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# Spectral Correspondences for Locally Symmetric Spaces – The Case of Exceptional Parameters

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# Zusammenfassung

Ziel dieser Arbeit ist es, die Korrespondenz des Laplace-Spektrums eines kompakten lokal symmetrischen Rang-1-Raumes mit dem ersten Band der Ruelle-Pollicott-Resonanzen des geodätischen Flusses auf dessen Einheitssphärenbündel zu vervollständigen. Die Erforschung dieser Fragestellung wurde von Flaminio und Forni im Kontext hyperbolischer Flächen begonnen und von Dyatlov, Faure und Guillarmou für reell hyperbolische Räume sowie von Guillarmou, Hilgert und Weich für allgemeine Rang-1-Räume fortgeführt. Mit Ausnahme des Falles hyperbolischer Flächen wurde in sämtlichen Arbeiten eine abzählbare Menge von Ausnahmepunkten ausgeschlossen, da die zugehörigen Poisson-Transformationen an diesen Punkten weder injektiv noch surjektiv sind. Wir benutzen vektorwertige Poisson-Transformationen, um auch die Ausnahmepunkte zu behandeln. Insbesondere werden explizite quanten-klassische Korrespondenzen bewiesen und die zugehörigen Darstellungen identifiziert. Während die Ausnahmepunkte im Fall hyperbolischer Flächen auf Darstellungen der diskreten Reihe von $\mathrm{SL}(2,\mathbb{R})$ führen, erweisen sich die resultierenden Darstellungen im Allgemeinen als Darstellungen relativer diskreter Reihen assoziierter nicht-Riemann'scher symmetrischer Räume.

# Abstract

The aim of this thesis is to complete the program of relating the Laplace spectrum for rank one compact locally symmetric spaces with the first band Ruelle-Pollicott resonances of the geodesic flow on its sphere bundle. This program was started by Flaminio and Forni for hyperbolic surfaces, continued by Dyatlov, Faure and Guillarmou for real hyperbolic spaces and by Guillarmou, Hilgert and Weich for general rank one spaces. Except for the case of hyperbolic surfaces a countable set of exceptional spectral parameters was always left untreated since the corresponding Poisson transforms are neither injective nor surjective. We use vector-valued Poisson transforms in order to treat also the exceptional spectral parameters. In particular, explicit quantum-classical correspondences are proven and the associated representations are identified. Whereas for hyperbolic surfaces the exceptional spectral parameters lead to discrete series representations of $\mathrm{SL}(2,\mathbb{R})$, the resulting representations turn out to be relative discrete series representations for associated non-Riemannian symmetric spaces in the general case.

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# Contents

# Introduction

Dynamical systems with additional symmetry are surprisingly rigid. One manifestation of this observation is the close connection between geodesic flows on locally symmetric spaces and their quantizations, the Laplace-Beltrami wave kernels. This was first observed for tori in the form of the Poisson summation formula and its non-commutative analog, the Selberg trace formula, where the length spectrum of closed geodesics and the spectrum of the Laplacian enter. In specific cases correspondences on the level of eigenfunctions were established about twenty years ago [LZ01, FF03, DH05, Müh06, Poh12].

In [DFG15] Dyatlov, Faure and Guillarmou showed that the spectrum of the geodesic flow on compact hyperbolic manifolds essentially decomposes into bands, the first of which is in one to one correspondence with the Laplace spectrum. For these spectral values they also constructed linear isomorphisms between the corresponding eigenspaces. In this context *essentially* means that there is a countable set of explicitly known spectral values for which the methods do not apply.

In [GHW18] the very explicit information available for hyperbolic surfaces was used to establish spectral correspondences also for the exceptional spectral values. In these cases the quantum side turns out to be related to the discrete series representations of $\mathrm{SL}(2,\mathbb{R})$, whereas the regular spectral values were related to irreducible unitary spherical principal series representations.

The theory of quantum-classical spectral correspondences with spherical principal series representations on the quantum side was extended to all rank one compact locally symmetric spaces in [GHW21]. In this thesis we complete the program for these spaces by establishing quantum-classical spectral correspondences on the level of eigenvectors for all exceptional spectral values.

We describe the setting in a little more detail. Let $G$ be a non-compact simple Lie group of real rank one and $\Gamma$ be a co-compact discrete subgroup of $G$. For simplicity we assume that $G$ has finite center and $\Gamma$ is torsion free. We fix a maximal compact subgroup $K$ and observe that the locally symmetric space $\Gamma\backslash G/K$ is a compact Riemannian manifold. Therefore its (elliptic) Laplace-Beltrami operator has discrete spectrum on $L^2(\Gamma\backslash G/K)$ with smooth eigenfunctions lifting to $\Gamma$-invariant eigenfunctions on $G/K$. Note that on $G/K$ the Laplace-Beltrami operator comes from a Casimir element and generates the algebra of $G$-invariant differential operators. For generic spectral parameters $\mu$ the eigenfunctions generate an irreducible $G$-representation which is equivalent to a spherical principal series representation $H_\mu$. The corresponding intertwiner is the Poisson transform $P_\mu$. So, generically the Laplace-Beltrami eigenspaces ${}^\Gamma E_{-\mu}$ can be identified with the $\Gamma$-invariant distribution vectors ${}^\Gamma H_\mu^{-\infty}$ in the corresponding spherical principal series representation, where the normalization of the spectral parameters is taken from [GHW21].

The word *generic* in the previous paragraph can be given a precise meaning. Let $\mathfrak{g}_0$ be the Lie algebra of $G$ and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$ (we use the analogous convention for all subspaces of $\mathfrak{g}_0$). The eigenvalues of the Laplace-Beltrami operator on $G/K$ are parameterized by elements of $\mathfrak{a}^*$ via the Harish-Chandra isomorphism, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ fixed by the choice of $K$ and $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. The parameters are unique up to the action of the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. A spectral parameter $\mu$ is *generic* if and only if it is *not* a zero of the Harish-Chandra $\mathbf{e}$-function which in turn is equivalent to the bijectivity of the intertwining Poisson transform $P_\mu$. Thus the exceptional parameters alluded to in the title of the thesis are the zeros of the $\mathbf{e}$-function.

In the case of compact hyperbolic surfaces (see [GHW18]) the exceptional spectral parameters are related to discrete series representations, which can be realized as smooth (in fact, holomorphic or anti-holomorphic) sections of certain $G$-homogeneous vector bundles over $G/K$. In these spaces of sections one has the action of a suitable *Bochner-Laplace operator* (see [Olb94, Lemma 2.2]). While these representations are no longer completely determined by the action of the Bochner-Laplacian, they are still irreducible unitary representations of $G$ obtained by a suitable vector-valued Poisson transform. This part can be generalized and we view the $\Gamma$-invariant sections, which descend to the locally symmetric space, as part of the quantization of the cotangent bundle of this space.

The cotangent bundle $T^*(\Gamma\backslash G/K) = \Gamma\backslash G \times_K \mathfrak{p}_0^*$ of $\Gamma\backslash G/K$ is foliated into the cosphere bundles $\Gamma\backslash G/Z_K(\mathfrak{a}) \times \{r\}$ with $r \in \mathfrak{a}_0^* \equiv \mathbb{R}$ determining the radius and the zero section $\Gamma\backslash G/K$. Each leaf of the foliation is invariant under the geodesic flow. On the zero section it is trivial, whereas on the cosphere bundles it is given by the right action $\Gamma\backslash G/M \times A \to \Gamma\backslash G/M,\ (gM, a) \mapsto gaM$, where we use the standard abbreviation $M$ for the centralizer $Z_K(\mathfrak{a})$ and set $A = \exp(\mathfrak{a}_0)$. This decomposition reduces the spectral analysis of the geodesic flow to the $A$-action on $\Gamma\backslash G/M$. This action is Anosov as one sees from the Bruhat decomposition $T(\Gamma\backslash G/M) = G \times_M (\mathfrak{n}_0^+ + \mathfrak{a}_0 + \mathfrak{n}_0^-)$, where $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0^\pm$ are the two Iwasawa decompositions of $\mathfrak{g}_0$ associated with the two possible orderings of the set $\Sigma$ of restricted roots in $\mathfrak{a}_0^*$. The approach to Ruelle-Pollicott resonances for the geodesic flow used in [GHW21] makes use of the set $\mathcal{D}'_+(\Gamma\backslash G/M)$ consisting of the distributions $u \in \mathcal{D}'(\Gamma\backslash G/M)$ whose wavefront set $\mathrm{WF}(u)$ is contained in the annihilator $\Gamma\backslash G \times_M (\mathfrak{n}_0^+ + \mathfrak{a}_0)^\perp \subseteq T^*(\Gamma\backslash G/M)$. Then the set of *resonant states* for the spectral parameter $\mu \in \mathfrak{a}^*$ is defined as

$$\mathrm{Res}(\mu) := \{u \in \mathcal{D}'_+(\Gamma\backslash G/M) \mid \forall H \in \mathfrak{a}_0 :\ H \cdot u + \mu(H)u = 0\},$$

where $H$ acts as a left-invariant vector field on $G/M$ descending to $\Gamma\backslash G/M$. A spectral parameter $\mu \in \mathfrak{a}^*$ is called a *Ruelle-Pollicott resonance* if $\mathrm{Res}(\mu) \neq 0$. The Ruelle-Pollicott resonances form a discrete set and the corresponding spaces of resonant states are finite dimensional. A *first band resonant state* is a resonant state $u$ which satisfies $X \cdot u = 0$, where $X$ is any vector field on $\Gamma\backslash G/M$ which is a section of the subbundle $G \times_M \mathfrak{n}_0^- \subseteq T(\Gamma\backslash G/M)$. We denote the space of first band resonant states for the spectral parameter $\mu \in \mathfrak{a}^*$ by $\mathrm{Res}^0(\mu)$. In the case of generic spectral parameters the *quantum-classical spectral correspondence* says that the push-forward of the canonical projection pr : $\Gamma\backslash G/M \to \Gamma\backslash G/K$ is a linear isomorphism $\mathrm{pr}_* : \mathrm{Res}^0(\mu - \rho) \to {}^\Gamma E_\mu$, where $\rho \in \mathfrak{a}_0^*$ is the usual half-sum of positive restricted roots counted with multiplicity (see [GHW21, Theorem 4.5]).

The strategy for our extension of the quantum-classical correspondence to exceptional spectral parameters is as follows. As in the generic case (see [GHW21, § 3.2]) we start by lifting the first band Ruelle-Pollicott resonances to $\Gamma$-invariant distributions on the global symmetric space. The lifted spaces can be interpreted in terms of spherical principal series (that part works for all spectral parameters, see [GHW21, Proposition 3.8]) and the first band resonant states $\mathrm{Res}^0(-\mu - \rho)$ correspond to the space ${}^\Gamma H_\mu^{-\infty}$ of $\Gamma$-invariant distribution vectors of the corresponding principal series. For an exceptional spectral parameter $\mu$ the corresponding principal series $H_\mu$ is no longer irreducible. But it has a manageable composition series and it turns out that the $\Gamma$-invariant distribution vectors are all contained in the socle (i.e. the sum of all irreducible subrepresentations) of the representation, see Theorem 6.1.1. In each of the rank one cases except $\mathrm{SO}_0(2,1)$ (the case of surfaces, see [GHW18]) the socle turns out to be irreducible with a unique minimal $K$-type $\tau_\mu$ (see Theorem 6.2.1) and we can show that the vector-valued Poisson transform associated with this $K$-type (sum of $K$-types in the case of surfaces) is injective, see Proposition 5.1.3. The image consists of spaces of $\Gamma$-invariant sections of vector bundles over $\Gamma\backslash G/K$ and we have a quantum-classical correspondence as soon as we have characterized the image of this Poisson transform.

We achieve the characterization of the image of the minimal $K$-type Poisson transform via Fourier expansions of $M$-invariant functions with respect to $M$-spherical $K$-representations. More precisely, we determine necessary and sufficient conditions for a Fourier series to represent a distribution vector of the reducible spherical principal series $H_\mu$, see Theorem 7.4.11, where the conditions are given in terms of generalized gradients (see [BÓØ96]). In each of the cases it is possible to determine a $G$-invariant system of differential equations on the sections of the homogeneous bundle $G \times_K V_{\tau_\mu}$ given by the minimal $K$-type $(\tau_\mu, V_{\tau_\mu})$ of the socle such that on the space of $\Gamma$-invariant solutions we can write down an explicit boundary value on $K/M$ in terms of Fourier coefficients, see Theorems 8.2.3, 8.3.2 and 8.4.2. Then our Fourier characterization of $H_\mu^{-\infty}$ allows us to show that the boundary values are contained in ${}^\Gamma H_\mu^{-\infty}$. In the case of $\mathrm{SO}_0(n,1)$ and for most exceptional spectral parameters in the case of $\mathrm{SU}(n,1)$ we have an alternative (and simpler) characterization of the vector-valued Poisson transform, which is based on techniques developed in [Mea89] to study Cauchy-Szegö maps for $\mathrm{SU}(n,1)$, see Theorems 8.1.1 and 8.2.1.

We can explicitly determine the socle of all reducible spherical principal series representations in rank one (see Theorem 6.2.1), and we see that the surface case is quite untypical. Not only is it the only case where the socle is not irreducible, it is also one of the very few cases in which the representation generated by the resonant states belongs to the discrete series of $G$. This is only the case for $\mathrm{SO}_0(2,1)$ (surfaces), $\mathrm{SU}(2,1)$, $\mathrm{Sp}(2,1)$ and $F_{4(-20)}$, see Theorem 6.2.2. On the other hand it turns out that all of these representations are unitarizable, see Theorem 6.2.1. We can determine the Langlands parameters (see Theorem 6.2.2), and in some cases geometric realizations, e.g. as solution spaces of differential equations are well-known (see [Olb94, Gai88]). But for most cases we did not find such descriptions in the literature. From the detailed information on the $K$-types we can actually identify the representations as relative discrete series representations of non-Riemannian symmetric spaces $G/H$ associated with $G/K$ (Theorem 6.2.3). [TW89] provides a geometric interpretation of a generating vector of such a representation in terms of cohomology, but it gives no description of the representation space as such. So our geometric realization as solution spaces of differential equations describing the images of minimal $K$-type Poisson transforms might actually be new.

As mentioned above, our results complete the picture of first band quantum-classical correspondences for compact locally symmetric spaces of rank one. In higher rank an analogous quantum-classical correspondence for generic spectral parameters has been established in [HWW21]. Extending that result to exceptional spectral parameters will be substantially harder as the information available on composition series of spherical principal series is much less explicit in higher rank. Moreover, some of the multiplicity one results we use (Propositions 2.4.3, 6.1.2, 7.3.2) or prove (Proposition 7.3.1) here are not always available in higher rank. As far as non-compact locally symmetric spaces are concerned, one has to replace the (discrete) spectrum of the algebra of invariant differential operators by a suitable concept of quantum resonances. So far one only has quantum-classical correspondences for convex co-compact real hyperbolic spaces and, for dimensions larger than two, only generic spectral parameters [GHW18, Had20]. For locally symmetric spaces with cusps the results on record are either very special (e.g. [LZ01, Müh06]) or else give only very rough information (e.g. [DH05]). In view of [GW22, Poh12], however, a quantum-classical correspondence for surfaces seems to be within reach. Finally, we mention [KW21], where quantum-classical correspondences for lifts of geodesic flows on compact locally symmetric spaces of rank one are treated for generic spectral parameters. That exceptional spectral parameters occur also in such situations can be seen from [KW20], where the authors have to leave out the case of three dimensional hyperbolic spaces because the Gaillard Poisson transform they use is not bijective.

## Outline of the thesis

We conclude this introduction with a brief description of the way the thesis is organized.

In the first two chapters we introduce the notion of Ruelle resonances and explain how they are related to principal series representations. After giving the relevant definitions we discuss several realizations and properties of principal series representations and investigate their $K$-types.

In Chapter 3 we recall the scalar Poisson transforms for symmetric spaces and introduce vector-valued analogs of it. Moreover, we define the exceptional parameters and relate them to reducible principal series representations.

Then, in Chapter 4, we consider the instructive example of surfaces. From [GHW18] we first recall the quantum-classical correspondence for exceptional parameters. In order to extend its proof to other cases, we reformulate several – mostly geometrically defined – objects into representation theoretic terms and discuss how vector-valued

Poisson transforms enter the picture.

Chapter 5 is concerned with mapping properties of vector-valued Poisson transforms leading to the choice of minimal $K$-types. Moreover, we determine the socles and their minimal $K$-types in all cases.

Chapter 6 deals with $\Gamma$-invariant distribution vectors in principal series representations. We show that these have to be contained in the socle of the representation.

In Chapter 7 we study Fourier expansions of $M$-invariant functions with respect to $M$-spherical $K$-representations. Apart from convergence issues we deal with the technicalities needed to characterize the spherical principal series representations in terms of Fourier expansions.

In Chapter 8 we complete the determination of the spectral correspondences by describing the $\Gamma$-invariant vectors in the image of the minimal $K$-type Poisson transform.

Finally, the last chapter discusses the real hyperbolic case in more detail and provides explicit forms of all occurring objects in this case.

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$.

# 1. Ruelle resonances

In this chapter we describe the classical side of the quantum-classical correspondence, i.e. the first band Ruelle resonances and its associated Ruelle resonance states. After introducing some basic notation we give a short overview of the definition, structure and properties of these resonances.

## 1.1. Basic notation

Let $G$ be a noncompact, connected, real, semisimple Lie group with finite center and $\Gamma \leq G$ a co-compact, torsion free lattice. We denote the Iwasawa decomposition of G by $G = KAN$. The $K$-, $A$-, or $N$-component in the Iwasawa decomposition is denoted by $k_I$, $a_I$, or $n_I$, respectively. Let $M \coloneqq Z_K(A)$ denote the centralizer of $A$ in $K$. The corresponding Lie algebras will be denoted by $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{a}_0, \mathfrak{n}_0, \mathfrak{m}_0$ with complexifications $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \mathfrak{m}$. Moreover, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition and denote the corresponding Cartan involution by $\theta$ (on $\mathfrak{g}_0$ and on $G$). Associated with the $\mathfrak{a}_0$-action we define the restricted root spaces

$$\mathfrak{g}_0^\alpha \coloneqq \{X \in \mathfrak{g}_0 \colon [H, X] = \alpha(H)X\}, \qquad \alpha \in \Sigma,$$

corresponding to the restricted roots $\Sigma \subset \mathfrak{a}_0^*$. Furthermore, we have the Bruhat decomposition given by $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_0^\alpha$. The Iwasawa decomposition determines a positive system $\Sigma^+ \subset \Sigma$. The half-sum of positive roots is denoted by $\rho \coloneqq \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$ with the multiplicities $m_\alpha \coloneqq \dim_{\mathbb{R}} \mathfrak{g}_0^\alpha$. If $\log : A \to \mathfrak{a}_0$ denotes the logarithm on $A$ and $\mu \in \mathfrak{a}^*$ we define $a^\mu \coloneqq e^{\mu(\log a)}$. By $\hat{K}$ (resp. $\hat{G}$, $\hat{M}$) we denote the equivalence classes of irreducible unitary representations of $K$ (resp. $G$, $M$). The Weyl group of $(\mathfrak{g}_0, \mathfrak{a}_0)$ is denoted by $W$. Let $\kappa$ denote the Killing form of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. For $\Upsilon \in \{K, M\}$ and a finite-dimensional representation $(\tau, V)$ of $\Upsilon$ we define the associated vector bundle $G \times_\Upsilon V$ as the quotient $(G \times V)/\sim$, where

$$\forall g \in G,\, x \in \Upsilon,\, v \in V \colon (g, v) \sim (gx, \tau(x^{-1})v).$$

We always identify the space of smooth sections of this bundle with

$$C^\infty(G \times_\Upsilon V) \coloneqq \{f \in C^\infty(G, V) \mid \forall g \in G,\, x \in \Upsilon \colon f(gx) = \tau(x^{-1})f(g)\}.$$

On each occurring Lie group, we always use a fixed Haar measure and, if not stated otherwise, normalize it in the compact case. For each smooth manifold $X$ we denote the space of distributions by $\mathcal{D}'(X)$.

## 1.2. Ruelle resonances on rank one locally symmetric spaces

In this section we assume $G$ to be of real rank one, i.e. that $\dim_{\mathbb{R}}(A) = 1$.

Resonances assign intrinsically defined discrete spectra to operators that do not have a discrete $L^2$-spectrum. They can be defined in many different equivalent ways using a wide variety of tools. Let us consider the example of a smooth vector field $X$ on a compact Riemannian manifold $\mathcal{M}$. Then, resonances can be defined as poles of a meromorphic continuation – as a family of continuous operators $R(\lambda)\colon C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ – of the *resolvent* $R(\lambda) \coloneqq (-X - \lambda)^{-1}\colon L^2(\mathcal{M}) \to L^2(\mathcal{M})$ to $\mathbb{C}$. In order to achieve such a continuation, one has to impose some conditions on the vector field. If $X$ generates an *Anosov flow*, i.e. if there exists a continuous flow-invariant splitting $T\mathcal{M} = E_0 \oplus E_s \oplus E_u$ of the tangent bundle of $\mathcal{M}$ such that the flow acts exponentially contracting resp. expanding on $E_s$ resp. $E_u$ and $E_0 = \mathbb{R}X$, such a continuation can be established (see e.g. [GHW21] and the literature cited therein). From now on, we investigate the case of the geodesic flow as follows.

**Notation 1.2.1** (cf. [GHW21, Section 3.1])**.** Under our assumptions on $G$ the quotient $G/K$ is a hyperbolic space (of rank one) over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$, where the latter two denote the quaternions and the octonions. We write

$$\mathbf{H}^n \coloneqq \mathbb{K}\mathbf{H}^n \coloneqq G/K,$$

where $n$ is the real dimension of $G/K$. Since $G/M \cong G/K \times K/M \cong \mathbf{H}^n \times \mathbb{S}^{n-1}$ by the $NAK$-decomposition we can identify $G/M$ with the unit sphere bundle

$$S\mathbf{H}^n \cong G/M.$$

Let $\mathbf{M} \coloneqq \Gamma\backslash G/K = \Gamma\backslash\mathbf{H}^n$. Then $\mathbf{M}$ is a smooth compact Riemannian locally symmetric space of rank 1 with unit sphere bundle $\mathcal{M} \coloneqq S\mathbf{M} \cong \Gamma\backslash G/M$. Under this isomorphism the geodesic flow on $\mathcal{M}$ is given by the natural right action of $A$. This flow is Anosov and the corresponding Anosov splitting of the tangent bundle $T\mathcal{M}$ is given by

$$T\mathcal{M} = E_0 \oplus E_s \oplus E_u,$$

where each subbundle can be expressed as an associated vector bundle:

$$E_0 \coloneqq \Gamma\backslash G \times_M \mathfrak{a}_0, \quad E_s \coloneqq \Gamma\backslash G \times_M \mathfrak{n}_0, \quad E_u \coloneqq \Gamma\backslash G \times_M \theta\mathfrak{n}_0.$$

We call these subbundles the *neutral*, *stable*, and *unstable bundle*. Similarly, the unit sphere bundle $S\mathbf{H}^n = G/M$ admits an Anosov splitting into

$$\widetilde{E}_0 \coloneqq G \times_M \mathfrak{a}_0, \quad \widetilde{E}_s \coloneqq G \times_M \mathfrak{n}_0, \quad \widetilde{E}_u \coloneqq G \times_M \theta\mathfrak{n}_0.$$

Each splitting induces a splitting of the cotangent bundle

$$T^*\mathcal{M} = E_0^* \oplus E_s^* \oplus E_u^*$$

defined (pointwise) by $E_0^*(E_s \oplus E_u) = 0$, $E_u^*(E_0 \oplus E_u) = 0$ and $E_s^*(E_0 \oplus E_s) = 0$.

By our rank one assumption, there exists a unique simple positive restricted root of $(\mathfrak{g}_0, \mathfrak{a}_0)$ that we denote by $\alpha$. Let $H \in \mathfrak{a}_0$ be defined by $\alpha(H) = 1$. Finally, we introduce the rescaled natural positive-definite scalar product

$$\langle \cdot, \cdot \rangle := -\frac{\kappa(\cdot, \theta \cdot)}{\kappa(H, H)} \tag{1.1}$$

on $\mathfrak{g}_0$ (and thus on $\mathfrak{g}_0^*$), so that the map

$$\mathfrak{a}^* \cong \mathbb{C}, \quad \lambda \mapsto \lambda(H)$$

becomes isometric. The generator $X$ of the geodesic flow on $\mathcal{M}$ is given by $H \in \mathfrak{a}_0$.

The following microlocal description of Ruelle resonance states turns out to be the most convenient in our setting.

**Definition 1.2.2** (cf. [GHW21, Lemma 2.2])**.** Let

$$\mathrm{Res}_X(\lambda) := \{u \in \mathcal{D}'(\mathcal{M}) \mid (X + \lambda)u = 0,\ \mathrm{WF}(u) \subset E_u^*\},$$

where $\mathrm{WF}(u) \subset T^*\mathcal{M}$ denotes the wave-front set of the distribution $u$. If $\mathrm{Res}_X(\lambda) \neq 0$, we call $\lambda$ a *Ruelle resonance* (or simply *resonance*) and $\mathrm{Res}_X(\lambda)$ the *space of Ruelle resonant states for* $\lambda$. By duality, we define

$$\mathrm{Res}_{X^*}(\lambda) := \{u \in \mathcal{D}'(\mathcal{M}) \mid (X - \lambda)u = 0,\ \mathrm{WF}(u) \subset E_s^*\}.$$

If $\mathrm{Res}_{X^*}(\lambda) \neq 0$, we call $\lambda$ a *co-resonance* and $\mathrm{Res}_{X^*}(\lambda)$ the *space of co-resonant states for* $\lambda$.

### 1.2.1. The first band resonances

In [DFG15] resp. [KW21, Theorem 4.1] it is shown for compact real hyperbolic spaces resp. compact locally symmetric spaces of rank one that the Ruelle resonances form an exact band structure. More precisely, if $\lambda$ is a Ruelle resonance with $\mathrm{Im}(\lambda) \neq 0$, then the real part $\mathrm{Re}(\lambda)$ of $\lambda$ is contained in $-\rho(H) - \mathbb{N}_0$ (see also [GHW21, Remark 3.2]). The first of these bands allows a particularly beautiful description.

**Definition 1.2.3** (cf. [GHW21, Definition 3.1])**.** A Ruelle resonant state $u$ is said to belong to the *first band* if it is annihilated by each smooth section of the unstable bundle $E_u$. We write $\mathrm{Res}_X^0(\lambda)$ for the first band resonant states at the resonance $\lambda \in \mathbb{C}$.

Similarly, we define the *first band co-resonant states* as the space of all Ruelle resonant states which are annihilated by each smooth section $U_+$ of the stable bundle $E_s$, i.e.

$$\mathrm{Res}_{X^*}^0(\lambda) := \{u \mid u \in \mathrm{Res}_{X^*}(\lambda),\ \forall U_+ \in C^\infty(G \times_M \mathfrak{n}_0) \colon U_+ u = 0\}.$$

By the canonical projection

$$\pi_\Gamma : G/M \to \Gamma\backslash G/M, \quad gM \mapsto \Gamma gM,$$

we can lift the (co-)resonant states to the cover $S\mathbf{H}^n = G/M$ of $S\mathbf{M} = \Gamma\backslash G/M$. We define for $\mu \in \mathfrak{a}^*$, with $\mathfrak{n}_{0,+} := \mathfrak{n}_0,\ \mathfrak{n}_{0,-} := \theta\mathfrak{n}_0$,

$$\mathcal{R}_\pm(\mu) := \{u \in \mathcal{D}'(G/M) \mid (X \mp \mu(H))u = 0,\ \forall U_\pm \in C^\infty(G \times_M \mathfrak{n}_{0,\pm})\colon U_\pm u = 0\}.$$

By [GHW21, Remark 3.3], the pullback $(\pi_\Gamma)^*$ induces linear isomorphisms ($\mu \in \mathfrak{a}^*$)

$$(\pi_\Gamma)^* : \mathrm{Res}^0_X(\mu(H)) \to {}^\Gamma\mathcal{R}_-(\mu), \quad (\pi_\Gamma)^* : \mathrm{Res}^0_{X^*}(\mu(H)) \to {}^\Gamma\mathcal{R}_+(\mu).$$

This uses the fact that the wave-front condition may be removed in the definition of the first band resonant states (see [DFG15, p. 9]).

The resonant states of the first band can be related to the representation theory of $G$. This relation is described in the following chapter.

# 2. Reducible principal series

In the representation theory of real reductive groups, principal series provide representations of great importance. One manifestation of this fact is Casselman's subrepresentation theorem (see e.g. [Wal88, 3.8.3]) that states that, in a certain sense, every representation of the group occurs as a subrepresentation of such a representation.

In this chapter we recall the main facts about the principal series we use in this thesis. In the context of the quantum-classical correspondence, *spherical* principal series representations appear as intermediate objects between the classical and quantum side (see Proposition 2.2.3). Since the exceptional parameters will lead to *reducible* principal series representations, we need a precise knowledge of their structure. Therefore, apart from different realizations and general properties, we inter alia discuss composition series and describe the occurring $K$-types in great detail (see e.g. Proposition 2.4.4).

While the general properties are also available in higher rank, the more explicit results on the composition series and the $K$-types are generally not. Therefore, we first consider $G$ as described in Section 1.1 and later on restrict to the rank one case for more explicit results.

## 2.1. Realizations of principal series representations

The principal series representations can be realized in different ways ("pictures") all of which have their advantages. Let $(\sigma, V_\sigma) \in \hat{M}$ with inner product $\langle \cdot, \cdot \rangle_\sigma$ and $\mu \in \mathfrak{a}^*$. Denote by $L^2(K, V_\sigma)$ the space of $V_\sigma$-valued functions which are $L^2$ with respect to the normalized Haar measure $\mathrm{d}k$ on $K$.

In the *induced picture* the representation space $H_{\sigma,\mu}$ is given by all measurable functions $f : G \to V_\sigma$ such that

i) $f(gman) = a^{\mu-\rho}\sigma(m^{-1})f(g)$ for all $g \in G, m \in M, a \in A, n \in N$,

ii) $f\big|_K \in L^2(K, V_\sigma)$.

The representation is given by

$$(\pi_{\sigma,\mu}(g)f)(x) \coloneqq f(g^{-1}x), \qquad g, x \in G,\ f \in H_{\sigma,\mu}.$$

Endowed with the norm $\|f\|^2 \coloneqq \int_K \|f(k)\|_\sigma^2 \,\mathrm{d}k$ this realization is a Hilbert space representation. The parametrization is chosen such that $H_{\sigma,\mu}$ is unitary if $\mu \in i\mathfrak{a}_0^*$ is imaginary. Note that this definition is well-defined up to equivalence since each intertwiner between two equivalent irreducible unitary representations $(\sigma_1, V_{\sigma_1})$, $(\sigma_2, V_{\sigma_2})$ of $M$ extends to an intertwiner of the corresponding principal series representations.

The Iwasawa decomposition shows that a function in $H_{\sigma,\mu}$ is completely determined by its restriction to $K$. Thus, the surjective isometry

$$H_{\sigma,\mu} \cong H^{\mathrm{cpt}}_{\sigma,\mu}, \tag{2.1}$$

where $H^{\mathrm{cpt}}_{\sigma,\mu}$ denotes the Hilbert space of all functions $f$ in $L^2(K, V_\sigma)$ such that $f(km) = \sigma(m^{-1})f(k)$ for all $k \in K,\ m \in M$, endowed with the same norm as above, gives another realization of the principal series representation. This realization is called the *compact picture*. Note that the representation space does not depend on $\mu$. However, in this picture the $G$-action is more complicated compared to the induced picture. It is induced by the action $\pi_{\sigma,\mu}$ via the isometry above and given by

$$(\pi^{\mathrm{cpt}}_{\sigma,\mu}(g)f)(k) := a_I(g^{-1}k)^{\mu-\rho} f(k_I(g^{-1}k)),$$

where $k \in K, g \in G$ and $f \in H^{\mathrm{cpt}}_{\sigma,\mu}$. In the following, we will simply write $H_{\sigma,\mu}$ for both realizations for the sake of simplicity. If $\sigma$ is the trivial representation we write $(\pi_\mu, H_\mu)$ and refer to these representations as the *spherical principal series.* The representation spaces in the spherical case naturally factor through the quotient $G/M$, respectively $K/M$, in the induced or compact picture and we will use these realizations in that case.

The same definitions can be made with respect to the opposite order on $\mathfrak{a}_0$, i.e. by choosing $-\Sigma^+$ as the positive system. More precisely, denoting $N_- := \theta N$, we define the *opposite principal series representation* on the space $H^{\mathrm{opp}}_{\sigma,\mu}$ given by all measurable functions $f : G \to V_\sigma$ such that

i) $f(gman) = a^{-(\mu-\rho)}\sigma(m^{-1})f(g)$ for all $g \in G, m \in M, a \in A, n \in N_-$,

ii) $f\big|_K \in L^2(K, V_\sigma)$

equipped with the left regular representation $\pi^{\mathrm{opp}}_{\sigma,\mu}$ where $\rho$ is still defined with respect to $\Sigma^+$. By restricting to $K$ we again obtain a compact picture on $H^{\mathrm{cpt,opp}}_{\sigma,\mu} := H^{\mathrm{cpt}}_{\sigma,\mu}$ given by

$$(\pi^{\mathrm{cpt,opp}}_{\sigma,\mu}(g)f)(k) := a_I^-(g^{-1}k)^{-(\mu-\rho)} f(k_I^-(g^{-1}k)),$$

where $G \ni g = k_I^-(g)a_I^-(g)n_I^-(g) \in KAN_-$ denotes the opposite Iwasawa decomposition. We also abbreviate $H(g) := \log a_I(g)$ and $H_-(g) := \log a_I^-(g)$. The Iwasawa decomposition is related to its opposite analog by the *longest Weyl group element* $w_-$, i.e. the unique element in $W$ which maps $\Sigma^+$ to $-\Sigma^+$. More precisely we have, for each $g \in G$,

$$k_I(gw_-) = k_I^-(g)w_- \qquad \text{and} \qquad a_I(gw_-) = w_-^{-1}a_I^-(g)w_- \tag{2.2}$$

for a fixed representative of $w_-$ in the normalizer $N_K(A)$ of $A$ in $K$ since

$$gw_- = k_I^-(g)w_-w_-^{-1}a_I^-(g)w_-w_-^{-1}n_I^-(g)w_- \in KAN.$$

Note that the Weyl group acts on $\hat{M}$ by $w\sigma(m) := \sigma(w^{-1}mw)$ where $w$ denotes a representative in $N_K(A)$. There is an interesting connection between the two introduced principal series.

**Proposition 2.1.1.** *The map*

$$\Phi : H^{\mathrm{cpt}}_{w_-^{-1}\sigma,-w_-^{-1}\mu+\rho} \to H^{\mathrm{cpt,opp}}_{\sigma,\mu+\rho}, \qquad \Phi(f)(k) \coloneqq f(kw_-)$$

*defines a bijective intertwiner between the associated representations.*

*Proof.* Since $w_- \in N_K(A)$ we have $w_-^{-1}mw_- \in M$ for each $m \in M$ and thus

$$\forall k \in K\colon \quad \Phi(f)(km) = w_-^{-1}\sigma(w_-^{-1}m^{-1}w_-)f(kw_-) = \sigma(m^{-1})\Phi(f)(k)$$

so that $\Phi(f) \in H^{\mathrm{cpt,opp}}_{\sigma,\mu}$. For the intertwining part we first note that for each $a \in A$

$$\exp(\mathrm{Ad}(w_-^{-1})\log a) = w_-^{-1}aw_- \quad \Longrightarrow \quad \mathrm{Ad}(w_-^{-1})\log a = \log(w_-^{-1}aw_-). \tag{2.3}$$

Now we calculate for $g \in G$ and $k \in K$ using (2.2) and (2.3)

$$\begin{aligned}
\Phi(\pi^{\mathrm{cpt}}_{w_-^{-1}\sigma,-w_-^{-1}\mu+\rho}(g)f)(k) &= a_I(g^{-1}kw_-)^{-w_-^{-1}\mu} f(k_I(g^{-1}kw_-)) \\
&= e^{-(w_-^{-1}\mu)(\log(w_-^{-1}a_I^-(g^{-1}k)w_-))} f(k_I^-(g^{-1}k)w_-) \\
&= e^{-(w_-^{-1}\mu)(\mathrm{Ad}(w_-^{-1})\log a_I^-(g^{-1}k))} f(k_I^-(g^{-1}k)w_-) \\
&= e^{-\mu(\log a_I^-(g^{-1}k))} f(k_I^-(g^{-1}k)w_-) \\
&= (\pi^{\mathrm{cpt,opp}}_{\sigma,\mu+\rho}(g)\Phi(f))(k).
\end{aligned}$$

□

**Remark 2.1.2.** Note that $-w_-^{-1}\mu + \rho = \mu + \rho$ so that $H^{\mathrm{cpt}}_{w_-^{-1}\sigma,\mu} \cong H^{\mathrm{cpt,opp}}_{\sigma,\mu}$ for each $\mu \in \mathfrak{a}^*$ and $w_-^{-1}\sigma = w_-\sigma$. However, in the stated form Proposition 2.1.1 works for every element $w$ of the Weyl group if one uses the Iwasawa decomposition associated to the positive system induced by $w$. Moreover, note that in the spherical case (i.e. $\sigma$ is trivial) we have $H_\mu \cong H_\mu^{\mathrm{opp}}$.

## 2.2. Principal series and the first band

In this section we briefly return to the rank one case and describe the relation between Ruelle resonant states and distributional vectors in the spherical principal series representations. Geometrically, this relation is established by associating to each point in the sphere bundle its two boundary values at infinity.

**Definition 2.2.1** (cf. [GHW21, Section 3.3])**.** The *initial* resp. *end point map* $B_-$ resp. $B_+$ which assigns to any point $y$ in the sphere bundle $G/M$ of $G/K$ the limiting point at $-\infty$ resp. $+\infty$ of the geodesic passing through $y$ is given by

$$B_\pm : G/M \to K/M, \; B(gM) \coloneqq k_I(gw_\pm)M$$

with the (non)trivial Weyl group element ($w_-$ resp.) $w_+ = eM \in W = N_K(A)/M$. We denote the pullback of $B_\pm$ by

$$\mathcal{Q}_\pm : \mathcal{D}'(K/M) \to \mathcal{D}'(G/M), \; \mathcal{Q}_\pm(T) \coloneqq B_\pm^* T$$

and introduce the map

$$\Phi_\pm : G/M \to \mathbb{R},\ gM \mapsto e^{-\alpha(H(g^{-1}B_\pm(gM)))}.$$

Finally we define the *initial* resp. *end point transform* $\mathcal{Q}_{\mu,\pm}$ for any $\mu \in \mathfrak{a}^*$ by

$$\mathcal{Q}_{\mu,\pm} : \mathcal{D}'(K/M) \to \mathcal{D}'(G/M),\ \mathcal{Q}_{\mu,\pm}(T) := \Phi_\pm^{\mu(H)}\mathcal{Q}_\pm(T)$$

and abbreviate $\mathcal{Q} := \mathcal{Q}_+$ and $\mathcal{Q}_\mu := \mathcal{Q}_{\mu,+}$.

**Proposition 2.2.2** (cf. [GHW21, Proposition 3.7]). *Let $\mathcal{Q}_{\mu,\pm}$ denote the initial and end point transforms from Definition 2.2.1. If we extend the $G$-representation $\pi_\mu^{\mathrm{cpt}}$ for $\mu \in \mathfrak{a}^*$ to $\mathcal{D}'(K/M)$ via the pullback obtained by the left $G$-action on $K/M \cong G/P$, the maps*

$$\mathcal{Q}_{\mu,\pm} : (\mathcal{D}'(K/M), \pi_{\mu+\rho}^{\mathrm{cpt}}) \to \mathcal{R}_\pm(\mu) \subseteq \mathcal{D}'(G/M)$$

*are equivariant topological isomorphisms.*

Composing $\mathcal{Q}_{\mu,\pm}$ with the pullback $(\pi_\Gamma)^*$ gives the following

**Proposition 2.2.3** (cf. [GHW21, Proposition 3.8]). *There are isomorphisms of finite dimensional vector spaces*

$$\mathrm{Res}_X^0(\mu(H)) \cong {}^\Gamma(\mathcal{D}'(K/M), \pi_{\mu+\rho}^{\mathrm{cpt}}) \text{ and } \mathrm{Res}_{X^*}^0(\mu(H)) \cong {}^\Gamma(\mathcal{D}'(K/M), \pi_{\mu+\rho}^{\mathrm{cpt}})$$

*where ${}^\Gamma()$ denotes the subspace of $\Gamma$-invariant elements.*

The following lemma connects the initial and end point transforms to the (opposite) principal series.

**Lemma 2.2.4.**

*i) The maps $\Phi_\pm$ are given by*

$$\Phi_+(gM) = e^{\alpha(H(g))} \text{ and } \Phi_-(gM) = e^{-\alpha(H_-(g))}.$$

*In particular, $\Phi_+ \in H_{\alpha+\rho}$ and $\Phi_- \in H_{\alpha+\rho}^{\mathrm{opp}}$.*

*ii) Let $f \in C^\infty(K/M) \cong C^\infty(K)^M$. Then we have for every $\mu \in \mathfrak{a}^*$*

*a) $\mathcal{Q}_+(f) \in H_\rho$, $\mathcal{Q}_-(f) \in H_\rho^{\mathrm{opp}}$,*

*b) $\mathcal{Q}_{\mu,+}(f) \in H_{\mu+\rho}$, $\mathcal{Q}_{\mu,-}(f) \in H_{\mu+\rho}^{\mathrm{opp}}$ and*

*c) $\mathcal{Q}_{\mu,+}(f)(kM) = f(k)$, $\mathcal{Q}_{\mu,-}(f)(kM) = f(kw_-)$.*

*Thus, $\mathcal{Q}_{\mu,+}(f)$ is the (unique) extension of $f$ to a function in $H_{\mu+\rho}$ and $\mathcal{Q}_{\mu,-}(f)$ is the (unique) extension of $f(\bullet\, w_-)$ to a function in $H_{\mu+\rho}^{\mathrm{opp}}$.*

*Proof.* By definition we have

$$\begin{aligned} H(g^{-1}B_+(gM)) &= H(g^{-1}k_I(g)) = H(n_I(g)^{-1}a_I(g)^{-1}) \\ &= H(a_I(g)^{-1}) = -H(a_I(g)) = -H(g) \end{aligned}$$

where the third equality follows since $A$ normalizes $N$. For $\Phi_-$ note that conjugation $C_{w_-}$ with $w_-$ acts as inversion on $A$ so that by (2.2)

$$H(gw_-) = \log a_I(gw_-) = \log a_I^-(g)^{-1} = -H_-(g)$$

independent of the representative of $w_- \in W$. This implies that

$$\begin{aligned} H(g^{-1}B_-(gM)) &= H(g^{-1}k_I(gw_-)) = H((gw_-)^{-1}k_I(gw_-)) \\ &= H(n_I(gw_-)^{-1}a_I(gw_-)^{-1}) = -H(gw_-) = H_-(g) \end{aligned}$$

and the claimed expressions for $\Phi_\pm$ follow. Now

$$gman = k_I(g)a_I(g)n_I(g)man = \underbrace{k_I(g)m}_{\in K}\underbrace{a_I(g)a}_{\in A}\underbrace{(ma)^{-1}n_I(g)man}_{\in N}$$

implies

$$\Phi_+(gmanM) = e^{\alpha(H(a_I(g)a))} = e^{\alpha(H(a))}e^{\alpha(H(g))} = a^\alpha\Phi_+(gM).$$

This proves $\Phi_+ \in H_{\alpha+\rho}$ and $\Phi_- \in H_{\alpha+\rho}^{\mathrm{opp}}$ is obtained analogously.

For ii) let $f \in C^\infty(K)^M$. Then

$$\mathcal{Q}_+(f)(gmanM) = f(B_+(gmanM)) = f(k_I(g)m) = f(B_+(gM)) = \mathcal{Q}_+(f)(gM)$$

shows $\mathcal{Q}_+(f) \in H_\rho$ and $\mathcal{Q}_-(f) \in H_\rho^{\mathrm{opp}}$ follows similarly. For ii)b note that

$$\begin{aligned} \mathcal{Q}_{\mu,+}(f)(gmanM) &= \Phi_+^{\mu(H)}(gmanM)\mathcal{Q}_+(f)(gmanM) \\ &= a^{\mu(H)\alpha}\Phi_+^{\mu(H)}(gM)\mathcal{Q}_+(f)(gM) \\ &= a^{\mu}\Phi_+^{\mu(H)}(gM)\mathcal{Q}_+(f)(gM) = a^\mu\mathcal{Q}_{\mu,+}(f)(gM). \end{aligned}$$

Finally we have

$$\begin{aligned} \mathcal{Q}_{\mu,+}(f)(kM) &= \Phi_+^{\mu(H)}(kM)\mathcal{Q}_+(f)(kM) = \mathcal{Q}_+(f)(kM) = f(k), \\ \mathcal{Q}_{\mu,-}(f)(kM) &= \Phi_-^{\mu(H)}(kM)\mathcal{Q}_-(f)(kM) = \mathcal{Q}_-(f)(kM) = f(kw_-). \end{aligned}$$

□

**Remark 2.2.5.** In view of Lemma 2.2.4, ii)c the isomorphism $\mathcal{Q}_\mu = \mathcal{Q}_{\mu,+}$ should be considered as the map (2.1) between different realizations of the spherical principal series representations, namely the compact picture and the induced picture extended to distributions. More precisely we have that $\mathcal{R}_+(\mu)$ is the space of *distributional elements of* $H_{\mu+\rho}$ (we will be more precise about this terminology in Section 2.3). We also have the corresponding statement for the opposite principal series but in that case, by Lemma 2.2.4, ii)c, the map $\mathcal{Q}_{\mu,-}$ is not consistent with the unique extension of a function $f \in C^\infty(K/M)$ to a function in $H_{\mu+\rho}^{\mathrm{opp}}$ but one has to twist $f$ with $w_-$ first.

## 2.3. Globalizations and infinitesimal character

Let $(\pi, \mathcal{H})$ denote a Hilbert space realization of a (subrepresentation of a) principal series representation. In this paragraph we define smooth and distribution vectors in $(\pi, \mathcal{H})$. We call a vector $v \in \mathcal{H}$ a *smooth* or $C^\infty$-vector for $\pi$ if

$$G \to \mathcal{H}, \qquad g \mapsto \pi(g)v$$

is smooth. Let $\mathcal{H}^\infty \subseteq \mathcal{H}$ denote the vector space of all smooth vectors in $\mathcal{H}$. For $\pi = \pi_{\sigma,\mu}^{\mathrm{cpt}}$ the smooth vectors are actually smooth functions (see e.g. [Vog08, Equation (5.15)(a)]):

$$\mathcal{H}^\infty = \{f : K \to V_\sigma \text{ smooth} \mid \forall k \in K,\ m \in M \colon f(km) = \sigma(m^{-1})f(k)\}.$$

The *distributional vectors* $\mathcal{H}^{-\infty}$ are given by the elements of the dual representation of the smooth vectors in the dual representation of $(\pi, \mathcal{H})$. We give an alternative description which often is more convenient. Let $\tilde{\sigma}$ denote the dual representation of $\sigma$. Then, using [Hel00, Chapter I, §5.3, Equation (25)], we see that

$$\langle \cdot, \cdot \rangle_{\sigma,\mu} \colon H_{\sigma,\mu} \times H_{\tilde{\sigma},-\mu} \to \mathbb{C}, \quad \langle f_1, f_2 \rangle_{\sigma,\mu} := \int_K f_2(k)(f_1(k))\,\mathrm{d}k$$

is a nondegenerate, bilinear, and $G$-invariant pairing between $H_{\sigma,\mu}$ and $H_{\tilde{\sigma},-\mu}$ for each $\sigma \in \hat{M}$ and $\mu \in \mathfrak{a}^*$. By this pairing, we see that the distributional vectors $H_{\sigma,\mu}^{-\infty}$ of the principal series representation $H_{\sigma,\mu}$ are given by the contragredient representation of $H_{\tilde{\sigma},-\mu}^\infty$. If $\sigma$ is trivial, the distributional vectors can be realized on $\mathcal{D}'(K/M)$, the space of distributions on $K/M$.

In rank one, as mentioned in Remark 2.2.5, the distributional vectors in the induced picture of $H_\mu$ are given by $\mathcal{R}(\mu - \rho) := \mathcal{R}_+(\mu - \rho)$ since

$$\mathcal{Q}_\mu \colon (\mathcal{D}'(K/M), \pi_\mu^{\mathrm{cpt}}) \xrightarrow{\sim} \mathcal{R}(\mu - \rho) \tag{2.4}$$

intertwines the $G$-actions and continuously extends (2.1) (see Proposition 2.2.2).

Note that we always have the linear embedding

$$\iota_{\sigma,\mu} \colon H_{\sigma,\mu} \hookrightarrow H_{\sigma,\mu}^{-\infty}, \quad \iota_{\sigma,\mu}(f_1)(f_2) := \langle f_1, f_2 \rangle_{\sigma,\mu}.$$

For each subrepresentation $V \leq H_{\sigma,\mu}$ we have the restricted pairing

$$V \times (H_{\tilde{\sigma},-\mu}/V^{\perp_{\sigma,\mu}}) \to \mathbb{C}, \quad \langle f_1, f_2 + V^{\perp_{\sigma,\mu}} \rangle_{\sigma,\mu} := \int_K f_2(k)(f_1(k))\,\mathrm{d}k,$$

where

$$V^{\perp_{\sigma,\mu}} := \{f_2 \in H_{\tilde{\sigma},-\mu} \mid \forall f_1 \in V \colon \langle f_1, f_2 \rangle_{\sigma,\mu} = 0\}. \tag{2.5}$$

This implies that $V^{-\infty}$ is the contragredient representation of $(H_{\tilde{\sigma},-\mu}/V^{\perp_{\sigma,\mu}})^\infty$.

Any principal series representation has an infinitesimal character. In order to describe the infinitesimal character of $H_{\sigma,\mu}$ we first fix some notation. Let $\mathfrak{t} \leq \mathfrak{m}$ denote a $\theta$-stable

Cartan subalgebra of $\mathfrak{m}$, $\lambda_\sigma$ be the highest weight of $\sigma$ with respect to some ordering in $\mathfrak{t}^*$ and $\rho_\mathfrak{m}$ denote the half-sum of positive roots for $(\mathfrak{m}, \mathfrak{t})$. Then $H_{\sigma,\mu}$ has infinitesimal character $\lambda_\sigma + \rho_\mathfrak{m} - \mu$ relative to $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}$ (cf. [Kna86, Proposition 8.22]). We recall the *Casimir element* $\Omega_\mathfrak{g}$, an important element of the center $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ of $\mathcal{U}(\mathfrak{g})$. Let $B$ be a fixed multiple of the Killing form $\kappa$. For a basis $X_1, \ldots, X_{\dim \mathfrak{g}_0}$ of $\mathfrak{g}_0$ let $(g^{ij})_{ij}$ denote the inverse matrix of $(B(X_i, X_j))_{i,j}$. Then the dual basis $(X^i)_i$ is given by $X^i = \sum g^{ij} X_j$ and the Casimir element is defined by

$$\Omega_\mathfrak{g} := \sum\nolimits_i X^i X_i = \sum\nolimits_{i,j} g^{ij} X_j X_i \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})).$$

Since $B$ is nondegenerate, there are unique elements $X_\varphi \in \mathfrak{g}_0$ for each $\varphi \in \mathfrak{g}_0^*$ such that $\varphi(X) = B(X, X_\varphi)$ for each $X \in \mathfrak{g}_0$. We put $\langle \varphi, \psi \rangle := B(X_\varphi, X_\psi)$ for $\varphi, \psi \in \mathfrak{g}_0^*$ resp. $\mathfrak{g}^*$. Let us extend the ordering on $\mathfrak{a}$ to $\mathfrak{h}$ such that $\Sigma^+$ arises by restriction from the positive roots of $(\mathfrak{g}, \mathfrak{h})$. By [Kna86, Lemma 12.28], the action of the Casimir element is then given by the scalar

$$\begin{aligned} \pi_{\sigma,\mu}(\Omega_\mathfrak{g}) &= \langle \lambda_\sigma + \rho_\mathfrak{m}, \lambda_\sigma + \rho_\mathfrak{m} \rangle + \langle \mu, \mu \rangle - \langle \rho + \rho_\mathfrak{m}, \rho + \rho_\mathfrak{m} \rangle \\ &= \langle \lambda_\sigma, \lambda_\sigma + 2\rho_\mathfrak{m} \rangle + \langle \mu, \mu \rangle - \langle \rho, \rho \rangle. \end{aligned} \tag{2.6}$$

## 2.4. Reducibility

We are particularly interested in reducible principal series representations, i.e. in the set

$$\mathcal{A}' := \{(\sigma, \mu) \in \hat{M} \times \mathfrak{a}^* \mid H_{\sigma,\mu} \text{ reducible}\}.$$

In this section we introduce the representation theoretic tools we need to describe the structure of these reducible representations.

### 2.4.1. Composition series, minimal $K$-types and socle

In general, principal series representations are not completely reducible. However, they are all of *finite length* (cf. [Kra78]). This means, there exists a finite *composition series*, i.e. a chain of subrepresentations of $H_{\sigma,\mu}$ of the form

$$0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = H_{\sigma,\mu}$$

such that the quotients $V_{i+1}/V_i$, the *composition factors*, are irreducible. By the Jordan-Hölder theorem, any two composition series have the same length and the same composition factors up to permutation and isomorphism. For rank one groups, a detailed description of the composition series of spherical principal series representations can be found in Appendix B.

Let $\pi$ denote an admissible Hilbert representation of $G$ (i.e. a continuous representation such that each $K$-isotypic component has finite dimension) and fix a Cartan subalgebra $\mathfrak{b}_0$ of $\mathfrak{k}_0$. With respect to some ordering, we define $\rho_\mathfrak{k}$ as the half-sum of the positive roots of $(\mathfrak{k}, \mathfrak{b})$. We say that $Y \in \hat{K}$ with highest weight $\lambda$ is a *minimal $K$-type* of $\pi$ if $Y$ occurs in $\pi$ restricted to $K$ and

$$\langle \lambda + 2\rho_{\mathfrak{k}}, \lambda + 2\rho_{\mathfrak{k}} \rangle$$

is minimal with respect to this property. The set of minimal $K$-types is independent of the choice of the ordering and its cardinality is finite and at least one. For principal series representations $\pi_{\sigma,\mu}$ each minimal $K$-type of $\pi_{\sigma,\mu}$ occurs in $\pi_{\sigma,\mu}|_K$ with multiplicity one (cf. [Vog79, Theorem 1.1]).

For any Hilbert representation $(\pi, \mathcal{H})$ of $G$ we define $\operatorname{soc}\pi$, the *socle of* $\pi$, as the closure (in the sense of [Kna86, Theorem 8.9]) of the sum of all completely reducible $(\mathfrak{g}, K)$-submodules of the underlying $(\mathfrak{g}, K)$-module of $(\pi, \mathcal{H})$ (see [KV95, p. 538]).

### 2.4.2. $K$-representations

We begin this section with a brief discussion of the decomposition of $\pi_{\sigma,\mu}|_K$ in general and then give some more precise results of this decomposition in the rank one case. Moreover, we define so-called generalized gradients which will be of great importance later on.

For the decomposition as $K$-representation we consider the compact picture $H^{\mathrm{cpt}}_{\sigma,\mu}$. As $K$-representation this coincides with the induced representation $\operatorname{Ind}_M^K \sigma$ of $\sigma$ to $K$. By Frobenius reciprocity we thus obtain for each $Y \in \hat{K}$ that

$$\operatorname{Hom}_K(H^{\mathrm{cpt}}_{\sigma,\mu}, Y) = \operatorname{Hom}_K(\operatorname{Ind}_M^K \sigma, Y) \cong \operatorname{Hom}_M(V_\sigma, Y).$$

Denote the multiplicity of $V_\sigma$ in $Y$ (and similarly for other groups and spaces) by

$$\operatorname{mult}_M(V_\sigma, Y) := \dim_{\mathbb{C}} \operatorname{Hom}_M(V_\sigma, Y).$$

Then, writing

$$\hat{K}_\sigma := \{Y \in \hat{K} \colon \operatorname{mult}_M(V_\sigma, Y) \neq 0\},$$

we have that, denoting equivalence as $K$-representations by $\cong_K$ and the Hilbert space direct sum by $\widehat{\bigoplus}$,

$$H^{\mathrm{cpt}}_{\sigma,\mu} \cong_K \widehat{\bigoplus}_{Y \in \hat{K}_\sigma} \operatorname{mult}_M(V_\sigma, Y) Y.$$

In the spherical case we will abbreviate $\hat{K}_M := \hat{K}_{\mathrm{triv}_M}$. If not stated otherwise, we will always realize $Y \in \hat{K}_M$ as a subrepresentation of $H^{\mathrm{cpt}}_{\sigma,\mu} = L^2(K/M)$. Note that $L^2(K/M)$ carries the left regular representation $L$. We denote the derived representation of $L$ by $\ell$.

#### Intertwiner

In the following, we describe a procedure to obtain $G$-equivariant maps between sections of associated vector bundles. As we shall see in Chapter 4, these *generalized gradients* generalize the classical raising and lowering operators of $\mathrm{PSL}(2, \mathbb{R})$. In the literature similar operators, so-called Schmid operators, occur in realizations of discrete series representations (see e.g. [KW76]). To define the gradients, we need the following fact.

**Proposition 2.4.1** (cf. [Ørs00, Proposition 3.1])**.** *Let $K$ act on $\mathfrak{p}^*$ by the coadjoint representation. The following map is defined for every $(\tau, Y) \in \hat{K}$:*

$$\nabla : C^\infty(G \times_K Y) \to C^\infty(G \times_K (Y \otimes \mathfrak{p}^*)),$$
$$(\nabla f)(g) \in \operatorname{Hom}(\mathfrak{p}, Y) \cong Y \otimes \mathfrak{p}^*, \ (\nabla f)(g)(X) := \left.\frac{d}{dt}\right|_{t=0} f(g \exp tX).$$

*Moreover, it defines a $G$-equivariant covariant derivative with zero torsion.*

**Definition 2.4.2.** Let $(\tau_i, Y_{\tau_i}) \in \hat{K}, \ i \in \{1,2\}$, be such that $Y_{\tau_2} \leq Y_{\tau_1} \otimes \mathfrak{p}^*$. Then, for $T \in \operatorname{Hom}_K(Y_{\tau_1} \otimes \mathfrak{p}^*, Y_{\tau_2})$, we define the *generalized gradient*

$$T \circ \nabla : C^\infty(G \times_K Y_{\tau_1}) \to C^\infty(G \times_K Y_{\tau_2}).$$

If not stated otherwise, we choose $T = \operatorname{pr}_{\tau_2}$, the orthogonal projection onto $Y_{\tau_2}$.

#### $M$-spherical functions in rank one

Let us now assume that $G$ has real rank one. In this case some more precise results on the $K$-types of the spherical principal series can be achieved. Most importantly, $(K, M)$ is a Gelfand pair in this case (cf. [Hel94, Chapter II, §6, Corollary 6.8]). This implies:

**Proposition 2.4.3.** *Let $\mathbb{C}$ denote the trivial $M$-representation. Then*

$$\forall Y \in \hat{K}_M : \ \operatorname{mult}_K(Y, H_\mu) = \operatorname{mult}_M(\mathbb{C}, Y) = \dim_{\mathbb{C}} Y^M = 1, \tag{2.7}$$

*where $Y^M := \{v \in Y \mid \forall m \in M : m.v = v\} \subseteq Y$ denotes the subspace of $M$-invariant elements. In particular, we have the multiplicity free decomposition*

$$H_\mu \cong_K \widehat{\bigoplus}_{Y \in \hat{K}_M} Y.$$

*Proof.* The first equality follows from Frobenius reciprocity and the last equality follows from [Hel00, Chapter V, Theorem 3.5 (iv)]. □

Note that $\operatorname{Ind}_M^K(\operatorname{triv}_M) \cong L^2(K/M)$ is isomorphic to $L^2(K)^M$, the $M$-invariant elements of $L^2(K)$ with respect to the right regular representation. The following proposition describes the $M$-spherical elements $Y^M$ for each $Y \in \hat{K}_M$ and is well-known to specialists. Since it turns out to be difficult to find a precise reference in the literature, we give a proof for the convenience of the reader.

**Proposition 2.4.4** (cf. [Hel00, Introduction, Proposition 3.2])**.** *Let $0 \neq (\tau, Y) \leq L^2(K)^M$ be an irreducible representation. Then*

*i) there exists a unique $\phi_Y \in Y^M$ such that $\phi_Y(e) = 1$ and $Y^M = \mathbb{C}\phi_Y$,*

*ii) $\varphi(k)\langle \phi_Y, \phi_Y \rangle_{L^2(K)} = \langle \varphi, \tau(k)\phi_Y \rangle_{L^2(K)}$ for $k \in K, \ \varphi \in Y$,*

*iii) $\langle \phi_Y, \phi_Y \rangle_{L^2(K)} = \frac{1}{\dim Y}, \ \phi_Y(k^{-1}) = \overline{\phi_Y(k)}, \ |\phi_Y(k)| \leq 1$ for $k \in K$. Moreover, for each $k \in N_K(A)$ we have $|\phi_Y(k)| = 1$.*

*Proof.* i) By Equation (2.7) we have $\dim_{\mathbb{C}} Y^M = 1$. Let $0 \neq \psi \in Y$ and choose some $k \in K$ such that $\psi(k) \neq 0$. Replacing $\psi$ by $\tau(k^{-1})\psi$ we may assume that $\psi(e) \neq 0$. The function

$$\Psi : K \to \mathbb{C}, \quad k \mapsto \int_M \tau(m)\psi(k)\,\mathrm{d}m$$

is contained in $Y^M$ with $\Psi(e) = \psi(e) \neq 0$. This proves the first part.

ii) For each $m \in M$ we have by the $K$-invariance of the Haar measure

$$\begin{aligned}\langle \varphi, \phi_Y \rangle_{L^2(K)} &= \int_K \varphi(k)\overline{\phi_Y(k)}\,\mathrm{d}k = \int_K \varphi(k)\overline{\phi_Y(m^{-1}k)}\,\mathrm{d}k \\ &= \int_K \varphi(mk)\overline{\phi_Y(k)}\,\mathrm{d}k = \int_K \overline{\phi_Y(k)} \int_M \varphi(mk)\,\mathrm{d}m\,\mathrm{d}k.\end{aligned}$$

Note that the map

$$\theta : K \to \mathbb{C}, \quad k \mapsto \int_M \varphi(mk)\,\mathrm{d}m = \int_M \tau(m^{-1})\varphi(k)\,\mathrm{d}m$$

is contained in $V^M = \mathbb{C}\phi_Y$. We infer that $\theta = \theta(e)\phi_Y = \varphi(e)\phi_Y$ and thus

$$\langle \varphi, \phi_Y \rangle_{L^2(K)} = \varphi(e) \int_K \overline{\phi_Y(k)}\phi_Y(k)\,\mathrm{d}k = \varphi(e)\langle \phi_Y, \phi_Y \rangle_{L^2(K)}.$$

Replacing $\varphi$ by $\tau(k^{-1})\varphi$ we obtain ii).

iii) By the Schur orthogonality relations we have

$$\begin{aligned}\frac{1}{\dim Y}\langle \varphi, \varphi \rangle_{L^2(K)}\langle \phi_Y, \phi_Y \rangle_{L^2(K)} &= \int_K \langle \tau(k)\phi_Y, \varphi \rangle_{L^2(K)}\overline{\langle \tau(k)\phi_Y, \varphi \rangle_{L^2(K)}}\,\mathrm{d}k \\ &\overset{ii)}{=} \int_K \overline{\varphi(k)\langle \phi_Y, \phi_Y \rangle_{L^2(K)}}\varphi(k)\langle \phi_Y, \phi_Y \rangle_{L^2(K)}\,\mathrm{d}k \\ &= \langle \phi_Y, \phi_Y \rangle^2_{L^2(K)} \int_K \overline{\varphi(k)}\varphi(k)\,\mathrm{d}k \\ &= \langle \phi_Y, \phi_Y \rangle^2_{L^2(K)}\langle \varphi, \varphi \rangle_{L^2(K)}.\end{aligned}$$

This proves $\langle \phi_Y, \phi_Y \rangle_{L^2(K)} = \frac{1}{\dim Y}$. By ii) we deduce

$$\phi_Y(k) = \dim Y \langle \phi_Y, \tau(k)\phi_Y \rangle_{L^2(K)} = \dim Y \overline{\langle \phi_Y, \tau(k^{-1})\phi_Y \rangle_{L^2(K)}} = \overline{\phi_Y(k^{-1})}$$

and, using the Cauchy-Schwarz inequality,

$$|\phi_Y(k)| = \dim Y |\langle \phi_Y, \tau(k)\phi_Y \rangle_{L^2(K)}| \leq \dim Y \langle \phi_Y, \phi_Y \rangle_{L^2(K)} = 1.$$

If $k \in N_K(A)$ we have $k^{-1}mk \in M$ for each $m \in M$ ($W$ is a group) and thus $\tau(k)\phi_Y \in Y^M = \mathbb{C}\phi_Y$ since $\tau(mk)\phi_Y = \tau(kk^{-1}mk)\phi_Y = \tau(k)\phi_Y$. Therefore, $\tau(k)\phi_Y$ and $\phi_Y$ are linearly dependent and we have an equality in the Cauchy-Schwarz inequality. $\square$

# 3. Poisson transforms

In the generic case, the quantum-classical correspondence is established by *scalar* Poisson transforms. These are given by certain $G$-equivariant mappings defined on spherical principal series representations whose images are contained in common eigenspaces of invariant differential operators. In rank one, these common eigenspaces are given by eigenspaces of the Laplacian so that bijective Poisson transforms lead to quantum-classical correspondences. We call parameters for which the Poisson transform defines an isomorphism *regular*. In the case of *exceptional* – i.e. non-regular – parameters it turns out that the Poisson transforms are not even injective (Theorem 3.2.2), so that we need some alternative. For this we introduce *vector-valued* Poisson transforms based on [Olb94], which admit similar properties to the scalar ones and generalize the latter in a natural way (Section 3.3). Moreover, they can be characterized by a universal property (Lemma 3.3.3), which – along with its corollaries – will be one of our main tools for proving spectral correspondences.

## 3.1. Invariant differential operators and eigensections

Let $(\tau, Y) \in \hat{K}$. A differential operator $D$ on $C^\infty(G \times_K Y)$ is called *invariant* if it commutes with the left regular representation $L$ on $C^\infty(G \times_K Y)$. Let $\mathbb{D}(G,\tau)$ denote the algebra of all invariant differential operators on $C^\infty(G \times_K Y)$ and abbreviate $\mathbb{D}(G/K) := \mathbb{D}(G, \mathrm{triv})$. Then $\mathbb{D}(G,\tau)$ is isomorphic to $\mathcal{U}(\mathfrak{g})^K/(\mathcal{U}(\mathfrak{g})I_{\tilde{\tau}})^K$ via the right regular representation $r$, where $I_{\tilde{\tau}} := \ker \tilde{\tau} \subset \mathfrak{k}$ denotes the kernel of $\tilde{\tau}$, the dual representation of $\tau$ (see [Olb94, Satz 2.4]).

For the trivial bundle the *Harish-Chandra homomorphism* $\chi : \mathbb{D}(G/K) \to S(\mathfrak{a}_0)^W$ allows us to identify $\mathbb{D}(G/K)$ with the $W$-invariants $S(\mathfrak{a}_0)^W$ of the symmetric algebra $S(\mathfrak{a}_0)$ of $\mathfrak{a}_0$ (see [Hel00, Chapter II, Theorems 4.3, 5.18]). Moreover, every character of $\mathbb{D}(G/K)$ is of the form

$$\chi_\mu \colon \mathbb{D}(G/K) \to \mathbb{C}, \quad \chi_\mu(D) := \chi(D)(\mu)$$

for some $\mu \in \mathfrak{a}^*$ and $\chi_\nu = \chi_\mu$ if and only if $\nu \in W\mu$ (cf. [Hel00, Chapter III, Lemma 3.11]). Let us denote the space of joint eigenfunctions of $\mathbb{D}(G/K)$ by

$$\mathcal{E}_\mu := \{f \in C^\infty(G/K) \mid \forall D \in \mathbb{D}(G/K) \colon Df = \chi_\mu(D)f\},$$

and, with the Riemannian distance function $d_{G/K}$ on $G/K$, for each $r \geq 0$

$$\mathcal{E}_{\mu,r}(G/K) := \{f \in \mathcal{E}_\mu \mid \sup_{g\in G}|e^{-rd_{G/K}(eK,gK)}f(g)| < \infty\}. \tag{3.1}$$

We put $\mathcal{E}_{\mu,\infty}(G/K) := \bigcup_{r\geq 0} \mathcal{E}_{\mu,r}(G/K)$, equipped with the direct limit topology.

For arbitrary $(\tau, Y) \in \hat{K}$ we define a representation $\chi_{\sigma,\mu}$ of $\mathbb{D}(G,\tau)$ for each $\mu \in \mathfrak{a}^*$ and $(\sigma, V) \in \hat{M}$ with $\mathrm{mult}_M(V,Y) \neq 0$ by

$$\chi_{\sigma,\mu} : \mathbb{D}(G,\tau) \to \mathrm{End}(\mathrm{Hom}_K(H_{\sigma,\mu}, Y)), \quad \chi_{\sigma,\mu}(r(u))(T) := T \circ \pi_{\sigma,\mu}(\mathrm{opp}\, u),$$

where $u \in \mathcal{U}(\mathfrak{g})^K$ and $\mathrm{opp} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is defined by $\mathrm{opp}(X) := -X$ for $X \in \mathfrak{g}$ (see [Olb94, Definition 2.10]). If $\mathrm{mult}_M(V,Y) = 1$ these representations are one dimensional and we can define the space of joint eigensections

$$E_{\sigma,\mu} := \{f \in C^\infty(G \times_K Y) \mid \forall D \in \mathbb{D}(G,\tau)\colon Df = \chi_{\sigma,\mu}(D)f\},$$

where we identified $\mathrm{End}(\mathrm{Hom}_K(H_{\sigma,\mu}, Y))$ with $\mathbb{C}$. Each $E_{\sigma,\mu}$ has an infinitesimal character and it coincides with that of $H_{\sigma,\mu}$ (see [Olb94, Folgerung 2.15]).

## 3.2. Mapping properties of scalar Poisson transforms

The asymptotics of joint eigenfunctions in $\mathcal{E}_\mu$ can be described by a specific meromorphic function on $\mathfrak{a}^*$, the *Harish-Chandra c-function* $\mathbf{c}(\mu)$. We define its “denominator”, the meromorphic function $\mathbf{e}(\mu)^{-1}$, by ($\mu \in \mathfrak{a}^*$)

$$\mathbf{e}(\mu)^{-1} := \prod_{\alpha\in\Sigma^+} \Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_\alpha + 1 + \frac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_\alpha + m_{2\alpha} + \frac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right)\right),$$

see e.g. [Sch84, Equation (5.17)]. Then $\mathbf{e}$ is an entire function on $\mathfrak{a}^*$ without zeros on the closure of the positive Weyl chamber.

**Definition 3.2.1.** For $\mu \in \mathfrak{a}^*$ we define the *scalar Poisson transform* by

$$P_\mu : \mathcal{D}'(K/M) \to \mathcal{E}_{\mu,\infty}(G/K), \quad P_\mu(T)(gK) := T(kM \mapsto a_I(g^{-1}k)^{-(\mu+\rho)}).$$

The mapping properties of these maps turn out to be closely related to the $\mathbf{e}$-function.

**Theorem 3.2.2** (cf. [vdBS87, Theorems 10.6, 12.2])**.** *$P_\mu$ is a topological isomorphism if and only if $\mathbf{e}(\mu) \neq 0$. If $\mathbf{e}(\mu) = 0$ then $P_\mu$ is neither injective nor surjective.*

This leads to the following definition.

**Definition 3.2.3.** We call

$$\mathbf{Ex} := \{\mu \in \mathfrak{a}^* \mid \mathbf{e}(\mu) = 0\}$$

the set of *exceptional parameters*.

In the rank one case the algebra $\mathbb{D}(G/K)$ is generated by the Laplacian on $G/K$ ([Hel00, Chapter II, Theorem 5.18]). Note that the scalar by which it acts on $\mathcal{E}_{\mu,\infty}(G/K)$ resp. $E_{\mathrm{triv}_M,\mu}$ can be deduced from Equation (2.6). For regular parameters $\mu \in \mathfrak{a}^* \setminus \mathbf{Ex}$ this leads to the following correspondence between first band (co-)resonant states and eigenfunctions of the (positive) Laplacian $\Delta_{\mathbf{M}}$ on $L^2(\mathbf{M})$, which is obtained by composing the isomorphisms from Proposition 2.2.3 with the bijective Poisson transform.

**Theorem 3.2.4** (cf. [GHW21, Theorem 4.5])**.** *If $\mu \in \mathfrak{a}^* \setminus \mathbf{Ex}$ is a regular spectral parameter, then there are natural isomorphisms*

$$\mathrm{Res}^0_X((\mu-\rho)(H)) \cong \mathrm{Eig}_{\Delta_\mathbf{M}}(\mu), \quad \mathrm{Res}^0_{X^*}((\mu-\rho)(H)) \cong \mathrm{Eig}_{\Delta_\mathbf{M}}(\mu),$$

*induced by the scalar Poisson transform $P_\mu$, where*

$$\mathrm{Eig}_{\Delta_\mathbf{M}}(\mu) \coloneqq \{u \in L^2(\mathbf{M})\colon (\Delta_\mathbf{M} - \rho(H)^2 + \mu(H)^2)u = 0\}.$$

The aim of this thesis is to generalize this result to the case of exceptional parameters.

**Remark 3.2.5.** Note that our definition of exceptional parameters agrees with the parameters which were excluded in [DFG15] and [GHW21]. Indeed, let $G$ be of real rank one. Then the $\mathbf{e}$-function is zero if and only if one of the Gamma functions has a pole which is the case if and only if

$$\mu \in \left(-\frac{1}{2}m_\alpha - 1 - 2\mathbb{N}_0\right)\alpha \cup \left(-\frac{1}{2}m_\alpha - m_{2\alpha} - 2\mathbb{N}_0\right)\alpha,$$

with the simple positive restricted root $\alpha$. Moreover, by [Hel70, Chapter IV, Theorem 1.1],

$$H_\mu \text{ irreducible} \quad \Leftrightarrow \quad \mathbf{e}(\mu)\mathbf{e}(-\mu) \neq 0.$$

Therefore, irreducibility of $H_\mu$ is sufficient but not necessary for the bijectivity of $P_\mu$.

## 3.3. Vector-valued Poisson transforms

In this section we describe generalized Poisson transforms based on [Olb94], which will serve as a substitute for the scalar Poisson transform for the exceptional parameters.

**Definition 3.3.1** (cf. [Olb94, Definition 3.2/Satz 3.4])**.** Let $\tau \in \hat{K}$, $\sigma \in \hat{M}$ and $\mu \in \mathfrak{a}^*$. Then we define the *(vector-valued) Poisson transform* by

$$P^\tau_{\sigma,\mu} : \mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau) \otimes H^{-\infty}_{\sigma,\mu} \to C^\infty(G \times_K V_\tau), \quad P^\tau_{\sigma,\mu}(T \otimes f)(g) = T(\pi_{\sigma,\mu}(g^{-1})f). \tag{3.2}$$

If $F : \mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau) \cong \mathrm{Hom}_M(V_\sigma, V_\tau)$ denotes the Frobenius isomorphism we have

$$P^\tau_{\sigma,\mu}(T \otimes f)(g) = \int_K \tau(k)F(T)(f(gk))\,\mathrm{d}k \tag{3.3}$$

$$= \int_K a_I(g^{-1}k)^{-(\mu+\rho)}\tau(k_I(g^{-1}k))F(T)(f(k))\,\mathrm{d}k \tag{3.4}$$

for $T \in \mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau)$, $f \in H_{\sigma,\mu}$ and $g \in G$. The image of $P^\tau_{\sigma,\mu}$ is contained in $E_{\sigma,\mu}$ and $P^\tau_{\sigma,\mu}$ is $\mathbb{D}(G,\tau) \times G$-equivariant, where $\mathbb{D}(G,\tau)$ acts on $\mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau)$ by $\chi_{\sigma,\mu}$. We abbreviate $P^\tau_{\sigma,\mu}$ by $P^\tau_\mu$ if $\sigma$ is the trivial representation of $M$. We may also define the Poisson transform in the compact picture where the definition agrees with Equation (3.4). Especially we have

$$P^{\tau,\mathrm{cpt}}_\mu(T \otimes f) = P^\tau_\mu(T \otimes \mathcal{Q}_{\mu-\rho,+}(f)) \tag{3.5}$$

for each $f \in \mathcal{D}'(K/M)$. For convenience of notations we simply write $P^\tau_\mu$ for $P^{\tau,\mathrm{cpt}}_\mu$ if the definition space is clear from the context.

**Remark 3.3.2** (Scalar vs. vector-valued)**.** If $\tau = \mathrm{triv}_K$ is the trivial $K$-representation we have $\mathrm{Hom}_K(H_\mu, V_\tau) \cong \mathrm{Hom}_M(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$. Let $t \in \mathrm{Hom}_M(\mathbb{C}, \mathbb{C})$ be the identity and $T := F^{-1}(t)$. Then

$$P^\tau_\mu(T \otimes f)(g) = \int_K a_I(g^{-1}k)^{-(\mu+\rho)} f(k)\,\mathrm{d}k = P_\mu(f)(gK).$$

The following lemma illustrates the naturalness of Olbrich's Poisson transforms.

**Lemma 3.3.3** (cf. [Olb94, Remark after Lemma 3.3])**.** *Let $\Psi : H_{\sigma,\mu} \to C^\infty(G \times_K V_\tau)$ be a $G$-equivariant map. Then*

$$\Psi = P^\tau_{\sigma,\mu}(T \otimes \bullet)$$

*where $T \in \mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau)$ is defined by $T(f) := \Psi(f)(e)$.*

*Proof.* For every $k \in K$ we have

$$T(\pi_{\sigma,\mu}(k)f) = \Psi(\pi_{\sigma,\mu}(k)f)(e) = \Psi(f)(k^{-1}) = \tau(k)\Psi(f)(e) = \tau(k)T(f)$$

and thus $T \in \mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau)$. Moreover we have for every $g \in G$ and $f \in H_{\sigma,\mu}$

$$P^\tau_{\sigma,\mu}(T \otimes f)(g) = \Psi(\pi_{\sigma,\mu}(g^{-1})f)(e) = \Psi(f)(g).$$ □

This lemma admits the following important implications.

**Corollary 3.3.4.** *Let $\Psi : H_{\sigma,\mu} \to C^\infty(G \times_K V_\tau)$ be a $G$-equivariant map where $V_\tau$ does not contain the $M$-representation $V_\sigma$. Then $\Psi = 0$.*

*Proof.* By Lemma 3.3.3 there exists $T \in \mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau)$ such that $\Psi = P^\tau_{\sigma,\mu}(T \otimes \bullet)$. But $\mathrm{Hom}_K(H_{\sigma,\mu}, V_\tau) \cong \mathrm{Hom}_M(V_\sigma, V_\tau) = 0$ by Frobenius reciprocity. □

**Corollary 3.3.5.** *Let $(\tau_i, V_{\tau_i}) \in \hat{K},\ i \in \{1, 2\}$, be such that*

$$\mathrm{mult}_K(V_{\tau_i}, H_{\sigma,\mu}) = \dim_\mathbb{C} \mathrm{Hom}_K(H_{\sigma,\mu}, V_{\tau_i}) = 1$$

*and let $\Phi : C^\infty(G \times_K V_{\tau_1}) \to C^\infty(G \times_K V_{\tau_2})$ be a $G$-equivariant map. By choosing $0 \neq T_i \in \mathrm{Hom}_K(H_{\sigma,\mu}, V_{\tau_i})$ we consider the Poisson transforms $P^{\tau_i}_{\sigma,\mu}$ as maps from $H_{\sigma,\mu}$ to $C^\infty(G \times_K V_{\tau_i})$. Then there exists some $c \in \mathbb{C}$ such that*

$$\Phi \circ P^{\tau_1}_{\sigma,\mu} = c \cdot P^{\tau_2}_{\sigma,\mu}.$$

*Proof.* Since

$$\Phi \circ P^{\tau_1}_{\sigma,\mu}(T_1 \otimes \bullet) : H_{\sigma,\mu} \to C^\infty(G \times_K V_{\tau_2})$$

is a $G$-equivariant map there exists, by Lemma 3.3.3, some $T \in \mathrm{Hom}_K(H_{\sigma,\mu}, V_{\tau_2})$ with

$$\Phi \circ P^{\tau_1}_{\sigma,\mu}(T_1 \otimes \bullet) = P^{\tau_2}_{\sigma,\mu}(T \otimes \bullet).$$

Since $\dim_\mathbb{C} \mathrm{Hom}_K(H_{\sigma,\mu}, V_{\tau_i}) = 1$ there exists some $c \in \mathbb{C}$ with $T = c \cdot T_2$. □

We may also define the analog of the Poisson transform for the opposite principal series, i.e.

$$P_{\sigma,\mu}^{\tau,\mathrm{opp}} : \mathrm{Hom}_K(H_{\sigma,\mu}^{\mathrm{opp}}, V_\tau) \otimes H_{\sigma,\mu}^{\mathrm{opp},-\infty} \to C^\infty(G \times_K V_\tau),$$
$$P_{\sigma,\mu}^{\tau,\mathrm{opp}}(T \otimes f)(g) := T(\pi_{\sigma,\mu}^{\mathrm{opp}}(g^{-1})f).$$

As in Definition 3.3.1 we get

$$\begin{aligned} P_{\sigma,\mu}^{\tau,\mathrm{opp}}(T \otimes f)(g) &= \int_K \tau(k)F(T)(f(gk))\,\mathrm{d}k \\ &= \int_K a_I^-(g^{-1}k)^{\mu+\rho}\tau(k_I^-(g^{-1}k))F(T)(f(k))\,\mathrm{d}k \end{aligned}$$

for $T \in \mathrm{Hom}_K(H_{\sigma,\mu}^{\mathrm{cpt},\mathrm{opp}}, V_\tau)$, $f \in H_{\sigma,\mu}^{\mathrm{cpt},\mathrm{opp}}$ and $g \in G$. Here the Frobenius isomorphism $F$ is defined as before by realizing the principal series in the compact picture.

**Corollary 3.3.6.** *Let $(\tau, V_\tau) \in \hat{K}$ be such that*

$$\mathrm{mult}_K(V_\tau, H_\mu) = \dim_{\mathbb{C}} \mathrm{Hom}_K(H_\mu, V_\tau) = 1.$$

*Let $0 \neq T \in \mathrm{Hom}_K(H_\mu^{\mathrm{cpt}}, V_\tau)$ and consider the Poisson transforms $P_\mu^\tau$ resp. $P_\mu^{\tau,\mathrm{opp}}$ as maps from $H_\mu^{\mathrm{cpt}}$ to $C^\infty(G \times_K V_\tau)$. Then, with the isomorphism $\Phi$ from Proposition 2.1.1,*

$$P_\mu^{\tau,\mathrm{opp}} \circ \Phi = c \cdot P_\mu^\tau,$$

*where $c$ is given by $\frac{\langle \tau(w_-)\phi_\tau, \phi_\tau\rangle_\tau}{\langle \phi_\tau, \phi_\tau\rangle_\tau}$ for some arbitrary $0 \neq \phi_\tau \in V_\tau^M$.*

*Proof.* Note that

$$P_\mu^{\tau,\mathrm{opp}} \circ \Phi\colon H_\mu^{\mathrm{cpt}} \to C^\infty(G \times_K V_\tau)$$

is $G$-equivariant. Thus, by Lemma 3.3.3, $P_\mu^{\tau,\mathrm{opp}} \circ \Phi$ equals $c \cdot P_\mu^\tau$ for some $c \in \mathbb{C}$. In order to compute $c$ we evaluate both sides at the delta distribution $\delta_{eM}$ at $eM$. Using Equation 2.2 we obtain

$$\begin{aligned} P_\mu^{\tau,\mathrm{opp}}(\Phi(\delta_{eM})) = P_\mu^{\tau,\mathrm{opp}}(\delta_{w_-M}) &= a_I^-(g^{-1}w_-)^{\mu+\rho}\tau(k_I^-(g^{-1}w_-))F(T)(1) \\ &= a_I(g^{-1})^{-(\mu+\rho)}\tau(k_I(g^{-1})w_-)F(T)(1). \end{aligned}$$

Note that $\mathbb{C}F(T)(1) = V_\tau^M$ by the multiplicity one assumption. Moreover we have $\tau(w_-)F(T)(1) \in V_\tau^M$ since

$$\tau(m)\tau(w_-)F(T)(1) = \tau(w_-)\tau(w_-^{-1}mw_-)F(T)(1) = \tau(w_-)F(T)(1)$$

for each $m \in M$ since $w_-^{-1}mw_- \in M$. Thus,

$$\tau(w_-)F(T)(1) = \frac{\langle \tau(w_-)F(T)(1), F(T)(1)\rangle_\tau}{\langle F(T)(1), F(T)(1)\rangle_\tau} F(T)(1) = cF(T)(1)$$

and therefore

$$P_\mu^{\tau,\mathrm{opp}}(\Phi(\delta_{eM})) = a_I(g^{-1})^{-(\mu+\rho)}\tau(k_I(g^{-1}))F(T)(1)c = P_\mu^\tau(\delta_{eM})c. \qquad \square$$

Note that in rank one, where we realized each $Y \in \hat{K}_M$ in $C^\infty(K/M)$, we have $\phi_Y \in Y^M$ for the $\phi_Y$ from Proposition 2.4.4. In this case Proposition 2.4.4 ii) yields

$$c = \frac{\langle \tau(w_-)\phi_Y, \phi_Y \rangle_{L^2(K)}}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} = \phi_Y(w_-^{-1}) = \phi_Y(w_-),$$

since $w_-^2 \in M$ and $\phi_Y$ is $M$-invariant. Note that since $\phi_Y(k^{-1}) = \overline{\phi_Y(k)}$ for each $k \in K$, this implies that $c$ is real. Moreover, $|\phi_Y(w_-)| = 1$ (again by Proposition 2.4.4 ii)) and thus $c \in \{\pm 1\}$.

# 4. An example: The case of surfaces

This chapter is devoted to the quantum-classical correspondence for exceptional parameters in the case of hyperbolic surfaces – corresponding to $G = \mathrm{PSL}(2,\mathbb{R})$ – as described in [GHW18]. In order to generalize the proof of [GHW18] to other rank one cases, our goal is to replace it with a more representation theoretic one. This will be instructive for the general rank one case where various challenges arise which are not present in the case of surfaces.

We start by introducing some notation and stating the correspondence. After relating the occurring objects to the representation theory of $\mathrm{PSL}(2,\mathbb{R})$, we see that the correspondence is actually induced by vector-valued Poisson transforms (Lemma 4.1.7). Section 4.2 is concerned with the proof of the correspondence, which uses Fourier decompositions to describe the spherical principal series. In the last two sections we discuss how the proof may be extended to more general situations. To this end, we first address the question of which Poisson transforms might be chosen in general, or rather, what distinguishes the occurring ones from other choices. As the raising and lowering operators used in the proof are restricted to the case of surfaces, we need some replacement for them as well. In the last section we see that generalized gradients provide such a replacement.

## 4.1. The quantum-classical correspondence

For $g \in \mathrm{SL}(2,\mathbb{R})$ we denote the equivalence class of $g$ in $G = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ by $[g]$.

