Dissertation

# The classical and the soft-killing Inverse First-Passage Time Problem: A Stochastic Order Approach

von

Alexander Klump

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Prof. Dr. Martin Kolb

vorgelegt der

Fakultät für Elektrotechnik,
Informatik und Mathematik,
Universität Paderborn

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## Abstract

We study the classical and the soft-killing variant of the inverse first-passage time problem for Brownian motion. Given a distribution on the positive real line, the (soft-killing) inverse first-passage time problem consists of asking for an unknown function, such that the (soft-killing) first-passage time of this function has the given distribution. By the use of stochastic order relations we provide new probabilistic and more elementary approaches to these problems, which were hitherto mostly tackled in the context of partial differential equations.

In the classical problem, at the one hand we obtain the known uniqueness result for solutions, but on the other hand establish new results, such as a comparison principle and sufficient conditions for monotonicity and Lipschitz continuity. Using these results we study the special case of the exponential distribution and other examples. Further, given a distribution, we study an interacting particle system, whose hydrodynamic limit finds the solution of the inverse first-passage time problem.

In the soft-killing problem we show a stronger version of the known existence and uniqueness result for continuous solutions, assuming only the necessary condition for existence, and extend the result to a more general class of Markov processes.



## Zusammenfassung

Wir untersuchen die klassische und die sogenannte soft-killing Variante des inversen first-passage time Problems für die Brownsche Bewegung. Zu einer gegebenen Verteilung auf den positiven reellen Zahlen, besteht das inverse (soft-killing) first-passage time Problem in der Suche nach einer unbekannten Funktion, deren (soft-killing) first-passage time die zuvor gegebene Verteilung hat. Durch die Verwendung von stochastischen Ordnungsrelationen erhalten wir neue probabilistische und elementarere Ansätze für diese Probleme, die bisher meist im Zusammenhang mit partiellen Differentialgleichungen behandelt wurden.

Bei dem klassischen Problem erhalten wir einerseits das bekannte Eindeutigkeitsresultat für Lösungen, erzielen aber andererseits auch neue Resultate, wie zum Beispiel ein Vergleichsprinzip und hinreichende Bedingungen für Monotonie und Lipschitzstetigkeit. Unter Anwendung dieser Resultate untersuchen wir den Spezialfall der Exponentialverteilung und andere Beispiele. Außerdem untersuchen wir ein interagierendes Teilchensystem, dessen hydrodynamischer Grenzwert bei gegebener Verteilung die Lösung des inversen first-passage time Problems findet.

Für das soft-killing Problem zeigen wir eine stärkere Version des bekannten Existenz- und Eindeutigkeitsresultates für stetige Lösungen, indem wir nur die notwendige Bedingung für die Existenz annehmen und das Ergebnis auf eine allgemeinere Klasse von Markov-Prozessen erweitern.

# Contents

# Chapter 1

# Introduction

The search for distribution constrained stopping times for Brownian motion presumably began with a question of Shiryaev during a Banach center meeting in 1976. According to [Pes02a], the question was posed, whether there exists a continuous function whose first-passage time of a Brownian motion is exponentially distributed. Later, this question was often formulated more generally, namely, whether there is an exponentially distributed stopping time with respect to Brownian motion. Subsequently, the even more general problem of finding stopping times with a given arbitrary distribution was studied. In [DG77] the authors characterize the existence of stopping times with given distributions for general stochastic processes by an equivalent condition, which applies for Brownian motion. From this point of view one could ask, within which classes of explicit realizations of stopping times the solutions can be found. In this work we will be concerned with the particular cases of realizing the solutions as first-passage times and soft-killed first-passage times.

## 1.1 Problems and motivation

**The inverse first-passage time problem** consists of the following problem. Let $(X_t)_{t\geq 0}$ be a Brownian motion. Given a distribution on $(0,\infty)$, or equivalently a random variable $\xi > 0$, find a function $b : (0,\infty) \to [-\infty,\infty]$, such that

$$\tau_b := \inf\{t > 0 : |X_t| \geq b(t)\} \tag{1.1}$$

has the given distribution of $\xi$, this means $\mathbb{P}(\tau_b > t) = \mathbb{P}(\xi > t) =: g(t)$ for all $t \geq 0$. Here the problem is stated with given distributions on the positive numbers, since for standard Brownian motion the probability of $\tau_b = 0$ is trivial by Blumenthal's law. The terminology comes from the classical first-passage time problem, where one wants to find the distribution of $\tau_b$, when $b$ is a given function. The direct problem is a classical problem in probability and only a few explicit examples are known. The inverse problem has another structure and induces, for example, the following questions:

- Do solutions always exist? If not, to what extend do they exist?
- If a solution exists, is it unique in some sense?
- If a solution exists, which properties does the solution have? More precisely, what can we say about regularity, shape or asymptotics?

The inverse first-passage time problem has a long history, which apparently started with the question of Shiryaev. In 1980 Anulova [Anu80] gave a positive answer to this question by in fact showing the existence of lower semicontinuous solutions to the general inverse first-passage time problem for reflected Brownian motion. Some time passed until the problem found new attention in [HW01] and [AZ01] as a possibility to model the default time of a firm in the context of credit risk modeling. This revived the research on the question of uniqueness, which was partially established in [Che+06] and [Che+11] in terms of a free boundary problem and finally answered in [EJ16] by a general uniqueness result concerning the lower semicontinuous solutions of Anulova in connection with an optimal stopping problem. Furthermore, as by-product a uniqueness result dropped from the much more general consideration of optimal stopping problems in [Bei+18].

Conditions for continuity were given in [Che+11] and [EJ16] and as implication of the study of existence of continuous solutions in [Pot21]. Asymptotics at zero have been studied in [Che+06]. Higher order regularity has been tackled in [CCS21].

Parallely, strongly related research has been accomplished, which project into the inverse first-passage time problem from several directions. To begin with, integral equations relating to the inverse and/or the direct problem have been found in [Pes02b], [Che+06], [EJ16] and [JKV09a]. Methods to solve the problem numerically have been presented in [ZS09], [Abu06], [SZ11] and [GP21]. A modification of the problem seeing the inverse unkown in the random starting position has been studied in [JKZ09], [JKV09b], [Abu13b] and [JKV14]. The branch of research of [De +19a], [BBP19], [Bec19], [Lee20], [Ber+20] and [Ber+21], studying certain branching particle systems with selection and corresponding free boundary problems, is related to the particular case of the inverse first-passage time problem where $\xi$ has exponential distribution. A more detailed overview over the existent literature mentioned above is given further below.

What strikes the eye is, that the inverse first-passage time problem has not been directly tackled by staying in or extending its own probabilistic scope, except for [EJ16] at most. The author of [Anu80] constructed solutions directly by a compactness argument in a probabilistic setting, but the work of

[Che+06], [Che+11] and [CCS21] relies on the connection to a free boundary problem and is based on purely analytical methods. The work of [EJ16] may be based on the setting of [Anu80] in combination with an optimal stopping representation but does not develop further tools to analyze solutions or its approximants from [Anu80]. Instead, the analysis of properties of solutions, such as in [EJ16], [Che+06] and [Che+11], mainly rely on directly deriving necessities for the survival function $g$. The branch concerning only the special case $g(t) = \exp(-t)$ is situated in the setting of a free boundary problem and mainly uses methods from partial differential equations, but in addition draws from a therein commonly method, which gives rise to the usual stochastic order and encourages the following approach pursued in this thesis.

In this thesis we aim to add a new method to the tools in the inverse first-passage time problem, which in regard to [Anu80] only uses stochastic orders and probability metrics as elementary supplements and can be shortly motivated by the following. In the setting of (1.1) with a distribution constriction on $\tau_b$ it is intuitive that the mass distribution of the conditioned time marginal

$$\mathbb{P}\left(X_t \in \cdot \ | \tau_b > t\right) \tag{1.2}$$

at time $t > 0$ in turn affects the further behavior of the solution $b$ after that time. Regarding this perception, in the study of the measure in (1.2) and certain approximants of it, the notion of stochastic order relations comes into play, yielding a tool by which we can control the appearing measures and thereby the boundaries to a suitable extend. We will use stochastic order relations in order to compare solutions to its approximants, compare solutions corresponding to different survival functions and compare solutions corresponding to different initial distributions. We will see that this approach enables us to deduce the known uniqueness result as well as new qualitative results, such as a comparison principle and sufficient conditions for monotonicity or continuity. The results are applied for some examples such as the exponential distribution.

**The inverse first-passage time problem with soft-killing** is in some sense a generalization of the inverse first-passage time problem for Brownian motion. Let $(X_t)_{t\geq 0}$ be a Brownian motion with $X_0 \sim \mu$ independent from the increments and $U$ an independent exponentially distributed random variable with mean 1. Moreover, for a killing rate $\lambda > 0$ and a measurable function $b : (0, \infty) \to [-\infty, \infty]$ let

$$\tau_b^{\mathrm{sk}} := \inf\left\{t \geq 0 : \lambda \int_0^t \mathbb{1}_{(-\infty, b(s))}(X_s)\,\mathrm{d}s > U\right\}. \tag{1.3}$$

The stopping time $\tau_b^{\mathrm{sk}}$ can be seen as the soft-killing variant of $\tau_b$, where one does not stop at the first-passage time directly, but after the process has spent a random memoryless time below the boundary.

For a given survival distribution $g$, the inverse first-passage time with soft-killing is then to find a function $b$ such that $\mathbb{P}\left(\tau_b^{\mathrm{sk}} > t\right) = g(t)$. Regarding this, by substitution and Brownian scaling one can assume without loss of generality that $\lambda = 1$. Apparently this modification of the original inverse first-passage time problem was considered the first time in [EEH14], where the authors showed that there exists a solution $b$ under the condition, that $g$ is twice continuously differentiable and satisfies

$$0 < -g'(t) < g(t) \qquad \forall t \geq 0 \tag{1.4}$$

and $\mu$ has a bounded, twice continuously differentiable density, which is positive everywhere and has bounded derivatives up to order 2.
The question of uniqueness has been tackled in the subsequent work [EHW20], where the authors show that there is a unique continuous solution, if $g$ is continuously differentiable, fulfills (1.4) and $\mu$ admits a density, which is contained in the Sobolev space $H^2$ and is positive everywhere.

The works [EEH14] and [EHW20] use the connection to a partial differential equation and a free boundary problem, respectively, and apply analytical methods from partial differential equations to approach the inverse first-passage time problem with soft-killing.

In this thesis we aim to approach the inverse first-passage time problem with soft-killing in a probabilistic and more elementary way and prove a stronger existence and uniqueness result for continuous solutions. On the one hand, we are able to impose weaker conditions on the initial distribution and to remove the condition of (1.4) at zero. On the other hand, our result is in fact true for a more general class of Markov processes, which was conjectured for diffusions in [EHW20]. On top of that, we obtain an additional comparison principle for the soft-killing problem. Similar to the hard-killing case, the key idea of our approach is to study the marginal measure

$$\mathbb{P}_\mu\left(X_t \in \cdot\,, \tau_b^{\mathrm{sk}} > t\right) \tag{1.5}$$

and a certain approximation with respect to the usual stochastic order and probability distances.

## 1.2 Main results and structure of the thesis

In this section we aim to give an overview of the results achieved in this work, where it is important to keep in mind that they emerge from the uniform approach discussed above. We will first introduce some notation and state our main results. Subsequently, we give a description of the structure of this work with regard to the main results. We begin with a throughout relevant and simplifying definition.

**Definition 1.2.1.** *We call a function* $g : [0, \infty) \to [0, 1]$ survival distribution*, if* $g(t) = \mathbb{P}(\xi > t)$ *for a real random variable* $\xi > 0$*. Additionally, we call a function* $b : [0, \infty] \to [0, \infty]$ boundary function*, if* $b$ *is lower semicontinuous.*

Let $\mathcal{P}$ denote the space of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Given $\mu \in \mathcal{P}$ and a standard Brownian motion $(W_t)_{t \geq 0}$ independent from $X_0$, denote with $\mathbb{P}_\mu$ a measure under which

$$X_t := W_t + X_0$$

is a Brownian motion with initial state $X_0 \sim \mu$.

### The inverse first-passage time problem

For $\mu \in \mathcal{P}$ and a survival distribution $g$ we denote the set of all boundary functions which solve the inverse first-passage time problem for the survival distribution $g$ and a Brownian motion with initial condition $\mu$ with

$$\mathrm{ifpt}(g, \mu) := \{b \text{ boundary function} : \mathbb{P}_\mu(\tau_b > t) = g(t)\ \forall t \geq 0\},$$

where $\tau_b$ is the first-passage time from (1.1). We denote the extinction time of $g$ with $t^g := \sup\{t \geq 0 : g(t) > 0\}$. The following uniqueness result is to be found in the thesis as Theorem 2.3.33, and states essentially the same as the uniqueness result from [EJ16] but with arbitrary initial distribution.

**Theorem 1.2.2** (Uniqueness)**.** *For every survival distribution* $g$ *and initial measure* $\mu \in \mathcal{P}$*, the solution in* $\mathrm{ifpt}(g, \mu)$ *is unique in the sense that all* $b \in \mathrm{ifpt}(g, \mu)$ *coincide on* $(0, t^g)$*.*

Define $P_t$ as the operator of convolution of measures with the Gaussian probability kernel, this is

$$P_t\mu(\mathrm{d}x) := \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}\, \mathrm{d}\mu(y)\, \mathrm{d}x \tag{1.6}$$

for $t \geq 0$ and $\mu \in \mathcal{P}$. Define the quantile-truncation $T_\alpha$ of measures by

$$T_\alpha(\mu) := \mu(\,\cdot \mid [-q_\alpha, q_\alpha]) = \frac{\mu(\cdot \cap [-q_\alpha, q_\alpha])}{\mu([-q_\alpha, q_\alpha])} \tag{1.7}$$

for $\alpha \in (0, 1]$ and $\mu \in \mathcal{P}$, where $q_\alpha := q_\alpha(\mu) := \inf\{c \geq 0 : \mu([-c, c]) \geq \alpha\}$.

For a probability measure $\mu$ and a survival distribution $g$ we make the following construction. We fix a timepoint $h \in (0, t^g)$. For $n \in \mathbb{N}$ let $\delta := \delta^{(n)} := h2^{-n}$ and for $k \in \mathbb{N}$ with $k\delta < t^g$ set

$$\alpha_k := \alpha_k^{(n)} := \frac{g(k\delta^{(n)})}{g((k-1)\delta^{(n)})}$$

and define

$$\mu_k^{+,n} \coloneqq \mu_k^+ \coloneqq T_{\alpha_k} \circ P_\delta \circ \ldots \circ T_{\alpha_1} \circ P_\delta(\mu) \tag{1.8}$$

and

$$\mu_k^{-,n} \coloneqq \mu_k^- \coloneqq P_\delta \circ T_{\alpha_k} \circ \ldots \circ P_\delta \circ T_{\alpha_1}(\mu). \tag{1.9}$$

An approximation in Wasserstein sense of the marginal measure (1.2) is given by Theorem 2.3.32.

**Theorem 1.2.3** (Approximation in Wasserstein)**.** *Let $\mu$ be symmetric with finite first absolute moment and $g$ a survival distribution. Let $b \in \mathrm{ifpt}(g, \mu)$. Then*

$$\mu_{2^n}^{\pm,n} \to \mathbb{P}_\mu \left( X_h \in \cdot \,|\tau_b > h \right)$$

*as $n \to \infty$ in the 1-Wasserstein distance.*

The techniques used are based on the following stochastic order relation. Recall that $\mathcal{P}$ denotes the space of probability measures on $\mathbb{R}$. For any two measures $\mu, \nu \in \mathcal{P}$ on $\mathbb{R}$ we say $\mu$ *is dominated by* $\nu$ *in the two-sided stochastic order*, and write $\mu \preceq \nu$, if

$$\mu([-c, c]) \geq \nu([-c, c]) \quad \forall c \geq 0.$$

For the unique solutions of the inverse first-passage time problem hold the following comparison principle to be found as Theorem 2.3.34 in the thesis.

**Theorem 1.2.4.** *Let $\mu_1, \mu_2 \in \mathcal{P}$, such that $\mu_1 \preceq \mu_2$. Let $g^1$ and $g^2$ be two survival distributions, such that $g^2/g^1$ is non-decreasing on $[0, t^{g^1})$. Then the solutions $b^i \in \mathrm{ifpt}(g^i, \mu_i)$ for $i \in \{1, 2\}$ satisfy*

$$b^1 \leq b^2$$

*pointwise on $(0, t^{g^1})$.*

Note that in the special case of $g^1(t) = g^2(t) = e^{-t}$ a corresponding comparison principle has been established in the free boundary problem context in [BBP19] and [Ber+21].

For a survival distribution $g$ and a number $\lambda > 0$ define $g^\lambda(t) \coloneqq g(\lambda t)$. We can give the following sufficient condition for Lipschitz continuity by Proposition 2.3.43 in combination with Theorem 2.3.47.

**Theorem 1.2.5.** *Let $g$ be a survival distribution, which is logarithmically convex on $(0, c) \subseteq (0, t^g)$, and $b \in \mathrm{ifpt}(g, 0)$. Then $b$ is non-decreasing on $[0, c)$. Additionally, if for all $\lambda \in (0, 1)$ it holds that $g^\lambda/g$ is non-decreasing on $(0, t^g/\lambda)$, then*

$$|b(t) - b(s)| \leq |t - s| \frac{b(s)}{s}$$

*for $0 < s < t < c$.*

Finally, partly by applying the criteria above, in Subsection 2.3.5 we obtain the following properties of the boundary function corresponding to the exponential distribution and a standard Brownian motion.

**Proposition 1.2.6.** *Fix* $\lambda > 0$ *and set* $g(t) := \exp(-\lambda t)$. *Let* $b \in \operatorname{ifpt}(g, \delta_0)$. *Then*

*(i)* $b$ *is non-decreasing and bounded from above by* $\pi/(2\sqrt{2\lambda})$,

*(ii)* $\lim_{t\to\infty} b(t) = \pi/(2\sqrt{2\lambda})$ *and*

*(iii)* $b$ *is locally Lipschitz continuous on* $(0, \infty)$ *and continuous on* $[0, \infty)$ *with* $b(0) = 0$.

For the special case of $g(t) = e^{-t}$ we already mentioned the work of [Ber+21], which in the free boundary context also comes to the conclusion of the asymptotic limit in (ii) and that $b$ is continuous on $(0, \infty)$.

Versions of the results of Theorem 1.2.2, Theorem 1.2.3, Theorem 1.2.4, Theorem 1.2.5 and Proposition 1.2.6 are published in the journal article [KK22a].

Now, let us introduce a simple particle system, whose hydrodynamic limit is of the form (1.2). For this let $g$ be a survival distribution and for a particle number $N \in \mathbb{N}$ let $B = (B^1, \ldots, B^N)$ be an $N$ dimensional Brownian motion. Further let $T_1, \ldots, T_N$ be independent and identically distributed random variables with $T_1 \sim g$ and let $T_{(1)} \leq \ldots \leq T_{(N)}$ be the corresponding order statistics. Let $(X_t^i)_{i\in A(t), t\geq 0}$ be the process, which results from the following scheme. At every timepoint $T_{(i)}$ we remove the particle with the greatest absolute value from the system and define the index set $A(t)$ of surviving particles up to a time $t$ as the particles, which have not been removed up to this time. The following follows from the more general Theorem 2.4.4.

**Theorem 1.2.7.** *Let* $b \in \operatorname{ifpt}(g, \delta_0)$. *Then for* $t \in (0, t^g)$ *holds*

$$\lim_{N\to\infty} \frac{1}{A(t)} \sum_{i\in A(t)} \delta_{X_t^i}([-a, a]) = \mathbb{P}_0\left(X_t \in [-a, a] \,|\tau_b > t\right), \quad \forall a \geq 0,$$

*almost surely, where* $(X_t)_{t\geq 0}$ *denotes the Brownian motion.*

### The inverse first-passage time problem with soft-killing

Now we turn our attention to the inverse first-passage time problem with soft-killing for Markov processes, where we state our main result in the Brownian case. For Brownian motion the result of Theorem 3.0.1 is the following.

**Theorem 1.2.8.** *Let $\mu \in \mathcal{P}$ be a probability measure. Let $g$ be a survival distribution continuously differentiable on $(0,\infty)$ and fulfilling*

$$0 < -g'(t) < g(t) \quad \forall t > 0. \tag{1.10}$$

*Then there is a unique continuous function $b : (0,\infty) \to \mathbb{R}$ such that for all $t > 0$ we have $g(t) = \mathbb{P}_\mu\left(\tau_b^{\mathrm{sk}} > t\right)$, where $\tau_b^{\mathrm{sk}}$ is the soft-killing time from* (1.3). *If additionally*

$$\begin{cases} \mu \ll \textit{Lebesgue measure, } \operatorname{supp}(\mu) \textit{ is connected and} \\ g'(0) := \lim_{h \searrow 0} \frac{g(h)-g(0)}{h} \textit{ exists with } g'(0) = g'(0+) \textit{ and } 0 < -g'(0) < 1, \end{cases} \tag{1.11}$$

*then $b(0+)$ exists and is the unique value such that $\mu((-\infty, b(0+))) = -g'(0)$.*

Note that this result allows more general initial measures and the condition (1.10) is a relaxation of the condition (1.4) in [EHW20].

The statement of Theorem 1.2.8 but with the condition of (1.11) is under review in the preprint article [KK22b].

### Structure of the thesis

The thesis is organized as follows. As the main part of this work, Chapter 2 is concerned with the classical inverse first-passage time problem for reflected Brownian motion.

In Section 2.1 we begin with studying the compact metric space of boundary functions and carrying out the proof of existence from [Anu80] in terms of boundary functions in the slightly more general case of an arbitrary initial distribution.

In Section 2.2 we prepare the analysis of the approximants of (1.2) and thereby the analysis of solutions, by studying the operators $P_t$ and $T_\alpha$, their relation to the usual stochastic order, the likelihood ratio order and to the Wasserstein and total variation distance.

In Section 2.3 we finally focus on quantitative and qualitative properties of solutions. We begin with auxiliary statements concerning general properties of boundary functions and marginal distributions. Subsequently, we carry out a study of the sequences $\mu_k^{\pm,n}$ from (1.8) and (1.9) and their relation to the general marginal measure in (1.2), leading to stochastic inequalities with respect to the usual stochastic order and thereby to the convergence result Theorem 1.2.3 and the uniqueness result of Theorem 1.2.2. We then deduce the comparison principle of Theorem 1.2.4. Furthermore, we utilize the comparison statement in order to achieve qualitative properties of solutions, including Theorem 1.2.5. Finally, we study the special case of exponential distribution and apply the previous results to further examples of survival distributions.

In Section 2.4 we first discuss the possibility to obtain the marginal measure (1.2) as a hydrodynamic limit from an interacting particle system, which a priori only depends on the given survival distribution. Regarding this we propose a generalization of the system studied in [De +19a], which therein corresponds to the exponential case of the inverse first-passage time problem, and conjecture that the hydrodynamic procedure should work for more general survival distributions. We then analyze the simpler but even more general system of Theorem 1.2.7. We conclude with a discussion on how to obtain proper visualizations of the unknown boundary solution and produce some examples.

Chapter 3 is concerned with the inverse first-passage time problem with soft-killing for a class of Markov processes, where the focus lies on the existence and uniqueness of a continuous solution under appropriate conditions.

In Section 3.1 we prepare the analysis of the measure (1.5) by studying the involved Markov kernel and a reweighting mechanism, which is the counterpart of $T_\alpha$ in the soft-killing setting, and their relation to the usual stochastic order and the total variation distance.

In Section 3.2 we focus on the proof of the existence and uniqueness of a continuous solution. We begin with auxiliary statements about the marginal measure from (1.5) and its approximation. Subsequently, we study their relation and convergence properties of the approximation, leading to stochastic inequalities with respect to the usual stochastic order and thereby to the uniqueness result of Theorem 1.2.8.

In Section 3.3 we shortly discuss how to extract a Monte-Carlo method from the achieved results in order to visualize solutions for the inverse soft-killing problem and produce some examples.

## 1.3 Related results for the inverse first-passage time problem

The inverse first-passage time problem for a stochastic process $(X_t)_{t\geq 0}$ with initial distribution $X_0 \sim \mu$ can be posed in a similar way as above. Given a random variable $\xi$ with values in $(0,\infty)$, find a function $b:(0,\infty)\to[-\infty,\infty]$, such that the (up-crossing) first-passage time

$$\tau_b := \inf\{t > 0 : X_t \geq b(t)\} \tag{1.12}$$

of $b$ by the process $(X_t)_{t\geq 0}$ has the same distribution as $\xi$. We will be mainly concerned with the case that $(X_t)_{t\geq 0}$ is a (reflected) Brownian motion, where we also refer to the reflected case as two-sided situation. In the following we give a detailed overview over the related results in the literature.

### Existence and uniqueness

**Anulova's barrier type solution:** The work of [Anu80] gave the first affirmative answer to the inverse first-passage time problem by showing that for any random variable $\xi$ with values in $(0,\infty]$ one can find a suitable so-called barrier as a closed set $B \subset [0,\infty] \times [-\infty,\infty]$ such that the hitting time

$$\tau_B = \inf\{t > 0 : (t, X_t) \in B\} \tag{1.13}$$

has the given distribution of $\xi$, where $(X_t)_{t\geq 0}$ is a standard Brownian motion. It turned out in [EJ16] that the barriers $B$ are associated with the epigraphs of lower semicontinuous functions. The key of this existence result is the compact Hausdorff topology on the set of barriers, which makes it possible to extract a converging sequence of discrete boundary functions, whose hitting times converge simultaneously in distribution to $g$ and to the hitting time of the limit.

**Continuous, piecewise linear approximation:** Under certain assumptions on $g$ the existence of a continuous limit of an approximation by continuous, piecewise linear functions has been established in [Pot21] for the standard and the reflected case.

**Free boundary problem and viscosity solution:** The first contribution to this topic has been made by the subsequent analytical works [Che+06], originating from the thesis [Che05], and [Che+11]. The authors essentially considered the following free boundary problem, where we translated it to our up-crossing first-passage time and reduced it from diffusion processes to Brownian motion. Given the survival function $g(t) \coloneqq \mathbb{P}(\xi > t)$ for $t \geq 0$ of a random variable $\xi$ with values in $(0,\infty)$ and a random variable $X_0$, find a function $w : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ and a boundary $b : [0,\infty) \to [-\infty,\infty]$ such that

$$\begin{cases} \partial_t w(x,t) = \frac{1}{2}\partial_x^2 w(x,t) & : x < b(t), t > 0, \\ w(x,t) = g(t) & : x \geq b(t), t > 0, \\ 0 \leq w(x,t) < g(t) & : x < b(t), t > 0, \\ w(x,0) = \mathbb{P}(X_0 < x) & : x \in \mathbb{R}, \end{cases} \tag{1.14}$$

where in [Che+11] the third constraint is substituted by $\frac{\partial}{\partial x} w(x,t) = 0$ for $x \geq b(t), t > 0$ and in [Che+06] the function $w_0$ is equal to $\mathbb{1}_{(0,\infty)}$ throughout. In order to explain the connection to the inverse first-passage time problem let $(X_t)_{t\geq 0}$ such that $(X_t - X_0)_{t\geq 0}$ is a standard Brownian motion. Then a formal candidate for a solution to (1.14) is the function

$$\hat{w}(x,t) \coloneqq \mathbb{P}(X_t < x, \tau_b > t). \tag{1.15}$$

In [Che+11] it is used that the problem in (1.14) can also be seen from the point of view of the direct first-passage time problem, this is, given $b$, one wants to find a function $w$ and a survival function $g$ such that (1.14) is fulfilled. Instead of using the usual generalization of solutions, the authors use the analytical notion of a so-called viscosity solution of a corresponding variational inequality for the inverse and the direct problem of (1.14), respectively. In [Che+06] it is shown that the inverse problem of (1.14) has a unique viscosity solution $w$, from which the function

$$b^w(t) := \sup\{x \in \mathbb{R} : w(x,t) < g(t)\} \tag{1.16}$$

can be extracted. Subsequently, [Che+11] proves that, if $g$ is continuous, $b^w$ is the unique lower semicontinuous solution to the inverse first-passage time problem of the Brownian $(X_t)_{t\geq 0}$.

**Optimal stopping:** The next step was done by [EJ16], where the inverse first-passage time of standard Brownian motion is connected with the following optimal stopping problem. Given the survival function $g(t) := \mathbb{P}(\xi > t)$ for $t \geq 0$ of a random variable $\xi$ with values in $(0,\infty)$, define the value function

$$v(t,x) := \inf_{\gamma \in \mathcal{T}[0,t]} \mathbb{E}\left[g(t-\gamma)\mathbb{1}_{\{\gamma<t\}} + \mathbb{1}_{\{\gamma=t,x+X_t\geq 0\}}\right], \quad t \geq 0, x \in \mathbb{R}, \tag{1.17}$$

where $\mathcal{T}[0,t]$ denotes the set of stopping times of the Brownian motion $(X_t)_{t\geq 0}$ taking values in $[0,t]$. Reusing the existence result of Anulova in [Anu80], it is shown that the function $b^v : [0,\infty) \to [-\infty,\infty]$ given by

$$b^v(t) := \inf\{x \in \mathbb{R} : v(t,x) = g(t)\} \tag{1.18}$$

is a lower semi-continuous solution. Now the key idea is that for discrete boundaries a discrete optimal stopping problem can be associated, which only depends on the discrete survival function. By passing a certain discrete approximation from [Che+11] to the limit, it turns out that in fact

$$v(t,x) = \mathbb{P}(X_t \leq x, \tau_b > t)$$

holds for every solution $b$. It follows that $b^v$ is the unique lower semicontinuous solution.

Related but coming from a much more general setting of optimal stopping is the work of [Bei+18]. As by-product from the consideration of a very general optimal stopping problem under a distribution constraint, they obtain in the case that $g$ has finite first moment the existence and uniqueness of lower semicontinuous solutions.

## Properties of solutions

Since the solutions to the inverse first-passage time problem are innately unkown, the further question arises how to derive properties of solutions in or without dependence on $g(t) = \mathbb{P}(\tau_b > t)$. We will only state the results relevant for this thesis.

**Continuity:** In [Che+11] it is shown that for a continuous survival function $g$ the solution $b$ fulfills $\liminf_{s \nearrow t} b(s) = b(t)$ for every $t > 0$. The work of [EJ16] extends this by showing that the converse is also true and by adding a further equivalent condition, namely, that $X_{\tau_b} = b(\tau_b)$ almost surely if $\tau_b < \infty$. Furthermore, both works establish the following criteria on $g$ for the continuity of the solution $b$. If $g$ is continuous, then

$$\inf_{\ell \leq s < t \leq u} -\frac{g(t) - g(s)}{t - s} > 0 \quad \Rightarrow \quad b \text{ continuous on } (\ell, u)$$

for $0 < \ell < u$ and, if the condition holds for $\ell = 0$, setting $b(0) = 0$ makes $b$ continuous on $[0, u)$.
Whereas [Che+11] partially draws from their setting for this purpose, the work of [EJ16] argues by elementary but subtle analysis and in fact only uses the essential infimum.

Further, the criteria on $g$ for the existence of continuous solutions from [Pot21] are criteria for the continuity of $b$.

**Other properties:** In [Che+06] the limit at zero of $b^w$ introduced in (1.16) is studied, which in light of [Che+11] is the unique solution of the inverse first-passage time problem if $g$ is continuous. The subsequent analytical work [CCS21] connects the higher order regularity of $g$ with the higher order regularity of the solution.

## The special case of exponential distribution

Since Shiryaev's problem originated as question about the exponential distribution, it is not surprising that the choice of $g(t) = \exp(-\lambda t)$ has found particular attention in the literature or has been used to evaluate the obtained results of the general case. A first visualization by simulation of the one-sided case is to be found in [ZS09] and for the two-sided case in [Abu06] and [Abu15]. We list some known properties of the solution $b$ corresponding to the exponential distribution and the case that $(X_t)_{t \geq 0}$ is a (standard) Brownian motion.

- The conditions about the limit at zero from [Che+06] yield (stated here for the up-crossing variant) that

$$\lim_{t \searrow 0} \frac{b(t)}{\sqrt{2t \log(1 - \exp(-t))}} = 1.$$

- The conditions for continuity of [Che+11] and [EJ16], and the existence result of [Pot21], yield individually that $b$ is continuous on $[0, \infty)$.

- The work of [CCS21] provides sufficient criteria for smoothness on $(0, \infty)$, which include the exponential distribution.

The condition for continuity of [EJ16] can be transferred to the case of reflected Brownian motion, and thus yields as well as the existence result of [Pot21], that $b$ is continuous on $[0, \infty)$ in the two-sided situation.

Apart from this, there exists a branch of research in the literature, which is strongly related to this special case of the inverse first-passage time problem. This connection has not been pointed out before, which shall be done below in a formal way. We will begin with the one-sided situation.

**A special free boundary problem:** The work of [BBP19] studies a free boundary problem equivalent to the following. Find functions $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $b : [0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_t v(x,t) = \frac{1}{2}\partial_x^2 v(x,t) + v(x,t) & : x < b(t), t > 0, \\ v(x,t) = 1 & : x \geq b(t), t > 0, \\ \partial_x v(b(t),t) = 0 & : t > 0, \\ v(x,0) = v_0(x) & : x \in \mathbb{R}, \end{cases} \tag{1.19}$$

where $v_0 : \mathbb{R} \to [0,1]$ is a non-decreasing function with $\lim_{x\to\infty} v_0(x) = 0$ and $\lim_{x\to-\infty} v_0(x) = 1$. The original free boundary problem in [BBP19] is formulated as down-crossing variant and omits the scalar $\frac{1}{2}$ in front of the Laplacian, thus the problems can be equivalently transformed into each other by a scaling of the spatial variable by factor $-\sqrt{2}$. In the following we will explain the connection to the more general free boundary problem (1.14) and therefore a connection to the inverse first-passage time problem. For $v_0(x) = \mathbb{P}(X_0 < x)$ and if $b$ is the solution to the inverse first-passage time problem for the standard exponential distribution, i.e. $\mathbb{P}(\tau_b > t) = \exp(-t)$, a formal candidate to (1.19) could be the function

$$\hat{v}(x,t) := \mathbb{P}(X_t < x \mid \tau_b > t)$$

by noticing that $\hat{v}(x,t) = e^t \hat{w}(x,t)$ with the notation from (1.15). Thus, differentiation in time and the product rule yield the connection of the two free boundary problems, which is, generally speaking, dividing the desired function $w$ of (1.15) by the normalizing factor $g(t)$. Hence, for given differentiable $g$, we obtain the generalized free boundary problem of (1.19) by substituting the first line of (1.19) by

$$\partial_t v = \frac{1}{2}\partial_x^2 v + hv, \quad x > b(t), t > 0,$$

where $h(t) := -\partial_t \log(g(t)) = -g'(t)/g(t)$ is the hazard rate of $g$.

In particular, the authors show in [BBP19] that (1.19) has a unique classical solution $(u, b)$, which includes that $b$ is continuous on $[0, \infty)$. Furthermore, they establish the comparison principle, that, if $v_0^{(1)} \geq v_0^{(2)}$ as initial conditions, the corresponding solutions stay ordered, i.e. $u^{(1)} \geq u^{(2)}$ and $b^{(1)} \leq b^{(2)}$, which gives rise to the usual stochastic order.

The free boundary problem considered in [De +19a] and [Lee20] is essentially the problem of (1.19) differentiated in the spatial variable. In [Lee20] it is shown that there exists $T > 0$ such that a solution with $C^1([0, T])$ exists. In [De +19a] uniqueness of solutions is shown. The approach to uniqueness of [De +19a], [BBP19] and [Ber+21] below uses a method, which gives rise to the discretization technique of (1.8) and (1.9) used in this thesis.

On the side, further results are to be found in this special situation. Referring to the analysis in [BBD18] the work of [BBP19] mentions a long time behavior of $b$. Corresponding to this, the question for a travelling wave solution can be answered by the sometimes called randomized first-passage time problem. For example, from [JKZ09] one can deduce that, if $1 - v_0(-x)$ is the distribution function of the Gamma$(2, \sqrt{2})$ distribution, then $v_0(x - b(t))$ with $b(t) = -\sqrt{2}t$ is the unique solution to (1.19). In view of the comparison principle this can be used to establish lower bounds for solutions corresponding to other initial distributions.

**The special free boundary problem for the two-sided case:** The authors of [Ber+21] consider a free boundary problem equivalent to finding a continuous $u : \mathbb{R}^d \times (0, \infty) \to [0, \infty)$ and a measurable $b : [0, \infty) \to [0, \infty]$ such that

$$\begin{cases} \partial_t u(x,t) = \frac{1}{2}\Delta u(x,t) + u(x,t) & : t > 0, \|x\| < b(t), \\ u(x,t) = 0 & : t > 0, \|x\| \geq b(t), \\ \int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x = 1 & : t > 0, \\ u(\cdot, t) \to \mu \text{ weakly as } t \to 0, \end{cases} \tag{1.20}$$

where $\mu$ is a probability measure and again the original problem of [Ber+21] omits the $\frac{1}{2}$. For $d = 1$ this free boundary problem is the two-sided analogue of (1.19). A corresponding comparison principle is shown, which is a special case of Theorem 1.2.4. The result of [Ber+21] is that (1.20) has a unique classical solution with the following properties. The boundary $b$ is finite on $(0, \infty)$ and continuous on $[0, \infty)$. Further, for $\alpha < \frac{1}{2}$ there exists a constant $C_\alpha < \infty$ such that $b(t) - b(s) \leq C_\alpha (t - s)^\alpha$ for all $s \geq 0$ and $t \in (s, s+1]$. Finally, it is stated that $b(\infty) := \lim_{t\to\infty} b(t)$ exists with convergence rate in $O(t^{-1})$. For $d = 1$ it holds $b(\infty) = \frac{\pi}{2\sqrt{2}}$. For $d \in \mathbb{N}$, the boundary $b$ is therefore the solution to the inverse first-passage time problem for Bessel processes of natural order or reflected Brownian motion in the case $g(t) = \exp(-t)$.

**Interacting particle representation:** The following general model was proposed in [Ber+20] and covers the particle systems we will discuss below. Let $(X_t^1, \dots, X_t^N)_{t\geq 0}$ be an $N$-particle process, which starts from $(X_0^1, \dots, X_0^N)$ in $(\mathbb{R}^d)^N$ for $d \in \mathbb{N}, N \in \mathbb{N}$. Every particle moves independently as $d$-dimensional Brownian motion and branches independently with rate one. At any branching time the particle with the lowest fitness is removed from the system, where fitness is measured by a fitness function $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}$. Hence, the number of particles is kept constant equal to $N$.
For $d = 1$ in [De +19a] it was shown for a strictly increasing fitness function, e.g. $\mathcal{F}(x) = x$, that the solution of (1.19) appears as so-called hydrodynamic limit of this particle system, therein called $N$-BBM. The work of [Bec19] transfers this result to the case $\mathcal{F}(x) = -|x|$. More generally for $d \in \mathbb{N}$, the authors in [Ber+20] establish the hydrodynamic limit for $\mathcal{F}(x) = -\|x\|$.

# Chapter 2

# The inverse first-passage time problem

In this chapter we will be concerned with the inverse first-passage time problem for reflected Brownian motion, but with the slight generalization from the existing literature, that the Brownian motion can start from a random initial point. More precisely, let $\mu$ be a probability measure on $\mathbb{R}$ and $(X_t)_{t\geq 0}$ be a Brownian motion with initial distribution $X_0 \sim \mu$, such that the increments of the Brownian motion are independent of $X_0$. Recall that we denoted the first-passage time by the reflected Brownian motion of a function $b$ with

$$\tau_b := \inf\{t > 0 : |X_t| \geq b(t)\}.$$

Now, the inverse first-passage time problem for reflected Brownian motion with initial distribution $\mu$ with respect to a survival distribution $g$ is to find a function $b : (0, \infty) \to \mathbb{R}$ such that $\mathbb{P}_\mu(\tau_b > t) = g(t)$ for all $t \geq 0$.

**Motivation of the approach:** Let us present and motivate our approach by the following. The work [Anu80] of Anulova proves that there is always a solution in form of a lower semicontinuous function $b$. It is intuitive that the mass distribution of the conditioned time marginal

$$\mathbb{P}\left(X_t \in \cdot\ |\tau_b > t\right) \tag{2.1}$$

in turn affects the further behavior of the solution $b$ after time $t$, which lets it seem natural to start an analysis of this conditioned measure. For this purpose, the setting of [Anu80] motivates the discretization of the time. By interpreting the conditioning on the survival $\tau_b > t$ as an inhomogeneous operator, for given timesteps $(t_k^n)_{k=1}^{m_n}$ with $0 < t_1^n < \ldots < t_{m_n}^n$, it is natural to discretize the marginal distribution in (2.1) by a type of Trotter product formula written as

$$T^{b(t_k^n)} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots \circ T^{b(t_1^n)} \circ P_{t_1^n}(\mu),$$

where $P_t$ denotes the Markov kernel of Brownian motion and $T^c$ denotes the conditioning to the interval $[-c, c]$. From the view of the inverse first-passage time problem the values $b(t_k^n)$ are unknown, but the approach of Anulova provides us with the approximate candidates $b^n(t_k^n)$ chosen successively such that

$$\mathbb{P}\left(|X_{t_k^n}| < b^n(t_k^n), \ldots, |X_{t_1^n}| < b^n(t_1^n)\right) = g(t_k^n) \tag{2.2}$$

with the aim to substitute $b(t_k^n)$. In order to proceed to the limit we use a convergence of lower semicontinuous functions, which can be derived from the Hausdorff topology used in [Anu80]. It is not far to seek for a method to analyze these quantiles and their behavior and try to pass them to the limit. In order to study $b^n(t_k^n)$, $k = 1, \ldots, m_n$, and the corresponding

$$T^{b^n(t_k^n)} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots \circ T^{b^n(t_1^n)} \circ P_{t_1^n}(\mu) \tag{2.3}$$

we will use stochastic order relations and probability distances.

## 2.1 Semicontinuous functions and existence of solutions

In this section we will revise the setting and approach of Anulova in [Anu80], which we will lead us to the fact that for a survival function $g$ with $g(0) = 1$, there is always a solution in form of a lower semicontinuous function. Motivated by this we recall the following definition.

**Definition 2.1.1.** *We call a function $g : [0, \infty) \to [0, 1]$ survival distribution if there exists a real random variable $\xi > 0$, such that $g(t) = \mathbb{P}(\xi > t)$ for all $t \in [0, \infty)$. A function $b : [0, \infty] \to [0, \infty]$ is called boundary function if it is lower semicontinuous.*

The set of the solutions of this form was defined by

$$\operatorname{ifpt}(g, \mu) := \{b \text{ boundary function} : \mathbb{P}_\mu(\tau_b > t) = g(t)\ \forall t \geq 0\}.$$

In this section we will first develop a useful connection between boundary functions and the so-called barriers, which were used by Anulova, in order to work only with boundary functions outside of this section. Afterwards we will establish the existence of solutions in $\operatorname{ifpt}(g, \mu)$ by carrying out the approach of Anulova.

### 2.1.1 Barriers and boundary functions

In order to construct certain stopping times for a Brownian motion $(X_t)_{t \geq 0}$ Anulova introduced a certain metric space of closed subsets of the product of time and space $H := [0, \infty] \times [-\infty, \infty]$. The stopping times are then realized as hitting times of those closed sets by the process $(t, X_t)_{t \geq 0}$. It turns out that in the same time this enables us to equip the set of boundary functions with a convenient metric, which can be characterized further. Essentially, this subsection deals with specific properties of this topology. We begin with an analysis of the specific class of subsets, which is a slightly adapted version of the class in [Anu80].

We call $B \subseteq H$ a (two-sided) barrier, if

(i) $B$ is closed,

(ii) $[0, \infty] \times \{-\infty\} \cup [0, \infty] \times \{\infty\} \subseteq B$,

(iii) $(t, x) \in B$ implies $(t, -x) \in B$ and

(iv) $x \geq 0$ and $(t, x) \in B$ imply $(t, y) \in B$ for all $y > x$.

We denote the set of all barriers with $\mathcal{B}$. We see that a barrier is spatially symmetric. Basically, it is described by its boundary, which can be seen as a function of time. In order to link both concepts we show that barriers and boundary functions are in fact the same, which was already pointed out without proof in [EJ16].

**Lemma 2.1.2.** *The mapping $B \mapsto (t \mapsto b(t) := \inf\{x \geq 0 : (t, x) \in B\})$, is a bijection between $\mathcal{B}$ and the set of boundary functions. Its inverse is $b \mapsto B := \{(t, x) \in H : |x| \geq b(t)\}$.*

*Proof.* First we check if those $b$ are lower semi-continuous. Since $B$ is closed we have $(t, b(t)) \in B$ for all $t \in [0, \infty]$. Let $t_0 \in [0, \infty]$ and $(t_n)_{n \in \mathbb{N}}$ be a sequence with limit $t_0$ such that $b(t_n) \to \liminf_{t \to t_0} b(t) =: y$. Then $(t_n, b(t_n)) \in B$ for all $n \in \mathbb{N}$ and $(t_n, b(t_n))$ is convergent with limit point $(t_0, y)$. Due to the closure of $B$ this limit point lies in $B$. By the definition of $b$ we see that

$$b(t_0) = \inf\{x \geq 0 : (t_0, x) \in B\} \leq y = \liminf_{t \to t_0} b(t).$$

Therefore $b$ is lower semi-continuous.

Conversely, let $b$ be a boundary function. In order to prove that the set $B := \{(t, x) \in H : |x| \geq b(t)\}$ is indeed a barrier, by the definition of a boundary function, the only property left to show is the closure of $B$. Let $(t_n, x_n)$ a convergent sequence in $B$. We denote its limit point with $(t_0, x)$. Without loss of generality we assume that $x \geq 0$. The definition of $B$ implies that $x_n \geq b(t_n)$ for all $n \in \mathbb{N}$. Thus

$$x = \liminf_{n \to \infty} x_n \geq \liminf_{n \to \infty} b(t_n) \geq \liminf_{t \to t_0} b(t) \geq b(t_0).$$

Consequently, $(t_0, x) \in B$.
Now let $B$ be a barrier. We want to show that

$$B = \{(t, x) \in H : |x| \geq b(t)\},$$

where $b$ is the boundary function we obtain by $B$. Let $(t, x) \in B$ then $(t, |x|) \in B$. This yields $|x| \geq b(t)$ by the definition of $b$. Thus $(t, x)$ is contained in the right-hand side above. Conversely, let $(t, x)$ be contained in the right-hand side. Then $|x| \geq b(t)$. The fact $(t, b(t)) \in B$ immediately implies that $(t, |x|) \in B$, which means that $(t, x) \in B$. □

Anulova equipped $\mathcal{B}$ in [Anu80] with the so-called Hausdorff topology induced by the Hausdorff-distance $r_H$ on the compact subsets of $H$ from (B.3), whose characterization has been further developed in later years, but was apparently not applied to the subject matter of barriers since then. A crucial property of this topology is the compactness of the set of barriers, which we will show by the Kuratowski-convergence from Definition B.2.6, which is equivalent to the convergence in the Hausdorff-distance.

**Proposition 2.1.3.** *$\mathcal{B}$ is compact with respect to the Hausdorff topology.*

*Proof.* Let $K(H)$ denote the set of compact subsets of $H$ equipped with the Hausdorff topology. By Theorem 3.2.4 from [Bee93] the space $K(H)$ is compact. We claim that $\mathcal{B} \subset K(H)$ is closed, hence compact. For this, let $B_n \to B$ be a converging sequence with respect to the Hausdorff-distance such that $B_n \in \mathcal{B}$ for all $n$. Since $B$ has to be closed it remains to show properties (ii)-(iv) of the definition of a barrier. Due to Theorem B.2.7 we have $B = \mathrm{Li}_{n\to\infty} B_n$ in the sense of the Kuratowski-convergence.

Let $t \in [0, \infty]$. Then $\{(t, -\infty), (t, \infty)\} \subset B_n$ for all $n$. By the definition of the Kuratowski limit inferior we see that $\{(t, -\infty), (t, \infty)\} \subset \mathrm{Li}_{n\to\infty} B_n = B$. This shows (ii).

Further, let $(t, x) \in B$ and let $U$ be a neighborhood of $(t, -x)$ in $H$. Consider the map $(-) : H \to H$, $(t, x) \mapsto (t, -x)$. Then $W := (-)(U)$ is a neighborhood of $(t, x)$. Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the intersection $B_n \cap W$ is non-empty. Since $B_n$ is a barrier, for those $n$ we have that $(-)(B_n \cap W) = B_n \cap U$ is non-empty. Therefore, $(t, -x) \in \mathrm{Li}_{n\to\infty} B_n = B$. This shows (iii).

At last, let $(t, x) \in B$ and without loss of generality assume $x \in [0, \infty)$. Let $y \in [x, \infty)$ and let $U$ be a neighborhood of $(t, y)$. Then $V := U - (0, y - x)$ is a neighborhood of $(t, x)$. Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $V \cap B_n \neq \emptyset$. For such $n$ take $(t_n, x_n) \in V \cap B_n$. Since $B_n$ is a barrier it holds that $(t_n, x_n + y - x) \in U \cap B_n$. Therefore $(t, y) \in B$. This yields (iv). □

By Lemma 2.1.2 we see that we can identify barriers with boundary functions. By this we transfer the topology on $\mathcal{B}$ to the set of boundary functions.

As soon as we want to work with boundary functions it becomes important to understand the transferred topology or convergence. The following statement is a corollary of Theorem 4.16 in [Dal93] and Theorem B.2.7 and identifies the convergence in Hausdorff sense as the so-called Γ-convergence. The notion of Γ-convergence is well-studied, for example see [Dal93].

**Theorem 2.1.4.** *Let $b$ and $b_n$ be boundary functions for $n \in \mathbb{N}$. Then $b_n \to b$ in the Hausdorff topology if and only if for every $t \in [0, \infty]$*

*(i) there exists a sequence $(t_n)_{n\in\mathbb{N}}$ converging to $t$ such that*

$$b(t) = \lim_{n\to\infty} b_n(t_n)$$

*(ii) and for every sequence $(t_n)_{n\in\mathbb{N}}$ converging to $t$ holds*

$$b(t) \leq \liminf_{n\to\infty} b_n(t_n).$$

*Proof.* By Theorem B.2.7 the convergence $b_n \to b$ in the Hausdorff distance is equivalent to the Kuratowski convergence of the corresponding barriers. But due to the symmetry of the barriers and the fact that the topology of $[0, \infty] \times [0, \infty)$ is the same as the induced topology of $H$, this convergence of barriers in the Kuratowski sense in $H$ is equivalent to the convergence of the epigraphs

$$\operatorname{epi}(b_n) \to \operatorname{epi}(b) := \{(t, x) \in [0, \infty] \times [0, \infty) : x \geq b(t)\}$$

in the Kuratowski sense in $[0, \infty] \times [0, \infty)$. But this is in turn equivalent to the convergence given by the two conditions of the statement, known as Γ-convergence, by Theorem 4.16 in [Dal93]. □

We want to mention that it is possible to avoid the notions of barriers and Hausdorff convergence, and to work directly with boundary functions and the Γ-convergence instead. From now on we are doing so, but can interpret the subject always in the light of the original work of [Anu80] in the sense that the previous consideration of barriers represents the historical origin of the approach.

Since the Γ-convergence is used throughout the thesis we use a separate notation.

**Definition 2.1.5.** *We say that a sequence of boundary functions $b_n$ converges in the sense of Γ-convergence (or equivalently in Hausdorff distance) to a boundary function $b$, write $b_n \xrightarrow{\Gamma} b$, if (i) and (ii) of Theorem 2.1.4 are fulfilled.*

This characterization of the Hausdorff-convergence is extremely useful when working with sequences of boundary functions, which we will see in the following. We begin with an alternative proof for a statement from [Anu80]. For $s \geq 0$ and a boundary function $b$ let

$$b^s := \infty \cdot \mathbb{1}_{[0,s)} + b \cdot \mathbb{1}_{[s,\infty]}.$$

**Lemma 2.1.6.** *Let $b_n \xrightarrow{\Gamma} b$ and $s > 0$. Then $b_n^s \xrightarrow{\Gamma} b^s$.*

*Proof.* By Theorem 2.1.4 we have to check conditions (i) and (ii) for every time point. Let $t \in [0, s)$. Since for large $n$ the remaining elements of any sequence $(t_n)_{n\in\mathbb{N}}$ converging to $t$ are smaller than $s$ we get

$$\liminf_{n\to\infty} b_n^s(t_n) = \infty = b^s(t).$$

Let $t \geq s$. Condition (i) is obviously fulfilled. Further

$$b^s(t) = b(t) \leq \liminf_{n\to\infty} b_n(t_n) \leq \liminf_{n\to\infty} b_n^s(t_n)$$

for every sequence $(t_n)_{n\in\mathbb{N}}$ converging to $t$. □

Now we will collect some direct deductions concerning the Γ-convergence of boundary functions, which can partly also be found or deduced in a more general setting in [Dal93].

**Lemma 2.1.7.** *Let $(b_n^\ell)_{n\in\mathbb{N}}$ and $(b_n^u)_{n\in\mathbb{N}}$ be sequences of boundary functions with $b_n^\ell \xrightarrow{\Gamma} b^\ell$ and $b_n^u \xrightarrow{\Gamma} b^u$, respectivly. Assume that $b_n^\ell \leq b_n^u$ for all $n$. Then $b^\ell \leq b^u$.*

*Proof.* For $t \in [0, \infty]$ let $t_n \to t$ be a converging sequence such that $b_n^u(t_n) \to b^u(t)$. Then it holds

$$b^\ell(t) \leq \liminf_{n\to\infty} b_n^\ell(t_n) \leq \liminf_{n\to\infty} b_n^u(t_n) = b^u(t).$$

Therefore it holds $b^\ell \leq b^u$. □

We will use the following relationship to pointwise convergence, which also can be found in Remark 5.5 of [Dal93].

**Lemma 2.1.8.** *Let $b_n$, $n \in \mathbb{N}$, be a sequence of boundary functions, such that $b_n \leq b_{n+1}$ pointwise. Then $b_n \xrightarrow{\Gamma} b$, where $b(t) := \lim_{n\to\infty} b_n(t)$ for $t \in [0, \infty]$. In particular, $b$ is lower-semicontinuous.*

*Proof.* Let $b^1$ and $b^2$ be accumulation points of $(b_n)_{n\in\mathbb{N}}$ with limiting subsequences $b_{n_k} \to b^1$ and $b_{n_\ell} \to b^2$. Then for any $t \in [0, \infty]$ there exists a sequence $(t_k)_{k\in\mathbb{N}}$ converging to $t$ such that $b_{n_k}(t_k) \to b^1(t)$. Thus

$$b^1(t) = \liminf_{k\to\infty} b_{n_k}(t_k) \geq \liminf_{k\to\infty} b_{n_\ell}(t_k) \geq \liminf_{s\to t} b_{n_\ell}(s) \geq b_{n_\ell}(t).$$

Since $b_{n_\ell}$ converges in the Hausdorff distance to $b^2$ this ordering implies by Lemma 2.1.7 that $b^1 \geq b^2$. Analogously, one can establish that $b^1 \leq b^2$. We end up with $b^1 = b^2$, and thus every accumulation point coincides. Due to the compactness we get $b_n \xrightarrow{\Gamma} b^1$. It is left to show that $b^1 = b$. Let $t \in [0,\infty]$ and $t_n \to t$ such that $b_n(t_n) \to b^1(t)$. Then for $m \in \mathbb{N}$ we have

$$b^1(t) = \liminf_{n\to\infty} b_n(t_n) \geq \liminf_{n\to\infty} b_m(t_n) \geq b_m(t).$$

By letting $m \to \infty$ it follows that $b^1(t) \geq b(t)$. By the definition of the $\Gamma$-convergence we have $b^1(t) \leq \liminf_{n\to\infty} b_n(t) = b(t)$. Hence $b^1 = b$. □

By Theorem B.2.2 for every boundary function such a sequence, consisting of continuous functions, exists. Thus this means in the Hausdorff topology every boundary function can be approximated by continuous functions from below.

We can also pass shape properties through the limit.

**Lemma 2.1.9.** *Let $U \subseteq [0,\infty]$ be open. If $b_n \xrightarrow{\Gamma} b$ and every $b_n$ is non-decreasing on $U \cap \{t \in [0,\infty] : b_n(t) < \infty\}$, then $b$ is non-decreasing on $U \cap \{t \in [0,\infty] : b(t) < \infty\}$.*

*Proof.* Define $F_b := \{t \in [0,\infty] : b(t) < \infty\}$. Let $s,t \in F_b \cap U$ with $s < t$. Then Theorem 2.1.4 implies that there exist sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ with $s_n \to s$ plus $b_n(s_n) \to b(s)$ and $t_n \to t$ plus $b_n(t_n) \to b(t)$. Since $s < t$ we can assume without loss of generality that $s_n < t_n$. Furthermore, since $s,t \in F_b$ and $U$ is open it follows that for all $n$ large enough we have $s_n, t_n \in F_{b_n} \cap U$. Consequently,

$$b(s) = \lim_{n\to\infty} b_n(s_n) \leq \lim_{n\to\infty} b_n(t_n) = b(t)$$

which completes the proof. □

**Lemma 2.1.10.** *Assume $b_n \xrightarrow{\Gamma} b$ and that every $b_n$ is concave (convex). Then $b$ is concave (convex).*

*Proof.* Let $s,t \in [0,\infty]$ with $s < t$ and $\alpha \in (0,1)$. Define $r := (1-\alpha)s + \alpha t$. As first case, let $b_n$ be concave for all $n$. By Theorem 2.1.4 there exists a sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_n \to r$ and $b_n(r_n) \to b(r)$. Without loss of generality we assume that $r_n \in (s,t)$ for all $n$. Further, set $\alpha_n := \frac{r_n - s}{t-s}$. By the concavity of $b_n$ it holds

$$\begin{aligned} b((1-\alpha)s + \alpha t) &= \lim_{n\to\infty} b_n((1-\alpha_n)s + \alpha_n t) \\ &\geq \liminf_{n\to\infty} (1-\alpha_n) b_n(s) + \alpha_n b_n(t) \\ &\geq (1-\alpha) \liminf_{n\to\infty} b_n(s) + \alpha \liminf_{n\to\infty} b_n(t) \\ &\geq (1-\alpha) b(s) + \alpha b(t) \end{aligned}$$

which means that $b$ is concave. Now, assume the case that $b_n$ is convex for all $n$. Then there exist sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ such that $t_n \to t$ and

$s_n \to s$ with $b_n(t_n) \to b(t)$ and $b_n(s_n) \to b(s)$. With this at hand follows in view of Theorem 2.1.4 that

$$\begin{aligned} b((1-\alpha)s+\alpha t) &\leq \liminf_{n\to\infty} b_n((1-\alpha)s_n+\alpha t_n) \\ &\leq \liminf_{n\to\infty}(1-\alpha)b_n(s_n)+\alpha b_n(t_n) = (1-\alpha)b(s)+\alpha b(t) \end{aligned}$$

which implies that $b$ is convex. □

### 2.1.2 Existence of solutions by discrete approximation

The purpose of this subsection is to extend the result of [Anu80] to the case where the Brownian motion starts from an arbitrary (random) starting point. In fact, it is straightforward to apply the techniques used by Anulova. We will label all statements which appeared directly in [Anu80] for the special case of standard Brownian motion as taken from [Anu80].

In the following let $g$ be a survival distribution and $\mu$ a probability distribution on $\mathbb{R}$. The following main theorem by Anulova states that $\mathrm{ifpt}(g,\mu)$ is not empty.

**Theorem 2.1.11** ([Anu80]). *For a probability distribution $\mu$ and every survival distribution $g$ there exists a boundary function $b$ such that $\tau_b$ is distributed according to $g$, i.e. $\mathbb{P}_\mu(\tau_b > t) = g(t)$ for all $t \geq 0$.*

In the previous subsection we have already mentioned that Anulova originally worked with barriers instead of boundary functions. Since we want to work in the direct setting of the first-passage time, we will carry out the approach of [Anu80] in terms of boundary functions and use the Γ-convergence instead of the Hausdorff distance. This has also the advantage that some technicalities of [Anu80] can be avoided.

In order to prove Theorem 2.1.11 the key step for using the compactness of the set of boundary functions is the following proposition.

**Proposition 2.1.12.** *Let $b$ be a boundary function and $b_n \xrightarrow{\Gamma} b$. Suppose one of the following conditions is fulfilled:*

*(i)* $\liminf_{n\to\infty} \tau_{b_n} > 0$ *almost surely,*

*(ii) there is a survival distribution $g$, such that $\tau_{b_n} \to g$.*

*Then $\tau_{b_n} \xrightarrow{\mathbb{P}} \tau_b$.*

The proof of Proposition 2.1.12 concerning (ii) is a generalization of an argument of Anulova used in the proof of Theorem 1 of [Anu80] to an arbitrary starting point, but the argument appeared to be valid also for (i) as condition, and proved to be very useful in the further analysis of the inverse first-passage time problem.

In the following we will prepare the proof of Proposition 2.1.12 by several individual statements. We define

$$\bar{\tau}_b := \inf\{t \geq 0 : |X_t| \geq b(t)\}.$$

The next statement shows that $\mathbb{P}(\bar{\tau}_b = 0) = 0$ implies condition (i) of Proposition 2.1.12.

**Lemma 2.1.13** ([Anu80])**.** *For $n \in \mathbb{N}$ let $b_n, b$ be boundary functions, such that $b_n \xrightarrow{\Gamma} b$. Then*

$$\bar{\tau}_b \leq \liminf_{n\to\infty} \tau_{b_n}$$

*almost surely.*

*Proof.* Abbreviate $s_n := \tau_{b_n} = \inf\{t > 0 : |X_t| \geq b_n(t)\}$. Since $b_n$ is lower semicontinuous and $t \mapsto X_t$ is almost surely continuous, we have that $|X_{s_n}| \geq b_n(s_n)$ almost surely. Let $(n_k)_{k\in\mathbb{N}}$ be a (possibly random) sequence such that $s_{n_k} \to s := \liminf_{n\to\infty} s_n$. Then we have by the $\Gamma$-convergence that

$$b(s) \leq \liminf_{n\to\infty} b_n(s_n) \leq \liminf_{n\to\infty} |X_{s_n}| = |X_s| \quad \text{a.s.},$$

which implies that $\bar{\tau}_b \leq s$. □

For $\varepsilon > 0$ and a boundary function $b$, note that $t \mapsto b(t) + \varepsilon$ is again a boundary function.

**Lemma 2.1.14** ([Anu80])**.** *Let $\varepsilon > 0$. For $n \in \mathbb{N}$ let $b_n, b$ be boundary functions, such that $b_n \xrightarrow{\Gamma} b$. Then*

$$\tau_{b+\varepsilon} \geq \limsup_{n\to\infty} \tau_{b_n}$$

*almost surely.*

*Proof.* Without loss of generality, assume that $t := \tau_{b+\varepsilon} < \infty$. Due to the lower semicontinuity we have almost surely that $|X_t| \geq b(t) + \varepsilon$. By the $\Gamma$-convergence let $t_n \to t$ be a (possibly random) sequence such that $b_n(t_n) \to b(t)$. Let $N \in \mathbb{N}$ be large enough such that for $n \geq N$ on the one hand $b_n(t_n) \leq b(t) + \frac{\varepsilon}{2}$ and on the other hand $|X_{t_n}| \geq b(t) + \frac{\varepsilon}{2}$. It follows

$$\sup_{n\geq N} \tau_{b_n} \leq \sup_{n\geq N} t_n.$$

By letting $N \to \infty$ we obtain $\limsup_{n\to\infty} \tau_{b_n} \leq t = \tau_{b+\varepsilon}$. □

For $s > 0$ and a boundary function $b$ recall the definition $b^s(t) = \mathbb{1}_{[0,s)}(t)\infty + \mathbb{1}_{[s,\infty]}(t)b(t)$.

**Lemma 2.1.15** ([Anu80]). *Let $b$ be a boundary function such that there is $s > 0$ such that $b = b^s$, this is $b(t) = \infty$ for all $t < s$. Then*

$$\tau_b = \lim_{\varepsilon \searrow 0} \tau_{b+\varepsilon}$$

*almost surely.*

*Proof.* In the following denote $W_t := X_t - X_0$. For $b$ write

$$\begin{aligned}\tau_b^+ &:= \inf\{t > 0 : X_t \geq b(t)\},\\ \tau_b^- &:= \inf\{t > 0 : -X_t \leq b(t)\}.\end{aligned}$$

Note that $\tau_b = \tau_b^+ \wedge \tau_b^-$. In view of symmetry in suffices to prove $\tau_b^+ = \lim_{\varepsilon \searrow 0} \tau_{c+\varepsilon}^+$ a.s.. Note that $\tau_{b+\varepsilon}^+$ is increasing in $\varepsilon$. Thus the limit $\lim_{\varepsilon \searrow 0} \tau_{b+\varepsilon}^+$ exist and $\tau_b^+ \leq \lim_{\varepsilon \searrow 0} \tau_{b+\varepsilon}^+$. Assume that $\mathbb{P}\left(\tau_b^+ < \lim_{\varepsilon \searrow 0} \tau_{b+\varepsilon}^+\right) > 0$. Then, since $\tau_{b+\varepsilon}^+, \tau_b^+ \geq s$, there exists $u \geq s$ with

$$0 < \mathbb{P}\left(\tau_b^+ \leq u < \lim_{\varepsilon \searrow 0} \tau_{b+\varepsilon}^+\right) = \mathbb{P}\left(\lim_{\varepsilon \searrow 0} \tau_{b+\varepsilon}^+ > u\right) - \mathbb{P}\left(\tau_b^+ > u\right). \tag{2.4}$$

Hence $\mathbb{P}\left(\tau_b^+ > u\right) < \mathbb{P}\left(\lim_{\varepsilon \searrow 0} \tau_{b+\varepsilon}^+ > u\right)$. Write

$$\begin{aligned}\mathbb{P}\left(\tau_{b+\varepsilon}^+ > u\right) &= \mathbb{P}\left(X_t < b(t) + \varepsilon \ \forall t \leq u\right) = \mathbb{P}\left(X_0 + W_t < b(t) + \varepsilon \ \forall t \in [s, u]\right)\\ &= \int_{\mathbb{R}} \mathbb{P}\left(W_t < b(t) + \varepsilon - x \ \forall t \in [s, u]\right) \mathrm{d}\mu(x).\end{aligned}$$

Defining $\theta_r := \mathbb{1}_{\{r \leq s\}} \frac{\varepsilon}{s}$ we we can verify that the Novikov condition is fulfilled by

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^u \theta_r^2 \,\mathrm{d}r\right)\right] = \mathbb{E}\left[\exp\left(\frac{\varepsilon^2}{2s}\right)\right] < \infty.$$

Set $\tilde{W}_t = W_t - \int_0^t \theta_r \,\mathrm{d}r$ and let $\tilde{\mathbb{P}}$ be the measure on $\sigma(X_s : s \in [0, u])$ given by

$$\tilde{\mathbb{P}} = \exp\left(\int_0^u \theta_r \,\mathrm{d}\tilde{W}_r + \frac{1}{2}\int_0^u \theta_r^2 \,\mathrm{d}r\right) \mathrm{d}\mathbb{P}.$$

We find for any $\varepsilon > 0$ by Girsanov's theorem that

$$\begin{aligned}&\mathbb{P}\left(W_t < b(t) + \varepsilon - x \ \forall t \in [s, u]\right)\\ &= \mathbb{P}\left(W_t - \int_0^t \theta_r \,\mathrm{d}r < b(t) - x \ \forall t \in [s, u]\right)\\ &= \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tilde{W}_t < b(t) - x \ \forall t \in [s,u]\}} \exp\left(-\int_0^u \theta_r \,\mathrm{d}\tilde{W}_r - \frac{1}{2}\int_0^u \theta_r^2 \,\mathrm{d}r\right)\right]\\ &= \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tilde{W}_t < b(t) - x \ \forall t \in [s,u]\}} \exp\left(-\frac{\varepsilon}{s}\tilde{W}_s - \frac{\varepsilon^2}{2s}\right)\right]\\ &= \mathbb{E}\left[\mathbb{1}_{\{W_t < b(t) - x \ \forall t \in [s,u]\}} \exp\left(-\frac{\varepsilon}{s}W_s - \frac{\varepsilon^2}{2s}\right)\right].\end{aligned}$$

But, as $\varepsilon \to 0$ by the dominated convergence theorem, where we use as bound that the normal distribution has exponential moments, we obtain

$$\mathbb{E}\left[\mathbb{1}_{\{W_t<b(t)-x \ \forall t\in[s,u]\}} \exp\left(-\frac{\varepsilon}{s}W_s - \frac{\varepsilon^2}{2s}\right)\right] \\ \to \mathbb{P}\left(W_t + x < b(t) \ \forall t \in [s,u]\right).$$

In conclusion, by the Fatou lemma

$$\begin{aligned}\mathbb{P}\left(\lim_{\varepsilon\searrow 0} \tau^+_{b+\varepsilon} > u\right) &\leq \mathbb{P}\left(\liminf_{\varepsilon\searrow 0}\{\tau^+_{b+\varepsilon} > u\}\right) \leq \liminf_{\varepsilon\searrow 0} \mathbb{P}\left(\tau^+_{b+\varepsilon} > u\right) \\ &\leq \limsup_{\varepsilon\searrow 0} \int_{\mathbb{R}} \mathbb{P}\left(W_t < b(t) + \varepsilon - x \ \forall t \in [s,u]\right) \mathrm{d}\mu(x) \\ &\leq \int_{\mathbb{R}} \limsup_{\varepsilon\searrow 0} \mathbb{P}\left(W_t < b(t) + \varepsilon - x \ \forall t \in [s,u]\right) \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}} \mathbb{P}\left(W_t + x < b(t) \ \forall t \in [s,u]\right) \mathrm{d}\mu(x) \\ &= \mathbb{P}\left(X_t < b(t) \ \forall t \in [s,u]\right) = \mathbb{P}\left(\tau^+_b > u\right),\end{aligned}$$

which is a contradiction to (2.4). Eventually, $\mathbb{P}\left(\tau^+_b = \lim_{\varepsilon\searrow 0} \tau^+_{b+\varepsilon}\right) = 1$. □

**Lemma 2.1.16** ([Anu80])**.** *Let $b$ be a boundary function with the property that there exists $s > 0$ such that $b = b^s$. Then, if $b_n \xrightarrow{\Gamma} b$,*

$$\tau_b = \lim_{n\to\infty} \tau_{b_n}$$

*almost surely.*

*Proof.* Let $m \in \mathbb{N}$. Then, since $\tau_b = \bar{\tau}_b$, we have

$$\tau_b \leq \liminf_{n\to\infty} \tau_{b_n} \leq \limsup_{n\to\infty} \tau_{b_n} \leq \tau_{b+\frac{1}{m}}$$

almost surely. By letting $m \to \infty$ we obtain by Lemma 2.1.15 that $\tau_b = \lim_{n\to\infty} \tau_{b_n}$ almost surely. □

**Lemma 2.1.17** ([Anu80])**.** *Let $b$ be a boundary function. Then*

$$\tau_b = \lim_{s\searrow 0} \tau_{b^s}$$

*almost surely.*

*Proof.* For $0 < s_1 \leq s_2$ we have $\tau_b \leq \tau_{b^{s_1}} \leq \tau_{b^{s_2}}$ the limit exists and $\tau_b \leq \lim_{s\searrow 0} \tau_{b^s}$. Further, let $m \in \mathbb{N}$. Since in the definition of $\tau_b$ the infimum is taken within a subset of $(0, \infty)$, we have that, almost surely, there exist a random time $T > 0$ with $T \in [\tau_b, \tau_b + \frac{1}{m})$ such that $|X_T| \geq b(T)$. For any $s_T < \frac{T}{2}$, this implies $|X_T| > b^{s_T}(T)$ and consequently $\tau_b \leq \lim_{s\searrow 0} \tau_{b^s} \leq \tau_{b^{s_T}} \leq \tau_b + \frac{1}{m}$. By $m \to \infty$ we obtain $\lim_{s\searrow 0} \tau_{b^s} = \tau_b$ almost surely. □

We can now assemble the statements above to prove Proposition 2.1.12.

*Proof of Proposition 2.1.12.* Let $\varepsilon > 0$ and $s > 0$. Then

$$\begin{aligned}
&\mathbb{P}\left(|\tau_{b_n} - \tau_b| > \varepsilon\right) \\
&\leq \mathbb{P}\left(|\tau_{b_n^s} - \tau_{b_n}| > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(|\tau_{b_n^s} - \tau_{b^s}| > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(|\tau_{b^s} - \tau_b| > \frac{\varepsilon}{3}\right) \\
&\leq \mathbb{P}\left(\tau_{b_n} \leq s\right) + \mathbb{P}\left(|\tau_{b_n^s} - \tau_{b^s}| > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(|\tau_{b^s} - \tau_b| > \frac{\varepsilon}{3}\right),
\end{aligned}$$

where we have used that

$$\mathbb{P}\left(|\tau_{b_n^s} - \tau_{b_n}| > \frac{\varepsilon}{3}\right) \leq \mathbb{P}\left(\tau_{b_n^s} \neq \tau_{b_n}\right) \leq \mathbb{P}\left(\tau_{b_n} \leq s\right).$$

By using the fact that by Lemma 2.1.6 and Lemma 2.1.16 we have that $\tau_{b_n^s} \to \tau_{b^s}$ a.s., we obtain

$$\limsup_{n\to\infty} \mathbb{P}\left(|\tau_{b_n} - \tau_b| > \varepsilon\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(\tau_{b_n} \leq s\right) + \mathbb{P}\left(|\tau_{b^s} - \tau_b| > \frac{\varepsilon}{3}\right).$$

By letting $s \searrow 0$ and using Lemma 2.1.17 we obtain

$$\limsup_{n\to\infty} \mathbb{P}\left(|\tau_{b_n} - \tau_b| > \varepsilon\right) \leq \lim_{s\searrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\tau_{b_n} \leq s\right).$$

Thus it is left to show that $\lim_{s\searrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\tau_{b_n} \leq s\right) = 0$.

Assume condition (i). With the Fatou lemma and in view of Lemma 2.1.13 it holds that

$$\limsup_{n\to\infty} \mathbb{P}\left(\tau_{b_n} \leq s\right) \leq \mathbb{P}\left(\limsup_{n\to\infty}\{\tau_{b_n} \leq s\}\right) = \mathbb{P}\left(\liminf_{n\to\infty} \tau_{b_n} \leq s\right).$$

Letting $s \to 0$ yields the desired statement.

Assume condition (ii). Define $g_n(t) := \mathbb{P}\left(\tau_{b_n} > t\right)$. For $s > 0$ take a sequence of continuity points $t_\ell \searrow s$ of $g$. For all $\ell \in \mathbb{N}$ it holds that

$$\limsup_{n\to\infty} \mathbb{P}\left(\tau_{b_n} \leq s\right) = \limsup_{n\to\infty} 1 - g_n(s) \leq \limsup_{n\to\infty} 1 - g_n(t_\ell) = 1 - g(t_\ell).$$

By letting $\ell \to \infty$ and using the right continuity of $g$ we obtain that the limit $\limsup_{n\to\infty} \mathbb{P}\left(\tau_{b_n} \leq s\right)$ is bounded from above by $1 - g(s)$, which tends to 0 as $s \to 0$. Thus the desired statement follows. □

Before we continue with the proof of Theorem 2.1.11, we extract a further argument of Anulova as general statement in order to use it later in this thesis.

**Lemma 2.1.18.** *Let $g$ be a survival distribution. For $n \in \mathbb{N}$ let $\delta^{(n)} > 0$, such that $\delta^{(n)} \to 0$ as $n \to \infty$. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of boundary functions such that $\mathbb{P}_\mu\left(\tau_{b_n} > k\delta^{(n)}\right) = g(k\delta^{(n)})$ for all $1 \leq k \leq \lfloor t^g/\delta^{(n)}\rfloor$. Then $\tau_{b_n} \to g$ in distribution as $n \to \infty$.*

*Proof.* Let $t \in (0, t^g)$ be a continuity point of $g$. Let $t_n^- := \lfloor t/\delta^{(n)} \rfloor \delta^{(n)}$ and $t_n^+ := \lceil t/\delta^{(n)} \rceil \delta^{(n)}$. For $n \in \mathbb{N}$ large enough we have $0 < t_n^- \leq t \leq t_n^+ < t^g$ and we have $t_n^\pm \to t$. Observe that by assumption we have

$$g(t_n^+) = \mathbb{P}_\mu \left( \tau_{b_n} > t_n^+ \right) \leq \mathbb{P}_\mu \left( \tau_{b_n} > t \right) \leq \mathbb{P}_\mu \left( \tau_{b_n} > t_n^- \right) = g(t_n^-).$$

By $t_n^+ \to t$ and $t_n^- \to t$ it follows that

$$\mathbb{P} \left( \tau_{b_n^{\mathrm{pc}}} > t \right) \to g(t),$$

since $t$ was a continuity point. The cases $t \in \{0, t^g\}$ are analogous with only $t_n^+$ or $t_n^-$, respectively. □

Now we are ready to prove the generalized version of the theorem of Anulova.

*Proof of Theorem 2.1.11.* For $n \in \mathbb{N}$ let $b_n$ be a boundary function inductively defined as follows. Set $b_n(t) := \infty$ for $t \notin \{k2^{-n} : k \in \mathbb{N}\}$ and given $b_n(k2^{-n})$ let $b_n((k+1)2^{-n})$ be such that

$$\mathbb{P}_\mu \left( |X_{2^{-n}}| < b_n(2^{-n}), \ldots, |X_{(k+1)2^{-n}}| < b_n((k+1)2^{-n}) \right) = g((k+1)2^{-n}).$$

By construction we have then $\mathbb{P}_\mu \left( \tau_{b_n} > k2^{-n} \right) = g(k2^{-n})$ for every $k \in \mathbb{N}$ and thus the requirements of Lemma 2.1.18 are fulfilled with $\delta^{(n)} = 2^{-n}$. Hence $\tau_{b_n} \to g$ in distribution. The compactness of $\mathcal{B}$ implies that there exists a subsequence $(b_{n_k})_{k \in \mathbb{N}}$ with a limit point with respect to the Hausdorff topology, which we denote with $b$. Since $\tau_{b_{n_k}}$ as a subsequence also converges in distribution to $g$, condition (ii) of Proposition 2.1.12 is fulfilled, and thus we obtain that $\tau_{b_{n_k}} \to \tau_b$ in probability. From this it is clear that $\tau_{b_{n_k}} \to \tau_b$ in distribution. Therefore, the law of $\tau_b$ must be the law given by $g$, which yields $b \in \mathrm{ifpt}(g, \mu)$. □

## 2.2 Properties of Gaussian convolution and truncation

In the beginning of this chapter we have motivated that our aim is to discretize the marginal distribution of the conditioned process given by (2.1) for a possible solution $b$ of the inverse first-passage time problem. This discretization was proposed in terms of the truncating function $T^c$, $c \in [0, \infty]$, given by

$$T^c(\mu) := \mu(\,\cdot\,|[-c, c]) \tag{2.5}$$

for a probability measure $\mu$ with $\mu([-c, c]) > 0$.

A slightly different representation of the proposed discretization in (2.3), which is more suitable from the view of the inverse problem, can be found by taking into account the fact that the approximants of the solution $b$ are constructed as certain quantiles. Denote with $\mathcal{P}$ the Borel probability measures on $\mathbb{R}$. Let us recall that the $\alpha$-truncating function $T_\alpha : \mathcal{P} \to \mathcal{P}$, $\alpha \in (0, 1]$ from (1.7) is given by

$$T_\alpha(\mu) := \mu(\,\cdot\,|[-q_\alpha, q_\alpha]) = T^{q_\alpha}(\mu), \tag{2.6}$$

where $q_\alpha := q_\alpha(\mu) := \inf\{c \geq 0 : \mu([-c, c]) \geq \alpha\}$. Furthermore, denote the convolution operator on $\mathcal{P}$ corresponding to a centered normal distribution with $P_t$, i.e.

$$P_t\mu := \mu * \mathcal{N}(0, t) = \mathbb{P}_\mu\left(X_t \in \cdot\,\right), \tag{2.7}$$

where $(X_t)_{t\geq 0}$ is a Brownian motion with $X_0 \sim \mu$ and $\mathcal{N}(0, t)$ denotes the normal distribution with mean 0 and variance $t$. Then another representation of (2.3) is given by

$$T_{\alpha_k^n} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots \circ T_{\alpha_1^n} \circ P_{t_1^n}(\mu), \tag{2.8}$$

where $\alpha_k^n := g(t_k^n)/g(t_{k-1}^n)$. In order to analyze the quantity of (2.8), we will study the individual and joint effects of $T^c$, $T_\alpha$ and $P_t$ on the two-sided usual stochastic order, on the two-sided likelihood ratio order, on probability distances and related aspects.

### 2.2.1 Usual stochastic order: Gaussian convolution, truncation

Recall that $\mathcal{P}$ denotes the set of probability measures on $\mathbb{R}$. We begin with the definition of the well-known usual stochastic order of probability measures and a two-sided modification of it. In the context of the inverse first-passage time problem we will be mainly concerned with the two-sided version. In this subsection we study the effect of the convolution operator and the $\alpha$-truncation on this two-sided stochastic order.

**Definition 2.2.1.** *Let $\mu, \nu \in \mathcal{P}$.*

*(i) $\mu$ is smaller than $\nu$ in the usual stochastic order, we write $\mu \preceq_{\mathrm{st}} \nu$, if*

$$\mu((-\infty, c]) \geq \nu((-\infty, c]) \ \forall c \geq 0.$$

*(ii) We say that $\nu$ dominates $\mu$ in the two-sided stochastic order, write $\mu \preceq \nu$, if*

$$\mu([-c, c]) \geq \nu([-c, c]) \ \forall c \geq 0,$$

*and call a mapping $F : \mathcal{P} \to \mathcal{P}$ dominance preserving for $\mu$ and $\nu$ with respect to $\preceq$, if $\mu \preceq \nu$ implies $F(\mu) \preceq F(\nu)$.*

*(iii) Define the* modulus law *of $\mu$ by*

$$\overline{\mu}(A) := \mu(-A \cap (-\infty, 0)) + \mu(A \cap \{0\}) + \mu(A \cap (0, \infty)).$$

*(iv) We call a measure $\mu \in \mathcal{P}$ symmetric, if $\mu(-A) = \mu(A)$ for any measurable $A \subseteq \mathbb{R}$.*

Note that for $c \geq 0$ we have

$$\overline{\mu}((-\infty, c]) = \overline{\mu}([-c, c]) = \mu([-c, 0)) + \mu(\{0\}) + \mu((0, c]) = \mu([-c, c]).$$

It follows that $\mu \preceq \nu$ if and only if $\overline{\mu} \preceq \overline{\nu}$ if and only if $\overline{\mu} \preceq_{\mathrm{st}} \overline{\nu}$. Further, note that $\overline{\mu}$ is the law of $|X|$, if $X \sim \mu$.

The well-known usual stochastic order admits the intuitive property, that $\mu$ is smaller that $\nu$ in the usual stochastic order, if and only if one can find a coupling $X \sim \mu$ and $Y \sim \nu$ such that $X \leq Y$ almost surely, see e.g. [Tho00]. The two-sided stochastic order has an analogous property.

**Lemma 2.2.2.** *Let $\mu$ and $\nu$ be probability measures. Then $\mu \preceq \nu$ if and only if there exists a pair of random variables $(X, Y)$ such that $X \sim \mu$ and $Y \sim \nu$ with $|X| \leq |Y|$.*

*Proof.* First, let $X, Y$ be random variables such that $X \sim \mu$ and $Y \sim \nu$ with $|X| \leq |Y|$. Then for $c \geq 0$ we have

$$\mu([-c, c]) = \mathbb{P}\left(X \in [-c, c]\right) = \mathbb{P}\left(|X| \leq c\right) \geq \mathbb{P}\left(|Y| \leq c\right) = \nu([-c, c]).$$

Conversely, let $\mu \preceq \nu$. Now as first step we assume that both $\mu$ and $\nu$ have finite support and write them in the following form

$$\mu = \sum_{i=1}^{n} p_i \delta_{x_i}, \quad \nu = \sum_{j=1}^{m} q_j \delta_{y_j},$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $|x_1| \leq \ldots \leq |x_n|$ and $|y_1| \leq \ldots \leq |y_m|$ and $p_1 + \ldots + p_n = 1 = q_1 + \ldots + q_m$ with $p_i, q_j > 0$. Define $P_k \coloneqq \sum_{i=1}^k p_i$ and $Q_\ell \coloneqq \sum_{j=1}^\ell q_j$. Let $U \sim \mathcal{U}(0,1)$ and define

$$X \coloneqq \sum_{k=1}^n \mathbb{1}_{(P_{k-1}, P_k]}(U) \cdot x_k, \quad Y \coloneqq \sum_{\ell=1}^m \mathbb{1}_{(Q_{\ell-1}, Q_\ell]}(U) \cdot y_\ell.$$

We observe that $X \sim \mu$ and $Y \sim \nu$. Furthermore, observe that the distribution functions of $|X|$ and $|Y|$ are given by

$$F_{|X|}(c) = \mathbb{P}\left(|X| \leq c\right) = \begin{cases} \mu([-c, c]) & : c \geq 0, \\ 0 & : c < 0, \end{cases}$$
$$F_{|Y|}(c) = \mathbb{P}\left(|Y| \leq c\right) = \begin{cases} \nu([-c, c]) & : c \geq 0, \\ 0 & : c < 0. \end{cases}$$

Thus $F_{|X|} \geq F_{|Y|}$. This implies that

$$\sum_{k=1}^n \mathbb{1}_{(P_{k-1}, P_k]}(t) \cdot |x_k| = \inf\{c \in \mathbb{R} : F_{|X|}(c) \geq t\}$$
$$\leq \inf\{c \in \mathbb{R} : F_{|Y|}(c) \geq t\} = \sum_{\ell=1}^m \mathbb{1}_{(Q_{\ell-1}, Q_\ell]}(t) \cdot |y_\ell|.$$

Therefore we have that

$$|X| = \sum_{k=1}^n \mathbb{1}_{(P_{k-1}, P_k]}(U) \cdot |x_k| \leq \sum_{\ell=1}^m \mathbb{1}_{(Q_{\ell-1}, Q_\ell]}(U) \cdot |y_\ell| = Y.$$

For the next step let $\mu, \nu$ be arbitrary and $X \sim \mu$ and $Y \sim \nu$ on the same probability space. Define

$$[X]_n \coloneqq \operatorname{sgn}(X) \sup\left\{\frac{k}{n} \leq |X| : k = 1, \ldots, n^2\right\},$$
$$]Y[_n \coloneqq \operatorname{sgn}(Y) \left(\inf\left\{\frac{k}{n} \geq |Y| : k = 1, \ldots, n^2\right\} \wedge n\right),$$

where $\operatorname{sgn}(0) \coloneqq 0$. Let $\mu_n \coloneqq \mathbb{P}\left([X]_n \in \cdot\,\right)$ and $\nu_n \coloneqq \mathbb{P}\left(]Y[_n \in \cdot\,\right)$. Since $|[X]_n| \leq |X|$ and $|]Y[_n| \geq |Y|$ we have for $c \geq 0$ that

$$\mu_n([-c, c]) = \mathbb{P}\left(|[X]_n| \leq c\right) \geq \mathbb{P}\left(|X| \leq c\right) = \mu([-c, c])$$
$$\geq \nu([-c, c]) = \mathbb{P}\left(|Y| \leq c\right) = \mathbb{P}\left(|]Y[_n| \leq c\right) = \nu_n([-c, c]).$$

Therefore $\mu_n \preceq \nu_n$. But additionally we have $[X]_n \to X$ and $]Y[_n \to Y$ almost surely, hence $\mu_n \to \mu$ and $\nu_n \to \nu$. By the result of the first step there exist pairs of random variables $(X_n, Y_n)$ such that $X_n \sim \mu_n$ and $Y_n \sim \nu_n$ with

$|X_n| \leq |Y_n|$. Define $\varrho_n := \mathbb{P}((X_n, Y_n) \in \cdot\,)$. Since $\mu_n \to \mu$ and $\nu_n \to \nu$ we have by Prokhorov's theorem that the families $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ are tight. Let $\varepsilon > 0$. Choose compact sets $K_1, K_2 \subseteq \mathbb{R}$ such that $\mathbb{P}(X_n \notin K_1) = \mu_n(\mathbb{R} \setminus K_1) < \frac{\varepsilon}{2}$ and $\mathbb{P}(Y_n \notin K_2) = \nu_n(\mathbb{R} \setminus K_2) < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\varrho_n(K_1 \times K_2) &= \mathbb{P}(X_n \in K_1, Y_n \in K_2)\\ &\geq 1 - \mathbb{P}(X_n \notin K_1) - \mathbb{P}(Y_n \notin K_2)\\ &> 1 - \varepsilon\end{aligned}$$

and thus $(\varrho_n)_{n\in\mathbb{N}}$ is tight. Let $(\varrho_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence with limit $\varrho$. Let $\pi_i : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_i$. We obtain

$$\mu = \lim_{k\to\infty} \mu_{n_k} = \lim_{k\to\infty} \varrho_{n_k} \circ \pi_1^{-1} = \varrho \circ \pi_1^{-1}$$

as well as

$$\nu = \lim_{k\to\infty} \nu_{n_k} = \lim_{k\to\infty} \varrho_{n_k} \circ \pi_2^{-1} = \varrho \circ \pi_2^{-1}.$$

Furthermore, note that the set $A := \{(x, y) \in \mathbb{R}^2 : |x| \leq |y|\}$ is closed. Therefore by the Portmanteau theorem we get

$$\begin{aligned}\varrho(A) &\geq \limsup_{k\to\infty} \varrho_{n_k}(A) = \limsup_{k\to\infty} \mathbb{P}((X_n, Y_n) \in A)\\ &= \limsup_{k\to\infty} \mathbb{P}(|X_n| \leq |Y_n|) = 1.\end{aligned}$$

Therefore, on the probability space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \varrho)$ the pair of random variables $(\pi_1, \pi_2)$ has the desired properties. □

We now begin with an analysis of the dominance preservation properties of the operators, which appear in our approach.

