(t)} + \frac{1}{\varphi_j(t)} \left| \sum_{i=1}^{k-1} \varphi_j(t)\gamma_{j,i}^{(k-1-i)}(t) \right| \\
&\le \frac{\sqrt{\varepsilon_{j,k}} + \sum_{i=1}^{k-1} \hat{\gamma}_{j,i}^{k-1-i}}{\varphi_j(t)}.
\end{aligned}$$

From this, since $\omega = T$ by *Step six*, and $\lim_{t\to T} \varphi_j(t) = \infty$ we obtain (4.48) for all $k = 1,\ldots,r_j$ with $j = 1,\ldots,m$, which shows assertion (iv) of the theorem and completes the proof. □

Remark 4.21. Assertion (i) in Theorem 4.20, namely $[0,T)$ being the maximal solution interval, naturally raises the question of a global solution in time.

(i) If the system's equations (4.28) are available and y_{ref} is defined on $\mathbb{R}_{\ge 0}$, the application of a suitable feedforward control scheme may achieve $y(t) - y_{\text{ref}}(t) \equiv 0$ for all $t \ge T$.

(ii) If equations (4.28) are available and the reference y_{ref} is defined on $\mathbb{R}_{\ge 0}$, then $y(t) - y_{\text{ref}}(t) \equiv 0$ for all time $t \ge T$ can be achieved by asking the reference to satisfy (4.28) for $t \ge T$, with no control input, i.e., $u \equiv 0$, and "initial conditions" $y_{j,\text{ref}}(T) = y_j(T), \dot{y}_{j,\text{ref}}(T) = \dot{y}_j(T), \ldots, y_{j,\text{ref}}^{(r_j-1)}(T) = y_j^{(r_j-1)}(T)$, for $j = 1,\ldots,m$.

(iii) If the system's equations are not available, the application of a "standard" funnel control scheme, for instance as in [21], for $t \ge T$ achieves that the error $y(t) - y_{\text{ref}}(t)$ evolves within an arbitrary small neighbourhood of zero for all $t \ge T$.

Remark 4.22. We highlight the following aspect of Theorem 4.20 which is of interest from a practical point of view. For any given arbitrary small $\varepsilon > 0$ there exists a time $T_\varepsilon < T$ such that each of the first $r_j - 1$ derivatives of the error $e_j = y_j - y_{j,\text{ref}}$

is smaller than ε for all $t \in [T_\varepsilon, T)$, i.e., for all $k = 0, \ldots, r_j - 1$, $j = 1, \ldots, m$, we have

$$\forall\, \varepsilon > 0\ \exists\, T_\varepsilon < T\ \forall t \in [T_\varepsilon, T) \;:\; |e_j^{(k)}(t)| \le \varepsilon.$$

This property is relevant, for instance, if during a docking manoeuvre the demanded accuracy at $t = T$ changes. In this situation, a standard funnel control scheme would not be able to guarantee that the error is within an arbitrary small neighbourhood of zero at $t = T$, since the funnel function is chosen in advance, and so the change of demanded accuracy cannot be taken into account.

Remark 4.23. We record the following interconnection of the results found in this chapter with results in [123] and [21].

(i) If a system (4.28) has *strict relative degree* $r = r_1 = \ldots = r_m \in \mathbb{N}$, i.e., it is of the form

$$\begin{aligned} y^{(r)}(t) &= f\left(d(t), \mathbf{T}(y, \ldots, y^{(r-1)})(t), u(t)\right), \\ y|_{[-\tau,0]} &= y^0 \in \mathcal{C}^{r-1}([-\tau, 0]; \mathbb{R}^m), \end{aligned}$$

then the system class is the same as in [21]. In this case, the error variables $e_{j,k}$ are collected in respective vectors e_k, the initial condition simplifies to the well known condition $\|e_k(0)\| < 1$ for all $k = 1, \ldots, r$, and it suffices to choose one bijection $\alpha : [0, 1) \to [c(r + 1), \infty)$, which also can replace β.

(ii) In the case of strict relative degree one, the control scheme (4.32) coincides with the controller proposed in [123].