**Definition 4.1.1.** Let $\mathbf{M} = \Gamma\backslash\mathbf{H}^2$ for some co-compact discrete torsion free subgroup $\Gamma \leq G$ be a quotient of the upper half plane so that $\mathbf{M}$ is a smooth oriented compact hyperbolic surface. Let $\mathcal{K}_\Gamma := (T^*\mathbf{M})^{1,0}$ denote the canonical line bundle on $\mathbf{M}$ and $\mathcal{K}_\Gamma^{-1} := (T^*\mathbf{M})^{0,1}$ be its dual, where the complex structure on $\mathbf{M}$ is chosen such that the canonical projection $\pi_\Gamma \colon \mathbf{H}^2 \to \mathbf{M}$ is holomorphic. For each $n \in \mathbb{N}$ we denote the tensor powers $\mathcal{K}_\Gamma^{\otimes n} := \mathcal{K}_\Gamma \otimes \ldots \otimes \mathcal{K}_\Gamma$ resp. ${\mathcal{K}_\Gamma^{-1}}^{\otimes n} := \mathcal{K}_\Gamma^{-1} \otimes \ldots \otimes \mathcal{K}_\Gamma^{-1}$ by $\mathcal{K}_\Gamma^{n}$ resp. $\mathcal{K}_\Gamma^{-n}$. For $v \in \mathcal{K}_\Gamma^{\pm 1}$ we denote the $n$-fold tensor product $v \otimes \ldots \otimes v \in \mathcal{K}_\Gamma^{\pm n}$ by $v^{\otimes n}$. For each $m \in \mathbb{Z} \setminus \{0\}$ we consider the map

$$\pi_m^* : C_c^\infty(\mathbf{M}, \mathcal{K}_\Gamma^m) \to C_c^\infty(S\mathbf{M}), \quad \pi_m^* u(z,v) := u(z)(v^{\otimes m})$$

and its dual operator $\pi_{m*} \colon \mathcal{D}'(S\mathbf{M}) \to \mathcal{D}'(\mathbf{M}, \mathcal{K}_\Gamma^m)$ as in [GHW18]. Moreover, we consider the Dolbeault operators $\partial$ and $\overline{\partial}$.

We can now state the quantum-classical correspondence for the exceptional parameters in the case of surfaces.

**Theorem 4.1.2** (Quantum-classical correspondence for exceptional spectral parameters, see [GHW18, Theorem 3.3])**.** *Let $\Gamma \leq G$ be a co-compact torsion free discrete subgroup, $\mathcal{M} \coloneqq \Gamma\backslash\mathbf{H}^2$ be a smooth oriented compact hyperbolic surface and $S\mathbf{M}$ be its unit tangent bundle. For each $n \in \mathbb{N}$ there is an isomorphism*

$$\pi_{n*} \oplus \pi_{-n*} \colon \operatorname{Res}^0_X(-n) \xrightarrow{\sim} H_n(\mathbf{M}) \oplus H_{-n}(\mathbf{M}),$$

*where $H_n(\mathbf{M})$ and $H_{-n}(\mathbf{M})$ are given by*

$$H_n(\mathbf{M}) \coloneqq \{u \in C^\infty(\mathbf{M}, \mathcal{K}^n) \mid \overline{\partial} u = 0\},\ H_{-n}(\mathbf{M}) \coloneqq \{u \in C^\infty(\mathbf{M}, \mathcal{K}^{-n}) \mid \partial u = 0\}.$$

In order to give a version of Theorem 4.1.2 that fits better with representation theory, we first give some introductory definitions of the structure theory of $G$.

**Definition 4.1.3.** Let $K \coloneqq \mathrm{PSO}(2) \leq G$ be maximal compact and $A$ resp. $N$ denote the projections of the diagonal resp. the unipotent upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$ to $G$ so that $G = KAN$ is an Iwasawa decomposition of $G$. Note that $M$ is trivial in this case. We abbreviate

$$k_\varphi \coloneqq \left[\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\right] \in K,\ \varphi \in [0,\pi[.$$

For each $m \in \mathbb{Z}$ we obtain a representation of $K$ on $\mathbb{C}_m \cong \mathbb{C}$ given by $L_m(k_\varphi) \coloneqq e^{2m\varphi i}$. We have $\widehat{K} = \{L_m \mid m \in \mathbb{Z}\}$ and let $G \times_K \mathbb{C}_m,\ m \in \mathbb{Z}$, be the associated line bundles. As on $\mathbf{M}$, we denote the canonical line bundle on $\mathbf{H}^2$ by $\mathcal{K} \coloneqq (T^*\mathbf{H}^2)^{1,0}$, its dual by $\mathcal{K}^{-1} \coloneqq (T^*\mathbf{H}^2)^{1,0}$ and their respective tensor products for $n \in \mathbb{N}$ by $\mathcal{K}^n$ resp. $\mathcal{K}^{-n}$. For $g \in G$ we define the pullback of the differential form $\mathrm{d}z$ by

$$g^*(\mathrm{d}z)\big|_{z=z_0}(X) \coloneqq \mathrm{d}z\big|_{z=g.z_0}(g'(z_0)X)$$

for every $X \in T_{z_0}\mathbf{H}^2$ and every $z_0 \in \mathbf{H}^2$, where $G$ acts on $\mathbf{H}^2$ by Möbius transformations. Furthermore, we use the natural projections

$$\pi_1 : G \times_K \mathbb{C}_m \to G/K,\ [g,\lambda] \mapsto gK, \quad \pi_2 : \mathcal{K}^n \to \mathbf{H}^2,\ (T^*_{z_0}\mathbf{H}^2)^{\otimes n} \ni \otimes_{j=1}^n \omega_{z_0,j} \mapsto z_0$$

and trivialize $G \times_K \mathbb{C}_m$ resp. $\mathcal{K}^n$ via

$$\begin{aligned} \pi_1^{-1}(\{gK\}) &\cong \mathbb{C}, \quad [g,\lambda] = [n(g)a(g), \lambda'] \mapsto \lambda' \\ \pi_2^{-1}(\{gK.i\}) &\cong \mathbb{C}, \quad \lambda\left((n(g)a(g))^{-1*}(\mathrm{d}z)\Big|_{z=g.i}\right)^{\otimes n} \mapsto \lambda \end{aligned}$$

for $g \in G$ with Iwasawa decomposition $g = n(g)a(g)k(g)$.

There is the following close connection between the bundles $\mathcal{K}^n$ and $G \times_K \mathbb{C}_{-n}$ (and analogously between $\mathcal{K}^{-n}$ and $G \times_K \mathbb{C}_n$). Note that $G \times_K \mathbb{C}_0 \cong G/K$ is the trivial bundle.

**Lemma 4.1.4.** *For every $n \in \mathbb{N}$ there is an isomorphism of homogeneous line bundles*

$$G \times_K \mathbb{C}_{-n} \cong \mathcal{K}^n$$

*given by $(f_1, f_2)$ where*

$$f_1 : G \times_K \mathbb{C}_{-n} \to \mathcal{K}^n, \quad [g,\lambda] \mapsto \lambda(g.\mathrm{d}z|_{z=i})^{\otimes n} := \lambda\left(g^{-1*}(\mathrm{d}z)\Big|_{z=g.i}\right)^{\otimes n}$$
$$f_2 : G/K \to \mathbf{H}^2, \quad gK \mapsto g.i.$$

*Proof.* First note that $f_1$ is well-defined: For $g \in G,\ k_\varphi \in K$ and $\lambda \in \mathbb{C}$ it holds that

$$\begin{aligned}
L_{-n}(k_\varphi^{-1})\lambda\left((gk_\varphi)^{-1*}(\mathrm{d}z)\Big|_{z=g.i}\right)^{\otimes n} &= e^{2n\varphi i}\lambda\left(\mathrm{d}z\big|_{z=i}(k_\varphi^{-1}g^{-1})'(g.i)\right)^{\otimes n}\\
&= e^{2n\varphi i}\lambda\left(\mathrm{d}z\big|_{z=i}(k_\varphi^{-1})'(i)(g^{-1})'(g.i)\right)^{\otimes n}\\
&= e^{2n\varphi i}k'_{-\varphi}(i)^n\lambda\left(\mathrm{d}z\big|_{z=i}(g^{-1})'(g.i)\right)^{\otimes n}\\
&= \lambda\left(\mathrm{d}z\big|_{z=i}(g^{-1})'(g.i)\right)^{\otimes n}\\
&= \lambda\left(g^{-1*}(\mathrm{d}z)\Big|_{z=g.i}\right)^{\otimes n},
\end{aligned}$$

where the fourth equality follows from $h'(i) = \frac{1}{(ci+d)^2}$ for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Moreover, we have a commuting diagram

$$\begin{array}{ccc}
G \times_K \mathbb{C}_{-n} & \xrightarrow{f_1} & \mathcal{K}^n \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\
G/K & \xrightarrow[f_2]{} & \mathbf{H}^2
\end{array}$$

since $\pi_2(f_1([g,\lambda])) = g.i = f_2(gK) = f_2(\pi_1([g,\lambda]))$. Furthermore, for every $gK \in G/K$ the mapping

$$\pi_1^{-1}(\{gK\}) \to \pi_2^{-1}(\{f_2(gK)\}), \quad [g,\lambda] \mapsto f_1([g,\lambda]) = \lambda\left(g^{-1*}(\mathrm{d}z)\Big|_{z=g.i}\right)^{\otimes n}$$

is the identity in the trivializations from Definition 4.1.1 and thus clearly linear. Finally note that $f_1$ maps $g_0[g,\lambda] = [g_0 g,\lambda]$ to $\lambda(g_0 g.\mathrm{d}z|_{z=i})^{\otimes n} = g_0.f_1([g,\lambda])$ (here $G$ acts by pulling back by the inverse of $g_0$) and $f_2$ is $G$-equivariant since the Möbius transformations define a group action. $\square$

Having identified the line bundles, we are now in a position to describe the analogs of the maps $\pi_m^*$ and $\pi_{m*}$ for the bundles $G \times_K \mathbb{C}_{-m}$. We first lift the definitions to $\mathbf{H}^2$.

**Definition 4.1.5.** For every $m \in \mathbb{Z} \setminus \{0\}$ let

$$\tilde{\pi}_m^* : C_c^\infty(\mathbf{H}^2, \mathcal{K}^m) \to C_c^\infty(S\mathbf{H}^2), \quad \tilde{\pi}_m^* u(z,v) := u(z)(v^{\otimes m})$$

and $\tilde{\pi}_{m*} \colon \mathcal{D}'(S\mathbf{H}^2) \to \mathcal{D}'(\mathbf{H}^2, \mathcal{K}^m)$ be its dual operator.

We will now use Lemma 4.1.4 and the fact that $S\mathbf{H}^2 \cong G$ to understand $\tilde{\pi}_n^*$ as a mapping from $C_c^\infty(G/K, G \times_K \mathbb{C}_{-n})$ to $C_c^\infty(G)$ (here we again concentrate on the case of $n \in \mathbb{N}$ to avoid cluttered notation; the case of $-n$ can be treated in an analogous manner). At first we identify $S\mathbf{H}^2$ with $G$ via

$$G \to S\mathbf{H}^2, \quad g \mapsto (g.i, g'(i)\ i),$$

where $i = \partial z \in T_i\mathbf{H}^2$ is the element with $\mathrm{d}z\big|_{z=i}(\partial z) = 1$. With this identification $\tilde{\pi}_n^*$ becomes

$$\tilde{\pi}_n^* : C_c^\infty(\mathbf{H}^2, \mathcal{K}^n) \to C_c^\infty(G), \quad \pi_n^* u(g) = u(g.i)\left((g'(i)\ i)^{\otimes n}\right).$$

Now we use Lemma 4.1.4 to obtain

$$\begin{aligned} \tilde{\pi}_n^* : C_c^\infty(G/K, G \times_K \mathbb{C}_{-n}) &\to C_c^\infty(\mathbf{H}^2, \mathcal{K}^n) \to C_c^\infty(G), \\ f \mapsto f_1 \circ f \circ f_2^{-1} &\mapsto (g \mapsto (f_1 \circ f \circ f_2^{-1})(g.i)(g'(i)\ i)^{\otimes n}). \end{aligned}$$

Fixing $g \in G$ and writing $f(gK) = [g, \lambda]$ we note that

$$\begin{aligned} (\tilde{\pi}_n^* f)(g) &= (f_1 \circ f \circ f_2^{-1})(g.i)(g'(i)\ i)^{\otimes n} \\ &= (f_1 \circ f)(gK)(g'(i)\ i)^{\otimes n} \\ &= f_1([g, \lambda])(g'(i)\ i)^{\otimes n} \\ &= \lambda \left( g^{-1*}(\mathrm{d}z)\Big|_{z=g.i} \right)^{\otimes n} (g'(i)\ i)^{\otimes n} \\ &= \lambda \left( \mathrm{d}z\big|_{z=i} (g^{-1})'(g.i) g'(i)\ i \right)^n \\ &= \lambda \left( \mathrm{d}z\big|_{z=i} e'(i)\ i \right)^n \\ &= \lambda. \end{aligned}$$

For every $k_\varphi \in K$ we also obtain

$$(\tilde{\pi}_n^* f)(g k_\varphi) = L_{-n}(k_\varphi^{-1})\lambda = L_{-n}(k_\varphi^{-1})(\tilde{\pi}_n^* f)(g)$$

since $[g, L_{-n}(k_\varphi)\delta] = [g k_\varphi, \delta] = f(g k_\varphi K) = f(gK) = [g, \lambda]$ if $f(gK) = [g k_\varphi, \delta]$. This shows that $\tilde{\pi}_n^*$ is just the natural mapping between different realizations of sections of the associated vector bundle $G \times_K \mathbb{C}_{-n}$: Indeed, note that we can identify the sections $C_c^\infty(G/K, G \times_K \mathbb{C}_{-n})$ with

$$C_c^\infty(G \times_K \mathbb{C}_{-n}) := \{ f \in C_c^\infty(G) \mid f(gk) = L_{-n}(k)^{-1} f(g) \}$$

via the trivialization of $G \times_K \mathbb{C}_{-n}$ from Definition 4.1.1, i.e. $f \in C_c^\infty(G/K, G \times_K \mathbb{C}_{-n})$ gets mapped to $\Phi(f) \in C_c^\infty(G \times_K \mathbb{C}_{-n})$ with

$$\Phi(f)(na) := \lambda \text{ for } n \in N,\ a \in A \text{ where } f(naK) = [na, \lambda].$$

In this realization we have

$$\tilde{\pi}_n^* : C^\infty_{(c)}(G \times_K \mathbb{C}_{-n}) \to C^\infty_{(c)}(G), \quad f \mapsto f.$$

Let us now describe the pullback $\tilde{\pi}_{n*} : \mathcal{D}'(G) \to \mathcal{D}'(G \times_K \mathbb{C}_{-n}) = (C^\infty_c(G \times_K \mathbb{C}_n))'$ of $\tilde{\pi}^*_{-n}$. We embed $C^\infty(G \times_K \mathbb{C}_n)$ into $\mathcal{D}'(G \times_K \mathbb{C}_n)$ by

$$\iota_n \colon C^\infty(G \times_K \mathbb{C}_n) \hookrightarrow \mathcal{D}'(G \times_K \mathbb{C}_n), \quad \iota_n(f)(\phi) := \int_{G/K} f(gK)\phi(gK)\, \mathrm{d}gK,$$

where we choose the invariant measure on $G/K$ to be compatible with the fixed Haar measures on $K$ and $G$.

**Lemma 4.1.6.** *For $f \in C^\infty(G)$ we have $\tilde{\pi}_{-n*}(f) \in C^\infty(G \times_K \mathbb{C}_n) \subset \mathcal{D}'(G \times_K \mathbb{C}_n)$ and*

$$\tilde{\pi}_{-n*} f(g) = \int_K f(gk) L_{-n}(k)^{-1}\, \mathrm{d}k = \int_K f(gk) L_n(k)\, \mathrm{d}k$$

*for all $g \in G$.*

*Proof.* Let $f \in C^\infty(G)$ and $\phi \in C^\infty_c(G \times_K \mathbb{C}_{-n})$. Then

$$\begin{aligned}
\int_{G/K} \tilde{\pi}_{-n*} f \cdot \phi\, \mathrm{d}gK &= (\tilde{\pi}_{-n*} f)(\phi) = f(\tilde{\pi}_n^*(\phi)) = \int_G f \cdot \tilde{\pi}_n^*(\phi)\, \mathrm{d}g \\
&= \int_{G/K} \int_K f(gk)\phi(gk)\, \mathrm{d}k\, \mathrm{d}gK \\
&= \int_{G/K} \int_K f(gk) L_{-n}(k)^{-1}\, \mathrm{d}k\, \phi(g)\, \mathrm{d}gK.
\end{aligned}$$

Thus, $\tilde{\pi}_{-n*} f$ is represented by the smooth function $g \mapsto \int_K f(gk) L_{-n}(k)^{-1}\, \mathrm{d}k$. □

We are now able to see that the quantum-classical correspondence can be described by vector-valued Poisson transforms.

**Lemma 4.1.7.** *Let $\mu \in \mathfrak{a}^*_\mathbb{C}$ and recall the definitions from Corollary 3.3.6. Then*

$$P_\mu^{L_n} = F(T)(1)(\tilde{\pi}_{-n*} \circ \mathcal{Q}_{\mu-\rho,+}) \quad \text{and} \quad P_\mu^{L_n,\mathrm{opp}} \circ \Phi = F(T)(1)(\tilde{\pi}_{-n*} \circ \mathcal{Q}_{\mu-\rho,-})$$

*and in particular*

$$P_\mu^{L_n} = c^{-1}(P_\mu^{L_n,\mathrm{opp}} \circ \Phi) = c^{-1} F(T)(1)(\tilde{\pi}_{-n*} \circ \mathcal{Q}_{\mu-\rho,-})$$

*with $c = L_n(w_-) = (-1)^n$.*

*Proof.* By a density argument we restrict our attention to smooth functions. In this case the first two equations of the Lemma are immediate from the integral description (3.3) of the Poisson transform resp. its analog for the opposite Poisson transform (recall from Lemma 2.2.4, ii)c that $\mathcal{Q}_{\mu-\rho,-}(f)$ is the unique extension of $\Phi \circ f$ to a function in $H_\mu^{\mathrm{opp}}$). The last equation follows from Corollary 3.3.6 and, choosing $\phi_{L_n} = 1$, we infer $c = L_n(w_-) = (-1)^n$ from the definition of $L_n$. □

**Remark 4.1.8.** Lemma 4.1.7 generalizes the description of the scalar Poisson transform as given in [GHW18, Lemma 2.1] (which was later generalized to general rank one groups in [GHW21, Proposition 4.4]) to the case of vector-valued Poisson transforms respectively nontrivial $K$-types.

We conclude this section by comparing different normalizations of scalar Poisson transforms as given in [GHW18] and [GHW21].

**Remark 4.1.9** (Comparison of different Poisson transforms and exceptional points)**.**
(i) For the comparison it suffices to investigate the different versions of the Schwartz kernel of the Poisson transform. For this purpose we view $G$ as $\mathrm{PSU}(1,1)$ and $\mathbf{H}^2$ as the Poincaré disk. In [GHW18, Lemma 2.1], the Poisson transform $P_\lambda : \mathcal{D}'(\mathbb{S}^1) \to C^\infty(\mathbf{H}^2)$ for $\lambda \in \mathbb{C}$ is defined by

$$P_\lambda(\omega)(x) \coloneqq \langle \omega, P^{1+\lambda}(x,\bullet)\rangle_{\mathbb{S}^1} \coloneqq \int_{\mathbb{S}^1} \omega(\nu) P^{1+\lambda}(x,\nu)\,\mathrm{d}\mu_{\mathbb{S}^1}(\nu)$$

where $\mu_{\mathbb{S}^1}$ is the standard measure on $\mathbb{S}^1$ and the kernel $P^{1+\lambda}$ is given by

$$P^{1+\lambda}(x,\nu) \coloneqq \left(\frac{1-|x|^2}{|x-\nu|^2}\right)^{1+\lambda}, \ x \in \mathbf{H}^2,\ \nu \in \mathbb{S}^1. \tag{4.1}$$

In [GHW21, Definition 4.1], the kernel $p_\mu$ for $\mu \in \mathfrak{a}^*$ of the Poisson transform is defined as in Definition 3.3.1 (choosing $\tau$ as the trivial $K$-representation)

$$p_\mu(gK, kM) \coloneqq a_I(g^{-1}k)^{-(\mu+\rho)} \in C^\infty(G/K \times K/M).$$

In our case $H$ is given by $\mathrm{diag}(\frac{1}{2}, -\frac{1}{2}) \in \mathfrak{a}_0$. We identify $\mathfrak{a}^* \cong \mathbb{C},\ \mu = \mu(H)\alpha \mapsto \mu(H)$ where $\rho$ gets mapped to $\frac{1}{2}$. We have (see e.g. [JW77, p. 148])

$$a_I(g^{-1}k)^{-2\rho} = \left(\frac{1-|g.0|^2}{|1-\langle g.0, k.1\rangle|^2}\right)^{2\rho(H)} = \frac{1-|g.0|^2}{|1-g.0\cdot\overline{k.1}|^2} = \frac{1-|g.0|^2}{|g.0-k.1|^2}. \tag{4.2}$$

Moreover, note that

$$p_\mu(gK,kM) = e^{-(\mu+\rho)(H(g^{-1}k))} = e^{-(\mu(H)+\frac{1}{2})\alpha(H(g^{-1}k))} = (e^{-2\rho(H(g^{-1}k))})^{\mu(H)+\frac{1}{2}}.$$

Therefore, Equation (4.2) implies

$$p_\mu(gK,kM) = \left(\frac{1-|g.0|^2}{|k.1-g.0|^2}\right)^{\mu(H)+\frac{1}{2}}.$$

Comparing this with (4.1) shows that for each $g \in \mathrm{PSU}(1,1)$ and $k \in K$

$$p_\mu(gK,kM) = P^{1+(\mu(H)-\frac{1}{2})}(g.0, k.1).$$

(ii) By Remark 3.2.5, the exceptional parameters for $G = \mathrm{PSL}(2,\mathbb{R})$ are given by

$$\begin{aligned}\mathbf{Ex} &= \left(-\frac{m_\alpha}{2} - 1 - 2\mathbb{N}_0\right)\alpha \cup \left(-\frac{m_\alpha}{2} - m_{2\alpha} - 2\mathbb{N}_0\right)\alpha \\ &= \left(-\frac{1}{2} - 1 - 2\mathbb{N}_0\right)\alpha \cup \left(-\frac{1}{2} - 2\mathbb{N}_0\right)\alpha = \left(-\frac{1}{2} - \mathbb{N}_0\right)\alpha\end{aligned}$$

so that $\mu \in \mathbf{Ex} \Leftrightarrow \mu(H) \in -\frac{1}{2} - \mathbb{N}_0$ resp. $\lambda = \mu(H) - \frac{1}{2} \in -\mathbb{N}$.

## 4.2. Proof of the quantum-classical correspondence

In this section we outline the proof of the quantum-classical correspondence from [GHW18] (Theorem 4.1.2). We will already use our reformulations of the objects used in the correspondence and see how generalized gradients fit into the overall picture. We first note that for each fixed $f \in C^\infty(G)$ and each $g \in G$ the function

$$\varphi \mapsto f(gk_\varphi)$$

is $\pi$-periodic and the formula for $\tilde{\pi}_{n*}f(g)$ from Lemma 4.1.6 corresponds to the $n$-th Fourier coefficient of that function (see also Lemma 4.1.7); viewing $G$ as $S\mathbf{H}^2$, these Fourier coefficients are given by the Fourier expansion in the fibers of $S\mathbf{H}^2$. Evaluating at $\varphi = 0$, we can thus write

$$f = \sum_{k\in\mathbb{Z}} f_k \text{ with } f_k := \tilde{\pi}_k^*(\tilde{\pi}_{k*}(f)) \in \tilde{\pi}_k^*(C^\infty(G \times_K \mathbb{C}_{-k})), \tag{4.3}$$

where the series converges at least pointwise. By [GHW18, §2.4] the series and its derivatives even converge uniformly on compact sets and one has an analogous decomposition for distributions (converging in the distribution sense; note that the dual of $\tilde{\pi}_{-k*}\colon C_c^\infty(G) \to C_c^\infty(G \times_K \mathbb{C}_k)$ extends $\tilde{\pi}_k^*$ to $\mathcal{D}'(G \times_K \mathbb{C}_{-k})$). Defining the infinitesimal rotation matrix

$$V := \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \in \mathfrak{k} \tag{4.4}$$

we note that $Vf_k = ikf_k$ where $V$ acts from the right. We further define

$$\eta_\pm = \frac{1}{2}(H \pm iB) = \frac{1}{4}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \text{ where } B := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.5}$$

These operators are called *raising and lowering operators* as they increase resp. decrease the eigenvalues of $V$ by $\pm i$ since they fulfill the commutation relation $[V, \eta_\pm] = \pm i\eta_\pm$ and thus, in the universal enveloping algebra of $\mathfrak{g}$,

$$V\eta_\pm = \eta_\pm V + [V, \eta_\pm] = \eta_\pm V \pm i\eta_\pm.$$

In particular, they induce operators

$$\eta_\pm : C^\infty(G \times_K \mathbb{C}_{-k}) \to C^\infty(G \times_K \mathbb{C}_{-(k\pm1)}), \tag{4.6}$$

where $\eta_\pm$ acts by the derived right regular representation. The idea now is to characterize the resonant states resp. the principal series by relations between the Fourier coefficients $f_k$ using the raising and lowering operators. More precisely, we have the following

**Lemma 4.2.1** (see [GHW18, Lemma 2.2])**.** *Let $\mu \in \mathfrak{a}^*$ and $f \in \mathcal{D}'(G)$. Then $f$ is an element of $\mathcal{R}(\mu) = H^{-\infty}_{\mu+\rho}$ (see Equation (2.4)) if and only if*

$$2\eta_\pm f_\ell = (\mu(H) + 1 \pm \ell)f_{\ell\pm1}$$

*for every $\ell \in \mathbb{Z}$.*

*Proof.* For $f \in \mathcal{R}(\mu)$ we have $Hf = \mu(H)f$ and $U_+ f = 0$ with $U_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since $H = \eta_+ + \eta_-$ this yields

$$0 = (\eta_+ + \eta_- - \mu(H))f \quad \Rightarrow \quad \forall \ell \in \mathbb{Z}\colon\; \eta_+ f_{\ell-1} + \eta_- f_{\ell+1} - \mu(H) f_\ell = 0 \tag{4.7}$$

by applying $\tilde{\pi}_\ell^* \circ \tilde{\pi}_{\ell *}$ to both sides and using that $\tilde{\pi}_{\ell *} \circ \tilde{\pi}_k^* = 0$ for $\ell \neq k$ and $\tilde{\pi}_{\ell *} \circ \tilde{\pi}_\ell^*$ is the identity. Similarly $U_+ = -i(\eta_+ - \eta_-) + V$ and $V f_\ell = i\ell f_\ell$ imply

$$0 = (-i(\eta_+ - \eta_-) + V)f \quad \Rightarrow \quad \forall \ell \in \mathbb{Z}\colon\; 0 = -i(\eta_+ f_{\ell-1} - \eta_- f_{\ell+1}) + i\ell f_\ell. \tag{4.8}$$

Combining (4.7) and (4.8) finishes the proof. □

**Remark 4.2.2.** Note that [GHW18, Lemma 2.2] uses the opposite principal series. We get the original result of that lemma as follows. By Lemma 4.1.7, Lemma 4.2.1 shows that

$$2\eta_\pm P_{\mu+\rho}^{L_{-\ell}} = (\mu(H) + 1 \pm \ell) P_{\mu+\rho}^{L_{-(\ell\pm1)}}$$

for each $\ell \in \mathbb{Z}$ if we choose $T$ such that $F(T)(1) = 1$. Thus, by the same lemma,

$$(-1)^\ell 2\eta_\pm(\tilde{\pi}_{\ell *} \circ \mathcal{Q}_{\mu,-}) = (-1)^{\ell\pm1}(\mu(H) + 1 \pm \ell)(\tilde{\pi}_{\ell\pm1 *} \circ \mathcal{Q}_{\mu,-})$$

resp. $2\eta_\pm f_\ell = -(\mu(H) + 1 \pm \ell) f_{\ell\pm1}$ for each $f \in H_{\mu+\rho}^{-\infty,\mathrm{opp}} = \mathcal{R}_-(\mu)$. Therefore,

$$2\eta_\pm f_{\ell\mp1} = -(\mu(H) + 1 \pm (\ell \mp 1)) f_{\ell\mp1\pm1} = (-\mu(H) \mp \ell) f_\ell$$

as in [GHW18, Lemma 2.2]. Alternatively, one can also directly use the fact that the right action of $w_-$ on $K = \mathbb{S}^1$ is given by $-1$ and consider the $\mathbb{C}_m$'s as spaces of functions on $\mathbb{S}^1$.

The main idea of the proof of the quantum classical correspondence is as follows:

i) Fix a finite set of $K$-types such that the direct sum of the corresponding "Fourier component maps" is injective on $\mathrm{Res}_X^0(-n)$.

ii) Determine the image of the direct sum:

   a) Use the Fourier characterization from Lemma 4.2.1 to find necessary conditions for the image and to reconstruct all the other Fourier coefficients from the fixed ones.

   b) Define $f$ as the formal sum of the Fourier coefficients and show that it gives rise to an element of $\mathrm{Res}_X^0(-n)$.

Let us describe the proof in some more detail.

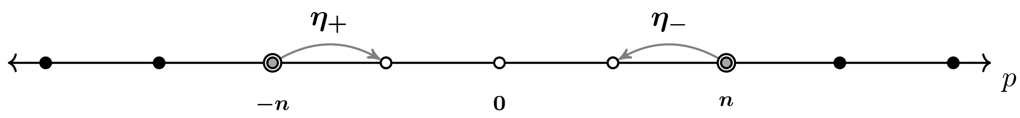



Figure 4.1.: Fourier components of $f \in {}^\Gamma H_\mu^{-\infty,\mathrm{opp}}$. White dots represent Fourier components that are zero.

*Idea of the proof of Theorem 4.1.2.* Let $\mu \in \mathfrak{a}^*$ with

$$\mu(H) = -n + \rho(H) = -n + \frac{1}{2} \in -\mathbb{N} + \frac{1}{2}$$

be an exceptional parameter and $f = \mathcal{Q}_{\mu-\rho,-}(\psi) \in {}^\Gamma\mathcal{R}_-(\mu - \rho) = {}^\Gamma H_\mu^{-\infty,\mathrm{opp}}$ for some $\psi \in \mathcal{D}'(K/M)$. Let all Poisson transforms be defined by normalizing $T$ by $F(T)(1) = 1$. Then Lemma 4.1.7 implies

$$\tilde{\pi}_{0*}(f) = \tilde{\pi}_{0*}(\mathcal{Q}_{\mu-\rho,-}(\psi)) = P_\mu^{L_0}(\psi) \in {}^\Gamma\mathcal{E}_{\mu,\infty}(G/K).$$

Thus, $\tilde{\pi}_{0*}(f)$ descends to an eigenfunction of $\Delta_{\mathbf{M}}$ with eigenvalue

$$\rho(H)^2 - \mu(H)^2 = -n(n-1)$$

by Equation (2.6). By the positivity of the Laplacian we get $\tilde{\pi}_{0*}(f) = 0$ and thus $f_0 = 0$ (for $n = 1$ one has to use a slightly modified argument, see [GHW18, p. 20]). In particular, the image of the scalar Poisson transform restricted to the $\Gamma$-invariant elements is zero in all exceptional cases. By Remark 4.2.2 we have

$$2\eta_+ f_{k-1} = (n-k)f_k \quad \text{and} \quad 2\eta_- f_{k+1} = (n+k)f_k \tag{4.9}$$

for each $k \in \mathbb{Z}$ and thus $f_k = 0$ for $|k| < n$. Therefore, $\eta_+ f_{-n} = 0$ and $\eta_- f_n = 0$. This implies that $\pi_{n*}(f) \in H_n(\mathbf{M})$ and $\pi_{-n*}(f) \in H_{-n}(\mathbf{M})$ if we consider $f$ as an element of $\mathrm{Res}_X^0(-n)$. Thus, the image of $\pi_{n*} \oplus \pi_{-n*}$ is really contained in the claimed space. For the injectivity let $f_n$ and $f_{-n}$ be zero. Then (4.9) implies that $f$ has to be zero.

For the surjectivity let $u \in H_n(\mathbf{M})$ and denote its $\Gamma$-invariant lift on $C^\infty(G \times_K \mathbb{C}_{-n})$ by $\tilde{u}$. Then we define $f_n \coloneqq \tilde{\pi}_n^*(\tilde{u})$, $f_k = 0$ for $k < n$ and (see (4.9))

$$f_k \coloneqq \frac{2}{n-k}\eta_+ f_{k-1}$$

for all $k > n$. By the same calculation as in the proof of [GHW18, Theorem 3.3] we see that

$$f \coloneqq \sum_{k \in \mathbb{Z}} f_k$$

defines a distribution. Moreover, it fulfills both relations from Equation (4.9) (also at $k = n$) and is an element of ${}^\Gamma H_{\mu+\rho}^{-\infty,\mathrm{opp}} = {}^\Gamma\mathcal{R}_-(\mu)$ by Remark 4.2.2. Thus, it descends to an element of $\mathrm{Res}_X^0(-n)$. The case $u \in H_{-n}(\mathbf{M})$ is analogous. □

## 4.3. Vector-valued Poisson transforms

In this section we determine the kernel of the vector-valued Poisson transform $P_\mu^{L_n}$. As these Poisson transforms are given by the "Fourier component maps" $\pi_{-n*}$ on the principal series, discussing the injectivity will shed some light on which $K$-types we could choose for the quantum-classical correspondence. In particular, we will see that in contrast to the generic parameters there does not exist an injective Poisson transform in the exceptional cases for $G = \mathrm{PSL}(2,\mathbb{R})$ (see Lemma 4.3.1 for $(\mu-\rho)(H) \in -\mathbb{N}$). This shows that we have to take the direct sum of *two* Poisson transforms as we did in Theorem 4.1.2. However, the injectivity of the used sum has nothing to do with the $\Gamma$-invariant elements (Lemma 4.3.3). Later on, in Proposition 5.1.3 and Theorem 5.1.6, we will see that the non-existence of injective Poisson transforms in the exceptional cases is rather special to the case $G = \mathrm{PSL}(2,\mathbb{R})$.

By the definition of $P_\mu^{L_n}$ (Definition 3.3.1) we can study its kernel by determining invariant subspaces of the principal series representation $H_\mu$. Let us first identify $\mathrm{SL}(2,\mathbb{R})$ with $\mathrm{SU}(1,1)$ using the Cayley transform

$$\Xi\colon \mathrm{SL}(2,\mathbb{R}) \ni g \mapsto CgC^{-1} \in \mathrm{SU}(1,1), \quad C \coloneqq \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Now realizing $(\pi_\mu^{\mathrm{cpt}}, H_\mu^{\mathrm{cpt}})$ on $L^2(\mathbb{S}^1) \cong L^2(K/M)$ by identifying

$$K/M \cong \mathbb{S}^1, \quad kM \mapsto \Xi(k).1 \tag{4.10}$$

via Möbius transformations on the closed Poincaré disk $\overline{\mathbf{H}^2} = \overline{B}_1(0)$ gives

$$(\pi_\mu^{\mathrm{cpt}}(g)f)(z) = P^{(\rho-\mu)(H)}(g.0,z)f(g^{-1}.z), \quad f \in L^2(\mathbb{S}^1),\, z \in \mathbb{S}^1$$

with the standard Poisson kernel $P$ (see Remark 4.1.9). The images of the raising and lowering operators from Equation (4.5) in $\mathfrak{su}(1,1)$ (complexified) are given by

$$\eta_+ \coloneqq \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \in \mathfrak{su}(1,1), \quad \eta_- \coloneqq \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \in \mathfrak{su}(1,1).$$

Their action on functions $e_p : \mathbb{S}^1 \to \mathbb{C},\ z \mapsto z^p$ $(p \in \mathbb{Z})$ is

$$\begin{aligned} 2\pi_\mu^{\mathrm{cpt}}(\eta_+)e_p &= ((\rho-\mu)(H) - p)e_{p-1} \\ 2\pi_\mu^{\mathrm{cpt}}(\eta_-)e_p &= ((\rho-\mu)(H) + p)e_{p+1} \\ \pi_\mu^{\mathrm{cpt}}(H)e_p = \pi_\mu^{\mathrm{cpt}}(\eta_+ + \eta_-)e_p &= \frac{(\rho-\mu)(H) - p}{2}e_{p-1} + \frac{(\rho-\mu)(H) + p}{2}e_{p+1}. \end{aligned} \tag{4.11}$$

By the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{p} = \cup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}$ we have that the $\mathfrak{a}$-action determines the $\mathfrak{g}$-action on each $K$-type. Since the $K$-finite vectors of $L^2(\mathbb{S}^1)$ are given by (the algebraic direct sum) $\bigoplus_{p\in\mathbb{Z}} e_p$ and $\mathfrak{a} = \mathbb{R}H$, the $\mathfrak{g}$-action is completely given by formula (4.11). Realizing the vector-valued Poisson transforms on $\mathcal{D}'(\mathbb{S}^1)$ we can now determine their kernels.

**Lemma 4.3.1** (Kernel of vector-valued Poisson transforms)**.** *Let $L_m \in \widehat{K}$ $(m \in \mathbb{Z})$ be arbitrary (see Definition 4.1.3). Identifying $K = K/M \cong \mathbb{S}^1$ via* (4.10)*, $L_m$ corresponds to the representation $e_m$ of $\mathbb{S}^1$. Then:*

*(i) The Poisson transform*

$$P_\mu^{e_m} : \mathcal{D}'(\mathbb{S}^1) \cong H_\mu^{-\infty} \to C^\infty(G \times_K \mathbb{C}_m)$$

*is injective if and only if $(\rho - \mu)(H) \notin -|m| + \mathbb{N} = \{-|m|+1, -|m|+2, \dots\}$.*

*(ii) If $(\rho - \mu)(H) \in -|m| + \mathbb{N}$ and $l \in \mathbb{Z}$ we have*

$$e_l \in \ker P_\mu^{e_{|m|}} \Leftrightarrow \begin{cases} l \leq (\mu - \rho)(H) & : -|m| + 1 \leq (\rho - \mu)(H) \leq |m| \\ |l| \geq (\rho - \mu)(H) & : |m| < (\rho - \mu)(H) \end{cases}$$

*and*

$$e_l \in \ker P_\mu^{e_{-|m|}} \Leftrightarrow \begin{cases} l \geq (\rho - \mu)(H) & : -|m| + 1 \leq (\rho - \mu)(H) \leq |m| \\ |l| \geq (\rho - \mu)(H) & : |m| < (\rho - \mu)(H). \end{cases}$$

*Proof.* Let us abbreviate equation (4.11) by

$$\pi_\mu(H)e_p = a_\mu(p)e_{p+1} + b_\mu(p)e_{p-1}.$$

If $(\rho - \mu)(H) \notin \mathbb{Z}$ we have $a_\mu(p) \neq 0 \neq b_\mu(p)$ for every $p \in \mathbb{Z}$. Thus, iteratively applying $\pi_\mu(H)$ to $e_l$ (for some arbitrary $l \in \mathbb{Z}$) will eventually have a non-zero $e_m$-part, i.e. $\mathrm{pr}_{e_m}(\pi_\mu(H)^q e_l) \neq 0$ for some $q \in \mathbb{N}$. This implies that $P_\mu^{e_m}$ is injective if $(\rho - \mu)(H) \notin \mathbb{Z}$.

Now let $(\rho - \mu)(H) = -|m| - k \in -|m| - \mathbb{N}_0$. We have to show that $P_\mu^{e_m}$ is injective. Note that $a_\mu(p) = 0 \Leftrightarrow p = |m| + k$ and $b_\mu(p) = 0 \Leftrightarrow p = -(|m| + k)$. This gives the $K$-type picture described in Figure 4.2a


(a) $(\rho - \mu)(H) = -|m| - k, \; k \in \mathbb{N}_0$

(b) $(\rho - \mu)(H) = -|m| + k, \; 1 \leq k \leq |m|$

Figure 4.2.: $K$-type pictures with kernel of $P_\mu^{e_{|m|}}$ (dotted) and arrows indicating the $\mathfrak{g}$-action

and shows that iteratively applying $\pi_\mu(H)$ to $e_l$ (for some arbitrary $l \in \mathbb{Z}$) will eventually have a non-zero $e_m$-part, i.e. $P_\mu^{e_m}$ is injective.

We are left to prove that $P_\mu^{e_m}$ is not injective if $(\rho - \mu)(H) = -|m| + k \in -|m| + \mathbb{N}$ and determine the kernel in this case. Note that

$$a_\mu(p) = 0 \Leftrightarrow p = |m| - k \text{ and } b_\mu(p) = 0 \Leftrightarrow p = -(|m| - k).$$


(a) $|m| < k \leq 2|m|$ (b) $k > 2|m|$

Figure 4.3.: $K$-type pictures for $(\rho - \mu)(H) = -|m| + k$ with kernel of $P_\mu^{e_{|m|}}$

Let us first consider the case $1 \leq k \leq |m|$:
Then the situation is described in Figure 4.2b. Thus, every $e_l$ with $l \leq |m| - k$ is in the kernel of $P_\mu^{e_{|m|}}$ and these are the only ones. Moreover, $e_l$ is in the kernel of $P_\mu^{e_{-|m|}}$ if and only if $l \geq -|m| + k$.
Let us finally consider the case $k > |m|$:

We distinguish between the two cases $k \leq 2|m|$ and $k > 2|m|$, see Figure 4.3. In the first case, we infer that $e_l \in \ker P_\mu^{e_{|m|}}$ if and only if $l \leq |m| - k$ and in the second case we have $e_l \in \ker P_\mu^{e_{|m|}}$ if and only if $|l| \geq -|m| + k$. Similarly, we have $e_l \in \ker P_\mu^{e_{-|m|}}$ if and only if $l \geq -|m| + k$ and in the second case $e_l \in \ker P_\mu^{e_{-|m|}}$ if and only if $|l| \geq -|m| + k$. □

**Example 4.3.2.** Applying Lemma 4.3.1 to $m = 0$ shows that the scalar Poisson transform $P_\mu$ is injective if and only if $(\rho - \mu)(H) \notin \mathbb{N} \Leftrightarrow \mu(H) \notin -\frac{1}{2} - \mathbb{N}_0$ (as mentioned in Remark 4.1.9(ii)). The kernel of $P_\mu$ agrees with the kernel given in [GHW18, Equation (2.19)].

Lemma 4.3.1 in particular shows that there do not exist injective vector-valued Poisson transforms in the exceptional cases. To obtain an injective map, we thus have to consider direct sums of them. The following lemma distinguishes the sum used in Theorem 4.1.2 from other ones and suggests that this choice is more related to the structure of the principal series than to the $\Gamma$-invariant elements.

**Lemma 4.3.3.** *For every $n \in \mathbb{N}$ and $\mu \in \mathfrak{a}^*$ with $\mu(H) = -n + \frac{1}{2}$ the mapping*

$$P_\mu^{L_n} \oplus P_\mu^{L_{-n}} \colon H_\mu^{\mathrm{cpt},-\infty} \cong \mathcal{D}'(\mathbb{S}^1) \to C^\infty(G \times_K \mathbb{C}_n) \oplus C^\infty(G \times_K \mathbb{C}_{-n})$$

*is injective. Moreover, $P_\mu^{L_m} \oplus P_\mu^{L_{-m}}$ is not injective for each $m \in \mathbb{N}_0$ with $m < n$.*

*Proof.* We first consider the case of $m < n$. Since $(\rho - \mu)(H) = n$, Lemma 4.3.1 implies

$$e_\ell \in \ker P_\mu^{L_m} \Leftrightarrow |\ell| \geq n \Leftrightarrow e_\ell \in \ker P_\mu^{L_{-m}}$$

so that in particular the direct sum is not injective. However, in the case of $P_\mu^{L_n} \oplus P_\mu^{L_{-n}}$, Lemma 4.3.1 implies that $e_l \in C^\infty(\mathbb{S}^1)$ is in the kernel of $P_\mu^{L_n}$ resp. $P_\mu^{L_{-n}}$ if and only if $l \leq -n$ resp. $l \geq n$. This implies the injectivity since the $e_l$ are the $K$-finite vectors in $H_\mu^{\mathrm{cpt},-\infty} \cong \mathcal{D}'(\mathbb{S}^1)$. □

## 4.4. The role of generalized gradients

This section is devoted to the question of which operators could play the role of the raising and lowering operators in the general case. More precisely, we discuss how generalized gradients come into play and how they can be used to give another proof of the Fourier relations from Lemma 4.2.1 which suggests a generalization beyond the case of $\mathrm{PSL}(2,\mathbb{R})$.

We define the Poisson transform $P_\mu^{L_\ell,\mathrm{cpt}}$ (see Definition 3.3.1) by $t \in \mathrm{Hom}_M(\mathbb{C},\mathbb{C}_\ell)$ with $t(1) := 1_\ell$ where $1_\ell$ denotes the 1 in $\mathbb{C}_\ell$, i.e.

$$P_\mu^{L_\ell,\mathrm{cpt}} : \mathcal{D}'(K/M) \to C^\infty(G \times_K \mathbb{C}_\ell),$$
$$P_\mu^{L_\ell,\mathrm{cpt}}(\phi)(g) := \int_K a_I(g^{-1}k)^{-(\mu+\rho)} L_\ell(k_I(g^{-1}k))t(\phi(k))\,\mathrm{d}k.$$

By Equation 3.5 we have $P_\mu^{L_\ell,\mathrm{cpt}} = P_\mu^{L_\ell} \circ \mathcal{Q}_{\mu-\rho}$ with

$$P_\mu^{L_\ell} : H_\mu^{-\infty} \to C^\infty(G \times_K \mathbb{C}_\ell), \quad P_\mu^{L_\ell}(f)(g) := F^{-1}(t)(\pi_\mu(g^{-1})f).$$

Lemma 4.1.7 implies

$$(\mathcal{Q}_{\mu-\rho}(\phi))_\ell = \tilde{\pi}_\ell^*(\tilde{\pi}_{\ell*}(\mathcal{Q}_{\mu-\rho}(\phi))) = \tilde{\pi}_\ell^*(P_\mu^{L_{-\ell},\mathrm{cpt}}(\phi)) = P_\mu^{L_{-\ell}}(\mathcal{Q}_{\mu-\rho}(\phi))$$

and thus $f_\ell = P_\mu^{L_{-\ell}}(f)$ for each $f \in H_\mu^{-\infty}$.

Note that $\mathfrak{p}$ decomposes as a $K$-representation into $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ where

$$\mathfrak{p}_+ := \left\{ \begin{pmatrix} z & iz \\ iz & -z \end{pmatrix} : z \in \mathbb{C} \right\} \quad \text{and} \quad \mathfrak{p}_- := \left\{ \begin{pmatrix} z & -iz \\ -iz & -z \end{pmatrix} : z \in \mathbb{C} \right\}.$$

Thus, for every $[(L_\ell, \mathbb{C}_\ell)] \in \widehat{K}$,

$$\mathbb{C}_\ell \otimes \mathfrak{p}^* = (\mathbb{C}_\ell \otimes \mathfrak{p}_+^*) \oplus (\mathbb{C}_\ell \otimes \mathfrak{p}_-^*)$$

with associated projections $T_\pm$ given by the restriction to $\mathfrak{p}_\pm$ if $\mathbb{C}_\ell \otimes \mathfrak{p}^*$ is viewed as $\mathrm{Hom}(\mathfrak{p}, \mathbb{C}_\ell)$. This gives rise to two generalized gradients $d_\pm := T_\pm \circ \nabla$, i.e.

$$d_\pm \colon C^\infty(G \times_K \mathbb{C}_\ell) \to C^\infty(G \times_K (\mathbb{C}_\ell \otimes \mathfrak{p}_+^*)) \cong C^\infty(G \times_K \mathrm{Hom}(\mathfrak{p}_\pm, \mathbb{C}_\ell))$$
$$f \mapsto \left[ g \mapsto \left.\frac{d}{dt}\right|_{t=0} f(g \exp t\,\bullet) \right].$$

Since $\mathfrak{p}_\pm = \mathbb{C}\eta_\pm$, we may write the *operators* $\eta_\pm$ from Equation (4.6) more complicated as the composition of the evaluation map at $\eta_\pm$ with $d_\pm$. We claim that

$$\mathrm{ev}_{\eta_\pm} \colon \mathrm{Hom}(\mathfrak{p}_\pm, \mathbb{C}_{-\ell}) \to \mathbb{C}_{-(\ell\pm1)}, \quad f \mapsto f(\eta_\pm)$$

is a $K$-equivariant isomorphism. Note first that the $K$-module structure on $\mathrm{Hom}(\mathfrak{p}_\pm, \mathbb{C}_\ell)$ induced by the structure on $\mathbb{C}_\ell \otimes \mathfrak{p}^*$ is given by

$$(k.f)(X) = \tau(k)(\mathrm{Ad}^*(k)f)(X).$$

The infinitesimal rotation matrix $V$ (see Equation (4.4)) acts on $\mathbb{C}_\ell$ by the scalar $i\ell$. For $f \in \mathrm{Hom}(\mathfrak{p}_\pm, \mathbb{C}_{-\ell})$ this gives

$$\begin{aligned}(V.f)(\eta_\pm) &= \left.\frac{d}{dt}\right|_{t=0} L_{-\ell}(\exp tV) f(\mathrm{Ad}(\exp(-tV))\eta_\pm) \\ &= \left.\frac{d}{dt}\right|_{t=0} e^{-\ell ti} f(e^{\mp ti}\eta_\pm) = -(\ell \pm 1)if(\eta_\pm)\end{aligned}$$

and the claim follows. This proves $\mathrm{ev}_{\eta_\pm} \circ T_\pm \in \mathrm{Hom}_K(\mathbb{C}_\ell \otimes \mathfrak{p}^*, \mathbb{C}_{\ell\mp1})$ and in particular that the $\eta_\pm$ are (up to scalar multiples all) generalized gradients in the case of $\mathrm{PSL}(2,\mathbb{R})$.

We are now ready to give an alternative proof of the Fourier relations of principal series representations from Lemma 4.2.1 using generalized gradients.

**Lemma 4.4.1.** *Let $\mu \in \mathfrak{a}^*$ and $f \in H^{-\infty}_{\mu+\rho}$. Then*

$$2\eta_\pm f_\ell = (\mu(H) + 1 \pm \ell) f_{\ell\pm1}.$$

*Proof.* By Corollary 3.3.5 there exist constants $c_\pm(\ell) \in \mathbb{C}$ such that

$$\eta_\pm \circ P^{L_{-\ell}}_{\mu+\rho} = c_\pm(\ell) \cdot P^{L_{-(\ell\pm1)}}_{\mu+\rho}.$$

Especially we have $\eta_\pm f_\ell = c_\pm(\ell) \cdot f_{\ell\pm1}$ for each $f \in H^{-\infty}_{\mu+\rho}$. In order to compute the scalars, let $\delta_{eM}$ denote the delta distribution at $eM$ on $K/M$. Then we have

$$P^{L_\ell,\mathrm{cpt}}_{\mu+\rho}(\delta_{eM})(g) = a_I(g^{-1})^{-(\mu+2\rho)} L_\ell(k_I(g^{-1}))1_\ell$$

and thus

$$(\eta_\pm \circ P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho})(\delta_{eM})(e) = c_\pm(\ell) \cdot P^{L_{-(\ell\pm1)},\mathrm{cpt}}_{\mu+\rho}(\delta_{eM})(e) = c_\pm(\ell) \cdot 1_{-(\ell\pm1)}.$$

Let us compute the differential $(\nabla \circ P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho})(\delta_{eM})(e) \in \mathbb{C}_{-\ell} \otimes \mathfrak{p}^* \cong \mathrm{Hom}(\mathfrak{p}, \mathbb{C}_{-\ell})$. A basis of $\mathfrak{p}$ is given by $\{H, B\}$ with $B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have

$$\begin{aligned}
(\nabla \circ P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho})(\delta_{eM})(e)(H) &= \frac{d}{dt}\Big|_{t=0} P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho}(\delta_{eM})(\exp tH) \\
&= \frac{d}{dt}\Big|_{t=0} a_I(\exp -tH)^{-(\mu+2\rho)} 1_{-\ell} \\
&= \frac{d}{dt}\Big|_{t=0} e^{t(\mu+2\rho)(H)} 1_{-\ell} = (\mu+2\rho)(H) 1_{-\ell}.
\end{aligned}$$

Moreover, $B = U_+ - V \in \mathfrak{n} \oplus \mathfrak{k}$ implies

$$\begin{aligned}
(\nabla \circ P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho})(\delta_{eM})(e)(B) &= -(\nabla \circ P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho}(\delta_{eM}))(e)(V) \\
&= -\frac{d}{dt}\Big|_{t=0} L_{-\ell}(\exp -tV) 1_{-\ell} = -i\ell 1_{-\ell}.
\end{aligned}$$

Now we can use $\eta_\pm = \frac{1}{2}(H \pm iB)$ to compute $\mathrm{ev}_{\eta_\pm}((d_\pm \circ P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho})(\delta_{eM})(e))$:

$$(\eta_\pm \circ P^{L_{-\ell},\mathrm{cpt}}_{\mu+\rho})(\delta_{eM})(e) = \frac{1}{2}((\mu+2\rho)(H) \pm i(-i\ell)) 1_{-(\ell\pm1)} = \frac{\mu(H) + 1 \pm \ell}{2} 1_{-(\ell\pm1)}.$$

□

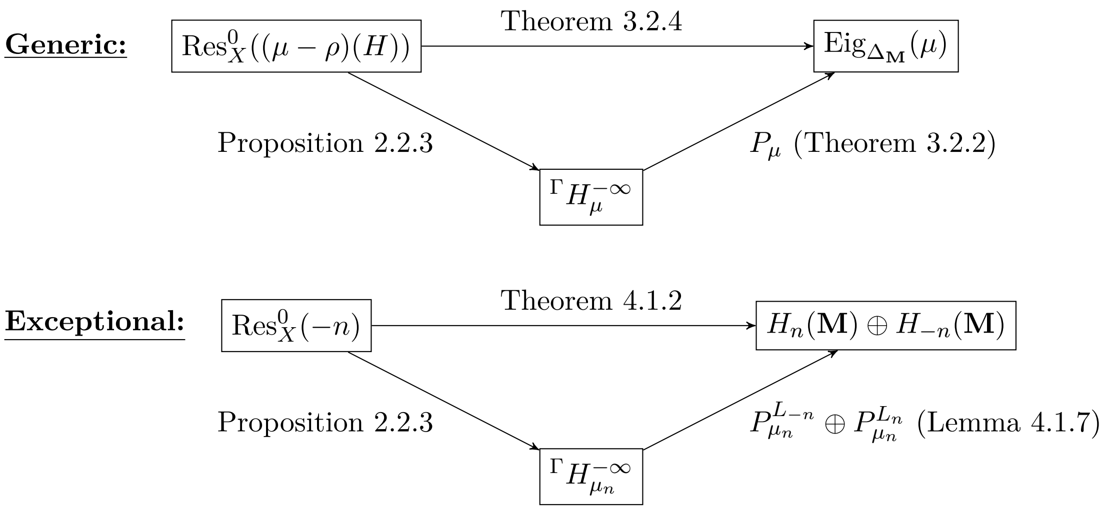



Figure 4.4.: Comparison of the generic with the exceptional case ($\mu_n(H) \coloneqq \frac{1}{2} - n,\ n \in \mathbb{N}$)

Figure 4.4 summarizes the case of $\mathrm{PSL}(2,\mathbb{R})$ schematically. In order to generalize the proof of the exceptional quantum-classical correspondence to other groups we have to address several questions. We give a short list of necessary steps for this approach.

i) Determine the kernels of vector-valued Poisson transforms and find injective direct sums of them.

ii) Prove an analog of the Fourier expansion in the fibers from Equation (4.3).

iii) Characterize the principal series by relations between the Fourier components (generalizing Lemma 4.2.1).

iv) Determine the image of the injective (sums of) Poisson transforms by using the Fourier relations.

Another more representation theoretic approach arises from the fact that the spaces $H_{\pm n}(\mathbf{M})$ are given by the $\Gamma$-invariant elements of *(anti-)holomorphic discrete series representations* (see e.g. [Kna86, Chapter VI]). Since the resonant states are also given by the $\Gamma$-invariant elements of the principal series one might try the following approach to establish a quantum-classical correspondence:

i) Why do holomorphic discrete series occur and how are they connected to the principal series?

ii) Do holomorphic discrete series occur in general or, if not, are there any substitutes for them?

iii) Find geometric realizations in sections of vector bundles of the representations from the previous step in order to describe the images of the (direct sums of) Poisson transforms.

In this thesis we will use a combination of these two approaches.

# 5. Mapping properties of vector-valued Poisson transforms

In the case of surfaces, it turned out that a quantum-classical correspondence can be established by an injective direct sum of vector-valued Poisson transforms. In this chapter we prove the analogous result in the general rank one case (Proposition 5.1.3). In higher rank, one has to take care of higher multiplicities of $K$-representations. Nevertheless, choosing appropriate Poisson transforms for each copy of the $K$-representation in the principal series, the same proof should also work in this case. However, to determine the direct sum explicitly (Theorem 5.1.6), we need to know the $K$-types of the irreducible constituents of the principal series representations. Besides injectivity we investigate necessary conditions for the images of vector-valued Poisson transforms. The main result in this context is Proposition 5.3.2, which uses generalized gradients to connect different Poisson transforms.

## 5.1. Injectivity of vector-valued Poisson transforms

In this section we investigate specific vector-valued Poisson transforms if $G$ is of rank one. We will see that if we pick a minimal $K$-type for each irreducible subspace of the representation, the direct sum of the associated Poisson transforms is injective. By our rank one assumption each spherical principal series representation $H_\mu$ decomposes multiplicity-freely as a $K$-representation (see Proposition 2.4.3). Therefore, we have the following

**Lemma 5.1.1.** *Let* $0 \neq (\tau, V)$ *be an irreducible* $K$*-representation with* $\mathrm{mult}_K(V, H_\mu) \neq 0$ *and* $t \in \mathrm{Hom}_M(\mathbb{C}, V)$*. Then*

$$P_\mu^\tau(F^{-1}(t) \otimes \bullet) = \frac{t(1)(e)}{\dim V} P_\mu^\tau(\mathrm{pr}_V \otimes \bullet),$$

*where* $F$ *denotes the Frobenius isomorphism (see Definition 3.3.1).*

*Proof.* By Equation (3.2) we have for each $f \in H_\mu^{-\infty}$ and $g \in G$ that

$$P_\mu^\tau(F^{-1}(t) \otimes f)(g) = F^{-1}(t)(\pi_\mu(g)^{-1} f) \text{ and } P_\mu^\tau(\mathrm{pr}_V \otimes f)(g) = \mathrm{pr}_V(\pi_\mu(g)^{-1} f).$$

By Proposition 2.4.3 we obtain that $\mathrm{Hom}_K(H_\mu, V) = \mathbb{C}\,\mathrm{pr}_V$ is one-dimensional since $(\tau, V) \in \hat{K}_M$. This proves that there exists some $c \in \mathbb{C}$ such that

$$P_\mu^\tau(F^{-1}(t) \otimes \bullet) = c P_\mu^\tau(\mathrm{pr}_V \otimes \bullet).$$

Recall the $M$-invariant function $\phi_V$ from Proposition 2.4.4. By Definition 3.3.1 we have

$$P^\tau_\mu(F^{-1}(t)\otimes\phi_V)(e) = \int_K \tau(k)t(\phi_V(k))\,\mathrm{d}k = \int_K \phi_V(k)\tau(k)t(1)\,\mathrm{d}k$$
$$= \int_K \phi_V(k)\tau(k)t(1)(e)\phi_V\,\mathrm{d}k,$$

where we recall that $V$ is realized in $C^\infty(K)^M$ so that $t(1)(e)$ makes sense, and we used Proposition 2.4.4 i) for the last equality. Using Proposition 2.4.4 iii) we infer

$$P^\tau_\mu(F^{-1}(t)\otimes\phi_V)(e)(e) = \int_K \phi_V(k)t(1)(e)\phi_V(k^{-1})\,\mathrm{d}k = \int_K \phi_V(k)t(1)(e)\overline{\phi_V(k)}\,\mathrm{d}k$$
$$= t(1)(e)\langle\phi_V,\phi_V\rangle_{L^2(K)} = \frac{t(1)(e)}{\dim V}.$$

On the other hand (3.2) yields

$$P^\tau_\mu(\mathrm{pr}_V\otimes\phi_V)(e) = \mathrm{pr}_V(\phi_V) = \phi_V.$$

Thus,

$$c = \frac{P^\tau_\mu(F^{-1}(t)\otimes\phi_V)(e)(e)}{P^\tau_\mu(\mathrm{pr}_V\otimes\phi_V)(e)(e)} = \frac{t(1)(e)}{\phi_V(e)\dim V} = \frac{t(1)(e)}{\dim V}. \qquad \square$$

From now on we choose $t\in\mathrm{Hom}_M(\mathbb{C},V)$ for each $(\tau,V)\in\hat{K}_M$ by $t(1) := \phi_V$ and define

$$P^\tau_\mu : H^{-\infty}_\mu \to C^\infty(G\times_K V), \qquad P^\tau_\mu(f) := P^\tau_\mu(F^{-1}(t)\otimes f).$$

Note that, by Lemma 5.1.1, we have for each $f\in H^{-\infty}_\mu$ and $g\in G$

$$P^\tau_\mu(f)(g) = \frac{1}{\dim V}\,\mathrm{pr}_V(\pi_\mu(g)^{-1}f). \tag{5.1}$$

**Proposition 5.1.2.** *Let $[(\tau,V_\tau)]\in\hat{K}_M$ and $\mu\in\mathfrak{a}^*$. Then the Poisson transform*

$$P^\tau_\mu : H^{-\infty}_\mu \to C^\infty(G\times_K V)$$

*is injective if and only if every non-trivial $G$-invariant subspace of $H^{-\infty}_\mu$ contains $\tau$. Moreover, the kernel is given by the distributional elements in the closure of the sum of all $G$-invariant subspaces $V\le H_\mu$ with $\mathrm{mult}_K(\tau,V)=0$.*

*Proof.* Since $P^\tau_\mu$ is $G$-equivariant, the kernel $\ker P^\tau_\mu$ is $G$-invariant. We claim that it equals the closure of the sum of all invariant subspaces of $H_\mu$ which do not contain the $K$-representation $(\tau,V_\tau)$:
If $\{0\}\neq W\le H_\mu$ is an invariant subspace of $H_\mu$ which does not contain the $K$-representation $\tau$, by (5.1) we have

$$P^\tau_\mu(f)(g) = \frac{1}{\dim V_\tau}\,\mathrm{pr}_{V_\tau}(\pi_\mu(g^{-1})f) = 0$$

for every $f \in W$ and $g \in G$ since $\pi_\mu(g^{-1})f \in W$. Thus, $f \in \ker P_\mu^\tau$. This proves the first inclusion because the kernel is closed.
Conversely, let $f \in \ker P_\mu^\tau$. Since the kernel is invariant, the distributional elements in the $G$-cyclic space $W_f$ of $f$ are also contained in the kernel of $P_\mu^\tau$. Therefore, $f$ is contained in an invariant space which does not contain $\tau$ (if $W_f$ contains $\tau$ we can choose $g = e$ to get a contradiction to $W_f \subseteq \ker P_\mu^\tau$). □

We now use this result to construct injective direct sums of Poisson transforms, which allow us to prove the spectral correspondences later on.

**Proposition 5.1.3.** *Let $\mu \in \mathfrak{a}^*$ and $\mathbf{Irr}(\mu)$ be the set of all non-zero irreducible subrepresentations of $H_\mu$. Then, if $(\tau_U, V_{\tau_U})$ is any non-zero $K$-type of $U$ for $U \in \mathbf{Irr}(\mu)$, the direct sum of the corresponding Poisson transforms*

$$\oplus_{U\in\mathbf{Irr}(\mu)} P_\mu^{\tau_U} : H_\mu^{-\infty} \to \bigoplus_{U\in\mathbf{Irr}(\mu)} C^\infty(G \times_K V_{\tau_U})$$

*is injective. A natural choice of $(\tau_U, V_{\tau_U})$ is given by a minimal $K$-type of $U$.*

*Proof.* Since the kernel of the direct sum $\oplus_{U\in\mathbf{Irr}(\mu)} P_\mu^{\tau_U}$ is the intersection of the kernels of $P_\mu^{\tau_U}$, $U \in \mathbf{Irr}(\mu)$, we can apply Proposition 5.1.2 to deduce

$$\oplus_{U\in\mathbf{Irr}(\mu)} P_\mu^{\tau_U} \text{ injective } \Leftrightarrow \forall\, \{0\} \neq V \leq H_\mu \ \exists U \in \mathbf{Irr}(\mu)\colon\ \mathrm{mult}_K(\tau_U, V) \neq 0.$$

Let $\{0\} \neq V \leq H_\mu$ be a non-trivial (closed) $G$-invariant subspace. We claim that there exists some $U \in \mathbf{Irr}(\mu)$ such that $\mathrm{mult}_K(\tau_U, V) \neq 0$. In fact, since $H_\mu$ has a composition series, $V$ also has a composition series by [KV95, p. 815]. In particular, there exists an irreducible subrepresentation $\{0\} \neq I \leq V$. But $I \in \mathbf{Irr}(\mu)$ by the definition of $\mathbf{Irr}(\mu)$ and $\mathrm{mult}_K(\tau_I, V) \neq 0$ since $I \leq V$. □

**Corollary 5.1.4.** *With the notation from Proposition 5.1.3 we have*

$$\mathrm{Res}_X^0((\mu - \rho)(H)) \cong \bigoplus_{U\in\mathbf{Irr}(\mu)} {}^\Gamma(\operatorname{im} P_\mu^{\tau_U}) \subseteq \bigoplus_{U\in\mathbf{Irr}(\mu)} {}^\Gamma C^\infty(G \times_K V_{\tau_U}).$$

As already mentioned in Proposition 5.1.3, a natural choice of the $K$-types is given by so-called minimal $K$-types, which are defined as follows.

**Definition 5.1.5** (cf. [Vog79, Definition 5.1]). Let $T \leq K$ be a maximal torus with Lie algebra $\mathfrak{t}_0$ (and $\mathfrak{t} := (\mathfrak{t}_0)_\mathbb{C}$), root system $\Delta_\mathfrak{k} := \Delta(\mathfrak{k}, \mathfrak{t})$ and positive system $\Delta_\mathfrak{k}^+ := \Delta^+(\mathfrak{k}, \mathfrak{t})$. If $X$ is a Harish-Chandra module for $\mathfrak{g}_0$, the set of *minimal $K$-types of $X$* is given by

$$\{\mu \in \hat{\mathfrak{k}}\colon\ \mathrm{mult}_K(\mu, X) \neq 0 \text{ and } \kappa(\mu + 2\rho_c,\ \mu + 2\rho_c) \text{ minimal with this property}\},$$

where

- $\hat{\mathfrak{k}} \subset (i\mathfrak{t}_0)^*$ denotes the set of the highest weights of the elements of $\hat{K}$ w.r.t. the positive system $\Delta_\mathfrak{k}^+$,

- $\kappa(\bullet, \bullet)$ denotes the form on $\mathfrak{g}^*$ induced by the Cartan-Killing form of $\mathfrak{g}$,
- $\rho_c$ denotes the half sum of the positive roots $\Delta_{\mathfrak{k}}^+$.

We call $\kappa(\bullet + 2\rho_c, \ \bullet + 2\rho_c)$ the *Vogan norm*.

The following theorem shows that among the rank one Riemannian symmetric spaces of non-compact type, the only case in which we really have a direct sum in Proposition 5.1.3 is given by the case of surfaces. Thus, in every other case, we obtain an injective Poisson transform, which we will call the *minimal $K$-type Poisson transform*. The proof of the theorem is done case by case in the following section.

**Theorem 5.1.6.** *With the notation from Proposition 5.1.3 every $U \in \mathbf{Irr}(\mu)$ has a unique minimal $K$-type $(\tau_U, V_{\tau_U})$. Moreover, $\mathbf{Irr}(\mu)$ is a singleton if $\mathfrak{g}_0 \neq \mathfrak{sl}(2, \mathbb{R})$. If $\mu \notin \mathbf{Ex}$, $\mathbf{Irr}(\mu) = \{U\}$ is always a singleton[1] (also for $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$) and $(\tau_U, V_{\tau_U})$ is the trivial $K$-representation.*

**Remark 5.1.7** (Connection to the exceptional set of [Olb94])**.** For exceptional parameters $\mu \in \mathbf{Ex}$ in the case of $\mathfrak{g}_0 \neq \mathfrak{sl}(2, \mathbb{R})$, $\mathbf{Irr}(\mu) = \{U\}$ is a singleton and the minimal $K$-type Poisson transform

$$P_\mu^{\tau_U} : H_\mu^{-\infty} \to C^\infty(G \times_K V_{\tau_U})$$

is injective by Proposition 5.1.3. As a first approach one might try to use [Olb94] to deduce whether this map is also surjective onto the corresponding space of joint eigensections. Unfortunately, [Olb94] itself defines some exceptional sets $\mathcal{A}'_\tau(\sigma)$ that are excluded in most of the results concerning mapping properties and into which all our minimal $K$-type Poisson transforms from above fall (for $\mathrm{SL}(2, \mathbb{R})$ they are empty). For $\tau \in \hat{K}, \ \sigma \in \tau|_M$ these sets are defined by (see [Olb94, Definition 4.7])

$$\mathcal{A}'_\tau(\sigma) := \{\mu \in \mathfrak{a}^* \mid \exists \mu' \notin W \cdot \mu, \ \sigma' \in \tau\big|_M, \ w \in W \colon E_{w\sigma, w\mu} \cap E_{\sigma', \mu'} \neq \{0\}, \ s\mu' > \mu\},$$

where $s$ denotes the maximal element of $W$. We prove that indeed $\mu \in \mathcal{A}'_{\tau_U}(\mathrm{triv}_M)$ for each exceptional parameter $\mu \in \mathbf{Ex}$ as above. Since $U \leq H_\mu$ corresponds to an irreducible subquotient of $H_{-\mu}$, we first infer that $P_{-\mu}^{\tau_U}$ is not injective by Proposition 5.1.2. We claim that this proves $\mu \in \mathcal{A}'_{\tau_U}(\mathrm{triv}_M)$. Indeed, by [Olb94, Satz 3.17], $\det \mathbf{c}_{\tau_U}(\mathrm{triv}_M, -\mu) = 0$ and [Olb94, Lemma 4.29, 2] implies $\mu \in \mathcal{A}'_{\tau_U}(\mathrm{triv}_M)$ since $P_\mu^{\tau_U}$ is injective.

The proof of Theorem 5.1.6 is done case by case (see Section 5.2) and organized as follows:

i) Calculate the highest weights of the representations in $\hat{K}_M$ w.r.t. some positive system (see Appendix A).

ii) Specify the composition series of the spherical principal series (see Appendix B).

[1]This part also follows from the uniqueness of the Langlands quotient (see [Kna86, Theorem 8.54]).

iii) Calculate the minimal $K$-types of all irreducible subrepresentations of the spherical principal series representations.

In each case, we also compare Proposition 5.1.2 to the injectivity of the scalar Poisson transform from Theorem 3.2.2.

Let us stress that our choice of Poisson transforms in Proposition 5.1.3 generalizes all known cases. More precisely, for $\mu \notin \mathbf{Ex}$, Proposition 5.1.3 always gives the scalar Poisson transform since the minimal $K$-type of the unique irreducible subrepresentation is the trivial representation by Theorem 5.1.6. Therefore, our choice is consistent with the generic quantum-classical correspondence from Theorem 3.2.4 (see also Remark 3.3.2). Moreover, if $\mu \in \mathbf{Ex}$ and $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$ we have $\mu(H) = -n + \frac{1}{2}$ for some $n \in \mathbb{N}$ by Remark 4.1.9(ii) and $H_\mu$ has irreducible subrepresentations $D_{n-1,+}$ and $D_{n-1,-}$ by Lemma B.1.1. The (unique) minimal $K$-type of $D_{n-1,+}$ resp. $D_{n-1,-}$ is given by $e_n$ resp. $e_{-n}$ (or $L_n$ resp. $L_{-n}$ when defined on functions on $K = \mathrm{PSO}(2,\mathbb{R})$). Thus, our choice gives the same maps as in the exceptional quantum-classical correspondence from Theorem 4.1.2 (see also Lemmas 4.1.7 and 4.3.3).

## 5.2. Minimal $K$-types and proofs of injectivity results

We now prove Theorem 5.1.6 in each case.

### The case of $\mathrm{SO}_0(n,1)$, $n \geq 3$

We first describe the exceptional set in this case.

**Lemma 5.2.1.** *For $G = \mathrm{SO}_0(n,1)$ we have*

$$\mathbf{Ex} = -(\rho + \mathbb{N}_0\alpha).$$

*Proof.* Since $m_{2\alpha} = 0$ and $\rho = \frac{1}{2}(m_\alpha\alpha + m_{2\alpha}2\alpha)$ we obtain

$$-(\rho + \mathbb{N}_0\alpha) = -\left(\frac{m_\alpha}{2} + \mathbb{N}_0\right)\alpha = \left(-\frac{m_\alpha}{2} - 1 - 2\mathbb{N}_0\right)\alpha \cup \left(-\frac{m_\alpha}{2} - 0 - 2\mathbb{N}_0\right)\alpha. \quad \square$$

For the scalar Poisson transform, Proposition 5.1.2 agrees with Theorem 3.2.2.

*Comparison to Theorem 3.2.2.* Let $\mu \in \mathfrak{a}^*$. If $H_\mu$ is irreducible, $P_\mu$ is injective since the kernel is an invariant subspace and $P_\mu$ is not the zero map since $H_\mu$ contains the trivial $K$-representation $Y_0$ (compare with Equation (5.1) and choose $f \in Y_0,\ g = e$). Thus, we may assume $\mu(H) \in \pm(\rho(H) + \mathbb{N}_0)$ by Lemma B.2.1.

Case $\mu(H) = \rho(H) + k \in \rho(H) + \mathbb{N}_0$ : Every non-trivial invariant subspace of $H_\mu$ (i.e. $V_k$) contains the trivial representation $Y_0$ and $P_\mu$ is injective by Proposition 5.1.2.

Case $\mu(H) = -(\rho(H) + k) \in -(\rho(H) + \mathbb{N}_0)$ : $W_k$ is a $G$-invariant subspace of $H_\mu$ which does not contain the trivial representation and $P_\mu$ is not injective. Thus, $P_\mu$ is not injective if and only if (recall $\alpha(H) = 1$)

$$\mu \in -(\rho + \mathbb{N}_0\alpha) = \mathbf{Ex}. \quad \square$$

### Minimal $K$-types of irreducible subrepresentations

Recall the half-sum of positive roots in the odd resp. even case from Equation (A.1)

$$\rho_c = \left(m - \frac{1}{2}\right) e_1 + \left(m - \frac{3}{2}\right) e_2 + \ldots + \frac{1}{2} e_m \text{ resp. } \rho_c = (m-1)e_1 + \ldots + e_{m-1}.$$

Using that the highest weight of $Y_\ell$ is $\ell e_1$ (see Appendix B.2.2 for the notation) we are now able to compute the minimal $K$-types of the irreducible subrepresentations $V_k$ and $W_k$ from Lemma B.2.1. Since the Cartan-Killing form of $\mathfrak{so}(2m+2,\mathbb{C})$ resp. $\mathfrak{g} = \mathfrak{so}(2m+1,\mathbb{C})$ is a multiple of the trace form $(X,Y) \mapsto \operatorname{tr}(XY)$, we can use the explicit definitions of the roots of $\mathfrak{k}$ from [Kna02, Chapter II, §1, Example 4 resp. 2] to see that the $e_i$ are orthogonal w.r.t. the Killing form and that every $e_i$ has the same length.

**Lemma 5.2.2.** *Let $U \subseteq \hat{K}_M = \{Y_\ell \colon \ell \in \mathbb{N}_0\}$ be a set of $M$-spherical $K$-types. Then the Vogan norm is minimal for $Y_\ell \in U$ if and only if $\ell$ is minimal for $Y_\ell \in U$.*

*Proof.* By the definition of minimal $K$-types we have to minimize

$$\kappa(\ell e_1 + 2\rho_c, \ell e_1 + 2\rho_c)$$

for $Y_\ell \in M$ (since $\ell e_1$ is the highest weight of $Y_\ell$). We first consider the even case, i.e. $G = \mathrm{SO}_0(2m,1)$ with $m > 1$. We have

$$\begin{aligned}\kappa(\ell e_1 + 2\rho_c, \ell e_1 + 2\rho_c) &= \kappa(\ell e_1, \ell e_1) + 4\kappa(\ell e_1, \rho_c) + 4\kappa(\rho_c, \rho_c) \\ &= (\ell^2 + 4\ell(m-1))\kappa(e_1, e_1) + 4\kappa(\rho_c, \rho_c)\end{aligned}$$

which is clearly minimal if and only if $\ell$ is minimal. The proof for $G = \mathrm{SO}_0(2m+1,1)$ is analogous (changing $m-1$ to $m - \frac{1}{2}$). ☐

**Proposition 5.2.3.** *Let $k \in \mathbb{N}_0$. Then $V_k$ and $W_k$ from Lemma B.2.1 have unique minimal $K$-types which are given by $Y_0$ and $Y_{k+1}$ respectively (note that $Y_0$ is the trivial $K$-representation).*

*Proof.* By Lemma B.2.1 the $K$-types of $V_k$ resp. $W_k$ are given by $\{Y_\ell \colon 0 \leq \ell \leq k\}$ resp. $\{Y_\ell \colon k+1 \leq \ell\}$. The Proposition now follows from Lemma 5.2.2. ☐

*Proof of Theorem 5.1.6.* If $\mu \notin \mathbf{Ex}$, the spherical principal series $H_\mu$ is irreducible or the unique irreducible subrepresentation of $H_\mu$ is given by $V_k$ for some $k \in \mathbb{N}_0$ by Lemma B.2.1. In both cases the unique minimal $K$-type is given by the trivial representation by Proposition 5.2.3.

If $\mu \in \mathbf{Ex}$, there exists some $k \in \mathbb{N}_0$ such that $\mu(H) = -(\rho(H) + k)$. By Lemma B.2.1 we see that there is only one irreducible subspace, namely $W_k$. The proposition thus follows from Proposition 5.2.3. ☐

### The case of $\mathrm{SU}(n,1)$, $n \neq 1$

**Lemma 5.2.4.** *For $G = \mathrm{SU}(n,1)$ we have*

$$\mathbf{Ex} = -(\rho + 2\mathbb{N}_0\alpha).$$

*Proof.* Since $\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha)$ and $m_{2\alpha} = 1$ we obtain

$$-(\rho + 2\mathbb{N}_0\alpha) = -\left(\frac{m_\alpha}{2} + m_{2\alpha} + 2\mathbb{N}_0\right)\alpha = \mathbf{Ex}. \qquad \square$$

*Comparison to Theorem 3.2.2.* Let $\mu \in \mathfrak{a}^*$ such that $H_\mu$ is reducible.

Case $\mu(H) = \rho(H) + 2k \in \rho(H) + 2\mathbb{N}_0$ : The non-trivial subspaces are $F_k, H_{k,\pm}$ and $H_{k,+} + H_{k,-}$. Each of these contains the trivial representation $Y_{0,0}$ and $P_\mu$ is injective by Proposition 5.1.2.

Case $\mu(H) = -\rho(H) - 2k \in -(\rho(H) + 2\mathbb{N}_0)$ : $I_k$ is an invariant subspace of $H_\mu$ which does not contain the trivial representation. Thus, $P_\mu$ is not injective in this case. Altogether, $P_\mu$ is not injective if and only if (recall $\alpha(H) = 1$)

$$\mu \in -(\rho + 2\mathbb{N}_0\alpha) = \mathbf{Ex}. \qquad \square$$

#### Minimal $K$-types of irreducible subrepresentations

We will now compute the minimal $K$-types of the irreducible subrepresentations $F_k$ and $I_k$ occurring in Lemma B.3.2. Recall the half sum of positive roots from Equations (A.2.3)

$$\rho_c = \frac{n-1}{2}e_1 + \frac{n-3}{2}e_2 + \ldots - \frac{n-1}{2}e_n.$$

Since the Cartan-Killing form of $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C}) = (\mathfrak{su}(n,1))_\mathbb{C}$ is a multiple of the trace form $(X,Y) \mapsto \mathrm{tr}(XY)$ we see that the $e_i$ are orthogonal w.r.t. the Killing form and that every $e_i$ has the same length. We rescale the Killing form such that the $e_i$ are orthonormal and denote the resulting form by $\tilde{\kappa}$.

**Lemma 5.2.5.** *Let $U \subseteq \hat{K}_M = \{Y_{p,q} \colon p,q \in \mathbb{N}_0\}$ be a set of $M$-spherical $K$-types. Then the Vogan norm is minimal for $Y_{p,q} \in U$ if and only if $(p+n-1)^2 + (q+n-1)^2 + (p-q)^2$ is minimal for $Y_{p,q} \in U$.*

*Proof.* By Lemma B.3.1, the highest weight of a $K$-type $Y_{p,q}$ is given by $qe_1 - pe_n + (p-q)e_{n+1}$. We have that

$$\begin{aligned}
&\tilde{\kappa}(qe_1 - pe_n + (p-q)e_{n+1} + 2\rho_c,\ qe_1 - pe_n + (p-q)e_{n+1} + 2\rho_c)\\
&= \|(q+n-1)e_1 + (n-3)e_2 + \ldots - (n-3)e_{n-1} - (p+n-1)e_n + (p-q)e_{n+1}\|_{\tilde{\kappa}}^2\\
&= (q+n-1)^2 + (n-3)^2 + (n-5)^2 + \ldots + (n-3)^2 + (p+n-1)^2 + (p-q)^2. \qquad \square
\end{aligned}$$

**Proposition 5.2.6.** *Let $k \in \mathbb{N}_0$. Then $F_k$ and $I_k$ from Lemma B.3.2 have unique minimal $K$-types which are given by $Y_{0,0}$ and $Y_{k+1,k+1}$ respectively (note that $Y_{0,0}$ is the trivial $K$-representation).*

*Proof.* By Lemma B.3.2 the $K$-types occurring in $F_k$ resp. $I_k$ are given by

$$\{Y_{p,q} \mid p,q \in \{1,\dots,k\}\} \quad \text{respectively} \quad \{Y_{p,q} \mid p,q \in \mathbb{N} \text{ with } p,q \geq k+1\}.$$

By Lemma 5.2.5 it suffices to minimize $(p+n-1)^2+(q+n-1)^2+(p-q)^2$. The minimum is attained at $p=q=0$ in the case of $F_k$ and at $p=q=k+1$ in the case of $I_k$. □

*Proof of Theorem 5.1.6.* If $\mu \notin \mathbf{Ex}$ the spherical principal series $H_\mu$ is irreducible or the unique irreducible subrepresentation of $H_\mu$ is given by $F_k$ for some $k \in \mathbb{N}_0$ by Lemma B.3.2. In both cases the unique minimal $K$-type is given by the trivial representation by Proposition 5.2.6.

If $\mu \in \mathbf{Ex}$ there exists some $k \in \mathbb{N}_0$ such that $\mu(H) = -(\rho(H)+2k)$. Lemma B.3.2 shows that $I_k$ is the only irreducible subspace of $H_\mu$. By Proposition 5.2.6 the minimal $K$-type of $I_k$ is uniquely given by $Y_{k+1,k+1}$. □

## The case of $\mathrm{Sp}(n,1),\ n \geq 2$

**Lemma 5.2.7.** *For $G = \mathrm{Sp}(n,1)$ we have*

$$\mathbf{Ex} = -(\rho + (2\mathbb{N}_0 - 2)\alpha).$$

*Proof.* Since $\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha)$ and $m_{2\alpha} = 3$ we obtain

$$\begin{aligned} -(\rho + (2\mathbb{N}_0 - 2)\alpha) &= -\left(\frac{m_\alpha}{2} + 1 + 2\mathbb{N}_0\right)\alpha \\ &= -\left(\frac{m_\alpha}{2} + 1 + 2\mathbb{N}_0\right)\alpha \cup -\left(\frac{m_\alpha}{2} + 3 + 2\mathbb{N}_0\right)\alpha = \mathbf{Ex}. \end{aligned}$$ □

*Comparison to Theorem 3.2.2.* Let $\mu \in \mathfrak{a}^*$ be such that $H_\mu$ is reducible.

Case $\mu(H) = \rho(H) - 2 + 2k,\ k \in \mathbb{N}$ : The non-trivial subspaces are $W_k$ and $M_k$. Each of these contains the trivial representation $V_{0,0}$. By Proposition 5.1.2, $P_\mu$ is injective.

Case $\mu(H) = -(\rho(H) - 2 + 2k),\ k \in \mathbb{N}$ : $\widetilde{M_k}$ is an invariant subspace of $H_\mu$ which does not contain $V_{0,0}$. Thus, $P_\mu$ is not injective in this case.

Case $\mu(H) = \rho(H) - 2$ : $T$ is the only invariant subspace of $H_\mu$. Since it contains the trivial representation, $P_\mu$ is injective in this case.

Case $\mu(H) = -(\rho(H) - 2)$ : $\widetilde{T}$ is an invariant subspace of $H_\mu$ which does not contain $V_{0,0}$. Thus, $P_\mu$ is not injective.

Altogether, $P_\mu$ is not injective if and only if (recall $\alpha(H) = 1$)

$$\mu \in -(\rho - 2\alpha + 2\mathbb{N}_0\alpha) = \mathbf{Ex}.$$ □

### Minimal $K$-types of irreducible subrepresentations

Recall the positive system from Equation (A.3.9) and the fundamental weights from Equation (A.3.11). Recall the half sum of positive roots from Equation (A.3.10)

$$\rho_c = ne_1 + (n-1)e_2 + \ldots + 2e_{n-1} + e_n + e_{n+1}.$$

Since the Cartan-Killing form of $\mathfrak{g} = \mathfrak{sp}(n+1,\mathbb{C})$ is a multiple of the trace form $(X,Y) \mapsto \operatorname{tr}(XY)$ we can use the explicit definitions of the roots of $\mathfrak{k}$ from [Kna02, Chapter II, §2, Example 3] to see that the $e_i$ are orthogonal w.r.t. the Killing form and that every $e_i$ has the same length. We rescale the Killing form such that the $e_i$ are orthonormal and denote the resulting form by $\tilde{\kappa}$.

**Lemma 5.2.8.** *Let $U \subseteq \hat{K}_M = \{V_{a,b}\colon a,b \in \mathbb{N}_0,\ a \geq b\}$ be a set of $M$-spherical $K$-types. Then the Vogan norm is minimal for $V_{a,b} \in U$ if and only if $(a+n)^2+(b+n-1)^2+(a-b+1)^2$ is minimal for $V_{a,b} \in U$.*

*Proof.* By Section A.3, the highest weight of a $K$-type $V_{a,b}$ is given by $ae_1+be_2+(a-b)e_{n+1}$. We have that

$$\begin{aligned}
&\tilde{\kappa}(ae_1 + be_2 + (a-b)e_{n+1} + 2\rho_c,\ ae_1 + be_2 + (a-b)e_{n+1} + 2\rho_c)\\
&= \|(a+n)e_1 + (b+n-1)e_2 + (n-2)e_3 + \ldots + e_n + (a-b+1)e_{n+1}\|_{\tilde{\kappa}}^2\\
&= (a+n)^2 + (b+n-1)^2 + (n-2)^2 + \ldots + 1 + (a-b+1)^2.
\end{aligned}$$ □

**Proposition 5.2.9.** *Let $k \in \mathbb{N}$. Then the irreducible representations $W_k$, $\widetilde{M_k}$, $T$ and $\widetilde{T}$ from Lemma B.4.1 have unique minimal $K$-types which are given by $V_{0,0}$, $V_{k+1,k+1}$, $V_{0,0}$ or $V_{1,1}$ respectively (where $V_{0,0}$ is the trivial $K$-representation) with highest weights $0$, $(k+1)\lambda_2$, $0$ or $\lambda_2$.*

*Proof.* By definition of the representations the occurring $K$-types are (with $a,b \in \mathbb{N}_0$)

$$\begin{aligned}
A(W_k) &:= \{V_{a,b}\colon b \leq a \leq k-1\}, & A(\widetilde{M_k}) &:= \{V_{a,b}\colon a \geq b > k\},\\
A(T) &:= \{V_{a,0}\colon a \in \mathbb{N}_0\}, & A(\widetilde{T}) &:= \{V_{a,b}\colon a \geq b > 0\}.
\end{aligned}$$

By Lemma 5.2.8 it suffices to minimize $(a+n)^2+(b+n-1)^2+(a-b+1)^2$. In each case, minimizing $a$ and $b$ also minimizes $(a-b+1)^2$ so that the minima are attained at $a=b=0$ for $W_k$ and $T$ and at $a=b=k+1$ resp. $a=b=1$ for $\widetilde{M_k}$ resp. $\widetilde{T}$. □

*Proof of Theorem 5.1.6.* If $\mu \notin \mathbf{Ex}$, the spherical principal series $H_\mu$ is irreducible or the unique irreducible subrepresentation of $H_\mu$ is given by $W_k$, for some $k \in \mathbb{N}$, or by $T$ by Lemma B.4.1. In every case, the unique minimal $K$-type is given by the trivial representation by Proposition 5.2.9.