**Proposition 2.2.3.** *For every $t > 0$ the operator $P_t$ is dominance preserving, i.e. if $\mu \preceq \nu$ then*

$$P_t\mu \preceq P_t\nu.$$

*Proof.* Let $\phi_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$. We have to show that $P_t\mu([-c, c]) \geq P_t\nu([-c, c])$, which is equivalent to

$$\begin{aligned}0 &\leq \int_{-c}^{c} \int_{\mathbb{R}} \phi_t(x - y)\,\mathrm{d}(\mu - \nu)(y)\,\mathrm{d}x = \int_{\mathbb{R}} \int_{-c}^{c} \phi_t(y - x)\,\mathrm{d}x\,\mathrm{d}(\mu - \nu)(y)\\ &= \int_{\mathbb{R}} \int_{y-c}^{y+c} \phi_t(x)\,\mathrm{d}x\,\mathrm{d}(\mu - \nu)(y) = \int_{\mathbb{R}} \omega(y)\,\mathrm{d}\mu(y) - \int_{\mathbb{R}} \omega(y)\,\mathrm{d}\nu(y)\\ &= \int_{[0,\infty)} \omega(y)\,\mathrm{d}\overline{\mu}(y) - \int_{[0,\infty)} \omega(y)\,\mathrm{d}\overline{\nu}(y)\end{aligned}$$

with the symmetric function $\omega(y) := \int_{y-c}^{y+c} \phi_t(x)\,\mathrm{d}x$. Additionally $\omega$ is nonincreasing on $[0,\infty)$, which can be seen by

$$\omega'(|y|) = \phi_t(|y|+c) - \phi_t(|y|-c) \le 0.$$

The desired inequality is a consequence of Lemma B.1.3. □

The statement of Proposition 2.2.3 could also be shown by a coupling argument, which in fact enables us to deduce a more general statement.

**Lemma 2.2.4.** *Let $\mu_1 \preceq \mu_2$ and $b : [0,\infty] \to [0,\infty]$ a lower-semicontinuous function. Then we have*

$$\mathbb{P}_{\mu_2}(\tau_b \in \cdot\,) \preceq_{\mathrm{st}} \mathbb{P}_{\mu_1}(\tau_b \in \cdot\,).$$

*Proof.* If $\mu_1 \preceq \mu_2$ then by Lemma 2.2.2 we can assume that there are random variables $X \sim \mu_1$ and $Y \sim \mu_2$ with $|X| \le |Y|$. Let $B - X$, $B^{(2)} - Y$ be two independent standard Brownian motions independent of $X$, $Y$. Let

$$T := \inf\{t > 0 : |B_t| = |B_t^{(2)}|\}.$$

Then

$$B_t^{(1)} := \begin{cases} B_t & : t < T, \\ B_t^{(2)} = B_T + (B_t^{(2)} - B_T^{(2)}) & : t \ge T, \operatorname{sgn}(B_T) = \operatorname{sgn}(B_T^{(2)}), \\ -B_t^{(2)} = B_T - (B_t^{(2)} - B_T^{(2)}) & : t \ge T, \operatorname{sgn}(B_T) \ne \operatorname{sgn}(B_T^{(2)}) \end{cases}$$

defines a Brownian motion with initial datum $B_0^{(1)} = X$.

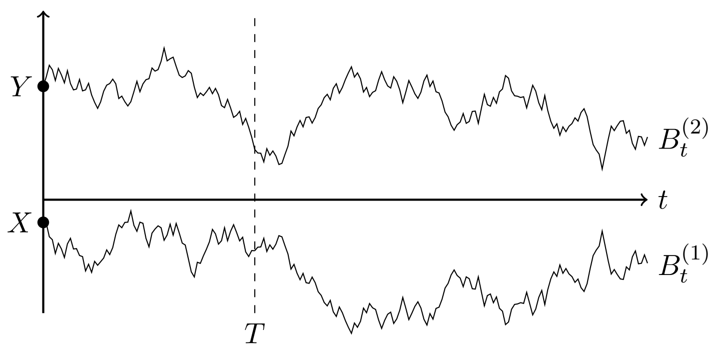


Figure 2.1: The coupling of $B^{(1)}$ and $B^{(2)}$ in the proof of Lemma 2.2.4.

By construction and due to the coupling it follows that $|B_t^{(1)}| \le |B_t^{(2)}|$ for $t \ge 0$. Now, if $b : [0,\infty] \to [0,\infty]$ is a lower-semicontinuous function, $|B_t^{(1)}| \le |B_t^{(2)}|$ for $t \ge 0$ means that

$$\inf\{t > 0 : |B_t^{(1)}| \ge b(t)\} \ge \inf\{t > 0 : |B_t^{(2)}| \ge b(t)\},$$

which implies that $\mathbb{P}_{\mu_2}(\tau_b \in \cdot\,) \preceq_{\mathrm{st}} \mathbb{P}_{\mu_1}(\tau_b \in \cdot\,)$. □

**Remark 2.2.5.** The proof of Lemma 2.2.4 showed that $|B_t^{(1)}| \leq |B_t^{(2)}|$ for $t \geq 0$. By Lemma 2.2.2 and the fact that $B_t^{(i)} \sim P_t\mu_i$ this gives a coupling proof for Proposition 2.2.3.

Now we turn to the order preserving properties of $T_\alpha$. At first, we prove that the restriction to non-atomic measures the function $T_\alpha$ is dominance preserving, which is sufficient in our situation, since the output of $P_t$ is always non-atomic. Note that in general we have $\mu([-q_\alpha(\mu), q_\alpha(\mu)]) \geq \alpha$ and for non-atomic $\mu$ we have $\mu([-q_\alpha(\mu), q_\alpha(\mu)]) = \alpha$, where $q_\alpha$ is defined within (2.6).

**Lemma 2.2.6.** *Let $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 \leq \alpha_2$. Let $\mu, \nu \in \mathcal{P}$, such that $\mu([-q_{\alpha_1}(\mu), q_{\alpha_1}(\mu)]) \leq \nu([-q_{\alpha_2}(\nu), q_{\alpha_2}(\nu)])$ and $\mu \preceq \nu$. Then $T_{\alpha_1}(\mu) \preceq T_{\alpha_2}(\nu)$.*

*Proof.* We have

$$\begin{aligned}\{c \geq 0 : \mu([-c,c]) \geq \alpha_1\} &\supseteq \{c \geq 0 : \nu([-c,c]) \geq \alpha_1\} \\ &\supseteq \{c \geq 0 : \nu([-c,c]) \geq \alpha_2\}.\end{aligned}$$

Thus

$$\begin{aligned}q_{\alpha_1}(\mu) &= \inf\{c \geq 0 : \mu([-c,c]) \geq \alpha_1\} \\ &\leq \inf\{c \geq 0 : \nu([-c,c]) \geq \alpha_2\} = q_{\alpha_2}(\nu).\end{aligned}$$

Let $c \geq 0$. If $c \geq q_{\alpha_1}(\mu)$ we have

$$T_{\alpha_1}(\mu)([-c,c]) = 1 \geq T_{\alpha_2}(\nu)([-c,c]).$$

If $c < q_{\alpha_1}(\mu)$, then

$$\begin{aligned}T_{\alpha_1}(\mu)([-c,c]) &= \frac{\mu([-c,c])}{\mu([-q_{\alpha_1}(\mu), q_{\alpha_1}(\mu)])} \geq \frac{\mu([-c,c])}{\nu([-q_{\alpha_2}(\nu), q_{\alpha_2}(\nu)])} \\ &\geq \frac{\nu([-c,c])}{\nu([-q_{\alpha_2}(\nu), q_{\alpha_2}(\nu)])} = T_{\alpha_2}(\nu)([-c,c]),\end{aligned}$$

which yields that $T_{\alpha_1}(\mu) \preceq T_{\alpha_2}(\nu)$. □

**Remark 2.2.7.** For any $\alpha \in (0,1)$, the mapping $T_\alpha$ is not dominance preserving. We present the following artificial counterexample. Let $\alpha \in (0,1)$ and $\varepsilon \in (0,\alpha)$. Let

$$\begin{aligned}\mu &:= (\alpha - \varepsilon)\delta_0 + (1 + \varepsilon - \alpha)\delta_1, \\ \nu &:= (\alpha - \varepsilon)\delta_0 + \varepsilon\delta_1 + (1 - \alpha)\delta_2.\end{aligned}$$

Then $\mu \preceq \nu$, but

$$\begin{aligned}T_\alpha(\mu) &= \mu, \\ T_\alpha(\nu) &= \frac{\alpha - \varepsilon}{\alpha}\delta_0 + \frac{\varepsilon}{\alpha}\delta_1.\end{aligned}$$

Therefore, $T_\alpha(\mu)(\{0\}) = \alpha - \varepsilon < \frac{\alpha - \varepsilon}{\alpha} = T_\alpha(\nu)(\{0\})$.

For the most situations the following technically simpler corollaries of Lemma 2.2.6 are sufficient.

**Corollary 2.2.8.** *Let $\alpha \in (0,1]$ and $\mu \in \mathcal{P}$, such that $c \mapsto \mu((-\infty,c])$ is continuous. Then $\mu \preceq \nu$ implies $T_\alpha(\mu) \preceq T_\alpha(\nu)$.*

**Corollary 2.2.9.** *Let $\mu \in \mathcal{P}$ and $\alpha \in (0,1]$. Then $T_\alpha(\mu) \preceq \mu$.*

Furthermore the truncation functions have some kind of semigroup property.

**Lemma 2.2.10.** *Let $\alpha, \beta \in (0,1]$ and $\mu \in \mathcal{P}$ such that $c \mapsto \mu((-\infty,c])$ is continuous. Then we have that $T_\alpha \circ T_\beta(\mu) = T_{\alpha\beta}(\mu)$.*

*Proof.* Since for every $c \geq q_\beta(\mu)$ it holds that $T_\beta(\mu)([-c,c]) = 1$, we have that $q_\alpha^\beta := q_\alpha(T_\beta(\mu)) \leq q_\beta(\mu) =: q_\beta$. Furthermore, due to the assumption on $\mu$ and since $T_\beta(\mu)$ is absolutely continuous with respect to $\mu$ we have that $c \mapsto T_\beta(\mu)((-\infty,c])$ is continuous and therefore, as in the proof of Lemma 2.2.6, we have that $T_\beta(\mu)([-q_\alpha^\beta, q_\alpha^\beta]) = \alpha$, $\mu([-q_\beta, q_\beta]) = \beta$ and $\mu([-q_{\alpha\beta}(\mu), q_{\alpha\beta}(\mu)]) = \alpha\beta$. Let $A \subset \mathbb{R}$ be measurable. Then

$$\begin{aligned} T_\alpha \circ T_\beta(\mu)(A) &= \frac{1}{\alpha} T_\beta(\mu)(A \cap [-q_\alpha^\beta, q_\alpha^\beta]) = \frac{1}{\alpha\beta}\mu(A \cap [-q_\alpha^\beta, q_\alpha^\beta] \cap [-q_\beta, q_\beta]) \\ &= \frac{1}{\alpha\beta}\mu(A \cap [-q_\alpha^\beta, q_\alpha^\beta]). \end{aligned}$$

It follows that $q_\alpha^\beta = q_{\alpha\beta}(\mu)$. And thus the statement is proven. ☐

For arbitrary measures we have the following result.

**Proposition 2.2.11.** *Let $\alpha, \beta \in (0,1]$ and $\mu \in \mathcal{P}$. Then we have that $T_{\alpha\beta}(\mu) \preceq T_\alpha \circ T_\beta(\mu)$.*

*Proof.* By definition we have

$$q_\alpha(T_\beta(\mu)) = \inf\{c \geq 0 : T_\beta(\mu)([-c,c]) \geq \alpha\}$$

and $T_\beta([-q_\beta(\mu), q_\beta(\mu)]) = 1 \geq \alpha$, which implies that $q_\alpha(T_\beta(\mu)) \leq q_\beta(\mu)$. Furthermore, we have for $c \geq 0$

$$T_\beta(\mu)([-,c,c]) = \frac{\mu([-c,c] \cap [-q_\beta(\mu), q_\beta(\mu)])}{\mu([-q_\beta(\mu), q_\beta(\mu)])} \leq \frac{\mu([-c,c])}{\beta}.$$

Thus, if $c \geq 0$, the inequality $T_\beta(\mu)([-c,c]) \geq \alpha$ implies $\mu([-c,c]) \geq \alpha\beta$. It follows that $q_{\alpha\beta}(T_\beta(\mu)) \leq q_\alpha(T_\beta(\mu)) \leq q_\beta(\mu)$. By this follows

$$\begin{aligned} T_\alpha \circ T_\beta(\mu) &= T_\beta(\mu)(\cdot \,|[-q_\alpha(T_\beta(\mu)), q_\alpha(T_\beta(\mu))]) \\ &= \mu(\cdot \,|[-q_\alpha(T_\beta(\mu)), q_\alpha(T_\beta(\mu))]). \end{aligned}$$

Additionally, for $c \geq 0$ with $c \leq q_{\alpha\beta}(\mu)$ follows

$$\begin{aligned}
T_{\alpha\beta}(\mu)([-c,c]) &= \frac{\mu([-c,c])}{\mu([-q_{\alpha\beta}(\mu), q_{\alpha\beta}(\mu)])} \\
&\geq \frac{\mu([-c,c])}{\mu([-q_\alpha(T_\beta(\mu)), q_\alpha(T_\beta(\mu))])} \\
&= \mu([-c,c] | [-q_{\alpha\beta}(\mu), q_{\alpha\beta}(\mu)]) = T_\alpha \circ T_\beta(\mu)([-c,c]).
\end{aligned}$$

Since for $c \geq q_{\alpha\beta}(\mu)$ we have $T_{\alpha\beta}(\mu)([-c,c]) = 1 \geq T_\alpha \circ T_\beta(\mu)([-c,c])$ the statement is proved. □

Now we begin with a study of the interaction of $T_\alpha$ and $P_t$.

**Lemma 2.2.12.** *Let $\alpha \in (0,1]$ and $\mu \in \mathcal{P}$ and $t > 0$. Then it holds that*

$$T_\alpha \circ P_t(\mu) \preceq P_t \circ T_\alpha(\mu).$$

*Proof.* As before we have that $\mu([-q_\alpha(\mu), q_\alpha(\mu)]) \geq \alpha$ and, since the mapping $c \mapsto P_t\mu((-\infty, c])$ is continuous, that $P_t\mu([-q_\alpha(P_t\mu), q_\alpha(P_t\mu)]) = \alpha$. Let $c \geq q_\alpha(P_t\mu)$. Then

$$T_\alpha \circ P_t(\mu)([-c,c]) = 1 \geq P_t \circ T_\alpha(\mu)([-c,c]).$$

For $0 \leq c < q_\alpha(P_t\mu)$ we have

$$\begin{aligned}
T_\alpha \circ P_t(\mu)([-c,c]) &= \frac{1}{\alpha} P_t\mu([-c,c]) = \frac{1}{\alpha} \int_{-c}^{c} \int_{-\infty}^{\infty} p_t(x,y) \, \mathrm{d}\mu(x) \, \mathrm{d}y \\
&\geq \frac{1}{\alpha} \int_{-c}^{c} \int_{-q_\alpha(\mu)}^{q_\alpha(\mu)} p_t(x,y) \, \mathrm{d}\mu(x) \, \mathrm{d}y \\
&\geq \frac{1}{\mu([-q_\alpha(\mu), q_\alpha(\mu)])} \int_{-c}^{c} \int_{-q_\alpha(\mu)}^{q_\alpha(\mu)} p_t(x,y) \, \mathrm{d}\mu(x) \, \mathrm{d}y \\
&= \int_{-c}^{c} \int_{-\infty}^{\infty} p_t(x,y) \, \mathrm{d}T_\alpha(\mu)(x) = P_t \circ T_\alpha(\mu)([-c,c]),
\end{aligned}$$

where $p_t(x,y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$ denotes the density of the Gaussian kernel. □

The statement of Lemma 2.2.12 meets the intuition by having in mind that paths which end up far-away from the origin are only punished in the left-hand side of the inequality.

### 2.2.2 Likelihood ratio order: Gaussian convolution, truncation

Recall that our general objective consists of studying the discretization of the marginal distribution from (2.1). The motivation to use the two-sided usual stochastic order was based on the viewpoint of the inverse problem to represent the discretization in terms of the mass truncating operator $T_\alpha$. However, in some applications one has to work with given boundary functions. This corresponds to a viewpoint of the direct problem, and in a study of the discretization of the marginal distribution we have to deal with the level truncation operator $T^c$ from (2.5). In contrary to $T_\alpha$, the truncation of $T^c$ fails to preserve the two-sided usual stochastic order very generally, as we will discuss in more detail in Remark 2.2.22.

In order to work with $T^c$ we introduce the likelihood ratio order, to which we conduct a similar study as in the case of the two-sided usual stochastic order. For sets $A, B \subseteq \mathbb{R}$ we write $A \leq B$ if $(x, y) \in A \times B$ implies $x \leq y$.

A measure $\mu \in \mathcal{P}$ is smaller than $\nu \in \mathcal{P}$ in the usual *likelihood ratio order*, write $\mu \preceq_{\mathrm{lr}} \nu$, if for all measurable $A \leq B$, we have

$$\mu(A)\nu(B) \geq \mu(B)\nu(A).$$

**Definition 2.2.13.** *Let $\mu, \nu \in \mathcal{P}$. We say $\nu$ dominates $\mu$ in the two-sided likelihood ratio order, write $\mu \preceq_{|\mathrm{lr}|} \nu$, if $\overline{\mu} \preceq_{\mathrm{lr}} \overline{\nu}$.*

We will introduce some operations which preserve this ordering. Let $P_t$ denote the operator defined in (2.7). In order to establish the counterpart of Proposition 2.2.3 we will investigate some general properties of the likelihood ratio order and use some results from the theory of total positivity. We begin with the following important characterization of the likelihood ration order from Theorem 1.C.20 in [SS07]. Since the proof in [SS07] was only carried out for absolutely continuous $\mu, \nu$, we will give a general proof.

**Theorem 2.2.14.** *Let $\mu, \nu \in \mathcal{P}$ and $X \sim \mu, Y \sim \nu$ be independent. Then we have $\mu \preceq_{\mathrm{lr}} \nu$ if and only if*

$$\mathbb{P}\left(\phi(X,Y) \in \cdot\,\right) \preceq_{\mathrm{st}} \mathbb{P}\left(\phi(Y,X) \in \cdot\,\right)$$

*for all $\phi \in \{\psi : \mathbb{R}^2 \to \mathbb{R} \text{ measurable} : x \leq y \Rightarrow \psi(x,y) \leq \psi(y,x)\}$.*

As preparation we will first characterize the likelihood ratio order by the following.

**Lemma 2.2.15.** *Let $\mu, \nu \in \mathcal{P}$ and $X \sim \mu, Y \sim \nu$ be independent. Then $\mu \preceq_{\mathrm{lr}} \nu$ if and only if $\pi := \mathbb{P}\left((X,Y) \in \cdot\,\right) - \mathbb{P}\left((Y,X) \in \cdot\,\right)$ is a non-negative measure on $\nabla := \{(x,y) \in \mathbb{R}^2 : x \leq y\}$.*

*Proof.* Suppose $\pi$ is such a measure. Then, $A \leq B$ implies $A \times B \subseteq \nabla$, and it follows immediately, that $\mu \preceq_{\mathrm{lr}} \nu$. Conversely, assume that $\mu \preceq_{\mathrm{lr}} \nu$. $\pi$ is a signed measure on $\mathbb{R}^2$ and for any $F \subseteq \Delta := \{(x,x) : x \in \mathbb{R}\}$ we have $\pi(F) = 0$. Further, we have $\pi(A \times B) \geq 0$ for $A \leq B$. Now, let $\pi_+, \pi_-$ the Hahn decomposition of $\pi$. Since $\pi_+, \pi_-$ are both absolutely continuous with respect to $|\pi|$, by the Radon-Nikodým-theorem there exist $|\pi|$-integrable functions $f_+, f_- \geq 0$, such that $\pi_+ = f_+ \,\mathrm{d}|\pi|$ and $\pi_- = f_- \,\mathrm{d}|\pi|$. Corollary 11.4 from [DiB02] yields that $|\pi|$-almost all points in $\mathbb{R}^2$ are Lebesgue points for $f := f_+ - f_-$. At first, we claim that for any Lebesgue point $(x,y) \in \nabla \setminus \Delta$ we have that $\pi(\overline{B_r(x,y)}) \geq 0$ for $r > 0$ small enough, where $B_r(x,y) := \{(u,v) \in \mathbb{R}^2 : \sqrt{(u-x)^2 + (v-y)^2} < r\}$. If this was true, by the definition of Lebesgue point it would follow that for $(x,y)$ being a Lebesgue point of $\nabla \setminus \Delta$ we have

$$0 \leq \frac{1}{|\pi|(\overline{B_r(x,y)})}\pi(\overline{B_r(x,y)}) = \frac{1}{|\pi|(\overline{B_r(x,y)})}\int_{\overline{B_r(x,y)}} f \,\mathrm{d}|\pi| \to f(x,y),$$

as $r \to 0$, which means that $f(x,y) \geq 0$. Since $\pi \equiv 0$ on $\Delta$, for $A \subset \nabla$ measurable would follow that $\pi(A) = \int_{A\setminus\Delta} f \,\mathrm{d}|\pi| \geq 0$, which means that $\pi$ is a non-negative measure on $\nabla$.

Let us now prove the claim. Let $(x,y) \in \nabla \setminus \Delta$ such a point and $r > 0$, such that $\overline{B_r(x,y)} \subset \nabla\setminus\Delta$. For $n \in \mathbb{N}$ define $\mathcal{D}_n := \{[k2^{-n}, (k+1)2^{-n}) \times [l2^{-n}, (l+1)2^{-n}) : (k,l) \in \mathbb{Z}^2\}$ and $\mathcal{S}_n := \{S \in \mathcal{D}_n : S \subseteq B_r(x,y)\}$. By this construction we obtain that

- $K_n := \bigcup_{S \in \mathcal{S}_n} S \subseteq B_r(x,y)$,
- $S_1, S_2 \in \mathcal{S}_n$ with $S_1 \neq S_2$ implies $S_1 \cap S_2 = \emptyset$,
- $K_n \subset K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = B_r(x,y)$,
- $\mathcal{S}_n \subseteq \{A \times B : A \leq B\}$ and $|\mathcal{S}_n| < \infty$.

Since $\pi$ is a finite measure we have by $\sigma$-continuity

$$\pi(B_r(x,y)) = \lim_{n\to\infty} \pi(K_n) = \lim_{n\to\infty} \sum_{S\in\mathcal{S}_n} \pi(S) \geq 0.$$

By observing that

$$\pi(\overline{B_r(x,y)}) = \lim_{n\to\infty} \pi(B_{r+\frac{1}{n}}(x,y)) \geq 0$$

the claim follows. □

*Proof of Theorem 2.2.14.* Suppose $\mu \preceq_{\mathrm{lr}} \nu$. Let $\pi$ be defined as in Lemma 2.2.15. Let $\phi$ be a function such that $x \leq y$ implies $\phi(x,y) \leq \phi(y,x)$ and $c \in \mathbb{R}$. The function $f(x) := \mathbb{1}_{(-\infty,c]}$ is measurable and non-increasing. Therefore, by Lemma 2.2.15 we have

$$\begin{aligned}
&\mathbb{P}\left(\phi(X,Y) \leq c\right) - \mathbb{P}\left(\phi(Y,X) \leq c\right) = \mathbb{E}\left[f(\phi(X,Y)) - f(\phi(Y,X))\right] \\
&= \mathbb{E}\left[f(\phi(X,Y)) - f(\phi(Y,X))\mathbb{1}_{\{X<Y\}}\right] + \mathbb{E}\left[f(\phi(X,Y)) - f(\phi(Y,X))\mathbb{1}_{\{X>Y\}}\right] \\
&= \int_{\nabla\setminus\Delta} \left(f(\phi(x,y)) - f(\phi(y,x))\right) \mathrm{d}\pi(x,y) \geq 0,
\end{aligned}$$

since for $x \leq y$ holds $f(\phi(x,y)) - f(\phi(y,x)) \geq 0$. This shows that

$$\mathbb{P}\left(\phi(X,Y) \in \cdot\,\right) \preceq_{\mathrm{st}} \mathbb{P}\left(\phi(Y,X) \in \cdot\,\right).$$

Conversely, let $A, B \subseteq \mathbb{R}$ be measurable with $A \leq B$. As in [SS07] define $\phi(x,y) := \mathbb{1}_A(y)\mathbb{1}_B(x)$. For $x < y$ holds $(y,x) \notin A \times B$ and therefore $\phi(x,y) = \mathbb{1}_{A\times B}(y,x) = 0 \leq \phi(y,x)$. Of course $\phi$ is measurable, hence by assumption $\phi(X,Y) \preceq_{\mathrm{st}} \phi(Y,X)$, and thus

$$\begin{aligned}\mu(A)\nu(B) &= \mathbb{P}\left(X \in A\right)\mathbb{P}\left(Y \in B\right) = \mathbb{E}\left[\phi(Y,X)\right] \\ &\geq \mathbb{E}\left[\phi(X,Y)\right] = \mathbb{P}\left(X \in B\right)\mathbb{P}\left(Y \in A\right) = \mu(B)\nu(A).\end{aligned}$$

Altogether it follows that $\mu \preceq_{\mathrm{lr}} \nu$. □

By choosing $\phi(x,y) := x$ we obtain the following important relation between the likelihood ratio order and the usual stochastic order as corollary from Theorem 2.2.14.

**Corollary 2.2.16.** *Let $\mu, \nu \in \mathcal{P}$. If $\mu \preceq_{|\mathrm{lr}|} \nu$, then $\mu \preceq \nu$.*

For absolutely continuous measures we obtain a very convenient characterization of the likelihood ratio order. This characterization was given as definition of the likelihood ratio order in [SS07], where the equivalence to our definition was stated without proof.

**Lemma 2.2.17.** *Let $\mu, \nu \in \mathcal{P}$ be absolutely continuous with respect to Lebesgue measure and $\mu = f\,\mathrm{d}x$ and $\nu = g\,\mathrm{d}x$. Then $\mu \preceq_{\mathrm{lr}} \nu$ if and only if*

$$f(x)g(y) \geq f(y)g(x)$$

*for Lebesgue-almost all $x \leq y$.*

*Proof.* Assume that $\mu \preceq_{\mathrm{lr}} \nu$. Let $x < y$ and $h_1, h_2 > 0$ small enough such that $A := [x, x+h_1] \leq B := [y, y+h_2]$. By the definition of the likelihood ratio order follows that $\mu(A)\nu(B) \geq \mu(B)\nu(A)$. Dividing by $h_1h_2$ yields

$$\frac{1}{h_1}\int_x^{x+h_1} f(t)\,\mathrm{d}t \frac{1}{h_2}\int_y^{y+h_2} g(s)\,\mathrm{d}s \geq \frac{1}{h_2}\int_y^{y+h_2} f(t)\,\mathrm{d}t \frac{1}{h_1}\int_x^{x+h_1} g(s)\,\mathrm{d}s.$$

By letting $h_1, h_2 \to 0$ we obtain $f(x)g(y) \geq g(x)f(y)$ for Lebesgue-almost all $x \leq y$.
Conversely, the assumption means that $\pi$ defined in Lemma 2.2.15 is a positive measure on $\nabla = \{(x,y) : x \leq y\}$, which implies that $\mu \preceq_{\mathrm{lr}} \nu$. □

The next statement gives a necessary and sufficient condition for the preservation of the two-sided likelihood ratio order under convolution operations. For absolutely continuous measures the counterpart of this statement concerning the one-sided likelihood ratio order is Theorem 1.C.9 in [SS07].

As preparation, for a density function $f : \mathbb{R} \to \mathbb{R}$ we define the *generalized absolute value density* by

$$f^*(v, u) \coloneqq f(v + u) + f(v - u) \tag{2.9}$$

for $v, u \geq 0$. Furthermore, we call a function $K : \mathbb{R}^2 \to \mathbb{R}$ totally positive of order 2 (TP$_2$ ) if

$$K(x_1, y_1)K(x_2, y_2) - K(x_1, y_2)K(x_2, y_1) \geq 0 \tag{2.10}$$

for all $x_1, x_2 \in \mathbb{R}$, $y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2, y_1 < y_2$.

**Theorem 2.2.18.** *Let $\eta \in \mathcal{P}$ be symmetric. Then the following properties are equivalent:*

*(i) $\mu \preceq_{|\mathrm{lr}|} \nu$ implies $\eta * \mu \preceq_{|\mathrm{lr}|} \eta * \nu$ for all $\mu, \nu \in \mathcal{P}$.*

*(ii) $\mathbb{P}(Z + u_1 \in \cdot) \preceq_{|\mathrm{lr}|} \mathbb{P}(Z + u_2 \in \cdot)$ whenever $Z \sim \eta$ and $|u_1| \leq |u_2|$.*

*If $\eta$ has a density $f$ with respect to the Lebesgue measure, the following is also an equivalent condition:*

*(iii) $f^*$ is almost everywhere totally positive of order 2 (TP$_2$).*

*Proof.* In the following, for $\sigma \in \{\mu, \nu\}$ let $X_\sigma \sim \sigma$ and $Z \sim \eta$ be independent. At first, as preparation, we observe that by symmetry

$$\begin{aligned}
&\mathbb{P}(|X_\sigma + Z| \leq v) \\
&= \mathbb{P}(|X_\sigma + Z| \leq v, X_\sigma > 0) + \mathbb{P}(|X_\sigma + Z| \leq v, X_\sigma = 0) \\
&\quad + \mathbb{P}(|X_\sigma + Z| \leq v, X_\sigma < 0) \\
&= \mathbb{P}(||X_\sigma| + Z| \leq v, X_\sigma > 0) + \mathbb{P}(||X_\sigma| + Z| \leq v, X_\sigma = 0) \\
&\quad + \mathbb{P}(||X_\sigma| - Z| \leq v, X_\sigma < 0) \\
&= \mathbb{P}(||X_\sigma| + Z| \leq v, X_\sigma > 0) + \mathbb{P}(||X_\sigma| + Z| \leq v, X_\sigma = 0) \\
&\quad + \mathbb{P}(||X_\sigma| + Z| \leq v, X_\sigma < 0) \\
&= \mathbb{P}(||X_\sigma| + Z| \leq v)
\end{aligned}$$

for $v \geq 0$, which implies that $\overline{\eta * \sigma} = \overline{\eta * \overline{\sigma}}$.

It is clear that (i) implies (ii) by observing that $\delta_{u_1} \preceq_{|\mathrm{lr}|} \delta_{u_2}$ whenever $|u_1| \leq |u_2|$.

Conversely, assume (ii) and $\mu \preceq_{|\mathrm{lr}|} \nu$. Let $A, B$ be measurable sets with $A \leq B$. Define the function $\phi(x, y) \coloneqq \mathbb{P}(|y + Z| \in A)\, \mathbb{P}(|x + Z| \in B)$. Due to the assumption $x \leq y$ implies $\phi(x, y) \leq \phi(y, x)$. We have by Theorem 2.2.14 that

$$\begin{aligned}
&\overline{\eta * \mu}(A)\overline{\eta * \nu}(B) = \overline{\eta * \overline{\mu}}(A)\overline{\eta * \overline{\nu}}(B) \\
&= \mathbb{P}(||X_\mu| + Z| \in A)\, \mathbb{P}(||X_\nu| + Z| \in B) = \mathbb{E}[\phi(|X_\nu|, |X_\mu|)] \\
&\geq \mathbb{E}[\phi(|X_\mu|, |X_\nu|)] = \mathbb{P}(||X_\mu| + Z| \in B)\, \mathbb{P}(||X_\nu| + Z| \in A) \\
&= \overline{\eta * \mu}(B)\overline{\eta * \nu}(A),
\end{aligned}$$

which implies $\eta * \mu \preceq_{|\mathrm{lr}|} \eta * \nu$.

Now assume $\eta$ has density $f$. If condition (ii) is true, for $0 \le u_1 \le u_2$ by Lemma 2.2.17 it follows that

$$f^*(v_1, u_1) f^*(v_2, u_2) \ge f^*(v_2, u_1) f^*(v_1, u_2)$$

for almost all $0 \le v_1 \le v_2$, which means that $f^*$ is totally positive of order 2 almost everywhere.

Conversely, if $f^*$ is totally positive of order 2 almost everywhere, then for $v_1 \le v_2$ the function $\phi(x, y) := f^*(v_1, y) f^*(v_2, x)$ is measurable and $x \le y$ implies $\phi(x, y) \le \phi(y, x)$ almost everywhere. Therefore, by Theorem 2.2.14 for $\mu \preceq_{|\mathrm{lr}|} \nu$ it follows that

$$\begin{aligned} h_\mu(v_1) h_\nu(v_2) &= \mathbb{E}\left[f^*(v_1, |X_\mu|) f^*(v_2, |X_\nu|)\right] = \mathbb{E}\left[\phi(|X_\nu|, |X_\mu|)\right] \\ &\ge \mathbb{E}\left[\phi(|X_\mu|, |X_\nu|)\right] = \mathbb{E}\left[f^*(v_1, |X_\nu|) f^*(v_2, |X_\mu|)\right] = h_\mu(v_2) h_\nu(v_1). \end{aligned}$$

But by Lemma 2.2.17 this means that $\overline{\eta * \mu} \preceq_{\mathrm{lr}} \overline{\eta * \nu}$. □

In §5 in Chapter 1 of [Kar68] we find that the density of a normal distribution is a Pólya frequency function of all orders, thus by Theorem 9.1 in Chapter 7 therein has a generalized absolute value density which is totally positive of all orders. This yields the useful fact that the Gaussian convolution operator also preserves the two-sided likelihood ratio order.

**Corollary 2.2.19.** *Let $\mu \preceq_{|\mathrm{lr}|} \nu$ and $t \ge 0$. Then $P_t \mu \preceq_{|\mathrm{lr}|} P_t \nu$.*

In the situation of the one-sided likelihood ratio order, the counterpart of Corollary 2.2.19 follows from the proof of Theorem 1.C.9 from [SS07], since the convolution density of $P_t$ is logconcave.

We will now study the properties of $T_\alpha$ and $T^c$ in terms of the two-sided likelihood ratio order.

**Lemma 2.2.20.** *Let $\mu, \nu$ absolutely continuous with respect to Lebesgue measure. Let $\alpha \in (0, 1]$ and assume that $\mu \preceq_{|\mathrm{lr}|} \nu$. Then $T_\alpha(\mu) \preceq_{|\mathrm{lr}|} T_\alpha(\nu)$.*

*Proof.* By Theorem 2.2.14 follows $\mu \preceq \nu$. This means that $q_\alpha(\mu) \le q_\alpha(\nu)$. Let $\overline{f}$ and $\overline{g}$ being the densities of $\overline{\mu}$ and $\overline{\nu}$, respectively. Note that then the densities of $\overline{T_\alpha(\mu)}$ and $\overline{T_\alpha(\nu)}$ are given by

$$\tilde{f} = \frac{1}{\alpha} \overline{f} \mathbb{1}_{[0, q_\alpha(\mu)]}, \quad \tilde{g} = \frac{1}{\alpha} \overline{g} \mathbb{1}_{[0, q_\alpha(\nu)]},$$

respectively. Note that by Lemma 2.2.17 we have that $\overline{f}(s)\overline{g}(t) \ge \overline{f}(t)\overline{g}(s)$ for almost all $s \le t$. Thus for $0 \le s \le t \le q_\alpha(\mu)$ we have

$$\tilde{f}(s)\tilde{g}(t) = \frac{1}{\alpha^2} \overline{f}(s)\overline{g}(t) \ge \frac{1}{\alpha^2} \overline{f}(t)\overline{g}(s) = \tilde{f}(t)\tilde{g}(s).$$

If $t > q_\alpha(\mu)$ or $s < 0$ the inequality is trivially fulfilled. By Lemma 2.2.17 follows that $T_\alpha(\mu) \preceq_{|\mathrm{lr}|} T_\alpha(\nu)$. □

Recall the definition of the operator in (2.5). The next result is a direct consequence of Theorem 1.C.6 of [SS07], but for completeness we give a proof here.

**Lemma 2.2.21.** *Let $\mu, \nu \in \mathcal{P}$ and $0 \leq p \leq q \leq \infty$, such that $\alpha_\mu := \mu([-p,p]) > 0$ and $\alpha_\nu := \nu([-q,q]) > 0$. Assume that $\mu \preceq_{|\mathrm{lr}|} \nu$. Then $T^p(\mu) \preceq_{|\mathrm{lr}|} T^q(\nu)$.*

*Proof.* At first consider the case $p = q$. Let $A \leq B$. Then $A \cap [-q,q] \leq B \cap [-q,q]$. Note that $\overline{T^q(\sigma)} = T^q(\overline{\sigma})$ for $\sigma \in \{\mu, \nu\}$. Set $\alpha_\sigma := \sigma([-q,q])$. It holds

$$\begin{aligned}\overline{T^q(\mu)}(A)\overline{T^q(\nu)}(B) &= \frac{1}{\alpha_\mu}\frac{1}{\alpha_\nu}\overline{\mu}(A \cap [-q,q])\overline{\nu}(B \cap [-q,q]) \\ &\geq \frac{1}{\alpha_\mu}\frac{1}{\alpha_\nu}\overline{\mu}(B \cap [-q,q])\overline{\nu}(A \cap [-q,q]) = \overline{T^q(\mu)}(B)\overline{T^q(\nu)}(A),\end{aligned}$$

which shows the statement in this case.
Now assume that $q = \infty$ and $\mu = \nu$. We have to show that $T^p(\overline{\mu}) \preceq_{\mathrm{lr}} \overline{\mu}$. Let $A \leq B$. First of all assume $A \subseteq (-\infty, p]$. Then

$$\begin{aligned}T^p(\overline{\mu})(A)\overline{\mu}(B) &= \frac{1}{\alpha_\mu}\overline{\mu}(A \cap [-p,p])\overline{\mu}(B) = \frac{1}{\alpha_\mu}\overline{\mu}(A \cap (-\infty,p])\overline{\mu}(B) \\ &= \frac{1}{\alpha_\mu}\overline{\mu}(A)\overline{\mu}(B) \geq \frac{1}{\alpha_\mu}\overline{\mu}(B \cap [-p,p])\overline{\mu}(A) = T^p(\overline{\mu})(B)\overline{\mu}(A).\end{aligned}$$

If $A \not\subseteq (-\infty, p]$, there exists $a \in A$, such that $a > p$. But since for every $b \in B$ we have $b \geq a > p$, this means that $B \cap (-\infty, p] = \emptyset$, which implies that $T^p(\overline{\mu})(B) = 0$, which makes the desired inequality trivially fulfilled.
Now consider the general case $0 \leq p \leq q \leq \infty$. We obtain

$$T^p(\mu) = T^p(T^q(\mu)) \preceq_{|\mathrm{lr}|} T^q(\mu) \preceq_{|\mathrm{lr}|} T^q(\nu),$$

which yields the statement. □

Finally, we want to emphasize that the statement of Lemma 2.2.21 relies on working with the likelihood ratio order.

**Remark 2.2.22.** For any $c > 0$ the function $T^c$ does not preserve the two-sided stochastic order. For specific pairs of measures, the problem can occur that the dominating measure looses more mass in the truncating mechanism, before reweighting, which is then distributed too close to the origin by the reweighting. We give the following general construction as counterexample. Let $\mu \in \mathcal{P}$ with $\mu([-c,c]) = 1$ such that there exists $x \in (0,c)$ with $0 < \mu([-x,x]) < 1$. Let $\nu$ be given by $\nu(A) := \mu(A \cap [-x,x]) + \mu(\mathbb{R} \setminus [-x,x])\sigma(A)$, where $\sigma$ is a measure with $\sigma(\mathbb{R} \setminus [-c,c]) = 1$. Then $\mu \preceq \nu$, but $T^c(\mu)([-x,x]) = \mu([-x,x]) < \mu([-x,x])/\mu(\mathbb{R} \setminus [-x,x]) = T^c(\nu)([-x,x])$. This counterexample shows that $T^c$ fails to be dominance preserving very generally.

### 2.2.3 Wasserstein distance: Gaussian convolution, truncation

At several stages in this work we will be concerned with measures of the form of (2.8) and similar measures where the Gaussian convolution and the truncation is interchanged. When varying the mesh size of time discretization it becomes important to gain control over the induced sequences of measures. For this purpose we will establish some results concerning the effect of $P_t$ and $T_\alpha$ on the Wasserstein distances.

Denote the set of all probability measures with finite first absolute moment with

$$\mathcal{P}^1 := \left\{ \mu \in \mathcal{P} : \int_{\mathbb{R}} |x| \,\mathrm{d}\mu(x) < \infty \right\}.$$

**Definition 2.2.23.** *For two probability measures $\mu, \nu \in \mathcal{P}^1$ we define the* Wasserstein distance *as*

$$d_{\mathrm{W}}(\mu,\nu) := \inf\{\mathbb{E}[|X-Y|] : X \sim \mu, Y \sim \nu\}. \tag{2.11}$$

This distance has several different representations. On the one hand by the Kantorovich-Rubenstein theorem we have that

$$d_{\mathrm{W}}(\mu,\nu) = \sup\left\{ \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}(\mu-\nu)(x) : \|\varphi\|_{\mathrm{Lip}} \leq 1, \|\varphi\|_\infty < \infty \right\}, \tag{2.12}$$

where $\|\varphi\|_{\mathrm{Lip}} := \sup\{|x-y|^{-1}|\varphi(x)-\varphi(y)| : x \neq y\}$ for a function $\varphi : \mathbb{R} \to \mathbb{R}$. For example see Theorem 1.14 in [Vil03]. On the other hand we have

$$d_{\mathrm{W}}(\mu,\nu) = \int_{\mathbb{R}} |\mu((-\infty,x]) - \nu((-\infty,x])|\,\mathrm{d}x \tag{2.13}$$

as it is stated in Remarks 2.19 in [Vil03].

Recall the definitions of the Gaussian convolution operator in (2.7) and the truncation operator in (2.6). Let $\mu \in \mathcal{P}^1$. Since for $t \geq 0$ we can sample from $P_t\mu$ by adding an independent $B_t \sim \mathcal{N}(0,t)$ to $X \sim \mu$, we see that also $P_t\mu \in \mathcal{P}^1$. For $\alpha \in (0,1]$ we always have $T_\alpha(\mu) \preceq \mu$. As a consequence of Lemma 2.2.2 this implies that $T_\alpha(\mu) \in \mathcal{P}^1$.

Firstly, we will collect the statements concerning the effect of $P_t$ on the Wasserstein distance. These comparatively simple statements may be found in the existing literature, but we have not been able to locate them directly and will give their proofs instead.

**Lemma 2.2.24.** *Let $\mu, \nu \in \mathcal{P}^1$. Then for $t \geq 0$ holds*

*(i)* $d_{\mathrm{W}}(P_t\mu, P_t\nu) \leq d_{\mathrm{W}}(\mu,\nu)$ *and*

*(ii)* $|d_{\mathrm{W}}(P_t\mu,\nu) - d_{\mathrm{W}}(\mu,\nu)| \leq \sqrt{t}$.

*Proof.* Let $(X, Y)$ be a coupling of $\mu$ and $\nu$, this is $X \sim \mu$ and $Y \sim \nu$. Let $B_t \sim \mathcal{N}(0, t)$ be independent from $X$ and $Y$. Then we have

$$d_{\mathrm{W}}\left(P_t\mu, P_t\nu\right) \leq \mathbb{E}\left[|X + B_t - (Y + B_t)|\right] = \mathbb{E}\left[|X - Y|\right]$$

and

$$d_{\mathrm{W}}\left(P_t\mu, \nu\right) \leq \mathbb{E}\left[|X + B_t - Y|\right] \leq \mathbb{E}\left[|X - Y|\right] + \mathbb{E}\left[|B_t|\right] \leq \mathbb{E}\left[|X - Y|\right] + \sqrt{t}.$$

Taking the infimum over all possible couplings on the right-hand side of both inequalities yields (i) and $d_{\mathrm{W}}\left(P_t\mu, \nu\right) \leq d_{\mathrm{W}}\left(\mu, \nu\right) + \sqrt{t}$. For the inequality left we use (2.12) and consider

$$\begin{aligned} d_{\mathrm{W}}\left(\mu, \nu\right) &= \sup_{\|\varphi\|_{\mathrm{Lip}} \leq 1} \mathbb{E}\left[\varphi(X) - \varphi(Y)\right] \\ &\leq \sup_{\|\varphi\|_{\mathrm{Lip}} \leq 1} \mathbb{E}\left[|\varphi(X) - \varphi(X + B_t)|\right] + \sup_{\|\varphi\|_{\mathrm{Lip}} \leq 1} \mathbb{E}\left[\varphi(X + B_t) - \varphi(Y)\right] \\ &\leq \mathbb{E}\left[|B_t|\right] + d_{\mathrm{W}}\left(P_t\mu, \nu\right) \leq d_{\mathrm{W}}\left(P_t\mu, \nu\right) + \sqrt{t}, \end{aligned}$$

which yields the statement. □

We now turn to the effect of $T_\alpha$ on the Wasserstein distance.

**Lemma 2.2.25.** *Let $\mu, \nu \in \mathcal{P}^1$. For $\alpha \in (0, 1]$ holds*

*(i)* $d_{\mathrm{W}}\left(T_\alpha(\mu), \nu\right) \leq d_{\mathrm{W}}\left(\mu, \nu\right) + 2\int_{|x| > q_\alpha(\mu)} |x| \,\mathrm{d}\mu(x)$ *and*

*(ii) if we assume that* $\mu((-\infty, -q_\alpha(\mu))) = \nu((-\infty, -q_\alpha(\nu)))$ *and in addition* $\mu([-q_\alpha(\mu), q_\alpha(\mu)]) = \nu([-q_\alpha(\nu), q_\alpha(\nu)])$ *then we have*

$$d_{\mathrm{W}}\left(T_\alpha(\mu), T_\alpha(\nu)\right) \leq \frac{1}{\alpha} d_{\mathrm{W}}\left(\mu, \nu\right).$$

*Proof.* For (i) we only have to consider the case $\mu([-q_\alpha(\mu), q_\alpha(\mu)]) < 1$. Let $(X, Y)$ be a coupling of $\mu$ and $\nu$, and independent of $X$ and $Y$ take $X_0 \sim T_\alpha(\mu)$. Define

$$\bar{X} \coloneqq X_0 + (X - X_0)\mathbb{1}_{\{|X| \leq q_\alpha(\mu)\}}.$$

We have that $\bar{X} \sim T_\alpha(\mu)$. Thus

$$d_{\mathrm{W}}\left(T_\alpha(\mu), \nu\right) \leq \mathbb{E}\left[|\bar{X} - Y|\right] \leq \mathbb{E}\left[|\bar{X} - X|\right] + \mathbb{E}\left[|X - Y|\right].$$

Further, using that $\mu([-q_\alpha(\mu), q_\alpha(\mu)]) < 1$ and that $T_\alpha(\mu) \preceq \mu$, we have

$$\begin{aligned} \mathbb{E}\left[|\bar{X} - X|\right] &= \mathbb{E}\left[|X_0 - X|\mathbb{1}_{\{|X| > q_\alpha(\mu)\}}\right] \\ &\leq \mathbb{E}\left[|X_0|\right]\mathbb{P}\left(|X| > q_\alpha(\mu)\right) + \mathbb{E}\left[|X|\mathbb{1}_{\{|X| > q_\alpha(\mu)\}}\right] \\ &\leq \mathbb{E}\left[|X|\right]\mathbb{P}\left(|X| > q_\alpha(\mu)\right) + \mathbb{E}\left[|X| \,\middle|\, |X| > q_\alpha(\mu)\right]\mathbb{P}\left(|X| > q_\alpha(\mu)\right) \\ &\leq 2\mathbb{E}\left[|X| \,\middle|\, |X| > q_\alpha(\mu)\right]\mathbb{P}\left(|X| > q_\alpha(\mu)\right) \\ &\leq 2\int_{|x| > q_\alpha(\mu)} |x| \,\mathrm{d}\mu(x). \end{aligned}$$

Taking the infimum over all possible coupling yields (i).

Let $F, G$ be the distribution functions of $\mu$ and $\nu$, and $\bar{F}, \bar{G}$ the distribution functions of $T_\alpha(\mu), T_\alpha(\nu)$, respectivly. Set $\hat{\alpha} := \mu([-q_\alpha(\mu), q_\alpha(\mu)])$ and $c_\mu := q_\alpha(\mu)$, $c_\nu := q_\alpha(\nu)$. The distribution functions satisfy

$$\bar{F}(x) = \begin{cases} 0 & : x < -c_\mu, \\ \frac{F(x)-F(-c_\mu)}{\hat{\alpha}} & : |x| \leq c_\mu, \\ 1 & : \text{else}, \end{cases} \qquad \bar{G}(x) = \begin{cases} 0 & : x < -c_\nu, \\ \frac{G(x)-G(-c_\nu)}{\hat{\alpha}} & : |x| \leq c_\nu, \\ 1 & : \text{else}. \end{cases}$$

Without loss of generality $c_\mu > c_\nu$. By (2.13) we can write

$$\begin{aligned} &d_{\mathrm{W}}\left(T_\alpha(\mu), T_\alpha(\nu)\right) = \int_{\mathbb{R}} |\bar{F}(x) - \bar{G}(x)| \,\mathrm{d}x \\ &= \frac{1}{\hat{\alpha}} \int_{-c_\mu}^{-c_\nu} |F(x) - F(-c_\mu)| \,\mathrm{d}x + \frac{1}{\hat{\alpha}} \int_{c_\nu}^{c_\mu} |F(x) - F(-c_\mu) - \hat{\alpha}| \,\mathrm{d}x \\ &\quad + \frac{1}{\hat{\alpha}} \int_{-c_\nu}^{c_\nu} |F(x) - F(-c_\mu) - G(x) + G(-c_\nu)| \,\mathrm{d}x. \end{aligned}$$

Since for $x \in (c_\nu, c_\mu)$ holds $F(x) - F(-c_\mu) - \hat{\alpha} = F(x) - (\hat{\alpha} + F(-c_\mu)) = F(x) - F(c_\mu) \leq 0$, we have that the first two summands of the quantity above equal

$$\begin{aligned} &\frac{1}{\hat{\alpha}} \left( \int_{-c_\mu}^{-c_\nu} (F(x) - F(-c_\mu)) \,\mathrm{d}x + \int_{c_\nu}^{c_\mu} (F(c_\mu) - F(x)) \,\mathrm{d}x \right) \\ &= \frac{1}{\hat{\alpha}} \left( \int_{-c_\mu}^{-c_\nu} F(x) \,\mathrm{d}x - \int_{c_\nu}^{c_\mu} F(x) \,\mathrm{d}x + (c_\mu - c_\nu)(F(c_\mu) - F(-c_\mu)) \right) \\ &= (c_\mu - c_\nu) + \frac{1}{\hat{\alpha}} \left( \int_{-c_\mu}^{-c_\nu} F(x) \,\mathrm{d}x - \int_{c_\nu}^{c_\mu} F(x) \,\mathrm{d}x \right). \end{aligned} \tag{2.14}$$

Now observe that

$$\begin{aligned} &\int_{-c_\mu}^{-c_\nu} F(x) \,\mathrm{d}x - \int_{c_\nu}^{c_\mu} F(x) \,\mathrm{d}x \\ &= \int_{-c_\mu}^{-c_\nu} (F(x) - G(x)) \,\mathrm{d}x + \int_{c_\nu}^{c_\mu} (G(x) - F(x)) \,\mathrm{d}x \\ &\quad - \left( \int_{c_\nu}^{c_\mu} G(x) \,\mathrm{d}x - \int_{-c_\mu}^{-c_\nu} G(x) \,\mathrm{d}x \right) \\ &\leq \int_{-c_\mu}^{-c_\nu} \big|F(x) - G(x)\big| \,\mathrm{d}x + \int_{c_\nu}^{c_\mu} \big|G(x) - F(x)\big| \,\mathrm{d}x \\ &\quad - (c_\mu - c_\nu)(G(c_\nu) - G(-c_\nu)) \\ &= \int_{-c_\mu}^{-c_\nu} \big|F(x) - G(x)\big| \,\mathrm{d}x + \int_{c_\nu}^{c_\mu} \big|F(x) - G(x)\big| \,\mathrm{d}x - (c_\mu - c_\nu)(\hat{\alpha}) \end{aligned}$$

which means that (2.14) can be bounded from above by

$$\frac{1}{\hat{\alpha}}\left(\int_{-c_\mu}^{-c_\nu} \big|F(x)-G(x)\big|\,\mathrm{d}x + \int_{c_\nu}^{c_\mu} \big|F(x)-G(x)\big|\,\mathrm{d}x\right).$$

Putting it together, using that $F(-c_\mu) = G(-c_\nu)$, yields

$$\begin{aligned}
&W(\bar{\mu},\bar{\nu}) \\
&\le \frac{1}{\hat{\alpha}}\bigg(W(\mu,\nu) + \int_{-c_\nu}^{c_\nu} |F(x)-F(-c_\mu)-G(x)+G(-c_\nu)|\,\mathrm{d}x - \int_{-c_\nu}^{c_\nu} |F(x)-G(x)|\,\mathrm{d}x \\
&\qquad\qquad - \int_{-\infty}^{-c_\mu} |F(x)-G(x)|\,\mathrm{d}x - \int_{c_\mu}^{\infty} |F(x)-G(x)|\,\mathrm{d}x\bigg) \\
&\le \frac{1}{\hat{\alpha}}\left(W(\mu,\nu) - \int_{-\infty}^{-c_\mu} |F(x)-G(x)|\,\mathrm{d}x - \int_{c_\mu}^{\infty} |F(x)-G(x)|\,\mathrm{d}x\right) \\
&\le \frac{1}{\hat{\alpha}} W(\mu,\nu),
\end{aligned}$$

which finishes the proof. □

**Remark 2.2.26.** Corresponding statements to Lemma 2.2.25 for the total variation distance and further statements are to be found in the appendix as Lemma A.4.2 and Lemma A.4.3.

## 2.3 Uniqueness, properties and examples

We now turn our focus back to the inverse first-passage time problem and will complete our analysis of solutions in this section. For this purpose our main tool is the discrete approximation

$$T_{\alpha_k^n} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots \circ T_{\alpha_1^n} \circ P_{t_1^n}(\mu), \tag{2.15}$$

where $\alpha_k^n := g(t_k^n)/g(t_{k-1}^n)$, already introduced in (2.8). From this we will extract a discrete boundary function, which serves us as approximant of solutions. The results of the previous section allow us to compare solutions to these and other approximants, compare solutions corresponding to different survival functions ordered in the hazard rate order and compare solutions corresponding to ordered initial distributions. The results with respect to the probability distances will let us study the convergence of the approximants, and the results of Section 2.1.1 will let us pass properties through the limit. This approach enables us to deduce the known uniqueness result by a Wasserstein convergence result of the approximants (2.8) to the marginal measure in (1.2). Furthermore we derive new qualitative results, such as a comparison principle and sufficient conditions on $g$ for monotonicity or Lipschitz continuity by using Brownian scaling. We demonstrate the results for some classical examples such as the exponential distribution.

### 2.3.1 Auxiliary results: boundary functions, survival distribution and marginal distributions

Before we begin carrying out our main approach of this section, we will prepare our analysis with a study of the direct properties of the objects which appear in the context of boundary functions, survival distributions and the corresponding marginal distribution. We begin with the intuitively clear statement that we cannot have the occasion for the Brownian paths to hit the boundary solution instantaneously.

Recall that for a measure $\mu \in \mathcal{P}$ its *support* is defined as

$$\operatorname{supp}(\mu) := \{x \in \mathbb{R} : U \subseteq \mathbb{R} \text{ open }, x \in U \Rightarrow \mu(U) > 0\}.$$

**Lemma 2.3.1.** *Let* $b \in \operatorname{ifpt}(g, \mu)$. *Then*

$$\liminf_{t \searrow 0} b(t) \geq \sup \operatorname{supp}(\overline{\mu}).$$

*Proof.* Without loss of generality we suppose that $\sup \operatorname{supp}(\overline{\mu}) = \sup \operatorname{supp}(\mu)$. Assume that

$$\xi := \liminf_{t \searrow 0} b(t) < \sup \operatorname{supp}(\mu).$$

Let $\delta > 0$ such that $\xi + \delta < \sup \mathrm{supp}(\mu)$. Let $t_n \to 0$ be a decreasing sequence such that $b(t_n) < \xi + \delta$ for every $n$. Now choose $x \in \mathrm{supp}(\mu)$ with $x > \xi + \delta$. Let $U$ be a neighbourhood of $x$ with $U \subseteq (\xi + \delta, \infty)$. For every $y > \xi + \delta$ we have

$$\mathbb{P}_y(\tau_b = 0) \geq \mathbb{P}_y(|X_{t_n}| \geq b(t_n) \text{ i.o.}) = 1,$$

since $X_{t_n} \to y$ almost surely. Therefore

$$0 = 1 - g(0) = \mathbb{P}_\mu(\tau_b = 0) \geq \int_U \mathbb{P}_y(\tau_b = 0)\,\mathrm{d}\mu(y) = \mu(U) > 0,$$

since $U$ was a neighbourhood of $x \in \mathrm{supp}(\mu)$. This contradiction yields the statement. □

For a boundary function $b$ we call

$$t^b := \inf\{t > 0 : b(t) = 0\}$$

the extinction time of $b$. Further denote $t^g := \sup\{t \geq 0 : g(t) > 0\}$.

**Lemma 2.3.2.** *Let $b \in \mathrm{ifpt}(g, \mu)$. Then*

$$t^b = \sup\{t \geq 0 : g(t) > 0\} = t^g.$$

*with $b(t_b) = 0$.*

*Proof.* By the definition of $\tau_b$ we see that $\tau_b \leq t^b$ almost surely. This implies $g(t_b) = 0$ and therefore we have $t^b \geq t^g$. Assume that $t^b > t^g$. Choose $\varepsilon > 0$ such that $\varepsilon < t^g < t^g + \varepsilon < t^b$. By Lemma B.2.1 we get that

$$\inf_{s \in [\varepsilon, t^g + \varepsilon]} b(s) > 0.$$

By Lemma B.2.4 we obtain a continuous function $f : [\varepsilon, t^g + \varepsilon] \to (0, \infty)$ such that $f(\varepsilon) = b(\varepsilon)$ and $f(t^g + \varepsilon) = b(t^g + \varepsilon)$ and $f \leq b$. By Lévy's forgery theorem B.1.1 we see that

$$\begin{aligned} 0 &< \mathbb{P}_\mu(|X_s| < f(s)\ \forall s \in [\varepsilon, t^g + \varepsilon]\ |\tau_b > \varepsilon) \\ &\leq \mathbb{P}_\mu(\tau_b > t^g + \varepsilon) = g(t^g + \varepsilon) = 0. \end{aligned}$$

This contradiction yields that $t^b = t^g$. Let $t_n \searrow t_b$ such that $b(t_n) = 0$. Since $b$ is lower semicontinuous we have $b(t^b) \leq \liminf_{n\to\infty} b(t_n) = 0$. □

If we ask for uniqueness of boundary functions we have to shrink the general set of boundary functions. For instance, suppose we have a boundary function $b$ such that at one time point $t > 0$ the function's value is 0. Thus, Brownian paths will run into the boundary function by no later than $t$. Therefore, the corresponding survival distribution does not depend on the values of the boundary function after time $t$. In view of this there is a natural way of standardizing a boundary function, which is a generalized version of [EJ16], and motivated by Lemma 2.3.1 and Lemma 2.3.2.

**Definition 2.3.3.** *We say a boundary function $b$ is* standard *if $b$ satisfies*

*(i)* $b(0) = \liminf_{t\to 0} b(t)$, $b(\infty) = \liminf_{t\to\infty} b(t)$

*(ii)* $b(t) = 0$ *for* $t > 0$ *implies* $b(s) = 0$ *for all* $s > t$

*and define* $\bar{b}(t) \coloneqq \liminf_{s\to 0} b(s)\mathbb{1}_{\{0\}}(t) + b(t)\mathbb{1}_{(0,t^b)}(t)$ *as the standardized version of $b$.*

The reason, why (i) differs from the definition in [EJ16] is that $\bar{b}$ should remain continuous if $b$ is.

For $\mu$ with unbounded support we have that $\liminf_{t\searrow 0} b(t) = \infty$. For $\mu$ with compact support we have at least the following assertions.

**Lemma 2.3.4.** *Let $\mu \in \mathcal{P}$ and $g$ be a survival distribution. Assume that* $\mathrm{supp}(\mu)$ *is bounded and*

$$\limsup_{t\to 0}(1-g(t))t^{-1/2}e^{\frac{\varepsilon^2}{2t}} = \infty \quad \forall \varepsilon > 0.$$

*Then for $b \in \mathrm{ifpt}(g,\mu)$ it follows*

$$\liminf_{t\searrow 0} b(t) = \sup \mathrm{supp}(\overline{\mu}).$$

*If $\overline{\mu}$ has an atom at $K \coloneqq \sup\mathrm{supp}(\overline{\mu})$ then*

$$\limsup_{t\searrow 0} \frac{b(t)-K}{\sqrt{2t\log(1/t)}} \geq 1.$$

*If $\overline{\mu}$ is absolutely continuous with density $\overline{f}$, $\limsup_{t\to 0} t^{-1}(1-g(t)) = 0$ and $\liminf_{x\searrow 0} x^{-1}\overline{f}(K-x) > 0$, we have*

$$\limsup_{t\searrow 0} t^{-1/2}(b(t)-K) = \infty.$$

*Proof.* For the first part assume $\liminf_{t\searrow 0} b(t) > K$. For every $0 < \varepsilon < \liminf_{t\searrow 0} b(t) - K$, using Lemma 2.2.4 and the reflection principle, we have for $t$ small enough that

$$\begin{aligned} 1-g(t) &= \mathbb{P}_\mu(\tau_b \leq t) \leq \mathbb{P}_{\delta_K}(\tau_b \leq t) = \mathbb{P}_{\delta_K}(\exists s \leq t : |X_s| \geq b(s)) \\ &\leq 2\mathbb{P}_0(\exists s \leq t : X_s \geq \varepsilon) \leq 4\mathbb{P}_0(|X_t| \geq \varepsilon) = 8\left(1-\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right)\right) \\ &\leq 8\frac{\sqrt{t}}{\varepsilon}\phi\left(\frac{\varepsilon}{\sqrt{t}}\right), \end{aligned}$$

where $\phi(x) \coloneqq \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $\Phi(x) = \int_{-\infty}^{x}\phi(z)\,\mathrm{d}z$. But this means

$$\infty = \limsup_{t\to 0}(1-g(t))t^{-1/2}e^{\frac{\varepsilon^2}{2t}} \leq \frac{8}{\varepsilon\sqrt{2\pi}}.$$

Together with Lemma 2.3.1 this contradiction implies that

$$\liminf_{t\searrow 0} b(t) = \sup \operatorname{supp}(\overline{\mu}).$$

Now assume additionally that $\overline{\mu}$ has an atom at $\sup\operatorname{supp}(\overline{\mu})$. Without loss of generality let $\mu$ have the atom at $K$. Suppose that

$$\limsup_{t\searrow 0} \frac{b(t) - K}{\sqrt{2t\log(1/t)}} < 1.$$

Then for $\varepsilon > 0$ small enough there is $t > 0$ such that for all $s \leq t$ we have $b(s) - K \leq (1-\varepsilon)\sqrt{2s\log(1/s)}$. But Lévy's modulus of continuity yields that almost surely there are arbitrary small $s \leq t$, such that $X_s - X_0 \geq (1-\varepsilon)\sqrt{2s\log(1/s)}$. Since there is a positive probability $p > 0$ to start from $K$, we had that $1 - g(t) = \mathbb{P}_\mu(\tau_b \leq t) \geq p$ for arbitrarily small $t > 0$. This contradiction shows $\limsup_{t\searrow 0} \frac{b(t)-K}{\sqrt{2t\log(1/t)}} \geq 1$.

For the last case we can assume without loss of generality that $\mu$ is symmetric and has density $f$. Note that it holds $2f(K-x) = \overline{f}(K-x)$. Now suppose $\limsup_{t\searrow 0} t^{-1/2}(b(t)-K) =: R < \infty$. Let $t_n \to 0$ be a sequence such that $b(t_n) \to K$. By Lemma B.1.2 we obtain

$$\begin{aligned}
0 &= \liminf_{n\to\infty} \frac{1}{t_n}\mathbb{P}_\mu(\tau_b \leq t_n) \geq \liminf_{n\to\infty} \frac{1}{t_n}\mathbb{P}_\mu(|X_{t_n}| \geq b(t_n)) \\
&\geq \liminf_{x\searrow 0} \frac{f(K-x)}{x} 2\int_0^\infty y\left(1 - \Phi\left(y + \limsup_{n\to\infty} \frac{b(t_n)-K}{\sqrt{t_n}}\right)\right) \mathrm{d}y \\
&\geq \liminf_{x\searrow 0} \frac{f(K-x)}{x} 2\int_0^\infty y\left(1 - \Phi(y+R)\right) \mathrm{d}y > 0
\end{aligned}$$

with $\Phi(x) := \int_{-\infty}^x \phi(z)\,\mathrm{d}z$ and the convention $\Phi(-\infty) = 0$. This contradiction yields the desired statement. $\square$

**Remark 2.3.5.** In Lemma 2.3.4 we did not aim to carry out a full study of the asymptotic behavior at zero and only used naive arguments. In the one-sided case and initial position at the origin more elaborated results concerning the limit at zero have been achieved in [Che+06].

The following intuitively clear but important result shows that we can recover the boundary function from the distribution of a Brownian motion conditioned to stay below the boundary function and is a reformulation of Proposition 3.1 from [EJ16] for the two-sided case.

**Lemma 2.3.6.** *Let $b$ a boundary function and define $\mu_t := \mathbb{P}_\mu(X_t \in \cdot \,|\tau_b > t)$ for $t > 0$ such that $\mathbb{P}_\mu(\tau_b > t) > 0$. Then holds*

$$\operatorname{supp}(\mu_t) = [-b(t), b(t)]$$

*for those $t > 0$.*

*Proof.* Since $\tau_b > t$ implies $|X_t| < b(t)$ we have $\operatorname{supp}(\mu_t) \subseteq [-b(t), b(t)]$. Conversely, let $x \in (-b(t), b(t))$. We will show that every neighbourhood of $x$ has positive mass in $\mu_t$. For this assume without loss of generality that $x \geq 0$. Otherwise we can mirror the initial measure $\mu$. Let $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq (-b(t), b(t))$. Set $s := t/2$. We have $\mu_s((-b(s), b(s))) = 1$, thus there exists $\delta \in (0, b(s))$ such that $\mu_s((-\delta, \delta)) > 0$. The function $b_1 : [s, t] \to [0, \infty]$ given by

$$b_1(r) := \begin{cases} \delta & : r = s, \\ b(r) & : r \in (s, t), \\ x + \varepsilon & : r = t \end{cases}$$

is lower semicontinuous and does not vanish on $[s, t]$. Corollary B.2.4 yields the existence of a function $f_1 : [s, t] \to (0, \infty)$ with $b_1 \geq f_1$ pointwise as well as $f_1(s) = \delta$ and $f_1(t) = x+\varepsilon$. Further, let $f_2 : [s, t] \to \mathbb{R}$ be a continuous function such that $f_1 > f_2 \geq -f_1$ pointwise as well as $f_2(s) = -\delta$ and $f_2(t) = x - \varepsilon$. We have that

$$\begin{aligned} \mu_t((x-\varepsilon, x+\varepsilon))g(t) &= \mathbb{P}_\mu \left( X_t \in (x-\varepsilon, x+\varepsilon), \tau_b > t \right) \\ &\geq \mathbb{P}_\mu \left( f_2(r) < X_r < f_1(r) \ \forall r \in [s, t], \tau_b > t \right) \\ &= \mathbb{P}_\mu \left( f_2(r) < X_r < f_1(r) \ \forall r \in [s, t], \tau_b > s \right) \\ &> 0 \end{aligned}$$

by Lévy's forgery theorem B.1.1. Since $x$ and $\varepsilon$ were arbitrary this means that $\overline{(-b(t), b(t))} \subseteq \operatorname{supp}(\mu_t)$, which finishes the proof. □

Of course the several typical properties of Brownian motion are reflected in the properties of solutions for the inverse problem. As first example, this results in the following invariance among suitable parameter families of distributions.

For a survival distribution $g$ and a number $\lambda > 0$ define $g^\lambda(t) := g(\lambda t)$ and

$$b^\lambda(t) := \frac{1}{\sqrt{\lambda}} b(\lambda t)$$

for $b \in \operatorname{ifpt}(g, \mu)$.

**Lemma 2.3.7.** *Let $g$ be a survival distribution and $\mu \in \mathcal{P}$. Let $b \in \operatorname{ifpt}(g, \mu)$. Then for $\lambda > 0$ it holds that $b^\lambda \in \operatorname{ifpt}(g^\lambda, \nu)$, where $\nu(\mathrm{d}x) := \mu(\sqrt{\lambda}\,\mathrm{d}x)$.*

*Proof.* Since $b \in \text{ifpt}(g,\mu)$ and by Brownian scaling we have that

$$\begin{aligned}
\mathbb{P}_\nu\left(\tau_{b^\lambda} > t\right) &= \int_{\mathbb{R}} \mathbb{P}\left(|W_s + x| < \frac{1}{\sqrt{\lambda}} b(\lambda s)\ \forall s \in [0,t]\right) \nu(\mathrm{d}x) \\
&= \int_{\mathbb{R}} \mathbb{P}\left(\left|\frac{1}{\sqrt{\lambda}} W_{\lambda s} + x\right| < \frac{1}{\sqrt{\lambda}} b(\lambda s)\ \forall s \in [0,t]\right) \nu(\mathrm{d}x) \\
&= \int_{\mathbb{R}} \mathbb{P}\left(\left|\frac{1}{\sqrt{\lambda}} W_s + \frac{1}{\sqrt{\lambda}} y\right| < \frac{1}{\sqrt{\lambda}} b(s)\ \forall s \in [0,\lambda t]\right) \mu(\mathrm{d}y) \\
&= \int_{\mathbb{R}} \mathbb{P}\left(\left|W_s + y\right| < b(s)\ \forall s \in [0,\lambda t]\right) \mu(\mathrm{d}y) \\
&= \mathbb{P}_\mu\left(\tau_b > \lambda t\right) = g(\lambda t) = g^\lambda(t),
\end{aligned}$$

which shows the statement. □

We have the following connection to last-exit times.

**Remark 2.3.8.** For a boundary function $b$, by the time inversion property the last exit time

$$\sigma_{\tilde{b}} := \sup\{t \geq 0 : |X_t| \geq \tilde{b}(t)\}$$

of $\tilde{b}(t) = t \cdot b\left(\frac{1}{t}\right)$ is connected to the first-passage time of $b$, by

$$\begin{aligned}
\mathbb{P}_0\left(\tau_b > t\right) = \mathbb{P}_0\left(|X_s| < b(s)\ \forall s \leq t\right) &= \mathbb{P}_0\left(|sX_{\frac{1}{s}}| < b(s)\ \forall s \leq t\right) \\
&= \mathbb{P}_0\left(\frac{1}{u}|X_u| < b\left(\frac{1}{u}\right)\ \forall u \geq \frac{1}{t}\right) \\
&= \mathbb{P}_0\left(|X_u| < \tilde{b}(u)\ \forall u \geq \frac{1}{t}\right) = \mathbb{P}_0\left(\sigma_{\tilde{b}} < \frac{1}{t}\right).
\end{aligned}$$

Thus first-passage time problems can be solved by finding the distribution of the corresponding last exit time and vice versa. See [Sal88] for example.

We now turn to the approach of dividing Gaussian convolution and mass truncation. We begin with a preparing statement.

**Lemma 2.3.9.** *Let $\mu$ be a probability measure and $b$ a boundary function, such that $D := \{0\} \cup \{t \in [0,\infty] : 0 < b(t) < \infty\}$ is a discrete set. Write $D = \{t_0, t_1, \ldots\}$ with $t_k < t_{k+1}$. Then for $k \in \{1,\ldots,|D|\}$ we have*

$$\mathbb{P}_\mu\left(X_{t_k} \in \cdot\ |\tau_b > t_k\right) = T_{\alpha_k} \circ P_{t_k - t_{k-1}} \circ \ldots \circ T_{\alpha_1} \circ P_{t_1}(\mu),$$

*where $\alpha_k := \mathbb{P}_\mu\left(\tau_b > t_k\ |\tau_b > t_{k-1}\right)$.*

*Proof.* The equality is clear for $k=0$, since $0=t_0<t_1$ and $\mathbb{P}_\mu(\tau_b>0)=1$. Abbreviate $\mu_k \coloneqq \mathbb{P}_\mu(X_{t_k}\in\cdot\ |\tau_b>t_k)$. We have by the Markov property

$$\begin{aligned}&P_{t_{k+1}-t_k}(\mu_k)([-b(t_{k+1}),b(t_{k+1})]) = \mathbb{P}_{\mu_k}\left(|X_{t_{k+1}-t_k}|\le b(t_{k+1})\right)\\&=\mathbb{P}_\mu\left(|X_{t_{k+1}}|\le b(t_{k+1})\,\middle|\tau_b>t_k\right)=\mathbb{P}_\mu(\tau_b>t_{k+1}\,|\tau_b>t_k)\\&=\alpha_{k+1}.\end{aligned}$$

This implies $q_{\alpha_{k+1}}(P_{t_{k+1}-t_k}(\mu_k))=b(t_{k+1})$. Consequently, for measurable $A$,

$$\begin{aligned}\mu_{k+1}(A)&=\mathbb{P}_\mu\left(X_{t_{k+1}}\in A\,\middle|\tau_b>t_{k+1}\right)\\&=\mathbb{P}_\mu\left(X_{t_{k+1}}\in A\,\middle||X_{t_{k+1}}|\le b(t_{k+1}),\tau_b>t_k\right)\\&=\frac{\mathbb{P}_\mu\left(X_{t_{k+1}}\in A,|X_{t_{k+1}}|\le q_{\alpha_{k+1}}(P_{t_{k+1}-t_k}(\mu_k))\,\middle|\tau_b>t_k\right)}{\mathbb{P}_\mu\left(|X_{t_{k+1}}|\le q_{\alpha_{k+1}}(P_{t_{k+1}-t_k}(\mu_k))\,\middle|\tau_b>t_k\right)}\\&=\frac{P_{t_{k+1}-t_k}(\mu_k)(A\cap[-q_{\alpha_{k+1}}(P_{t_{k+1}-t_k}(\mu_k)),q_{\alpha_{k+1}}(P_{t_{k+1}-t_k}(\mu_k))])}{P_{t_{k+1}-t_k}(\mu_k)([-q_{\alpha_{k+1}}(P_{t_{k+1}-t_k}(\mu_k)),q_{\alpha_{k+1}}(P_{t_{k+1}-t_k}(\mu_k))])}\\&=T_{\alpha_{k+1}}\circ P_{t_{k+1}-t_k}(\mu_k)(A).\end{aligned}$$

Hence, the statement follows by induction. □

The next statement transfers the ordering of the starting measures to the corresponding ordering of some of the existing boundary functions. At a later stage we will be able to extend this result to the more general Theorem 2.3.34.

**Theorem 2.3.10.** *Let $\mu_1,\mu_2\in\mathcal{P}$ such that $\mu_1\preceq\mu_2$ and $g$ a survival distribution. Let $i\in\{1,2\}$ and let $b^i\in\operatorname{ifpt}(g,\mu_i)$. Then for $j\in\{1,2\}\setminus\{i\}$ there exists a barrier $b^j\in\operatorname{ifpt}(g,\mu_j)$ with $b^1\le b^2$.*

**Remark 2.3.11.** In the special case $g(t)=e^{-t}$ a corresponding comparison principle is also to be found for the unique solutions of a free boundary problem in [Ber+21] and [BBP19].

In order to prove Theorem 2.3.10 we will use an approximation of boundary functions, which was already used in [EJ16] and [Che+11]. For the application at a later stage in the proof of Theorem 2.3.17 we present a slightly adapted version.

Let $b$ be a standard boundary function and $D(b)$ an arbitrary countable set and $D_n(b)\subseteq D_{n+1}(b)$ finite such that $\bigcup_{n\in\mathbb{N}}D_n(b)=D(b)$. For $n\in\mathbb{N}$ define for every $k\in\mathbb{N}$

$$t_k^n \coloneqq \inf\left\{t\in[k2^{-n},(k+1)2^{-n}]:b(t)=\inf_{s\in[k2^{-n},(k+1)2^{-n}]}b(s)\right\}.$$

Set $A_n^1(b)\coloneqq\{t_k^n : k\in\{1,\dots,n2^n\}\}$. Furthermore, let $(t_n)_{n\in\mathbb{N}}$ be an enumeration of $\{t\in[0,\infty):\mathbb{P}_\mu(\tau_b=t)>0\}$. Set $A_n^2(b)\coloneqq\{t_1,\dots,t_n\}$ and

$A_n(b) := A_n^1(b) \cup A_n^2(b) \cup D_n(b)$. The mentioned adaption of the construction in comparison to [EJ16] is merely the addition of $D_n(b)$ to the discrete timesteps. By this we are able to take into account other required dependencies in the construction. If not especially defined, one can take $D(b) = \emptyset$ as default value and ends up with the construction of [EJ16]. Since

$$\inf_{s\in[k2^{-n},(k+1)2^{-n}]} b(s) = \min\left(\inf_{s\in\left[\frac{2k}{2^{n+1}},\frac{2k+1}{2^{n+1}}\right]} b(s), \inf_{s\in\left[\frac{2k+1}{2^{n+1}},\frac{2(k+1)}{2^{n+1}}\right]} b(s)\right)$$

we have that $A_n^1(b) \subseteq A_{n+1}^1(b)$ and thus $A_n(b) \subseteq A_{n+1}(b)$. For $n \in \mathbb{N}$ we define the boundary function

$$b_n(t) := \begin{cases} b(t) & : t \in A_n(b), \\ \infty & : t \notin A_n(b). \end{cases} \tag{2.16}$$

Note that $b_n \geq b_{n+1} \geq b$ and thus $\tau_{b_n} \geq \tau_{b_{n+1}} \geq \tau_b$.

In the following we collect adapted versions of results in [EJ16], which where achieved in the context of the approximation of (2.16). The proofs follow the lines of the proofs in [EJ16].

**Proposition 2.3.12** ([EJ16])**.** *Let $b$ be a boundary function. Then*

$$\tau_b = \tau_b' := \inf\{t > 0 : |X_t| > b(t)\}$$

*almost surely.*

*Proof.* By definition we have $\tau_b' \geq \tau_b$. Let $0 < s < t$ and $Z_{s,t}^1 := \sup_{u\in[s,t]}(X_u - b(u))$ and $Z_{s,t}^2 := \sup_{u\in[s,t]}(-X_u - b(u))$. It holds

$$Z_{s,t}^1 = X_s + \sup_{u\in[s,t]} (X_u - X_s - b(u)),$$

where the terms of the sum are independent and $X_s$ has a density. Hence $Z_{s,t}^1$ has no atoms. In the same manner one sees that $Z_{s,t}^2$ has no atoms. Consequently,

$$\begin{aligned} Z_{s,t} := \sup_{u\in[s,t]} (|X_u| - b(u)) &= \sup_{u\in[s,t]} (\max(X_u, -X_u) - b(u)) \\ &= \sup_{u\in[s,t]} \max(X_u - b(u), -X_u - b(u)) = \max(Z_{s,t}^1, Z_{s,t}^2) \end{aligned}$$

has no atoms. Now, note that on the event $\{\tau_b \in [s,t]\}$ we have

$$Z_{s,t} \geq |X_{\tau_b}| - b(\tau_b) \geq 0,$$

since $b$ is lower semicontinuous. Thus

$$\begin{aligned} &\mathbb{P}_\mu(\tau_b \in [s,t]) = \mathbb{P}_\mu(\tau_b \in [s,t], Z_{s,t} \geq 0) \\ &= \mathbb{P}_\mu(\tau_b \in [s,t], Z_{s,t} > 0) = \mathbb{P}_\mu\left(\tau_b \in [s,t], \tau_b' \in [s,t]\right), \end{aligned}$$

which shows that $\mathbb{P}_\mu(s \leq \tau_b \leq t < \tau_b') = 0$. Since this holds for all such rational $s, t$, we have $\tau_b = \tau_b'$ almost surely. □

**Lemma 2.3.13** ([EJ16]). *We have*

*(i)* $\lim_{n\to\infty} \tau_{b_n} = \tau_b$ *almost surely and*

*(ii)* $\lim_{n\to\infty} \mathbb{P}_\mu(\tau_{b_n} > t) = \mathbb{P}_\mu(\tau_b > t)$ *for every* $t \geq 0$.

*Proof.* We begin with (i). Since $\tau_{b_n} \geq \tau_b$, it suffices to show that $\limsup_{n\to\infty} \tau_{b_n} \leq \tau_b$. In the following we fix a path. By Lemma 2.3.12 we can assume that $\tau_b = \tau_b'$ and since for $\tau_b = \infty$ there is nothing left to show, we assume $\tau_b < \infty$. Let $\varepsilon > 0$. Since $\tau_b = \tau_b'$ there is a $t < \tau_b + \varepsilon$, such that $|X_t| > b(t)$. Since $s \mapsto |X_s|$ is continuous, there exists $\delta > 0$ with $|X_u| > b(t)$ for all $u \in (t-\delta, t+\delta)$. For $n$ large enough we have $2^{-n} < \delta$ and $2^{-n} < \min(t,\varepsilon) \leq \max(t,\varepsilon) < n$. For $k := \lfloor 2^n t \rfloor$ holds $t \in [k2^{-n}, (k+1)2^{-n}]$. This implies $|t_k^n - t| < \delta$, and thus

$$|X_{t_k^n}| > b(t) \geq b(t_k^n).$$

Since $t_k^n \in A_n(b)$, we have $\tau_{b_n} \leq t_k^n \leq t + 2^{-n} \leq \tau_b + 2\varepsilon$. Thus, by letting $\varepsilon \to 0$, for any path of the event $\{\tau_b = \tau_b'\}$ we have $\limsup_{n\to\infty} \tau_{b_n} \leq \tau_b$.
Finally we prove (ii). For $t \geq 0$ define $g_n(t) := \mathbb{P}_\mu(\tau_{b_n} > t)$ and $g(t) := \mathbb{P}_\mu(\tau_b > t)$. We have $g(t) \leq g_{n+1}(t) \leq g_n(t)$. In particular, $\lim_{n\to\infty} g_n(t)$ exists. From $\tau_{b_n} \to \tau_b$ almost surely follows $g_n(s) \to g(s)$ for all continuity points of $g$. Fix $t \geq 0$. If $\mathbb{P}_\mu(\tau_b = t) = g(t-) - g(t) > 0$, for $n$ large enough we have $t \in A_n(b)$, and thus $\tau_b = t$ implies that $\tau_{b_n} = t$. This means that $\tau_{b_n} > t$ for almost all $n$ implies $\tau_b \neq t$ and $t \leq \lim_{n\to\infty} \tau_{b_n} = \tau_b$, hence $\tau_b > t$. On the other hand, by $\lim_{n\to\infty} \tau_{b_n} = \tau_b$, one sees that $\tau_b > t$ implies $\tau_{b_n} > t$ for almost all $n$. Consequently,

$$\lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} \mathbb{P}_\mu(\tau_{b_n} > t) = \mathbb{P}_\mu\left(\bigcap_{n\in\mathbb{N}} \{\tau_{b_n} > t\}\right) = \mathbb{P}_\mu(\tau_b > t) = g(t),$$

which finishes the proof. □

*Proof of Theorem 2.3.10.* We will execute the proof for the case of $i = 2$. The other case is analoguous. Hence, assume $b^2 \in \mathrm{ifpt}(g, \mu_2)$. Let $(b_n^2)_{n\in\mathbb{N}}$ be the construction of (2.16) with $D(b^2) = \emptyset$. At first we claim that $b_n^2 \xrightarrow{\Gamma} b^2$. Since $A_n(b^2)$ is a finite set it is clear that $b_n^2$ is a lower semi-continuous function. Let $\tilde{b}^2$ be any accumulation point of the sequence and let $(b_{n_k}^2)_{k\in\mathbb{N}}$ be a subsequence such that $b_{n_k}^2 \xrightarrow{\Gamma} \tilde{b}^2$ in the Hausdorff metric. Since $b_{n_k}^2 \geq b^2$ by construction, we have $\liminf_{k\to\infty} \tau_{b_{n_k}^2} \geq \tau_{b^2} > 0$ almost surely. From Proposition 2.1.12 we have that $\tau_{b_{n_k}^2} \to \tau_{\tilde{b}^2}$ in probability. Due to the almost sure convergence of Lemma 2.3.13 we obtain $\tau_{b^2} = \tau_{\tilde{b}^2}$ almost surely. By Lemma 2.3.6 we get $b^2 = \tilde{b}^2$. Thus every accumulation point coincides with $b$ and thus the claim is proved.

Now define $g_n(t) := \mathbb{P}_{\mu^2}\left(\tau_{b_n^2} > t\right)$ and $g_n(\infty) := g(\infty)$. By Lemma 2.3.13 we obtain in particular $g_n(t) \to g(t)$ for every continuity point $t \in [0,\infty]$ of $g$.

Write $A_n(b^2) = \{t_1^n, \dots, t_{m_n}^n\}$ with $m_n := |A_n(b^2)|$ and set $t_0^n := 0$. Define $\alpha_k^n := g_n(t_k^n)/g_n(t_{k-1}^n)$ for $k$ such that $g_n(t_{k-1}^n) > 0$.

Recall $T_\alpha$ from (2.6) and the definition $q_\alpha(\mu) := \inf\{c \geq 0 : \mu([-c, c]) \geq \alpha\}$. Further, let $P_t$ be the operator defined in (2.7).

Now define

$$\mu_k^i := T_{\alpha_k^n} \circ P_{t_k^n - t_{k-1}^n} \circ \dots \circ T_{\alpha_1^n} \circ P_{t_1^n}(\mu^i)$$

for $k$ such that $g_n(t_k^n) > 0$. By Proposition 2.2.3 and Corollary 2.2.8 we have that $T_{\alpha_k^n} \circ P_{t_k^n - t_{k-1}^n}$ preserves the order and thus it is a direct consequence that

$$\mu_k^1 \preceq \mu_k^2. \tag{2.17}$$

Now, we claim that $b^2(t_k^n) = b_n^2(t_k^n) = q_{\alpha_k^n}(P_{t_k^n - t_{k-1}^n}\mu_{k-1}^2)$. If we assume the statement for $k-1$ by Lemma 2.3.9 we can identify

$$\mu_{k-1}^2 = \mathbb{P}_{\mu_2}\left(X_{t_{k-1}^n} \in \cdot \ \middle| \tau_{b_n^2} > t_{k-1}^n\right)$$

and thus we can compute

$$\begin{aligned} P_{t_k^n - t_{k-1}^n}\mu_{k-1}^2([-b^2(t_k^n), b^2(t_k^n)]) &= \mathbb{P}_{\mu_2}\left(|X_{t_k^n}| \leq b^2(t_k^n) \ \middle| \tau_{b_n^2} > t_{k-1}^n\right) \\ = \mathbb{P}_{\mu_2}\left(\tau_{b_n^2} > t_k^n \ \middle| \tau_{b_n^2} > t_{k-1}^n\right) &= g_n(t_k^n)/g_n(t_{k-1}^n) = \alpha_k^n, \end{aligned}$$

which means that $b^2(t_k^n) = q_{\alpha_k^n}(P_{t_k^n - t_{k-1}^n}\mu_{k-1}^2)$. Hence the claim follows by induction. Now set $b_n^1(t_k^n) := q_{\alpha_k^n}(P_{t_k^n - t_{k-1}^n}\mu_{k-1}^1)$ and $b_n^1(t) := \infty$ for $t \notin A_n(b^2)$. All in all we can identify by Lemma 2.3.9 that

$$\mu_k^i = \mathbb{P}_{\mu_i}\left(X_{t_k^n} \in \cdot \ \middle| \tau_{b_n^i} > t_k^n\right).$$

By construction we have for $t \in [t_k^n, t_{k+1}^n)$ that

$$\begin{aligned} \mathbb{P}_{\mu_1}\left(\tau_{b_n^1} > t\right) &= \mathbb{P}_{\mu_1}\left(|X_{t_k^n}| \leq b_n^1(t_k^n), \dots, |X_{t_1^n}| \leq b_n^1(t_1^n)\right) \\ &= \alpha_k^n \cdot \dots \cdot \alpha_1^n = g_n(t_k^n) = g_n(t). \end{aligned}$$

Therefore, since $g_n \to g$ in distribution, by Proposition 2.1.12, for an accumulation point $b^1$ of $(b_n^1)_{n\in\mathbb{N}}$ we have that $\tau_{b^1}$ has distribution according to $g$, which means that $b^1 \in \text{ifpt}(g, \mu_1)$. In addition, by (2.17) and Lemma 2.3.6 we see that $b_n^1 \leq b_n^2$. By Lemma 2.1.7 follows $b^1 \leq b^2$. □

In the context of finding ordered solutions we have the following intuitive statement, which emphasizes once again that the choice to search for solutions among lower semicontinuous functions is suitable for the inverse first-passage time problem.