(iii) Combining the findings in the proof of Theorem 4.20 with the result found in [21, Thm. 1.9], it is clear that the controller from [21] can be straightforwardly extended to the case of vector relative degree.

Example 4.24. We simulate an application of the control scheme (4.32) to a system (4.28) with strict relative degree $r = r_1 = \cdots = r_m$ and $(d, f, \mathbf{T}) \in \mathcal{N}^{m,r}$. As indicated in the preliminary text, docking of spaceship is a red-hot application of the findings of the present section. Therefore, we look at such a situation as an example. Consider a passive target space station, e.g., the international space station ISS, in a circular orbit around the earth and an active spacecraft chasing the first. For simulation purposes, we assume the passive space station to be on a constant altitude r_s with constant angular velocity $\omega = \sqrt{\mu/(r_e + r_s)^3}$, where $\mu \approx 3.986 \cdot 10^{14}\,\mathrm{m}^3/\mathrm{s}^2$ is the standard gravitational parameter and $r_e = 6378137\,\mathrm{m}$ the radius of the earth. To analyse the motion of the spacecraft, we use Hill's local-vertical-local-horizontal coordinate frame introduced in [81], see Figure 4.8.

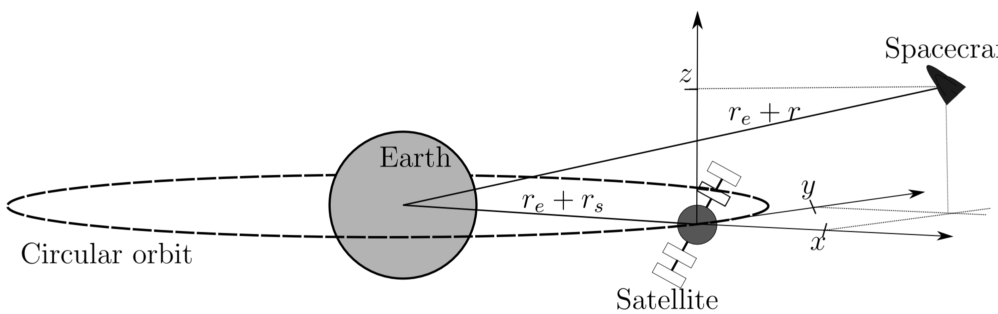


Figure 4.8: Hill's local-vertical-local-horizontal coordinate frame.

Within this frame we use the commonly used Clohessy-Wiltshire model for satellite rendezvous proposed in [51]. A detailed and comprehensive application of this model in the context of spaceship rendezvous can be found in [110]. Let $r(t)$ denote the altitude of the chasing spacecraft at time t. In virtue of Hill's coordinate frame let $x := r - r_s$ be the component of relative distance along the radial direction, y be the downtrack component along satellite's circular orbit, and z be the distance component along the satellite's angular momentum. Then, setting $\zeta = (\zeta_1, \zeta_2, \zeta_3)^\top := (x, y, z)^\top$ we obtain the Clohessy-Wiltshire equations

$$\begin{aligned}
\ddot{\zeta}_1(t) &= 3\omega^2\zeta_1(t) + 2\omega\dot{\zeta}_2(t) + u_x(t),\\
\ddot{\zeta}_2(t) &= -2\omega\dot{\zeta}_1(t) + u_y(t),\\
\ddot{\zeta}_3(t) &= -\omega^2\zeta_3(t) + u_z(t).
\end{aligned}$$

With

$$f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3,\ (\xi, \nu) \mapsto \begin{pmatrix} 3\omega^2\xi_1 + 2\omega\nu_2 \\ -2\omega\nu_1 \\ -\omega^2\xi_3 \end{pmatrix},$$