Now let $\mu \in \mathbf{Ex}$. We distinguish the following two cases:

Case $\mu(H) = -(\rho(H) - 2 + 2k),\ k \in \mathbb{N}$ : By Lemma B.4.1, $\widetilde{M_k}$ is the only irreducible subspace of $H_\mu$. By Proposition 5.2.9, the minimal $K$-type of $\widetilde{M_k}$ is uniquely given by $V_{k+1,k+1}$.

Case $\mu(H) = -(\rho(H) - 2)$ : By Lemma B.4.1, $\widetilde{T}$ is the only irreducible subspace of $H_\mu$. By Proposition 5.2.9, the minimal $K$-type of $\widetilde{T}$ is uniquely given by $V_{1,1}$. □

## The case of $\mathrm{F}_{4(-20)}$

**Lemma 5.2.10.** *For $G = \mathrm{F}_{4(-20)}$ we have*

$$\mathbf{Ex} = -(\rho + (2\mathbb{N}_0 - 6)\alpha) = -(5 + 2\mathbb{N}_0)\alpha.$$

*Proof.* Since $\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha)$ and $m_{2\alpha} = 7$ we obtain

$$\begin{aligned}-(\rho + (2\mathbb{N}_0 - 6)\alpha) &= -\left(\frac{m_\alpha}{2} + 1 + 2\mathbb{N}_0\right)\alpha \\ &= -\left(\frac{m_\alpha}{2} + 1 + 2\mathbb{N}_0\right)\alpha \cup -\left(\frac{m_\alpha}{2} + 7 + 2\mathbb{N}_0\right)\alpha = \mathbf{Ex}.\end{aligned}$$ □

*Comparison to Theorem 3.2.2.* Let $\mu \in \mathfrak{a}^*$ be such that $H_\mu$ is reducible.

Case $\mu(H) = \rho(H) - 6 + 2k,\ k \in \mathbb{N}_{\geq 3}$ : The non-trivial subspaces are $W_k$ and $M_k$. Each of these contains the trivial representation $V_{0,0}$. By Proposition 5.1.2, $P_\mu$ is injective.

Case $\mu(H) = -(\rho(H) - 6 + 2k),\ k \in \mathbb{N}_0$ : $\widetilde{M}_k$ is an invariant subspace of $H_\mu$ which does not contain $V_{0,0}$. Thus, $P_\mu$ is not injective in this case.

Case $\mu(H) = \rho(H) - 6 + 2k,\ k \in \{0, 1, 2\}$ : $M_k$ is the only invariant subspace of $H_\mu$. Since it contains the trivial representation, $P_\mu$ is injective in this case.

Altogether, $P_\mu$ is not injective if and only if (recall $\alpha(H) = 1$)

$$\mu \in -(\rho - 6\alpha + 2\mathbb{N}_0\alpha) = \mathbf{Ex}.$$ □

#### Minimal $K$-types of irreducible subrepresentations

Recall the positive system from Section A.4 and the associated half sum of positive roots from Equation (A.4.16)

$$\rho_c = \frac{7}{2}e_1 + \frac{5}{2}e_2 + \frac{3}{2}e_3 + \frac{1}{2}e_4.$$

By Section A.4 the $e_i$ are orthogonal and have the same length w.r.t. the Killing form. We again rescale the Killing form such that the $e_i$ are orthonormal and denote the resulting form by $\tilde{\kappa}$.

**Lemma 5.2.11.** *Let $U \subseteq \hat{K}_M = \{V_{m,\ell} \colon m, \ell \in \mathbb{N}_0,\ m \geq \ell,\ m \equiv \ell \bmod 2\}$ be a set of $M$-spherical $K$-types. Then the Vogan norm is minimal for $V_{m,\ell} \in U$ if and only if $(m+7)^2 + (\ell+5)^2 + (\ell+3)^2 + (\ell+1)^2$ is minimal for $V_{m,\ell} \in U$.*

*Proof.* By Section A.4, the highest weight of a $K$-type $V_{m,\ell}$ is given by $\frac{m}{2}e_1 + \frac{\ell}{2}e_2 + \frac{\ell}{2}e_3 + \frac{\ell}{2}e_4$. We have that

$$\begin{aligned}&\tilde{\kappa}\left(\frac{m}{2}e_1 + \frac{\ell}{2}e_2 + \frac{\ell}{2}e_3 + \frac{\ell}{2}e_4 + 2\rho_c,\ \frac{m}{2}e_1 + \frac{\ell}{2}e_2 + \frac{\ell}{2}e_3 + \frac{\ell}{2}e_4 + 2\rho_c\right) \\ &= \tilde{\kappa}\left(\frac{m+7}{2}e_1 + \frac{\ell+5}{2}e_2 + \frac{\ell+3}{2}e_3 + \frac{\ell+1}{2}e_4,\ \frac{m+7}{2}e_1 + \frac{\ell+5}{2}e_2 + \frac{\ell+3}{2}e_3 + \frac{\ell+1}{2}e_4\right) \\ &= \left(\frac{m+7}{2}\right)^2 + \left(\frac{\ell+5}{2}\right)^2 + \left(\frac{\ell+3}{2}\right)^2 + \left(\frac{\ell+1}{2}\right)^2.\end{aligned}$$ □

**Proposition 5.2.12.** *Let $k \in \mathbb{N}_0$. Then the irreducible subrepresentations $W_k$ (for $k \geq 3$), $M_k$ (for $k \leq 2$) and $\widetilde{M}_k$ from Lemma B.5.1 have unique minimal $K$-types which are given by $V_{0,0}$ resp. $V_{2k+2,0}$ (where $V_{0,0}$ is the trivial $K$-representation).*

*Proof.* By definition of the representations, the occurring $K$-types are (with $m, \ell \in \mathbb{N}_0,\ m \geq \ell,\ m \equiv \ell \bmod 2$)

$$A(W_k) := \{V_{m,\ell} \colon m + \ell \leq 2k - 6\}, \quad A(M_k) := \{V_{m,\ell} \colon m - \ell \leq 2k\},$$
$$A(\widetilde{M_k}) := \{V_{m,\ell} \colon m - \ell > 2k\}.$$

By Lemma 5.2.11 it suffices to minimize the quantity $(m+7)^2+(\ell+5)^2+(\ell+3)^2+(\ell+1)^2$. Choosing $m = \ell = 0$ resp. $m = 2k + 2$, $\ell = 0$ minimizes $m$ and $\ell$ simultaneously so that the minima are attained at these and only these points. ☐

*Proof of Theorem 5.1.6.* If $\mu \notin \mathbf{Ex}$, the spherical principal series $H_\mu$ is irreducible or the unique irreducible subrepresentation of $H_\mu$ is given by $W_k$ (if $\mu(H) \geq \rho(H)$) or by $M_k$ (if $\mu(H) < \rho(H)$) by Lemma B.5.1. In every case, the unique minimal $K$-type is given by the trivial representation by Proposition 5.2.12.

Now let $\mu \in \mathbf{Ex}$. Then, by Lemma B.5.1, $\widetilde{M_k}$ is the only irreducible subspace of $H_\mu$. By Proposition 5.2.12, the minimal $K$-type of $\widetilde{M_k}$ is uniquely given by $V_{2(k+1),0}$. ☐

## 5.3. The role of generalized gradients

In this section we use generalized gradients to connect different Poisson transforms associated with inequivalent $K$-representations for $G$ of rank one (Proposition 5.3.2). This connection is one of the main ingredients for the Fourier characterization we prove in Chapter 7, which will allow us to characterize the images of the minimal $K$-type Poisson transforms. We first introduce some notation.

**Notation 5.3.1.** Recall the inner product $\langle \cdot, \cdot \rangle = -\frac{\kappa(\cdot, \theta \cdot)}{\kappa(H,H)}$ on $\mathfrak{g}_0$ from Equation (1.1) and extend it to $\mathfrak{g}$ using complex linearity. We identify

$$\mathbf{I} : \mathfrak{p} \to \mathfrak{p}^*, \quad X \mapsto \langle X, \cdot \rangle.$$

If $X_1, \ldots, X_{\dim \mathfrak{p}}$ is a basis of $\mathfrak{p}$ we denote the dual basis with respect to $\langle \cdot, \cdot \rangle$ by $\tilde{X}_1, \ldots, \tilde{X}_{\dim \mathfrak{p}}$, i.e.

$$\mathbf{I}(\tilde{X}_i)(X_j) = \langle \tilde{X}_i, X_j \rangle = \delta_{ij}.$$

We now relate Poisson transforms using generalized gradients.

**Proposition 5.3.2.** *For $Y \in \hat{K}_M$ let $\mathrm{d}_V^Y := T_V^Y \circ \nabla$ with $T_V^Y \in \mathrm{Hom}_K(Y \otimes \mathfrak{p}^*, V)$, where $V \leq L^2(K)$ denotes an irreducible subrepresentation of $Y \otimes \mathfrak{p}^*$, be a generalized gradient and $\mu \in \mathfrak{a}^*$. Choose a basis $X_1, \ldots, X_{\dim \mathfrak{p}}$ of $\mathfrak{p}_0$ such that $X_1 \in \mathfrak{a}$ and $X_j \in \mathfrak{k} \oplus \mathfrak{n}$ (e.g. an orthonormal basis of $\mathfrak{p}$ with $X_1 \in \mathfrak{a}$). Let*

$$p_{Y,\mu} := (\mu + \rho)(X_1)\phi_Y \otimes \mathbf{I}(\tilde{X}_1) - \sum_{j=2}^{\dim \mathfrak{p}} \ell(k_I(X_j))\phi_Y \otimes \mathbf{I}(\tilde{X}_j) \in Y \otimes \mathfrak{p}^*,$$

*where $k_I(X_j) \in \mathfrak{k}$ denotes the $\mathfrak{k}$-component in the $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$-decomposition of $X_j$. Then* *i)* $p_{Y,\mu}$ *is independent of the basis and $M$-invariant,*

*ii)* $\mathrm{d}_V^Y \circ P_\mu^Y = T_V^Y(p_{Y,\mu})(e)P_\mu^V$ *if $V$ is $M$-spherical, i.e.* $V \leq L^2(K)^M$,

*iii)* $\mathrm{d}_V^Y \circ P_\mu^Y = 0$ *if $V$ is not $M$-spherical, i.e.* $V^M = 0$.

*Proof.* i) Identifying

$$Y \otimes \mathfrak{p}^* \cong \mathrm{Hom}(\mathfrak{p}, Y), \quad f \otimes \lambda \mapsto (X \mapsto \lambda(X)f),$$

the tensor $p_{Y,\mu}$ corresponds to the homomorphism given by

$$\begin{aligned} p_{Y,\mu}(X) &= (\mu+\rho)(X)\phi_Y \quad && \forall X \in \mathfrak{a}, \\ p_{Y,\mu}(X) &= \ell(k_I(X))\phi_Y && \forall X \in \mathfrak{p} \cap (\mathfrak{k} \oplus \mathfrak{n}), \end{aligned}$$

which is independent of the basis. For the $M$-invariance note first that the $K$-action on $\mathrm{Hom}(\mathfrak{p}, Y)$ is given by

$$(k.\Phi)(X) = k.\Phi(k^{-1}.X) = L(k)\Phi(\mathrm{Ad}(k^{-1})X), \quad X \in \mathfrak{p},\ \Phi \in \mathrm{Hom}(\mathfrak{p}, Y).$$

Since $M$ stabilizes $\mathfrak{a}$ and $\phi_Y$ is $M$-invariant we have for each $X \in \mathfrak{a}$,

$$\begin{aligned} (m.p_{Y,\mu})(X) &= L(m)p_{Y,\mu}(\mathrm{Ad}(m^{-1})X) = L(m)p_{Y,\mu}(X) \\ &= (\mu+\rho)(X)L(m)\phi_Y = (\mu+\rho)(X)\phi_Y = p_{Y,\mu}(X). \end{aligned}$$

Moreover, since $M$ leaves $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ invariant, we have for each $X \in \mathfrak{p} \cap (\mathfrak{k} \oplus \mathfrak{n})$,

$$\begin{aligned} (m.p_{Y,\mu})(X) &= L(m)p_{Y,\mu}(\mathrm{Ad}(m^{-1})X) = L(m)\ell(k_I(\mathrm{Ad}(m^{-1})X))\phi_Y \\ &= L(m)\ell(\mathrm{Ad}(m^{-1})k_I(X))\phi_Y \\ &= L(m)L(m^{-1})\ell(k_I(X))L(m)\phi_Y \\ &= \ell(k_I(X))\phi_Y = p_{Y,\mu}(X). \end{aligned}$$

This proves the first part.

ii), iii) Let $\delta_{eM}$ denote the Delta distribution at $eM$ on $K/M$. Then

$$P_\mu^Y(\delta_{eM})(g) = a_I(g^{-1})^{-(\mu+\rho)}\tau(k_I(g^{-1}))\phi_Y \in C^\infty(G \times_K Y). \tag{5.2}$$

We first obtain

$$\begin{aligned} (\nabla \circ P_\mu^Y(\delta_{eM}))(e)(X_1) &= \frac{d}{dt}\Big|_{t=0} P_\mu^Y(\delta_{eM})(\exp tX_1) \\ &= \frac{d}{dt}\Big|_{t=0} a_I(\exp -tX_1)^{-(\mu+\rho)}\phi_Y \\ &= \frac{d}{dt}\Big|_{t=0} e^{t(\mu+\rho)(X_1)}\phi_Y = (\mu+\rho)(X_1)\phi_Y. \end{aligned}$$

For $j \in \{2, \ldots, \dim \mathfrak{p}\}$ we write $X_j = k_I(X_j) + n_I(X_j) \in \mathfrak{k}_0 \oplus \mathfrak{n}_0$ and obtain

$$\begin{aligned}(\nabla \circ P_\mu^Y(\delta_{eM}))(e)(X_j) &= (\nabla \circ P_\mu^Y(\delta_{eM}))(e)(k_I(X_j)) + (\nabla \circ P_\mu^Y(\delta_{eM}))(e)(n_I(X_j)) \\ &= (\nabla \circ P_\mu^Y(\delta_{eM}))(e)(k_I(X_j)) \\ &= \frac{d}{dt}\Big|_{t=0} \tau(\exp -tk_I(X_j))\phi_Y = -\ell(k_I(X_j))\phi_Y,\end{aligned}$$

where we used in the second step that $P_\mu^Y(\delta_{eM})(n) = \phi_Y$ for $n \in N$ by (5.2). Thus,

$$(\nabla \circ P_\mu^Y(\delta_{eM}))(e) = (\mu+\rho)(X_1)\phi_Y \otimes \mathbf{I}(\tilde{X}_1) - \sum_{j=2}^{\dim \mathfrak{p}} \ell(k_I(X_j))\phi_Y \otimes \mathbf{I}(\tilde{X}_j)$$

and therefore

$$\begin{aligned}(\mathrm{d}_V^Y \circ P_\mu^Y(\delta_{eM}))(e) &= T_V^Y((\nabla \circ P_\mu^Y(\delta_{eM}))(e)) \\ &= T_V^Y\left((\mu+\rho)(X_1)\phi_Y \otimes \mathbf{I}(\tilde{X}_1) - \sum_{j=2}^{\dim \mathfrak{p}} \ell(k_I(X_j))\phi_Y \otimes \mathbf{I}(\tilde{X}_j)\right).\end{aligned}$$

By Corollary 3.3.4 and 3.3.5, $\mathrm{d}_V^Y \circ P_\mu^Y$ has to be a multiple of $P_\mu^V$ if $V$ is $M$-spherical and 0 otherwise. In particular, we deduce that

$$T_V^Y\left((\mu+\rho)(X_1)\phi_Y \otimes \mathbf{I}(\tilde{X}_1) - \sum_{j=2}^{\dim \mathfrak{p}} \ell(k_I(X_j))\phi_Y \otimes \mathbf{I}(\tilde{X}_j)\right)$$

is a multiple of $P_\mu^V(\delta_{eM})(e) = \phi_V$. Since $\phi_V(e) = 1$ this multiple is given by

$$T_V^Y\left((\mu+\rho)(X_1)\phi_Y \otimes \mathbf{I}(\tilde{X}_1) - \sum_{j=2}^{\dim \mathfrak{p}} \ell(k_I(X_j))\phi_Y \otimes \mathbf{I}(\tilde{X}_j)\right)(e). \qquad \square$$

## 5.4. Želobenko operators

As the minimal $K$-type Poisson transforms appear in the exceptional sets from [Olb94] (see Remark 5.1.7), we consider these sets in some more detail. Their existence is closely related to so-called *Želobenko operators* or *discrete symmetry operators* (see [Olb94, Definition 4.7]). Like the Knapp-Stein intertwiners, these are non-trivial intertwiners between different principal series representations, where, in contrast to the Knapp-Stein intertwiners for the spherical principal series, the associated $M$-representation may change. Moreover, they are induced by the "big" Weyl group $W(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{g}$, instead of the real Weyl group $W(\mathfrak{g}, \mathfrak{a})$. These operators can be used to relate the minimal $K$-type Poisson transforms to Poisson transforms of the intertwined representations. Unfortunately, we do not leave the exceptional sets from [Olb94] so that these relations do not allow us to describe the images. Therefore, the results of this section will have no further application in the following chapters and are intended for completeness only. We define the Želobenko operators as in [Ž76] and determine the principal series representations they relate to our exceptional ones for the classical rank one groups.

We fix a $\theta$-stable Cartan subalgebra $\mathfrak{t}_0$ in $\mathfrak{m}_0$ and let $\mathfrak{h}_0$ denote the Cartan subalgebra $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ of $\mathfrak{g}_0$. We order the nonzero roots $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$ of $(\mathfrak{g}, \mathfrak{h})$ in such a way that we have $\{\alpha|_{\mathfrak{a}_0} \mid \alpha \in \Delta^+\} = \Sigma^+$ for the set $\Delta^+$ of positive roots in $\Delta$, denote the sum of all positive/negative root spaces by $\mathfrak{u}^\pm$ and the half sum of positive roots by $\delta$.

Setting $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{u}^+$, the *Verma module* $\mathcal{M}(\Lambda)$ of some $\Lambda \in \mathfrak{h}^*$ is defined by

$$\mathcal{M}(\Lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\Lambda-\delta},$$

where $\mathbb{C}_{\Lambda-\delta}$ denotes the (one dimensional) $\mathfrak{b}$-representation in which $\mathfrak{h}$ acts by $\Lambda - \delta$ and $\mathfrak{u}^+$ acts trivially.

We fix a $W(\mathfrak{g}, \mathfrak{h})$-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{h}$. Let $\beta \in \Delta^+$ and suppose that $2\frac{(\Lambda,\beta)}{(\beta,\beta)} =: N \in \mathbb{N}$. According to [Ž76, Definition 3.1], there exists an element $\Omega_{\beta,N} \in \mathcal{U}(\mathfrak{u}^-)$ and a non-trivial, $\mathfrak{g}$-equivariant map

$$S_\beta(\Lambda)\colon \mathcal{M}(\Lambda - N\beta) \to \mathcal{M}(\Lambda), \qquad S_\beta(\Lambda)(u \otimes z) := u\Omega_{\beta,N} \otimes z.$$

Moreover, both $S_\beta(\Lambda)$ and $\Omega_{\beta,N}$ are uniquely defined up to a constant.

**Definition 5.4.1** (cf. [Ž76, §3])**.** Let $\beta \in \Delta^+, N \in \mathbb{N}$ and $\mathcal{P}_\beta^N$ denote the set of all parameters $(\sigma, \mu) \in \hat{M} \times \mathfrak{a}^*$ such that

- $2\frac{\langle \mathrm{hw}(\tilde{\sigma})+\rho_\mathfrak{m}+\mu,\beta\rangle}{\langle\beta,\beta\rangle} = N$, where $\mathrm{hw}(\tilde{\sigma}) \in i\mathfrak{t}_0^* \subseteq \mathfrak{h}^*$ denotes the highest weight of $\tilde{\sigma}$,
- $\mathrm{hw}(\tilde{\sigma}) - N\beta|_{\mathfrak{t}_0}$ is the highest weight of an $M$-representation.

Write $M = M_0 Z$, where $M_0$ denotes the analytic subgroup of $\mathfrak{m}_0$ and $Z$ is a finite abelian subgroup which is central in $M$ and generated by elements of order two ([Ž76, p. 1006]). For each $(\sigma, \mu) \in \mathcal{P}_\beta^N$ let $(\sigma_\beta, \mu_\beta) \in \hat{M} \times \mathfrak{a}^*$ be defined by

i) $\sigma_\beta|_{M_0}$ is dual to the $M_0$-representation of highest weight $\mathrm{hw}(\tilde{\sigma}) - N\beta|_{\mathfrak{t}_0}$ and $Z$ acts through $\sigma$,

ii) $\mu_\beta := \mu - N\beta|_\mathfrak{a}$.

Then there exists a unique $G$-equivariant map $L_{\beta,\sigma,\mu}\colon H^\infty_{\sigma,\mu} \to H^\infty_{\sigma_\beta,\mu_\beta}$ such that

$$\langle v_{\tilde{\sigma}_\beta}, L_{\beta,\sigma,\mu} f(e)\rangle = \langle v_{\tilde{\sigma}}, r(\Omega_{\beta,N}) f(e)\rangle,$$

where $v_{\tilde{\sigma}_\beta}$ resp. $v_{\tilde{\sigma}}$ denote highest weight vectors of $\tilde{\sigma}_\beta$ resp. $\tilde{\sigma}$, $r$ denotes the right regular representation and $\langle\cdot,\cdot\rangle$ is the natural pairing. We call $L_{\beta,\sigma,\mu}$ a *discrete symmetry operator* or *Želobenko operator* of $H^\infty_{\sigma,\mu}$.

*Proof.* See [Ž76, Proposition 3.2] with $\mathrm{hw}(\tilde{\sigma}) + \rho_\mathfrak{m} + \mu \in \Sigma_\beta^N$ and [Ž76, Definition 3.1] for the definition of $L_{\beta,\sigma,\mu}$ (which is called $S_\beta(\mathrm{hw}(\tilde{\sigma}) + \rho_\mathfrak{m} + \mu)$ there). □

Using the Frobenius isomorphism from Definition 3.3.1 we define

$$l_{\beta,\sigma,\mu}\colon \operatorname{Hom}_M(V_{\sigma_\beta}, V_\tau) \to \operatorname{Hom}_M(V_\sigma, V_\tau), \quad t \mapsto F(F^{-1}(t) \circ L_{\beta,\sigma,\mu}),$$

where $V_\diamond$ denotes a finite dimensional vector space assigned to the highest weight $\diamond$. From [Olb94, Satz 3.16] we cite the following result relating Poisson transforms.

**Theorem 5.4.2** (cf. [Olb94, Satz 3.16]). *Let $\beta \in \Sigma^+$ and $(\sigma, \mu) \in \mathcal{P}_\beta^N$ for some $N \in \mathbb{N}$. Then, for each $t \in \operatorname{Hom}_M(V_{\sigma_\beta}, V_\tau)$ and $f \in H_{\sigma,\mu}^\infty$,*

$$P^\tau_{\sigma_\beta,\mu_\beta}(t \otimes L_{\beta,\sigma,\mu}(f)) = P^\tau_{\sigma,\mu}(l_{\beta,\sigma,\mu}(t) \otimes f).$$

By [Ž76, Corollary 3.4] we have $L_{\beta,\sigma,\mu} = 0$ for each *imaginary root* $\beta$, i.e. for each $\beta \in \Delta^+$ with $\beta|_{\mathfrak{a}} = 0$ (note that there are no noncompact imaginary roots since $\mathfrak{h}_0$ is maximally noncompact, see [Kna02, Proposition 6.70]). Therefore, we only consider roots $\beta \in \Delta^+$ such that $\beta|_{\mathfrak{a}} \neq 0$. Moreover, since the maps appearing in Theorem 5.4.2 otherwise vanish, we are particularly interested in $K$-types $\tau$ such that $\sigma$ and $\sigma_\beta$ both occur in $\tau|_M$. The following proposition investigates Želobenko operators for classical rank one groups in the case of exceptional parameters (note that our minimal $K$-types are always in $U$).

**Proposition 5.4.3.** *Let $\mu_\ell \in \mathbf{Ex}$ be an exceptional parameter. Then, for $\mu_\ell \leq -\rho$, there always exist parameters $(\sigma, \mu) \in \hat{M} \times \mathfrak{a}^*$ such that there exists a Želobenko operator $Z_{\mu_\ell} := L_{\beta,\sigma,\mu}$ of $H^\infty_{\sigma,\mu}$ mapping into $H^\infty_{\mu_\ell}$ where $\beta$ is a complex root, i.e. neither an imaginary nor a real root (i.e. $\beta|_{\mathfrak{t}_0} \neq 0$). The following table lists all these parameters. Here, the highest weight of the $M$-representations are denoted as in [Bal79, Lemmas 4.3, 5.3] for $G \in \{\mathrm{SU}(n,1), \mathrm{Sp}(n,1)\}$ and as in Appendix A for $G = \mathrm{SO}_0(n,1)$ (then $M \cong \mathrm{SO}(n-1)$). Moreover, the common $K$-types of $H_{\sigma,\mu}$ and $H_{\mu_\ell}$ determine a $(\mathfrak{g}, K)$-submodule $U$ of $H_{\mu_\ell}$. For $\mathrm{SL}(2,\mathbb{R})$ complex roots do not exist so that there is no such operator in this case.*

| | $\mu_\beta = \mu_\ell \in \mathbf{Ex}$ | $\sigma$ | $\mu$ | $U$ |
|---|---|---|---|---|
| $\mathrm{SO}_0(n,1)$ | $-\rho - \ell\alpha$ | $(\ell+1)e_1$ | $\alpha - \rho$ | $W_\ell$ |
| $\mathrm{SU}(n,1)$ | $-\rho - 2\ell\alpha$ | $\frac{\ell+1}{2}(\overline{\overline{\varepsilon}}_1 + \overline{\overline{\varepsilon}}_{n+1}) - (\ell+1)\overline{\overline{\varepsilon}}_n$<br>$-\frac{\ell+1}{2}(\overline{\overline{\varepsilon}}_1 + \overline{\overline{\varepsilon}}_{n+1}) + (\ell+1)\overline{\overline{\varepsilon}}_2$ | $-\rho - (\ell-1)\alpha$ | $W_{\ell,-}$<br>$W_{\ell,+}$ |
| $\mathrm{Sp}(n,1)$ | $-\rho - (2\ell-2)\alpha$ | $\ell(e_3 + \frac{e_1-e_2}{2})$ | $-\rho - (\ell-2)\alpha$ | $\widetilde{W}_\ell$ |

*Proof.* Let us first consider the case of $G = \mathrm{SO}_0(n,1)$, $n \geq 3$. In the notation of Definition 5.4.1 we want to have $\sigma_\beta = 0$ and $\mu_\beta = \mu_\ell = -\rho - \ell\alpha$ for some $\beta \in \Delta^+$. Since $0 = \sigma_\beta|_{M_0}$ is dual to the $M_0$-representation of highest weight $\mathrm{hw}(\tilde{\sigma}) - N\beta|_{\mathfrak{t}_0}$ we must have $\mathrm{hw}(\tilde{\sigma}) = N\beta|_{\mathfrak{t}_0}$. By the dominance of $\mathrm{hw}(\tilde{\sigma})$ we deduce that $\beta|_{\mathfrak{t}_0}$ has to be dominant. However, the only positive complex root with this property is given by $e_0 + e_1$, where $e_0$ denotes the real root $\alpha$ extended trivially to $\mathfrak{h}$ and $e_1$ is as in Appendix A (extended to $\mathfrak{h}$). Moreover, since $\mu_\beta = \mu - N\beta|_{\mathfrak{a}}$, we infer that

$$\mu = N\beta|_{\mathfrak{a}} + \mu_\beta = N(e_0 + e_1)|_{\mathfrak{a}} - \rho - \ell\alpha = (N - \ell)\alpha - \rho.$$

Furthermore,

$$N = 2\frac{\langle \mathrm{hw}(\tilde\sigma) + \rho_{\mathfrak{m}} + \mu, \beta\rangle}{\langle \beta,\beta\rangle} = 2\frac{\langle N\beta|_{\mathfrak{t}_0} + \rho_{\mathfrak{m}} + N\beta|_{\mathfrak{a}} + \mu_\ell, \beta\rangle}{\langle \beta,\beta\rangle} = 2N + 2\frac{\langle \rho_{\mathfrak{m}} + \mu_\ell, \beta\rangle}{\langle \beta,\beta\rangle}$$

implies that ($\rho_{\mathfrak{m}} = \frac{n-3}{2}e_1 + \ldots + \frac{n-(2m-1)}{2}e_{m-1}$ with $m = \lfloor \frac{n}{2}\rfloor$ and $\rho = \frac{n-1}{2}\alpha$)

$$N = -2\frac{\langle \rho_{\mathfrak{m}} + \mu_\ell, \beta\rangle}{\langle \beta,\beta\rangle} = -\langle \rho_{\mathfrak{m}} + \mu_\ell, \beta\rangle = \ell + 1.$$

Thus, $\mathrm{hw}(\sigma) = (\ell+1)e_1$ and $\mu = \alpha - \rho$. Finally, we see that $(\sigma,\mu) \in \mathcal{P}^{\ell+1}_{e_1+e_2}$ and thus that there exists a Želobenko operator of $H^\infty_{\sigma,\mu}$ mapping into $H^\infty_{\mu_\ell}$.

In the case of $G = \mathrm{SU}(n,1),\ n \geq 2$, we use [Bal79, §4] to describe the roots. We use the same notation as in that paper but add a 'B' to the index of the Cartan subalgebras. With respect to her maximally compact Cartan subalgebra $\mathfrak{h}_B$ (diagonal matrices), the roots of $\mathfrak{g}$ are given by $\pm(\overline{\varepsilon}_i - \overline{\varepsilon}_j),\ 1 \leq i < j \leq n+1$. We use a Cayley transform associated to the imaginary noncompact root $\beta := \overline{\varepsilon}_1 - \overline{\varepsilon}_{n+1}$ to obtain a maximally noncompact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. More precisely, in the notation of [Kna02, Chapter VI, §7] we choose $E_\beta = E_{1,n+1}$ so that $\overline{E_\beta} = E_{n+1,1}$, where $E_{i,j}$ denotes the matrix which is 1 in the $(i,j)$ entry and zero elsewhere. Then we have the Cayley transform

$$c_\beta := \mathrm{Ad}(\exp\frac{\pi}{4}(\overline{E_\beta} - E_\beta)) = \mathrm{Ad}(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & & -1 \\ & \sqrt{2} & & & \\ & & \ddots & & \\ & & & \sqrt{2} & \\ 1 & & & & 1 \end{pmatrix})$$

and the (real form of the) new Cartan subalgebra $\mathfrak{h}$ is given by

$$\mathfrak{h}_0 := \mathfrak{g}_0 \cap c_\beta(\mathfrak{h}_B) = \mathfrak{h}_B^- \oplus \mathbb{R}(E_\beta + \overline{E_\beta})$$

with $H = E_\beta + \overline{E_\beta},\ \mathbb{R}H = \mathfrak{a}_0$. Let $e_j := \overline{\varepsilon}_j \circ c_\beta^{-1} \in \mathfrak{h}^*$. Then the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are given by $\pm(e_i - e_j),\ 1 \leq i < j \leq n+1$ and there are two positive complex roots such that the restriction to $\mathfrak{m}$ is dominant, given by $e_2 - e_{n+1}$ and $e_1 - e_n$.

We first consider $e_2 - e_{n+1}$. Note that the half sum of positive roots is given by

$$\begin{aligned}\rho(H)\frac{e_1 - e_{n+1}}{2} + \rho_{\mathfrak{m}} &= \rho(H)\frac{e_1 - e_{n+1}}{2} + \frac{n-2}{2}e_2 + \frac{n-4}{2}e_3 + \ldots - \frac{n-2}{2}e_n \\ &= \frac{n}{2}e_1 + \frac{n-2}{2}e_2 + \frac{n-4}{2}e_3 + \ldots - \frac{n-2}{2}e_n - \frac{n}{2}e_{n+1}.\end{aligned}$$

As in the real case we calculate

$$\begin{aligned}N &= -2\frac{\langle \rho_{\mathfrak{m}} + \mu_\ell, e_2 - e_{n+1}\rangle}{\langle e_2 - e_{n+1}, e_2 - e_{n+1}\rangle} \\ &= -\langle \frac{n-2}{2}e_2 - (n+2\ell)\frac{e_1 - e_{n+1}}{2}, e_2 - e_{n+1}\rangle \\ &= \langle (\frac{n}{2} + \ell)(e_1 - e_{n+1}) - \frac{n-2}{2}e_2, e_2 - e_{n+1}\rangle = \ell + 1\end{aligned}$$

which is always in $\mathbb{N}$. We infer that $\tilde{\sigma}$ has highest weight $N(e_2 - e_{n+1})|_{\mathfrak{h}_B^-} = (\ell+1)\overline{\overline{\varepsilon}}_2 - \frac{\ell+1}{2}(\overline{\overline{\varepsilon}}_1 + \overline{\overline{\varepsilon}}_{n+1}) \in D_M$ in the notation of [Bal79, Lemma 4.3] so that its dual representation $\sigma$ corresponds to $\frac{\ell+1}{2}(\overline{\overline{\varepsilon}}_1 + \overline{\overline{\varepsilon}}_{n+1}) - (\ell+1)\overline{\overline{\varepsilon}}_n$. Moreover we obtain

$$\mu = \mu_\ell + (\ell+1)(e_2 - e_{n+1})|_{\mathfrak{a}} = -\rho - 2\ell\alpha + (\ell+1)\alpha = -\rho - (\ell-1)\alpha.$$

Let us finally determine the joint $K$-types of $H_{\sigma,\mu}$ and $H_{\mu_\ell}$. The $K$-types of the latter are the $Y_{p,q}$ with highest weight $p\overline{\varepsilon}_1 - q\overline{\varepsilon}_2 + (q-p)\overline{\varepsilon}_{n+1}$. We use [Bal79, Theorem 4.4] to determine which of these are also $K$-types of $H_{\sigma,\mu}$. In that notation we have $b_0 = \frac{\ell+1}{2}$, $b_n = -(\ell+1)$ and $b_i = 0$ for $2 \le i < n$. Moreover, $a_1 = p$, $a_n = -q$ and $a_{n+1} = q - p$. Then we have $b_1 = a_1 + a_n + (\ell+1)$ and thus

$$b_0 = \frac{\ell+1}{2} = \frac{a_1 + a_n + a_{n+1} + \ell + 1}{2} = \frac{b_1 + a_{n+1}}{2}$$

since $a_1 + a_n + a_{n+1} = 0$. We obtain that $Y_{p,q}$ occurs in $H_{\sigma,\mu}$ if and only if $b_n = -(\ell+1) \ge a_n \Leftrightarrow q \ge \ell + 1$.

In the case of $e_1 - e_n$ the calculations are similar. We obtain

$$\begin{aligned} N &= -2\frac{\langle \rho_{\mathfrak{m}} + \mu_\ell, e_1 - e_n\rangle}{\langle e_1 - e_n, e_1 - e_n\rangle} \\ &= -\langle -\frac{n-2}{2}e_n - (n+2\ell)\frac{e_1 - e_{n+1}}{2}, e_1 - e_n\rangle \\ &= \langle (\frac{n}{2} + \ell)(e_1 - e_{n+1}) + \frac{n-2}{2}e_n, e_1 - e_n\rangle = \ell + 1. \end{aligned}$$

In the notation of [Bal79, Lemma 4.3], $\sigma$ corresponds to $(\ell+1)\overline{\overline{\varepsilon}}_2 - \frac{\ell+1}{2}(\overline{\overline{\varepsilon}}_1 + \overline{\overline{\varepsilon}}_{n+1}) \in D_M$ and $\mu = -\rho - (\ell-1)\alpha$. By the branching rule [Bal79, Theorem 4.4] we obtain that the joint $K$-types of $H_{\sigma,\mu}$ and $H_{\mu_\ell}$ are given by the $Y_{p,q}$ with $p \ge \ell + 1$.

Let us now turn to the case of $\mathrm{Sp}(n,1)$, $n \ge 2$. Again we use the notation from [Bal79, §5] resp. [BSK80]. This time the Cayley transform we use and the roots are described in [BSK80, §1], the latter given by

$$\Phi := \{\pm e_i \pm e_j \mid 1 \le i < j \le n+1\} \cup \{\pm 2e_i \mid 1 \le i \le n+1\}.$$

The real roots are $\pm(e_1 + e_2)$ and the unique positive complex root with dominant restriction to $\mathfrak{m}$ is given by $e_1 + e_3$. We have $\rho_{\mathfrak{m}} = \frac{1}{2}(e_1 - e_2) + (n-1)e_3 + \ldots + e_{n+1}$, $\mu_\ell = -(2n + 2\ell - 1)\frac{e_1+e_2}{2}$ and calculate

$$\begin{aligned} N &= -\langle \rho_{\mathfrak{m}} + \mu_\ell, \beta\rangle \\ &= -\langle \frac{1}{2}(e_1 - e_2) + (n-1)e_3 + \ldots + e_{n+1} - (n + \ell - \frac{1}{2})(e_1 + e_2), e_1 + e_3\rangle \\ &= \langle (n + \ell - 1)e_1 - (n-1)e_3, e_1 + e_3\rangle = \ell, \end{aligned}$$

which is in $\mathbb{N}$ if and only if $\mu_\ell \le -(2n+1)\alpha = -\rho$. The highest weight of $\sigma$ is given by $\ell(e_3 + \frac{e_1 - e_2}{2})$ and $\mu = \ell\alpha - (2n + 2\ell - 1)\alpha = -(2n + \ell - 1)\alpha$. For the joint $K$-types of $H_{\sigma,\mu}$ and $H_{\mu_\ell}$ we use the branching rule [Bal79, Theorem 5.5]. In that notation we have $b_0 = \frac{\ell}{2}$ and obtain the restriction $a_1 \ge b_2 = \ell$ so that the joint $K$-types are given by $V_{a,b}$, with highest weight $a\varepsilon_1 + b\varepsilon_2 + (a-b)\varepsilon_{n+1}$, where $a \ge \ell$. □

# 6. $\Gamma$-invariant elements

In this chapter we investigate which principal series representations admit $\Gamma$-invariant distributional elements and, if the representation is reducible, in which composition factors they can occur. As these turn out to be given by the socle, it then suffices to determine the images of the injective minimal $K$-type Poisson transforms from Proposition 5.1.3 and Theorem 5.1.6 restricted to the socle to obtain spectral correspondences (recall Proposition 2.2.3). In rank one we investigate the socle in more detail. In particular, we obtain an interesting relationship between the socles of the principal series attached to the exceptional parameters and the representations of the relative discrete series of the associated pseudo-Riemannian symmetric spaces in Theorem 6.2.3. We do not have to assume that the co-compact lattice $\Gamma \leq G$ is torsion free in this chapter.

## 6.1. Location of $\Gamma$-invariant elements

Let $G$ be as in Section 1.1 (i.e. not necessarily of rank one).

**Theorem 6.1.1** (Location of **$\Gamma$**-invariant elements)**.** *Let $\sigma \in \hat{M}$ and $\mu \in \mathfrak{a}^*$. Assume that the socle of $H_{\sigma,\mu}$ decomposes multiplicity-freely. Then*

$$^{\Gamma}H_{\sigma,\mu}^{-\infty} \cong {}^{\Gamma}(\operatorname{soc} H_{\sigma,\mu})^{-\infty} = \bigoplus_{V \leq H_{\sigma,\mu} \text{ irred.}} {}^{\Gamma}V^{-\infty},$$

*where the sum on the right hand side is finite. Moreover, for each irreducible $V \leq H_{\sigma,\mu}$, the existence of $\Gamma$-invariant distributional elements in $V$ implies that $V$ is infinitesimally unitary.*

*Proof.* Note first that $H_{\sigma,\mu}$ has finitely many irreducible subrepresentations by the finite length of $H_{\sigma,\mu}$ and our multiplicity one assumption. We claim that the dual principal series representation $H_{\tilde{\sigma},-\mu}$ has finitely many irreducible quotients. Indeed, let $H_{\tilde{\sigma},-\mu}/V$, for some subrepresentation $V \leq H_{\tilde{\sigma},-\mu}$, denote an irreducible quotient of $H_{\tilde{\sigma},-\mu}$. Then we have that $V^{\perp_{\tilde{\sigma},-\mu}} \leq H_{\sigma,\mu}$ is a subrepresentation (see Equation (2.5) for the notation). Moreover, $V^{\perp_{\tilde{\sigma},-\mu}} \leq H_{\sigma,\mu}$ is the dual representation of $H_{\tilde{\sigma},-\mu}/V$ and therefore irreducible. If $H_{\tilde{\sigma},-\mu}/V_1 \neq H_{\tilde{\sigma},-\mu}/V_2$ are two different irreducible quotients, we obtain two different irreducible subrepresentations $V_1^{\perp_{\tilde{\sigma},-\mu}} \neq V_2^{\perp_{\tilde{\sigma},-\mu}} \leq H_{\sigma,\mu}$ by the non-degeneracy of $\langle \cdot, \cdot \rangle_{\tilde{\sigma},-\mu}$. Since there are only finitely many of the latter, $H_{\tilde{\sigma},-\mu}$ resp. $H_{\tilde{\sigma},-\mu}^{\infty}$ has finitely many irreducible quotients $H_{\tilde{\sigma},-\mu}/V_j,\ j = 1, \ldots, n$ resp. $H_{\tilde{\sigma},-\mu}^{\infty}/V_j^{\infty},\ j = 1, \ldots, n$.

By definition we have that $H_{\sigma,\mu}^{-\infty} = \operatorname{Hom}_{\mathbb{C}}(H_{\tilde{\sigma},-\mu}^{\infty}, \mathbb{C})$ is the space of continuous linear maps from $H_{\tilde{\sigma},-\mu}^{\infty}$ to $\mathbb{C}$, equipped with the dual representation of $H_{\tilde{\sigma},-\mu}^{\infty}$. This implies

$$^{\Gamma}H_{\sigma,\mu}^{-\infty} = {}^{\Gamma}\operatorname{Hom}_{\mathbb{C}}(H_{\tilde{\sigma},-\mu}^{\infty}, \mathbb{C}) = \operatorname{Hom}_{\Gamma}(H_{\tilde{\sigma},-\mu}^{\infty}, \mathbb{C}). \tag{6.1}$$

Note that $H^\infty_{\tilde\sigma,-\mu}$ is a nuclear Fréchet space (consider the compact picture and see e.g. [CHM00, §2]) and a differentiable $G$-module. Moreover, $\mathbb{C}$ is a differentiable nuclear $\Gamma$-module. Therefore we may use Frobenius reciprocity to obtain (see [Zuc78, Lemma 1.3])

$$^{\Gamma}H^{-\infty}_{\sigma,\mu} = \mathrm{Hom}_\Gamma(H^\infty_{\tilde\sigma,-\mu},\mathbb{C}) \cong \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}, \mathrm{Ind}_\Gamma^{G,\infty}(\mathbb{C})),$$

where $\mathrm{Ind}_\Gamma^{G,\infty}(\mathbb{C}) \cong C^\infty(\Gamma\backslash G)$ denotes the representation smoothly induced by the trivial representation of $\Gamma$. By [GGPS69, Chapter 1, §2.3], there exists a countable subset $\hat G_\Gamma \subset \hat G$ such that $\mathrm{Ind}_\Gamma^G(\mathbb{C})$ decomposes as a direct sum

$$\mathrm{Ind}_\Gamma^G(\mathbb{C}) \cong \widehat{\bigoplus}_{\pi\in\hat G_\Gamma} m_\Gamma(\pi)\pi,$$

where each multiplicity $m_\Gamma(\pi) \geq 1$ is finite. Therefore, if $0 \neq \varphi \in {}^\Gamma H^{-\infty}_{\sigma,\mu}$ with corresponding $\varphi_F \in \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}, \mathrm{Ind}_\Gamma^{G,\infty}(\mathbb{C}))$, there exists some $\pi \in \hat G_\Gamma$ such that $\mathrm{pr}_\pi \circ \varphi_F \neq 0$, where $\mathrm{pr}_\pi$ denotes the orthogonal projection onto one copy of $\pi$ in $\mathrm{Ind}_\Gamma^G(\mathbb{C})$. Since $\varphi_F$ and $\mathrm{pr}_\pi$ are continuous and linear they are smooth. Therefore, $\mathrm{pr}_\pi \circ \varphi_F$ maps $H^\infty_{\tilde\sigma,-\mu}$ into $\pi^\infty$. By [War72, §4.4, p. 253], $H^\infty_{\tilde\sigma,-\mu}$ and $\pi^\infty$ are smooth Fréchet representations. Therefore, the image of $\mathrm{pr}_\pi \circ \varphi_F$ is closed and a topological summand of $\pi^\infty$ [Wal92, Lemma 11.5.1, Theorem 11.6.7(2)]. Since $\pi$ is irreducible, $\pi^\infty$ is irreducible (see e.g. [War72, p. 254]) and therefore $\mathrm{pr}_\pi \circ \varphi_F$ is surjective. Now [Die70, Theorem 12.16.8] implies that the canonical factorization $H^\infty_{\tilde\sigma,-\mu}/\ker(\mathrm{pr}_\pi \circ \varphi_F) \to \pi^\infty$ is a topological isomorphism. Since $\pi^\infty$ is irreducible, $H^\infty_{\tilde\sigma,-\mu}/\ker(\mathrm{pr}_\pi \circ \varphi_F)$ is irreducible. It follows that $\ker(\mathrm{pr}_\pi \circ \varphi_F) = V_j^\infty$ for some $j \in \{1,\ldots,n\}$. Thus we proved that if $\mathrm{pr}_\pi \circ \varphi_F \neq 0$, then it factors through an irreducible quotient of $H^\infty_{\tilde\sigma,-\mu}$.

Consider the finite set

$$F := \{\pi \in \hat G_\Gamma \mid \exists\, j \in \{1,\ldots,n\} \colon \pi^\infty \cong H^\infty_{\tilde\sigma,-\mu}/V_j^\infty\}.$$

For $\pi \in F$ with $\pi^\infty \cong H^\infty_{\tilde\sigma,-\mu}/V_j^\infty$ we set $j(\pi) := j$. Moreover, let

$$I_\Gamma := \{j \in \{1,\ldots,n\} \mid \exists\, \pi_j := \pi \in F \colon j(\pi) = j\}.$$

Then $\mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}, \mathrm{Ind}_\Gamma^{G,\infty}(\mathbb{C})) \cong \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}, \bigoplus_{\pi\in F} m_\Gamma(\pi)\pi)$ is isomorphic to

$$\begin{aligned}
\bigoplus_{\pi\in F}\bigoplus_{k=1}^{m_\Gamma(\pi)} \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu},\pi) &\cong \bigoplus_{\pi\in F}\bigoplus_{k=1}^{m_\Gamma(\pi)} \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}/V^\infty_{j(\pi)},\pi)\\
&\cong \bigoplus_{\pi\in F} \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}/V^\infty_{j(\pi)}, m_\Gamma(\pi)\pi)\\
&\cong \bigoplus_{j\in I_\Gamma} \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}/V^\infty_j, m_\Gamma(\pi_j)\pi_j)\\
&\cong \bigoplus_{j\in I_\Gamma} \mathrm{Hom}_G(H^\infty_{\tilde\sigma,-\mu}/V^\infty_j, \mathrm{Ind}_\Gamma^{G,\infty}(\mathbb{C}))\\
&\cong \bigoplus_{j\in I_\Gamma} \mathrm{Hom}_\Gamma(H^\infty_{\tilde\sigma,-\mu}/V^\infty_j, \mathbb{C})\\
&\cong \bigoplus_{j\in I_\Gamma} \mathrm{Hom}_\Gamma((H_{\tilde\sigma,-\mu}/V_j)^\infty, \mathbb{C}).
\end{aligned}$$

Note that the dual representation of $H_{\tilde{\sigma},-\mu}/V_j$ is given by $W_j := V_j^{\perp_{\tilde{\sigma},-\mu}} \leq H_{\sigma,\mu}$. Therefore, as in (6.1),

$$\bigoplus_{j\in I_\Gamma} \mathrm{Hom}_\Gamma((H_{\tilde{\sigma},-\mu}/V_j)^\infty, \mathbb{C}) = \bigoplus_{j\in I_\Gamma} {}^\Gamma W_j^{-\infty}.$$

This proves the first part. We now prove the second part concerning the infinitesimal unitarity. Let $\varphi_F$ and $\pi$ as above. Then, denoting the $K$-finite elements by $\cdot_K$, we have (cf. [Wal92, Corollary 11.6.8])

$$\left(H^\infty_{\tilde{\sigma},-\mu}/\ker(\mathrm{pr}_\pi \circ \varphi_F)\right)_K \cong \pi_K$$

as $(\mathfrak{g}, K)$-modules. Since $\pi$ is unitary we infer that $H_{\tilde{\sigma},-\mu}/\ker(\mathrm{pr}_\pi \circ\varphi_F)$ is infinitesimally unitary. □

Note that Theorem 6.1.1 applies if $H_{\sigma,\mu}$ is irreducible. The following proposition shows that the hypotheses of Theorem 6.1.1 are in particular satisfied in the rank one case.

**Proposition 6.1.2.** *Let $G$ be of real rank one. Then the socle of $H_{\sigma,\mu}$ decomposes multiplicity-freely for each $\sigma \in \hat{M}$ and $\mu \in \mathfrak{a}^*$.*

*Proof.* See [Col85, Theorem (6.1.3)]. □

**Example 6.1.3.** Figure 6.1 describes the spherical principal series representations which can possibly contain Γ-invariant elements for $G = \mathrm{SO}_0(n, 1)$, $n \geq 2$, and $G = \mathrm{Sp}(n, 1)$, $n \geq 2$. The unitary principal series is given by $\mu \in i\mathfrak{a}_0^*$ in both cases and the complementary series consists of the parameters $\mu$ with $\mu(H) \in ]-\rho(H), \rho(H)[$ resp. $\mu(H) \in ]-\rho(H)+2, \rho(H)-2[$, where $H \in \mathfrak{a}_0$ as before denotes the unique element with $\alpha(H) = 1$ for the unique simple positive real root $\alpha$. Moreover, $H_\mu$ is reducible if and only if $\mu \in \pm(\rho + \mathbb{N}_0\alpha)$ resp. $\mu \in \pm(\rho + (2\mathbb{N}_0 - 2)\alpha)$ and $\mu$ is exceptional if and only if $H_\mu \neq H_\rho$ is reducible and has a unitarizable subrepresentation. In each case, the constant functions form an irreducible subspace of $H_\rho$ and thus ${}^\Gamma(\mathrm{soc}\, H_\rho)^{-\infty} \neq \{0\}$.

**Remark 6.1.4.** Recall from Theorem 6.1.1 that

$${}^\Gamma H_\mu^{-\infty} = \bigoplus_{U\in\mathbf{Irr}(\mu)} {}^\Gamma U^{-\infty}.$$

Choosing $(\tau_U, V_{\tau_U})$ as in Proposition 5.1.3 (e.g. a minimal $K$-type of $U$) we have by Proposition 5.1.2 that each $P_\mu^{\tau_U}\Big|_{U^{-\infty}}$ is injective and therefore

$${}^\Gamma H_\mu^{-\infty} \cong \bigoplus_{U\in\mathbf{Irr}(\mu)} {}^\Gamma P_\mu^{\tau_U}(U^{-\infty}) \subseteq \bigoplus_{U\in\mathbf{Irr}(\mu)} {}^\Gamma C^\infty(G\times_K V_{\tau_U}).$$

Moreover, since the socle never contains the trivial representation in the exceptional cases, the scalar Poisson transform maps ${}^\Gamma H_\mu^{-\infty}$ to zero in these cases.

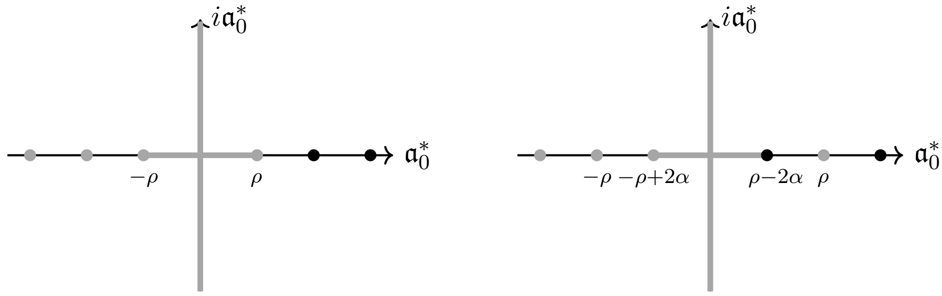



Figure 6.1.: Parameters $\mu$ for which $H_\mu$ has a unitarizable subrepresentation (gray) resp. is reducible (dots) for $G = \mathrm{SO}_0(n,1),\ n \geq 2$, (left) resp. $G = \mathrm{Sp}(n,1),\ n \geq 2$, (right). The exceptional set is given by the gray dots except for $\mu = \rho$.

## 6.2. The socle

In rank one we can describe the socle of spherical principal series representations in more detail. Let us first summarize what we know so far.

**Theorem 6.2.1.** *Denoting the set of minimal $K$-types by $\tau_{\min}$ and the Harish-Chandra module of $\mathrm{soc}(H_\mu)$ by $\mathrm{soc}(H_\mu)_K$ we have (see Appendix A for the notation)*

| $G$ | $\mathbf{Ex} = \{\mu_\ell \mid \ell \in \mathbb{N}_0\}$ | $\mathrm{soc}(H_{\mu_\ell})_K$ | $\tau_{\min}(\mathrm{soc}(H_{\mu_\ell}))$ |
|---|---|---|---|
| $\mathrm{SO}_0(2,1)$ | $\mu_\ell = -\rho - \ell\alpha$ | $\bigoplus_{k\geq\ell+1} Y_k \oplus Y_{-k}$ | $\{Y_{-(\ell+1)}, Y_{(\ell+1)}\}$ |
| $\mathrm{SO}_0(n,1),\ n \geq 3$ | $\mu_\ell = -\rho - \ell\alpha$ | $\bigoplus_{k=\ell+1}^{\infty} Y_k$ | $\{Y_{\ell+1}\}$ |
| $\mathrm{SU}(n,1),\ n \geq 2$ | $\mu_\ell = -\rho - 2\ell\alpha$ | $\bigoplus_{p,q=\ell+1}^{\infty} Y_{p,q}$ | $\{Y_{\ell+1,\ell+1}\}$ |
| $\mathrm{Sp}(n,1),\ n \geq 2$ | $\mu_\ell = -\rho - (2\ell-2)\alpha$ | $\bigoplus_{a\geq b\geq\ell+1} V_{a,b}$ | $\{V_{\ell+1,\ell+1}\}$ |
| $\mathrm{F}_{4(-20)}$ | $\mu_\ell = -\rho - (2\ell-6)\alpha$ | $\bigoplus_{\substack{m-k\geq 2\ell+2 \\ m\equiv k \bmod 2}} V_{m,k}$ | $\{V_{2\ell+2,0}\}$ |

*In each case, every irreducible subrepresentation of $\mathrm{soc}(H_\mu)$ is unitarizable and has a unique minimal $K$-type. For $G \neq \mathrm{SO}_0(2,1)$ the socle is irreducible for all exceptional parameters. For $G = \mathrm{SO}_0(2,1)$ the socle decomposes into two irreducible subrepresentations which are given by discrete series representations.*

*Proof.* For the exceptional parameters and the minimal $K$-types of the socles we refer the reader to Chapter 5.1 – precise information about the socles and its $K$-types can be found in Appendix B. Moreover, [JW77, Theorem 6.3 (1-3)] resp. [Joh76, Theorem 5.3 (2)] show that the socles are unitarizable. For $G = \mathrm{SO}_0(2,1)$ the decomposition of the socle follows from [Kna86, p. 38] with $n = 2(\ell+1)$, where two (unitary, irreducible) discrete series representations $\mathcal{D}^+_{2(\ell+1)}$ and $\mathcal{D}^-_{2(\ell+1)}$ occur. □

In each case we can compute the Langlands parameters of the socle representations. Similar calculations can be found in [Rob22].

**Theorem 6.2.2** (Langlands parameters)**.** *We have the following Langlands parameters for* $\mathrm{soc}(H_{\mu_\ell})$*,* $\mu_\ell \in \mathbf{Ex}$ *(see Theorem 6.2.1), in the notation of [Kna86, Theorem 8.54]*

| $G$ | $S$ | $\omega \in \hat{M}$ | $\nu \in \mathfrak{a}^*$ |
|---|---|---|---|
| $\mathrm{SO}_0(n,1),\ n \geq 2$ | $G$ *if* $n=2$ | $-$ | $-$ |
| | $P$ *if* $n \neq 2$ | $(\ell+1)e_1$ | $\frac{n-3}{2}\alpha$ |
| $\mathrm{SU}(n,1),\ n \geq 2$ | $G$ *if* $n=2$ | $-$ | $-$ |
| | $P$ *if* $n \neq 2$ | $(\ell+1)(\bar{\bar{\varepsilon}}_2 - \bar{\bar{\varepsilon}}_n)$ | $(n-2)\alpha$ |
| $\mathrm{Sp}(n,1),\ n \geq 2$ | $G$ *if* $n=2$ | $-$ | $-$ |
| | $P$ *if* $n \neq 2$ | $(\ell+1)(\bar{\varepsilon}_2 + \bar{\varepsilon}_3)$ | $(2n-3)\alpha$ |
| $\mathrm{F}_{4(-20)}$ | $G$ | $-$ | $-$ |

*Here, the highest weight of the* $M$*-representation* $\omega$ *is denoted as in [Bal79, Lemmas 4.3, 5.3] for* $G \in \{\mathrm{SU}(n,1), \mathrm{Sp}(n,1)\}$ *and as in Appendix A for* $G = \mathrm{SO}_0(n,1)$ *(then* $M \cong \mathrm{SO}(n-1)$*). By definition, if* $S = G$*, the socle* $\mathrm{soc}(H_{\mu_\ell})$ *is tempered. Moreover, in these cases, it is a discrete series representation if and only if* $\mu_\ell(H) \leq -\rho(H)$*. The Blattner parameter of the discrete series (see [Kna86, Terminology p. 310]) is given by its minimal* $K$*-type. If* $\mu_\ell(H) > -\rho(H)$*, the socle is a limit of discrete series representation (this case only occurs for* $G = \mathrm{Sp}(2,1)$ *and* $G = \mathrm{F}_{4(-20)}$*).*

*Proof.* Using the branching rules described in [Bal79] and [Kna02, Theorem 9.16] we first try to find $\omega \in \hat{M}$ such that the minimal $K$-type of $\mathrm{soc}(H_{\mu_\ell})$ is also minimal for the induced representation $\mathrm{Ind}_M^K(\omega)$. To determine $\nu \in \mathfrak{a}^*$ we compare the infinitesimal character of the socle, which is the same as that of $H_{\mu_\ell}$, with the infinitesimal character of the principal series representation corresponding to the pair $(\omega, \nu)$. They have to coincide up to the action of an element of the Weyl group and can be calculated using [Kna86, Proposition 8.22]. If one of the two steps above does not work, we must have $S = G$, i.e. the socle is tempered. In this case [KZ82, Theorem 14.2] shows that it has to be a discrete series representation or a limit of discrete series representation depending on the infinitesimal character being regular or singular. The connection to the Blattner parameter follows from [Kna86, Chapter XV, §1, Example (1)]. □

In the real case, for $\mathrm{soc}(H_{\mu_\ell})$, we get the same parameters as obtained by the Želobenko operator in Proposition 5.4.3 since $(\alpha - \rho)(H) = 1 - \frac{n-1}{2} = -\frac{n-3}{2}$ (note that the $\mathfrak{a}$-parameter of [Kna86, Theorem 8.54] has a minus sign in comparison to our $\mathfrak{a}$-parameter).

Figure 6.2 schematically summarizes the main results on the exceptional case so far.

In the case of surfaces, the exceptional parameters lead to discrete series representations of $\mathrm{SL}(2,\mathbb{R})$. This phenomenon generalizes to the rank one case in the following way.

**Theorem 6.2.3.** *There is a one-to-one correspondence between the representations* $\mathrm{soc}(H_{\mu_\ell})$*,* $\mu_\ell \in \mathbf{Ex}$ *(see Theorem 6.2.1), and the relative discrete series of the associated pseudo-Riemannian symmetric spaces* $G/H$ *starting at the end of the complementary series*[1]*. More precisely, each of these representations corresponds to a minimal closed invari-*

[1] More precisely, if the complementary series corresponds to the real parameters $s \in [-s_0, s_0]$, we only consider representations associated to parameters $s \in \mathbb{R}$ with $|s| \geq s_0$.

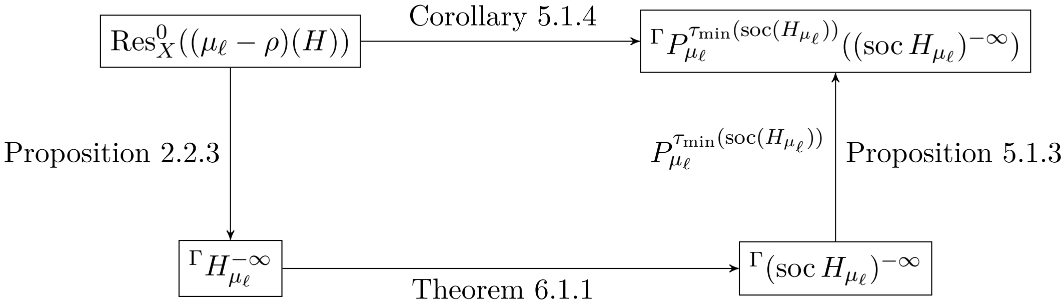



Figure 6.2.: The exceptional case for $G \neq \mathrm{SO}_0(2,1)$ and $\mu_\ell$ as in Theorem 6.2.1

*ant subspace of* $L^2(G/H)$ *with* $H = \mathrm{SO}_0(n-1,1)$, $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1,1)) \cong \mathrm{U}(n-1,1)$, $\mathrm{Sp}(1) \times \mathrm{Sp}(n-1,1)$, *or* $\mathrm{Spin}(1,8)$ *respectively.*

*Proof.* In the classical cases the Plancherel formula for $G/H$ is determined in [Far79, Theorem 10 ($q = 1$)], where the representations occurring in its discrete part are described in [Far79, proof of Theorem 9.2] (note that $c(s)c(-s) = 0$ for $s \geq s_0$ – where $[-s_0, s_0]$ corresponds to the complementary series (see [Far79, p. 417]) – if and only if $\mu(H) := -s$ defines an exceptional parameter). Comparing the $K$-types one recovers our socle representations, where $\mathscr{Y}_{\ell m}$ in [Far79, p. 399] corresponds to our $Y_\ell$, $Y_{p,q} \oplus Y_{q,p}$ with $2p := \ell + m$, $2q := \ell - m$, or $V_{a,b}$ with $2a := \ell + m$, $2b := \ell - m$, for $G = \mathrm{SO}_0(n,1)$, $n \geq 3$, or $G = \mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, $n \geq 2$, respectively (note that $\mathrm{O}(n,1)$, $\mathrm{U}(n,1)$ are used instead of $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$ in [Far79]). For $G = \mathrm{SO}_0(2,1)$ the $\mathscr{Y}_{\ell m} = \mathscr{Y}^2_\ell \otimes \mathscr{Y}^1_m$ in [Far79] is two-dimensional ($\mathscr{Y}^2_\ell$ is spanned by $(x \pm iy)^\ell$) and corresponds to $Y_\ell \oplus Y_{-\ell}$ in our notation.

For the exceptional case we refer to Appendix C for details and only present an outline of the proof here. The Plancherel formula can be found in [Kos83, p. 85], where $\theta_r$ should also occur for $r = 0$. Again, the exceptional points occur in the discrete part of this formula and thus again lead to relative discrete series representations by [Kos83, Theorem 3.12.1] (in [Kos83, Remark 3.13.4] $\zeta_5$ and $-\theta_0$ are missing). By the definitions of the spherical distributions $\theta_r$ and $\zeta_s$ in [Kos83, pp. 62, 81] we see that their associated representations are subquotients of spherical principal series representations and, comparing the occurring $K$-types (see [Kos83, Proposition 3.9.4, pp. 71, 82]), that they are given by our socle representations. □

# 7. Fourier characterization

From now on we consider spherical principal series representations for exceptional parameters in the rank one case. Our aim is to find explicit realizations of the unitary irreducible subrepresentations occurring in Theorem 6.1.1 in the space of smooth sections of a specific vector bundle. In the case of surfaces this realization relied heavily on the decomposition of distributions in the fibres of the sphere bundle (see Equation (4.3)). We prove an analog of this decomposition – which, taking care of higher multiplicities, also works in higher rank – in the first two sections (see especially Propositions 7.2.3 and 7.2.4). Apart from being interesting itself, it is closely related to Poisson transforms (Lemma 7.1.4) and allows a nice characterization of spherical principal series representations using the conditions determined by Proposition 5.3.2 (see Theorem 7.4.11). While we formulate everything on the cover $G/K$, the same proofs – by taking the quotient by $\Gamma$ – show the same characterization for the locally symmetric space $\mathbf{M}$. We use this characterization to give explicit descriptions of the images of $\Gamma$-invariant elements under the injective vector-valued Poisson transforms from Section 5.1 in each of the cases listed in Chapter 8. Furthermore, we prove a number of results concerning decompositions of tensor products and generalized gradients which may be of independent interest.

## 7.1. Generalized Fourier series

In the following we describe a generalized Fourier series that is closely related to the Poisson transform and essentially gives that, properly interpreted, each $f \in H_\mu$ is the sum of all its Poisson transform images.

**Definition 7.1.1.** For each $Y \in \hat{K}_M$ let

$$\pi_Y : C^\infty(G \times_K Y) \hookrightarrow C^\infty(G)^M, \quad \pi_Y(\varphi)(g) := \varphi(g)(e),$$

where $C^\infty(G)^M$ denotes the right $M$-invariant elements in $C^\infty(G)$. Moreover, let $\mathcal{D}'(G \times_K Y)$ denote the dual of $C_c^\infty(G \times_K \tilde{Y})$, where we realize the dual representation $\tilde{Y}$ of $Y$ as the complex conjugate representation of $Y$. We embed $C^\infty(G/M)$ into $\mathcal{D}'(G/M)$ by

$$\iota_{G/M} \colon C^\infty(G/M) \hookrightarrow \mathcal{D}'(G/M), \quad \iota_{G/M}(f)(\varphi) := \int_G f(gM)\varphi(gM)\,\mathrm{d}g$$

and $C^\infty(G \times_K Y)$ into $\mathcal{D}'(G \times_K Y)$ by

$$\iota_Y : C^\infty(G \times_K Y) \hookrightarrow \mathcal{D}'(G \times_K Y), \quad \iota_Y(f)(\varphi) := \int_G \pi_Y(f)(g)\pi_{\tilde{Y}}(\varphi)(g)\,\mathrm{d}g.$$

If it is clear from the context we omit the embeddings $\iota_*$ for the sake of readability. Let us further define the pullback

$$\pi_Y^*: \mathcal{D}'(G/M) \to \mathcal{D}'(G \times_K Y), \quad \pi_Y^*(f)(\varphi) := f(\pi_{\tilde{Y}}(\varphi)).$$

We now give a first version of generalized Fourier series.

**Lemma 7.1.2.** *Let $f \in C^\infty(G)^M$ be a right $M$-invariant smooth function and*

$$\mathrm{pr}_{Y_\tau}: L^2(K/M) \to Y_\tau$$

*denote the orthogonal projection onto $Y_\tau \in \hat{K}_M$. For every fixed $g \in G$, the series*

$$\sum_{\tau \in \hat{K}_M} \mathrm{pr}_{Y_\tau}(f(g\bullet)),$$

*where $f(g\bullet) \in C^\infty(K/M)$ is defined by*

$$f(g\bullet): K/M \to \mathbb{C}, \quad kM \mapsto f(gk),$$

*converges absolutely and uniformly to $f(g\bullet)$. Moreover, we can uniquely decompose $f$ into the* generalized Fourier series

$$f = \sum_{\tau \in \hat{K}_M} f_{Y_\tau},$$

*where $f_{Y_\tau} \in \pi_{Y_\tau}(C^\infty(G \times_K Y_\tau))$ and the series converges pointwise. The functions $f_{Y_\tau}$ are given by*

$$f_{Y_\tau} = \pi_{Y_\tau}(g \mapsto \mathrm{pr}_{Y_\tau}(f(g\bullet))).$$

*Proof.* We decompose $f$ according to the right regular $K$-representation: For fixed $g \in G$ consider the function

$$f_g: K \to \mathbb{C}, \quad f_g(k) := f(gk).$$

Then $f_g \in C^\infty(K)^M \cong C^\infty(K/M)$. Since, by definition of $\hat{K}_M$,

$$L^2(K/M) \cong \widehat{\bigoplus}_{\tau \in \hat{K}_M} Y_\tau$$

we can decompose

$$f_g = \sum_{\tau \in \hat{K}_M} \mathrm{pr}_{Y_\tau}(f_g) \tag{7.1}$$

which converges in the Hilbert sense. By [Hel00, Chapter V, Theorem 3.5 (iii)] this convergence is absolute and uniform. Thus, we have for every $k \in K$

$$f(gk) = f_g(k) = \sum_{\tau \in \hat{K}_M} \mathrm{pr}_{Y_\tau}(f_g)(k).$$

Note that $(g \mapsto \mathrm{pr}_{Y_\tau}(f_g)) \in C^\infty(G \times_K Y_\tau)$; indeed

$$f_{g\tilde{k}}(k) = f(g\tilde{k}k) = f_g(\tilde{k}k) = \sum_{\tau \in \hat{K}_M} \mathrm{pr}_{Y_\tau}(f_g)(\tilde{k}k) = \sum_{\tau \in \hat{K}_M} (\tau(\tilde{k})^{-1} \mathrm{pr}_{Y_\tau}(f_g))(k)$$

implies $\mathrm{pr}_{Y_\tau}(f_{g\tilde{k}}) = \tau(\tilde{k})^{-1} \mathrm{pr}_{Y_\tau}(f_g)$ for every $g \in G$, $\tilde{k} \in K$. We can write

$$f = \sum_{\tau \in \hat{K}_M} f_\tau,$$

where $f_\tau := \pi_{Y_\tau}(g \mapsto \mathrm{pr}_{Y_\tau}(f_g)) \in C^\infty(G)^M$ since, by Equation (7.1),

$$f(g) = f_g(e) = \sum_{\tau \in \hat{K}_M} \mathrm{pr}_{Y_\tau}(f_g)(e) = \sum_{\tau \in \hat{K}_M} f_\tau(g)$$

for every $g \in G$.

Conversely, for proving uniqueness, let $f = \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\varphi_\tau)$ with $\varphi_\tau \in C^\infty(G \times_K Y_\tau)$. We need to show that $\varphi_\tau(g) = \mathrm{pr}_{Y_\tau}(f_g)$ for every $g \in G$. We calculate for $k \in K,\ g \in G$

$$\begin{aligned} f_g(k) = f(gk) &= \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\varphi_\tau)(gk) = \sum_{\tau \in \hat{K}_M} \varphi_\tau(gk)(e) \\ &= \sum_{\tau \in \hat{K}_M} (\tau(k)^{-1}\varphi_\tau(g))(e) = \sum_{\tau \in \hat{K}_M} \varphi_\tau(g)(k). \end{aligned}$$

This yields $\mathrm{pr}_{Y_\tau}(f_g) = \varphi_\tau(g)$ and proves the uniqueness. □

**Notation 7.1.3.** Let

$$\pi_Y \colon \mathcal{D}'(G \times_K Y) \to \mathcal{D}'(G/M), \quad \pi_Y(f)(\varphi) := f(\pi_Y^*(\varphi)).$$

In Lemma 7.1.4 iii) we see that this extends the definition of $\pi_Y$ from Definition 7.1.1.

The following lemma also provides an alternative form of the generalized Fourier series, which we mainly use later on, and connects it to Poisson transforms.

**Lemma 7.1.4.** *Let $Y \in \hat{K}_M$ and recall the maps $\iota_{G/M}$, $\iota_Y$ from Definition 7.1.1. Then*

*i) $\pi_Y^*(f)(g) = \mathrm{pr}_Y(f(g\bullet))$ for each $f \in C^\infty(G/M),\ g \in G$, so that $\pi_Y^*(C^\infty(G/M)) \subseteq C^\infty(G \times_K Y)$ and $\pi_Y^*(C_c^\infty(G/M)) \subseteq C_c^\infty(G \times_K Y)$,*

*ii) $f = \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\pi_{Y_\tau}^*(f))$ pointwise for each $f \in C^\infty(G/M)$,*

*iii) $\pi_Y(\iota_Y(f)) = \iota_{G/M}(\pi_Y(f))$ for each $f \in C^\infty(G \times_K Y)$ and*

*iv) $\forall \mu \in \mathfrak{a}^*\colon P_\mu^Y = \frac{1}{\dim Y}\pi_Y^* \circ \mathcal{Q}_{\mu-\rho}$ on $\mathcal{D}'(K/M)$.*

*Proof.* i) By Lemma 7.1.2 we can write $f = \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(u_\tau)$, where $u_\tau \in C^\infty(G \times_K Y_\tau)$ is given by $u_\tau(g) = \mathrm{pr}_{Y_\tau}(f(g\bullet))$. For each $\varphi \in C_c^\infty(G \times_K \tilde{Y})$ we use the orthogonality of the $Y_\tau$ to obtain

$$\begin{aligned}
\pi_Y^*(f)(\varphi) = f(\pi_{\tilde{Y}}(\varphi)) &= \int_G \pi_{\tilde{Y}}(\varphi)(g) f(g)\,\mathrm{d}g \\
&= \int_{G/K} \int_K \pi_{\tilde{Y}}(\varphi)(gk) f(gk)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_{G/K} \int_K \varphi(g)(k) \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(u_\tau)(gk)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_{G/K} \sum_{\tau \in \hat{K}_M} \int_K \varphi(g)(k) u_\tau(g)(k)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_{G/K} \int_K \varphi(g)(k) u_Y(g)(k)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_{G/K} \int_K \pi_{\tilde{Y}}(\varphi)(gk) \pi_Y(u_Y)(gk)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_G \pi_{\tilde{Y}}(\varphi)(g) \pi_Y(u_Y)(g)\,\mathrm{d}g \\
&= \iota_Y(u_Y)(\varphi).
\end{aligned}$$

Note that if $f$ has compact support $\operatorname{supp} f \subset G/M$ and $\mathrm{pr} : G \to G/M$ denotes the canonical projection, we have that $\operatorname{supp}(\pi_Y^*(f)) \subseteq \mathrm{pr}^{-1}(\operatorname{supp} f) \cdot K$ is compact since $M$ is compact.

ii) follows from Lemma 7.1.2 and i).

iii) Let $f \in C^\infty(G \times_K Y)$ and $\varphi \in C_c^\infty(G/M)$. By ii) we decompose

$$\varphi = \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\pi_{Y_\tau}^*(\varphi))$$

where $\pi_Y^*(\varphi) \in C^\infty(G \times_K Y)$. By the orthogonality of the $Y_\tau$ we have

$$\begin{aligned}
\iota_{G/M}(\pi_Y(f))(\varphi) = \int_G \pi_Y(f)(gM)\varphi(gM)\,\mathrm{d}g &= \int_{G/K} \int_K \pi_Y(f)(gkM)\varphi(gkM)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_{G/K} \sum_{\tau \in \hat{K}_M} \int_K f(g)(kM) \pi_{Y_\tau}^*(\varphi)(g)(k)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_{G/K} \int_K f(g)(kM) \pi_{\tilde{Y}}^*(\varphi)(g)(k)\,\mathrm{d}k\,\mathrm{d}gK \\
&= \int_G \pi_Y(f)(g) \pi_{\tilde{Y}}(\pi_{\tilde{Y}}^*(\varphi))(g)\,\mathrm{d}g = \iota_Y(f)(\pi_{\tilde{Y}}^*(\varphi)) = \pi_Y(\iota_Y(f))(\varphi).
\end{aligned}$$

iv) By continuity (recall Proposition 2.2.2) we restrict our attention to smooth functions $\phi \in C^\infty(K/M)$. In this case the equality follows from Lemma 5.1.1 and i) (recall that $\phi_{Y_\tau}(e) = 1$). □

## 7.2. Convergence of generalized Fourier series

In the following we will prove that the convergence in Lemma 7.1.4 ii) is uniform on compact sets and that the same is true for each derivative. Therefore the convergence is a convergence in $C_c^\infty(G/M)$ for $f \in C_c^\infty(G/M)$, where we equip $C_c^\infty(G/M)$ with the inductive limit topology $C_c^\infty(G/M) = \lim_{C \subseteq G/M} C_C^\infty(G/M)$, where the limit runs over all compact subsets $C \subseteq G/M$ and we denote by $C_C^\infty(G/M) \subseteq C_c^\infty(G/M)$ the subset of all functions which are supported in $C$. We describe the topology on the spaces $C_C^\infty(G/M)$ in some more detail by defining a family of norms.

Let $\mathcal{B} := \{X_1, \ldots, X_n\} \subseteq \mathfrak{g}_0$ be a basis of $\mathfrak{g}_0$. For $\ell \in \mathbb{N}_0$ and $C' \subset G$ compact we introduce the following norm on $C^\infty(G/M)$:

$$\|f\|_{H^\ell(C')} := \sum_{k=0}^{\ell} \sum_{X_1,\ldots,X_k \in \mathcal{B}} \sup_{g \in C'} |(X_1 \cdots X_k f)(gM)|,$$

where $X \in \mathfrak{g}_0$ acts on $f \in C^\infty(G/M)$ by the derived left regular representation

$$\forall g \in G\colon (Xf)(gM) := \frac{d}{dt}\Big|_{t=0} f(\exp(-tX)gM).$$

The summand for $k = 0$ is understood as not differentiating, i.e. as $\sup_{g \in C'} |f(gM)|$. Identifying $C^\infty(\Gamma\backslash G/M) \cong {}^\Gamma C^\infty(G/M)$ using the pullback of the canonical projection $\pi_\Gamma$, we can transfer the topology generated by these norms to $C^\infty(\Gamma\backslash G/M)$.

We have the following lemma related to the Riemann-Lebesgue lemma.

**Lemma 7.2.1.** *Let $f \in C^\infty(G/M)$. For each $C \subset G$ compact, $\ell \in \mathbb{N}_0$ and $N \in \mathbb{N}$ there exists a constant $C_{f,C,N,\ell} > 0$ independent of $Y_\tau$ such that*

$$\forall Y_\tau \in \hat{K}_M\colon \|\pi_{Y_\tau}(\pi^*_{Y_\tau}(f))\|_{H^\ell(C)} \leq C_{f,C,N,\ell} \cdot (1 + \|\tau\|^2)^{-N},$$

*where $\|\tau\|$ denotes the length of the highest weight of $Y_\tau$. Moreover, if $f_n \to 0$ in $C^\infty(G/M)$ we can find $C_{f_n,C,N,\ell}$ such that $\lim_{n\to\infty} C_{f_n,C,N,\ell} = 0$.*

*Proof.* For each $g \in G$ we have $f(g\bullet) \in C^\infty(K/M)$. By a slight abuse of notation we will write $\tau$ also for the highest weight of $(\tau, Y_\tau)$. Applying [Hel00, Chapter V, Lemma 3.2] to $C^\infty(K/M)$ with the uniform norm $\|\bullet\|_\infty$ and the left regular representation $\lambda$ we obtain

$$\forall Y_\tau \in \hat{K}_M, \forall m \in \mathbb{N}\colon \quad \|\pi^*_{Y_\tau}(f)(g)\|_\infty \leq C_1 c_\tau^{-m} \dim(Y_\tau)^2 \|\lambda(\Omega^m) f(g\bullet)\|_\infty, \tag{7.2}$$

where

i) $\Omega$ is a bi-invariant differential operator on $K$ with

$$\Omega\chi_\tau = c_\tau \chi_\tau$$

for the character $\chi_\tau$ of $Y_\tau$ (cf. proof of [Hel00, Chapter V, Theorem 3.1]),

ii) $c_\tau \geq 1+\langle \tau+\rho_{[\mathfrak{k},\mathfrak{k}]}, \tau+\rho_{[\mathfrak{k},\mathfrak{k}]}\rangle - \langle \rho_{[\mathfrak{k},\mathfrak{k}]}, \rho_{[\mathfrak{k},\mathfrak{k}]}\rangle = 1+\langle \tau, \tau+2\rho_{[\mathfrak{k},\mathfrak{k}]}\rangle$, where $\rho_{[\mathfrak{k},\mathfrak{k}]}$ denotes the half-sum of positive roots in the semisimple part $[\mathfrak{k},\mathfrak{k}]$ of $\mathfrak{k}$, (see [Hel00, Chapter V, Equation (16) of §1 & proof of Lemma 3.2])

iii) $C_1 > 0$ is some constant independent of $f, C, N, \ell$ and $g$ given by the continuity of $\lambda$ on $C^\infty(K/M)$.

By the Weyl dimension formula we have

$$\dim(Y_\tau) = \prod_{\alpha\in\Delta^+_{[\mathfrak{k},\mathfrak{k}]}} \frac{\langle \tau+\rho_{[\mathfrak{k},\mathfrak{k}]}, \alpha\rangle}{\langle \rho_{[\mathfrak{k},\mathfrak{k}]}, \alpha\rangle},$$

where $\Delta^+_{[\mathfrak{k},\mathfrak{k}]}$ denotes the positive roots in $[\mathfrak{k},\mathfrak{k}]$. Therefore, we can conclude that there exists a constant $\tilde{C}$ depending only on $\mathfrak{k}$ such that, for $m \geq m_N \in \mathbb{N}$ large enough,

$$c_\tau^{-m} \dim(Y_\tau)^2 \leq \tilde{C} \cdot (1+\|\tau\|^2)^{-N}$$

and thus, by Equation (7.2),

$$\forall\, Y_\tau \in \hat{K}_M\colon \quad \|\pi^*_{Y_\tau}(f)(g)\|_\infty \leq C_1\tilde{C}\|\lambda(\Omega^{m_N})f(g\bullet)\|_\infty \cdot (1+\|\tau\|^2)^{-N}.$$

Taking the supremum over $C$ on both sides we hence infer

$$\forall\, Y_\tau \in \hat{K}_M\colon \quad \sup_{g\in C}\|\pi^*_{Y_\tau}(f)(g)\|_\infty \leq C_1\tilde{C}\sup_{g\in C}\|\lambda(\Omega^{m_N})f(g\bullet)\|_\infty \cdot (1+\|\tau\|^2)^{-N}.$$

Note that since the map $g \mapsto \|\lambda(\Omega^{m_N})f(g\bullet)\|_\infty$ from $G$ to $\mathbb{R}_{\geq 0}$ is continuous by the smoothness of $f$, the suprema are actually finite. We abbreviate

$$C_{f,C,N,0} \coloneqq C_1\tilde{C}\sup_{g\in C}\|\lambda(\Omega^{m_N})f(g\bullet)\|_\infty < \infty.$$

Note that the procedure above also works for $X_1\cdots X_k f$ instead of $f$ for $X_1,\dots,X_k \in \mathcal{B}$ and $0 \leq k \leq \ell$. We set

$$C_{f,C,N,\ell} \coloneqq \max\{C_{\varphi,C,N,0} \mid \exists\, 0 \leq k \leq \ell,\ \exists X_1,\dots,X_k \in \mathcal{B}\colon \varphi = X_1\cdots X_k f\}.$$

By the definition of $\pi^*_{Y_\tau}$ we have $\pi^*_{Y_\tau}(X_1\cdots X_k f) = X_1\cdots X_k \pi^*_{Y_\tau}(f)$ for all $X_1,\dots,X_k$ as above. Finally we obtain that for each $Y_\tau \in \hat{K}_M$

$$\begin{aligned}\sup_{g\in C}|(X_1\cdots X_k \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)))(g)| &= \sup_{g\in C}|(\pi^*_{Y_\tau}(X_1\cdots X_k f))(g)(e)| \\ \leq \sup_{g\in C}\|\pi^*_{Y_\tau}(X_1\cdots X_k f)(g)\|_\infty &\leq C_{f,C,N,\ell}\cdot(1+\|\tau\|^2)^{-N}.\end{aligned}$$

This proves the first part and the second part follows from the definition of $C_{f,C,N,\ell}$. □

**Remark 7.2.2.** We remark that the proof of Lemma 7.2.1 also works for smooth functions $f \in C^\infty(G)$ on $G$. Moreover, since $\Omega$ is $K$-bi-invariant, all constants are invariant under right translations with elements of $K$, i.e. we obtain the same constants when considering $r(k)f$ instead of $f$ for $k \in K$.

**Proposition 7.2.3.** *Let $f \in C_c^\infty(G/M)$. Then*

$$\sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)) \tag{7.3}$$

*is absolutely convergent with respect to each* $\|\bullet\|_{H^\ell(C)}$ *and converges to* $f$ *in* $C_c^\infty(G/M)$.

*Proof.* Let $\mathrm{pr}\colon G \to G/M$ denote the canonical projection. By the definition of the inductive limit topology on $C_c^\infty(G/M)$ we have to find a compact set $C \subset G/M$ such that $\mathrm{supp}(\pi_{Y_\tau}(\pi^*_{Y_\tau}(f))) \subseteq C$ for each $Y_\tau \in \hat{K}_M$ and such that for each $\ell \in \mathbb{N}_0$ we have that $\sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))$ converges to $f$ with respect to $\|\bullet\|_{H^\ell(\mathrm{pr}^{-1}(C))}$. As in the proof of Lemma 7.1.4 i) we see that the condition on the supports is fulfilled if we choose $C := \mathrm{supp}(f) \cdot K$. Let $\ell \in \mathbb{N}_0$ and $N \in \mathbb{N}$ be fixed. By Lemma 7.2.1 there exists a constant $C_{f,C,N,\ell}$ independent of $Y_\tau$ such that

$$\forall\, Y_\tau \in \hat{K}_M\colon \|\pi_{Y_\tau}(\pi^*_{Y_\tau}(f))\|_{H^\ell(C)} \leq C_{f,C,N,\ell} \cdot (1 + \|\tau\|^2)^{-N}.$$

Thus we have for each finite subset $F \subseteq \hat{K}_M$ that

$$\|\sum_{\tau \in F} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))\|_{H^\ell(\mathrm{pr}^{-1}(C))} \leq \sum_{\tau \in F} \|\pi_{Y_\tau}(\pi^*_{Y_\tau}(f))\|_{H^\ell(\mathrm{pr}^{-1}(C))} \leq C_{f,C,N,\ell} \sum_{\tau \in F} (1 + \|\tau\|^2)^{-N}. \tag{7.4}$$

Let $\varepsilon > 0$. Note that the weight lattice of $[\mathfrak{k}, \mathfrak{k}]$ is a lattice in the finite dimensional space $(i\mathfrak{t}_0)^*$, where $\mathfrak{t}_0$ denotes the Lie algebra of a maximal torus $T$ in $\tilde{K}$, the analytic subgroup of $[\mathfrak{k}_0, \mathfrak{k}_0]$. Therefore, we may identify $\hat{K}_M$ with a subset of $\mathbb{Z}^d$ in $\mathbb{R}^d$ with $d := \dim \mathfrak{t}_0$. We infer that if $N$ is large enough, there exists a finite set $F_0 \subseteq \hat{K}_M$ such that the right hand side of (7.4) is smaller than $\varepsilon$ for each finite set $F \subseteq \hat{K}_M$ with $F \cap F_0 = \emptyset$. Therefore, for each such $F$,

$$\|\sum_{\tau \in F} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))\|_{H^\ell(\mathrm{pr}^{-1}(C))} \leq \sum_{\tau \in F} \|\pi_{Y_\tau}(\pi^*_{Y_\tau}(f))\|_{H^\ell(\mathrm{pr}^{-1}(C))} \leq C_{f,C,N,\ell} \cdot \varepsilon.$$

Hence, the series in (7.3) converges absolutely and to its pointwise limit $f$ (see Lemma 7.1.4 ii)) with respect to $\|\bullet\|_{H^\ell(\mathrm{pr}^{-1}(C))}$. □

We can also decompose distributions.