**Lemma 2.3.14.** *Let $\mu \in \mathcal{P}$ and $g$ a survival distribution. Let $b_1, b_2 \in \text{ifpt}(g, \mu)$ standard with $b_1 \leq b_2$. Then $b_1 = b_2$.*

*Proof.* Assume that there is $t > 0$ such that $b_1(t) < b_2(t)$. This means in particular that $t < t^b$. For $s \in (0,t)$ there is by Corollary B.2.4 a continuous function $f : [s,t] \to (0,\infty)$ such that $f \leq b_2$ and $f(s) = b_1(s)$ and $f(t) = b_2(t)$. This means

$$\begin{aligned} 0 &= \mathbb{P}_\mu(\tau_{b_2} > t) - \mathbb{P}_\mu(\tau_{b_1} > t) = \mathbb{P}_\mu(\tau_{b_2} > t) - \mathbb{P}_\mu(\tau_{b_1} > t, \tau_{b_2} > t) \\ &= \mathbb{P}_\mu(\tau_{b_2} > t, \tau_{b_1} \leq t) \geq \mathbb{P}_\mu(|X_u| < f(u)\ \forall u \in [s,t], |X_t| \geq b_1(t), \tau_{b_1} > s) \\ &> 0, \end{aligned}$$

where the positivity is due to Levy's forgery theorem B.1.1. This contradiction shows that $b_1 = b_2$. ☐

Since we are working with arbitrary initial measures, we could have allowed survival functions, which do not satisfy $g(0) = 1$. In particular, this would lead to allow the Brownian motion to start above $\liminf_{s\to 0} b(s)$. The following intuitively clear statements shows that this generalization is not necessary.

**Lemma 2.3.15.** *Let $\mu \in \mathcal{P}$ and $b$ be a standard boundary function. Then for $t > 0$ with $\mathbb{P}_\mu(\tau_b > t) > 0$ holds*

$$\mathbb{P}_\mu(X_t \in \cdot\ |\tau_b > t) = \mathbb{P}_{T^{b(0)}(\mu)}(X_t \in \cdot\ |\tau_b > t).$$

*Proof.* Due to the continuity of the path of $(X_t)_{t\geq 0}$ we have that $\{\tau_b > t\} \subseteq \{|X_0| \leq b(0)\}$, since $b$ is standard. This yields

$$\begin{aligned} \mathbb{P}_\mu(X_t \in A\ |\tau_b > t) &= \frac{\mathbb{P}_\mu(X_t \in A, \tau_b > t)}{\mathbb{P}_\mu(\tau_b > t)} \\ &= \frac{\mathbb{P}_\mu(X_t \in A, \tau_b > t, |X_0| \leq b(0))}{\mathbb{P}_\mu(\tau_b > t, |X_0| \leq b(0))} = \frac{\mathbb{P}_\mu(X_t \in A, \tau_b > t\ ||X_0| \leq b(0))}{\mathbb{P}_\mu(\tau_b > t\ ||X_0| \leq b(0))} \\ &= \frac{\mathbb{P}_{T^{b(0)}(\mu)}(X_t \in A, \tau_b > t)}{\mathbb{P}_{T^{b(0)}(\mu)}(\tau_b > t)} = \mathbb{P}_{T^{b(0)}(\mu)}(X_t \in A\ |\tau_b > t) \end{aligned}$$

for every measurable $A$. ☐

Staying in the context of arbitrary initial distributions we could ask how the initial distribution affects the survival distribution. By tools which were used in the proof of Proposition 2.1.12 we can deduce some sort of continuity in the initial distribution, which will be used in proof of the general uniqueness result of Theorem 2.3.33.

**Lemma 2.3.16.** *Let $\mu_n \to \mu$ weakly in $\mathcal{P}$. Further, let $b$ be a boundary function such that $\lim_{s\searrow 0}\limsup_{n\to\infty}\mathbb{P}_{\mu_n}(\tau_b \leq s) = 0 = \mathbb{P}_\mu(\tau_b = 0)$. Then it holds $\mathbb{P}_{\mu_n}(\tau_b > t) \to \mathbb{P}_\mu(\tau_b > t)$ for all $t \geq 0$.*

*Proof.* Let $s > 0$. First consider

$$\begin{aligned}&\limsup_{n\to\infty} |\mathbb{P}_{\mu_n}(\tau_b > 0) - \mathbb{P}_\mu(\tau_b > 0)| \\ &= \limsup_{n\to\infty} \mathbb{P}_{\mu_n}(\tau_b = 0) \le \limsup_{n\to\infty} \mathbb{P}_{\mu_n}(\tau_b \le s).\end{aligned}$$

Now let $t > 0$. For $0 < s < t$ let $b^s(u) = b(u)$ for $u \ge s$ and $b^s(u) = \infty$ for $0 \le u < s$. Then we have

$$\begin{aligned}&|\mathbb{P}_{\mu_n}(\tau_b > t) - \mathbb{P}_\mu(\tau_b > t)| \\ &= |\mathbb{P}_{\mu_n}(\tau_b > t, \tau_{b^s} = \tau_b) - \mathbb{P}_\mu(\tau_b > t, \tau_{b^s} = \tau_b)| \\ &= |\mathbb{P}_{\mu_n}(\tau_{b^s} > t) - \mathbb{P}_{\mu_n}(\tau_{b^s} > t, \tau_{b^s} \ne \tau_b) - \mathbb{P}_\mu(\tau_{b^s} > t) + \mathbb{P}_\mu(\tau_{b^s} > t, \tau_{b^s} \ne \tau_b)| \\ &\le |\mathbb{P}_{\mu_n}(\tau_{b^s} > t) - \mathbb{P}_\mu(\tau_{b^s} > t)| + \mathbb{P}_{\mu_n}(\tau_{b^s} \ne \tau_b) + \mathbb{P}_\mu(\tau_{b^s} \ne \tau_b) \\ &\le |\mathbb{P}_{\mu_n}(\tau_{b^s} > t) - \mathbb{P}_\mu(\tau_{b^s} > t)| + \mathbb{P}_{\mu_n}(\tau_b \le s) + \mathbb{P}_\mu(\tau_b \le s).\end{aligned}$$

By writing

$$\begin{aligned}\mathbb{P}_x(\tau_{b^s} > t) &= \mathbb{P}_x(|X_u| < b(u)\ \forall u \in [s,t]) \\ &= \int_{\mathbb{R}} \mathbb{P}_y\left(|X_u| < b\left(u + \frac{s}{2}\right)\ \forall u \in \left[\frac{s}{2}, t - \frac{s}{2}\right]\right) P_{\frac{s}{2}}\delta_x(\mathrm{d}y)\end{aligned}$$

we see that the mapping $x \mapsto \mathbb{P}_x(\tau_{b^s} > t)$ is continuous. Thus by the weak convergence we have $\mathbb{P}_{\mu_n}(\tau_{b^s} > t) \to \mathbb{P}_\mu(\tau_{b^s} > t)$. Hence,

$$\limsup_{n\to\infty} |\mathbb{P}_{\mu_n}(\tau_b > t) - \mathbb{P}_\mu(\tau_b > t)| \le \limsup_{n\to\infty} \mathbb{P}_{\mu_n}(\tau_b \le s) + \mathbb{P}_\mu(\tau_b \le s).$$

Letting $s \searrow 0$, the assumption yields the statement. □

When working with boundary functions, the distribution of the Brownian motion conditioned to stay below the boundary function plays an important role. The following theorem provides a useful tool to connect the ordering of boundary functions to the ordering of the corresponding conditioned distributions. It is strongly related to Lemma 2.1.2 from [Rob91b] and Theorem 2.8 from [Rob91a]. For more details regarding these relations see Remark 2.3.18.

**Theorem 2.3.17.** *For $i \in \{1,2\}$ let $b^i$ be a standard boundary function and $\mu_i \in \mathcal{P}$, such that $\mathbb{P}_{T^{b^i(0)}(\mu^i)}(\tau_{b^i} = 0) = 0$. Assume that $\mu^1 \preceq_{|\mathrm{lr}|} \mu^2$ and $b^1 \le b^2$. Then for $t \ge 0$ with $\mathbb{P}_{\mu^i}(\tau_{b^i} > t) > 0$ holds*

$$\mathbb{P}_{\mu^1}(X_t \in \cdot\ |\tau_{b^1} > t) \preceq_{|\mathrm{lr}|} \mathbb{P}_{\mu^2}(X_t \in \cdot\ |\tau_{b^2} > t). \tag{2.18}$$

**Remark 2.3.18.** Similar, related or more general problems as Theorem 2.3.17 have been considered for a larger class of processes or functions in the past. In the following we try to give a few details of related statements.
In terms of the usual stochastic order, in [Rob91b] it is stated, combining

Lemma 2.1.1 and Lemma 2.1.2 therein, that for suitable strictly positive functions $f \leq g$ it holds

$$\mathbb{P}_x \left( Y_t \in \cdot \ \middle| \tau_f > t \right) \preceq \mathbb{P}_y \left( Y_t \in \cdot \ \middle| \tau_g > t \right),$$

for $x < y$, where $(Y_t)_{t \geq 0}$ is a time-homogeneous one-dimensional diffusion process with point symmetric drift coefficient and symmetric diffusion coefficient. In terms of the likelihood ratio order, in [Rob91a] it is stated in Theorem 2.8 that for cadlag functions $f \leq g$ it holds

$$\mathbb{P}_x \left( Z_t \in \cdot \ \middle| \tau_f > t \right) \preceq_{\mathrm{lr}} \mathbb{P}_x \left( Z_t \in \cdot \ \middle| \tau_g > t \right),$$

where $(Z_t)_{t \geq 0}$ is a real-valued, possibly time-inhomogeneous Markov process with initial value $x \in \mathbb{R}$, whose transition preserves the likelihood ratio order in the following sense. Whenever $y < z, 0 \leq s < t$ it shall hold that

$$\mathbb{P}_x \left( Z_t \in \cdot \ \middle| Z_s = y \right) \preceq_{\mathrm{lr}} \mathbb{P}_x \left( Z_t \in \cdot \ \middle| Z_s = z \right).$$

In an application as first step this preservation has to be established. For example, in the case of $Z_t = |X_t|$ being the reflected Brownian motion it would follow from Corollary 2.2.19. The work of [Rob91a] refers to the likelihood ratio order between processes as strong stochastic monotonocity.
Another approach to establish an ordering in the usual stochastic order is to construct realizations of the conditioned processes which are ordered pathwise. In the case of Itô diffusions this is done for sufficiently smooth boundaries $f \leq g$ in the introduction of [Rob91a]. In the case of squared Bessel processes of dimension 2 this approach is carried out for positive measurable boundaries $f \leq g$ in [BB10].

In order to prove Theorem 2.3.17 we will use the approximation of boundary functions, which we introduced in (2.16) and which is an adapted version of an approximation already used in [EJ16] and [Che+11].

*Proof of Theorem 2.3.17.* At first note, that by Lemma 2.3.15 we can assume without loss of generality that already $\mathbb{P}_{\mu^i} (\tau_{b^i} = 0) = 0$.
For the boundary functions $b^i$ let $(b^i_n)_{n \in \mathbb{N}}$ the construction of (2.16), where we choose $D_n(b^i) := A^1_n(b^j) \cup A^2_n(b^j)$ with $j \in \{1, 2\} \setminus \{i\}$.
We claim that for fixed $n \in \mathbb{N}$ the boundary functions $b^1_n$ and $b^2_n$ fulfill (2.18). For this note that by construction $A_n := A_n(b^1) = A_n(b^2)$. Let $m := |A_n|$ and write $A_n = \{a_1, \ldots, a_m\}$ and $a_{m+1} := \infty$ such that $a_k < a_{k+1}$ for $k \in \{1, \ldots, m\}$. Let $k$ such that $t \in [a_k, a_{k+1})$. Observe that by the Markov property we have the representation

$$\mathbb{P}_{\mu^i} \left( X_t \in \cdot \ \middle| \tau_{b^i_n} > t \right) = P_{t-a_k} \circ T^{b^i(a_k)} \circ P_{a_k - a_{k-1}} \circ \ldots \circ T^{b^i(a_1)} \circ P_{a_1}(\mu^i).$$

Since $\mu^1 \preceq_{|\mathrm{lr}|} \mu^2$ and $b^1(a_\ell) \leq b^2(a_\ell)$ we obtain by alternating application of Proposition 2.2.19 and Lemma 2.2.21 that

$$\begin{aligned}\mathbb{P}_{\mu^1}\left(X_t \in \cdot \ \middle| \tau_{b_n^1} > t\right) &= P_{t-a_k} \circ T^{b^1(a_k)} \circ P_{a_k - a_{k-1}} \circ \ldots \circ T^{b^1(a_1)} \circ P_{a_1}(\mu^1) \\ &\preceq_{|\mathrm{lr}|} P_{t-a_k} \circ T^{b^2(a_k)} \circ P_{a_k - a_{k-1}} \circ \ldots \circ T^{b^2(a_1)} \circ P_{a_1}(\mu^2) \\ &= \mathbb{P}_{\mu^1}\left(X_t \in \cdot \ \middle| \tau_{b_n^1} > t\right).\end{aligned}$$

This means that $b_n^1$ and $b_n^2$ fulfill (2.18). We will know pass this property through the limit.
By Lemma 2.3.13 we have that $\tau_{b_n^i} \to \tau_{b^i}$ almost surely. This implies that for any $t \in (0, t^{b^i})$ it holds

$$\{\tau_{b^i} > t\} \subseteq \liminf_{n\to\infty} \{\tau_{b_n^i} > t\}$$

holds almost surely. Consequently, using Fatou's Lemma this yields

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t \geq \tau_{b_n^i}\right) &= \limsup_{n\to\infty} \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t\right) - \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t, \tau_{b_n^i} > t\right) \\ &\leq \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t\right) - \mathbb{P}_{\mu^i}\left(\{\tau_{b^i} > t\} \cap \liminf_{n\to\infty}\{\tau_{b_n^i} > t\}\right) \\ &= \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t\right) - \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t\right) = 0.\end{aligned}$$

Further, for $t > 0$ by Lemma 2.3.13 we have that $\mathbb{P}_{\mu^i}\left(\tau_{b_n^i} > t\right) \to \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t\right)$ as $n \to \infty$ and using this we obtain

$$\begin{aligned}\mathbb{P}_{\mu^i}\left(\tau_{b_n^i} > t \geq \tau_{b^i}\right) &= \mathbb{P}_{\mu^i}\left(\tau_{b_n^i} > t\right) - \mathbb{P}_{\mu^i}\left(\tau_{b_n^i} > t, \tau_{b^i} > t\right) \\ &= \mathbb{P}_{\mu^i}\left(\tau_{b_n^i} > t\right) - \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t\right) + \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t \geq \tau_{b_n^i}\right) \\ &\to 0\end{aligned}$$

as $n \to \infty$. Therefore, for every $t > 0$ we get for every $c \geq 0$ that

$$\begin{aligned}&|\mathbb{P}_{\mu^i}\left(|X_t| \leq c, \tau_{b_n^i} > t\right) - \mathbb{P}_{\mu^i}\left(|X_t| \leq c, \tau_{b^i} > t\right)| \\ &= |\mathbb{P}_{\mu^i}\left(|X_t| \leq c, \tau_{b_n^i} > t, \tau_{b^i} \leq t\right) - \mathbb{P}_{\mu^i}\left(|X_t| \leq c, \tau_{b^i} > t, \tau_{b_n^i} \leq t\right)| \\ &\leq \mathbb{P}_{\mu^i}\left(\tau_{b_n^i} > t \geq \tau_{b^i}\right) + \mathbb{P}_{\mu^i}\left(\tau_{b^i} > t \geq \tau_{b_n^i}\right) \to 0.\end{aligned}$$

This means that

$$\mathbb{P}_{\mu^i}\left(|X_t| \in \cdot \ \middle| \tau_{b_n^i} > t\right) \to \mathbb{P}_{\mu^i}\left(|X_t| \in \cdot \ \middle| \tau_{b^i} > t\right)$$

in the sense of convergence of distributions. Now, since the ordering (2.18) holds for $b_n^1$ and $b_n^2$ for all $n \in \mathbb{N}$, by Theorem B.1.4 follows that (2.18) holds also for $b^1$ and $b^2$. □

### 2.3.2 The lower approximation and uniqueness of continuous solutions

We have seen that Anulova's approach contains the construction of a sequence of barriers whose first-passage times converge in distribution to the given survival function. In our setting this construction can be done as follows.

Let $\mu \in \mathcal{P}$ and $g$ be a survival distribution. Recall $t^g := \sup\{t \in \mathbb{R} : g(t) > 0\}$ as the final extinction time. From now on we fix a timepoint $h \in (0, t^g)$ and set $\delta := \delta^{(n)} := h2^{-n}$.
For every $k \in \mathbb{N}$ with $k\delta < t^g$ set

$$\alpha_k := \alpha_k^{(n)} := \frac{g(k\delta)}{g((k-1)\delta)}$$

and define

$$\mu_k^+ := \mu_k^{+,n} := T_{\alpha_k} \circ P_\delta \circ \ldots \circ T_{\alpha_1} \circ P_\delta(\mu)$$

and $\mu_0^+ := \mu$, where the last equality above is a consequence of Lemma 2.3.9. We define

$$b_n^+(k\delta) = \sup \operatorname{supp}(\mu_k^+),$$

$b_n^+(t) := \infty$ for all $t \in (0, t^g) \setminus \delta\mathbb{N}$ and $b_n^+(t) := 0$ else. In the following this construction will be called the *lower barrier approximation*. By Lemma 2.3.9 we have

$$\mu_k^{+,n} = \mathbb{P}_\mu\left(X_{k\delta} \in \cdot \,\Big|\tau_{b_n^+} > k\delta\right).$$

Further, note that $b_n^+(k\delta)$ is the unique value such that

$$\mathbb{P}_\mu\left(|X_{k\delta}| \leq b_n^+(k\delta) \,\big||X_{\ell\delta}| \leq b_n^+(\ell\delta)\ \forall l \in \{1, \ldots, k-1\}\right) = \alpha_k$$

and thus essentially coincides with the construction of Anulova.

In the special case of $g$ corresponding to the exponential distribution several properties of the sequence $(\mu_k^{+,n})_{n,k\in\mathbb{N}}$ already appeared implicitly in [De +19a] in the context of a free boundary problem. More precisely, in the special setting $g(t) = \exp(-t)$ and under the assumption that the initial distribution admits a Lebesgue density with bounded support, analogues of Lemma 2.3.19 and Lemma 2.3.20 are to be found in [De +19a] for the one-sided case.

Now, note that by construction we have

$$\mathbb{P}_\mu\left(\tau_{b_n^+} > k\delta^{(n)}\right) = g(k\delta^{(n)}).$$

Hence by Lemma 2.1.18 we have $\tau_{b_n^+} \to g$ in distribution as $n \to \infty$.

Although the lower barrier approximation only depends on the survival distribution $g$ and the initial measure $\mu$, its relation to arbitrary solutions of the inverse first-passage time problem always contains the following remarkable property.

**Lemma 2.3.19.** *Let $b \in \operatorname{ifpt}(g, \mu)$ and $\mu_t := \mathbb{P}_\mu\left(X_t \in \cdot \,|\tau_b > t\right)$. Then*

$$\mu_k^{+,n} \preceq \mu_{k\delta}$$

*for all $k \in \mathbb{N}$ such that $k\delta < t^g$, and $n \in \mathbb{N}$.*

*Proof.* Let $n \in \mathbb{N}$ be fixed. In the following the dependency on $n$ is dropped in the notation. For $s \geq 0$ we introduce the shifted boundary function $b^s(r) := b(r+s)$. For $s < t < t^g$ and $\nu \in \mathcal{P}$ set

$$S_{t,s}(\nu) := \mathbb{P}_\nu\left(X_{t-s} \in \cdot \,|\tau_{b^s} > t-s\right),$$

as long as it is defined. Furthermore, for $k > \ell$ such that $k\delta < t^g$, define

$$S_{k,\ell}^+(\nu) := T_{\alpha_k} \circ P_\delta \circ \ldots \circ T_{\alpha_{\ell+1}} \circ P_\delta(\nu).$$

By the definition of $\mu_\ell^+$ we see that

$$S_{k,\ell}^+(\mu_\ell^+) = T_{\alpha_k} \circ P_\delta \circ \ldots \circ T_{\alpha_{\ell+1}}(P_\delta \circ T_{\alpha_\ell} \circ P_\delta \circ \ldots \circ T_{\alpha_1} \circ P_\delta(\mu)) = \mu_k^+.$$

By the Markov property we have that

$$\begin{aligned}
S_{t,s}(\mu_s) = \mathbb{P}_{\mu_s}\left(X_{t-s} \in \cdot \,|\tau_{b^s} > t-s\right) &= \frac{\mathbb{P}_{\mu_s}\left(X_{t-s} \in \cdot\,, \tau_{b^s} > t-s\right)}{\mathbb{P}_{\mu_s}\left(\tau_{b^s} > t-s\right)} \\
&= \frac{\mathbb{E}_\mu\left[\mathbb{P}_{X_s}\left(X_{t-s} \in \cdot\,, \tau_{b^s} > t-s\right) \mathbb{1}_{\{\tau_b > s\}}\right]}{\mathbb{E}_\mu\left[\mathbb{P}_{X_s}\left(\tau_{b^s} > t-s\right)\right] \mathbb{1}_{\{\tau_b > s\}}} \\
&= \frac{\mathbb{P}_\mu\left(X_t \in \cdot\,, \tau_b > t\right)}{\mathbb{P}_\mu\left(\tau_b > t\right)} = \mathbb{P}_\mu\left(X_t \in \cdot \ |\tau_b > t\right) = \mu_t.
\end{aligned}$$

In order to make an induction argument, let $k \in \mathbb{N}$ such that $k\delta < t^g$ and assume that $\mu_{k-1}^+ \preceq \mu_{(k-1)\delta}$. By Proposition 2.2.3 and Corollary 2.2.8 we see that then $S_{k,k-1}^+(\mu_{k-1}^+) \preceq S_{k,k-1}^+(\mu_{(k-1)\delta})$. We claim that

$$S_{k,k-1}^+(\mu_{(k-1)\delta}) \preceq S_{k\delta,(k-1)\delta}(\mu_{(k-1)\delta}).$$

For this, note that by the Markov property, similar as above,

$$\begin{aligned}
\mathbb{P}_{\mu_{(k-1)\delta}}\left(\tau_{b^{(k-1)\delta}} > \delta\right) &= \frac{\mathbb{E}_\mu\left[\mathbb{P}_{X_{(k-1)\delta}}\left(\tau_{b^{(k-1)\delta}} > \delta\right) \mathbb{1}_{\{\tau_b > (k-1)\delta\}}\right]}{\mathbb{P}_\mu\left(\tau_b > (k-1)\delta\right)} \\
&= \frac{\mathbb{P}_\mu\left(\tau_b > k\delta\right)}{\mathbb{P}_\mu\left(\tau_b > (k-1)\delta\right)} = \frac{g(k\delta)}{g((k-1)\delta)} = \alpha_k.
\end{aligned}$$

Now let $c \geq 0$. If $c \geq q_{\alpha_k}(P_\delta \mu_{(k-1)\delta})$, then

$$S^+_{k,k-1}(\mu_{(k-1)\delta})([-c,c]) = 1 \geq S_{k\delta,(k-1)\delta}(\mu_{(k-1)\delta})([-c,c]).$$

If $c < q_{\alpha_k}(P_\delta \mu_{(k-1)\delta})$ we have

$$\begin{aligned} S^+_{k,k-1}(\mu_{(k-1)\delta})([-c,c]) &= \frac{1}{\alpha_k} P_\delta \mu_{(k-1)\delta}([-c,c]) = \frac{1}{\alpha_k} \mathbb{P}_{\mu_{(k-1)\delta}}(|X_\delta| \leq c) \\ &\geq \frac{1}{\alpha_k} \mathbb{P}_{\mu_{(k-1)\delta}}(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} > \delta) = \mathbb{P}_{\mu_{(k-1)\delta}}(|X_\delta| \leq c \,|\tau_{b^{(k-1)\delta}} > \delta) \\ &= S_{k\delta,(k-1)\delta}(\mu_{(k-1)\delta})([-c,c]), \end{aligned}$$

which shows the claim. Altogether it follows that

$$\mu^+_k = S^+_{k,k-1}(\mu^+_{k-1}) \preceq S^+_{k,k-1}(\mu_{(k-1)\delta}) \preceq S_{k\delta,(k-1)\delta}(\mu_{(k-1)\delta}) = \mu_{k\delta}.$$

Since $\mu^+_0 = \mu = \mu_0$, the desired statement follows by induction. □

**Lemma 2.3.20.** *Let $n \in \mathbb{N}$. Then it holds that*

$$\mu^{+,n}_k \preceq \mu^{+,n+1}_{2k}$$

*for all $k \in \mathbb{N}$ such that $k\delta^{(n)} < t^g$.*

*Proof.* Let $k$ such as in the statement's condition. For $\nu \in \mathcal{P}$ define $H^{+,n}_k(\nu) := T_{\alpha^{(n)}_k} \circ P_{\delta^{(n)}}(\nu)$. We first claim that

$$H^{+,n}_k(\nu) \preceq H^{+,n+1}_{2k} \circ H^{+,n+1}_{2k-1}(\nu).$$

Note that $P_{\delta^{(n+1)}} \circ P_{\delta^{(n+1)}} = P_{\delta^{(n)}}$ due to the semigroup property of the Gaussian kernel and the fact that $\delta^{(n)} = \delta^{(n+1)} + \delta^{(n+1)}$. Thus the claim is equivalent to

$$T_{\alpha^{(n)}_k} \circ P_{\delta^{(n+1)}} \circ P_{\delta^{(n+1)}}(\nu) \preceq T_{\alpha^{(n+1)}_{2k}} \circ P_{\delta^{(n+1)}} \circ T_{\alpha^{(n+1)}_{2k-1}} \circ P_{\delta^{(n+1)}}(\nu).$$

Using Lemma 2.2.12 and Lemma 2.2.10 we obtain for arbitrary $\tilde{\nu} \in \mathcal{P}$ that

$$\begin{aligned} T_{\alpha^{(n+1)}_{2k}} \circ P_{\delta^{(n+1)}} \circ T_{\alpha^{(n+1)}_{2k-1}}(\tilde{\nu}) &\succeq T_{\alpha^{(n+1)}_{2k}} \circ T_{\alpha^{(n+1)}_{2k-1}} \circ P_{\delta^{(n+1)}}(\tilde{\nu}) \\ &= T_{\alpha^{(n+1)}_{2k}\alpha^{(n+1)}_{2k-1}} \circ P_{\delta^{(n+1)}}(\tilde{\nu}) \\ &= T_{\alpha^{(n)}_k} \circ P_{\delta^{(n+1)}}(\tilde{\nu}), \end{aligned}$$

where we have used that $\alpha^{(n+1)}_{2k}\alpha^{(n+1)}_{2k-1} = \alpha^{(n)}_k$ by definition. Choosing $\tilde{\nu} = P_{\delta^{(n+1)}}(\nu)$ yields the claim. For an induction, assume that $\mu^{+,n}_{(k-1)} \preceq \mu^{+,n+1}_{2(k-1)}$.

We obtain by Lemma 2.2.3 and Corollary 2.2.8 that $H_k^{+,n}$ is dominance preserving. Using this and observing that

$$\mu_{\ell\delta}^{+} = H_\ell^{+,n} \circ \ldots \circ H_1^{+,n}(\mu)$$

we get by using the assumption and the statement of the claim that

$$\begin{aligned}\mu_k^{+,n} &= H_k^{+,n}(\mu_{k-1}^{+,n}) \preceq H_k^{+,n}(\mu_{2(k-1)}^{+,n+1}) \\ &\preceq H_{2k}^{+,n+1} \circ H_{2k-1}^{+,n+1}(\mu_{2(k-1)}^{+,n+1}) = \mu_{2k}^{+,n+1}.\end{aligned}$$

By induction the statement follows. □

The statements above yield the following corollary.

**Corollary 2.3.21.** *Let $N \in \mathbb{N}$, $h > 0$ and recall $\delta^{(n)} = h2^{-n}$. Let $b \in \operatorname{ifpt}(g, \mu)$. Then for all $n \geq N$ holds*

$$b_n^{+}(t) \leq b_{n+1}^{+}(t) \leq b(t)$$

*for every $t \in h2^{-N}\mathbb{N} \cap [0, t^g]$. Every accumulation point $b^+$ of $(b_n^+)_{n\in\mathbb{N}}$ with respect to the Hausdorff metric fulfills $\tau_{b^+} \sim g$.*

*Proof.* By Lemma 2.3.19 and Lemma 2.3.20 we obtain the ordering

$$\mu^{+,n}_{\frac{t}{\delta^{(n)}}} \preceq \mu^{+,n+1}_{\frac{t}{\delta^{(n+1)}}} \preceq \mu_t.$$

By recovering the boundary functions by Lemma 2.3.6 we obtain the desired inequalities, since the supports of the measures are ordered in the same way. By the fact that $\tau_{b_n^+} \to g$ in distribution as $n \to \infty$ and Proposition 2.1.12 we obtain the last part. □

At this stage, by the lower barrier approximation, it is already possible to deduce a uniqueness property for continuous solutions.

**Proposition 2.3.22** (uniqueness of continuous boundary functions)**.** *There is at most one standard $b \in \operatorname{ifpt}(g, \mu)$ which is continuous on $(0, t^g)$.*

*Proof.* Assume $b \in \operatorname{ifpt}(g, \mu)$ is standard and continuous on $(0, t^g)$. Let $h = 1$. Due to the compactness of the set of boundary functions there exists an accumulation point $b_0$ of the lower barrier approximation with respect to the Hausdorff topology. Let $b_{n_k}^+$ be a subsequence with limit point $b_0$. Let $t \in (0, t^g)$. For $k \in \mathbb{N}$ choose $t_k \in 2^{-n_k}\mathbb{N}\cap[0, t^g]$ such that $t_k \to t$. By Lemma 2.3.21 and Theorem 2.1.4 we deduce that

$$b_0(t) \leq \liminf_{k\to\infty} b_{n_k}^{+}(t_k) \leq \liminf_{k\to\infty} b(t_k) = b(t)$$

since $b$ is continuous at $t$. Since $b_0 \in \operatorname{ifpt}(g, \mu)$, by Lemma 2.3.14 the boundary function $b$ has to coincide with the standard version of $b_0$. This proves the desired statement. □

### 2.3.3 The upper approximation and uniqueness

In the previous section we established a lower bound for the marginal measure $\mu_t$, which gives rise to search for an upper bound. Recalling the statement from Lemma 2.2.12 we observe that adopting the setting from the previous section, the measure resulting from $\mu_k^+$, when interchanging the truncation and the convolution, namely

$$\mu_k^- := \mu_k^{-,n} := P_\delta \circ T_{\alpha_k} \circ \ldots \circ P_\delta \circ T_{\alpha_2} \circ P_\delta \circ T_{\alpha_1}(\mu) \tag{2.19}$$

is larger than the measure $\mu_k^+$ in the two-sided stochastic order. In the special case of $g(t) = e^{-t}$ and $\mu$ admitting a density, a one-sided version of this sequence would coincide with a construction made in [De +19a] in the context of a free boundary problem. In our general setting we have to work with a slight modification of this sequence. For fixed $n \in \mathbb{N}$ define for all $k \geq 2$ the sequence of measures

$$\tilde{\mu}_k^- := \tilde{\mu}_k^{-,n} := P_\delta \circ T_{\alpha_k} \circ \ldots \circ P_\delta \circ T_{\alpha_2} \circ P_\delta(\mu)$$

and $\tilde{\mu}_1^- := \tilde{\mu}_1^{-,n} := P_\delta\mu$ and $\tilde{\mu}_0^- := \tilde{\mu}_0^{-,n} := \mu$. In comparison to $\mu_k^{-,n}$ we removed the first application of $T_{\alpha_1}$.

This family of sequences will be called *the upper barrier approximation* and will serve us as the desired upper bound.

Again, for the special case $g(t) = e^{-t}$ the properties of $\mu_k^-$ obtained by Lemma 2.3.23 and Lemma 2.3.27 implicitely appeared in [De +19a] in terms of (2.19) and for our proofs we draw from ideas therein.

**Lemma 2.3.23.** *Let $b \in \operatorname{ifpt}(g, \mu)$ and $\mu_t := \mathbb{P}_\mu(X_t \in \cdot\,|\tau_b > t)$. Then*

$$\mu_{k\delta} \preceq \tilde{\mu}_k^{-,n}$$

*for all $k \in \mathbb{N}$ such that $k\delta < t^g$, and $n \in \mathbb{N}$.*

In order to prove Lemma 2.3.23 we will prove an auxiliary statement by the use of the stochastic inequality of Theorem 2.3.17 with respect to the two-sided usual stochastic order, which already appeared in the context of the first-passage time problem in [Rob91b]. In fact, we could also use Lemma 2.1.2 of [Rob91b] instead of Theorem 2.3.17 in the following proof of Proposition 2.3.24. The formulation of the following statement is optimized for the application in the proof of Lemma 2.3.23.

**Proposition 2.3.24.** *Let $t > 0$. Let $b$ be a boundary function and $\nu \in \mathcal{P}$ such that $\mathbb{P}_\nu(\tau_b > t) > 0$. Let $s \in (0, t]$. Choose $x \in \mathbb{R}$ such that $|x| \geq b(s)$. Then*

$$\mathbb{P}_\nu(X_t \in \cdot\,|\tau_b > t) \preceq \mathbb{P}_x(X_{t-s} \in \cdot\,)$$

*Proof.* Define $\nu_s := \mathbb{P}_\nu(X_s \in \cdot \,|\tau_b > s)$ and $b^s(t) := b(t+s)$. We have that $\mathrm{supp}(\nu_s) \subseteq [-b(s), b(s)]$. This means that for $\nu_s$-almost every $y \in \mathrm{supp}(\nu_s)$ we have $|y| \leq b(s) \leq |x|$ and thus using Theorem 2.3.17 for the first inequality and Lemma 2.2.3 for the second, we have

$$\mathbb{P}_y(X_{t-s} \in \cdot \,|\tau_{b^s} > t-s) \preceq \mathbb{P}_y(X_{t-s} \in \cdot) \preceq \mathbb{P}_x(X_{t-s} \in \cdot).$$

Now we have

$$\begin{aligned}
&\mathbb{P}_\nu(X_t \in \cdot \,|\tau_b > t) = \mathbb{P}_{\nu_s}(X_{t-s} \in \cdot \,|\tau_{b^s} > t-s) \\
&= \frac{\int_{\mathbb{R}} \mathbb{P}_y(X_{t-s} \in \cdot\,, \tau_{b^s} > t-s)\,\mathrm{d}\nu_s(y)}{\mathbb{P}_{\nu_s}(\tau_{b^s} > t-s)} \\
&= \frac{\int_{\mathbb{R}} \mathbb{P}_y(X_{t-s} \in \cdot \,|\tau_{b^s} > t-s)\,\mathbb{P}_y(\tau_{b^s} > t-s)\,\mathrm{d}\nu_s(y)}{\mathbb{P}_{\nu_s}(\tau_{b^s} > t-s)} \\
&\preceq \frac{\int_{\mathbb{R}} \mathbb{P}_x(X_{t-s} \in \cdot\,)\,\mathbb{P}_y(\tau_{b^s} > t-s)\,\mathrm{d}\nu_s(y)}{\mathbb{P}_{\nu_s}(\tau_{b^s} > t-s)} = \mathbb{P}_x(X_{t-s} \in \cdot\,),
\end{aligned}$$

which yields the statement. □

The following statement will yield the induction basis in the proof of Lemma 2.3.23.

**Lemma 2.3.25.** *Let $b$ be a boundary function and $\nu \in \mathcal{P}$ such that $\mathbb{P}_\nu(\tau_b = 0) = 0$. Let $t > 0$ such that $\mathbb{P}_\nu(\tau_b > t) > 0$. Then*

$$\mathbb{P}_\nu(X_t \in \cdot \,|\tau_b > t) \preceq \mathbb{P}_\nu(X_t \in \cdot\,).$$

*Proof.* For the following note that $|X_{\tau_b}| \geq b(\tau_b)$ and $\tau_b > 0$ almost surely. For $c \geq 0$ we have by Proposition 2.3.24 with $s = \tau_b$ that

$$\begin{aligned}
&\mathbb{P}_\nu(|X_t| \leq c, \tau_b \leq t) = \mathbb{E}_\nu\left[\mathbb{1}_{\{\tau_b \leq t\}} \mathbb{P}_{X_{\tau_b}}(|X_{t-s}| \leq c)_{s=\tau_b}\right] \\
&\leq \mathbb{E}_\nu\left[\mathbb{1}_{\{\tau_b \leq t\}} \mathbb{P}_\nu(|X_t| \leq c\,|\tau_b > t)\right] = \mathbb{P}_\nu(\tau_b \leq t)\,\mathbb{P}_\nu(|X_t| \leq c\,|\tau_b > t).
\end{aligned}$$

Now we can deduce

$$\begin{aligned}
&\mathbb{P}_\nu(|X_t| \leq c) \\
&= \mathbb{P}_\nu(|X_t| \leq c, \tau_b \leq t) + \mathbb{P}_\nu(|X_t| \leq c, \tau_b > t) \\
&\leq \mathbb{P}_\nu(\tau_b \leq t)\,\mathbb{P}_\nu(|X_t| \leq c\,|\tau_b > t) + \mathbb{P}_\nu(\tau_b > t)\,\mathbb{P}_\nu(|X_t| \leq c\,|\tau_b > t) \\
&= \mathbb{P}_\nu(|X_t| \leq c\,|\tau_b > t),
\end{aligned}$$

which shows the desired statement. □

**Remark 2.3.26.** On the first glance the statement of Lemma 2.3.25 could seem like a special case of Lemma 2.1.2 in [Rob91b], but due to a different definition of the random initial value in equation (2.1.7) in [Rob91b] this is only the case if $\nu = \delta_x$.

The statement of Lemma 2.3.25 can also be shown directly in an elementary but technical way, by first showing the statement for discrete boundary functions, then for continuous boundary functions, and then using the approximation of general boundary functions by continuous ones in the Hausdorff metric. The proof is given in the appendix as Lemma A.1.1.

*Proof of Lemma 2.3.23.* Fix $n \in \mathbb{N}$. Since $\tilde{\mu}_1^- = P_\delta \mu$ we obtain the statement for $k = 1$ as consequence of Lemma 2.3.25. Let $k \geq 2$ with $k\delta < t^g$ and assume that $\mu_{(k-1)\delta} \preceq \tilde{\mu}_{k-1}^-$. Let $\nu \in \mathcal{P}$ such that $\operatorname{supp}(\nu) \subseteq [-b((k-1)\delta), b((k-1)\delta)]$ and $\nu$ has no point mass at the boundary. Note that this already implies $\mathbb{P}_\nu\left(\tau_{b^{(k-1)\delta}} = 0\right) = 0$, where $b^{(k-1)\delta}$ denotes the shifted boundary function given by $b^{(k-1)\delta}(t) = b(t + (k-1)\delta)$. Set $\beta_k := \mathbb{P}_\nu\left(\tau_{b^{(k-1)\delta}} > \delta\right)$ and assume that $\beta_k \leq \nu([-q_{\alpha_k}(\nu), q_{\alpha_k}(\nu)])$. We claim that

$$H_k(\nu) := S_{k\delta,(k-1)\delta}(\nu) \preceq H_k^-(\nu) := P_\delta \circ T_{\alpha_k}(\nu), \tag{2.20}$$

where we use the notation from the proof of Lemma 2.3.19. We have to show that for $c \geq 0$ holds

$$H_k^-(\nu)([-c, c]) \leq H_k(\nu)([-c, c]). \tag{2.21}$$

We can rewrite both sides as follows. On the one hand

$$\begin{aligned} H_k(\nu)([-c, c]) &= \mathbb{P}_\nu\left(|X_\delta| \leq c \,|\tau_{b^{(k-1)\delta}} > \delta\right) \\ &= \frac{1}{\beta_k} \int_{\mathbb{R}} \mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}\nu(x), \end{aligned}$$

on the other hand

$$\begin{aligned} &H_k^-(\nu)([-c, c]) = P_\delta \circ T_{\alpha_k}(\nu)([-c, c]) = \mathbb{P}_{T_{\alpha_k}(\nu)}\left(|X_\delta| \leq c\right) \\ &= \int_{\mathbb{R}} \mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} \leq \delta\right) + \mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}T_{\alpha_k}(\nu)(x). \end{aligned}$$

which makes the inequality (2.21) above equivalent to

$$\begin{aligned} &\int_{\mathbb{R}} \mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} \leq \delta\right) \mathrm{d}T_{\alpha_k}(\nu)(x) \\ &\leq \int_{\mathbb{R}} \mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}\left(\frac{\nu}{\beta_k} - T_{\alpha_k}(\nu)\right)(x). \end{aligned} \tag{2.22}$$

Now observe that

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} \leq \delta\right) \mathrm{d}T_{\alpha_k}(\nu)(x)\\
&= 1 - \int_{\mathbb{R}} \mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}T_{\alpha_k}(\nu)(x)\\
&= \frac{1}{\beta_k}\int_{\mathbb{R}} \mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}\nu(x) - \int_{\mathbb{R}} \mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}T_{\alpha_k}(\nu)(x)\\
&= \int_{\mathbb{R}} \mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}\left(\frac{\nu}{\beta_k} - T_{\alpha_k}(\nu)\right)(x). \qquad (2.23)
\end{aligned}$$

Essentially we have shown the desired inequality (2.22) for $c = \infty$, which means that the appearing measures corresponding to the distribution functions depending on $c$ have the same mass. As next step we will establish an inequality between the integrands by the help of Corollary 2.3.24. The idea is to stop the process at the time point, when it passes the boundary and use that at this timepoint it has to be located over any point the process conditioned to survival could be at. Since $|X_{\tau_{b^{(k-1)\delta}}}| \geq b(\tau_{b^{(k-1)\delta}})$ almost surely, by using Corollary 2.3.24 we obtain

$$\begin{aligned}
&\mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} \leq \delta\right) = \mathbb{E}_x\left[\mathbb{E}_x\left[\mathbb{1}_{\{|X_\delta|\leq c\}}\middle|\mathcal{F}_{\tau_{b^{(k-1)\delta}}}\right]\mathbb{1}_{\{\tau_{b^{(k-1)\delta}}\leq\delta\}}\right]\\
&= \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_{b^{(k-1)\delta}}}}\left(|X_{\delta-s}| \leq c\right)_{s=\tau_{b^{(k-1)\delta}}} \cdot \mathbb{1}_{\{\tau_{b^{(k-1)\delta}}\leq\delta\}}\right]\\
&\leq \mathbb{P}_z\left(|X_\delta| \leq c \,|\tau_{b^{(k-1)\delta}} > \delta\right)\mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} \leq \delta\right)
\end{aligned}$$

for any $z$ with $|z| < b((k-1)\delta)$. Hence, the bound also stays true, if we take the infimum over those $z$. Recall that $\beta_k \leq \nu([-q_{\alpha_k}(\nu), q_{\alpha_k}(\nu)])$. Additionally, recall that we assumed $\mathrm{supp}(\nu) \subseteq [-b((k-1)\delta), b((k-1)\delta)]$ and that $\nu$ has no point mass at the boundary. By the definition of $T_{\alpha_k}$ we see that $\frac{\nu}{\beta_k} - T_{\alpha_k}(\nu)$ is a positive measure with mass completely in $(-b((k-1)\delta), b((k-1)\delta))$. Therefore, due to the mass equality from (2.23), we have

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} \leq \delta\right) \mathrm{d}T_{\alpha_k}(\nu)(x)\\
&\leq \inf_{|z|<b((k-1)\delta)} \mathbb{P}_z\left(|X_\delta| \leq c\,|\tau_{b^{(k-1)\delta}} > \delta\right) \int_{\mathbb{R}} \mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} \leq \delta\right) \mathrm{d}T_{\alpha_k}(\nu)(x)\\
&= \inf_{|z|<b((k-1)\delta)} \mathbb{P}_z\left(|X_\delta| \leq c\,|\tau_{b^{(k-1)\delta}} > \delta\right) \int_{\mathbb{R}} \mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}\left(\frac{\nu}{\beta_k} - T_{\alpha_k}(\nu)\right)(x)\\
&\leq \int_{\mathbb{R}} \mathbb{P}_x\left(|X_\delta| \leq c\,|\tau_{b^{(k-1)\delta}} > \delta\right)\mathbb{P}_x\left(\tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}\left(\frac{\nu}{\beta_k} - T_{\alpha_k}(\nu)\right)(x)\\
&= \int_{\mathbb{R}} \mathbb{P}_x\left(|X_\delta| \leq c, \tau_{b^{(k-1)\delta}} > \delta\right) \mathrm{d}\left(\frac{\nu}{\beta_k} - T_{\alpha_k}(\nu)\right)(x),
\end{aligned}$$

which proves the claim. Observe that $\nu = \mu_{(k-1)\delta}$ fulfilles the required conditions and we have indeed $\beta_k = \alpha_k \leq \nu([-q_{\alpha_k}(\nu), q_{\alpha_k}(\nu)])$. By combining

Corollary 2.2.8 and Proposition 2.2.3 that $H_k^-$ is dominance preserving for measures, which are absolutely continuous with respect to Lebesgue measure. Since $\mu_{(k-1)\delta}$ is absolutely continuous with respect to Lebesgue measure, together with the result (2.20) above this yields

$$\mu_{k\delta} = H_k(\mu_{(k-1)\delta}) \preceq H_k^-(\mu_{(k-1)\delta}) \preceq H_k^-(\tilde{\mu}_{k-1}^-) = \tilde{\mu}_k^-.$$

The statement follows therefore by induction. □

Similar as in the case of the lower barrier approximation we can establish a corresponding monotonicity property.

**Lemma 2.3.27.** *Let $n \in \mathbb{N}$. Then it holds that*

$$\tilde{\mu}_k^{-,n} \succeq \tilde{\mu}_{2k}^{-,n+1}$$

*for all $k \in \mathbb{N}$ such that $k\delta^{(n)} < t^g$.*

*Proof.* For $\nu \in \mathcal{P}$ define $H_k^{-,n}(\nu) := P_{\delta^{(n)}} \circ T_{\alpha_k^{(n)}}(\nu)$ for $k \in \mathbb{N}$ such that $k\delta^{(n)} < t^g$. We claim that for $\nu$ such that $c \mapsto \nu((-\infty, c])$ is continuous, we have

$$H_k^{-,n}(\nu) \succeq H_{2k}^{-,n+1} \circ H_{2k-1}^{-,n+1}(\nu).$$

By Lemma 2.2.12 and Lemma 2.2.10 we obtain

$$\begin{aligned} &T_{\alpha_{2k}^{(n+1)}} \circ P_{\delta^{(n+1)}} \circ T_{\alpha_{2k-1}^{(n+1)}}(\nu) \\ &\preceq P_{\delta^{(n+1)}} \circ T_{\alpha_{2k}^{(n+1)}} \circ T_{\alpha_{2k-1}^{(n+1)}}(\nu) \\ &= P_{\delta^{(n+1)}} \circ T_{\alpha_k^{(n)}}(\nu). \end{aligned}$$

Applying $P_{\delta^{(n+1)}}$ on both sides yields the claim by using Lemma 2.2.3 and the semigroup property of the Gaussian kernel. For $k = 1$ we obtain

$$\begin{aligned} \tilde{\mu}_2^{-,n+1} &= P_{\delta^{(n+1)}} \circ T_{\alpha_2^{(n+1)}} \circ P_{\delta^{(n+1)}}(\mu) \\ &\preceq P_{\delta^{(n+1)}} \circ P_{\delta^{(n+1)}}(\mu) = P_{\delta^{(n)}}(\mu) = \tilde{\mu}_1^{-,n}. \end{aligned}$$

Now assume that $\tilde{\mu}_{k-1}^{-,n} \succeq \tilde{\mu}_{2(k-1)}^{-,n+1}$ for fixed $k \geq 2$. Both measures are absolutely continuous with respect to Lebesgue measure. Therefore, by combining Corollary 2.2.8 and Proposition 2.2.3, and using the claim above we have

$$\tilde{\mu}_k^{-,n} = H_k^{-,n}(\tilde{\mu}_{k-1}^{-,n}) \succeq H_k^{-,n}(\tilde{\mu}_{2(k-1)}^{-,n+1}) \succeq H_{2k}^{-,n+1} \circ H_{2k-1}^{-,n+1}(\tilde{\mu}_{2(k-1)}^{-,n+1}) = \tilde{\mu}_{2k}^{-,n+1}.$$

Thus, the statement follows by induction. □

**Remark 2.3.28.** The construction of the upper barrier approximation would be more intuitive, if we worked with $\mu_k^-$ instead of $\tilde{\mu}_k^-$ as in the special case of the exponential distribution. In the one-sided situation this was done in [De +19a] in the context of a free boundary problem. But the desired direction of $\preceq$ in the step of the proof of Lemma 2.3.27, where Lemma 2.2.10 is applied, can only be obtained in the extreme case $T_\alpha \circ T_\beta(\nu) = T_{\alpha\beta}(\nu)$, as Proposition 2.2.11 shows. For arbitrary $\mu$ as starting measure the inequality may fail to be true for $k = 1$. The actual construction is a work around for this technical problem. However, the more intuitive way of construction above would work for every starting measure $\mu$ which satisfies $T_\alpha \circ T_\beta(\mu) = T_{\alpha\beta}(\mu)$, which includes for example $\mu = \delta_0$ and non-atomic measures.

**Remark 2.3.29.** The statements Lemma 2.3.19, Lemma 2.3.20, Lemma 2.3.23 and Lemma 2.3.27 add up to a monotonic squeezing type statement by which any marginal distribution $\mathbb{P}_\mu\left(X_t \in \cdot \,|\tau_b > t\right)$ corresponding to a $b \in \mathrm{ifpt}(g, \mu)$ is affected. It is natural to strive after an argument about some sort of distance of the lower and the upper barrier approximation. In the special case of [De +19a], the approach in terms of the $L^1$-distance of the involved densities would give rise to the total variation distance of measures in the general case. The general setting differs from the special case in the cruical point that in the special case the sequence of $(\alpha_k)_{k\in\mathbb{N}}$ is constant for fixed $n$, namely $\alpha_k = e^{-\delta}$ for every $k \in \mathbb{N}$, which would lead in the general case to a comparison of truncating operators with different mass truncation. Unfortunately, this comparison is inconvenient for our purpose, for example see Lemma A.4.3 related to the total variation distance. In order to get rid of this problem, we align the lower and upper barrier approximation as

$$\begin{aligned}
\mu_k^{+,n} &= \phantom{P_\delta \circ{}} T_{\alpha_k} \circ P_\delta \circ \ldots \circ T_{\alpha_2} \circ P_\delta \circ T_{\alpha_1} \circ P_\delta \ (\mu),\\
\tilde{\mu}_k^{-,n} &= P_\delta \circ T_{\alpha_k} \circ \ldots \circ P_\delta \circ T_{\alpha_2} \circ P_\delta \qquad\qquad\quad (\mu),
\end{aligned}$$

which leads to the natural comparison of truncation with same amounts of mass and some single extra applications of truncation and convolution.

Let $d_{\mathrm{W}}$ denote the Wasserstein distance from (2.11). With the comparing of Remark 2.3.29 in mind we obtain the following behavior with respect to the Wasserstein distance.

**Lemma 2.3.30.** *Assume that $\mu \in \mathcal{P}^1$ is symmetric and let $g$ be a survival distribution. The upper and lower barrier approximation satisfy*

$$d_{\mathrm{W}}\left(\mu_k^{+,n}, \tilde{\mu}_k^{-,n}\right) \leq \frac{\varepsilon_n(\mu, g)}{g(k\delta^{(n)})} + \sqrt{\delta^{(n)}},$$

*for all $n, k \in \mathbb{N}$, where $(\varepsilon_n(\mu, g))_{n\in\mathbb{N}}$ is a sequence converging to zero only depending on $\mu$ and $g$.*

*Proof.* Recall the definition of $S^+_{k,1}$ and $H^+_1$ from the proof of Lemma 2.3.19. Note that

$$\begin{aligned}\mu_k^{+,n} &= S^+_{k,1} \circ H^+_1(\mu) \text{ and}\\ \tilde{\mu}_k^{-,n} &= P_\delta \circ S^+_{k,1}(\mu).\end{aligned}$$

Since all of the operations on $\mu$ preserve the symmetricness of $\mu$, we obtain by the second item of Lemma 2.2.24 and alternated application of Lemma 2.2.25 and Lemma 2.2.24 that

$$\begin{aligned}d_{\mathrm{W}}\left(\mu_k^{+,n}, \tilde{\mu}_k^{-,n}\right) &\leq d_{\mathrm{W}}\left(S^+_{k,1} \circ H^+_1(\mu), S^+_{k,1}(\mu)\right) + \sqrt{\delta}\\ &\leq \left(\prod_{\ell=2}^{k} \alpha_\ell\right)^{-1} d_{\mathrm{W}}\left(H^+_1(\mu), \mu\right) + \sqrt{\delta}\\ &= \frac{g(\delta)}{g(k\delta)} d_{\mathrm{W}}\left(T_{\alpha_1} \circ P_\delta(\mu), \mu\right) + \sqrt{\delta}.\end{aligned}$$

Take $\varepsilon_n(\mu, g) := g(\delta^{(n)}) d_{\mathrm{W}}\left(T_{\alpha_1^{(n)}} \circ P_{\delta^{(n)}}(\mu), \mu\right)$. We claim that this sequence converges to zero. For the following, note that $W_t := X_t - X_0$ defines a standard Brownian motion. Another application of Lemma 2.2.25 and Lemma 2.2.24 yields

$$\begin{aligned}d_{\mathrm{W}}\left(T_{\alpha_1} \circ P_\delta(\mu), \mu\right) &\leq d_{\mathrm{W}}\left(P_\delta\mu, \mu\right) + 2\int_{\mathbb{R}\setminus[-q_{\alpha_1}(P_\delta\mu), q_{\alpha_1}(P_\delta\mu)]} |x| \, \mathrm{d}P_\delta\mu(x)\\ &\leq \sqrt{\delta} + 2\mathbb{E}_\mu\left[|X_0 + W_\delta| \mathbb{1}_{\{|X_0+W_\delta| > q_{\alpha_1}(P_\delta\mu)\}}\right]\\ &\leq \sqrt{\delta} + 2\mathbb{E}_\mu\left[|W_\delta|\right] + 2\mathbb{E}_\mu\left[|X_0| \mathbb{1}_{\{|X_0+W_\delta| > q_{\alpha_1}(P_\delta\mu)\}}\right].\end{aligned}$$

On the one hand we have $\mathbb{E}_\mu\left[|W_\delta|\right] \leq \sqrt{\delta}$. On the other hand observe that the finite measure $|X_0| \, \mathrm{d}\mathbb{P}_\mu$ is absolutely continuous with respect to $\mathbb{P}_\mu$. Consequently, since for $n \to \infty$ we have

$$\begin{aligned}\mathbb{P}_\mu\left(|X_0 + W_\delta| > q_{\alpha_1}(P_\delta\mu)\right) &= 1 - P_\delta\mu([-q_{\alpha_1}(P_\delta\mu), q_{\alpha_1}(P_\delta\mu)])\\ &= 1 - \alpha_1 = 1 - g(\delta^{(n)}) \to 0,\end{aligned}$$

it follows that

$$\mathbb{E}_\mu\left[|X_0| \mathbb{1}_{\{|X_0+W_\delta| > q_\alpha(P_\delta\mu)\}}\right] \to 0$$

as $n \to \infty$. This eventually yields $\varepsilon_n(\mu, g) \to 0$. □

**Remark 2.3.31.** In the general case and the aligning of Remark 2.3.29 the use of the total variation distance is inconvenient. We already mentioned that by another alignment in the special case of [De +19a] it was worked with the total variation distance, which was suitable since the truncated mass was constantly $1-e^{-\delta}$. Working with their alignment, for continuous $g$ it is possible to derive the total variation result Lemma A.5.2 in the spirit of Lemma 2.3.30 by altering the discrete timesteps. For an application similar as in the following Theorem 2.3.32 see Proposition A.5.4.

We will now connect the result of Lemma 2.3.30 to solutions of the inverse first-passage time problem.

**Theorem 2.3.32.** *Assume $\mu \in \mathcal{P}^1$ is symmetric and $b \in \operatorname{ifpt}(g, \mu)$. Then we have both $\mu_{2^n}^{+,n}, \mu_{2^n}^{-,n}$ converge to $\mu_h := \mathbb{P}_\mu\left(X_h \in \cdot \,|\tau_b > h\right)$ in Wasserstein distance.*

*Proof.* For abbreviation set $\mu^{+,n} := \mu_{2^n}^{+,n}$ and $\mu^{-,n} := \mu_{2^n}^{-,n}$ and $\tilde{\mu}^{-,n} := \tilde{\mu}_{2^n}^{-,n}$. Combining Lemma 2.3.19 and Lemma 2.3.23 and considerig the fact that dropping truncation makes a measure at most wider, this is applying Corollary 2.2.9, we have

$$\mu^{+,n} \preceq \tilde{\mu}^{-,n} \preceq \mu * \mathcal{N}(0, h) \tag{2.24}$$

for every $n \in \mathbb{N}$. Therefore the sequences of probability measures $(\mu^{+,n})_{n\in\mathbb{N}}$ and $(\tilde{\mu}^{-,n})_{n\in\mathbb{N}}$ are tight. Let $(\mu^n)_{n\in\mathbb{N}} \in \{(\mu^{+,n})_{n\in\mathbb{N}}, (\tilde{\mu}^{-,n})_{n\in\mathbb{N}}\}$. In view of Prohorov's theorem let $\mu^0$ be an accumulation point such that $\mu^0 = \lim_{\ell\to\infty} \mu^{n_\ell}$ in the sense of weak convergence. For every $c \geq 0$ we have that by either Lemma 2.3.20 or Lemma 2.3.27 the sequence $\mu^n([-c, c])$ is monotone. But this means that it has a limit and thus

$$\lim_{n\to\infty} \mu^n([-c, c]) = \lim_{\ell\to\infty} \mu^{n_\ell}([-c, c]).$$

Let $c \in \mathbb{R}$ be a continuity point of $\mathbb{R} \to [0, 1],\ x \mapsto \mu^0((-\infty, x])$. Since $\mu$ was symmetric, so is $\mu^n$ for all $n \in \mathbb{N}$. Hence, by using that $\mu^n$ has no atoms,

$$\begin{aligned}
\lim_{n\to\infty} \mu^n((-\infty, c]) &= \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(c) \lim_{n\to\infty} \mu^n([-c, c]) \\
&= \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(c) \lim_{\ell\to\infty} \mu^{n_\ell}([-c, c]) \\
&= \lim_{\ell\to\infty} \mu^{n_\ell}((-\infty, c]) = \mu^0((-\infty, c]),
\end{aligned}$$

which means that actually $\mu^n \to \mu^0$ in the sense of weak convergence. Denote with $\mu^+$ and $\tilde{\mu}^-$ the weak limits of $\mu^{+,n}$ and $\tilde{\mu}^{-,n}$, respectively. Now observe that by (2.24) and the gluing lemma B.1.13 in combination with Lemma 2.2.2 there exist random variables $Z_n^+ \sim \mu^{+,n}$, $Z_n^- \sim \tilde{\mu}^{-,n}$ and $Z \sim \mu * \mathcal{N}(0, h)$ such that

$$|Z_n^+| \leq |Z_n^-| \leq |Z|,$$

which implies that for $R > 0$ we have

$$\mathbb{E}\left[|Z_n^+|\mathbb{1}_{\{|Z_n^+|>R\}}\right] \leq \mathbb{E}\left[|Z_n^-|\mathbb{1}_{\{|Z_n^-|>R\}}\right] \leq \mathbb{E}\left[|Z|\mathbb{1}_{\{|Z|>R\}}\right] \stackrel{R\to\infty}{\to} 0,$$

since $Z$ is integrable. Now Theorem B.1.10 yields that $d_{\mathrm{W}}\left(\mu^{+,n}, \mu^+\right) \to 0$ and $d_{\mathrm{W}}\left(\tilde{\mu}^{-,n}, \tilde{\mu}^-\right) \to 0$. By Lemma 2.3.30 we can conclude that $\mu^+ = \tilde{\mu}^- = \mu^0$. Now let $c \in \mathbb{R}$ be again a continuity point of $\mu^0$.
Let $b \in \operatorname{ifpt}(\mu, g)$ and define $\mu_t := \mathbb{P}_\mu\left(X_t \in \cdot\, | \tau_b > t\right)$. Observe that $\mu_t$ is symmetric, since $\mu$ is. Due to Lemma 2.3.19 and Lemma 2.3.23 in the case $c \geq 0$ we have

$$\begin{aligned}\mu^0((-\infty, c]) &= \frac{1}{2} + \frac{1}{2}\lim_{n\to\infty} \mu^{+,n}([-c, c]) \geq \frac{1}{2} + \frac{1}{2}\mu_h([-c, c])\\ &\geq \frac{1}{2} + \frac{1}{2}\lim_{n\to\infty} \tilde{\mu}^{-,n}([-c, c]) = \mu^0((-\infty, c])\end{aligned}$$

and in the case $c < 0$ we have

$$\begin{aligned}\mu^0((-\infty, c]) &= \frac{1}{2} - \frac{1}{2}\lim_{n\to\infty} \tilde{\mu}^{-,n}([-c, c]) \geq \frac{1}{2} - \frac{1}{2}\mu_h([-c, c])\\ &\geq \frac{1}{2} - \frac{1}{2}\lim_{n\to\infty} \mu^{+,n}([-c, c]) = \mu^0((-\infty, c]).\end{aligned}$$

Since the set of continuity points is dense in $\mathbb{R}$ we obtain that $\mu_h = \mu^0$. Furthermore, note that we have clearly $\mu_k^{-,n} \preceq \tilde{\mu}_k^{-,n}$ and by Lemma 2.2.12 it follows inductively that $\mu_k^{+,n} \preceq \mu_k^{-,n}$ for all $k\delta < t^g$. Therefore, we have analogously to above

$$\mu^{+,n}([-c, c]) \geq \mu^{-,n}([-c, c]) \geq \tilde{\mu}^{-,n}([-c, c]),$$

which implies in the same manner as above, that $d_{\mathrm{W}}\left(\mu^{-,n}, \mu_h\right) \to 0$. □

Let us now finalize the goal of uniqueness.

**Theorem 2.3.33** (Uniqueness)**.** *Let $\mu \in \mathcal{P}$ and $g$ be a survival distribution. Then the standard boundary function $b \in \operatorname{ifpt}(g, \mu)$ is unique.*

*Proof.* As first step assume that $\mu \in \mathcal{P}^1$ is symmetric. Let $b \in \operatorname{ifpt}(g, \mu)$ and let $\mu_0$ be the limit of $\mu_{2^n}^{+,n}$. By Lemma 2.3.6 and Theorem 2.3.32 we see that $b(h) = \sup\operatorname{supp}(\mu^0)$, but since $h \in (0, t^g)$ was arbitrarily chosen and $\mu^0$ did not depend on the choice of $b$ we obtain that every boundary function in $\operatorname{ifpt}(\mu, g)$ has to coincide with $b$ on $(0, t^g)$ and thus there is only one standard boundary function.

As second step let $\mu \in \mathcal{P}$ be arbitrary. We can assume without loss of generality that $\mu$ is symmetric, since $\operatorname{ifpt}(g, \mu) = \operatorname{ifpt}(g, \tilde{\mu})$, where $\tilde{\mu}$ is the symmetrized version of $\mu$, i.e. $\tilde{\mu}(A) = \mu(A \cap \{0\}) + \frac{1}{2}\mu(A \setminus \{0\}) + \frac{1}{2}\mu(-A \setminus \{0\})$. We define $\mu_n := \mu(\cdot\, |[-n, n])$ for every $n \in \mathbb{N}$ such that $\mu([-n, n]) > 0$. Then

$\mu_n$ is symmetric and it holds $\mu_n \in \mathcal{P}^1$. Furthermore, we have $\mu_n \preceq \mu_{n+1} \preceq \mu$. By Theorem 2.3.10 we have that for $b$ there exist $b_n \in \operatorname{ifpt}(g, \mu_n)$ such that $b_n \leq b_{n+1} \leq b$. Without loss of generality we can assume that $b_n$ is standard. But then since $\mu_n \in \mathcal{P}^1$, the boundary functions $b_n$ are the unique standard boundary functions in $\operatorname{ifpt}(g, \mu_n)$, and thus do not depend on $b$. By the monotonicity and Lemma 2.1.8 we obtain that there exists $\underline{b}$ such that $b_n \xrightarrow{\Gamma} \underline{b}$ in Hausdorff distance. At first we claim that $\underline{b} \in \operatorname{ifpt}(g, \mu)$.
By Lemma 2.1.7 we have $\underline{b} \leq b$. For $n \in \mathbb{N}$ we have $b_n \leq \underline{b}$ and thus on the one hand

$$\lim_{s \searrow 0} \limsup_{n \to \infty} \mathbb{P}_{\mu_n}\left(\tau_{\underline{b}} \leq s\right) \leq \lim_{s \searrow 0} \limsup_{n \to \infty} \mathbb{P}_{\mu_n}\left(\tau_{b_n} \leq s\right) = \lim_{s \searrow 0} 1 - g(s) = 0.$$

For $\varepsilon > 0$ let $n$ be large enough such that $\mu([-n, n]) \geq 1 - \varepsilon$. Then

$$\mathbb{P}_{\mu}\left(\tau_{\underline{b}} \leq s\right) \leq \varepsilon + \mu([-n, n])\mathbb{P}_{\mu_n}\left(\tau_{\underline{b}} \leq s\right).$$

In the view of the above we have $\lim_{s \searrow 0} \mathbb{P}_{\mu}\left(\tau_{\underline{b}} \leq s\right) \leq \varepsilon$, which means on the other hand $\mathbb{P}_{\mu}\left(\tau_{\underline{b}} = 0\right) = 0$ by $\varepsilon \searrow 0$. Now by Lemma 2.3.16 we obtain that for every $t \geq 0$ we have

$$\begin{aligned} g(t) = \mathbb{P}_{\mu}\left(\tau_{b} > t\right) &\geq \mathbb{P}_{\mu}\left(\tau_{\underline{b}} > t\right) \\ &= \lim_{n \to \infty} \mathbb{P}_{\mu_n}\left(\tau_{\underline{b}} > t\right) \geq \lim_{n \to \infty} \mathbb{P}_{\mu_n}\left(\tau_{b_n} > t\right) = g(t), \end{aligned}$$

which means that $\underline{b} \in \operatorname{ifpt}(g, \mu)$. Since $\underline{b} \leq b$, by Lemma 2.3.14 this implies that every standard solution $b$ coincides with the standardized version of $\underline{b}$. □

### 2.3.4 Comparison principle and properties of solutions

If we pose the inverse first-passage time problem with initial measures, which are ordered in the two-sided stochastic order, it is natural to expect that the marginal distributions of the Brownian motions conditioned to not have hit the corresponding boundary solutions stay ordered in the two-sided stochastic order. Indeed, one can fix two different survival distributions which are ordered in the well-known hazard rate order of random variables, to get the following strengthened version of Theorem 2.3.10.

**Theorem 2.3.34** (Comparison principle). *Let $\mu_1, \mu_2 \in \mathcal{P}$, such that $\mu_1 \preceq \mu_2$. Let $g^1$ and $g^2$ be two survival distributions, such that $g^2/g^1$ is non-decreasing on $[0, t^{g^1})$. Then the unique standard solutions $b^i \in \operatorname{ifpt}(g^i, \mu_i)$ for $i \in \{1, 2\}$ satisfy*

$$b^1 \leq b^2$$

*pointwise.*

In order to prove Theorem 2.3.34 we will use the lower barrier approximation from Subsection 2.3.2. The uniqueness guarantees us the following statement about convergence in the Hausdorff distance, by which we will be able to pass properties of the approximants to the limit.

**Lemma 2.3.35.** *Let $\mu \in \mathcal{P}$. Let $g_n$, $n \in \mathbb{N}$ and $g$ be survival distributions such that $g_n \to g$ in distribution. Let $b_n \in \mathrm{ifpt}(\mu, g_n)$ and $b \in \mathrm{ifpt}(\mu, g)$. Then $b_n \mathbb{1}_{(0,t^g)} \xrightarrow{\Gamma} b\mathbb{1}_{(0,t^g)}$.*

*Proof.* Let $N' \subseteq \mathbb{N}$ be a subsequence. Then, due to compactness, there exists a subsequence $N'' \subseteq N'$, such that $b_n \to b_0$ along $N''$, where $b_0$ is a boundary function. The assumptions on $b_n$ yield that $\tau_{b_n} \to g$ in distribution. Thus by Proposition 2.1.12 it follows that $b_0 \in \mathrm{ifpt}(\mu, g)$. Therefore, due to uniqueness, the standardized versions of $b_0$ and $b$ coincide. In particular, $b_0\mathbb{1}_{(0,t^g)} = b\mathbb{1}_{(0,t^g)}$. If $t \notin (0, t^g)$, for any sequence $t_n \to t$ along $N''$ it holds

$$\liminf_{n\in N''} b_n(t_n)\mathbb{1}_{(0,t^g)}(t_n) \geq 0 = b(t_n)\mathbb{1}_{(0,t^g)}(t)$$

and the constant sequence $t_n = t$ fulfills $\lim_{n\in N''} b_n(t_n)\mathbb{1}_{(0,t^g)}(t_n) = 0 = b(t_n)\mathbb{1}_{(0,t^g)}(t)$. Else if $t \in (0, t^g)$ and $t_n \to t$ along $N''$, then for $n$ large enough we have

$$b(t) = b_0(t) \leq \liminf_{n\in N''} b_n(t_n) = \liminf_{n\in N''} b_n(t_n)\mathbb{1}_{(0,t^g)}(t_n),$$

since $b_n \xrightarrow{\Gamma} b_0$ along $N''$. Furthermore, there exists a sequence $t_n \to t$ such that

$$b_n(t_n)\mathbb{1}_{(0,t^g)}(t_n) = b_n(t_n) \to b_0(t) = b(t) = b\mathbb{1}_{(0,t^g)}(t).$$

Hence, by Theorem 2.1.4, we have $b_n\mathbb{1}_{(0,t^g)} \xrightarrow{\Gamma} b\mathbb{1}_{(0,t^g)}$ along $N''$. Since the space of boundary functions is a metric space and $N'$ was arbitrary we obtain the statement. ☐

Since the value of the lower barrier approximants $b_n^+$ is defined to be 0 at $t \in \{0, \infty\}$ we obtain the following corollary.

**Corollary 2.3.36.** *Let $\mu \in \mathcal{P}$ and $b \in \mathrm{ifpt}(g, \mu)$ the unique standard boundary function. Then $b_n^+ \xrightarrow{\Gamma} b\mathbb{1}_{(0,\infty)}$.*

*Proof of Theorem 2.3.34.* At first note, that by the monotonicity we have $t^{g^1} \leq t^{g^2}$. Let $(b_n^i)_{n\in\mathbb{N}}$ be the lower barrier approximation from Subsection 2.3.2 with $h = 1$ corresponding to $g^i$ and $\mu_i$. Fix $n \in \mathbb{N}$ and recall that for $k2^{-n} < t^{g^i}$

$$\alpha_k^i = \frac{g^i(k2^{-n})}{g^i((k-1)2^{-n})}.$$

Now Lemma 2.3.9 yields, that for $k2^{-n} < t^{g^1}$ holds

$$\mathbb{P}_{\mu_i}\left(X_{k2^{-n}} \in \cdot \ \middle| \tau_{b_n^i} > k2^{-n}\right) = T_{\alpha_k^i} \circ P_{2^{-n}} \circ \ldots \circ T_{\alpha_1^i} \circ P_{2^{-n}}(\mu_i)$$

and by the assumption on the survival distributions we obtain

$$\frac{g^2((k-1)2^{-n})}{g^1((k-1)2^{-n})} \leq \frac{g^2(k2^{-n})}{g^1(k2^{-n})}$$

which implies $\alpha_k^1 \leq \alpha_k^2$. By iteratively applying the statements of Proposition 2.2.3 and Lemma 2.2.6 we get

$$\mathbb{P}_{\mu_1}\left(X_{k2^{-n}} \in \cdot \ \middle| \tau_{b_n^1} > k2^{-n}\right) \preceq \mathbb{P}_{\mu_2}\left(X_{k2^{-n}} \in \cdot \ \middle| \tau_{b_n^2} > k2^{-n}\right).$$

Hence, Lemma 2.3.6 implies that $b_n^1(k2^{-n}) \leq b_n^2(k2^{-n})$ for all $k2^{-n} < t^{g^1}$. By the definition of the lower barrier approximation and $t^{g^1} \leq t^{g^2}$ follows that $b_n^1 \leq b_n^2$. By Corollary 2.3.36 we have that $b_n^i \xrightarrow{\Gamma} b^i \mathbb{1}_{(0,\infty)}$. By Lemma 2.1.7 we deduce that $b^1 \mathbb{1}_{(0,\infty)} \leq b^2 \mathbb{1}_{(0,\infty)}$, and thus, since $b^1$ and $b^2$ were standard, $b^1 \leq b^2$. □

**Remark 2.3.37.** The assumption on $g^1$ and $g^2$ in Theorem 2.3.34 that $t \mapsto g^2(t)/g^1(t)$ is non-decreasing on $[0, t^{g^1})$ coincides with the notion of the hazard rate order of random variables as defined in Section 1.B of [SS07].

**Remark 2.3.38.** For $g^1 = g^2 = g$ the comparison principle of Theorem 2.3.34 motivates the following question. The solution in $\text{ifpt}(g, \delta_0)$ is dominated by any other solution $\text{ifpt}(g, \mu_2)$. Thus, one could fix $b$ and ask whether $\mu$ exists, such that $b \in \text{ifpt}(g, \mu)$. This problem is sometimes called randomized first-passage time problem. For the one-sided situation see for example [JKZ09], [JKV09b], [JKV14]. In the two-sided situation this question has been studied for the special case of constant boundaries in [Abu13b]. A partial overview is given in [Abu13a].

**Remark 2.3.39.** In the special case $g^1(t) = g^2(t) = e^{-t}$ a corresponding comparison principle appears for solutions of a free boundary problem in [Ber+21] and [BBP19].

**Remark 2.3.40.** For certain survival distributions, it is possible to show by Theorem 2.3.34 that the corresponding solutions are bounded. For example, see Corollary 2.3.54 in the exponential case. In this case that the boundary is bounded, the integral equation from Proposition A.2.1 connects the initial distribution $\mu$, the survival distribution $g$ and the solution $b$.

The assumption in Theorem 2.3.34 that $g^1$ and $g^2$ are ordered in the hazard rate order cannot be weakened to the usual stochastic order as the following example demonstrates.

**Example 2.3.41.** Let $\mu \in \mathcal{P}$. We merely prove the existence of an artificial example of two boundary functions such that their survival distributions are ordered in the usual stochastic order but the boundary functions are not ordered pointwise. For this purpose let $K, c, \varepsilon > 0$ and $s_2 > s_1 > s_0 > 0$. First, set $b^i(t) = \infty$ for $t \notin [s_0, s_2]$. Let one boundary function be given by $b^2(t) := K$ for $t \in [s_0, s_2]$. Let the other one be given by

$$b^1(t) := \begin{cases} K + \frac{(c-K)}{s_1} \cdot t & : t \in [s_0, s_1], \\ c + \frac{K+\varepsilon-c}{s_2-s_1} \cdot (t - s_1) & : t \in (s_1, s_2]. \end{cases}$$

For an illustration compare Figure 2.2. Thus on $[s_0, s_2]$ the function $b^1$ is continuous and piecewise linear, with $b^1(s_1) = c$. Observe that

$$\lim_{c \searrow 0} \mathbb{P}_\mu (\tau_{b^1} > s_1) \leq \lim_{c \searrow 0} \mathbb{P}_\mu (|X_{s_1}| \leq c) = 0.$$

Thus we can choose $c \in (0, K)$ such that $\mathbb{P}_\mu (\tau_{b^1} > s_1) \leq \mathbb{P}_\mu (\tau_{b^2} > s_2)$ since $\mathbb{P}_\mu (\tau_{b^2} > s_2) > 0$. We have $\mathbb{P}_\mu (\tau_{b^1} > t) \leq \mathbb{P}_\mu (\tau_{b^2} > t)$ for $t \in [0, s_1]$ and

$$\mathbb{P}_\mu (\tau_{b^1} > t) \leq \mathbb{P}_\mu (\tau_{b^1} > s_1) \leq \mathbb{P}_\mu (\tau_{b^2} > s_2) \leq \mathbb{P}_\mu (\tau_{b^2} > t)$$

for $t \in [s_1, s_2]$ and thus $\tau_{b^1}$ is smaller in the usual stochastic order than $\tau_{b^2}$, but $b^1(s_2) = K + \varepsilon > K = b^2(s_2)$.

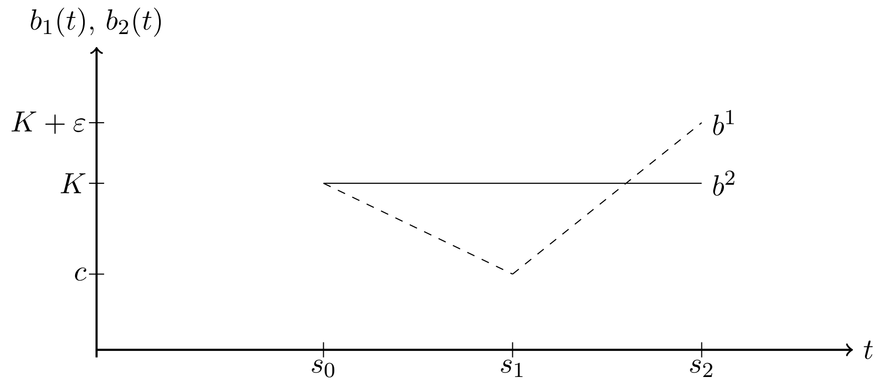


Figure 2.2: Illustration of the boundary functions in Example 2.3.41.

Concerning the converse direction, it is clear that $b^1 \leq b^2$ implies that $\tau_{b^1}$ is smaller in the usual stochastic order than $\tau_{b^2}$. Similarly to above, this cannot be strengthened to an implication of the hazard rate order of the survival distributions as the next example shows. Thus the converse of Theorem 2.3.34 is not true.

**Example 2.3.42.** Let $\mu \in \mathcal{P}$. Again we merely prove the existence of an artificial example of two boundary functions, which are ordered pointwise, but their survival distributions are not ordered in the hazard rate ordering. For this purpose let $K, c$ and $s_2 > s_1 > s_0 > 0$. Again, set $b^i(t) = \infty$ for $t \notin [s_0, s_2]$. Then, let one boundary function be given again by $b^2(t) := K$ for $t \in [s_0, s_2]$. Let the other one be given by

$$b^1(t) := \begin{cases} K + \frac{(c-K)}{s_1} \cdot t & : t \in [s_0, s_1], \\ K & : t \in (s_1, s_2]. \end{cases}$$

Now, denote $g^i(t) := \mathbb{P}_\mu(\tau_{b^i} > t)$ and $\mu^i_{s_1} := \mathbb{P}_\mu(X_{s_1} \in \cdot \ |\tau_{b^i} > s_1)$. We have

$$\mathbb{P}_\mu(\tau_{b^2} > s_2 \, |\tau_{b^2} > s_1) = \mathbb{P}_{\mu^2_{s_1}}(\tau_K > s_2 - s_1) < \mathbb{P}_0(\tau_K > s_2 - s_1).$$

Furthermore, by $\lim_{c \searrow 0} \mu^1_{s_1} = \delta_0$ weakly, we can deduce that

$$\lim_{c \searrow 0} \mathbb{P}_\mu(\tau_{b^1} > s_2 \, |\tau_{b^1} > s_1) = \lim_{c \searrow 0} \mathbb{P}_{\mu^1_{s_1}}(\tau_K > s_2 - s_1) = \mathbb{P}_0(\tau_K > s_2 - s_1).$$

Thus we can choose $c > 0$ such that

$$\frac{g^1(s_1)g^2(s_2)}{g^1(s_2)g^2(s_1)} = \frac{\mathbb{P}_\mu(\tau_{b^2} > s_2 \, |\tau_{b^2} > s_1)}{\mathbb{P}_\mu(\tau_{b^1} > s_2 \, |\tau_{b^1} > s_1)} < 1,$$

which means

$$\frac{g^2(s_2)}{g^1(s_2)} < \frac{g^2(s_1)}{g^1(s_1)}$$

and shows that $g^1$ is not smaller than $g^2$ in the hazard rate order.

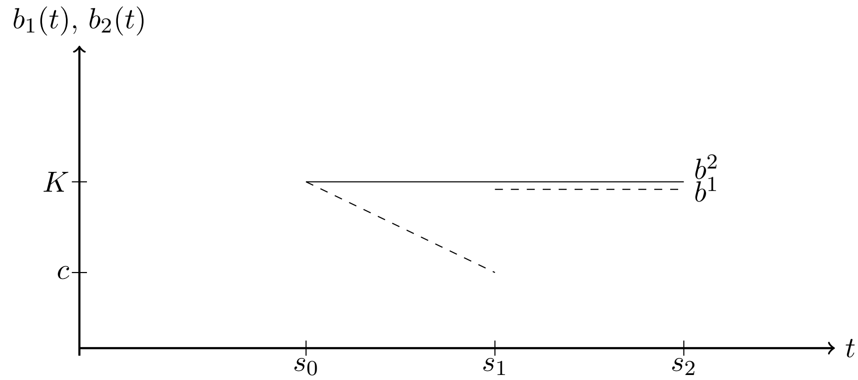


Figure 2.3: Illustration of the boundary functions in Example 2.3.42.

The following is devoted to apply the comparison principle above in order to deduce properties of solutions.

**Regularity**

By making use of Brownian scaling a first application of Theorem 2.3.34 is the following, giving a sufficient but strong condition for local Lipschitz continuity of a non-decreasing boundary function. Recall the notation $g^\lambda(t) := g(\lambda t)$ for a survival distribution $g$ and a number $\lambda > 0$. Note that then $t^{g^\lambda} = t^g/\lambda$.

**Proposition 2.3.43.** *Let $g$ be a survival distribution and $b \in \operatorname{ifpt}(g, 0)$ standard. Assume that for all $\lambda \in (0,1)$ holds that $g^\lambda/g$ is non-decreasing on $(0, t^g)$. Then $t \mapsto \frac{b(t)}{\sqrt{t}}$ is non-increasing. In particular,*

$$b(t) - b(s) \le |t-s|\frac{b(s)}{s}$$

*for $0 < s < t$.*

*Proof.* Recall the notation $b^\lambda(t) := \frac{1}{\sqrt{\lambda}} b(\lambda t)$. Let $\lambda_1 > \lambda_2 > 0$. Then Lemma 2.3.7 we have $b^{\lambda_i} \in \operatorname{ifpt}(g^{\lambda_i}, 0)$. Now set $\lambda := \frac{\lambda_2}{\lambda_1}$. For $s, t \in (0, t^g/\lambda_1)$ with $s < t$ we have

$$\frac{g(\lambda_2 s)}{g(\lambda_1 s)} = \frac{g(\lambda\lambda_1 s)}{g(\lambda_1 s)} \le \frac{g(\lambda\lambda_1 t)}{g(\lambda_1 t)} = \frac{g(\lambda_2 t)}{g(\lambda_1 t)}.$$

Thus $g^{\lambda_2}/g^{\lambda_1}$ is non-decreasing on $(0, t^{g_1^\lambda})$. Consequently, Theorem 2.3.34 yields that we have $b^{\lambda_1} \le b^{\lambda_2}$. For $0 < s < t$ and $r > t$ this means and choosing $\lambda_2 = \frac{s}{r}$ and $\lambda_1 = \frac{t}{r}$ that

$$\frac{b(s)}{\sqrt{s}} = \frac{b(\lambda_2 r)}{\sqrt{\lambda_2 r}} = \frac{1}{\sqrt{r}} b^{\lambda_2}(r) \ge \frac{1}{\sqrt{r}} b^{\lambda_1}(r) = \frac{b(\lambda_1 r)}{\sqrt{\lambda_1 r}} = \frac{b(t)}{\sqrt{t}}, \tag{2.25}$$

which yields the first part of the statement. By $b(t) \le \sqrt{\frac{t}{s}} b(s)$ we can deduce

$$b(t) - b(s) \le \left(\sqrt{\frac{t}{s}} - 1\right) b(s) \le \left(\sqrt{\frac{t}{s}} + 1\right)\left(\sqrt{\frac{t}{s}} - 1\right) b(s) = \frac{t-s}{s} b(s),$$

which finishes the proof. □

**Remark 2.3.44.**

- In terms of the hazard rate $h(t) := -\frac{\partial}{\partial_t}\log(g(t))$ of a differentiable survival function $g$ the condition that $g^\lambda(t)/g(t)$ is non-decreasing in $t$ for $\lambda \in (0,1)$ is equivalent to $h(t) \ge \lambda h(\lambda t)$ for $\lambda \in (0,1)$. By this, the condition is often to be checked conveniently, if $g$ belongs to a suitable parameter family of distributions.
- The statement of Proposition 2.3.43 exploits the Brownian scaling for a boundary function of Lemma 2.3.7. This property was already used in the context of first-passage times in [Val09, p.112]. Reversed to (2.25),

it is shown that under the assumption $b^{\lambda_1} \geq b^{\lambda_2}$ for all $\lambda_1 \leq \lambda_2 \leq 1$, it holds that $t \mapsto \frac{b(t)}{\sqrt{t}}$ is non-decreasing. Regarding this, the complementary condition that $g^\lambda/g$ shall be non-increasing for all $\lambda \in (0,1)$ comes into mind, but this cannot be fulfilled by any survival distribution as the following computation shows. If we assumed this condition we had

$$1 \leq \frac{g(\lambda t)}{g(t)} \leq \frac{g(\lambda 0)}{g(0)} = 1$$

for all $\lambda \in (0,1)$ and $t \in (0, t^g)$. This would imply that $g \equiv 1$, which is a degenerate survival function.

The applicable part of Proposition 2.3.43 is the following corollary.

**Corollary 2.3.45.** *Let $g$ be a survival distribution and $b \in \mathrm{ifpt}(g,0)$ is non-decreasing. Assume that for all $\lambda \in (0,1)$ holds that $g^\lambda/g$ is non-decreasing on $(0,t^g)$. Then $b$ is locally Lipschitz continuous on $(0,\infty)$.*

A weaker condition for continuity of non-decreasing boundary functions is given by the following refinement of Theorem 8.2 of [EJ16] in the one-sided inverse first-passage time problem. In the situation that the boundary function is non-decreasing, the proof of Theorem 8.2 of [EJ16] shows Hölder continuity for any parameter smaller than 1/2, but the arguments of [EJ16] can be refined to yield Hölder continuity with parameter 1/2.