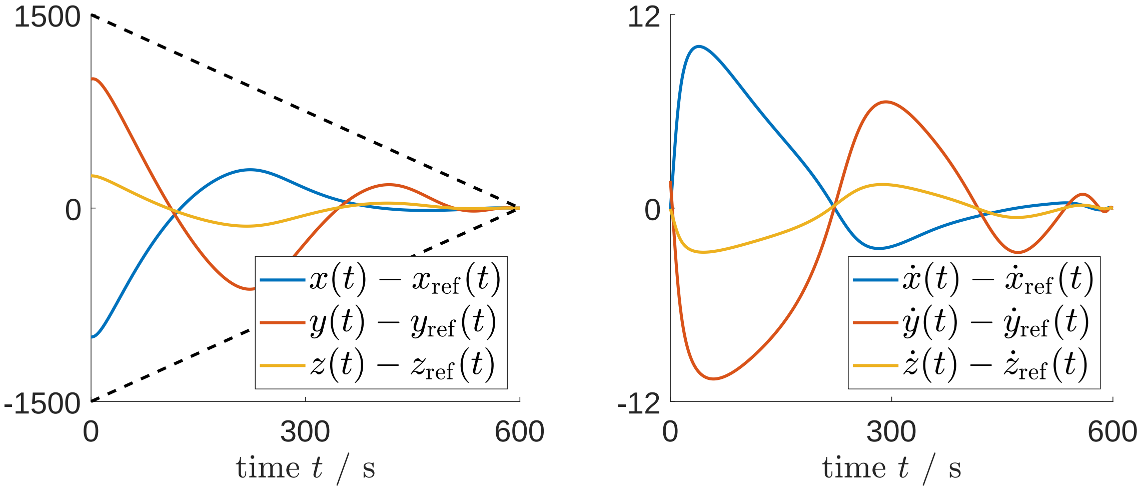
and $u = (u_1, u_2, u_3)^\top := (u_x, u_y, u_z)^\top$, with $B = I_3 \in \mathbb{R}^{3\times3}$ the equations of motion above with output ζ can compactly be written as

$$\ddot{\zeta}(t) = f(\zeta(t), \dot{\zeta}(t)) + Bu(t), \quad \zeta(0) = \zeta_0^0 \in \mathbb{R}^3,\ \dot{\zeta}(0) = \zeta_1^0 \in \mathbb{R}^3,$$

which is a system of relative degree two and belongs to the system class $\mathcal{N}^{3,2}$. For simulation purposes we choose $r_s = 415000\,\text{m}$ (approximately the altitude of the ISS), which yields $\omega \approx 0.00113\,\text{s}^{-1}$ corresponding to an orbital period of approximately 93 minutes. Since we aim to simulate a docking manoeuvre, we choose the reference trajectory $\zeta_{\text{ref}} : t \mapsto (0,0,0)^\top$ in Hill's coordinate frame, which in particular yields $\zeta(T) = (0,0,0)^\top$, i.e., docking at $t = T$. We take the initial conditions as in [110] $x(0) = -y(0) = 1000\,\text{m}$, and additionally $z(0) = 250\,\text{m}$; and $\dot{x}(0) = -0.1\,\text{ms}^{-1}$, $\dot{y}(0) = 1.69\,\text{ms}^{-1}$ and $\dot{z}(0) = -0.05\,\text{ms}^{-1}$. As docking time we choose $T = 600\,\text{s}$, which is docking within ten minutes. For the funnel parameters we choose, $N : s \mapsto -s\cos(10^{-2}\,s)$ and $\alpha : s \mapsto (r+1)c/(1-s)$, where with $c = 1$ the initial conditions (4.34) are satisfied. In accordance with 4.30a we choose $\varphi(t) = 1/(T-t)$ as funnel function. Note that, since $\lim_{t\to T}\varphi(t) = \infty$ simulation is possible only for $[0, t_{\max}]$ with $t_{\max} = T - \mathsf{eps}$ for a predefined $\mathsf{eps} > 0$. Since $\varphi(t)\|e(t)\| < 1$ for all $t \in [0, T)$ we choose eps such that a certain upper bound of the spatial error at final simulation time $t_{\max}$ is guaranteed, i.e.,

$$\|e(t_{\max})\| < \frac{1}{\varphi(t_{\max})} = c(T - t_{\max}) \le \mathsf{eps} \ \Rightarrow\ t_{\max} \ge T - \mathsf{eps}/c.$$

For the simulation we choose $\mathsf{eps} = 10^{-10}\,\text{m}$ which means a spatial accuracy of Ångström (range of size of atoms). This seems to be a unnecessary high accuracy since in real applications the required rendezvous distance is about centimetres, then magnetic docking structures become active; however, if these fail unexpectedly, the feedback control is still capable to perform a docking manoeuvre. The results of the simulation are shown in the figures below. Figure 4.9a shows the element-wise error between the reference and the respective coordinates. It can be seen that the docking manoeuvre is successful within the predefined finite time T and the errors evolve within the prescribed funnel boundary.