**Proposition 7.2.4.** *Let $u \in \mathcal{D}'(G/M)$ be a distribution. Then the sum*

$$\sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\pi^*_{Y_\tau}(u))$$

*converges absolutely and to $u$ in the weak sense.*

*Proof.* Let $f \in C_c^\infty(G/M)$. For each $Y_\tau \in \hat{K}_M$ we have (see Definition 7.1.1 and Notation 7.1.3)

$$\pi_{Y_\tau}(\pi^*_{Y_\tau}(u))(f) = \pi^*_{Y_\tau}(u)(\pi^*_{\tilde{Y}}(f)) = u(\pi_{\tilde{Y}}(\pi^*_{\tilde{Y}}(f)))$$

and therefore, by Proposition 7.2.3 and the continuity of $u$,

$$\sum_{\tau\in\hat{K}_M} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)) = f \text{ in } C_c^\infty(G/M) \Rightarrow \sum_{\tau\in\hat{K}_M} u(\pi_{\tilde{Y}}(\pi^*_{\tilde{Y}}(f))) = u(f).$$

For the absolute convergence note that (see [Hör90, Definition 2.1.1]) the restriction of $u$ to $C^\infty(\operatorname{supp}(f)K)$ is of finite order, i.e. there exist $\ell \in \mathbb{N}_0$ and $C > 0$ with

$$\forall \varphi \in C^\infty(\operatorname{supp}(f)K)\colon \qquad |u(\varphi)| \leq C\|\varphi\|_{H^\ell(\operatorname{supp}(f)K)}.$$

Then

$$|\pi_{Y_\tau}(\pi^*_{Y_\tau}(u)(f))| = |u(\pi_{\tilde{Y}}(\pi^*_{\tilde{Y}}(f)))| \leq C\|\pi_{\tilde{Y}}(\pi^*_{\tilde{Y}}(f))\|_{H^\ell(\operatorname{supp}(f)K)}.$$

The absolute convergence now follows from Lemma 7.2.1. □

**Lemma 7.2.5.** *Fix $c > 0$ and $N \in \mathbb{N}$. If $\psi_\tau \in C^\infty(G \times_K Y_\tau)$ for $\tau \in \hat{K}_M$ are chosen such that*

$$\iota_{G/M}(\pi_{Y_\tau}(\psi_\tau))(\pi_{\tilde{Y}_\tau}(\overline{\psi_\tau})) \leq c \cdot (1 + \|\tau\|^2)^N,$$

*then $\psi := \sum_{\tau\in\hat{K}_M} \iota_{G/M}(\pi_{Y_\tau}(\psi_\tau))$ is absolutely convergent in the weak sense and defines a distribution on $G/M$.*

*Proof.* We first prove the pointwise convergence of $\psi$ on $C_c^\infty(G/M)$. For each test function $f \in C_c^\infty(G/M)$ we have by Lemma 7.1.4 iii), Notation 7.1.3 and Definition 7.1.1

$$\iota_{G/M}(\pi_{Y_\tau}(\psi_\tau))(f) = \pi_{Y_\tau}(\iota_{Y_\tau}(\psi_\tau))(f) = \iota_{Y_\tau}(\psi_\tau)(\pi^*_{\tilde{Y}_\tau}(f)) = \int_G \pi_{Y_\tau}(\psi_\tau)(g)\pi_{\tilde{Y}_\tau}(\pi^*_{\tilde{Y}_\tau}(f))(g)\,\mathrm{d}g.$$

The Cauchy-Schwarz inequality thus implies that

$$|\iota_{G/M}(\pi_{Y_\tau}(\psi_\tau))(f)|^2 \leq \int_G |\pi_{Y_\tau}(\psi_\tau)(g)|^2\,\mathrm{d}g \cdot \int_G |\pi_{\tilde{Y}_\tau}(\pi^*_{\tilde{Y}_\tau}(f))(g)|^2\,\mathrm{d}g.$$

For the first factor we obtain

$$\int_G |\pi_{Y_\tau}(\psi_\tau)(g)|^2\,\mathrm{d}g = \iota_{G/M}(\pi_{Y_\tau}(\psi_\tau))(\pi_{\tilde{Y}_\tau}(\overline{\psi_\tau})) \leq c \cdot (1 + \|\tau\|^2)^N.$$

For the second factor Lemma 7.2.1 implies that for each $m \in \mathbb{N}$ there exists a constant $\tilde{C} := C_{\varphi,\mathrm{pr}^{-1}(\operatorname{supp}(f))K,m,0}$ independent of $Y_\tau$ such that

$$\forall\, Y_\tau \in \hat{K}_M\colon \|\pi_{Y_\tau}(\pi^*_{Y_\tau}(f))\|_{H^0(\mathrm{pr}^{-1}(\operatorname{supp}(f))K)} \leq \tilde{C} \cdot (1 + \|\tau\|^2)^{-m}.$$

Choosing $m$ sufficiently large we thus obtain that

$$\sum_{\tau\in\hat{K}_M} |\iota_{G/M}(\pi_{Y_\tau}(\psi_\tau))(f)| < \infty$$

converges absolutely. We now prove the continuity of $\psi$. Let $C \subset G$ be a compact set and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n \in C_c^\infty(G/M)$ such that $\operatorname{supp}(f_n) \subseteq CM$ for each $n \in \mathbb{N}$ and $\|f_n\|_{H^\ell(CM)}$ converges to 0 for each fixed $\ell \in \mathbb{N}_0$. We have to prove that $\psi(f_n) \to 0$ (see [Hör90, Theorem 2.1.4]). Again by Lemma 7.2.1 we may choose for each $m \in \mathbb{N}$ constants $\tilde{C}_n$ independent of $Y_\tau$ such that

$$\forall\, Y_\tau \in \hat{K}_M \colon \|\pi_{Y_\tau}(\pi^*_{Y_\tau}(f_n))\|_{H^0(CM)} \leq \tilde{C}_n \cdot (1 + \|\tau\|^2)^{-m}.$$

Moreover, by the second part of Lemma 7.2.1 we may choose the constants $\tilde{C}_n$ such that $\lim_{n\to\infty} \tilde{C}_n = 0$. Proceeding as above we arrive at

$$\sum_{\tau\in\hat{K}_M} |\iota_{G/M}(\pi_{Y_\tau}(\psi_\tau))(f_n)| \leq \sqrt{c \cdot \tilde{C}_n} \sum_{\tau\in\hat{K}_M} (1 + \|\tau\|^2)^{\frac{N-m}{2}} \to 0,$$

since the series on the right hand side converges for $m$ large enough. □

## 7.3. Tensor product decompositions

In order to use the conditions from Proposition 5.3.2 for computations, we need to describe the generalized gradients more precisely. It turns out that this can be done rather uniformly and the corresponding scalars can be computed explicitly.

In this section we describe the general results in the rank one case. Since the definition of generalized gradients involves the $K$-decomposition of $Y \otimes \mathfrak{p}$ for $Y \in \hat{K}$, we first prove some results on it. Then, we define and describe the generalized gradients we will use in the proof of the Fourier characterization (Definition 7.3.4 and Proposition 7.3.9) and depict a method to determine the corresponding scalars from Proposition 5.3.2 (Lemma 7.3.10 and 7.3.11). Explicit forms of all decompositions and scalars are stated and computed in Appendix A.

We start with the following multiplicity one result.

**Proposition 7.3.1.** *Let $Y \in \hat{K}$. Then $Y \otimes \mathfrak{p}^*$ decomposes multiplicity-freely.*

*Proof.* By [Kna02, Chapter IX, §8, Problem 15] it suffices to prove that all weights of $\mathfrak{p} \cong_K \mathfrak{p}^*$ have multiplicity one, i.e. if $\mathfrak{t}_0 \leq \mathfrak{k}_0$ is a maximal torus we have that $\mathfrak{t}$ acts multiplicity-freely on $\mathfrak{p}$.

Let us first assume that the ranks $\operatorname{rk}\mathfrak{k}_0$ and $\operatorname{rk}\mathfrak{g}_0$ coincide. Then $\mathfrak{t} \leq \mathfrak{k} \leq \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and we have the root-space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{t})} \mathfrak{g}_\alpha,$$

where each $\mathfrak{g}_\alpha$ is one-dimensional. We note that the root spaces $\mathfrak{g}_\alpha$ are invariant under the ($\mathbb{C}$-linear continuation of the) Cartan involution $\theta$; indeed we have for each $X \in \mathfrak{g}_\alpha$

$$\forall H \in \mathfrak{t}\colon \quad [H, \theta X] = \theta[\theta H, X] = \theta[H, X] = \alpha(H)\theta X \qquad \Rightarrow \qquad \theta X \in \mathfrak{g}_\alpha.$$

Therefore, writing $X = \frac{X+\theta X}{2} + \frac{X-\theta X}{2}$, we obtain $\mathfrak{g}_\alpha = (\mathfrak{k}\cap\mathfrak{g}_\alpha)\oplus(\mathfrak{p}\cap\mathfrak{g}_\alpha)$ and thus

$$\mathfrak{p} = \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{t})} (\mathfrak{p}\cap\mathfrak{g}_\alpha).$$

Since $\dim_{\mathbb{C}}(\mathfrak{p}\cap\mathfrak{g}_\alpha)\in\{0,1\}$ we see that $\mathfrak{t}$ acts multiplicity-freely on $\mathfrak{p}$.

Let us now consider the case $\operatorname{rk}\mathfrak{k}_0 < \operatorname{rk}\mathfrak{g}_0$. By [Kna02, Proposition 6.60] the centralizer $\mathfrak{h}_0 := Z_{\mathfrak{g}_0}(\mathfrak{t}_0) = \mathfrak{t}_0 \oplus Z_{\mathfrak{p}_0}(\mathfrak{t}_0)$ is a $\theta$-stable Cartan subalgebra of $\mathfrak{g}_0$. Our real rank one assumption shows that $\mathfrak{a}_0 := Z_{\mathfrak{p}_0}(\mathfrak{t}_0)$ is one-dimensional. For $\alpha\in\Delta$ we first note that

$$X\in\mathfrak{g}_\alpha \qquad\Rightarrow\qquad \theta X\in\mathfrak{g}_{\theta\alpha},$$

where we define $(\theta\alpha)(H) := \alpha(\theta H)$. Thus, $\mathfrak{g}_\alpha + \mathfrak{g}_{\theta\alpha}$ is $\theta$-stable and decomposes into a $\mathfrak{k}$- and $\mathfrak{p}$-part.

We claim that if $\alpha$, $\alpha'\in\Delta$ are two roots with $\alpha\big|_{\mathfrak{t}} = \alpha'\big|_{\mathfrak{t}}$, then $\alpha'=\alpha$ or $\alpha'=\theta\alpha$. If this is true we obtain the result as follows. Let $\beta\in\mathfrak{t}^*$. For $\beta=0$ the weight space of $\beta$ in $\mathfrak{p}$ is given by $\mathfrak{a}$, which is one-dimensional. For $\beta\neq 0$ the weight space of $\beta$ in $\mathfrak{p}$ is given by

$$\sum_{\substack{\alpha\in\Delta\\ \alpha|_{\mathfrak{t}}=\beta}} \pi(\mathfrak{g}_\alpha+\mathfrak{g}_{\theta\alpha}),$$

where $\pi\colon \mathfrak{g}\to\mathfrak{p}$, $\quad X\mapsto\frac{X-\theta X}{2}$ denotes the projection onto $\mathfrak{p}$. Then our claim implies that there are at most two roots $\alpha$, $\theta\alpha\in\Delta$ with $\alpha\big|_{\mathfrak{t}} = \theta\alpha\big|_{\mathfrak{t}} = \beta$. Therefore, the weight space of $\beta$ in $\mathfrak{p}$ is given by the one-dimensional space $\pi(\mathfrak{g}_\alpha+\mathfrak{g}_{\theta\alpha})$.

Let us finally prove our claim in the rank one case. By the classification of real forms it suffices to consider the groups $\mathrm{SO}_0(n,1)$ with $n=2p+1$ odd (recall that we are in the case $\operatorname{rk}\mathfrak{k}_0 < \operatorname{rk}\mathfrak{g}_0$). In this case all roots have the same length and this implies our claim since every root $\alpha\in\Delta$ is determined by its restrictions to $\mathfrak{t}$ and $\mathfrak{a}$. ☐

Note that the proof of Proposition 7.3.1 does not use our rank one assumption if $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{k}$. In this case we can say more.

**Proposition 7.3.2.** *Let $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{k}$ and $Y_\tau\in\hat{K}$ with highest weight $\tau$. Denote the non-compact roots by $\Delta_n$. Then the tensor product $Y_\tau\otimes\mathfrak{p}^*$ decomposes into*

$$Y_\tau\otimes\mathfrak{p}^* \cong \bigoplus_{\beta\in\Delta_n} m(\beta)Y_{\tau,\beta},$$

*where the multiplicities $m(\beta)$ are at most $1$ and $Y_{\tau,\beta}$ has weight $\tau+\beta$. Moreover, we have*

$$m(\beta)=1 \qquad\Longrightarrow\qquad \beta\in S,$$

*with $S := \{\beta\in\Delta_n \mid \tau+\beta \text{ dominant}\}\subseteq\Delta_n$.*

*Proof.* First we note that $\mathfrak{p} \cong_K \mathfrak{p}^*$ by the Killing form. By [Kna02, Proposition 9.72] the highest weight of each irreducible constituent of $Y_\tau \otimes \mathfrak{p}$ is of the form $\tau + \beta$, where $\beta$ is a weight of $\mathfrak{p}$, i.e. $\beta \in \Delta_n$. Moreover each irreducible constituent occurs at most with multiplicity one by [Kna02, Chapter IX, §8, Problem 15] since the weight spaces of $\mathfrak{p}$ have multiplicity one by the root space decomposition. Since the highest weight $\tau + \beta$ has to be dominant we can restrict the sum to the subset $S \subseteq \Delta_n$. □

**Notation 7.3.3.** For $V,\ Y \in \hat{K}$ we write

$$V \leftrightarrow Y :\Leftrightarrow V \leq Y \otimes \mathfrak{p} \Leftrightarrow Y \leq V \otimes \mathfrak{p},$$

if $V$ is reachable from $Y$ by tensoring with $\mathfrak{p}$. Here, the second equivalence follows from [BÓØ96, Remark 2.8].

We also introduce a more refined version of reachability using matrix coefficients of $\mathfrak{p}$ and use it to define the generalized gradients we will use to prove the Fourier characterization.

**Definition 7.3.4.** Define the $K$-equivariant map

$$\omega : \mathfrak{p} \to C^\infty(K/M), \quad \omega(X)(kM) := \langle \mathrm{Ad}(k^{-1})X, H \rangle,$$

where $\langle \cdot, \cdot \rangle$ is as in Notation 5.3.1 and $H \in \mathfrak{a}_0$ is defined on page 9. Note that $\omega(H)(eM) = 1$. For each $Y \in \hat{K}_M$ we further define the $K$-equivariant map

$$\omega_Y : Y \otimes \mathfrak{p} \to C^\infty(K/M), \quad \omega_Y(\varphi \otimes X) := \omega(X)\varphi.$$

For $V \in \hat{K}$ with $V \leftrightarrow Y$ we write

$$V \overset{\omega}{\leftrightarrow} Y :\Leftrightarrow V \leq \omega_Y(Y \otimes \mathfrak{p}).$$

Note that $V \overset{\omega}{\leftrightarrow} Y$ implies $V \in \hat{K}_M$ since the image of $\omega_Y$ is contained in $C^\infty(K/M)$. By [BÓØ96, Lemma 4.4 (c)] we have

$$V \overset{\omega}{\leftrightarrow} Y \Leftrightarrow Y \overset{\omega}{\leftrightarrow} V.$$

In this case we realize $V \leq L^2(K/M)$ and define $T_V^Y \in \mathrm{Hom}_K(Y \otimes \mathfrak{p}^*, V)$ by

$$T_V^Y : Y \otimes \mathfrak{p}^* \to V, \quad T_V^Y(\varphi \otimes \psi) := \mathrm{pr}_V(\omega_Y(\varphi \otimes \mathbf{I}^{-1}(\psi))),$$

where $\mathbf{I}$ is as in Notation 5.3.1 and $\mathrm{pr}_V$ denotes the orthogonal projection

$$\mathrm{pr}_V : L^2(K/M) \cong \widehat{\bigoplus}_{W \in \hat{K}_M} W \to V.$$

If $V \leftrightarrow Y$ but not $V \overset{\omega}{\leftrightarrow} Y$ we define

$$T_V^Y : Y \otimes \mathfrak{p}^* \to V, \quad T_V^Y := \mathrm{pr}_V \circ (\mathrm{id}_Y \otimes \mathbf{I}^{-1}), \tag{7.5}$$

with the orthogonal projection $\mathrm{pr}_V : Y \otimes \mathfrak{p} \to V$. In each case, let $\mathrm{d}_V^Y$ denote the associated generalized gradients given by $T_V^Y \circ \nabla$. Since the tensor product decomposes multiplicity-freely by Proposition 7.3.1, there exist uniquely determined homomorphisms $\iota_Y^V \in \mathrm{Hom}_K(V, Y \otimes \mathfrak{p}^*)$ such that

$$T_V^Y \circ \iota_Y^V = \mathrm{id}_V \text{ and } T_V^Y \circ \iota_Y^W = 0 \tag{7.6}$$

for each $W \leftrightarrow Y$ with $V \ncong W$. In Proposition 7.3.9 we give an explicit formula for $\iota_Y^V$ in the case $V \in \hat{K}_M$.

**Example 7.3.5.** In the case $G = \mathrm{PSL}(2, \mathbb{R})$ we have

$$\omega(H)(k_\varphi) = 2\,\mathrm{tr}(H\,\mathrm{Ad}(k_\varphi)H) = \frac{z + z^{-1}}{2}, \quad \omega(B)(k_\varphi) = 2\,\mathrm{tr}(B\,\mathrm{Ad}(k_\varphi)H) = \frac{z^{-1} - z}{2i}$$

with $z := e^{2\varphi i}$, $H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $k_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \in G$. Hence, we obtain for $f \in C^\infty(G \times_K \mathbb{C}_n)$ – by recalling $\mathrm{d}_{L_{n+1}}^{L_n} = T_{L_{n+1}}^{L_n} \circ \nabla$ –

$$\begin{aligned}(\mathrm{d}_{L_{n+1}}^{L_n} f)(g) &= \mathrm{pr}_{L_{n+1}}(\omega_{L_n}((H.f)(g) \otimes H + (B.f)(g) \otimes B)) \\ &= \mathrm{pr}_{L_{n+1}}\left(\frac{z + z^{-1}}{2}(H.f)(g) + \frac{z^{-1} - z}{2i}(B.f)(g)\right) \\ &= \frac{z}{2}((H + iB).f)(g) = z\eta_+ f(g),\end{aligned}$$

where we considered $\mathbb{C}_\ell$ as the one dimensional space spanned by $z^\ell \in C^\infty(\mathbb{S}^{n-1})$.

**Remark 7.3.6.** By definition we have for each $Y \in \hat{K}_M$

$$\sum_{V \overset{\omega}{\leftrightarrow} Y} T_V^Y = \omega_Y \circ (\mathrm{id}_Y \otimes \mathbf{I}^{-1}).$$

In the following we describe the embeddings $\iota_Y^V$ from Definition 7.3.4 in more detail.

**Lemma 7.3.7.** *Let $Y, V \in \hat{K}_M$ with $V \leftrightarrow Y$. Then the operator*

$$\Phi : V \to Y \otimes \mathfrak{p}^*, \quad \Phi(f) := \sum_{j=1}^{\dim \mathfrak{p}} \mathrm{pr}_Y(\omega(X_j)f) \otimes \mathbf{I}(\tilde{X}_j)$$

*is independent of the basis $(X_j)_j$ of $\mathfrak{p}$ and $K$-equivariant. Moreover, the map*

$$V \to V, \quad f \mapsto \sum_{j=1}^{\dim \mathfrak{p}} \mathrm{pr}_V(\omega(\tilde{X}_j)\,\mathrm{pr}_Y(\omega(X_j)f))$$

*is a multiple of the identity. We denote the corresponding scalar by $\lambda(V, Y)$.*

*Proof.* Let $k \in K$ and consider $Y \otimes \mathfrak{p}^*$ as $\operatorname{Hom}(\mathfrak{p}, Y)$ by

$$Y \otimes \mathfrak{p}^* \cong \operatorname{Hom}(\mathfrak{p}, Y), \quad f \otimes \lambda \mapsto (X \mapsto \lambda(X)f).$$

Then, for $f \in V$,

$$\Phi(k.f)(X_i) = \sum_{j=1}^{\dim \mathfrak{p}} \operatorname{pr}_Y(\omega(X_j)(k.f))\mathbf{I}(\tilde{X}_j)(X_i) = \operatorname{pr}_Y(\omega(X_i)(k.f)).$$

By linearity we obtain $\Phi(k.f)(X) = \operatorname{pr}_Y(\omega(X)(k.f))$ for each $X \in \mathfrak{p}$. Note that this expression and thus $\Phi$ is independent of the basis. On the other hand, note that

$$k.\Phi(f) = \sum_{j=1}^{\dim \mathfrak{p}} k. \operatorname{pr}_Y(\omega(X_j)f) \otimes \operatorname{Ad}^*(k)\mathbf{I}(\tilde{X}_j)$$

and thus

$$(k.\Phi(f))(\operatorname{Ad}(k)X_i) = k. \operatorname{pr}_Y(\omega(X_i)f) = \operatorname{pr}_Y((k.\omega(X_i))(k.f)) = \operatorname{pr}_Y(\omega(\operatorname{Ad}(k)X_i)(k.f)).$$

Since $\operatorname{Ad}(k)X_1, \ldots, \operatorname{Ad}(k)X_{\dim \mathfrak{p}}$ is a basis of $\mathfrak{p}$ we have $(k.\Phi(f))(X) = \operatorname{pr}_Y(\omega(X)(k.f))$ for each $X \in \mathfrak{p}$. This proves $\Phi(k.f) = k.\Phi(f)$ and thus the first part of the lemma. From Definition 7.3.4 we recall that

$$\Psi := \operatorname{pr}_V \circ \omega_Y \circ (\operatorname{id}_Y \otimes \mathbf{I}^{-1}) \colon Y \otimes \mathfrak{p}^* \to V$$

is $K$-equivariant. The map in the lemma is given by the composition $\Psi \circ \Phi$. It is scalar by Schur's lemma. ☐

The scalar $\lambda(V, Y)$ has the following properties.

**Proposition 7.3.8** (cf. [BÓØ96, Lemma 4.4, Theorem 4.6])**.** *Let $V$, $Y \in \hat{K}_M$ such that $V \leftrightarrow Y$. Then*

*i)* $\lambda(V, Y) \geq 0$,

*ii)* $V \overset{\omega}{\leftrightarrow} Y \Leftrightarrow \lambda(V, Y) \neq 0 \Leftrightarrow \lambda(Y, V) \neq 0$,

*iii)* $\sum_{W \overset{\omega}{\leftrightarrow} Y} \lambda(Y, W) = 1$,

*iv)* $\lambda(V, Y) \dim V = \lambda(Y, V) \dim Y$.

The embeddings associated with the generalized gradients also admit a nice description using matrix coefficients.

**Proposition 7.3.9.** *Let $Y$, $V \in \hat{K}_M$ with $V \overset{\omega}{\leftrightarrow} Y$. Then we have for each $f \in V$*

$$\iota_Y^V(f) = \frac{1}{\lambda(V, Y)} \sum_{j=1}^{\dim \mathfrak{p}} \operatorname{pr}_Y(\omega(X_j)f) \otimes \mathbf{I}(\tilde{X}_j). \tag{7.7}$$

*Proof.* By Lemma 7.3.7 we know that the rand hand side of (7.7) is $K$-equivariant as a function in $f$. The scalar $\lambda(V,Y)$ is non-zero by Proposition 7.3.8. For each $W \in \hat{K}$ with $W \leftrightarrow Y$ and $V \ncong W$, the map $T_W^Y \circ \iota_Y^V$ is an intertwiner between $V$ and $W$ and thus zero by Schur's lemma. The normalization by $\lambda(V,Y)$ ensures that $T_V^Y \circ \iota_Y^V$ is the identity on $V$. This finishes the proof since we have multiplicity one by Proposition 7.3.1. □

The following lemma provides a method to calculate the scalars $\lambda(V,Y)$ (for explicit forms see Appendix A).

**Lemma 7.3.10.** *The scalar* $\lambda(V,Y)$ *from Lemma 7.3.7 is given by*

$$\lambda(V,Y) = \mathrm{pr}_Y(\omega(H)\phi_V)(eM).$$

*Proof.* If $H = X_1, \ldots, X_{\dim \mathfrak{p}}$ is as in Proposition 5.3.2 and $H = \tilde{X}_1, \ldots, \tilde{X}_{\dim p}$ its dual basis (see Notation 5.3.1) we may write, for each $f \in V$,

$$\iota_Y^V(f) = \sum_{j=1}^{\dim \mathfrak{p}} f_j \otimes \mathbf{I}(\tilde{X}_j) \in Y \otimes \mathfrak{p}^* \tag{7.8}$$

for some $f_1, \ldots, f_{\dim \mathfrak{p}} \in Y$. In particular, we have $\iota_Y^V(f)(H)(eM) = f_1(eM)$ by considering $\iota_Y^V(f)$ as an element of $\mathrm{Hom}(\mathfrak{p}, Y)$. By Definition 7.3.4 and Remark 7.3.6 we infer

$$f = \sum_{W \overset{\omega}{\leftrightarrow} Y} T_W^Y(\iota_Y^V(f)) = \omega_Y((\mathrm{id}_Y \otimes \mathbf{I}^{-1})(\iota_Y^V(f))) = \sum_{j=1}^{\dim \mathfrak{p}} \omega_Y(f_j \otimes \tilde{X}_j) = \sum_{j=1}^{\dim \mathfrak{p}} \omega(\tilde{X}_j) f_j.$$

Note that, since $X_j \in \mathfrak{k} \oplus \mathfrak{n}$ for $j = 2, \ldots, \dim \mathfrak{p}$ and $X_1 \in \mathfrak{a}$, the orthogonality of $\mathfrak{a}$ and $\mathfrak{k} \oplus \mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle$ implies $\omega(\tilde{X}_j)(eM) = \langle \tilde{X}_j, H \rangle = 0$ for each $j = 2, \ldots, \dim \mathfrak{p}$ and therefore

$$f(eM) = \sum_{j=1}^{\dim \mathfrak{p}} \omega(\tilde{X}_j)(eM) f_j(eM) = f_1(eM) = \iota_Y^V(f)(H)(eM).$$

In particular, we have for $f = \phi_V$

$$\iota_Y^V(\phi_V)(H)(eM) = \phi_V(eM) = 1.$$

On the other hand, Proposition 7.3.9 shows that

$$\iota_Y^V(\phi_V)(H)(eM) = \frac{1}{\lambda(V,Y)} \mathrm{pr}_Y(\omega(H)\phi_V)(eM).$$ □

Note that, in the situation of Proposition 5.3.2, we have for $V, Y \in \hat{K}_M$ with $V \overset{\omega}{\leftrightarrow} Y$ that

$$T_Y^V(p_{V,\mu})(e) = (\mu + \rho)(H)\lambda(V,Y) + \nu(V,Y) \text{ with } \nu(V,Y) := T_Y^V(p_{V,-\rho})(e). \tag{7.9}$$

The following lemma allows us to compute the scalars $T_Y^V(p_{V,\mu})(e)$ from Proposition 5.3.2 explicitly in all the rank one cases (see Appendix A).

**Lemma 7.3.11.** *Let $V, Y \in \hat{K}_M$ be such that $V \overset{\omega}{\leftrightarrow} Y$ and $\{0\} \neq U \leq H_\mu$ be a closed $G$-invariant subspace with* $\mathrm{mult}_K(Y, U) \neq 0$ *and* $\mathrm{mult}_K(V, U) = 0$. *Then we have $T_Y^V(p_{V,\mu})(e) = 0$ and thus*

$$\nu(V, Y) = -(\mu + \rho)(H)\lambda(V, Y).$$

*Moreover, for $V \in \hat{K}_M$ with $V \overset{\omega}{\leftrightarrow} \mathbb{C}$ we have*

$$T_{\mathbb{C}}^V(p_{V,\mu})(e) = 0 \Leftrightarrow \mu(H) = \rho(H).$$

*Proof.* Let $0 \neq f \in Y \leq U$. Then, by Equation (5.1), $P_\mu^Y(f)(e) = \frac{1}{\dim Y}\,\mathrm{pr}_V(f) \neq 0$. On the other hand Proposition 5.1.2 implies that $P_\mu^V(f) = 0$. Therefore,

$$0 = \mathrm{d}_Y^V(P_\mu^V(f))(e) = T_Y^V(p_{V,\mu})(e)P_\mu^Y(f)(e)$$

implies that $T_Y^V(p_{V,\mu})(e) = 0$. For $\mu(H) = \rho(H)$ we have that the constant functions form an invariant subspace, proving one direction. For the equivalence note that for each $V \in \hat{K}_M$ with $V \overset{\omega}{\leftrightarrow} \mathbb{C}$, $T_{\mathbb{C}}^V(p_{V,\mu})(e) = \nu(V, \mathbb{C}) + (\mu + \rho)(H)\lambda(V, \mathbb{C})$ is an affine map in $\mu(H)$ with $\lambda(V, \mathbb{C}) \neq 0$ (by Proposition 7.3.8.ii)). ☐

**Proposition 7.3.12.** *Let $Y \in \hat{K}_M$ and $V \in \hat{K}$ with $V \leftrightarrow Y$. Then, for each $\mu \in \mathfrak{a}$,*

$$\mathrm{d}_V^Y \circ P_\mu^Y \neq 0 \Rightarrow V \overset{\omega}{\leftrightarrow} Y.$$

*Proof for $G \neq \mathrm{SO}_0(3,1)$*[1]*.* By Proposition 5.3.2.iii) we see that $\mathrm{d}_V^Y \circ P_\mu^Y \neq 0$ implies that $V \in \hat{K}_M$. Using Proposition 7.3.8.ii), Lemma 7.3.10 and Lemma A.1.2, A.2.1, A.3.1 resp. A.4.1 we infer that $V \overset{\omega}{\leftrightarrow} Y$ if and only if $V \leftrightarrow Y$ and $V \in \hat{K}_M$. ☐

## 7.4. Computations for the Fourier characterization

The aim of this section is proving the converse direction in Proposition 5.3.2, i.e. we want to prove that if the equations derived from Proposition 5.3.2 are satisfied for some distribution $f \in \mathcal{D}'(G/M)$ we already have $f \in H_\mu^{-\infty}$. The precise result is given in Theorem 7.4.11. It provides a technique to determine images for Poisson transforms. We start with the following reformulation of Proposition 5.3.2.

**Lemma 7.4.1.** *Assume the setting from Proposition 5.3.2. Then, for each $f \in H_\mu^{-\infty}$,*

*i)* $(\mathrm{d}_V^Y \circ \pi_Y^*)(f) = T_V^Y(p_{Y,\mu})(e)\frac{\dim Y}{\dim V}\pi_V^*(f)$ *if $V$ is $M$-spherical, i.e. $V \leq L^2(K)^M$,*

*ii)* $(\mathrm{d}_V^Y \circ \pi_Y^*)(f) = 0$ *if $V$ is not $M$-spherical, i.e. $V^M = 0$.*

[1] For $G = \mathrm{SO}_0(3,1)$ we have, for $k \in \mathbb{N}_0$, $Y_k \leftrightarrow Y_k$ but $Y_k \overset{\omega}{\not\leftrightarrow} Y_k$ by Proposition A.1.4 and Lemma A.1.2. Realizing $Y_k$ explicitly as a subrepresentation of $Y_k \otimes \mathfrak{p}^*$ one can prove that $\mathrm{pr}_{Y_k}((\mathrm{id}_Y \otimes \mathbf{I}^{-1})(p_{Y_k,\mu}))(e) = 0$ for each $\mu \in \mathfrak{a}$ and thus $\mathrm{d}_{Y_k}^{Y_k} \circ P_\mu^{Y_k} = 0$ by Proposition 5.3.2.ii). This is done in Section B.2.4.

*Proof.* This is a direct consequence of Proposition 5.3.2 and Lemma 7.1.4 iv). □

We consider the $\mathfrak{a}_0$- and $\mathfrak{n}_0$-action separately and start with the first one.

**Lemma 7.4.2.** *Let $\mu \in \mathfrak{a}^*$ and $f = \sum_{\tau\in\hat{K}_M} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)) \in \mathcal{D}'(G/M)$ (recall Proposition 7.2.4) with $\pi^*_{Y_\tau}(f) \in C^\infty(G \times_K Y_\tau)$ such that the equations from Lemma 7.4.1 i) and ii) hold for $f$ for every irreducible constituent of $Y_\tau \otimes \mathfrak{p}^*$ and every $Y_\tau \in \hat{K}_M$. Let $X \in \mathfrak{a}_0$. For each $V$, $Y_\tau \in \hat{K}_M$ with $V \leftrightarrow Y_\tau$ we define*

$$f_{V,\tau,X} \in C^\infty(G/M), \qquad f_{V,\tau,X}(gM) \coloneqq \iota_V^{Y_\tau}(\pi^*_{Y_\tau}(f)(g))(X)(e).$$

*Then, in the weak sense,*

$$r(X)f = \sum_{\tau\in\hat{K}_M} \sum_{\substack{V \overset{\omega}{\leftrightarrow} Y_\tau \\ V\in\hat{K}_M}} \frac{\dim V}{\dim Y_\tau} T^V_{Y_\tau}(p_{V,\mu})(e) f_{V,\tau,X},$$

*where $r$ denotes the right regular representation of $\mathfrak{a}_0$ on $\mathcal{D}'(G/M)$.*

*Proof.* Note first that, as in the proof of Lemma 7.3.10, $f_{V,\tau,X}(gM) = \pi^*_{Y_\tau}(f)(g)(e)t$ if $X = tH$. For each $\varphi \in C_c^\infty(G/M)$ we have (denoting $f(\varphi)$ by $\langle f, \varphi\rangle$)

$$\langle r(X)f, \varphi\rangle = -\langle f, r(X)\varphi\rangle = -\sum_{\tau\in\hat{K}_M} \langle \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)), r(X)\varphi\rangle = \sum_{\tau\in\hat{K}_M} \langle r(X)\pi_{Y_\tau}(\pi^*_{Y_\tau}(f)), \varphi\rangle.$$

In particular, by the absolute convergence from Proposition 7.2.4, we obtain that

$$\sum_{\tau\in\hat{K}_M} r(X)\pi_{Y_\tau}(\pi^*_{Y_\tau}(f))$$

converges absolutely to $r(X)f$ in the weak sense. We will now compute the summands explicitly. Note first that for each $g \in G$

$$(r(X)\pi_{Y_\tau}(\pi^*_{Y_\tau}(f)))(g) = \frac{d}{dt}\Big|_{t=0} \pi^*_{Y_\tau}(f)(g \exp tX)(e) = (((\nabla \circ \pi^*_{Y_\tau}(f))(g))(X))(e). \tag{7.10}$$

We claim that

$$(\nabla \circ \pi^*_{Y_\tau}(f))(g) = \sum_{V\leftrightarrow Y_\tau} (\iota^V_{Y_\tau} \circ T^{Y_\tau}_V)((\nabla \circ \pi^*_{Y_\tau}(f))(g)). \tag{7.11}$$

Indeed, both sides are elements of $Y_\tau \otimes \mathfrak{p}^*$ and by Definition 7.3.4 they are equal if

$$T^{Y_\tau}_W((\nabla \circ \pi^*_{Y_\tau}(f))(g)) = T^{Y_\tau}_W\left(\sum_{V\leftrightarrow Y_\tau} (\iota^V_{Y_\tau} \circ T^{Y_\tau}_V)((\nabla \circ \pi^*_{Y_\tau}(f))(g))\right)$$

for each irreducible subrepresentation $W$ with $W \leftrightarrow Y_\tau$. But this follows from the definition of the $\iota^V_{Y_\tau}$.

Note that, since $\mathrm{d}_V^{Y_\tau} = T_V^{Y_\tau} \circ \nabla$,

$$\sum_{V \leftrightarrow Y_\tau} (\iota_{Y_\tau}^V \circ T_V^{Y_\tau})((\nabla \circ \pi_{Y_\tau}^*(f))(g)) = \sum_{V \leftrightarrow Y_\tau} \iota_{Y_\tau}^V(\mathrm{d}_V^{Y_\tau}(\pi_{Y_\tau}^*(f))(g)).$$

The equations from Lemma 7.4.1 yield

$$\begin{aligned}(\nabla \circ \pi_{Y_\tau}^*(f))(g) &= \sum_{V \leftrightarrow Y_\tau} \iota_{Y_\tau}^V(\mathrm{d}_V^{Y_\tau}(\pi_{Y_\tau}^*(f))(g)) \\ &= \sum_{\substack{V \leftrightarrow Y_\tau \\ V \in \hat{K}_M}} \iota_{Y_\tau}^V\left(\frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e)\pi_V^*(f)(g)\right) \\ &= \sum_{\substack{V \leftrightarrow Y_\tau \\ V \in \hat{K}_M}} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e)\iota_{Y_\tau}^V(\pi_V^*(f)(g)).\end{aligned}$$

By Proposition 7.3.12 it suffices to sum over all $V \in \hat{K}_M$ with $V \overset{\omega}{\leftrightarrow} Y_\tau$. Using Equation (7.10) we thus obtain

$$\begin{aligned}(r(X)\pi_{Y_\tau}(\pi_{Y_\tau}^*(f)))(g) &= \sum_{\substack{V \overset{\omega}{\leftrightarrow} Y_\tau \\ V \in \hat{K}_M}} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e)((\iota_{Y_\tau}^V(\pi_V^*(f)(g)))(X))(e) \\ &= \sum_{\substack{V \overset{\omega}{\leftrightarrow} Y_\tau \\ V \in \hat{K}_M}} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e) f_{Y_\tau,V,X}(gM)\end{aligned}$$

and $r(X)f = \sum_{\tau \in \hat{K}_M} r(X)\pi_{Y_\tau}(\pi_{Y_\tau}^*(f))$ equals

$$\sum_{\tau \in \hat{K}_M} \sum_{\substack{V \overset{\omega}{\leftrightarrow} Y_\tau \\ V \in \hat{K}_M}} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e) f_{Y_\tau,V,X} = \sum_{V \in \hat{K}_M} \sum_{\substack{V \overset{\omega}{\leftrightarrow} Y_\tau \\ \tau \in \hat{K}_M}} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e) f_{Y_\tau,V,X}.$$

□

In order to compute the sums occurring in Lemma 7.4.2 we write

$$p_{V,\mu} = (\mu + \rho)(H)\phi_V \otimes \mathbf{I}(H) + p_{V,-\rho}. \tag{7.12}$$

Let us first consider the contribution of the first summand in this decomposition.

**Lemma 7.4.3.** *Let $Y \in \hat{K}_M$, $X \in \mathfrak{p}$ and $\varphi \in Y$. Then*

$$\sum_{V \overset{\omega}{\leftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(\phi_V \otimes \mathbf{I}(H))(e)\iota_V^Y(\varphi)(X)(e) = (\omega(X)\varphi)(e).$$

*Proof.* By Definition 7.3.4 and Lemma 7.3.10 we have for each $V \in \hat{K}$ with $V \overset{\omega}{\leftrightarrow} Y$

$$T_Y^V(\phi_V \otimes \mathbf{I}(H))(e) = \mathrm{pr}_Y(\omega(H)\phi_V)(e) = \lambda(V, Y).$$

Using Proposition 7.3.8 iv) and 7.3.9 we calculate

$$\begin{aligned}
&\sum_{V \overset{\omega}{\leftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(\phi_V \otimes \mathbf{I}(H))(e)\iota_V^Y(\varphi)(X)(e) \\
&= \sum_{V \overset{\omega}{\leftrightarrow} Y} \frac{\lambda(Y,V)}{\lambda(V,Y)}\lambda(V,Y)\frac{1}{\lambda(Y,V)} \sum_{j=1}^{\dim \mathfrak{p}} \mathrm{pr}_V(\omega(X_j)\varphi)(e)\mathbf{I}(\tilde{X}_j)(X) \\
&= \sum_{V \overset{\omega}{\leftrightarrow} Y} \sum_{j=1}^{\dim \mathfrak{p}} \mathrm{pr}_V(\omega(X_j)\varphi)(e)\mathbf{I}(\tilde{X}_j)(X) \\
&= \sum_{V \overset{\omega}{\leftrightarrow} Y} \mathrm{pr}_V(\omega(X)\varphi)(e) = (\omega(X)\varphi)(e). \qquad \square
\end{aligned}$$

For the contribution of the second summand in (7.12) we need some preparation. This is the content of the following three lemmas.

**Lemma 7.4.4.** *Let $\mathfrak{g}_0$ be a semisimple Lie algebra, $B$ be some non-zero multiple of the Killing form $\kappa$. If $X_1, \ldots, X_{\dim(\mathfrak{p}_0/\mathfrak{a}_0)}$ is a basis of $\mathfrak{p}_0 \cap (\mathfrak{k}_0 \oplus \mathfrak{n}_0)$ let $\tilde{X}_1, \ldots, \tilde{X}_{\dim(\mathfrak{p}_0/\mathfrak{a}_0)}$ denote the dual basis defined by $B(\tilde{X}_i, X_j) = \delta_{ij}$. Then $\sum_{j=1}^{\dim(\mathfrak{p}_0/\mathfrak{a}_0)}[\tilde{X}_j, k_I(X_j)] \in \mathfrak{a}_0$ and*

$$\sum_{j=1}^{\dim(\mathfrak{p}_0/\mathfrak{a}_0)} B([\tilde{X}_j, k_I(X_j)], H) = 2\rho(H) \quad \forall H \in \mathfrak{a}_0.$$

*Proof of Lemma 7.4.4.* We first claim that $\sum_{j=1}^{\dim(\mathfrak{p}_0/\mathfrak{a}_0)}[\tilde{X}_j, k_I(X_j)] \in \mathfrak{p}_0$ is independent of the basis. Let $X_1', \ldots, X_{\dim(\mathfrak{p}_0/\mathfrak{a}_0)}'$ be another basis with base change matrix $(a_{ij})$, i.e. $X_j' = \sum_m a_{mj}X_m$. If $(b_{ij})$ denotes the inverse of $(a_{ij})$ we claim that $\tilde{X}_j' = \sum_\ell b_{j\ell}\tilde{X}_\ell$. Indeed,

$$B(\sum_\ell b_{j\ell}\tilde{X}_\ell, X_i') = B(\sum_\ell b_{j\ell}\tilde{X}_\ell, \sum_m a_{mi}X_m) = \sum_\ell\sum_m b_{j\ell}a_{mi}B(\tilde{X}_\ell, X_m) = \sum_m b_{jm}a_{mi} = \delta_{ij}.$$

Thus,

$$\begin{aligned}
\sum_j [\tilde{X}_j', k_I(X_j')] &= \sum_j \left[\sum_\ell b_{j\ell}\tilde{X}_\ell, k_I(\sum_m a_{mj}X_m)\right] = \sum_m\sum_\ell [\tilde{X}_\ell, k_I(X_m)]\sum_j a_{mj}b_{j\ell} \\
&= \sum_m\sum_\ell [\tilde{X}_\ell, k_I(X_m)]\delta_{m\ell} = \sum_m [\tilde{X}_m, k_I(X_m)]
\end{aligned}$$

is independent of the basis.

We will now construct a convenient basis of $\mathfrak{p}_0 \cap (\mathfrak{k}_0 \oplus \mathfrak{n}_0)$. Let $\Sigma^+$ denote the set of positive restricted roots. We may assume that $B$ is a positive multiple of the Killing form (otherwise $-B$ is of this form and the signs of the $\tilde{X}_j$'s are flipped). For each $\lambda \in \Sigma^+$ we choose a basis $Y_1^\lambda, \ldots, Y_{\dim \mathfrak{g}_0^\lambda}^\lambda$ of the restricted root space $\mathfrak{g}_0^\lambda$ such that $B(Y_j^\lambda, \theta Y_k^\lambda) = -\frac{1}{2}\delta_{jk}$, where $\theta$ denotes the Cartan involution, and define

$$X_j^\lambda := Y_j^\lambda - \theta Y_j^\lambda, \quad j \in \{1, \ldots, \dim \mathfrak{g}_0^\lambda\}.$$

Note that, since

$$B(X_j^\lambda, X_k^\mu) = -2B(Y_j^\lambda, \theta Y_k^\mu) = -2B(Y_j^\lambda, \theta Y_k^\mu)\delta_{\lambda\mu},$$

we have that the $X_j^\lambda$'s are orthonormal, i.e. $\tilde{X}_j^\lambda = X_j^\lambda$. By the restricted root space decomposition, every $X \in \mathfrak{p}_0 \cap (\mathfrak{k}_0 \oplus \mathfrak{n}_0)$ is of the form $\sum_{\lambda \in \Sigma^+} X_\lambda - \theta X_\lambda$ for some $X_\lambda \in \mathfrak{g}_0^\lambda$. Therefore, the $X_j^\lambda$, $\lambda \in \Sigma^+$, form a basis of $\mathfrak{p}_0 \cap (\mathfrak{k}_0 \oplus \mathfrak{n}_0)$. Note that

$$X_j^\lambda = 2Y_j^\lambda - (Y_j^\lambda + \theta Y_j^\lambda) \in \mathfrak{n}_0 \oplus \mathfrak{k}_0 \qquad \Longrightarrow \qquad k_I(X_j^\lambda) = -(Y_j^\lambda + \theta Y_j^\lambda).$$

By the invariance of the Killing form we deduce for each $H \in \mathfrak{a}_0$

$$\begin{aligned} B([\tilde{X}_j^\lambda, k_I(X_j^\lambda)], H) = B(\tilde{X}_j^\lambda, [k_I(X_j^\lambda), H]) \qquad &= B(\tilde{X}_j^\lambda, [H, Y_j^\lambda + \theta Y_j^\lambda]) \\ = B(\tilde{X}_j^\lambda, \lambda(H)(Y_j^\lambda - \theta Y_j^\lambda)) &= \lambda(H) B(\tilde{X}_j^\lambda, X_j^\lambda) = \lambda(H). \end{aligned}$$

Thus,

$$\sum_{\lambda \in \Sigma^+} \sum_{j=1}^{\dim \mathfrak{g}_0^\lambda} B([\tilde{X}_j^\lambda, k_I(X_j^\lambda)], H) = \sum_{\lambda \in \Sigma^+} \lambda(H) \dim \mathfrak{g}_0^\lambda = 2\rho(H).$$

Moreover,

$$[\tilde{X}_j^\lambda, k_I(X_j^\lambda)] = [X_j^\lambda, k_I(X_j^\lambda)] = [Y_j^\lambda - \theta Y_j^\lambda, -(Y_j^\lambda + \theta Y_j^\lambda)] = 2[\theta Y_j^\lambda, Y_j^\lambda] \in \mathfrak{g}_0^0 \cap \mathfrak{p}_0$$

implies that $[\tilde{X}_j^\lambda, k_I(X_j^\lambda)] \in \mathfrak{a}_0$ since $\mathfrak{g}_0^0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0$. □

**Lemma 7.4.5.** *Let $X_1, \ldots, X_{\dim \mathfrak{p}}$ be as in Proposition 5.3.2. Then $\sum_{j=2}^{\dim \mathfrak{p}} \ell(k_I(X_j))\omega(\tilde{X}_j) = -2\rho(H)\omega(H)$.*

*Proof.* Since $\omega : \mathfrak{p} \to C^\infty(K/M)$ is $K$-equivariant we have

$$\sum_{j=2}^{\dim \mathfrak{p}} \ell(k_I(X_j))\omega(\tilde{X}_j) = \sum_{j=2}^{\dim \mathfrak{p}} \omega([k_I(X_j), \tilde{X}_j])$$

By Lemma 7.4.4, $\sum_{j=2}^{\dim \mathfrak{p}} [k_I(X_j), \tilde{X}_j]$ is an element of $\mathfrak{a}_0$ and therefore a multiple of $H$. Let $\lambda \in \mathbb{R}$ denote this multiple. Then Lemma 7.4.4 implies that

$$\lambda = \langle \lambda H, H \rangle = \sum_{j=2}^{\dim \mathfrak{p}} \langle [k_I(X_j), \tilde{X}_j], H \rangle = -2\rho(H). \qquad \square$$

**Lemma 7.4.6.** *Let $Y \in \hat{K}_M$ and $X \in \mathfrak{p}$. Then*

$$\sum_{V \overset{\omega}{\leftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \iota_Y^V(\phi_V)(X) = \begin{cases} -2\rho(H)\phi_Y & : X = H \\ \ell(k_I(X))\phi_Y & : X \perp \mathfrak{a} \end{cases},$$

*where the bar denotes complex conjugation.*

*Proof.* For each $\psi \in L^2(K/M)$ we have by orthogonality and Proposition 2.4.4 ii),

$$\mathrm{pr}_Y(\psi)(e) = \langle \mathrm{pr}_Y(\psi), \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)} = \langle \psi, \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)}. \tag{7.13}$$

Therefore, since $\omega(X)$, $X \in \mathfrak{p}_0$, is real valued (second step) and using the product rule and Lemma 7.4.5 (third step), $T_Y^V(p_{V,-\rho})(e)$ equals

$$\begin{aligned} -\sum_{j=2}^{\dim \mathfrak{p}} \mathrm{pr}_Y(\omega(\tilde{X}_j)\ell(k_I(X_j))\phi_V)(e) &= -\langle \sum_{j=2}^{\dim \mathfrak{p}} \omega(\tilde{X}_j)\ell(k_I(X_j))\phi_V, \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)} \\ &= -\sum_{j=2}^{\dim \mathfrak{p}} \langle \ell(k_I(X_j))\phi_V, \omega(\tilde{X}_j) \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)} \\ &= \langle \phi_V, -2\rho(H)\omega(H) \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)} \\ &\quad + \langle \phi_V, \sum_{j=2}^{\dim \mathfrak{p}} \omega(\tilde{X}_j)\ell(k_I(X_j)) \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)}. \end{aligned}$$

Applying Equation (7.13) for $V$ and Proposition 2.4.4 iii) to this equation, we infer that $\dim V \cdot T_Y^V(p_{V,-\rho})(e) = \dim Y \cdot \overline{T_V^Y(-p_{Y,\rho})(e)}$ and thus

$$\sum_{V \overset{\omega}{\leftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \iota_Y^V(\phi_V)(X) = \sum_{V \overset{\omega}{\leftrightarrow} Y} T_V^Y(-p_{Y,\rho})(e) \iota_Y^V(\phi_V)(X).$$

Note that $T_V^Y(-p_{Y,\rho}) \in V$ is left $M$-invariant since $p_{Y,\rho}$ is left $M$-invariant by Proposition 5.3.2 i) and $T_V^Y : Y \otimes \mathfrak{p}^* \to V$ is $K$-equivariant. Therefore it is a multiple of $\phi_V$ and we have $T_V^Y(-p_{Y,\rho}) = T_V^Y(-p_{Y,\rho})(e)\phi_V$. We infer that

$$\sum_{V \overset{\omega}{\leftrightarrow} Y} T_V^Y(-p_{Y,\rho})(e) \iota_Y^V(\phi_V)(X) = \sum_{V \overset{\omega}{\leftrightarrow} Y} \iota_Y^V(T_V^Y(-p_{Y,\rho}))(X) = -p_{Y,\rho}(X).$$

The lemma now follows from the definition of $p_{Y,\rho}(X)$. □

We are now able to compute the contribution of the second part in (7.12).

**Lemma 7.4.7.** *Let $Y \in \hat{K}_M$, $X \in \mathfrak{p}$ and $\varphi \in Y$. Then*

$$\sum_{V \overset{\omega}{\leftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(p_{V,-\rho})(e) \iota_V^Y(\varphi)(X)(e) = \begin{cases} -2\rho(H)\varphi(e) & : X = H \\ -(\ell(k_I(X))\varphi)(e) & : X \perp \mathfrak{a} \end{cases}.$$

*Proof.* Note first that Proposition 7.3.9 implies that

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(p_{V,-\rho})(e) \iota_V^Y(\varphi)(X)(e) = \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(p_{V,-\rho})(e) \frac{1}{\lambda(Y,V)} \operatorname{pr}_V(\omega(X)\varphi)(e).$$

By Equation (7.13) we infer that

$$\begin{aligned}
&\sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(p_{V,-\rho})(e) \frac{1}{\lambda(Y,V)} \operatorname{pr}_V(\omega(X)\varphi)(e) \\
&= \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(p_{V,-\rho})(e) \frac{1}{\lambda(Y,V)} \langle \omega(X)\varphi, \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} \rangle_{L^2(K)} \\
&= \langle \varphi, \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \frac{1}{\lambda(Y,V)} \omega(X) \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} \rangle_{L^2(K)} \\
&= \langle \varphi, \operatorname{pr}_Y( \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \frac{1}{\lambda(Y,V)} \omega(X) \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} ) \rangle_{L^2(K)},
\end{aligned}$$

where the last equation follows from $\varphi \in Y$ and the orthogonality of the $K$-types. Using Propositions 7.3.9 and 7.3.8 we deduce that

$$\begin{aligned}
&\operatorname{pr}_Y( \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \frac{1}{\lambda(Y,V)} \omega(X) \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} ) \\
&= \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \frac{1}{\lambda(Y,V)} \operatorname{pr}_Y(\omega(X) \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} ) \\
&= \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \frac{\lambda(V,Y)}{\lambda(Y,V)} \iota_Y^V( \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} )(X) \\
&= \sum_{V \overset{\omega}{\longleftrightarrow} Y} \overline{T_Y^V(p_{V,-\rho})(e)} \iota_Y^V( \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} )(X).
\end{aligned}$$

Finally Proposition 2.4.4 iii) and Lemma 7.4.6 imply that

$$\begin{aligned}
&\sum_{V \overset{\omega}{\longleftrightarrow} Y} \overline{T_Y^V(p_{V,-\rho})(e)} \iota_Y^V( \frac{\phi_V}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} )(X) \\
&= \frac{1}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}}{\langle \phi_V, \phi_V \rangle_{L^2(K)}} \overline{T_Y^V(p_{V,-\rho})(e)} \iota_Y^V(\phi_V)(X) \\
&= \frac{1}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} \overline{T_Y^V(p_{V,-\rho})(e)} \iota_Y^V(\phi_V)(X) \\
&= \frac{1}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \begin{cases} -2\rho(H)\phi_Y & : X = H \\ \ell(k_I(X))\phi_Y & : X \perp \mathfrak{a} \end{cases}.
\end{aligned}$$

Summarizing, we have for $X = H$

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(p_{V,-\rho})(e)\iota_V^Y(\varphi)(X)(e)$$
$$= -2\rho(H)\langle \varphi, \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)} = -2\rho(H)\varphi(e)$$

and for $X \in \mathfrak{p}$ with $X \perp \mathfrak{a}$

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y} \frac{\dim V}{\dim Y} T_Y^V(p_{V,-\rho})(e)\iota_V^Y(\varphi)(X)(e) = \langle \varphi, \ell(k_I(X)) \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)}$$
$$= -\langle \ell(k_I(X))\varphi, \frac{\phi_Y}{\langle \phi_Y, \phi_Y \rangle_{L^2(K)}} \rangle_{L^2(K)} = -(\ell(k_I(X))\varphi)(e). \qquad \square$$

We are now ready to prove the Fourier characterization.

**Proposition 7.4.8.** *In the setting of Lemma 7.4.2 we have*

$$r(H)f = (\mu - \rho)(H)f.$$

*Proof.* By Lemma 7.4.2 we have

$$r(H)f = \sum_{\tau \in \hat{K}_M} \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim V}{\dim Y_\tau} T_{Y_\tau}^V(p_{V,\mu})(e) f_{V,\tau,H},$$

with (for $g \in G$) $f_{V,\tau,H}(gM) \coloneqq \iota_V^{Y_\tau}(\pi_{Y_\tau}^*(f)(g))(H)(e)$. Lemma 7.4.3 and 7.4.7 imply that

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim V}{\dim Y_\tau} T_{Y_\tau}^V(p_{V,\mu})(e)\iota_V^{Y_\tau}(\pi_{Y_\tau}^*(f)(g))(H)(e)$$
$$= (\mu + \rho)(H)\pi_{Y_\tau}^*(f)(g)(e) - 2\rho(H)\pi_{Y_\tau}^*(f)(g)(e)$$
$$= (\mu - \rho)(H)\pi_{Y_\tau}^*(f)(g)(e)$$
$$= (\mu - \rho)(H)\pi_{Y_\tau}(\pi_{Y_\tau}^*(f))(g).$$

Thus, $r(H)f = \sum_{\tau \in \hat{K}_M} (\mu - \rho)(H)\pi_{Y_\tau}(\pi_{Y_\tau}^*(f)) = (\mu - \rho)(H)f.$ $\square$

**Remark 7.4.9.** In view of $f_{V,\tau,H}(gM) = \pi_{Y_\tau}^*(f)(g)(e)$ (as in the proof of Lemma 7.3.10), Proposition 7.4.8 proves

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim V}{\dim Y_\tau} T_{Y_\tau}^V(p_{V,\mu})(e) = \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\lambda(Y_\tau, V)}{\lambda(V, Y_\tau)} T_{Y_\tau}^V(p_{V,\mu})(e) = (\mu - \rho)(H).$$

**Proposition 7.4.10.** *Let $\mu \in \mathfrak{a}^*$ and $f = \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\pi_{Y_\tau}^*(f)) \in \mathcal{D}'(G/M)$ be as in Lemma 7.2.5 such that the equations from Lemma 7.4.1 i) and ii) hold for $f$ for every irreducible constituent of $Y_\tau \otimes \mathfrak{p}^*$ and every $Y_\tau \in \hat{K}_M$. Let $U_+ \in C^\infty(G \times_M \mathfrak{n})$ be a smooth section. Then $U_+ f = 0$.*

*Proof.* Note first that

$$U_+ f = \sum_{\tau \in \hat{K}_M} U_+ \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)).$$

Let $X_1, \ldots, X_{\dim \mathfrak{n}}$ be a basis of $\mathfrak{n}_0$. Then there exist functions $\kappa_j \in C^\infty(G)$ such that

$$U_+(g) = \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) X_j \quad \forall g \in G.$$

Writing $k_C(X_j)$ resp. $p_C(X_j)$ for the $\mathfrak{k}$- resp. $\mathfrak{p}$-part of the Cartan decomposition of $Y_j$ we define

$$U_+^{\mathfrak{k}}(g) \coloneqq \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) k_C(X_j), \quad U_+^{\mathfrak{p}}(g) \coloneqq \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) p_C(X_j).$$

Note that, by definition of $U_+$ and since $M$ preserves the Cartan decomposition, we have

$$U_+(gm) = \mathrm{Ad}(m^{-1}) U_+(g),\ U_+^{\mathfrak{k}}(gm) = \mathrm{Ad}(m^{-1}) U_+^{\mathfrak{k}}(g),\ U_+^{\mathfrak{p}}(gm) = \mathrm{Ad}(m^{-1}) U_+^{\mathfrak{p}}(g)$$

for each $g \in G$ and $m \in M$. We have

$$\begin{aligned} U_+^{\mathfrak{k}} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))(gM) &= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) \left.\frac{d}{dt}\right|_{t=0} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))(g \exp t k_C(X_j) M) \\ &= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) \left.\frac{d}{dt}\right|_{t=0} \pi^*_{Y_\tau}(f)(g \exp t k_C(X_j))(e) \\ &= -\sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) (\ell(k_C(X_j)) \pi^*_{Y_\tau}(f)(g))(e). \end{aligned} \tag{7.14}$$

For the $\mathfrak{p}$-part we obtain

$$\begin{aligned} U_+^{\mathfrak{p}} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))(gM) &= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) \left.\frac{d}{dt}\right|_{t=0} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))(g \exp t p_C(X_j) M) \\ &= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) \left.\frac{d}{dt}\right|_{t=0} \pi^*_{Y_\tau}(f)(g \exp t p_C(X_j))(e) \\ &= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) (((\nabla \circ \pi^*_{Y_\tau}(f))(g))(p_C(X_j)))(e). \end{aligned}$$

As in the proof of Lemma 7.4.2 we infer that

$$U_+^{\mathfrak{p}} \pi_{Y_\tau}(\pi^*_{Y_\tau}(f))(gM) = \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e) \iota_{Y_\tau}^V(\pi^*_V(f)(g))(p_C(X_j))(e).$$

If we define

$$\Psi_{V,Y_\tau} \in C^\infty(G/M), \quad \Psi_{V,Y_\tau}(gM) := \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g)\iota^V_{Y_\tau}(\pi^*_V(f)(g))(p_C(X_j))(e)$$

we thus have

$$U^{\mathfrak{p}}_+ \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)) = \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim Y_\tau}{\dim V} T^{Y_\tau}_V(p_{Y_\tau,\mu})(e)\Psi_{V,Y_\tau}.$$

Before we consider the series over these expressions, we make sure that each summand converges faster to zero than any polynomial when tested against some test function $\varphi \in C_c^\infty(G)^M$. Indeed, it suffices to consider, for some $j \in \{1, \dots, \dim \mathfrak{n}\}$,

$$\begin{aligned}
&\left| \int_G \iota^V_{Y_\tau}(\pi^*_V(f)(g))(p_C(X_j))(e)\varphi(g)\,\mathrm{d}g \right| \\
&= \frac{1}{\lambda(V, Y_\tau)} \left| \int_G \mathrm{pr}_{Y_\tau}(\omega(p_C(X_j))\pi^*_V(f)(g))(e)\varphi(g)\,\mathrm{d}g \right| \\
&= \frac{1}{\lambda(V, Y_\tau)} \left| \langle \int_G \omega(p_C(X_j))\varphi(g)\pi^*_V(f)(g)\,\mathrm{d}g, \frac{\phi_{Y_\tau}}{\langle \phi_{Y_\tau}, \phi_{Y_\tau}\rangle} \rangle \right| \\
&\leq \frac{\sqrt{\dim Y_\tau}}{\lambda(V, Y_\tau)} \max_{k\in K} |\omega(p_C(X_j))(k)| \left\| \int_G \varphi(g)\pi^*_V(f)(g)\,\mathrm{d}g \right\|_{L^2(K)},
\end{aligned}$$

where we used Proposition 7.3.9 in the first, Equation (7.13) in the second and Proposition 2.4.4 and the Cauchy-Schwarz inequality in the third step. For $k \in K$ we set $\varphi_k(g) := \varphi(gk^{-1})$ and write

$$\begin{aligned}
\left\| \int_G \varphi(g)\pi^*_V(f)(g)\,\mathrm{d}g \right\|_{L^2(K)} &\leq \max_{k\in K} \left| \int_G \varphi(g)\pi^*_V(f)(g)(k)\,\mathrm{d}g \right| \\
&= \max_{k\in K} \left| \int_G \varphi(g)\pi^*_V(f)(gk)(e)\,\mathrm{d}g \right| \\
&= \max_{k\in K} \left| \int_G \varphi_k(g)\pi^*_V(f)(g)(e)\,\mathrm{d}g \right|.
\end{aligned}$$

Now, using Remark 7.2.2 we may proceed as in Lemma 7.2.5 to prove the convergence. Hence, we can consider

$$\begin{aligned}
U^{\mathfrak{p}}_+ f &= \sum_{Y_\tau \in \hat{K}_M} U^{\mathfrak{p}}_+ \pi_{Y_\tau}(\pi^*_{Y_\tau}(f)) = \sum_{\tau \in \hat{K}_M} \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim Y_\tau}{\dim V} T^{Y_\tau}_V(p_{Y_\tau,\mu})(e)\Psi_{V,Y_\tau} \\
&= \sum_{V \in \hat{K}_M} \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim Y_\tau}{\dim V} T^{Y_\tau}_V(p_{Y_\tau,\mu})(e)\Psi_{V,Y_\tau}.
\end{aligned}$$

Finally Lemmas 7.4.3 and 7.4.7 imply that, for $V \in \hat{K}_M$ fixed,

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim Y_\tau}{\dim V} T^{Y_\tau}_V(p_{Y_\tau,\mu})(e)\Psi_{V,Y_\tau}(gM)$$

$$= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g) \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e) \iota_{Y_\tau}^V(\pi_V^*(f)(g))(p_C(X_j))(e)$$
$$= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g)(-\ell(k_I(p_C(X_j)))\pi_V^*(f)(g))(e)$$
$$= \sum_{j=1}^{\dim \mathfrak{n}} \kappa_j(g)(\ell(k_C(X_j))\pi_V^*(f)(g))(e). \tag{7.15}$$

Combining Equation (7.14) and (7.15) we infer

$$U_+ f = U_+^{\mathfrak{k}} f + U_+^{\mathfrak{p}} f = \sum_{V \in \hat{K}_M} U_+^{\mathfrak{k}} \pi_V(\pi_V^*(f)) + \sum_{V \overset{\omega}{\longleftrightarrow} Y_\tau} \frac{\dim Y_\tau}{\dim V} T_V^{Y_\tau}(p_{Y_\tau,\mu})(e) \Psi_{V,Y_\tau} = 0. \quad \square$$

**Theorem 7.4.11** (Fourier characterization of spherical principal series)**.** *Let $\mu \in \mathfrak{a}^*$ and $f = \sum_{\tau \in \hat{K}_M} \pi_{Y_\tau}(\pi_{Y_\tau}^*(f)) \in \mathcal{D}'(G/M)$ be as in Lemma 7.2.5 such that the equations from Lemma 7.4.1 i) and ii) hold for $f$ for every irreducible constituent of $Y_\tau \otimes \mathfrak{p}^*$ and every $Y_\tau \in \hat{K}_M$. Then $f \in H_\mu^{-\infty}$.*

*Proof.* This follows from Propositions 7.4.8, Proposition 7.4.10 and the characterization $\mathcal{R}(\mu - \rho)$ of $H_\mu^{-\infty}$ from (2.4). $\square$

## 7.5. Properties of generalized gradients

We conclude this chapter with some results on the structure of generalized gradients. The following result is independent of the basis $(X_j)_j$ of $\mathfrak{p}$.

**Lemma 7.5.1.** *Let $Y \in \hat{K}_M$ and $f \in C^\infty(G \times_K Y)$. Then, for each $g \in G$,*

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y} \mathrm{d}_V^Y(f)(g) = \sum_{j=1}^{\dim \mathfrak{p}} \omega(\widetilde{X}_j)(r(X_j)f)(g).$$

*Proof.* By definition we have $(\nabla f)(g) = \sum_{j=1}^{\dim \mathfrak{p}} (r(X_j)f)(g) \otimes \mathbf{I}(\widetilde{X}_j) \in Y \otimes \mathfrak{p}^*$. Therefore,

$$(\omega_Y \circ (\mathrm{id}_Y \otimes \mathbf{I}^{-1}))((\nabla f)(g)) = \sum_{j=1}^{\dim \mathfrak{p}} \omega(\widetilde{X}_j)(r(X_j)f)(g).$$

By Remark 7.3.6 and $\mathrm{d}_V^Y = T_V^Y \circ \nabla$ we obtain

$$\sum_{V \overset{\omega}{\longleftrightarrow} Y} \mathrm{d}_V^Y(f)(g) = \sum_{j=1}^{\dim \mathfrak{p}} \omega(\widetilde{X}_j)(r(X_j)f)(g). \quad \square$$

**Lemma 7.5.2.** *Let $V, Y \in \hat{K}_M$ with $V \overset{\omega}{\longleftrightarrow} Y$, $\varphi \in C^\infty(G \times_K Y)$ and $\psi \in C^\infty(G \times_K V)$. Then, if one side exists,*

$$\langle \pi_Y(\varphi), \pi_Y(\mathrm{d}_Y^V(\psi)) \rangle_{L^2(G)} = -\langle \pi_V(\mathrm{d}_V^Y(\varphi)), \pi_V(\psi) \rangle_{L^2(G)}.$$

*Proof.* Note first that if $Y \neq W \in \hat{K}$ and $\eta \in C^\infty(G \times_K W)$ we have

$$\langle \pi_V(\varphi), \pi_W(\eta) \rangle_{L^2(G)} = 0$$

by splitting the integral into $G/K$ and $K$. Therefore we obtain

$$\langle \pi_Y(\varphi), \pi_Y(\mathrm{d}_Y^V(\psi)) \rangle_{L^2(G)} = \langle \pi_Y(\varphi), \sum_{W \overset{\omega}{\leftrightarrow} V} \pi_W(\mathrm{d}_W^V(\psi)) \rangle_{L^2(G)}.$$

Evaluating Lemma 7.5.1 at $eM \in K/M$ yields (since $\omega(\widetilde{X}_j)(eM) = 0$ for $j \geq 2$)

$$\sum_{W \overset{\omega}{\leftrightarrow} V} \pi_W(\mathrm{d}_W^V(\psi)) = r(H)\pi_V(\psi).$$

Together we conclude that

$$\langle \pi_Y(\varphi), \pi_Y(\mathrm{d}_Y^V(\psi)) \rangle_{L^2(G)} = \langle \pi_Y(\varphi), r(H)\pi_V(\psi) \rangle_{L^2(G)} = -\langle r(H)\pi_Y(\varphi), \pi_V(\psi) \rangle_{L^2(G)},$$

where we used the right-invariance of the Haar measure on $G$. The same argument yields

$$\langle r(H)\pi_Y(\varphi), \pi_V(\psi) \rangle_{L^2(G)} = \langle \pi_V(\mathrm{d}_V^Y(\varphi)), \pi_V(\psi) \rangle_{L^2(G)}.$$

□

We may also define operators $T_W^V$ and $\iota_V^W$ for each $V \in \hat{K}$ (and not just $V \in \hat{K}_M$) in analogy to Equations (7.5) and (7.6) by Proposition 7.3.1. With this we can formulate the following lemma, which is especially useful if $\mathrm{d}_V^Y f = 0$ for all $V \in \hat{K}$ with $V \leftrightarrow Y$ but not $V \overset{\omega}{\leftrightarrow} Y$ (e.g. as in Proposition 7.3.12).

**Lemma 7.5.3.** *Let $Y \in \hat{K}_M$ and $\Omega_\mathfrak{p} := \sum_{i=1}^{\dim \mathfrak{p}} \tilde{X}_i X_i \in \mathcal{U}(\mathfrak{g})$, for some basis $(X_i)_{i=1}^{\dim \mathfrak{p}}$ of $\mathfrak{p}_0$, denote the "Casimir element of $\mathfrak{p}$" with respect to $\langle \cdot, \cdot \rangle$ from Equation* (1.1)*. Then, for $V \leftrightarrow Y$ and $W \leftrightarrow V$,*

$$\mathcal{F}_Y^{W,V} : W \to Y, \quad \varphi \mapsto \sum_{j=1}^{\dim \mathfrak{p}} \iota_Y^V(\iota_V^W(\varphi)(\tilde{X}_j))(X_j)$$

*is $K$-equivariant and*

$$\forall f \in C^\infty(G \times_K Y),\ g \in G\colon \quad (r(\Omega_\mathfrak{p})f)(g) = \sum_{V \leftrightarrow Y} \mathcal{F}_Y^{Y,V}(\mathrm{d}_Y^V \mathrm{d}_V^Y f(g)).$$

*Each $\mathcal{F}_Y^{Y,V}$ is a multiple of the identity on $Y$. For $V \overset{\omega}{\leftrightarrow} Y$ this multiple is given by $\frac{1}{\lambda(V,Y)}$.*

*Proof.* Note first that for each $f \in C^\infty(G \times_K Y)$ and $g \in G$

$$\nabla f(g) = \sum_{V \leftrightarrow Y} \iota_Y^V \mathrm{d}_V^Y f(g) \in Y \otimes \mathfrak{p}^*,$$

since, by definition, both sides agree after applying $T_V^Y$ for some $V \in \hat{K}$ with $V \leftrightarrow Y$. In particular, we infer for $X \in \mathfrak{p}$

$$r(X)f(g) = \sum_{V \leftrightarrow Y} \iota_Y^V(\mathrm{d}_V^Y f(g))(X).$$

Moreover, for another element $Z \in \mathfrak{p}$, by applying the previous equation twice,

$$\begin{aligned}(r(Z)r(X)f)(g) &= \sum_{V\leftrightarrow Y} \iota_Y^V(r(Z)\mathrm{d}_V^Y f(g))(X) \\ &= \sum_{V\leftrightarrow Y} \iota_Y^V(\sum_{W\leftrightarrow V} \iota_V^W(\mathrm{d}_W^V \mathrm{d}_V^Y f(g))(Z))(X) \\ &= \sum_{V\leftrightarrow Y}\sum_{W\leftrightarrow V} \iota_Y^V(\iota_V^W(\mathrm{d}_W^V \mathrm{d}_V^Y f(g))(Z))(X).\end{aligned}$$

Hence, we obtain

$$(r(\Omega_\mathfrak{p})f)(g) = \sum_{V\leftrightarrow Y}\sum_{W\leftrightarrow V} \mathcal{F}_Y^{W,V}(\mathrm{d}_W^V \mathrm{d}_V^Y f(g)).$$

We now prove that $\mathcal{F}_Y^{W,V}$ is $K$-equivariant. Indeed, note first that by the $K$-equivariance of $\iota_V^W$

$$\forall X \in \mathfrak{p},\ k \in K\colon \quad \iota_V^W(\tau_W(k)\varphi)(X) = \tau_V(k)\iota_V^W(\varphi)(\mathrm{Ad}(k^{-1})(X)),$$

where $\tau_V$ resp. $\tau_W$ denote the $K$-actions on $V$ resp. $W$. Thus,

$$\begin{aligned}\mathcal{F}_Y^{W,V}(\tau_W(k)\varphi) &= \sum_{j=1}^{\dim\mathfrak{p}} \iota_Y^V(\tau_V(k)\iota_V^W(\varphi)(\mathrm{Ad}(k^{-1})\tilde{X}_j))(X_j) \\ &= \tau_Y(k)\sum_{j=1}^{\dim\mathfrak{p}} \iota_Y^V(\iota_V^W(\varphi)(\mathrm{Ad}(k^{-1})\tilde{X}_j))(\mathrm{Ad}(k^{-1})X_j).\end{aligned}$$

Therefore, to prove the claim, it suffices to show the independence of $\mathcal{F}_Y^{W,V}(\varphi)$ from the chosen basis. However, this follows as in the proof of Lemma 7.4.4.

By Schur's lemma, the $K$-equivariance implies

$$(r(\Omega_\mathfrak{p})f)(g) = \sum_{V\leftrightarrow Y} \mathcal{F}_Y^{Y,V}(\mathrm{d}_Y^V \mathrm{d}_V^Y f(g))$$

and each $\mathcal{F}_Y^{Y,V}$ is a multiple of the identity. Now let $V \in \hat{K}$ with $V \overset{\omega}{\leftrightarrow} Y$ be fixed. Using Proposition 7.3.9 in the first two and Lemma 7.3.7 in the last step, we have for $\varphi \in Y$

$$\begin{aligned}\mathcal{F}_Y^{Y,V}(\varphi) &= \sum_{j=1}^{\dim\mathfrak{p}} \frac{1}{\lambda(V,Y)}\,\mathrm{pr}_Y(\omega(X_j)\iota_V^Y(\varphi)(\tilde{X}_j)) \\ &= \frac{1}{\lambda(V,Y)\lambda(Y,V)}\sum_{j=1}^{\dim\mathfrak{p}} \mathrm{pr}_Y(\omega(X_j)\,\mathrm{pr}_V(\omega(\tilde{X}_j)\varphi)) \\ &= \frac{1}{\lambda(V,Y)}\varphi.\end{aligned}$$

□

For $G = \mathrm{PSL}(2,\mathbb{R})$, Lemma 7.5.3 reduces to a calculation in $\mathcal{U}(\mathfrak{g})$. Indeed, realizing the elements of $\hat{K}$ on $\mathbb{S}^{n-1}$ (as in Lemma 4.3.1), we have

$$\lambda(L_{n-1}, L_n) = \mathrm{pr}_{L_n}(\omega(H)z^{n-1})(1) = \mathrm{pr}_{L_n}\left(\frac{z+z^{-1}}{2}z^{n-1}\right)(1) = \frac{1}{2} = \lambda(L_{n+1}, L_n)$$

by Example 7.3.5. Thus, a direct calculation shows

$$\frac{1}{\lambda(L_{n+1}, L_n)}\eta_-\eta_+ + \frac{1}{\lambda(L_{n-1}, L_n)}\eta_+\eta_- = H^2 + B^2 = \Omega_{\mathfrak{p}}.$$

We can also check Lemma 4.2.1 by obtaining – recall $\rho(H) = \frac{1}{2}$ –

$$2(\eta_+\eta_- + \eta_-\eta_+)f_\ell = ((\mu+\rho)(H)^2 - \rho(H)^2 - \ell^2)f_\ell,$$

where the first two summands correspond to the action of the Casimir operator $\Omega_{\mathfrak{g}} = H^2 + B^2 - V^2$ on $H_{\mu+\rho}$ by Equation (2.6) and the last one to the action of $V^2 \in \mathcal{U}(\mathfrak{k})$.

# 8. Spectral correspondence

Let us now, finally, state and prove the quantum-classical correspondences for the exceptional parameters. By Remark 6.1.4 we obtain such correspondences as soon as we have determined the images of the minimal $K$-type Poisson transforms restricted to the socle (see also Proposition 5.1.3 and Theorem 5.1.6). The characterization of the Poisson images requires the case by case calculations for decompositions of tensor products from Appendix A.

## 8.1. The case of $G = \mathrm{SO}_0(n,1),\ n \geq 3$

By Propositions A.1.4 and A.1.6 we have for each $k \in \mathbb{N}_0$

$$Y_k \otimes \mathfrak{p}^* \cong Y_{k-1} \oplus Y_{k+1} \oplus V_k \quad \text{if } n \neq 3, \qquad Y_k \otimes \mathfrak{p}^* \cong Y_{k-1} \oplus Y_{k+1} \oplus Y_k \quad \text{if } n = 3,$$

where $V_k$ is not $M$-spherical. We define generalized gradients $\mathrm{d}_V^{Y_k} := T_V^{Y_k} \circ \nabla$ with $T_V^{Y_k} \in \mathrm{Hom}_K(Y_k \otimes \mathfrak{p}^*, V)$ as in Definitions 2.4.2, 7.3.4 and abbreviate

$$\mathrm{d}_\pm := \mathrm{d}_{Y_{k\pm1}}^{Y_k}, \qquad \mathrm{D} := \mathrm{d}_{V_k}^{Y_k} \quad \text{resp.} \quad \mathrm{D} := \mathrm{d}_{Y_k}^{Y_k}$$

for $n \neq 3$ resp. $n = 3$. Let $\mu = -\rho - \ell\alpha \in \mathbf{Ex}$, see Theorem 6.2.1, be an exceptional parameter and recall the structure and properties of $\mathrm{soc}(H_\mu)$ from Theorem 6.2.1. Using Proposition 7.3.12, Proposition 7.3.8.ii) and Remark A.1.3 we infer for each $k \in \mathbb{N}_0$

$$V \overset{\omega}{\not\leftrightarrow} Y_k \implies \mathrm{d}_V^{Y_k} \circ P_\mu^{Y_k} = 0 \qquad \text{and} \qquad V \overset{\omega}{\leftrightarrow} Y_k \iff V \in \{Y_{k-1}, Y_{k+1}\}$$

if $Y_{k-1}$ exists[1]. Therefore,

$$\mathrm{D} \circ P_\mu^{Y_k} = 0.$$

By Theorem 6.2.1 the minimal $K$-type of $\mathrm{soc}(H_\mu)$ is $Y_{\ell+1}$. Since

$$\mathrm{d}_- \circ P_\mu^{Y_{\ell+1}} = T_{Y_\ell}^{Y_{\ell+1}}(p_{Y_{\ell+1},\mu})(e) P_\mu^{Y_\ell}$$

by Proposition 5.3.2.ii) and Proposition 5.1.2 implies $P_\mu^{Y_\ell}\Big|_{(\mathrm{soc}(H_\mu))^{-\infty}} = 0$, we obtain

$$\mathrm{d}_- \circ P_\mu^{Y_{\ell+1}}\Big|_{(\mathrm{soc}(H_\mu))^{-\infty}} = 0.$$

[1]I.e.: For $k = 0$ we only have $Y_1$ on the right hand side of the second equivalence.

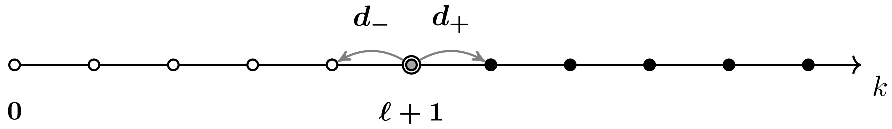



Figure 8.1.: Fourier components of $f \in {}^\Gamma H_\mu^{-\infty}$. White dots represent Fourier components that are zero; see also Figure B.2.

Summarizing, we have

$$P_\mu^{Y_{\ell+1}} : (\mathrm{soc}(H_\mu))^{-\infty} \to \{f \in C^\infty(G \times_K Y_{\ell+1}) \mid \mathrm{d}_- f = 0,\ \mathrm{D} f = 0\}.$$

We will now investigate which $K$-types $\mu$ with highest weight $\mu_1 e_1 + \ldots + \mu_m e_m$, $m \coloneqq \mathrm{rk}\,\mathfrak{k} = \lfloor \frac{n}{2} \rfloor$, occur on the right hand side. Applying [DGK88, Theorem 6] to the minimal $K$-type $\tau \coloneqq Y_{\ell+1}$ (with highest weight $(\ell+1)e_1$) of $\mathrm{soc}(H_\mu)$, we find that $\mu_j = 0$ for $j > 1$, $\mu_1 \geq \ell + 1$ and that each $\mu$ of this form occurs with multiplicity one. Therefore, the highest weights of the $K$-types in $\{f \in C^\infty(G \times_K Y_{\ell+1}) \mid \mathrm{d}_- f = 0,\ \mathrm{D} f = 0\}$ are given by $ke_1$ for $k \geq \ell + 1$. Since $Y_k$ has highest weight $ke_1$, these $K$-types are exactly the same as the $K$-types of $\mathrm{soc}(H_\mu)$ (see Theorem 6.2.1). Hence, we have

$$(\mathrm{soc}(H_\mu))_K \cong \{f \in C^\infty(G \times_K Y_{\ell+1}) \mid \mathrm{d}_- f = 0,\ \mathrm{D} f = 0\}_K,$$

where the $K$ in the index denotes the Harish-Chandra module. Proceeding as in [Olb94, Satz 4.13] we infer that the Poisson transform $P_\mu^{Y_{\ell+1}}$ yields an isomorphism (similar to the scalar case, see Equation (3.1), Theorem 3.2.2) from $(\mathrm{soc}(H_\mu))^{-\infty}$ to

$$\{f \in C^\infty(G \times_K Y_{\ell+1}) \mid \mathrm{d}_- f = 0,\ \mathrm{D} f = 0, \exists\, r \geq 0\colon \sup_{g \in G} |e^{-r d_{G/K}(eK, gK)} f(g)| < \infty\}.$$

In particular, we have the following correspondence for the $\Gamma$-invariant elements.

**Theorem 8.1.1** (Spectral Correspondence)**.** *Let* $\mathbf{Ex} \ni \mu = -(\rho + \ell\alpha)$, $\ell \in \mathbb{N}_0$, *be an exceptional parameter. Then the socle* $\mathrm{soc}(H_\mu)$ *of* $H_\mu$ *is irreducible, unitary and its* $K$*-types are given by* $Y_k$ *for* $k \geq \ell+1$. *The minimal* $K$*-type is* $Y_{\ell+1}$ *and the corresponding Poisson transform induces an isomorphism*

$$P_\mu^{Y_{\ell+1}} \colon {}^\Gamma(\mathrm{soc}(H_\mu))^{-\infty} \cong {}^\Gamma\{f \in C^\infty(G \times_K Y_{\ell+1}) \mid \mathrm{d}_- f = 0,\ \mathrm{D} f = 0\}.$$

*Proof.* This follows from the discussion above and the fact that each $\Gamma$-invariant function fulfills the growth condition (for each $r \geq 0$)

$$\sup_{g \in G} |e^{-r d_{G/K}(eK, gK)} f(g)| = \sup_{g \in \mathcal{F}} |e^{-r d_{G/K}(eK, gK)} f(g)| < \infty,$$

where $\mathcal{F}$ denotes a fundamental domain of $\Gamma \backslash G$ (note that the latter is compact by assumption). ☐

**Example 8.1.2** (see Section 9.4.2 for more details)**.** For the first exceptional parameter $\mu = -\rho$ we get ($Y_1 \cong \mathfrak{p}^*$)

$$P_{-\rho}^{Y_1}\colon {}^{\Gamma}(\mathrm{soc}(H_{-\rho}))^{-\infty} \cong \{f \in C^\infty(\Lambda^1(\Gamma\backslash G/K)) \mid \delta f = 0,\ df = 0\},$$

where $\Lambda^1(\Gamma\backslash G/K)$ denotes the bundle of one forms and ($\delta$ resp.) $d$ is the (co)-differential. The dimension is given by the first Betti number $b_1(\Gamma\backslash G/K)$.