**Proposition 2.3.46.** *Let $I := (\eta, T)$ with $0 < \eta < T$. Let $g$ be a survival distribution, which satisfies $-g'(t) \geq C$ for almost all $t \in I$ for some $C > 0$. Let $b \in \mathrm{ifpt}(g,\mu)$ for a symmetric $\mu \in \mathcal{P}$ and assume that $b$ is non-decreasing on $I$. Then there is a constant $K > 0$ such that*

$$|b(t) - b(s)| \leq K|t-s|^{\frac{1}{2}}$$

*for $s,t \in I$.*

*Proof.* Note that since $b$ is non-decreasing on $I$ we have $g(T-) > 0$. As in the proof of Theorem 8.2 of [EJ16], let $t_0 \in I$, $a > 0$ and $d > 0$ and define

$$E := \{s \in (t_0, t_0 + a) : b(s) > b(t_0) + d\}.$$

As preparation we consider the densities $f$ and $h$ of the probability measures

$$\begin{aligned} f(x)\,\mathrm{d}x &:= \mathbb{P}_\mu\left(X_{t_0} \in \mathrm{d}x \,|\tau_b > t_0\right), \\ h(x)\,\mathrm{d}x &:= \mathbb{P}_\mu\left(X_{t_0} \in \mathrm{d}x \,||X_s| \leq b(t_0)\ \forall s \leq t_0\right), \end{aligned}$$

respectively. By Theorem 2.3.17, since $b(t_0) \geq b(s)$ for all $s \leq t_0$, we obtain

$$\begin{aligned} \mathbb{P}_\mu\left(X_{t_0} \in \mathrm{d}x \,|\tau_b > t_0\right) &\preceq_{|\mathrm{lr}|} \mathbb{P}_\mu\left(X_{t_0} \in \mathrm{d}x \,\middle|\tau_{b(t_0)} > t_0\right) \\ &= \mathbb{P}_\mu\left(X_{t_0} \in \mathrm{d}x \,||X_s| \leq b(t_0)\ \forall s \leq t_0\right). \end{aligned}$$

Regarding this note that $b(t_0) < \infty$ since otherwise $-g' = 0$ in a neighbourhood of $t_0$. We have by Lemma 2.2.17 that

$$\frac{f(x)}{h(x)} \text{ is non-increasing for almost all } x \in [0, b(t_0)].$$

Since off the nullset there has to exist $y \in (0, b(t_0))$ such that $f(y) \leq h(y)$, we also have that there is $\varepsilon > 0$ such that $f(x) \leq h(x)$ for almost all $x \in [b(t_0) - \varepsilon, b(t_0)]$. The density $h$ is given explicitly in Proposition B.1.5. In particular, we have that $h(b(t_0)) = 0$ and that the following limits exist and are finite. In view of the above it holds

$$\limsup_{z \searrow 0} \frac{f(b(t_0) - z)}{z} \leq \lim_{z \searrow 0} \frac{h(b(t_0) - z)}{z} = -h'(b(t_0))$$

off a nullset. The value $-h'(b(t_0))$ can be bounded from above. For this denote with $h_{t,m}$, $t, m > 0$, the density of the measure $\mathbb{P}_\mu\left(X_t \in \cdot \,|\tau_m > t\right)$. We arrive at

$$\limsup_{z \searrow 0} \frac{f(b(t_0) - z)}{z} \leq -h'(b(t_0)) \leq \sup_{t \in [\eta, T], m \in [b(\eta), b(T)]} -h'_{t,m}(m) < \infty, \quad (2.26)$$

where the supremum over this compact set is finite due to continuity in the parameters $t$ and $m$, which follows from the representation in Proposition B.1.5. On the other hand we have

$$\sup_{y \in \mathbb{R}} f(y) = \frac{1}{g(t_0)} \sup_{y \in \mathbb{R}} \frac{\mathbb{P}_\mu\left(X_{t_0} \in \mathrm{d}y, \tau_b > t_0\right)}{\mathrm{d}y} \leq \frac{1}{g(T-)} \sup_{y \in \mathbb{R}} \frac{\mathbb{P}_\mu\left(X_\eta \in \mathrm{d}y\right)}{\mathrm{d}y} < \infty. \quad (2.27)$$

The bounds in (2.26) and (2.26) imply that there is a constant $L > 0$ only depending on $\eta$ and $T$ such that $f(y) \leq L \cdot (b(t_0) - y)$ for almost all $y \leq b(t_0)$. Further, define the boundary function

$$b_1(t) := b(t)\mathbb{1}_{[0,t_0]}(t) + (b(t_0) + d)\mathbb{1}_{(t_0,\infty)}(t).$$

The corresponding hitting time is

$$\tau_{b_1} = \inf\{t > 0 : |X_t| \geq b_1(t)\}.$$

As in Theorem 8.2 of [EJ16] we have that $\tau_b \in E$ implies $\tau_{b_1} \in E$. Hence

$$\mathbb{P}_\mu\left(\tau_{b_1} \in E\right) \geq \mathbb{P}_\mu\left(\tau_b \in E\right) = \int_E (-g'(t))\,\mathrm{d}t \geq C\lambda(E),$$

where $\lambda$ denotes the Lebesgue measure. Let us introduce the notation

$$\tau_{b_1}^+ := \inf\{t > 0 : X_t \geq b_1(t)\}, \quad \tau_{b_1}^- := \inf\{t > 0 : X_t \leq -b_1(t)\}$$

and note that by symmetry

$$\begin{aligned}\mathbb{P}_\mu\left(\tau_{b_1}\in E\right) &= \mathbb{P}_\mu\left(\tau_{b_1}\in E, \tau_{b_1}>t_0\right)\\ &\le \mathbb{P}_\mu\left(\tau_{b_1}^+\in E, \tau_{b_1}>t_0\right)+\mathbb{P}_\mu\left(\tau_{b_1}^-\in E, \tau_{b_1}>t_0\right)\\ &= 2\mathbb{P}_\mu\left(\tau_{b_1}^+\in E, \tau_{b}>t_0\right)\le 2\mathbb{P}_\mu\left(\tau_{b_1}^+\in E \,\middle|\, \tau_{b}>t_0\right).\end{aligned}$$

Before we make the refinement step, note that under $\mathbb{P}_y$ the hitting time $\tau^+_{b(t_0)+d}$ of the constant level $b(t_0)+d$ has the following well-known density by the reflection principle. We have

$$\mathbb{P}_y\left(\tau^+_{b(t_0)+d}\in \mathrm{d}t\right)=\frac{b(t_0)+d-y}{\sqrt{2\pi(t-t_0)^3}}e^{-\frac{(b(t_0)+d-y)^2}{2(t-t_0)}}\,\mathrm{d}t.$$

Now the refinement is to observe that for measurable $A\subseteq(t_0,t_0+a)$

$$\begin{aligned}&\mathbb{P}_\mu\left(\tau_1^+\in A\,\middle|\,\tau_b>t_0\right)\\ &=\int_{-b(t_0)}^{b(t_0)}\mathbb{P}_y\left(\tau^+_{b(t_0)+d}\in A\right)\mathbb{P}\left(X_{t_0}\in \mathrm{d}y\,\middle|\,\tau_b>t_0\right)\\ &=\int_{-b(t_0)}^{b(t_0)}\mathbb{P}_y\left(\tau^+_{b(t_0)+d}\in A\right)f(y)\,\mathrm{d}y\\ &=\int_{-b(t_0)}^{b(t_0)}\int_A\frac{b(t_0)+d-y}{\sqrt{2\pi(t-t_0)^3}}e^{-\frac{(b(t_0)+d-y)^2}{2(t-t_0)}}\,\mathrm{d}tf(y)\,\mathrm{d}y\\ &\le\int_{-b(t_0)}^{b(t_0)}\int_A\frac{b(t_0)+d-y}{\sqrt{2\pi(t-t_0)^3}}e^{-\frac{(b(t_0)+d-y)^2}{2(t-t_0)}}\,\mathrm{d}tL(b(t_0)-y)\,\mathrm{d}y\\ &=L\int_0^{2b(t_0)}\int_A\frac{y+d}{\sqrt{2\pi(t-t_0)^3}}e^{-\frac{(y+d)^2}{2(t-t_0)}}\,\mathrm{d}ty\,\mathrm{d}y\\ &=\int_A\left[-\frac{L}{\sqrt{2\pi(t-t_0)}}\int_0^{2b(t_0)}y\frac{\partial}{\partial y}e^{-\frac{(y+d)^2}{2(t-t_0)}}\,\mathrm{d}y\right]\mathrm{d}t\\ &=\int_A\left[-\frac{L}{\sqrt{2\pi(t-t_0)}}2b(t_0)e^{-\frac{(2b(t_0)+d)^2}{2(t-t_0)}}+\frac{L}{\sqrt{2\pi(t-t_0)}}\int_0^{2b(t_0)}e^{-\frac{(y+d)^2}{2(t-t_0)}}\,\mathrm{d}y\right]\mathrm{d}t\\ &\le\int_A\frac{L}{\sqrt{2\pi(t-t_0)}}\int_d^\infty e^{-\frac{y^2}{2(t-t_0)}}\,\mathrm{d}y\,\mathrm{d}t=\int_A L\int_{\frac{d}{\sqrt{t-t_0}}}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,\mathrm{d}u\,\mathrm{d}t\\ &\le\int_A\frac{L\sqrt{t-t_0}}{d\sqrt{2\pi}}e^{-\frac{d^2}{2(t-t_0)}}\,\mathrm{d}t\le\frac{L\sqrt{a}}{d\sqrt{2\pi}}e^{-\frac{d^2}{2a}}\cdot\lambda(A).\end{aligned}$$

With this at hand, we can now deduce that

$$C\lambda(E)\le\mathbb{P}\left(\tau_1\in E\right)\le 2\frac{L\sqrt{a}}{d\sqrt{2\pi}}e^{-\frac{d^2}{2a}}\lambda(E).$$

If $E$ is non-empty we have $\lambda(E) > 0$, which then implies that by the inequality above either $d \leq \sqrt{a}$ or $e^{\frac{d^2}{2a}} \leq \frac{2L}{C\sqrt{2\pi}}$, yielding altogether that

$$d \leq \sqrt{2\log(C_0)a}$$

with $C_0 = \max(\frac{2L}{C\sqrt{2\pi}}, e)$. For $t > t_0$ such that $b(t) - b(t_0) > 0$ the inequality above is true for all $a > t - t_0$ and $d \in (0, b(t) - b(t_0))$ since $E$ is non-empty. Letting $d \nearrow b(t) - b(t_0)$ yields $b(t) - b(t_0) \leq \sqrt{2\log(C_0)a}$ and letting $a \searrow t - t_0$ afterwards yields then

$$b(t) - b(t_0) \leq \sqrt{2\log(C_0)}|t - t_0|^{\frac{1}{2}},$$

which finishes the proof. □

**Monotonicity**

The previous statements become useful, if the boundary function is known to be non-decreasing. In order to make this more accessible we will prove that the following conditions are sufficient for non-decreasingness.

**Theorem 2.3.47.** *Let $g$ be a survival distribution which is logarithmically convex on $(0, c) \subseteq (0, t^g)$. Let $\mu \in \mathcal{P}$ fulfill one of the following conditions:*

*(i)* $\mu = \delta_0$ *or*

*(ii)* $\mu$ *is symmetric, has compact support* $[-K, K]$ *and is absolutely continuous with respect to the Lebesgue measure with a version* $f$ *of a density of* $\mu$ *such that*

  *(a)* $\liminf_{x \searrow 0} \frac{f(K-x)}{x} > \liminf_{t \searrow 0} \frac{1-g(t)}{t}$ *and*

  *(b)* $\mu \preceq_{|\mathrm{lr}|} P_t\mu$ *for every* $t > 0$.

*Then* $b \in \mathrm{ifpt}(g, \mu)$ *is non-decreasing on* $[0, c)$.

**Remark 2.3.48.** Further below Lemma 2.3.51 gives sufficient conditions for the property that $\mu \preceq_{|\mathrm{lr}|} P_t\mu$ for all $t > 0$.

**Remark 2.3.49.** The condition of Theorem 2.3.47 is not expected to be sharp, which is suggested at a later stage by the example given in Corollary 2.3.65.

Before we begin with the proof of Theorem 2.3.47, we start with some preparational statements, which will be used for the situation of non-trivial initial distributions. In the case of standard Brownian motion starting in zero the proof of Theorem 2.3.47 could be carried out already at this stage by working with the lower barrier approximation.

In the situation of a non-trivial initial distribution we will work with the following sequence of piecewise constant boundary functions, which is a more suitable construction for this situation.

Let $\mu \in \mathcal{P}$ be absolutely continuous with respect to the Lebesgue measure and $g$ be a survival distribution. For $n \in \mathbb{N}$ let $\delta := \delta^{(n)} > 0$ be a positive step width. For every $k \in \mathbb{N}$ set

$$\alpha_k := \alpha_k^{(n)} := \begin{cases} \frac{g(k\delta^{(n)})}{g((k-1)\delta^{(n)})} & : k\delta^{(n)} \leq t^g, \\ 0 & : \text{else.} \end{cases}$$

Now we define a sequence $(c_k^n)_{k\in\mathbb{N}_0}$ of non-negative numbers in the following way. Set $c_0^n := \sup \operatorname{supp}(\mu)$. For $k \in \mathbb{N}_0$ assume $c_0^n, c_1^n, \ldots, c_k^n$ are already defined. Then, if $\alpha_{k+1}^{(n)} > 0$ choose $c_{k+1}^n \in [0, \infty]$ as the unique value such that

$$\alpha_{k+1}^{(n)} = \mathbb{P}_\mu \left( \sup_{s\in(k\delta,(k+1)\delta]} |X_s| \leq c_{k+1}^n \,\middle|\, \sup_{s\in((\ell-1)\delta,\ell\delta]} |X_s| \leq c_\ell^n \ \forall \ell \in \{1, \ldots, k\} \right)$$

and set $c_{k+1}^n = 0$, if $\alpha_{k+1}^{(n)} = 0$. Now define the piecewise constant boundary function

$$b_n^{\text{pc}} := \sum_{\ell=0}^{\infty} c_{\ell+1}^n \mathbb{1}_{(\ell\delta,(\ell+1)\delta)} + \min(c_\ell^n, c_{\ell+1}^n) \mathbb{1}_{\{\ell\delta\}}. \tag{2.28}$$

The extra definition at the discrete lattice points ensures that $b_n^{\text{pc}}$ is lower semicontinuous.

The key idea to prove Theorem 2.3.47 will be to establish inductively the monotonicity of the approximating boundary functions. For the base case of the induction we will use the following statement.

**Lemma 2.3.50.** *Let $\mu \in \mathcal{P}$ be symmetric, such that $\mu \preceq_{|\text{lr}|} P_t\mu$ for all $t > 0$. Further, let $b$ be a standard boundary function such that* $\inf_{t\in[0,\infty]} b(t) \geq \sup \operatorname{supp}(\mu)$*. Then we have*

$$\mu \preceq_{|\text{lr}|} \mathbb{P}_\mu \left( X_t \in \cdot \,|\tau_b > t \right).$$

*Proof.* Let $K := \sup \operatorname{supp}(\mu)$ and $t > 0$. Then in view of Theorem 2.3.17 it suffices to prove the statement for $b \equiv K$. From now on assume therefore that $b \equiv K$. Define for $n \in \mathbb{N}$ the boundary function

$$b_n(s) := \begin{cases} K & : s \in \{tk2^{-n} : k \in \{1, \ldots, 2^n\}\} \\ \infty & : \text{else.} \end{cases}$$

Then we have

$$\mathbb{P}_\mu \left( X_t \in \cdot \,|\tau_{b_n} > t \right) = \left( T^K \circ P_{t2^{-n}} \right)^{2^n} (\mu).$$

We claim that $(T^K \circ P_{t2^{-n}})^{k-1}(\mu) \preceq_{|\mathrm{lr}|} (T^K \circ P_{t2^{-n}})^k(\mu)$ for $k \in \mathbb{N}$. Assuming the claim for $k$, it follows for $k+1$ by Corollary 2.2.19 and Lemma 2.2.21. By the assumption of the statement we have $\mu \preceq_{|\mathrm{lr}|} P_{t2^{-n}}\mu$. Moreover, Lemma 2.2.21 yields $\mu = T^K(\mu) \preceq_{|\mathrm{lr}|} T^K \circ P_{t2^{-n}}(\mu)$. Hence, the claim follows by induction. It follows therefore, that

$$\mu \preceq_{|\mathrm{lr}|} \mathbb{P}_\mu \left(X_t \in \cdot \,|\tau_{b_n} > t\right)$$

for every $n$. We have that $\mathbb{P}_\mu \left(X_t \in \cdot \,|\tau_{b_n} > t\right) \to \mathbb{P}_\mu \left(X_t \in \cdot \,|\tau_b > t\right)$ in distribution. Since the likelihood ratio order is preserved in the limit, see Lemma B.1.4, the desired statement follows. □

We are now ready to proof Theorem 2.3.47 also in the situation of non-trivial initial distributions.

*Proof of Theorem 2.3.47.* We claim that it suffices to find $b_n$ non-decreasing on $\{b_n < \infty\} \cap (0,c)$ such that $\tau_{b_n} \to g$ in distribution.

As first step, we will prove the claim. Subsequently, we will prove the existence of such approximations distinguished by the cases (i) and (ii).

Assume we have found a sequence $b_n$ non-decreasing on $\{b_n < \infty\} \cap (0,c)$ and $\tau_{b_n} \to g$ in distribution. By Lemma 2.3.35 we obtain that $b_n \mathbb{1}_{(0,t^g)} \xrightarrow{\Gamma} b\mathbb{1}_{(0,t^g)}$ in the Hausdorff distance. Since $(0,c) \subseteq (0,t^g)$, by Lemma 2.1.9 we get that $b$ is non-decreasing on $\{b < \infty\} \cap (0,c)$. It is left to extend this to $(0,c)$. By the convexity of $\log \circ g$ it follows that $g$ is continuous on $(0,c)$. As in Theorem 8.1 of [EJ16] this implies that for every $t \in (0,c)$ we have $\liminf_{s \nearrow t} b(s) = b(t)$. Assume there is $t \in (0,c)$ such that $b(t) = \infty$. Now, suppose that there is $r \in (t,c)$ such that $b(r) < \infty$. Since $\liminf_{s\nearrow t} b(s) = \infty$ there has to be $0 < s < t$ such that $b(s) > b(r)$. This contradiction to $b$ being non-decreasing on $(0,c) \cap \{b < \infty\}$ shows that $b(t) = \infty$ for all $t \in (a,c)$, where $a := \inf\{t \in (0,c) : b(t) = \infty\} \wedge c$. Therefore, $b$ is non-decreasing on $(0,a)$ and hence, since $b(0) = \liminf_{s\to 0} b(s)$, we have $b$ is non-decreasing on $[0,c)$.

Now, we focus on the existence on the approximating sequences. We will treat both cases differently, but make a common preparation first.

We define $\alpha_k = g(k\delta^{(n)})/g((k-1)\delta^{(n)})$ for $k\delta^{(n)} < t^g$. Since $\log \circ g$ is convex on $(0,c)$ we have by Lemma B.2.8 that

$$\begin{aligned}\log(\alpha_k) &= \log \circ g((k-1)\delta^{(n)} + \delta^{(n)}) - \log \circ g((k-1)\delta^{(n)}) \\ &\leq \log \circ g(k\delta^{(n)} + \delta^{(n)}) - \log \circ g(k\delta^{(n)}) = \log(\alpha_{k+1}),\end{aligned}$$

if $(k+1)\delta^{(n)} < c$. Therefore $\alpha_k \leq \alpha_{k+1}$ for such $k$. We will use this property in the following.

We first consider the case (i). In this case we set $\delta^{(n)} = 2^{-n}$ and let $(b_n)_{n\in\mathbb{N}}$ be the lower barrier approximation from Subsection 2.3.2 for $g$ and $\mu = \delta_0$, which fulfills $\tau_{b_n} \to g$. We denote

$$\mu_k^n := \mathbb{P}_\mu\left(X_{k\delta^{(n)}} \in \cdot \,\middle|\, \tau_{b_n} > k\delta^{(n)}\right)$$

for $k \in \mathbb{N}$ with $k\delta^{(n)} < t^g$. Now, we claim that for $n$ large enough we have

$$\mu_{k-1}^n \preceq \mu_k^n$$

for all $k \in \mathbb{N}$ with $k\delta^{(n)} < c$. If we assume the claim for $k \in \mathbb{N}$ with $(k+1)\delta^{(n)} < c$, then by Proposition 2.2.3 and Lemma 2.2.6 it follows that

$$\mu_k^n = T_{\alpha_k} \circ P_{\delta^{(n)}}(\mu_{k-1}^n) \preceq T_{\alpha_{k+1}} \circ P_{\delta^{(n)}}(\mu_k^n) = \mu_{k+1}^n.$$

Hence, it is left to show that $\mu_0^n \preceq \mu_1^n$, which is clear since $\mu_0 = \delta_0$. From the statement of the claim together with Lemma 2.3.6 it follows that $b_n(k\delta^{(n)}) \leq b_n((k+1)\delta^{(n)})$ for $k \in \mathbb{N}$ with $(k+1)t_n < c$. By the construction of the lower barrier approximation this means that $b_n$ is non-decreasing on $\{b_n < \infty\} \cap (0, c)$.

Consider now case (ii). Without loss of generality we can assume $c = t^g$ by replacing $g$ by $g\mathbb{1}_{(-\infty,c)}$. Let $t_n \to 0$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty}(1-g(t_n))/t_n = \liminf_{t\searrow 0}(1-g(t))/t$. Let $(b_n^{\mathrm{pc}})_{n\in\mathbb{N}}$ be the piecewise constant barrier approximation with $\delta^{(n)} = t_n$ corresponding to $g$ and $\mu$. Now for $k \in \mathbb{N}$ define the boundary function

$$b_n^k := b_n^{\mathrm{pc}}\mathbb{1}_{[0,k\delta^{(n)})} + b_n^{\mathrm{pc}}(k\delta^{(n)}-)\mathbb{1}_{[k\delta^{(n)},\infty]}$$

and denote

$$\mu_k^n := \mathbb{P}_\mu\left(X_{k\delta^{(n)}} \in \cdot \,\middle|\, \tau_{b_n^k} > k\delta^{(n)}\right).$$

The use of the boundary function $b_n^k$ has technical reasons, since we want to study the quantities $b_n^{\mathrm{pc}}(k\delta^{(n)}-) = \sup\operatorname{supp}(\mu_k^n)$. In the definition of $\mu_k^n$ we would rather use directly $b_n^{\mathrm{pc}}$ instead, but apriori by the construction of $b_n^{\mathrm{pc}}$ it is still possible that $b_n^{\mathrm{pc}}(k\delta^{(n)}) < b_n^{\mathrm{pc}}(k\delta^{(n)}-) = b_n^k(k\delta^{(n)})$, which would result in $\sup\operatorname{supp}(\mu_k^n) = b_n^{\mathrm{pc}}(k\delta^{(n)}+)$. Now, by construction, we have $\mathbb{P}_\mu\left(\tau_{b_n^k} > k\delta^{(n)}\right) = g(k\delta^{(n)})$.

Let us continue with the claim that for $n$ large enough we have

$$\mu_{k-1}^n \preceq_{|\mathrm{lr}|} \mu_k^n \tag{2.29}$$

for all $k \in \mathbb{N}$ with $k\delta^{(n)} < c$. If we assume the claim for $k \in \mathbb{N}$ with $(k+1)\delta^{(n)} < c$, then in particular it holds $\mu_{k-1}^n \preceq \mu_k^n$. Assume $b_n^k(k\delta^{(n)}) > b_n^{k+1}((k+1)\delta^{(n)})$.

By Lemma 2.2.4 it follows that

$$\begin{aligned}\alpha_k &= \mathbb{P}_{\mu^n_{k-1}}\left(\sup_{s\in[0,t_n]}|X_s| \le b^k_n(k\delta^{(n)})\right)\\ &\ge \mathbb{P}_{\mu^n_k}\left(\sup_{s\in[0,\delta^{(n)}]}|X_s| \le b^k_n(k\delta^{(n)})\right)\\ &> \mathbb{P}_{\mu^n_k}\left(\sup_{s\in[0,\delta^{(n)}]}|X_s| \le b^{k+1}_n((k+1)\delta^{(n)})\right) = \alpha_{k+1}.\end{aligned}$$

This contradiction shows $b^k_n(k\delta^{(n)}) \le b^{k+1}_n((k+1)\delta^{(n)})$. By Theorem 2.3.17 it follows

$$\begin{aligned}\mu^n_k &= \mathbb{P}_{\mu^n_{k-1}}\left(X_{\delta^{(n)}} \in \cdot \,\middle|\, \tau_{b^k_n(k\delta^{(n)})} > \delta^{(n)}\right)\\ &\preceq_{|\mathrm{lr}|} \mathbb{P}_{\mu^n_k}\left(X_{\delta^{(n)}} \in \cdot \,\middle|\, \tau_{b^{k+1}_n((k+1)\delta^{(n)})} > \delta^{(n)}\right) = \mu^n_{k+1}.\end{aligned}$$

Hence, it is left to show that $\mu^n_0 \preceq_{|\mathrm{lr}|} \mu^n_1$. For this it suffices to show that for $n$ large enough it holds that $b_n(0) = b^1_n(\delta^{(n)}) \ge K$, because in this case we could deduce by the assumption (b) that $\mu \preceq_{|\mathrm{lr}|} P_t\mu$ for every $t > 0$ and Lemma 2.3.50 that

$$\mu \preceq_{|\mathrm{lr}|} \mathbb{P}_\mu\left(X_{t_n} \in \cdot \,\middle|\, \tau_{b^1_n} > t_n\right) = \mu^n_1.$$

Therefore, we want to show that for $n$ large enough it holds that $b_n(0) = b^1_n(\delta^{(n)}) \ge K$. As preparation for this, recall that $\delta^{(n)} = t_n$ and define $\tau^{\pm}_K := \inf\{s > 0 : \pm X_s \ge K\}$ and observe that by the reflection principle we have that

$$\frac{1}{t}\mathbb{P}_\mu\left(\tau^+_K \vee \tau^-_K \le t\right) \le \frac{1}{t}\mathbb{P}_{-K}\left(\tau^+_K \le t\right) = \frac{1}{t}\mathbb{P}\left(|X_t - X_0| \ge 2K\right) \to 0 \quad \text{as } t \to 0.$$

Thus, we have by the symmetry of $\mu$ and the reflection principle that

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{t_n}\mathbb{P}_\mu\left(\tau_K \le t_n\right) &= \liminf_{n\to\infty}\frac{1}{t_n}\left(2\mathbb{P}_\mu\left(\tau^+_K \le t_n\right) - \mathbb{P}_\mu\left(\tau^+_K \vee \tau^-_K \le t_n\right)\right)\\ &= 2\liminf_{n\to\infty}\frac{1}{t_n}\mathbb{P}_\mu\left(\tau^+_K \le t_n\right)\\ &= 2\liminf_{n\to\infty}\frac{1}{t_n}\mathbb{P}_\mu\left(|X_{t_n}| \ge K\right).\end{aligned}$$

By Lemma B.1.2 we have that

$$\liminf_{n\to\infty}\frac{1}{t_n}\mathbb{P}_\mu\left(|X_{t_n}| \ge K\right) \ge \frac{1}{2}\liminf_{x\searrow 0}\frac{f(K-x)}{x}.$$

Assume now that $b_n^1(t_n) < K$ along a subsequence $N \subseteq \mathbb{N}$. Then would follow

$$\begin{aligned}
&\liminf_{t\searrow 0} \frac{1-g(t)}{t} = \lim_{n\in\mathbb{N}} \frac{1-g(t_n)}{t_n} = \liminf_{n\in N} \frac{1-g(t_n)}{t_n} \\
&= \liminf_{n\in N} \frac{1}{t_n}\mathbb{P}_\mu\left(\tau_{b_n^1} \le t_n\right) \ge \liminf_{n\to\infty} \frac{1}{t_n}\mathbb{P}_\mu\left(\tau_K \le t_n\right) \\
&\ge \liminf_{x\searrow 0} \frac{f(K-x)}{x}.
\end{aligned}$$

By recalling that $\delta^{(n)} = t_n$, this contradiction shows that $b_n^1(t_n) \ge K$ for all $n$ large enough, which finally establishes the claim from (2.29). By this, together with Lemma 2.3.6 it follows that $b_n^k(k\delta^{(n)}) \le b_n^{k+1}((k+1)\delta^{(n)})$ for $k \in \mathbb{N}$ with $(k+1)\delta^{(n)} < c$, which in turn implies that $b_n^k(k\delta^{(n)}) = b_n^{k+1}(k\delta^{(n)})$ for every $k \in \mathbb{N}$ with $(k+1)\delta^{(n)} < c$. Thus, for $n \in \mathbb{N}$ we have that the boundary function $b_n^{\lfloor c/\delta^{(n)}\rfloor}$ is non-decreasing on $(0,c)$. Hence the boundary function

$$b_n := b_n^{\lfloor c/\delta^{(n)}\rfloor}\mathbb{1}_{[0,c)}$$

inherits this property. It is left to show that $\tau_{b_n}$ converges to $g$ in distribution. In order to show this, we check the requirements of Lemma 2.1.18. We have that $b_n^{pc}$ is non-decreasing on $[0, \lfloor c/\delta^{(n)}\rfloor\delta^{(n)})$ and that $b_n^{\lfloor c/\delta^{(n)}\rfloor}(t) = b_n^{pc}(t)$ for every $t \in [0, \lfloor c/\delta^{(n)}\rfloor\delta^{(n)})$. By this we have for $k \le \lfloor c/\delta^{(n)}\rfloor = \lfloor t^g/\delta^{(n)}\rfloor$ that

$$\begin{aligned}
&\mathbb{P}_\mu\left(\tau_{b_n} > k\delta^{(n)}\right) = \mathbb{P}_\mu\left(\tau_{b_n^{\mathrm{pc}}} > k\delta^{(n)}\right) \\
&= \mathbb{P}_\mu\left(\sup_{s\in((\ell-1)\delta,\ell\delta]} |X_s| \le c_\ell^n \ \forall \ell \in \{1,\dots,k\}\right) = \prod_{\ell=1}^k \alpha_k^{(n)} = g(k\delta^{(n)}).
\end{aligned}$$

Hence, since $\delta^{(n)} = t_n \to 0$, by Lemma 2.1.18 we have $\tau_{b_n} \to g$ in distribution as $n \to \infty$, which yields the desired statement. □

In order to make Theorem 2.3.47 more accessible we give the following characterization for the condition (b) in case (ii). For this purpose recall the generalized absolute value density from (2.9) and the notion of total positivity of order 2 from (2.10).

**Lemma 2.3.51.** *For a symmetric probability measure the property $\mu \preceq_{|\mathrm{lr}|} P_t\mu$ is fulfilled, if one of the following conditions, ordered by descending strength, is fulfilled:*

*(i)* $\mu * \delta_{u_1} \preceq_{|\mathrm{lr}|} \mu * \delta_{u_2}$ *whenever* $|u_1| \le |u_2|$.

*(ii)* $\mu \preceq_{|\mathrm{lr}|} \mu * \delta_u$ *for any* $u \in \mathbb{R}$.

*If $\mu$ has a density $f$ we have the following list of sufficient conditions, ordered by descending strength,* *(i)'* $f^*$ *is totally positive of order* 2 *almost everywhere.*

*(ii)'* $[0,\infty) \cap \operatorname{supp}(f) \to \mathbb{R}, v \mapsto \frac{f^*(v,u)}{f(v)}$ *is non-decreasing almost everywhere for any* $u \in \mathbb{R}$.

*(iii)'* $[0,\infty) \cap \operatorname{supp}(f) \to \mathbb{R}, v \mapsto \mathbb{E}_v\left[\frac{f(X_t)}{f(X_0)}\right]$ *is non-decreasing almost everywhere,*

*where the last condition is also necessary.*

*Proof.* We denote $W_t := X_t - X_0$, which is a standard Brownian motion. Regarding the first part, it is clear that (i) implies (ii). Recall that for $A \leq B$, if $x \in A, y \in B$ implies $x \leq y$. In order to see that (ii) is sufficient, let $A, B \subseteq \mathbb{R}$ be measurable with $A \leq B$. Recall $\overline{\mu}$ from Definition 2.2.1 and consider

$$\begin{aligned}\overline{\mu}(A)\overline{P_t\mu}(B) &= \int_{\mathbb{R}} \overline{\mu}(A)\overline{\mu * \delta_u}(B)\mathbb{P}\left(W_t \in \mathrm{d}u\right) \\ &\geq \int_{\mathbb{R}} \overline{\mu}(B)\overline{\mu * \delta_u}(A)\mathbb{P}\left(W_t \in \mathrm{d}u\right) = \overline{\mu}(B)\overline{P_t\mu}(A),\end{aligned}$$

which means that $\mu \preceq_{|\mathrm{lr}|} P_t\mu$.

In the case when $\mu$ has a density $f$ it it folllows by Theorem 2.2.18 that (i) is equivalent to (i)' and by Lemma 2.2.17 it follows that (ii) is equivalent to (ii)'. In order to see the remaining part, observe that by Lemma 2.2.17 the property $\mu \preceq_{|\mathrm{lr}|} P_t\mu$ is equivalent to

$$f^*(v_1,0)\mathbb{E}\left[f^*(v_2,W_t)\right] \geq f^*(v_2,0)\mathbb{E}\left[f^*(v_1,W_t)\right]$$

for almost all $v_1, v_2 \in [0,\infty) \cap \operatorname{supp}(f)$ with $v_1 \leq v_2$, which exactly means that

$$\frac{\mathbb{E}\left[f^*(v,W_t)\right]}{f^*(v,0)} = \frac{\mathbb{E}\left[f(v+W_t)+f(v-W_t)\right]}{2f(v)} = \mathbb{E}_v\left[\frac{f(X_t)}{f(X_0)}\right]$$

is non-decreasing for almost all $v \in [0,\infty) \cap \operatorname{supp}(f)$. □

### 2.3.5 The shape of the exponential boundary and further examples

This subsection is devoted to prove properties of the exponential boundary and other examples. The visualizations were done by a Monte-Carlo method, which we will discuss in Subsection 2.4.3.

For $\lambda > 0$ set $g_\lambda(t) := e^{-\lambda t}$ throughout this subsection. Obviously, we have $t^{g_\lambda} = \infty$. By the uniqueness of Theorem 2.3.33 it is justified to speak of the *Shiryaev boundary* as the unique standard boundary function contained in $\operatorname{ifpt}(g_\lambda, 0)$. A visualization can be found later in Figure 2.4. Furthermore, let

$$M_\lambda := \frac{\pi}{2\sqrt{2\lambda}}$$

and $\nu_\lambda$ be the probability measure on $[-M_\lambda, M_\lambda]$ given by

$$\nu_\lambda(\mathrm{d}x) := \varphi_\lambda(x)\,\mathrm{d}x := \sqrt{\lambda/2}\cos(\sqrt{2\lambda}x)\,\mathrm{d}x.$$

If the situation is clear, we write $M := M_\lambda$, $g := g_\lambda$, $\varphi := \varphi_\lambda$ and $\nu := \nu_\lambda$. In order to begin our analysis we give a proof of the following well-known result reformulated for the context of the inverse first-passage time problem.

**Proposition 2.3.52.** *Let $\lambda > 0$. The constant boundary function $b \equiv M_\lambda$ is a solution to the inverse first-passage problem with respect to $g_\lambda$ and initial distribution $\nu_\lambda$, i.e. $\mathbb{P}_{\nu_\lambda}(\tau_b > t) = e^{-\lambda t}$.*

*Proof.* Define $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $f(t,x) := e^{\lambda t}\cos(mx)$ for $m > 0$. We introduce the process $Y_t := f(t, X_t)$. Then, by the Itô formula,

$$Y_t - Y_0 = \int_0^t (\lambda - \frac{m^2}{2})e^{\lambda s}\cos(mX_s)\,\mathrm{d}s - \int_0^t e^{\lambda s} m \sin(mX_s)\,\mathrm{d}W_s.$$

We observe that setting $m := \sqrt{2\lambda}$ makes $Y_t - Y_0$ a martingale, and hence $Z_t := Y_{t\wedge\tau_b} - Y_0$ defines a martingale. Note that

$$\tau_b = \tau_M = \inf\{t > 0 : |X_t| = M\}$$

is almost surely finite under $\mathbb{P}_x$ for $|x| \le M$ and therefore, using $\cos(mX_{\tau_b}) = \cos(mM) = 0$, we have

$$\begin{aligned}
\cos(mx) &= \mathbb{E}_x\left[Z_t + Y_0\right] = \mathbb{E}_x\left[Y_t \mathbb{1}_{\{\tau_M > t\}}\right] + \mathbb{E}_x\left[Y_{\tau_M}\mathbb{1}_{\{\tau_M \le t\}}\right] \\
&= \mathbb{E}_x\left[e^{\lambda t}\cos(mX_t)\mathbb{1}_{\{\tau_M > t\}}\right] + \mathbb{E}_x\left[e^{\lambda \tau_M}\cos(mX_{\tau_M})\mathbb{1}_{\{\tau_M \le t\}}\right] \\
&= e^{\lambda t}\mathbb{E}_x\left[\cos(mX_t)\mathbb{1}_{\{\tau_M > t\}}\right] \\
&= e^{\lambda t}\int_{-M}^{M} p_-(t,x,y)\cos(my)\,\mathrm{d}y,
\end{aligned}$$

where $p_-(t,x,y)$ denotes the transition density of $\mathbb{P}_x(X_t \in \mathrm{d}y, \tau_M > t)$. It holds that $p_-(t,x,y) = p_-(t,y,x)$, for example this can be seen by the representation of $p_-(t,x,y)$ from Proposition 8.2 in Chapter 2 of [PS78], which is written down explicitly in Proposition B.1.5. By this we obtain

$$\cos(mx) = e^{\lambda t}\int_{-M}^{M} p_-(t,y,x)\cos(my)\,\mathrm{d}y. \tag{2.30}$$

Now consider

$$\begin{aligned}1 &= \int_{-M}^{M} \nu(\mathrm{d}x) = \int_{-M}^{M} \frac{m}{2}\cos(mx)\,\mathrm{d}x \\ &= e^{\lambda t}\int_{-M}^{M}\frac{m}{2}\int_{-M}^{M} p_{\overline{-}}(t,y,x)\cos(my)\,\mathrm{d}y\,\mathrm{d}x \\ &= e^{\lambda t}\int_{-M}^{M}\int_{-M}^{M} p_{\overline{-}}(t,y,x)\,\mathrm{d}x\,\nu(\mathrm{d}y) \\ &= e^{\lambda t}\int_{-M}^{M}\mathbb{P}_y\left(\tau_M > t\right)\nu(\mathrm{d}y) = e^{\lambda t}\mathbb{P}_\nu\left(\tau_M > t\right).\end{aligned}$$

Consequently, $\mathbb{P}_\nu\left(\tau_M > t\right) = g(t)$. □

**Remark 2.3.53.** Note that in view of $\mathbb{P}_{\nu_\lambda}\left(\tau_{M_\lambda} > t\right) = e^{-\lambda t}$ by (2.30) in fact we have shown that

$$\mathbb{P}_{\nu_\lambda}\left(X_t \in \mathrm{d}x \,|\tau_{M_\lambda} > t\right) = \nu_\lambda(\mathrm{d}x), \tag{2.31}$$

which means that $\nu_\lambda$ is a quasi-stationary distribution. The equation (2.31) can be also established by direct computation by the representation of the transition density from Proposition B.1.5. The quasi-stationarity of $\nu_\lambda$ is a well-known result and essentially one of the simplest examples in the context of quasi-stationary distributions of diffusions in a bounded interval. For example see the more general result Theorem 6.4 in [CMS13]. This property of $\nu_\lambda$ is the same as the fact that $\varphi_\lambda$ corresponds to a time-independent solution of the one-dimensional free boundary problem of [Ber+21].

By the comparison principle, this leads to the following statement about the solutions corresponding to the exponential distribution.

**Corollary 2.3.54.** *Let $\lambda > 0$ and $\mu \preceq \nu_\lambda$. The standard boundary function $b \in \mathrm{ifpt}(g_\lambda, \mu)$ satisfies $b \le M_\lambda$.*

*Proof.* By Proposition 2.3.52 the constant barrier $b^\nu \equiv \pi/(2\sqrt{2\lambda})$ is the standard solution corresponding to $g$ and $\nu$. Therefore, by Theorem 2.3.34, we have $b \le \frac{\pi}{2\sqrt{2\lambda}}$. □

In view of Corollary 2.3.54 for $\mu \preceq \nu_\lambda$ the integral equation of Proposition A.2.1 holds for $b \in \mathrm{ifpt}(g_\lambda, \mu)$. In the context of the free boundary problem in [BBP19] a one-sided version of the integral equation is stated as tool to study the asymptotic behavior of the free boundary. For the mere asymptotic long time limit we can carry out a direct analysis using methods from quasi-stationary distributions.

**Proposition 2.3.55.** *Let $b \in \mathrm{ifpt}(g_\lambda, \mu)$, such that $\mathrm{supp}(\mu) \subseteq [-M_\lambda, M_\lambda]$ with $\mu(\{-M_\lambda, M_\lambda\}) = 0$ and $b \le M_\lambda$. Then*

$$\lim_{t\to\infty} b(t) = M_\lambda.$$

*Proof.* Write $\tau_M \coloneqq \inf\{t > 0 : |X_t| \geq M\}$. By Lemma B.1.7 we have that $\mathbb{P}_x\left(X_t \in \mathrm{d}y \,|\tau_M > t\right) \to \nu$ in total variation exponentially fast and uniformly in $x \in (-M, M)$. By Theorem B.1.8 this means that the so-called $Q$-process given by $\mathbb{Q}_x \coloneqq \lim_{t\to\infty} \mathbb{P}_x\left(\,\cdot\,\,|\tau_M > t\right)$ exists and has the property that $\mathbb{Q}_x(X_t \in \cdot)$ converges for all $x \in (-M, M)$ in total variation to a measure $\beta$. By Proposition B.1.6 and the formula from Theorem B.1.8 we obtain that $\beta$ has the density $\frac{4}{\pi}\sqrt{\frac{2}{\lambda}}\varphi^2$. We now assume that

$$\liminf_{t\to\infty} b(t) < M,$$

which implies that there exists $\delta \in (0, M)$ and a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \nearrow \infty$ and $b(t_n) < M - \delta$ for all $n \in \mathbb{N}$. Define the boundary function $\gamma(t) = M - \delta \cdot \mathbb{1}_{\{t_n : n\in\mathbb{N}\}}(t)$ and let $I_\delta \coloneqq [-M + \delta, M - \delta]$. Note that we have $b(t) \leq \gamma(t) \leq M$ for $t \geq 0$. Let $m \in \mathbb{N}$. Then for $t \geq t_m$ holds

$$\begin{aligned}
\mathbb{P}_\mu\left(\tau_b > t\right) \leq \mathbb{P}_\mu\left(\tau_\gamma > t\right) &= \mathbb{P}_\mu\left(\tau_\gamma > t, \tau_M > t\right)\\
&= \mathbb{P}_\mu\left(\tau_M > t\right)\mathbb{P}_\mu\left(\tau_\gamma > t \,|\tau_M > t\right)\\
&\leq \mathbb{P}_\mu\left(\tau_M > t\right)\mathbb{P}_\mu\left(\tau_\gamma > t_m \,|\tau_M > t\right)\\
&= \mathbb{P}_\mu\left(\tau_M > t\right)\mathbb{P}_\mu\left(X_{t_1} \in I_\delta, \ldots, X_{t_m} \in I_\delta \,|\tau_M > t\right).
\end{aligned}$$

Now note, that by Theorem B.1.8 we have in particular that

$$e^{\lambda t}\mathbb{P}_x\left(\tau_M > t\right) = \frac{\mathbb{P}_x\left(\tau_M > t\right)}{\mathbb{P}_\nu\left(\tau_M > t\right)} \to \frac{4}{\pi}\sqrt{\frac{2}{\lambda}}\varphi(x).$$

Consequently, using the definition of the $Q$-process of Theorem B.1.8, we have

$$\begin{aligned}
1 &= \lim_{t\to\infty} e^{\lambda t}\mathbb{P}_\mu\left(\tau_b > t\right)\\
&\leq \lim_{t\to\infty} e^{\lambda t}\mathbb{P}_\mu\left(\tau_M > t\right)\mathbb{P}_\mu\left(X_{t_1} \in I_\delta, \ldots, X_{t_m} \in I_\delta \,|\tau_M > t\right)\\
&= \int_{\mathbb{R}} \frac{2}{M\lambda}\varphi(x)\,\mathrm{d}\mu(x)\mathbb{Q}_\mu(X_{t_1} \in I_\delta, \ldots, X_{t_m} \in I_\delta). \qquad (2.32)
\end{aligned}$$

We claim that

$$\mathbb{Q}_\mu(X_{t_1} \in I_\delta, \ldots, X_{t_m} \in I_\delta) \to 0$$

as $m \to \infty$. By Theorem B.1.8 we have also that

$$d_{\mathrm{TV}}\left(\mathbb{Q}_x(X_t \in \cdot\,), \beta\right) \to 0 \qquad (2.33)$$

for every $x \in (-M, M)$. Now, by Theorem 4.1 in Chapter 6 of [Tho00] the tail-$\sigma$-algebra $\mathcal{T}$ of $X$ is $\mathbb{Q}_x$-trivial for every $x \in (-M, M)$. The set

$$E \coloneqq \bigcup_{l\in\mathbb{N}}\bigcap_{k\geq l}\{X_{t_k} \in I_\delta\}$$

lies in $\mathcal{T}$ and thus we have $\mathbb{Q}_x(E) \in \{0,1\}$. Note that $\beta(I_\delta) < 1$, and thus $\varepsilon := 1 - \beta(I_\delta) > 0$. By (2.33) choose $L_x \in \mathbb{N}$ large enough such that

$$\mathbb{Q}_x(X_s \in I_\delta) - \beta(I_\delta) \leq \sup_{t \geq t_{L_x}} d_{\mathrm{TV}}\left(\mathbb{Q}_x(X_t \in \cdot\,), \beta\right) \leq \frac{\varepsilon}{2}$$

for $s \geq t_{L_x}$. This implies

$$\mathbb{Q}_x(X_s \in I_\delta) \leq \frac{\varepsilon}{2} + \beta(I_\delta) = \frac{\varepsilon}{2} + 1 - (1 - \beta(I_\delta)) = 1 - \frac{\varepsilon}{2}$$

for all $s \geq t_{L_x}$. Now for $l \geq L_x$ holds

$$\mathbb{Q}_x\left(\bigcap_{k \geq l}\{X_{t_k} \in I_\delta\}\right) \leq \mathbb{Q}_x(X_{t_l} \in I_\delta) \leq 1 - \frac{\varepsilon}{2}.$$

We obtain

$$\mathbb{Q}_x(E) = \lim_{l \to \infty} \mathbb{Q}_x\left(\bigcap_{k \geq l}\{X_{t_k} \in I_\delta\}\right) \leq 1 - \frac{\varepsilon}{2} < 1$$

and therefore $\mathbb{Q}_x(E) = 0$, hence

$$\int_{\mathbb{R}} \mathbb{Q}_x(X_{t_k} \in I_\delta \ \forall k \in \mathbb{N})\, \mathrm{d}\mu(x) \leq \int_{\mathbb{R}} \mathbb{Q}_x(E)\, \mathrm{d}\mu(x) = 0.$$

But in view of (2.32) this yields a contradiction. Eventually, this means our assumption was false, therefore $\lim_{t\to\infty} b(t) = M$. □

Recall that in the context of the Shiryaev boundary the Brownian motion starts in the origin.

**Proposition 2.3.56.** *The solution* $b \in \mathrm{ifpt}(g_\lambda, 0)$ *is non-decreasing.*

*Proof.* Let $s > 0$ and set $\mu_s := \mathbb{P}_0\left(X_s \in \cdot \ \middle| \tau_b > s\right)$. Further define the shifted boundary function $b_s(t) := b(s+t)$. By the Markov property one sees that

$$\mathbb{P}_{\mu_s}\left(\tau_{b_s} > t\right) = \mathbb{P}_0\left(\tau_b > s + t \,|\tau_b > s\right) = e^{-\lambda t}$$

which means that $b_s \in \mathrm{ifpt}(g, \mu_s)$. Note that $b_s$ is standard. By Theorem 2.3.34 we deduce that $b_s \geq b\mathbb{1}_{(0,\infty]}$. Since $s$ was arbitrary this implies $b(t) \leq b(t+s)$ for all $t > 0$ and $s \geq 0$, which proves the statement. □

Using the monotonicity and exploiting the explicit formula of the density given in Proposition B.1.5 in a naive way we obtain the following speed of convergence.

**Proposition 2.3.57.** *Let $b \in \operatorname{ifpt}(g_\lambda, 0)$. Then*

$$|M_\lambda - b(t)| \leq M_\lambda - \left(\frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t} + \frac{1}{M_\lambda^2}\right)^{-\frac{1}{2}} \leq \frac{M_\lambda^4 \cdot 8\log\left(\frac{4}{\pi}+1\right)}{\pi^2} \cdot \frac{1}{t}$$

*for $t > 0$.*

*Proof.* Define $\lambda(m) := \frac{\pi^2}{8m^2}$ for $m > 0$. Let $t > 0$. Suppose for a fixed $h > 0$ holds $|M - b(t)| \geq h$, this is $b(t) \leq M - h$. Then since $b$ is non-decreasing we have by the integrated density of Proposition B.1.6 that

$$\begin{aligned}
&e^{-\lambda(M)t} = \mathbb{P}_0\left(\tau_b > t\right) \\
&\leq \mathbb{P}_0\left(\tau_{M-h} > t\right) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\lambda(M-h)t}\sin\left((2k+1)\frac{\pi}{2}\right) \\
&= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\lambda(M-h)t}(-1)^k \\
&\leq \frac{4}{\pi}\sum_{k=0}^{\infty}e^{-(4k+1)^2\lambda(M-h)t} \\
&\leq \frac{4}{\pi}e^{-\lambda(M-h)t}\sum_{k=0}^{\infty}e^{-8k\lambda(M-h)t} \\
&\leq \frac{4}{\pi}e^{-\lambda(M-h)t}\left(1-e^{-8\lambda(M-h)t}\right)^{-1}.
\end{aligned}$$

From this we obtain the inequality

$$1 - e^{-8\lambda(M-h)t} \leq \frac{4}{\pi}e^{-(\lambda(M-h)-\lambda(M))t},$$

which transforms into the inequality

$$\begin{aligned}
1 &\leq \frac{4}{\pi}e^{-(\lambda(M-h)-\lambda(M))t} + e^{-8\lambda(M-h)t} \\
&= e^{-(\lambda(M-h)-\lambda(M))t}\left(\frac{4}{\pi} + e^{-(7\lambda(M-h)+\lambda(M))t}\right) \\
&\leq \left(\frac{4}{\pi}+1\right)e^{-(\lambda(M-h)-\lambda(M))t},
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{\log\left(\frac{4}{\pi}+1\right)}{t} \geq \lambda(M-h) - \lambda(M) \\
&= \frac{\pi^2}{8(M-h)^2} - \frac{\pi^2}{8M^2} = \frac{\pi^2}{8}\left(\frac{1}{(M-h)^2} - \frac{1}{M^2}\right).
\end{aligned}$$

We obtain

$$\frac{1}{(M-h)^2} \leq \frac{8}{\pi^2} \frac{\log\left(\frac{4}{\pi}+1\right)}{t} + \frac{1}{M^2},$$

which transforms into

$$h \leq M - \left(\frac{8}{\pi^2} \frac{\log\left(\frac{4}{\pi}+1\right)}{t} + \frac{1}{M^2}\right)^{-\frac{1}{2}}.$$

The first inequality of the statement follows by choosing $h = M - b(t)$.
For the remaining inequality we compute

$$\begin{aligned}
&M - \left(\frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t} + \frac{1}{M^2}\right)^{-\frac{1}{2}} \\
&\leq \left(M - \left(\frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t} + \frac{1}{M^2}\right)^{-\frac{1}{2}}\right)\left(M + \left(\frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t} + \frac{1}{M^2}\right)^{-\frac{1}{2}}\right) \\
&\leq M^2 - \left(\frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t} + \frac{1}{M^2}\right)^{-1} \\
&= \left(M^2\left(\frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t} + \frac{1}{M^2}\right) - 1\right) \cdot \left(\frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t} + \frac{1}{M^2}\right)^{-1} \\
&\leq M^4 \frac{8\log\left(\frac{4}{\pi}+1\right)}{\pi^2 t}
\end{aligned}$$

which finishes the proof. □

**Remark 2.3.58.** In the situation of Proposition 2.3.57, by which we mean the convergence from below, in the context of a free boundary problem, Theorem 2.2 of [Ber+21] proves an exponential convergence rate.

Another way to establish the statement of Proposition 2.3.56 would be an application of Theorem 2.3.47, which will be done below for $\nu_\lambda$ as initial measure.

**Corollary 2.3.59.** *For $\lambda \geq \lambda_0$ every $b \in \mathrm{ifpt}(g_{\lambda_0}, \nu_\lambda)$ is non-decreasing.*

*Proof.* At first observe that by Theorem 2.3.17 we have

$$\nu_\lambda = \mathbb{P}_{\nu_\lambda}\left(X_t \in \cdot \,|\tau_{M_\lambda} > t\right) \preceq_{|\mathrm{lr}|} \mathbb{P}_{\nu_\lambda}\left(X_t \in \cdot\right) = P_t \nu_\lambda. \tag{2.34}$$

Futhermore, we have that

$$\lim_{x \searrow 0} \frac{\varphi_\lambda(M_\lambda - x)}{x} = \lambda > \lambda_0 = \lim_{t \searrow 0} \frac{1 - e^{-\lambda_0 t}}{t}.$$

By Theorem 2.3.47 the statement follows. □

**Remark 2.3.60.** At this point it is worth mentioning that the generalized absolute value density $\varphi_\lambda^*$ of $\varphi_\lambda$ is in fact a totally positive function of order 2, which is shown in Proposition A.3 and which also implies the condition (2.34) of Theorem 2.3.47 by Lemma 2.2.18.

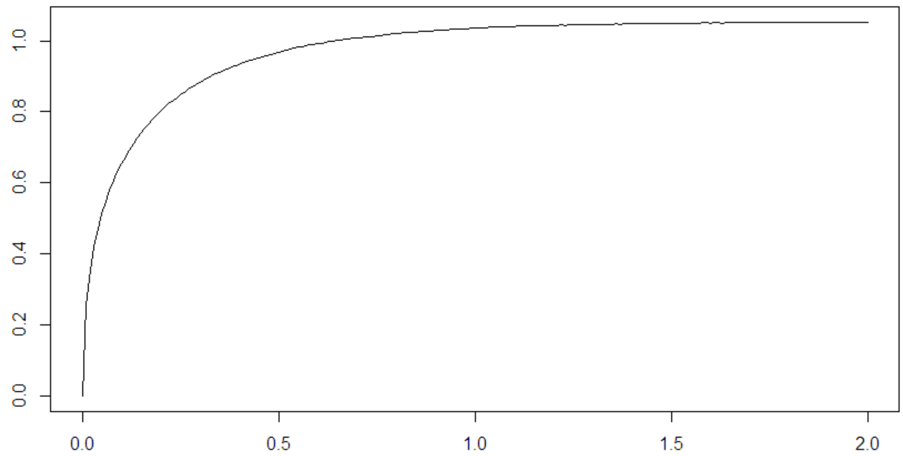


Figure 2.4: Shiryaev boundary corresponding to Exp(1) and $\mu = \delta_0$.

The following statement collects the properties of the Shiryaev boundary, which follow from the scaling property of the exponential distribution. A visualization of the Shiryaev boundary can be found in Figure 2.4.

**Proposition 2.3.61.** *Let* $b^\lambda \in \text{ifpt}(g_\lambda, 0)$ *and* $\lambda_1 > \lambda_2 > 0$. *Then*

*(i)* $b^{\lambda_1} \le b^{\lambda_2}$,

*(ii)* $b^\lambda$ *is locally Lipschitz continuous on* $(0, \infty)$,

*(iii)* $\sup_{t \ge 0} |b^{\lambda_2}(t) - b^{\lambda_1}(t)| \le M_{\lambda_2} - M_{\lambda_1}$.

*Proof.* We have that $g_{\lambda_2}(t)/g_{\lambda_1}(t) = e^{(\lambda_1 - \lambda_2)t}$ is increasing. Combining the statements of Lemma 2.3.7 and Theorem 2.3.34 yields (i). Corollary 2.3.45 yields (ii).
Let $t \ge 0$. By (i) and the fact that $b$ is non-decreasing we have

$$\begin{aligned}
|b^{\lambda_2}(t) - b^{\lambda_1}(t)| = b^{\lambda_2}(t) - b^{\lambda_1}(t) &= \frac{1}{\sqrt{\lambda_2}} b(\lambda_2 t) - \frac{1}{\sqrt{\lambda_1}} b(\lambda_1 t) \\
&\le \frac{1}{\sqrt{\lambda_2}} b(\lambda_1 t) - \frac{1}{\sqrt{\lambda_1}} b(\lambda_1 t) = \left( \frac{1}{\sqrt{\lambda_2}} - \frac{1}{\sqrt{\lambda_1}} \right) b(\lambda_1 t) \\
&\le \left( \frac{1}{\sqrt{\lambda_2}} - \frac{1}{\sqrt{\lambda_1}} \right) M_1 = M_{\lambda_2} - M_{\lambda_1},
\end{aligned}$$

which yields (iii). □

**Remark 2.3.62.** By Lemma 2.3.4 it follows $\lim_{t\to 0} b(t) = b(0) = 0$. Thus we have shown that $b$ is continuous and even almost everywhere differentiable on $(0, \infty)$. Regularity properties of the boundary corresponding to the exponential distribution have been also derived indirectly in the literature. Depending on the context the considered problem relates to reflected or standard Brownian motion or different initial distributions. For the two-sided case in [Ber+21] the continuity of the solution of a certain free boundary problem corresponding to the inverse first-passage time problem for the exponential distribution is shown. Further, the authors establish a local Hölder continuity for non-decreasing solutions. In the one-sided situation, continuity follows from several more general criteria for continuity. For example, see [Che+11], [EJ16] and [Pot21]. The work of [CCS21] provides sufficient criteria for smoothness on $(0, \infty)$, which include the exponential distribution. In the context of the one-sided free boundary problem corresponding to the exponential distribution [BBP19] showed continuity of the boundary solution, and in [Lee20] the local existence of a continuously differentiable solution was established. More details were given in the introduction.

### Results on further survival distributions

In the following we will consider other examples of survival distributions $g$, for which properties of the solution to the inverse first-passage time problem can be derived. The numerical visualizations are done by a method we will discuss in Subsection 2.4.3.

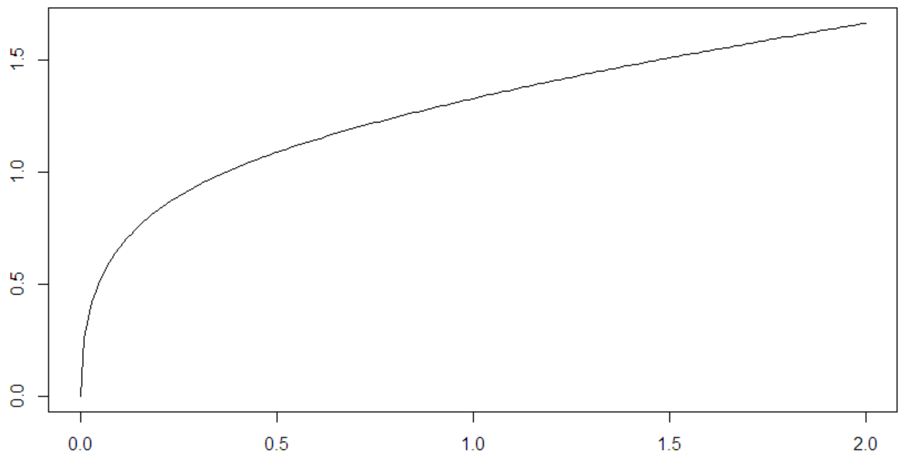


Figure 2.5: Boundary function corresponding to Lomax$(1, 1)$.

**Corollary 2.3.63** (cf. Figure 2.5)**.** *Let $g$ be the survival distribution corresponding to the Lomax distribution with scale $\lambda > 0$ and shape $\alpha > 0$, this* *is* $g(t) = \left(1+\frac{t}{\lambda}\right)^{-\alpha}$. *The standard boundary function* $b \in \operatorname{ifpt}(g, 0)$ *is non-decreasing and locally Lipschitz continuous.*

*Proof.* Note that $t^g = \infty$. We have

$$\frac{\partial^2}{\partial t^2}\log(g(t)) = \frac{\alpha}{\lambda^2}\frac{1}{\left(a+\frac{t}{\lambda}\right)^2} \geq 0.$$

Thus $g$ is logconvex and non-constant on $(0, \infty)$. Hence, Theorem 2.3.47 yields that $b$ is non-decreasing. Further, let $\eta \in (0, 1)$. Then

$$\frac{g_\eta(t)}{g(t)} = \left(\frac{\lambda+\eta t}{\lambda+t}\right)^{-\alpha}.$$

Thus

$$\frac{\partial}{\partial t}\log\left(\frac{g_\eta(t)}{g(t)}\right) = -\alpha\left(\frac{1}{\lambda/\eta+t} - \frac{1}{\lambda+t}\right) \geq 0.$$

Theorem 2.3.43 yields that $b$ is Lipschitz continuous. □

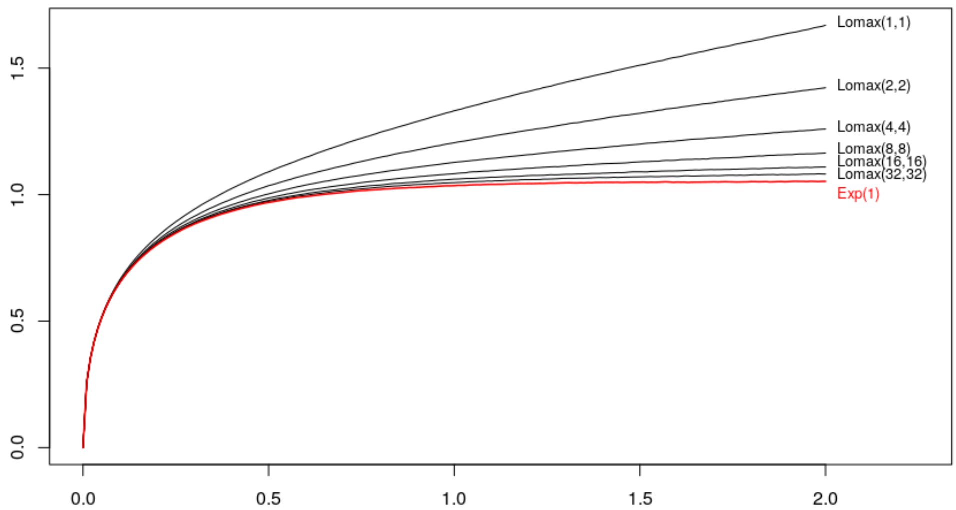


Figure 2.6: Lomax boundaries approximating the Shiryaev boundary.

**Corollary 2.3.64** (cf. Figure 2.6)**.** *Let* $g_n(t) := (1+\frac{t}{n})^{-\lambda n}$ *with* $\lambda > 0$*, this is* $g_n$ *corresponds to the Lomax distribution with scale* $n \in \mathbb{N}$ *and shape* $\lambda n$*, and* $g(t) = e^{-\lambda t}$*. Let* $b_n \in \operatorname{ifpt}(\mu, g_n)$ *and* $b \in \operatorname{ifpt}(\mu, g)$ *be standard. Then* $b_n \searrow b$ *pointwise.*

*Proof.* Since $g_n/g_{n+1}$ and $g_n/g$ are non-decreasing and $t^g = \infty$ Theorem 2.3.34 implies

$$b \leq b_{n+1} \leq b_n.$$

Now Lemma 2.1.8 yields that $b_n$ converges in Hausdorff distance to its pointwise limit $\tilde{b}$ given by $\tilde{b}(t) = \lim_{n\to\infty} b_n(t)$. By the inequality above we also obtain $b \leq \tilde{b}$. On the other hand we have by Lemma 2.3.35 that

$$b_n \mathbb{1}_{(0,\infty)} \xrightarrow{\Gamma} b\mathbb{1}_{(0,\infty)}.$$

Thus $b$ coincides with $\tilde{b}$ on $(0,\infty)$. But since $b$ is standard we obtain

$$b(0) = \liminf_{t\to 0} b(t) = \liminf_{t\to 0} \tilde{b}(t) \geq \tilde{b}(0)$$

and analogously $\tilde{b}(\infty) \leq b(\infty)$. All in all this yields $b = \tilde{b}$. □

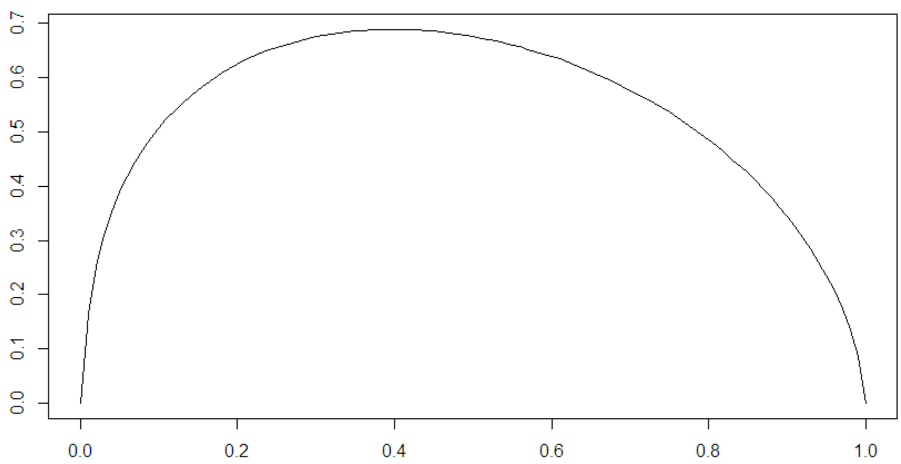


Figure 2.7: Boundary function corresponding to Kumaraswamy(1/2, 1).

Regarding the next statement cf. Figure 2.7, which suggests that the sufficient condition of Theorem 2.3.47 is not sharp. Intuitively, this is due to the fact that a positive slope of the boundary function is not only induced by the decreasing hazard rate of $g$, but also by the concentration of mass away from 0.

**Corollary 2.3.65.** *Let $g$ be the survival distribution corresponding to the Kumaraswamy distribution with parameters $\alpha, \beta > 0$, this is $g(t) = (1-t^\alpha)^\beta$ for $t \in [0,1]$. The standard boundary function $b \in \mathrm{ifpt}(g, 0)$ is non-decreasing on $[0,(1-\alpha)^{1/\alpha})$.*

*Proof.* We have for $t \in (0,1)$

$$\frac{\partial^2}{\partial t^2}\log(g(t)) = -\frac{\beta\alpha t^{\alpha-2}(\alpha-1+t^{\alpha})}{(1-t^{\alpha})^2}.$$

Thus $\frac{\partial^2}{\partial t^2}\log(g(t)) \geq 0$ for $t \leq (1-\alpha)^{1/\alpha}$. Theorem 2.3.47 yields the statement. □

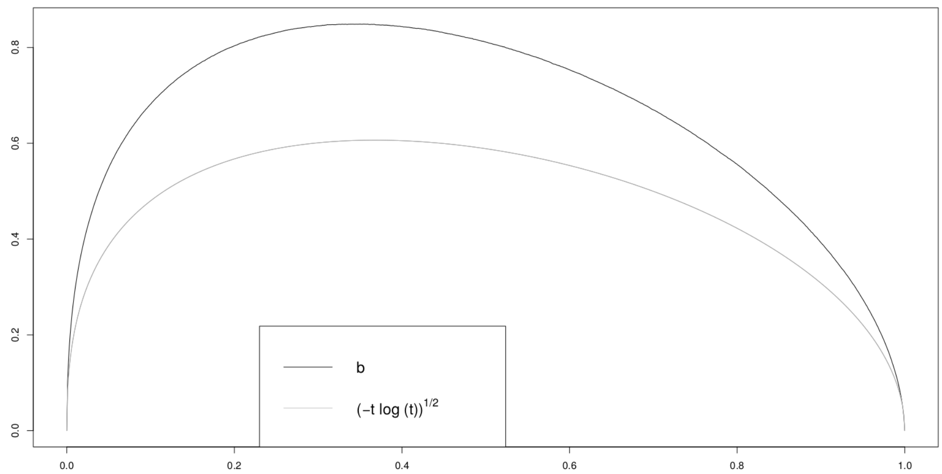


Figure 2.8: The uniform boundary and Lerche's boundary $b_{\mathrm{L}}$ on $(0,1)$.

The next example shows that by the direct first-passage time problem we can deduce statements about certain inverse first passage-time problems via Theorem 2.3.34.

**Corollary 2.3.66** (cf. Figure 2.8)**.** *Let* $g(t) = \max(1-t/a, 0)$ *be the survival distribution of the uniform distribution on* $(0,a)$*. Then for* $b \in \mathrm{ifpt}(g, \delta_0)$ *holds*

$$b(t) \geq \sqrt{t\log\left(\frac{a}{t}\right)}$$

*for all* $t \in (0,a)$*.*

*Proof.* By Example 4 in Section 1 of Chapter 1 in [Ler86] we have that the first-passage time $\tau_{b_{\mathrm{L}}}$ of the boundary function

$$b_{\mathrm{L}}(t) := \mathbb{1}_{(0,a)}(t)\sqrt{t\log\left(\frac{a}{t}\right)}, \tag{2.35}$$

has density

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\log\left(\frac{a}{t}\right)}{at}}, \quad t \in (0,a)$$

which is non-increasing on $(0,a)$. This implies that the distribution of $\tau_{b_{\mathrm{L}}}$ is smaller in the likelihood ratio order that the distribution of $\tau_b$, which implies that this ordering holds also with respect to the hazard rate order, for example see Theorem 1.C.1 from [SS07]. By Theorem 2.3.34 this means that $b \geq b_{\mathrm{L}}$. □

We will end this subsection with an example concerning the connection between the first-passage and the last-exit time.

**Proposition 2.3.67.** *Let $\gamma > 0$ and $\sigma_b := \sup\{t > 0 : |X_t| \geq b(t)\}$ and*

$$b(t) = \sqrt{t \log\left(\frac{t}{\gamma}\right)}\mathbb{1}_{(\gamma,\infty)}(t).$$

*Then under $\mathbb{P}_0$ the last-exit time $\sigma_b$ has the density*

$$\frac{1}{\sqrt{2\pi}}t^{-\frac{3}{2}}\sqrt{\gamma \log\left(\frac{t}{\gamma}\right)}$$

*for $t \in (\gamma, \infty)$.*

*Proof.* Recall the boundary $b_{\mathrm{L}}$ from (2.35). As above, by Example 4 in Section 1 of Chapter 1 in [Ler86] we know the density of the first-passage time $\tau_{b_{\mathrm{L}}}$ of the boundary function $b_L(t)$. It holds for $t > \frac{1}{a}$ that

$$t \cdot b_{\mathrm{L}}\left(\frac{1}{t}\right) = \mathbb{1}_{(0,a)}\left(\frac{1}{t}\right)\sqrt{t\log(at)} = \mathbb{1}_{\left(\frac{1}{a},\infty\right)}(t)\sqrt{t\log(at)}.$$

By the computation of Remark 2.3.8 it follows for $t > \frac{1}{a}$ that

$$\int_{\frac{1}{t}}^{a} \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\log\left(\frac{a}{u}\right)}{au}}\,\mathrm{d}u = \mathbb{P}_0\left(\tau_{b_{\mathrm{L}}} > \frac{1}{t}\right) = \mathbb{P}_0\left(\sigma_b < t\right).$$

Differentiating in $t$ yields that $\sigma_b$ has the density

$$\frac{1}{\sqrt{2\pi}}t^{-\frac{3}{2}}\sqrt{\frac{\log(ta)}{a}}.$$

Choosing $\gamma = \frac{1}{a}$ yields the statement. □

## 2.4 Simulation and interacting particle representation

In the introduction we presented a connection of the inverse first-passage time problem with a free boundary problem. For appropriate partial differential equations sometimes solutions can be found to correspond to so-called hydrodynamic limits of interacting particle systems. Roughly speaking, these cases often have in common that the macroscopic, continuous dynamic described by the partial differential equation corresponds to microscopic effects between finitely many particles and a driving dynamic of the particles.

In this section we will discuss two possible choices of interacting particle systems, which give rise to the inverse first-passage time problem in this manner. First of all we will shortly address the mass-preserving particle system with branching and selection analyzed in [De +19a] and propose a generalization in order to connect the system with the general inverse first-passage time problem. For this generalized particle system we conjecture the corresponding generalized result of [De +19a]. Subsequently, we analyze a particle system without branching, which could be seen as a simplification of the system in [De +19a], since it only holds a similar selection mechanism. We prove that the hydrodynamic limit exists and corresponds to the general inverse first-passage time problem. At last, motivated by and connected to the non-branching particle system, but usable without it, we specify the Monte-Carlo approximation of Anulova-type boundary approximants, which is used in this thesis to generate visualizations of solutions.

An overview about related work on particle systems is to be found in Remark 2.4.6.

### 2.4.1 Generalization of the $N$-Branching Brownian motion

As discussed in the introduction the work of [De +19a] studies the so-called $N$-Branching Brownian motion, abbreviated with $N$-BBM, and shows that the hydrodynamic limit can be identified as a solution to a certain free boundary problem, which is a description of the inverse first-passage time problem for the exponential distribution. Let us recall the model of the $N$-BBM in $\mathbb{R}$ with respect to the general description by [Ber+20]. The particle system starts with $N$ particles at positions $(X_0^1, \dots, X_0^N)$ in $\mathbb{R}^N$, each independently sampled from $\mu \in \mathcal{P}$. Every particle moves independently as Brownian motion and branches independently at rate one. At any branching time the particle with the lowest fitness is removed from the system, where fitness is measured by a function $\mathcal{F} : \mathbb{R} \to \mathbb{R}$. The particle system is illustrated in Figure 2.9. The work [De +19a] studied the particular case $\mathcal{F}(x) = x$, but [Bec19] pointed out that their proof also works for the case $\mathcal{F}(x) = -|x|$. In order to connect this with the inverse first-passage time problem for a lower semicontinuous function $b$ we

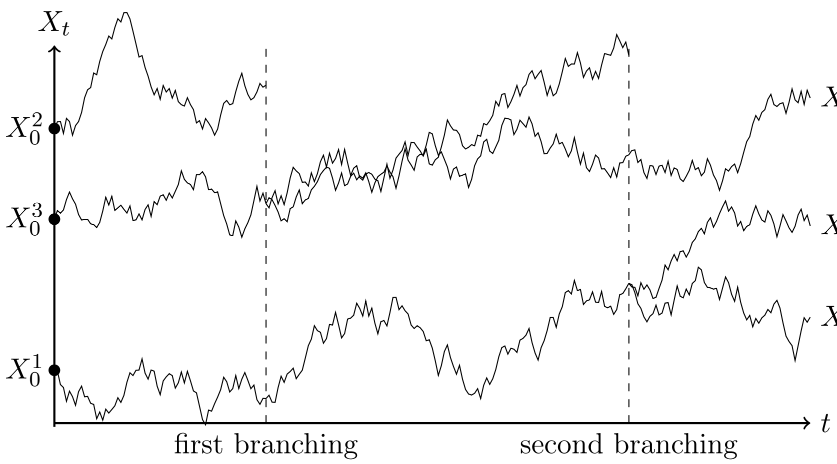


Figure 2.9: Illustration of the $N$-BBM with $N = 3$, $\mathcal{F}(x) = -|x|$.

introduce the notation

$$\tau_b^{\mathcal{F}} := \inf\{t > 0 : \mathcal{F}(X_t) \leq -b(t)\},$$

where $X_t$ is the Brownian motion with initial distribution $\mu$.

In [De +19a] the following hydrodynamic limit was identified as solution of a free boundary problem.

**Theorem 2.4.1** ([De +19a])**.** *Let $\mathcal{F}(x) = x$ and $g(t) = e^{-t}$. Let $\mu$ be absolutely continuous with support bounded from below. Let $b : (0, \infty) \to \mathbb{R}$ be the unique lower semicontinuous solution such that $\mathbb{P}_\mu\left(\tau_b^{\mathcal{F}} > t\right) = g(t)$. Then for $t \in (0, t^g)$ and $a \in \mathbb{R}$ it holds almost surely*

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\mathcal{F}(X_t^i)\geq a\}} \overset{n\to\infty}{\longrightarrow} \mathbb{P}_\mu\left(\mathcal{F}(X_t) \geq a \,\middle|\, \tau_b^{\mathcal{F}} > t\right).$$

The statement of Theorem 2.4.1 gives rise to a connection to the general inverse first-passage time problem. Regarding this we propose a modification of the $N$-BBM to generalize this model in order for the hydrodynamic limit to correspond to more general survival distributions.

Let $g$ be a survival distribution which is absolutely continuous on $(0, t^g)$. Note that $g$ is differentiable almost everywhere. Let

$$h(t) := -\frac{\partial}{\partial t}\log(g(t)), \quad \text{for } t \in (0, t^g) \text{ almost everywhere},$$

denote the hazard rate of $g$. Then the generalized model for the time horizon $[0, t^g)$ is as follows, where we only modify the rate of branching. Again the particle system starts with $N$ particles at random positions $(X_0^1, \ldots, X_0^N)$ in $\mathbb{R}^N$. Every particle moves independently as a Brownian motion and branches independently with rate $h$. In other words, the process of the number of branching events of one particle is given by a Poisson process on $[0, t^g)$ with intensity measure $h(t)\,\mathrm{d}t$. The remaining part of the model remains unchanged: at any branching time the particle with the lowest fitness is removed from the system, where fitness is measured by a function $\mathcal{F} : \mathbb{R} \to \mathbb{R}$. For an illustration see again Figure 2.9.

We make the following conjecture.

**Conjecture 2.4.2.** *Let $\mathcal{F}(x) = x$ or $\mathcal{F}(x) = -|x|$. Furthermore, let $\mu$ be a probability measure and $g$ be an absolutely continuous survival distribution. With these more general assumptions the statement of Theorem 2.4.1 is true.*

From a computational point of view, the $N$-BBM with generalized branching rate would be hard to realize since the successive computations of the branching times afford the computation of new conditioned survival distributions within every branching time step. Therefore, we want to turn our attention to a simpler particle system, which has the additional advantage of being not restricted to differentiable survival distributions.

### 2.4.2 A particle system without branching

In the following we present a simpler type of particle system without branching, which also gives rise to the inverse first-passage time problem.

The matter of this section is motivated by the following illustrative process. Let $g$ be a survival distribution. For a particle number $N \in \mathbb{N}$ consider $N$ independent Brownian motions, which we will refer to as particles in the following. Independently from the particles, let $T_1, \ldots, T_N$ be independent and identically distributed random variables with $T_1 \sim g$. Let

$$T_{(1)} \leq \ldots \leq T_{(N)}$$

be the corresponding order statistics. Let $(X_t^i)_{i \in A(t), t \geq 0}$ be the process, which results from the following scheme. At every timepoint $T_{(i)}$ we remove the particle with the greatest absolute value from the system and define the index set $A(t)$ of surviving particles up to a time $t$ as the particles, which have not been removed up to this time. For an illustration compare Figure 2.10.

A more formal definition will be given later on. The main result of this section will be a generalized version of the following result.

**Theorem 2.4.3.** *Let $b \in \mathrm{ifpt}(g, 0)$. Then for $t \in (0, t^g)$ holds*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i \in A(t)} \delta_{X_t^i}([-a, a]) = \mathbb{P}_0 \left( X_t \in [-a, a], \tau_b > t \right), \quad \forall a \in \geq 0,$$

*almost surely, where $(X_t)_{t \geq 0}$ denotes the Brownian motion.*

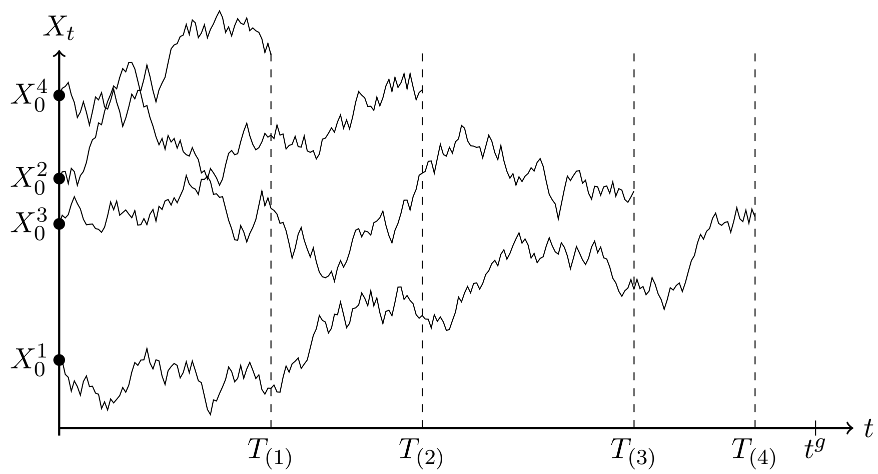


Figure 2.10: Illustration of the non-branching system with $N = 4$.

Let us now prepare the formal definitions. For a particle number $N$ and a initial distribution $\mu \in \mathcal{P}$ let $(B_0^1, \ldots, B_0^N) \sim \mu^{\otimes N}$. Let $B^1, \ldots, B^N$ be independent Brownian motions with initial configurations $B_0^1, \ldots, B_0^N$ independent from the increments. Furthermore, from now on let

$$0 =: t_0^N < t_1^N \leq \ldots \leq t_N^N \tag{2.36}$$

be $N + 1$ fixed timepoints. Let the number of timepoints up to a time $t$ be denoted with

$$k^N(t) \coloneqq \sup\{k \in \{1, \ldots, N\} : t_k^N \leq t\}.$$

Set $A_0 \coloneqq \{1, \ldots, N\}$ and define inductively for $\ell \in \{1, \ldots, N\}$

$$A_\ell \coloneqq A_{\ell-1} \setminus \{\underset{i \in A_{\ell-1}}{\arg\max} |B^i_{t_\ell^N}|\}.$$

The continuous time particle system we want to consider is then the system with empirical measure

$$\chi_t^N \coloneqq \frac{1}{|A_{k^N(t)}|} \sum_{i \in A_{k^N(t)}} \delta_{B_t^i}.$$

To describe it in words, the change of $A_{k^N(t)}$ over time only consists of removing the particle with highest absolute value from the system at the timepoints $t_k^N$. Note that we have $|A_{k^N(t)}| = N - k^N(t)$. We will prove the following statement, which implies Theorem 2.4.3 due to the law of large numbers. It is worth mentioning that we do not impose conditions on $g$.

**Theorem 2.4.4.** *Assume that for every $t \in (0, t^g)$ holds*

$$\lim_{N\to\infty} \frac{k^N(t)}{N} = 1 - g(t). \tag{2.37}$$

*Let $b \in \operatorname{ifpt}(g, \mu)$ with symmetric $\mu \in \mathcal{P}^1$ and fix $t \in (0, t^g)$. Then*

$$\lim_{N\to\infty} \chi_t^N([-a, a]) = \mathbb{P}_\mu \left( X_t \in [-a, a] \,|\tau_b > t \right), \quad \forall a \geq 0,$$

*almost surely, where $(X_t)_{t\geq 0}$ denotes the Brownian motion.*

**Remark 2.4.5.** The assumption (2.37) is to be understood as an assumption on the sequence of ordered timepoints

$$t_1^N \leq \ldots \leq t_N^N$$

from (2.36). The two main situations, which we want to cover up with this assumption are the following.

- As in the situation of Theorem 2.4.3 let $T_{(k)}$ denote the $k$-order statistic of the $N$ samples $T_1, \ldots, T_N$ of the distribution given by $g$. Then if we choose $t_k^N := T_{(k)}$ for fixed $t \in (0, t^g)$ we have

  $$\frac{k^N(t)}{N} = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\{T_k \leq t\}} \to \mathbb{P}\left(T_1 \leq t\right) = 1 - g(t)$$

  almost surely as $N \to \infty$.

- A deterministic choice of timepoints is given by $t_k^N := t_k^{(N)}$, where

  $$t_k^{(N)} := g^{-1}\left(\frac{N-k}{N}\right), \quad k \in \{1, \ldots, N\},$$

  and $g^{-1}$ denotes the generalized inverse as defined in (A.4). The property $\lim_{N\to\infty} N^{-1} k^N(t) = 1 - g(t)$ is established in Lemma A.5.1.

Before we begin with the preparation for the proof of Theorem 2.4.4 we give a short overview on the related work in the context of this particle system.

**Remark 2.4.6.** The system presented in Theorem 2.4.4 can be seen as a very simple case of a more general class of particle systems with topological interactions. Prototypes of this generic term are the basic model presented in [Car+16] and the $N$-BBM model of [De +19a] discussed in the previous subsection, in which removed particles are re-injected into the system. The latter model has been further modified in [De +19b] and generalized in [GS21], where the generalization consists of a branching rate dependent on the position of the particles. On top of that, the work of [Ata20] presents two very general models. Concerning the first one, the injection of new particles is governed by a given function $I : [0,\infty) \to [0,\infty)$ and a so-called injection measure, and the removal of particles is also governed by a function $J : [0,\infty) \to [0,\infty)$, but restricted to the right-most particle. Under suitable conditions existence of the hydrodynamic limit is proven, where it is identified as a solution to a partial differential equation with an additional so-called order-respecting absorption condition.

Let $g$ be a given survival distribution. Then, if we choose in the model of [Ata20] the injections to be zero, i.e. $I \equiv 0$, and the removal function to be $J(t) := 1 - g(t)$, we end up with the one-sided version of the system from Theorem 2.4.4 with the specific removal times

$$t_k^N = \inf\{t \geq 0 : \lfloor N(1 - g(t)) \rfloor \geq k\},$$

which correspond to the second point of Remark 2.4.5 and meet the condition of Theorem 2.4.4 by Lemma A.5.1. The conditions of the result of [Ata20] are fulfilled, if $g$ is absolutely continuous and Hölder continuous with exponent larger than $1/2$. Therefore, the intersection of the work of [Ata20] and this thesis is that, for survival distributions fulfilling the conditions mentioned above and the timepoints of the second point of Remark 2.4.5, the hydrodynamic limit exists, but is characterized in different ways.

In the second model of [Ata20], there are also no injections, but the removal is restricted to empirical quantiles among the particles, where the target quantile does not depend on the number of particles. Hence, the system cannot be adjusted to remove the right-most particle.

Let us begin with the analysis of the system in Theorem 2.4.4. From now on for $m \in \mathbb{N}$ let $(t_k^{(m)})_{k\in\{1,\ldots,n_m\}}$ be another fixed sequence such that

$$0 =: t_0^{(m)} < t_1^{(m)} \leq t_2^{(m)} \leq \ldots \leq t_{n_m}^{(m)} \leq t^g \tag{2.38}$$

with $n_m \in \mathbb{N} \cup \{\infty\}$. We will use these timepoints as fixed time lattice, when the number of particles goes to infinity.

We denote the index of the largest lattice point left from a timepoint $t \in (0, t^g)$ again with

$$k^{(m)}(t) := \sup\{k \in \mathbb{N}_0 : t_k^{(m)} \leq t\}$$

and the corresponding lattice point by $(t)_m := t^{(m)}_{k^{(m)}(t)}$. The largest index which results in the same lattice point as the lattice point of an index $k \in \{1, \ldots, n_m\}$ is then $[k]_n := k^{(m)}(t_k^{(m)})$.

Parametrized with $m \in \mathbb{N}$ we will construct two processes whose empirical measures serve as almost sure lower and upper bounds of $\chi_t^N$ in the two-sided stochastic order for every $N \geq m$. The following technique of notation and construction for the particle system is inspired from [De +19a].

Define $A_0^+ := \{1, \dots, N\}$ and inductively for $k \in \{1, \dots, n_m\}$

$$A_k^+ := A_k^{+,m} := \{i \in A_{k-1}^+ : |B^i_{t_k^{(m)}}| \leq q_k^{+,N}\}$$

where

$$q_k^{+,N} := q_k^{+,N,m} := \inf \left\{ a \geq 0 : \sum_{i \in A_{k-1}^+} \mathbb{1}_{\{|B^i_{t_k^{(m)}}| \leq a\}} \geq N - k^N(t_k^{(m)}) \right\}. \quad (2.39)$$

In words, for the construction of $A_k^+$, from lattice point $t_{k-1}^{(m)}$ to $t_k^{(m)}$ we count the amount of particles, which would have been removed in between these lattice points in the non-branching process and remove this amount at time $t_k^{(m)}$ at once from the system by cutting off the particles with largest absolute value at time $t_k^{(m)}$. For an illustration compare Figure 2.11.

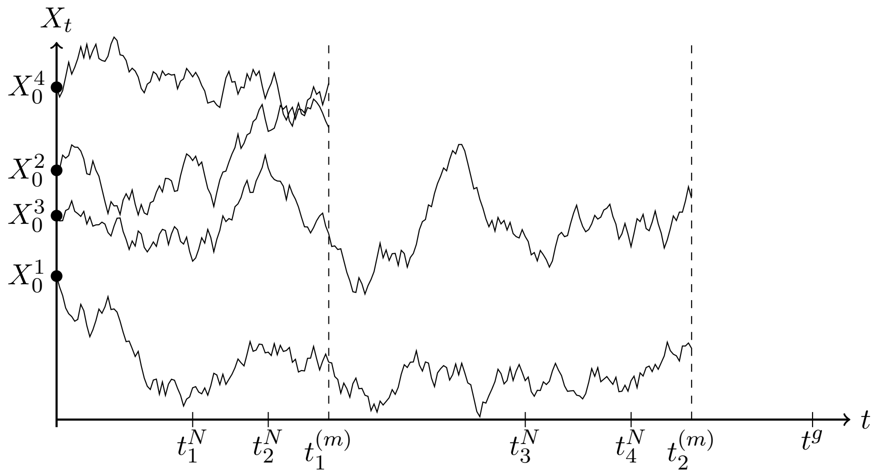


Figure 2.11: Illustration of the $A^+$-process for the non-branching system with $N = 4$.

Further let $A_1^-$ be an independently, uniformly chosen random subset of $\{1, \dots, N\}$ with $N - k^{(N)}(t_1^{(m)})$ elements and for $k \in \{2, \dots, n_m\}$ define inductively

$$A_k^- := A_k^{-,N} := \{i \in A_{k-1}^- : |B^i_{t_{k-1}^{(m)}}| \leq q_k^{-,N}\}$$

where

$$q_k^{-,N} := q_k^{-,N,m} := \inf \left\{ a \geq 0 : \sum_{i \in A_{k-1}^-} \mathbb{1}_{\{|B^i_{t_{k-1}^{(m)}}| \leq a\}} \geq N - k^N(t_k^{(m)}) \right\}.$$

In words, for the construction of $A_k^-$, we again count the amount of particles which are removed from the non-branching process between $t_{k-1}^{(m)}$ and $t_k^{(m)}$, but in contrary to $A_k^+$ we remove this amount of particles from the system by cutting off the particles with largest absolute value at time $t_{k-1}^{(m)}$. For an illustration compare Figure 2.12.

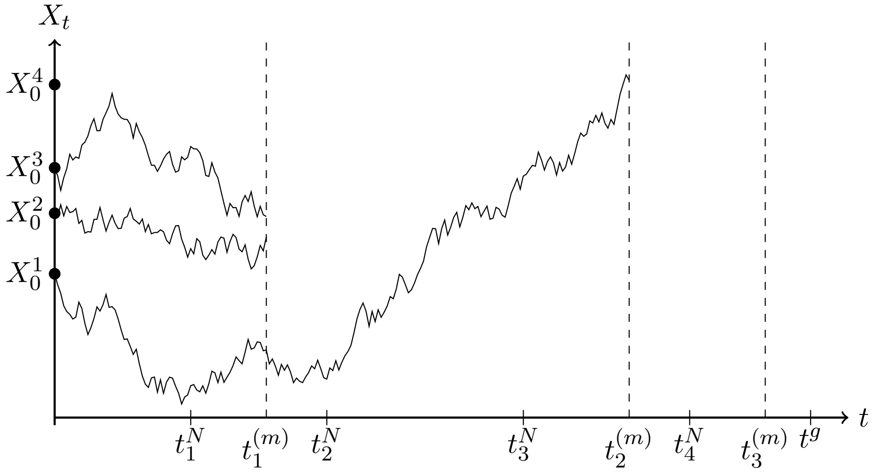


Figure 2.12: Illustration of the $A^-$-process for the non-branching system with $N = 4$.