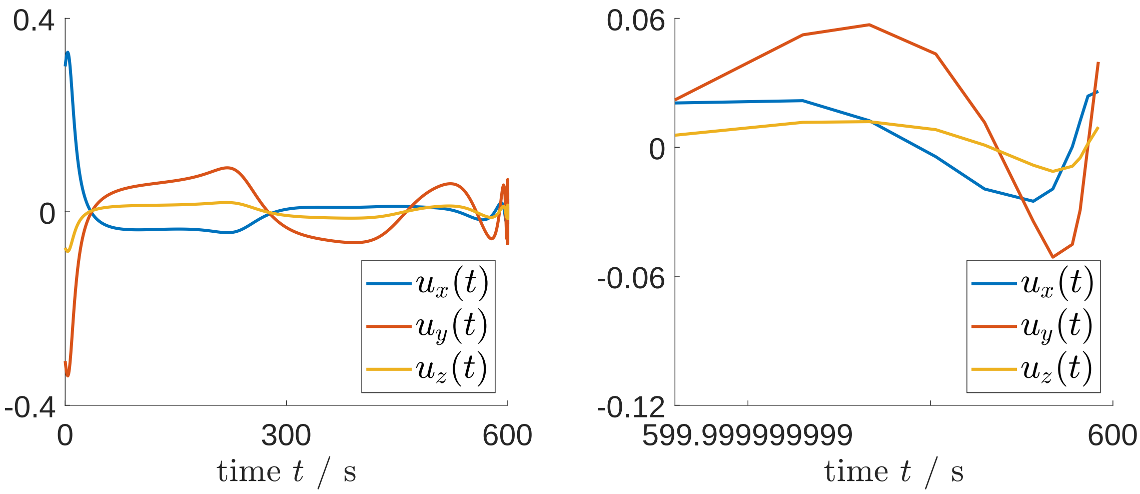


(a) Element-wise tracking errors. The dashed line represents the funnel boundary $1/\varphi$.

(b) Error of velocities.

Figure 4.9: Tracking error, funnel boundary and errors of the velocities.

Figure 4.9b shows the respective errors of the velocities. Note that for $t \to T$, the error of velocities are zero, i.e., the passive space station and the active spaceship have the same velocity at $t = T$, which is desirable performing a docking manoeuvre. Figure 4.10 shows the control input during the docking manoeuvre.



(a) Control input.

(b) Detailed control input in the very last moments.

Figure 4.10: Control input.

In Figure 4.10b the control input in the very last moments before docking is depicted. Note that in the last moments before docking, the funnel boundary is close to zero. Therefore, even small deviations from the reference and its derivative cause (relatively) high input values. The simulation has been performed in MATLAB (solver: ode23tb, RelTol $= 10^{-10}$, AbsTol$= 10^{-10}$). ◇

We conclude this section with a second example to illustrate Theorem 4.20. To this end, we consider an academic nonlinear system which has a vector relative degree. Moreover, the distribution of the control input is unlike the previous example, and we make use of the possibility to choose different funnel boundaries for the output channels, as discussed in Remark 4.16 (iv).