## 8.2. The case of $G = \mathrm{SU}(n,1),\ n \geq 2$

By Proposition 7.3.2 and Remark A.2.3 we have for $p, q \in \mathbb{N}_0$

$$Y_{p,q} \otimes \mathfrak{p}^* \cong \bigoplus_{\beta \in S} Y_{p,q,\beta},$$

where $S := \{\pm(e_1 - e_{n+1}), e_2 - e_{n+1}, -e_{n-1} + e_{n+1}, \pm(e_n - e_{n+1})\} \subseteq \Delta_n$. The representations $V_1$ resp. $V_2$ with highest weights $qe_1 + e_2 - pe_n + (p-q-1)e_{n+1}$ resp. $qe_1 - e_{n-1} - pe_n + (p-q+1)e_{n+1}$ are not $M$-spherical. In this notation we have

$$Y_{p,q} \otimes \mathfrak{p}^* \cong Y_{p-1,q} \oplus Y_{p+1,q} \oplus Y_{p,q-1} \oplus Y_{p,q+1} \oplus V_1 \oplus V_2,$$

whenever these representations exist (i.e. whenever the corresponding weights of $Y_{p,q,\beta}$ are indeed dominant). We define generalized gradients $\mathrm{d}_V^{Y_{p,q}} := T_V^{Y_{p,q}} \circ \nabla$ with $T_V^{Y_{p,q}} \in \mathrm{Hom}_K(Y_{p,q} \otimes \mathfrak{p}^*, V)$ as in Definition 7.3.4 and abbreviate

$$\mathrm{d}_{\pm,1} := \mathrm{d}_{Y_{p\pm1,q}}^{Y_{p,q}}, \qquad \mathrm{d}_{\pm,2} := \mathrm{d}_{Y_{p,q\pm1}}^{Y_{p,q}}, \qquad \mathrm{D}_j := \mathrm{d}_{V_j}^{Y_{p,q}}, \quad j = 1,2.$$

Let $\mu = -(\rho + 2\ell\alpha) \in \mathbf{Ex},\ \ell \in \mathbb{N}_0$, be an exceptional parameter and recall the structure and properties of $\mathrm{soc}(H_\mu)$ from Theorem 6.2.1. Using Proposition 7.3.12, Proposition 7.3.8.ii) and Remark A.2.2 we infer

$$V \overset{\omega}{\not\leftrightarrow} Y_{p,q} \implies \mathrm{d}_V^{Y_{p,q}} \circ P_\mu^{Y_{p,q}} = 0 \qquad \text{and} \qquad V \overset{\omega}{\leftrightarrow} Y_{p,q} \iff V \in \{Y_{p\pm1,q}, Y_{p,q\pm1}\}$$

whenever the occurring representations exist. Therefore, for $j \in \{1,2\}$,

$$\mathrm{D}_j \circ P_\mu^{Y_{p,q}} = 0. \tag{8.1}$$

The minimal $K$-type of $\mathrm{soc}(H_\mu)$ is $Y_{\ell+1,\ell+1}$ (see Theorem 6.2.1). By Proposition 5.3.2.ii),

$$\begin{aligned} \mathrm{d}_{-,1} \circ P_\mu^{Y_{\ell+1,\ell+1}} &= T_{Y_{\ell,\ell+1}}^{Y_{\ell+1,\ell+1}}(p_{Y_{\ell+1,\ell+1},\mu})(e) P_\mu^{Y_{\ell,\ell+1}} \\ \mathrm{d}_{-,2} \circ P_\mu^{Y_{\ell+1,\ell+1}} &= T_{Y_{\ell+1,\ell}}^{Y_{\ell+1,\ell+1}}(p_{Y_{\ell+1,\ell+1},\mu})(e) P_\mu^{Y_{\ell+1,\ell}}. \end{aligned}$$

Since Proposition 5.1.2 implies that $P_\mu^{Y_{\ell,\ell+1}}|_{(\mathrm{soc}(H_\mu))^{-\infty}} = 0$ and $P_\mu^{Y_{\ell+1,\ell}}|_{(\mathrm{soc}(H_\mu))^{-\infty}} = 0$, we obtain that, for $j \in \{1,2\}$,

$$\mathrm{d}_{-,j} \circ P_\mu^{Y_{\ell+1,\ell+1}}\Big|_{(\mathrm{soc}(H_\mu))^{-\infty}} = 0.$$

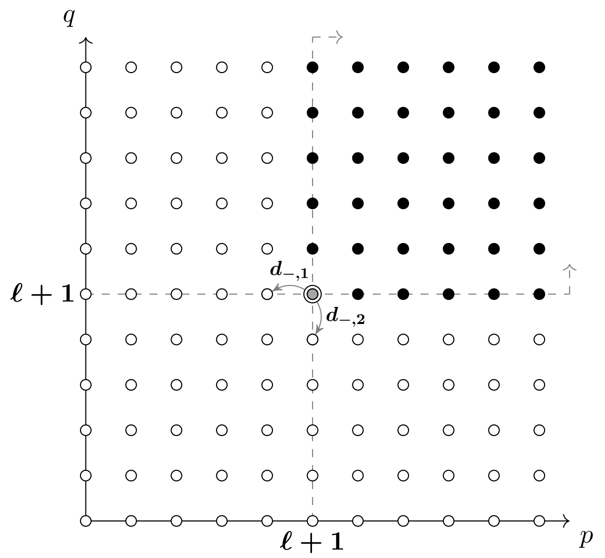



Figure 8.2.: Fourier components of $f \in {}^{\Gamma}H_{\mu}^{-\infty}$. White dots represent Fourier components that are zero; see also Figure B.4.

Summarizing, we have

$$P_{\mu}^{Y_{\ell+1,\ell+1}} : (\mathrm{soc}(H_{\mu}))^{-\infty} \to \{f \in C^{\infty}(G \times_K Y_{\ell+1,\ell+1}) \mid \mathrm{d}_{-,j} f = 0,\ \mathrm{D}_j f = 0,\ j \in \{1,2\}\}. \tag{8.2}$$

We will first present a method similar to the case of $G = \mathrm{SO}_0(n,1)$. For this method we have to assume $n \neq 2$ and $\ell \neq 0$. Then [Mea89, Equations (2.7.3), (2.7.4), Lemma 6.2.1, Proposition 6.4.6] imply that the highest weights of the $K$-types on the right hand side of (8.2) are given by $p'e_1 - q'e_n + (q'-p')e_{n+1}$ with $p' \geq \ell+1$ and $q' \geq \ell+1$, each occurring with multiplicity at most one. By definition, the corresponding representations are $Y_{p,q}$ for $p, q \geq \ell+1$. Since the Poisson transform $P_{\mu}^{Y_{\ell+1,\ell+1}}$ is injective by Proposition 5.1.3, each $K$-type of the socle (see Theorem 6.2.1) has to occur in its image (restricted to the socle). Therefore the $K$-types of $\{f \in C^{\infty}(G \times_K Y_{\ell+1,\ell+1}) \mid \mathrm{d}_{-,j} f = 0,\ \mathrm{D}_j f = 0,\ j \in \{1,2\}\}$ are given by $Y_{p,q},\ p, q \geq \ell+1$, each one occurring with multiplicity one. Hence, we obtain

$$(\mathrm{soc}(H_{\mu}))_K \cong \{f \in C^{\infty}(G \times_K Y_{\ell+1,\ell+1}) \mid \mathrm{d}_{-,j} f = 0,\ \mathrm{D}_j f = 0,\ j \in \{1,2\}\}_K.$$

Proceeding as in the case of $G = \mathrm{SO}_0(n,1)$ we find

**Theorem 8.2.1** (Spectral Correspondence 1)**.** *Let $n \neq 2$ and $\mathbf{Ex} \ni \mu = -(\rho + 2\ell\alpha)$, for $\ell \in \mathbb{N}$ (i.e. $\ell \neq 0$), be an exceptional parameter. Then the socle $\mathrm{soc}(H_{\mu})$ of $H_{\mu}$ is irreducible, unitary and its $K$-types are given by $Y_{p,q}$ for $p, q \geq \ell+1$. The minimal $K$-type is $Y_{\ell+1,\ell+1}$ and the corresponding Poisson transform induces an isomorphism*

$$P_{\mu}^{Y_{\ell+1,\ell+1}} \colon {}^{\Gamma}(\mathrm{soc}(H_{\mu}))^{-\infty} \cong {}^{\Gamma}\{f \in C^{\infty}(G \times_K Y_{\ell+1,\ell+1}) \colon \mathrm{d}_{-,j} f = 0,\ \mathrm{D}_j f = 0,\ j \in \{1,2\}\}.$$

*Proof.* See Theorem 8.1.1. □

In order to treat the remaining parameters ($n = 2$ or $\ell = 0$) we will use the Fourier characterization of the principal series. The following lemma is based on Lemma 7.2.5.

**Lemma 8.2.2.** *Let $\mu := -(\rho + 2\ell\alpha)$, $\ell \in \mathbb{N}_0$, an exceptional parameter. Let $\psi_{p,q} \in C^\infty(G \times_K Y_{p,q})$ for $p, q \geq \ell + 1$ be such that the equations from Lemma 7.4.1 are fulfilled (with $\psi_{p,q}$ instead of $\pi^*_{Y_{p,q}}(f)$). Assume that $\pi_{Y_{\ell+1,\ell+1}}(\psi_{Y_{\ell+1,\ell+1}}) \in C^\infty(G)$ has finite $L^2$-norm. Then the formal sum*

$$f := \sum_{p,q\geq \ell+1} \iota_{G/M}(\pi_{Y_{p,q}}(\psi_{p,q}))$$

*defines a distribution on $G/M$.*

*Proof.* We abbreviate $T^{p_1,q_1}_{p_2,q_2} := T^{Y_{p_1,q_1}}_{Y_{p_2,q_2}}(p_{Y_{p_1,q_1,\mu}})(e) \in \mathbb{C}$. It suffices to prove the estimate in Lemma 7.2.5. Using Lemma 7.5.2 (second step) and the equations from Lemma 7.4.1 (first and third step) we infer for the $L^2$ inner product

$$\begin{aligned}
\|\pi_{Y_{p,q}}(\psi_{p,q})\|^2 &= \frac{\dim Y_{p,q}}{\dim Y_{p-1,q}} \frac{1}{T^{p-1,q}_{p,q}} \langle \pi_{Y_{p,q}}(\psi_{p,q}), \pi_{Y_{p,q}}(\mathrm{d}_{+,1}\psi_{p-1,q})\rangle \\
&= -\frac{\dim Y_{p,q}}{\dim Y_{p-1,q}} \frac{1}{T^{p-1,q}_{p,q}} \langle \pi_{Y_{p-1,q}}(\mathrm{d}_{-,1}\psi_{p,q}), \pi_{Y_{p-1,q}}(\psi_{p-1,q})\rangle \\
&= -\left(\frac{\dim Y_{p,q}}{\dim Y_{p-1,q}}\right)^2 \frac{T^{p,q}_{p-1,q}}{T^{p-1,q}_{p,q}} \langle \pi_{Y_{p-1,q}}(\psi_{p-1,q}), \pi_{Y_{p-1,q}}(\psi_{p-1,q})\rangle.
\end{aligned}$$

By Proposition A.5.1, Remark A.2.2 and Remark A.2.3 this equals

$$\frac{(n+p-2)(n+p+q-1)}{p(n+p+q-2)} \frac{n+p}{p-1-\ell} \|\pi_{Y_{p-1,q}}(\psi_{p-1,q})\|^2.$$

Iteratively applying this equation we find that for each $m \in \mathbb{N}_0$

$$\|\pi_{Y_{\ell+m,q}}(\psi_{\ell+m,q})\|^2 = \prod_{r=2}^{m} \frac{(n+\ell+r-2)(n+\ell+r+q-1)}{(\ell+r)(n+\ell+r+q-2)} \frac{n+\ell+r}{r-1} \|\pi_{Y_{\ell+1,q}}(\psi_{\ell+1,q})\|^2.$$

The latter product equals

$$\frac{(n+\ell+m+q-1)(n+\ell+m-2)!(\ell+1)!(n+\ell+m)!}{(n+\ell+q)(n+\ell-1)!(\ell+m)!(m-1)!(n+\ell+1)!} \|\pi_{Y_{\ell+1,q}}(\psi_{\ell+1,q})\|^2,$$

which grows polynomially in $m$ (in fact it is $\mathcal{O}(m^{2n+\ell})$). Interchanging the roles of $p$ and $q$ this proves the estimate in Lemma 7.2.5 and therefore the lemma. □

If the $\psi_{p,q}$ from Lemma 8.2.2 happen to be $\Gamma$-invariant we can consider the $\pi_{Y_{p,q}}(\psi_{p,q})$ as functions in $C^\infty(\Gamma\backslash G)$. Choosing a (compact) fundamental domain for the action of $\Gamma$ in $G$, we can thus use Lemma 8.2.2 to construct a $\Gamma$-invariant distribution on $G/M$ respectively a distribution on the locally symmetric space $\Gamma\backslash G/M$.

**Theorem 8.2.3** (Spectral Correspondence 2)**.** *Let* $\mathbf{Ex} \ni \mu = -(\rho + 2\ell\alpha)$, $\ell \in \mathbb{N}_0$, *be an exceptional parameter. Then the socle* $\operatorname{soc}(H_\mu)$ *of* $H_\mu$ *is irreducible, unitary and its* $K$*-types are given by* $Y_{p,q}$ *for* $p, q \geq \ell + 1$*. The minimal* $K$*-type is* $Y_{\ell+1,\ell+1}$ *and the corresponding Poisson transform induces an isomorphism from* ${}^\Gamma(\operatorname{soc}(H_\mu))^{-\infty}$ *onto*

$$^\Gamma\{u \in C^\infty(G \times_K Y_{\ell+1,\ell+1}) \mid \textit{properties } i) - vi) \textit{ below}\},$$

*where the properties are as follows.*

*For* $u \in C^\infty(G \times_K Y_{\ell+1,\ell+1})$ *let* $\psi_{\ell+1,\ell+1} := \dim Y_{\ell+1,\ell+1} \cdot u$ *and define recursively for* $p, q \geq \ell + 1$ *(see Lemma 7.4.1)*

$$\psi_{p+1,\ell+1} := \frac{\dim Y_{p+1,\ell+1}}{\dim Y_{p,\ell+1}} \frac{1}{T^{p,\ell+1}_{p+1,\ell+1}} \mathrm{d}_{+,1}\psi_{p,\ell+1}, \qquad \psi_{p,q} := \frac{\dim Y_{p,q}}{\dim Y_{p,q-1}} \frac{1}{T^{p,q-1}_{p,q}} \mathrm{d}_{+,2}\psi_{p,q-1},$$

*where we abbreviate* $T^{p_1,q_1}_{p_2,q_2} := T^{Y_{p_1,q_1}}_{Y_{p_2,q_2}}(p_{Y_{p_1,q_1},\mu})(e) \in \mathbb{C}$*. Then we define the properties*

*i)* $\mathrm{d}_{+,1}\psi_{p,q} = T^{p,q}_{p+1,q} \frac{\dim Y_{p,q}}{\dim Y_{p+1,q}} \psi_{p+1,q}$, $\qquad (p \geq \ell + 1,\ q \geq \ell + 2)$,

*ii)* $\mathrm{d}_{-,1}\psi_{p,q} = T^{p,q}_{p-1,q} \frac{\dim Y_{p,q}}{\dim Y_{p-1,q}} \psi_{p-1,q}$, $\qquad (p \geq \ell + 2,\ q \geq \ell + 1)$,

*iii)* $\mathrm{d}_{-,1}\psi_{\ell+1,q} = 0$, $\qquad (q \geq \ell + 1)$,

*iv)* $\mathrm{d}_{-,2}\psi_{p,q} = T^{p,q}_{p,q-1} \frac{\dim Y_{p,q}}{\dim Y_{p,q-1}} \psi_{p,q-1}$, $\qquad (p \geq \ell + 1,\ q \geq \ell + 2)$,

*v)* $\mathrm{d}_{-,2}\psi_{p,\ell+1} = 0$, $\qquad (p \geq \ell + 1)$,

*vi)* $\mathrm{D}_j\psi_{p,q} = 0$, $\qquad (p, q \geq \ell + 1,\ j \in \{1, 2\})$.

*Proof.* We first prove that the Poisson transform maps into the claimed space. If $u = P^{Y_{\ell+1,\ell+1}}_\mu(f)$ for some $f \in (\operatorname{soc}(H_\mu))^{-\infty}$ we have $\psi_{\ell+1,\ell+1} = \pi^*_{Y_{\ell+1,\ell+1}}(f)$ by Lemma 7.1.4.iv). Properties $i), ii), iv)$ and $vi)$ are exactly the equations from Lemma 7.4.1. To prove the third property we note that

$$\mathrm{d}_{-,1}\psi_{\ell+1,q} = \mathrm{d}_{-,1}\pi^*_{Y_{\ell+1,q}}(f) = T^{\ell+1,q}_{\ell,q} \frac{\dim Y_{\ell+1,q}}{\dim Y_{\ell,q}} \pi^*_{Y_{\ell,q}}(f) = 0,$$

since the socle does not contain the $K$-type $Y_{\ell,q}$. Similarly we see that property $v)$ is fulfilled. Since the Poisson transform is $G$-equivariant it preserves $\Gamma$-invariant elements.

For the surjectivity let $u \in {}^\Gamma C^\infty(G \times_K Y_{\ell+1,\ell+1})$ with the desired properties. Define

$$f := \sum_{p,q \geq \ell+1} \iota_{G/M}(\pi_{Y_{p,q}}(\psi_{p,q})).$$

Note first that each $\psi_{p,q}$ is $\Gamma$-invariant since $\psi_{\ell+1,\ell+1}$ is $\Gamma$-invariant and each involved map is $G$-equivariant. Lemma 8.2.2 thus implies that $f$ defines a $\Gamma$-invariant distribution on $G/M$. By Theorem 7.4.11 we have $f \in {}^\Gamma H^{-\infty}_\mu$ and, since there are only terms for

$p, q \geq \ell + 1$ in the defining sum of $f$, we also have $f \in {}^{\Gamma}(\mathrm{soc}(H_\mu))^{-\infty}$. The orthogonality of the $K$-types implies (see Definition 7.1.1 for the relevant definitions)

$$\pi^*_{Y_{\ell+1,\ell+1}}\big(\sum_{p,q\in J} \iota_{G/M}(\pi_{Y_{p,q}}(\psi_{p,q}))\big) = \sum_{p,q\in J} \iota_{G/M}(\pi_{Y_{p,q}}(\psi_{p,q})) \circ \pi_{Y_{\ell+1,\ell+1}} = 0$$

for $J := \{(p,q) \in \mathbb{N}_0^2 \colon (p,q) \neq (\ell+1, \ell+1)\}$ and

$$\pi^*_{Y_{\ell+1,\ell+1}}(\iota_{G/M}(\pi_{Y_{\ell+1,\ell+1}}(\psi_{\ell+1,\ell+1}))) = \iota_{Y_{\ell+1,\ell+1}}(\psi_{\ell+1,\ell+1}),$$

where we used Lemma 7.1.4.iii) and i). Using Lemma 7.1.4.iv) we obtain

$$P_\mu^{Y_{\ell+1,\ell+1}}(f) = \frac{1}{\dim Y_{\ell+1,\ell+1}} \pi^*_{Y_{\ell+1,\ell+1}}(f) = \frac{1}{\dim Y_{\ell+1,\ell+1}} \psi_{\ell+1,\ell+1} = u. \qquad \square$$

## 8.3. The case of $G = \mathrm{Sp}(n,1),\ n \geq 2$

By Proposition 7.3.2 and Remark A.3.3 we have for each $a, b \in \mathbb{N}_0$ with $a \geq b$

$$V_{a,b} \otimes \mathfrak{p}^* \cong V_{a+1,b} \oplus V_{a-1,b} \oplus V_{a,b+1} \oplus V_{a,b-1} \oplus \bigoplus_{\substack{\beta\in S \\ V_{a,b,\beta}\notin \hat{K}_M}} V_{a,b,\beta}.$$

We define generalized gradients $\mathrm{d}_V^{V_{a,b}} := T_V^{V_{a,b}} \circ \nabla$ with $T_V^{V_{a,b}} \in \mathrm{Hom}_K(V_{a,b} \otimes \mathfrak{p}^*, V)$ as in Definition 7.3.4 and abbreviate

$$\mathrm{d}_{\pm,1} := \mathrm{d}^{V_{a,b}}_{V_{a\pm1,b}}, \qquad \mathrm{d}_{\pm,2} := \mathrm{d}^{V_{a,b}}_{V_{a,b\pm1}}, \qquad \mathrm{D}_\beta := \mathrm{d}^{V_{a,b}}_{V_{a,b,\beta}}$$

for each $\beta \in S$ with $V_{a,b,\beta} \notin \hat{K}_M$. Let $\mu = -(\rho + (2\ell - 2)\alpha) \in \mathbf{Ex}$ be an exceptional parameter and recall the structure and properties of $\mathrm{soc}(H_\mu)$ from Theorem 6.2.1. Using Proposition 7.3.12, Proposition 7.3.8.ii) and Remark A.3.2 we infer for each $a, b \in \mathbb{N}_0$ with $a \geq b$

$$V \overset{\omega}{\not\leftrightarrow} V_{a,b} \implies \mathrm{d}_V^{V_{a,b}} \circ P_\mu^{V_{a,b}} = 0 \text{ and } V \overset{\omega}{\leftrightarrow} V_{a,b} \iff V \in \{V_{a+1,b}, V_{a-1,b}, V_{a,b+1}, V_{a,b-1}\}$$

whenever the occurring representations exist. The minimal $K$-type of $\mathrm{soc}(H_\mu)$ is given by $V_{\ell+1,\ell+1}$ (see Theorem 6.2.1).

The spectral correspondence in the quaternionic case is established by using the Fourier characterization of the principal series as in Theorem 8.2.3. By Lemma 7.2.5 we obtain the following result.

**Lemma 8.3.1.** *Let $\mu := -(\rho + (2\ell - 2)\alpha)$, $\ell \in \mathbb{N}_0$, an exceptional parameter. Let $\psi_{a,b} \in C^\infty(G \times_K V_{a,b})$ for $a, b \geq \ell + 1$ be such that the equations from Lemma 7.4.1 are fulfilled (with $\psi_{a,b}$ instead of $\pi^*_{V_{a,b}}(f)$). Assume that $\pi_{V_{\ell+1,\ell+1}}(\psi_{V_{\ell+1,\ell+1}}) \in C^\infty(G)$ has finite $L^2$-norm. Then the formal sum*

$$f := \sum_{a\geq b\geq \ell+1} \iota_{G/M}(\pi_{V_{a,b}}(\psi_{a,b}))$$

*defines a distribution on $G/M$.*

*Proof.* We abbreviate $T^{a_1,b_1}_{a_2,b_2} := T^{V_{a_1,b_1}}_{V_{a_2,b_2}}(p_{V_{a_1,b_1,\mu}})(e) \in \mathbb{C}$. It suffices to prove the estimate in Lemma 7.2.5. Using Lemma 7.5.2 (second step) and the equations from Lemma 7.4.1 (first and third step) we infer for the $L^2$-norm as in Lemma 8.2.2

$$\|\pi_{V_{a,b}}(\psi_{a,b})\|^2 = -\left(\frac{\dim V_{a,b}}{\dim V_{a-1,b}}\right)^2 \frac{T^{a,b}_{a-1,b}}{T^{a-1,b}_{a,b}} \|\pi_{V_{a-1,b}}(\psi_{a-1,b})\|^2.$$

By Equation (7.9), Proposition A.5.1 and Proposition 7.3.8.iv) we have

$$\frac{T^{a,b}_{a-1,b}}{T^{a-1,b}_{a,b}} = \frac{-2n+1-a-\ell}{a-\ell}\frac{\lambda(V_{a,b},V_{a-1,b})}{\lambda(V_{a,b},V_{a-1,b})} = \frac{-2n+1-a-\ell}{a-\ell}\frac{\dim V_{a-1,b}}{\dim V_{a,b}}$$

and thus

$$\|\pi_{V_{a,b}}(\psi_{a,b})\|^2 = \frac{2n-1+a+\ell}{a-\ell}\frac{\dim V_{a,b}}{\dim V_{a-1,b}} \|\pi_{V_{a-1,b}}(\psi_{a-1,b})\|^2.$$

Iteratively applying this equation we infer that for each $m \in \mathbb{N}_0$

$$\begin{aligned}\|\pi_{V_{\ell+m,b}}(\psi_{\ell+m,b})\|^2 &= \prod_{r=2}^{m} \frac{2n-1+2\ell+r}{r}\frac{\dim V_{\ell+r,b}}{\dim V_{\ell+r-1,b}} \|\pi_{V_{\ell+1,b}}(\psi_{\ell+1,b})\|^2 \\ &= \frac{\dim V_{\ell+m,b}}{\dim V_{\ell+1,b}} \prod_{r=2}^{m} \frac{2n-1+2\ell+r}{r} \|\pi_{V_{\ell+1,b}}(\psi_{\ell+1,b})\|^2.\end{aligned}$$

Note that $\prod_{r=2}^{m} \frac{2n-1+2\ell+r}{r} = \frac{(2n-1+2\ell+m)!}{m!(2n+2\ell)!}$ is $\mathcal{O}(m^{2n-1+2\ell})$. Moreover, the dimension formula from Remark A.3.3 shows that $\dim V_{\ell+m,b}$ grows at most polynomially in $m$. A similar argument works for the $b$-variable. $\square$

**Theorem 8.3.2** (Spectral Correspondence)**.** *Let* $\mathbf{Ex} \ni \mu = -(\rho + (2\ell-2)\alpha)$, $\ell \in \mathbb{N}_0$, *be an exceptional parameter. Then the socle* $\mathrm{soc}(H_\mu)$ *of* $H_\mu$ *is irreducible, unitary and its* $K$*-types are given by* $V_{a,b}$ *for* $a \geq b \geq \ell+1$*. The minimal* $K$*-type is* $V_{\ell+1,\ell+1}$ *and the corresponding Poisson transform induces an isomorphism from* ${}^\Gamma(\mathrm{soc}(H_\mu))^{-\infty}$ *onto*

$${}^\Gamma\{u \in C^\infty(G \times_K V_{\ell+1,\ell+1}) \colon \textit{properties } i) - v) \textit{ below}\},$$

*where the properties are as follows.*

*For* $u \in C^\infty(G \times_K V_{\ell+1,\ell+1})$ *let* $\psi_{\ell+1,\ell+1} := \dim V_{\ell+1,\ell+1} \cdot u$ *and define recursively for* $a \geq b \geq \ell+1$ *(see Lemma 7.4.1)*

$$\psi_{a+1,\ell+1} := \frac{\dim V_{a+1,\ell+1}}{\dim V_{a,\ell+1}}\frac{1}{T^{a,\ell+1}_{a+1,\ell+1}} \mathrm{d}_{+,1}\psi_{a,\ell+1}, \qquad \psi_{a,b} := \frac{\dim V_{a,b}}{\dim V_{a,b-1}}\frac{1}{T^{a,b-1}_{a,b}} \mathrm{d}_{+,2}\psi_{a,b-1},$$

*where we abbreviate* $T^{a_1,b_1}_{a_2,b_2} := T^{V_{a_1,b_1}}_{V_{a_2,b_2}}(p_{V_{a_1,b_1,\mu}})(e) \in \mathbb{C}$*. Then we define the properties*

*i)* $\mathrm{d}_{+,1}\psi_{a,b} = T^{a,b}_{a+1,b}\frac{\dim V_{a,b}}{\dim V_{a+1,b}}\psi_{a+1,b}$, $\qquad (a \geq b \geq \ell+2)$,

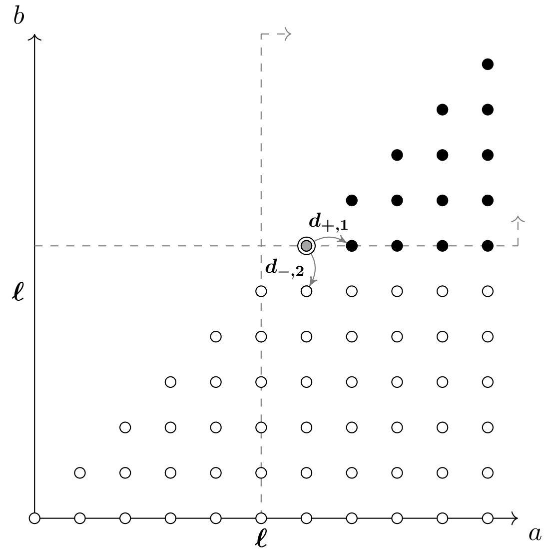



Figure 8.3.: Fourier components of $f \in {}^{\Gamma}H_{\mu}^{-\infty}$. White dots represent Fourier components that are zero; see also Figure B.6.

*ii)* $\mathrm{d}_{-,1}\psi_{a,b} = T^{a,b}_{a-1,b}\frac{\dim V_{a,b}}{\dim V_{a-1,b}}\psi_{a-1,b}, \qquad (a \geq \ell+2,\ b \geq \ell+1),$

*iii)* $\mathrm{d}_{-,2}\psi_{a,b} = T^{a,b}_{a,b-1}\frac{\dim V_{a,b}}{\dim V_{a,b-1}}\psi_{a,b-1}, \qquad (a \geq b \geq \ell+2),$

*iv)* $\mathrm{d}_{-,2}\psi_{a,\ell+1} = 0, \qquad (a \geq \ell+1),$

*v)* $\mathrm{d}_V^{V_{a,b}}\psi_{a,b} = 0, \qquad (a \geq b \geq \ell+1,\ V \leftrightarrow V_{a,b},\ V \notin \hat{K}_M).$

*Proof.* The proof is analogous to the proof of Theorem 8.2.3. □

## 8.4. The case of $G = \mathrm{F}_{4(-20)}$

By Proposition 7.3.2 and Remark A.4.3 we have for each $m, k \in \mathbb{N}_0$ with $m \geq k$ and $m \equiv k \bmod 2$

$$V_{m,k} \otimes \mathfrak{p}^* \cong V_{m+1,k+1} \oplus V_{m-1,k-1} \oplus V_{m+1,k-1} \oplus V_{m-1,k+1} \oplus \bigoplus_{\substack{\beta \in S \\ V_{m,k,\beta} \notin \hat{K}_M}} V_{m,k,\beta}.$$

We define generalized gradients $\mathrm{d}_V^{V_{m,k}} := T_V^{V_{m,k}} \circ \nabla$ with $T_V^{V_{m,k}} \in \mathrm{Hom}_K(V_{m,k} \otimes \mathfrak{p}^*, V)$ as in Definition 7.3.4 and abbreviate

$$\mathrm{d}_{\pm,1} := \mathrm{d}^{V_{m,k}}_{V_{m\pm1,k\pm1}}, \qquad \mathrm{d}_{\pm,2} := \mathrm{d}^{V_{m,k}}_{V_{m\pm1,k\mp1}}, \qquad \mathrm{D}_\beta := \mathrm{d}^{V_{m,k}}_{V_{m,k,\beta}}$$

for each $\beta \in S$ with $V_{m,k,\beta} \notin \hat{K}_M$.

Let $\mu = -(\rho + (2\ell - 6)\alpha) \in \mathbf{Ex}$, $\ell \in \mathbb{N}_0$, be an exceptional parameter and recall the structure and properties of $\mathrm{soc}(H_\mu)$ from Theorem 6.2.1. Using Proposition 7.3.12, Proposition 7.3.8.ii) and Remark A.4.2 we infer for each $m \geq k \in \mathbb{N}_0$ with $m \equiv k \bmod 2$

$$V \overset{\omega}{\not\leftrightarrow} V_{m,k} \implies \mathrm{d}_V^{V_{m,k}} \circ P_\mu^{V_{m,k}} = 0 \qquad \text{and} \qquad V \overset{\omega}{\leftrightarrow} V_{m,k} \iff V \in \{V_{m\pm1,k\pm1}\}$$

whenever the occurring representations exist. The minimal $K$-type of $\mathrm{soc}(H_\mu)$ is given by $V_{2\ell+2,0}$ (see Theorem 6.2.1).

As in the quaternionic case we use Theorem 7.4.11 to prove a spectral correspondence. By Lemma 7.2.5 we obtain

**Lemma 8.4.1.** *Let $\mu := -(\rho+(2\ell-6)\alpha)$, $\ell \in \mathbb{N}_0$, an exceptional parameter. Let $\psi_{m,k} \in C^\infty(G\times_K V_{m,k})$ for $m \equiv k \bmod 2$, $m-k \geq 2(\ell+1)$, be such that the equations from Lemma 7.4.1 are satisfied (with $\psi_{m,k}$ instead of $\pi^*_{V_{m,k}}(f)$). Assume that $\pi_{V_{2\ell+2,0}}(\psi_{V_{2\ell+2,0}}) \in C^\infty(G)$ has finite $L^2$-norm. Then the formal sum*

$$f := \sum_{\substack{m-k\geq 2\ell+2 \\ m\equiv k \bmod 2}} \iota_{G/M}(\pi_{V_{m,k}}(\psi_{m,k}))$$

*defines a distribution on $G/M$.*

*Proof.* We abbreviate $T^{m_1,k_1}_{m_2,k_2} := T^{V_{m_1,k_1}}_{V_{m_2,k_2}}(p_{V_{m_1,k_1},\mu})(e) \in \mathbb{C}$. It suffices to prove the estimate in Lemma 7.2.5. Using Lemma 7.5.2 (second step) and the equations from Lemma 7.4.1 (first and third step) we infer for the $L^2$-norm as in Lemma 8.2.2

$$\|\pi_{V_{m,k}}(\psi_{m,k})\|^2 = -\left(\frac{\dim V_{m,k}}{\dim V_{m-1,k-1}}\right)^2 \frac{T^{m,k}_{m-1,k-1}}{T^{m-1,k-1}_{m,k}} \|\pi_{V_{m-1,k-1}}(\psi_{m-1,k-1})\|^2.$$

By Equation (7.9), Proposition A.5.1 and Proposition 7.3.8.iv) we have

$$\frac{T^{m,k}_{m-1,k-1}}{T^{m-1,k-1}_{m,k}} = \frac{-14-2\ell-m-k}{4-2\ell+m+k}\frac{\lambda(V_{m,k},V_{m-1,k-1})}{\lambda(V_{m,k},V_{m-1,k-1})} = \frac{-14-2\ell-m-k}{4-2\ell+m+k}\frac{\dim V_{m-1,k-1}}{\dim V_{m,k}}$$

and thus

$$\|\pi_{V_{m,k}}(\psi_{m,k})\|^2 = \frac{14+2\ell+m+k}{4-2\ell+m+k}\frac{\dim V_{m,k}}{\dim V_{m-1,k-1}} \|\pi_{V_{m-1,k-1}}(\psi_{m-1,k-1})\|^2.$$

Iteratively applying this equation we infer for $a(m,k) := \frac{m+k}{2}$ and $p := a(m,k) - (\ell+1)$

$$\begin{aligned} \|\pi_{V_{m,k}}(\psi_{m,k})\|^2 &= \prod_{r=1}^{p-1} \frac{7+\ell+a(m,k)-r}{2-\ell+a(m,k)-r}\frac{\dim V_{m-r,k-r}}{\dim V_{m-r-1,k-r-1}} \|\pi_{V_{m-p,k-p}}(\psi_{m-p,k-p})\|^2 \\ &= \frac{\dim V_{m,k}}{\dim V_{m-p,k-p}} \prod_{r=1}^{p-1} \frac{7+\ell+a(m,k)-r}{2-\ell+a(m,k)-r} \|\pi_{V_{m-p,k-p}}(\psi_{m-p,k-p})\|^2, \end{aligned}$$

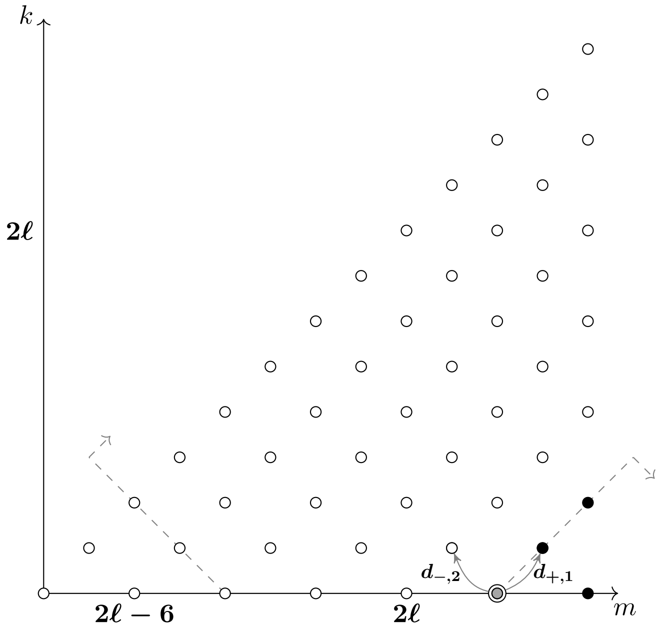



Figure 8.4.: Fourier components of $f \in {}^{\Gamma}H_{\mu}^{-\infty}$. White dots represent Fourier components that are zero; see also Figure B.9.

with $a(m-p, k-p) = \ell+1$. Note that $\prod_{r=1}^{p-1} \frac{7+\ell+a(m,k)-r}{2-\ell+a(m,k)-r} = \frac{(7+2\ell+p)!\cdot 6}{(7+2\ell+1)!\cdot(2+p)!}$ is $\mathcal{O}(p^{7+2\ell-2})$. Moreover, the dimension formula from Remark A.4.3 shows that $\dim V_{m,k}$ grows at most polynomially in $m$ and $k$. A similar argument works for the step from $V_{m,k}$ with $a(m,k) = \ell+1$ to $V_{2(\ell+1),0}$ by decreasing $b(m,k) := \frac{m-k}{2}$ (by going from $V_{m,k}$ to $V_{m-1,k+1}$). □

**Theorem 8.4.2** (Spectral Correspondence)**.** *Let* $\mathbf{Ex} \ni \mu = -(\rho + (2\ell-6)\alpha)$, $\ell \in \mathbb{N}_0$, *be an exceptional parameter. Then the socle* $\mathrm{soc}(H_\mu)$ *of* $H_\mu$ *is irreducible, unitary and its* $K$*-types are given by* $V_{m,k}$ *for* $m \equiv k \bmod 2$, $m-k \geq 2(\ell+1)$*. The minimal* $K$*-type is* $V_{2\ell+2,0}$ *and the corresponding Poisson transform induces an isomorphism from* ${}^{\Gamma}(\mathrm{soc}(H_\mu))^{-\infty}$ *onto*

$$^{\Gamma}\{u \in C^\infty(G \times_K V_{2\ell+2,0})\colon \textit{properties i) – v) below}\},$$

*where the properties are as follows. Let* $a(m,k) := \frac{m+k}{2}$ *and* $b(m,k) := \frac{m-k}{2}$*. For* $u \in C^\infty(G \times_K V_{2\ell+2,0})$ *let* $\psi_{\ell+1,\ell+1} := \dim V_{2\ell+2,0} \cdot u$ *and define recursively for* $m \equiv k \bmod 2$, $m-k \geq 2(\ell+1)$ *(see Lemma 7.4.1)*

$$\psi_{a,\ell+1} := \frac{\dim V_{m,m-2\ell-2}}{\dim V_{m-1,m-2\ell-3}} \frac{1}{T_{a,\ell+1}^{a-1,\ell+1}} \mathrm{d}_{+,1}\psi_{a-1,\ell+1}, \;\; \psi_{a,b} := \frac{\dim V_{m,k}}{\dim V_{m-1,k+1}} \frac{1}{T_{a,b}^{a,b-1}} \mathrm{d}_{+,2}\psi_{a,b-1},$$

*where we abbreviate* $T_{a_2,b_2}^{a_1,b_1} := T_{V_{a_2+b_2,a_2-b_2}}^{V_{a_1+b_1,a_1-b_1}}(p_{V_{a_1+b_1,a_1-b_1},\mu})(e) \in \mathbb{C}$*. Then we define the properties* *i)* $\mathrm{d}_{+,1}\psi_{a,b} = T^{a,b}_{a+1,b}\frac{\dim V_{m,k}}{\dim V_{m+1,k+1}}\psi_{a+1,b}, \qquad (a \geq b \geq \ell+2),$

*ii)* $\mathrm{d}_{-,1}\psi_{a,b} = T^{a,b}_{a-1,b}\frac{\dim V_{m,k}}{\dim V_{m-1,k-1}}\psi_{a-1,b}, \qquad (a \geq \ell+2,\ b \geq \ell+1),$

*iii)* $\mathrm{d}_{-,2}\psi_{a,b} = T^{a,b}_{a,b-1}\frac{\dim V_{m,k}}{\dim V_{m-1,k+1}}\psi_{a,b-1}, \qquad (a \geq b \geq \ell+2),$

*iv)* $\mathrm{d}_{-,2}\psi_{a,\ell+1} = 0, \qquad (a \geq \ell+1),$

*v)* $\mathrm{d}^{V_{m,k}}_{V}\psi_{a,b} = 0, \qquad (a \geq b \geq \ell+1,\ V \leftrightarrow V_{m,k},\ V \notin \hat{K}_M).$

*Proof.* The proof is analogous to the proof of Theorem 8.2.3. □

# 9. An example: The real hyperbolic case

In this chapter we describe the case of $G = \mathrm{SO}_0(n,1),\ n \geq 3$, in some more detail. In particular, we explicitly describe the projections and embeddings for tensor products, the Fourier characterization of the spherical principal series and the generalized gradients in this case. For the general structure of $G$ we refer the reader to Appendix B.2.1. Recall also the decomposition of the spherical principal series into spherical harmonics from Appendix B.2.2.

## 9.1. Generalized gradients

In order to describe the generalized gradients, we first need to decompose $Y_\ell \otimes \mathfrak{p}^*$ for $[(\tau_\ell, Y_\ell)] \in \widehat{K}$ as a $K$-representation. This is done in Proposition A.1.4 and A.1.6. By these, we have $Y_\ell \otimes \mathfrak{p}^* \cong Y_{\ell+1} \oplus Y_{\ell-1} \oplus V_\ell$ for $\ell \in \mathbb{N}$, where $V_\ell$ does not contain the trivial $M$-representation if $n \neq 3$ and we write $V_\ell := Y_\ell$ if $n = 3$, and $Y_0 \otimes \mathfrak{p}^* \cong Y_1$. The associated generalized gradients are given by

$$\begin{aligned} \mathbf{d}_+ &:= T^{Y_\ell}_{Y_{\ell+1}} \circ \nabla : C^\infty(G \times_K Y_\ell) \to C^\infty(G \times_K Y_{\ell+1}), \\ \mathbf{d}_- &:= T^{Y_\ell}_{Y_{\ell-1}} \circ \nabla : C^\infty(G \times_K Y_\ell) \to C^\infty(G \times_K Y_{\ell-1}), \\ \mathbf{D} &:= \mathrm{pr}_{V_\ell} \circ \nabla : C^\infty(G \times_K Y_\ell) \to C^\infty(G \times_K V_\ell), \end{aligned} \tag{9.1}$$

where the maps $T^{Y_\ell}_{Y_{\ell\pm1}} \in \mathrm{Hom}_K(Y_\ell \otimes \mathfrak{p}^*, Y_{\ell\pm1})$ are given as follows: We have $\mathfrak{p} \cong \mathbb{C}^n$ as $K$-representation, i.e.

$$\mathrm{Ad}(\mathrm{diag}(A,1))X_v = X_{Av}$$

for every $v \in \mathbb{C}^n,\ \mathrm{diag}(A,1) \in K$ (see Appendix B.2.1 for the notation). Thus, we may identify

$$Y_\ell \otimes \mathfrak{p}^* \cong Y_\ell \otimes (\mathbb{C}^n)^*, \quad \phi \otimes X^*_{e_j} \mapsto \phi \otimes e^*_j, \tag{9.2}$$

where $X^*_{e_j}(X_{e_i}) := \delta_{ji} =: e^*_j(e_i)$. This shows that the map

$$Y_\ell \otimes \mathfrak{p}^* \to C^\infty(\mathbb{S}^{n-1}), \quad \phi \otimes X^*_{e_j} \mapsto x_j \phi,$$

is $K$-equivariant, where $x_j$ is the monomial on $\mathbb{S}^{n-1}$ corresponding to $e^*_j$. By Equation (B.2.3) we can decompose

$$x_j \phi = \phi^+_j + |x|^2 \phi^-_j$$

where $\phi_j^{\pm} \in Y_{\ell\pm 1}$ are explicitly given by (cf. [FØ19, Equation (B.4)])

$$\phi_j^{+} = \mathrm{P}(x_j\phi) = x_j\phi - \frac{|x|^2}{n+2\ell-2}\frac{\partial\phi}{\partial x_j}, \qquad \phi_j^{-} = \frac{1}{n+2\ell-2}\frac{\partial\phi}{\partial x_j}. \tag{9.3}$$

This leads to the definition $T^{Y_\ell}_{Y_{\ell-1}}(\phi\otimes X^*_{e_j}) := \phi_j^-$ and $T^{Y_\ell}_{Y_{\ell+1}}(\phi\otimes X^*_{e_j}) := \phi_j^+$ for $\phi\otimes X^*_{e_j} \in Y_\ell\otimes\mathfrak{p}^*\cong Y_\ell\otimes(\mathbb{C}^n)^*$. We also define $\mathbf{d}_-$ for $\ell = 0$ for convenience of notation (note that it is the zero map because of the derivative).

Note that the definition of the projections coincides with the general one from Definition 7.3.4 since the identification of $K/M$ with $\mathbb{S}^{n-1}$ comes from the adjoint action on $\mathfrak{p}_0$ (see Appendix B.2.2), $\omega(X_{e_j}) = x_j \in C^\infty(\mathbb{S}^{n-1})$ and $\mathbf{I}(X_{e_j}) = X_{e_j}$. The following lemma turns out to be very useful for computations involving spherical harmonics.

**Lemma 9.1.1.** *Let $f$ be a homogeneous polynomial of degree $\ell$. Then*

$$\mathrm{P}(x_i f) = \mathrm{P}(x_i\mathrm{P}(f)).$$

*Proof.* Writing $f = \mathrm{P}(f) + |x|^2 q$ with $q$ homogeneous of degree $\ell-2$ implies

$$x_i f = x_i\mathrm{P}(f) + |x|^2 x_i q \Rightarrow \mathrm{P}(x_i f) = \mathrm{P}(x_i\mathrm{P}(f))$$

by applying P to both sides since multiples of $|x|^2$ are in the kernel of P. □

We now describe the embeddings of $Y_{\ell\pm1}$ into $Y_\ell\otimes\mathfrak{p}^*\cong Y_\ell\otimes(\mathbb{C}^n)^*$ corresponding to $T^{Y_\ell}_{Y_{\ell-1}}$ and $T^{Y_\ell}_{Y_{\ell+1}}$.

**Lemma 9.1.2.** *Let*

$$\iota^{Y_{\ell+1}}_{Y_\ell} : Y_{\ell+1}\to Y_\ell\otimes(\mathbb{C}^n)^*, \quad f\mapsto \frac{1}{\ell+1}\sum_{j=1}^n \frac{\partial f}{\partial x_j}\otimes e_j^*$$

*and*

$$\iota^{Y_{\ell-1}}_{Y_\ell} : Y_{\ell-1}\to Y_\ell\otimes(\mathbb{C}^n)^*, \quad f\mapsto \frac{n+2\ell-4}{\ell+n-3}\sum_{j=1}^n \mathrm{P}(x_j f)\otimes e_j^*.$$

*Then*

*i)* $\iota^{Y_{\ell+1}}_{Y_\ell}$ *and* $\iota^{Y_{\ell-1}}_{Y_\ell}$ *are $K$-equivariant,*

*ii)* $T^{Y_\ell}_{Y_{\ell+1}}\circ\iota^{Y_{\ell+1}}_{Y_\ell} = \mathrm{id}_{Y_{\ell+1}}$ *and* $T^{Y_\ell}_{Y_{\ell-1}}\circ\iota^{Y_{\ell-1}}_{Y_\ell} = \mathrm{id}_{Y_{\ell-1}}$,

*iii)* $T^{Y_\ell}_{Y_{\ell+1}}\circ\iota^{Y_{\ell-1}}_{Y_\ell} = 0$ *and* $T^{Y_\ell}_{Y_{\ell-1}}\circ\iota^{Y_{\ell+1}}_{Y_\ell} = 0$.

Let us first compare these definitions with Proposition 7.3.9. We shall confine ourselves to the case $\iota^{Y_{\ell-1}}_{Y_\ell}$ and obtain – due to Remark A.1.3 –

$$\iota^{Y_{\ell-1}}_{Y_\ell}(f)(X_{e_j}) = \frac{1}{\lambda(Y_{\ell-1},Y_\ell)}\mathrm{pr}_{Y_\ell}(\omega(X_{e_j})f) = \frac{1}{\lambda(Y_{\ell-1},Y_\ell)}\mathrm{P}(x_j f) = \frac{n+2\ell-4}{\ell+n-3}\mathrm{P}(x_j f).$$

A similar argument works for the projections $T^{Y_\ell}_W$. Lemma 9.1.2 may also be proved in the following direct way using only the structure of spherical harmonics.

*Proof.* i) We first prove the $K$-equivariance of $\iota_{Y_\ell}^{Y_{\ell+1}}$. Let $f \in Y_{\ell+1}$ and $k \in K$. Then

$$\iota_{Y_\ell}^{Y_{\ell+1}}(k.f) = \iota_{Y_\ell}^{Y_{\ell+1}}(f \circ k^{-1}) = \frac{1}{\ell+1} \sum_{j=1}^{n} \frac{\partial (f \circ k^{-1})}{\partial x_j} \otimes e_j^*$$

and

$$k.\iota_{Y_\ell}^{Y_{\ell+1}}(f) = \frac{1}{\ell+1} \sum_{j=1}^{n} k.\frac{\partial f}{\partial x_j} \otimes k.e_j^* = \frac{1}{\ell+1} \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \circ k^{-1} \otimes e_j^* \circ k^{-1}.$$

Using the canonical isomorphism

$$\Phi : Y_\ell \otimes (\mathbb{C}^n)^* \cong \mathrm{Hom}(\mathbb{C}^n, Y_\ell), \quad \Phi(f \otimes \lambda)(v) := \lambda(v) f$$

we have

$$\begin{aligned} \Phi(\iota_{Y_\ell}^{Y_{\ell+1}}(k.f))(e_i) &= \frac{1}{\ell+1} \frac{\partial (f \circ k^{-1})}{\partial x_i}, \\ \Phi(k.\iota_{Y_\ell}^{Y_{\ell+1}}(f))(e_i) &= \frac{1}{\ell+1} \sum_{j=1}^{n} \left( \frac{\partial f}{\partial x_j} \circ k^{-1} \right) e_j^*(k^{-1} e_i). \end{aligned}$$

By the chain rule we obtain

$$\Phi(\iota_{Y_\ell}^{Y_{\ell+1}}(k.f))(e_i) = \frac{1}{\ell+1} \sum_{j=1}^{n} \left( \frac{\partial f}{\partial x_j} \circ k^{-1} \right) \frac{\partial k_j^{-1}}{\partial x_i},$$

where $k_j^{-1}$ denotes the $j$-th component of $k^{-1} \in \mathrm{Aut}(\mathbb{C}^n)$. For each $x \in \mathbb{C}^n$ we have

$$\frac{\partial k_j^{-1}}{\partial x_i}(x) = \lim_{h \to 0} \frac{k_j^{-1}(x + h e_i) - k_j^{-1}(x)}{h} = k_j^{-1}(e_i) = e_j^*(k^{-1} e_i)$$

since $k^{-1}$ is linear. This proves the equivariance of $\iota_{Y_\ell}^{Y_{\ell+1}}$.

Let us now prove the equivariance of $\iota_{Y_\ell}^{Y_{\ell-1}}$. For $f \in Y_{\ell-1}$ and $k \in K$ we have to show that

$$\sum_{j=1}^{n} \mathrm{P}(x_j(k.f)) \otimes e_j^* = \sum_{j=1}^{n} k.\mathrm{P}(x_j f) \otimes e_j^* \circ k^{-1}.$$

Evaluating at $e_i$ using $\Phi$ this is equivalent to

$$\mathrm{P}(x_i(k.f)) = \sum_{j=1}^{n} k.\mathrm{P}(x_j f) e_j^*(k^{-1} e_i) \tag{9.4}$$

for each $i \in \{1, \ldots, n\}$. Note that

$$k^{-1}.e_i^* = \sum_{j=1}^{n} e_i^*(k e_j) = \sum_{j=1}^{n} e_j^*(k^{-1} e_i) e_j^* \tag{9.5}$$

since $k$ is orthogonal. By the equivariance of P the left hand side of Equation (9.4) reads

$$\mathrm{P}(x_i(k.f)) = \mathrm{P}(k.((k^{-1}.x_i)f)) = k.\mathrm{P}((k^{-1}.x_i)f).$$

Since the monomial $x_i$ equals $e_i^*$ we can use Equation (9.5) to get

$$\begin{aligned}\mathrm{P}(x_i(k.f)) = k.\mathrm{P}((k^{-1}.e_i^*)f) &= k.\mathrm{P}\left(\sum_{j=1}^n e_j^*(k^{-1}e_i)e_j^* f\right)\\ &= \sum_{j=1}^n k.\mathrm{P}(e_j^* f)e_j^*(k^{-1}e_i) = \sum_{j=1}^n k.\mathrm{P}(x_j f)e_j^*(k^{-1}e_i)\end{aligned}$$

by the linearity of P and the linearity of the action of $k$. This proves Equation (9.4) and thus the equivariance of $\iota_{Y_\ell}^{Y_{\ell-1}}$.

ii) We first prove $T_{Y_{\ell+1}}^{Y_\ell} \circ \iota_{Y_\ell}^{Y_{\ell+1}} = \mathrm{id}_{Y_{\ell+1}}$. For each $f \in Y_{\ell+1}$ we have

$$\begin{aligned}T_{Y_{\ell+1}}^{Y_\ell}(\iota_{Y_\ell}^{Y_{\ell+1}}(f)) = T_{Y_{\ell+1}}^{Y_\ell}\left(\frac{1}{\ell+1}\sum_{j=1}^n \frac{\partial f}{\partial x_j} \otimes e_j^*\right) &= \frac{1}{\ell+1}\sum_{j=1}^n T_{Y_{\ell+1}}^{Y_\ell}\left(\frac{\partial f}{\partial x_j} \otimes e_j^*\right)\\ = \frac{1}{\ell+1}\sum_{j=1}^n \mathrm{P}\left(x_j \frac{\partial f}{\partial x_j}\right) &= \frac{1}{\ell+1}\mathrm{P}\left(\sum_{j=1}^n x_j \frac{\partial f}{\partial x_j}\right).\end{aligned}$$

Since $f$ is a homogeneous polynomial of degree $\ell+1$ we have

$$\sum_{j=1}^n x_j \frac{\partial f}{\partial x_j} = (\ell+1)f.$$

Since $f$ is harmonic this yields

$$T_{Y_{\ell+1}}^{Y_\ell}(\iota_{Y_\ell}^{Y_{\ell+1}}(f)) = \frac{1}{\ell+1}\mathrm{P}\left((\ell+1)f\right) = \mathrm{P}(f) = f.$$

Let us now prove $T_{Y_{\ell-1}}^{Y_\ell} \circ \iota_{Y_\ell}^{Y_{\ell-1}} = \mathrm{id}_{Y_{\ell-1}}$. For each $f \in Y_{\ell-1}$ we have

$$\begin{aligned}T_{Y_{\ell-1}}^{Y_\ell}(\iota_{Y_\ell}^{Y_{\ell-1}}(f)) &= \frac{n+2\ell-4}{\ell+n-3}\sum_{j=1}^n T_{Y_{\ell-1}}^{Y_\ell}(\mathrm{P}(x_j f) \otimes e_j^*)\\ &= \frac{n+2\ell-4}{\ell+n-3}\sum_{j=1}^n \frac{1}{n+2\ell-2}\frac{\partial \mathrm{P}(x_j f)}{\partial x_j}.\end{aligned} \tag{9.6}$$

By Equation (9.3) we have

$$\mathrm{P}(x_j f) = x_j f - \frac{|x|^2}{n+2(\ell-1)-2}\frac{\partial f}{\partial x_j} = x_j f - \frac{|x|^2}{n+2\ell-4}\frac{\partial f}{\partial x_j}$$

and thus

$$\begin{aligned}\frac{\partial \mathrm{P}(x_j f)}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(x_j f - \frac{|x|^2}{n+2\ell-4}\frac{\partial f}{\partial x_j}\right)\\ &= f + x_j\frac{\partial f}{\partial x_j} - \frac{1}{n+2\ell-4}\left(2x_j\frac{\partial f}{\partial x_j} + |x|^2\frac{\partial^2 f}{\partial x_j^2}\right).\end{aligned}$$

This implies

$$\sum_{j=1}^n \frac{\partial \mathrm{P}(x_j f)}{\partial x_j} = nf + \sum_{j=1}^n x_j\frac{\partial f}{\partial x_j} - \frac{1}{n+2\ell-4}\left(2\sum_{j=1}^n x_j\frac{\partial f}{\partial x_j} + |x|^2\Delta f\right),$$

where $\Delta$ denotes the standard Laplacian on $\mathbb{R}^n$. Since $f$ is homogeneous of degree $\ell-1$ and harmonic this simplifies to

$$\sum_{j=1}^n \frac{\partial \mathrm{P}(x_j f)}{\partial x_j} = nf + (\ell-1)f - \frac{1}{n+2\ell-4}2(\ell-1)f = \frac{(\ell+n-3)(n+2\ell-2)}{n+2\ell-4}f.$$

Plugging this into Equation (9.6) finishes the proof of ii).

iii) We first prove $T^{Y_\ell}_{Y_{\ell+1}} \circ \iota^{Y_{\ell-1}}_{Y_\ell} = 0$. For $f \in Y_{\ell-1}$ we have

$$\begin{aligned}T^{Y_\ell}_{Y_{\ell+1}}(\iota^{Y_{\ell-1}}_{Y_\ell}(f)) &= T^{Y_\ell}_{Y_{\ell+1}}\left(\frac{n+2\ell-4}{\ell+n-3}\sum_{j=1}^n \mathrm{P}(x_j f)\otimes e_j^*\right)\\ &= \frac{n+2\ell-4}{\ell+n-3}\sum_{j=1}^n T^{Y_\ell}_{Y_{\ell+1}}(\mathrm{P}(x_j f)\otimes e_j^*)\\ &= \frac{n+2\ell-4}{\ell+n-3}\sum_{j=1}^n \mathrm{P}(x_j\mathrm{P}(x_j f)).\end{aligned}$$

By Lemma 9.1.1 we have $\mathrm{P}(x_j\mathrm{P}(x_j f)) = \mathrm{P}(x_j^2 f)$. This yields

$$T^{Y_\ell}_{Y_{\ell+1}}(\iota^{Y_{\ell-1}}_{Y_\ell}(f)) = \frac{n+2\ell-4}{\ell+n-3}\sum_{j=1}^n \mathrm{P}(x_j^2 f) = \frac{n+2\ell-4}{\ell+n-3}\mathrm{P}(|x|^2 f) = 0.$$

Let us finally prove $T^{Y_\ell}_{Y_{\ell-1}} \circ \iota^{Y_{\ell+1}}_{Y_\ell} = 0$. For each $f \in Y_{\ell+1}$ we calculate

$$\begin{aligned}T^{Y_\ell}_{Y_{\ell-1}}(\iota^{Y_{\ell+1}}_{Y_\ell}(f)) &= T^{Y_\ell}_{Y_{\ell-1}}\left(\frac{1}{\ell+1}\sum_{j=1}^n \frac{\partial f}{\partial x_j}\otimes e_j^*\right) = \frac{1}{\ell+1}\sum_{j=1}^n T^{Y_\ell}_{Y_{\ell-1}}\left(\frac{\partial f}{\partial x_j}\otimes e_j^*\right)\\ &= \frac{1}{\ell+1}\sum_{j=1}^n \frac{1}{n+2\ell-2}\frac{\partial^2 f}{\partial x_j^2} = \frac{1}{(\ell+1)(n+2\ell-2)}\Delta f = 0\end{aligned}$$

since $f$ is harmonic. □

For explicit calculations using generalized gradients we need the following two lemmas.

**Lemma 9.1.3.** *For each $f \in Y_\ell$ we have*

$$\frac{\partial f}{\partial x_1}(e_1) = \ell f(e_1) \text{ and } \mathrm{P}(x_1 f)(e_1) = \frac{n+\ell-2}{n+2\ell-2} f(e_1).$$

*Proof.* For the first equality note that

$$\frac{\partial f}{\partial x_1}(e_1) = \left( x_1 \frac{\partial f}{\partial x_1} \right)(e_1) = \left( \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} \right)(e_1) = \ell f(e_1).$$

For the second equality we use Equation (9.3) to get

$$\mathrm{P}(x_1 f)(e_1) = \left( x_1 f - \frac{|x|^2}{n+2\ell-2} \frac{\partial f}{\partial x_1} \right)(e_1) = f(e_1) - \frac{1}{n+2\ell-2} \frac{\partial f}{\partial x_1}(e_1).$$

By the first equality this simplifies to

$$\mathrm{P}(x_1 f)(e_1) = f(e_1) - \frac{\ell}{n+2\ell-2} f(e_1) = \frac{n+\ell-2}{n+2\ell-2} f(e_1). \qquad \square$$

**Lemma 9.1.4.** *Let $\ell \in \mathbb{N}$. Then*

*i)* $\mathrm{P}(x_1^\ell)$ *is a linear combination of* $x_1^{\ell-2j}|x|^{2j}$, $j = 0, \ldots, \frac{\ell}{2}$,

*ii)* $\mathrm{P}(x_1^\ell)(e_1) = \frac{n+\ell-3}{n+2\ell-4} \mathrm{P}(x_1^{\ell-1})(e_1)$ *and* $\frac{\partial}{\partial x_1} \mathrm{P}(x_1^\ell) = \frac{\ell(n+\ell-3)}{n+2\ell-4} \mathrm{P}(x_1^{\ell-1})$.

*Proof.* i) By Lemma 9.1.1 and Equation (9.3) we have

$$\mathrm{P}(x_1^\ell) = \mathrm{P}(x_1 \mathrm{P}(x_1^{\ell-1})) = x_1 \mathrm{P}(x_1^{\ell-1}) - \frac{|x|^2}{n+2\ell-4} \frac{\partial}{\partial x_1} \mathrm{P}(x_1^{\ell-1}).$$

The first part now follows by induction.

ii) Since $\frac{\partial}{\partial x_1} \mathrm{P}(x_1^\ell)$ is an $M$-invariant, harmonic, homogeneous polynomial of degree $\ell - 1$ it has to be a multiple of $\mathrm{P}(x_1^{\ell-1})$ (recall $Y_{\ell-1}^M = \mathbb{C}\mathrm{P}(x_1^{\ell-1})$ from Equation B.2.4). We compute this multiple by evaluating at $e_1$. By Lemma 9.1.3 and Lemma 9.1.1

$$\frac{\partial}{\partial x_1} \mathrm{P}(x_1^\ell)(e_1) = \ell \mathrm{P}(x_1^\ell)(e_1) = \ell \mathrm{P}(x_1 \mathrm{P}(x_1^{\ell-1}))(e_1) = \ell \frac{n+\ell-3}{n+2\ell-4} \mathrm{P}(x_1^{\ell-1})(e_1). \qquad \square$$

We define the Poisson transform $P_\mu^{Y_\ell,\mathrm{cpt}} : \mathcal{D}'(K/M) \to C^\infty(G \times_K Y_\ell)$ by continuous extension of (see Definition 3.3.1)

$$P_\mu^{Y_\ell,\mathrm{cpt}}(\phi)(g) := \int_K a_I(g^{-1}k)^{-(\mu+\rho)} \tau_\ell(k_I(g^{-1}k)) t(\phi(k)) \, \mathrm{d}k,$$

where $t \in \mathrm{Hom}_M(\mathbb{C}, Y_\ell)$ is given by $t(1) := \mathrm{P}(x_1^\ell)$[1]. By Equation (3.5) we have

$$P_\mu^{Y_\ell} \circ \mathcal{Q}_{\mu-\rho} = P_\mu^{Y_\ell,\mathrm{cpt}}$$

[1]Note that this normalization deviates slightly from $t(1) := \phi_{Y_\ell} = \frac{\mathrm{P}(x_1^\ell)}{\mathrm{P}(x_1^\ell)(e_1)}$ as introduced in Section 5.1.

on $\mathcal{D}'(K/M)$ where $P_\mu^{Y_\ell} : H_\mu^{-\infty} \to C^\infty(G \times_K Y_\ell)$ is given by continuous extension of

$$P_\mu^{Y_\ell}(f)(g) \coloneqq \int_K \tau_\ell(k) t(f(gk))\, \mathrm{d}k = F^{-1}(t)(\pi_\mu(g^{-1})f). \tag{9.7}$$

The following lemma is the direct generalization of Lemma 4.4.1 to $G = \mathrm{SO}_0(n, 1)$.

**Lemma 9.1.5.** *Let $\mu \in \mathfrak{a}^*$ and $f \in H_\mu^{-\infty}$. Then*

$$\begin{aligned}(\mathbf{d}_+ \circ P_\mu^{Y_\ell})(f) &= ((\mu+\rho)(H) + \ell) P_\mu^{Y_{\ell+1}}(f)\ \forall \ell \in \mathbb{N}_0,\\ (\mathbf{d}_- \circ P_\mu^{Y_\ell})(f) &= \lambda_\ell \cdot (\mu(H) - (\rho(H) + \ell - 1)) P_\mu^{Y_{\ell-1}}(f)\ \forall \ell \in \mathbb{N},\\ (\mathbf{D} \circ P_\mu^{Y_\ell})(f) &= 0\ \forall \ell \in \mathbb{N},\end{aligned}$$

*where* $\lambda_\ell \coloneqq \frac{\ell(\ell+n-3)}{(2\ell+n-4)(2\ell+n-2)}$.

**Remark 9.1.6.** Note that $\lambda_\ell \neq 0$ since $n \geq 3$.

Again we provide an explicit proof of Lemma 9.1.5. We first consider the first two equalities and postpone the proof of the last equality, which completes the proof of Proposition 7.3.12, to Section B.2.4.

*Proof of the first two equalities of Lemma 9.1.5.* Let $\delta_{eM}$ denote the Delta distribution at $eM$ on $K/M$. Then

$$P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM})(g) = a_I(g^{-1})^{-(\mu+\rho)} \tau_\ell(k_I(g^{-1})) \mathrm{P}(x_1^\ell) \in C^\infty(G \times_K Y_\ell).$$

A basis of $\mathfrak{p}$ is given by $H, X_{e_2}, \ldots, X_{e_n}$ (see Appendix B.2.1). We have

$$\begin{aligned}(\nabla \circ P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM}))(e)(H) &= \frac{d}{dt}\Big|_{t=0} P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM})(\exp tH)\\ &= \frac{d}{dt}\Big|_{t=0} a_I(\exp -tH)^{-(\mu+\rho)} \mathrm{P}(x_1^\ell)\\ &= \frac{d}{dt}\Big|_{t=0} e^{t(\mu+\rho)(H)} \mathrm{P}(x_1^\ell) = (\mu+\rho)(H) \mathrm{P}(x_1^\ell).\end{aligned}$$

Note that, for $j \in \{2, \ldots, n\}$,

$$X_{e_j} = \begin{pmatrix} & e_j^T & \\ -e_j & & e_j \\ & e_j^T & \end{pmatrix} - \begin{pmatrix} & e_j^T & \\ -e_j & & \\ & & \end{pmatrix} \in \mathfrak{n} \oplus \mathfrak{k}.$$

Let the latter matrix (without the minus sign) be denoted by $A_j \in \mathfrak{k}$. Then

$$(\nabla \circ P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM}))(e)(X_{e_j}) = -(\nabla \circ P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM}))(e)(A_j) = A_j.\mathrm{P}(x_1^\ell) = \mathrm{P}(A_j.x_1^\ell).$$

A standard calculation shows that the elementary $(n \times n)$-matrix $E_{ij}$ with $(E_{ij})_{kl} \coloneqq \delta_{ki}\delta_{lj}$ acts on functions $f : \mathbb{R}^n \to \mathbb{R}$ via the derived left regular representation by

$$E_{ij}.f = -x_j \frac{\partial f}{\partial x_i}. \tag{9.8}$$

Therefore

$$(\nabla \circ P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM}))(e)(X_{e_j}) = \mathrm{P}(-x_j \ell x_1^{\ell-1}) = (-\ell)\mathrm{P}(x_j x_1^{\ell-1}).$$

Altogether we see that $(\nabla \circ P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM}))(e) \in Y_\ell \otimes \mathfrak{p}^* \cong \mathrm{Hom}(\mathfrak{p}, Y_\ell)$ is given by

$$(\nabla \circ P_\mu^{Y_\ell,\mathrm{cpt}}(\delta_{eM}))(e) = (\mu+\rho)(H)\mathrm{P}(x_1^\ell) \otimes H^* - \ell \sum_{j=2}^n \mathrm{P}(x_j x_1^{\ell-1}) \otimes X_{e_j}^*.$$

In order to calculate the generalized gradients we have to project this tensor onto $Y_{\ell\pm1}$. More precisely, we have to decompose the corresponding polynomial

$$(\mu+\rho)(H)\mathrm{P}(x_1^\ell)x_1 - \ell \sum_{j=2}^n \mathrm{P}(x_j x_1^{\ell-1})x_j \tag{9.9}$$

according to Equation (B.2.3). By Lemma 9.1.1 we obtain $\mathrm{P}(x_j \mathrm{P}(x_j x_1^{\ell-1})) = \mathrm{P}(x_j^2 x_1^{\ell-1})$. Plugging this into Equation (9.9) yields

$$\begin{aligned}
&\mathrm{P}((\mu+\rho)(H)\mathrm{P}(x_1^\ell)x_1 - \ell \sum_{j=2}^n \mathrm{P}(x_j x_1^{\ell-1})x_j) = (\mu+\rho)(H)\mathrm{P}(x_1^{\ell+1}) - \ell \sum_{j=2}^n \mathrm{P}(x_j^2 x_1^{\ell-1}) \\
&= \mathrm{P}((\mu+\rho)(H)x_1^{\ell+1} - \ell \sum_{j=2}^n x_j^2 x_1^{\ell-1}) = \mathrm{P}(((\mu+\rho)(H)+\ell)x_1^{\ell+1} - \ell|x|^2 x_1^{\ell-1}) \\
&= \mathrm{P}(((\mu+\rho)(H)+\ell)x_1^{\ell+1}) = ((\mu+\rho)(H)+\ell)\mathrm{P}(x_1^{\ell+1}) \\
&= ((\mu+\rho)(H)+\ell)P_\mu^{Y_{\ell+1},\mathrm{cpt}}(\delta_{eM})(e).
\end{aligned}$$

This shows

$$(\mathbf{d}_+ \circ P_\mu^{Y_\ell,\mathrm{cpt}})(\delta_{eM})(e) = ((\mu+\rho)(H)+\ell)P_\mu^{Y_{\ell+1},\mathrm{cpt}}(\delta_{eM})(e)$$

and the first equality of the lemma follows from Corollary 3.3.5.

For the second equality of the lemma we first claim that (9.9) can be written as

$$((\mu+\rho)(H)+\ell)\mathrm{P}(x_1^\ell)x_1 - |x|^2 \frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell). \tag{9.10}$$

Indeed, by the $K$-equivariance of P and Equation (9.8)

$$-\ell\mathrm{P}(x_j x_1^{\ell-1}) = \mathrm{P}(A_j.x_1^\ell) = A_j.\mathrm{P}(x_1^\ell) = \left(x_1 \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_1}\right)\mathrm{P}(x_1^\ell).$$

This yields

$$\begin{aligned}-\ell\sum_{j=2}^{n}\mathrm{P}(x_jx_1^{\ell-1})x_j &= \sum_{j=2}^{n}x_j\left(x_1\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_1}\right)\mathrm{P}(x_1^\ell)\\ &= \sum_{j=2}^{n}x_jx_1\frac{\partial}{\partial x_j}\mathrm{P}(x_1^\ell)-\sum_{j=2}^{n}x_j^2\frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell)\\ &= x_1\left(\sum_{j=1}^{n}x_j\frac{\partial}{\partial x_j}\mathrm{P}(x_1^\ell)-x_1\frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell)\right)-\sum_{j=2}^{n}x_j^2\frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell)\\ &= \ell x_1\mathrm{P}(x_1^\ell)-|x|^2\frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell),\end{aligned}$$

where we used the fact that the Euler operator acts on homogeneous polynomials of degree $\ell$ by the scalar $\ell$. This proves Equation (9.10).
We apply Equation (9.3) on (9.10) to deduce that $(\mathbf{d}_-\circ P_\mu^{Y_\ell,\mathrm{cpt}})(\delta_{eM})(e)$ is given by

$$\frac{(\mu+\rho)(H)+\ell}{n+2\ell-2}\frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell)-\frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell)=\frac{\mu(H)-(\rho(H)+\ell-1)}{n+2\ell-2}\frac{\partial}{\partial x_1}\mathrm{P}(x_1^\ell)$$

where we used $\rho(H)=\frac{n-1}{2}$. Finally Lemma 9.1.4 ii) implies

$$(\mathbf{d}_-\circ P_\mu^{Y_\ell,\mathrm{cpt}})(\delta_{eM})(e)=\frac{\ell(\ell+n-3)}{n+2\ell-4}\frac{\mu(H)-(\rho(H)+\ell-1)}{n+2\ell-2}\mathrm{P}(x_1^{\ell-1})$$

which finishes the proof since $\mathrm{P}(x_1^{\ell-1})=P_\mu^{Y_{\ell-1},\mathrm{cpt}}(\delta_{eM})(e)$. □

## 9.2. Fourier characterization

In this section we characterize the spherical principal series representations by relations between their Fourier components (Proposition 9.2.10), generalizing Lemma 4.2.1 from the $\mathrm{PSL}(2,\mathbb{R})$ case. The explicit structure of $\mathrm{SO}_0(n,1)$ allows a more transparent proof compared to the general one given in Section 7.4. We first recall the important objects and their properties from the general case.

**Definition 9.2.1.** We embed $C^\infty(G\times_K Y_\ell)$ into the smooth right $M$-invariant functions $C^\infty(G)^M$ by the map

$$\pi_{Y_\ell}: C^\infty(G\times_K Y_\ell)\to C^\infty(G)^M,\quad \pi_{Y_\ell}(\varphi)(nak):=\varphi(na)(k.e_1),\ n\in N,a\in A,k\in K,$$

where $e_1=(1,0,\dots,0)\in\mathbb{S}^{n-1}$. Note that $\pi_{Y_\ell}$ is indeed injective since $K$ acts transitively on the sphere $\mathbb{S}^{n-1}\subseteq\mathbb{R}^n$ and $Y_\ell$ consists of polynomials on $\mathbb{S}^{n-1}$. Since $\varphi\in C^\infty(G\times_K Y_\ell)$ we have for each $g=nak\in G=NAK$

$$\pi_{Y_\ell}(\varphi)(g)=\varphi(na)(k.e_1)=(\tau_\ell(k^{-1})\varphi(na))(e_1)=\varphi(g)(e_1).$$

We further denote by

$$\pi_{Y_\ell}^*: \mathcal{D}'(G/M) \to \mathcal{D}'(G \times_K Y_\ell), \quad \pi_{Y_\ell}^*(f)(\varphi) := f(\pi_{Y_\ell}(\varphi))$$

the transpose of $\pi_{Y_\ell}$ defined by duality where $\mathcal{D}'(G \times_K Y_\ell)$ is given by the dual of $C_c^\infty(G \times_K Y_\ell)$. We embed smooth sections into distributional sections by

$$\iota_{Y_\ell}\colon C^\infty(G \times_K Y_\ell) \to \mathcal{D}'(G \times_K Y_\ell), \quad \iota(f)(\varphi) := \int_G \pi_{Y_\ell}(f)(g)\pi_{Y_\ell}(\varphi)(g)\,\mathrm{d}g$$

and $C^\infty(G/M)$ into $\mathcal{D}'(G/M)$ by

$$\iota_{G/M}\colon C^\infty(G/M) \to \mathcal{D}'(G/M), \quad \iota_{G/M}(f)(\varphi) := \int_G f(gM)\varphi(gM)\,\mathrm{d}g.$$

As in Lemma 7.1.2, we can uniquely expand each $f \in C^\infty(G)^M$ into a Fourier series. We extend the definition of $\pi_{Y_\ell}$ to distributional sections as in Notation 7.1.3 and can also decompose distributions (see Proposition 7.2.4). Let us introduce some scalars relating the Poisson transforms to orthogonal projections.

**Remark 9.2.2.** There exist constants $c_\ell \in \mathbb{C} \setminus \{0\}$ such that

$$c_\ell \cdot P_\mu^{Y_\ell}(f)(g) = \mathrm{pr}_{Y_\ell}(f(g\bullet))$$

for all $\mu \in \mathfrak{a}^*$, $f \in C^\infty(G)^M$, $g \in G$ and $\ell \in \mathbb{N}_0$ where we extend the Poisson transform $P_\mu^{Y_\ell}$ to $C^\infty(G)^M$ by the same formula.

*Proof.* By the definition of the Poisson transform from Equation (9.7) we have

$$P_\mu^{Y_\ell}(f)(g) = F^{-1}(t)(f(g\bullet)).$$

Now $F^{-1}(t)$ and $\mathrm{pr}_{Y_\ell}$ are non-zero elements of $\mathrm{Hom}_K(L^2(K/M), Y_\ell)$ which is one-dimensional by Proposition 2.4.3. Thus, there exists a constant $c_\ell \in \mathbb{C} \setminus \{0\}$ such that $\mathrm{pr}_{Y_\ell} = c_\ell \cdot F^{-1}(t)$. □

As in Lemma 7.1.4 we have the following

**Lemma 9.2.3.** *Recall the maps $\iota_{G/M}$ and $\iota_{Y_\ell}$ from Definition 9.2.1. We have*

*i)* $\pi_{Y_\ell}^*(f)(g) = \mathrm{pr}_{Y_\ell}(f(g\bullet))$ *for each* $f \in C^\infty(G/M)$, $g \in G$, *so that* $\pi_{Y_\ell}^*(C^\infty(G/M)) \subseteq C^\infty(G \times_K Y_\ell)$ *and* $\pi_{Y_\ell}^*(C_c^\infty(G/M)) \subseteq C_c^\infty(G \times_K Y_\ell)$,

*ii)* $f = \sum_{\ell\in\mathbb{N}_0} \pi_{Y_\ell}(\pi_{Y_\ell}^*(f))$ *pointwise for each* $f \in C^\infty(G/M)$,

*iii)* $\pi_{Y_\ell}(\iota_{Y_\ell}(f)) = \iota_{G/M}(\pi_{Y_\ell}(f))$ *for each* $f \in C^\infty(G \times_K Y_\ell)$ *and*

*iv)* $\forall \mu \in \mathfrak{a}^*\colon P_\mu^{Y_\ell,\mathrm{cpt}} = P_\mu^{Y_\ell} \circ \mathcal{Q}_{\mu-\rho} = \frac{1}{c_\ell}\pi_{Y_\ell}^* \circ \mathcal{Q}_{\mu-\rho}$ *on* $\mathcal{D}'(K/M)$.

*Proof.* The first three parts follow as in Lemma 7.1.4. We prove part iv). By continuity (recall Proposition 2.2.2) we restrict our attention to smooth functions $\phi \in C^\infty(K/M)$. The first equality follows from Equation (3.5). For the second equality we combine Remark 9.2.2 and i). □

Lemma 9.2.3 allows the following reformulation of Lemma 9.1.5.

**Lemma 9.2.4.** *Let $\mu \in \mathfrak{a}^*$ and $f \in H_\mu^{-\infty}$. Then*

$$\begin{aligned}(\mathbf{d}_+ \circ \pi_{Y_\ell}^*)(f) &= \frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H)+\ell)\pi_{Y_{\ell+1}}^*(f)\ \forall \ell \in \mathbb{N}_0,\\ (\mathbf{d}_- \circ \pi_{Y_\ell}^*)(f) &= \frac{c_\ell}{c_{\ell-1}}\lambda_\ell(\mu(H)-(\rho(H)+\ell-1))\pi_{Y_{\ell-1}}^*(f)\ \forall \ell \in \mathbb{N},\\ (\mathbf{D} \circ \pi_{Y_\ell}^*)(f) &= 0\ \forall \ell \in \mathbb{N},\end{aligned}$$

*where $\lambda_\ell = \frac{\ell(\ell+n-3)}{(2\ell+n-4)(2\ell+n-2)}$.*

In particular, we obtain

**Remark 9.2.5.** For each $\mu \in \mathfrak{a}^*$ and $f \in H_\mu^{-\infty}$ we have

$$\mathbf{d}_+\mathbf{d}_-\pi_{Y_\ell}^*(f) = \xi_\ell\, \pi_{Y_\ell}^*(f) \qquad \text{and} \qquad \mathbf{d}_-\mathbf{d}_+\pi_{Y_\ell}^*(f) = \xi_{\ell+1}\, \pi_{Y_\ell}^*(f),$$

where $\xi_\ell := \lambda_\ell \cdot (\mu(H)^2 - (\rho(H)+\ell-1)^2)$.

In order to use Lemma 9.2.4 for proving the Fourier characterization, we need to determine the constants $c_\ell$ explicitly. Indeed, these constants follow a nice recursion.

**Lemma 9.2.6.** *The scalar $c_\ell$ from Remark 9.2.2 is given by*

$$c_\ell = \frac{\mathrm{P}(x_1^\ell)(e_1)}{\|\mathrm{P}(x_1^\ell)\|^2_{L^2(\mathbb{S}^{n-1})}} = \frac{\dim Y_\ell}{\mathrm{P}(x_1^\ell)(e_1)},$$

*where $\mathbb{S}^{n-1}$ is equipped with the normalized Lebesgue measure. Moreover we have the recursion formula*

$$c_{\ell+1} = \frac{n+2\ell}{\ell+1}c_\ell, \quad c_0 = 1.$$

*Proof.* Let $f := \pi_{Y_\ell}(\varphi) \in C^\infty(G)^M$ where $\varphi \in C^\infty(G \times_K Y_\ell)$ is given by

$$\varphi(g) := \tau_\ell(k_{NAK}(g)^{-1})\mathrm{P}(x_1^\ell).$$

By Equation (9.7) we have

$$P_\mu^{Y_\ell}(f)(e) = \int_K \tau_\ell(k)t(f(k))\,\mathrm{d}k = \int_K f(k)\tau_\ell(k)t(1)\,\mathrm{d}k.$$

Since $t(1) = \mathrm{P}(x_1^\ell)$ by definition we get

$$\begin{aligned}P_\mu^{Y_\ell}(f)(e)(e_1) &= \int_K f(k)\tau_\ell(k)t(1)(e_1)\,\mathrm{d}k = \int_K f(k)\mathrm{P}(x_1^\ell)(k^{-1}.e_1)\,\mathrm{d}k\\ &= \int_K \varphi(k)(e_1)\mathrm{P}(x_1^\ell)(k^{-1}.e_1)\,\mathrm{d}k = \int_K \mathrm{P}(x_1^\ell)(k.e_1)\mathrm{P}(x_1^\ell)(k^{-1}.e_1)\,\mathrm{d}k\\ &= \int_{\mathrm{SO}(n)} \mathrm{P}(x_1^\ell)(ke_1)\mathrm{P}(x_1^\ell)(k^{-1}e_1)\,\mathrm{d}k\\ &= \int_{\mathrm{SO}(n)} \mathrm{P}(x_1^\ell)(ke_1)\mathrm{P}(x_1^\ell)(k^{T}e_1)\,\mathrm{d}k.\end{aligned}$$

Now note that the first component of $ke_1$ equals the first component of $k^T e_1$ (i.e. $\langle ke_1, e_1\rangle = \langle k^T e_1, e_1\rangle$). By Lemma 9.1.4 i) we infer

$$\mathrm{P}(x_1^\ell)(ke_1) = \mathrm{P}(x_1^\ell)(k^T e_1).$$

Hence, since $\mathrm{P}(x_1^\ell)$ is real,

$$P_\mu^{Y_\ell}(f)(e)(e_1) = \|\mathrm{P}(x_1^\ell)\|^2_{L^2(K)} = \|\mathrm{P}(x_1^\ell)\|^2_{L^2(\mathbb{S}^{n-1})}.$$

On the other hand we have for each $k \in K$

$$f(k) = f(ek) = \pi_{Y_\ell}(\varphi)(ek) = \varphi(e)(k.e_1) = \mathrm{P}(x_1^\ell)(k.e_1)$$

so that $f\big|_K \in L^2(K)^M \cong L^2(K/M)$ corresponds to $\mathrm{P}(x_1^\ell)$ under the isomorphism $L^2(K/M) \cong L^2(\mathbb{S}^{n-1})$ induced by (B.2.1). We obtain

$$\mathrm{pr}_{Y_\ell}(f(e\bullet)) = \mathrm{pr}_{Y_\ell}(\mathrm{P}(x_1^\ell)) = \mathrm{P}(x_1^\ell)$$

since $\mathrm{P}(x_1^\ell) \in Y_\ell$ and the orthogonal projection $\mathrm{pr}_{Y_\ell}$ is the identity on $Y_\ell$. In particular we see that

$$\mathrm{pr}_{Y_\ell}(f(e\bullet))(e_1) = \mathrm{P}(x_1^\ell)(e_1).$$

By Remark 9.2.2 the scalar $c_\ell$ is therefore given by

$$c_\ell = \frac{\mathrm{pr}_{Y_\ell}(f(e\bullet))(e_1)}{P_\mu^{Y_\ell}(f)(e)(e_1)} = \frac{\mathrm{P}(x_1^\ell)(e_1)}{\|\mathrm{P}(x_1^\ell)\|^2_{L^2(\mathbb{S}^{n-1})}}.$$

Now [Hel00, Introduction, Proposition 3.2 (ii)] yields

$$\left\| \frac{\mathrm{P}(x_1^\ell)}{\mathrm{P}(x_1^\ell)(e_1)} \right\|^2_{L^2(\mathbb{S}^{n-1})} = \frac{1}{\dim Y_\ell}$$

and thus

$$c_\ell = \frac{\mathrm{P}(x_1^\ell)(e_1)\dim Y_\ell}{\mathrm{P}(x_1^\ell)(e_1)^2} = \frac{\dim Y_\ell}{\mathrm{P}(x_1^\ell)(e_1)}.$$

As a result Lemma 9.1.4 ii) yields

$$\frac{c_{\ell+1}}{c_\ell} = \frac{\dim Y_{\ell+1}}{\dim Y_\ell}\frac{\mathrm{P}(x_1^\ell)(e_1)}{\mathrm{P}(x_1^{\ell+1})(e_1)} = \frac{\dim Y_{\ell+1}}{\dim Y_\ell}\frac{n+2\ell-2}{n+\ell-2}.$$

Remark A.1.7 implies

$$\frac{\dim Y_{\ell+1}}{\dim Y_\ell} = \frac{\binom{n+\ell-2}{\ell+1}\frac{n+2\ell}{n-2}}{\binom{n+\ell-3}{\ell}\frac{n+2\ell-2}{n-2}} = \frac{n+\ell-2}{\ell+1}\frac{n+2\ell}{n+2\ell-2}$$

and finally

$$\frac{c_{\ell+1}}{c_\ell} = \frac{n+\ell-2}{\ell+1}\frac{n+2\ell}{n+2\ell-2}\frac{n+2\ell-2}{n+\ell-2} = \frac{n+2\ell}{\ell+1}. \qquad \square$$

We have the following expression for the sum of the generalized gradients.

**Lemma 9.2.7.** *Let $\varphi \in C^\infty(G \times_K Y_\ell)$. Then, for each $g \in G$,*

$$\mathbf{d}_-(\varphi)(g) + \mathbf{d}_+(\varphi)(g) = \sum_{j=1}^n x_j(r(X_{e_j})\varphi)(g).$$

*In particular, evaluating both sides at $e_1$ yields*

$$r(H)\pi_{Y_\ell}(\varphi) = \pi_{Y_{\ell-1}}(\mathbf{d}_-\varphi) + \pi_{Y_{\ell+1}}(\mathbf{d}_+\varphi).$$

*Proof.* This follows directly from Lemma 7.5.1 since $\omega(X_{e_j}) = x_j \in C^\infty(\mathbb{S}^{n-1})$. □

The remainder of this section is devoted to the Fourier characterization of $H_\mu^{-\infty}$.