Define the empirical measures

$$\xi_k^{\pm,N} := \frac{1}{|A_k^\pm|} \sum_{i \in A_k^\pm} \delta_{B^i_{t_k^{(m)}}}$$

and

$$\chi_t^{\pm,N} := \frac{1}{|A^\pm_{k^{(m)}(t)}|} \sum_{i \in A^\pm_{k^{(m)}(t)}} \delta_{B_t^i}, \tag{2.40}$$

as long as $|A_k^\pm|, |A^\pm_{k^{(m)}(t)}| > 0$. We want to compare the empirical measures of the processes at the timepoints $(t_k^{(m)})_{k\in\{1,\ldots,n_m\}}$ by a suitable coupling, which in our case demands that the particle numbers of both processes are equal. Note that by the definitions it follows immediately that we have

$$|A_\ell| = N - \ell, \quad \ell \in \{1, \ldots, N\}$$

and

$$|A_k^\pm| = N - k^N(t_k^{(m)}), \quad k \in \{1, \ldots, n_m\}.$$

Therefore, we have at time $t_k^{(m)}$ that

$$|A^{\pm}_{k^{(m)}(t_k^{(m)})}| = N - k^N(t_k^{(m)}) = |A_{k^{(N)}(t_k^{(m)})}|.$$

It is more convenient to couple rather the discrete time step processes instead of just the continuous time process, even if it is the case that for a $k \in \{1, \dots, n_m - 1\}$ holds $t_k^{(m)} = t_{k+1}^{(m)}$, which then implies that in the discrete time process from $k$ to $k+1$ does not happen anything. We define

$$\xi_k^{m,N} := \frac{1}{|A_{k^N(t_k^{(m)})}|} \sum_{i \in A_{k^N(t_k^{(m)})}} \delta_{B^i_{t_k^{(m)}}}$$

and

$$\chi_t^{m,N} := \frac{1}{|A_{k^N((t)_m)}|} \sum_{i \in A_{k^N((t)_m)}} \delta_{B_t^i},$$

as long as $|A_{k^N(t_k^{(m)})}|, |A_{k^N((t)_m)}| > 0$.

We will prove that the processes $\chi_t^{\pm,N}$ can serve us as stochastic barriers for $\chi_t^{m,N}$.

**Lemma 2.4.7.** *There is a coupling $(\tilde{\xi}^{+,N}, \xi^{m,N}, \tilde{\xi}^{-,N})$ of the tripel of random measures $(\xi^{+,N}, \xi^{m,N}, \xi^{-,N})$ such that for every $k$*

$$\tilde{\xi}_k^{+,N} \preceq \xi_k^{m,N} \preceq \tilde{\xi}_k^{-,N}$$

*almost surely, and thus there is also a coupling $\tilde{\chi}_t^{+,N} \preceq \chi_t^{m,N} \preceq \tilde{\chi}_t^{-,N}$.*

Before we prove Lemma 2.4.7 we make some preparations. Our strategy will be an induction from discrete time step to discrete time step. Concerning such a single time step, the only thing we have to ensure about our constructed coupling is then, that the ordering property is preserved by the dynamic of the involved processes. In order to achieve this in a rigorous way, we will first introduce notation, which enables us to crystallize out the dynamic of the involved processes in one discrete step. Then we state some auxiliary statements about set orderings, which will help us to construct the desired couplings.

Let $M \in \mathbb{N}$ and $y, x, z \in \mathbb{R}^M$. Let $B^y, B^x, B^z$ denote $M$-dimensional Brownian motions with $B_0^{y,i} = y_i$, $B_0^{x,i} = x_i$ and $B_0^{z,i} = z_i$. Fix $t \geq 0$ and timepoints $s_0 := 0 \leq s_1 \leq \ldots \leq s_j \leq t$ with $j \leq M$. Define

$$A_t^{+,y}(j) := \{i \in \{1, \dots, M\} : |B_t^{y,i}| \leq q_t^{+,y}(j)\}$$

where

$$q_t^{+,y}(j) := q_t^{+,y} := \inf\{a \in \mathbb{R} : \sum_{i=1}^{M} \mathbb{1}_{\{|B_t^{y,i}| \leq a\}} \geq M - j\}.$$

Let $A^x := \{1, \dots, M\}$ and for $\ell \in \{2, \dots, j\}$

$$A^x(s_1, \dots, s_\ell) := A^x(s_1, \dots, s_{\ell-1}) \setminus \{ \underset{i \in A^x(s_1,\dots,s_{\ell-1})}{\arg\max} |B_{s_\ell}^{x,i}|\}.$$

Further define

$$A^{-,z}(j) := \{i \in \{1, \dots, M\} : |z_i| \leq q^{-,z}(j)\}$$

where

$$q^{-,z}(j) := q^{-,z} := \inf\{a \geq 0 : \sum_{i=1}^{M} \mathbb{1}_{\{|z_i| \leq a\}} \geq M - j\}.$$

### Set ordering

For a tuple $x \in \mathbb{R}^n$ and a subset $A \subseteq \mathbb{R}$ let us introduce the notation

$$|x \cap A| := \sum_{i=1}^{n} \mathbb{1}_A(x_i) = |\{i \in \{1, \dots, n\} : x_i \in A\}|.$$

**Definition 2.4.8.** *For tuples $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ define the partial order*

$$x \preceq y :\Leftrightarrow |x \cap [-a, a]| \geq |y \cap [-a, a]| \quad \forall a \geq 0. \tag{2.41}$$

*In this case, call $x$ dominated by $y$.*

Note that, if $x \preceq y$, it directly follows that $n \geq m$.

**Lemma 2.4.9.** *Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ with $n \geq m$, such that $|x_i| \leq |y_i|$ for all $i \in \{1, \dots, m\}$. Then $x \preceq y$.*

*Proof.* Let $a \in \mathbb{R}$. We have

$$I := \{i \in \{1, \dots, m\} : |y_i| \leq a\} \subseteq J := \{j \in \{1, \dots, m\} : |x_j| \leq a\}.$$

Thus

$$|x \cap [-a, a]| = |\{i \in \{1, \dots, n\} : |x_i| \leq a\}| \geq |J| \geq |I| = |y \cap [-a, a]|,$$

which shows the statement. □

**Lemma 2.4.10.** *Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then the following conditions are equivalent:*

*(i)* $x \preceq y$

*(ii)* $n \geq m$ *and there exists an injective function* $\pi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ *such that* $|x_{\pi(j)}| \leq |y_j|$ *for every* $j \in \{1, \ldots, m\}$.

*If* $|y_i|$ *and* $|x_i|$ *are non-decreasing in* $i$ *then* $\pi$ *can be chosen as identity.*

*Proof.* Without loss of generality we can assume that $|y_i|$ and $|x_i|$ are non-decreasing in $i$. Then the implication from (ii) to (i) follows from Lemma 2.4.9 above. For the remaining direction note that (i) directly implies $n \geq m$. By our initial assumption it is left to show that $|x_j| \leq |y_j|$ for every $\{j \in \{1, \ldots, m\}\}$. For this we will carry out an induction over $m$. For $m = 1$ the statement is clear. Let now $m \geq 2$, $x \preceq y$, and assume the implication from (i) to (ii) holds for all tuples $\tilde{x}$ and $(m-1)$-tuples $\tilde{y}$. We will first show that $|x_1| \leq |y_1|$. In order to see this consider

$$1 \leq |y \cap [-|y_1|, |y_1|]| \leq |x \cap [-|y_1|, |y_1|]|.$$

Since $|x_i|$ is non-decreasing in $i$ this implies $|x_1| \leq |y_1|$. Now define

$$\tilde{x} = (x_2, \ldots, x_n) \qquad \text{and} \qquad \tilde{y} = (y_2, \ldots, y_m).$$

Let $a \geq 0$. If $a \geq |y_1|$ we have

$$\begin{aligned} |(y_2, \ldots, y_k) \cap [-a, a]| &= |y \cap [-a, a]| - 1 \\ &\leq |x \cap [-a, a]| - 1 = |(x_2, \ldots, x_n) \cap [-a, a]|, \end{aligned}$$

where the last equality holds since $|x_1| \leq |y_1| \leq a$. This means $\tilde{x} = \{x_2, \ldots, x_n\} \preceq \tilde{y} = \{y_2, \ldots, y_m\}$. But by the assumption of the induction for $(m-1)$-tuples this means that for $j \in \{2, \ldots, m\}$ we have $|x_j| \leq |y_j|$. All in all we have shown that $|x_j| \leq |y_j|$ for all $j \in \{1, \ldots, m\}$. □

**Definition 2.4.11.** *For a tuple* $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ *define* $\varrho_k$ *with* $k \leq m$ *as the function, which assigns to* $z$ *the* $k$*-tuple consisting of the first* $k$ *entries of* $z$*, which have the smallest absolute value. This means* $\varrho_k(z)$ *is defined by*

- $\varrho_k(z) = (z_{\iota(1)}, \ldots, z_{\iota(k)})$ *for an increasing, injective map* $\iota : \{1, \ldots, k\} \to \{1, \ldots, m\}$,
- *for* $i \in \{1, \ldots, k\}$ *and* $j \in \{1, \ldots, m\}$, *if* $|x_{\iota(i)}| = |x_j|$ *for* $\iota(i) \geq j$, *then* $j \in \iota(\{1, \ldots, k\})$,
- $|x_i| \leq |x_j|$ *for all* $i \in \iota(\{1, \ldots, k\})$, $j \notin \iota(\{1, \ldots, k\})$.

**Lemma 2.4.12.** *Let* $x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}^m$ *be tuples with* $n \geq m$ *and* $x \preceq y$. *Then for every* $m \leq k \leq n$ *holds* $\varrho_k(x) \preceq y$.

*Proof.* By Lemma 2.4.10 we can assume without loss of generality that $|x_i|$ and $|y_i|$ are non-decreasing in $i$. Note that then $\varrho_k(x) = (x_1, \dots, x_k)$. For every $j \in \{1, \dots, m\}$ we have by the lemma above that $|x_j| \le |y_j|$. By Lemma 2.4.9 above the statement follows. □

In the following situation the set ordering can be expressed by the two-sided stochastic order.

**Lemma 2.4.13.** *Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and $n = m$. Then $x \preceq y$ if and only if $\frac{1}{n}\sum_{i=1}^n \delta_{x_i} \preceq \frac{1}{m}\sum_{j=1}^m \delta_{y_j}$.*

*Proof.* Since $n = m$, it holds for $a \ge 0$ that

$$|x \cap [-a,a]| \ge |y \cap [-a,a]| \Leftrightarrow \sum_{i=1}^n \mathbb{1}_{[-a,a]}(x_i) \ge \sum_{i=1}^m \mathbb{1}_{[-a,a]}(y_i)$$
$$\Leftrightarrow \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{[-a,a]}(x_i) \ge \frac{1}{m}\sum_{i=1}^n \mathbb{1}_{[-a,a]}(y_i).$$

This was to be shown. □

### Coupling of the stochastic barriers

Now we begin with the preparation of the coupling, where the key idea is to imitate the dynamic of the particle process with coupled Brownian paths. The existence of the required couplings in the following statements can be seen from the explicit construction of Lemma 2.2.4. Between tuples, for the formulation of the following lemmas we use the partial order from (2.41).

Recall that $B^y, B^x, B^z$ denote $M$-dimensional Brownian motions with $B_0^{y,i} = y_i$, $B_0^{x,i} = x_i$ and $B_0^{z,i} = z_i$.

**Lemma 2.4.14** (cf. Figure 2.13)**.** *Let $x, y \in \mathbb{R}^M$. Assume that $B^y$ and $B^x$ are coupled such that $|B_t^{y,i}| \le |B_t^{x,i}|$ for all $t \ge 0$ and $i \in \{1, \dots, M\}$. Then*

$$(B_t^{y,i})_{i \in A_t^{+,y}(j)} \preceq (B_t^{x,i})_{i \in A^x(s_1,\dots,s_j)}.$$

*Proof.* Since

$$(B_t^{y,i})_{i\in\{1,\dots,M\}} \preceq (B_t^{x,i})_{i\in\{1,\dots,M\}} \preceq (B_t^{x,i})_{i \in A^x(s_1,\dots,s_j)}$$

and $|A^x(s_1,\dots,s_j)| = M - j$ by Lemma 2.4.12 we can deduce that

$$(B_t^{y,i})_{i \in A_t^{+,y}(j)} = \varrho_{M-j}((B_t^{y,i})_{i\in\{1,\dots,M\}}) \preceq (B_t^{x,i})_{i \in A^x(s_1,\dots,s_j)}$$

which proves the statement. □

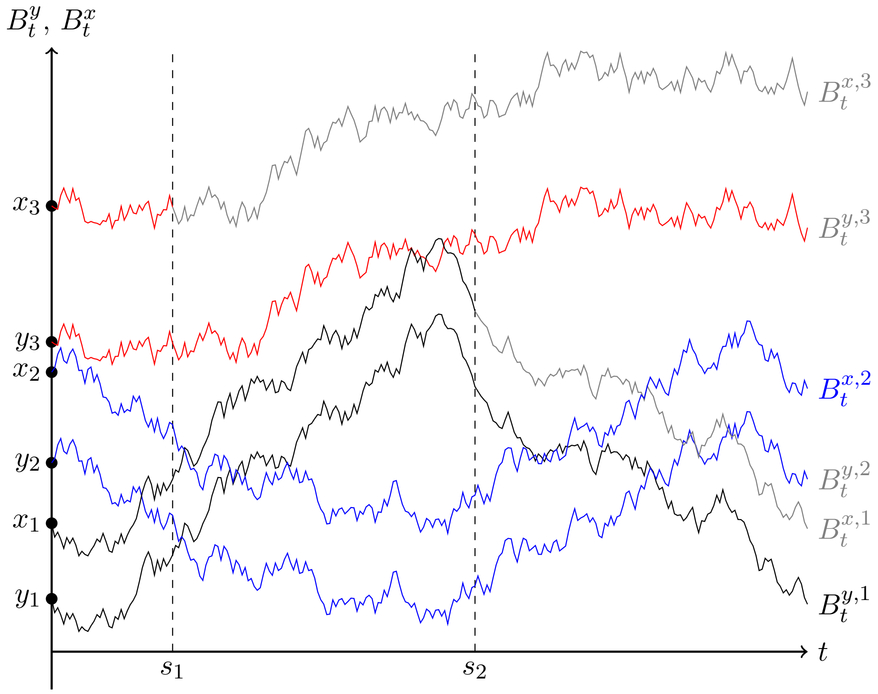


Figure 2.13: Illustration of the coupling construction from Lemma 2.4.14 for $M = 3$ and $j = 2$. The common colors black, blue and red indicate the pairing of the coupling, where the gray colorization represents the removal from $A^x(s_1, \dots, s_j)$ and $A^{+,y}$, respectively. Note that the removal from $A^{+,y}$ takes place at time $t$.

**Lemma 2.4.15** (cf. Figure 2.14)**.** *Let $x, z \in \mathbb{R}^M$. Let $B^x$ be given as in Lemma 2.4.14 and assume that $x \preceq z$. Then there exists a Brownian motion $B^z$ such that*

$$(B_t^{x,i})_{i \in A^x(s_1,\dots,s_j)} \preceq (B_t^{z,i})_{i \in A^{-,z}(j)}.$$

*Proof.* Define $s_{j+1} \coloneqq t$. Assume that for $\ell \in \{0, 1, \dots, j\}$ up to time $s_\ell$ we have found $B^z$ and the ordering is already fulfilled, this is we assume

$$\tilde{x} \coloneqq (B_{s_\ell}^{x,i})_{i \in A^x(s_1,\dots,s_\ell)} \preceq (B_{s_\ell}^{z,i})_{i \in A^{-,z}(j)} \eqqcolon \tilde{z}.$$

By Lemma 2.4.10 there exists an injective map $\pi_0 : A^{-,z}(j) \to A^x(s_1, \dots, s_\ell)$ such that $|\tilde{z}_i| \geq |\tilde{x}_{\pi_0(i)}|$ for all $i \in A^{-,z}(j)$. Let $W^{\tilde{z}}$ be a $|\tilde{z}|$-dimensional Brownian motion with $W_0^{\tilde{z}} = \tilde{z}$ such that $|W_u^{\tilde{z},i}| \geq |B_{u+s_\ell}^{x,\pi_0(i)}|$ for all $u \geq 0$ and $i \in A^{-,z}(j)$. Set

$$B_s^{z,i} \coloneqq W_{s-s_\ell}^{\tilde{z},i}, \quad s \in (s_\ell, s_{\ell+1}], \quad i \in A^{-,z}(j).$$

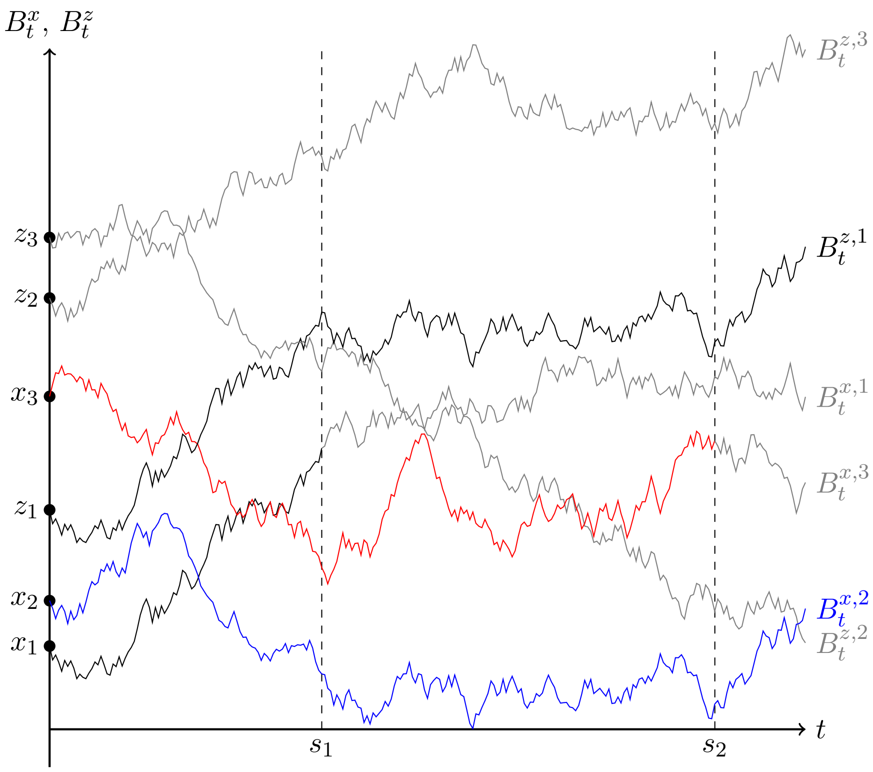


Figure 2.14: Illustration of the coupling construction from Lemma 2.4.15 for $M = 3$ and $j = 2$. The common color black indicates the pairing of the coupling, where the gray colorization represents the removal from $A^x(s_1, \dots, s_j)$ and $A^{-,z}$, respectively. Note that the removal from $A^{-,z}$ takes place at time zero.

We have then by Lemma 2.4.10 that $(B^{x,i}_{s_{\ell+1}})_{i\in A^x(s_1,\dots,s_\ell)} \preceq (B^{z,i}_{s_{\ell+1}})_{i\in A^{-,z}(j)}$ since

$$|B^{z,i}_{s_{\ell+1}}| = |W^{\tilde{z},i}_{s_{\ell+1}-s_\ell}| \geq |B^{x,\pi_0(i)}_{s_{\ell+1}}|$$

for all $i \in A^{-,z}$. By Lemma 2.4.12 follows that

$$(B^{x,i}_{s_{\ell+1}})_{i\in A^x(s_1,\dots,s_{\ell+1})} = \varrho_{A^x(s_1,\dots,s_\ell)-1}((B^{x,i}_{s_{\ell+1}})_{i\in A^x(s_1,\dots,s_\ell)}) \preceq (B^{z,i}_{s_{\ell+1}})_{i\in A^{-,z}(j)},$$

where $\varrho_k$ is defined in Definition 2.4.11. By induction the statement follows. $\square$

With these coupling lemmas we are now ready to proof Lemma 2.4.7.

*Proof of Lemma 2.4.7.* Assume the statement for $k \in \{0,\dots,n_m-1\}$. Let $M := N - k^N(t_k^{(m)}) = |A_{k^{(N)}(t_k^{(m)})}|$ and $x, y, z \in \mathbb{R}^M$ such that

$$\frac{1}{M}\sum_{i=1}^M \delta_{y_i} = \tilde{\xi}^{+,N}_{t_k^{(m)}}, \quad \frac{1}{M}\sum_{i=1}^M \delta_{x_i} = \tilde{\xi}^{m,N}_{t_k^{(m)}}, \quad \frac{1}{M}\sum_{i=1}^M \delta_{z_i} = \tilde{\xi}^{-,N}_{t_k^{(m)}}.$$

By the asssumption of the induction we have $y \preceq x \preceq z$. The underlying process from $\xi_k^{m,N}$ to $\xi_{k+1}^N$ has its particles removed at the timepoints

$$s_1 := t^N_{k^N(t_k^{(m)})+1} - t_k^{(m)}, \dots, s_j := t^N_{k^N(t_{k+1}^{(m)})} - t_k^{(m)}$$

with altogether

$$j := k^N(t_{k+1}^{(m)}) - k^N(t_k^{(m)})$$

particles removed. (As mentioned before $j = 0$ is possible.) To achieve a coupling between $\xi_{k+1}^{m,N}$ and $\xi_{k+1}^{-,N}$ we can take an arbitrary Brownian motion $B^x$ and corresponding to that, $B^z$ as produced in Lemma 2.4.15 with start in $x$ and $z$. For the coupling between $\xi_{k+1}^{m,N}$ and $\xi_{k+1}^{+,N}$ we take a Brownian motion $B^y$ with start in $y$ coupled to $B^x$ in the way required by Lemma 2.4.14. We obtain a coupling of $\xi_{k+1}^{+,N}, \xi_{k+1}^{m,N}, \xi_{k+1}^{-,N}$ by defining

$$\tilde{\xi}_{k+1}^{+,N} := \frac{1}{\left|A^{+,y}_{t_{k+1}^{(m)}-t_k^{(m)}}(j)\right|} \sum_{i \in A^{+,y}_{t_{k+1}^{(m)}-t_k^{(m)}}(j)} \delta_{B^{y,i}_{t_{k+1}^{(m)}-t_k^{(m)}}},$$

$$\xi_{k+1}^{m,N} := \frac{1}{\left|A^x(s_1,\dots,s_j)\right|} \sum_{i \in A^x(s_1,\dots,s_j)} \delta_{B^{x,i}_{t_{k+1}^{(m)}-t_k^{(m)}}},$$

$$\tilde{\xi}_{k+1}^{-,N} := \frac{1}{\left|A^{-,z}(j)\right|} \sum_{i \in A^{-,z}(j)} \delta_{B^{z,i}_{t_{k+1}^{(m)}-t_k^{(m)}}},$$

where we use the notation of Lemma 2.4.15 and Lemma 2.4.14. Since

$$\begin{aligned} N - k^{(N)}(t_{k+1}^{(m)}) &= \left|A^{+,y}_{t_{k+1}^{(m)}-t_k^{(m)}}(j)\right| \\ &= |A^x(s_1,\dots,s_j)| = |A^{-,z}(j)| \end{aligned}$$

we have by Lemma 2.4.14 and Lemma 2.4.15 combined with Lemma 2.4.13 that this coupling fulfills the desired ordering property, namely that

$$\tilde{\xi}_{k+1}^{+,N} \preceq \xi_{k+1}^{m,N} \preceq \tilde{\xi}_{k+1}^{-,N}.$$

The statement for the discrete time processes follows therefore by induction.

Now observe, if for $t \geq 0$ we have $t > t^{(m)}_{k^{(m)}(t)}$ we can start with the coupled configurations of $\tilde{\xi}^{+,N}_{k^{(m)}(t)}, \xi^{m,N}_{k^{(m)}(t)}, \tilde{\xi}^{-,N}_{k^{(m)}(t)}$ and choose the increments of $B^y$ and $B^z$ in such a way that the particle systems stay ordered up to time $t$. See for example the proof of Lemma 2.2.5 for such a coupling. □

**Hydrodynamic limit of stochastic barriers**

As next step we will establish a hydrodynamic limit for the lower and upper stochastic barriers $\chi_t^{\pm,N}$ from (2.40).

We define $\alpha_k^{(m)} := g(t_k^{(m)})/g(t_{k-1}^{(m)})$ for $k \in \{1,\dots,n_m\}$ such that $t_k^{(m)} < t^g$. Let $\nu_1^{-,m} := P_{t_1^{(m)}}(\mu)$ and

$$\begin{aligned}
\nu_k^{+,m} &:= T_{\alpha_k^{(m)}} \circ P_{t_k^{(m)}-t_{k-1}^{(m)}} \circ \dots \circ T_{\alpha_1^{(m)}} \circ P_{t_1^{(m)}}(\mu),\\
\nu_k^{-,m} &:= P_{t_k^{(m)}-t_{k-1}^{(m)}} \circ T_{\alpha_k^{(m)}} \circ \dots \circ P_{t_2^{(m)}-t_1^{(m)}} \circ T_{\alpha_2^{(m)}} \circ P_{t_1^{(m)}}(\mu)
\end{aligned}$$

and

$$\begin{aligned}
q_k^+ &:= q_k^{+,m} := \sup\operatorname{supp}(\nu_k^{+,m}), \quad k \in \{1,\dots,n_m\},\\
q_k^- &:= q_k^{-,m} := \sup\operatorname{supp}(T_{\alpha_k^{(m)}} \circ \nu_{k-1}^{-,m}), \quad k \in \{2,\dots,n_m\},
\end{aligned} \tag{2.42}$$

for $k$ such that $t_k^{(m)} < t^g$. Set $q_k^+ := 0$ if $t_k^{(m)} = t^g$.

**Theorem 2.4.16.** *Assume that for every $t \in (0, t^g)$ holds*

$$\lim_{N\to\infty} \frac{k^N(t)}{N} = 1 - g(t).$$

*Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable and bounded function and $t \in (0, t^g)$. Then*

$$\chi_t^{\pm,N}(\varphi) \to P_{t-(t)_m} \circ \nu_{k^{(m)}(t)}^{\pm,m}(\varphi)$$

*almost surely as $N \to \infty$.*

*Proof.* In order to bring the empirical process together with the deterministic quantiles, in the following we use ideas from the proof of Proposition 3 in [De +19a]. Define $F_0^{\pm,N}(i) := F_0^{\pm}(i) := 1$ and

$$\begin{aligned}
F_k^{+,N}(i) &:= \mathbb{1}_{\{|B^i_{t_1^{(m)}}| \le q_1^{+,N},\dots,|B^i_{t_k^{(m)}}| \le q_k^{+,N}\}}, \quad k \ge 1\\
F_k^{+}(i) &:= \mathbb{1}_{\{|B^i_{t_1^{(m)}}| \le q_1^{+},\dots,|B^i_{t_k^{(m)}}| \le q_k^{+}\}}, \quad k \ge 1\\
F_1^{-,N}(i) &:= F_1^{-}(i) := \mathbb{1}_{\{i \in A_1^-\}}\\
F_k^{-,N}(i) &:= F_1^{-,N}(i)\mathbb{1}_{\{|B^i_{t_1^{(m)}}| \le q_2^{-,N},\dots,|B^i_{t_{k-1}^{(m)}}| \le q_k^{-,N}\}}, \quad k \ge 2\\
F_k^{-}(i) &:= F_1^{-}(i)\mathbb{1}_{\{|B^i_{t_1^{(m)}}| \le q_2^{-},\dots,|B^i_{t_{k-1}^{(m)}}| \le q_k^{-}\}}, \quad k \ge 2.
\end{aligned}$$

The definition for $F_1^{-,N}$ differs since we only defined $A_1^-$ to be a independently, uniformly chosen random subset of $\{1,\dots,N\}$ with $N - k^{(N)}(t_1^{(m)})$ elements.

We first claim that for every $k \in \{0, \dots, k^{(m)}(t)\}$ we have that

$$\frac{1}{N}\sum_{i=1}^{N} \left|F_k^{\pm,N}(i) - F_k^{\pm}(i)\right| \to 0 \tag{2.43}$$

almost surely as $N \to \infty$ and will deduce the desired statement by using the assertion of the claim. We have for any $s \geq t_k^{(m)}$ that

$$\begin{aligned}
&\Big|\frac{|A_k^{\pm}|}{N} \cdot \frac{1}{|A_k^{\pm}|}\sum_{i\in A_k^{\pm}} \varphi(B_s^i) - g(t_k^{(m)})P_{s-t_k^{(m)}}\nu_k^{\pm,m}(\varphi)\Big| \\
&\leq \Big|\frac{1}{N}\sum_{i\in A_k^{\pm}} \varphi(B_s^i) - \frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^{\pm}(i)\Big| \\
&\quad + \Big|\frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^{\pm}(i) - g(t_k^{(m)})P_{s-t_k^{(m)}}\nu_k^{\pm,m}(\varphi)\Big|.
\end{aligned} \tag{2.44}$$

The first term tends to zero as $N \to \infty$ by observing

$$\begin{aligned}
&\Big|\frac{1}{N}\sum_{i\in A_k^{\pm}} \varphi(B_s^i) - \frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^{\pm}(i)\Big| \\
&= \Big|\frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^{\pm,N}(i) - \frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^{\pm}(i)\Big| \\
&\leq \|\varphi\|_\infty \frac{1}{N}\sum_{i=1}^{N}\left|F_k^{\pm,N}(i) - F_k^{\pm}(i)\right| \to 0
\end{aligned}$$

almost surely. For the second term we treat the lower and upper process differently. By the law of large numbers we have that

$$\begin{aligned}
&\Big|\frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^{+}(i) - g(t_k^{(m)})P_{s-t_k^{(m)}}\nu_k^{+,m}(\varphi)\Big| \\
&= \Big|\frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^{+}(i) - \mathbb{E}_\mu\left[\varphi(B_s^1)F_k^{+}(1)\right]\Big| \to 0
\end{aligned} \tag{2.45}$$

almost surely as $N \to \infty$. For the upper barrier recall that $A_1^-$ was defined to be a independently, uniformly chosen random subset of $\{1, \dots, N\}$ with $N - k^{(N)}(t_1^{(m)})$ elements. It follows that

$$\frac{1}{N}\sum_{i=1}^{N} F_1^-(i) = \frac{N - k^N(t_1^{(m)})}{N} \to g(t_1^{(m)})$$

and thus by Lemma B.1.9 and the law of large numbers it follows that

$$\frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_k^-(i)=\frac{1}{N}\sum_{i=1}^{N}\varphi(B_s^i)F_1^-(i)\mathbb{1}_{\{|B^i_{t_1^{(m)}}|\le q_2^-,\dots,|B^i_{t_{k-1}^{(m)}}|\le q_k^-\}}$$
$$\overset{N\to\infty}{\longrightarrow} g(t_1^{(m)})\mathbb{E}_\mu\left[\varphi(B_s^1)\mathbb{1}_{\{|B^1_{t_1^{(m)}}|\le q_2^-,\dots,|B^1_{t_{k-1}^{(m)}}|\le q_k^-\}}\right]$$
$$=g(t_k^{(m)})P_{s-t_k^{(m)}}\nu_k^{-,m}(\varphi) \tag{2.46}$$

almost surely as $N\to\infty$. Thus the quantity in (2.44) converges to zero almost surely. Therefore, by noting that $|A_k^\pm|/N\to g(t_k^{(m)})$ almost surely, the desired statement follows by choosing $k=k^{(m)}(t)$ and $s=t\ge t^{(m)}_{k^{(m)}(t)}=(t)_m$.

We will now prove the claim from (2.43) by induction. Note that by the definitions the claim is true for $k=0$ and in the case of the upper barrier also for $k=1$. For this assume that the convergence of (2.43) is true for fixed $k\in\{0,\dots,k^{(m)}(t)-1\}$. We will treat the lower and the upper barrier separately. We have

$$\frac{1}{N}\sum_{i=1}^{N}\left|F_{k+1}^{+,N}(i)-F_{k+1}^{+}(i)\right|$$
$$=\frac{1}{N}\sum_{i=1}^{N}\left|F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+,N}\}}-F_k^{+}(i)\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+}\}}\right|$$
$$\le\frac{1}{N}\sum_{i=1}^{N}\left|F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+,N}\}}-F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+}\}}\right|$$
$$+\frac{1}{N}\sum_{i=1}^{N}\left|F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+}\}}-F_k^{+}(i)\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+}\}}\right|$$
$$\le\frac{1}{N}\sum_{i=1}^{N}F_k^{+,N}(i)\left|\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+,N}\}}-\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+}\}}\right|$$
$$+\frac{1}{N}\sum_{i=1}^{N}\left|F_k^{+,N}(i)-F_k^{+}(i)\right|.$$

The last term is tending to zero by assumption while the remaining term can be written as follows.

$$\frac{1}{N}\sum_{i=1}^{N}F_k^{+,N}(i)\left|\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+,N}\}}-\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+}\}}\right|$$
$$=\frac{1}{N}\sum_{i=1}^{N}\operatorname{sgn}(q_{k+1}^{+,N}-q_{k+1}^{+})F_k^{+,N}(i)(\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+,N}\}}-\mathbb{1}_{\{|B^i_{t_{k+1}^{(m)}}|\le q_{k+1}^{+}\}})$$

$$
\begin{aligned}
&= \Big|\frac{1}{N}\sum_{i=1}^{N} F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k+1}}|\le q^{+,N}_{k+1}\}} - F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k+1}}|\le q^{+}_{k+1}\}}\Big| \\
&\le \Big|\frac{1}{N}\sum_{i=1}^{N} F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k+1}}|\le q^{+,N}_{k+1}\}} - F_k^{+}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k+1}}|\le q^{+}_{k+1}\}}\Big| \\
&\quad + \Big|\frac{1}{N}\sum_{i=1}^{N} F_k^{+}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k+1}}|\le q^{+}_{k+1}\}} - F_k^{+,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k+1}}|\le q^{+}_{k+1}\}}\Big| \\
&\le \Big|\frac{1}{N}\sum_{i=1}^{N} F_{k+1}^{+}(i) - \frac{1}{N}\sum_{i=1}^{N} F_{k+1}^{+,N}(i)\Big| + \frac{1}{N}\sum_{i=1}^{N}\big|F_k^{+,N}(i) - F_k^{+}(i)\big|.
\end{aligned}
$$

Again by assumption the last term tends to zero. For the upper barrier, we assume $k \ge 2$ and we have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\big|F_{k+1}^{-,N}(i) - F_{k+1}^{-}(i)\big| \\
&= \frac{1}{N}\sum_{i=1}^{N}\Big|F_k^{-,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-,N}_{k+1}\}} - F_k^{-}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\Big| \\
&\le \frac{1}{N}\sum_{i=1}^{N}\Big|F_k^{-,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-,N}_{k+1}\}} - F_k^{-,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\Big| \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\Big|F_k^{-,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}} - F_k^{-}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\Big| \\
&\le \frac{1}{N}\sum_{i=1}^{N} F_k^{-,N}(i)\Big|\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-,N}_{k+1}\}} - \mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\Big| \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\big|F_k^{-,N}(i) - F_k^{-}(i)\big|.
\end{aligned}
$$

The last term is tending to zero by assumption while the remaining term can be written as follows.

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} F_k^{-,N}(i)\Big|\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-,N}_{k+1}\}} - \mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\Big| \\
&= \frac{1}{N}\sum_{i=1}^{N} \operatorname{sgn}(q^{-,N}_{k+1} - q^{-}_{k+1}) F_k^{-,N}(i)\big(\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-,N}_{k+1}\}} - \mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\big) \\
&= \Big|\frac{1}{N}\sum_{i=1}^{N} F_k^{-,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-,N}_{k+1}\}} - F_k^{-,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\Big| \\
&\le \Big|\frac{1}{N}\sum_{i=1}^{N} F_k^{-,N}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-,N}_{k+1}\}} - F_k^{-}(i)\mathbb{1}_{\{|B^i_{t^{(m)}_{k}}|\le q^{-}_{k+1}\}}\Big|
\end{aligned}
$$

$$+ \Big| \frac{1}{N} \sum_{i=1}^{N} F_k^-(i) \mathbb{1}_{\{|B^i_{t_k^{(m)}}| \le q^-_{k+1}\}} - F_k^{-,N}(i) \mathbb{1}_{\{|B^i_{t_k^{(m)}}| \le q^-_{k+1}\}} \Big|$$

$$\le \Big| \frac{1}{N} \sum_{i=1}^{N} F^-_{k+1}(i) - \frac{1}{N} \sum_{i=1}^{N} F^{-,N}_{k+1}(i) \Big| + \frac{1}{N} \sum_{i=1}^{N} \big| F_k^{-,N}(i) - F_k^-(i) \big|.$$

Again by assumption the last term tends to zero. Thus the assertion left to show for the claim is

$$\Big| \frac{1}{N} \sum_{i=1}^{N} F^{\pm}_{k+1}(i) - \frac{1}{N} \sum_{i=1}^{N} F^{\pm,N}_{k+1}(i) \Big| \to 0$$

almost surely as $N \to \infty$. But on the one hand we have almost surely

$$\frac{1}{N} \sum_{i=1}^{N} F^{\pm,N}_{k+1}(i) = \frac{|A^{\pm}_{k+1}|}{N} = \frac{N - k^N(t^{(m)}_{k+1})}{N} \to g(t^{(m)}_{k+1}).$$

On the other hand it holds almost surely

$$\frac{1}{N} \sum_{i=1}^{N} F^{\pm}_{k+1}(i) \to g(t^{(m)}_{k+1}),$$

which is clear by the arguments in (2.45) and (2.46) for $\varphi \equiv 1$. This finishes the proof. □

**Lemma 2.4.17.** *Fix $t \in (0, t^g)$ and recall the notation $(t)_m = t^{(m)}_{k^{(m)}(t)}$. Assume that*

$$\lim_{N \to \infty} \frac{k^N(t)}{N} = 1 - g(t)$$

*and $\lim_{m\to\infty} g((t)_m) = g(t)$. Then, for measurable, bounded $\varphi : \mathbb{R} \to \mathbb{R}$ holds*

$$\lim_{m\to\infty} \limsup_{N\to\infty} \big| \chi_t^N(\varphi) - \chi_t^{m,N}(\varphi) \big| = 0$$

*almost surely.*

*Proof.* We have that

$$\begin{aligned}
&\left|\chi_t^N(\varphi) - \chi_t^{m,N}(\varphi)\right| \\
&= \left| \frac{1}{|A_{k^N(t)}|} \sum_{i \in A_{k^N(t)}} \varphi(B_t^i) - \frac{1}{|A_{k^N((t)_m)}|} \sum_{i \in A_{k^N((t)_m)}} \varphi(B_t^i) \right| \\
&\le \frac{1}{|A_{k^N(t)}|} \sum_{i \in A_{k^N((t)_m)}} \left| \varphi(B_t^i) - \frac{|A_{k^N(t)}|}{|A_{k^N((t)_m)}|} \varphi(B_t^i) \right| \\
&\quad + \frac{1}{|A_{k^N(t)}|} \sum_{i \in A_{k^N((t)_m)} \setminus A_{k^N(t)}} \left| \varphi(B_t^i) \right| \\
&\le 2\|\varphi\|_\infty \frac{1}{|A_{k^N(t)}|} \Big( |A_{k^N((t)_m)}| - |A_{k^N(t)}| \Big) \\
&= 2\|\varphi\|_\infty \frac{1}{N - k^N(t)} \left( k^N(t) - k^N((t)_m) \right) \\
&\overset{N\to\infty}{\longrightarrow} 2\|\varphi\|_\infty \frac{1}{g(t)} \left( g((t)_m) - g(t) \right)
\end{aligned}$$

which converges to 0 as $m \to \infty$ by assumption. □

Now we will put all results together to yield Theorem 2.4.4.

*Proof of Theorem 2.4.4.* Let $t \in (0, t^g)$. Without loss of generality we assume that $g(t) < 1$. Using the coupling of Lemma 2.4.7 we get that for $a \ge 0$ we have almost surely

$$\tilde{\chi}_t^{+,N}([-a,a]) \ge \chi_t^{m,N}([-a,a]) \ge \tilde{\chi}_t^{-,N}([-a,a])$$

yielding by Theorem 2.4.16 that

$$\begin{aligned}
P_{t-(t)_m} \nu^{+,m}_{k^{(m)}(t)}([-a,a]) &\ge \limsup_{N\to\infty} \chi_t^{m,N}([-a,a]) \\
&\ge \liminf_{N\to\infty} \chi_t^{m,N}([-a,a]) \ge P_{t-(t)_m} \nu^{-,m}_{k^{(m)}(t)}([-a,a]) \qquad (2.47)
\end{aligned}$$

almost surely. Now we make the specific choice $t_k^{(m)} := k2^{-n}t$. Note that then $(t)_m = t$ and $\nu_k^{\pm,m}$ coincide with $\mu_k^{+,m}$ and $\tilde{\mu}_k^{-,m}$ from Theorem 2.3.33. By the weak convergence implied by Theorem 2.3.32 we have then, since $\mu_t$ is non-atomic, that

$$\lim_{m\to\infty} \nu^{+,m}_{k^{(m)}(t)}([-a,a]) = \lim_{m\to\infty} \nu^{-,m}_{k^{(m)}(t)}([-a,a]) = \mu_t([-a,a]).$$

Thus for $a \ge 0$, as a consequence of (2.47) and Lemma 2.4.17 we obtain that

$$\lim_{N\to\infty} \chi_t^N([-a,a]) = \mu_t([-a,a]) \qquad (2.48)$$

almost surely. In order to see that the almost sure event can be chosen independent from $a$, note that we get that the limit of (2.48) holds for all rational $a \geq 0$ on one almost sure event. By the continuity of $a \mapsto \mu_t([-a,a])$ it follows that on this almost sure event, the limit of (2.48) holds for all $a \geq 0$. □

**Remark 2.4.18.** The proof of Theorem 2.4.4 relied on the convergence of the sequences $\nu^{\pm,m}_{k^{(m)}(t)}$, which was achieved by choosing dyadic numbers as lattice points and using the convergence result Theorem 2.3.33. When $g$ is continuous this convergence is also guaranteed by Proposition A.5.4 for the lattice points given by (2.51).

### 2.4.3 Simulation of solutions

In the inverse first-passage time problem some research has focused on solving the problem numerically, which in particular leads to visualization techniques for the boundary function. This is particularly interesting since in general the existing and unique boundary can be expected to not have a certain closed form which can be determined.

In the following we will give a short overview of the methods in the literature. In the context of credit risk modeling the work of [HW01] proposes a numerical approximation of a discretization scheme similar to the scheme in [Anu80]. The work of [AZ01] proposes to solve a free boundary problem related to the problem in [Che+06] numerically. The mathematical work of [ZS09] presents two approaches. The one approach is based on a continuous, piecewise approximation, which is estimated by a Monte-Carlo method. The other approach numerically approximates the solution of an integral equation, the so-called Master equation (cf. Section 14, [PS06]). The author of [Abu06] transfers the latter to the case of reflected Brownian motion. The authors of [GP21] propose a modified method by estimating the integral of the equation by using the empirical distribution of $g$. Another approach can be found in [SZ11], which is sometimes called tangent-method.

In the following we present a Monte-Carlo approach resulting from Subsection 2.4.2, which we used in this thesis to visualize the unknown solution. The method will be specified in Definition 2.4.20. Essentially, it is a Monte-Carlo approximation of Anulova's scheme of quantiles, but with a different choice of discrete time points.

Let a survival distribution $g$ and an initial measure $\mu$, and timepoints

$$0 < t_1^{(m)} \leq t_2^{(m)} \leq \ldots t_{n_m}^{(m)} \leq t^g$$

be given. We recall the quantiles

$$q_k^{+,m} := \sup \operatorname{supp}(\nu_k^{+,m}),$$

where

$$\nu_k^{+,m} := T_{\alpha_k^{(m)}} \circ P_{t_k^{(m)} - t_{k-1}^{(m)}} \circ \ldots \circ T_{\alpha_1^{(m)}} \circ P_{t_1^{(m)}}(\mu)$$

with $\alpha_k^{(m)} := g(t_k^{(m)})/g(t_{k-1}^{(m)})$. In the case of dyadic timesteps, the discrete boundary functions

$$b_m(s) := \begin{cases} q_k^{+,m} & : s = t_k^{(m)}, k \in \{1, \dots, n_m\}, \\ \infty & : \text{else}, \end{cases} \qquad s \in [0, t^g],$$

served us as approximants of the solution of ifpt$(g, \mu)$ as $m \to \infty$ in the Hausdorff distance. Here we use the general time points from (2.38). Now, Theorem 2.4.16 suggests the alternative approach to approximate the deterministic quantiles $q_k^{+,m}$ by the random quantiles from (2.39). Let us recall the setting. For $N \in \mathbb{N}$ we had timepoints $t_0^N < t_1^N \leq \ldots \leq t_N^N$ and the number of timepoints up to a time $t$ was denoted with

$$k^N(t) := \sup\{k \in \{0, 1, \dots, N\} : t_k^N \leq t\}.$$

For independent Brownian motions $B^1, \dots, B^N$ we defined the empirical quantiles

$$q_k^{+,N,m} := \inf\left\{ a \geq 0 : \sum_{i \in A_{k-1}^+} \mathbb{1}_{\{|B^i_{t_k^{(m)}}| \leq a\}} \geq N - k^N(t_k^{(m)}) \right\}, \tag{2.49}$$

where inductively for $k \in \{1, \dots, n_m\}$ we defined

$$A_{k-1}^+ := \{i \in \{1, \dots, n\} : |B^i_{t_{k-1}^{(m)}}| \leq q_{k-1}^{+,N}, \dots, |B^i_{t_1^{(m)}}| \leq q_1^{+,N}\}.$$

Eventually, we obtain the sequence of random boundary functions

$$b_m^N(s) := \begin{cases} q_k^{+,N,m} & : s = t_k^{(m)}, k \in \{1, \dots, n_m\}, \\ \infty & : \text{else}, \end{cases} \qquad s \in [0, t^g], \tag{2.50}$$

which can be seen as the corresponding Monte-Carlo method. The next lemma justifies that this yields an appropriate candidate for an approximation of $b_m$.

**Lemma 2.4.19.** *Assume that for every $t \in (0, t^g)$ holds*

$$\lim_{N \to \infty} \frac{k^N(t)}{N} = 1 - g(t).$$

*Then, for every $k \in \{1, \dots, n_m\}$ it holds that*

$$\lim_{N \to \infty} b_m^N(t_k^m) = b_m(t_k^m)$$

*almost surely.*

*Proof.* Assume that $\liminf_{N\to\infty} q_k^{+,N,m} < R < q_k^{+,m}$. We obtain by (2.43) in the proof of Theorem 2.4.16 that

$$\begin{aligned}
0 &= \limsup_{N\to\infty} \frac{1}{N}(|\{i \in A_k^{+,N} : |B^i_{t_k^{(m)}}| \leq q_k^+\}| - |\{i \in A_k^{+,N} : |B^i_{t_k^{(m)}}| \leq q_k^{+,N}\}|) \\
&\geq \liminf_{N\to\infty} \frac{1}{N}(|\{i \in A_k^{+,N} : |B^i_{t_k^{(m)}}| \leq q_k^+\}| - |\{i \in A_k^{+,N} : |B^i_{t_k^{(m)}}| \leq R\}|) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{i\in A_k^{+,N}} \mathbb{1}_{(R,q_k^+)}(|B^i_{t_k^{(m)}}|) = \mathbb{P}_\mu\left(|X_{t_k^{(m)}}| \in (R, q_k^+), \tau_{b_m} > t_k^{(m)}\right) > 0.
\end{aligned}$$

If we assume that $\limsup_{N\to\infty} q_k^{+,N,m} > R > q_k^{+,m}$, we analogously obtain a contradiction. These contradictions show that $\lim_{N\to\infty} q_k^{+,N,m} = q_k^{+,m}$ almost surely. □

For our simulations we work with specific lattice points, which are adapted on the structure of $g$ and result in a simpler algorithm. Additionally, it can be hoped that this choice yields better accuracy and efficiency of the Monte-Carlo simulation.

**The lower barrier approximation using $g$-quantiles**

For the rest of the subsection we choose the lattice points $t_0 := 0$ and

$$t_k^{(m)} := g^{-1}\left(\frac{m-k}{m}\right), \quad k \in \{1, \ldots, m\} \tag{2.51}$$

where $g^{-1}$ denotes the generalized inverse of $g$, this is

$$g^{-1}(q) := \inf\{t \geq 0 : g(t) \leq q\}, \quad q \in [0,1]. \tag{2.52}$$

For the discrete boundary function $b_m$ we have due to the definition and Lemma A.5.1 that

$$\mathbb{P}_\mu(\tau_{b_m} > t) = \mathbb{P}_\mu(\tau_{b_m} > (t)_m) = g((t)_m) \to g(t)$$

for $t \in (0, t^g)$, which implies that $\tau_{b_m} \to g$ in distribution. Using Lemma 2.1.17, the compactness of the set of boundary functions and the uniqueness it follows that $b_m \xrightarrow{\Gamma} b$ in the Hausdorff distance. Furthermore, note that by Lemma A.5.1 we have $\frac{k^m(t)}{m} \to 1 - g(t)$. This justifies the use of the following approximation.

**Definition 2.4.20.** *Let $g$ be a survival distribution and $\mu \in \mathcal{P}$. For $m \in \mathbb{N}$ let $(t_k^{(m)})_{k\in\{1,\ldots,m\}}$ be given by (2.51). For $N \geq m$ set $t_k^N := t_k^{(N)}$. Our Monte-Carlo approximation of $b \in \mathrm{ifpt}(g,\mu)$ consists of generating the random functions*

$$b_m^N : \{t_0^{(m)}, t_1^{(m)}, \ldots, t_m^{(m)}\} \to [0,\infty],\ t_k^{(m)} \mapsto q_k^{+,N,m}$$

*from* (2.50).

**Remark 2.4.21.** The specific choice of $t_k^N$ and $t_k^{(m)}$ in Definition 2.50 simplifies the definition of $b_m^N(t_k^{(m)})$ for those $N \geq m$ such that $m|N$. Namely, if $N = m\ell$, using the definition (2.39) we have

$$b_m^N(t_k^{(m)}) = q_k^{+,m} = \inf\left\{a \geq 0 : \sum_{i \in A_{k-1}^+} \mathbb{1}_{\{|B^i_{t_k^{(m)}}| \leq a\}} \geq N - k^N(t_k^{(m)})\right\}$$
$$= \inf\left\{a \geq 0 : \sum_{i \in A_{k-1}^+} \mathbb{1}_{\{|B^i_{t_k^{(m)}}| \leq a\}} \geq m\ell - k\ell\right\}.$$

Hence, in the procedure at every time step the constant amount of $\ell$ particles is removed.

A common practice to present the approximation methods both for the inverse and the direct first-passage time problem is the application for a particular example of a solution to the direct first-passage time problem, for which the pair of the boundary function and survival distribution is known in an explicit way and given by a closed-form expression. For the one-sided problem the commonly used example is the so-called Daniel's boundary, see [Dan69], which was constructed with the method of images. Usual examples for the two-sided problem are affine linear boundaries, see for example [And60], [Abu02] and [SY11], and square-root boundaries as in [NFK99]. In these cases the simplicity of the boundaries is paid with very cumbersome survival functions. Moreover, other specific boundaries for the two-sided case can be constructed from the already mentioned method of images, see for example [Dan82] and [Ler86]. From the latter we will take the following pair in order to apply our approximation.

**Example 2.4.22.** For $b(t) = \mathbb{1}_{(0,1)}(t)\sqrt{-t\log(t)}$ and initial distribution $\delta_0$ the corresponding survival distribution is according to [Ler86] given by

$$\mathbb{P}_0(\tau_b > t) = 1 - 2\Phi(-b(t)/\sqrt{t}) - \frac{2}{\pi}b(t)$$
$$= 1 - 2\Phi(-\sqrt{-\log(t)}) - \frac{2}{\pi}\sqrt{-t\log(t)}.$$

Since the exact boundary is known, in Figure 2.15 we can make a comparison between the approximation and $b$, which we will call Lerche's boundary, where the generalized inverse of $g$ is obtained numerically from the density of $\tau_b$.

In Figure 2.15 we observe that this approximation seems to have similar monotonicity and boundedness properties as the lower barrier approximation of Anulova in Section 2.3.2. This behaviour is made rigorous by Proposition A.5.5, which in particular states that for $n, m \in \mathbb{N}$ with $m \mid n$ we have

$$b_m^+(t_k^{(m)}) \leq b_n^+(t_k^{(m)}) \leq b(t_k^{(m)}), \qquad k \in \{1, \ldots, m\}. \tag{2.53}$$

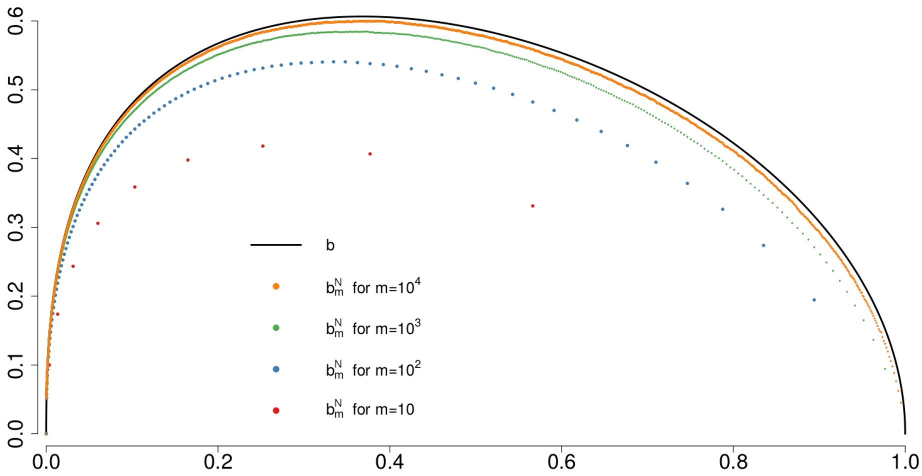


Figure 2.15: The approximated boundary for Lerche's boundary with $N = 10^7$.

For the case when the exact boundary function is not known and shall be visualized it is a problem not to know how large one has to choose $m$ to obtain an appropriate visualization. In the following we provide at least a heuristic solution to this problem.

### The path of the maximal particle

Since the empirical measure of the non-branching process is converging to the marginal distribution of the Brownian motion conditioned to not have crossed the boundary solution to the inverse first-passage time problem, the trajectory of the maximal absolute value of the particles is also a canonical candidate to be a visualization of the actual boundary function. To be more precise, we define the process

$$M_t^N := \max\{|B_t^i| : i \in A_{k^{(N)}(t)}\}$$

for $t \in (0, t^g)$, where $A_{k^{(N)}(t)}$ is the set of particles alive in the system introduced in the Subsection 2.4.2.

We can observe the following by Theorem 2.4.4.

**Proposition 2.4.23.** *Let $g$ be a survival distribution and $\mu \in \mathcal{P}^1$. Let $b \in$ ifpt$(g, \mu)$. For $t \in (0, t^g)$ we have*

$$\liminf_{N\to\infty} M_t^N \geq b(t)$$

*almost surely.*

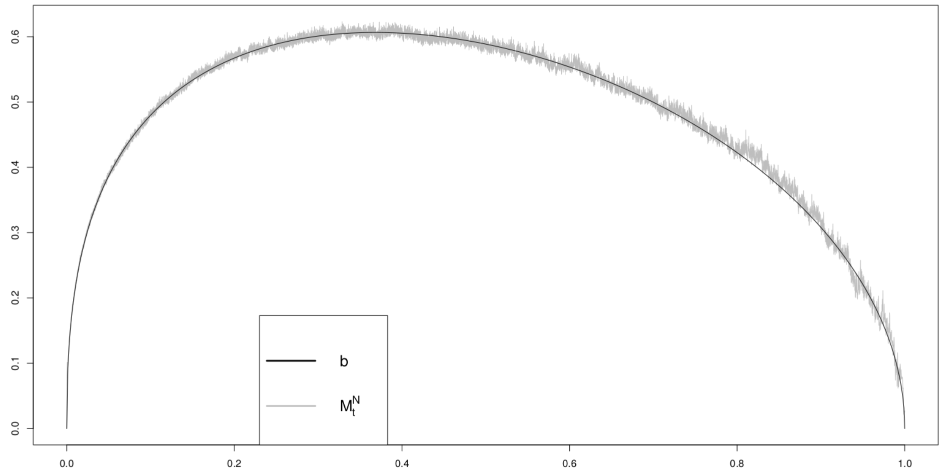


Figure 2.16: Path of the maximal particle $M_t^N$ of the non-branching system for the survival distribution corresponding to Lerche's boundary with $N = 10^5$.

*Proof.* Assume that $\liminf_{N\to\infty} M_t^N < R < b(t)$, we obtain

$$\begin{aligned}
0 &= \limsup_{N\to\infty} \frac{|A_{k^{(N)}(t)}|}{N} \left( \frac{1}{|A_{k^{(N)}(t)}|} \sum_{i \in A_{k^{(N)}(t)}} \delta_{B_t^i}(-b(t), b(t)) - 1 \right) \\
&= \limsup_{N\to\infty} \frac{1}{N} (|\{i \in A_{k^{(N)}(t)} : |B_t^i| \le b(t)\}| - |\{i \in A_{k^{(N)}(t)} : |B_t^i| \le M_t^N\}|) \\
&\ge \liminf_{N\to\infty} \frac{1}{N} (|\{i \in A_{k^{(N)}(t)} : |B_t^i| \le b(t)\}| - |\{i \in A_{k^{(N)}(t)} : |B_t^i| \le R\}|) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{i \in A_{k^{(N)}(t)}} \mathbb{1}_{(R, b(t))}(|B_t^i|) = \mathbb{P}_\mu\left(|X_t| \in (R, b(t)), \tau_b > t\right) > 0,
\end{aligned}$$

which shows that $\liminf_{N\to\infty} M_t^N \ge b(t)$ almost surely. □

As example, compare Figure 2.16, where we can see that the trajectory of the maximal absolute value of the particles behaves similar to the known boundary.

**Remark 2.4.24.** From this we can derive a visual criterion to decide whether $m$ is large enough. Note that by (2.53) we have that

$$q_k^{+,m} = b_m(t_k^{(m)}) \le b(t_k^{(m)}).$$

Thus, if we can assume that the Monte-Carlo method performs well enough, i.e. assuming $q_k^{+,N,m} \approx b_m(t_k^{(m)})$ and $b(t_k^{(m)}) \lessapprox M^N_{t_k^{(m)}}$, we have a visual criterion for whether $m$ is large enough, i.e. an appropriate simulation of $b$, by demanding

$$q_k^{+,N,m} \approx M^N_{t_k^{(m)}}.$$

For this we compare the following example regarding Figure 2.17.

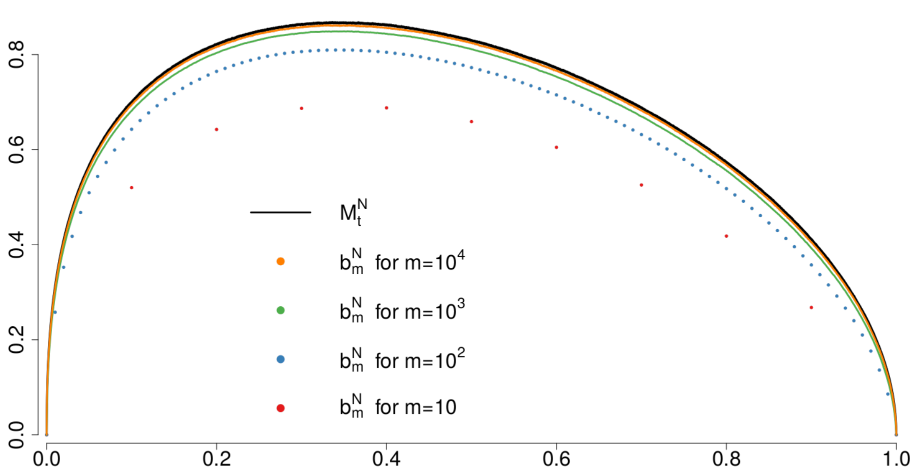


Figure 2.17: Approximated boundaries for the uniform distribution with $N = 10^7$.

**Example 2.4.25.** We consider $g(t) = \max(1-t, 0)$, which corresponds to the uniform distribution. Note that $t_k^{(m)} = g^{-1}((m-k)/m) = k/m$. For $m = 10^4$ the affinity of the paths of $M_t^N$ and $b_m^N$ indicates that both pathes are good visualizations of the unknown boundary corresponding to the uniform distribution.

### An example with a singular survival distribution via order statistics

Another possible choice of $t_k^{(m)}$ and $t_k^N$ are the order statistics of the distribution $g$, which in some sense approximate the timepoints $g^{-1}((m-k)/m)$. Again $g((t)_m) \to g(t)$, at least a.s., yields that $b_m \to b$ in Hausdorff distance.

**Example 2.4.26.** In this example let $g$ correspond to the Cantor distribution, this means $g$ corresponds to the weak limit $\lim_{n\to\infty} \text{Uniform}(C_n)$, where $C_n$ is the $n$-th step in the construction of the Cantor set. An approximative sample of $g$ can be obtained by

$$2\sum_{j=1}^{K} 3^{-j} U_j,$$

where $U_j$ are independent Bernoulli variables with parameter 1/2. Setting $K = \infty$ would result in an exact sample of $g$. Generating approximated order statistics in this way for $g$ yields the approximation in Figure 2.18 of $b \in \mathrm{ifpt}(g, 0)$.

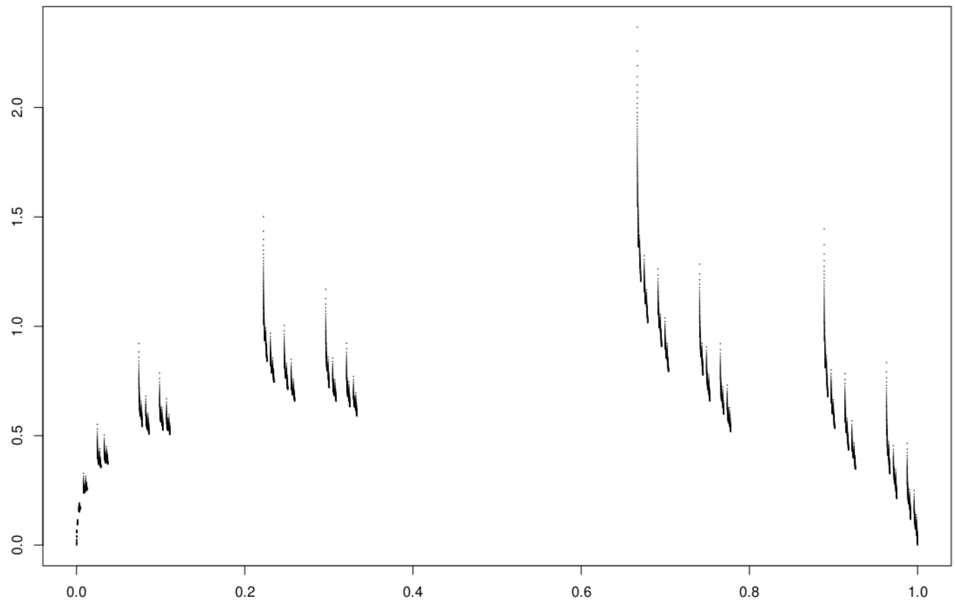


Figure 2.18: Approximated boundary for the Cantor distribution with $N = 10^7$, $m = 10^4$ and $K = 10^4$.

# Chapter 3

# The inverse first-passage time problem with soft-killing

In this chapter we will be concerned with the inverse first-passage time problem with soft-killing, which we directly formulate in terms of Markov processes. Suppose that $(X_t)_{t\geq 0}$ is a Markov process on a filtrated probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with transition semigroup $(P_t)_{t\geq 0}$ and cadlag paths. For a measurable function $b : (0, \infty) \to [-\infty, \infty]$ let

$$\tau_b^{\mathrm{sk}} := \inf \left\{ t \geq 0 : \int_0^t \mathbb{1}_{(-\infty, b(s))}(X_s) \,\mathrm{d}s > U \right\} \tag{3.1}$$

be the soft-killing time of the first-passage of $b$, where $U$ is an independent and exponentially distributed random variable. Note that with this definition we already set $\lambda = 1$ in comparison with (1.3). Given a random variable $\xi$ with values in $(0, \infty)$ we search for a function $b : (0, \infty) \to \mathbb{R}$ such that

$$\mathbb{P}_\mu \left( \tau_b^{\mathrm{sk}} > t \right) = \mathbb{P}(\xi > t) \qquad \forall t \geq 0, \tag{3.2}$$

where $\mu$ is the initial distribution of $(X_t)_{t\geq 0}$. Denote the set of solutions to the inverse-first passage time problem with soft-killing with

$$\operatorname{ifptk}(g, \mu) := \left\{ b : (0, \infty) \to \mathbb{R} \text{ measurable} : g(t) = \mathbb{P}_\mu \left( \tau_b^{\mathrm{sk}} > t \right) \forall t > 0 \right\}.$$

We will impose some requirements on the process $(X_t)_{t\geq 0}$ and its semigroup $(P_t)_{t\geq 0}$, where we understand $P_t$ as an operator on the space of (sub-)probability measures by the relation

$$P_t\mu(f) := \int_{\mathbb{R}} \mathbb{E}\left[ f(X_t) | X_0 = x \right] \mu(\mathrm{d}x)$$

for measurable and bounded functions $f : \mathbb{R} \to \mathbb{R}$. In this chapter we prove the following general statement.

**Theorem 3.0.1.** *Assume that $(X_t)_{t\geq 0}$ is such that*

*(i)* $P_t\delta_x$ *is equivalent to the Lebesgue measure for every* $x \in \mathbb{R}$ *and* $t > 0$*,*

*(ii)* $(X_t)_{t\geq 0}$ *has almost surely continuous sample paths,*

*(iii) for any tight collection* $\mathcal{S}$ *of probability measures on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *we have*

$$\sup_{\mu\in\mathcal{S}} d_{\mathrm{P}}(P_t\mu, \mu) \to 0 \qquad \text{as } t \to 0, \tag{3.3}$$

*(iv)* $P_t$ *preserves the usual stochastic order, i.e.* $\mu \preceq_{\mathrm{st}} \nu$ *implies* $P_t\mu \preceq_{\mathrm{st}} P_t\mu$*.*

*Let* $\mu \in \mathcal{P}$*. Furthermore, let* $g$ *be a survival distribution continuously differentiable on* $(0, \infty)$ *and fulfilling*

$$0 < -g'(t) < g(t) \quad \forall t > 0 \tag{3.4}$$

*Then there is exactly one continuous* $b : (0, \infty) \to \mathbb{R}$ *such that* (3.2) *is fulfilled. If additionally*

$$\begin{cases} \mu \ll \textit{Lebesgue measure, } \operatorname{supp}(\mu) \textit{ is connected and} \\ g'(0) := \lim_{h\searrow 0} \frac{g(h)-g(0)}{h} \textit{ exists with } g'(0) = g'(0+) \textit{ and } 0 < -g'(0) < 1, \end{cases} \tag{3.5}$$

*then* $b(0+)$ *exists and is the unique value such that* $\mu((-\infty, b(0+))) = -g'(0)$*.*

The uniqueness result is proven in Theorem 3.2.13. The statement of the remaining part follows from Corollary 3.2.10.

**Remark 3.0.2.** The condition (3.4) can be rewritten as

$$0 < h(t) := -\frac{\partial}{\partial t}\log(g(t)) < 1 \quad \forall t > 0$$

in terms of the hazard rate function $h$ of $g$. The soft-killing time from (3.1) essentially waits for the time, when the exponential random variable is exceeded by the amount of time spent below the boundary. Thus, the rate of the soft-killing time cannot exceed the rate of the exponential random variable, which is 1. On the other hand, the Brownian motion can spend time below the boundary in any time-interval, thus can exceed the exponential random variable at any time. This means the rate of the soft-killing time should be larger than zero at any time. Hence, intuitively, for the existence of a continuous solution, the condition of (3.4) is necessary for $g$. Later, this will be made rigorous in Lemma 3.2.1.

**Remark 3.0.3.** The conditions of Theorem 3.0.1 are fulfilled by Brownian motion, of which the transition kernel is equivalent to Lebesgue measure and the sample paths are continuous. We already know that the Gaussian convolution preserves the usual stochastic order. For point (iii) of Theorem 3.0.1 consider Lemma B.1.11, from which follows that

$$d_{\mathrm{P}}\left(P_t\mu, \mu\right) \leq \sqrt{\mathbb{E}\left[|B_t|\right]} \leq t^{1/4},$$

where $B_t \sim \mathcal{N}(0,t)$. Since this bound is independent from $\mu$ the property (iii) is fulfilled.

In the general situation naturally the question arises, which processes fulfill the properties (iii) and (iv) from Theorem 3.0.1. We give the following sufficient criteria, where the proofs are to be found as Lemma A.6.1, Lemma A.6.2 and Lemma A.6.3.

**Remark 3.0.4.** Consider again the conditions on the process in Theorem 3.0.1.

1. If $(X_t)_{t\geq 0}$ has the strong Markov property and fulfills (ii), then (iv) is fulfilled.
2. If $(X_t)_{t\geq 0}$ is locally uniformly continuous in probability, then (iii) is fulfilled.
3. If $(X_t)_{t\geq 0}$ is a $C_b$-Feller process, then (iii) is fulfilled.

**Related work in the soft-killing problem for the case of Brownian motion:** The inverse first-passage time problem with soft-killing for Brownian motion was considered the first time in [EEH14], where existence of a solution under stronger conditions on $\mu$ and $g$ was shown. In the subsequent work [EHW20] the existence and uniqueness of continuous solutions were shown under similar conditions, which were already mentioned in the introduction. A common condition of both works is the requirement on $g$ to be continuously differentiable on $[0,\infty)$ and to fulfill (3.4) for $t = 0$ as well. Furthermore, regularity assumptions on $\mu$ are required. Roughly speaking, these additional conditions at $t = 0$ are due to the use of analytical methods from partial differential equations. Namely, the approach of [EHW20] consists of a connection to the following free boundary problem, where, given a survival distribution $g$, one wants to find a function $u : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ and a boundary $b : [0,\infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) - \mathbb{1}_{(-\infty,b(t)]}(x)\cdot u(t,x) & : x\in\mathbb{R}, t>0, \\ u(0,x) = u_0(x) > 0 & : x\in\mathbb{R}, \\ b(0) = b_0, & \\ \int_{\mathbb{R}} u(t,x)\,\mathrm{d}x = g(t) & : t\geq 0, \end{cases} \tag{3.6}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ and $b_0 \in \mathbb{R}$ is given initial data. Heuristically, if $b$ is a solution to the inverse first-passage time problem with soft-killing corresponding to $g$, then the density of

$$Q_t^b(\mu) := \mathbb{P}_\mu\left(X_t \in \cdot\,, \tau_b^{\text{sk}} > t\right)$$

is a solution candidate of the free boundary problem. The work of [EHW20] shows that this is indeed the unique solution.

For diffusion processes, in [EHW20] it is conjectured under imposing conditions on the diffusion coefficients, that the inverse first-passage time problem should have also a unique continuous solution. Theorem 3.0.1 shows that this is indeed true for a large class of diffusion processes.

**Motivation of the stochastic order approach:** Before we get into the details, let us motivate our approach. If $b$ is a measurable function and $\mu$ a probability measure it follows directly from the identity

$$\{\tau_b^{\text{sk}} > t\} = \{U > \int_0^t \mathbb{1}_{(-\infty, b(s))}(X_s)\,\mathrm{d}s\}$$

that

$$Q_t^b(\mu) = \mathbb{E}_\mu\left[\mathbb{1}_{\{X_t \in \,\cdot\,\}} e^{-\int_0^t \mathbb{1}_{(-\infty,b(s))}(X_s)\,\mathrm{d}s}\right]. \tag{3.7}$$

Taking a partition $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0,t]$ we can approximate the integral appearing in $Q_t^b(\mu)$ by a Riemann-type sum, which results in approximating

$$\begin{aligned} Q_t^b(\mu) \approx Q_t^{b,n}(\mu) &:= \mathbb{E}_\mu\left[\mathbb{1}_{\{X_t \in \,\cdot\,\}} e^{-\sum_{k=1}^n (t_k - t_{k-1})\mathbb{1}_{(-\infty,b(t_k))}(X_{t_k})}\right] \\ &= \mathbb{E}_\mu\left[\mathbb{1}_{\{X_t \in \,\cdot\,\}} \prod_{k=1}^n e^{-(t_k - t_{k-1})\mathbb{1}_{(-\infty,b(t_k))}(X_{t_k})}\right], \end{aligned}$$

which, by the Markov property can be written as an inductive scheme by

$$Q_{t_k}^{b,n}(\mu)(\mathrm{d}x) = e^{-(t_k - t_{k-1})\mathbb{1}_{(-\infty,b(t_k))}(x)} P_{t_k - t_{k-1}}(Q_{t_{k-1}}^{b,n}(\mu))(\mathrm{d}x) \tag{3.8}$$

and $Q_{t_0}^{b,n}(\mu) = \mu$.

Seen from the perspective of the inverse problem the only part in this inductive scheme depending on the knowledge of $b$ is the multiplication with $e^{-(t_k - t_{k-1})\mathbb{1}_{(-\infty,b(t_k))}(x)}$ and thus depends on the values $(b(t_k))_{k=1}^n$. A possible approach is now to substitute these values by values obtained from $g$, which is originally the only information we ought to make use of. A natural choice consists of chosing the substitutes $(q_k)_{k=1}^n$ iteratively such that given the first $k-1$ values and $Q_{t_{k-1}}^{q,n}(\mu)$ we choose $q_k$ as the value such that

$$\int_{\mathbb{R}} e^{-(t_k - t_{k-1})\mathbb{1}_{(-\infty,q_k)}(x)} P_{t_k - t_{k-1}}(Q_{t_{k-1}}^{q,n}(\mu))(\mathrm{d}x) = g(t_k) = Q_{t_k}^b(\mu)(\mathbb{R}) \tag{3.9}$$

and set

$$Q^{q,n}_{t_k}(\mu)(\mathrm{d}x) = e^{-(t_k-t_{k-1})\mathbb{1}_{(-\infty,q_k)}(x)} P_{t_k-t_{k-1}}(Q^{q,n}_{t_{k-1}}(\mu))(\mathrm{d}x), \tag{3.10}$$

which is now only depending on $\mu$ and $g$. For the endpoint of the interval $[0,t]$ we obtain a value $q(t) \coloneqq q_n$. It is now very reasonable to ask, whether the function $q$ is an approximative solution to the inverse first-passage time problem for soft-killed Brownian motion, when letting the mesh of the partition going to zero. This will be approached by the use of the usual stochastic order relation and probability distances on the measures appearing from (3.10).

## 3.1 Properties of Markovian evolution and reweighting

In this section, our aim is to prepare the analysis of the discretization technique in (3.10). In order to formalize the appearing quantities in the previous motivation let us introduce the following abstract description of the reweighting mechanism from (3.10) and (3.9).

Let $\mu \in \mathcal{P}$. For $t > 0$ define for $\alpha \in (e^{-t}\mu(\mathbb{R}), \mu(\mathbb{R})]$ the reweighting operator by

$$R^t_\alpha(\mu) \coloneqq e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}(x)}\mu(\mathrm{d}x), \tag{3.11}$$

where

$$q^t_\alpha(\mu) \coloneqq \sup\{q \in \mathbb{R} : \int_{\mathbb{R}} e^{-t\mathbb{1}_{(-\infty,q)}(x)}\mu(\mathrm{d}x) \geq \alpha\}$$

with $\sup\emptyset \coloneqq -\infty$ is the reweighting threshold. Note that, if $\mu$ is non-atomic, $R^t_\alpha(\mu)$ is again non-atomic and we have $R^t_\alpha(\mu)(\mathbb{R}) = \alpha$ and

$$\begin{aligned} R^t_\alpha(\mu)((-\infty, q^t_\alpha(\mu))) &= e^{-t}\mu((-\infty, q^t_\alpha(\mu))) \\ &= \frac{1}{e^t - 1}(1 - e^{-t})\mu((-\infty, q^t_\alpha(\mu))) = \frac{\mu(\mathbb{R}) - \alpha}{e^t - 1}. \end{aligned} \tag{3.12}$$

Also, it follows directly from the definition that, if $\alpha \in (e^{-t}\beta\mu(\mathbb{R}), \beta\mu(\mathbb{R})]$ with $\beta > 0$, we have

$$R^t_\alpha(\beta\mu) = \beta R^t_{\alpha/\beta}(\mu). \tag{3.13}$$

Further, let $(P_t)_{t\geq 0}$ merely be a semi-group of Markov kernels with $P_0 = \mathrm{id}$. Additionally required assumptions will be made within the statements.

Note, that for $g$ fulfilling (3.4), by $R^t_\alpha$ it is now possible to formalize the quantity in (3.10) into

$$R^{t_k-t_{k-1}}_{g(t_k)} \circ P_{t_k-t_{k-1}} \circ \ldots \circ R^{t_1}_{g(t_1)} \circ P_{t_1}(\mu),$$

which we will analyze by first carrying out a study of the individual and joint effects of $R^t_\alpha$ and $P_t$ on the usual stochastic order and probability distances.

### 3.1.1 Usual stochastic order: Markovian evolution, reweighting

As a first observation we will state that the already mentioned reweighting operator preserves the usual stochastic ordering under appropriate conditions.

**Lemma 3.1.1.** *Let $\mu, \nu$ be sub-probabilty measures with $\mu(\mathbb{R}) = \nu(\mathbb{R})$ and $\mu \preceq_{\mathrm{st}} \nu$. Let $t > 0$ and $\alpha \in (e^{-t}\mu(\mathbb{R}), \mu(\mathbb{R}))$ and assume that $R^t_\alpha(\nu)(\mathbb{R}) = \alpha$. Then it holds that $R^t_\alpha(\mu) \preceq_{\mathrm{st}} R^t_\alpha(\nu)$.*

*Proof.* For $q \in \mathbb{R}$ is $x \mapsto e^{-t\mathbb{1}_{(-\infty,q)}(x)}$ a non-decreasing function. Thus it holds

$$\int_{\mathbb{R}} e^{-t\mathbb{1}_{(-\infty,q)}(x)}\,\mathrm{d}\mu(x) \le \int_{\mathbb{R}} e^{-t\mathbb{1}_{(-\infty,q)}(x)}\,\mathrm{d}\nu(x).$$

Therefore it follows $q^t_\alpha(\nu) \ge q^t_\alpha(\mu)$ and thus $e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}(x)} \ge e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\nu))}(x)}$. Now, observe that for $c < q^t_\alpha(\nu)$ it holds

$$\begin{aligned} R^t_\alpha(\mu)((-\infty,c]) &= \int_{(-\infty,c]} e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}(x)}\,\mathrm{d}\mu(x) \ge \int_{(-\infty,c]} e^{-t}\,\mathrm{d}\mu(x)) \\ &= e^{-t}\mu((-\infty,c]) \ge e^{-t}\nu((-\infty,c]) = \int_{(-\infty,c]} e^{-t}\,\mathrm{d}\nu(x) \\ &= \int_{(-\infty,c]} e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\nu))}(x)}\,\mathrm{d}\nu(x) = R^t_\alpha(\nu)((-\infty,c]). \end{aligned}$$

On the other hand, for $c \ge q^t_\alpha(\nu)$ we find

$$\begin{aligned} R^t_\alpha(\mu)((c,\infty)) &= \int_{(c,\infty)} e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}(x)}\,\mathrm{d}\mu(x) = \int_{(c,\infty)} \mathrm{d}\mu(x) = \mu((c,\infty)) \\ &\le \nu((c,\infty)) = \int_{(c,\infty)} e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\nu)(x))}(x)}\,\mathrm{d}\nu(x) = R^t_\alpha(\nu)((c,\infty)), \end{aligned}$$

which means that

$$\begin{aligned} R^t_\alpha(\mu)((-\infty,c]) &= R^t_\alpha(\mu) - R^t_\alpha(\mu)((c,\infty)) \\ &\ge \alpha - R^t_\alpha(\nu)((c,\infty)) = R^t_\alpha(\nu)((-\infty,c]). \end{aligned}$$

All in all, this shows $R^t_\alpha(\mu) \preceq_{\mathrm{st}} R^t_\alpha(\nu)$. □

**Remark 3.1.2.** We want to emphasize that for $\alpha \ge \beta$, the stochastic inequality $\alpha^{-1}R^t_\alpha(\mu) \preceq_{\mathrm{st}} \beta^{-1}R^t_\beta(\mu)$ is not true, in contrast to the hard killing case. For instance, let $\mu$ be equivalent to the Lebesgue measure and $\alpha > \beta$ in the situation of Lemma 3.1.1. Then for $c < q^t_\alpha(\mu) \le q^t_\beta(\mu)$ it holds

$$\begin{aligned} \alpha^{-1}R^t_\alpha(\mu)((-\infty,c]) &= \frac{e^{-t}}{\alpha}\mu((-\infty,c]) \\ &< \frac{e^{-t}}{\beta}\mu((-\infty,c]) = \beta^{-1}R^t_\beta(\mu)((-\infty,c]). \end{aligned}$$

Since the mass below the reweighting threshold is not truncated but reweighted with the same weighting factor, the mass below the lower threshold gets more weight in the normalized measure.

For $t > 0$ recall $P_t\mu := \int_{\mathbb{R}} \mathbb{P}_x(X_t \in \cdot\,)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}} P_t\delta_x\,\mathrm{d}\mu(x)$.

**Lemma 3.1.3.** *Let $\mu \in \mathcal{P}$, $t, s > 0$ and $\alpha \in (e^{-t}, 1]$. Assume that $\alpha = R^t_\alpha(P_s\mu)(\mathbb{R})$. Then we have*

$$P_s R^t_\alpha(\mu) \preceq_{\mathrm{st}} R^t_\alpha(P_s\mu).$$

*Proof.* Abbreviate $q := q^t_\alpha(P_s\mu)$. First let $c \geq q$. It holds

$$\begin{aligned}
&R^t_\alpha(P_s\mu)((c,\infty)) = \int_{(c,\infty)} e^{-t\mathbb{1}_{(-\infty,q)}(x)} P_s\mu(\mathrm{d}x) \\
&= P_s\mu((c,\infty)) = \int_{\mathbb{R}} P_s\delta_x((c,\infty))\mu(\mathrm{d}x) \\
&\geq \int_{\mathbb{R}} P_s\delta_x((c,\infty)) e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))(x)}}\mu(\mathrm{d}x) = P_sR^t_\alpha(\mu)((c,\infty)).
\end{aligned}$$

Due to the fact that $P_sR^t_\alpha(\mu)(\mathbb{R}) \geq \alpha = R^t_\alpha(P_s\mu)(\mathbb{R})$ it holds that

$$\begin{aligned}
P_sR^t_\alpha(\mu)((-\infty,c]) &= P_sR^t_\alpha(\mu)(\mathbb{R}) - P_sR^t_\alpha(\mu)((c,\infty)) \\
&\geq \alpha - R^t_\alpha(P_s\mu)((c,\infty)) = R^t_\alpha(P_s\mu)((-\infty,c]).
\end{aligned}$$

Now let $c < q$. We have

$$\begin{aligned}
R^t_\alpha(P_s\mu)((-\infty,c]) &= \int_{(-\infty,c]} e^{-t\mathbb{1}_{(-\infty,q)}(x)} P_s\mu(\mathrm{d}x) = e^{-t}P_t\mu((-\infty,c]) \\
&= \int_{\mathbb{R}} e^{-t}P_s\delta_x((-\infty,c])\mu(\mathrm{d}x) \\
&\leq \int_{\mathbb{R}} e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}(x)} P_s\delta_x((-\infty,c])\mu(\mathrm{d}x) \\
&= P_sR^t_\alpha(\mu)((-\infty,c])
\end{aligned}$$

which finishes the proof. □

**Lemma 3.1.4.** *Let $t, s > 0$, $\mu$ a non-atomic probability measure and $\beta \in (e^{-t}\mu(\mathbb{R}), \mu(\mathbb{R}))$, $\alpha \in (e^{-s}\beta, \beta)$. Then it holds*

$$R^s_\alpha(R^t_\beta(\mu)) \preceq_{\mathrm{st}} R^{s+t}_\alpha(\mu).$$

*Proof.* At first note that

$$\begin{aligned}
R^s_\alpha(R^t_\beta(\mu)) &= e^{-s\mathbb{1}_{(-\infty,q^s_\alpha(R^t_\beta(\mu)))}(x)} R^t_\beta(\mu)(\mathrm{d}x) \\
&= e^{-s\mathbb{1}_{(-\infty,q^s_\alpha(R^t_\beta(\mu)))}(x) - t\mathbb{1}_{(-\infty,q^t_\beta(\mu))}(x)}\mu(\mathrm{d}x).
\end{aligned}$$

Let $c < q_\alpha^{s+t}(\mu)$. Then

$$\begin{aligned}
&R_\alpha^s(R_\beta^t(\mu))((-\infty,c]) \\
&= \int_{(-\infty,c]} e^{-s\mathbb{1}_{(-\infty,q_\alpha^s(R_\beta^t(\mu)))}(x)-t\mathbb{1}_{(-\infty,q_\beta^t(\mu))}(x)}\mu(\mathrm{d}x) \\
&\geq \int_{(-\infty,c]} e^{-(s+t)}\mu(\mathrm{d}x) = \int_{(-\infty,c]} e^{-(s+t)\mathbb{1}_{(-\infty,q_\alpha^{s+t}(\mu))}(x)}\mu(\mathrm{d}x) \\
&= R_\alpha^{s+t}(\mu)((-\infty,c]).
\end{aligned}$$

On the other hand, since $R_\alpha^s(R_\beta^t(\mu))(\mathbb{R}) = \alpha = R_\alpha^{s+t}(\mu)(\mathbb{R})$, we have for $c \geq q_\alpha^{s+t}(\mu)$, that

$$\begin{aligned}
&R_\alpha^{s+t}(\mu)((-\infty,c]) = \alpha - R_\alpha^{s+t}(\mu)((c,\infty)) \\
&= \alpha - \mu((c,\infty)) \\
&\leq \alpha - \int_{(c,\infty)} e^{-s\mathbb{1}_{(-\infty,q_\alpha^s(R_\beta^t(\mu)))}(x)-t\mathbb{1}_{(-\infty,q_\beta^t(\mu))}(x)}\mu(\mathrm{d}x) \\
&= R_\alpha^s(R_\beta^t(\mu))((-\infty,c]),
\end{aligned}$$

which shows the desired statement. □

**Corollary 3.1.5.** *Let $t,s,u,v > 0$, $\mu \in \mathcal{P}$ and $\beta \in (e^{-t}\mu(\mathbb{R}),\mu(\mathbb{R}))$, $\alpha \in (e^{-s}\beta,\beta)$. Assume that $P_v\mu$ is non-atomic. Then it holds*

$$R_\alpha^s \circ P_u \circ R_\beta^t \circ P_v(\mu) \preceq_{\mathrm{st}} R_\alpha^{s+t} \circ P_{u+v}(\mu).$$

*Proof.* Note that by Lemma 3.1.1 $R_\alpha^s$ preserves the order in the following situation, such that by using Lemma 3.1.3, Lemma 3.1.4 we can deduce

$$\begin{aligned}
R_\alpha^s \circ P_u \circ R_\beta^t \circ P_v(\mu) &\preceq_{\mathrm{st}} R_\alpha^s \circ R_\beta^t \circ P_u \circ P_v(\mu) \\
&\preceq_{\mathrm{st}} R_\alpha^{s+t} \circ P_u \circ P_v(\mu) = R_\alpha^{s+t} \circ P_{u+v}(\mu),
\end{aligned}$$

which finishes the proof. □

### 3.1.2 Total variation distance: Markovian evolution, reweighting

**Definition 3.1.6.** *We define the* total variation distance *between two measures $\mu,\nu \in \mathcal{P}$ by*

$$d_{\mathrm{TV}}(\mu,\nu) := \sup_{A\in\mathcal{B}(\mathbb{R})} |\mu(A)-\nu(A)|. \tag{3.14}$$

If $\mu$ and $\nu$ are absolutely continuous with respect to the Lebesgue measure with densities $f$ and $g$, respectivly, we see that by taking $A = \{f \geq g\}$

the supremum in the definition of the total variation is attained. Since from $\mu(\mathbb{R}) = \nu(\mathbb{R})$ follows that $(\mu - \nu)(\{f \geq g\}) = -(\mu - \nu)(\{f < g\})$ we have

$$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \int_{\mathbb{R}} |f(x) - g(x)| \,\mathrm{d}x. \tag{3.15}$$

This means, the total variation distance of $\mu$ and $\nu$ coincides with the half of the $L^1$-distance of their densities. Furthermore, the identity $|f-g| = f+g-2(f \wedge g)$ implies

$$d_{\mathrm{TV}}(\mu, \nu) = \left(1 - \int_{\mathbb{R}} (f \wedge g)(x) \,\mathrm{d}x\right). \tag{3.16}$$

Another representation is

$$d_{\mathrm{TV}}(\mu, \nu) = \inf\{\mathbb{P}(X \neq Y) : X \sim \mu, Y \sim \nu\}, \tag{3.17}$$

which is a consequence of Theorem 1.27 from [Vil03].

It is well-known that Markov kernels are contractive for the total variation distance. For completeness we give a short proof by the independent-then-forever coupling, e.g. see [Dou+18].