Example 4.25. We illustrate Theorem 4.20 by a simulation of an academic example. Consider the nonlinear system

$$\begin{aligned} y_1^{(1)}(t) &= -y_1(t) - \eta(t)^2 + u_1(t) - 0.5u_3(t), \\ y_2^{(2)}(t) &= -5\tanh(y_2(t)^2) - 0.3u_1(t) + u_2(t) + 0.1u_3(t), \\ y_3^{(3)}(t) &= \|(y_1(t), y_2(t), y_3(t))^\top\|^2 + 0.8u_1(t) + u_3(t), \\ \dot\eta(t) &= -\eta(t) + y_1(t)^2, \end{aligned}$$

which obviously has vector relative degree $\bar r = (1, 2, 3)$ and belongs to the system class $\mathcal{N}^{3,\bar r}$. Note that for demonstration purposes, unlike the standard case, the control distribution is chosen so that the dynamics of each output channel are affected by more than only one input channel. As reference signal we choose the smooth trajectory

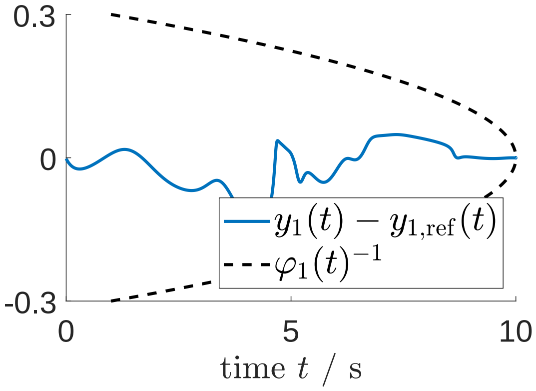
$$y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} e^{\left(\frac{T/2-t}{2}\right)^2} \\ \cos(t) \\ \sin(t) \end{pmatrix}.$$

As final time we choose $T = 10\,\text{s}$, and with the parameters $c_1 = 0.1, c_2 = 0.5, c_3 = 0.3$ and $\rho_1 = 0.5, \rho_2 = 1.5, \rho_3 = 1$ we choose the control functions for $j = 1, 2, 3$

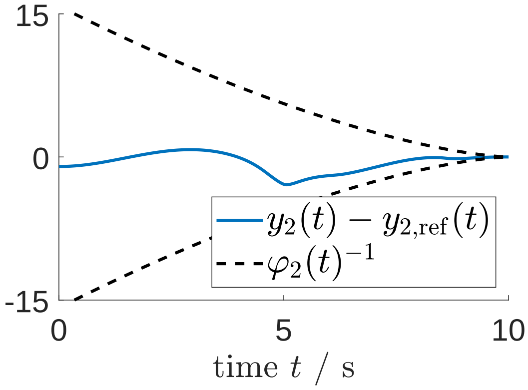
$$\varphi_j(t) = \frac{1}{c_j}\frac{1}{(T-t)^{\rho_j}},\ \alpha_j(s) = \frac{c_j(r_j+1)}{1-sm},\ \beta(\sigma) = \frac{1}{1-\sigma},\ N(\tau) = -\tau\cos\left(10^{-2}\tau\right),$$

where $t \in [0, T)$, $s \in [0, m)$, $\sigma \in [0, 1)$ and $\tau \in \mathbb{R}_{\geq 0}$.

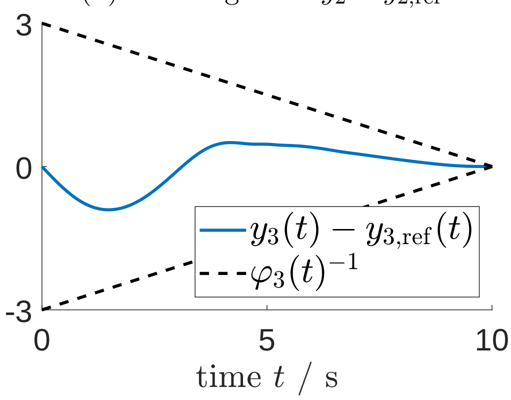
In virtue of Remark 4.22 we choose $\mathsf{eps} = 10^{-9}$. Determined by the final time $T = 10\,\text{s}$ we, simulate tracking on the interval $0 - 10$ seconds, where the very last moments cannot be simulated as already discussed in Example 4.24. The results of the simulation are shown in Figures 4.11 & 4.12. Figure 4.11 shows the tracking error (componentwise) and the respective funnel boundary. Note that the funnel boundaries $1/\varphi_i$, $i = 1, 2, 3$, have different shapes due to the variation of ρ_i as discussed in Remark 4.16 (iv). Figure 4.12 shows the control inputs generated by the control scheme 4.32 to achieve exact tracking. As discussed in Example 4.24, in the very last moments the errors are close to the funnel boundary and hence, even the smallest dynamic causes a reacting control. The simulation has been performed in MATLAB (solver: ode15s, RelTol $= 10^{-7}$, AbsTol $= 10^{-9}$). ◇



(a) Tracking error $y_1 - y_{1,\text{ref}}$.