**Lemma 9.2.8.** *Let $\mu \in \mathfrak{a}^*$ and $f = \sum_{\ell\in\mathbb{N}_0} f_\ell \in \mathcal{D}'(G/M)$ with $f_\ell = \pi_{Y_\ell}(\psi_\ell) \in \pi_{Y_\ell}(C^\infty(G \times_K Y_\ell))$ such that the equations from Lemma 9.2.4 hold for $f$. Then $r(X)f = (\mu - \rho)(X)f$ for every $X \in \mathfrak{a}_0$, where $r$ denotes the right regular representation of $\mathfrak{a}_0$ on $\mathcal{D}'(G/M)$.*

*Proof.* By definition we have for every $\varphi \in C_c^\infty(G/M)$

$$(r(X)f)(\varphi) = -f(r(X)\varphi) = -\sum_{\ell\in\mathbb{N}_0} f_\ell(r(X)\varphi) = \sum_{\ell\in\mathbb{N}_0}(r(X)f_\ell)(\varphi).$$

In particular, we infer that $\sum_{\ell\in\mathbb{N}_0} r(X)f_\ell$ converges absolutely to $r(X)f$ in the weak sense by Proposition 7.2.4. Let us prove that $\psi_\ell = \pi_{Y_\ell}^*(f)$. Indeed, we infer by continuity

$$\pi_{Y_\ell}^*(f) = \pi_{Y_\ell}^*(\sum_{k\in\mathbb{N}_0}\pi_{Y_k}(\psi_k)) = \sum_{k\in\mathbb{N}_0}\pi_{Y_\ell}^*(\pi_{Y_k}(\psi_k)),$$

where, for each $g \in G$,

$$\pi_{Y_\ell}^*(\pi_{Y_k}(\psi_k))(g) = \mathrm{pr}_{Y_\ell}(\psi_k(g)) = \delta_{k\ell}\psi_k(g)$$

by Lemma 9.2.3 i). This implies $\psi_\ell = \pi_{Y_\ell}^*(f)$. By the last equation of Lemma 9.2.4 we may apply Lemma 9.2.7 to get

$$r(H)f_\ell = r(H)\pi_{Y_\ell}(\pi_{Y_\ell}^*(f)) = \pi_{Y_{\ell-1}}(\mathbf{d}_-\pi_{Y_\ell}^*(f)) + \pi_{Y_{\ell+1}}(\mathbf{d}_+\pi_{Y_\ell}^*(f)).$$

The first two equations of Lemma 9.2.4 imply that

$$\begin{aligned} r(H)f_\ell &= \frac{c_\ell}{c_{\ell-1}}\lambda_\ell(\mu(H) - (\rho(H) + \ell - 1))\pi_{Y_{\ell-1}}(\pi_{Y_{\ell-1}}^*(f)) \\ &\quad + \frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H) + \ell)\pi_{Y_{\ell+1}}(\pi_{Y_{\ell+1}}^*(f)) \\ &= \frac{c_\ell}{c_{\ell-1}}\lambda_\ell(\mu(H) - (\rho(H) + \ell - 1))f_{\ell-1} + \frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H) + \ell)f_{\ell+1}. \end{aligned}$$

Summarizing we infer that

$$\sum_{\ell=0}^{m} r(H)f_\ell = \sum_{\ell=0}^{m-1} a_\ell f_\ell + \frac{c_{m-1}}{c_m}((\mu+\rho)(H)+m-1)f_m + \frac{c_m}{c_{m+1}}((\mu+\rho)(H)+m)f_{m+1},$$

with

$$a_\ell = \frac{c_{\ell+1}}{c_\ell}\lambda_{\ell+1}(\mu(H)-(\rho(H)+\ell)) + \frac{c_{\ell-1}}{c_\ell}((\mu+\rho)(H)+\ell-1),$$

where we define $c_{-1} := 0$ for $\ell = 0$. We claim that $a_\ell = (\mu-\rho)(H)$. By Lemma 9.2.6 and the definition of $\lambda_\ell$ we see that $a_\ell$ equals

$$\begin{aligned}
&\frac{n+2\ell}{\ell+1}\frac{(\ell+1)(\ell+n-2)}{(n+2\ell-2)(n+2\ell)}(\mu(H)-(\rho(H)+\ell)) + \frac{\ell}{n+2\ell-2}((\mu+\rho)(H)+\ell-1)\\
&= \frac{\ell+n-2}{n+2\ell-2}(\mu(H)-(\rho(H)+\ell)) + \frac{\ell}{n+2\ell-2}((\mu+\rho)(H)+\ell-1)\\
&= \mu(H) - \frac{\ell+n-2}{n+2\ell-2}(\rho(H)+\ell) + \frac{\ell}{n+2\ell-2}(\rho(H)+\ell-1)\\
&= \mu(H)-\rho(H),
\end{aligned}$$

where we used $\rho(H) = \frac{n-1}{2}$ for the last equality. This proves the claim and we infer that

$$\begin{aligned}
\sum_{\ell=0}^{m} r(H)f_\ell(\varphi) =&(\mu-\rho)(H)\sum_{\ell=0}^{m-1} f_\ell(\varphi) + \frac{c_{m-1}}{c_m}((\mu+\rho)(H)+m-1)f_m(\varphi)\\
&+\frac{c_m}{c_{m+1}}((\mu+\rho)(H)+m)f_{m+1}(\varphi).
\end{aligned}$$

We finally claim that the last two summands converge to 0 as $m \to \infty$. By Lemma 9.2.6 and the orthogonality of the $Y_\ell$ we have

$$\frac{c_{m-1}}{c_m}f_m(\varphi) = \frac{c_{m-1}}{c_m}f_m(\varphi_m) = \frac{c_{m-1}}{c_m}f(\varphi_m) = \frac{m}{n+2m-2}f(\varphi_m).$$

As in the proof of Proposition 7.2.4 we see that for each $N > 0$ there exists a constant $C$ independent of $m$ such that

$$|f(\varphi_m)| \le C(1+m^2)^{-N}.$$

Choosing $N$ large enough implies the claim. □

**Lemma 9.2.9.** *Let $\mu \in \mathfrak{a}^*$ and $f = \sum_{\ell\in\mathbb{N}_0} f_\ell \in \mathcal{D}'(G/M)$ with $f_\ell \in \pi_{Y_\ell}(C^\infty(G\times_K Y_\ell))$ such that the equations from Lemma 9.2.4 hold for $f$. Then $U_+ f = 0$ for every $U_+ \in C^\infty(G\times_M \mathfrak{n}_0)$.*

*Proof.* Recall from Appendix B.2.1 the basis $\{Y_{e_1},\ldots,Y_{e_{n-1}}\}$ of $\mathfrak{n}_0$. We write $U_+ \in C^\infty(G\times_M \mathfrak{n}_0)$ as

$$U_+(g) = \kappa_1(g)Y_{e_1} + \ldots + \kappa_{n-1}(g)Y_{e_{n-1}}, \quad g \in G,$$

for some real-valued smooth functions $\kappa_j \in C^\infty(G)$. By definition we have for each $\varphi \in C_c^\infty(G)^M$

$$(U_+ f)(\varphi) = f(U_+^* \varphi) = \sum_{\ell\in\mathbb{N}_0} f_\ell(U_+^*\varphi) = \sum_{\ell\in\mathbb{N}_0} (U_+ f_\ell)(\varphi) = \sum_{\ell\in\mathbb{N}_0} \sum_{j=1}^{n-1} (r(Y_{e_j}) f_\ell)(\kappa_j \varphi). \tag{9.11}$$

Note that for each $j \in \{2, \dots, n\}$

$$\mathfrak{n}_0 \ni Y_{e_{j-1}} = \begin{pmatrix} 0 & e_{j-1}^T & 0 \\ -e_{j-1} & \mathbf{0}_{n-1} & e_{j-1} \\ 0 & e_{j-1}^T & 0 \end{pmatrix} = X_{e_j} + \begin{pmatrix} & e_j^T & \\ -e_j & & \\ & & \end{pmatrix} \in \mathfrak{p}_0 \oplus \mathfrak{k}_0$$

and denote the latter matrix by $A_j \in \mathfrak{k}_0$. We have

$$r(Y_{e_{j-1}}) f_\ell = r(X_{e_j}) f_\ell + r(A_j) f_\ell \tag{9.12}$$

and first investigate the first summand. For each $g \in G$ it evaluates to

$$\begin{aligned} r(X_{e_j}) f_\ell(g) = r(X_{e_j}) \pi_{Y_\ell}(\pi_{Y_\ell}^*(f))(g) &= \frac{d}{dt}\Big|_{t=0} \pi_{Y_\ell}^*(f)(g \exp tX_{e_j})(e_1) \\ &= (\nabla \pi_{Y_\ell}^*(f))(g)(X_{e_j})(e_1). \end{aligned} \tag{9.13}$$

According to Equation (7.11) we write

$$\begin{aligned} (\nabla \pi_{Y_\ell}^*(f))(g) &= \iota_{Y_\ell}^{Y_{\ell-1}}(\mathbf{d}_- \pi_{Y_\ell}^*(f)(g)) + \iota_{Y_\ell}^{Y_{\ell+1}}(\mathbf{d}_+ \pi_{Y_\ell}^*(f)(g)) \\ &= \frac{c_\ell}{c_{\ell-1}} \lambda_\ell(\mu(H) - (\rho(H) + \ell - 1)) \iota_{Y_\ell}^{Y_{\ell-1}}(\pi_{Y_{\ell-1}}^*(f)(g)) \\ &\quad + \frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H) + \ell) \iota_{Y_\ell}^{Y_{\ell+1}}(\pi_{Y_{\ell+1}}^*(f)(g)), \end{aligned}$$

where we used the first two equations of Lemma 9.2.4. Moreover, we obtain

$$\begin{aligned} (\nabla \pi_{Y_\ell}^*(f))(g)(X_{e_j}) &= \frac{c_\ell}{c_{\ell-1}} \lambda_\ell(\mu(H) - (\rho(H) + \ell - 1)) \iota_{Y_\ell}^{Y_{\ell-1}}(\pi_{Y_{\ell-1}}^*(f)(g))(X_{e_j}) \\ &\quad + \frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H) + \ell) \iota_{Y_\ell}^{Y_{\ell+1}}(\pi_{Y_{\ell+1}}^*(f)(g))(X_{e_j}) \end{aligned}$$

using the isomorphism $Y_\ell \otimes \mathfrak{p}^* \cong \mathrm{Hom}(\mathfrak{p}, Y_\ell)$. Since $X_{e_j} \in \mathfrak{p}$ corresponds to $e_j \in \mathbb{C}^n$ under the isomorphism $\mathfrak{p} \cong \mathbb{C}^n$ we have by the definition of $\iota_{Y_\ell}^{Y_{\ell-1}}$ and $\iota_{Y_\ell}^{Y_{\ell+1}}$ that

$$\begin{aligned} (\nabla \pi_{Y_\ell}^*(f))(g)(X_{e_j}) &= \frac{c_\ell}{c_{\ell-1}} \lambda_\ell(\mu(H) - (\rho(H) + \ell - 1)) \frac{n + 2\ell - 4}{n + \ell - 3} \mathrm{P}(x_j \pi_{Y_{\ell-1}}^*(f)(g)) \\ &\quad + \frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H) + \ell) \frac{1}{\ell+1} \frac{\partial \pi_{Y_{\ell+1}}^*(f)(g)}{\partial x_j}. \end{aligned}$$

By Equation (9.3) we infer

$$\mathrm{P}(x_j\pi^*_{Y_{\ell-1}}(f)(g)) = x_j\pi^*_{Y_{\ell-1}}(f)(g) - \frac{|x|^2}{n+2(\ell-1)-2}\frac{\partial\pi^*_{Y_{\ell-1}}(f)(g)}{\partial x_j}$$

and therefore

$$\mathrm{P}(x_j\pi^*_{Y_{\ell-1}}(f)(g))(e_1) = -\frac{1}{n+2\ell-4}\frac{\partial\pi^*_{Y_{\ell-1}}(f)(g)}{\partial x_j}(e_1).$$

As a result we obtain

$$\begin{aligned}(\nabla\pi^*_{Y_\ell}(f))(g)(X_{e_j})(e_1) &= \frac{c_\ell}{c_{\ell-1}}\lambda_\ell(\mu(H)-(\rho(H)+\ell-1))\frac{n+2\ell-4}{n+\ell-3}\mathrm{P}(x_j\pi^*_{Y_{\ell-1}}(f)(g))(e_1)\\ &\quad+\frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H)+\ell)\frac{1}{\ell+1}\frac{\partial\pi^*_{Y_{\ell+1}}(f)(g)}{\partial x_j}(e_1)\\ &= -\frac{c_\ell}{c_{\ell-1}}\lambda_\ell(\mu(H)-(\rho(H)+\ell-1))\frac{1}{n+\ell-3}\frac{\partial\pi^*_{Y_{\ell-1}}(f)(g)}{\partial x_j}(e_1)\\ &\quad+\frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H)+\ell)\frac{1}{\ell+1}\frac{\partial\pi^*_{Y_{\ell+1}}(f)(g)}{\partial x_j}(e_1). \qquad (9.14)\end{aligned}$$

Let us now discuss the second summand of Equation (9.12). We first note that

$$r(A_j)f_\ell(g) = r(A_j)\pi_{Y_\ell}(\pi^*_{Y_\ell}(f))(g) = \left.\frac{d}{dt}\right|_{t=0}\pi_{Y_\ell}(\pi^*_{Y_\ell}(f))(g\exp tA_j).$$

Now $\pi^*_{Y_\ell}(f)\in C^\infty(G\times_K Y_\ell)$ and $\exp tA_j\in K$ imply that

$$\pi_{Y_\ell}(\pi^*_{Y_\ell}(f))(g\exp tA_j) = \pi^*_{Y_\ell}(f)(g\exp tA_j)(e_1) = \tau_\ell(\exp -tA_j)(\pi^*_{Y_\ell}(f)(g))(e_1).$$

We conclude that

$$r(A_j)f_\ell(g) = \left.\frac{d}{dt}\right|_{t=0}\tau_\ell(\exp -tA_j)(\pi^*_{Y_\ell}(f)(g))(e_1).$$

Using Equation (9.8) we obtain that the matrix $E_{j1}-E_{1j}\in\mathfrak{so}(n)$ acts on functions $\psi:\mathbb{R}^n\to\mathbb{R}$ via the derived left regular representation by

$$(E_{j1}-E_{1j}).\psi = -x_1\frac{\partial\psi}{\partial x_j}+x_j\frac{\partial\psi}{\partial x_1}.$$

Since $\tau_\ell$ is the left regular representation and $\exp tA_j.\vartheta = \exp t(E_{1j}-E_{j1})\vartheta,\ \vartheta\in\mathbb{S}^{n-1}$, by the definition of the action of $K$ on the sphere (see Appendix B.2.2) we deduce

$$r(A_j)f_\ell(g) = \left(x_j\frac{\partial\pi^*_{Y_\ell}(f)(g)}{\partial x_1}-x_1\frac{\partial\pi^*_{Y_\ell}(f)(g)}{\partial x_j}\right)(e_1) = -\frac{\partial\pi^*_{Y_\ell}(f)(g)}{\partial x_j}(e_1). \qquad (9.15)$$

Combining the Equations (9.12), (9.13), (9.14) and (9.15) we infer that

$$\begin{aligned} r(Y_{e_{j-1}})f_\ell(g) =& -\frac{c_\ell}{c_{\ell-1}}\lambda_\ell(\mu(H)-(\rho(H)+\ell-1))\frac{1}{n+\ell-3}\frac{\partial \pi^*_{Y_{\ell-1}}(f)(g)}{\partial x_j}(e_1) \\ &+\frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H)+\ell)\frac{1}{\ell+1}\frac{\partial \pi^*_{Y_{\ell+1}}(f)(g)}{\partial x_j}(e_1)-\frac{\partial \pi^*_{Y_\ell}(f)(g)}{\partial x_j}(e_1) \\ =&\frac{c_\ell}{c_{\ell-1}}\lambda_\ell(\mu(H)-(\rho(H)+\ell-1))\frac{1}{n+\ell-3}r(A_j)f_{\ell-1}(g) \\ &-\frac{c_\ell}{c_{\ell+1}}((\mu+\rho)(H)+\ell)\frac{1}{\ell+1}r(A_j)f_{\ell+1}(g)+r(A_j)f_\ell(g). \end{aligned}$$

Summarizing we conclude that $\sum_{\ell=0}^m (r(Y_{e_{j-1}})f_\ell)(\varphi)$ equals

$$\begin{aligned} &\sum_{\ell=0}^{m-1} a_\ell(r(A_j)f_\ell)(\varphi)+\left(1-\frac{c_{m-1}}{c_m}\frac{(\mu+\rho)(H)+m-1}{m}\right)(r(A_j)f_m)(\varphi) \\ &-\frac{c_m}{c_{m+1}}\frac{(\mu+\rho)(H)+m}{m+1}(r(A_j)f_{m+1})(\varphi), \end{aligned} \tag{9.16}$$

where the coefficient $a_\ell\in\mathbb{C}$ is given by

$$a_\ell=\frac{c_{\ell+1}}{c_\ell}\lambda_{\ell+1}\frac{\mu(H)-(\rho(H)+\ell)}{n+\ell-2}-\frac{c_{\ell-1}}{c_\ell}\frac{(\mu+\rho)(H)+\ell-1}{\ell}+1$$

for $\ell\neq 0$ and

$$a_0=\frac{c_1}{c_0}\lambda_1\frac{\mu(H)-\rho(H)}{n-2}+1.$$

Note that $\pi_{0*}(f)(g)\in Y_0$ is the restriction of a homogeneous polynomial of degree zero, i.e. of a constant function. In particular, the partial derivative of $\pi_{0*}(f)(g)$ with respect to $x_j$ is zero and thus

$$(r(A_j)f_0)(\psi)=0.$$

We will now prove that $a_\ell=0$ for each $\ell>0$, i.e.

$$a_\ell=\frac{c_{\ell+1}}{c_\ell}\lambda_{\ell+1}\frac{\mu(H)-(\rho(H)+\ell)}{n+\ell-2}-\frac{c_{\ell-1}}{c_\ell}\frac{(\mu+\rho)(H)+\ell-1}{\ell}+1=0.$$

Applying Lemma 9.2.6 and using the definition of $\lambda_\ell$ it follows that the first summand is given by

$$\begin{aligned} &\frac{n+2\ell}{\ell+1}\lambda_{\ell+1}\frac{\mu(H)-(\rho(H)+\ell)}{n+\ell-2} \\ =&\frac{n+2\ell}{\ell+1}\frac{(\ell+1)(\ell+n-2)}{(2(\ell+1)+n-4)(2(\ell+1)+n-2)}\frac{\mu(H)-(\rho(H)+\ell)}{n+\ell-2} \\ =&\frac{n+2\ell}{\ell+1}\frac{(\ell+1)(\ell+n-2)}{(n+2\ell-2)(n+2\ell)}\frac{\mu(H)-(\rho(H)+\ell)}{n+\ell-2} \\ =&\frac{\mu(H)-(\rho(H)+\ell)}{n+2\ell-2}=-\frac{\rho(H)+\ell-\mu(H)}{n+2\ell-2}. \end{aligned}$$

The second summand simplifies to

$$-\frac{\ell}{n+2(\ell-1)}\frac{(\mu+\rho)(H)+\ell-1}{\ell}=-\frac{(\mu+\rho)(H)+\ell-1}{n+2\ell-2}.$$

Finally, we deduce that

$$\begin{aligned}
a_\ell &= -\frac{\rho(H)+\ell-\mu(H)}{n+2\ell-2}-\frac{(\mu+\rho)(H)+\ell-1}{n+2\ell-2}+1\\
&= -\frac{\rho(H)+\ell-\mu(H)+(\mu+\rho)(H)+\ell-1}{n+2\ell-2}+1\\
&= -\frac{2\rho(H)+2\ell-1}{n+2\ell-2}+1=-\frac{n-1+2\ell-1}{n+2\ell-2}+1=0.
\end{aligned}$$

By Equation (9.16) and Lemma 9.2.6 we thus infer that

$$\begin{aligned}
\sum_{\ell=0}^{m}(r(Y_{e_{j-1}})f_\ell)(\varphi) &= \left(1-\frac{c_{m-1}}{c_m}\frac{(\mu+\rho)(H)+m-1}{m}\right)(r(A_j)f_m)(\varphi)\\
&\quad -\frac{c_m}{c_{m+1}}\frac{(\mu+\rho)(H)+m}{m+1}(r(A_j)f_{m+1})(\varphi)\\
&= \left(1-\frac{m}{n+2m-2}\frac{(\mu+\rho)(H)+m-1}{m}\right)(r(A_j)f_m)(\varphi)\\
&\quad -\frac{m+1}{n+2m}\frac{(\mu+\rho)(H)+m}{m+1}(r(A_j)f_{m+1})(\varphi)\\
&= -\left(1-\frac{m}{n+2m-2}\frac{(\mu+\rho)(H)+m-1}{m}\right)f_m(r(A_j)\varphi)\\
&\quad +\frac{m+1}{n+2m}\frac{(\mu+\rho)(H)+m}{m+1}f_{m+1}(r(A_j)\varphi).
\end{aligned}$$

Equation (9.11) therefore yields

$$\begin{aligned}
(U_+f)(\varphi) &= \lim_{m\to\infty}\sum_{\ell=0}^{m}\sum_{j=2}^{n}(r(Y_{e_{j-1}})f_\ell)(\kappa_{j-1}\varphi)\\
&= -\lim_{m\to\infty}\left(1-\frac{m}{n+2m-2}\frac{(\mu+\rho)(H)+m-1}{m}\right)f_m\left(\sum_{j=2}^{n}r(A_j)(\kappa_{j-1}\varphi)\right)\\
&\quad +\lim_{m\to\infty}\frac{m+1}{n+2m}\frac{(\mu+\rho)(H)+m}{m+1}f_{m+1}\left(\sum_{j=2}^{n}r(A_j)(\kappa_{j-1}\varphi)\right).
\end{aligned}$$

Since $f$ defines a distribution, the series

$$\sum_{\ell\in\mathbb{N}_0}f_\ell\left(\sum_{j=2}^{n}r(A_j)(\kappa_{j-1}\varphi)\right)=f\left(\sum_{j=2}^{n}r(A_j)(\kappa_{j-1}\varphi)\right)$$

converges and especially $f_\ell(\sum_{j=2}^{n}r(A_j)(\kappa_{j-1}\varphi))\to 0$ as $\ell\to\infty$. This implies that

$$(U_+f)(\varphi)=-\left(1-\frac{1}{2}\right)\cdot 0+\frac{1}{2}\cdot 1\cdot 0=0. \qquad \square$$

**Proposition 9.2.10.** *Let $\mu \in \mathfrak{a}^*$ and $f = \sum_{\ell\in\mathbb{N}_0} f_\ell \in \mathcal{D}'(G/M)$ with $f_\ell \in \pi_{Y_\ell}(C^\infty(G \times_K Y_\ell))$. Then $f \in H_\mu^{-\infty}$ if and only if the equations from Lemma 9.2.4 hold for $f$.*

*Proof.* The if-part follows from Lemma 9.2.8 and Lemma 9.2.9. The only-if-part is given by Lemma 9.2.4. □

## 9.3. Explicit formulas for generalized gradients

The following two lemmas provide explicit formulas for the generalized gradients (recall that the embeddings $\pi_{Y_\ell}$ are injective).

**Lemma 9.3.1.** *Let $u_{\ell-1} \in C^\infty(G \times_K Y_{\ell-1})$. Then*

$$(n + 2\ell - 4)\pi_{Y_\ell}(\mathbf{d}_+ u_{\ell-1})(g) = (n + \ell - 3)(r(H)u_{\ell-1}(g))(e_1) - \sum_{j=2}^{n} \frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j}(e_1)$$

*for each $g \in G$.*

*Proof.* By definition $\pi_{Y_\ell}(\mathbf{d}_+ u_{\ell-1})(g) = (\mathbf{d}_+ u_{\ell-1})(g)(e_1)$ equals

$$\begin{aligned} T_{Y_\ell}^{Y_{\ell-1}}((\nabla u_{\ell-1})(g))(e_1) &= T_{Y_\ell}^{Y_{\ell-1}}\left(\sum_{j=1}^{n}(\nabla u_{\ell-1})(g)(X_{e_j}) \otimes e_j^*\right)(e_1) \\ &= \sum_{j=1}^{n} T_{Y_\ell}^{Y_{\ell-1}}(r(X_{e_j})u_{\ell-1}(g) \otimes e_j^*)(e_1) \\ &= \sum_{j=1}^{n} \mathrm{P}(x_j(r(X_{e_j})u_{\ell-1}(g)))(e_1). \end{aligned}$$

Equation (9.3) yields

$$\mathrm{P}(x_j(r(X_{e_j})u_{\ell-1}(g))) = x_j r(X_{e_j})u_{\ell-1}(g) - \frac{|x|^2}{n + 2\ell - 4}\frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j}.$$

For $j \in \{2, \dots, n\}$ we obtain

$$\mathrm{P}(x_j(r(X_{e_j})u_{\ell-1}(g)))(e_1) = -\frac{1}{n + 2\ell - 4}\frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j}(e_1)$$

and for $j = 1$ by Lemma 9.1.3

$$\mathrm{P}(x_1(r(X_{e_1})u_{\ell-1}(g)))(e_1) = \frac{n + \ell - 3}{n + 2\ell - 4} r(X_{e_1})u_{\ell-1}(g)(e_1).$$

The lemma now follows from $X_{e_1} = H$. □

**Lemma 9.3.2.** *Let $u_\ell \in C^\infty(G \times_K Y_\ell)$. Then*

$$(n+2\ell-2)\pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell)(g) = \ell(r(H)u_\ell(g))(e_1) + \sum_{j=2}^{n} \frac{\partial(r(X_{e_j})u_\ell(g))}{\partial x_j}(e_1)$$

*for each $g \in G$.*

*Proof.* By definition $\pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell)(g) = (\mathbf{d}_- u_\ell)(g)(e_1)$ equals

$$\begin{aligned} T^{Y_\ell}_{Y_{\ell-1}}((\nabla u_\ell)(g))(e_1) &= T^{Y_\ell}_{Y_{\ell-1}}\left(\sum_{j=1}^{n}(\nabla u_\ell)(g)(X_{e_j}) \otimes e_j^*\right)(e_1) \\ &= \sum_{j=1}^{n} T^{Y_\ell}_{Y_{\ell-1}}(r(X_{e_j})u_\ell(g) \otimes e_j^*)(e_1) \\ &= \sum_{j=1}^{n} \frac{1}{n+2\ell-2}\frac{\partial(r(X_{e_j})u_\ell(g))}{\partial x_j}(e_1), \end{aligned}$$

where we obtain

$$\frac{\partial(r(X_{e_1})u_\ell(g))}{\partial x_1}(e_1) = \frac{\partial(r(H)u_\ell(g))}{\partial x_1}(e_1) = \ell(r(H)u_\ell(g))(e_1)$$

using Lemma 9.1.3. □

We can use these explicit forms to give a direct proof of Lemma 7.5.2 for $\mathrm{SO}_0(n,1)$.

**Lemma 9.3.3.** *Let $\ell \in \mathbb{N}$ and $u_\ell \in C^\infty(G \times_K Y_\ell)$. Then*

$$\langle \pi_{Y_\ell}(u_\ell), \pi_{Y_\ell}(\mathbf{d}_+ u_{\ell-1})\rangle_{L^2(G)} = -\langle \pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell), \pi_{Y_{\ell-1}}(u_{\ell-1})\rangle_{L^2(G)}$$

*for each $u_{\ell-1} \in C^\infty(G \times_K Y_{\ell-1})$ if one side exists.*

*Proof.* By Lemma 9.3.1 we obtain

$$\begin{aligned} (n+2\ell-4)\langle \pi_{Y_\ell}(u_\ell), \pi_{Y_\ell}(\mathbf{d}_+ u_{\ell-1})\rangle_{L^2(G)} &= \int_G u_\ell(g)(e_1)(n+\ell-3)\overline{(r(H)u_{\ell-1}(g))(e_1)}\,\mathrm{d}g \\ &\quad - \int_G u_\ell(g)(e_1)\sum_{j=2}^{n}\overline{\frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j}(e_1)}\,\mathrm{d}g. \end{aligned}$$

For the first summand we have

$$\int_G u_\ell(g)(e_1)\overline{(r(H)u_{\ell-1}(g))(e_1)}\,\mathrm{d}g = -\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g.$$

For the second summand recall from Equation (9.15) that the element $A_j := \begin{pmatrix} & e_j^T & \\ -e_j & & \\ & & \end{pmatrix}$ of $\mathfrak{k}_0$ acts via the left regular representation on $r(X_{e_j})u_{\ell-1}(g)$ by

$$A_j.r(X_{e_j})u_{\ell-1}(g) = x_1\frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j} - x_j\frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_1}$$

and thus $(A_j.r(X_{e_j})u_{\ell-1}(g))(e_1) = \frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j}(e_1)$. Moreover, $u_{\ell-1} \in C^\infty(G \times_K Y_{\ell-1})$ implies

$$\begin{aligned}(A_j.r(X_{e_j})u_{\ell-1}(g))(e_1) &= \frac{d}{dt}\Big|_{t=0} r(X_{e_j})u_{\ell-1}(g)(\exp -tA_j.e_1)\\ &= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} u_{\ell-1}(g\exp sX_{e_j})(\exp -tA_j.e_1)\\ &= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} u_{\ell-1}(g\exp sX_{e_j}\exp -tA_j)(e_1)\\ &= -(r(X_{e_j}A_j)u_{\ell-1}(g))(e_1),\end{aligned}$$

where we extend $r$ to the universal enveloping algebra of $\mathfrak{g}$. We infer that

$$\begin{aligned}&\int_G u_\ell(g)(e_1)\sum_{j=2}^n \overline{\frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j}(e_1)}\,\mathrm{d}g\\ &= -\sum_{j=2}^n \int_G u_\ell(g)(e_1)\overline{(r(X_{e_j}A_j)u_{\ell-1}(g))(e_1)}\,\mathrm{d}g\\ &= -\sum_{j=2}^n \int_G (r(A_jX_{e_j})u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g\\ &= -\sum_{j=2}^n \int_G (r(X_{e_j}A_j - [X_{e_j},A_j])u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g.\end{aligned}$$

Note that the commutator $[A_j, X_{e_j}]$ is given by $H$. It follows that

$$\begin{aligned}&\int_G u_\ell(g)(e_1)\sum_{j=2}^n \overline{\frac{\partial(r(X_{e_j})u_{\ell-1}(g))}{\partial x_j}(e_1)}\,\mathrm{d}g\\ &= -\sum_{j=2}^n \int_G (r(X_{e_j}A_j)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g - (n-1)\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g\\ &= \sum_{j=2}^n \int_G \frac{\partial(r(X_{e_j})u_\ell(g))}{\partial x_j}(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g - (n-1)\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g.\end{aligned}$$

Therefore $(n+2\ell-4)\langle \pi_{Y_\ell}(u_\ell), \pi_{Y_\ell}(\mathbf{d}_+ u_{\ell-1})\rangle_{L^2(G)}$ equals

$$\begin{aligned}&-(n+\ell-3)\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g - \sum_{j=2}^n \int_G \frac{\partial(r(X_{e_j})u_\ell(g))}{\partial x_j}(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g\\ &\qquad + (n-1)\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g\\ &= -(\ell-2)\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g - \sum_{j=2}^n \int_G \frac{\partial(r(X_{e_j})u_\ell(g))}{\partial x_j}(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g.\end{aligned}$$

Using Lemma 9.3.2 we may rewrite this expression as

$$-(n+2\ell-2)\langle \pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell), \pi_{Y_{\ell-1}}(u_{\ell-1})\rangle_{L^2(G)} + 2\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g. \tag{9.17}$$

According to Lemma 9.2.7 we obtain

$$\begin{aligned}\int_G (r(H)u_\ell(g))(e_1)\overline{u_{\ell-1}(g)(e_1)}\,\mathrm{d}g &= \langle \pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell) + \pi_{Y_{\ell+1}}(\mathbf{d}_+ u_\ell), \pi_{Y_{\ell-1}}(u_{\ell-1})\rangle_{L^2(G)}\\ &= \langle \pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell), \pi_{Y_{\ell-1}}(u_{\ell-1})\rangle_{L^2(G)}\end{aligned}$$

since – using the orthogonality of the $Y_\ell$ –

$$\begin{aligned}\langle \pi_{Y_{\ell+1}}(\mathbf{d}_+ u_\ell), \pi_{Y_{\ell-1}}(u_{\ell-1})\rangle_{L^2(G)} &= \int_G \pi_{Y_{\ell+1}}(\mathbf{d}_+ u_\ell)(g)\overline{\pi_{Y_{\ell-1}}(u_{\ell-1})(g)}\,\mathrm{d}g\\ &= \int_{G/K}\int_K (\mathbf{d}_+ u_\ell)(nak)(e_1)\overline{(u_{\ell-1})(nak)(e_1)}\,\mathrm{d}k\,\mathrm{d}naK\\ &= \int_{G/K}\int_K (\mathbf{d}_+ u_\ell)(na)(k.e_1)\overline{(u_{\ell-1})(na)(k.e_1)}\,\mathrm{d}k\,\mathrm{d}naK\\ &= 0.\end{aligned}$$

Finally, Equation (9.17) yields

$$(n+2\ell-4)\langle \pi_{Y_\ell}(u_\ell), \pi_{Y_\ell}(\mathbf{d}_+ u_{\ell-1})\rangle_{L^2(G)} = -(n+2\ell-4)\langle \pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell), \pi_{Y_{\ell-1}}(u_{\ell-1})\rangle_{L^2(G)}$$

with $n+2\ell-4 \geq 3+2-4 = 1 > 0$. □

## 9.4. Spectral correspondence

In this section we state and prove a correspondence using only our Fourier characterization. We first investigate the general case and then consider the first exceptional parameter in some more detail. In the latter case, the minimal $K$-type Poisson transform is related to one-forms and some more precise results can be shown.

### 9.4.1. The case of arbitrary exceptional parameters

We start with a direct proof of Theorem 6.1.1 in the case of $\mathrm{SO}_0(n,1)$.

**Proposition 9.4.1** (see Theorem 6.1.1). *Let $\mu := -\rho - k\alpha \in \mathfrak{a}^*$, for some $k \in \mathbb{N}_0$, be an exceptional spectral parameter (see Lemma 5.2.1). Recall the unique irreducible subrepresentation $W_k$ of $H_\mu$ from Lemma B.2.1. Then*

$${}^\Gamma H_\mu^* = {}^\Gamma(W_k)^*,$$

*where ${}^\Gamma()^*$ denotes the $\Gamma$-invariant smooth, $L^2$ or distribution vectors.*

*Proof.* Since $(W_k)^* \subset H_\mu^*$, one inclusion is trivial. To prove ${}^\Gamma H_\mu^* \subseteq {}^\Gamma(W_k)^*$ we need to show that

$$\mathrm{pr}_{Y_\ell}(f) = 0$$

for every $f \in {}^\Gamma H_\mu^*$ and every $\ell \leq k$. By Remark 9.2.2 we have

$$\mathrm{pr}_{Y_\ell}(f) = \mathrm{pr}_{Y_\ell}(\pi_\mu(e)f) = c_\ell \cdot P_\mu^{Y_\ell}(f)(e)$$

for some $c_\ell \in \mathbb{C} \setminus \{0\}$. Therefore, it suffices to prove $P_\mu^{Y_\ell}\Big|_{{}^\Gamma H_\mu^*} = 0$ for $\ell \leq k$. We start with the case $\ell = 0$. Recall from Section 3.2 that the scalar Poisson transform $P_\mu^{Y_0}$ maps $H_\mu^*$ into

$$\left\{ f \in C^\infty(\mathbf{H}^n) \colon \Delta f = \left(\rho(H)^2 - \mu(H)^2\right) f \right\},$$

where $\mathbf{H}^n = G \cdot [e_{n+1}] \cong G/K$ denotes the real hyperbolic space of dimension $n$ and $\Delta$ denotes the positive Laplacian. Taking $\Gamma$-invariants on both sides implies that the $G$-equivariant Poisson transform $P_\mu^{Y_0}$ maps ${}^\Gamma H_\mu^*$ into

$$\left\{ f \in C^\infty(\mathbf{M}) \colon \Delta_{\mathbf{M}} f = \left(\rho(H)^2 - \mu(H)^2\right) f \right\} \tag{9.18}$$

by identifying ${}^\Gamma C^\infty(\mathbf{H}^n) \cong C^\infty(\Gamma \backslash \mathbf{H}^n)$. By the definition of $\Gamma$ we have that $\mathbf{M} = \Gamma \backslash \mathbf{H}^n$ is a smooth compact Riemannian manifold. For $k \neq 0$, the positivity of the Laplacian and

$$\rho(H)^2 - \mu(H)^2 = -2\rho(H)k - k^2 < 0 \tag{9.19}$$

therefore imply that the space in (9.18) is the zero space and $P_\mu^{Y_0}\Big|_{{}^\Gamma H_\mu^*} = 0$. Let us prove the same equality for $k = 0$, i.e. $\mu = -\rho$: In this case, by Equation (9.19), $P_{-\rho}^{Y_0}(f)$ descends to a harmonic function on the compact Riemannian manifold $\Gamma \backslash \mathbf{H}^n$ and thus has to be constant. Lemma 9.1.5 yields

$$(\mathbf{d}_- \circ P_{-\rho}^{Y_1})(f) = \lambda_1 \cdot (-2\rho(H)) P_{-\rho}^{Y_0}(f) = \frac{n-1}{n} P_{-\rho}^{Y_0}(f).$$

Using Lemma 9.3.3 (which is applicable by the last equation of Lemma 9.1.5) we obtain

$$\begin{aligned} 0 \leq \|\pi_{Y_0}(P_{-\rho}^{Y_0}(f))\|_{L^2}^2 &= \frac{n}{n-1} \langle \pi_{Y_0}(\mathbf{d}_- P_{-\rho}^{Y_1}(f)), \pi_{Y_0}(P_{-\rho}^{Y_0}(f)) \rangle_{L^2} \\ &= -\frac{n}{n-1} \langle \pi_{Y_1}(P_{-\rho}^{Y_1}(f)), \pi_{Y_1}(\mathbf{d}_+ P_{-\rho}^{Y_0}(f)) \rangle_{L^2} = 0, \end{aligned}$$

since the derivative of the constant function $P_{-\rho}^{Y_0}(f)$ is zero. Thus, $\pi_{Y_0}(P_{-\rho}^{Y_0}(f)) = 0$ and $P_{-\rho}^{Y_0}(f) = 0$ follows from the injectivity of $\pi_{Y_0}$. The first equation of Lemma 9.1.5 reads

$$\forall \ell \in \mathbb{N}_0,\ f \in H_\mu^* \colon \quad (\mathbf{d}_+ \circ P_\mu^{Y_\ell})(f) = (\ell - k) P_\mu^{Y_{\ell+1}}(f)$$

and recursion yields $P_\mu^{Y_\ell}\Big|_{{}^\Gamma H_\mu^*} = 0$ for $\ell \leq k$. □

In analogy to Proposition 2.2.3 we have the following

**Proposition 9.4.2.** *Let $\mu := -\rho - k\alpha \in \mathfrak{a}^*$, for some $k \in \mathbb{N}_0$, be an exceptional spectral parameter. Then there are isomorphisms of finite dimensional vector spaces*

$$\mathrm{Res}^0_X((\mu-\rho)(H)) \cong {}^\Gamma(W_k)^{-\infty} \text{ and } \mathrm{Res}^0_{X^*}((\mu-\rho)(H)) \cong {}^\Gamma(W_k)^{-\infty}$$

*where ${}^\Gamma()^{-\infty}$ denotes the space of $\Gamma$-invariant distribution vectors.*

To prepare for the quantum-classical correspondence, we need one more step.

**Lemma 9.4.3.** *Let $k \in \mathbb{N}_0$, $u_{k+1} \in C_c^\infty(G \times_K Y_{k+1})$,*

$$u_\ell := \frac{c_\ell}{c_{\ell-1}} \frac{1}{\ell-(k+1)} \mathbf{d}_+ u_{\ell-1} \tag{9.20}$$

*for each $\ell > k+1$ and $u_\ell := 0$ for each $\ell < k+1$. Define the formal sum*

$$f := \sum_{\ell\in\mathbb{N}_0} \pi_{Y_\ell}(u_\ell).$$

*Then:*

*i) If $\mathbf{D}u_\ell = 0$ for each $\ell \in \mathbb{N}_0$, then $f$ defines a distribution on $G$.*

*ii) If $\mathbf{D}u_\ell = 0$ and*

$$\mathbf{d}_- u_\ell = \frac{c_\ell}{c_{\ell-1}} \lambda_\ell(-2\rho(H) - k - \ell + 1) u_{\ell-1} \tag{9.21}$$

*for each $\ell \in \mathbb{N}_0$, then $f$ defines a distribution in $H_\mu^{-\infty}$ with $\mu := -\rho - k\alpha \in \mathfrak{a}^*$.*

*Proof.* We start with the first part. By Lemma 7.2.5 it suffices to show that $\|\pi_{Y_\ell}(u_\ell)\|^2$ is $\mathcal{O}((1+\ell)^N)$ for some $N \in \mathbb{N}$ as $\ell \to \infty$. Using Lemma 9.3.3 and Lemma 9.2.6 we obtain

$$\begin{aligned}
\|\pi_{Y_\ell}(u_\ell)\|^2 &\overset{(9.20)}{=} \frac{c_\ell}{c_{\ell-1}} \frac{1}{\ell-(k+1)} \langle \pi_{Y_\ell}(u_\ell), \pi_{Y_\ell}(\mathbf{d}_+ u_{\ell-1})\rangle \\
&= -\frac{c_\ell}{c_{\ell-1}} \frac{1}{\ell-(k+1)} \langle \pi_{Y_{\ell-1}}(\mathbf{d}_- u_\ell), \pi_{Y_{\ell-1}}(u_{\ell-1})\rangle \\
&\overset{(9.21)}{=} -\left(\frac{c_\ell}{c_{\ell-1}}\right)^2 \frac{\lambda_\ell(-2\rho(H)-k-\ell+1)}{\ell-(k+1)} \|\pi_{Y_{\ell-1}}(u_{\ell-1})\|^2 \\
&= -\left(\frac{n+2\ell-2}{\ell}\right)^2 \frac{\ell(\ell+n-3)(-(n-1)-k-\ell+1)}{(n+2\ell-4)(n+2\ell-2)(\ell-(k+1))} \|\pi_{Y_{\ell-1}}(u_{\ell-1})\|^2 \\
&= \frac{(n+2\ell-2)(\ell+n-3)(n+k+\ell-2)}{\ell(n+2\ell-4)(\ell-(k+1))} \|\pi_{Y_{\ell-1}}(u_{\ell-1})\|^2.
\end{aligned}$$

For every $m \in \mathbb{N}$, iterating this equation yields that $\|\pi_{Y_{k+m}}(u_{k+m})\|^2$ equals

$$\begin{aligned}
&\prod_{r=2}^{m} \frac{(n+2(k+r)-2)(k+r+n-3)(n-2+k+k+r)}{(k+r)(n+2(k+r)-4)(k+r-(k+1))} \|\pi_{Y_{k+1}}(u_{k+1})\|^2 \\
&= \prod_{r=2}^{m} \frac{(n+2(k+r)-2)(k+r+n-3)(n-2+2k+r)}{(k+r)(n+2(k+r)-4)(r-1)} \|\pi_{Y_{k+1}}(u_{k+1})\|^2 \\
&= \frac{2k+2m+n-2}{2k+n} \prod_{r=2}^{m} \frac{(k+r+n-3)(n-2+2k+r)}{(k+r)(r-1)} \|\pi_{Y_{k+1}}(u_{k+1})\|^2 \\
&= \frac{2k+2m+n-2}{2k+n} \frac{(k+m+n-3)!(k+1)!(n+2k+m-2)!}{(k+n-2)!(k+m)!(m-1)!(n+2k-1)!} \|\pi_{Y_{k+1}}(u_{k+1})\|^2.
\end{aligned}$$

Summarizing all constants that are independent of $m$ into a constant

$$C := \frac{(k+1)!}{(2k+n)(k+n-2)!(n+2k-1)!}$$

yields that $\|\pi_{Y_{k+m}}(u_{k+m})\|^2$ equals

$$(2k+2m+n-2)\frac{(k+m+n-3)!(n+2k+m-2)!}{(k+m)!(m-1)!} C\|\pi_{Y_{k+1}}(u_{k+1})\|^2.$$

Note that

$$\frac{(k+m+n-3)!}{(k+m)!} = (k+m+n-3)\cdots(k+m+1)$$

is $\mathcal{O}(m^{n-3})$ (there are $n-3$ factors) and

$$\frac{(n+2k+m-2)!}{(m-1)!} = (n+2k+m-2)\cdots m$$

is $\mathcal{O}(m^{n+2k-1})$. Therefore $\|\pi_{Y_{k+m}}(u_{k+m})\|^2$ is $\mathcal{O}(m^{2n+2k-3})$. This proves i).

Let us now prove the second part. By definition of $u_\ell$ we have

$$\mathbf{d}_+ u_{\ell-1} = \frac{c_{\ell-1}}{c_\ell}(-k+(\ell-1))u_\ell \tag{9.22}$$

for every $\ell > k+1$. Note that this equation also holds for every $\ell \leq k+1$. Together with the assumptions of the second part each equation of Lemma 9.2.4 is fulfilled for $\mu(H) = -\rho(H) - k$. The lemma now follows from the first part and Proposition 9.2.10. □

We can gain some more information about the action of the generalized gradients by connecting them to the action of Casimir elements. This makes use of the fact that we know the infinitesimal characters of principal series representations.

**Remark 9.4.4.** For $\varphi \in C^\infty(G \times_K Y_\ell)$ with $\mathbf{D}\varphi = 0$ Lemma 7.5.3 implies

$$\frac{1}{\lambda(Y_{\ell+1}, Y_\ell)}\mathbf{d}_-\mathbf{d}_+\varphi + \frac{1}{\lambda(Y_{\ell-1}, Y_\ell)}\mathbf{d}_+\mathbf{d}_-\varphi = r(\Omega_\mathfrak{p})\varphi.$$

If $\varphi = P_\mu^{Y_\ell}(f)$ for some $f \in H_\mu$ we thus obtain – by Remark A.1.3 and Equation (2.6) –

$$\frac{n+2\ell}{\ell+1}\mathbf{d}_-\mathbf{d}_+\varphi + \frac{n+2\ell-4}{n+\ell-3}\mathbf{d}_+\mathbf{d}_-\varphi = (\mu(H)^2 - \rho(H)^2 - \ell(\ell+n-2))\varphi,$$

where we used the fact that the Casimir operator of $\mathfrak{k}$ acts by $\langle \ell e_1 + 2\rho_c, \ell e_1\rangle$ (see e.g. [Kna86, Lemma 12.28] for $\nu = \ell e_1 + \rho_c$ with the highest weight $\ell e_1$ of $Y_\ell$).

We can now prove a quantum-classical correspondence which, in contrast to the correspondence from Theorem 8.1.1, does not use the multiplicity-one result of [DGK88].

**Theorem 9.4.5** (Quantum-classical correspondence for exceptional spectral parameters)**.** *Let $\mu := -\rho - k\alpha \in \mathfrak{a}^*$, for some $k \in \mathbb{N}_0$, be an exceptional spectral parameter (see Lemma 5.2.1). Recall the unique irreducible subrepresentation $W_k$ of $H_\mu$ from Lemma B.2.1. By Proposition 5.2.3, the minimal $K$-type of $W_k$ is given by $Y_{k+1}$. Then the map*

$$P_\mu^{Y_{k+1}}\Big|_{{}^\Gamma H_\mu^{-\infty}} : {}^\Gamma H_\mu^{-\infty} \to \mathcal{H},$$

*where $\mathcal{H}$ denotes the space*

$$\{u \in {}^\Gamma C^\infty(G \times_K Y_{k+1}) \mid \mathbf{d}_- u = 0,\ r(\Omega_\mathfrak{g})u = (\mu(H)^2 - \rho(H)^2)u,\ \mathbf{D}\mathbf{d}_+^m u = 0\ \forall m \in \mathbb{N}_0\},$$

*is an isomorphism (with $\mu(H)^2 - \rho(H)^2 = k(k+n-1)$). Moreover, we have*

$$\mathcal{H} \cong {}^\Gamma(W_k)^{-\infty},$$

*where ${}^\Gamma(W_k)^{-\infty}$ denotes the $\Gamma$-invariant distribution vectors in $W_k$.*

*Proof.* The proof is separated into the following steps:

i) $\operatorname{im}\left(P_\mu^{Y_{k+1}}\Big|_{{}^\Gamma H_\mu^{-\infty}}\right) \subseteq \mathcal{H}$,

ii) $P_\mu^{Y_{k+1}}\Big|_{{}^\Gamma H_\mu^{-\infty}}$ is injective, and

iii) $\operatorname{im}\left(P_\mu^{Y_{k+1}}\Big|_{{}^\Gamma H_\mu^{-\infty}}\right) = \mathcal{H}$.

i) The second equation of Lemma 9.2.4 shows

$$(\mathbf{d}_- \circ P_\mu^{Y_{k+1}})(f) = -2\lambda_{k+1} \cdot (\rho(H) + k)P_\mu^{Y_k}(f)$$

for each $f \in H_\mu^{-\infty}$ and therefore $\mathbf{d}_-(\mathrm{im}(P_\mu^{Y_{k+1}}\big|_{{}^\Gamma H_\mu^{-\infty}})) = 0$ by Proposition 9.4.1. Using Lemma 9.2.4 repeatedly proves that for each $f \in {}^\Gamma H_\mu^{-\infty}$, $\mathbf{d}_+^m P_\mu^{Y_{k+1}}(f)$ is a multiple of $P_\mu^{Y_{k+1+m}}(f)$ and

$$\mathbf{D}\mathbf{d}_+^m P_\mu^{Y_{k+1}}(f) = 0.$$

Finally, the scalar action of $\Omega_\mathfrak{g}$ follows from the infinitesimal character of $H_\mu$ (see Equation (2.6)) and the $G$-equivariance of $P_\mu^{Y_{k+1}}$. This proves i).

ii) The injectivity follows from Proposition 5.1.3 since $\mathbf{Irr}(\mu) = \{W_k\}$ by Lemma B.2.1 and $\mathrm{mult}_K(Y_{k+1}, W_k) = 1 \neq 0$.

iii) For the surjectivity let $u \in \mathcal{H}$. Recall the constants $c_\ell \in \mathbb{C} \setminus \{0\}$ from Remark 9.2.2. Define $u_\ell := 0$ for $\ell < k+1$, $u_{k+1} := c_{k+1} \cdot u$ and recursively for $\ell > k+1$

$$u_\ell := \frac{c_\ell}{c_{\ell-1}} \frac{1}{\ell - (k+1)} \mathbf{d}_+ u_{\ell-1}. \tag{9.23}$$

We define the formal sum

$$f := \sum_{\ell \in \mathbb{N}_0} \pi_{Y_\ell}(u_\ell). \tag{9.24}$$

Note first that – by the third property – the first equation of Remark 9.4.4 is fulfilled for each $u_\ell$. Moreover, by the second property and since $u_{k+1} \in C^\infty(G \times_K Y_{k+1})$, we infer the second equation of Remark 9.4.4, which simplifies to

$$\frac{n+2k+2}{k+2} \mathbf{d}_- \mathbf{d}_+ u_{k+1} = (\mu(H)^2 - \rho(H)^2 - (k+1)(k+n-1)) u_{k+1},$$

since $\mathbf{d}_- u_{k+1} = 0$. Substituting $\mathbf{d}_+ u_{k+1}$ for the corresponding scalar multiple of $u_{k+2}$ (by Equation (9.23)) we obtain

$$\mathbf{d}_- u_\ell = \frac{c_\ell}{c_{\ell-1}} \lambda_\ell (-2\rho(H) - k - \ell + 1) u_{\ell-1}. \tag{9.25}$$

for $\ell = k+2$. Let us now iterate that argument to obtain this equation for all $\ell \in \mathbb{N}_0$ so that we can use Lemma 9.4.3. Note that, as $G$-equivariant maps, the generalized gradients commute with $\Omega_\mathfrak{g}$ so that the latter acts by $\mu(H)^2 - \rho(H)^2$ on each of the $u_\ell$. As above, we thus obtain the second equation of Remark 9.4.4 for each $\ell$. Replacing again $\mathbf{d}_+ u_{k+2}$ with the corresponding scalar multiple of $u_{k+3}$ in that equation for $\ell = k+2$, $\mathbf{d}_+ u_{k+3}$ by the one for $u_{k+4}$ and so on, we iteratively infer Equation (9.25) for all $\ell \geq k+2$. As $\mathbf{d}_- u = 0$ implies that equation for $\ell = k+1$ and as it is trivially fulfilled for $\ell \leq k$ (both sides are zero), it holds for each $\ell \in \mathbb{N}_0$.

By Lemma 9.4.3, $f$ defines a function in $H_\mu^{-\infty}$. Since every $u_\ell$ is left $\Gamma$-invariant (note that the generalized gradients are $G$-equivariant and $u_{k+1}$ is $\Gamma$-invariant by definition), $f \in {}^\Gamma H_\mu^{-\infty}$. In order to prove the surjectivity we are left to prove that $P_\mu^{Y_{k+1}}(f) = u$.

Equation (2.4) resp. Proposition 2.2.2 implies that there exists some $\tilde{f} \in \mathcal{D}'(K/M)$ such that $\mathcal{Q}_{\mu-\rho}(\tilde{f}) = f$. By Lemma 9.2.3 we infer

$$P_\mu^{Y_\ell}(f) = P_\mu^{Y_\ell}(\mathcal{Q}_{\mu-\rho}(\tilde{f})) = \frac{1}{c_\ell}\pi_{Y_\ell}^*(\mathcal{Q}_{\mu-\rho}(\tilde{f})) = \frac{1}{c_\ell}\pi_{Y_\ell}^*(f) = \frac{u_\ell}{c_\ell}.$$

For $\ell = k+1$ we especially infer

$$P_\mu^{Y_{k+1}}(f) = \frac{u_{k+1}}{c_{k+1}} = u.$$

All in all we constructed $f \in {}^\Gamma H_\mu^{-\infty}$ such that $P_\mu^{Y_{k+1}}(f) = u$. This finishes the proof of the surjectivity. The isomorphism

$$\mathcal{H} \cong {}^\Gamma(W_k)^{-\infty}$$

follows directly from Proposition 9.4.1. □

### 9.4.2. One-forms on the real hyperbolic space

For the first exceptional parameter $-\rho \in \mathbf{Ex}$ we can say slightly more about the correspondence. Note first that the minimal $K$-type of $W_0$ is given by $Y_1 \cong \mathfrak{p}$ such that

$$G \times_K Y_1 \cong G \times_K \mathfrak{p}^* \cong \Lambda^1 \mathbf{H}^n := T^*\mathbf{H}^n,$$

where $T^*\mathbf{H}^n$ denotes the cotangent bundle of $\mathbf{H}^n$. The irreducible $M$-representations occurring in $\tau := (\mathrm{Ad}, \mathfrak{p})$ are given by

$$\mathfrak{p}|_M = \mathfrak{a} \oplus \{X_v \colon v_1 = 0\} =: (\sigma_0, V_{\sigma_0}) \oplus (\sigma_1, V_{\sigma_1}).$$

Here, $\sigma_0$ is the trivial representation and $V_{\sigma_1} \cong \mathfrak{n}$ as $M$-representation. As in Definition 3.3.1 we define the Poisson transform $P_\mu^\tau \colon H_\mu^{-\infty} \to C^\infty(G \times_K \mathfrak{p}) \cong C^\infty(\Lambda^1\mathbf{H}^n)$ by continuous extension of

$$P_\mu^\tau(f)(g) := \int_K a_I(g^{-1}k)^{-(\mu+\rho)} f(k)\, \mathrm{Ad}(k_I(g^{-1}k))H \,\mathrm{d}k.$$

Note that the tensor product $Y_1 \otimes \mathfrak{p}^*$ is given by

$$Y_1 \otimes \mathfrak{p}^* \cong \mathrm{Hom}(\mathfrak{p}, \mathfrak{p}) \cong \mathrm{Mat}_n(\mathbb{C}),$$

where $\mathrm{SO}(n) \cong K$ acts on $\mathrm{Mat}_n(\mathbb{C})$ by conjugation. Using this isomorphism we infer the decomposition

$$\mathrm{Mat}_n(\mathbb{C}) = \left\{\begin{smallmatrix}\text{diagonal}\\ \text{matrices}\end{smallmatrix}\right\} \oplus \left\{\begin{smallmatrix}\text{symmetric and traceless}\\ \text{matrices}\end{smallmatrix}\right\} \oplus \left\{\begin{smallmatrix}\text{skew symmetric}\\ \text{matrices}\end{smallmatrix}\right\} = Y_0 \oplus V_1 \oplus Y_2$$

into irreducible $K$-representations.

**Lemma 9.4.6.** *Let* $\tilde{\tau} := (\mathrm{Ad}, \Lambda^2\mathfrak{p})$. *Then* $\tilde{\tau} \cong_K V_1$ *and especially* $\mathrm{mult}_M(\sigma_0, \tilde{\tau}|_M) = 0$.

*Proof.* Identifying $\mathfrak{p}$ with $\mathbb{C}^n$, the adjoint representation corresponds to the defining representation of $\mathrm{SO}(n) \cong K$ on $\mathbb{C}^n$. The highest weights of the representation of $\mathrm{SO}(n)$ on $\Lambda^2\mathbb{C}^n$ are given in [Kna02, Problems 7-10, p. 340f.] and coincide with those of $V_1$ (note that both representations are reducible in the case of $n = 4$). By the branching rules for $\mathrm{SO}(n)$ (cf. [Kna02, Theorem 9.16]), we see that the trivial $M$-representation $\sigma_0$ does not occur in $\Lambda^2\mathbb{C}^n$ (we always have $c_1 = 1 \neq 0$ in the notation of [Kna02]). □

Thus, it is natural to consider the exterior derivative and the codifferential

$$d : C^\infty(\Lambda^*\mathbf{H}^n) \to C^\infty(\Lambda^{*+1}\mathbf{H}^n) \qquad \text{and} \qquad \delta : C^\infty(\Lambda^{*+1}\mathbf{H}^n) \to C^\infty(\Lambda^*\mathbf{H}^n).$$

Note that Lemma 9.4.6 and Corollary 3.3.4 imply that $d \circ P^\tau_\mu$ is the zero map. Moreover, since $\delta \circ P^\tau_\mu$ is a multiple of the scalar Poisson transform by Corollary 3.3.5 and the latter one restricted to the socle $\mathrm{soc}(H_{-\rho})$ is zero by Remark 6.1.4, we infer that $P^\tau_{-\rho}|_{{}^\Gamma H^{-\infty}_{-\rho}}$ maps into

$$\mathcal{H}^1 := \{\eta \in C^\infty(\Lambda^1\mathbf{H}^n) \mid d\eta = 0,\, \delta\eta = 0\}.$$

It now follows from [Gai88, Theorem 2(e)] that $\mathcal{H}^1$ is irreducible (as well as the socle) and we conclude as in Section 8.1 that $P^\tau_{-\rho}$ is indeed an isomorphism from ${}^\Gamma H^{-\infty}_{-\rho}$ onto ${}^\Gamma\mathcal{H}^1$. In particular, we arrive at

**Proposition 9.4.7.** *The Ruelle resonance states associated with the first exceptional parameter $\mu = -\rho$ are given by the $\Gamma$-invariant elements in $\mathcal{H}^1$:*

$$\mathrm{Res}^0_X((\mu-\rho)(H)) = \mathrm{Res}^0_X(1-n) \cong {}^\Gamma\{\eta \in C^\infty(\Lambda^1\mathbf{H}^n) \mid d\eta = 0,\ \delta\eta = 0\}.$$

*Moreover, we may also identify (recall $\mathbf{M} = \Gamma\backslash\mathbf{H}^n$)*

$$\mathrm{Res}^0_X(-2\rho(H)) \cong \{\eta \in C^\infty(\Lambda^1\mathbf{M})\colon \Delta^1\eta = 0\} \cong \{\eta \in C^\infty(\Lambda^1\mathbf{M})\colon d\eta = 0,\ \delta\eta = 0\},$$

*where $\Delta^1 := d\delta + \delta d$ denotes the Laplace operator on one forms. In particular, the corresponding dimension is given by the first Betti number $b_1(\mathbf{M})$.*

*Proof.* The first isomorphism follows from Proposition 2.2.3. We prove the relation to the Laplace operator. Since $\mathbf{M} = \Gamma\backslash\mathbf{H}^n$ is a smooth compact manifold, we have an $L^2$ inner product on the one forms given by

$$\langle \eta, \zeta\rangle_{\Lambda^1} := \int_{\mathbf{M}} \eta \wedge \star\zeta,$$

where $\star$ denotes the Hodge star operator. It is well-known that the codifferential $\delta$ is the adjoint operator of the differential $d$ with respect to this inner product. Thus, since $\Delta^1 = d\delta + \delta d$, we obtain

$$\langle \Delta^1\eta, \eta\rangle_{\Lambda^1} = \langle d\delta\eta, \eta\rangle_{\Lambda^1} + \langle \delta d\eta, \eta\rangle_{\Lambda^1} = \langle \delta\eta, \delta\eta\rangle_{\Lambda^1} + \langle d\eta, d\eta\rangle_{\Lambda^1}.$$

Especially, we see that $d\eta = 0$ and $\delta\eta = 0$ if $\Delta^1\eta = 0$. By the definition of $\Delta^1$, this is even an equivalence. For the dimension note that there is an isomorphism between harmonic one forms and the first de Rham cohomology group $H^1(\mathbf{M};\mathbb{R})$ of $\mathbf{M}$ by the Hodge theorem. Finally the dimension of $H^1(\mathbf{M};\mathbb{R})$ is given by $b_1(\mathbf{M})$. □

Following the lines of [Olb94, §5.2], we may also remark that

$$\mathcal{A}'_{\tau}(\sigma_0) = \{-\rho\} \qquad \text{and} \qquad \mathcal{A}'_{\tau}(\sigma_1) = \{-(\rho - \alpha)\}.$$

# A. Computations of scalars relating Poisson transforms

| $G$ | $K$ | $K/M$ | $m_\alpha$ | $m_{2\alpha}$ | $\rho(H)$ |
|---|---|---|---|---|---|
| $\mathrm{SO}_0(n,1),\ n \geq 2$ | $\mathrm{S}(\mathrm{O}(n)\times\mathrm{O}(1)) \cong \mathrm{SO}(n)$ | $\mathbb{S}^{n-1}$ | $n-1$ | $0$ | $\frac{n-1}{2}$ |
| $\mathrm{SU}(n,1),\ n \geq 2$ | $\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1)) \cong \mathrm{U}(n)$ | $\mathbb{S}^{2n-1}$ | $2n-2$ | $1$ | $n$ |
| $\mathrm{Sp}(n,1),\ n \geq 2$ | $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ | $\mathbb{S}^{4n-1}$ | $4n-4$ | $3$ | $2n+1$ |
| $\mathrm{F}_{4(-20)}$ | $\mathrm{Spin}(9)$ | $\mathbb{S}^{15}$ | $8$ | $7$ | $11$ |

Table A.1.: Structural data of rank one groups (recall that $\alpha(H) = 1$ for the unique simple positive restricted root $\alpha$ of $(\mathfrak{g}, \mathfrak{a})$). The isomorphism of $K/M$ with a sphere is given by the adjoint action of $K$ on $H \in \mathfrak{a}_0 \subseteq \mathfrak{p}$.

In order to compute the scalars $T_V^Y(p_{Y,\mu})$ occurring in Proposition 5.3.2 we first compute the scalars $\lambda(V,Y)$ in each case and then conclude by using Lemma 7.3.11 and Equation (7.9). For the explicit calculations we will use *hypergeometric functions*.

**Definition A.0.1.** The (Gaussian, ordinary) *hypergeometric function* $F$ (of type $(2,1)$) is defined by (if the series converges)

$$F(a,b,c,z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $a,b,c,z \in \mathbb{R},\ c > 0$, and

$$(q)_n := \begin{cases} 1 & : n = 0 \\ q(q+1)\dots(q+n-1) & : n > 0 \end{cases}$$

denotes the *Pochhammer symbol*. Note that $F$ is a polynomial in $z$ if $a$ or $b$ is a non-positive integer.

**Lemma A.0.2.** *(cf. [JW77, Lemma 4.1]) Assume $|z| < 1$ or $a \in -\mathbb{N}_0$ or $b \in -\mathbb{N}_0$. Then $F$ has the following properties:*

*(i)* $\frac{d}{dz}F(a,b,c,z) = \frac{ab}{c}F(a+1,b+1,c+1,z)$,

*(ii)* $(c-b-a)F(a,b,c,z) = (c-b)F(a,b-1,c,z) + a(z-1)F(a+1,b,c,z)$,

*(iii)* $(c-b-a)F(a,b,c,z) = (c-a)F(a-1,b,c,z) + b(z-1)F(a,b+1,c,z)$,

*(iv)* $F(a,b+1,c,z) - F(a,b,c,z) = \frac{az}{c}F(a+1,b+1,c+1,z)$,

*(v)* $F(a+1,b,c,z) - F(a,b,c,z) = \frac{bz}{c}F(a+1,b+1,c+1,z)$.

## A.1. The Case of $G = \mathrm{SO}_0(n,1)$, $n \geq 3$

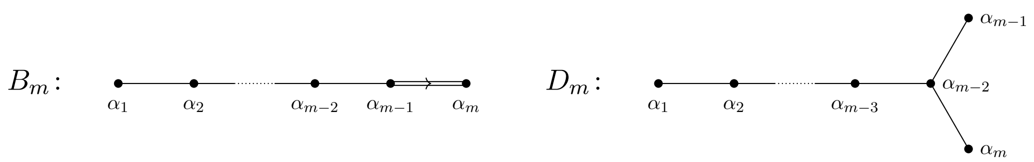
Considering the compact picture and the isomorphism $K/M \cong \mathbb{S}^{n-1}$ we see that (see Equation (B.2.2)) $H_\mu$ decomposes as the Hilbert space direct sum

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{n-1}) \cong_K \widehat{\bigoplus}_{\ell\in\mathbb{N}_0} Y_\ell,$$

where $Y_\ell$ denotes the space of all spherical harmonics of degree $\ell$.

**Remark A.1.1.** For $G = \mathrm{SO}_0(2,1)$ we have $H_\mu \cong_K \widehat{\bigoplus}_{\ell\in\mathbb{Z}} Y_\ell$, with $Y_\ell := \mathbb{C}\cdot z^\ell \subset C^\infty(\mathbb{S}^1)$.

The Dynkin diagram of $K$ is of type $B_m$ if $n = 2m+1$ is odd and of type $D_m$ if $n = 2m$ is even:




We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ as in [Kna02, Chapter II, §1, Example 2, 4] with roots

$\Delta_\mathfrak{k} = \{\pm e_i \pm e_j : 1 \leq i \neq j \leq m\} \cup \{\pm e_i : 1 \leq i \leq m\}$ resp. $\Delta_\mathfrak{k} = \{\pm e_i \pm e_j : 1 \leq i \neq j \leq m\}$

if $K \cong \mathrm{SO}(2m+1)$ resp. $K \cong \mathrm{SO}(2m)$ for some $m \in \mathbb{N}$. As positive systems we consider

$\Delta_\mathfrak{k}^+ = \{e_i \pm e_j : 1 \leq i < j \leq m\} \cup \{e_i : 1 \leq i \leq m\}$ resp. $\Delta_\mathfrak{k}^+ = \{e_i \pm e_j : 1 \leq i < j \leq m\}$

with simple systems $\Pi$ given by

$$\Pi := \{\alpha_1, \ldots, \alpha_m\} \quad \text{with} \quad \alpha_j := \begin{cases} e_j - e_{j+1} & : j < m \\ e_m & : j = m \end{cases}$$

in the odd case and

$$\Pi := \{\alpha_1, \ldots, \alpha_m\} \quad \text{with} \quad \alpha_j := \begin{cases} e_j - e_{j+1} & : j < m \\ e_{m-1} + e_m & : j = m \end{cases}$$

in the even case. The corresponding half sum of positive roots is given by

$$\rho_c = \left(m - \frac{1}{2}\right) e_1 + \left(m - \frac{3}{2}\right) e_2 + \ldots + \frac{1}{2} e_m \text{ resp. } \rho_c = (m-1)e_1 + \ldots + e_{m-1}.$$

The highest weight of $Y_\ell$ is $\ell e_1$ (see e.g. [Kna02, Example 1 of §V.1, p. 277]). Introducing the angular coordinates

$$x_1 = r\cos(\xi), \quad x_i = r\sin(\xi)\omega_i,\ i \geq 2,$$

where $\sum_{i=2}^n \omega_i^2 = 1,\ 0 \leq \xi \leq \pi$, we infer by [JW77, Theorem 3.1(2)] that

$$\phi_{Y_k} = \cos^k(\xi) F\left(-\frac{k}{2}, \frac{1-k}{2}, \frac{n-1}{2}, -\tan^2(\xi)\right).$$

In order to compute the scalars $\lambda(V,Y)$ for $Y, V \in \hat{K}_M = \{[Y_\ell] : \ell \in \mathbb{N}_0\}$ it suffices to decompose $\omega(H)\phi_V$ by Lemma 7.3.10.

**Lemma A.1.2.** *For each $k \in \mathbb{N}_0$ we have*

$$\omega(H)\phi_{Y_k} = \frac{k}{n+2k-2}\phi_{Y_{k-1}} + \frac{n+k-2}{n+2k-2}\phi_{Y_{k+1}}.$$

*Proof.* Recall that the identification from Equation (B.2.1) comes from the $K$-action on $\mathfrak{p}$, where $e_1 \in \mathbb{S}^{n-1}$ corresponds to $H \in \mathfrak{a}$. This implies that

$$\omega(H) = x_1 = \cos(\xi)$$

as a function in $C^\infty(\mathbb{S}^{n-1})$. Therefore,

$$\omega(H)\phi_{Y_k} = \cos^{k+1}(\xi) F\left(-\frac{k}{2}, \frac{1-k}{2}, \frac{n-1}{2}, -\tan^2\xi\right).$$

By Lemma A.0.2.(ii) with $a = -\frac{k}{2},\ b = \frac{1-k}{2},\ c = \frac{n-1}{2}$ and $z = -\tan^2\xi$ we infer that $(n+2k-2)F\left(-\frac{k}{2}, \frac{1-k}{2}, \frac{n-1}{2}, z\right)$ equals

$$(n+k-2)F\left(-\frac{k+1}{2}, -\frac{k}{2}, \frac{n-1}{2}, z\right) + \frac{k}{\cos^2\xi}F\left(\frac{1-k}{2}, \frac{2-k}{2}, \frac{n-1}{2}, z\right).$$

Multiplying by $\cos^{k+1}\xi$ yields the result. □

**Remark A.1.3.** Note that Lemma 7.3.10 implies that

$$\lambda(Y_k, Y_{k+1}) = \mathrm{pr}_{Y_{k+1}}(\omega(H)\phi_{Y_k})(eM) = \frac{n+k-2}{n+2k-2}\phi_{Y_{k+1}}(eM) = \frac{n+k-2}{n+2k-2}.$$

Similarly, we have $\lambda(Y_k, Y_{k-1}) = \frac{k}{n+2k-2}$. The scalars $T_{Y_{k\pm1}}^{Y_k}(p_{Y_k,\mu})(e)$ will be computed in Proposition A.5.1.

In order to describe the generalized gradients properly we will now decompose the relevant tensor products.

**Proposition A.1.4.** *Let $K = \mathrm{SO}(2m+1)$, $m \geq 1$. For $m > 1$ the tensor product $Y_k \otimes \mathfrak{p}^*$ decomposes for $k \in \mathbb{N}$ into*

$$Y_k \otimes \mathfrak{p}^* \cong Y_{k-1} \oplus Y_{k+1} \oplus V_k$$

*where $V_k$ is the $K$-representation with highest weight $ke_1 + e_2$. Moreover, we have $Y_k \otimes \mathfrak{p}^* \cong Y_{k-1} \oplus Y_k \oplus Y_{k+1}$ if $m = 1$.*

*Proof.* The coadjoint representation of $K$ on $\mathfrak{p}^* \cong \mathbb{C}^{2m+1}$ is equivalent to the defining representation (as well as $Y_1$) and has weights $\pm e_i$, $i \in \{1, \ldots, m\}$, and 0. Writing

$$Y_k \otimes \mathfrak{p}^* \cong Y_k \otimes Y_1 \cong \bigoplus_{\Lambda_i \in \hat{K}} \mathcal{L}_i \Lambda_i,$$

where $\mathcal{L}_i \coloneqq \mathrm{mult}(\Lambda_i, Y_k \otimes Y_1)$ denotes the multiplicity, we have by [FS97, p.274]

$$\mathcal{L}_i = \sum_{w \in W} \mathrm{sign}(w)\, \mathrm{mult}_{Y_1}(w(\Lambda_i + \rho_c) - \rho_c - ke_1),$$

where $\mathrm{mult}_{Y_1}(\mu) \in \mathbb{N}_0$ denotes the multiplicity of the weight $\mu$ in $Y_1$ and $W$ denotes the Weyl group of $\mathfrak{k}$. If $\mathcal{L}_i \neq 0$ there has to exist some $w \in W$ such that $w(\Lambda_i + \rho_c) - \rho_c - ke_1$ is a weight of $Y_1$, i.e.

$$w(\Lambda_i + \rho_c) - \rho_c - ke_1 = \pm e_j \Leftrightarrow \Lambda_i = w^{-1}(\rho_c + ke_1 \pm e_j) - \rho_c$$

for some $j \in \{1, \ldots, m\}$ or

$$w(\Lambda_i + \rho_c) - \rho_c - ke_1 = 0 \Leftrightarrow \Lambda_i = w^{-1}(\rho_c + ke_1) - \rho_c.$$

Let us first consider the case $m \neq 1$. Since $\Lambda_i$ is a highest weight it is dominant. Thus, $\rho_c + ke_1 \pm e_j$ resp. $\rho_c + ke_1$ must not lie on the boundary of any Weyl chamber. This is the case if and only if the weight of $Y_1$ is contained in $\{0, \pm e_1, e_2, -e_m\}$. In the first three cases we obtain for $\Lambda_i + \rho_c$

$$\begin{aligned}
w^{-1}(\rho_c + ke_1) &= w^{-1}\left(\left(k + m - \frac{1}{2}\right)e_1 + \left(m - \frac{3}{2}\right)e_2 + \ldots + \frac{1}{2}e_m\right) \\
w^{-1}(\rho_c + ke_1 \pm e_1) &= w^{-1}\left(\left(k \pm 1 + m - \frac{1}{2}\right)e_1 + \left(m - \frac{3}{2}\right)e_2 + \ldots + \frac{1}{2}e_m\right) \\
w^{-1}(\rho_c + ke_1 + e_2) &= w^{-1}\left(\left(k + m - \frac{1}{2}\right)e_1 + \left(m - \frac{1}{2}\right)e_2 + \ldots + \frac{1}{2}e_m\right)
\end{aligned}$$

which is dominant if and only if $w = id$ yielding $\Lambda_i = ke_1, (k \pm 1)e_1, ke_1 + e_2$ respectively. For $\Lambda_i + \rho_c = w^{-1}(\rho_c + ke_1 - e_m)$ we have

$$\Lambda_i + \rho_c = w^{-1}\left(\left(k + m - \frac{1}{2}\right)e_1 + \left(m - \frac{3}{2}\right)e_2 + \ldots + \frac{3}{2}e_{m-1} - \frac{1}{2}e_m\right)$$

which is dominant if and only if $w = s_{e_m}$ is the reflection along $e_m$. For this $w$ we have $\Lambda_i = ke_1$. Altogether we have

$$\begin{aligned}\mathrm{mult}(ke_1, Y_k \otimes Y_1) &= \sum_{w\in W} \mathrm{sign}(w)\,\mathrm{mult}_{Y_1}(w(\Lambda_i + \rho_c) - \rho_c - ke_1)\\ &= \mathrm{sign}(id)\,\mathrm{mult}_{Y_1}(0) + \mathrm{sign}(s_{e_m})\,\mathrm{mult}_{Y_1}(-e_m) = 0\end{aligned}$$

and similarly that the representations with highest weights $(k \pm 1)e_1$ resp. $ke_1 + e_2$ occur with multiplicity one. For $m = 1$ the weights of $Y_1$ are $-e_1$, $0$ and $e_1$. We get $\Lambda_i = (k-1)e_1$, $ke_1$ resp. $(k+1)e_1$ in this case, each with multiplicity one. □

**Remark A.1.5.** Using the Weyl dimension formula we see that (in the notation of Proposition A.1.4)

$$\dim Y_k = \binom{2m+k-2}{k}\frac{m+k-\frac{1}{2}}{m-\frac{1}{2}},\ \dim V_k = \frac{k(2k+2m-1)}{k+2m-2}\binom{2m+k-1}{k+1}.$$

**Proposition A.1.6.** *Let $K = \mathrm{SO}(2m),\ m \geq 2$. The tensor product $Y_k \otimes \mathfrak{p}^*$ decomposes for $k \in \mathbb{N}$ into*

$$Y_k \otimes \mathfrak{p}^* \cong Y_{k-1} \oplus Y_{k+1} \oplus V_k,$$

*where $V_k$ is the $K$-representation with highest weight $ke_1 + e_2$ for $m \neq 2$ and the sum of the $K$-representations with highest weight $ke_1 + e_2$ resp. $ke_1 - e_2$ for $m = 2$.*

*Proof.* The coadjoint representation of $K$ on $\mathfrak{p}^* \cong \mathbb{C}^{2m}$ is equivalent to the defining representation (as well as $Y_1$) and has weights $\pm e_i,\ i \in \{1,\dots,m\}$. Each weight occurs with multiplicity one. We can now decompose $Y_k \otimes \mathfrak{p}^* \cong Y_k \otimes Y_1$ using the Racah-Speiser algorithm. Let

$$Y_k \otimes Y_1 \cong \bigoplus_{\Lambda_i \in \hat{K}} \mathcal{L}_i \Lambda_i$$

with $\mathcal{L}_i := \mathrm{mult}(\Lambda_i, Y_k \otimes Y_1) = \sum_{w\in W} \mathrm{sign}(w)\,\mathrm{mult}_{Y_1}(w(\Lambda_i + \rho_c) - \rho_c - ke_1)$ as in the odd case. Since $w(\Lambda_i + \rho_c) - \rho_c - ke_1 = \pm e_i \Leftrightarrow \Lambda_i = w^{-1}(\rho_c + ke_1 \pm e_i) - \rho_c$, the latter expression has to be dominant since $\Lambda_i$ is a highest weight. Thus, the weight $\rho_c + ke_1 \pm e_i$ must not lie on the boundary of any Weyl chamber. For $m \neq 2$, this is the case if and only if the weight $\pm e_i$ is $\pm e_1$ or $e_2$. For $m = 2$ we get $\pm e_1$ or $\pm e_2$. In each case the weight $\rho_c + ke_1 \pm e_i$ is dominant, so $w = id$. Moreover, the weight $w^{-1}(\rho_c + ke_1 \pm e_i) - \rho_c$ is given by $ke_1 \pm e_1 = (k \pm 1)e_1$ resp. $ke_1 + e_2$ for $m \neq 2$. For $m = 2$ we additionally get $ke_1 - e_2$. □

**Remark A.1.7.** Using the Weyl dimension formula we see that (in the notation of Proposition A.1.6)

$$\dim Y_k = \binom{2m+k-3}{k}\frac{m+k-1}{m-1},\ \dim V_k = \frac{(4m+2k-4)(m+k-1)}{k+1}\binom{2m+k-4}{k-1}.$$

We remark that the dimensions of the constituents of $V_k$ are both given by $k(k+2)$ in the case of $m = 2$. The formula

$$\dim Y_k = \binom{n+k-3}{k} \frac{\frac{n}{2}+k-1}{\frac{n}{2}-1} = \binom{n+k-3}{k} \frac{n+2k-2}{n-2}$$

summarizes the odd and the even case.

## A.2. The Case of $G = \mathrm{SU}(n,1),\ n \geq 2$

By Equation (B.3.12) we can decompose $H_\mu$ as the Hilbert space direct sum

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{2n-1}) \cong_K \widehat{\bigoplus}_{p,q \in \mathbb{N}_0} Y_{p,q}. \tag{A.2.1}$$

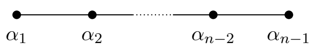
A maximal torus $T$ in $\mathrm{U}(n) \cong K$ is given by the diagonal matrices

$$T := \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \colon \theta_j \in \mathbb{R}\}.$$

Although $\mathfrak{k}_0$ is not semisimple, it is still reductive and can be decomposed as

$$\mathfrak{k}_0 = \mathfrak{z}(\mathfrak{k}_0) \oplus [\mathfrak{k}_0, \mathfrak{k}_0] \cong (i\mathbb{R} \cdot \mathrm{I}_n) \oplus \mathfrak{su}(n).$$

Let $\mathfrak{t}_0 = \mathrm{Lie}(T)$ denote the diagonal matrices in $\mathfrak{su}(n,1)$. Then $\mathfrak{t}_0 = \mathfrak{z}(\mathfrak{k}_0) \oplus \mathfrak{h}_0$ where $\mathfrak{h}_0$ is a Cartan subalgebra of $[\mathfrak{k}_0, \mathfrak{k}_0] \cong \mathfrak{su}(n)$ (traceless diagonal matrices). Note that $\mathfrak{h}$ is a Cartan subalgebra of $[\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(n)_\mathbb{C} \cong \mathfrak{sl}(n, \mathbb{C})$ and that the roots of $\mathfrak{h}$ in $[\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{sl}(n, \mathbb{C})$ determine the roots of $\mathfrak{t}$ in $\mathfrak{k}$ (by extending them to $\mathfrak{t}$ by defining them to be 0 on the center $\mathfrak{z}(\mathfrak{k}) = \mathbb{C} \cdot \mathrm{I}_n$). The Dynkin diagram of $\mathfrak{sl}(n, \mathbb{C})$ is of type $A_{n-1}$:




Denoting the dual basis of the standard diagonal matrix basis $E_{ii},\ 1 \leq i \leq n+1$, by $(e_i)_i$ we obtain that the roots $\Delta_\mathfrak{k}$ of $(\mathfrak{k}, \mathfrak{t})$ resp. $\Delta$ of $(\mathfrak{g}, \mathfrak{t})$ are given by

$$\Delta_\mathfrak{k} = \{e_i - e_j \colon 1 \leq i \neq j \leq n\} \text{ resp. } \Delta = \{e_i - e_j \colon 1 \leq i \neq j \leq n+1\}. \tag{A.2.2}$$

We choose the positive system $\Delta_\mathfrak{k}^+ = \{e_i - e_j \colon 1 \leq i < j \leq n\}$ with simple roots $\Pi := \{\alpha_1, \ldots, \alpha_{n-1}\},\ \alpha_j := e_j - e_{j+1}$, and half sum

$$\rho_c = \left(\frac{n-1}{2}\right) e_1 + \left(\frac{n-3}{2}\right) e_2 + \ldots - \frac{n-1}{2} e_n. \tag{A.2.3}$$

By Lemma B.3.1, the highest weight of $Y_{p,q}$ is given by $qe_1 - pe_n + (p-q)e_{n+1}$. Introducing the angular coordinates (on $\mathbb{C}^n \cong \mathbb{R}^{2n}$)

$$z_1 = r\cos(\xi)e^{i\varphi}, \quad z_j = r\sin(\xi)\omega_j,\ 2 \leq j \leq n$$

where $\sum_{j=2}^n |\omega_j|^2 = 1,\ 0 \leq \varphi \leq 2\pi$ and $0 \leq \xi \leq \frac{\pi}{2}$ we have (see [JW77, Theorem 3.1(3)])

$$\phi_{Y_{p,q}} = e^{i(p-q)\varphi} \cos^{p+q}(\xi) F(-p, -q, n-1, -\tan^2(\xi)).$$

We can now compute the scalars. In this case (for $G = \mathrm{SU}(n,1)$), similar computations can be found in [Mea89, Theorem 5.6.6].

**Lemma A.2.1.** *For each $p, q \in \mathbb{N}_0$ we have*

$$\begin{aligned}2(p+q+n-1)\omega(H)\phi_{Y_{p,q}} &= (p+n-1)\phi_{Y_{p+1,q}} + q\phi_{Y_{p,q-1}} \\ &\quad + (q+n-1)\phi_{Y_{p,q+1}} + p\phi_{Y_{p-1,q}}.\end{aligned}$$

*Proof.* Write $\phi_{Y_{p,q}} = e^{i(p-q)\varphi}h_{p,q}(\xi)$. In the angular coordinates from above we have

$$\omega(H) = \mathrm{Re}(z_1) = \cos(\xi)\cos(\varphi)$$

as a function in $C^\infty(\mathbb{S}^{2n-1})$. Therefore,

$$\begin{aligned}\omega(H)\phi_{Y_{p,q}} &= \cos(\xi)\cos(\varphi)e^{i(p-q)\varphi}h_{p,q}(\xi) \\ &= \frac{\cos(\xi)h_{p,q}(\xi)}{2}e^{i(p-q+1)\varphi} + \frac{\cos(\xi)h_{p,q}(\xi)}{2}e^{i(p-q-1)\varphi}.\end{aligned} \tag{A.2.4}$$

Lemma A.0.2.(iii) implies that

$$\cos(\xi)h_{p,q}(\xi) = \frac{p+n-1}{p+q+n-1}h_{p+1,q}(\xi) + \frac{q}{p+q+n-1}h_{p,q-1}(\xi) \tag{A.2.5}$$

and Lemma A.0.2.(ii) implies that

$$\cos(\xi)h_{p,q}(\xi) = \frac{q+n-1}{p+q+n-1}h_{p,q+1}(\xi) + \frac{p}{p+q+n-1}h_{p-1,q}(\xi). \tag{A.2.6}$$

Combining the equations (A.2.4), (A.2.5) and (A.2.6) yields the result. □

**Remark A.2.2.** As in Remark A.1.3, Lemma A.2.1 determines the scalars $\lambda(Y_{p,q}, V)$ for each $V \in \hat{K}_M$ with $V \leftrightarrow Y_{p,q}$.

To decompose the relevant tensor products we use Proposition 7.3.2. By Equation (A.2.2) we infer that the non-compact roots are given by

$$\Delta_n = \{\pm(e_i - e_{n+1})\colon 1 \leq i \leq n\}.$$

The following remark ensures that each representation $Y_{\tau,\beta},\ \beta \in S$, in Proposition 7.3.2 actually occurs.

**Remark A.2.3.** Using the Weyl dimension formula we see that

$$\begin{aligned}\dim Y_{p,q} &= \binom{q+n-2}{n-2}\binom{p+n-2}{n-2}\frac{n+p+q-1}{n-1} = \dim Y_{q,p}, \\ \dim Y_{p,q,-e_{n-1}+e_{n+1}} &= \binom{q+n-1}{q}\binom{p+n-2}{p}\frac{(n+p+q-1)p(n-2)}{(n+q-2)(p+1)} = \dim Y_{q,p,e_2-e_{n+1}}.\end{aligned}$$

For $n = 2$ this has to be read as $\dim Y_{p,0,-e_1+e_3} = p = \dim Y_{0,p,e_2-e_3}$. We get that

$$\sum_{\beta \in S \subseteq \Delta_n} \dim Y_{p,q,\beta} = \dim \mathfrak{p} \cdot \dim Y_{p,q} = 2n \cdot \dim Y_{p,q},$$

which implies that $m(\beta) = 1$ if and only if the corresponding formula for the dimension of $Y_{p,q,\beta}$ is not zero.

### A.2.1. Alternative proof of tensor product decomposition

In this section we give another proof of the tensor product decomposition from Proposition 7.3.2 for $\mathrm{SU}(n,1)$, $n \geq 2$, using the Racah-Speiser algorithm. We first identify the irreducible components of $\mathfrak{p}$ in $\hat{K}_M = \{Y_{p,q} \mid p, q \in \mathbb{N}_0\}$.