**Lemma 3.1.7.** *Let $\mu, \nu \in \mathcal{P}$ and $t > 0$. Then $d_{\mathrm{TV}}(P_t\mu, P_t\nu) \leq d_{\mathrm{TV}}(\mu, \nu)$.*

*Proof.* Let $X \sim \mu$ and $Y \sim \nu$. Let the distribution $\pi$ on $\mathbb{R}^2$ given by

$$\pi(A \times B) = \mathbb{E}\left[\mathbb{1}_{\{X=Y\}} P_t\delta_X(A \cap B) + \mathbb{1}_{\{X \neq Y\}} P_t\delta_X(A) P_t\delta_Y(B)\right].$$

Let $(X_t, Y_t) \sim \pi$. It holds $X_t \sim P_t\mu$ and $Y_t \sim P_t\nu$. Note that $X_t \neq Y_t$ implies $X \neq Y$. Hence

$$d_{\mathrm{TV}}(P_t\mu, P_t\nu) \leq \mathbb{P}(X_t \neq Y_t) \leq \mathbb{P}(X \neq Y).$$

Now, taking the infimum over all possible couplings of $X$ and $Y$ yields the statement. □

At next we study the effect of the reweighting on the total variation distance.

**Lemma 3.1.8.** *Let $\mu, \nu \in \mathcal{P}$ be absolutely continuous with respect to the Lebesgue measure and $t > 0$. Then for all $\alpha, \beta \in [e^{-t}, 1]$ it holds*

$$d_{\mathrm{TV}}\left(\alpha^{-1} R_\alpha^t(\mu), \beta^{-1} R_\beta^t(\nu)\right) \leq (\alpha \vee \beta)^{-1} d_{\mathrm{TV}}(\mu, \nu) + \left(1 - \frac{\alpha \wedge \beta}{\alpha \vee \beta}\right).$$

*Proof.* As first step let $\alpha = \beta$. Let $\mu = f\,\mathrm{d}x$ and $\nu = g\,\mathrm{d}x$ and without loss of generality $q^t_\alpha(\mu) \le q^t_\alpha(\nu)$. We have that the densities of $\alpha^{-1}R^t_\alpha(\mu)$ and $\alpha^{-1}R^t_\alpha(\nu)$ are given by $\alpha^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}}f$ and $\alpha^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\nu))}}g$. Thus we have

$$
\begin{aligned}
&\min(\alpha^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))(x)}}f(x), \alpha^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\nu))(x)}}g(x))\\
&= \frac{1}{\alpha}\begin{cases} \min(f(x), g(x)) & : x \ge q^t_\alpha(\nu),\\ \min(f(x), e^{-t}g(x)) & : q^t_\alpha(\mu) \le x < q^t_\alpha(\nu),\\ e^{-t}\min(f(x), g(x)) & : x < q^t_\alpha(\mu). \end{cases}
\end{aligned}
$$

Note that by $\min(f(x), g(x)) - \min(f(x), e^{-t}g(x)) \le g(x) - e^{-t}g(x)$ we have

$$
\begin{aligned}
&\int_{q^t_\alpha(\mu)}^{q^t_\alpha(\nu)} (\min(f(x), g(x)) - \min(f(x), e^{-t}g(x)))\,\mathrm{d}x\\
&+ (1-e^{-t})\int_{-\infty}^{q^t_\alpha(\mu)} \min(f(x), g(x))\,\mathrm{d}x\\
&\le (1-e^{-t})\int_{q^t_\alpha(\mu)}^{q^t_\alpha(\nu)} g(x)\,\mathrm{d}x + (1-e^{-t})\int_{-\infty}^{q^t_\alpha(\mu)} g(x)\,\mathrm{d}x\\
&= (1-e^{-t})\nu((-\infty, q^t_\alpha(\nu))) = 1-\alpha.
\end{aligned}
$$

Thus, by the respresentation of the total variation distance from (3.16) we obtain

$$
\begin{aligned}
&d_{\mathrm{TV}}\left(\alpha^{-1}R^t_\alpha(\mu), \alpha^{-1}R^t_\alpha(\nu)\right)\\
&= \frac{1}{\alpha}\Bigg(\alpha - \int_{q^t_\alpha(\nu)}^{\infty} \min(f(x), g(x))\,\mathrm{d}x - \int_{q^t_\alpha(\mu)}^{q^t_\alpha(\nu)} \min(f(x), e^{-t}g(x))\,\mathrm{d}x\\
&\qquad\qquad - \int_{-\infty}^{q^t_\alpha(\mu)} e^{-t}\min(f(x), g(x))\,\mathrm{d}x\Bigg)\\
&= \frac{1}{\alpha}\Bigg(1 - \int_{\mathbb{R}} \min(f(x), g(x))\,\mathrm{d}x - (1-\alpha)\\
&\qquad + \int_{q^t_\alpha(\mu)}^{q^t_\alpha(\nu)} (\min(f(x), g(x)) - \min(f(x), e^{-t}g(x)))\,\mathrm{d}x\\
&\qquad\qquad + (1-e^{-t})\int_{-\infty}^{q^t_\alpha(\mu)} \min(f(x), g(x))\,\mathrm{d}x\Bigg)\\
&\le \frac{1}{\alpha}\left(1 - \int_{\mathbb{R}} \min(f(x), g(x))\,\mathrm{d}x\right)\\
&= \frac{1}{\alpha} d_{\mathrm{TV}}(\mu, \nu)
\end{aligned}
$$

which shows the desired inequality for the first step.
As second step let $\alpha, \beta \in (e^{-t}, 1]$ and $\mu = \nu = f\,\mathrm{d}x$. Then the densities of

$\alpha^{-1}R^t_\alpha(\mu)$ and $\beta^{-1}R^t_\beta(\mu)$ are given by $\alpha^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}}f$ and $\beta^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\beta(\mu))}}f$. Without loss of generality assume that $\alpha > \beta$. In particular, this implies $q_\alpha(\mu) < q_\beta(\mu)$. By the representation in (3.15) we have

$$\begin{aligned}
&2d_{\mathrm{TV}}\left(\alpha^{-1}R^t_\alpha(\mu), \beta^{-1}R^t_\beta(\mu)\right)\\
&= \int_{\mathbb{R}} \left|\alpha^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\alpha(\mu))}(x)}f(x) - \beta^{-1}e^{-t\mathbb{1}_{(-\infty,q^t_\beta(\mu))}(x)}f(x)\right| \mathrm{d}x\\
&= \left|\frac{e^{-t}}{\alpha} - \frac{e^{-t}}{\beta}\right| \int_{-\infty}^{q^t_\alpha(\mu)} f(x)\,\mathrm{d}x + \left|\frac{1}{\alpha} - \frac{e^{-t}}{\beta}\right| \int_{q^t_\alpha(\mu)}^{q^t_\beta(\mu)} f(x)\,\mathrm{d}x\\
&\quad + \left|\frac{1}{\alpha} - \frac{1}{\beta}\right| \int_{q^t_\beta(\mu)}^{\infty} f(x)\,\mathrm{d}x\\
&= \left(\frac{1}{\beta} - \frac{1}{\alpha}\right) e^{-t}\mu((-\infty, q^t_\alpha(\mu)))\\
&\quad + \left(\frac{1}{\alpha} - \frac{e^{-t}}{\beta}\right)(\mu((-\infty, q^t_\beta(\mu))) - \mu((-\infty, q^t_\alpha(\mu))))\\
&\quad + \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)(1 - \mu((-\infty, q^t_\beta(\mu))))\\
&= \frac{\alpha-\beta}{\alpha\beta}e^{-t}\frac{1-\alpha}{1-e^{-t}} + \frac{\beta-\alpha e^{-t}}{\alpha\beta}\left(\frac{1-\beta}{1-e^{-t}} - \frac{1-\alpha}{1-e^{-t}}\right)\\
&\quad + \frac{\alpha-\beta}{\alpha\beta}\left(1 - \frac{1-\beta}{1-e^{-t}}\right)\\
&= \frac{\alpha-\beta}{\alpha\beta}\frac{e^{-t}-\alpha e^{-t}}{1-e^{-t}} + \frac{\beta-\alpha e^{-t}}{\alpha\beta}\frac{\alpha-\beta}{1-e^{-t}} + \frac{\alpha-\beta}{\alpha\beta}\frac{\beta-e^{-t}}{1-e^{-t}}\\
&= \frac{\alpha-\beta}{\alpha\beta}\frac{1}{1-e^{-t}}\left(e^{-t} - \alpha e^{-t} + \beta - \alpha e^{-t} + \beta - e^{-t}\right)\\
&= \frac{\alpha-\beta}{\alpha\beta}2\frac{\beta-\alpha e^{-t}}{1-e^{-t}} = 2\left(1 - \frac{\beta}{\alpha}\right)\frac{1-\frac{\alpha}{\beta}e^{-t}}{1-e^{-t}}\\
&\leq 2\left(1 - \frac{\beta}{\alpha}\right).
\end{aligned}$$

Combining the results of the first and the second step and employing the triangle inequality of the total variation distance finishes the proof. □

**Remark 3.1.9.** An analogous statement of Lemma 3.1.8 for the Wasserstein distance in the case that $\beta = \alpha$ is to be found in the appendix as Lemma A.4.4.

**Corollary 3.1.10.** *Assume that $P_s\delta_x$ is absolutely continuous with respect to Lebesgue measure for every $x \in \mathbb{R}$. Let $\mu, \nu \in \mathcal{P}$ and $t, s > 0$. Then for all $\alpha \in [e^{-t}, 1]$ it holds*

$$d_{\mathrm{TV}}\left(\alpha^{-1}R^t_\alpha \circ P_s(\mu), P_s(\nu)\right) \leq d_{\mathrm{TV}}(\mu, \nu) + 1 - \alpha.$$

*Proof.* By using that $P_s(\nu) = R_1^t(P_s(\nu))$ is absolutely continuous we have by Lemma 3.1.8 that

$$d_{\mathrm{TV}}\left(\alpha^{-1} R_\alpha^t \circ P_s(\mu), P_s(\nu)\right) \leq d_{\mathrm{TV}}\left(P_s(\mu), P_s(\nu)\right) + 1 - \alpha$$

and by Lemma 3.1.7 that

$$d_{\mathrm{TV}}\left(P_s(\mu), P_s(\nu)\right) + 1 - \alpha \leq d_{\mathrm{TV}}\left(\mu, \nu\right) + 1 - \alpha,$$

which finishes the proof. □

## 3.2 Existence and uniqueness of continuous solutions

This section is devoted to prove Theorem 3.0.1. Therefore, throughout this section, we will assume that the Markov process $(X_t)_{t\geq 0}$ admits the properties required by Theorem 3.0.1, this means that

(i) $P_t\delta_x$ is equivalent to the Lebesgue measure for every $x \in \mathbb{R}$ and $t > 0$,

(ii) $(X_t)_{t\geq 0}$ has almost surely continuous sample pathes,

(iii) for any tight collection $\mathcal{S}$ of probability measures we have

$$\sup_{\mu\in\mathcal{S}} d_{\mathrm{P}}(P_t\mu, \mu) \to 0 \quad \text{as } t \to 0,$$

(iv) $P_t$ preserves the usual stochastic order, i.e. $\mu \preceq_{\mathrm{st}} \nu$ implies $P_t\mu \preceq_{\mathrm{st}} P_t\mu$.

We will begin with an analysis of the properties of the marginal measure $Q^b_t(\mu)$ from (3.7) and its approximation. Subsequently, we arrive at Theorem 3.0.1 by comparing $Q^b_t(\mu)$ with its approximation in the usual stochastic order, by proving convergence properties of the approximation and by extracting a candidate of solution to the inverse first-passage time problem with soft-killing from a suitable limit.

### 3.2.1 Auxiliary results: boundary functions, survival distribution and marginal distributions

For $b : (0, \infty) \to \mathbb{R}$ measurable and a sub-probability measure $\mu$ we define $Q^b_{t,s}(\mu)$ by

$$Q^b_{t,s}(\mu)(f) := Q_{t,s}(\mu)(f) := \mathbb{E}_\mu\left[f(X_{t-s})e^{-\int_s^t \mathbb{1}_{(-\infty,b(r))}(X_{r-s})\,\mathrm{d}r}\right] \tag{3.18}$$

for measurable and bounded functions $f : \mathbb{R} \to \mathbb{R}$. Furthermore denote the time the Markov process spends below the boundary function with

$$\Gamma^b_t := \Gamma_t := \int_0^t \mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r.$$

By $\{\tau^{\mathrm{sk}}_b > t\} = \{U > \Gamma_t\}$, we observe that

$$\mathbb{E}_\mu\left[f(X_t)\mathbb{1}_{\{\tau^{\mathrm{sk}}_b>t\}}\right] = \mathbb{E}_\mu\left[f(X_t)e^{-\Gamma_t}\right] = Q_{t,0}(\mu)(f)$$

and thus we have

$$Q^b_t(\mu) := Q_t(\mu) := Q_{t,0}(\mu) = \mathbb{P}_\mu\left(X_t \in \cdot, \tau^{\mathrm{sk}}_b > t\right).$$

The next statement shows that under appropriate conditions on $b$, we can recover the boundary function $b$, if we only know the pair consisting of $g$ and $Q^b_t(\mu)$. The statement is essentially a reformulation of Lemma 4.2 in [EHW20].

**Lemma 3.2.1.** *Assume that* $b : (0, \infty) \to \mathbb{R}$ *is continuous,* $\mu \in \mathcal{P}$ *and* $g$ *a survival distribution, such that* $b \in \mathrm{ifptk}(g, \mu)$*. Then* $g$ *is continuously differentiable on* $(0, \infty)$ *and it holds*

$$\frac{\partial}{\partial t} g(t) = -Q_t^b(\mu)((-\infty, b(t)))$$

*for all* $t > 0$*. In particular* $g$ *fulfills* (3.4)*. If* $b(0+)$ *exists and* $\mu$ *has no atoms,* $g$ *has a right-derivative* $g'(0)$ *in* $0$ *and* $g'(0+) = g'(0) = -\mu((-\infty, b(0+)))$*.*

*Proof.* For fixed $t > 0$, since then $X_t \neq b(t)$ almost surely, we have by continuity of the boundary function and the paths that almost surely

$$\int_t^{t+h} \mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r = h\mathbb{1}_{(-\infty,b(t))}(X_t)$$

for $h > 0$ small enough. Therefore, we can deduce by the dominated convergence theorem with the bound

$$\begin{aligned} |h^{-1}\left(e^{-\int_t^{t+h}\mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r} - 1\right)| &\leq h^{-1}\left(1 - e^{-h}\right) \\ &\leq h^{-1}\left(1 - (1-h)\right) = 1 \end{aligned}$$

for $h > 0$, that

$$\begin{aligned} &\lim_{h\searrow 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\searrow 0} \mathbb{E}_\mu\left[h^{-1}\left(e^{-\int_t^{t+h}\mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r} - 1\right)e^{-\Gamma_t}\right] \\ &= \mathbb{E}_\mu\left[\lim_{h\searrow 0} h^{-1}\left(e^{-\int_t^{t+h}\mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r} - 1\right)e^{-\Gamma_t}\right] \\ &= \mathbb{E}_\mu\left[\lim_{h\searrow 0} h^{-1}(e^{-h\mathbb{1}_{(-\infty,b(t))}(X_t)} - 1)e^{-\Gamma_t}\right] = \mathbb{E}_\mu\left[-\mathbb{1}_{(-\infty,b(t))}(X_t)e^{-\Gamma_t}\right] \\ &= -Q_t(\mu)((-\infty, b(t))). \end{aligned}$$

For the left-derivative fix $t > 0$. We have again that, since then $X_t \neq b(t)$ almost surely, we have by continuity of the boundary function and the paths that almost surely

$$\int_{t-h}^{t} \mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r = h\mathbb{1}_{(-\infty,b(t))}(X_t)$$

for $h > 0$ small enough. With the bound

$$\begin{aligned} &|h^{-1}\left(e^{\int_{t-h}^{t}\mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r} - 1\right)| \\ &\leq h^{-1}\left(e^h - 1\right) \leq h^{-1}\left(\frac{1}{1-h} - 1\right) = \frac{1}{1-h} \leq 2 \end{aligned}$$

for $h \in (0, 1/2)$, we obtain by the dominated convergence theorem that

$$\lim_{h\searrow 0} \frac{g(t-h)-g(t)}{h} = \lim_{h\searrow 0} \mathbb{E}_\mu \left[ h^{-1} \left( e^{\int_{t-h}^t \mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r} - 1 \right) e^{-\Gamma_t} \right]$$
$$= \mathbb{E}_\mu \left[ \lim_{h\searrow 0} h^{-1} \left( e^{\int_{t-h}^t \mathbb{1}_{(-\infty,b(r))}(X_r)\,\mathrm{d}r} - 1 \right) e^{-\Gamma_t} \right]$$
$$= \mathbb{E}_\mu \left[ \lim_{h\searrow 0} h^{-1} (e^{h\mathbb{1}_{(-\infty,b(t))}(X_t)} - 1) e^{-\Gamma_t} \right] = \mathbb{E}_\mu \left[ \mathbb{1}_{(-\infty,b(t))}(X_t) e^{-\Gamma_t} \right]$$
$$= Q_t(\mu)((-\infty, b(t))).$$

By the continuity of the paths and the boundary function and the fact that $X_t \neq b(t)$ almost surely for fixed $t > 0$, it follows that

$$g'(t) = -\mathbb{E}_\mu \left[ \mathbb{1}_{(-\infty,b(t))}(X_t) e^{-\Gamma_t} \right]$$

is continuous in $t > 0$. With the additional conditions at $t = 0$, the arguments above are true for $t = 0$ and the convention $b(0) := b(0+)$. This finishes the proof. □

We will now begin with our study of the discrete approximations of the time marginal 3.7.

Let $g$ be a survival distribution differentiable on $(0, \infty)$ and fulfilling (3.4). For a probability measure $\mu$ and $n \in \mathbb{N}$ let $\delta := \delta^{(n)}$ be a dyadic sequence, i.e. $\delta^{(n)} = 2\delta^{(n+1)}$, and define

$$S_k^{g,+,n}(\mu) := R^{\delta^{(n)}}_{g(k\delta^{(n)})} \circ P_{\delta^{(n)}} \circ \ldots \circ R^{\delta^{(n)}}_{g(\delta^{(n)})} \circ P_{\delta^{(n)}}(\mu)$$

for $k \in \mathbb{N}$, where $R^t_\alpha$ is the reweighting operation defined in (3.11). Further set $q_k^n := q^\delta_{g(k\delta)}(P_\delta S^{g,+,n}_{k-1}(\mu))$ and $D := \cup_{n\in\mathbb{N}} D_n$ with $D_n := \{k\delta^{(n)} : k \in \mathbb{N}_0\}$. As abbreviation we set $\lfloor t \rfloor_n := \lfloor t/\delta^{(n)} \rfloor \delta^{(n)}$.

With respect to the survival distribution $g$, for $t \geq 0$ and $n \in \mathbb{N}$ we define

$$Q_t^{+,n}(\mu) := R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n} \circ S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor}(\mu)$$
$$= R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n} \circ R^{\delta^{(n)}}_{g(\lfloor t\rfloor_n)} \circ P_{\delta^{(n)}} \circ \ldots \circ R^{\delta^{(n)}}_{g(\delta^{(n)})} \circ P_{\delta^{(n)}}(\mu).$$

Further we define

$$q^{(n)}(t) := \begin{cases} q^n_{\lfloor t/\delta^{(n)}\rfloor} & : t \in D_n \setminus \{0\}, \\ q^{t-\lfloor t\rfloor_n}_{g(t)}(P_{t-\lfloor t\rfloor_n} S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor}(\mu)) & : t \notin D_n. \end{cases} \tag{3.19}$$

Additionally, if the right-derivative $g'(0)$ exists we define $q^{(n)}(0) := \inf\{q \in \mathbb{R} : \mu((-\infty, q)) \geq -g'(0)\}$.

We want to emphasize that $Q_t^{+,n}(\mu)$ is a sort of discrete approximation for $Q_t\mu$ as it was in the hard killing case. We have $Q_t(\mu)(\mathbb{R}) = g(t) = Q_t^{+,n}(\mu)(\mathbb{R})$ and have additionally the following lemma, which gives us the interpretation that $q^{(n)}(t)$ is a sort of approximation to a possible solution $b$.

**Lemma 3.2.2.** *Let $g$ be a survival distribution continuously differentiable on $(0,\infty)$ and fulfilling (3.4). Let $0<\eta<T$. As $n\to\infty$ we have*

$$Q_t^{+,n}(\mu)((-\infty,q^{(n)}(t)))\to -g'(t)$$

*uniformly in $t\in[\eta,T]$. If $\mu$ has no atoms and $g$ has right-derivative in 0 with $g'(0+)=g'(0)$ the statement holds for $\eta=0$.*

*Proof.* With the additional conditions the following arguments also apply for the case $\eta=0$. In view of (3.12) we have for $t>0$ that

$$Q_t^{+,n}(\mu)((-\infty,q^{(n)}(t)))=\begin{cases}\frac{S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor-1}(\mu)(\mathbb{R})-g(t)}{e^{\delta^{(n)}}-1}=\frac{g(t-\delta^{(n)})-g(t)}{e^{\delta^{(n)}}-1} & : t\in D_n,\\ \frac{S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor}(\mu)(\mathbb{R})-g(t)}{e^{t-\lfloor t\rfloor_n}-1}=\frac{g(\lfloor t\rfloor_n)-g(t)}{e^{t-\lfloor t\rfloor_n}-1} & : t\notin D_n.\end{cases}$$

Since $g$ fulfilles (3.4) we have that $-g'(t)\le g(t)$ for $t\ge\eta$. Furthermore, $g'$ is uniformly continuous on $[\frac{\eta}{2},T]$. If $\eta>0$ let $n$ large enough such that $\delta^{(n)}<\frac{\eta}{2}$. Further, let $n$ be large enough such that for $\varepsilon>0$ we have $|u-r|<\delta^{(n)}$, $u,r\in[\frac{\eta}{2},T]$, implies $|g'(u)-g'(r)|\le\varepsilon$. We can deduce by the mean value theorem and the inequality $|1-h/(e^h-1)|\le h$ that

$$\begin{aligned}&|Q_t^{+,n}(\mu)((-\infty,q^{(n)}(t)))-(-g'(t))|\\ &=\begin{cases}|g'(t)-g'(\xi)\frac{\delta^{(n)}}{e^{\delta^{(n)}}-1}| & : t\in D_n \text{ with } \xi\in[t-\delta^{(n)},t],\\ |g'(t)-g'(\xi)\frac{t-\lfloor t\rfloor_n}{e^{t-\lfloor t\rfloor_n}-1}| & : t\notin D_n \text{ with } \xi\in[\lfloor t\rfloor_n,t]\end{cases}\\ &\le\begin{cases}|g'(t)-g'(\xi)|+|g'(\xi)\left(1-\frac{\delta^{(n)}}{e^{\delta^{(n)}}-1}\right)| & : t\in D_n \text{ with } \xi\in[t-\delta^{(n)},t],\\ |g'(t)-g'(\xi)|+|g'(\xi)\left(1-\frac{t-\lfloor t\rfloor_n}{e^{t-\lfloor t\rfloor_n}-1}\right)| & : t\notin D_n \text{ with } \xi\in[\lfloor t\rfloor_n,t]\end{cases}\\ &\le\varepsilon+g(T)\delta^{(n)}\end{aligned}$$

for $t\in[\eta,T]$. Letting $n\to\infty$ yields the statement, since $\varepsilon$ can be chosen arbitrarily small. ☐

We will make use of the following alternative representation for $Q_t^{+,n}(\mu)$.

**Lemma 3.2.3.** *For $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$ we have that*

$$\begin{aligned}&Q_t^{+,n}(\mu)\\ &=\mathbb{E}_\mu\left[\mathbb{1}_{\{X_t\in\cdot\}}e^{-(t-\lfloor t\rfloor_n)\mathbb{1}_{(-\infty,q^{(n)}(t))}(X_t)}e^{-\sum_{\ell=1}^{\lfloor t/\delta^{(n)}\rfloor}\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})}\right]\\ &=g(\lfloor t\rfloor_n)R^{t-\lfloor t\rfloor_n}_{\frac{g(t)}{g(\lfloor t\rfloor_n)}}\circ P_{t-\lfloor t\rfloor_n}\\ &\quad\circ\frac{1}{\alpha^{(n)}_{\lfloor t/\delta^{(n)}\rfloor}}R^{\delta^{(n)}}_{\alpha^{(n)}_{\lfloor t/\delta^{(n)}\rfloor}}\circ P_{\delta^{(n)}}\circ\ldots\circ\frac{1}{\alpha^{(n)}_1}R^{\delta^{(n)}}_{\alpha^{(n)}_1}\circ P_{\delta^{(n)}}(\mu),\end{aligned}$$

*where $\alpha_k^{(n)}=g(k\delta)/g((k-1)\delta)$.*

*Proof.* We abbreviate $q_\ell^n \coloneqq q^{(n)}(\ell\delta^{(n)})$. We first claim that

$$S_k^{g,+,n}(\mu) = \mathbb{E}_\mu\left[\mathbb{1}_{\{X_{k\delta}\in\cdot\,\}} e^{-\sum_{\ell=1}^k \delta \mathbb{1}_{(-\infty,q_\ell^n)}(X_{\ell\delta})}\right]$$
$$= g(k\delta)\frac{1}{\alpha_k^{(n)}} R^{\delta^{(n)}}_{\alpha_k^{(n)}} \circ P_{\delta^{(n)}} \circ \ldots \circ \frac{1}{\alpha_1^{(n)}} R^{\delta^{(n)}}_{\alpha_1^{(n)}} \circ P_{\delta^{(n)}}(\mu).$$

The statement is clear for $k = 0$. Assume the statement is true for $k$. Regarding the first equality, we have by the Markov property that

$$P_\delta S_k^{g,+,n}(\mu) = \mathbb{E}_{S_k^{g,+,n}(\mu)}\left[\mathbb{1}_{\{X_\delta\in\cdot\,\}}\right]$$
$$= \mathbb{E}_\mu\left[\mathbb{E}_{X_{k\delta}}\left[\mathbb{1}_{\{X_\delta\in\cdot\,\}}\right] e^{-\sum_{\ell=1}^k \delta \mathbb{1}_{(-\infty,q_\ell^n)}(X_{\ell\delta})}\right]$$
$$= \mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\mathbb{1}_{\{X_{(k+1)\delta}\in\cdot\,\}}\middle|\sigma(X_s : 0 \le s \le k\delta)\right] e^{-\sum_{\ell=1}^k \delta \mathbb{1}_{(-\infty,q_\ell^n)}(X_{\ell\delta})}\right]$$
$$= \mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\mathbb{1}_{\{X_{(k+1)\delta}\in\cdot\,\}} e^{-\sum_{\ell=1}^k \delta \mathbb{1}_{(-\infty,q_\ell^n)}(X_{\ell\delta})}\middle|\sigma(X_s : 0 \le s \le k\delta)\right]\right]$$
$$= \mathbb{E}_\mu\left[\mathbb{1}_{\{X_{(k+1)\delta}\in\cdot\,\}} e^{-\sum_{\ell=1}^k \delta \mathbb{1}_{(-\infty,q_\ell^n)}(X_{\ell\delta})}\right].$$

Therefore we have

$$R^\delta_{g((k+1)\delta)}(P_\delta S_k^{g,+,n}(\mu))$$
$$= \mathbb{E}_\mu\left[e^{-\delta \mathbb{1}_{(-\infty,q_{k+1}^n)}(X_{(k+1)\delta})} \mathbb{1}_{\{X_{(k+1)\delta}\in\cdot\,\}} e^{-\sum_{\ell=1}^k \delta \mathbb{1}_{(-\infty,q_\ell^n)}(X_{\ell\delta})}\right]$$
$$= \mathbb{E}_\mu\left[\mathbb{1}_{\{X_{(k+1)\delta}\in\cdot\,\}} e^{-\sum_{\ell=1}^{k+1} \delta \mathbb{1}_{(-\infty,q_\ell^n)}(X_{\ell\delta})}\right].$$

Furthermore, regarding the second equality, we have by the assumption that

$$\int_{\mathbb{R}} e^{-\delta \mathbb{1}_{(-\infty,q_{k+1}^n)}(x)} P_\delta \frac{1}{\alpha_k^{(n)}} R^{\delta^{(n)}}_{\alpha_k^{(n)}} \circ P_{\delta^{(n)}} \circ \ldots \circ \frac{1}{\alpha_1^{(n)}} R^{\delta^{(n)}}_{\alpha_1^{(n)}} \circ P_{\delta^{(n)}}(\mu)(\mathrm{d}x)$$
$$= \int_{\mathbb{R}} e^{-\delta \mathbb{1}_{(-\infty,q_{k+1}^n)}(x)} \frac{1}{g(k\delta)} P_\delta S_k^{g,+,n}(\mu)(\mathrm{d}x) = \frac{g((k+1)\delta)}{g(k\delta)} = \alpha_{k+1}^{(n)},$$

which implies

$$q_{k+1}^n = q^\delta_{\alpha_{k+1}^{(n)}}(P_\delta \frac{1}{\alpha_k^{(n)}} R^{\delta^{(n)}}_{\alpha_k^{(n)}} \circ P_{\delta^{(n)}} \circ \ldots \circ \frac{1}{\alpha_1^{(n)}} R^{\delta^{(n)}}_{\alpha_1^{(n)}} \circ P_{\delta^{(n)}}(\mu)).$$

Therefore

$$R^\delta_{\alpha_{k+1}^{(n)}}(P_\delta \frac{1}{\alpha_k^{(n)}} R^{\delta^{(n)}}_{\alpha_k^{(n)}} \circ P_{\delta^{(n)}} \circ \ldots \circ \frac{1}{\alpha_1^{(n)}} R^{\delta^{(n)}}_{\alpha_1^{(n)}} \circ P_{\delta^{(n)}}(\mu))$$
$$= \frac{1}{g((k+1)\delta)} \alpha_{k+1}^{(n)} R^\delta_{g((k+1)\delta)}(P_\delta S_k^{g,+,n}(\mu)).$$

The claim follows thus by induction. The full statement follows by the claim combined with (3.13), the definition of $Q_t^{+,n}$ and the definition of the reweighting operator. □

### 3.2.2 The upper approximation: existence, uniqueness and comparison principle

We now concentrate on the goal to find a candidate of solution by the discrete approximation. In order to obtain existence and uniqueness all at once we will constantly compare our approximation to possible solutions. A key step towards our goal is the following monotonicity and bounding statement, which can be understood as counterpart of the statements regarding the lower barrier approximation in the hard-killing case in Subsection 2.3.2.

**Lemma 3.2.4.** *Let $\mu$ be a probability measure and $g$ survival distribution differentiable on $(0,\infty)$ fulfilling (3.4). Then for $b \in \operatorname{ifptk}(g,\mu)$ it holds*

$$Q_t^b(\mu) \preceq_{\mathrm{st}} Q_t^{+,n+1}(\mu) \preceq_{\mathrm{st}} Q_t^{+,n}(\mu).$$

*for $t \geq 0$.*

*Proof.* As preparational step we claim that for $t \geq s > 0$ we have

$$Q_t(\mu) \preceq_{\mathrm{st}} R_{g(t)}^{t-s} \circ P_{t-s}(Q_s(\mu)). \tag{3.20}$$

For the following recall $Q_{t,s}(\mu)$ from (3.18). We abbreviate $\delta = \delta^{(n)}$ and denote $q := q_{g(k\delta)}^{\delta}(P_\delta Q_s(\mu))$. Let $c \geq q$. In this case we have

$$\begin{aligned}
&R_{g(t)}^{t-s}(P_{t-s}Q_s(\mu))((c,\infty)) = \int_{(c,\infty)} e^{-(t-s)\mathbb{1}_{(-\infty,q)}(x)} \mathbb{P}_{Q_s(\mu)}\left(X_{t-s} \in \mathrm{d}x\right) \\
&= \mathbb{P}_{Q_s(\mu)}\left(X_{t-s} > c\right) \geq \mathbb{E}_{Q_s(\mu)}\left[\mathbb{1}_{\{X_{t-s}>c\}} e^{-\int_0^{t-s} \mathbb{1}_{(-\infty,b(r+s))}(X_r)\,\mathrm{d}r}\right] \\
&= Q_{t,s}(Q_s(\mu))((c,\infty)) = Q_t\mu(c,\infty))
\end{aligned}$$

by the Markov property. From this follows, since $R_{g(t)}^{t-s}(P_{t-s}Q_s(\mu))(\mathbb{R}) = g(t) = Q_t\mu(\mathbb{R})$, that

$$R_{g(t)}^{t-s}(P_{t-s}Q_s(\mu))((-\infty,c]) \leq Q_t(\mu)((-\infty,c]).$$

Now let $c < q$. We find

$$\begin{aligned}
&Q_t(\mu)((-\infty,c]) = Q_{t,s}(Q_s(\mu))((-\infty,c]) \\
&= \mathbb{E}_{Q_s(\mu)}\left[\mathbb{1}_{\{X_{t-s}\leq c\}} e^{-\int_0^{t-s} \mathbb{1}_{(-\infty,b(r+s))}(X_r)\,\mathrm{d}r}\right] \geq \mathbb{E}_\nu\left[\mathbb{1}_{\{X_{t-s}\leq c\}} e^{-(t-s)}\right] \\
&= \mathbb{E}_{Q_s(\mu)}\left[\mathbb{1}_{\{X_{t-s}\leq c\}} e^{-(t-s)\mathbb{1}_{(-\infty,q)}(X_{t-s})}\right] = R_{g(t)}^{t-s}(P_{t-s}Q_s(\mu))((-\infty,c]).
\end{aligned}$$

This shows $Q_t(\mu) \preceq_{\mathrm{st}} R^{t-s}_{g(t)}(P_{t-s}Q_s(\mu))$.

Now as first step we show that

$$Q_{k\delta}(\mu) \preceq_{\mathrm{st}} S^{g,+,n+1}_{2k}(\mu) \preceq_{\mathrm{st}} S^{g,+,n}_{k}(\mu) \tag{3.21}$$

for $k \in \mathbb{N}$. For this assume that it holds $\nu := Q_{(k-1)\delta^{(n)}}(\mu) \preceq_{\mathrm{st}} S^{g,+,n}_{k-1}(\mu)$.
Thus, we can deduce by (3.20) and Lemma 3.1.1 that

$$Q_{k\delta}(\mu) \preceq_{\mathrm{st}} R^{\delta}_{g(k\delta)}(P_\delta \nu) \preceq_{\mathrm{st}} R^{\delta}_{g(k\delta)}(P_\delta S^{g,+,n}_{k-1}(\mu)) = S^{g,+,n}_{k}(\mu).$$

For the second inequality assume that $S^{g,+,n+1}_{2(k-1)}(\mu) \preceq_{\mathrm{st}} S^{g,+,n}_{k-1}(\mu)$. We then have by using Corollary 3.1.5 and Lemma 3.1.3 that

$$\begin{aligned} S^{g,+,n+1}_{2k}(\mu) &= R^{\delta^{(n+1)}}_{g(2k\delta^{(n+1)})} \circ P_{\delta^{(n+1)}} \circ R^{\delta^{(n+1)}}_{g((2k-1)\delta^{(n+1)})} \circ P_{\delta^{(n+1)}} \circ S^{g,+,n+1}_{2(k-1)}(\mu) \\ &\preceq_{\mathrm{st}} R^{\delta^{(n)}}_{g(k\delta^{(n)})} \circ P_{\delta^{(n)}} \circ S^{g,+,n+1}_{2(k-1)}(\mu) \\ &\preceq_{\mathrm{st}} R^{\delta^{(n)}}_{g(k\delta^{(n)})} \circ P_{\delta^{(n)}} \circ S^{g,+,n}_{k-1}(\mu) = S^{\mathrm{sk},+,n}_{k}(\mu). \end{aligned}$$

The desired inequality (3.21) follows inductively, since for $k = 0$ all inequalities are fulfilled. For $t > 0$ we have now by (3.20) and (3.21) that

$$\begin{aligned} Q_t(\mu) = Q_{t,\lfloor t\rfloor_n}(Q_{\lfloor t\rfloor_n}(\mu)) &\preceq_{\mathrm{st}} R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n}(Q_{\lfloor t\rfloor_n}(\mu)) \\ &\preceq_{\mathrm{st}} R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n}(S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor}(\mu)) = Q^{+,n}_t(\mu). \end{aligned}$$

Furthermore, we have by Corollary 3.1.5 and (3.21) that

$$\begin{aligned} &Q^{+,n+1}_t(\mu) = R^{t-\lfloor t\rfloor_{n+1}}_{g(t)} \circ P_{t-\lfloor t\rfloor_{n+1}}(S^{g,+,n+1}_{\lfloor t/\delta^{(n+1)}\rfloor}(\mu)) \\ &= R^{t-\lfloor t\rfloor_{n+1}}_{g(t)} \circ P_{t-\lfloor t\rfloor_{n+1}} \circ R^{\lfloor t\rfloor_{n+1}-\lfloor t\rfloor_n}_{g(\lfloor t\rfloor_{n+1})} \circ P_{\lfloor t\rfloor_{n+1}-\lfloor t\rfloor_n}(S^{g,+,n+1}_{2\lfloor t/\delta^{(n)}\rfloor}(\mu)) \\ &\preceq_{\mathrm{st}} R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n}(S^{g,+,n+1}_{2\lfloor t/\delta^{(n)}\rfloor}(\mu)) \\ &\preceq_{\mathrm{st}} R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n}(S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor}(\mu)) = Q^{+,n}_t(\mu), \end{aligned}$$

which finishes the proof. □

**Remark 3.2.5.** Contrary to the hard killing case, the stochastic inequalities of Lemma 3.2.4 do not imply inequalities of the reweighting thresholds of the operators $R^{\delta^{(n)}}$ and $R^{\delta^{(n+1)}}$ appearing in $S^{g,+,n}(\mu)$. This can be seen by the following artificial manipulation of $g$. For fixed $n \in \mathbb{N}$ let $\beta = g(\delta^{(n+1)})$ and $\alpha = g(\delta^{(n)})$. Further, let $\mu$ be a non-atomic probability measure. We have $q^{\delta^{(n+1)}}_{\beta}(P_{\delta^{(n+1)}}\mu) \to -\infty$ for $\beta \to 1$. Thus $R^{\delta^{(n+1)}}_{\beta}(P_{\delta^{(n+1)}}\mu)(f) \to P_{\delta^{(n+1)}}\mu(f)$

for every bounded measurable $f$ as $\beta \to 1$. Hence

$$\lim_{\beta \to 1} \int_{\mathbb{R}} e^{-\delta^{(n+1)} \mathbb{1}_{(-\infty,q)}(x)} P_{\delta^{(n+1)}} R_\beta^{\delta^{(n+1)}} (P_{\delta^{(n+1)}} \mu)(\mathrm{d}x)$$
$$= \int_{\mathbb{R}} e^{-\delta^{(n+1)} \mathbb{1}_{(-\infty,q)}(x)} P_{\delta^{(n)}} \mu(\mathrm{d}x)$$
$$> \int_{\mathbb{R}} e^{-\delta^{(n)} \mathbb{1}_{(-\infty,q)}(x)} P_{\delta^{(n)}} \mu(\mathrm{d}x).$$

By the definition of the reweighting threshold, we can deduce that there is a $\beta$ such that

$$q_\alpha^{\delta^{(n+1)}} (P_{\delta^{(n+1)}} R_\beta^{\delta^{(n+1)}} (P_{\delta^{(n+1)}} \mu)) > q_\alpha^{\delta^{(n)}} (P_{\delta^{(n)}} \mu).$$

By letting $\beta \to e^{-\delta^{(n+1)}}$ we get in analogy the possibility of observing the case

$$q_\alpha^{\delta^{(n+1)}} (P_{\delta^{(n+1)}} R_\beta^{\delta^{(n+1)}} (P_{\delta^{(n+1)}} \mu)) < q_\alpha^{\delta^{(n)}} (P_{\delta^{(n)}} \mu).$$

**Remark 3.2.6.** Additional to the stochastic inequalities from above, we could seek for stochastic inequalities from below as we have done in the hard killing case. It turns out that in the soft-killing problem this type of inequality is more inaccessible. The crucial ingredient of the desired bound from below would be $P_t R_{g(t)}^t(\mu) \preceq_{\mathrm{st}} Q_t^b \mu$, which is equivalent to

$$\mathbb{P}_{\mu|_{[q_{g(t)}(\mu),\infty)}} (X_t \in \cdot \,|\Gamma_t \geq U) \preceq_{\mathrm{st}} \mathbb{P}_{\mu|_{(-\infty,q_{g(t)}(\mu))}} (X_t \in \cdot \,|t \geq U > \Gamma_t)$$

We were not able to show this inequality and the difficulty seems to lie in the counteracting behavior of the initial positions and the conditioning.

Now we leave the path we have taken in the hard killing case. From now on we will concentrate on the upper approximation, which we obtain by the stochastic inequalities of Lemma 3.2.4. The next statement establishes the convergence of $Q_t^{+,n}$ and lets us extract a candidate of solution from the limit.

**Theorem 3.2.7.** *Let $\mu$ be a probability measure, $g$ a survival distribution differentiable on $(0,\infty)$ and fulfilling (3.4). For every $t > 0$ there exists a sub-probability measure $Q_t^+(\mu)$ such that*

*(i)* $Q_t^{+,n}(\mu) \to Q_t^+(\mu)$ *in the sense of weak convergence as $n \to \infty$,*

*(ii) If* $b \in \mathrm{ifptk}(g,\mu)$, *it holds* $Q_t^b(\mu) \preceq_{\mathrm{st}} Q_t^+(\mu)$,

*(iii)* $Q_t^+(\mu)$ *is equivalent to the Lebesgue measure and* $Q_t^+(\mu)(\mathbb{R}) = g(t)$,

*(iv)* $q^{(n)}(t) \to q(t)$ *as $n \to \infty$ for every $t > 0$, where $q(t)$ is the unique value determined by* $Q_t^+(\mu)((-\infty, q(t))) = -g'(t)$ *and*

*(v) If* $b \in \mathrm{ifptk}(g,\mu)$ *is continuous, then* $q(t) \geq b(t)$.

*Proof.* Let $t > 0$. By Lemma 3.2.3 we have

$$e^{-t}\mathbb{P}_\mu\left(X_t \in A\right) \le Q_t^{+,n}(\mu)(A) \le \mathbb{P}_\mu\left(X_t \in A\right) \tag{3.22}$$

for every measurable $A \subseteq \mathbb{R}$. By the upper bound of these inequalities we have that, the collection $(Q_t^{+,n}(\mu))_{n\in\mathbb{N}}$, seen as finite measures, is tight since $\mathbb{P}_\mu\left(X_t \in \cdot\,\right)$ is tight. Since $Q_t^{+,n}(\mu)(\mathbb{R}) = g(t)$ we can deduce by Prokhorov's theorem that $(Q_t^{+,n}(\mu))_{n\in\mathbb{N}}$ is relatively compact. Let $\sigma_t$ be an accumulation point of $(Q_t^{+,n}(\mu))_{n\in\mathbb{N}}$ in the sense of weak convergence. Then by the Portmanteau theorem and (3.22) we have that for all closed sets $F \subseteq \mathbb{R}$ it holds

$$\sigma_t(F) \ge \limsup_{n\to\infty} Q_t^{+,n}(\mu)(F) \ge e^{-t}\mathbb{P}_\mu\left(X_t \in F\right)$$

and for all open sets $U \subseteq \mathbb{R}$ we have

$$\sigma_t(U) \le \liminf_{n\to\infty} Q_t^{+,n}(\mu)(U) \le \mathbb{P}_\mu\left(X_t \in U\right).$$

Since the measures are regular it follows that

$$e^{-t}\mathbb{P}_\mu\left(X_t \in A\right) \le \sigma_t(A) \le \mathbb{P}_\mu\left(X_t \in A\right)$$

for every measurable $A \subseteq \mathbb{R}$, which implies that $\sigma_t$ is equivalent to the Lebesgue measure.

Then by Lemma 3.2.4, we have that for every $c \in \mathbb{R}$ we have that the sequence $Q_t^{+,n}(\mu)((-\infty, c])$ is monotonic in $n$ and thus, by the equivalence to the Lebesgue measure and the Portmanteu theorem, must converge to $\sigma_t((-\infty, c])$. But in view of the equivalence to the Lebesgue measure and the Portmanteau theorem, this already means that $Q_t^{+,n}(\mu)$ converges weakly to $Q_t^{+}(\mu) := \sigma_t$.

Let $b \in \operatorname{ifptk}(g, \mu)$. By Lemma 3.2.4 we have $Q_t^b(\mu) \preceq_{\mathrm{st}} Q_t^{+,n}(\mu)$. This ordering is preserved in the limit $n \to \infty$, and thus $Q_t^b(\mu) \preceq_{\mathrm{st}} Q_t^{+}(\mu)$.

For $t > 0$, due to the fact that $Q_t^{+}(\mu)$ is equivalent to the Lebesgue measure and $g$ fulfills (3.4), we can find a unique value $q(t)$, such that

$$Q_t^{+}(\mu)((-\infty, q(t)]) = -g'(t).$$

If $b \in \operatorname{ifptk}(g, \mu)$ is continuous we have due to Lemma 3.2.1 that

$$Q_t\mu((-\infty, b(t)) = -g'(t),$$

By the inequality $Q_t^b\mu \preceq_{\mathrm{st}} Q_t^{+}(\mu)$ it follows that $b(t) \le q(t)$. Since

$$Q_t^{+}(\mu)((-\infty, c]) = \lim_{n\to\infty} Q_t^{+,n}(\mu)((-\infty, c]))$$

for all $c \in \mathbb{R}$ and $Q_t^{+}(\mu)$ is equivalent to Lebesgue measure, for every $(c_n)_{n\in\mathbb{N}}$ with $Q_t^{+,n}(\mu)((-\infty, c_n]) \to -g'(t)$ follows $c_n \to q(t)$. Hence, considering Lemma 3.2.2 we have $q^{(n)}(t) \to q(t)$. □

**Theorem 3.2.8.** *Let* $\mu \preceq_{\mathrm{st}} \nu$. *Then* $Q_t^+(\mu) \preceq_{\mathrm{st}} Q_t^+(\nu)$.

*Proof.* By Lemma 3.1.1 and the fact that the Markov operator preserves the order we obtain that

$$R^{\delta^{(n)}}_{g(k\delta^{(n)})} \circ P_{\delta^{(n)}} \quad \text{and} \quad R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n}$$

preserve the usual stochastic order. In order to begin with an induction, we have

$$S_0^{g,+,n}(\mu) = \mu \preceq_{\mathrm{st}} \nu = S_0^{g,+,n}(\nu).$$

Let us assume $S_{k-1}^{g,+,n}(\mu) \preceq_{\mathrm{st}} S_{k-1}^{g,+,n}(\nu)$. Then it follows that

$$\begin{aligned} S_k^{g,+,n}(\mu) &= R^{\delta^{(n)}}_{g(k\delta^{(n)})} \circ P_{\delta^{(n)}} \circ S_{k-1}^{g,+,n}(\mu) \\ &\preceq_{\mathrm{st}} R^{\delta^{(n)}}_{g(k\delta^{(n)})} \circ P_{\delta^{(n)}} \circ S_{k-1}^{g,+,n}(\nu) = S_k^{g,+,n}(\nu). \end{aligned}$$

By induction it follows that $S_k^{g,+,n}(\mu) \preceq_{\mathrm{st}} S_k^{g,+,n}(\nu)$. Altogether it follows that

$$\begin{aligned} Q_t^{+,n}(\mu) &= R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n} \circ S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor}(\mu) \\ &\preceq_{\mathrm{st}} R^{t-\lfloor t\rfloor_n}_{g(t)} \circ P_{t-\lfloor t\rfloor_n} \circ S^{g,+,n}_{\lfloor t/\delta^{(n)}\rfloor}(\nu) = Q_t^{+.n}(\nu). \end{aligned}$$

The ordering is preserved in the weak limit. □

From Theorem 3.2.8 we can deduce the ordering between the corresponding quantile functions $q$. This will be done later in Theorem 3.2.14.

We continue with a study of the function $q$. We will be able to obtain the continuity of $q$ by the following continuity of the measure $Q_t^+(\mu)$.

**Lemma 3.2.9.** *Let* $\mu \in \mathcal{P}$. *Let* $g$ *be a survival distribution continuously differentiable on* $(0,\infty)$ *and fulfilling* (3.4). *Let* $d_{\mathrm{P}}$ *denote the Prohorov metric from* (B.1). *Let* $0 < \eta < T$ *and* $t, s \in [\eta, T]$. *Then there is a tight collection of probability measures* $\mathcal{S}_T$ *only depending on* $T$ *and* $\mu$ *such that*

$$d_{\mathrm{P}}\left(g(t)^{-1}Q_t^+(\mu), g(s)^{-1}Q_s^+(\mu)\right) \le |t-s| + \sup_{\sigma\in\mathcal{S}_T} d_{\mathrm{P}}\left(P_{|t-s|}\sigma, \sigma\right).$$

*If* $\mu$ *is absolutely continuous to Lebesgue measure the statement holds also for* $\eta = 0$.

*Proof.* By the fact that the Prohorov metric is bounded by 1, we can assume without loss of generality that $|t-s| \le 1$. Since the Prohorov metric metrizes the weak convergence it suffices to show a bound of the type

$$d_{\mathrm{P}}\left(g(t)^{-1}Q_t^{+,n}(\mu), g(s)^{-1}Q_s^{+,n}(\mu)\right) \le |t-s| + \sup_{\sigma\in\mathcal{S}_T} d_{\mathrm{P}}\left(P_{|t-s|}\sigma, \sigma\right) + \varepsilon_n,$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence converging to zero. Without loss of generality assume that $t \geq s \geq \eta$. By the triangle inequality we observe

$$\begin{aligned}
&d_{\mathrm{P}}\left(g(t)^{-1}Q_t^{+,n}(\mu), g(s)^{-1}Q_s^{+,n}(\mu)\right)\\
&\leq d_{\mathrm{P}}\left(g(t)^{-1}Q_t^{+,n}(\mu), P_{t-\lfloor s\rfloor_n}(g(\lfloor s\rfloor_n)^{-1}Q_{\lfloor s\rfloor_n}^{+,n}(\mu))\right)\\
&\quad + d_{\mathrm{P}}\left(P_{t-s}\circ P_{s-\lfloor s\rfloor_n}(g(\lfloor s\rfloor_n)^{-1}Q_{\lfloor s\rfloor_n}^{+,n}(\mu)), P_{s-\lfloor s\rfloor_n}(g(\lfloor s\rfloor_n)^{-1}Q_{\lfloor s\rfloor_n}^{+,n}(\mu))\right)\\
&\quad + d_{\mathrm{P}}\left(P_{s-\lfloor s\rfloor_n}(g(\lfloor s\rfloor_n)^{-1}Q_{\lfloor s\rfloor_n}^{+,n}(\mu)), g(s)^{-1}Q_s^{+,n}(\mu)\right). \qquad (3.23)
\end{aligned}$$

In the following we abbreviate

$$\nu := g(\lfloor s\rfloor_n)^{-1}Q_{\lfloor s\rfloor_n}^{+,n}(\mu).$$

Observe that in view of Remark 3.2.3 we obtain

$$\begin{aligned}
&g(t)^{-1}Q_t^{+,n}(\mu)\\
&= \frac{g(\lfloor t\rfloor_n)}{g(t)} R^{t-\lfloor t\rfloor_n}_{\frac{g(t)}{g(\lfloor t\rfloor_n)}} \circ P_{t-\lfloor t\rfloor_n} \circ \frac{1}{\alpha_{\lfloor t/\delta^{(n)}\rfloor}} R^{\delta^{(n)}}_{\alpha_{\lfloor t/\delta^{(n)}\rfloor}} \circ P_{\delta^{(n)}} \circ \ldots \circ \frac{1}{\alpha_1} R^{\delta^{(n)}}_{\alpha_1} \circ P_{\delta^{(n)}}(\mu)\\
&= \frac{g(\lfloor t\rfloor_n)}{g(t)} R^{t-\lfloor t\rfloor_n}_{\frac{g(t)}{g(\lfloor t\rfloor_n)}} \circ P_{t-\lfloor t\rfloor_n} \circ \frac{1}{\alpha_{\lfloor t/\delta^{(n)}\rfloor}} R^{\delta^{(n)}}_{\alpha_{\lfloor t/\delta^{(n)}\rfloor}} \circ P_{\delta^{(n)}} \circ \ldots\\
&\qquad\qquad \ldots \circ \frac{1}{\alpha_{\lfloor s/\delta^{(n)}\rfloor+1}} R^{\delta^{(n)}}_{\alpha_{\lfloor s/\delta^{(n)}\rfloor+1}} \circ P_{\delta^{(n)}}(\nu), \qquad (3.24)
\end{aligned}$$

where $\alpha_k := g(k\delta)/g((k-1)\delta)$. Further we can write

$$P_{t-\lfloor s\rfloor_n}\nu = P_{t-\lfloor t\rfloor_n} \circ P_{\delta^{(n)}} \circ \ldots \circ P_{\delta^{(n)}}(\nu). \qquad (3.25)$$

Denote $h(t) := -\frac{\partial}{\partial t}\log(g(t))$. Recall that by (3.4) we have $0 < h(t) < 1$ for all $t > 0$. It holds

$$\alpha_k = \frac{g(k\delta^{(n)})}{g((k-1)\delta^{(n)})} = e^{-\int_{(k-1)\delta^{(n)}}^{k\delta^{(n)}} h(y)\,\mathrm{d}y}.$$

In view of (3.24) compared with (3.25), by Lemma B.1.11 and Corollary 3.1.10 we have

$$\begin{aligned}
&d_{\mathrm{P}}\left(g(t)^{-1}Q_t^{+,n}(\mu), P_{t-\lfloor s\rfloor_n} g(\lfloor s\rfloor_n)^{-1}Q_{\lfloor s\rfloor_n}^{+,n}(\mu)\right)\\
&\leq d_{\mathrm{TV}}\left(g(t)^{-1}Q_t^{+,n}(\mu), P_{t-\lfloor s\rfloor_n} g(\lfloor s\rfloor_n)^{-1}Q_{\lfloor s\rfloor_n}^{+,n}(\mu)\right)\\
&\leq 1 - \frac{g(t)}{g(\lfloor t\rfloor_n)} + \sum_{k=\lfloor s/\delta^{(n)}\rfloor+1}^{\lfloor t/\delta^{(n)}\rfloor}(1-\alpha_k)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - e^{-\int_{\lfloor t \rfloor_n}^{t} h(y)\,\mathrm{d}y} + \sum_{k=\lfloor s/\delta^{(n)} \rfloor+1}^{\lfloor t/\delta^{(n)} \rfloor} 1 - e^{-\int_{(k-1)\delta^{(n)}}^{k\delta^{(n)}} h(y)\,\mathrm{d}y} \\
&\leq \int_{\lfloor t \rfloor_n}^{t} h(y)\,\mathrm{d}y + \sum_{k=\lfloor s/\delta^{(n)} \rfloor+1}^{\lfloor t/\delta^{(n)} \rfloor} \int_{(k-1)\delta^{(n)}}^{k\delta^{(n)}} h(y)\,\mathrm{d}y = \int_{\lfloor s/\delta^{(n)} \rfloor \delta^{(n)}}^{t} h(y)\,\mathrm{d}y \\
&\leq (t - \lfloor s/\delta^{(n)} \rfloor \delta^{(n)}) \leq |t-s| + \delta^{(n)} \qquad (3.26)
\end{aligned}
$$

using that $0 \leq h \leq 1$, since $g$ fulfilles (3.4). In the case $\eta = 0$ here we have to use that $\mu$ has a density, since then we can use Lemma 3.1.8 even in the case $s = 0$. In a similar manner as above we have

$$
g(s)^{-1} Q_s^{+,n}(\mu) = \frac{g(\lfloor s \rfloor_n)}{g(s)} R^{s-\lfloor s \rfloor_n}_{\frac{g(s)}{g(\lfloor s \rfloor_n)}} \circ P_{s-\lfloor s \rfloor_n}(g(\lfloor s \rfloor_n)^{-1} Q_{\lfloor s \rfloor_n}^{+,n}(\mu)).
$$

This means we have, with a further application of Lemma 3.1.10, that

$$
\begin{aligned}
&d_{\mathrm{P}} \left( P_{s-\lfloor s \rfloor_n}(g(\lfloor s \rfloor_n)^{-1} Q_{\lfloor s \rfloor_n}^{+,n}(\mu)), g(s)^{-1} Q_s^{+,n}(\mu) \right) \\
&\leq d_{\mathrm{TV}} \left( P_{s-\lfloor s \rfloor_n}(g(\lfloor s \rfloor_n)^{-1} Q_{\lfloor s \rfloor_n}^{+,n}(\mu)), g(s)^{-1} Q_s^{+,n}(\mu) \right) \leq 1 - \frac{g(s)}{g(\lfloor s \rfloor_n)}. \qquad (3.27)
\end{aligned}
$$

As next step observe that for all $n \in \mathbb{N}$ and $s \in [0, T]$ we have

$$
\begin{aligned}
&P_{s-\lfloor s \rfloor_n} \nu(A) = P_{s-\lfloor s \rfloor_n} g(\lfloor s \rfloor_n)^{-1} Q_{\lfloor s \rfloor_n}^{+,n}(\mu)(A) \\
&= \int_{\mathbb{R}} \mathbb{E}_x \left[ 1_{\{X_s \in A\}} e^{-\sum_{k=0}^{\lfloor s \rfloor_n 2^n} 2^{-n} 1_{(-\infty, q^{(n)}(k2^{-n}))}(X_{k2^{-n}})} \right] \mu(dx) \\
&\leq g(\lfloor s \rfloor_n)^{-1} \int_{\mathbb{R}} \mathbb{P}_x(X_s \in A) \mu(dx) \\
&\leq g(T)^{-1} \int_{\mathbb{R}} \mathbb{P}_x(X_s \in A) \mu(dx).
\end{aligned}
$$

This implies that for all $n \in \mathbb{N}$ and $s \in [0, T]$ we can see $P_{s-\lfloor s \rfloor_n} \nu$ as part of a bigger collection $\mathcal{S}_T$ of tight probability measures, since $(P_s \mu)_{s \in [0,T]}$ is tight due to the continuity of the pathes. Now we have

$$
\begin{aligned}
&d_{\mathrm{P}} \left( P_{t-s} \circ P_{s-\lfloor s \rfloor_n}(g(\lfloor s \rfloor_n)^{-1} Q_{\lfloor s \rfloor_n}^{+,n}(\mu)), P_{s-\lfloor s \rfloor_n}(g(\lfloor s \rfloor_n)^{-1} Q_{\lfloor s \rfloor_n}^{+,n}(\mu)) \right) \\
&= d_{\mathrm{P}} \left( P_{t-s} \circ P_{s-\lfloor s \rfloor_n}(\nu), P_{s-\lfloor s \rfloor_n}(\nu) \right) \\
&\leq \sup_{\sigma \in \mathcal{S}_T} d_{\mathrm{P}} \left( P_{t-s}\sigma, \sigma \right).
\end{aligned}
$$

As last step, by putting the above, (3.26) and (3.27) together, we obtain in view of the triangle bound in (3.23) that

$$\begin{aligned}
&d_{\mathrm{P}}\left(g(t)^{-1}Q_t^{+}(\mu), g(s)^{-1}Q_s^{+}(\mu)\right)\\
&= \lim_{n\to\infty} d_{\mathrm{P}}\left(g(t)^{-1}Q_t^{+,n}(\mu), g(s)^{-1}Q_s^{+,n}(\mu)\right)\\
&\le \lim_{n\to\infty} |t-s| + \delta^{(n)} + \sup_{\sigma\in\mathcal{S}_T} d_{\mathrm{P}}\left(P_{t-s}\sigma,\sigma\right) + \left(1 - \frac{g_u(s)}{g_u(\lfloor s\rfloor_n)}\right)\\
&= |t-s| + \sup_{\sigma\in\mathcal{S}_T} d_{\mathrm{P}}\left(P_{t-s}\sigma,\sigma\right),
\end{aligned}$$

which shows the statement. □

**Corollary 3.2.10.** *Let $\mu \in \mathcal{P}$. Let $g$ be a survival distribution continuously differentiable on $(0,\infty)$ and fulfilling (3.4). Then the function $q : (0,\infty) \to \mathbb{R}$ implicitly defined by $Q_t^{+}(\mu)((-\infty, q(t))) = -g'(t)$ is continuous. If additionally (3.5) is fulfilled, then $q(0+)$ exists and is the unique value such that $\mu((-\infty, q(0+))) = -g'(0)$.*

*Proof.* Let $\eta > 0$. Employing the third point of the imposed assumptions on the Markov process the previous Lemma 3.2.9 yields in particular that $t \to Q_t^{+}(\mu)$ is continuous in $t \ge \eta$ in the sense of weak convergence. Assume there is a sequence $(t_m)_{m\in\mathbb{N}}$ of non-negative numbers such that $t_m \to t \ge \eta$ and $x := \lim_{m\to\infty} a(t_m) \ne q(t)$. Assume that then $x > q(t)$. By the above mentioned continuity it holds $Q_{t_m}^{+}(\mu) \to Q_t^{+}(\mu)$ in the sense of weak convergence. For $c \in (q(t), x)$ we would have

$$\begin{aligned}
-g'(t) = \lim_{m\to\infty} -g'(t_m) &= \lim_{m\to\infty} Q_{t_m}^{+}(\mu)((-\infty, q(t_m)))\\
&\ge \lim_{m\to\infty} Q_{t_m}^{+}(\mu)((-\infty, c)) = Q_t^{+}(\mu)((-\infty, c))\\
&> Q_t^{+}(\mu)((-\infty, q(t))) = -g'(t),
\end{aligned}$$

which is a contradiction. The case $x < q(t)$ is analogous, and thus $q$ has to be continuous in $t$. With the additional assumption, the arguments above apply for the case $t = 0$. □

Regarding the representation of $Q_t^{+,n}(\mu)$ by Lemma 3.2.3 in terms of a type of Riemann sums, it makes sense to demand more control over the convergence of $q^{(n)}$ to $q$.

**Lemma 3.2.11.** *Let $g$ be a survival distribution continuously differentiable on $(0,\infty)$ and fulfilling (3.4) and $\mu \in \mathcal{P}$. Recall the function $q : (0,\infty) \to \mathbb{R}$ implicitely defined by $Q_t^{+}(\mu)((-\infty, q(t))) = -g'(t)$. Then for $0 < \eta < T$ the function $q^{(n)}$ converges uniformly in $t \in [\eta, T]$ to the function $q$.*

In order to prove Lemma 3.2.11 we will use Dini's theorem on certain approximants and therefore need the continuity of the approximants. This continuity will be provided by the following auxiliary statement.

**Lemma 3.2.12.** *Let $g$ be a survival distribution continuously differentiable on $(0,\infty)$ and fulfilling (3.4) and $\mu \in \mathcal{P}$. Then the mapping*

$$[0,\infty) \to \mathcal{P},\ t \mapsto g(t)^{-1} R^t_{g(t)}(P_t\mu)$$

*is continuous in the sense of weak convergence, where we identify $R^0_1(\mu) = \mu$.*

*Proof.* Recall that we have that $[0,\infty) \to \mathcal{P},\ t \mapsto P_t\mu$ is continuous in the sense of weak convergence and $P_t\mu$ is equivalent to the Lebesgue measure for every $t > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ continuous and bounded. For simplicity of notation abbreviate $q_t := q^t_{g(t)}(P_t\mu)$. For the continuity in $t = 0$ consider

$$\begin{aligned}&\left|\int_{\mathbb{R}} f(x) R^t_{g(t)}(P_t\mu)(\mathrm{d}x) - P_t\mu(f)\right| \\ &= \left|\mathbb{E}_\mu\left[f(X_t)e^{-t 1_{(-\infty,q_t)}(X_t)} - f(X_t)\right]\right| \le \|f\|_\infty |e^{-t} - 1| \to 0,\end{aligned}$$

as $t \to 0$. Thus the continuity in $t = 0$ follows from the assumption that $P_t\mu \to \mu$ in the sense of weak convergence and that $g$ is continuous in $t = 0$. Now consider $t > 0$ and let $t_m \to t$ as $m \to \infty$. We have $P_{t_m}\mu \to P_t\mu$ in the sense of weak convergence. As first case assume that there is $\varepsilon > 0$ such that $\limsup_{m\to\infty} q_{t_m} \ge q_t + \varepsilon$. Then we had, since $P_t\mu$ has no atoms, that

$$\begin{aligned}\frac{1-g(t)}{1-e^{-t}} &= \lim_{m\to\infty} \frac{1-g(t_m)}{1-e^{-t_m}} = \lim_{m\to\infty} P_{t_m}\mu((-\infty, q_{t_m})) \\ &\ge \lim_{m\to\infty} P_{t_m}\mu((-\infty, q_t+\varepsilon)) = P_t\mu((-\infty, q_t+\varepsilon)) \\ &> P_t\mu((-\infty, q_t)) = \frac{1-g(t)}{1-e^{-t}},\end{aligned}$$

which is a contradiction. Analogously, the case that there is $\varepsilon > 0$ with $\liminf_{m\to\infty} q_{t_m} \le q_t - \varepsilon$ leads to a contradiction. Hence, we obtain that

$$\lim_{s\to t} q_s = q_t.$$

Therefore we have, as $s \to t$,

$$\begin{aligned}&\left|\int_{\mathbb{R}} f(x) R^s_{g(s)}(P_s\mu)(\mathrm{d}x) - \mathbb{E}_\mu\left[f(X_s)e^{-t 1_{(-\infty,q_t)}(X_s)}\right]\right| \\ &= \left|\mathbb{E}_\mu\left[f(X_s)\left(e^{-s 1_{(-\infty,q_s)}(X_s)} - e^{-t 1_{(-\infty,q_t)}(X_s)}\right]\right)\right| \\ &\le \|f\|_\infty \cdot \left(|e^{-s} - e^{-t}| + P_s\mu([q_s \wedge q_t, q_s \vee s_t))\right)\end{aligned}$$

due to the weak convergence of $P_s\mu \to P_t$ and the assumption that $P_t\mu$ has no atoms. By the same weak convergence in follows that

$$\mathbb{E}_\mu\left[f(X_s)e^{-t 1_{(-\infty,q_t)}(X_s)}\right] \to \int_{\mathbb{R}} f(x) R^t_{g(t)}(P_t\mu)(\mathrm{d}x),$$

since $x \mapsto f(x)e^{-t 1_{(-\infty,q_t)}(x)}$ has only one point of discontinuity and $P_t\mu$ is non-atomic. This proves the statement. □

We are now ready to prove Lemma 3.2.11.

*Proof of Lemma 3.2.11.* With the additional assumption the following arguments are also true for $\eta = 0$. For $t \geq \eta$ let $a^{(n)}(t)$ implicitly defined by

$$Q_t^{+,n}(\mu)((-\infty, a^{(n)}(t))) = -g'(t),$$

which is possible since $Q_t^{+,n}(\mu)$ is equivalent to the Lebesgue measure and $g$ fulfills (3.4). By Lemma 3.2.12 we can deduce that $t \mapsto Q_t^{+,n}$ is continuous in $t \geq \eta$ in the sense of weak convergence. Now, analogously to the prove of Corollary 3.2.10, it can be seen that $a^{(n)}$ is continuous for $t \geq \eta$. By the ordering of Lemma 3.2.4 we have that

$$q(t) \leq a^{(n+1)}(t) \leq a^{(n)}(t)$$

for every $t \geq \eta$. Since $Q_t^{+,n}((-\infty, a^{(n)}(t))) \to Q_t^{+}((-\infty, q(t))$, it follows as in proof of Theorem 3.2.7, that $a^{(n)}(t) \to q(t)$. In view of this and by Dini's theorem, for example see Theorem 7.3 from [Rud76], by continuity of $q$ and $a^{(n)}$ it follows that $\sup_{t\in[\eta,T]} |a^{(n)}(t) - q(t)| \to 0$ as $n \to \infty$. We will finish the proof by showing that

$$\sup_{t\in[\eta,T]} |q^{(n)}(t) - a^{(n)}(t)| \to 0$$

as $n \to \infty$. As preparation for this, we claim that there exists a compact set $K_T \subset \mathbb{R}$ only depending on $\eta$ and $T$, such that for all $n$ large enough we have

$$q^{(n)}(t), a^{(n)}(t) \in K_T$$

for all $t \in [\eta, T]$. In order to see this, we begin as follows. By Prohorov's theorem the collection of measures $(P_t\mu)_{t\in[\eta,T]}$ is tight. Thus for $\varepsilon > 0$, by Lemma 3.2.3 we can find $k(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ and $t \in [\eta, T]$

$$Q_t^{+,n}(\mu)(\mathbb{R} \setminus [-k(\varepsilon), k(\varepsilon)]) \leq P_t\mu(\mathbb{R} \setminus [-k(\varepsilon), k(\varepsilon)]) \leq \varepsilon.$$

Thus, we have $Q_t^{+,n}(\mu)((q^{(n)}(t), \infty)) \leq \varepsilon$, whenever $q^{(n)}(t) > k(\varepsilon)$, and similarly $Q_t^{+,n}(\mu)((-\infty, q^{(n)}(t)))) \leq \varepsilon$, whenever $q^{(n)}(t) < -k(\varepsilon)$. For a function $f$ denote $\|f\|_{[\eta,T]} \coloneqq \sup_{t\in[\eta,T]} |f(t)|$. We have by Lemma 3.2.2 for $t \in [\eta, T]$ that

$$\begin{aligned}
&Q_t^{+,n}((q^{(n)}(t), \infty)) = |g(t) - Q_t^{+,n}((-\infty, q^{(n)}(t)))| \\
&= |g(t) + g'(t) + (-g'(t) - Q_t^{+,n}((-\infty, q^{(n)}(t))))| \\
&\geq |g(t) + g'(t)| - |Q_t^{+,n}((-\infty, q^{(n)}(t))) - (-g'(t))| \\
&\geq g(T) \left|1 + \frac{g'(t)}{g(t)}\right| - \sup_{s\in[\eta,T]} |Q_s^{+,n}((-\infty, q^{(n)}(s))) - (-g'(s))| \\
&\geq g(T) \inf_{s\in[\eta,T]} \left|1 + \frac{g'(s)}{g(s)}\right| - \sup_{s\in[\eta,T]} |Q_s^{+,n}((-\infty, q^{(n)}(s))) - (-g'(s))| \\
&\to g(T) \inf_{s\in[\eta,T]} \left|1 + \frac{g'(s)}{g(s)}\right|
\end{aligned}$$

as $n \to \infty$. On the other hand we have

$$
\begin{aligned}
&Q_t^{+,n}(\mu)((-\infty, q^{(n)}(t)))) \geq |g'(t)| - |Q_t^{+,n}(\mu)((-\infty, q^{(n)}(t)))) + g'(t)| \\
&\geq g(T) \left| \frac{g'(t)}{g(t)} \right| - \sup_{s\in[\eta,T]} |Q_s^{+,n}((-\infty, q^{(n)}(s))) - (-g'(s))| \\
&\geq g(T) \inf_{s\in[\eta,T]} \left| \frac{g'(s)}{g(s)} \right| - \sup_{s\in[\eta,T]} |Q_s^{+,n}((-\infty, q^{(n)}(s))) - (-g'(s))| \\
&\to g(T) \inf_{s\in[\eta,T]} \left| \frac{g'(s)}{g(s)} \right|.
\end{aligned}
$$

Now, note that due to (3.4) and the continuity of $g$ and $g'$ we have

$$
\varepsilon_T := \frac{1}{2} g(T) \min \left( \inf_{s\in[\eta,T]} \left| 1 + \frac{g'(s)}{g(s)} \right|, \inf_{s\in[\eta,T]} \left| \frac{g'(s)}{g(s)} \right| \right) > 0.
$$

In view of the above, for $n$ large enough we have necessarily that $q^{(n)}(t) \leq k(\varepsilon_T)$ and $q^{(n)}(t) \geq -k(\varepsilon_T)$ for all $t \in [0,T]$. For $a^{(n)}(t)$ holds

$$
Q_t^{+,n}((a^{(n)}(t), \infty)) = g(t) + g'(t) \geq g(T) \inf_{s\in[\eta,T]} \left| 1 + \frac{g'(s)}{g(s)} \right|
$$

and

$$
Q_t^{+,n}(\mu)((-\infty, a^{(n)}(t)))) = -g'(t) \geq g(T) \inf_{s\in[\eta,T]} \left| \frac{g'(s)}{g(s)} \right|.
$$

Hence, analogously to above we have necessarily that $|a^{(n)}(t)| \leq k(\varepsilon_T)$ for all $t \in [\eta, T]$. This yields the claim by setting $K_T := [-k(\varepsilon_T), k(\varepsilon_T)]$.
As next step assume that

$$
\limsup_{n\to\infty} \sup_{t\in[\eta,T]} |q^{(n)}(t) - a^{(n)}(t)| \neq 0. \tag{3.28}
$$

Then there would exist $\theta > 0$, a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ and a converging sequence $(t_k)_{k\in\mathbb{N}}$ contained in $[\eta, T]$, such that

$$
|q^{(n_k)}(t_k) - a^{(n_k)}(t_k)| \geq \theta \tag{3.29}
$$

for all $k \in \mathbb{N}$. Denote $t_0 := \lim_{k\to\infty} t_k$ and observe that, since $q^{(n)}(t), a^{(n)}(t) \in K_T$ for all $t \in [\eta, T]$, we can assume without loss of generality that the limits $\lim_{k\to\infty} q^{(n_k)}(t_k)$ and $\lim_{k\to\infty} a^{(n_k)}(t_k)$ exist. We denote

$$
c_1 := \min(\lim_{k\to\infty} q^{(n_k)}(t_k), \lim_{k\to\infty} a^{(n_k)}(t_k))
$$

and

$$
c_2 := \max(\lim_{k\to\infty} q^{(n_k)}(t_k), \lim_{k\to\infty} a^{(n_k)}(t_k)).
$$

By (3.29) it would follow that $|c_1 - c_2| \geq \theta > 0$. Now let

$$A_n(t) := (\min(q^{(n)}(t), a^{(n)}(t)), \max(q^{(n)}(t), a^{(n)}(t)))$$

and observe, that on the one hand we have by Lemma 3.2.3 that

$$Q^{+,n}_{t_k}(\mu)(A_n) \geq e^{-T}\mathbb{P}_\mu\left(X_{t_k} \in A_{n_k}(t_k)\right) \to \mathbb{P}_\mu\left(X_{t_0} \in (c_1, c_2)\right) > 0,$$

since $P_t\mu$ is continuous in $t \in [\eta, T]$ and is equivalent to Lebesgue measure for every $t \geq \eta$. But on the other hand we have in view of Lemma 3.2.2 that

$$\begin{aligned}\sup_{t\in[\eta,T]} Q^{+,n}_t(\mu)(A_n) &= \sup_{t\in[\eta,T]} |Q^{+,n}_t(\mu)((-\infty, q^{(n)}(t))) - Q^{+,n}_t(\mu)((-\infty, a^{(n)}(t)))| \\ &= \sup_{t\in[\eta,T]} |Q^{+,n}_t(\mu)((-\infty, q^{(n)}(t))) - (-g'(t))| \\ &\to 0.\end{aligned}$$

Consequently, the assumption in (3.28) has to be false, and it follows

$$\sup_{t\in[\eta,T]} |q^{(n)}(t) - a^{(n)}(t)| \to 0,$$

which finishes the proof. □

With the uniform convergence at hand, we can finalize our goal by establishing the convergence of the Riemann type sum appearing in $Q^{+,n}_t(\mu)$ to an integral of the desired form $\Gamma^q_t$.

**Theorem 3.2.13.** *Let $\mu \in \mathcal{P}$. Let $g$ be a survival distribution continuously differentiable on $(0, \infty)$ fulfilling (3.4). Let $q$ be the continuous function implicitly defined by $Q^+_t(\mu)((-\infty, q(t))) = -g'(t)$. We have*

$$Q^+_t(\mu) = Q^q_t(\mu) = \mathbb{E}_\mu\left[\mathbb{1}_{\{X_t\in\ \cdot\}} e^{-\int_0^t \mathbb{1}_{(-\infty,q(s))}(X_s)\,\mathrm{d}s}\right]$$

*for every $t \geq 0$, thus $q \in \mathrm{ifptk}(g, \mu)$. Additionally, if $b \in \mathrm{ifptk}(g, \mu)$ is continuous it follows $b = q$.*

*Proof.* Let $t > 0$. By Lemma 3.2.11 we have that $q^{(n)}$ converges to $q$ uniformly on $[\eta, t]$ for any $0 < \eta < t$. In order to use Lemma B.2.9 we consider

$$\begin{aligned}\mathbb{E}_\mu\left[\int_\eta^t \mathbb{1}_{\{0\}}(q(s) - X_s)\,\mathrm{d}s\right] &= \int_\eta^t \mathbb{E}_\mu\left[\mathbb{1}_{\{0\}}(q(s) - X_s)\right]\mathrm{d}s \\ &= \int_\eta^t \mathbb{P}_\mu\left(X_s = q(s)\right)\mathrm{d}s = 0.\end{aligned}$$

Since the integral is non-negative it follows that almost surely

$$\int_\eta^t \mathbb{1}_{\{0\}}(q(s) - X_s)\,\mathrm{d}s = 0.$$

By Lemma B.2.9 we can now deduce that almost surely

$$\lim_{n\to\infty} \sum_{\ell=\lfloor \eta/\delta^{(n)}\rfloor+1}^{\lfloor t/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta}) = \int_\eta^t \mathbb{1}_{(-\infty,q(s))}(X_s)\,\mathrm{d}s.$$

Thus by the representation of Lemma 3.2.3 and the dominated convergence theorem we get for measurable $A \subseteq \mathbb{R}$ that

$$\begin{aligned}
&\limsup_{n\to\infty} \left| Q_t^{+,n}(\mu)(A) - \mathbb{E}_\mu \left[ \mathbb{1}_{\{X_t\in A\}} e^{-\int_\eta^t \mathbb{1}_{(-\infty,q(s))}(X_s)\,\mathrm{d}s} \right] \right| \\
&\le \limsup_{n\to\infty} \mathbb{E}_\mu \Bigg[ \Bigg| e^{-(t-\lfloor t\rfloor_n)\mathbb{1}_{(-\infty,q^{(n)}(t))}(X_t)} e^{-\sum_{\ell=1}^{\lfloor t/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} \\
&\qquad\qquad - e^{-\int_\eta^t \mathbb{1}_{(-\infty,q(s))}(X_s)\,\mathrm{d}s} \Bigg| \mathbb{1}_{\{X_t\in A\}} \Bigg] \\
&\le \limsup_{n\to\infty} \mathbb{E}_\mu \left[ \left| e^{-\sum_{\ell=\lfloor \eta/\delta^{(n)}\rfloor+1}^{\lfloor t/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} - e^{-\int_\eta^t \mathbb{1}_{(-\infty,q(s))}(X_s)\,\mathrm{d}s} \right| \right] \\
&\quad + \limsup_{n\to\infty} \mathbb{E}_\mu \Bigg[ \Bigg| e^{-(t-\lfloor t\rfloor_n)\mathbb{1}_{(-\infty,q^{(n)}(t))}(X_t)} e^{-\sum_{\ell=1}^{\lfloor t/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} \\
&\qquad\qquad - e^{-\sum_{\ell=\lfloor \eta/\delta^{(n)}\rfloor+1}^{\lfloor t/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} \Bigg| \Bigg] \\
&= \mathbb{E}_\mu \left[ \lim_{n\to\infty} \left| e^{-\sum_{\ell=\lfloor \eta/\delta^{(n)}\rfloor+1}^{\lfloor t/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} - e^{-\int_\eta^t \mathbb{1}_{(-\infty,q(s))}(X_s)\,\mathrm{d}s} \right| \right] \\
&\quad + \limsup_{n\to\infty} \mathbb{E}_\mu \Bigg[ e^{-\sum_{\ell=\lfloor \eta/\delta^{(n)}\rfloor+1}^{\lfloor t/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} \cdot \\
&\qquad\qquad \cdot \left| e^{-(t-\lfloor t\rfloor_n)\mathbb{1}_{(-\infty,q^{(n)}(t))}(X_t)} e^{-\sum_{\ell=1}^{\lfloor \eta/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} - 1 \right| \Bigg] \\
&\le \limsup_{n\to\infty} \mathbb{E}_\mu \left[ \left| e^{-(t-\lfloor t\rfloor_n)\mathbb{1}_{(-\infty,q^{(n)}(t))}(X_t)} e^{-\sum_{\ell=1}^{\lfloor \eta/\delta^{(n)}\rfloor} \delta^{(n)} \mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta})} - 1 \right| \right] \\
&\le \left| e^{-\eta} - 1 \right|
\end{aligned}$$

for $\eta > 0$. By letting $\eta \to 0$ we obtain

$$\lim_{n\to\infty} Q_t^{+,n}(\mu)(A) = \mathbb{E}_\mu \left[ \mathbb{1}_{\{X_t\in A\}} e^{-\int_0^t \mathbb{1}_{(-\infty,q(s))}(X_s)\,\mathrm{d}s} \right] = Q_t^q(\mu)(A).$$

This means that $Q_t^+(\mu) = Q_t^q(\mu)$.

Let $b \in \mathrm{ifptk}(g,\mu)$ be continuous. We will prove that necessarily $b = q$. In view of Theorem 3.2.7 we have $q \ge b$ pointwise. Consequently, we have also

$\Gamma_t^q \geq \Gamma_t^b$. By using that $Q_t^+(\mu) = g(t)$ given by Theorem 3.2.7 this yields

$$0 \leq \mathbb{E}_\mu \left[ e^{-\Gamma_t^b} - e^{-\Gamma_t^q} \right] = \mathbb{P}_\mu \left( \tau_b^{\text{sk}} > t \right) - Q_t^+(\mu)(\mathbb{R}) = g(t) - g(t) = 0,$$

which implies that $\Gamma_t^q = \Gamma_t^b$ almost surely.

In the following let us fix a continuous path such that $\Gamma_t^q = \Gamma_t^b$. For such a path we have that

$$\int_0^T \left( \mathbb{1}_{(-\infty,q(s))}(X_s) - \mathbb{1}_{(-\infty,b(s))}(X_s) \right) \mathrm{d}s = \Gamma_t^q - \Gamma_t^b = 0. \tag{3.30}$$

Since by $q \geq b$ for every $0 \leq s \leq t$ holds $\mathbb{1}_{(-\infty,q(s))}(X_s) \geq \mathbb{1}_{(-\infty,b(s))}(X_s)$ it follows by (3.30) that $\mathbb{1}_{(-\infty,q(s))}(X_s) = \mathbb{1}_{(-\infty,b(s))}(X_s)$ for Lebesgue almost all $s \in (0, t]$, which means that $X_s \notin [b(s), q(s))$ for Lebesgue almost all $s \in (0, t]$. By the continuity of the path and the continuity of $b$ and $q$ this implies that we have $X_s \notin (b(s), q(s))$ for all $s \in (0, t]$.

Altogether, this implies, on the almost sure event that the Brownian motion is continuous and it holds that $\Gamma_t^q = \Gamma_t^b$, we have $X_t \notin (b(t), q(t))$. This implies that $\mathbb{P}_\mu(X_t \in (b(t), q(t))) = 0$. Since by our general assumption the measure $\mathbb{P}_\mu(X_t \in \cdot\,)$ is equivalent to the Lebesgue measure it follows that $b(t) = q(t)$. Since $t$ was arbitrary we obtain $b = q$. □

We conclude by stating a comparison principle for solutions corresponding to ordered initial distributions.

**Theorem 3.2.14.** *Let $g$ be a survival distribution continuously differentiable on $(0, \infty)$ fulfilling (3.4). Let $\mu_1, \mu_2 \in \mathcal{P}$ with $\mu_1 \preceq_{\text{st}} \mu_2$. Let $b_i \in \text{ifptk}(g, \mu_i)$ be continuous. Then it holds*

$$b_1 \leq b_2$$

*pointwise.*

*Proof.* By Theorem 3.2.13 and Theorem 3.2.8 we obtain that

$$Q_t^{b_1}(\mu_1) = Q_t^+(\mu_1) \preceq_{\text{st}} Q_t^+(\mu_2) = Q_t^{b_2}(\mu_2).$$

By the uniqueness of Theorem 3.2.13 we obtain that $b_1$ and $b_2$ fulfill

$$Q_t^+(\mu_i)((-\infty, b_i(t))) = -g'(t)$$

for $t > 0$. By the stochastic ordering it follows that $b_1(t) \leq b_2(t)$ for all $t > 0$. □

## 3.3 Simulation of solutions for the soft-killing problem

In this section we shortly discuss how to achieve a simulation of the unknown but unique continuous boundary function $b \in \text{ifptk}(g, \mu)$ in the case of Brownian motion. In Section 2.4 we already discussed an interacting particle representation for the inverse first passage-time problem for reflected Brownian motion, which will be referred to as hard-killing problem in the following. Regarding this, a first idea for the soft-killing problem could be to begin with $N$ particles, which perform Brownian motion independently between certain timepoints and employ a certain jumping effect at these specific timepoints.

In the soft killing problem moving particles do not have to be necessarily killed but could be reweighted instead. This makes it possible to introduce an approach which makes use of a weighting factor for every particle, depending on the time a particle has approximately spend under the boundary function. This leads to the following Monte-Carlo method which approximates for $n \in \mathbb{N}$ the function $q^{(n)}$ from (3.19) at the timepoints $\mathbb{N}_0 2^{-n}$.

Let $g$ be a survival distribution continuously differentiable on $(0, \infty)$, and recall that the hazard rate of $g$ is given by $h(t) = -\frac{\partial}{\partial t} \log(g(t))$. Let $\mu \in \mathcal{P}$ and assume that $g$ fulfills the hazard rate condition of (3.4).

Let $(X_t^1, \dots, X_t^N)_{t \geq 0}$ be an $N$-dimensional Brownian motion with initial configuration $(X_0^1, \dots, X_0^N) \sim \mu^{\otimes N}$. For $n \in \mathbb{N}$ let timepoints $(t_k^n)_{k \in \mathbb{N}}$ be given by

$$t_k^n := k \cdot 2^{-n} = k\delta^{(n)}.$$

We define the weighting process $(\widehat{w}_k)_{k \in \mathbb{N}_0} = (\widehat{w}_k^1, \dots, \widehat{w}_k^N)_{k \in \mathbb{N}_0}$ inductively by $\widehat{w}_0^i := \frac{1}{N}$ for any $i \in \{1, \dots, N\}$ and for $k \in \mathbb{N}$

$$\widehat{w}_k^i := \widehat{w}_{k-1}^i \cdot e^{-\delta^{(n)} \mathbb{1}_{(-\infty, \widehat{q}_k^{(n)})}(X^i_{k\delta^{(n)}})},$$

where

$$\widehat{q}_k^{(n)} := \sup \left\{ q \in \mathbb{R} : \sum_{i=1}^N \widehat{w}_{k-1}^i e^{-\delta^{(n)} \mathbb{1}_{(-\infty, q)}(X^i_{k\delta^{(n)}})} \geq \frac{g(k\delta^{(n)})}{g((k-1)\delta^{(n)})} \sum_{i=1}^N \widehat{w}_{k-1}^i \right\}.$$

Heuristically, $\widehat{q}^{(n)}$ is the empirical version of $q^{(n)}$ from (3.19). In order to study this empirical approximation rigorously, we begin with the following.