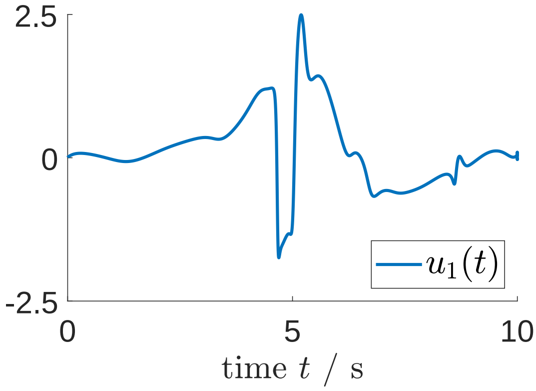


(b) Tracking error $y_2 - y_{2,\text{ref}}$.

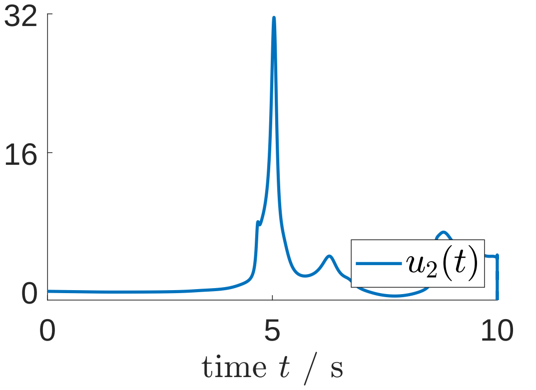


(c) Tracking error $y_3 - y_{3,\text{ref}}$.

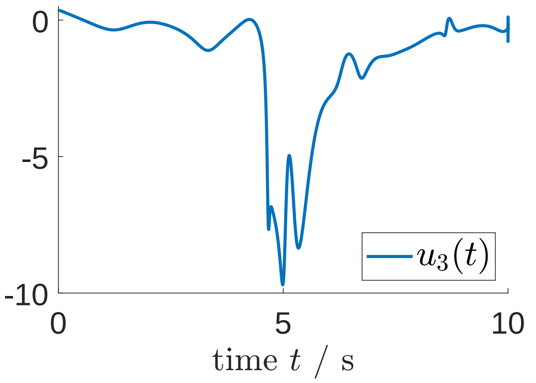
Figure 4.11: Tracking errors with funnel boundaries.



(a) Control input u_1.



(b) Control input u_2.



(c) Control input u_3.

Figure 4.12: Control inputs.

5 Conclusion and outlook

In the present thesis, three main contributions to output reference tracking of nonlinear multibody systems via feedback control are presented. We summarize the main results and give a brief outlook on related future research topics.

Internal dynamics In Chapter 2 we introduced a structurally novel ansatz for the representation of the internal dynamics of nonlinear multibody systems in terms of the output and the internal variable. In the first part of that chapter we used this ansatz to derive an explicit representation of a coordinate transformation, which allows to decouple the internal dynamics completely algorithmically without the need to compute the Byrnes-Isidori form. In particular, in contrast to the Byrnes-Isidori form, the novel representation does not include solutions of partial differential equations but only algebraic relations and symbolic differentiation. The decoupled internal dynamics are then, for instance, open to stability analysis. Moreover, since the internal dynamics are given in terms of the system's output and the internal variable, the system's state can be represented explicitly via these respective terms, i.e., beside the explicit representation of the coordinate transformation, its inverse is given as well. In the second part of Chapter 2 we performed a stability analysis of a subclass of multibody systems with structured right-hand side. We derived explicit conditions on the system parameters, which allow to determine the stability of the internal dynamics in advance without the need to decouple these. Both results, the decoupling procedure and the stability conditions, were demonstrated via illustrative examples. Future research will be, to extend the presented results to systems with vector relative degree. Another aim is, to find a structurally similar ansatz of representing the internal dynamics such that the straightforward algorithmic decoupling can be extended to more general systems, e.g., systems which are not affine linear in the control term.