**Lemma A.2.4.** *We have*

$$\Phi : \mathfrak{p}_1^* \cong Y_{1,0}$$

*as $K$-representations.*

*Proof.* First recall from the proof of Lemma B.3.1 that $Y_{1,0}$ is spanned by $z_1, \dots, z_n$. We claim that

$$\Phi : \mathfrak{p}_1^* \to Y_{1,0}, \quad \Phi(\varphi)(v) := \varphi\left(\begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix}\right),$$
$$\Phi^{-1} : Y_{1,0} \to \mathfrak{p}_1^*, \quad \Phi^{-1}(\psi)\left(\begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix}\right) := \psi(v)$$

are $K$-equivariant isomorphisms. One easily sees that both maps map into the indicated spaces and are inverses of each other. Let us prove the $K$-equivariance. Note that for each $k = \mathrm{diag}(A, \lambda) \in K$ and $v \in \mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ we have

$$\mathrm{Ad}(k) \begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & \lambda^{-1} A v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & k.v \\ 0 & 0 \end{pmatrix}$$

with the action from Equation (B.3.11). Thus,

$$\Phi(\mathrm{Ad}^*(k)\varphi)(v) = \varphi\left(\mathrm{Ad}(k^{-1}) \begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix}\right) = \Phi(\varphi)(k^{-1}.v) = (k\Phi(\varphi))(v). \qquad \square$$

**Lemma A.2.5.** *We have*

$$\Psi : \mathfrak{p}_2^* \cong Y_{0,1}$$

*as $K$-representations.*

*Proof.* By the proof of Lemma B.3.1 we infer that $Y_{0,1}$ is spanned by $\overline{z_1}, \dots, \overline{z_n}$. We claim that

$$\Psi : \mathfrak{p}_2^* \to Y_{0,1}, \quad \Psi(\varphi)(v) := \varphi\left(\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right),$$
$$\Psi^{-1} : Y_{0,1} \to \mathfrak{p}_2^*, \quad \Psi^{-1}(\psi)\left(\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right) = \psi(v)$$

are $K$-equivariant isomorphisms. Note first that $\Psi$ maps into $Y_{0,1}$ since, for $\lambda \in \mathbb{C}$,

$$\Psi(\varphi)(\lambda v) = \varphi\left(\begin{pmatrix} \mathbf{0}_n & 0 \\ (\lambda v)^* & 0 \end{pmatrix}\right) = \varphi\left(\overline{\lambda}\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right) = \overline{\lambda}\Psi(\varphi)(v).$$

Similarly we see that $\Psi^{-1}$ maps into $\mathfrak{p}_2^*$ since

$$\begin{aligned}\Psi^{-1}(\psi)\left(\lambda\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right) &= \Psi^{-1}(\psi)\left(\begin{pmatrix} \mathbf{0}_n & 0 \\ (\overline{\lambda} v)^* & 0 \end{pmatrix}\right) = \psi(\overline{\lambda} v) = \lambda\psi(v) \\ &= \lambda\Psi^{-1}(\psi)\left(\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right).\end{aligned}$$

One easily sees that $\Psi$ and $\Psi^{-1}$ are inverse to each other. Let us finally prove the $K$-equivariance. Note that for each $k = \mathrm{diag}(A, \lambda) \in K$ and $v \in \mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ we have

$$\mathrm{Ad}(k)\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & 0 \\ \lambda v^* A^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & 0 \\ (\lambda^{-1} A v)^* & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & 0 \\ (k.v)^* & 0 \end{pmatrix},$$

where we used $A^* = A^{-1}$ and $\overline{\lambda} = \lambda^{-1}$. Thus,

$$\Psi(\mathrm{Ad}^*(k)\varphi)(v) = \varphi\left(\mathrm{Ad}(k^{-1})\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right) = \begin{pmatrix} \mathbf{0}_n & 0 \\ (k^{-1}.v)^* & 0 \end{pmatrix} = (k\Psi(\varphi))(v). \qquad \square$$

**Proposition A.2.6.** *For $n \neq 2$ we have: For $p, q \neq 0$ the tensor product $Y_{p,q} \otimes \mathfrak{p}_1^*$ decomposes into*

$$Y_{p,q} \otimes \mathfrak{p}_1^* \cong Y_{p,q-1} \oplus Y_{p+1,q} \oplus V_{p,q}^1,$$

*where $V_{p,q}^1$ is the $K$-representation with highest weight $qe_1 - e_{n-1} - pe_n + (p-q+1)e_{n+1}$. If $p = 0$ we have*

$$Y_{0,q} \otimes \mathfrak{p}_1^* \cong Y_{0,q-1} \oplus Y_{1,q}$$

*and for $q = 0$*

$$Y_{p,0} \otimes \mathfrak{p}_1^* \cong Y_{p+1,0} \oplus V_{p,0}^1,$$

*with $Y_{0,0} \otimes \mathfrak{p}_1^* \cong Y_{1,0}$. For $n = 2$ we have*

$$Y_{p,q} \otimes \mathfrak{p}_1^* \cong Y_{p,q-1} \oplus Y_{p+1,q}$$

*for $q \neq 0$,*

$$Y_{p,0} \otimes \mathfrak{p}_1^* \cong Y_{p+1,0} \oplus V_{p,0}^1$$

*for $p \neq 0$, where $V_{p,0}^1$ has highest weight $-e_1 - pe_2 + (p+1)e_3$, and $Y_{0,0} \otimes \mathfrak{p}_1^* \cong Y_{1,0}$.*

*Proof.* We first decompose the tensor products with respect to $[\mathfrak{k},\mathfrak{k}]$ (i.e. we omit the action of the center). By the proof of Lemma B.3.1 we infer that $\mathfrak{p}_1^* \cong Y_{1,0}$ has weights $-e_j,\ j \in \{1,\dots,n\}$, with respect to $[\mathfrak{k},\mathfrak{k}]$. Writing

$$Y_{p,q} \otimes Y_{1,0} \cong \bigoplus_{\Lambda_i \in \widehat{K}} \mathcal{L}_i \Lambda_i,$$

where $\mathcal{L}_i \coloneqq \mathrm{mult}(\Lambda_i, Y_{p,q} \otimes Y_{1,0})$ denotes the multiplicity, we have by [FS97, p. 274]

$$\mathcal{L}_i = \sum_{w \in W} \mathrm{sign}(w)\, \mathrm{mult}_{Y_{1,0}}(w(\Lambda_i + \rho_c) - \rho_c - (qe_1 - pe_n)),$$

where $\mathrm{mult}_{Y_{1,0}}(\mu) \in \mathbb{N}_0$ denotes the multiplicity of the weight $\mu$ in $Y_{1,0}$,

$$\rho_c = \frac{n-1}{2}e_1 + \frac{n-3}{2}e_2 + \ldots - \frac{n-1}{2}e_n$$

by Equation (A.2.3) and $W$ denotes the Weyl group of $\mathfrak{k}$. Note that

$$\begin{aligned} &w(\Lambda_i + \rho_c) - \rho_c - (qe_1 - pe_n) = -e_j \\ \Leftrightarrow\ &\Lambda_i = w^{-1}(\rho_c + qe_1 - pe_n - e_j) - \rho_c. \end{aligned}$$

Since $\Lambda_i$ has to be dominant, $\rho_c + qe_1 - pe_n - e_j$ must not lie on the boundary of any Weyl chamber. We first discuss the case $n \neq 2$. Then $\rho_c + qe_1 - pe_n - e_j$ is annihilated by $e_j - e_{j+1}$ for $j \in \{2,\dots,n-2\}$. For $j = 1$ it is on the boundary if and only if $q = 0$ and for $j = n-1$ if and only if $p = 0$. For $j = n$ it is never on the boundary. In each case the weight is dominant so that we have $w = id$ in each case. This leads to

$$\Lambda_i \in \{(q-1)e_1 - pe_n, qe_1 - e_{n-1} - pe_n, qe_1 - (p+1)e_n\}$$

corresponding to the representations $Y_{p,q-1}$, $V^1_{p,q}$ resp. $Y_{p+1,q}$, where $Y_{p,q-1}$ does not occur if $q = 0$ and $V^1_{p,q}$ does not occur if $p = 0$. For the highest weights w.r.t. $\mathfrak{k}$ we may use [Kna02, Proposition 9.72] to obtain that each irreducible component of the tensor product, with highest weight $\sum_{i=1}^{n+1} a_i e_i$, has to fulfill $\sum_{i=1}^{n+1} a_i = 0$ (for the weights of $Y_{p,q}$ see Lemma B.3.1).

Let us finally discuss the case $n = 2$. Analogous to the previous calculations we obtain

$$\rho_c + qe_1 - pe_2 - e_j \in \left\{ \left(q - \frac{1}{2}\right)e_1 - \left(p + \frac{1}{2}\right)e_2, \left(q + \frac{1}{2}\right)e_1 - \left(p + \frac{3}{2}\right)e_2 \right\},$$

where the first one lies on the boundary if $p = q = 0$ and thus

$$\Lambda_i \in \{(q-1)e_1 - pe_2, qe_1 - (p+1)e_2\},$$

where the first one does not occur if $p = q = 0$. The second weight corresponds to the representation $Y_{p+1,q}$. The first weight corresponds to the representation $Y_{p,q-1}$ for $q \neq 0$. For $q = 0$ and $0 \neq p$ it is given by $-e_1 - pe_2$. For the highest weights w.r.t. $\mathfrak{k}$ we proceed as above. $\square$

**Proposition A.2.7.** *For $n \neq 2$ the tensor product $Y_{p,q} \otimes \mathfrak{p}_2^*$ decomposes as follows. If $p, q \neq 0$ we have*

$$Y_{p,q} \otimes \mathfrak{p}_2^* \cong Y_{p,q+1} \oplus Y_{p-1,q} \oplus V_{p,q}^2,$$

*where $V_{p,q}^2$ is the $K$-representation with highest weight $qe_1 + e_2 - pe_n + (p-q-1)e_{n+1}$. If $p = 0$ we have*

$$Y_{0,q} \otimes \mathfrak{p}_2^* \cong Y_{0,q+1} \oplus V_{0,q}^2$$

*and if $q = 0$*

$$Y_{p,0} \otimes \mathfrak{p}_2^* \cong Y_{p,1} \oplus Y_{p-1,0},$$

*with $Y_{0,0} \otimes \mathfrak{p}_2^* \cong Y_{0,1}$. For $n = 2$ we obtain*

$$Y_{p,q} \otimes \mathfrak{p}_2^* \cong Y_{p,q+1} \oplus Y_{p-1,q}$$

*for $p \neq 0$ and*

$$Y_{0,q} \otimes \mathfrak{p}_2^* \cong Y_{0,q+1} \oplus V_{0,q}^2,$$

*where $V_{0,q}^2$ has highest weight $qe_1 + e_2 - (q+1)e_{n+1}$ and $Y_{0,0} \otimes \mathfrak{p}_2^* \cong Y_{0,1}$.*

*Proof.* Again it suffices to decompose the tensor product with respect to $[\mathfrak{k}, \mathfrak{k}]$ (in the following we always denote weights w.r.t. $[\mathfrak{k}, \mathfrak{k}]$; for the center we proceed as in the proof of Proposition A.2.6). By the proof of Lemma B.3.1 we infer that $\mathfrak{p}_2^* \cong Y_{0,1}$ has weights $e_j$ for $j \in \{1, \dots, n\}$. Writing

$$Y_{p,q} \otimes Y_{0,1} \cong \bigoplus_{\Lambda_i \in \widehat{K}} \mathcal{L}_i \Lambda_i,$$

where $\mathcal{L}_i \coloneqq \mathrm{mult}(\Lambda_i, Y_{p,q} \otimes Y_{0,1})$ denotes the multiplicity, we have by [FS97, p. 274]

$$\mathcal{L}_i = \sum_{w \in W} \mathrm{sign}(w) \, \mathrm{mult}_{Y_{0,1}}(w(\Lambda_i + \rho_c) - \rho_c - (qe_1 - pe_n)),$$

where $\mathrm{mult}_{Y_{0,1}}(\mu) \in \mathbb{N}_0$ denotes the multiplicity of the weight $\mu$ in $Y_{0,1}$,

$$\rho_c = \frac{n-1}{2} e_1 + \frac{n-3}{2} e_2 + \ldots - \frac{n-1}{2} e_n$$

by Equation (A.2.3) and $W$ denotes the Weyl group of $\mathfrak{k}$. Note that

$$\begin{aligned} w(\Lambda_i + \rho_c) - \rho_c - (qe_1 - pe_n) &= e_j \\ \Leftrightarrow \Lambda_i &= w^{-1}(\rho_c + qe_1 - pe_n + e_j) - \rho_c. \end{aligned}$$

Since $\Lambda_i$ has to be dominant, $\rho_c + qe_1 - pe_n + e_j$ must not lie on the boundary of any Weyl chamber. We first discuss the case $n \neq 2$. Then $\rho_c + qe_1 - pe_n + e_j$ is annihilated by $e_{j-1} - e_j$ for $j \in \{3, \dots, n-1\}$. For $j = 1$ it is never on the boundary. For $j = 2$ it is on the boundary if and only if $q = 0$ and for $j = n$ if and only if $p = 0$. In each case the weight is dominant so that we have $w = id$ in each case. This leads to

$$\Lambda_i \in \{(q+1)e_1 - pe_n, qe_1 + e_2 - pe_n, qe_1 - (p-1)e_n\}$$

corresponding to the representations $Y_{p,q+1}$, $V_{p,q}^2$ resp. $Y_{p-1,q}$, where $V_{p,q}^2$ does not occur if $q = 0$ and $Y_{p-1,q}$ does not occur if $p = 0$.

Let us finally discuss the case $n = 2$. Analogous to the previous calculations we obtain

$$\rho_c + qe_1 - pe_2 + e_j \in \left\{ \left(q + \frac{3}{2}\right) e_1 - \left(p + \frac{1}{2}\right) e_2, \left(q + \frac{1}{2}\right) e_1 - \left(p - \frac{1}{2}\right) e_2 \right\},$$

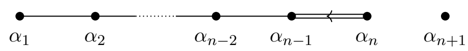
where the second one lies on the boundary if $p = q = 0$, and thus

$$\Lambda_i \in \{(q+1)e_1 - pe_2, qe_1 - (p-1)e_2\},$$

where the second summand does not occur if $p = q = 0$. The first weight corresponds to the representation $Y_{p,q+1}$. The second weight corresponds to the representation $Y_{p-1,q}$ for $p \neq 0$. For $p = 0$ and $q \neq 0$ it is given by $qe_1 + e_2$. □

## A.3. The Case of $G = \mathrm{Sp}(n, 1)$, $n \geq 2$

In this case we have $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and $\mathfrak{g} = \mathfrak{sp}(n,1)_{\mathbb{C}} = \mathfrak{sp}(n+1, \mathbb{C})$. The Dynkin diagram of $K$ is of type $C_n \times C_1$:




We choose a Cartan subalgebra of $\mathfrak{sp}(n, \mathbb{C}) \times \mathfrak{sp}(1, \mathbb{C})$ and introduce notation as in [Kna02, Chapter II, §1, Example 3 & p. 685] such that we have for the roots $\Delta_{\mathfrak{k}}$ of $(\mathfrak{k}, \mathfrak{h})$ resp. $\Delta$ of $(\mathfrak{g}, \mathfrak{h})$

$$\Delta_{\mathfrak{k}} = \{\pm e_i \pm e_j \colon 1 \leq i \neq j \leq n\} \cup \{\pm 2e_i \colon 1 \leq i \leq n+1\} \tag{A.3.7}$$

$$\Delta = \{\pm e_i \pm e_j \colon 1 \leq i \neq j \leq n+1\} \cup \{\pm 2e_i \colon 1 \leq i \leq n+1\}. \tag{A.3.8}$$

We choose the positive system

$$\Delta_{\mathfrak{k}}^+ = \{e_i \pm e_j \colon 1 \leq i < j \leq n\} \cup \{2e_i \colon 1 \leq i \leq n+1\}. \tag{A.3.9}$$

with simple positive roots $\Pi \coloneqq \{\alpha_1, \dots, \alpha_{n+1}\}$ given by

$$\alpha_i \coloneqq e_i - e_{i+1} \text{ for } 1 \leq i \leq n-1 \qquad \text{and} \qquad \alpha_j \coloneqq 2e_j \text{ for } j \in \{n, n+1\}.$$

The corresponding half sum of positive roots is given by

$$\rho_c = ne_1 + (n-1)e_2 + \ldots + 2e_{n-1} + e_n + e_{n+1}. \tag{A.3.10}$$

The fundamental weights $\lambda_1, \ldots, \lambda_{n+1}$ defined by $2\frac{\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$ are given by

$$\lambda_j = \sum_{i=1}^{j} e_i \text{ for } 1 \leq j \leq n \qquad \text{and} \qquad \lambda_{n+1} = e_{n+1}. \tag{A.3.11}$$

By Equation (B.4.15) we have

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{4n-1}) \cong_K \widehat{\bigoplus}_{a \geq b \geq 0} V_{a,b},$$

where $V_{a,b}$ has highest weight $ae_1 + be_2 + (a-b)e_{n+1}$. We now introduce angular coordinates on $\mathbb{H}^n \cong \mathbb{R}^{4n}$ as in [JW77, Theorem 3.1(4)]. For $(w_1, \ldots, w_n) \in \mathbb{H}^n$ we write

$$w_1 = r\cos(\xi)(\cos(t) + y\sin(t)), \quad w_i = r\sin(\xi)q_i,\ i \geq 2$$

where $q_i, y \in \mathbb{H}$ such that $|y|^2 = 1 = \sum_{i=2}^{n} |q_i|^2$, $\mathrm{Re}(y) = 0$ and $0 \leq \xi \leq \frac{\pi}{2}$, $0 \leq t \leq \pi$. Then we have by [JW77, Theorem 3.1(4)][1] (our $V_{a,b}$ corresponds to $V^{p,q}$ of [JW77] with $p \coloneqq a+b$ and $q \coloneqq a-b$ by [JW77, Lemma 3.3])

$$\phi_{V_{a,b}} = \frac{1}{a-b+1} \frac{\sin((a-b+1)t)}{\sin(t)} \cos^{a+b}(\xi) F\left(-b, -(a+1), 2(n-1), -\tan^2(\xi)\right),$$

where the normalizing factor $\frac{1}{a-b+1}$ follows from $\phi_{V_{a,b}}(eM) = 1$, where $eM$ corresponds to $t = \xi = 0$, and using $\lim_{t \to 0} \frac{\sin((a-b+1)t)}{\sin(t)} = a-b+1$.

**Lemma A.3.1.** *For $a, b \in \mathbb{N}_0$ with $a \geq b$ we have*

$$\begin{aligned} 2(a-b+1)(2n-1+a+b)\omega(H)\phi_{V_{a,b}} &= (a-b+2)(2n-1+a)\phi_{V_{a+1,b}} \\ &\quad + b(a-b+2)\phi_{V_{a,b-1}} \\ &\quad + (a-b)(2n-2+b)\phi_{V_{a,b+1}} \\ &\quad + (a-b)(a+1)\phi_{V_{a-1,b}}. \end{aligned}$$

*Proof.* Write $\phi_{V_{a,b}} = \frac{1}{a-b+1}\chi_q(t)h_{a,b}(\xi)$ such that $\chi_q(t) = \frac{\sin((q+1)t)}{\sin(t)}$. In the angular coordinates above we have

$$\omega(H) = \mathrm{Re}(w_1) = \cos(\xi)\cos(t)$$

as a function in $C^\infty(\mathbb{S}^{4n-1})$. Note that $2\cos(t)\chi_q(t) = \chi_{q+1}(t) + \chi_{q-1}(t)$. Therefore,

$$\begin{aligned} \omega(H)\phi_{V_{a,b}} &= \cos(\xi)\cos(t)\chi_q(t)h_{a,b}(\xi) \\ &= \frac{\cos(\xi)h_{a,b}(\xi)}{2}\chi_{q+1}(t) + \frac{\cos(\xi)h_{a,b}(\xi)}{2}\chi_{q-1}(t). \end{aligned} \tag{A.3.12}$$

[1] There is a sign error in [JW77, Theorem 3.1(4)]; solving the differential equation in [JW77, p.147] actually gives $\frac{\sin((q+1)t)}{\sin(t)} \cos^p(\xi) F\left(\frac{-p+q}{2}, -\frac{p+q+2}{2}, 2(n-1), -\tan^2(\xi)\right)$.

Lemma A.0.2.(iii) implies

$$\cos(\xi)h_{a,b}(\xi) = \frac{2n-2+b}{2n+a+b-1}h_{a,b+1}(\xi) + \frac{a+1}{2n+a+b-1}h_{a-1,b}(\xi) \tag{A.3.13}$$

and Lemma A.0.2.(ii) implies that

$$\cos(\xi)h_{a,b}(\xi) = \frac{2n-1+a}{2n+a+b-1}h_{a+1,b}(\xi) + \frac{b}{2n+a+b-1}h_{a,b-1}(\xi). \tag{A.3.14}$$

Inserting Equation (A.3.13) and (A.3.14) into Equation (A.3.12) proves the result. □

**Remark A.3.2.** As in Remark A.1.3, Lemma A.3.1 determines the scalars $\lambda(V_{a,b}, V)$ for each $V \in \hat{K}_M$ with $V \leftrightarrow V_{a,b}$.

To decompose the relevant tensor products we use Proposition 7.3.2. By Equation (A.3.8) we infer that the non-compact roots are given by

$$\Delta_n = \{\pm e_i \pm e_{n+1} \colon 1 \leq i \leq n\}.$$

The following remark ensures that each representation $Y_{\tau,\beta},\ \beta \in S$, in Proposition 7.3.2 actually occurs.

**Remark A.3.3.** Using the Weyl dimension formula we see that the representation $W_{\xi_1,\xi_2,\xi_3}$ with highest weight $\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_{n+1}$ has dimension

$$\dim W_{\xi_1,\xi_2,\xi_3} = \frac{\xi_1+\xi_2+2n-1}{(2n-1)(2n-2)}(\xi_1-\xi_2+1)(\xi_3+1)\binom{\xi_1+2n-2}{2n-3}\binom{\xi_2+2n-3}{2n-3}$$

and the representation $W^1_{\xi_1,\xi_2,\xi_3}$ with highest weight $\xi_1 e_1+\xi_2 e_2+e_3+\xi_3 e_{n+1}$ has dimension

$$\begin{aligned}\dim W^1_{\xi_1,\xi_2,\xi_3} = &\binom{\xi_1+2n-1}{2n-3}\binom{\xi_2+2n-2}{2n-1}\frac{(\xi_1+\xi_2+2n-1)(2n-4)(\xi_1-\xi_2+1)}{2(\xi_1+2n-2)(\xi_2+2n-3)}\cdot \\ &\cdot\frac{(\xi_1+1)(\xi_3+1)}{\xi_2+1}.\end{aligned}$$

Using these dimension formulas we get (note that $\mathfrak{p} \cong \mathbb{H}^n_{\mathbb{C}} \cong V_{1,0}$)

$$\sum_{\beta\in S\subseteq\Delta_n} \dim V_{a,b,\beta} = \dim\mathfrak{p}\cdot\dim V_{a,b} = 4n\cdot\dim V_{a,b},$$

so that $m(\beta) = 1$ if and only if the corresponding formula for the dimension of $V_{a,b,\beta}$ is not zero. Alternatively, the algorithm we used in the case of $\mathrm{SO}_0(n,1)$ can be applied to verify this result.

We can also compute the roots of $\mathfrak{p}$ directly. For this we use the following

**Definition A.3.4** ([Bou05, Chapter VIII, § 7, no. 2])**.** A subset $A$ of the weight lattice is called $\Delta_{\mathfrak{k}}$-*saturated* if for all $\lambda \in A$ and all $\alpha \in \Delta_{\mathfrak{k}}$ we have $\lambda - t\alpha \in A$ for all integers $t \in \mathbb{Z} \cap [0, \lambda(H_\alpha)]$, where $H_\alpha$ denotes the unique element of the commutator $[\mathfrak{g}_0^\alpha, \mathfrak{g}_0^{-\alpha}]$ with $\alpha(H_\alpha) = 2$.

**Lemma A.3.5.** *Every weight of* $\mathfrak{p}$ *has multiplicity one. There are* $4n$ *weights, given by*

$$\{\pm e_i \pm e_{n+1} \colon 1 \leq i \leq n\}.$$

*Proof.* Recall from Section B.4.2 that the highest weight of $\mathfrak{p} \cong V_{1,0}$ is given by $e_1 + e_{n+1}$. Following [Bou05, Chapter VIII, §7, no. 2, Proposition 5(i)], the weights $\mathscr{H}$ of $\mathfrak{p}$ are the smallest $\Delta_{\mathfrak{k}}$-saturated subset of the weight lattice containing $e_1 + e_{n+1}$.

We use the realization of $\mathfrak{k} \cong \mathfrak{sp}(n,\mathbb{C}) \times \mathfrak{sp}(1,\mathbb{C})$ and its Cartan subalgebra $\mathfrak{h}$ from [Kna02, Chapter II, §1, Example 3]:

$$\begin{gathered} \mathfrak{h} = \{H(h) \colon h \in \mathbb{C}^{n+1}\}, \text{ with} \\ H(h) \coloneqq \operatorname{diag}(h_1, \dots, h_n, -h_1, \dots, -h_n, h_{n+1}, -h_{n+1}), \\ e_j(H(h)) \coloneqq h_j, \quad 1 \leq j \leq n+1,\ h \in \mathbb{C}^{n+1}, \end{gathered}$$

where the signs can be chosen independently. Recall the roots $\Delta_{\mathfrak{k}}$ from Equation (A.3.7). The root normals $H_\alpha$ from Definition A.3.4 are given by

$$\begin{aligned} H_{\pm e_i \pm e_j} &= H(\pm e_i \pm e_j), \quad && 1 \leq i < j \leq n, \\ H_{\pm 2e_i} &= H(\pm e_i), && 1 \leq i \leq n+1. \end{aligned}$$

We will now determine $\mathscr{H}$. Since $e_1 + e_{n+1} \in \mathscr{H}$ and $2e_{n+1}$ is a root we have

$$\forall t \in \mathbb{Z} \cap [0, (e_1 + e_{n+1})(H_{2e_{n+1}})] \colon e_1 + e_{n+1} - t(2e_{n+1}) \in \mathscr{H} \Rightarrow e_1 - e_{n+1} \in \mathscr{H},$$

since $(e_1 + e_{n+1})(H_{2e_{n+1}}) = (e_1 + e_{n+1})(H(e_{n+1})) = 1$. Similarly we have $e_1 - e_j \in \Delta_{\mathfrak{k}}$ for $2 \leq j \leq n$ with $(e_1 \pm e_{n+1})(H_{e_1 - e_j}) = (e_1 \pm e_{n+1})(H(e_1 - e_j)) = 1$ and thus

$$e_1 \pm e_{n+1} - (e_1 - e_j) = e_j \pm e_{n+1} \in \mathscr{H}.$$

Finally, since $e_i + e_j \in \Delta_{\mathfrak{k}}$ for $i \neq j,\ 1 \leq i \leq n$, satisfies $(e_j \pm e_{n+1})(H(e_i + e_j)) = 1$ we infer that

$$e_j \pm e_{n+1} - (e_i + e_j) = -e_i \pm e_{n+1} \in \mathscr{H}.$$

Summarizing, we have

$$\forall\, 1 \leq i \leq n \colon\ \pm e_i \pm e_{n+1} \in \mathscr{H},$$

where the signs can be chosen independently. Hence we obtained $4n$ different weights. Since $\dim \mathfrak{p} = 4n$ there can be no more weights and every weight has to occur with multiplicity one. ☐

## A.4. The Case of $G = \mathrm{F}_{4(-20)}$

In this case we have $K = \mathrm{Spin}(9)$ with $\mathfrak{k}_0 = \mathfrak{so}(9)$ and $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{k} = 4$. Therefore, we may choose a Cartan subalgebra $\mathfrak{t}$ of both $\mathfrak{k}$ and $\mathfrak{g}$. The root system can be realized in $V = \mathbb{R}^4$ with the standard basis $e_1, e_2, e_3, e_4$ in the following way (see [Bou02, Plate VIII])

$$\Delta = \{\pm e_i \colon 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j \colon 1 \leq i < j \leq 4\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\} \quad \text{(A.4.15)}$$
$$\Delta_{\mathfrak{k}} = \{\pm e_i \colon 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j \colon 1 \leq i < j \leq 4\}.$$

We choose the positive system $\Delta_{\mathfrak{k}}^+ = \{e_i - e_j \colon 1 \leq i < j \leq 4\} \cup \{e_i \colon 1 \leq i \leq 4\}$ with

$$\rho_c = \frac{7}{2}e_1 + \frac{5}{2}e_2 + \frac{3}{2}e_3 + \frac{1}{2}e_4. \quad \text{(A.4.16)}$$

By (B.5.17) we have

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{15}) \cong_K \widehat{\bigoplus_{\substack{m \geq \ell \geq 0 \\ m \equiv \ell \bmod 2}}} V_{m,\ell}, \quad \text{(A.4.17)}$$

where $V_{m,\ell}$ has highest weight $\frac{m}{2}e_1 + \frac{\ell}{2}e_2 + \frac{\ell}{2}e_3 + \frac{\ell}{2}e_4$. Introducing angular coordinates on $\mathbb{R}^{16}$ as in [Joh76, p. 275] we can write (see [Joh76, Theorem 3.1])

$$\phi_{V_{m,\ell}} = \chi_\ell(\varphi) h_{m,\ell}(\xi)$$

with

$$\chi_\ell(\varphi) \coloneqq \cos(\varphi)^\ell F\left(-\frac{\ell}{2}, \frac{-\ell+1}{2}, \frac{7}{2}, -\tan(\varphi)^2\right),$$
$$h_{m,\ell}(\xi) \coloneqq \cos(\xi)^m F\left(\frac{\ell-m}{2}, \frac{-m-\ell-6}{2}, 4, -\tan(\xi)^2\right).$$

**Lemma A.4.1.** *For $m, \ell \in \mathbb{N}_0,\ \ell \leq m,\ m \equiv \ell \bmod 2$, we have*

$$(6+2\ell)(14+2m)\omega(H)\phi_{V_{m,\ell}} = (6+\ell)(14+m+\ell)\phi_{V_{m+1,\ell+1}} + (6+\ell)(m-\ell)\phi_{V_{m-1,\ell+1}}$$
$$+ \ell(8+m-\ell)\phi_{V_{m+1,\ell-1}} + \ell(m+\ell+6)\phi_{V_{m-1,\ell-1}}.$$

*Proof.* In the angular coordinates of [Joh76, p. 275] we have

$$\omega(H) = x = \cos(\xi)\cos(\varphi)$$

as a function in $C^\infty(\mathbb{S}^{15})$. We claim that

$$\cos(\varphi)\chi_\ell(\varphi) = \frac{6+\ell}{6+2\ell}\chi_{\ell+1}(\varphi) + \frac{\ell}{6+2\ell}\chi_{\ell-1}(\varphi). \quad \text{(A.4.18)}$$

Using Lemma A.0.2.(ii) and the symmetry of the hypergeometric function in the first two variables we infer that for $z \coloneqq -\tan(\varphi)^2$

$$(6+2\ell)F\left(-\frac{\ell}{2}, \frac{-\ell+1}{2}, \frac{7}{2}, z\right) = (6+\ell)F\left(\frac{-(\ell+1)}{2}, -\frac{\ell}{2}, \frac{7}{2}, z\right)$$
$$+ \frac{\ell}{\cos(\varphi)^2}F\left(\frac{-\ell+1}{2}, \frac{-\ell+2}{2}, \frac{7}{2}, z\right).$$

Multiplying both sides by $\cos(\varphi)^{\ell+1}$ now proves the claim. We now express the product $\cos(\xi)h_{m,\ell}(\xi)$ in two different forms. By Lemma A.0.2.(iii) we have

$$\cos(\xi)h_{m,\ell}(\xi) = \frac{8+m-\ell}{14+2m}h_{m+1,\ell-1}(\xi) + \frac{m+\ell-6}{14+2m}h_{m-1,\ell-1}(\xi) \tag{A.4.19}$$

and by Lemma A.0.2.(ii) similarly

$$\cos(\xi)h_{m,\ell}(\xi) = \frac{14+m+\ell}{14+2m}h_{m+1,\ell+1}(\xi) + \frac{m-\ell}{14+2m}h_{m-1,\ell+1}(\xi). \tag{A.4.20}$$

Since $\omega(H)\phi_{V_{m,\ell}} = \cos(\varphi)\chi(\varphi)\cos(\xi)h_{m,\ell}(\xi)$ we arrive at the desired result by combining Equations (A.4.18), (A.4.19) and (A.4.20). □

**Remark A.4.2.** As in Remark A.1.3, Lemma A.4.1 determines the scalars $\lambda(Y_{m,\ell}, V)$ for each $V \in \hat{K}_M$ with $V \leftrightarrow Y_{m,\ell}$.

To decompose the relevant tensor products we use Proposition 7.3.2. By Equation (A.4.15) we infer that the non-compact roots are given by

$$\Delta_n = \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

The following remark ensures that each representation $Y_{\tau,\beta},\ \beta \in S$, in Proposition 7.3.2 actually occurs.

**Remark A.4.3.** Using the Weyl dimension formula we see that the representation $W_{a_1,a_2,a_3,a_4}$ with highest weight $a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ has dimension

$$\dim W_{a_1,a_2,a_3,a_4} = \frac{1}{6!\cdot 4!\cdot 2\cdot 7\cdot 5\cdot 3}\cdot\delta_1\cdot\delta_2\cdot\delta_3\cdot\prod_{i=1}^{4}(9+2(a_i-i)),$$

with $\delta_i \coloneqq \prod_{j=i+1}^{4}(a_i + a_j + 9 - i - j)(a_i - a_j + j - i)$. Using this dimension formula we get

$$\sum_{\beta\in S\subseteq\Delta_n} \dim V_{m,\ell,\beta} = \dim\mathfrak{p}\cdot\dim V_{m,\ell} = 16\cdot\dim V_{m,\ell},$$

so that $m(\beta) = 1$ if and only if the corresponding formula for the dimension of $V_{m,\ell,\beta}$ is not zero. Alternatively, the algorithm we used in the case of $\mathrm{SO}_0(n,1)$ can be applied to verify this result.

## A.5. The scalars relating Poisson transforms

We will now compute the scalars $T_Y^V(p_{V,\mu})(e)$ from Proposition 5.3.2. Since we already computed the scalars $\lambda(V, Y)$ in each case, it suffices to determine the scalars $\nu(V, Y)$ (see Equation (7.9) for the notation).

**Proposition A.5.1** (Scalars between Poisson transforms)**.**

*i)* $G = \mathrm{SO}_0(n,1),\ n \geq 3$*: For* $\ell \in \mathbb{N}_0$*,*

$$\nu(Y_\ell, Y_{\ell+1}) = \ell\lambda(Y_\ell, Y_{\ell+1}), \qquad \nu(Y_\ell, Y_{\ell-1}) = -(2\rho(H) + \ell - 1)\lambda(Y_\ell, Y_{\ell-1}),$$

*ii)* $G = \mathrm{SU}(n,1),\ n \geq 2$*: For* $p, q \in \mathbb{N}_0$*,*

$$\begin{aligned}
\nu(Y_{p,q}, Y_{p+1,q}) &= 2p\lambda(Y_{p,q}, Y_{p+1,q}),\\
\nu(Y_{p,q}, Y_{p,q-1}) &= -2(\rho(H) + q - 1)\lambda(Y_{p,q}, Y_{p,q-1}),\\
\nu(Y_{p,q}, Y_{p,q+1}) &= 2q\lambda(Y_{p,q}, Y_{p,q+1}),\\
\nu(Y_{p,q}, Y_{p-1,q}) &= -2(\rho(H) + p - 1)\lambda(Y_{p,q}, Y_{p-1,q}),
\end{aligned}$$

*iii)* $G = \mathrm{Sp}(n,1),\ n \geq 2$*: For* $a, b \in \mathbb{N}_0$ *with* $a \geq b$*,*

$$\begin{aligned}
\nu(V_{a,b}, V_{a+1,b}) &= 2a\lambda(V_{a,b}, V_{a+1,b}),\\
\nu(V_{a,b}, V_{a,b-1}) &= -(4n - 2 + 2b)\lambda(V_{a,b}, V_{a,b-1}),\\
\nu(V_{a,b}, V_{a,b+1}) &= 2(b-1)\lambda(V_{a,b}, V_{a,b+1}),\\
\nu(V_{a,b}, V_{a-1,b}) &= -(4n + 2a)\lambda(V_{a,b}, V_{a-1,b}),
\end{aligned}$$

*iv)* $G = \mathrm{F}_{4(-20)}$*: For* $m, \ell \in \mathbb{N}_0,\ \ell \leq m,\ m \equiv \ell \bmod 2$*,*

$$\begin{aligned}
\nu(V_{m,\ell}, V_{m+1,\ell+1}) &= (m + \ell)\lambda(V_{m,\ell}, V_{m+1,\ell+1}),\\
\nu(V_{m,\ell}, V_{m-1,\ell+1}) &= -(14 + m - \ell)\lambda(V_{m,\ell}, V_{m-1,\ell+1}),\\
\nu(V_{m,\ell}, V_{m+1,\ell-1}) &= (m - \ell - 6)\lambda(V_{m,\ell}, V_{m+1,\ell-1}),\\
\nu(V_{m,\ell}, V_{m-1,\ell-1}) &= -(20 + m + \ell)\lambda(V_{m,\ell}, V_{m-1,\ell-1}).
\end{aligned}$$

*Proof.* In view of Lemma 7.3.11 it suffices to find a closed $G$-invariant subspace $U \leq H_\mu$, for some $\mu \in \mathfrak{a}^*$, such that $\mathrm{mult}_K(V,U) = 0$ and $\mathrm{mult}_K(Y,U) \neq 0$. In this case we have $\nu(V,Y) = -(\mu + \rho)(H)\lambda(V,Y)$. The following table determines the Harish-Chandra module $U_K$ of $U$ in each case (see Appendix B).

| $G$ | $V$ | $Y$ | $U_K$ | $\mu(H)$ | $(\mu+\rho)(H)$ |
|---|---|---|---|---|---|
| $\mathrm{SO}_0(n,1)$ | $Y_\ell$ | $Y_{\ell+1}$ | $W_\ell$ | $-\rho(H)-\ell$ | $-\ell$ |
| | $Y_\ell$ | $Y_{\ell-1}$ | $V_{\ell-1}$ | $\rho(H)+\ell-1$ | $n+\ell-2$ |
| $\mathrm{SU}(n,1)$ | $Y_{p,q}$ | $Y_{p+1,q}$ | $W_{p,+}$ | $-2p-\rho(H)$ | $-2p$ |
| | $Y_{p,q}$ | $Y_{p,q-1}$ | $H_{q-1,+}$ | $\rho(H)+2(q-1)$ | $2(n+q-1)$ |
| | $Y_{p,q}$ | $Y_{p,q+1}$ | $W_{q,-}$ | $-2q-\rho(H)$ | $-2q$ |
| | $Y_{p,q}$ | $Y_{p-1,q}$ | $H_{p-1,-}$ | $\rho(H)+2(p-1)$ | $2(n+p-1)$ |
| $\mathrm{Sp}(n,1)$ | $V_{a,b}$ | $V_{a+1,b}$ | $\widetilde{W}_{a+1}$ | $-(\rho(H)+2a)$ | $-2a$ |
| | $V_{a,b}$ | $V_{a,b-1}$ | $M_{b-1}$ | $\rho(H)+2b-4$ | $4n+2(b-1)$ |
| | $V_{a,b}$ | $V_{a,b+1}$ | $\widetilde{M}_b$ | $-(\rho(H)-2+2b)$ | $-2(b-1)$ |
| | $V_{a,b}$ | $V_{a-1,b}$ | $W_a$ | $\rho(H)-2+2a$ | $4n+2a$ |
| $\mathrm{F}_{4(-20)}$ | $V_{m,\ell}$ | $V_{m+1,\ell+1}$ | $\widetilde{W}_{\frac{m+\ell}{2}+3}$ | $-(\rho(H)+m+\ell)$ | $-(m+\ell)$ |
| | $V_{m,\ell}$ | $V_{m-1,\ell+1}$ | $M_{\frac{m-\ell}{2}-1}$ | $\rho(H)+m-\ell-8$ | $14+m-\ell$ |
| | $V_{m,\ell}$ | $V_{m+1,\ell-1}$ | $\widetilde{M}_{\frac{m-\ell}{2}}$ | $-(\rho(H)-6+m-\ell)$ | $6-m+\ell$ |
| | $V_{m,\ell}$ | $V_{m-1,\ell-1}$ | $W_{\frac{m+\ell}{2}+2}$ | $\rho(H)-2+m+\ell$ | $20+m+\ell$ |

□

# B. Structure theory of rank one groups

## B.1. Structure theory of $\mathrm{PSL}(2,\mathbb{R})$

We have

$$\begin{aligned}
G &:= \mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm 1\},\\
K &:= \mathrm{PSO}(2) := \mathrm{SO}(2,\mathbb{R})/\{\pm 1\},\\
A &:= \left\{\begin{pmatrix} e^t & 0\\ 0 & e^{-t}\end{pmatrix} : t\in\mathbb{R}\right\}/\{\pm 1\},\\
N &:= \left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} : x\in\mathbb{R}\right\}/\{\pm 1\}
\end{aligned}$$

and $M = Z_K(A) = \{[I]\}$ where we denote the equivalence class of $g \in \mathrm{SL}(2,\mathbb{R})$ in $G$ by $[g]$. The corresponding Lie algebras (where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ denotes the Cartan decomposition) are given by

$$\begin{aligned}
\mathfrak{g}_0 &= \{X \in \mathrm{Mat}(2,\mathbb{R})\colon\ \operatorname{tr} X = 0\},\\
\mathfrak{k}_0 &= \left\{tV\colon t\in\mathbb{R},\ V := \begin{pmatrix} 0 & \frac12\\ -\frac12 & 0\end{pmatrix}\right\},\\
\mathfrak{p}_0 &= \{X\in\mathfrak{g}_0\colon X = X^T\},\\
\mathfrak{a}_0 &= \left\{tH\colon H = \begin{pmatrix} \frac12 & 0\\ 0 & -\frac12\end{pmatrix},\ t\in\mathbb{R}\right\},\\
\mathfrak{n}_0 &= \left\{\begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix} : x\in\mathbb{R}\right\}.
\end{aligned}$$

Note that $m_\alpha = 1$ so that $\rho(H) = \frac12\alpha(H) = \frac12$.

**Lemma B.1.1** (Composition series of the spherical principal series)**.** *The spherical principal series representation $H_\mu$ associated to $\mu\in\mathfrak{a}^*$ (see Section 2.1) is reducible if and only if $\mu(H)\in\pm(\rho(H)+\mathbb{N}_0) = \pm(\frac12+\mathbb{N}_0)$ where $H = \mathrm{diag}(\frac12,-\frac12)$. Moreover, let $e_p : \mathbb{S}^1\to\mathbb{S}^1$, $e_p(z) := z^p$. Then*

*i) if $\mu(H) = \rho(H)+k,\ k\in\mathbb{N}_0$, the spaces (see Figure B.1)*

$$F_k := \bigoplus_{p=-k}^{k}\mathbb{C}e_p,\ H_{k,+} := \bigoplus_{p=-k}^{\infty}\mathbb{C}e_p \text{ and } H_{k,-} := \bigoplus_{p=-k}^{\infty}\mathbb{C}e_{-p}$$

*are the only non-trivial $(\mathfrak{g},K)$-submodules of (the Harish-Chandra module of) $H_\mu$. Furthermore, $F_k$, $H_{k,+}/F_k$ and $H_{k,-}/F_k$ are irreducible.*

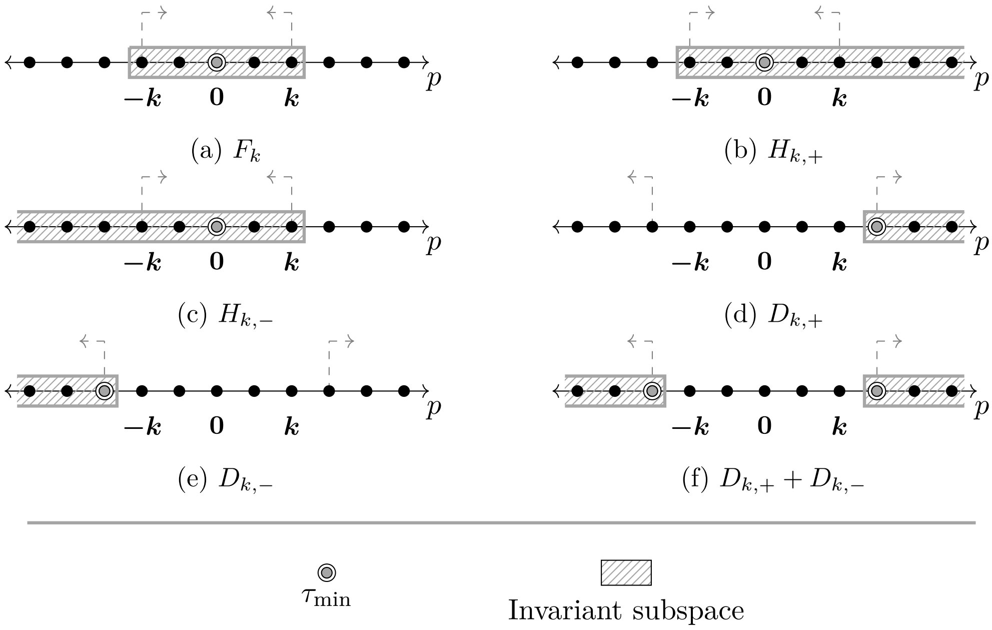



Figure B.1.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ for $\mathrm{PSL}(2,\mathbb{R})$

*ii) (dual case) if* $\mu(H) = -\rho(H) - k,\ k \in \mathbb{N}_0$, *the spaces (see Figure B.1)*

$$D_{k,+} \coloneqq \bigoplus_{p=k+1}^{\infty} \mathbb{C}e_p,\ \ D_{k,-} \coloneqq \bigoplus_{p=k+1}^{\infty} \mathbb{C}e_{-p} \text{ and } D_{k,+} \oplus D_{k,-}$$

*are the only non-trivial* $(\mathfrak{g}, K)$*-submodules of (the Harish-Chandra module of)* $H_\mu$*. Furthermore,* $D_{k,+}, D_{k,-}$ *and* $H_\mu/(D_{k,+} \oplus D_{k,-})$ *are irreducible.*

*Proof.* This follows from Equation (4.11). □

## B.2. Structure theory of $\mathrm{SO}_0(n,1),\ n \geq 3$

### B.2.1. General structure

Let $Q \coloneqq \mathrm{diag}(1, \dots, 1, -1) \in \mathrm{Mat}_{n+1}(\mathbb{R})$. Then we define

$$G \coloneqq \{g \in \mathrm{SL}_{n+1}(\mathbb{R})\colon g^T Q g = Q\}_0,$$

$$K \coloneqq \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(n) \right\} \cong \mathrm{SO}(n),$$

$$A \coloneqq \left\{ \begin{pmatrix} \cosh(t) & & \sinh(t) \\ & \mathrm{I}_{n-1} & \\ \sinh(t) & & \cosh(t) \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$N := \left\{ \begin{pmatrix} 1-\frac{\|v\|^2}{2} & v^T & \frac{\|v\|^2}{2} \\ -v & \mathrm{I}_{n-1} & v \\ -\frac{\|v\|^2}{2} & v^T & 1+\frac{\|v\|^2}{2} \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\}$$

and

$$M := Z_K(A) = \left\{ \begin{pmatrix} 1 & & \\ & B & \\ & & 1 \end{pmatrix} : B \in \mathrm{SO}(n-1) \right\} \cong \mathrm{SO}(n-1)$$

with Lie algebras (where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ denotes the Cartan decomposition)

$$\begin{aligned}
\mathfrak{g}_0 &= \{X \in \mathfrak{gl}(n+1,\mathbb{R})\colon X^T Q + QX = 0\} \\
&= \left\{ \begin{pmatrix} A & v \\ v^T & 0 \end{pmatrix} \in \mathfrak{gl}(n+1,\mathbb{R})\colon A + A^T = 0 \right\}, \\
\mathfrak{k}_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}(n+1,\mathbb{R})\colon A + A^T = 0 \right\} \cong \mathfrak{so}(n), \\
\mathfrak{p}_0 &= \left\{ X_v := \begin{pmatrix} \mathbf{0}_n & v \\ v^T & 0 \end{pmatrix} \in \mathfrak{gl}(n+1,\mathbb{R})\colon v \in \mathbb{R}^n \right\} \cong \mathbb{R}^n, \\
\mathfrak{a}_0 &= \left\{ tH = \begin{pmatrix} & & t \\ & \mathbf{0}_{n-1} & \\ t & & \end{pmatrix} \in \mathfrak{gl}(n+1,\mathbb{R})\colon t \in \mathbb{R} \right\} \cong \mathbb{R}, \\
\mathfrak{n}_0 &= \left\{ Y_v := \begin{pmatrix} 0 & v^T & 0 \\ -v & \mathbf{0}_{n-1} & v \\ 0 & v^T & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\} \cong \mathbb{R}^{n-1}, \\
\mathfrak{m}_0 &= \left\{ \begin{pmatrix} 0 & & \\ & B & \\ & & 0 \end{pmatrix} : B + B^T = 0 \right\} \cong \mathfrak{so}(n-1).
\end{aligned}$$

The restricted root spaces

$$\mathfrak{g}_0^\lambda = \{X \in \mathfrak{g}_0\colon [H,X] = \lambda(H)X\}, \quad \lambda \in \{\alpha, 2\alpha\},$$

are given by

$$\mathfrak{g}_0^\alpha = \mathfrak{n}_0,\ \mathfrak{g}_0^{2\alpha} = 0 \text{ with dimensions } m_\alpha = n-1,\ m_{2\alpha} = 0.$$

Especially we obtain

$$\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha) = \frac{n-1}{2}\alpha.$$

### B.2.2. Decomposition of $H_\mu$ into spherical harmonics

As a $K$-representation the spherical principal series representation $H_\mu$, $\mu \in \mathfrak{a}^*$, is equivalent to the left regular representation of $K$ on $L^2(K/M)$, independently of the parameter $\mu$. Letting $\operatorname{diag}(A, 1) \in K$ act on $\vartheta \in S^{n-1}$ by $A\vartheta$ defines a transitive action of $K$ on the unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$. This allows us to identify

$$K/M \cong \mathbb{S}^{n-1}, \quad kM \mapsto k.e_1, \tag{B.2.1}$$

where $e_1 := (1, 0, \dots, 0) \in \mathbb{S}^{n-1}$. In view of Lie theory the latter isomorphism is just the orbit-stabilizer theorem for the adjoint action of $K$ on $\mathfrak{p}_0$ at $H$. We obtain

$$H_\mu \cong L^2(\mathbb{S}^{n-1})$$

as $K$-representation. By the theory of spherical harmonics we therefore deduce that $H_\mu$ decomposes as the Hilbert space direct sum (denoted by $\widehat{\bigoplus}$)

$$H_\mu \cong \widehat{\bigoplus}_{\ell \in \mathbb{N}_0} Y_\ell, \tag{B.2.2}$$

where

$$Y_\ell := \{p|_{\mathbb{S}^{n-1}} : p \text{ harmonic, homogeneous polynomial of degree } \ell\}$$

denotes the space of spherical harmonics of degree $\ell$. The highest weight of $Y_\ell$ is $\ell e_1$ (see e.g. [Kna02, Chapter V, §1, Example 1, p. 277]). Note that every homogeneous polynomial $p$ of degree $\ell$ can be uniquely decomposed into

$$p = q_1 + |x|^2 q_2 \tag{B.2.3}$$

where $|x|^2 := \sum_{i=1}^n x_i^2$, $q_1$ is harmonic and homogeneous of degree $\ell$ and $q_2$ is harmonic and homogeneous of degree $\ell - 2$. We write P for the $K$-equivariant projection onto the harmonic part. By [JW77, Theorem 3.1(2)] we have

$$Y_\ell^M = \mathbb{C}\mathrm{P}(x_1^\ell). \tag{B.2.4}$$

Introducing the angular coordinates

$$x_1 = r\cos(\xi), \quad x_i = r\sin(\xi)\omega_i, \ i \geq 2,$$

where $\sum_{i=2}^n \omega_i^2 = 1, \ 0 \leq \xi \leq \pi$, we have

$$\phi_{Y_\ell} = \cos^k(\xi) F\left(-\frac{k}{2}, \frac{1-k}{2}, \frac{n-1}{2}, -\tan^2(\xi)\right).$$

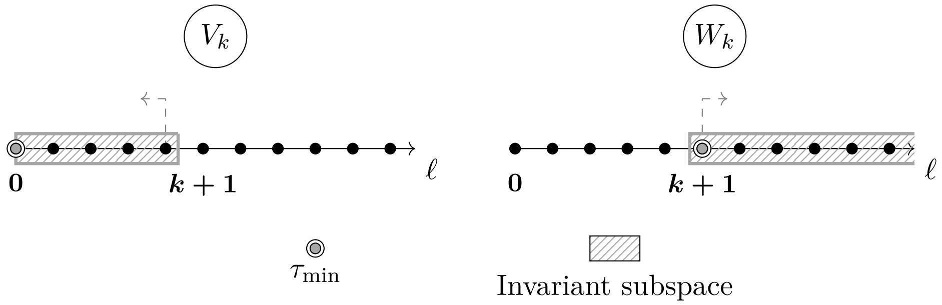



Figure B.2.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ with gray arrows indicating the $\mathfrak{g}$-action

### B.2.3. Composition series of $H_\mu$

Recall from (B.2.2) the decomposition of the spherical principal series representation $H_\mu$ into spherical harmonics:

$$H_\mu \cong L^2(K/M) \cong L^2(\mathbb{S}^{n-1}) \cong \widehat{\bigoplus}_{\ell \in \mathbb{N}_0} Y_\ell.$$

Recall the definition of $H \in \mathfrak{a}_0$ from page 9. We have $H = E_{1,n+1} + E_{n+1,1}$ with the standard $E_{i,j}$-basis of $\mathrm{Mat}_{n+1}(\mathbb{C})$.

**Lemma B.2.1** (Composition series of the spherical principal series)**.** *The spherical principal series representation $H_\mu$ associated to $\mu \in \mathfrak{a}^*$ (see Section 2.1) is reducible if and only if $\mu(H) \in \pm(\rho(H) + \mathbb{N}_0) = \pm(\frac{n-1}{2} + \mathbb{N}_0)$. Moreover, we have:*

*i) If $\mu(H) = \rho(H) + k,\ k \in \mathbb{N}_0$, the space*

$$V_k := \bigoplus_{\ell=0}^{k} Y_\ell$$

*is an irreducible $(\mathfrak{g}, K)$-submodule of (the Harish-Chandra module of) $H_\mu$. Furthermore, the quotient $H_\mu / V_k$ is irreducible.*

*ii) (dual case) If $\mu(H) = -\rho(H) - k,\ k \in \mathbb{N}_0$, the space*

$$W_k := \bigoplus_{\ell=k+1}^{\infty} Y_\ell$$

*is irreducible. Furthermore, the quotient $H_\mu / W_k$ is irreducible.*

*Proof.* See [JW77, Theorem 5.1 (2)] with $\nu = (\rho - \mu)(H) = \frac{n-1}{2} - \mu(H)$. □

### B.2.4. Tensor products and proof of Proposition 7.3.12

In this section we will make the tensor product decomposition

$$Y_m \otimes \mathfrak{p}^* = Y_{m-1} \oplus Y_{m+1} \oplus V_m$$

explicit and prove the last equality of Lemma 9.1.5 resp. Proposition 7.3.12 for $\mathrm{SO}_0(n,1)$ with $n \geq 3$. By Definition 7.3.4,

$$\mathrm{id}_{Y_{m\pm1}} = T^{Y_m}_{Y_{m\pm1}} \circ \iota^{Y_{m\pm1}}_{Y_m} : Y_{m\pm1} \to Y_m \otimes \mathfrak{p}^* \to Y_{m\pm1}.$$

Therefore,

$$Y_{m-1} \cong_K \iota^{Y_{m-1}}_{Y_m}(Y_{m-1}) = \left\{ \sum_{k=1}^{n} \mathrm{P}(x_k\varphi) \otimes \mathbf{I}(\tilde{X}_k) \mid \varphi \in Y_{m-1} \right\}, \tag{B.2.5}$$

$$Y_{m+1} \cong_K \iota^{Y_{m+1}}_{Y_m}(Y_{m+1}) = \left\{ \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\psi \otimes \mathbf{I}(\tilde{X}_k) \mid \psi \in Y_{m+1} \right\}. \tag{B.2.6}$$

On $Y := \bigoplus_{k\in\mathbb{N}_0} Y_k$ we have the Fisher inner product given by

$$\langle p, q\rangle := (\partial(p)\overline{q})(0),$$

where $\partial(p) := p(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Note that $Y_k \perp Y_m$ for $k \neq m$. We have $\langle fg, h\rangle = \langle g, \partial(f)h\rangle$ for each $f, g, h \in Y$ and especially $\langle |x|^2 g, h\rangle = \langle g, \Delta h\rangle = 0$. Moreover, $\langle f, g\rangle = \overline{\langle g, f\rangle}$. On $Y_m \otimes \mathfrak{p}^*$ we define an inner product by (identifying $\mathfrak{p}^* \cong Y_1$)

$$\langle \sum_{k=1}^{n} f_k \otimes \mathbf{I}(\tilde{X}_k), \sum_{j=1}^{n} \varphi_j \otimes \mathbf{I}(\tilde{X}_j)\rangle := \sum_{k=1}^{n}\sum_{j=1}^{n} \langle f_k, \varphi_j\rangle \langle \mathbf{I}(\tilde{X}_k), \mathbf{I}(\tilde{X}_j)\rangle = \sum_{k=1}^{n} \langle f_k, \varphi_k\rangle.$$

We obtain $\iota^{Y_{m+1}}_{Y_m}(Y_{m+1}) \perp \iota^{Y_{m-1}}_{Y_m}(Y_{m-1})$, since (for $\varphi \in Y_{m-1},\ \psi \in Y_{m+1}$)

$$\begin{aligned}\langle \sum_{k=1}^{n} \mathrm{P}(x_k\varphi) \otimes \mathbf{I}(\tilde{X}_k), \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\psi \otimes \mathbf{I}(\tilde{X}_k)\rangle &= \sum_{k=1}^{n} \langle \mathrm{P}(x_k\varphi), \frac{\partial}{\partial x_k}\psi\rangle \\ = \sum_{k=1}^{n} \langle x_k\varphi - \frac{|x|^2}{n+2m-4}, \frac{\partial}{\partial x_k}\psi\rangle &= \sum_{k=1}^{n} \langle x_k\varphi, \frac{\partial}{\partial x_k}\psi\rangle = \langle \varphi, \Delta\psi\rangle = 0,\end{aligned}$$

where we used Equation (9.3) in the second step. In order to describe $V_m$ as the orthogonal complement of $\iota^{Y_{m+1}}_{Y_m}(Y_{m+1}) \oplus \iota^{Y_{m-1}}_{Y_m}(Y_{m-1})$ in $Y_m \otimes \mathfrak{p}^*$ we use that the Fisher inner product is invariant under orthogonal transformations. We start with a preparatory lemma and prove the invariance in Proposition B.2.3.

**Lemma B.2.2.** *Let $m, n \in \mathbb{N}$ and $f_i,\ g_i \in \mathbb{R}^n$ for each $1 \leq i \leq m$. Then we have*

$$\langle f, g\rangle = \sum_{\sigma\in S_m} \prod_{i=1}^{m} \langle f_i, g_{\sigma(i)}\rangle_2$$

*for $f := \prod_{i=1}^{m} \langle x, f_i\rangle_2,\ g := \prod_{i=1}^{m} \langle x, g_i\rangle_2$, where $\langle\cdot,\cdot\rangle_2$ denotes the euclidean inner product.*

*Proof.* We first denote by $f_{i,j}$ the $j$-th component of $f_i \in \mathbb{R}^n$ and express $f$ as a linear combination of monomials by multiplying out the product

$$f(x) = \prod_{i=1}^{m} \sum_{j=1}^{n} x_j f_{i,j} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} x^\alpha \sum_{\varphi \in \mathcal{F}([m],[n],\alpha)} \prod_{i=1}^{m} f_{i,\varphi(i)}, \tag{B.2.7}$$

where $\mathcal{F}([m],[n],\alpha)$ denotes the set of all maps $\varphi\colon \{1,\ldots,m\} \to \{1,\ldots,n\}$ with $|\varphi^{-1}(k)| = \alpha_k$ for each $k \in \{1,\ldots,n\}$. Doing the same for $g$ and using that $\langle x^\alpha, x^\beta\rangle = 0$ for $\alpha \neq \beta \in \mathbb{N}^n$ and $\langle x^\alpha, x^\alpha \rangle = \alpha!$ for the Fisher inner product we obtain that

$$\langle f, g\rangle = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} \alpha! \sum_{\varphi \in \mathcal{F}([m],[n],\alpha)} \sum_{\tilde{\varphi} \in \mathcal{F}([m],[n],\alpha)} \prod_{i=1}^{m} f_{i,\varphi(i)} g_{i,\tilde{\varphi}(i)}.$$

By composing $\varphi$ as above with an element $\sigma$ of the symmetric group $S_m$, we may write $\tilde{\varphi} \circ \sigma = \varphi$. The choice of $\sigma$ is unique up to the action of the subgroup $S_{\alpha_1} \times \ldots \times S_{\alpha_n} \leq S_m$ which stabilizes $\varphi$. Thus, we obtain

$$\begin{aligned}
\langle f, g\rangle &= \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} \alpha! \sum_{\varphi \in \mathcal{F}([m],[n],\alpha)} \sum_{\sigma \in S_m/(S_{\alpha_1} \times \ldots \times S_{\alpha_n})} \prod_{i=1}^{m} f_{i,\varphi(i)} g_{i,\varphi(\sigma^{-1}(i))} \\
&= \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} \sum_{\varphi \in \mathcal{F}([m],[n],\alpha)} \sum_{\sigma \in S_m} \prod_{i=1}^{m} f_{i,\varphi(i)} g_{i,\varphi(\sigma^{-1}(i))} \\
&= \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} \sum_{\varphi \in \mathcal{F}([m],[n],\alpha)} \sum_{\sigma \in S_m} \prod_{i=1}^{m} f_{i,\varphi(i)} g_{\sigma(i),\varphi(i)},
\end{aligned}$$

since $\frac{\alpha!}{|S_{\alpha_1} \times \ldots \times S_{\alpha_n}|} = \frac{\alpha!}{\alpha!} = 1$. Factorizing the product (as in (B.2.7)) we arrive at

$$\langle f, g\rangle = \sum_{\sigma \in S_m} \prod_{i=1}^{m} \sum_{j=1}^{m} f_{i,j} g_{\sigma(i),j} = \sum_{\sigma \in S_m} \prod_{i=1}^{m} \langle f_i, g_{\sigma(i)}\rangle_2. \qquad \square$$

**Proposition B.2.3** (Invariance of Fisher inner product). *The Fisher inner product is invariant by orthogonal transformations, i.e.*

$$\langle A.f, A.g\rangle := \langle f \circ A^{-1}, g \circ A^{-1}\rangle = \langle f, g\rangle$$

*for each* $A \in \mathrm{O}(n)$ *and any two homogeneous polynomials* $f$, $g$ *in* $n$ *variables.*

*Proof.* This follows from Lemma B.2.2 by taking linear combinations of the special polynomials introduced in that lemma and using the invariance of the euclidean inner product under orthogonal transformations. $\square$

The invariance now implies that

$$V_m \cong_K (\iota_{Y_m}^{Y_{m+1}}(Y_{m+1}) \oplus \iota_{Y_m}^{Y_{m-1}}(Y_{m-1}))^\perp \leq Y_m \otimes \mathfrak{p}^*$$

and, more precisely,

$$V_m \cong_K \left\{ \sum_{k=1}^n f_k \otimes \mathbf{I}(\tilde{X}_k) \in Y_m \otimes \mathfrak{p}^* \mid \sum_{k=1}^n \frac{\partial}{\partial x_k} f_k = 0,\ \sum_{k=1}^n \mathrm{P}(x_k f_k) = 0 \right\},$$

since the inner product is non-degenerate and, for $\varphi \in Y_{m-1},\ f_k \in Y_m$, and $\psi \in Y_{m+1}$ (recall Equation (B.2.5) and (B.2.6)),

$$\begin{aligned} 0 &= \langle \sum_{k=1}^n \mathrm{P}(x_k\varphi) \otimes \mathbf{I}(\tilde{X}_k), \sum_{k=1}^n f_k \otimes \mathbf{I}(\tilde{X}_k)\rangle = \sum_{k=1}^n \langle \mathrm{P}(x_k\varphi), f_k\rangle = \sum_{k=1}^n \langle x_k\varphi, f_k\rangle \\ &= \langle \varphi, \sum_{k=1}^n \frac{\partial}{\partial x_k} f_k\rangle, \\ 0 &= \langle \sum_{k=1}^n \frac{\partial}{\partial x_k}\psi \otimes \mathbf{I}(\tilde{X}_k), \sum_{k=1}^n f_k \otimes \mathbf{I}(\tilde{X}_k)\rangle = \sum_{k=1}^n \langle \frac{\partial}{\partial x_k}\psi, f_k\rangle = \langle \psi, \sum_{k=1}^n x_k f_k\rangle. \end{aligned}$$

We will finally give another characterization of $\iota_{Y_m}^{Y_{m+1}}(Y_{m+1}) \oplus \iota_{Y_m}^{Y_{m-1}}(Y_{m-1}) \leq Y_m \otimes \mathfrak{p}^*$. An element $f \in Y_m \otimes \mathfrak{p}^*$ is contained in this set if and only if $f = \iota_{Y_m}^{Y_{m+1}}(T_{Y_{m+1}}^{Y_m}(f)) + \iota_{Y_m}^{Y_{m-1}}(T_{Y_{m-1}}^{Y_m}(f))$. For $f = \sum_{k=1}^n f_k \otimes \mathbf{I}(\tilde{X}_k)$ this gives, for each $k = 1, \ldots, n$,

$$f_k \overset{!}{=} \frac{1}{m+1}\frac{\partial}{\partial x_k}\sum_{j=1}^n \mathrm{P}(x_j f_j) + \frac{n+2m-4}{(m+n-3)(n+2m-2)}\mathrm{P}(x_k \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j),$$

which, using Equation (9.3) again (note that $\frac{\partial}{\partial x_j} f_j \in Y_{m-1}$), is seen to be equivalent to

$$m f_k \overset{!}{=} \sum_{j=1}^n x_j \frac{\partial}{\partial x_k} f_j + \frac{1}{m+n-3}\left((m-1)x_k \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j - |x|^2 \sum_{j=1}^n \frac{\partial^2}{\partial x_k \partial x_j} f_j\right). \quad \text{(B.2.8)}$$

We are now able to prove Proposition 7.3.12 for $G = \mathrm{SO}_0(n,1),\ n \geq 3$.

*Proof of 7.3.12 for $G = \mathrm{SO}_0(n,1)$.* We have to show (B.2.8) for

$$f = (\mu+\rho)(H)\phi_{Y_m} \otimes \mathbf{I}(H) - \sum_{j=2}^n \ell(k_I(X_j))\phi_{Y_m} \otimes \mathbf{I}(\tilde{X}_j),$$

where $m \in \mathbb{N}_0$ and $\ell$ denotes the derived left regular representation (see Proposition 5.3.2). We first consider the case $k = 1$ and abbreviate $a := (\mu+\rho)(H)$. Since $-\ell(k_I(X_j))$

acts as $x_1 \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_1}$ on $Y_m$ and by Euler's homogeneous function theorem we obtain

$$\begin{aligned}
\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_k} f_j &= x_1 \frac{\partial}{\partial x_1} a\phi_{Y_m} + \sum_{j=2}^{n} x_j \frac{\partial}{\partial x_1} \left( x_1 \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_1} \right) \phi_{Y_m} \\
&= a x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} + \frac{\partial}{\partial x_1} x_1 \sum_{j=2}^{n} x_j \frac{\partial}{\partial x_j} \phi_{Y_m} - \sum_{j=2}^{n} x_j^2 \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} \\
&= a x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} + \frac{\partial}{\partial x_1} x_1 \left( m\phi_{Y_m} - x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} \right) - (|x|^2 - x_1^2) \frac{\partial^2}{\partial x_1^2} \phi_{Y_m}.
\end{aligned}$$

Moreover, using $\Delta \phi_{Y_m} = 0$ in step three and $\frac{\partial}{\partial x_1} \phi_{Y_m} \in Y_{m-1}$ in step four, we infer

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j &= \frac{\partial}{\partial x_1} a\phi_{Y_m} + \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \left( x_1 \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_1} \right) \phi_{Y_m} \\
&= \frac{\partial}{\partial x_1} a\phi_{Y_m} + x_1 \sum_{j=2}^{n} \frac{\partial^2}{\partial x_j^2} \phi_{Y_m} - \sum_{j=2}^{n} \frac{\partial}{\partial x_j} x_j \frac{\partial}{\partial x_1} \phi_{Y_m} \\
&= \frac{\partial}{\partial x_1} a\phi_{Y_m} - x_1 \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} - \sum_{j=2}^{n} x_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_1} \phi_{Y_m} - (n-1) \frac{\partial}{\partial x_1} \phi_{Y_m} \\
&= \frac{\partial}{\partial x_1} a\phi_{Y_m} - (m-1) \frac{\partial}{\partial x_1} \phi_{Y_m} - (n-1) \frac{\partial}{\partial x_1} \phi_{Y_m} \\
&= (a - m - n + 2) \frac{\partial}{\partial x_1} \phi_{Y_m}. \qquad \text{(B.2.9)}
\end{aligned}$$

Therefore, the right hand side of (B.2.8) for $k = 1$ is given by

$$\begin{aligned}
& a x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} + \frac{\partial}{\partial x_1} x_1 \left( m\phi_{Y_m} - x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} \right) - (|x|^2 - x_1^2) \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} \\
&\qquad + \frac{a - m - n + 2}{m + n - 3} \left( (m-1) x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} - |x|^2 \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} \right) \\
&= a x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} + m\phi_{Y_m} + m x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} - \frac{\partial}{\partial x_1} x_1^2 \frac{\partial}{\partial x_1} \phi_{Y_m} - (|x|^2 - x_1^2) \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} \\
&\qquad + \frac{a - m - n + 2}{m + n - 3} \left( (m-1) x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} - |x|^2 \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} \right) \\
&= a x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} + m\phi_{Y_m} + m x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} - 2 x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} - |x|^2 \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} \\
&\qquad + \frac{a - m - n + 2}{m + n - 3} \left( (m-1) x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} - |x|^2 \frac{\partial^2}{\partial x_1^2} \phi_{Y_m} \right) \\
&= m\phi_{Y_m} + \frac{(a-1)(n + 2m - 4)}{m + n - 3} x_1 \frac{\partial}{\partial x_1} \phi_{Y_m} - \frac{a - 1}{m + n - 3} |x|^2 \frac{\partial^2}{\partial x_1^2} \phi_{Y_m}.
\end{aligned}$$

This equals $mf_1 = ma\phi_{Y_m}$ if and only if

$$m\phi_{Y_m} = \frac{n+2m-4}{m+n-3}x_1\frac{\partial}{\partial x_1}\phi_{Y_m} - \frac{1}{m+n-3}|x|^2\frac{\partial^2}{\partial x_1^2}\phi_{Y_m}$$
$$\Leftrightarrow m(m+n-3)\phi_{Y_m} = (n+2m-4)x_1\frac{\partial}{\partial x_1}\phi_{Y_m} - |x|^2\frac{\partial^2}{\partial x_1^2}\phi_{Y_m}. \tag{B.2.10}$$

To prove this equation we use Lemma 9.1.1 and Equation (9.3) to obtain

$$\mathrm{P}(x_1^{m+1}) = \mathrm{P}(x_1\mathrm{P}(x_1^m)) = x_1\mathrm{P}(x_1^m) - \frac{|x|^2}{n+2m-2}\frac{\partial}{\partial x_1}\mathrm{P}(x_1^m)$$

and thus

$$\frac{\partial}{\partial x_1}\mathrm{P}(x_1^{m+1}) = \mathrm{P}(x_1^m) + \frac{n+2m-4}{n+2m-2}x_1\frac{\partial}{\partial x_1}\mathrm{P}(x_1^m) - \frac{|x|^2}{n+2m-2}\frac{\partial^2}{\partial x_1^2}\mathrm{P}(x_1^m).$$

Together with Lemma 9.1.1 and 9.1.3 we infer that

$$\left(\frac{(m+1)(n+m-2)}{n+2m-2} - 1\right)\mathrm{P}(x_1^m) = \frac{n+2m-4}{n+2m-2}x_1\frac{\partial \mathrm{P}(x_1^m)}{\partial x_1} - \frac{|x|^2}{n+2m-2}\frac{\partial^2\mathrm{P}(x_1^m)}{\partial x_1^2},$$

which is equivalent to Equation (B.2.10) since $\phi_{Y_m}$ is a multiple of $\mathrm{P}(x_1^m)$.

Let us now consider the case $k \neq 1$. By the product rule (third step) and Euler's homogeneous function theorem (fourth step) we have

$$\begin{aligned}
\sum_{j=1}^n x_j\frac{\partial}{\partial x_k}f_j &= ax_1\frac{\partial}{\partial x_k}\phi_{Y_m} + \sum_{j=2}^n x_j\frac{\partial}{\partial x_k}(x_1\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_1})\phi_{Y_m}\\
&= ax_1\frac{\partial}{\partial x_k}\phi_{Y_m} + x_k\frac{\partial}{\partial x_k}(x_1\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_1})\phi_{Y_m}\\
&\qquad + \sum_{\substack{j=2\\ j\neq k}}^n x_j\frac{\partial}{\partial x_k}(x_1\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_1})\phi_{Y_m}\\
&= ax_1\frac{\partial}{\partial x_k}\phi_{Y_m} - (x_1\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_1})\phi_{Y_m}\\
&\qquad + \frac{\partial}{\partial x_k}\sum_{j=2}^n x_1x_j\frac{\partial}{\partial x_j}\phi_{Y_m} - \frac{\partial}{\partial x_k}\sum_{j=2}^n x_j^2\frac{\partial}{\partial x_1}\phi_{Y_m}\\
&= ax_1\frac{\partial}{\partial x_k}\phi_{Y_m} - (x_1\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_1})\phi_{Y_m}\\
&\qquad + \frac{\partial}{\partial x_k}x_1(m\phi_{Y_m} - x_1\frac{\partial}{\partial x_1}\phi_{Y_m}) - \frac{\partial}{\partial x_k}(|x|^2 - x_1^2)\frac{\partial}{\partial x_1}\phi_{Y_m}\\
&= (a-1+m)x_1\frac{\partial}{\partial x_k}\phi_{Y_m} + x_k\frac{\partial}{\partial x_1}\phi_{Y_m} - \frac{\partial}{\partial x_k}|x|^2\frac{\partial}{\partial x_1}\phi_{Y_m}\\
&= (a-1+m)x_1\frac{\partial}{\partial x_k}\phi_{Y_m} - x_k\frac{\partial}{\partial x_1}\phi_{Y_m} - |x|^2\frac{\partial^2}{\partial x_k\partial x_1}\phi_{Y_m}.
\end{aligned}$$

Moreover, using Equation (B.2.9) we obtain that the right hand side of (B.2.8) is given by

$$\begin{aligned}
&(a-1+m)x_1\frac{\partial}{\partial x_k}\phi_{Y_m} - x_k\frac{\partial}{\partial x_1}\phi_{Y_m} - |x|^2\frac{\partial^2}{\partial x_k\partial x_1}\phi_{Y_m}\\
&\quad + \frac{(m-1)(a-m-n+2)}{m+n-3}x_k\frac{\partial}{\partial x_1}\phi_{Y_m} - \frac{(a-m-n+2)\cdot|x|^2}{m+n-3}\frac{\partial^2}{\partial x_k\partial x_1}\phi_{Y_m}\\
&= (a-1+m)x_1\frac{\partial}{\partial x_k}\phi_{Y_m} + \left(\frac{(m-1)(a-m-n+2)}{m+n-3} - 1\right)x_k\frac{\partial}{\partial x_1}\phi_{Y_m}\\
&\quad - \frac{a-1}{m+n-3}|x|^2\frac{\partial^2}{\partial x_k\partial x_1}\phi_{Y_m}.
\end{aligned}$$

We prove that this equals the left hand side of (B.2.8) for $k \neq 1$, which is given by $mf_k = -m\ell(k_I(X_k))\phi_{Y_m} = m(x_1\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_1})\phi_{Y_m}$. For this it suffices to prove

**Lemma B.2.4.** *We have*

$$|x|^2\frac{\partial^2}{\partial x_k\partial x_1}\mathrm{P}(x_1^m) = (m+n-3)x_1\frac{\partial}{\partial x_k}\mathrm{P}(x_1^m) + (m-1)x_k\frac{\partial}{\partial x_1}\mathrm{P}(x_1^m).$$

Indeed, using the lemma in the first step yields ($\phi_{Y_m}$ is a multiple of $\mathrm{P}(x_1^m)$)

$$\begin{aligned}
&(a-1+m)x_1\frac{\partial}{\partial x_k}\phi_{Y_m} + \left(\frac{(m-1)(a-m-n+2)}{m+n-3} - 1\right)x_k\frac{\partial}{\partial x_1}\phi_{Y_m}\\
&\quad - \frac{a-1}{m+n-3}|x|^2\frac{\partial^2}{\partial x_k\partial x_1}\phi_{Y_m}\\
&= (a-1+m)x_1\frac{\partial}{\partial x_k}\phi_{Y_m} + \left(\frac{(m-1)(a-m-n+2)}{m+n-3} - 1\right)x_k\frac{\partial}{\partial x_1}\phi_{Y_m}\\
&\quad - \frac{a-1}{m+n-3}((m+n-3)x_1\frac{\partial}{\partial x_k}\phi_{Y_m} + (m-1)x_k\frac{\partial}{\partial x_1}\phi_{Y_m})\\
&= (a-1+m-(a-1))x_1\frac{\partial}{\partial x_k}\phi_{Y_m} +\\
&\qquad \left(\frac{(m-1)(a-m-n+2)}{m+n-3} - 1 - \frac{(a-1)(m-1)}{m+n-3}\right)x_k\frac{\partial}{\partial x_1}\phi_{Y_m}\\
&= m(x_1\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_1})\phi_{Y_m}.
\end{aligned}$$

It remains to prove the lemma. Using the $K$-equivariance of P (first step), Lemma 9.1.1 (fourth step) and Equation (9.3) (fifth step) we obtain

$$\begin{aligned}
(x_1\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_1})\mathrm{P}(x_1^m) &= \mathrm{P}((x_1\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_1})x_1^m) = -\mathrm{P}(mx_kx_1^{m-1})\\
&= -m\mathrm{P}(x_kx_1^{m-1}) = -m\mathrm{P}(x_k\mathrm{P}(x_1^{m-1}))\\
&= m(\frac{|x|^2}{n+2m-4}\frac{\partial}{\partial x_k}\mathrm{P}(x_1^{m-1}) - x_k\mathrm{P}(x_1^{m-1})).
\end{aligned}$$

Since $\mathrm{P}(x_1^{m-1}) = \frac{n+2m-4}{m(n+m-3)} \frac{\partial}{\partial x_1} \mathrm{P}(x_1^m)$ by Lemma 9.1.4.ii) this implies

$$\begin{aligned}
&(x_1 \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_1})\mathrm{P}(x_1^m) = \frac{|x|^2}{n+m-3} \frac{\partial^2}{\partial x_k \partial x_1} \mathrm{P}(x_1^m) - x_k \frac{n+2m-4}{n+m-3} \frac{\partial}{\partial x_1} \mathrm{P}(x_1^m) \\
\Leftrightarrow\ &(n+m-3)(x_1 \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_1})\mathrm{P}(x_1^m) = \\
&\qquad |x|^2 \frac{\partial^2}{\partial x_k \partial x_1} \mathrm{P}(x_1^m) - x_k(n+2m-4) \frac{\partial}{\partial x_1} \mathrm{P}(x_1^m) \\
\Leftrightarrow\ &|x|^2 \frac{\partial^2 \mathrm{P}(x_1^m)}{\partial x_k \partial x_1} = (n+m-3)(x_1 \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_1})\mathrm{P}(x_1^m) + x_k(n+2m-4) \frac{\partial}{\partial x_1} \mathrm{P}(x_1^m) \\
&\qquad = (n+m-3)x_1 \frac{\partial}{\partial x_k} \mathrm{P}(x_1^m) + (m-1)x_k \frac{\partial}{\partial x_1} \mathrm{P}(x_1^m).
\end{aligned}$$

This finishes the proof. □

## B.3. Structure theory of $\mathrm{SU}(n,1)$, $n \neq 1$

### B.3.1. General structure

With $Q = \mathrm{diag}(1, \dots, 1, -1) \in \mathrm{Mat}_{n+1}(\mathbb{R})$ we define

$$\begin{aligned}
G &:= \{g \in \mathrm{SL}_{n+1}(\mathbb{C}) \colon g^* Q g = Q\}, \\
K &:= \{\mathrm{diag}(A, \lambda) \in \mathrm{SL}_{n+1}(\mathbb{C}) \colon A \in \mathrm{U}(n),\ |\lambda| = 1\} \cong \mathrm{U}(n), \\
A &:= \left\{ \begin{pmatrix} \cosh(t) & & \sinh(t) \\ & \mathrm{I}_{n-1} & \\ \sinh(t) & & \cosh(t) \end{pmatrix} : t \in \mathbb{R} \right\}, \\
N &:= \langle \exp(X) \colon X \in \mathfrak{n}_0 \rangle, \\
M &:= Z_K(A) = \{\mathrm{diag}(b, B, b) \in \mathrm{SL}_{n+1}(\mathbb{C}) \colon B \in \mathrm{U}(n-1)\}
\end{aligned}$$

with Lie algebras (where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ denotes the Cartan decomposition)

$$\begin{aligned}
\mathfrak{g}_0 &= \left\{ X = \begin{pmatrix} A & v \\ v^* & w \end{pmatrix} \in \mathfrak{gl}(n+1, \mathbb{C}) \colon A + A^* = 0,\ w + \overline{w} = 0,\ \mathrm{tr}\, X = 0 \right\}, \\
\mathfrak{k}_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & w \end{pmatrix} \in \mathfrak{g}_0 \right\} \cong \mathfrak{u}(n) \text{ via } \begin{pmatrix} A & 0 \\ 0 & w \end{pmatrix} \mapsto A, \\
\mathfrak{p}_0 &= \left\{ X_v := \begin{pmatrix} \mathbf{0}_n & v \\ v^* & 0 \end{pmatrix} \in \mathfrak{gl}(n+1, \mathbb{C}) \colon v \in \mathbb{C}^n \right\} \cong \mathbb{C}^n, \\
\mathfrak{a}_0 &= \left\{ tH = \begin{pmatrix} & & t \\ & \mathbf{0}_{n-1} & \\ t & & \end{pmatrix} \in \mathfrak{gl}(n+1, \mathbb{R}) \colon t \in \mathbb{R} \right\} \cong \mathbb{R}, \\
\mathfrak{n}_0 &:= \mathfrak{g}_0^{\alpha} \oplus \mathfrak{g}_0^{2\alpha},
\end{aligned}$$

$$\mathfrak{g}_0^{\alpha} = \left\{ \begin{pmatrix} 0 & v^* & 0 \\ -v & \mathbf{0}_{n-1} & v \\ 0 & v^* & 0 \end{pmatrix} : v \in \mathbb{C}^{n-1} \right\} \cong \mathbb{C}^{n-1},$$
$$\mathfrak{g}_0^{2\alpha} = \left\{ \begin{pmatrix} w & 0 & -w \\ 0 & \mathbf{0}_{n-1} & 0 \\ w & 0 & -w \end{pmatrix} : w \in i\mathbb{R} \right\} \cong \mathbb{R},$$
$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} w & & \\ & B & \\ & & w \end{pmatrix} : B + B^* = 0, 2w + \operatorname{tr} B = 0 \right\} \cong \mathfrak{u}(n-1).$$

The dimensions of $\mathfrak{g}_0^{\alpha}$ resp. $\mathfrak{g}_0^{2\alpha}$ are given by

$$m_\alpha = 2n-2, \quad m_{2\alpha} = 1.$$

Thus, $\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha) = n\alpha$.

### B.3.2. Decomposition of $H_\mu$ into spherical harmonics

As in the orthogonal case we have

$$H_\mu \cong L^2(K/M)$$

as $K$-representation, where $L^2(K/M)$ carries the left regular representation. Consider the real $(2n-1)$-sphere

$$\left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 = 1 \right\} \cong \mathbb{S}^{2n-1}$$

in $\mathbb{C}^n$ embedded by the isomorphism

$$\mathbb{C}^n \cong \mathbb{R}^{2n}, \ (z_1, \dots, z_n) \mapsto (\operatorname{Re}(z_1), \operatorname{Im}(z_1), \dots, \operatorname{Re}(z_n), \operatorname{Im}(z_n)).$$

We also write $\mathbb{S}^{2n-1}$ for this sphere and define a transitive $K$-action on it by

$$k.\vartheta := \frac{1}{\lambda} A\vartheta, \ k = \operatorname{diag}(A, \lambda) \in K, \ \vartheta \in \mathbb{S}^{2n-1} \subseteq \mathbb{C}^n. \tag{B.3.11}$$

The stabilizer at $e_1 = (1, 0, \dots, 0) \in \mathbb{S}^{2n-1}$ is given by $M$. Hence,

$$K/M \cong \mathbb{S}^{2n-1}, \quad kM \mapsto k.e_1.$$

We can decompose $H_\mu$ as the Hilbert space direct sum (cf. [JW77, Theorem 3.1 (3)])

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{2n-1}) \cong_K \widehat{\bigoplus}_{k\in\mathbb{N}_0} Y_k \cong_K \widehat{\bigoplus}_{k\in\mathbb{N}_0} \bigoplus_{\substack{p,q\in\mathbb{N}_0 \\ p+q=k}} Y_{p,q}, \tag{B.3.12}$$

where

$$Y_{p,q} \coloneqq \{f \in Y_{p+q} \colon f(\alpha z) = \alpha^p \overline{\alpha}^q f(z) \ \forall \alpha \in \mathbb{C}, |\alpha| = 1, z \in \mathbb{S}^{2n-1}\} \tag{B.3.13}$$

with $f(z) \coloneqq f(\mathrm{Re}(z_1), \mathrm{Im}(z_1), \dots, \mathrm{Re}(z_n), \mathrm{Im}(z_n))$. Moreover, again by [JW77, Theorem 3.1(3)], each $Y_{p,q}$ is irreducible and

$$Y_{p,q}^M = \mathbb{C}\mathrm{P}(z_1^p \overline{z_1}^q). \tag{B.3.14}$$

For the convenience of the reader we give a detailed proof for the highest weights.

**Lemma B.3.1** (cf. [Kna02, Chapter V, §1, Example 1])**.** *The highest weight of the irreducible $K$-representation $Y_{p,q}$ from Equation B.3.13 with respect to $\mathfrak{t}$ is given by $qe_1 - pe_n + (p-q)e_{n+1}$.*

*Proof.* Let $f \in Y_{p,q}$. By definition, $f$ has the following two properties:

i) $f \in Y_{p+q}$, i.e. $f$ is a harmonic, homogeneous, complex-valued polynomial of degree $p+q$ in $2n$ real variables $x_1, \dots, x_{2n}$,

ii) defining $f(z) \coloneqq f(\mathrm{Re}(z_1), \mathrm{Im}(z_1), \dots, \mathrm{Re}(z_n), \mathrm{Im}(z_n))$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ it holds that $f(\alpha z) = \alpha^p \overline{\alpha}^q f(z)$ for every $\alpha \in \mathbb{C}$.

Defining $z_j \coloneqq x_{2j-1} + ix_{2j},\ j = 1, \dots, n$, we may consider $f$ as a polynomial in the variables $z_1, \dots, z_n, \overline{z_1}, \dots, \overline{z_n}$ which is still homogeneous of degree $p+q$. By the second property, $f$ is a linear combination of monomials of the form

$$z_1^{k_1} \cdot \ldots \cdot z_n^{k_n} \overline{z_1}^{\ell_1} \cdot \ldots \cdot \overline{z_n}^{\ell_n} \text{ with } p = \sum_{j=1}^n k_j \text{ and } q = \sum_{j=1}^n \ell_j.$$

By [Kna02, Chapter V, §1, Example 1] these monomials are weight vectors of weight $\sum_{j=1}^n (\ell_j - k_j) e_j$ with respect to $(\mathfrak{su}(n)_{\mathbb{C}}, \mathfrak{h})$. This expression is maximal with respect to $\Delta^+$ for $\ell_1 = q,\ k_n = p$ and $\ell_j = k_j = 0$ otherwise (so that the highest weight is $qe_1 - pe_n$ with respect to $(\mathfrak{su}(n)_{\mathbb{C}}, \mathfrak{h})$). The corresponding weight vector is given by the monomial

$$h(z) \coloneqq z_n^p \overline{z_1}^q = (x_{2n-1} + ix_{2n})^p (x_1 - ix_2)^q.$$

We finally check that $h \in Y_{p,q}$:

i) $h \in Y_{p,q}$: Every monomial occurring in the first factor of $h(z)$ is of degree $p$ and every monomial occurring in the second factor is of degree $q$. Thus, $f$ is a homogeneous polynomial of degree $p+q$. It is also harmonic because

$$\begin{aligned}(\Delta h)(x_1, \dots, x_{2n}) &= \left( \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_{2n-1}^2} + \frac{\partial^2}{\partial x_{2n}^2} \right) h \right) (x_1, \dots, x_{2n}) \\ &= q(q-1)(x_{2n-1} + ix_{2n})^p (x_1 - ix_2)^{q-2} \\ &\quad - q(q-1)(x_{2n-1} + ix_{2n})^p (x_1 - ix_2)^{q-2} \\ &\quad + p(p-1)(x_{2n-1} + ix_{2n})^{p-2} (x_1 - ix_2)^q \\ &\quad - p(p-1)(x_{2n-1} + ix_{2n})^{p-2} (x_1 - ix_2)^q \\ &= 0.\end{aligned}$$

ii) $h(\alpha z) = (\alpha z_n)^p \overline{\alpha z_1}^q = \alpha^p \overline{\alpha}^q h(z)$.