**Lemma 3.3.1.** *Fix $n \in \mathbb{N}$. Let $T > 0$ and $q^{(n)}$ be given as in* (3.19)*. For any measurable and bounded function $\varphi : C[0, T] \to \mathbb{R}$ we have that*

$$\sum_{i=1}^N \widehat{w}_k^i \varphi((X_t^i)_{t \in [0,T]}) \to \mathbb{E}_\mu \left[ \varphi((X_t)_{t \in [0,T]}) e^{-\sum_{\ell=1}^k \delta^{(n)} \mathbb{1}_{(-\infty, q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta^{(n)}})} \right]$$

*as $N \to \infty$ almost surely.*

*Proof.* The technique is similar to the idea used in Theorem 2.4.16. At first define $(w_k)_{k\in\mathbb{N}_0}$ by $w_0^i \coloneqq \frac{1}{N}$ and

$$w_k^i \coloneqq w_{k-1}^i \cdot e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})}.$$

We claim that

$$\sum_{i=1}^N |\widehat{w}_k^i - w_k^i| \to 0 \tag{3.31}$$

as $N \to \infty$ almost surely. In order to see this, consider

$$\begin{aligned}
&\sum_{i=1}^N |\widehat{w}_k^i - w_k^i| \\
&\leq \sum_{i=1}^N |\widehat{w}_k^i - \widehat{w}_{k-1}^i \cdot e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})}| \\
&\quad + \sum_{i=1}^N |\widehat{w}_{k-1}^i \cdot e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})} - w_k^i| \\
&\leq \sum_{i=1}^N \widehat{w}_{k-1}^i |e^{-\delta^{(n)}\mathbb{1}_{(-\infty,\widehat{q}_k^{(n)})}(X^i_{k\delta^{(n)}})} - e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})}| \\
&\quad + \sum_{i=1}^N |\widehat{w}_{k-1}^i - w_{k-1}^i|.
\end{aligned} \tag{3.32}$$

The sign of the first term does only depend on $\widehat{q}_k^{(n)}$ and $q^{(n)}(k\delta^{(n)})$, which can be exploited as follows.

$$\begin{aligned}
&\sum_{i=1}^N \widehat{w}_{k-1}^i |e^{-\delta^{(n)}\mathbb{1}_{(-\infty,\widehat{q}_k^{(n)})}(X^i_{k\delta^{(n)}})} - e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})}| \\
&= \sum_{i=1}^N \widehat{w}_{k-1}^i \operatorname{sgn}\left(q^{(n)}(k\delta^{(n)}) - \widehat{q}_k^{(n)}\right) \cdot \\
&\quad \cdot \left(e^{-\delta^{(n)}\mathbb{1}_{(-\infty,\widehat{q}_k^{(n)})}(X^i_{k\delta^{(n)}})} - e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})}\right) \\
&= \left|\sum_{i=1}^N \widehat{w}_{k-1}^i \left(e^{-\delta^{(n)}\mathbb{1}_{(-\infty,\widehat{q}_k^{(n)})}(X^i_{k\delta^{(n)}})} - e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})}\right)\right| \\
&\leq \left|\sum_{i=1}^N \widehat{w}_k^i - \sum_{i=1}^N w_k^i\right| + \left|\sum_{i=1}^N w_k^i - \sum_{i=1}^N \widehat{w}_{k-1}^i \cdot e^{-\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(k\delta^{(n)}))}(X^i_{k\delta^{(n)}})}\right| \\
&\leq \left|\sum_{i=1}^N \widehat{w}_k^i - \sum_{i=1}^N w_k^i\right| + \sum_{i=1}^N |w_{k-1}^i - \widehat{w}_{k-1}^i|.
\end{aligned} \tag{3.33}$$

Thus, let us now assume that

$$\sum_{i=1}^{N} |w_{k-1}^i - \widehat{w}_{k-1}^i| \to 0 \tag{3.34}$$

for $N \to \infty$ almost surely. As preparational step observe that by the definition of $\widehat{q}_k^{(n)}$ we have

$$\begin{aligned}
\sum_{i=1}^{N} \widehat{w}_{k-1}^i \cdot \frac{g(k\delta^{(n)})}{g((k-1)\delta^{(n)})} &\leq \sum_{i=1}^{N} \widehat{w}_k^i \\
&= \sum_{i=1}^{N} \widehat{w}_{k-1}^i \cdot \left( e^{-\delta^{(n)} \mathbb{1}_{\left(-\infty, \widehat{q}_k^{(n)}\right]}(X_{k\delta^{(n)}}^i)} + (1 - e^{-\delta^{(n)}}) \mathbb{1}_{\{X_{k\delta^{(n)}}^i = \widehat{q}_k^{(n)}\}} \right) \\
&= \sum_{i=1}^{N} \widehat{w}_{k-1}^i \cdot e^{-\delta^{(n)} \mathbb{1}_{\left(-\infty, \widehat{q}_k^{(n)}\right]}(X_{k\delta^{(n)}}^i)} + (1 - e^{-\delta^{(n)}}) \sum_{i=1}^{N} \widehat{w}_{k-1}^i \mathbb{1}_{\{X_{k\delta^{(n)}}^i = \widehat{q}_k^{(n)}\}} \\
&\leq \sum_{i=1}^{N} \widehat{w}_{k-1}^i \cdot \frac{g(k\delta^{(n)})}{g((k-1)\delta^{(n)})} + \frac{1}{N},
\end{aligned}$$

which implies by the assumption and $\lim_{N\to\infty} \sum_{i=1}^{N} w_{k-1}^i = g((k-1)\delta^{(n)})$ almost surely that

$$\lim_{N\to\infty} \sum_{i=1}^{N} \widehat{w}_k^i = g(k\delta^{(n)})$$

almost surely. By the law of large numbers we have $\lim_{N\to\infty} \sum_{i=1}^{N} w_k^i = g(k\delta^{(n)})$ almost surely. From this we can deduce by (3.32) and (3.33) and combined with the assumption from (3.34) that

$$\limsup_{N\to\infty} \sum_{i=1}^{N} |\widehat{w}_k^i - w_k^i| \leq \limsup_{N\to\infty} \left| \sum_{i=1}^{N} \widehat{w}_k^i - \sum_{i=1}^{N} w_k^i \right| = 0$$

almost surely. By induction, this yields the claim from (3.31). Since by the claim we have

$$\left| \sum_{i=1}^{N} \widehat{w}_k^i \varphi((X_t^i)_{t\in[0,T]}) - \sum_{i=1}^{N} w_k^i \varphi((X_t^i)_{t\in[0,T]}) \right| \leq \|\varphi\|_\infty \sum_{i=1}^{N} |\widehat{w}_k^i - w_k^i| \to 0$$

almost surely as $N \to \infty$, we obtain the desired statement by the law of large numbers. □

From this we can deduce the convergence of the Monte-Carlo method.

**Theorem 3.3.2.** *Let $n \in \mathbb{N}$. Then we have almost surely*

$$\widehat{q}_k^{(n)} \to q^{(n)}(k\delta^{(n)}) \qquad \forall k \in \mathbb{N}$$

*as $N \to \infty$.*

*Proof.* Fix $k \in \mathbb{N}$. Set $\alpha := g(k\delta^{(n)})/g((k-1)\delta^{(n)})$. From Lemma 3.3.1 follows that we have

$$\begin{aligned}\widehat{F}_N(x) &:= \left(\sum_{i=1}^N \widehat{w}_{k-1}^i\right)^{-1} \sum_{i=1}^N \widehat{w}_{k-1}^i \cdot \mathbb{1}_{(-\infty,x)}(X^i_{k\delta^{(n)}}) \\ &\to g((k-1)\delta^{(n)})^{-1}\mathbb{E}_\mu\left[\mathbb{1}_{\{X_{k\delta^{(n)}}<x\}} e^{-\sum_{\ell=1}^{k-1}\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta^{(n)}})}\right] \\ &=: F(x)\end{aligned}$$

for every $x \in \mathbb{Q}$ almost surely as $N \to \infty$. Since $\widehat{F}_N$ and $F$ are non-decreasing and $F$ is continuous we can extend this to the property that $\widehat{F}_N(x) \to F(x)$ for all $x \in \mathbb{R}$ as $N \to \infty$ almost surely. Furthermore, since $F$ is continuous, we can deduce by Lemma B.2.11 that the convergence is in fact uniform. This means $\sup_{x\in\mathbb{R}} |\widehat{F}_N - F| \to 0$ as $N \to \infty$ almost surely. Since the normalizing factor for $\widehat{F}_N$ is $\sum_{i=1}^N \widehat{w}_{k-1}^i$ we observe that

$$\begin{aligned}\widehat{q}_k^{(n)} &= \sup\left\{q \in \mathbb{R} : e^{-\delta^{(n)}}\widehat{F}_N(q) + 1 - \widehat{F}_N(q) \geq \frac{g(k\delta^{(n)})}{g((k-1)\delta^{(n)})}\right\} \\ &= \sup\left\{q \in \mathbb{R} : \widehat{F}_N(q) \leq \frac{1-\alpha}{1-e^{-\delta^{(n)}}}\right\}. \qquad (3.35)\end{aligned}$$

Further, with the definition of $q^{(n)}(k\delta^{(n)})$ in (3.19) we can use (3.12) and the representation from Lemma 3.2.3 and we have that

$$\begin{aligned}&g((k-1)\delta^{(n)})F(q^{(n)}(k\delta^{(n)})) \\ &= \mathbb{E}_\mu\left[\mathbb{1}_{\{X_{k\delta^{(n)}}<q^{(n)}(k\delta^{(n)})\}} e^{-\sum_{\ell=1}^{k-1}\delta^{(n)}\mathbb{1}_{(-\infty,q^{(n)}(\ell\delta^{(n)}))}(X_{\ell\delta^{(n)}})}\right] \\ &= P_{\delta^{(n)}}S_{k-1}^{g,+,n}(\mu)((-\infty, q^{(n)}(k\delta^{(n)}))) \\ &= \frac{g((k-1)\delta^{(n)}) - g(k\delta^{(n)})}{1-e^{-\delta^{(n)}}} = g((k-1)\delta^{(n)})\frac{1-\alpha}{1-e^{-\delta^{(n)}}}.\end{aligned}$$

Hence, since $F$ is strictly increasing and continuous $q^{(n)}(k\delta^{(n)})$ is the unique value with

$$F(q^{(n)}(k\delta^{(n)})) = \beta := \frac{1-\alpha}{1-e^{-\delta^{(n)}}}.$$

In the following we work on the event $\lim_{N\to\infty}\sup_{x\in\mathbb{R}}|\widehat{F}_N - F| = 0$. Let $\varepsilon > 0$ such that $\min(\beta, 1-\beta) > \varepsilon > 0$ and $N \in \mathbb{N}$ such that $\sup_{x\in\mathbb{R}}|\widehat{F}_N(x) - F(x)| < \varepsilon/2$. Then

$$\widehat{F}_N(F^{-1}(\beta-\varepsilon)) \leq \sup_{x\in\mathbb{R}}|\widehat{F}_N(x) - F(x)| + F(F^{-1}(\beta-\varepsilon)) \leq \beta - \frac{\varepsilon}{2}.$$

Analogously we have

$$\widehat{F}_N(F^{-1}(\beta+\varepsilon)) \geq -\sup_{x\in\mathbb{R}}|\widehat{F}_N(x) - F(x)| + F(F^{-1}(\beta+\varepsilon)) = \beta + \frac{\varepsilon}{2}.$$

Therefore, by (3.35) we see that $\widehat{q}_k^{(n)} \geq F^{-1}(\beta-\varepsilon)$. Since $\widehat{F}_N$ is left-continuous we have $\widehat{F}_N(\widehat{q}_k^{(n)}) \leq \beta$ and it also follows that $\widehat{q}_k^{(n)} \leq F^{-1}(\beta+\varepsilon)$, which means that

$$F^{-1}(\beta-\varepsilon) \leq \liminf_{N\to\infty} \widehat{q}_k^{(n)} \leq \limsup_{N\to\infty} \widehat{q}_k^{(n)} \leq F^{-1}(\beta+\varepsilon).$$

Since $F^{-1}$ is continuous, letting $\varepsilon \to 0$ yields the statement. □

**Simulation**

Now we turn the focus to the simulation of solutions of the inverse first-passage time problem for soft-killed Brownian motion. For $n \in \mathbb{N}$ and $N \in \mathbb{N}$ the random function

$$\widehat{q}^{(n)} : \mathbb{N}\delta^{(n)} \to \mathbb{R},\ k\delta^{(n)} \mapsto \widehat{q}_k^{(n)}$$

is a Monte-Carlo approximation of the discrete and deterministic approximation $q^{(n)}$ from (3.19).
Naturally, known distributions of $\tau_b^{\mathrm{sk}}$ for explicitly given boundaries $b$ are useful examples to test the Monte-Carlo algorithm. By the identity

$$\mathbb{P}_\mu\left(\tau_b^{\mathrm{sk}} > t\right) = \mathbb{E}_\mu\left[e^{-\Gamma_t^b}\right] = \mathbb{E}_\mu\left[e^{-\int_0^t \mathbb{1}_{(-\infty,b(s))}(X_s)\,\mathrm{d}s}\right].$$

one may computes the distribution of $\tau_b^{\mathrm{sk}}$ by the means of the distribution of the occupation time $\Gamma_t^b$.

At least one example for which the distribution of $\Gamma_t^b$ is in some sense explicitly known is the case $b(t) = c + \nu t$, where $c, \nu \in \mathbb{R}$. The explicit formulas of [Pec99a] and [Pec99b] can be used to compute $\mathbb{E}_\mu\left[e^{-\Gamma_t^b}\right]$, but lead to formulas which are inconvenient for a computational implementation. The special case of $\nu = 0$ is treated in a more elementary way in Lemma B.1.12 and leads to the following example.

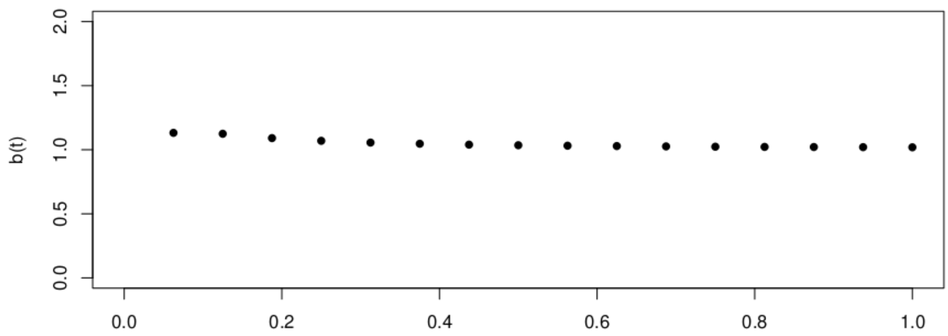


Figure 3.1: The approximated boundary $\hat{q}^{(n)}$ with $n = 4$ corresponding to $g_1$ from Example 3.3.3 with $N = 10^7$ and $\mu = \delta_0$. The exact solution is the constant function with value 1.

**Example 3.3.3.** Let $b(t) = 1$ for all $t \geq 0$ and set $\mu = \delta_0$. Then, due to Lemma B.1.12, the distribution of the soft-killing time $\tau_b^{\mathrm{sk}}$ is given by

$$g_1(t) = e^{-t}\left(2\Phi\left(\sqrt{\frac{1}{t}}\right) - 1\right) + \int_0^t e^{-u}{}_1F_1(1/2; 1; -(t-u))\sqrt{\frac{1}{2\pi u^3}}e^{-\frac{1}{2u}},$$

where ${}_1F_1$ denotes the confluent hypergeometric function of the first kind. Thus we can make a comparison of the constant boundary and the approximated solution (cf. Figure 3.1), where we have chosen $\mu = \delta_0$.

Visualizations for unknown boundaries can be seen in Figure 3.2.

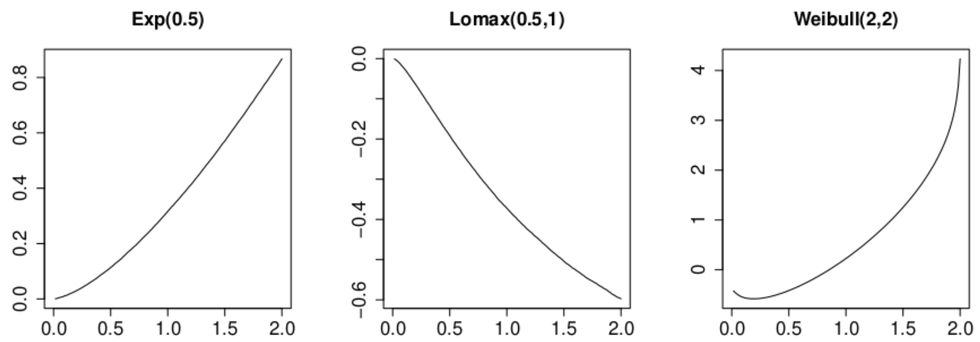


Figure 3.2: The approximated boundaries for $g(t) = e^{-\frac{1}{2}t}$ (Exp(1/2)), $g(t) = (1+t)^{-\frac{1}{2}}$ (Lomax(1/2, 1)) and $g(t) = e^{-\frac{1}{4}t^2}$ (Weibull(2, 2)).

# Chapter 4

# Outlook

In this chapter we want to address open questions, which are connected to the results in this thesis. We will divide the discussion into two parts concerning the hard-killing and the soft-killing problem.

## The hard-killing problem

### Regularity and further properties of solutions

In this thesis we approached the problem of finding properties of solutions from a probabilistic point of view. On the one hand we provided sufficient criteria for Lipschitz continuity and Hölder continuity in Subsection 2.3.4, where the probabilistic method for the latter came from [EJ16]. More subtle conditions for the regularity of solutions were obtained in the work of [CCS21] in the context of free boundary problems. This gives rise to the question, whether such results could be also obtained in a probabilistic way. On the other hand, by our stochastic order approach we studied properties of solutions such as the shape of solutions in terms of monotonicity. The question arises whether one can extend the approach in order to study further properties such as concavity. These properties can be particularly relevant in order to connect the inverse problem with results from the direct first-passage time problem. For example in [Pes02a] the author studies the behavior at zero of the first-passage time density under the assumption that the boundary is continuously differentiable, increasing and concave .

### Generalization to a larger class of Markov processes

In the inverse first-passage time problem for general processes as stated in terms of (1.12), the first issues are naturally the existence and uniqueness of solutions. In this thesis the approach for Brownian motion was build on the marginal distribution (2.1) and the discrete approximation

$$T_{\alpha_k^n} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots \circ T_{\alpha_1^n} \circ P_{t_1^n}(\mu)$$

from (2.15). For Markov processes, for which the truncation operator truncates at unique quantiles $q_{\alpha_k^n}$, the corresponding lower barrier approximation of [Anu80] is unique. As it was the case for Brownian motion it is reasonable to expect that a solution is to be found as limit point of the lower barrier approximation. More specifically, for the class of processes with continuous paths and the first-passage time property

$$\tau_b = \inf\{t > 0 : |X_t| \geq b(t)\} \stackrel{!}{=} \tau_b' := \inf\{t > 0 : |X_t| > b(t)\}$$

the arguments of Anulova in Subsection 2.1.2 can be applied. For Brownian motion this first-passage time property was directly proved by [EJ16], e.g. see Proposition 2.3.12, but it can also be seen by the proof of Lemma 2.1.15. The properties used therein apply for more general processes, which therefore suggests that a generalization of the discretization approach in this thesis is possible for more general diffusions. Related to this, in [EJ16] it is mentioned that their results for the case of Brownian motion, including uniqueness, are expected to hold for more general diffusion processes. As example, the specific generalization from reflected Brownian motion to Bessel processes would be particularly interesting, due to the correspondence to a multi-dimensional Brownian motion model as it was considered in the free boundary context of [Ber+21]. The problem of existence and uniqueness for processes with discontinuous paths remains open.

### The maximum process of the particle system without branching

Given a survival distribution $g$ as parameter, in Subsection 2.4.2 we introduced a particle system $(B_t^i)_{i \in A_{k^{(N)}(t)}}$, $t \geq 0$, for which we showed in Theorem 2.4.4 that its hydrodynamic limit exists and corresponds to the marginal distribution from (1.2) for the solution of the inverse first-passage time problem. Since this limit measure possibly has bounded support, the process given by of the maximal particle alive at time $t \geq 0$, i.e.

$$M_t^N := \max\{|B_t^i| : i \in A_{k^{(N)}(t)}\}, \quad t \geq 0,$$

is an interesting object. We obtained in Proposition 2.4.23 that almost surely $\liminf_{N\to\infty} M_t^N \geq b(t)$ and Figure 2.16 suggests that even

$$\lim_{N\to\infty} M_t^N = b(t)$$

almost surely. If this can be made rigorous, the question arises, whether the deviations are governed by a central limit theorem and in which sense the deviations depend on $t \geq 0$.

## The soft-killing problem

### Comparison principle and properties of solutions

A canonical continuation of the study of the soft-killing problem would be the study of properties of solutions. This is particularly relevant since the soft-killing first-passage time with arbitrary killing rate $\lambda > 0$ as in (1.3) converges to the first-passage time for $\lambda \to \infty$. Therefore, at least heuristically, the solutions to the soft-killing inverse first-passage time problem should approximate the solution of the hard-killing problem. Although this limiting behavior has not yet been provided rigorously, this connection would lead to an approach to pass properties from the soft-killing solutions to the hard-killing solutions.

As matters stand, there has as yet been no detailed study of the properties of solutions for the inverse first-passage time problem with soft-killing. On the one hand, the study of regularity in the hard-killing problem in the context of free boundary problems of [CCS21] may motivate a study of regularity in the soft-killing problem in the context of the free boundary problems as it was the setting in [EHW20]. On the other hand the question arises, whether a direct probabilistic approach is possible.

In this thesis, we began the study of properties of solutions in the hard-killing problem by establishing a comparison principle. In Theorem 3.2.14 we showed a related comparison principle for the soft-killing problem. However, a crucial difference is the fact that the reweighting mechanism is not monotone in the mass parameter as it was demonstrated in Remark 3.1.2. This prevented us from extending the comparison principle in the soft-killing problem to a comparison between different survival distributions as it was the case in the hard-killing problem. Nevertheless, the question arises which properties of solutions can be deduced by the comparison principle in the soft-killing problem.

### Extension of the existence and uniqueness result

For a general survival distribution, if there are solutions at all, reasonable equivalence classes of solutions would be given by the equivalence relation

$$b_1 \sim b_2 :\Leftrightarrow \int_0^\infty |b_1(s) - b_2(s)| \,\mathrm{d}s = 0.$$

Note that if the survival distribution does not meet the conditions of Theorem 3.0.1 on the survival distribution, it follows by Lemma 3.2.1 that we cannot have continuous solutions. The question arises how many different equivalence classes can be found in $\operatorname{ifptk}(g, \mu)$. In the setting of Theorem 3.0.1, this question means that, a priori, there may exist other discontinuous solutions $b \in \operatorname{ifptk}(g, \mu)$, but their relation to the unique continuous solution

$b_c \in \text{ifptk}(g, \mu)$ is unknown. More precisely, we have the inclusion

$$\{b \text{ measurable } : \int_0^\infty |b(s) - b_c(s)| \, ds = 0\} \subseteq \text{ifptk}(g, \mu),$$

but do not know whether there is equality or not.

Another direction in which Theorem 3.0.1 could be extended is to consider other classes of processes. We made the restriction to processes with continuous paths. The problem of existence and uniqueness for processes with discontinuous paths remains open.

# Appendix A

# Appended proofs

In this section we collect alternative proofs to statements from the thesis, or statements, which are not necessary for the main focus of the thesis but add to the overall picture.

## A.1 Alternative proof of Lemma 2.3.25

In Subsection 2.3.3 we made use of the statement of Lemma 2.3.25, which can also be deduced from a related statement in [Rob91b]. In the thesis, the statement of Lemma 2.3.25 follows directly by the more general likelihood ratio ordering result of Theorem 2.3.17. Here we present a more self-contained proof, which uses the usual stochastic order techniques of this thesis.

**Lemma A.1.1.** *Let $b$ be a standard boundary function and $\nu \in \mathcal{P}$ such that $\nu$ is absolutely continuous w.r.t. Lebesgue measure. Then*

$$\mathbb{P}_\nu \left( X_t \in \cdot \, | \tau_b > t \right) \preceq \mathbb{P}_\nu \left( X_t \in \cdot \, \right) \tag{A.1}$$

*for $t \in [0, t^b)$.*

*Proof.* First note, that without loss of generality we can assume that $b(0) > 0$ and, by Lemma 2.3.15, assume that $\mathbb{P}_\nu \left( \bar{\tau}_b = 0 \right) = 0$.
Now, as first step we claim, that it suffices to find a sequence $(b_n)_{n\in\mathbb{N}}$ of boundary functions for which (A.1) holds and $\tau_{b_n} \to \tau_b$ almost surely. The first steps of the following proof of this claim is done in the spirit of the proofs of Corollary 5.2 and Lemma 6.1 from [EJ16] and rely one ideas taken from there. In order to see this, assume that $(b_n)_{n\in\mathbb{N}}$ is such a sequence. Note that $\tau_{b_n} \to \tau_b$ a.s. implies that for any $s \in (0, t^b)$

$$\{\tau_b > s\} \subseteq \liminf_{n\to\infty} \{\tau_{b_n} > s\}$$

holds almost surely. Consequently, using Fatou's Lemma this yields

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}_\nu\left(\tau_b > s \geq \tau_{b_n}\right) &= \limsup_{n\to\infty} \mathbb{P}_\nu\left(\tau_b > s\right) - \mathbb{P}_\nu\left(\tau_b > s, \tau_{b_n} > s\right) \\ &\leq \mathbb{P}_\nu\left(\tau_b > s\right) - \mathbb{P}_\nu\Big(\{\tau_b > s\} \cap \liminf_{n\to\infty}\{\tau_{b_n} > s\}\Big) \\ &= \mathbb{P}_\nu\left(\tau_b > s\right) - \mathbb{P}_\nu\left(\tau_b > s\right) = 0.\end{aligned}$$

If additionally $s$ is an continuity point of $t \mapsto \mathbb{P}_\nu\left(\tau_b > t\right)$ it follows that $\mathbb{P}_\nu\left(\tau_{b_n} > s\right) \to \mathbb{P}_\nu\left(\tau_b > s\right)$ as $n \to \infty$ and using this we obtain

$$\begin{aligned}\mathbb{P}_\nu\left(\tau_{b_n} > s \geq \tau_b\right) &= \mathbb{P}_\nu\left(\tau_{b_n} > s\right) - \mathbb{P}_\nu\left(\tau_{b_n} > s, \tau_b > s\right) \\ &= \mathbb{P}_\nu\left(\tau_{b_n} > s\right) - \mathbb{P}_\nu\left(\tau_b > s\right) + \mathbb{P}_\nu\left(\tau_b > s \geq \tau_{b_n}\right) \\ &\to 0\end{aligned}$$

as $n \to \infty$. Therefore, for every $s > 0$ being a continuity point we get for every $c \geq 0$ that

$$\begin{aligned}&\left|\mathbb{P}_\nu\left(|X_s| \leq c, \tau_{b_n} > s\right) - \mathbb{P}_\nu\left(|X_s| \leq c, \tau_b > s\right)\right| \\ &= \left|\mathbb{P}_\nu\left(|X_s| \leq c, \tau_{b_n} > s, \tau_b \leq s\right) - \mathbb{P}_\nu\left(|X_s| \leq c, \tau_b > s, \tau_{b_n} \leq s\right)\right| \\ &\leq \mathbb{P}_\nu\left(\tau_{b_n} > s \geq \tau_b\right) + \mathbb{P}_\nu\left(\tau_b > s \geq \tau_{b_n}\right) \to 0.\end{aligned}$$

To have (A.1) being true for $s$ and $b_n$ is equivalent to have

$$\mathbb{P}_\nu\left(|X_s| \leq c, \tau_{b_n} > s\right) \geq \mathbb{P}_\nu\left(|X_t| \leq c\right) \mathbb{P}_\nu\left(\tau_{b_n} > s\right)$$

for every $c \geq 0$. By the results above we see that this already yields the same inequality, by letting $n \to \infty$, for every $c \geq 0$, which means that (A.1) holds for $b$ for every continuity point $s \in [0, t^b)$. Consequently, (A.1) would be true for arbitrary $s$ if the expression $\mathbb{P}_\nu\left(|X_t| \leq c, \tau_b > t\right)$ was right-continuous in $t$. To see this let $t > 0$. By continuity of measure and monotonicity we have

$$\lim_{s\searrow t} \mathbb{P}_\nu\left(|X_t| \leq c, \tau_b > s\right) = \mathbb{P}_\nu\left(|X_t| \leq c, \tau_b > t\right).$$

Furthermore, for $s > t$ we have

$$\begin{aligned}&\sup_{r>t} \left|\mathbb{P}_\nu\left(|X_s| \leq c, \tau_b > r\right) - \mathbb{P}_\nu\left(|X_t| \leq c, \tau_b > r\right)\right| \\ &\leq \sup_{r>t} \left|\mathbb{P}_\nu\left(|X_s| \leq c, \tau_b > r, |X_t| > c\right) - \mathbb{P}_\nu\left(|X_t| \leq c, \tau_b > r, |X_s| > c\right)\right| \\ &\leq \mathbb{P}_\nu\left(|X_s| \leq c, |X_t| > c\right) + \mathbb{P}_\nu\left(|X_t| \leq c, |X_s| > c\right) \to 0\end{aligned}$$

as $s \searrow t$. Altogether

$$\begin{aligned}
&|\mathbb{P}_\nu(|X_s| \leq c, \tau_b > s) - \mathbb{P}_\nu(|X_t| \leq c, \tau_b > t)| \\
&\leq |\mathbb{P}_\nu(|X_s| \leq c, \tau_b > s) - \mathbb{P}_\nu(|X_t| \leq c, \tau_b > s)| \\
&\quad + |\mathbb{P}_\nu(|X_t| \leq c, \tau_b > s) - \mathbb{P}_\nu(|X_t| \leq c, \tau_b > t)| \\
&\leq \sup_{r>t} |\mathbb{P}_\nu(|X_s| \leq c, \tau_b > r) - \mathbb{P}_\nu(|X_t| \leq c, \tau_b > r)| \\
&\quad + |\mathbb{P}_\nu(|X_t| \leq c, \tau_b > s) - \mathbb{P}_\nu(|X_t| \leq c, \tau_b > t)| \\
&\to 0
\end{aligned}$$

as $s \searrow t$, which proves the right-continuity in $t$, and thus the claim of the first step.
As second step we will assume that $b$ was already continuous. Denote with $D_n \coloneqq \{k2^{-n} : k \in \mathbb{N}\}$ the dyadic numbers. Define boundary functions $b_n$ by

$$b_n(t) \coloneqq \begin{cases} b(t) & : t \in D_n, \\ \infty & : t \notin D_n, t < t^b, \\ 0 & : \text{else}. \end{cases}$$

We claim that every $b_n$ fulfilles (A.1). In order to see this, denote $d_n \coloneqq 2^{-n}$, $\lfloor t \rfloor_n \coloneqq \sup\{kd_n \leq t : k \in \mathbb{N}_0\}$ and

$$\alpha_k^n \coloneqq \frac{\mathbb{P}_\nu(\tau_{b_n} > kd_n)}{\mathbb{P}_\nu(\tau_{b_n} > (k-1)d_n)}.$$

By the Markov property of Brownian motion, Corollary 2.2.9 and the semigroup property of the Gaussian convolution operator we have for every $t \in (0, t^b)$ that

$$\begin{aligned}
&\mathbb{P}_\nu(X_t \in \cdot\, | \tau_{b_n} > t) = \mathbb{P}_\nu(X_t \in \cdot\, | \tau_{b_n} > \lfloor t \rfloor_n) \\
&= P_{t-\lfloor t \rfloor_n} \circ T_{\alpha^n_{d_n \lfloor t \rfloor_n}} \circ P_{d_n} \circ \ldots \circ T_{\alpha_1^n} \circ P_{d_n}(\nu) \\
&\preceq P_{t-\lfloor t \rfloor_n} \circ P_{d_n} \circ \ldots \circ P_{d_n}(\nu) = P_{t-\lfloor t \rfloor_n} \circ P_{\lfloor t \rfloor_n}(\nu) \\
&= \mathbb{P}_\nu(X_t \in \cdot\,),
\end{aligned}$$

which proves the claim. Now observe, that in view of Theorem 2.1.4 we have for any sequence $(t_n)_{n\in\mathbb{N}}$ converging to $t$ that

$$\liminf_{n\to\infty} b_n(t_n) \geq \liminf_{n\to\infty} b(t_n) \geq b(t),$$

since $b$ is lower semicontinuous. Furthermore, we can take a sequence $(t_n)_{n\in\mathbb{N}}$ converging to $t$ such that $t_n \in D_n$. Then

$$\lim_{n\to\infty} b_n(t_n) = \lim_{n\to\infty} b(t_n) = b(t)$$

since $b$ is continuous. Hence, Theorem 2.1.4 yields that $b_n \xrightarrow{\Gamma} b$. Since $\mathbb{P}_\nu(\bar{\tau}_b = 0) = 0$, by Proposition 2.1.12 we obtain that $\tau_{b_n} \to \tau_b$ in probability. But since we can take almost surely convergent subsequence from this, the requirements of the first step are fullfilled and thus the statement holds also for $b$.
As third and last step assume that $b$ is as arbitrary as in the statement. Then by Corollary B.2.5 there exists a sequence $(b_n)_{n\in\mathbb{N}}$ of continuous functions such that $b_n \leq b_{n+1}$, $b_n \to b$ pointwise and $b_n(0) = b(0)$. First, we claim that $\mathbb{P}_\nu(\tau_{b_n} = 0) = 0$. For this note that by Lemma 2.3.1 it follows that $\operatorname{supp}(\nu) \subseteq [-b(0), b(0)]$, since $b$ is standard. Now by assumption on $\nu$ we have $|X_0| < b(0) = b_n(0) = \liminf_{s\searrow 0} b_n(s)$ almost surely, which indeed implies that $\tau_{b_n} > 0$ almost surely. This means that by the second step every $b_n$ fulfilles (A.1). Additionally, by Lemma 2.1.8 it follows that $b_n \xrightarrow{\Gamma} b$ in the sense of boundary functions. Furthermore, Proposition 2.1.12 implies again that $\tau_{b_n} \to \tau_b$ in probability, from which we can deduce that there is an almost surely convergent subsequence. But this means that by the first step that $b$ fulfilles (A.1), and thus the statement is proven. □

Staying within the self-contained path we can derive Proposition 2.3.24 from Lemma A.1.1.

**Corollary A.1.2.** *Let $t > 0$. Let $b$ be a boundary function and $\nu \in \mathcal{P}$ such that $\mathbb{P}_\nu(\tau_b > t) > 0$. Let $s \in (0, t]$. Choose $x \in \mathbb{R}$ such that $|x| \geq b(s)$. Then*

$$\mathbb{P}_\nu(X_t \in \cdot\, | \tau_b > t) \preceq \mathbb{P}_x(X_{t-s} \in \cdot\,).$$

*Proof.* It holds $\nu_s := \mathbb{P}_\nu(X_s \in \cdot\, | \tau_b > s) \preceq \delta_x$, because we have that $\operatorname{supp}(\nu_s) \subseteq [-b(s), b(s)]$. Since $\nu_s$ is absolutely continuous w.r.t. Lebesgue measure, this means that under the measure $\mathbb{P}_{\nu_s}$ we have almost surely $|X_0| < b(s) \leq \liminf_{r\searrow 0} b^s(r)$, which implies that $\mathbb{P}_{\nu_s}(\tau_{b^s} = 0) = 0$. By Lemma 2.3.25 and Proposition 2.2.3 we have

$$\begin{aligned}\mathbb{P}_\nu(X_t \in \cdot\, | \tau_b > t) &= \mathbb{P}_{\nu_s}(X_{t-s} \in \cdot\, | \tau_{b^s} > t - s) \\ &\preceq \mathbb{P}_{\nu_s}(X_{t-s} \in \cdot) \preceq \mathbb{P}_x(X_{t-s} \in \cdot),\end{aligned}$$

which yields the statement. □

## A.2 The Fredholm integral equation connecting $g$, $b$ and $\mu$

In the following we derive the integral equation, which was mentioned in Remark 2.3.40.

**Proposition A.2.1.** *Let $\mu \in \mathcal{P}$ be symmetric and $\alpha \in \mathbb{R}$ with $\int_{\mathbb{R}} e^{\alpha x}\,\mathrm{d}\mu(x) < \infty$. Further, let $g$ be a continuous survival distribution. Let $b \in \mathrm{ifpt}(g,\mu)$ and assume that $\limsup_{t\to\infty} b(t) < \infty$. Then*

$$\int_{\mathbb{R}} e^{\alpha x}\,\mathrm{d}\mu(x) = \int_0^\infty \cosh(\alpha b(t)) e^{-\frac{\alpha^2}{2}t} g(\mathrm{d}t).$$

*Proof.* Due to the symmetry of $\mu$ we have that $\mathrm{sgn}(X_{\tau_b})$ is independent from $\tau_b$ and has Rademacher distribution. A direct consequence of $\limsup_{t\to\infty} b(t) < \infty$ is that $\tau_b < \infty$ almost surely. Due to the continuity of $g$ we obtain as in Theorem 8.1 of [EJ16] that $|X_{\tau_b}| = b(\tau_b)$ almost surely. Let

$$N_t := e^{\alpha X_{t\wedge\tau_b} - \frac{\alpha^2}{2} t\wedge\tau_b}.$$

The Novikov condition and the optional stopping theorem yield that $(N_t)_{t\geq 0}$ is a martingale. Now, using that $\limsup_{t\to\infty} b(t) < \infty$ there is a $T > 0$ such that $M := \sup_{t\geq T} b(t) < \infty$. Thus we have

$$\begin{aligned} e^{\alpha X_{t\wedge\tau_b}} &\leq \max\left(e^{\sup_{t\leq T}\alpha X_{t\wedge\tau_b}}, e^{|\alpha| M}\right) \leq \max\left(e^{\sup_{t\leq T}\alpha X_t}, e^{|\alpha| M}\right) \\ &\leq e^{\sup_{t\leq T}\alpha X_t} + e^{|\alpha|M} = e^{\alpha X_0} e^{\alpha \sup_{t\leq T}(X_t - X_0)} + e^{|\alpha| M} \end{aligned}$$

The right-hand side is integrable due to the assumption on $\mu$ and the fact, that the moment generating function of the normal distribution exists, and therefore, by the dominated convergence theorem, using that $\tau_b \sim g$, we have

$$\begin{aligned} &\int_{\mathbb{R}} e^{\alpha x}\,\mathrm{d}\mu(x) = \mathbb{E}_\mu[N_0] = \lim_{t\to\infty}\mathbb{E}_\mu[N_t] = \mathbb{E}_\mu\left[e^{\alpha X_{\tau_b} - \frac{\alpha^2}{2}\tau_b}\right] \\ &= \mathbb{E}_\mu\left[e^{\alpha b(\tau_b) - \frac{\alpha^2}{2}\tau_b}\mathbb{1}_{\{\mathrm{sgn}(X_{\tau_b})=1\}}\right] + \mathbb{E}_\mu\left[e^{-\alpha b(\tau_b) - \frac{\alpha^2}{2}\tau_b}\mathbb{1}_{\{\mathrm{sgn}(X_{\tau_b})=-1\}}\right] \\ &= \frac{1}{2}\left(\mathbb{E}_\mu\left[e^{\alpha b(\tau_b) - \frac{\alpha^2}{2}\tau_b}\right] + \mathbb{E}_\mu\left[e^{-\alpha b(\tau_b) - \frac{\alpha^2}{2}\tau_b}\right]\right) \\ &= \mathbb{E}_\mu\left[\cosh(\alpha b(\tau_b)) e^{-\frac{\alpha^2}{2}\tau_b}\right] = \int_0^\infty \cosh(\alpha b(t)) e^{-\frac{\alpha^2}{2}t} g(\mathrm{d}t) \end{aligned}$$

which is the desired statement. □

**Remark A.2.2.** The integral equation of Proposition A.2.1 appeared in the one-sided situation for special cases of the parameters $\mu$, $g$ and $b$ and is seen from different viewpoints in the literature. In the context of the first-passage time problem, for example, see [JKV09a], where the proof also relies on the exponential martingale, and see [JKZ09], [JKV09b], [JKV14] in the context of the randomized first-passage time problem. Indirectly, the equation appeared also in [BBP19] in the special case that $g$ corresponds to the exponential distribution in the context of a free boundary problem. For the two-sided situation the integral equation appeared in the special case of a constant boundary in [Abu13b].

## A.3 Total positivity of the quasi-stationary distribution $\nu_\lambda$

In Subsection 2.3.5 we worked with the density function

$$\varphi(x) = \frac{\pi}{4a}\cos(\frac{\pi x}{2a})\mathbb{1}_{[-a,a]}(x)$$

of the quasi-stationary distribution $\nu_\lambda$. By the tools of quasi-stationarity in Corollary 2.3.59 it was possible to derive one of the required conditions for Theorem 2.3.47. A stronger sufficient condition is provided by Lemma 2.3.51, namely total positivity of the generalized absolute value density of $\varphi$. In Remark 2.3.60 it was stated that this is indeed fulfilled, but was not proved nor needed in the thesis.

**Proposition A.3.1.** *Let $\varphi$ be the density of the quasi-stationary distribution of the Brownian motion killed at $a > 0$ and $-a$, which is then given by $\varphi(x) = \frac{\pi}{4a}\cos(\frac{\pi x}{2a})\mathbb{1}_{[-a,a]}(x)$. Then the generalized absolute value density of $\varphi$, given by $\varphi^*(u, v) = \varphi(v+u) + \varphi(v-u)$ for $v, u \geq 0$, is TP$_2$ (totally positive of order 2).*

*Proof.* At first, note that $\varphi^*$ is totally positive of order 2 if and only if $(v, u) \mapsto a\varphi^*(bv, bu)$ is totally positive of order 2, where $a, b > 0$ are arbitrary constants. Hence, without loss of generality we can assume that $\varphi(x) = \cos(x)\mathbb{1}_{[-\pi/2,\pi/2]}(x)$.

Furthermore, due to the symmetry of $\varphi^*(u, v) = \varphi^*(v, u)$, it holds that

$$\det\begin{pmatrix}\varphi^*(v_1,u_1) & \varphi^*(v_1,u_2)\\ \varphi^*(v_2,u_1) & \varphi^*(v_2,u_2)\end{pmatrix} = \det\begin{pmatrix}\varphi^*(u_1,v_1) & \varphi^*(u_1,v_2)\\ \varphi^*(u_2,v_1) & \varphi^*(v_2,u_2).\end{pmatrix}$$

Thus it would suffice to show that

$$\det\begin{pmatrix}\varphi^*(v_1,u_1) & \varphi^*(v_1,u_2)\\ \varphi^*(v_2,u_1) & \varphi^*(v_2,u_2)\end{pmatrix} = \varphi^*(v_1,u_2)\varphi^*(v_2,u_2) - \varphi^*(v_1,u_2)\varphi^*(v_2,u_1) \tag{A.2}$$

$$\geq 0 \qquad \forall\, 0 \leq v_1 \leq v_2 \leq u_2, 0 \leq u_1 \leq u_2,$$

which is automatically fulfilled, if $|v_1 - u_2| \geq \frac{\pi}{2}$ or $|v_2 - u_1| \geq \frac{\pi}{2}$. From now on, let $0 \leq v_1 \leq v_2$ and $0 \leq u_1 \leq u_2$ such that $v_2 \leq u_2$ and $\max(|v_1 - u_2|, |v_2 - u_1|) \leq \frac{\pi}{2}$. We distinguish six cases.

1. case: $v_2 + u_2 < \frac{\pi}{2}$.

$$\begin{aligned}
&\det\begin{pmatrix}\varphi^*(v_1,u_1) & \varphi^*(v_1,u_2)\\ \varphi^*(v_2,u_1) & \varphi^*(v_2,u_2)\end{pmatrix}\\
&= (\cos(v_1+u_1) + \cos(v_1-u_1))(\cos(v_2+u_2) + \cos(v_2-u_2))\\
&\quad - (\cos(v_1+u_2) + \cos(v_1-u_2))(\cos(v_2+u_1) + \cos(v_2-u_1))\\
&= 4\cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2) - 4\cos(v_1)\cos(u_2)\cos(v_2)\cos(u_1)\\
&= 0,
\end{aligned}$$

which yields the desired statement.

2. case: $v_2 + u_2 \geq \frac{\pi}{2}$, $v_1 + u_2 < \frac{\pi}{2}$, $v_2 + u_1 < \frac{\pi}{2}$

Note that this case implies $\frac{\pi}{4} < v_2 \leq u_2 < \frac{\pi}{2}$, $|v_2 - u_2| = u_2 - v_2 < \frac{\pi}{2}$, which implies in particular $v_2 + u_2 < \pi$. Then

$$\begin{aligned}
&\det \begin{pmatrix} \varphi^*(v_1, u_1) & \varphi^*(v_1, u_2) \\ \varphi^*(v_2, u_1) & \varphi^*(v_2, u_2) \end{pmatrix} \\
&= (\cos(v_1 + u_1) + \cos(v_1 - u_1)) (0 + \cos(v_2 - u_2)) \\
&\quad - (\cos(v_1 + u_2) + \cos(v_1 - u_2)) (\cos(v_2 + u_1) + \cos(v_2 - u_1)) \\
&= 2\cos(v_1)\cos(u_1) (\cos(v_2)\cos(u_2) + \sin(v_2)\sin(u_2)) \\
&\quad - 4\cos(v_1)\cos(u_2)\cos(v_2)\cos(u_1) \\
&= 2\cos(v_1)\cos(u_1) (\cos(v_2)\cos(u_2) + \sin(v_2)\sin(u_2) - 2\cos(v_2)\cos(u_2)) \\
&= 2\cos(v_1)\cos(u_1) (-(\cos(v_2)\cos(u_2) - \sin(v_2)\sin(u_2))) \\
&= 2\cos(v_1)\cos(u_1)\sin\left(v_2 + u_2 - \frac{\pi}{2}\right) \geq 0,
\end{aligned}$$

since $v_2 + u_2 - \frac{\pi}{2} \in (0, \frac{\pi}{2})$ and $\cos(v_1)\cos(u_1) \geq 0$.

3. case: $v_1 + u_2 \geq \frac{\pi}{2}$, $v_2 + u_1 < \frac{\pi}{2}$.

It follows in this case, by the general assumption $v_2 \leq u_2$ that $v_1 - u_2 \leq v_2 - u_2 \leq 0$, hence $|v_1 - u_2| \geq |v_2 - u_2|$, which in turn implies $u_2 - v_2 = |v_2 - u_2| < \frac{\pi}{2}$, due to the general assumption $|v_1 - u_2| < \frac{\pi}{2}$. Thus we have

$$\begin{aligned}
&\det \begin{pmatrix} \varphi^*(v_1, u_1) & \varphi^*(v_1, u_2) \\ \varphi^*(v_2, u_1) & \varphi^*(v_2, u_2) \end{pmatrix} \\
&= (\cos(v_1 + u_1) + \cos(v_1 - u_1)) (0 + \cos(v_2 - u_2)) \\
&\quad - (0 + \cos(v_1 - u_2)) (\cos(v_2 + u_1) + \cos(v_2 - u_1)) \\
&= 2\cos(v_1)\cos(u_1)(\cos(v_2)\cos(u_2) + \sin(v_2)\sin(u_2)) \\
&\quad 2\cos(v_2)\cos(u_1)(\cos(v_1)\cos(u_2) + \sin(v_1)\sin(u_2)) \\
&= 2\cos(v_1)\cos(u_1)\sin(v_2)\sin(u_2) - 2\cos(v_2)\cos(u_1)\sin(v_1)\sin(u_2) \\
&= 2\cos(u_1)\sin(u_2)(\cos(v_1)\sin(v_2) - \cos(v_2)\sin(v_1)) \\
&= 2\cos(u_1)\sin(u_2)\cos(v_1)\cos(v_2)(\tan(v_2) - \tan(v_1)),
\end{aligned}$$

which is non-negative, since $v_2 \geq v_1$ and $0 \leq v_1 \leq v_2 \leq v_2 + u_1 < \frac{\pi}{2}$ and $0 \leq u_1 \leq u_2 < \frac{\pi}{2} + v_1 < \pi$, together with the fact that tan is increasing on $[0, \frac{\pi}{2})$.

4. case: $v_1 + u_2 < \frac{\pi}{2}$, $v_2 + u_1 \geq \frac{\pi}{2}$

In analogy to the computation of the 3. case we obtain

$$\begin{aligned}
&\det \begin{pmatrix} \varphi^*(v_1, u_1) & \varphi^*(v_1, u_2) \\ \varphi^*(v_2, u_1) & \varphi^*(v_2, u_2) \end{pmatrix} \\
&= (\cos(v_1 + u_1) + \cos(v_1 - u_1))\cos(v_2 - u_2) \\
&\quad - (\cos(v_1 + u_2) + \cos(v_1 - u_2))\cos(v_2 - u_1) \\
&= 2\cos(v_1)\sin(v_2)\cos(u_1)\cos(u_2)(\tan(u_2) - \tan(u_1)),
\end{aligned}$$

which is non-negative, since $u_2 \geq u_1$ and $0 \leq v_1, v_2, u_1 \leq u_2 \leq v_1 + u_2 < \frac{\pi}{2}$.

5. case: $v_1 + u_2 \geq \frac{\pi}{2}$, $v_2 + u_1 \geq \frac{\pi}{2}$, $v_1 + u_1 < \frac{\pi}{2}$

As in the 3. case we have $\frac{\pi}{2} > |v_1 - u_2| \geq |v_2 - u_2|$ and therefore $u_2 < \frac{\pi}{2} + v_1 < \pi$. We compute

$$
\begin{aligned}
&\det \begin{pmatrix} \varphi^*(v_1, u_1) & \varphi^*(v_1, u_2) \\ \varphi^*(v_2, u_1) & \varphi^*(v_2, u_2) \end{pmatrix} \\
&= (\cos(v_1 + u_1) + \cos(v_1 - u_1)) \cos(v_2 - u_2) - \cos(v_1 - u_2) \cos(v_2 - u_1) \\
&= 2\cos(v_1)\cos(u_1)(\cos(v_2)\cos(u_2) + \sin(v_2)\sin(u_2)) \\
&\quad - (\cos(v_1)\cos(u_2) + \sin(v_1)\sin(u_2))(\cos(v_2)\cos(u_1) + \sin(v_2)\sin(u_1)) \\
&= 2\cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2) + 2\cos(v_1)\cos(u_1)\sin(v_2)\sin(u_2) \\
&\quad - \cos(v_1)\cos(u_2)\cos(v_2)\cos(u_1) - \sin(v_1)\sin(u_2)\sin(v_2)\sin(u_1) \\
&\quad - \cos(v_1)\cos(u_2)\sin(v_2)\sin(u_1) - \cos(v_2)\cos(u_1)\sin(v_1)\sin(u_2) \\
&= \cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2) - \sin(v_1)\sin(u_1)\sin(v_2)\sin(u_2) \\
&\quad + \cos(v_1)\sin(v_2)\,(\cos(u_1)\sin(u_2) - \cos(u_2)\sin(u_1)) \\
&\quad + \cos(u_1)\sin(u_2)\,(\cos(v_1)\sin(v_2) - \cos(v_2)\sin(v_1))
\end{aligned}
$$

Assume $v_2, u_2 \neq \frac{\pi}{2}$. Then dividing the quantity above by the then non-zero value $\cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2)$ results in

$$
\begin{aligned}
&1 - \tan(v_1)\tan(u_1)\tan(v_2)\tan(u_2) \\
&\quad + \tan(v_2)(\tan(u_2) - \tan(u_1)) + \tan(u_2)(\tan(v_2) - \tan(v_1)) \\
&= (\tan(v_2) - \tan(v_1))(\tan(u_2) - \tan(u_1)) \\
&\quad + (1 - \tan(v_1)\tan(u_1))(1 + \tan(v_2)\tan(u_2)) \\
&= (\tan(v_2) - \tan(v_1))(\tan(u_2) - \tan(u_1)) \\
&\quad + \frac{\tan(v_1) + \tan(u_1)}{\tan(v_1 + u_1)}(1 + \tan(v_2)\tan(u_2)).
\end{aligned} \tag{A.3}
$$

Now we treat three possible cases. In the case $v_2, u_2 < \frac{\pi}{2}$, the divisor

$$\cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2)$$

used above is positive and the remaining quantity above in (A.3) is non-negative, since $v_1 \leq v_2, u_1 \leq u_2 < \frac{\pi}{2}$ and $v_1 + u_1 < \frac{\pi}{2}$. In the case that $v_2 < \frac{\pi}{2}$ and $u_2 > \frac{\pi}{2}$ the divisor $\cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2)$ is negative, thus we have to show that the quantity above in (A.3) is non-positive. For this note that $\tan(v_2) - \tan(v_1) \geq 0$, while

$$
\begin{aligned}
\tan(u_2) - \tan(u_1) &= -\tan(-u_2) - \tan(u_1) = -\tan(\pi - u_2) - \tan(u_1) \\
&= -(\tan(\pi - u_2) - \tan(u_1)) \leq 0,
\end{aligned}
$$

since $u_1, \pi - u_2 \in [0, \frac{\pi}{2})$. Recall that it holds $u_2 - v_2 < \frac{\pi}{2}$. We have that $(\tan(v_1) + \tan(u_1))/\tan(v_1 + u_1) \geq 0$, since $0 \leq v_1 + u_1 \leq \frac{\pi}{2}$, and

$1+\tan(v_2)\tan(u_2) = 1-\tan(v_2)\tan(\pi-u_2) = (\tan(v_2)+\tan(\pi-u_2))/\tan(v_2+\pi-u_2) \leq 0$, since $v_2, \pi-u_2 < \frac{\pi}{2}$ but $\pi-(u_2-v_2) > \frac{\pi}{2}$. This completes the argument in this case. The case left is to assume that $v_2, u_2 > \frac{\pi}{2}$. Then the divisor $\cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2)$ is positive and it is to show that (A.3) is non-negative. In the same manner as above $\tan(v_2)-\tan(v_1)$ and $\tan(u_2)-\tan(u_1)$ are both non-positive and $\tan(v_2)\tan(u_2) \geq 0$, since $v_2, u_2 \in (\frac{\pi}{2}, \pi)$, which completes the 5. case.

6. case: $v_1 + u_1 \geq \frac{\pi}{2}$
It holds

$$
\begin{aligned}
&\det\begin{pmatrix} \varphi^*(v_1,u_1) & \varphi^*(v_1,u_2) \\ \varphi^*(v_2,u_1) & \varphi^*(v_2,u_2) \end{pmatrix} \\
&= \cos(v_1-u_1)\cos(v_2-u_2) - \cos(v_1-u_2)\cos(v_2-u_1) \\
&= (\cos(v_1)\cos(u_1)+\sin(v_1)\sin(u_1))(\cos(v_2)\cos(u_2)+\sin(v_2)\sin(u_2)) \\
&\quad - (\cos(v_1)\cos(u_2)+\sin(v_1)\sin(u_2))(\cos(v_2)\cos(u_1)+\sin(v_2)\sin(u_1)) \\
&= \cos(v_1)\cos(u_1)\cos(v_2)\cos(u_2) + \cos(v_1)\cos(u_1)\sin(v_2)\sin(u_2) \\
&\quad + \cos(v_2)\cos(u_2)\sin(v_1)\sin(u_1) + \sin(v_1)\sin(u_1)\sin(v_2)\sin(u_2) \\
&\quad - \cos(v_1)\cos(u_2)\cos(v_2)\cos(u_1) - \cos(v_1)\cos(u_2)\sin(v_2)\sin(u_1) \\
&\quad - \sin(v_1)\sin(u_2)\cos(v_2)\cos(u_1) - \sin(v_1)\sin(u_2)\sin(v_2)\sin(u_1) \\
&= \cos(v_1)\cos(u_1)\sin(v_2)\sin(u_2) + \cos(v_2)\cos(u_2)\sin(v_1)\sin(u_1) \\
&\quad - \cos(v_1)\cos(u_2)\sin(v_2)\sin(u_1) - \sin(v_1)\sin(u_2)\cos(v_2)\cos(u_1) \\
&= (\cos(v_1)\sin(v_2) - \cos(v_2)\sin(v_1))(\cos(u_1)\sin(u_2) - \cos(u_2)\sin(u_1)) \\
&= \sin(v_2-v_1)\sin(u_2-u_1),
\end{aligned}
$$

which is non-negative, since $v_2 - v_1 \leq u_2 - v_1 < \frac{\pi}{2}$ and $u_2 - u_1 \leq u_2 - v_1 + v_1 - u_1 \leq u_2 - v_1 + |v_1 - u_1| \leq \pi$.

This finishes the proof. □

# A.4 Truncation and reweighting with respect to other probability distances

### Total variation distance and truncation

Recall the total variation distance from (3.14). A total variation counterpart for the Wasserstein distance related to the statements of Lemma 2.2.24 and Lemma 2.2.25 are given by the following.

**Lemma A.4.1.** *Let $\mu, \nu \in \mathcal{P}$. Then for all $t \geq 0$ we have*

$$d_{\mathrm{TV}}(P_t\mu, P_t\nu) \leq d_{\mathrm{TV}}(\mu, \nu).$$

*Proof.* Let $X \sim \mu$, $Y \sim \nu$ and $B_t \sim \mathcal{N}(0,t)$ independent from $X$ and $Y$. Then $X + B_t \sim P_t\mu$ and $Y + B_t \sim P_t\nu$. Thus by (3.17) we have

$$d_{\mathrm{TV}}\left(P_t\mu, P_t\nu\right) \leq \mathbb{P}\left(X + B_t \neq Y + B_t\right) = \mathbb{P}\left(X \neq Y\right).$$

Taking the infimum over all possible $X, Y$ yields the statement. □

**Lemma A.4.2.** *Let $\mu, \nu \in \mathcal{P}$ be absolutely continuous with respect to the Lebesgue measure with densities $f$ and $g$, respectivly. Then for all $\alpha \in (0,1]$ holds*

$$d_{\mathrm{TV}}\left(T_\alpha\mu, T_\alpha\nu\right) \leq \frac{1}{\alpha} d_{\mathrm{TV}}\left(\mu, \nu\right).$$

*Proof.* Assume $\alpha < 1$ and without loss of generality $q_\alpha(\mu) > q_\alpha(\nu)$. The densities of $T_\alpha\mu$ and $T_\alpha\nu$ are given by $\frac{1}{\alpha} f \mathbb{1}_{[-q_\alpha(\mu), q_\alpha(\mu)]}$ and$\frac{1}{\alpha} g \mathbb{1}_{[-q_\alpha(\nu), q_\alpha(\nu)]}$. Their minumum is given by

$$\frac{1}{\alpha}(f \wedge g)\mathbb{1}_{[-q_\alpha(\nu), q_\alpha(\nu)]}.$$

For the next step note that

$$\begin{aligned}\int_{\mathbb{R}\setminus[-q_\alpha(\nu), q_\alpha(\nu)]} (f \wedge g)(x)\,\mathrm{d}x &\leq \int_{\mathbb{R}\setminus[-q_\alpha(\nu), q_\alpha(\nu)]} g(x)\,\mathrm{d}x \\ &= 1 - \nu([-q_\alpha(\nu), q_\alpha(\nu)]) = 1 - \alpha.\end{aligned}$$

By (3.16) we obtain that

$$\begin{aligned}d_{\mathrm{TV}}\left(T_\alpha\mu, T_\alpha\nu\right) &= \left(1 - \frac{1}{\alpha}\int_{[-q_\alpha(\nu), q_\alpha(\nu)]} (f \wedge g)(x)\,\mathrm{d}x\right) \\ &= \frac{1}{\alpha}\left(1 - (1-\alpha) - \int_{[-q_\alpha(\nu), q_\alpha(\nu)]} (f \wedge g)(x)\,\mathrm{d}x\right) \\ &\leq \frac{1}{\alpha}\left(1 - \int_{\mathbb{R}} (f \wedge g)(x)\,\mathrm{d}x\right) \\ &= \frac{1}{\alpha} d_{\mathrm{TV}}\left(\mu, \nu\right),\end{aligned}$$

which completes the proof. □

**Lemma A.4.3.** *Let $\mu, \nu \in \mathcal{P}$ be absolutely continuous with respect to the Lebesgue measure with densities $f$ and $g$, respectivly. Then for all $\alpha \in (0,1]$ holds*

$$d_{\mathrm{TV}}\left(T_\alpha\mu, \nu\right) \leq d_{\mathrm{TV}}\left(\mu, \nu\right) + (1 - \alpha).$$

*Proof.* Recall the densities from the proof of Lemma A.4.2. Let $q := \min\{q_\alpha(\mu), \sup\operatorname{supp}(\nu)\}$. The minimum of the densities of $T_\alpha\mu$ and $\nu$ is then given by

$$(\alpha^{-1} f \wedge g)\mathbb{1}_{[-q,q]}.$$

Thus by (3.16) we can write

$$\begin{aligned} d_{\mathrm{TV}}\left(T_\alpha\mu, \nu\right) - d_{\mathrm{TV}}\left(\mu, \nu\right) &= \left(\int_{\mathbb{R}} (f \wedge g)(x)\,\mathrm{d}x - \int_{[-q,q]} (\alpha^{-1} f \wedge g)(x)\,\mathrm{d}x\right) \\ &\leq \left(\int_{\mathbb{R}} (f \wedge g)(x)\,\mathrm{d}x - \int_{[-q,q]} (f \wedge g)(x)\,\mathrm{d}x\right) \\ &= \int_{\mathbb{R}\setminus[-q,q]} (f \wedge g)(x)\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}\setminus[-q_\alpha(\mu),q_\alpha(\mu)]} f(x)\,\mathrm{d}x \leq (1-\alpha), \end{aligned}$$

which yields the statement. □

**Wasserstein distance and reweighting**

Recall the Wasserstein distance $d_{\mathrm{W}}$ from (2.11) and the reweighting operator $R_\alpha^t$ from (3.11). The behaviour of $R_\alpha^t$ with respect to this distance is described by the next lemma and is an analogous statement to Lemma 3.1.8 as mentioned in Remark 3.1.9.

**Lemma A.4.4.** *Let $\mu, \nu \in \mathcal{P}^1$ non-atomic with $R_\alpha^t(\mu)(\mathbb{R}) = R_\alpha^t(\nu) = \alpha$, where $t > 0$ and $\alpha \in (e^{-t}, 1)$. Then $\frac{1}{\alpha}R_\alpha^t(\mu), \frac{1}{\alpha}R_\alpha^t(\nu) \in \mathcal{P}^1$ and*

$$d_{\mathrm{W}}\left(\frac{1}{\alpha}R_\alpha^t(\mu), \frac{1}{\alpha}R_\alpha^t(\nu)\right) \leq \frac{1}{\alpha} d_{\mathrm{W}}\left(\mu, \nu\right).$$

*Proof.* By the definition of $R_\alpha^t$ it is clear that $\int_{\mathbb{R}} |x| R_\alpha^t(\mu)(\mathrm{d}x) \leq \int_{\mathbb{R}} |x|\mu(dx)$, and thus $\frac{1}{\alpha}R_\alpha^t(\mu) \in \mathcal{P}^1$ and analogous $\frac{1}{\alpha}R_\alpha^t(\nu) \in \mathcal{P}^1$.
For $\sigma \in \{\mu, \nu\}$ let $F_\sigma(x) := \sigma((-\infty, x])$. We have

$$\begin{aligned} F_{\frac{1}{\alpha}R_\alpha^t(\sigma)}(x) &:= \frac{1}{\alpha} R_\alpha^t(\sigma)((-\infty, x]) \\ &= \frac{1}{\alpha}\begin{cases} e^{-t}\sigma((-\infty, x]) & : x \leq q_\alpha^t(\sigma), \\ e^{-t}\sigma((-\infty, q_\alpha^t(\sigma)) + \sigma((q_\alpha^t(\sigma), x]) & : x > q_\alpha^t(\sigma), \end{cases} \\ &= \frac{1}{\alpha}\begin{cases} e^{-t}F_\sigma(x) & : x \leq q_\alpha^t(\sigma), \\ F_\sigma(x) - F_\sigma(q_\alpha^t(\sigma))(1 - e^{-t}) & : x > q_\alpha^t(\sigma), \end{cases} \\ &= \frac{1}{\alpha}\begin{cases} e^{-t}F_\sigma(x) & : x \leq q_\alpha^t(\sigma), \\ F_\sigma(x) - (1-\alpha) & : x > q_\alpha^t(\sigma), \end{cases} \end{aligned}$$

where in the last equation we used that by (3.12) we get that

$$F_\sigma(q_\alpha^t(\sigma)) = \frac{\sigma(\mathbb{R}) - \alpha}{1 - e^{-t}} = \frac{1-\alpha}{1-e^{-t}}.$$

For the inequality left to show, we assume without loss of generality that $q_\alpha^t(\mu) \leq q_\alpha^t(\nu)$. Observe that for $x \in [q_\alpha^t(\mu), q_\alpha^t(\nu)]$ it holds

$$\begin{aligned} e^{-t}F_\nu(x) &\leq e^{-t}F_\nu(q_\alpha^t(\nu)) = e^{-t}\frac{1-\alpha}{1-e^{-t}} = \frac{1-\alpha}{1-e^{-t}} - (1-e^{-t})\frac{1-\alpha}{1-e^{-t}} \\ &= F_\mu(q_\alpha^t(\mu)) - (1-\alpha) \leq F_\mu(x) - (1-\alpha). \end{aligned}$$

By the alternative representation of the Wasserstein distance given in (2.13) we obtain

$$\begin{aligned} &\alpha d_{\mathrm{W}}\left(\frac{1}{\alpha}R_\alpha^t(\mu), \frac{1}{\alpha}R_\alpha^t(\nu)\right) = \alpha \int_{\mathbb{R}} \left|F_{\frac{1}{\alpha}R_\alpha^t(\mu)}(x) - F_{\frac{1}{\alpha}R_\alpha^t(\nu)}(x)\right| \mathrm{d}x \\ &= \int_{-\infty}^{q_\alpha^t(\mu)} \left|e^{-t}F_\mu(x) - e^{-t}F_\nu(x)\right| \mathrm{d}x \\ &\quad + \int_{q_\alpha^t(\mu)}^{q_\alpha^t(\nu)} \left|F_\mu(x) - (1-\alpha) - e^{-t}F_\nu(x)\right| \mathrm{d}x \\ &\quad + \int_{q_\alpha^t(\nu)}^{\infty} \left|F_\mu(x) - (1-\alpha) - (F_\nu(x) - (1-\alpha))\right| \mathrm{d}x \\ &= e^{-t}\int_{-\infty}^{q_\alpha^t(\mu)} \left|F_\mu(x) - F_\nu(x)\right| \mathrm{d}x + \int_{q_\alpha^t(\nu)}^{\infty} \left|F_\mu(x) - F_\nu(x)\right| \mathrm{d}x \\ &\quad + \int_{q_\alpha^t(\mu)}^{q_\alpha^t(\nu)} \left(F_\mu(x) - (1-\alpha) - e^{-t}F_\nu(x)\right) \mathrm{d}x. \end{aligned}$$

Considering that $(1-e^{-t})F_\nu(x) \leq (1-e^{-t})F_\nu(q_\alpha^t(\nu)) = (1-\alpha)$ for $x \leq q_\alpha^t(\nu)$ we have

$$\begin{aligned} &\int_{q_\alpha^t(\mu)}^{q_\alpha^t(\nu)} \left(F_\mu(x) - (1-\alpha) - e^{-t}F_\nu(x)\right) \mathrm{d}x \\ &= \int_{q_\alpha^t(\mu)}^{q_\alpha^t(\nu)} \left(F_\mu(x) - F_\nu(x) + (1-e^{-t})F_\nu(x) - (1-\alpha)\right) \mathrm{d}x \\ &\leq \int_{q_\alpha^t(\mu)}^{q_\alpha^t(\nu)} \left|F_\mu(x) - F_\nu(x)\right| \mathrm{d}x. \end{aligned}$$

Added together this yields

$$
\begin{aligned}
&\alpha d_{\mathrm{W}}\left(\frac{1}{\alpha}R_\alpha^t(\mu), \frac{1}{\alpha}R_\alpha^t(\nu)\right) \\
&\le \int_{-\infty}^{q_\alpha^t(\mu)} \left|F_\mu(x) - F_\nu(x)\right| \mathrm{d}x + \int_{q_\alpha^t(\nu)}^{\infty} \left|F_\mu(x) - F_\nu(x)\right| \mathrm{d}x \\
&\quad + \int_{q_\alpha^t(\mu)}^{q_\alpha^t(\nu)} \left|F_\mu(x) - F_\nu(x)\right| \mathrm{d}x \\
&= \int_{\mathbb{R}} \left|F_\mu(x) - F_\nu(x)\right| \mathrm{d}x = d_{\mathrm{W}}\left(\mu, \nu\right),
\end{aligned}
$$

which gives the desired statement. □

## A.5 Total variation version of Lemma 2.3.30 for continuous survival distributions

If $g$ corresponds to the exponential distribution, it was mentioned in reference to [De +19a] that a total variation version of Lemma 2.3.30 is true. For more arbitrary survival distributions a similar result becomes more difficult since the truncated mass differs in the corresponding alignment. For general continuous $g$, in Remark 2.3.31 we already proposed an approach in order to deal with this difference. We chose discrete timesteps, which interact with the truncated mass. By this we are able to derive the total variation version of Lemma 2.3.30 for continuous survival distributions as stated in Remark 2.3.31. Below we will carry out the proof.

In the following let $t_k^{(m)}$ be given by the following fixed specific sequence. We define the lattice points $t_0 \coloneqq 0$ and

$$
t_k^{(m)} \coloneqq g^{-1}\left(\frac{m-k}{m}\right), \quad k \in \{1, \dots, m\}
$$

where $g^{-1}$ denotes the generalized inverse of $g$, this is

$$
g^{-1}(q) \coloneqq \inf\{t \ge 0 : g(t) \le q\}, \quad q \in [0,1]. \tag{A.4}
$$

Note that since $g$ is a survival distribution we have always $t_1^{(m)} > 0$ but it can happen that $t_k = t_{k+1}$ for some $k$.

We denote the index of the largest lattice point left from a timepoint $t \in (0, t^g)$ with

$$
k^{(m)}(t) \coloneqq \sup\{k \in \mathbb{N}_0 : t_k^{(m)} \le t\}
$$

and the corresponding lattice point by $(t)_m \coloneqq t_{k^{(m)}(t)}^{(m)}$. The largest index which results in the same lattice point as the lattice point of an index $k \in \{1, \dots, m\}$ is then $[k]_n \coloneqq k^{(m)}(t_k^{(m)})$.

**Lemma A.5.1.** *For $t \in (0, t^g)$ we have*

*(i)* $k^{(m)}(t) = \lfloor m(1 - g(t)) \rfloor$ *and*

*(ii)* $(t)_m \to g^{-1}(g(t))$ *as* $m \to \infty$ *and*

*(iii)* $\{t_k^{(m)} : k \in \mathbb{N}_0\} \cap (g^{-1}(g(t)), t) = \emptyset$ *and*

*(iv)* $g((t)_m) \to g(t)$.

*Proof.* We calculate

$$\begin{aligned} k^{(m)}(t) &= \sup\{k \in \mathbb{N}_0 : t_k^{(m)} \leq t\} = \sup\{k \in \mathbb{N}_0 : g^{-1}(\frac{m-k}{m}) \leq t\} \\ &= \sup\{k \in \mathbb{N}_0 : g(t) \leq \frac{m-k}{m}\} = \sup\{k \in \mathbb{N}_0 : k \leq m(1 - g(t))\} \\ &= \lfloor m(1 - g(t)) \rfloor \end{aligned}$$

and since $g^{-1}$ is right-continuous we have

$$(t)_m = t_{k^{(m)}(t)}^{(m)} = g^{-1}\left(\frac{m - k^{(m)}(t)}{m}\right) \to g^{-1}(g(t))$$

which shows (ii). Since $g^{-1}$ is non-increasing we have

$$(t)_m = t_{k^{(m)}(t)}^{(m)} = g^{-1}\left(\frac{m - k^{(m)}(t)}{m}\right) \leq g^{-1}(g(t))$$

which, combined with the definitions, implies (iii). For (iv), if $g^{-1}(g(t))$ is a continuity point of $g$ the statement follows from (ii). Now assume that $g$ is discontinuous at $u := g^{-1}(g(t))$. For $m$ large enough there exists $k \in \{1, \ldots, m\}$ such that

$$\lim_{s \nearrow u} g(s) > \frac{m-k}{m} \geq g(u),$$

which implies that $g(s) > \frac{m-k}{m}$ for all $s > u$. This implies that

$$t \geq t_k^{(m)} = \inf\{s \geq u : g(s) \leq \frac{m-k}{m}\} \geq u.$$

By (iii) follows that $t_{k^{(m)}(t)}^{(m)} = u$. Hence $g(t_{k^{(m)}(t)}^{(m)}) = g(g^{-1}(g(t))) = g(t)$, which eventually implies $\lim_{m\to\infty} g(t_{k^{(m)}(t)}^{(m)}) = g(t)$. □

Now let $\tilde\nu_1^{-,m} := P_{t_1^{(m)}}$ and

$$\begin{aligned}\tilde\nu_k^{+,m} &:= T_{\frac{m-k}{m-k+1}} \circ P_{t_k^{(m)}-t_{k-1}^{(m)}} \circ \dots \qquad\qquad\qquad \circ T_{\frac{m-1}{m}} \circ P_{t_1^{(m)}}(\mu),\\ \tilde\nu_k^{-,m} &:= \qquad\qquad P_{t_k^{(m)}-t_{k-1}^{(m)}} \circ T_{\frac{m-k}{m-k+1}} \circ \dots \circ P_{t_2^{(m)}-t_1^{(m)}} \circ T_{\frac{m-2}{m-1}} \circ P_{t_1^{(m)}}(\mu)\end{aligned}$$

and

$$\begin{aligned}\tilde q_k^{+,m} &:= \sup\operatorname{supp}(\tilde\nu_k^{+,m}), \quad k \in \{1,\dots,m\},\\ \tilde q_k^{-,m} &:= \sup\operatorname{supp}(T_{\frac{m-k}{m-k+1}}(\tilde\nu_{k-1}^{-,m})), \quad k \in \{2,\dots,m\}.\end{aligned}$$

Note that in the case that $g$ is continuous, we have $g(t_k^{(m)})/g(t_k^{(m)}) = \frac{m-k}{m-k+1}$ and $\tilde\nu_k^{\pm,m}$ coincides with $\nu_k^{\pm,m}$ from Theorem 2.4.16.

We have the following statement, which implies a total variation version of Lemma 2.3.30 for continuous survival distributions and timepoints as quantiles.

**Lemma A.5.2.** *For $m \in \mathbb{N}_{\geq 2}$ and $k \in \{1,\dots,m\}$ we have*

$$d_{\mathrm{TV}}\left(\tilde\nu_k^{+,m}, \tilde\nu_k^{-,m}\right) \leq \frac{1}{m-k+1} + \frac{(m-1)(k-1)}{(m-k+1)^3}.$$

*Proof.* We will begin by proving the following claim for $k \in \{1,\dots,m-1\}$.

$$d_{\mathrm{TV}}\left(\tilde\nu_k^{+,m}, T_{\frac{m-(k+1)}{m-k}}(\tilde\nu_k^{-,m})\right) \leq \sum_{\ell=1}^{k}\left(\prod_{j=l}^{k-1}\frac{m-j}{m-(j+1)}\right)\frac{1}{(m-\ell)^2},$$

where $\prod_{j=k}^{k-1} := 1$. For the following note that by Lemma 2.2.10 for $\ell \in \{1,\dots,k-1\}$ we can write

$$T_{\frac{m-(\ell+1)}{m-\ell}} = T_{\frac{(m-\ell)^2-1}{(m-\ell)^2}} \circ T_{\frac{m-\ell}{m-(\ell-1)}}$$

on the space on absolutely continuous probability measures. As first step of induction we have the following easy consequence of Lemma A.4.3.

$$\begin{aligned}d_{\mathrm{TV}}\left(\tilde\nu_1^{+,m}, T_{\frac{m-2}{m-1}}(\tilde\nu_1^{-,m})\right) &= d_{\mathrm{TV}}\left(\tilde\nu_1^{+,m}, T_{\frac{(m-1)^2-1}{(m-1)^2}}(\tilde\nu_1^{+,m})\right)\\ &\leq \left(1-\frac{(m-1)^2-1}{(m-1)^2}\right) = \frac{1}{(m-1)^2}.\end{aligned}$$

Now assume that the claim is true for $l \in \{1, \dots, k-1\}$. Then we have by applying Lemma A.4.3, Lemma A.4.2 and Lemma A.4.1 that

$$\begin{aligned}
& d_{\mathrm{TV}}\left(\tilde{\nu}_k^{+,m}, T_{\frac{m-(k+1)}{m-k}}(\tilde{\nu}_k^{-,m})\right) \\
&= d_{\mathrm{TV}}\left(T_{\frac{m-k}{m-k+1}} \circ P_{t_k^{(m)}-t_{k-1}^{(m)}}(\tilde{\nu}_{k-1}^{+,m}), T_{\frac{(m-k)^2-1}{(m-k)^2}} \circ T_{\frac{m-k}{m-k+1}}(\tilde{\nu}_k^{-,m})\right) \\
&\leq \frac{1}{(m-k)^2} + d_{\mathrm{TV}}\left(T_{\frac{m-k}{m-k+1}} \circ P_{t_k^{(m)}-t_{k-1}^{(m)}}(\tilde{\nu}_{k-1}^{+,m}), T_{\frac{m-k}{m-k+1}}(\tilde{\nu}_k^{-,m})\right) \\
&\leq \frac{1}{(m-k)^2} + \frac{m-k+1}{m-k} d_{\mathrm{TV}}\left(P_{t_k^{(m)}-t_{k-1}^{(m)}}(\tilde{\nu}_{k-1}^{+,m}), \tilde{\nu}_k^{-,m}\right) \\
&\leq \frac{1}{(m-k)^2} + \frac{m-k+1}{m-k} d_{\mathrm{TV}}\left(\tilde{\nu}_{k-1}^{+,m}, T_{\frac{m-k}{m-k+1}}(\tilde{\nu}_{k-1}^{-,m})\right) \\
&\leq \frac{1}{(m-k)^2} + \frac{m-k+1}{m-k} \sum_{\ell=1}^{k-1} \left(\prod_{j=\ell}^{k-2} \frac{m-j}{m-(j+1)}\right) \frac{1}{(m-\ell)^2} \\
&= \sum_{\ell=1}^{k} \left(\prod_{j=\ell}^{k-1} \frac{m-j}{m-(j+1)}\right) \frac{1}{(m-\ell)^2},
\end{aligned}$$

which proves the claim. Now observe that

$$\begin{aligned}
& \sum_{\ell=1}^{k} \left(\prod_{j=\ell}^{k-1} \frac{m-j}{m-(j+1)}\right) \frac{1}{(m-\ell)^2} \\
&\leq \sum_{\ell=1}^{k} \left(\prod_{j=1}^{k-1} \frac{m-j}{m-(j+1)}\right) \frac{1}{(m-\ell)^2} \\
&= \frac{m-1}{m-k} \sum_{\ell=1}^{k} \frac{2}{(m-\ell)^2} \leq \frac{m-1}{m-k} k \frac{1}{(m-k)^2}.
\end{aligned}$$

Altogether by applying Lemma A.4.3 and Lemma A.4.1 we obtain

$$\begin{aligned}
& d_{\mathrm{TV}}\left(\tilde{\nu}_k^{+,m}, \tilde{\nu}_k^{-,m}\right) \\
&= d_{\mathrm{TV}}\left(T_{\frac{m-k}{m-k+1}} \circ P_{t_k^{(m)}-t_{k-1}^{(m)}}(\tilde{\nu}_{k-1}^{+,m}), \tilde{\nu}_k^{-,m}\right) \\
&\leq \frac{1}{m-k+1} + d_{\mathrm{TV}}\left(P_{t_k^{(m)}-t_{k-1}^{(m)}}(\tilde{\nu}_{k-1}^{+,m}), \tilde{\nu}_k^{-,m}\right) \\
&\leq \frac{1}{m-k+1} + \mathbb{1}_{\{k\geq 2\}} d_{\mathrm{TV}}\left(\tilde{\nu}_{k-1}^{+,m}, T_{\frac{m-k}{m-k+1}}(\tilde{\nu}_{k-1}^{-,m})\right) \\
&\leq \frac{1}{m-k+1} + \sum_{\ell=1}^{k-1} \left(\prod_{j=\ell}^{k-2} \frac{m-j}{m-(j+1)}\right) \frac{1}{(m-\ell)^2}
\end{aligned}$$

$$\leq \frac{1}{m-k+1} + \frac{m-1}{m-k+1}(k-1)\frac{1}{(m-k+1)^2},$$

which yields the statement. □

For general $g$ we have at least the following behavior.

**Lemma A.5.3.** *For $t \in (0, t^g)$ holds*

$$d_{\mathrm{TV}}\left(\tilde{\nu}^{+,m}_{k^{(m)}(t)}, \nu^{+,m}_{k^{(m)}(t)}\right) \to 0,$$

*as $m \to \infty$.*

*Proof.* If $g$ is continuous the statement is trivial, because $\tilde{\nu}^{+,m}_k$ and $\nu^{+,m}_k$ agree. They key of this proof will be to reduce the definitions of $\tilde{\nu}^{+,m}_k$ and $\nu^{+,m}_k$ to the relevant steps and distinguish at which steps can be made a significant error in the total variation distance.

For this let $D = \{r_1, r_2, \ldots\}$ denote the points of discontinuities of $g$. It is clear that $p := \sum_{i=1}^{\infty} g(r_i-) - g(r_i) \leq 1$. For fixed $\varepsilon \in (0, p)$ we can choose $K \in \mathbb{N}$ such that

$$\sum_{i=1}^{K} g(r_i-) - g(r_i) \geq p - \varepsilon.$$

Now fix $m$ and let

$$\{k_1, \ldots, k_n\} = \{k^{(m)}(s) : s \in [0,t]\}$$

with $k_1 < \ldots < k_n$. Note that then, considering $P_0$ as identity and using Lemma 2.2.10, we have $\nu^{+,m}_{k^{(m)}(t)} = R^+_n$ and $\tilde{\nu}^{+,m}_{k^{(m)}(t)} = \tilde{R}^+_n$ with

$$\begin{aligned} R^+_\ell &:= T_{g(t^{(m)}_{k_\ell})/g(t^{(m)}_{k_{\ell-1}})} \circ P_{t^{(m)}_{k_\ell} - t^{(m)}_{k_{\ell-1}}} \circ \ldots \circ T_{g(t^{(m)}_{k_1})} \circ P_{t^{(m)}_{k_1}}(\mu) \\ \tilde{R}^+_\ell &:= T_{\frac{m-k_\ell}{m-k_{\ell-1}}} \circ P_{t^{(m)}_{k_\ell} - t^{(m)}_{k_{\ell-1}}} \circ \ldots \circ T_{\frac{m-k_1}{m}} \circ P_{t^{(m)}_{k_1}}(\mu) \end{aligned}$$

for $\ell \in \{1, \ldots, n\}$ and $R^+_0 = \tilde{R}^+_0 = \mu$. Further, define

$$a_\ell := \min\left(\frac{1}{m}, g(t^{(m)}_{k_\ell}-) - g(t^{(m)}_{k_\ell})\right) + \min\left(\frac{1}{m}, g(t^{(m)}_{k_{\ell-1}}-) - g(t^{(m)}_{k_{\ell-1}})\right)$$

and

$$\beta_\ell := \max\left(\frac{m-k_\ell}{m-k_{\ell-1}}, \frac{g(t^{(m)}_{k_\ell})}{g(t^{(m)}_{k_{\ell-1}})}\right).$$

As preparing step one we will compute the following bound.

$$1-\beta_\ell^{-1}\min\left(\frac{m-k_\ell}{m-k_{\ell-1}},\frac{g(t^{(m)}_{k_\ell})}{g(t^{(m)}_{k_{\ell-1}})}\right)\leq\frac{a_\ell}{g(t)^2}. \tag{A.5}$$

For this, note that for $\ell\in\{1,\dots,n\}$ it holds

$$\frac{m-(k_\ell+1)}{m}\leq g(t^{(m)}_{k_\ell})\leq\frac{m-k_\ell}{m}$$

and thus

$$\frac{m-(k_\ell+1)}{m-k_{\ell-1}}\leq\frac{g(t^{(m)}_{k_\ell})}{g(t^{(m)}_{k_{\ell-1}})}\leq\frac{m-k_\ell}{m-(k_{\ell-1}+1)}.$$

Thus it follows

$$\begin{aligned}
&1-\beta_\ell^{-1}\min\left(\frac{m-k_\ell}{m-k_{\ell-1}},\frac{g(t^{(m)}_{k_\ell})}{g(t^{(m)}_{k_{\ell-1}})}\right)\\
&=1-\min\left(\frac{(m-k_\ell)g(t^{(m)}_{k_{\ell-1}})}{(m-k_{\ell-1})g(t^{(m)}_{k_\ell})},\frac{(m-k_{\ell-1})g(t^{(m)}_{k_\ell})}{(m-k_\ell)g(t^{(m)}_{k_{\ell-1}})}\right)\\
&=\max\left(\frac{(m-k_{\ell-1})g(t^{(m)}_{k_\ell})-(m-k_\ell)g(t^{(m)}_{k_{\ell-1}})}{(m-k_{\ell-1})g(t^{(m)}_{k_\ell})},\frac{(m-k_\ell)g(t^{(m)}_{k_{\ell-1}})-(m-k_{\ell-1})g(t^{(m)}_{k_\ell})}{(m-k_\ell)g(t^{(m)}_{k_{\ell-1}})}\right)\\
&\leq\max\left(\frac{(m-k_{\ell-1})g(t^{(m)}_{k_\ell})-(m-k_\ell)g(t^{(m)}_{k_{\ell-1}})}{mg(t^{(m)}_{k_{\ell-1}})g(t^{(m)}_{k_\ell})},\frac{(m-k_\ell)g(t^{(m)}_{k_{\ell-1}})-(m-k_{\ell-1})g(t^{(m)}_{k_\ell})}{mg(t^{(m)}_{k_\ell})g(t^{(m)}_{k_{\ell-1}})}\right)\\
&\leq\frac{1}{g(t)^2}\left|\frac{m-k_{\ell-1}}{m}g(t^{(m)}_{k_\ell})-\frac{m-k_\ell}{m}g(t^{(m)}_{k_{\ell-1}})\right|\\
&\leq\frac{1}{g(t)^2}\left(\left|\frac{m-k_{\ell-1}}{m}g(t^{(m)}_{k_\ell})-g(t^{(m)}_{k_\ell})g(t^{(m)}_{k_{\ell-1}})\right|+\left|g(t^{(m)}_{k_\ell})g(t^{(m)}_{k_{\ell-1}})-\frac{m-k_\ell}{m}g(t^{(m)}_{k_{\ell-1}})\right|\right)\\
&\leq\frac{1}{g(t)^2}\left(\left|\frac{m-k_{\ell-1}}{m}-g(t^{(m)}_{k_{\ell-1}})\right|+\left|\frac{m-k_\ell}{m}-g(t^{(m)}_{k_\ell})\right|\right)\\
&\leq\frac{a_\ell}{g(t)^2}.
\end{aligned}$$

As preparating step two consider

$$\begin{aligned}
\prod_{j=1}^{\ell} \frac{1}{\beta_j} &= \prod_{j=1}^{\ell} \min\left( \frac{m-k_{j-1}}{m-k_j}, \frac{g(t_{k_{j-1}}^{(m)})}{g(t_{k_j}^{(m)})} \right) \\
&\leq \prod_{j=1}^{\ell} \min\left( \frac{m-k_{j-1}}{m-k_j}, \frac{m-k_{j-1}}{m-(k_j+1)} \right) \\
&\leq \prod_{j=1}^{\ell} \frac{m-k_{j-1}}{m-k_j} = \frac{m}{m-k_\ell} \\
&\leq \frac{1}{g(t_{k_\ell}^{(m)})} \leq \frac{1}{g(t)}.
\end{aligned}$$

Assume that for $\ell \in \{1, \ldots, n\}$ holds

$$d_{\mathrm{TV}}\left(R_{\ell-1}^+, \tilde{R}_{\ell-1}^+\right) \leq \frac{1}{g(t)^2} \sum_{h=1}^{\ell-1} \left( \prod_{j=h+1}^{\ell-1} \frac{1}{\beta_j} \right) a_h.$$

Then, using Lemma 2.2.10, Lemma A.4.3, Lemma A.4.2 and Lemma A.4.1 it follows

$$\begin{aligned}
&d_{\mathrm{TV}}\left(R_\ell^+, \tilde{R}_\ell^+\right) \\
&= d_{\mathrm{TV}}\left( T_{g(t_{k_\ell}^{(m)})/g(t_{k_{\ell-1}}^{(m)})} \circ P_{t_{k_\ell}^{(m)} - t_{k_{\ell-1}}^{(m)}}(R_{\ell-1}^+), T_{\frac{m-k_\ell}{m-k_{\ell-1}}} \circ P_{t_{k_\ell}^{(m)} - t_{k_{\ell-1}}^{(m)}}(\tilde{R}_{\ell-1}^+) \right) \\
&\leq d_{\mathrm{TV}}\left( T_{\beta_\ell} \circ P_{t_{k_\ell}^{(m)} - t_{k_{\ell-1}}^{(m)}}(R_{\ell-1}^+), T_{\beta_\ell} \circ P_{t_{k_\ell}^{(m)} - t_{k_{\ell-1}}^{(m)}}(\tilde{R}_{\ell-1}^+) \right) \\
&\quad + \left( 1 - \beta_\ell^{-1} \min\left( \frac{m-k_\ell}{m-k_{\ell-1}}, \frac{g(t_{k_\ell}^{(m)})}{g(t_{k_{\ell-1}}^{(m)})} \right) \right) \\
&\leq \frac{1}{\beta_\ell} d_{\mathrm{TV}}\left(R_{\ell-1}^+, \tilde{R}_{\ell-1}^+\right) + \frac{a_\ell}{g(t)^2} \\
&\leq \frac{1}{\beta_\ell} \frac{1}{g(t)^2} \sum_{h=1}^{\ell-1} \left( \prod_{j=h+1}^{\ell-1} \frac{1}{\beta_j} \right) a_h + \frac{a_\ell}{g(t)^2} \\
&= \frac{1}{g(t)^2} \sum_{h=1}^{\ell} \left( \prod_{j=h+1}^{\ell} \frac{1}{\beta_j} \right) a_h,
\end{aligned}$$

where we have used (A.5). Using that $R_0^+ = \tilde{R}_0^+$, it follows by induction that

$$\begin{aligned}
&d_{\mathrm{TV}}\left(\nu^{+,m}_{k^{(m)}(t)}, \tilde{\nu}^{+,m}_{k^{(m)}(t)}\right) = d_{\mathrm{TV}}\left(R_n^+, \tilde{R}_n^+\right) \\
&\leq \frac{1}{g(t)^2}\sum_{h=1}^{n}\left(\prod_{j=h+1}^{n}\frac{1}{\beta_j}\right)a_h\frac{1}{g(t)^2} \leq \sum_{h=1}^{n}\left(\prod_{j=1}^{n}\frac{1}{\beta_j}\right)a_h \leq \frac{1}{g(t)^3}\sum_{h=1}^{n}a_h \\
&= \frac{1}{g(t)^3}\sum_{h=1}^{n}\min\left(\frac{1}{m}, g(t^{(m)}_{k_h}-) - g(t^{(m)}_{k_h})\right) + \min\left(\frac{1}{m}, g(t^{(m)}_{k_{h-1}}-) - g(t^{(m)}_{k_{h-1}})\right) \\
&\leq \frac{1}{g(t)^3}2\sum_{i=1}^{\infty}\min\left(\frac{1}{m}, g(r_i-) - g(r_i)\right) \\
&\leq 2\sum_{i=1}^{K}\frac{1}{m} + 2\sum_{i=K+1}^{\infty} g(r_i-) - g(r_i) \\
&\leq \frac{2K}{m} + 2\varepsilon.
\end{aligned}$$

This shows that

$$\lim_{m\to\infty} d_{\mathrm{TV}}\left(\nu^{+,m}_{k^{(m)}(t)}, \tilde{\nu}^{+,m}_{k^{(m)}(t)}\right) \leq \varepsilon.$$

Since $\varepsilon$ can be chosen arbitrarily small, the statement follows. □

If $g$ is continuous one has $\nu_k^{\pm,m} = \tilde{\nu}_k^{\pm,m}$ and thus the required property of the proof of Theorem 2.4.4 was fulfilled by the following statement.