Funnel pre-compensator In Chapter 3 we showed that the conjunction of the funnel pre-compensator, introduced in [32], with a minimum phase system of arbitrary relative degree results in a minimum phase system of the same relative degree. This resolves the open question raised in [32] where the aforesaid was proven for the special case of relative degree two. Using the fact that the derivatives of the pre-compensator's output are known explicitly we showed that output reference tracking with prescribed transient behaviour using funnel based feedback control schemes is possible with output feedback only. This was illustrated by a numerical simulation of an academic example. Robustness of the conjunction was illustrated by a second example, where a chaotic reference was tracked. In particular, output tracking with unknown output derivatives is possible for the class of linear minimum phase systems (3.5); and moreover, for a class of linear non-minimum phase systems (single-input, single output systems as well as multi-input, multi-output systems). Since the investigations in the recent works [23] and [18] show applicability of existing control techniques to nonlinear non-minimum phase systems we are confident that an integration of the funnel pre-compensator into this particular context will also be fruitful. Future research will be focused on systems which are not linear affine in the control term, where we conjecture that a different proof technique is necessary, as a comparison of [25] and [21] suggests.

Funnel control In Chapter 4 we presented two new results in funnel control. First, we considered output reference tracking for linear minimum phase systems, where the output is subject to possible measurement losses. We proposed a feedback law, which achieves tracking of a reference signal with prescribed accuracy, i.e., whenever the output signal is available, the tracking error evolves within prescribed bounds. We derived explicit bounds on both intervals, the minimal required availability of the output signal, and the maximal allowable duration of losses. These bounds are quite conservative and it will be topic of future research to obtain relaxed conditions on both intervals. For systems with trivial internal dynamics, there are no restrictions on the duration of availability and absence of the output measurement, except for finiteness of the interval of absence. An application of the proposed controller was illustrated by a numerical simulation of the *mass on car system* first considered in [179]. Additional research is needed to further develop the presented idea of shifting the funnel function in the presence of signal losses to obtain more relaxed conditions for the allowable durations of signal losses. Moreover, the extension of the presented idea to nonlinear systems is an open problem. Regarding this, from Remark 4.2 it is clear that some kind of Lipschitz condition will be required for the system to prevent a blow up during the absence of the output measurement. In the second part of Chapter 4, we studied the long-standing problem of exact tracking in finite time via feedback control. We developed a novel funnel control law for a class of nonlinear systems with arbitrary vector relative degree. We showed that the proposed controller achieves that the output tracks a given reference signal and has its exact values at a predefined finite time. Moreover, until this final time, the tracking error evolves within prescribed bounds. This was illustrated by two numerical simulations. First, for the case of strict relative degree docking of a space shuttle to a satellite was simulated. Second, the case of vector relative degree was simulated for an academic example. Topics for future research in funnel control will be, e.g., the consideration of time-discrete systems and systems with output delay (the controller receives delayed signals), to name but two urgent problems.

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List of symbols

$\mathbb{N}$	set of non-negative integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0}$	$:= [0, \infty)$, set of non-negative real numbers
$[a, b], [a, b), (a, b)$	a closed, half open, and open interval for $a, b \in \mathbb{R}$ with $a < b$
$\mathbb{C}$	set of complex numbers
$\mathbb{C}_-$	$:= \{\, z \in \mathbb{C} \mid \mathrm{Re}(z) < 0 \,\}$, set of complex numbers with negative real part
$\mathbb{C}_+$	$:= \{\, z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0 \,\}$, set of complex numbers with non-negative real part
$n!$	$:= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, factorial of the integer $n \in \mathbb{N}$
$\langle \cdot, \cdot \rangle$	inner product in $\mathbb{R}^n$
$\|x\|$	$:= \sqrt{\langle x, x \rangle}$, Euclidean norm of $x \in \mathbb{R}^n$
$\mathbf{Gl}_n(\mathbb{R})$	group of invertible matrices $A \in \mathbb{R}^{n \times n}$
$A > 0$	$\iff \langle x, Ax \rangle > 0$ the matrix $A \in \mathbf{Gl}_n(\mathbb{R})$ is positive definite
$\sigma(A)$	$:= \{\, \lambda \in \mathbb{C} \mid \det(A - \lambda I) = 0 \,\}$, spectrum of a matrix $A \in \mathbb{R}^{n \times n}$
$\mathrm{im}\, A$	$:= \{\, y \in \mathbb{R}^m \mid y = Ax,\ x \in \mathbb{R}^n \,\}$, image of a matrix $A \in \mathbb{R}^{m \times n}$
$\ker A$	$:= \{\, x \in \mathbb{R}^n \mid Ax = 0 \in \mathbb{R}^m \,\}$, kernel of a matrix $A \in \mathbb{R}^{m \times n}$
$\mathrm{rk}\, A$	$:= \dim(\mathrm{im}\, A)$, rank of a matrix $A \in \mathbb{R}^{m \times n}$
$\|A\|$	$:= \max_{\|x\|=1} \|Ax\|$, spectral norm of a matrix $A \in \mathbb{R}^{m \times n}$
$\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$	diagonal matrix $A = (a_{i,j})_{i,j=1,\ldots,n}$ with entries $a_{i,j} = \alpha_i$ if $i = j$, and $a_{i,j} = 0$ if $i \neq j$
$A^\dagger$	Moore–Penrose pseudoinverse of the matrix $A \in \mathbb{R}^{m \times n}$, defined by $A^\dagger := (A^\top A)^{-1} A^\top$ if $\mathrm{rk}\, A = n$, or $A^\dagger := A^\top (A A^\top)^{-1}$ if $\mathrm{rk}\, A = m$
$\mathcal{C}^k(I; \mathbb{R}^n)$	set of k-times continuously differentiable functions $f : I \to \mathbb{R}^n$, $\mathcal{C}^0(I; \mathbb{R}^n) = \mathcal{C}(I; \mathbb{R}^n)$, $I \subseteq \mathbb{R}$ an interval

$AC_{loc}(\mathbb{R}_{\geq 0};\mathbb{R})$	set of locally absolute continuous functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$
$\mathcal{L}^1_{loc}(I;\mathbb{R}^n)$	set of locally Lebesgue integrable functions $f : I \to \mathbb{R}^n$, i.e., $\int_J \|f(t)\| \, dt < \infty$ for all compact $J \subseteq I$, $I \subseteq \mathbb{R}$ an interval
$\mathcal{L}^\infty_{loc}(I;\mathbb{R}^n)$	set of locally essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval
$\mathcal{L}^\infty(I;\mathbb{R}^n)$	Lebesgue space of measurable and essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval
$\|f\|_\infty$	$:= \operatorname{ess\,sup}_{s\in I} \|f(s)\|$, norm of $f \in \mathcal{L}^\infty(I;\mathbb{R}^n)$, $I \subseteq \mathbb{R}$ an interval
$\mathcal{W}^{k,\infty}(I;\mathbb{R}^n)$	Sobolev space of k-times weakly differentiable functions $f : I \to \mathbb{R}^n$, such that $\dot{f}, \ldots, f^{(k)} \in \mathcal{L}^\infty(I;\mathbb{R}^n)$, $I \subseteq \mathbb{R}$ an interval
$f\|_J$	restriction of the function $f : I \to \mathbb{R}^n$ to $J \subseteq I$, $I, J \subseteq \mathbb{R}$ intervals
a.a.	almost all
$\wedge$	logical conjunction “and” of two statements: $a \wedge b$ is true if, and only if, a is true and b is true
$\vee$	logical conjunction “or” of two statements: $a \vee b$ is false if, and only if, both a and b are false