**Proposition A.5.4.** *Let $\mu \in \mathcal{P}$. Let $b \in \mathrm{ifpt}(g,\mu)$. Then $P_{t-(t)_m}\tilde{\nu}^{\pm,m}_{k^{(m)}(t)} \to \mathbb{P}_\mu\left(X_t \in \cdot\ | \tau_b > t\right)$ weakly as $m \to \infty$.*

*Proof.* For the discrete boundary function

$$\tilde{b}_m(s) := \begin{cases} b(0) & : s = 0, \\ \tilde{q}^{+,m}_{k^{(m)}(s)} & : s = t_k^{(m)}, k \in \{1,\dots,n_m\}, \\ 0 & : s > t^g, \\ \infty & : \text{else}, \end{cases}$$

we see immediately that

$$\begin{aligned}
\mathbb{P}_\mu\left(\tau_{\tilde{b}_m} > s\right) = \mathbb{P}_\mu\left(\tau_{\tilde{b}_m} > (s)_m\right) &= \frac{m - k^{(m)}(s)}{m} \\
&= 1 - \frac{\lfloor m(1-g(s))\rfloor}{m} \to g(s)
\end{aligned}$$

for $s \in (0, t^g)$, which implies that $\tau_{\tilde{b}_m} \to g$ in distribution. By Lemma 2.1.17, the compactness of the set of boundary functions and the uniqueness of Theorem 2.3.33 it follows that $\tilde{b}_m \xrightarrow{\Gamma} b$ in the Hausdorff distance, which in turn implies by Lemma 2.1.17 that $\tau_{\tilde{b}_m} \to \tau_b$ in probability. We can then choose a subsequence $(\tau_{\tilde{b}_{m_n}})_{n\in\mathbb{N}}$, such that $\tau_{\tilde{b}_{m_n}} \to \tau_b$ almost surely. With the same reasoning as in the proof of Theorem 2.3.17 we obtain

$$P_{t-(t)_{m_n}}\tilde{\nu}^{+,m_n}_{k^{(m_n)}(t)} = \mathbb{P}_\mu\left(X_t \in \cdot \;\middle|\tau_{\tilde{b}_{m_n}} > t\right) \to \mathbb{P}_\mu\left(X_t \in \cdot \;|\tau_b > t\right)$$

in the sense of weak convergence of distributions. Further we have by Lemma A.5.2 that

$$\begin{aligned}
&d_{\mathrm{TV}}\left(P_{t-(t)_{m_n}}\tilde{\nu}^{+,m_n}_{k^{(m_n)}(t)}, P_{t-(t)_{m_n}}\tilde{\nu}^{-,m_n}_{k^{(m_n)}(t)}\right)\\
&\le d_{\mathrm{TV}}\left(\tilde{\nu}^{+,m_n}_{k^{(m_n)}(t)}, \tilde{\nu}^{-,m_n}_{k^{(m_n)}(t)}\right)\\
&\le \frac{1}{m_n - k^{(m_n)}(t)+1} + \frac{(m_n-1)(k^{(m_n)}(t)-1)}{(m_n-k^{(m_n)}(t)+1)^3}\\
&= \frac{1}{\lceil m_n g(t)\rceil + 1} + \frac{(m_n-1)(\lfloor m_n(1-g(t))\rfloor - 1)}{(\lceil m_n g(t)\rceil+1)^3}\\
&\overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

which proves the statement. □

For continuous $g$ we could therefore use the statement of Theorem 2.4.4 in the following way. We have that $\chi^N_{t_k^{(m)}} = \chi^{m,N}_{t_k^{(m)}}$. For continuous $g$ the statement of Lemma A.5.2 also yields the convergence of Theorem 2.4.4 for the timesteps $t_k^{(m)} = g^{-1}((m-k)/m)$. Hence, combining the coupling of Lemma 2.4.7 and the convergence of Theorem 2.4.16 yields for $a \ge 0$ that almost surely

$$\begin{aligned}
S^{+,m}_{k^{(m)}(t_k^{(m)})}(\mu)([-a,a]) &= \lim_{N\to\infty} \chi^{+,N}_{t_k^{(m)}}([-a,a])\\
&\ge \lim_{N\to\infty} \chi^{N}_{t_k^{(m)}}([-a,a]) = \mu_{t_k^{(m)}}([-a,a]),
\end{aligned}$$

which shows that for continuous $g$ we have $\nu_k^{+,m} \preceq \mu_{t_k^{(m)}}$. The following shows that this is true for general $g$ by a similar technique as used in Lemma 2.3.19.

**Proposition A.5.5.** *Let $g$ be a survival distribution and $\mu \in \mathcal{P}$. Let $b \in$ $\mathrm{ifpt}(g,\mu)$. Further, assume that $n, m \in \mathbb{N}$ with $m \mid n$. Then for $k \in \mathbb{N}$ holds*

$$\nu_k^{+,m} \preceq \nu^{+,n}_{kn/m} \preceq \mu_{t_k^{(m)}} := \mathbb{P}_\mu\left(X_{t_k^{(m)}} \in \cdot \;\middle|\tau_b > t_k^{(m)}\right)$$

*where $\mu \in \mathcal{P}$. In particular, $b_m^+(t_k^{(m)}) \le b_n^+(t_k^{(m)}) \le b(t_k^{(m)})$.*

*Proof.* We first prove that $\nu_k^{+,m} \preceq \mu_{t_k^{(m)}}$. The proof is similar to the proof of Lemma 2.3.19. Define $b^s(t) := b(t+s)$ and for $s < t$ and $\nu \in \mathcal{P}$ define

$$H_k^{+,m}(\nu) := T_{\alpha_k^{(m)}} \circ P_{t_k^{(m)} - t_{k-1}^{(m)}}(\nu).$$

Note that by the Markov property we have

$$\mathbb{P}_{\mu_{t_{k-1}^{(m)}}}\left(\tau_{b^{t_{k-1}^{(m)}}} > t_k^{(m)} - t_{k-1}^{(m)}\right) = \alpha_k^{(m)}.$$

If $c \geq q_{\alpha_k^{(m)}}(P_{t_k^{(m)}-t_{k-1}^{(m)}}\mu_{t_{k-1}^{(m)}})$, then

$$H_k^{+,m}(\mu_{t_{k-1}^{(m)}})([-c,c]) = 1 \geq S_{t_k^{(m)},t_{k-1}^{(m)}}(\mu_{t_{k-1}^{(m)}})([-c,c]).$$

If $c < q_{\alpha_k^{(m)}}(P_{t_k^{(m)}-t_{k-1}^{(m)}}\mu_{t_{k-1}^{(m)}})$, then

$$\begin{aligned}
&H_k^{+,m}(\mu_{t_{k-1}^{(m)}})([-c,c]) \\
&= \frac{1}{\alpha_k^{(m)}} P_{t_k^{(m)}-t_{k-1}^{(m)}}\mu_{t_{k-1}^{(m)}}([-c,c]) = \frac{1}{\alpha_k^{(m)}} \mathbb{P}_{\mu_{t_{k-1}^{(m)}}}\left(|X_{t_k^{(m)}-t_{k-1}^{(m)}}| \leq c\right) \\
&\geq \frac{1}{\alpha_k^{(m)}} \mathbb{P}_{\mu_{t_{k-1}^{(m)}}}\left(|X_{t_k^{(m)}-t_{k-1}^{(m)}}| \leq c, \tau_{b^{t_{k-1}^{(m)}}} > t_k^{(m)} - t_{k-1}^{(m)}\right) \\
&= \mathbb{P}_{\mu_{t_{k-1}^{(m)}}}\left(|X_{t_k^{(m)}-t_{k-1}^{(m)}}| \leq c \,\middle|\, \tau_{b^{t_{k-1}^{(m)}}} > t_k^{(m)} - t_{k-1}^{(m)}\right) = \mu_{t_k^{(m)}}([-c,c])
\end{aligned}$$

where the last equality follows again by the Markov property. Altogether, if we assume that $\nu_{k-1}^{+,m} \preceq \mu_{t_{k-1}^{(m)}}$ for $k \in \{1,\dots,m\}$, we obtain by the claim and the fact that $H_k^{+,m}$ is order preserving

$$\nu_k^{+,m} = H_k^{+,m}(\nu_{k-1}^{+,m}) \preceq H_k^{+,m}(\mu_{t_{k-1}^{(m)}}) \preceq \mu_{t_k^{(m)}}.$$

The desired statement follows by induction, since $\nu_0^{+,m} = \mu_{t_0}^{(m)} = \mu$.

Now we prove that $\nu_k^{+,m} \preceq \nu_{kn/m}^{+,n}$. The proof is similar to the proof of Lemma 2.3.20. Let $\ell \in \mathbb{N}$ such that $n = \ell m$. Suppose that $\nu_{k-1}^{+,m} \preceq \nu_{(k-1)\ell}^{+,n}$. By the identity $t_k^{(m)} = g^{-1}(\frac{m-k}{m}) = g^{-1}(\frac{n-k\ell}{n}) = t_{k\ell}^{(n)}$ we have

$$\alpha_{k\ell}^{(n)} \cdot \ldots \cdot \alpha_{(k-1)\ell+1}^{(n)} = \frac{g(t_{k\ell}^{(n)})}{g(t_{(k-1)\ell}^{(n)})} = \frac{g(t_k^{(m)})}{g(t_{k-1}^{(m)})} = \alpha_k^{(m)}.$$

Now observe that then by successive application of Lemma 2.2.12 and then using Lemma 2.2.10 we have

$$\begin{aligned}
&H_{k\ell}^{+,n} \circ \ldots \circ H_{(k-1)\ell+1}^{+,n}(\nu) \\
&= T_{\alpha_{k\ell}^{(n)}} \circ P_{t_{k\ell}^{(n)}-t_{k\ell-1}^{(n)}} \circ \ldots \circ T_{\alpha_{(k-1)\ell+1}^{(n)}} \circ P_{t_{(k-1)\ell+1}^{(n)}-t_{(k-1)\ell}^{(n)}}(\nu) \\
&\succeq T_{\alpha_{k\ell}^{(n)}} \circ \ldots \circ T_{\alpha_{(k-1)\ell+1}^{(n)}} \circ P_{t_{k\ell}^{(n)}-t_{k\ell-1}^{(n)}} \circ \ldots \circ P_{t_{(k-1)\ell+1}^{(n)}-t_{(k-1)\ell}^{(n)}}(\nu) \\
&= T_{\alpha_k^{(m)}} \circ P_{t_{k\ell}^{(n)}-t_{k\ell-1}^{(n)}} = H_k^{+,m}(\nu).
\end{aligned}$$

Hence, by the fact that $H_k^{+,m}$ is dominance preserving we get

$$\begin{aligned}\nu_k^{+,m} &= H_k^{+,m}(\nu_{k-1}^{+,m}) \preceq H_k^{+,m}(\nu_{(k-1)\ell}^{+,n}) \\ &\preceq H_{k\ell}^{+,n} \circ \ldots \circ H_{(k-1)\ell+1}^{+,n}(\nu_{(k-1)\ell}^{+,n}) = \nu_{k\ell}^{+,n}.\end{aligned}$$

By induction the desired statement follows. □

## A.6 Sufficient criteria on Markov processes for the conditions of Theorem 3.0.1

In the general situation of Theorem 3.0.1 naturally the question arises, which Markov processes fulfil the properties (iii) and (iv) from Theorem 3.0.1. In the following we will derive convenient sufficient criteria.

**Lemma A.6.1.** *Assume that $(X_t)_{t\geq 0}$ fulfills (ii) from Theorem 3.0.1 and has the strong Markov property. Then property (iv) from Theorem 3.0.1 is fulfilled, i.e. $P_t$ preserves the usual stochastic order.*

*Proof.* For $x \in \mathbb{R}$ let $(X_t^x)_{t\geq 0}$ denote a version of $(X_t)_{t\geq 0}$ such that $X_0^x = x$ almost surely. For $x \leq y$ define $T \coloneqq \inf\{t > 0 : X_t^x = X_t^y\}$. Further define

$$\tilde{X}_t^y \coloneqq X_t^y \mathbb{1}_{\{T>t\}} + X_t^x \mathbb{1}_{\{T\leq t\}}.$$

Since the processes have continuous pathes, we have $X_T^x = X_T^y$ almost surely on $\{T < \infty\}$. Note that for a continuous and bounded function $f$ by the strong Markov property we have

$$\begin{aligned}&\mathbb{E}\left[f(X_t^x)\mathbb{1}_{\{T\leq t\}}\right] = \mathbb{E}\left[P_{t-T}\delta_{X_T^x}(f)\mathbb{1}_{\{T\leq t\}}\right] \\ &= \mathbb{E}\left[P_{t-T}\delta_{X_T^y}(f)\mathbb{1}_{\{T\leq t\}}\right] = \mathbb{E}\left[f(X_t^y)\mathbb{1}_{\{T\leq t\}}\right].\end{aligned}$$

Thus

$$\mathbb{E}\left[f(\tilde{X}_t^y)\right] = \mathbb{E}\left[f(X_t^y)\mathbb{1}_{\{T>t\}} + f(X_t^x)\mathbb{1}_{\{T\leq t\}}\right] = \mathbb{E}\left[f(X_t^y)\right] = P_t\delta_y(f).$$

Since we have $X_t^x \leq \tilde{X}_t^y$ almost surely, this shows that $P_t\delta_x \preceq_{\mathrm{st}} P_t\delta_y$. For $c \in \mathbb{R}$ this means that $x \mapsto P_t\delta_x((-\infty, c])$ is a non-increasing function. Hence,

$$\begin{aligned}P_t\mu((-\infty, c]) &= \int_{\mathbb{R}} P_t\delta_x((-\infty, c])\,\mathrm{d}\mu(x) \\ &\geq \int_{\mathbb{R}} P_t\delta_x((-\infty, c])\,\mathrm{d}\nu(x) = P_t\nu((-\infty, c]),\end{aligned}$$

which shows the desired statement. □

We now give two different criteria for property (iii).

**Lemma A.6.2.** *Assume that* $(X_t)_{t\geq 0}$ *is locally uniformly continuous in probability, i.e. for every compact subset* $K \subset \mathbb{R}$ *we have*

$$\lim_{t\to 0}\sup_{x\in K}\mathbb{P}_x\left(|X_t - X_0| > \varepsilon\right) = 0.$$

*Let* $\mathcal{S}$ *be a tight collection of probability measures. Then* (3.3) *is fulfilled.*

*Proof.* If $\mathcal{S}$ is a tight family of probability measures, let $K \subset \mathbb{R}$ compact such that $\mu(K) \geq 1-\varepsilon$, $\varepsilon \in (0,1)$. We have by Lemma B.1.11 that

$$\begin{aligned} &d_{\mathrm{P}}(\mathbb{P}_\mu\left(X_t \in \cdot\,\right), \mu) \leq \left(\mathbb{E}_\mu\left[|X_t - X_0| \wedge 1\right]\right)^{\frac{1}{2}} \\ &\leq \left(\varepsilon + \sup_{x\in K}\mathbb{E}_x\left[|X_t - X_0| \wedge 1\right]\right)^{\frac{1}{2}} \leq \left(2\varepsilon + \sup_{x\in K}\mathbb{P}_x\left(|X_t - X_0| > \varepsilon\right)\right)^{\frac{1}{2}}. \end{aligned}$$

Since $(X_t)_{\geq 0}$ is locally uniformly continuous in probability this implies that

$$\lim_{t\to 0}\sup_{\mu\in\mathcal{S}} d_{\mathrm{P}}(\mathbb{P}_\mu\left(X_t \in \cdot\,\right), \mu) \leq (2\varepsilon)^{1/2},$$

which shows the statement, since $\varepsilon$ can be chosen arbitrarily small. ☐

For Feller processes property (iii) from Theorem 3.0.1 can also be deduced.

**Lemma A.6.3.** *Assume* $(P_t)_{t\geq 0}$ *is a* $C_b$*-Feller semigroup, i.e. for every* $f$ *continuous and bounded the function* $x \mapsto P_t f(x)$ *is continuous and bounded and for compact* $K \subseteq \mathbb{R}$ *it holds*

$$\sup_{x\in K}|P_t f(x) - f(x)| \to 0 \quad \text{as } t \to 0.$$

*Let* $\mathcal{S}$ *be a tight collection of probability measures. Then* (3.3) *is fulfilled.*

*Proof.* For a function $f : \mathbb{R} \to \mathbb{R}$ define $\|f\|_\infty := \sup_{x\in\mathbb{R}}|f(x)|$ and $\|f\|_{\mathrm{L}} := \sup_{x\neq x}\frac{|f(x)-f(y)|}{|x-y|}$. Further define $\|f\|_{\mathrm{BL}} := \|f\|_\infty + \|f\|_{\mathrm{L}}$.

$$d_\beta(\mu,\nu) := \sup\left\{\left|\int_{\mathbb{R}} f(x)\mu(dx) - \int_{\mathbb{R}} f(x)\nu(dx)\right| : \|f\|_{\mathrm{BL}} \leq 1\right\}$$

Since $d_{\mathrm{P}}(\mu,\nu) \leq \sqrt{\frac{3}{2}d_\beta(\mu,\nu)}$, see for example [Dud02], it would suffice to show

$$\sup_{\mu\in\mathcal{S}} d_\beta(P_t\mu, \mu) \to 0$$

as $t \to 0$.
In order to achieve this let $(t_n)_{n\in\mathbb{N}}$ be a sequence with $t_n \to 0$.
Step 1:
We claim that the family

$$\tilde{\mathcal{S}} := \{P_{t_n}\mu : n \in \mathbb{N}, \mu \in \mathcal{S}\}$$

is tight. For this let $(\nu_m)_{m\in\mathbb{N}}$ be a sequence contained in $\tilde{\mathcal{S}}$. Thus for $m \in \mathbb{N}$ a $\nu_m$ takes the form $\nu_m = P_{t_{n_m}}\mu_m$ with $n_m \in \mathbb{N}$ and $\mu_m \in \mathcal{S}$. Since $\mathcal{S}$ is tight we can find a subsequence $(m_k)_{k\in\mathbb{N}}$ such that $\mu_{m_k} \to \tilde{\mu}$ as $k \to \infty$ and $\tilde{\mu}$ is a probability measure. Since $(t_n)_{n\in\mathbb{N}}$ is compact we can assume without loss of generality that $t_{n_{m_k}}$ is convergent. We distinguish two cases:
Case 1: $\lim_{k\to\infty} t_{n_{m_k}} \neq 0$.
Then $t_{n_{m_k}} = t_{n_0}$ with $n_0 \in \mathbb{N}$ without loss of generality. Now for an arbitrary function $f \in C_b(\mathbb{R})$ we have

$$\begin{aligned}
&\int_{\mathbb{R}} f(x)\nu_{m_k}(dx) = \int_{\mathbb{R}} f(x)P_{t_{n_0}}\mu_{m_k}(dx) = \int_{\mathbb{R}} P_{t_{n_0}}f(x)\mu_{m_k}(dx) \\
&\to \int_{\mathbb{R}} P_{t_{n_0}}f(x)\tilde{\mu}(dx) = \int_{\mathbb{R}} f(x)P_{t_{n_0}}\tilde{\mu}(dx),
\end{aligned}$$

since $P_{t_{n_0}}f$ is continuous and bounded. This implies that $\nu_{m_k} \to P_{t_{n_0}}\tilde{\mu}$ weakly. Thus $(\nu_m)_{m\in\mathbb{N}}$ has a convergent subsequence.
Case 2: $\lim_{k\to\infty} t_{n_{m_k}} = 0$.
Since $\mu_{m_k} \to \tilde{\mu}$ we have that for $\varepsilon > 0$ there is a compact set $K \subset \mathbb{R}$, such that $\mu_{m_k}(\mathbb{R}\setminus K) \leq \varepsilon$ for all $k \in \mathbb{N}$. Then we have for $f \in C_b(\mathbb{R})$ that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} f(x)P_{t_{n_{m_k}}}\mu_{m_k}(dx) - \int_{\mathbb{R}} f(x)\tilde{\mu}(dx)\right| \\
&\leq \left|\int_{\mathbb{R}} f(x)P_{t_{n_{m_k}}}\mu_{m_k}(dx) - \int_{\mathbb{R}} f(x)\mu_{m_k}(dx)\right| + \left|\int_{\mathbb{R}} f(x)\mu_{m_k}(dx) - \int_{\mathbb{R}} f(x)\tilde{\mu}(dx)\right| \\
&\leq \int_{\mathbb{R}} |P_{t_{n_{m_k}}}f(x) - f(x)|\mu_{m_k}(dx) + \left|\int_{\mathbb{R}} f(x)\mu_{m_k}(dx) - \int_{\mathbb{R}} f(x)\tilde{\mu}(dx)\right| \\
&\leq 2\|f\|_\infty\varepsilon + \sup_{x\in K}|P_{t_{n_{m_k}}}f(x) - f(x)| + \left|\int_{\mathbb{R}} f(x)\mu_{m_k}(dx) - \int_{\mathbb{R}} f(x)\tilde{\mu}(dx)\right| \\
&\to 2\|f\|_\infty\varepsilon \quad \to 0 \quad (\varepsilon \to 0).
\end{aligned}$$

Again this implies that $(\nu_m)_{m\in\mathbb{N}}$ has a convergent subsequence.
All in all this means that $\tilde{\mathcal{S}}$ is tight.
Step 2:
Let $\varepsilon > 0$ and $K \subset \mathbb{R}$ such that for all $\nu \in \tilde{S}$ we have $\nu(\mathbb{R}\setminus K) \leq \varepsilon$. Now define $B \coloneqq \{f : \|f\|_{\mathrm{BL}} \leq 1\}$ and $B_K \coloneqq \{f|_K : f \in B\} \subset C(K)$. Then we have

- $\forall x \in K$ we have $\{|f(x)| : f \in B_K\} \subset [-1, 1]$ and
- $\forall x \in K$ we have that for $\eta > 0$ the condition $|x - y| < \eta$ implies $|f(x) - f(y)| \leq \|f\|_{\mathrm{BL}}|x - y| \leq \eta$ for all $f \in B_K$.

By the Arzela-Ascoli theorem we can deduce that $B_K$ is relatively compact in $C(K)$ with respect to $\|\cdot\|_\infty$. Thus there are finitely many $f_1, \ldots, f_p \in B$, such that

$$\overline{B} \subset \bigcup_{\ell=1}^{p} \{f \in C(K) : \sup_{x \in K} |f(x) - f_\ell(x)| \leq \varepsilon\}.$$

Now let $f \in B$ be arbitrary. Then there is $\ell = 1, \ldots, p$ such that $\sup_{x \in K} |f(x) - f_\ell(x)| \leq \varepsilon$. Hence it holds for all $\mu \in \mathcal{S}$ and all $n \in \mathbb{N}$ that

$$\begin{aligned}
&\left| \int_{\mathbb{R}} f(x) P_{t_n}\mu(dx) - \int_{\mathbb{R}} f(x)\mu(dx) \right| \\
&\leq \left| \int_{\mathbb{R}\setminus K} f(x) P_{t_n}\mu(dx) - \int_{\mathbb{R}\setminus K} f(x)\mu(dx) \right| \\
&\quad + \left| \int_K (f(x) - f_\ell(x)) P_{t_n}\mu(dx) - \int_K (f(x) - f_\ell(x))\mu(dx) \right| \\
&\quad + \left| \int_K f_\ell(x) P_{t_n}\mu(dx) - \int_K f_\ell(x)\mu(dx) \right| \\
&\leq \int_{\mathbb{R}\setminus K} |f(x)| P_{t_n}\mu(dx) + \int_{\mathbb{R}\setminus K} |f(x)|\mu(dx) \\
&\quad + \int_K \underbrace{|f(x) - f_\ell(x)|}_{\leq \varepsilon} P_{t_n}\mu(dx) + \int_K \underbrace{|f(x) - f_\ell(x)|}_{\leq \varepsilon} \mu(dx) \\
&\quad + \left| \int_K P_{t_n} f_\ell(x)\mu(dx) - \int_K f_\ell(x)\mu(dx) \right| \\
&\leq P_{t_n}\mu(\mathbb{R} \setminus K) + \mu(\mathbb{R} \setminus K) + 2\varepsilon + \int_K |P_{t_n} f_\ell(x) - f_\ell(x)| \, \mu(dx) \\
&\leq 4\varepsilon + \sup_{x \in K} |P_{t_n} f_\ell(x) - f_\ell(x)| \, .
\end{aligned}$$

But since $f \in B$ was arbitrary this implies that

$$\begin{aligned}
&\sup_{\mu \in \mathcal{S}} d_\beta(P_{t_n}\mu, \mu) = \sup_{\mu \in \mathcal{S}} \sup_{f \in B} \left\{ \left| \int_{\mathbb{R}} f(x)\mu(dx) - \int_{\mathbb{R}} f(x)\nu(dx) \right| \right\} \\
&\leq 4\varepsilon + \max_{\ell = 1, \ldots, p} \sup_{x \in K} |P_{t_n} f_\ell(x) - f_\ell(x)| \\
&\to 4\varepsilon
\end{aligned}$$

as $n \to \infty$. Since $\varepsilon > 0$ was arbitrary we have $\sup_{\mu \in \mathcal{S}} d_\beta(P_{t_n}\mu, \mu) \to 0$, which implies the statement. □

# Appendix B

# Background tools

In this section we collect background tools from the literature and auxiliary statements and their proofs, which are thematically independent from the issue in this thesis, but used to to get along in the proofs of the thesis. We separate the statements roughly by the mathematical branches probability and analysis.

## B.1 Probability

In this subsection we present the statements which have a probabilistic context.

The following well-known result is stated as Theorem 38 in [Fre83] and was used repeatedly in Subsection 2.3.1 for the purpose to study the behavior of Brownian paths below a semi-continuous boundary.

**Theorem B.1.1** (Levy's forgery theorem)**.** *Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = 0$. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Let $\varepsilon > 0$. Then*

$$\mathbb{P}\left(|B_t - f(t)| \leq \varepsilon \ \forall t \in [0,1]\right) > 0.$$

The following technical lemma was applied in Lemma 2.3.4, which considered the small time behavior of boundary functions.

**Proposition B.1.2.** *Let $(B_t)_{t\geq 0}$ be a Brownian motion, $K > 0$ and $f$ a symmetric probability density with $\operatorname{supp}(f) \subseteq [-K, K]$. Further let $K_n \to K$ and $t_n \to 0$ be converging sequences. Then*

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{t_n} \int_{-K}^{K} \mathbb{P}\left(|B_{t_n} + x| \geq K_n\right) f(x)\,\mathrm{d}x \\ &\geq \liminf_{x\searrow 0} \frac{f(K-x)}{x} 2 \int_0^\infty y \left(1 - \Phi\left(y + \limsup_{n\to\infty} \frac{K_n - K}{\sqrt{t_n}}\right)\right) \mathrm{d}y\end{aligned}$$

*with $\Phi(x) := \int_{-\infty}^{x} \phi(z)\,\mathrm{d}z$ and the conventions $\Phi(-\infty) = 1 - \Phi(\infty) = 0$.*

*Proof.* In the following we will often use the inequality $1 - \Phi(x) \leq \frac{1}{x}\phi(x)$ for $x > 0$. Consider

$$\begin{aligned}
&\int_{-K}^{K} \mathbb{P}\left(|B_{t_n} + y| \geq K_n\right) f(y)\,\mathrm{d}y \\
&= 2\int_0^K f(y) \int_{K_n}^{\infty} \frac{1}{\sqrt{t_n}} \phi((z-y)/\sqrt{t_n})\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + 2\int_0^K f(y) \int_{-\infty}^{-K_n} \frac{1}{\sqrt{t_n}} \phi((z-y)/\sqrt{t_n})\,\mathrm{d}z\,\mathrm{d}y
\end{aligned}$$

For the last term of the sum holds

$$\begin{aligned}
&2\int_0^K f(y) \int_{-\infty}^{-K_n} \frac{1}{\sqrt{t_n}} \phi((z-y)/\sqrt{t_n})\,\mathrm{d}z\,\mathrm{d}y \\
&= 2\int_0^K f(y) \int_{K_n+y}^{\infty} \frac{1}{\sqrt{t_n}} \phi(z/\sqrt{t_n})\,\mathrm{d}z\,\mathrm{d}y \\
&\leq 2\int_0^K f(y) \int_{K_n}^{\infty} \frac{1}{\sqrt{t_n}} \phi(z/\sqrt{t_n})\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_{K_n}^{\infty} \frac{1}{\sqrt{t_n}} \phi(z/\sqrt{t_n})\,\mathrm{d}z = 1 - \Phi(K/\sqrt{t_n}) \\
&\leq \frac{\sqrt{t_n}}{K_n} \phi(K/\sqrt{t_n}) = o(t_n).
\end{aligned}$$

Now choose $\varepsilon \in (0, K)$ and $\varepsilon_n$ such that $\varepsilon_n > 0$, $K_n - K < \varepsilon_n \leq K_n$ and $\varepsilon_n - (K_n - K) \nearrow \varepsilon$. It holds

$$\begin{aligned}
&2\int_0^K f(y) \int_{K_n}^{\infty} \frac{1}{\sqrt{t_n}} \phi((z-y)/\sqrt{t_n})\,\mathrm{d}z\,\mathrm{d}y = 2\int_0^K f(y)(1 - \Phi((K_n - y)/\sqrt{t_n}))\,\mathrm{d}y \\
&= 2\int_{(K_n-K)/\sqrt{t_n}}^{K_n/\sqrt{t_n}} \sqrt{t_n} f(K_n - y\sqrt{t_n})(1 - \Phi(y))\,\mathrm{d}y \\
&= 2\int_{\varepsilon_n/\sqrt{t_n}}^{K_n/\sqrt{t_n}} \sqrt{t_n} f(K_n - y\sqrt{t_n})(1 - \Phi(y))\,\mathrm{d}y \\
&\quad + 2\int_{(K_n-K)/\sqrt{t_n}}^{\varepsilon_n/\sqrt{t_n}} \sqrt{t_n} f(K_n - y\sqrt{t_n})(1 - \Phi(y))\,\mathrm{d}y.
\end{aligned}$$

On the one hand we have

$$\begin{aligned}
&\int_{\varepsilon_n/\sqrt{t_n}}^{K_n/\sqrt{t_n}} \sqrt{t_n} f(K_n - y\sqrt{t_n})(1-\Phi(y))\,\mathrm{d}y \\
&\le (1-\Phi(\varepsilon_n/\sqrt{t_n})) \int_{\varepsilon_n/\sqrt{t_n}}^{K_n/\sqrt{t_n}} \sqrt{t_n} f(K_n - y\sqrt{t_n})\,\mathrm{d}y \\
&= (1-\Phi(\varepsilon_n/\sqrt{t_n})) \int_{\varepsilon_n}^{K_n} f(K_n - y)\,\mathrm{d}y \\
&\le (1-\Phi(\varepsilon_n/\sqrt{t_n})) \le \frac{\sqrt{t_n}}{\varepsilon_n}\phi(\varepsilon_n/\sqrt{t_n}) = o(t_n).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\frac{1}{t_n} 2 \int_{(K_n-K)/\sqrt{t_n}}^{\varepsilon_n/\sqrt{t_n}} \sqrt{t_n} f(K_n - y\sqrt{t_n})(1-\Phi(y))\,\mathrm{d}y \\
&= 2\int_{(K_n-K)/\sqrt{t_n}}^{\varepsilon_n/\sqrt{t_n}} \frac{f(K_n - y\sqrt{t_n})}{y\sqrt{t_n} - (K_n - K)} \left(y - \frac{K_n-K}{\sqrt{t_n}}\right)(1-\Phi(y))\,\mathrm{d}y \\
&\ge \inf_{x\in(0,\varepsilon_n-(K_n-K))} \frac{f(K-x)}{x} 2\int_{\frac{K_n-K}{\sqrt{t_n}}}^{\varepsilon_n/\sqrt{t_n}} \left(y - \frac{K_n-K}{\sqrt{t_n}}\right)(1-\Phi(y))\,\mathrm{d}y
\end{aligned}$$

Further we have

$$\begin{aligned}
&\int_{\frac{K_n-K}{\sqrt{t_n}}}^{\varepsilon_n/\sqrt{t_n}} \left(y - \frac{K_n-K}{\sqrt{t_n}}\right)(1-\Phi(y))\,\mathrm{d}y \\
&= \int_0^{(\varepsilon_n-(K_n-K))/\sqrt{t_n}} y\left(1-\Phi\left(y+\frac{K_n-K}{\sqrt{t_n}}\right)\right)\mathrm{d}y \\
&= \int_0^{\infty} y\left(1-\Phi\left(y+\frac{K_n-K}{\sqrt{t_n}}\right)\right)\mathrm{d}y \\
&\quad - \int_{(\varepsilon_n-(K_n-K))/\sqrt{t_n}}^{\infty} y\left(1-\Phi\left(y+\frac{K_n-K}{\sqrt{t_n}}\right)\right)\mathrm{d}y.
\end{aligned}$$

By

$$\begin{aligned}
&\int_{(\varepsilon_n-(K_n-K))/\sqrt{t_n}}^{\infty} y\left(1-\Phi\left(y+\frac{K_n-K}{\sqrt{t_n}}\right)\right)\mathrm{d}y \\
&\le \int_{(\varepsilon_n-(K_n-K))/\sqrt{t_n}}^{\infty} \frac{y}{y+\frac{K_n-K}{\sqrt{t_n}}}\phi\left(y+\frac{K_n-K}{\sqrt{t_n}}\right)\mathrm{d}y \\
&\le \max\left(1, \frac{\varepsilon_n-(K_n-K))}{\varepsilon_n}\right)\int_{(\varepsilon_n-(K_n-K))/\sqrt{t_n}}^{\infty} \phi\left(y+\frac{K_n-K}{\sqrt{t_n}}\right)\mathrm{d}y \\
&\le \max\left(1, \frac{\varepsilon_n-(K_n-K))}{\varepsilon_n}\right)\left(1-\Phi\left(\frac{\varepsilon_n}{\sqrt{t_n}}\right)\right) \\
&\to 0
\end{aligned}$$

as $n \to \infty$ and the fact that $\int_0^\infty y(1-\Phi(y+a))\,\mathrm{d}y$ is continuous and decreasing in $a$, we can deduce that

$$\begin{aligned}\liminf_{n\to\infty} \int_{\frac{K_n-K}{\sqrt{t_n}}}^{\varepsilon_n/\sqrt{t_n}} \left(y - \frac{K_n-K}{\sqrt{t_n}}\right)(1-\Phi(y))\,\mathrm{d}y \\ = \int_0^\infty y\left(1-\Phi\left(y+\limsup_{n\to\infty}\frac{K_n-K}{\sqrt{t_n}}\right)\right)\mathrm{d}y.\end{aligned}$$

Altogether, it follows with $\varepsilon_n - (K_n - K) \nearrow \varepsilon$ that

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{t_n}\mathbb{P}_\mu\left(|X_{t_n}| \geq K_n\right) \\ &\geq \inf_{x\in(0,\varepsilon)} \frac{f(K-x)}{x} 2\int_0^\infty y\left(1-\Phi\left(y+\limsup_{n\to\infty}\frac{K_n-K}{\sqrt{t_n}}\right)\right)\mathrm{d}y\end{aligned}$$

Letting $\varepsilon \to 0$ yields the statement. □

The following result is a direct consequence of Theorem 1.A.3 (a) in [SS07].

**Lemma B.1.3.** *Let $\mu, \nu$ be probability measures with $\mu \preceq_{\mathrm{st}} \nu$. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Assume that $f$ is non-increasing. Then it holds $\nu \circ f^{-1} \preceq_{\mathrm{st}} \mu \circ f^{-1}$. In particular, if $|f|$ is bounded, we have*

$$\int_{\mathbb{R}} f(x)\mu(\mathrm{d}x) \geq \int_{\mathbb{R}} f(x)\nu(\mathrm{d}x).$$

The following result can be found as Theorem 1.C.7 in [SS07].

**Theorem B.1.4.** *Let $\mu$, $\nu$ and $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ be probability measures, such that for every $n \in \mathbb{N}$ holds $\mu_n \preceq_{\mathrm{lr}} \nu_n$. Assume that $\mu_n \to \mu$ and $\nu_n \to \nu$ in distribution as $n \to \infty$. Then we have $\mu \preceq_{\mathrm{lr}} \nu$.*

By Proposition 8.2 in Chapter 2 of [PS78] we have the following representation of the transition density of a Brownian motion absorbed at a two-sided boundary. This result is used in Subsection 2.3.5 for the study of the exponential case.

**Proposition B.1.5.** *Let $(X_t)_{t\geq 0}$ be a Brownian motion and $M > 0$. For $x, y \in [-M, M]$ we have*

$$\mathbb{P}_x\left(X_t \in \mathrm{d}y, \tau_M > t\right) = \frac{1}{M}\sum_{n=1}^\infty e^{-\frac{n^2\pi^2 t}{8M^2}} \sin\left(n\pi\frac{x+M}{2M}\right)\sin\left(n\pi\frac{y+M}{2M}\right)\mathrm{d}y.$$

By integration we obtain the following hitting time distribution.

**Proposition B.1.6.** *For $x \in [-M, M]$ we have*

$$\mathbb{P}_x\left(\tau_M > t\right) = \frac{4}{\pi}\sum_{k=0}^\infty \frac{1}{2k+1} e^{-\frac{(2k+1)^2\pi^2 t}{8M^2}} \sin\left((2k+1)\pi\frac{x+M}{2M}\right).$$

*Proof.* We have

$$\begin{aligned}
\mathbb{P}_x(\tau_M > t) &= \mathbb{P}_x(X_t \in [-M, M], \tau_M > t) \\
&= \frac{1}{M}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 t}{8M^2}} \sin\left(n\pi\frac{x+M}{2M}\right)\int_{-M}^{M} \sin\left(n\pi\frac{y+M}{2M}\right)\mathrm{d}y \\
&= \frac{1}{M}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 t}{8M^2}} \sin\left(n\pi\frac{x+M}{2M}\right)\frac{2M}{n\pi}(1-\cos(n\pi)) \\
&= \frac{1}{M}\sum_{k=0}^{\infty} e^{-\frac{(2k+1)^2\pi^2 t}{8M^2}} \sin\left((2k+1)\pi\frac{x+M}{2M}\right)\frac{2M}{(2k+1)\pi}2 \\
&= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1} e^{-\frac{(2k+1)^2\pi^2 t}{8M^2}} \sin\left((2k+1)\pi\frac{x+M}{2M}\right).
\end{aligned}$$

This shows the statement. □

From Proposition B.1.5 we can also deduce the following.

**Lemma B.1.7.** *Let $(X_t)_{t\geq 0}$ be a Brownian motion and $M > 0$. Let $\varphi(x) = \frac{\pi}{4M}\cos\left(\frac{\pi}{2M}x\right)$. It holds uniformly in $A \subseteq [-M, M]$ that*

$$\mathbb{P}_x(X_t \in A \,|\tau_M > t) \to \int_A \varphi(y)\,\mathrm{d}y$$

*exponentially fast and uniformly in $x \in (-M, M)$ as $t \to \infty$.*

*Proof.* Let $\lambda = \frac{\pi^2}{8M^2}$, note that $\varphi(x) = \sin\left(\pi\frac{x+M}{2M}\right)$ and consider

$$\begin{aligned}
&\left|\frac{\mathbb{P}_x(X_t \in A, \tau_M > t)}{e^{-\lambda t}\frac{16M}{\pi^2}\varphi(x)} - \int_A \varphi(y)\,\mathrm{d}y\right| \\
&= \left|\sum_{n=1}^{\infty} e^{-\frac{(n^2-1)\pi^2 t}{8M^2}} \frac{\sin\left(n\pi\frac{x+M}{2M}\right)}{\sin\left(\pi\frac{x+M}{2M}\right)}\int_A \frac{\pi}{4M}\sin\left(n\pi\frac{y+M}{2M}\right)\mathrm{d}y - \int_A\varphi(y)\,\mathrm{d}y\right| \\
&\leq \sum_{n=2}^{\infty} e^{-\frac{(n^2-1)\pi^2 t}{8M^2}}\left|\frac{\sin\left(n\pi\frac{x+M}{2M}\right)}{\sin\left(\pi\frac{x+M}{2M}\right)}\right|\frac{\pi}{4M} \\
&\leq \frac{\pi}{4M}\sum_{n=2}^{\infty} n e^{-(n^2-1)\lambda t},
\end{aligned}$$

which converges to 0 exponentially fast. Thus we have

$$\begin{aligned}
&\left|\mathbb{P}_x(X_t \in A \,|\tau_M > t) - \int_A\varphi(y)\,\mathrm{d}y\right| \\
&\leq \left|\frac{e^{-\lambda t}\frac{16M}{\pi^2}\varphi(x)}{\mathbb{P}_x(\tau_M > t)}\frac{\mathbb{P}_x(X_t \in A, \tau_M > t)}{e^{-\lambda t}\frac{16M}{\pi^2}\varphi(x)} - \int_A\varphi(y)\,\mathrm{d}y\right| \\
&\leq \frac{e^{-\lambda t}\frac{16M}{\pi^2}\varphi(x)}{\mathbb{P}_x(\tau_M > t)}\frac{\pi}{4M}\sum_{n=2}^{\infty} n e^{-(n^2-1)\lambda t} + \left|\frac{e^{-\lambda t}\frac{16M}{\pi^2}\varphi(x)}{\mathbb{P}_x(\tau_M > t)} - 1\right|,
\end{aligned}$$

which converges to 0 exponentially fast, since

$$\left|\frac{e^{-\lambda t}\frac{16M}{\pi^2}\varphi(x)}{\mathbb{P}_x\left(\tau_M > t\right)} - 1\right| \leq \frac{\pi}{4M}\sum_{n=2}^{\infty} n e^{-(n^2-1)\lambda t}$$

by the first computation. □

In the proof of Proposition 2.3.55 we will make use of the so-called $Q$-process, whose existence is described by the following statement, which follows from Theorem 1.1 and Theorem 1.3 in [CV14].

**Theorem B.1.8.** *Let $(X_t)_{t\geq 0}$ be a Markov process with state space $E \cup \{\partial\}$ and $\tau_\partial \coloneqq \inf\{t \geq 0 : X_t = \delta\}$. Assume that there exists a probability measure $\alpha$ on $E$ such that*

$$d_{\mathrm{TV}}\left(\mathbb{P}_x\left(X_t \in \cdot\ |\tau_\partial > t\right), \alpha\right) \to 0$$

*exponentially fast uniformly in $x \in E$. Then for any $x \in E$ there exists the measure*

$$\mathbb{Q}_x(A) = \lim_{t\to\infty} \mathbb{P}_x\left(A\,|\tau_\partial > t\right), \forall A \in \sigma(X_u : u \leq s), \forall s \geq 0.$$

*Under the family $(\mathbb{Q}_x)_{x\in E}$ the process $(X_t)_{t\geq 0}$ is a Markov process, has an invariant distribution $\beta$ and for any $x \in E$ it holds that $\mathbb{Q}_x(X_t \in \cdot\,)$ converges in total variation to $\beta$. The limit $\eta(x) = \lim_{t\to\infty} \frac{\mathbb{P}_x(\tau_\partial > t)}{\mathbb{P}_\alpha(\tau_\partial > t)}$ exists and*

$$\beta(\mathrm{d}x) = \frac{\eta(x)\alpha(\mathrm{d}x)}{\int_E \eta(y)\alpha(\mathrm{d}y)}.$$

The following statement is a purely auxiliary lemma for the application in the context of the non-branching particle system with selection and will be used in the proof of Theorem 2.4.16.

**Lemma B.1.9.** *For $N \in \mathbb{N}$ let $A_N \subset \{1, \ldots, N\}$ be a random set of deterministic cardinality $a_N$ with $N^{-1}a_N \to a > 0$ and $X_1^N, \ldots, X_N^N$ real-valued random variables. We assume that there exists $M < \infty$ such that $|X_i^N| \leq M$ for every $N$ and $i$ and that*

$$\frac{1}{a_N}\sum_{i\in A_N} X_i \to c$$

*almost surely as $N \to \infty$, where $c \in \mathbb{R}$. Let $Z_1, Z_2, \ldots$ be independent from each other and from all other randomness and identically distributed random variables with $\mathbb{E}\left[Z_1^4\right] < \infty$. Then it holds that*

$$S_N \coloneqq \frac{1}{a_N}\sum_{i\in A_N} Z_i X_i \to \mathbb{E}\left[Z_1\right] c$$

*almost surely as $N \to \infty$.*

*Proof.* Set $p := \mathbb{E}[Z_1]$. In the proof we drop the dependency of $X_i^N$ on $N$ in the notation. If we can prove that $\bar{S}_N := S_N - a_N^{-1} \sum_{i \in A_N} pX_i$ converges almost surely to zero the statement is proven. For this we define $\bar{Z}_i := Z_i - p$ and calculate

$$\bar{S}_N^4 = \frac{1}{a_N^4} \left( \left( \sum_{i \in A_N} X_i^2 \bar{Z}_i^2 \right)^2 + \left( \sum_{i \in A_N} \sum_{j \in A_N : j \neq i} X_i X_j \bar{Z}_i \bar{Z}_j \right)^2 \right.$$
$$\left. + 2 \sum_{k \in A_N} X_k^2 \bar{Z}_k^2 \sum_{i \in A_N} \sum_{j \in A_N : j \neq i} X_i X_j \bar{Z}_i \bar{Z}_j \right).$$

We have on the one hand

$$\mathbb{E}\left[ \sum_{k \in A_N} X_k^2 \bar{Z}_k^2 \sum_{i \in A_N} \sum_{j \in A_N : j \neq i} X_i X_j \bar{Z}_i \bar{Z}_j \right]$$
$$= \mathbb{E}\left[ \sum_{k \in A_N} \sum_{i \in A_N} \sum_{j \in A_N : j \neq i} X_k^2 X_i X_j \mathbb{E}\left[ \bar{Z}_k^2 \bar{Z}_i \bar{Z}_j \middle| X_1, \ldots, X_N, A_N \right] \right] = 0$$

and on the other hand

$$\mathbb{E}\left[ \left( \sum_{i \in A_N} X_i^2 \bar{Z}_i^2 \right)^2 \right]$$
$$= \mathbb{E}\left[ \sum_{i \in A_N} X_i^4 \mathbb{E}\left[ \bar{Z}_i^4 \right] \right] + \mathbb{E}\left[ \sum_{i \in A_N} \sum_{j \in A_N : j \neq i} X_i^2 X_j^2 \mathbb{E}\left[ \bar{Z}_i^2 \bar{Z}_j^2 \right] \right]$$
$$\leq N M^4 \mathbb{E}\left[ \bar{Z}_1^4 \right] + N^2 M^4 \mathbb{E}\left[ \bar{Z}_1^2 \right]^2$$

and

$$\mathbb{E}\left[ \left( \sum_{i \in A_N} \sum_{j \in A_N : j \neq i} X_i X_j \bar{Z}_i \bar{Z}_j \right)^2 \right] = \mathbb{E}\left[ 2 \sum_{i \in A_N} \sum_{j \in A_N : j \neq i} X_i^2 X_j^2 \mathbb{E}\left[ \bar{Z}_i^2 \bar{Z}_j^2 \right] \right]$$
$$\leq 2 N^2 M^4 \mathbb{E}\left[ \bar{Z}_1^2 \right]^2$$

Since for $\varepsilon > 0$ we have

$$\mathbb{P}\left( |\bar{S}_N| > \varepsilon \right) \leq \frac{\mathbb{E}\left[ \bar{S}_N^4 \right]}{\varepsilon^4}$$

the probability $\mathbb{P}\left( |\bar{S}_N| > \varepsilon \right)$ is summable over $N$ which shows that $\bar{S}_N \to 0$ almost surely by the Borel-Cantelli lemma. □

Recall the Wasserstein distance from (2.11). In the proof of Theorem 2.3.32 we will make use of the following statement, which is a consequence of Theorem 7.12 in [Vil03].

**Theorem B.1.10.** *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures in $\mathcal{P}^1$ and let $\mu \in \mathcal{P}^1$. Then the following are equivalent:*

*(i)* $d_{\mathrm{W}}(\mu_n, \mu) \to 0$ *as* $n \to \infty$,

*(ii)* $\mu_n \to \mu$ *in the sense of weak convergence and it holds*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}} |x| \mathbb{1}_{[R,\infty)}(|x|) \,\mathrm{d}\mu_n(x) = 0.$$

*(iii)* $\mu_n \to \mu$ *in the sense of weak convergence and it holds*

$$\lim_{n\to\infty} \int_{\mathbb{R}} |x| \,\mathrm{d}\mu_n(x) = \int_{\mathbb{R}} |x| \,\mathrm{d}\mu(x).$$

We will also make use of the Prohorov metric in Lemma 3.2.9. For $\mu, \nu \in \mathcal{P}$ the Prohorov metric defined by

$$d_{\mathrm{P}}(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(B) \le \nu(B^\varepsilon) + \varepsilon \text{ for all } B \in \mathcal{B}(\mathbb{R})\}, \tag{B.1}$$

where $B^\varepsilon := \{x \in \mathbb{R} : \inf_{y\in B} |x - y| \le \varepsilon\}$. This metric metrizes the weak convergence of probability measures as stated in Theorem 11.3.3 from [Dud02].

According to Corollary 11.6.4 in [Dud02] an alternative representation for the Prohorov metric is given by the following.

$$d_{\mathrm{P}}(\mu, \nu) = \inf\{\inf\{\varepsilon > 0 : \mathbb{P}(|X - Y| > \varepsilon) \le \varepsilon\} : X \sim \nu, Y \sim \nu\}. \tag{B.2}$$

By this we can deduce the following lemma.

**Lemma B.1.11.** *For $\mu, \nu \in \mathcal{P}$ it holds*

*(i)* $d_{\mathrm{P}}(\mu, \nu) \le d_{\mathrm{TV}}(\mu, \nu)$ *and*

*(ii)* $d_{\mathrm{P}}(\mu, \nu) \le \sqrt{\inf\{\mathbb{E}[|X - Y| \wedge 1] : X \sim \mu, Y \sim \nu\}} \le \sqrt{d_{\mathrm{W}}(\mu, \nu)}$.

*Proof.* (i) Let $X \sim \mu$ and $Y \sim \nu$. Without loss of generality assume $\mathbb{P}(X \ne Y) > 0$. For every $\varepsilon > 0$ we have $\mathbb{P}(|X - Y| > \varepsilon) \le \mathbb{P}(X \ne Y)$, thus in particular for $\varepsilon = \mathbb{P}(X \ne Y)$. We can deduce

$$\begin{aligned} d_{\mathrm{P}}(\mu, \nu) &\le \inf\{\varepsilon > 0 : \mathbb{P}(|X - Y| > \varepsilon) \le \varepsilon\} \\ &\le \mathbb{P}(X \ne Y). \end{aligned}$$

Together with (3.17), taking the infimum over all possible couplings yields the desired inequality.

(ii) We can assume without loss of generality that $0 < d_{\mathrm{P}}(\mu, \nu) < 1$, since otherwise this implies that $|X - Y| \geq 1$ almost surely for every coupling $X \sim \mu$ , $Y \sim \nu$. Now let $X \sim \mu$ and $Y \sim \nu$ be such a coupling. Further, assume that $0 < \mathbb{E}[|X - Y| \wedge 1] < 1$. Then we can deduce for $\varepsilon = \sqrt{\mathbb{E}[|X - Y| \wedge 1]}$ by the Markov inquality that

$$\begin{aligned}\mathbb{P}(|X - Y| > \varepsilon) &= \mathbb{P}(|X - Y| \wedge 1 > \varepsilon)\\ &\leq \varepsilon^{-1}\mathbb{E}[|X - Y| \wedge 1] = \sqrt{\mathbb{E}[|X - Y| \wedge 1]} = \varepsilon.\end{aligned}$$

Therefore, we have

$$\begin{aligned}d_{\mathrm{P}}(\mu, \nu) &\leq \inf\{\varepsilon > 0 : \mathbb{P}(|X - Y| > \varepsilon) \leq \varepsilon\}\\ &\leq \sqrt{\mathbb{E}[|X - Y| \wedge 1]},\end{aligned}$$

which finishes the proof by taking the infimum over all possible couplings. □

In order to check our simulation for the soft-killing solutions we use the following example of a distribution of a soft-killing first-passage time in Example 3.3.3.

**Lemma B.1.12.** *Let $b(t) = c$ for all $t \geq 0$, where $c \in \mathbb{R}$. Then*

$$\begin{aligned}\mathbb{E}_\mu\left[e^{-\int_0^t \mathbb{1}_{(-\infty,b(s))}(X_s)\,\mathrm{d}s}\right] &= \int_{\mathbb{R}} e^{-t\mathbb{1}_{(-\infty,c)}(x)}\left(2\Phi\left(\sqrt{\frac{|c-x|}{t}}\right) - 1\right)\mathrm{d}\mu(x)\\ &+ \int_{\mathbb{R}}\int_0^t e^{-u\mathbb{1}_{(-\infty,c)}(x)}\,{}_1F_1(1/2; 1; -(t-u))\sqrt{\frac{|c-x|}{2\pi u^3}}e^{-\frac{|c-x|}{2u}}\,\mathrm{d}u\,\mathrm{d}\mu(x),\end{aligned}$$

*where ${}_1F_1$ denotes the confluent hypergeometric function of the first kind.*

*Proof.* Define the hitting time

$$T_c \coloneqq \inf\{t > 0 : X_t = c\},$$

whose distribution is the well-known unshifted Lévy distribution, this means we have

$$\mathbb{P}_x(T_c \leq t) = \int_0^t \sqrt{\frac{|c-x|}{2\pi u^3}}e^{-\frac{|c-x|}{2u}}\,\mathrm{d}u = 2 - 2\Phi\left(\sqrt{\frac{|c-x|}{t}}\right)$$

Furthermore, it is clear by the definitions that

$$\int_0^t \mathbb{1}_{(-\infty,b(s))}(X_s)\,\mathrm{d}s = (T_c \wedge t)\mathbb{1}_{(-\infty,c)}(X_0) + \int_{T_c\wedge t}^t \mathbb{1}_{(-\infty,c)}(X_s)\,\mathrm{d}s.$$

Since $X_{T_c} = c$ almost surely, we have by the strong Markov property that

$$\begin{aligned}
&\mathbb{E}_\mu\left[e^{-\int_0^t \mathbb{1}_{(-\infty,b(s))}(X_s)\,\mathrm{d}s}\right]\\
&= \mathbb{E}_\mu\left[e^{-(T_c\wedge t)\mathbb{1}_{(-\infty,c)}(X_0)-\int_{T_c\wedge t}^t \mathbb{1}_{(-\infty,c)}(X_s)\,\mathrm{d}s}\right]\\
&= \mathbb{E}_\mu\left[e^{-(T_c\wedge t)\mathbb{1}_{(-\infty,c)}(X_0)}\mathbb{E}_0\left[e^{-\int_0^{t-u} \mathbb{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s}\right]_{u=T_c\wedge t}\right].
\end{aligned}$$

By the first arcsine law by Lévy it is well-known that under $\mathbb{P}_0$ the normed occupation time $\frac{1}{t-u}\int_0^{t-u} \mathbb{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s$ has the standard arcsine distribution, which coincides with the Beta$(1/2, 1/2)$ distribution. Since the moment generating function corresponding to the Beta$(\alpha,\beta)$ distribution is the confluent hypergeometric function ${}_1F_1(\alpha;\alpha+\beta;\cdot\,)$ of the first kind, we have that

$$\begin{aligned}
&\mathbb{E}_\mu\left[e^{-\int_0^t \mathbb{1}_{(-\infty,b(s))}(X_s)\,\mathrm{d}s}\right] = \mathbb{E}_\mu\left[e^{-(T_c\wedge t)\mathbb{1}_{(-\infty,c)}(X_0)}{}_1F_1(1/2;1;-(t-T_c\wedge t))\right]\\
&= \mathbb{E}_\mu\left[e^{-t\mathbb{1}_{(-\infty,c)}(X_0)}\mathbb{1}_{\{T_c>t\}}\right] + \mathbb{E}_\mu\left[e^{-T_c\mathbb{1}_{(-\infty,c)}(X_0)}{}_1F_1(1/2;1;-(t-T_c))\mathbb{1}_{\{T_c\le t\}}\right]\\
&= \int_{\mathbb{R}} e^{-t\mathbb{1}_{(-\infty,c)}(x)}\left(2\Phi\left(\sqrt{\frac{|c-x|}{t}}\right)-1\right)\mathrm{d}\mu(x)\\
&\quad + \int_{\mathbb{R}}\int_0^t e^{-u\mathbb{1}_{(-\infty,c)}(x)}{}_1F_1(1/2;1;-(t-u))\sqrt{\frac{|c-x|}{2\pi u^3}}e^{-\frac{|c-x|}{2u}}\,\mathrm{d}u\,\mathrm{d}\mu(x).
\end{aligned}$$

□

For two probability measures $\mu$ and $\nu$ on $\mathbb{R}$ a probability measure on $\mathbb{R}^2$ with marginals $\mu$ and $\nu$ is called a coupling of $\mu$ and $\nu$.

The following lemma can be found as Lemma 7.6 in [Vil03].

**Lemma B.1.13.** *Let $\mu_1, \mu_2, \mu_3$ be three probability measures supported in Polish spaces $\Omega_1, \Omega_2, \Omega_3$ respectively, and let $\pi_{12}$ a coupling of $\mu_1$ and $\mu_2$ and $\pi_{23}$ a coupling of $\mu_2$ and $\mu_3$. Then there exists a probability measure $\pi_{123}$ on $\Omega_1\times\Omega_2\times\Omega_3$ with marginals $\pi_{12}$ on $\Omega_1\times\Omega_2$ and $\pi_{23}$ on $\Omega_2\times\Omega_3$.*

## B.2 Analysis

In this subsection we present the auxiliary results, which have an analytical background. We will first collect multiple statements concerning semicontinuous functions, which are used in Subsection 2.1.1 and Subsection 2.3.1 in order to study the set of boundary functions and related aspects.

**Lemma B.2.1.** *A lower semi-continuous function $b : [0,\infty] \to [0,\infty]$ attains its infimum on compact sets $I \subset [0,\infty)$.*

*Proof.* Let $x := \inf_{s\in I} b(s)$ and $(t_n)_{n\in\mathbb{N}} \subset I$ a minimizing sequence, this means $b(t_n) \to x$. If we assume that $I$ is compact, we can assume without loss of generality that $(t_n)_{n\in\mathbb{N}}$ has a limit point $t$. Then due to the lower semi-continuity of $b$ we have

$$b(t) \leq \liminf_{s\to t} b(s) \leq \lim_{n\to\infty} b(t_n) = x$$

This shows that $b(t) = \inf_{s\in I} b(s)$. □

In [Hah21] we find in Kapitel II, §10 the following two statements as Satz III and IV.

**Theorem B.2.2.** *Let $S$ be a metric space and let $g : S \to \mathbb{R} \cup \{-\infty, \infty\}$ be lower semi-continuous. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions such that $g_n \leq g_{n+1}$ and $g_n \to g$ pointwise.*

**Theorem B.2.3.** *Let $S$ be a metric space. Further, let $g : S \to \mathbb{R} \cup \{-\infty, \infty\}$ be lower semi-continuous and $h : S \to \mathbb{R}$ be upper semi-continuous, such that $h \leq g$. Then there exists a continuous function $f : S \to \mathbb{R}$ such that $h \leq f \leq g$.*

We can deduce the following corollaries.

**Corollary B.2.4.** *Let $I \subset \mathbb{R}$ be compact and $b : I \to (0, \infty]$ be a lower semi-continuous function. Let $F \subset I$ be a finite set. Then there exist a continuous function $f : I \to (0, \infty)$ such that $f \leq b$ and $f(t) = b(t)$ for all $t \in F$.*

*Proof.* Let $x := \inf_{t\in I} b(t)$. By Lemma B.2.1 we obtain that $x > 0$. Define the upper semi-continuous function $h : I \to \mathbb{R}$ by

$$h(t) := \begin{cases} b(t) & : t \in F, \\ x & : \text{else.} \end{cases}$$

Then by Theorem B.2.3 above there exists a continuous function $f$ such that $h \leq f \leq b$. Since $f \geq x > 0$ the statement is proved. □

**Corollary B.2.5.** *Let $S$ be a metric space and let $g : S \to \mathbb{R} \cup \{-\infty, \infty\}$ be lower semi-continuous. Let $F \subseteq S$ be a finite set. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions such that $g_n \leq g_{n+1}$ and $g_n \to g$ pointwise and $g_n(t) = g(t)$ for all $t \in F$.*

*Proof.* We obtain a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous function with the first two properties by Theorem B.2.2. Since $f_1$ is continuous and $f_1 \leq g$ we have that the function

$$h : S \to \mathbb{R} \cup \{-\infty, \infty\},\ h(t) := \begin{cases} g(t) & : t \in F, \\ f_1(t) & : \text{else} \end{cases}$$

is upper semi-continuous. By Theorem B.2.3 there exists a continuous function $f$ such that $h \leq f \leq g$. Taking $g_n := \max(f, f_n)$ yields the statement. □

We will now collect some statements related to the Hausdorff topology, which is used in Subsection 2.1.1.

**Definition B.2.6.** *Let $H = [0,\infty] \times [-\infty,\infty]$ and denote with $K(H)$ the set of compact and non-empty subsets of $H$. Consider the bijective map $\varphi : H \to [0,1] \times [-1,1]$, $(t,x) \mapsto (t/(1+t), x/(1+|x|))$. Let $r$ be metric on $H$, which is induced by $\varphi$ from the euclidean metric on $[0,1] \times [-1,1]$.*

*a) We define the* Hausdorff-metric *on $K(H)$ by*

$$r_H(C,D) := \max\{\sup_{x\in C} r(x,D), \sup_{y\in D} r(y,C)\} \tag{B.3}$$

*We equip $K(H)$ with the so-called* Hausdorff-topology.

*b) Let $(B_n)_{n\in\mathbb{N}}$ be a sequence in $K(H)$. Denote with $\mathcal{U}_x$ the filter of neighbourhoods of a point $h \in H$. We define the* Kuratowski limit inferior *by*

$$\operatorname*{Li}_{n\to\infty} B_n := \{h \in H : U \in \mathcal{U}_h \Rightarrow U \cap B_n \neq \emptyset \text{ for almost all } n\}$$

*and the* Kuratowski limit superior *by*

$$\operatorname*{Ls}_{n\to\infty} B_n := \{h \in H : U \in \mathcal{U}_h \Rightarrow U \cap B_n \neq \emptyset \text{ for infinitely many } n\}.$$

*If $\operatorname*{Li}_{n\to\infty} B_n = \operatorname*{Ls}_{n\to\infty} B_n =: B$ we call $(B_n)_{n\in\mathbb{N}}$* Kuratowski-convergent *to the* Kuratowski-limit *$B$.*

The following statement links the convergence in Hausdorff-metric with the Kuratowski-convergence, which are in fact the same on compact ambient spaces.

**Theorem B.2.7.** *The Kuratowski-convergence on $K(H)$ coincides with the convergence in the Hausdorff-metric.*

*Proof.* Let the Fell topology on $K(H)$ be defined as in Definition 5.1.1 of [Bee93]. By Corollary 5.1.11 in [Bee93], since $H$ is a compact metric space, the Fell topology and the topology induced by the Hausdorff-metric coincide on the closed, non-empty subsets of $H$. Since $H$ is compact, those are $K(H)$. In addition by Theorem 5.2.10 in [Bee93] the Kuratowski-limit coincide with the limit of sequences with respect to the Fell topology. □

The following statement eases the use of convexity in the proof of Theorem 2.3.34.

**Lemma B.2.8.** *Let $f : (a,b) \to \mathbb{R}$ be a function. If $f$ is convex then for all $s,t \in (a,b)$ with $s < t$ and $c \in (0, b-t)$ it holds, that*

$$f(s+c) + f(t) \leq f(t+c) + f(s).$$

*If $f$ is measurable the converse is also true.*

*Proof.* Assume that $f$ is convex on $(a,b)$. For such $s,t,c$ we can choose

$$\theta_1 \coloneqq \frac{t-s}{t-s+c}, \quad \theta_2 \coloneqq \frac{c}{t-s+c}.$$

We have that $s+c = \theta_1 s + (1-\theta_1)(t+c)$ and $t = \theta_2 s + (1-\theta_2)(t+c)$, and $\theta_1 + \theta_2 = 1$. Due to the convexity of $f$ we get

$$\begin{aligned} f(s+c) + f(t) &\leq \theta_1 f(s) + (1-\theta_1) f(t+c) + \theta_2 f(s) + (1-\theta_2) f(t+c) \\ &= f(t+c) + f(s). \end{aligned}$$

Conversely, the by the equation in the statement we can deduce for two points $x,y \in (a,b)$ with $x<y$ by setting $s \coloneqq x$, $t \coloneqq (x+y)/2$ and $c \coloneqq (y-x)/2$, that

$$2f\left(\frac{x+y}{2}\right) \leq f(y) + f(x),$$

which means that $f$ is midpoint convex. If a function is measurable and midpoint convex it is convex, see for example [Don69, p.12], which finishes the proof. □

For completeness we give a proof for the following auxiliary lemma, for which have not found a suitable reference in the literature. It will be used in the proof of Theorem 3.2.13.

**Lemma B.2.9.** *Let $T > \eta > 0$. Let $f : [\eta, T] \to \mathbb{R}$ be a continuous function with*

$$\int_\eta^T \mathbb{1}_{f^{-1}(\{0\})}(s)\,\mathrm{d}s = 0.$$

*Furthermore, let $f_n \to f$ uniformly on $[\eta, T]$. Then for any sequence of partitions $(Z_n)$ of $[\eta, T]$ with mesh tending to zero (this means that $Z_n = \{t_0^n, \ldots, t_{m_n}^n\}$, where $\eta = t_0^n < \ldots < t_{m_n}^n = T$ and $\lim_{n\to\infty} \max_{i=1,\ldots,m_n} |t_i^n - t_{i-1}^n| = 0$) it holds*

$$\sum_{i=1}^{m_n} \mathbb{1}_{(-\infty,0)}(f_n(t_i^n))(t_i^n - t_{i-1}^n) \to \int_\eta^T \mathbb{1}_{(-\infty,0)}(f(s))\,\mathrm{d}s.$$

*Proof.* Let $(Z_n)_{n\in\mathbb{N}}$ be such a sequence of partitions. Let $D \coloneqq \{t \in [\eta, T] : s \mapsto \mathbb{1}_{(-\infty,0)}(f(s))$ is discontinuous at $t\}$. We have $D \subset f^{-1}(\{0\})$ and thus, by the assumption, the bounded function $s \mapsto \mathbb{1}_{(-\infty,0)}(f(s))$ is almost everywhere continuous on $[\eta, T]$. By Lebesgue's criterion for Riemann integrability, for example see Theorem 11.33 from [Rud76], it follows that we have

$$\sum_{i=1}^{m_n} \mathbb{1}_{(-\infty,0)}(f(t_i^n))(t_i^n - t_{i-1}^n) \to \int_\eta^T \mathbb{1}_{(-\infty,0)}(f(s))\,\mathrm{d}s.$$

For $\varepsilon > 0$ let

$$\phi_\varepsilon(x) := \begin{cases} 0 & : |x| \geq 2\varepsilon, \\ \frac{2\varepsilon - |x|}{\varepsilon} & : |x| \in (\varepsilon, 2\varepsilon), \\ 1 & : |x| \leq \varepsilon. \end{cases}$$

For $\varepsilon > 0$ let $n$ be large enough such that $\sup_{t\in[\eta,T]} |f_n(t) - f(t)| < \varepsilon$. Then we have

$$\begin{aligned}
&\left| \sum_{i=1}^{m_n} \mathbb{1}_{(-\infty,0)}(f_n(t_i^n))(t_i^n - t_{i-1}^n) - \sum_{i=1}^{m_n} \mathbb{1}_{(-\infty,0)}(f(t_i^n))(t_i^n - t_{i-1}^n) \right| \\
&\leq \sum_{i=1}^{m_n} |\mathbb{1}_{(-\infty,0)}(f_n(t_i^n)) - \mathbb{1}_{(-\infty,0)}(f(t_i^n))|(t_i^n - t_{i-1}^n) \\
&= \sum_{i=1}^{m_n} \left( \mathbb{1}_{(-\infty,0)\times(0,\infty)\cup(0,\infty)\times(-\infty,0)}(f_n(t_i^n), f(t_i^n)) \right) (t_i^n - t_{i-1}^n) \\
&= \sum_{i=1}^{m_n} \left( \mathbb{1}_{(-\varepsilon,0)\times(0,\varepsilon)\cup(0,\varepsilon)\times(-\varepsilon,0)}(f_n(t_i^n), f(t_i^n)) \right) (t_i^n - t_{i-1}^n) \\
&\leq \sum_{i=1}^{m_n} \left( \mathbb{1}_{(-\varepsilon,0)}(f(t_i^n)) + \mathbb{1}_{(0,\varepsilon)}(f(t_i^n)) \right) (t_i^n - t_{i-1}^n) \\
&\leq \sum_{i=1}^{m_n} \mathbb{1}_{(-\varepsilon,\varepsilon)}(f(t_i^n))(t_i^n - t_{i-1}^n) \leq \sum_{i=1}^{m_n} \phi_\varepsilon(f(t_i^n))(t_i^n - t_{i-1}^n) \to \int_\eta^T \phi_\varepsilon(f(s))\,\mathrm{d}s,
\end{aligned}$$

as $n \to \infty$, since $\phi_\varepsilon \circ f$ is continuous and thus Riemann-integrable. Now by letting $\varepsilon \to 0$ we get by the dominated convergence theorem that

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \sum_{i=1}^{m_n} \mathbb{1}_{(-\infty,0)}(f_n(t_i^n))(t_i^n - t_{i-1}^n) - \sum_{i=1}^{m_n} \mathbb{1}_{(-\infty,0)}(f(t_i^n))(t_i^n - t_{i-1}^n) \right| \\
&\leq \int_\eta^T \phi_\varepsilon(f(s))\,\mathrm{d}s \to \int_\eta^T \mathbb{1}_{\{0\}}(f(s))\,\mathrm{d}s = 0,
\end{aligned}$$

since $\int_\eta^T \mathbb{1}_{f^{-1}(\{0\})}(s)\,\mathrm{d}s = 0$. Thus the desired statement follows. □

The following can be found as Problem 127 in [PS72, p.81], thus we present it with a proof.

**Lemma B.2.10.** *Let $a < b$. Let $f_n : [a,b] \to \mathbb{R}$ be a non-decreasing functions for every $n \in \mathbb{N}$ and $f : [a,b] \to \mathbb{R}$ a continuous function. If $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in [a,b]$, then $\sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.*

*Proof.* We have that $f$ is also non-decreasing. Furthermore, since $f$ is continuous, it follows that $f$ is uniformly continuous on $[a,b]$. Let $\varepsilon > 0$. We can find finitely many points $a = x_0 < x_1 < \ldots < x_m = b$ such that $|f(x_k) - f(x_{k-1})| < \varepsilon$ for all $k \in \{1, \ldots, m\}$. Further, let $N \in \mathbb{N}$ be large enough such that for $n \geq N$ we have $|f(x_k) - f_n(x_k)| < \varepsilon$ for all $k \in \{0, 1, \ldots, m\}$. For $x \in [a, b]$ we can find $k \in \{1, \ldots, m\}$ such that $x_{k-1} \leq x \leq x_k$. Thus for $n \geq N$ we have

$$f(x) - 2\varepsilon < f(x_{k-1}) - \varepsilon < f_n(x_{k-1}) \leq f_n(x) \leq f_n(x_k) < f(x_k) + \varepsilon < f(x) + 2\varepsilon.$$

It follows that for any $x \in [a, b]$ we have

$$|f_n(x) - f(x)| < 2\varepsilon.$$

This shows the statement. □

The following consequence of Lemma B.2.10 above will be used in the proof of Theorem 3.3.2.

**Lemma B.2.11.** *Let $(F_n)_{n\in\mathbb{N}}$ and $F$ be distribution functions on $\mathbb{R}$. Assume that $F$ is continuous. Assume that $F_n(x) \to F(x)$ as $n \to \infty$ for every $x \in \mathbb{R}$. Then*

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0$$

*as $n \to \infty$.*

*Proof.* Let $\varepsilon > 0$. Choose $M > 0$ large enough such that $F(-M) \leq \varepsilon$ and $F(M) \geq 1 - \varepsilon$. We can choose $N \in \mathbb{N}$ large enough such that on the one hand we have $|F_n(-M) - F(-M)| < \varepsilon$ and $|F_n(M) - F(M)| < \varepsilon$ for all $n \geq N$, and on the other hand, taking Lemma B.2.10 into account, that

$$\sup_{x\in[-M,M]} |F_n(x) - F(x)| < \varepsilon$$

for all $n \geq N$. Altogether, we have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \\
&\leq \sup_{x\in[-M,M]} |F_n(x) - F(x)| + \sup_{x<-M} |F_n(x) - F(x)| + \sup_{x>M} |F_n(x) - F(x)| \\
&\leq \varepsilon + \sup_{x<-M} (F_n(x) + F(x)) + \sup_{x>M} (1 - F_n(x) + 1 - F(x)) \\
&\leq \varepsilon + F_n(-M) + F(-M) + 1 - F_n(M) + 1 - F(M) \\
&\leq \varepsilon + 2\varepsilon + \varepsilon + 2\varepsilon + \varepsilon = 7\varepsilon
\end{aligned}$$

for all $n \geq N$. This shows the statement. □

# List of Symbols

| | | |
|---|---|---|
| $b$ | boundary function | 5, 18 |
| $g$ | survival distribution | 5, 18 |
| $\overset{\Gamma}{\to}$ | $\Gamma$ convergence of boundary functions | 21 |
| $\sim$ | distributed according to; equal in distribution | 5 |
| supp | support of a probability measure or function | 48 |
| $\ll$ | absolutely continuous with respect to | 8, 134 |
| $d_{\mathrm{W}}$ | Wasserstein distance for probability measures | 44 |
| $d_{\mathrm{TV}}$ | total variation distance for probability measures | 140 |
| $d_{\mathrm{P}}$ | Prokhorov distance for probability measures | 208 |
| $\overline{\mu}$ | modulus law of a probability measure $\mu$ | 31 |
| $f^*$ | generalized absolute value density | 41 |
| $\mathrm{ifpt}(g, \mu)$ | set of solution for the inverse first-passage time problem for $g$ and $\mu$ | 5, 18 |
| $\mathrm{ifptk}(g, \mu)$ | set of solution for the inverse first-passage time problem with soft-killing for $g$ and $\mu$ | 133 |
| $P_t$ | Gaussian kernel, Markov kernel | 30, 133 |
| $T_\alpha$ | quantile-truncation operator | 5, 30 |
| $T^c$ | level-truncation operator | 30 |
| $R_\alpha^t$ | reweighting operator | 137 |
| $(W_t)_{t\geq 0}$ | standard Brownian motion | 5 |
| $(X_t)_{t\geq 0}$ | Brownian motion with initial state $X_0$, Markov process | 5, 133 |
| $\mathcal{P}$ | set of Borel probability measures on $\mathbb{R}$ | 5 |
| $\mathcal{P}^1$ | set of Borel probability measures on $\mathbb{R}$ with finite first absolute moment | 44 |

| | | |
|---|---|---|
| $\mathrm{TP}_2$ | totally positive of order 2 | 41 |
| $\preceq_{\mathrm{st}}$ | usual stochastic order | 31 |
| $\preceq$ | two-sided usual stochastic order | 6, 31 |
| $\preceq_{\mathrm{lr}}$ | likelihood ratio order | 38 |
| $\preceq_{\lvert\mathrm{lr}\rvert}$ | two-sided likelihood ratio order | 38 |
| $\tau_b$ | first-passage time of $b$ by reflected Brownian motion | 1, 17 |
| $\bar{\tau}_b$ | first-passage time of $b$ by reflected Brownian motion or its starting point | 25 |
| $\tau_b^{\mathrm{sk}}$ | soft-killing first-passage time of $b$ by Brownian motion | 3, 133 |
| $t^b$ | extinction time of a boundary function $b$ | 49 |
| $t^g$ | extinction time of a survival distribution $g$ | 5, 49 |

# Bibliography


[Abu02] Mario Abundo. “Some conditional crossing results of Brownian motion over a piecewise-linear boundary”. In: *Statist. Probab. Lett.* 58.2 (2002), pp. 131–145.

[Abu06] Mario Abundo. “Limit at zero of the first-passage time density and the inverse problem for one-dimensional diffusions”. In: *Stoch. Anal. Appl.* 24.6 (2006), pp. 1119–1145.

[Abu13a] Mario Abundo. “Some randomized first-passage problems for one-dimensional diffusion processes”. In: *Sci. Math. Jpn.* 76.1 (2013), pp. 33–46.

[Abu13b] Mario Abundo. “The double-barrier inverse first-passage problem for Wiener process with random starting point”. In: *Statist. Probab. Lett.* 83.1 (2013), pp. 168–176.

[Abu15] Mario Abundo. “An overview on inverse first-passage-time problems for one-dimensional diffusion processes”. In: *Recent advances in probability and statistics.* Vol. 12. Lect. Notes Semin. Interdiscip. Mat. Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2015, pp. 1–44.

[And60] T. W. Anderson. “A modification of the sequential probability ratio test to reduce the sample size”. In: *Ann. Math. Statist.* 31 (1960), pp. 165–197.

[Anu80] S.V. Anulova. “On Markov stopping times with a given distribution for a Wiener process.” In: *Theory Probab. Appl.* 25 (1980), pp. 362–366.

[Ata20] Rami Atar. *The heat equation with order-respecting absorption and particle systems with topological interaction.* 2020. arXiv: `2011.07535`. URL: `https://arxiv.org/abs/2011.07535`.

[AZ01] Marco Avellaneda and Jingyi Zhu. “Modeling the Distance-to-Default Process of a Firm”. In: *Risk* 14.12 (2001), pp. 125–129.

[BB10] Itai Benjamini and Nathanaël Berestycki. “Random paths with bounded local time”. In: *J. Eur. Math. Soc. (JEMS)* 12.4 (2010), pp. 819–854.

[BBD18] Julien Berestycki, Éric Brunet, and Bernard Derrida. "Exact solution and precise asymptotics of a Fisher-KPP type front". In: *J. Phys. A* 51.3 (2018), pp. 035204, 21.

[BBP19] Julien Berestycki, Éric Brunet, and Sarah Penington. "Global existence for a free boundary problem of Fisher-KPP type". In: *Nonlinearity* 32.10 (2019), pp. 3912–3939.

[Bec19] Erin Melissa Beckmann. *Asymptotic Behavior of Certain Branching Processes.* Thesis (Ph.D.)–Duke University. 2019.

[Bee93] Gerald A. Beer. *Topologies on Closed and Closed Convex Sets.* Mathematics and its applications. Kluwer Academic, 1993.

[Bei+18] Mathias Beiglböck, Manu Eder, Christiane Elgert, and Uwe Schmock. "Geometry of distribution-constrained optimal stopping problems". In: *Probab. Theory Related Fields* 172.1-2 (2018), pp. 71–101.

[Ber+20] Julien Berestycki, Éric Brunet, James Nolen, and Sarah Penington. *Brownian bees in the infinite swarm limit.* 2020. arXiv: `2006.06486`. URL: `https://arxiv.org/abs/2006.06486`.

[Ber+21] Julien Berestycki, Éric Brunet, James Nolen, and Sarah Penington. "A free boundary problem arising from branching Brownian motion with selection". In: *Trans. Amer. Math. Soc.* 374.9 (2021), pp. 6269–6329.

[Car+16] Gioia Carinci, Anna De Masi, Cristian Giardinà, and Errico Presutti. *Free boundary problems in PDEs and particle systems.* Vol. 12. SpringerBriefs in Mathematical Physics. Springer, Cham, 2016, pp. vii+110.

[CCS21] Xinfu Chen, John Chadam, and David Saunders. *Higher-Order Regularity of the Free Boundary in the Inverse First-Passage Problem.* 2021. arXiv: `2112.10918`. URL: `https://arxiv.org/abs/2112.10918`.

[Che+06] Lan Cheng, Xinfu Chen, John Chadam, and David Saunders. "Analysis of an inverse first passage problem from risk management". In: *SIAM J. Math. Anal.* 38.3 (2006), pp. 845–873.

[Che+11] Xinfu Chen, Lan Cheng, John Chadam, and David Saunders. "Existence and uniqueness of solutions to the inverse boundary crossing problem for diffusions". In: *Ann. Appl. Probab.* 21.5 (2011), pp. 1663–1693.

[Che05] Lan Cheng. *Analysis and numerical solution of an inverse first passage problem from risk management.* Thesis (Ph.D.)–University of Pittsburgh. ProQuest LLC, Ann Arbor, MI, 2005, p. 95.

[CMS13] Pierre Collet, Servet Martínez, and Jaime San Martín. *Quasi-Stationary Distributions : Markov Chains, Diffusions and Dynamical Systems.* Probability and Its Applications. Springer Berlin Heidelberg, 2013.

[CV14] Nicolas Champagnat and Denis Villemonais. "Exponential convergence to quasi-stationary distribution and Q-process". In: *Probab. Theory Related Fields* 164 (Apr. 2014).

[Dal93] Gianni Dal Maso. *An introduction to* $\Gamma$*-convergence.* Vol. 8. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, 1993, pp. xiv+340.

[Dan69] H. E. Daniels. "The minimum of a stationary Markov process superimposed on a $U$-shaped trend". In: *J. Appl. Probability* 6 (1969), pp. 399–408.

[Dan82] H. E. Daniels. "Sequential tests constructed from images". In: *Ann. Statist.* 10.2 (1982), pp. 394–400.

[De +19a] Anna De Masi, Pablo A. Ferrari, Errico Presutti, and Nahuel Soprano-Loto. "Hydrodynamics of the $N$-BBM process". In: *Stochastic dynamics out of equilibrium.* Vol. 282. Springer Proc. Math. Stat. Springer, Cham, 2019, pp. 523–549.

[De +19b] Anna De Masi, Pablo A. Ferrari, Errico Presutti, and Nahuel Soprano-Loto. "Non local branching Brownian motions with annihilation and free boundary problems". In: *Electron. J. Probab.* 24 (2019), Paper No. 63, 30.

[DG77] R. M. Dudley and Sam Gutmann. "Stopping times with given laws". In: *Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976).* 1977, 51–58. Lecture Notes in Math., Vol. 581.

[DiB02] Emmanuele DiBenedetto. *Real analysis.* Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Boston, Inc., Boston, MA, 2002, pp. xxiv+485.

[Don69] William F. Donoghue Jr. *Distributions and Fourier transforms.* Vol. 32. Pure and Applied Mathematics. Academic Press, New York, 1969, pp. viii+315.

[Dou+18] Randal Douc, Eric Moulines, Pierre Priouret, and Philippe Soulier. *Markov chains.* Springer Series in Operations Research and Financial Engineering. Springer, Cham, 2018, pp. xviii+757.

[Dud02] R. M. Dudley. *Real analysis and probability.* Vol. 74. Cambridge Studies in Advanced Mathematics. Revised reprint of the 1989 original. Cambridge University Press, Cambridge, 2002, pp. x+555.

[EEH14] Boris Ettinger, Steven N. Evans, and Alexandru Hening. "Killed Brownian motion with a prescribed lifetime distribution and models of default". In: *Ann. Appl. Probab.* 24.1 (2014), pp. 1–33.

[EHW20] Boris Ettinger, Alexandru Hening, and Tak Kwong Wong. "The inverse first passage time problem for killed Brownian motion". In: *Ann. Appl. Probab.* 30.3 (2020), pp. 1251–1275.

[EJ16] Erik Ekström and Svante Janson. "The inverse first-passage problem and optimal stopping". In: *Ann. Appl. Probab.* 26.5 (Oct. 2016), pp. 3154–3177.

[Fre83] David Freedman. *Brownian Motion and Diffusion.* New York, Springer, 1983.

[GP21] Sercan Gür and Klaus Pötzelberger. "On the empirical estimator of the boundary in inverse first-exit problems". In: *Comput. Statist.* 36.3 (2021), pp. 1809–1820.

[GS21] Pablo Groisman and Nahuel Soprano-Loto. "Rank dependent branching-selection particle systems". In: *Electron. J. Probab.* 26 (2021), Paper No. 158, 27.

[Hah21] Hans Hahn. *Theorie der reellen Funktionen.* Springer-Verlag Berlin Heidelberg, 1921.

[HW01] John C Hull and Alan D White. "Valuing credit default swaps II: Modeling default correlations". In: *The Journal of derivatives* 8.3 (2001), pp. 12–21.

[JKV09a] Sebastian Jaimungal, Alex Kreinin, and Angelo Valov. *Integral Equations and the First Passage Time of Brownian Motions.* 2009. arXiv: `0902.2569`. URL: `https://arxiv.org/abs/0902.2569`.

[JKV09b] Sebastian Jaimungal, Alex Kreinin, and Angelo Valov. *Randomized First Passage Times.* 2009. arXiv: `0911.4165`. URL: `https://arxiv.org/abs/0911.4165`.

[JKV14] S. Jaimungal, A. Kreinin, and A. Valov. "The generalized Shiryaev problem and Skorokhod embedding". In: *Theory Probab. Appl.* 58.3 (2014), pp. 493–502.

[JKZ09] Ken Jackson, Alexander Kreinin, and Wanhe Zhang. "Randomization in the first hitting time problem". In: *Statist. Probab. Lett.* 79.23 (2009), pp. 2422–2428.

[Kar68] Samuel Karlin. *Total positivity. Vol. I.* Stanford University Press, Stanford, Calif, 1968, pp. xii+576.

[KK22a] A. Klump and M. Kolb. "Uniqueness of the inverse first-passage time problem and the shape of the Shiryaev boundary". In: *Theory Probab. Appl.* (2022).

[KK22b] A. Klump and M. Kolb. *An elementary approach to the inverse first-passage time problem for soft-killed Brownian motion.* Submitted. 2022.

[Lee20] J. M. Lee. "Free boundary problems and biological systems with selection rules". In: *Arch. Math. (Basel)* 114.1 (2020), pp. 85–95.

[Ler86] Hans Rudolf Lerche. *Boundary crossing of Brownian motion.* Vol. 40. Lecture Notes in Statistics. Its relation to the law of the iterated logarithm and to sequential analysis. Springer-Verlag, Berlin, 1986, pp. vi+142.

[NFK99] Alex Novikov, Volf Frishling, and Nino Kordzakhia. "Approximations of boundary crossing probabilities for a Brownian motion". In: *J. Appl. Probab.* 36.4 (1999), pp. 1019–1030.

[Pec99a] Andreas Pechtl. "Distributions of occupation times of Brownian motion with drift". In: *J. Appl. Math. Decis. Sci.* 3.1 (1999), pp. 41–62.

[Pec99b] Andreas Pechtl. "Some applications of occupation times of Brownian motion with drift in mathematical finance". In: *J. Appl. Math. Decis. Sci.* 3.1 (1999), pp. 63–73.

[Pes02a] Goran Peskir. "Limit at zero of the Brownian first-passage density". In: *Probab. Theory Related Fields* 124.1 (2002), pp. 100–111.

[Pes02b] Goran Peskir. "On integral equations arising in the first-passage problem for Brownian motion". In: *J. Integral Equations Appl.* 14.4 (2002), pp. 397–423.

[Pot21] Yoann Potiron. *Existence in the inverse Shiryaev problem.* 2021. arXiv: `2106.11573`. URL: `https://arxiv.org/abs/2106.11573`.

[PS06] Goran Peskir and Albert Shiryaev. *Optimal stopping and free-boundary problems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2006, pp. xxii+500.

[PS72] G. Pólya and G. Szegő. *Problems and theorems in analysis. Vol. I: Series, integral calculus, theory of functions.* Die Grundlehren der mathematischen Wissenschaften, Band 193. Springer-Verlag, New York-Berlin, 1972, pp. xix+389.

[PS78] Sidney C. Port and Charles J. Stone. *Brownian motion and classical potential theory.* Probability and mathematical statistics. New York, Academic Press, 1978.

[Rob91a] G. O. Roberts. "A comparison theorem for conditioned Markov processes". In: *J. Appl. Probab.* 28.1 (1991), pp. 74–83.

[Rob91b] G. O. Roberts. "Asymptotic approximations for Brownian motion boundary hitting times". In: *Ann. Probab.* 19.4 (1991), pp. 1689–1731.

[Rud76] Walter Rudin. *Principles of mathematical analysis.* Third. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976, pp. x+342.

[Sal88] Paavo Salminen. "On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary". In: *Adv. in Appl. Probab.* 20.2 (1988), pp. 411–426.

[SS07] M. Shaked and J.G. Shanthikumar. *Stochastic Orders.* Springer Series in Statistics. Springer New York, 2007.

[SY11] Paavo Salminen and Marc Yor. "On hitting times of affine boundaries by reflecting Brownian motion and Bessel processes". In: *Period. Math. Hungar.* 62.1 (2011), pp. 75–101.

[SZ11] Jing-Sheng Song and Paul Zipkin. "An approximation for the inverse first passage time problem". In: *Adv. in Appl. Probab.* 43.1 (2011), pp. 264–275.

[Tho00] Hermann Thorisson. *Coupling, Stationarity, and Regeneration.* Probability and its Applications. New York, Springer, 2000.

[Val09] Angel Veskov Valov. *First passage times: Integral equations, randomization and analytical approximations.* Thesis (Ph.D.)–University of Toronto (Canada). ProQuest LLC, Ann Arbor, MI, 2009, p. 142.

[Vil03] Cédric Villani. *Topics in Optimal Transportation.* Graduate Studies in Mathematics. Providence, American Math. Soc., 2003.

[ZS09] Cristina Zucca and Laura Sacerdote. "On the inverse first-passage-time problem for a Wiener process". In: *Ann. Appl. Probab.* 19.4 (2009), pp. 1319–1346.