Calculating the action of $k \coloneqq \mathrm{diag}(it_1, \dots, it_n, -i\sum t_i)$ on $h$ as in [Kna02, Chapter V, §1, Example 1] using Equation (B.3.11) we obtain

$$\begin{aligned}(k.h)(\vartheta) &= \frac{d}{dr}\Big|_{r=0} f(e^{-it_1 r - i\sum t_i r}\vartheta_1, \dots, e^{-it_n r - i\sum t_i r}\vartheta_n) \\ &= q(it_1) - p(it_n) + (p-q)(-i\sum t_i)\end{aligned}$$

so that the highest weight of $Y_{p,q}$ with respect to $(\mathfrak{k}, \mathfrak{t})$ is given by $qe_1 - pe_n + (p-q)e_{n+1}$. □

### B.3.3. Composition series of $H_\mu$

Recall from (B.3.12) the decomposition of the spherical principal series representation $H_\mu$ into spherical harmonics:

$$H_\mu \cong L^2(K/M) \cong L^2(\mathbb{S}^{2n-1}) \cong \widehat{\bigoplus}_{k\in\mathbb{N}_0} \bigoplus_{\substack{p,q\in\mathbb{N}_0\\ p+q=k}} Y_{p,q}.$$

We have $H = E_{1,n+1} + E_{n+1,1} \in \mathfrak{a}$ with the standard $E_{i,j}$-basis of $\mathrm{Mat}_{n+1}(\mathbb{C})$.

**Lemma B.3.2** (Composition series of the spherical principal series)**.** *The spherical principal series representation $H_\mu$ associated to $\mu \in \mathfrak{a}^*$ (see Section 2.1) is reducible if and only if $\mu(H) \in \pm(\rho(H) + 2\mathbb{N}_0) = \pm(n + 2\mathbb{N}_0)$. Moreover we have*

*i) If $\mu(H) = \rho(H) + 2k,\ k \in \mathbb{N}_0$, the spaces (see Figure B.3)*

$$F_k \coloneqq \bigoplus_{p,q=0}^{k} Y_{p,q},\ \ H_{k,+} \coloneqq \bigoplus_{p=0}^{\infty}\bigoplus_{q=0}^{k} Y_{p,q},\ \ H_{k,-} \coloneqq \bigoplus_{p=0}^{k}\bigoplus_{q=0}^{\infty} Y_{p,q},\ \ H_{k,+} + H_{k,-}$$

*are the only non-trivial $(\mathfrak{g}, K)$-submodules of (the Harish-Chandra module of) $H_\mu$. Furthermore, $F_k$, $H_{k,+}/F_k, H_{k,-}/F_k$ and $H_\mu/(H_{k,+} + H_{k,-})$ are irreducible.*

*ii) (dual case) If $\mu(H) = -\rho(H) - 2k,\ k \in \mathbb{N}_0$, the spaces (see Figure B.4)*

$$I_k \coloneqq \bigoplus_{p,q=k+1}^{\infty} Y_{p,q},\ \ W_{k,+} \coloneqq \bigoplus_{p=k+1}^{\infty}\bigoplus_{q=0}^{\infty} Y_{p,q},\ \ W_{k,-} \coloneqq \bigoplus_{p=0}^{\infty}\bigoplus_{q=k+1}^{\infty} Y_{p,q},\ \ W_{k,+} + W_{k,-}$$

*are the only non-trivial $(\mathfrak{g}, K)$-submodules of (the Harish-Chandra module of) $H_\mu$. Furthermore, $I_k$, $W_{k,+}/I_k, W_{k,-}/I_k$ and $H_\mu/(W_{k,+} + W_{k,-})$ are irreducible.*

*Proof.* See [JW77, Theorem 5.1 (3)] with $\nu = (\rho - \mu)(H) = n - \mu(H)$. □

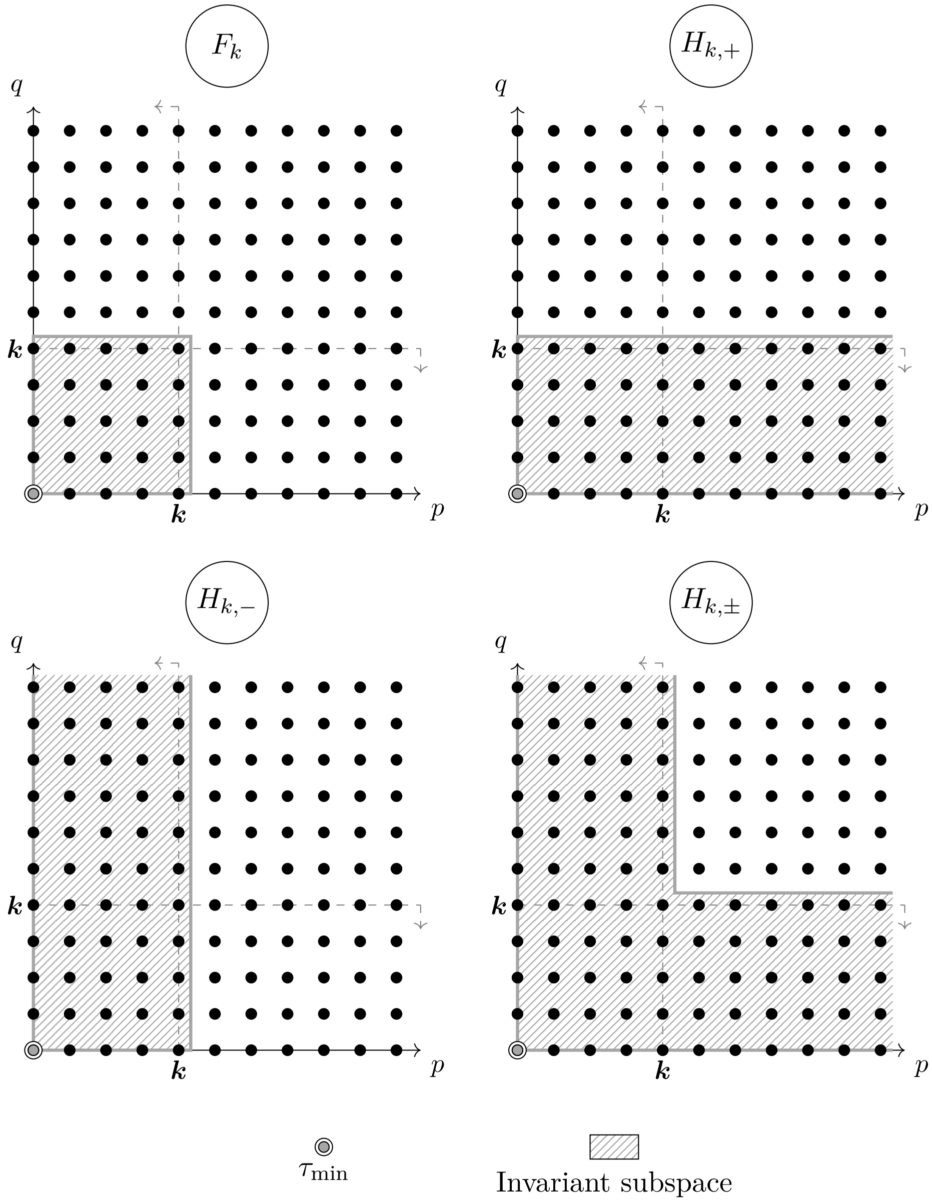



Figure B.3.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ in the case $\mu(H) = \rho(H) + 2k,\ k \in \mathbb{N}_0$, with $H_{k,\pm} := H_{k,+} + H_{k,-}$

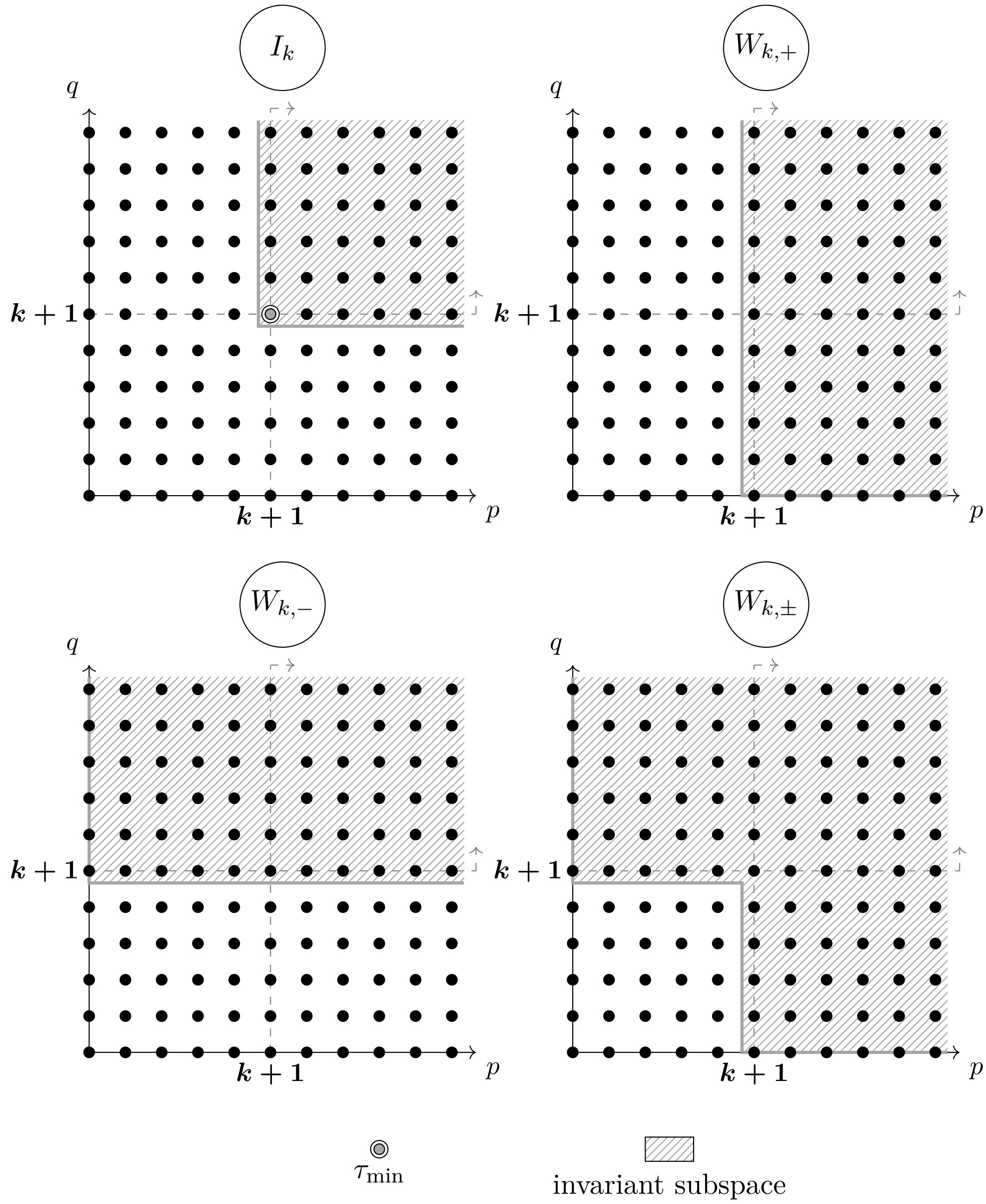



Figure B.4.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ in the case $\mu(H) = -(\rho(H) + 2k)$, $k \in \mathbb{N}_0$, with $W_{k,\pm} \coloneqq W_{k,+} + W_{k,-}$

## B.4. Structure theory of $\mathrm{Sp}(n,1)$, $n \neq 1$

### B.4.1. General structure

Let $\mathbb{H}$ denote the quaternions and recall $Q = \mathrm{diag}(1,\dots,1,-1) \in \mathrm{Mat}_{n+1}(\mathbb{R})$. If $g$ is a matrix over $\mathbb{H}$ we define $g^* := \overline{g}^T$ where the bar denotes the componentwise conjugation in $\mathbb{H}$. Then let

$$\begin{aligned}
G &:= \{g \in \mathrm{GL}_{n+1}(\mathbb{H})\colon g^*Qg = Q\},\\
K &:= \{\mathrm{diag}(A,\lambda) \in \mathrm{GL}_{n+1}(\mathbb{H})\colon AA^* = \mathrm{I}_n,\ |\lambda| = 1\} \cong \mathrm{Sp}(n)\times\mathrm{Sp}(1),\\
A &:= \left\{\begin{pmatrix}\cosh(t) & & \sinh(t)\\ & \mathrm{I}_{n-1} & \\ \sinh(t) & & \cosh(t)\end{pmatrix} : t\in\mathbb{R}\right\},\\
N &:= \langle \exp(X)\colon X\in\mathfrak{n}_0\rangle,\\
M &:= Z_K(A) = \{\mathrm{diag}(b,B,b)\in\mathrm{GL}_{n+1}(\mathbb{H})\colon B\in\mathrm{Sp}(n-1),\ |b|=1\},
\end{aligned}$$

where the symplectic group $\mathrm{Sp}(m)$, $m\in\mathbb{N}$, is defined by

$$\mathrm{Sp}(m) := \{A\in\mathrm{GL}_m(\mathbb{H})\colon AA^* = \mathrm{I}_m\}.$$

The corresponding Lie algebras (where $\mathfrak{g}_0 = \mathfrak{k}_0\oplus\mathfrak{p}_0$ denotes the Cartan decomposition) are given by

$$\begin{aligned}
\mathfrak{g}_0 &= \left\{X = \begin{pmatrix}A & v\\ v^* & w\end{pmatrix}\in\mathfrak{gl}(n+1,\mathbb{H})\colon A + A^* = 0,\ w+\overline{w} = 0\right\},\\
\mathfrak{k}_0 &= \left\{\begin{pmatrix}A & 0\\ 0 & w\end{pmatrix}\in\mathfrak{g}_0\right\}\cong\mathfrak{sp}(n)\oplus\mathfrak{sp}(1),\\
\mathfrak{p}_0 &= \left\{X_v := \begin{pmatrix}\mathbf{0}_n & v\\ v^* & 0\end{pmatrix}\in\mathfrak{gl}(n+1,\mathbb{H})\colon v\in\mathbb{H}^n\right\}\cong\mathbb{H}^n,\\
\mathfrak{a}_0 &= \left\{tH = \begin{pmatrix} & & t\\ & \mathbf{0}_{n-1} & \\ t & & \end{pmatrix}\in\mathfrak{gl}(n+1,\mathbb{R})\colon t\in\mathbb{R}\right\}\cong\mathbb{R},\\
\mathfrak{n}_0 &:= \mathfrak{g}_0^\alpha\oplus\mathfrak{g}_0^{2\alpha},\\
\mathfrak{g}_0^\alpha &= \left\{\begin{pmatrix}0 & v^* & 0\\ -v & \mathbf{0}_{n-1} & v\\ 0 & v^* & 0\end{pmatrix} : v\in\mathbb{H}^{n-1}\right\}\cong\mathbb{H}^{n-1},\\
\mathfrak{g}_0^{2\alpha} &= \left\{\begin{pmatrix}w & 0 & -w\\ 0 & \mathbf{0}_{n-1} & 0\\ w & 0 & -w\end{pmatrix} : w+\overline{w} = 0\right\}\cong\mathbb{R}^3,\\
\mathfrak{m}_0 &= \left\{\begin{pmatrix}w & & \\ & B & \\ & & w\end{pmatrix} : B+B^* = 0, w+\overline{w} = 0\right\}\cong\mathfrak{sp}(n-1)\oplus\mathfrak{sp}(1).
\end{aligned}$$

The dimensions of $\mathfrak{g}_0^\alpha$ resp. $\mathfrak{g}_0^{2\alpha}$ are given by

$$m_\alpha = 4n - 4, \quad m_{2\alpha} = 3.$$

Thus, $\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha) = (2n+1)\alpha$.

### B.4.2. Decomposition of $H_\mu$ as $K$-representation and description of $\hat{K}_M$

As before we have

$$H_\mu \cong L^2(K/M)$$

as $K$-representation, where $L^2(K/M)$ carries the left regular representation. We view $\mathbb{H}^n$ as a right $\mathbb{H}$-vector space and consider the real $(4n-1)$-sphere

$$\left\{ (w_1, \ldots, w_n) \in \mathbb{H}^n \colon \sum_{j=1}^n |w_j|^2 = 1 \right\} \cong \mathbb{S}^{4n-1}$$

in $\mathbb{H}^n$ embedded by the isomorphism

$$\mathbb{H}^n \cong \mathbb{R}^{4n},\ (w_1, \ldots, w_n) \mapsto (\Phi(w_1), \ldots, \Phi(w_n)),$$
$$\Phi : \mathbb{H} = \mathbb{C} \oplus j\mathbb{C} \cong \mathbb{R}^4,\ w = z_1 + jz_2 \mapsto (\mathrm{Re}(z_1), \mathrm{Im}(z_1), \mathrm{Re}(z_2), \mathrm{Im}(z_2))$$

where $\{1, i, j, k\} \subseteq \mathbb{H}$ is the standard $\mathbb{R}$-basis of $\mathbb{H}$. We also write $\mathbb{S}^{4n-1}$ for this sphere and define a transitive $K$-action on it by

$$k.\vartheta := A\vartheta \frac{1}{\lambda},\ k = \mathrm{diag}(A, \lambda) \in K, \vartheta \in \mathbb{S}^{4n-1} \subseteq \mathbb{H}^n.$$

The stabilizer at $e_1 = (1, 0, \ldots, 0) \in \mathbb{S}^{4n-1}$ is given by $M$ inducing the isomorphism

$$K/M \cong \mathbb{S}^{4n-1}, \quad kM \mapsto k.e_1.$$

By [Kna02, Chapter IX.8, Problem 12], $H_\mu$ decomposes as the Hilbert space direct sum

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{4n-1}) \cong_K \widehat{\bigoplus}_{a \geq b \geq 0} V_{a,b}, \tag{B.4.15}$$

where $V_{a,b}$ has highest weight $ae_1 + be_2 + (a-b)e_{n+1}$. By [JW77, Theorem 3.1] we moreover have (note that our $V_{a,b}$ corresponds to $V^{p,q}$ of [JW77] with $p := a + b$ and $q := a - b$ by [JW77, Lemma 3.3])

$$Y_k = \bigoplus_{\substack{a \geq b \geq 0 \\ a+b=k}} V_{a,b}$$

and the space $V_{a,b}$ with $k = a + b$ is given by the $K$-cyclic space

$$V_{a,b} = K \cdot \mathrm{P}(f_{a,b}) \subseteq Y_k, \tag{B.4.16}$$

where

$$f_{a,b} := \sum_{i=b}^{\lfloor k/2 \rfloor} (-1)^{i-b} \binom{k-b-i}{i-b} r_1^{k-2i} r_2^{2i}$$

with $r_1 := 2\,\mathrm{Re}(w_1)$ and $r_2^2 := |w_1|^2$ for $w = (w_1, \ldots, w_n) \in \mathbb{S}^{4n-1} \subseteq \mathbb{H}^n$.

### B.4.3. Composition series of $H_\mu$

Recall from (B.4.15) the decomposition of the spherical principal series representation $H_\mu$ as $K$-representation:

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{4n-1}) \cong_K \widehat{\bigoplus}_{a\geq b\geq 0} V_{a,b}.$$

We have $H = E_{1,n+1} + E_{n+1,1} \in \mathfrak{a}$ with the standard $E_{i,j}$-basis of $\mathrm{Mat}_{n+1}(\mathbb{H})$.

**Lemma B.4.1** (Composition series of the spherical principal series)**.** *The spherical principal series representation $H_\mu$ associated to $\mu \in \mathfrak{a}^*$ (see Section 2.1) is reducible if and only if $\mu(H) \in \pm(\rho(H) - 2 + 2\mathbb{N}_0) = \pm(2n - 1 + 2\mathbb{N}_0)$. Moreover, we have:*

*i) If $\mu(H) = \rho(H) - 2 + 2k,\ k \in \mathbb{N}$, the spaces (see Figure B.5)*

$$W_k := \bigoplus_{b\leq a\leq k-1} V_{a,b}, \qquad M_k := \bigoplus_{a\geq b\leq k} V_{a,b}$$

*are the only non-trivial $(\mathfrak{g}, K)$-submodules of (the Harish-Chandra module of) $H_\mu$. Furthermore, $W_k$, $M_k/W_k$ and $H_\mu/M_k$ are irreducible.*

*ii) (dual case) If $\mu(H) = -(\rho(H) - 2 + 2k),\ k \in \mathbb{N}$, the spaces (see Figure B.6)*

$$\widetilde{W}_k := \bigoplus_{b\leq a> k-1} V_{a,b}, \qquad \widetilde{M}_k := \bigoplus_{a\geq b> k} V_{a,b}$$

*are the only non-trivial $(\mathfrak{g}, K)$-submodules of (the Harish-Chandra module of) $H_\mu$. Furthermore, $\widetilde{M}_k$, $\widetilde{W}_k/\widetilde{M}_k$ and $H_\mu/\widetilde{W}_k$ are irreducible.*

*iii) If $\mu(H) = \rho(H) - 2$ the space (see Figure B.7a)*

$$T := \bigoplus_{a=0}^{\infty} V_{a,0}$$

*is the only non-trivial $(\mathfrak{g}, K)$-submodule of (the Harish-Chandra module of) $H_\mu$. Furthermore, $T$ and $H_\mu/T$ are irreducible.*

*iv) (dual case) If $\mu(H) = -(\rho(H) - 2)$ the space (see Figure B.7b)*

$$\widetilde{T} := \bigoplus_{a\geq b>0} V_{a,b}$$

*is the only non-trivial $(\mathfrak{g}, K)$-submodule of (the Harish-Chandra module of) $H_\mu$. Furthermore, $\widetilde{T}$ and $H_\mu/\widetilde{T}$ are irreducible.*

*Proof.* See [JW77, Theorem 5.1 (4)] with $\nu = (\rho - \mu)(H) = 2n + 1 - \mu(H)$. □

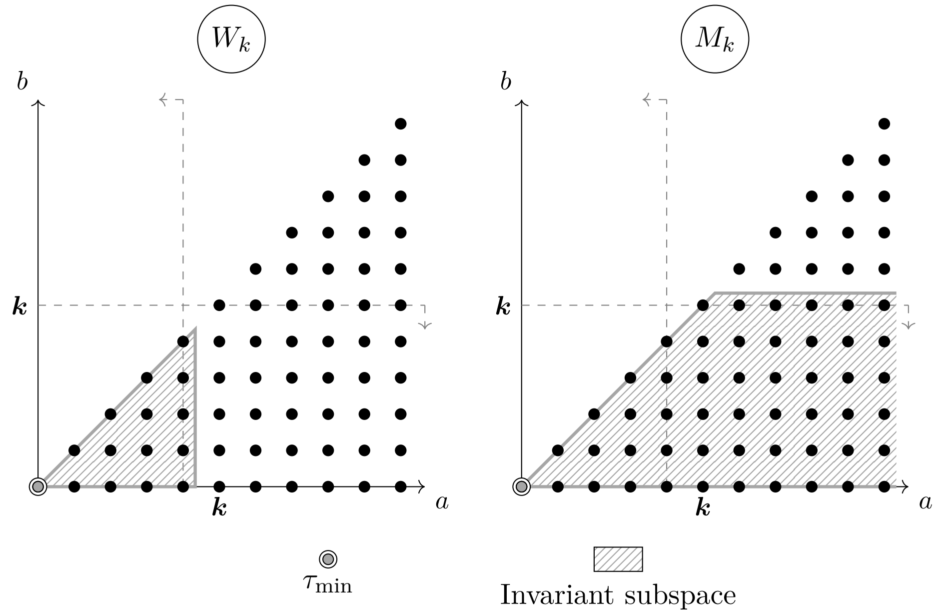



Figure B.5.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ in the case $\mu(H) = \rho(H) - 2 + 2k,\ k \in \mathbb{N}$

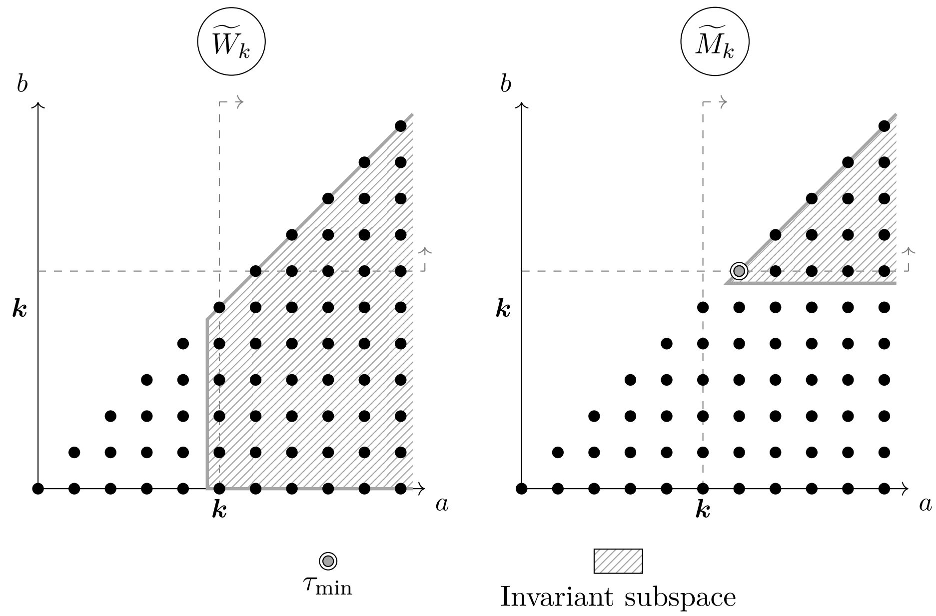
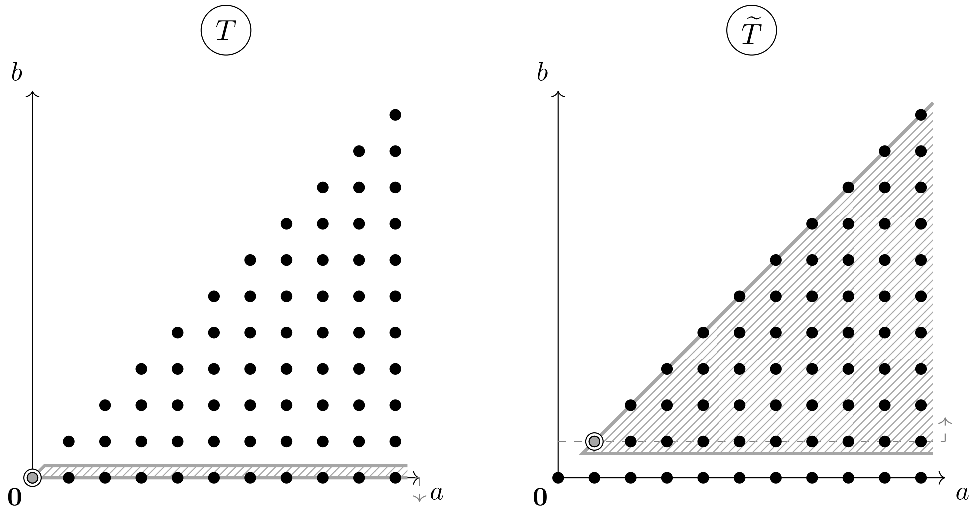



Figure B.6.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ in the case $\mu(H) = -(\rho(H) - 2 + 2k),\ k \in \mathbb{N}$




(a) $\mu(H) = \rho(H) - 2$ (b) $\mu(H) = -(\rho(H) - 2)$

Figure B.7.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ in the case $\mu(H) = \rho(H) - 2$ resp. $\mu(H) = -(\rho(H) - 2)$

## B.5. Structure theory of $\mathbf{F}_{4(-20)}$

### B.5.1. Decomposition of $H_\mu$ as $K$-representation and composition series

By [Joh76, Theorem 3.1] we see that $H_\mu$ decomposes as the Hilbert space direct sum

$$H_\mu \cong_K L^2(K/M) \cong_K L^2(\mathbb{S}^{15}) \cong_K \widehat{\bigoplus_{\substack{m \geq k \geq 0 \\ m \equiv k \bmod 2}}} V_{m,k}, \tag{B.5.17}$$

where $V_{m,k}$ is the $K$-representation with highest weight $\frac{m}{2}e_1 + \frac{k}{2}e_2 + \frac{k}{2}e_3 + \frac{k}{2}e_4$ (see [Joh76, p. 278]).

**Lemma B.5.1** (Composition series of the spherical principal series)**.** *The spherical principal series representation $H_\mu$ associated to $\mu \in \mathfrak{a}^*$ (see Section 2.1) is reducible if and only if $\mu(H) \in \pm(\rho(H) - 6 + 2\mathbb{N}_0) = \pm(5 + 2\mathbb{N}_0)$. Moreover, we have:*

*i) If $\mu(H) = \rho(H) - 6 + 2\ell$, $\ell \in \mathbb{N}_{\geq 3}$, the spaces (see Figure B.8)*

$$W_\ell := \bigoplus_{m+k \leq 2\ell - 6} V_{m,k}, \qquad M_\ell := \bigoplus_{m-k \leq 2\ell} V_{m,k}$$

*are the only non-trivial $(\mathfrak{g}, K)$-submodules of (the Harish-Chandra module of) $H_\mu$. Furthermore, $W_\ell$, $M_\ell/W_\ell$ and $H_\mu/M_\ell$ are irreducible.*

*ii) (dual case) If $\mu(H) = -(\rho(H) - 6 + 2\ell)$, $\ell \in \mathbb{N}_{\geq 3}$, the spaces (see Figure B.9)*

$$\widetilde{W}_\ell := \bigoplus_{m+k > 2\ell - 6} V_{m,k}, \qquad \widetilde{M}_\ell := \bigoplus_{m-k > 2\ell} V_{m,k}$$

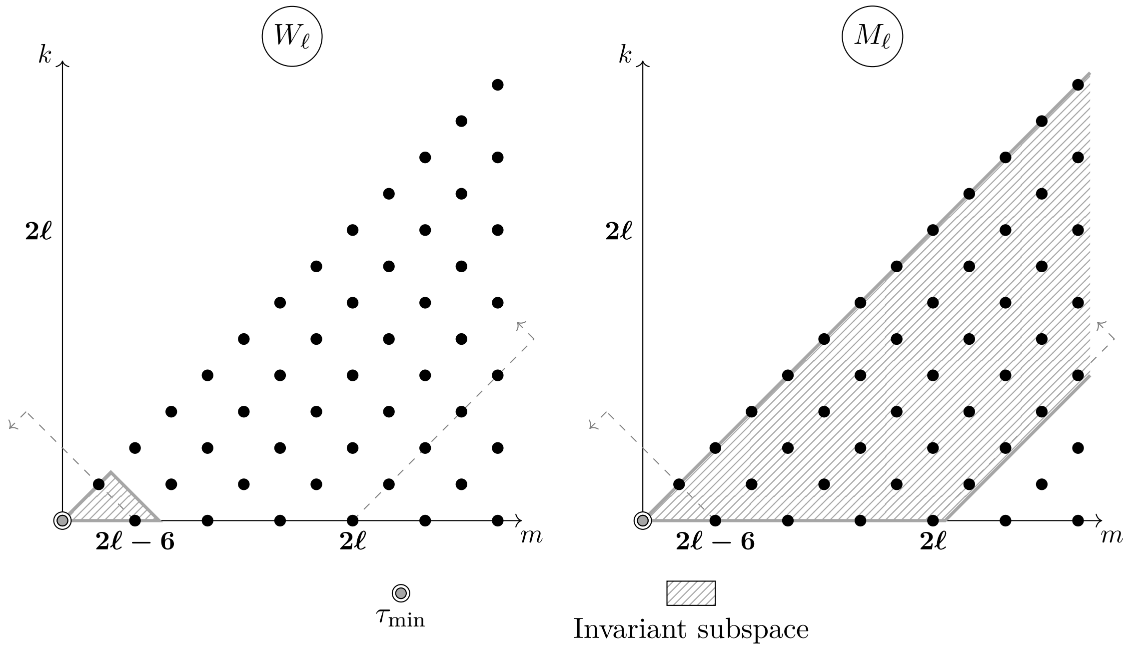



Figure B.8.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ in the case $\mu(H) = \rho(H) - 6 + 2\ell,\ \ell \in \mathbb{N}_{\geq 3}$

*are the only non-trivial $(\mathfrak{g}, K)$-submodules of (the Harish-Chandra module of) $H_\mu$. Furthermore, $\widetilde{M}_\ell$, $\widetilde{W}_\ell/\widetilde{M}_\ell$ and $H_\mu/\widetilde{W}_\ell$ are irreducible.*

*iii) If $\mu(H) = \rho(H) - 6 + 2\ell,\ \ell \in \{0, 1, 2\}$, the space (see Figure B.8)*

$$M_\ell := \bigoplus_{m-k \leq 2\ell} V_{m,k}$$

*is the only non-trivial $(\mathfrak{g}, K)$-submodule of (the Harish-Chandra module of) $H_\mu$. Furthermore, $M_\ell$ and $H_\mu/M_\ell$ are irreducible.*

*iv) (dual case) If $\mu(H) = -(\rho(H) - 6 + 2\ell),\ \ell \in \{0, 1, 2\}$, the space (see Figure B.9)*

$$\widetilde{M}_\ell := \bigoplus_{m-k > 2\ell} V_{m,k}$$

*is the only non-trivial $(\mathfrak{g}, K)$-submodule of (the Harish-Chandra module of) $H_\mu$. Furthermore, $\widetilde{M}_\ell$ and $H_\mu/\widetilde{M}_\ell$ are irreducible.*

*Proof.* See [JW77, Theorem 5.1 (4)] with $\nu = (\rho - \mu)(H) = 11 - \mu(H)$. □

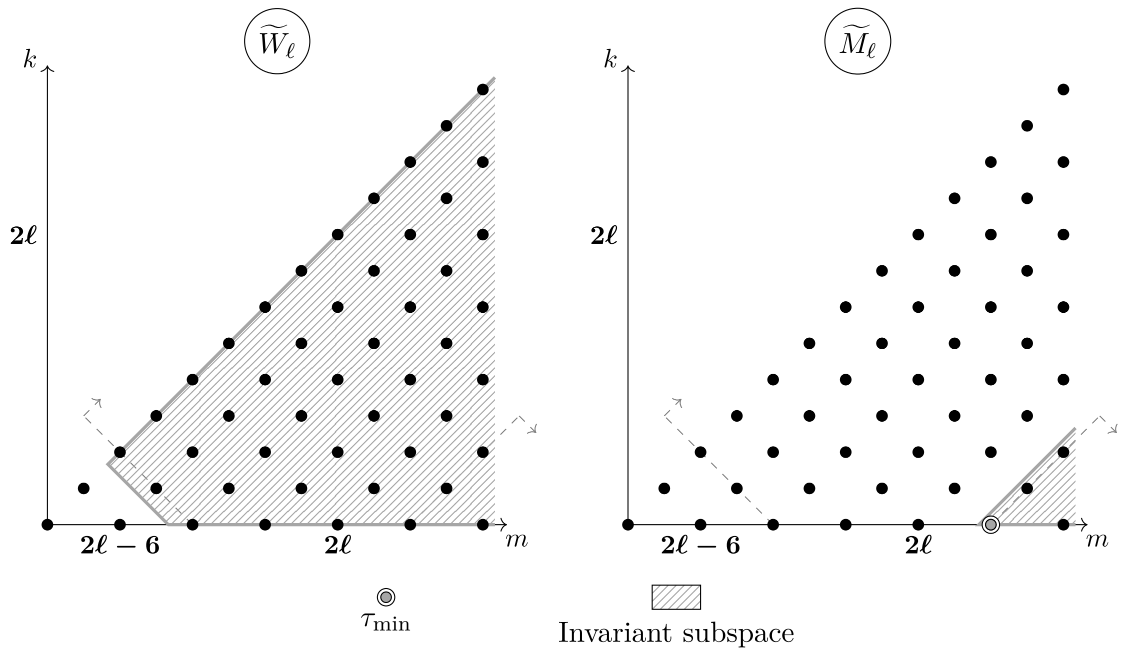



Figure B.9.: $K$-type images of the non-trivial invariant subspaces of $H_\mu$ in the case $\mu(H) = -(\rho(H) - 6 + 2\ell),\ \ell \in \mathbb{N}_{\geq 3}$

# C. Relative discrete series for $\mathrm{F}_{4(-20)}$

In this chapter we correct some misprints and fill in some omitted proofs from [Kos83] in the case of $G = \mathrm{F}_{4(-20)}$, $H = \mathrm{Spin}(1,8)$. There, the $K$-types are denoted by $H_{p,q}$ for $p,q \in \mathbb{N}_0$, corresponding to $V_{p+2q,p}$ in our notation. We first prove that $(-1)^{r+1}\theta_r$ resp. its corresponding kernel $(-1)^{r+1}\Theta_r$ is a positive definite spherical distribution resp. kernel for each $r \in \mathbb{N}_0$ (see [Kos83, p. 81] for the definition and [Far79, Theorem 1.1] for the correspondence). Indeed, by [Kos83, p. 84] we may write, for each $\phi(x) = F(t)Y(b) \in C_c^\infty(G/H)$ as in [Kos83, p. 68], i.e. $F \in C_c^\infty(\mathbb{R})$ even and $Y \in H_{p,q}$,

$$\Theta_r(\phi,\phi) = \lim_{s\to\rho+2r} \frac{\tau(-s)\alpha_{p,q}(s)}{s-\rho-2r}\beta_{p,q}(-\rho-2r)^2 \left|\int_0^\infty F(t)\Psi_{p,q}(t,\rho+2r)A(t)\,\mathrm{d}t\right|^2 \cdot \\ \cdot \int_B |Y(b)|^2\,\mathrm{d}b, \qquad \text{(C.0.1)}$$

where $\tau(s) \coloneqq \frac{\Gamma(\frac{1}{2}(s-\rho)+8)}{\Gamma(\frac{1}{2}(-s-\rho)+8)}$, $A(t) \coloneqq \sinh(t)^7\cosh(t)^{15}$,

$$\alpha_{p,q}(s) \coloneqq \prod_{j=1}^{p+q} \frac{-s+\rho+2j-2}{s+\rho+2j-2} \prod_{k=1}^{q} \frac{-s+\rho+2k-8}{s+\rho+2k-8},$$

$$\beta_{p,q}(s) \coloneqq \frac{7\cdot\Gamma(4)\cdot\Gamma(\frac{7}{2})\cdot 2^6}{\sqrt{\pi}\cdot\Gamma(p+4)} \frac{(\frac{1}{2}(s-\rho)-p-q+1)\cdots(\frac{1}{2}(s-\rho))}{\Gamma(\frac{1}{2}(s-\rho)+q+8)}$$

and

$$\Psi_{p,q}(t,s) \coloneqq \cosh(t)^{p+2q}\sinh(t)^p F\left(-\frac{s-\rho}{2}+p+q, \frac{s-\rho}{2}+p+q+11, p+4, -\sinh(t)^2\right).$$

Note first that $\beta_{p,q}(-\rho-2r) \neq 0$ if and only if $q \geq r+4$ (see also [Kos83, p. 71] with $h = r+3$). Therefore, and since the remaining terms are non-negative, it suffices to consider the sign of $\lim_{s\to\rho+2r} \frac{\tau(-s)\alpha_{p,q}(s)}{s-\rho-2r}$ for $q \geq r+4$ and prove that it is independent of $p$ and $q$. We have

$$\begin{aligned}
&\lim_{s\to\rho+2r} \frac{\tau(-s)\alpha_{p,q}(s)}{s-\rho-2r} \\
&= \lim_{s\to\rho+2r} \frac{1}{s-\rho-2r} \frac{\Gamma(\frac{1}{2}(-s-\rho)+8)}{\Gamma(\frac{1}{2}(s-\rho)+8)} \prod_{j=1}^{p+q} \frac{-s+\rho+2j-2}{s+\rho+2j-2} \prod_{k=1}^{q} \frac{-s+\rho+2k-8}{s+\rho+2k-8} \\
&= \lim_{s\to\rho+2r} \frac{1}{s-\rho-2r} \frac{\Gamma(\frac{1}{2}(-s-\rho)+8)}{\Gamma(r+8)} \frac{-s+\rho+2r}{s+\rho+2r} \prod_{\substack{j=1\\ j\neq r+1}}^{p+q} \frac{-2r+2j-2}{2\rho+2r+2j-2}.
\end{aligned}$$

$$\cdot \frac{-s+\rho+2r}{s+\rho+2r} \cdot \prod_{\substack{k=1\\k\neq r+4}}^{q} \frac{-2r+2k-8}{2\rho+2r+2k-8}$$

$$= \frac{1}{(2\rho+4r)^2} \frac{1}{\Gamma(r+8)} \prod_{\substack{j=1\\j\neq r+1}}^{p+q} \frac{-2r+2j-2}{2\rho+2r+2j-2} \prod_{\substack{k=1\\k\neq r+4}}^{q} \frac{-2r+2k-8}{2\rho+2r+2k-8} \cdot$$

$$\cdot \lim_{s\to\rho+2r} (s-\rho-2r)\Gamma(\frac{1}{2}(-s-\rho)+8).$$

Defining $\varphi(s) := \frac{1}{2}(-s-\rho)+8$ yields

$$\lim_{s\to\rho+2r} (s-\rho-2r)\Gamma(\frac{1}{2}(-s-\rho)+8) = \operatorname{Res}_{s=\rho+2r} \Gamma(\varphi(s)) = \frac{\operatorname{Res}_{z=\varphi(\rho+2r)} \Gamma(z)}{\varphi'(\rho+2r)}$$

$$= \frac{(-1)^{-r-3}}{(r+3)!(-\frac{1}{2})} = 2\frac{(-1)^r}{(r+3)!},$$

since $\varphi(\rho+2r) = \varphi(11+2r) = -r-3$. Thus, we obtain that

$$\lim_{s\to\rho+2r} \frac{\tau(-s)\alpha_{p,q}(s)}{s-\rho-2r}$$

$$= \frac{1}{(2\rho+4r)^2} \frac{1}{\Gamma(r+8)} 2\frac{(-1)^r}{(r+3)!} \prod_{k=1}^{r+3} \frac{-r+k-4}{\rho+r+k-4} \prod_{k=r+5}^{q} \frac{-2r+2k-8}{2\rho+2r+2k-8} \cdot$$

$$\cdot \prod_{\substack{j=1\\j\neq r+1}}^{p+q} \frac{-2r+2j-2}{2\rho+2r+2j-2}$$

$$= -\frac{1}{\rho+2r} \frac{1}{2\rho+4r} \frac{1}{\Gamma(r+8)} \frac{1}{(r+3)!} \frac{(r+3)!(r+7)!}{(\rho+2r-1)!} \prod_{k=r+5}^{q} \frac{-2r+2k-8}{2\rho+2r+2k-8} \cdot$$

$$\cdot \prod_{\substack{j=1\\j\neq r+1}}^{p+q} \frac{-2r+2j-2}{2\rho+2r+2j-2}$$

$$= -\frac{1}{2\rho+4r} \frac{1}{(\rho+2r)!} \prod_{k=r+5}^{q} \frac{-r+k-4}{\rho+r+k-4} \cdot \prod_{\substack{j=1\\j\neq r+1}}^{p+q} \frac{j-(r+1)}{\rho+r+j-1}$$

has sign $(-1)^{r+1}$ independent of $p$ and $q$ so that $(-1)^{r+1}\theta_r$ is a positive definite spherical distribution for each $r \in \mathbb{N}_0$.

We now state the Plancherel formula.

**Theorem C.0.1** ([Kos83, Theorem 3.13.1]). *For each $\phi \in C_c^\infty(G/H)$ we have*

$$\frac{1}{C^2}\int_X |\phi(x)|^2 \,\mathrm{d}x = \frac{1}{2\pi}\int_0^\infty Z_{i\nu}(\phi,\phi)\frac{\mathrm{d}\nu}{|c(i\nu)|^2} + \sum_{0<\rho+2r<\rho} Z_{\rho+2r}(\phi,\phi) \operatorname{Res}_{s=\rho+2r}\left(\frac{1}{c(s)c(-s)}\right) +$$

$$+ \sum_{\rho\leq\rho+2r} \Theta_r(\phi,\phi)c_{-2}\left(\frac{1}{c(s)c(-s)}, \rho+2r\right),$$

*where*

$$c(s) := \frac{2^{\rho-s}\cdot 7!\cdot\Gamma(s)}{\Gamma(\frac{s-\rho+8}{2})\Gamma(\frac{11+s}{2})\Gamma(\frac{5+s}{2})}, \qquad C := \frac{6!\cdot 2^5\cdot\sqrt{\pi}}{\Gamma(4)\Gamma(\frac{7}{2})},$$

$\mathrm{Res}_{s=s_0}(f)$ *denotes the residue of* $f$ *at* $s_0$ *and* $c_{-2}(f,s_0)$ *denotes the coefficient of* $(s-s_0)^{-2}$ *in the Laurent series for* $f$ *near* $s_0$.

**Remark C.0.2.** This formulation differs from Kosters's formulation only in that here the point $\rho$ is also allowed in the last summand.

*Proof.* The proof of [Kos83] mimics the proof of Faraut, [Far79, p. 428-431] and is based on the Plancherel formula of the *Fourier Jacobi transform*, which has been worked out by Flensted-Jensen in [FJ77, Appendix 1]. We first let $\phi$ denote a $K$-finite function, write

$$\int_X |\phi(x)|^2\,\mathrm{d}x = 2^{-2(\alpha+\beta+1)}\int_B |Y(b)|^2\,\mathrm{d}b\int_0^\infty |F(t)|^2\Delta_{\alpha,\beta}(t)\,\mathrm{d}t$$

with $\alpha := p+3$, $\beta := p+2q+7$ and put

$$\Delta_{\alpha,\beta}(t) := 2^{2(\alpha+\beta+1)}\sinh(t)^{2\alpha+1}\cosh(t)^{2\beta+1}$$

as in [Kos83, Proof of Theorem 3.13.1]. Then, we use the Plancherel formula from [FJ77, Equation (A.11)] to write (note the condition $s>0$ in contrast to [Kos83])

$$\int_0^\infty |F(t)|^2\Delta_{\alpha,\beta}(t)\,\mathrm{d}t = \frac{1}{2\pi}|\hat{F}(\nu)|^2\frac{\mathrm{d}\nu}{|C(\nu)|^2} - 2\pi i\sum_{\substack{\nu_0=is\in i\mathbb{R}\\ C(-\nu_0)=0\\ s>0}}|\hat{F}(\nu)|^2\,\mathrm{Res}_{\nu=\nu_0}\frac{1}{C(\nu)C(-\nu)},$$

where

$$C(\nu) := \frac{2^{\alpha+\beta+1-i\nu}\Gamma(i\nu)\Gamma(\alpha+1)}{\Gamma(\frac{1}{2}(\alpha+\beta+1+i\nu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\nu))}$$

and $\hat{F}$ denotes the Fourier Jacobi transform of $F$, given by

$$\hat{F}(\nu) := \frac{1}{2\pi}\int_0^\infty F(t)\phi_\nu^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,\mathrm{d}t$$

with the Jacobi function

$$\phi_\nu^{(\alpha,\beta)}(t) = \frac{\Psi_{p,q}(t,i\nu)}{\sinh(t)^p\cosh(t)^{p+2q}}.$$

In order to connect the discrete part of the Plancherel formula to the discrete part of the theorem, we need a last ingredient from [Kos83] and can then follow [Far79, p. 428-431] to fill in the omitted proof of Kosters. As in [Kos83], we first observe that for each $\nu\in[0,\infty[$

$$\frac{1}{|C(\nu)|^2}\frac{|c(i\nu)|^2}{2\pi 2^{-2(\alpha+\beta+1)}|\beta_{p,q}(i\nu)|^2} = C^2.$$

Since all involved functions are meromorphic on $\mathbb{C}$, we can use the identity theorem to obtain

$$\frac{c(i\nu)c(-i\nu)}{C(\nu)C(-\nu)2\pi 2^{-2(\alpha+\beta+1)}\beta_{p,q}(i\nu)\beta_{p,q}(-i\nu)} = C^2 \tag{C.0.2}$$

for each $\nu \in \mathbb{C}$. For the discrete part of the Plancherel formula we have to describe the residue of $\frac{1}{C(\nu)C(-\nu)}$ at some $\nu_0 \in i\mathbb{R}$. By Equation (C.0.2), we may write

$$\operatorname{Res}_{\nu=\nu_0}\left(\frac{1}{C(\nu)C(-\nu)}\right) = C^2 2\pi 2^{-2(\alpha+\beta+1)} \operatorname{Res}_{\nu=\nu_0}\left(\frac{\beta_{p,q}(i\nu)\beta_{p,q}(-i\nu)}{c(i\nu)c(-i\nu)}\right).$$

Now we can proceed as in [Far79, p. 431]. The zeros of $C(-\nu_0)$ for $\nu_0 = is \in i\mathbb{R}$ with $s > 0$ are given by $s = \rho + 2r$, $r \in \mathbb{Z}$, with (recall $\rho = 11$)

$$0 < \rho + 2r \leq 2q + 3 \Leftrightarrow r \leq q - 4.$$

If $r < 0$ then $\frac{1}{c(s)c(-s)}$ has a simple pole at $s = \rho + 2r$. Moreover, $\beta_{p,q}(-(\rho + 2r)) = 0$ if $\rho + 2r > 2q + 3 \Leftrightarrow r > q - 4$. For each $r < 0$ and each $p, q \in \mathbb{N}_0$ we may thus write

$$\operatorname{Res}_{\nu=\nu_0}\left(\frac{\beta_{p,q}(i\nu)\beta_{p,q}(-i\nu)}{c(i\nu)c(-i\nu)}\right) = \beta_{p,q}(\rho + 2r)\beta_{p,q}(-\rho - 2r) \operatorname{Res}_{\nu=\nu_0}\left(\frac{1}{c(i\nu)c(-i\nu)}\right)$$

and in turn get contributions of $Z_{\rho+2r}$ for $r < 0$. On the contrary, for $r \in \mathbb{N}_0$, $\frac{1}{c(s)c(-s)}$ has a pole of order two at $\rho + 2r$. For $r \leq q - 4$ we have

$$\beta_{p,q}(\rho + 2r) = 0 \qquad \text{and} \qquad \beta_{p,q}(-\rho - 2r) \neq 0.$$

Thus,

$$\operatorname{Res}_{\nu=\nu_0}\left(\frac{\beta_{p,q}(i\nu)\beta_{p,q}(-i\nu)}{c(i\nu)c(-i\nu)}\right) = \beta'_{p,q}(\rho + 2r)\beta_{p,q}(-\rho - 2r)c_{-2}\left(\frac{1}{c(i\nu)c(-i\nu)}, \nu_0\right)$$

and we get a contribution of $\Theta_r$ in these cases. □

We call invariant Hilbert subspaces in $L^2(G/H)$ resp. its corresponding positive definite distributions *relative discrete series* if they are not embedded into the complementary series. The corresponding distributions are then given by $\zeta_{\rho+2r}$ (corresponding to the kernel $Z_{\rho+2r}$) for $5 \leq \rho + 2r < \rho$ and $(-1)^{r+1}\theta_r$ (corresponding to $\Theta_r$) for $r \in \mathbb{N}_0$ (see also [Kos83, Remark 3.13.4], where $\zeta_5$ and $-\theta_0$ are excluded). We conclude this chapter by proving that the representations corresponding to these distributions are given by the socle representations of the spherical principal series representations in the exceptional cases. We first recall some notation from [Kos83]. The distribution $\theta_r$ is defined by

$$\forall \phi \in C_c^\infty(G)\colon \qquad \theta_r(\phi) := \int_B V_r(b)(\pi'_{-\rho-2r}(\phi)u_{-\rho-2r})(b)\,\mathrm{d}b,$$

where $V_r(\xi) := \frac{P_1(\xi)^r}{\Gamma(r+4)} \log(P_1(\xi))$ for $\xi \in B \cong K/M$ and $P_1 : G/MN \to \mathbb{R}$ is defined in [Kos83, p. 60]. Moreover, the spherical principal series is realized on $\Xi := G/MN$ by

$$E_s(\Xi) := \{f \in C^\infty(\Xi) \mid \forall t \in \mathbb{R},\, g \in G\colon f(ga_t\xi^0) = e^{(s-\rho)(t)} f(g\xi^0)\}$$

where $\xi^0 \in \Xi$ is the element used to identify $K/M \cong B$, $kM \mapsto k.\xi^0$ such that the stabilizer of $\xi^0$ in $G$ equals $MAN$. Finally, we define the $H$-spherical element $u_s$ of $E'_s(\Xi) := (E_{-s}(\Xi))'$ by

$$u_s(f) := \frac{1}{\Gamma(\frac{s-\rho+8}{2})} \int_B P_1(b)^{\frac{1}{2}(s-\rho)} f(b)\,\mathrm{d}b$$

and set

$$(\pi'_s(\phi)u_s)(f) := \int_G (\pi'_s(g)u_s)(f)\phi(g)\,\mathrm{d}g.$$

By [Kos83, p. 62], $\pi'_s(\phi)u_s \in E_s(\Xi)$ is actually smooth.

We now describe the realizations of the relative discrete series. Let

$$\Phi\colon E'_{\rho+2r}(\Xi) \to \mathcal{D}'(G), \quad \varphi \mapsto (\phi \mapsto \varphi(\pi'_{-\rho-2r}(\phi)u_{-\rho-2r}))$$

and

$$\varphi_0\colon E_{-\rho-2r}(\Xi) \to \mathbb{C}, \quad f \mapsto \int_B V_r(b)f(b)\,\mathrm{d}b.$$

Then we have $\theta_r = \Phi(\varphi_0)$. Note that $\Phi$ is $G$-equivariant since, for each $g \in G$,

$$\begin{aligned}\Phi(\pi'_{\rho+2r}(g)\varphi)(\phi) &= (\pi'_{\rho+2r}(g)\varphi)(\pi'_{-\rho-2r}(\phi)u_{-\rho-2r}) = \varphi(\pi_{-\rho-2r}(g^{-1})\pi'_{-\rho-2r}(\phi)u_{-\rho-2r})\\ &= \varphi(\pi'_{-\rho-2r}(g^{-1}\phi))u_{-\rho-2r} = \Phi(\varphi)(g^{-1}\phi) = (g\Phi(\varphi))(\phi).\end{aligned}$$

Therefore, the representation generated by $\theta_r$ is equal to the image under $\Phi$ of the representation $\overline{G\varphi_0}$ generated by $\varphi_0$, which in turn is given by $\overline{G\varphi_0}/\ker(\Phi|_{\overline{G\varphi_0}})$. Since $\overline{G\theta_r}$ is unitary and irreducible, the latter quotient is a unitary, irreducible subquotient of $E'_{\rho+2r}(\Xi)$. Hence, it has to be the unique maximal quotient of $E'_{\rho+2r}(\Xi) = (E_{-\rho-2r}(\Xi))'$. This however is isomorphic to the unique irreducible subrepresentation of $E_{-\rho-2r}$. The case of $Z_{\rho+2r}$ for $5 \leq \rho + 2r < \rho$ can be treated in an analogous fashion.

We can also verify that the occurring $K$-types match those of the socle. Let us first consider the case of $\theta_r$ for $r \in \mathbb{N}_0$. On the one hand we get from Equation (C.0.1) and since $\beta_{p,q}(-\rho - 2r) \neq 0$ if and only if $q \geq r + 4$ (by [Kos83, p. 71] with $h = r + 3$) that only $K$-types $H_{p,q} = V_{p+2q,p}$ with $q \geq r + 4$ can occur in the representation generated by $\theta_r$. On the other hand, in the notation of Lemma B.5.1, the exceptional parameter $-\rho - 2r = -(\rho - 6 + 2\ell)$ corresponds to $\ell = r + 3$, where the $K$-types in the socle $\widetilde{M_\ell}$ are (again) given by $V_{m,k}$ with $q = \frac{m-k}{2} > \ell = r + 3$.

Let us finally consider the case of $\zeta_{\rho+2r}$ for $5 \leq \rho + 2r < \rho$. In this case, [Kos83, Proposition 3.9.4] implies that $\zeta_s$ vanishes identically if and only if $\beta_{p,q}(s)\beta_{0,q}(-s)$ vanishes. By [Kos83, (i), (iii) on p. 70f.] we infer that

$$\beta_{p,q}(\rho + 2r) \neq 0 \text{ for all } p, q \in \mathbb{N}_0 \qquad \text{and} \qquad \beta_{0,q}(-\rho - 2r) \neq 0 \Leftrightarrow q \geq r + 4.$$

Thus, again only $K$-types $H_{p,q} = V_{p+2q,p}$ with $q \geq r + 4$ occur.

# D. An example: $G = \mathrm{SU}(n, 1)$, $n \geq 2$

In this chapter we consider the complex case in more detail. First, we describe the generalized gradients more explicitly using the structure of complex spherical harmonics and determine the corresponding embeddings $\iota_{Y_{p,q}}^{Y_{a,b}}$ from Definition 7.3.4. Then, after proving some facts about spherical harmonics, we give a proof of the Fourier characterization and provide a more explicit spectral correspondence in Theorem D.5.4. For the general structure of $G$ and the decomposition of the spherical principal series into spherical harmonics we refer to Appendix B.3.1 resp. Appendix B.3.2.

## D.1. Generalized gradients

In order to describe the generalized gradients, we have to decompose $Y_{p,q} \otimes \mathfrak{p}^*$ for $[(\tau_{p,q}, Y_{p,q})] \in \widehat{K}$ as a $K$-representation. Note first that $\mathfrak{p}^*$ is reducible and decomposes as

$$\mathfrak{p}^* = \mathfrak{p}_1^* \oplus \mathfrak{p}_2^*,$$

where

$$\mathfrak{p}_1 := \left\{ \begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix} : v \in \mathbb{C}^n \right\}, \quad \mathfrak{p}_2 := \left\{ \begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix} : v \in \mathbb{C}^n \right\}.$$

Then, by Proposition A.2.6 and A.2.7, $Y_{p,q} \otimes \mathfrak{p}^*$ generically (for some special choices of $p$ and $q$ some of the subrepresentations have dimension zero) decomposes as

$$Y_{p,q} \otimes \mathfrak{p}^* \cong Y_{p-1,q} \oplus Y_{p+1,q} \oplus Y_{p,q-1} \oplus Y_{p,q+1} \oplus V_{p,q}^1 \oplus V_{p,q}^2.$$

Before defining the associated generalized gradients we first introduce a useful notation.

**Notation D.1.1.**

$$z_j : \mathbb{C}^n \to \mathbb{C}, \quad (v_1, \ldots, v_n)^T \mapsto v_j,$$
$$\overline{z_j} : \mathbb{C}^n \to \mathbb{C}, \quad (v_1, \ldots, v_n)^T \mapsto \overline{v_j},$$
$$|z|^2 = \sum_{j=1}^n z_j \overline{z_j} : \mathbb{C}^n \to \mathbb{C}, \quad (v_1, \ldots, v_n)^T \mapsto \sum_{j=1}^n v_j \overline{v_j}.$$

Next we define the generalized gradients

$$\begin{aligned}
\mathbf{d}_{+,\mathrm{hol}} &:= T^{Y_{p,q}}_{Y_{p+1,q}} \circ \nabla : C^\infty(G \times_K Y_{p,q}) \to C^\infty(G \times_K Y_{p+1,q}),\\
\mathbf{d}_{-,\mathrm{ahol}} &:= T^{Y_{p,q}}_{Y_{p,q-1}} \circ \nabla : C^\infty(G \times_K Y_{p,q}) \to C^\infty(G \times_K Y_{p,q-1}),\\
\mathbf{d}_{+,\mathrm{ahol}} &:= T^{Y_{p,q}}_{Y_{p,q+1}} \circ \nabla : C^\infty(G \times_K Y_{p,q}) \to C^\infty(G \times_K Y_{p,q+1}),\\
\mathbf{d}_{-,\mathrm{hol}} &:= T^{Y_{p,q}}_{Y_{p-1,q}} \circ \nabla : C^\infty(G \times_K Y_{p,q}) \to C^\infty(G \times_K Y_{p-1,q}),\\
\mathbf{D}^j &:= \mathrm{pr}_{V^j_{p,q}} \circ \nabla : C^\infty(G \times_K Y_{p,q}) \to C^\infty(G \times_K V^j_{p,q}),\ j = 1, 2,
\end{aligned}$$

where $T^{Y_{p,q}}_{Y_{p\pm1,q}} \in \mathrm{Hom}_K(Y_{p,q} \otimes \mathfrak{p}^*, Y_{p\pm1,q})$ and $T^{Y_{p,q}}_{Y_{p,q\pm1}} \in \mathrm{Hom}_K(Y_{p,q} \otimes \mathfrak{p}^*, Y_{p,q\pm1})$ are chosen as follows: Using the $K$-equivariant maps

$$\begin{aligned}
Y_{p,q} \otimes Y_{1,0} &\to C^\infty(\mathbb{S}^{2n-1}), \qquad \varphi \otimes z_j \mapsto z_j\varphi,\\
Y_{p,q} \otimes Y_{0,1} &\to C^\infty(\mathbb{S}^{2n-1}), \qquad \varphi \otimes \overline{z_j} \mapsto \overline{z_j}\varphi,
\end{aligned}$$

and the isomorphisms $\Phi$, $\Psi$ from Lemma A.2.4 resp. A.2.5 we obtain $K$-equivariant maps

$$Y_{p,q} \otimes \mathfrak{p}_1^* \cong Y_{p,q} \otimes Y_{1,0} \to C^\infty(\mathbb{S}^{2n-1}), \qquad Y_{p,q} \otimes \mathfrak{p}_2^* \cong Y_{p,q} \otimes Y_{0,1} \to C^\infty(\mathbb{S}^{2n-1}).$$

By [FØ19, Equation (B.7)], we have for each $\varphi \in Y_{p,q}$

$$z_j\varphi = \varphi_j^{+,\mathrm{hol}} + |z|^2\varphi_j^{-,\mathrm{ahol}}, \qquad \overline{z_j}\varphi = \varphi_j^{+,\mathrm{ahol}} + |z|^2\varphi_j^{-,\mathrm{hol}}, \tag{D.1.1}$$

with $\varphi_j^{\pm,\mathrm{hol}} \in Y_{p\pm1,q}$, $\varphi_j^{\pm,\mathrm{ahol}} \in Y_{p,q\pm1}$ given by

$$\varphi_j^{+,\mathrm{hol}} = \mathrm{P}(z_j\varphi) = z_j\varphi - \frac{|z|^2}{p+q+n-1}\frac{\partial\varphi}{\partial\overline{z_j}}, \qquad \varphi_j^{-,\mathrm{hol}} = \frac{1}{p+q+n-1}\frac{\partial\varphi}{\partial z_j}, \tag{D.1.2}$$

$$\varphi_j^{+,\mathrm{ahol}} = \mathrm{P}(\overline{z_j}\varphi) = \overline{z_j}\varphi - \frac{|z|^2}{p+q+n-1}\frac{\partial\varphi}{\partial z_j}, \qquad \varphi_j^{-,\mathrm{ahol}} = \frac{1}{p+q+n-1}\frac{\partial\varphi}{\partial\overline{z_j}}. \tag{D.1.3}$$

Finally we set

$$\begin{aligned}
T^{Y_{p,q}}_{Y_{p+1,q}}(\varphi \otimes \Phi^{-1}(z_j)) &:= \varphi_j^{+,\mathrm{hol}}, \qquad T^{Y_{p,q}}_{Y_{p,q-1}}(\varphi \otimes \Phi^{-1}(z_j)) := \varphi_j^{-,\mathrm{ahol}},\\
T^{Y_{p,q}}_{Y_{p,q+1}}(\varphi \otimes \Psi^{-1}(\overline{z_j})) &:= \varphi_j^{+,\mathrm{ahol}}, \qquad T^{Y_{p,q}}_{Y_{p-1,q}}(\varphi \otimes \Psi^{-1}(\overline{z_j})) := \varphi_j^{-,\mathrm{hol}},
\end{aligned}$$

and extend the definitions to $Y_{p,q} \otimes \mathfrak{p}^*$ by defining $T^{Y_{p,q}}_{Y_{p+1,q}}$ and $T^{Y_{p,q}}_{Y_{p,q-1}}$ to be 0 on $Y_{p,q} \otimes \mathfrak{p}_2^*$ and $T^{Y_{p,q}}_{Y_{p,q+1}}$ and $T^{Y_{p,q}}_{Y_{p-1,q}}$ to be 0 on $Y_{p,q} \otimes \mathfrak{p}_1^*$.

In order to be able to do most of the necessary computations in $C^\infty(\mathbb{S}^{2n-1})$, we discuss how $\mathfrak{p}^*$ embeds into $C^\infty(\mathbb{S}^{2n-1})$ using Lemma A.2.4 and A.2.5.

**Lemma D.1.2.** *Let $X_{e_j}$, $\widetilde{X}_{e_j}$, $j \in \{1, \dots, n\}$, be the basis of $\mathfrak{p}$ defined by*

$$X_{e_j} := \begin{pmatrix} \mathbf{0}_n & e_j \\ e_j^T & 0 \end{pmatrix}, \quad \widetilde{X}_{e_j} := \begin{pmatrix} \mathbf{0}_n & ie_j \\ -ie_j^T & 0 \end{pmatrix}$$

*and consider the corresponding dual basis* $X_{e_j}^*$, $\widetilde{X}_{e_j}^* \in \mathfrak{p}^*$. *Then the isomorphism*

$$\mathfrak{p}^* \cong \mathfrak{p}_1^* \oplus \mathfrak{p}_2^* \cong Y_{1,0} \oplus Y_{0,1}$$

*maps*

$$X_{e_j}^* \mapsto \frac{1}{2}(z_j, \overline{z_j}), \quad \widetilde{X}_{e_j}^* \mapsto \frac{i}{2}(-z_j, \overline{z_j}).$$

*Proof.* For each $v \in \mathbb{C}^n$ we have

$$\begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & \frac{v}{2} \\ \frac{v^T}{2} & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_n & i(-i\frac{v}{2}) \\ -i(-i\frac{v^T}{2}) & 0 \end{pmatrix} = \sum_{m=1}^n \frac{v_m}{2} X_{e_m} - i\frac{v_m}{2}\widetilde{X}_{e_m}$$

and thus

$$X_{e_j}^*\left(\begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix}\right) = \frac{v_j}{2} \text{ and } \widetilde{X}_{e_j}^*\left(\begin{pmatrix} \mathbf{0}_n & v \\ 0 & 0 \end{pmatrix}\right) = -i\frac{v_j}{2}.$$

Moreover we obtain

$$\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & \frac{\overline{v}}{2} \\ \frac{v^*}{2} & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_n & i\left(\frac{i}{2}\overline{v}\right) \\ -i\left(\frac{i}{2}v^*\right) & 0 \end{pmatrix} = \sum_{m=1}^n \frac{\overline{v_m}}{2} X_{e_m} + i\frac{\overline{v_m}}{2}\widetilde{X}_{e_m}$$

and thus

$$X_{e_j}^*\left(\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right) = \frac{\overline{v_j}}{2} \text{ and } \widetilde{X}_{e_j}^*\left(\begin{pmatrix} \mathbf{0}_n & 0 \\ v^* & 0 \end{pmatrix}\right) = i\frac{\overline{v_j}}{2}.$$

With the isomorphisms from Lemma A.2.4 and A.2.5 this yields

$$X_{e_j}^* = \Phi^{-1}\left(\frac{z_j}{2}\right) + \Psi^{-1}\left(\frac{\overline{z_j}}{2}\right) \text{ and } \widetilde{X}_{e_j}^* = \Phi^{-1}\left(-i\frac{z_j}{2}\right) + \Psi^{-1}\left(i\frac{\overline{z_j}}{2}\right). \qquad \square$$

We can now describe the embeddings $\iota_{Y_{a,b}}^{Y_{p,q}}$ from Definition 7.3.4.

**Lemma D.1.3.** *Let*

$$\iota_{Y_{p,q}}^{Y_{p+1,q}} : Y_{p+1,q} \to Y_{p,q} \otimes \mathfrak{p}_1^*, \quad f \mapsto \frac{1}{p+1}\sum_{j=1}^n \frac{\partial f}{\partial z_j} \otimes \Phi^{-1}(z_j)$$

$$\iota_{Y_{p,q}}^{Y_{p,q+1}} : Y_{p,q+1} \to Y_{p,q} \otimes \mathfrak{p}_2^*, \quad f \mapsto \frac{1}{q+1}\sum_{j=1}^n \frac{\partial f}{\partial \overline{z_j}} \otimes \Psi^{-1}(\overline{z_j})$$

*and*

$$\iota_{Y_{p,q}}^{Y_{p-1,q}} : Y_{p-1,q} \to Y_{p,q} \otimes \mathfrak{p}_2^*, \quad f \mapsto \frac{n+p+q-2}{n+p-2}\sum_{j=1}^n \mathrm{P}(z_j f) \otimes \Psi^{-1}(\overline{z_j})$$

$$\iota_{Y_{p,q}}^{Y_{p,q-1}} : Y_{p,q-1} \to Y_{p,q} \otimes \mathfrak{p}_1^*, \quad f \mapsto \frac{n+p+q-2}{n+q-2}\sum_{j=1}^n \mathrm{P}(\overline{z_j} f) \otimes \Phi^{-1}(z_j).$$

*Then*

*i)* $\iota_{Y_{p,q}}^{Y_{p\pm1,q}}$ *and* $\iota_{Y_{p,q}}^{Y_{p,q\pm1}}$ *are K-equivariant,*

*ii)* $T_{Y_{p\pm1,q}}^{Y_{p,q}} \circ \iota_{Y_{p,q}}^{Y_{p\pm1,q}} = \mathrm{id}_{Y_{p\pm1,q}}$ *and* $T_{Y_{p,q\pm1}}^{Y_{p,q}} \circ \iota_{Y_{p,q}}^{Y_{p,q\pm1}} = \mathrm{id}_{Y_{p,q\pm1}}$,

*iii)* $T_{Y_{a,b}}^{Y_{p,q}} \circ \iota_{Y_{p,q}}^{Y_{c,d}} = 0$ *for each* $(a,b) \neq (c,d)$.

Let us again briefly explain how these embeddings are related to Proposition 7.3.9. We confine ourselves to the case $\iota_{Y_{p,q}}^{Y_{p+1,q}}$ and obtain

$$\begin{aligned}\iota_{Y_{p,q}}^{Y_{p+1,q}}(f) &= \frac{1}{\lambda(Y_{p+1,q}, Y_{p,q})} \sum_{j=1}^{n} \mathrm{pr}_{Y_{p,q}}\left(\frac{\overline{z_j}}{2} f\right) \otimes \Phi^{-1}(z_j) + \mathrm{pr}_{Y_{p,q}}\left(\frac{z_j}{2} f\right) \otimes \Psi^{-1}(\overline{z_j}) \\ &= \frac{2(p+q+n)}{p+1} \sum_{j=1}^{n} \mathrm{pr}_{Y_{p,q}}\left(\frac{\overline{z_j}}{2} f\right) \otimes \Phi^{-1}(z_j) \\ &= \frac{1}{p+1} \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \otimes \Phi^{-1}(z_j),\end{aligned}$$

where we used Remark A.2.2, Equation (D.1.1) and the fact that

$$\omega\left(\begin{pmatrix} \mathbf{0}_n & e_j \\ 0 & 0 \end{pmatrix}\right)(kM) = \frac{1}{2}\,\mathrm{tr}\left(\mathrm{Ad}(k^{-1})\begin{pmatrix} \mathbf{0}_n & e_j \\ 0 & 0 \end{pmatrix} H\right)$$

corresponds to $\frac{\overline{z_j}}{2} \in C^\infty(\mathbb{S}^{2n-1})$ after identifying $K/M \cong \mathbb{S}^{2n-1}$ (see Equation (B.3.11)). As in the case of $\mathrm{SO}_0(n,1)$, we give a direct proof of Lemma D.1.3 using only the structure of complex spherical harmonics.

*Proof.* i) We first show the $K$-equivariance of $\iota_{Y_{p,q}}^{Y_{p+1,q}}$. Let $k = \mathrm{diag}(A, \lambda) \in K$ and $f \in Y_{p+1,q}$. Then

$$\iota_{Y_{p,q}}^{Y_{p+1,q}}(k.f) = \iota_{Y_{p,q}}^{Y_{p+1,q}}(f \circ k^{-1}) = \frac{1}{p+1} \sum_{j=1}^{n} \frac{\partial (f \circ k^{-1})}{\partial z_j} \otimes \Phi^{-1}(z_j)$$

and

$$k.\iota_{Y_{p,q}}^{Y_{p+1,q}}(f) = \frac{1}{p+1} \sum_{j=1}^{n} k.\frac{\partial f}{\partial z_j} \otimes k.\Phi^{-1}(z_j) = \frac{1}{p+1} \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \circ k^{-1} \otimes \Phi^{-1}(k.z_j).$$

Note that there exists a canonical homomorphism

$$\Theta : Y_{p,q} \otimes \mathfrak{p}_1^* \cong \mathrm{Hom}(\mathbb{C}^n, Y_{p,q}), \quad \Theta(f \otimes \lambda)(v) = \Phi(\lambda)(v) f.$$

It thus suffices to prove that

$$\Theta(\iota_{Y_{p,q}}^{Y_{p+1,q}}(k.f))(e_m) = \frac{1}{p+1} \frac{\partial (f \circ k^{-1})}{\partial z_m}$$

equals

$$\Theta(k.\iota_{Y_{p,q}}^{Y_{p+1,q}}(f))(e_m) = \frac{1}{p+1}\sum_{j=1}^{n}(k.z_j)(e_m)\frac{\partial f}{\partial z_j} \circ k^{-1}$$

for each $m \in \{1,\ldots,n\}$. By the chain rule we obtain

$$\begin{aligned}\frac{\partial(f\circ k^{-1})}{\partial z_m} &= \sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}\circ k^{-1}\right)\frac{\partial k_j^{-1}}{\partial z_m} + \left(\frac{\partial f}{\partial \overline{z_j}}\circ k^{-1}\right)\frac{\partial \overline{k_j^{-1}}}{\partial z_m}\\ &= \sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}\circ k^{-1}\right)\frac{\partial k_j^{-1}}{\partial z_m},\end{aligned}$$

since $k_j^{-1}$ (the $j$-th component of $k^{-1} \in \mathrm{Aut}(\mathbb{C}^n)$) is linear. It remains to prove that

$$\frac{\partial k_j^{-1}}{\partial z_m} = (k.z_j)(e_m) = z_j(k^{-1}.e_m).$$

However, this follows directly from the definition of the derivative. This proves the equivariance of $\iota_{Y_{p,q}}^{Y_{p+1,q}}$ and the equivariance of $\iota_{Y_{p,q}}^{Y_{p,q+1}}$ is proven analogously.

For the $K$-equivariance of $\iota_{Y_{p,q}}^{Y_{p-1,q}}$ we have to show that for $f \in Y_{p-1,q}$

$$\sum_{j=1}^{n}\mathrm{P}(z_j k.f)\otimes\Psi^{-1}(\overline{z_j}) = \sum_{j=1}^{n}k.\mathrm{P}(z_j f)\otimes k.\Psi^{-1}(\overline{z_j}).$$

Let $\widetilde{\Theta}$ be the isomorphism given by

$$\widetilde{\Theta} : Y_{p,q}\otimes\mathfrak{p}_2^* \cong \mathrm{Hom}(\overline{\mathbb{C}^n}, Y_{p,q}), \quad \widetilde{\Theta}(f\otimes\lambda)(v) = \Psi(\lambda)(v)f,$$

where $\overline{\mathbb{C}^n}$ denotes $\mathbb{C}^n$ with the opposite complex structure. It suffices to prove

$$\mathrm{P}(z_m k.f) = \sum_{j=1}^{n}k.\mathrm{P}(z_j f)(k.\overline{z_j})(e_m)$$

for each $m \in \{1,\ldots,n\}$. Note that

$$k^{-1}.z_m = \sum_{j=1}^{n} z_m(k.e_j)z_j,$$

with, since $k$ acts unitary,

$$z_m(k.e_j) = \langle k.e_j, e_m\rangle_{\mathbb{C}^n} = \langle e_j, k^{-1}.e_m\rangle_{\mathbb{C}^n} = \overline{z_j}(k^{-1}.e_m) = k.\overline{z_j}(e_m).$$

By the $K$-equivariance of P we thus obtain that

$$\mathrm{P}(z_m k.f) = \mathrm{P}(k.(k^{-1}.z_m)f) = k.\mathrm{P}((k^{-1}.z_m)f) = k.\mathrm{P}\left(\sum_{j=1}^{n}(k.\overline{z_j})(e_m)z_j f\right)$$
$$= \sum_{j=1}^{n} k.\mathrm{P}(z_j f)(k.\overline{z_j})(e_m).$$

This finishes the proof of the first part.

ii) For each $f \in Y_{p+1,q}$ we have

$$T_{Y_{p+1,q}}^{Y_{p,q}}(\iota_{Y_{p,q}}^{Y_{p+1,q}}(f)) = \frac{1}{p+1} T_{Y_{p+1,q}}^{Y_{p,q}}\left(\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}\otimes \Phi^{-1}(z_j)\right)$$
$$= \frac{1}{p+1}\mathrm{P}\left(\sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}\right) = \mathrm{P}(f) = f.$$

The proof of $T_{Y_{p,q+1}}^{Y_{p,q}} \circ \iota_{Y_{p,q}}^{Y_{p,q+1}} = \mathrm{id}_{Y_{p,q+1}}$ is analogous.

For $T_{Y_{p-1,q}}^{Y_{p,q}} \circ \iota_{Y_{p,q}}^{Y_{p-1,q}} = \mathrm{id}_{Y_{p-1,q}}$ let $f \in Y_{p-1,q}$. Then

$$T_{Y_{p-1,q}}^{Y_{p,q}}(\iota_{Y_{p,q}}^{Y_{p-1,q}}(f)) = \frac{n+p+q-2}{n+p-2} T_{Y_{p-1,q}}^{Y_{p,q}}\left(\sum_{j=1}^{n}\mathrm{P}(z_j f)\otimes \Psi^{-1}(\overline{z_j})\right)$$
$$= \frac{n+p+q-2}{n+p-2}\,\frac{1}{p+q+n-1}\left(\sum_{j=1}^{n}\frac{\partial \mathrm{P}(z_j f)}{\partial z_j}\right).$$

By Equation (D.1.2) we infer

$$\sum_{j=1}^{n}\frac{\partial \mathrm{P}(z_j f)}{\partial z_j} = \sum_{j=1}^{n} f + z_j\frac{\partial f}{\partial z_j} - \frac{1}{p+q+n-2}\left(\overline{z_j}\frac{\partial f}{\partial \overline{z_j}} + |z|^2\frac{\partial^2 f}{\partial z_j \partial \overline{z_j}}\right)$$
$$= nf + (p-1)f - \frac{1}{p+q+n-2}(qf + |z|^2\Delta f)$$
$$= \frac{(n+p-2)(n+p+q-1)}{n+p+q-2} f.$$

The proof of $T_{Y_{p,q-1}}^{Y_{p,q}} \circ \iota_{Y_{p,q}}^{Y_{p,q-1}} = \mathrm{id}_{Y_{p,q-1}}$ is analogous.

iii) For each $f \in Y_{p,q-1}$ we have

$$T_{Y_{p+1,q}}^{Y_{p,q}}(\iota_{Y_{p,q}}^{Y_{p,q-1}}(f)) = T_{Y_{p+1,q}}^{Y_{p,q}}(\frac{n+p+q-2}{n+q-2}\sum_{j=1}^{n}\mathrm{P}(\overline{z_j}f)\otimes z_j)$$
$$= \frac{n+p+q-2}{n+q-2}\sum_{j=1}^{n}\mathrm{P}(z_j\mathrm{P}(\overline{z_j}f))$$
$$= \frac{n+p+q-2}{n+q-2}\mathrm{P}(|z|^2 f) = 0,$$

where we used Lemma 9.1.1. This also proves $T^{Y_{p,q}}_{Y_{p,q+1}} \circ \iota^{Y_{p-1,q}}_{Y_{p,q}} = 0$.

For each $f \in Y_{p,q+1}$ we have, since $f$ is harmonic,

$$\begin{aligned} T^{Y_{p,q}}_{Y_{p-1,q}}(\iota^{Y_{p,q+1}}_{Y_{p,q}}(f)) &= \frac{1}{q+1}\sum_{j=1}^{n} T^{Y_{p,q}}_{Y_{p-1,q}}\left(\frac{\partial f}{\partial \overline{z_j}} \otimes \Psi^{-1}(\overline{z_j})\right) \\ &= \frac{1}{(q+1)(p+q+n-1)}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial z_j \partial \overline{z_j}} = 0. \end{aligned}$$

This also proves $T^{Y_{p,q}}_{Y_{p,q-1}} \circ \iota^{Y_{p+1,q}}_{Y_{p,q}} = 0$ and the other equalities are clear. □

## D.2. Computations for spherical harmonics

In this section we prove some interesting relations in the spaces $Y_{p,q}$ of complex spherical harmonics. We will use these relations for the computation of the scalars relating the Poisson transforms to the generalized gradients and for the Fourier characterization of the principal series.

**Lemma D.2.1.** *We have the following expressions for the derived representation $\tau_{p,q}$ on $Y_{p,q}$ for $1 \leq j, m \leq n$*

$$\begin{aligned} \tau_{p,q}(\operatorname{diag}(E_{jm} - E_{mj}, 0)) &= z_j\frac{\partial}{\partial z_m} + \overline{z_j}\frac{\partial}{\partial \overline{z_m}} - z_m\frac{\partial}{\partial z_j} - \overline{z_m}\frac{\partial}{\partial \overline{z_j}}, \\ \tau_{p,q}(\operatorname{diag}(iE_{jm} + iE_{mj}, 0)) &= i\left(\overline{z_m}\frac{\partial}{\partial \overline{z_j}} - z_m\frac{\partial}{\partial z_j} + \overline{z_j}\frac{\partial}{\partial \overline{z_m}} - z_j\frac{\partial}{\partial z_m}\right), \ j \neq m, \\ \tau_{p,q}(\operatorname{diag}(-i, \mathbf{0}_{n-1}, i)) &= -i\left(\sum_{j=2}^{n}\overline{z_j}\frac{\partial}{\partial \overline{z_j}} - z_j\frac{\partial}{\partial z_j} + 2\left(\overline{z_1}\frac{\partial}{\partial \overline{z_1}} - z_1\frac{\partial}{\partial z_1}\right)\right). \end{aligned}$$

*Proof.* Let $\varphi \in Y_{p,q}$. We first assume that $j \neq m$ and extend the definition of $\tau_{p,q}$ to $\mathrm{Mat}_{n+1}(\mathbb{C})$. Then we have for every $z = (z_1, \ldots, z_n) \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$

$$\begin{aligned} \tau_{p,q}(\operatorname{diag}(E_{jm}, 0))\varphi(z) &= \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(-t\operatorname{diag}(E_{jm}, 0)).z) \\ &\overset{(B.3.11)}{=} \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(-tE_{jm})z) \\ &= \left.\frac{d}{dt}\right|_{t=0} \varphi(z - tz_m e_j). \end{aligned}$$

Writing $z_r = x_{2r-1} + ix_{2r} \in \mathbb{R} + i\mathbb{R}$ we obtain

$$\begin{aligned} \tau_{p,q}(\operatorname{diag}(E_{jm}, 0))\varphi(z) &= -i\left(x_{2m-1}\frac{\partial}{\partial x_{2j-1}} - x_{2m}\frac{\partial}{\partial x_{2j}}\right)\varphi(z) \\ &= -\left(z_m\frac{\partial}{\partial z_j} + \overline{z_m}\frac{\partial}{\partial \overline{z_j}}\right)\varphi(z). \end{aligned}$$

This proves the first equation. The second equation follows for $j \neq m$ from

$$\begin{aligned}
\tau_{p,q}(\mathrm{diag}(iE_{jm},0))\varphi(z) &= \frac{d}{dt}\Big|_{t=0} \varphi(\exp(-t\,\mathrm{diag}(iE_{jm},0)).z) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi(z - itz_m e_j) \\
&= \left( x_{2m}\frac{\partial}{\partial x_{2j-1}} - x_{2m-1}\frac{\partial}{\partial x_{2j}} \right)\varphi(z) \\
&= i\left( \overline{z_m}\frac{\partial}{\partial \overline{z_j}} - z_m\frac{\partial}{\partial z_j} \right)\varphi(z).
\end{aligned}$$

For the last equation we first obtain

$$\begin{aligned}
\tau_{p,q}(\mathrm{diag}(-i,\mathbf{0}_{n-1},i))\varphi(z) &= \frac{d}{dt}\Big|_{t=0} \varphi(\exp(-t\,\mathrm{diag}(-i,\mathbf{0}_{n-1},i)).z) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi(\mathrm{diag}(e^{2it},e^{it},\ldots,e^{it})z) \\
&= -\sum_{j=2}^{n} l(iE_{jj})\varphi(z) - 2l(iE_{11})\varphi(z),
\end{aligned}$$

where $l$ denotes the derived left regular representation. For each $j \in \{1,\ldots,n\}$ we have

$$\begin{aligned}
l(iE_{jj})\varphi(z) &= \frac{d}{dt}\Big|_{t=0} \varphi(z_1,\ldots,z_{j-1},e^{-it}z_j,z_{j+1},\ldots,z_n) \\
&= \left( x_{2j}\frac{\partial}{\partial x_{2j-1}} - x_{2j-1}\frac{\partial}{\partial x_{2j}} \right)\varphi(z) \\
&= i\left( \overline{z_j}\frac{\partial}{\partial \overline{z_j}} - z_j\frac{\partial}{\partial z_j} \right)\varphi(z).
\end{aligned}$$

This finishes the proof of the last two equations. □

**Lemma D.2.2.** *For each $f \in Y_{p,q}$ we have*

$$\frac{\partial f}{\partial z_1}(e_1) = pf(e_1), \qquad \frac{\partial f}{\partial \overline{z_1}}(e_1) = qf(e_1)$$

*and*

$$\mathrm{P}(z_1 f)(e_1) = \frac{p+n-1}{p+q+n-1}f(e_1), \qquad \mathrm{P}(\overline{z_1}f)(e_1) = \frac{q+n-1}{p+q+n-1}f(e_1).$$

*Proof.* We have

$$\frac{\partial f}{\partial z_1}(e_1) = \left( \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j} \right)(e_1) = pf(e_1)$$

and the second equality follows analogously. Using Equation (D.1.2) we obtain

$$\mathrm{P}(z_1 f)(e_1) = \left( z_1 f - \frac{|z|^2}{p+q+n-1} \frac{\partial f}{\partial \overline{z_1}} \right)(e_1) = f(e_1) - \frac{q}{p+q+n-1} f(e_1)$$

and the claimed equality follows. The formula for $\mathrm{P}(\overline{z_1} f)(e_1)$ follows similarly. □

**Lemma D.2.3.** *Let $p, q \in \mathbb{N}_0$. Then*

*i)*

$$\begin{aligned}
\mathrm{P}(z_1^p \overline{z_1}^q)(e_1) &= \frac{p+n-2}{p+q+n-2} \mathrm{P}(z_1^{p-1} \overline{z_1}^q)(e_1), \\
\mathrm{P}(z_1^p \overline{z_1}^q)(e_1) &= \frac{q+n-2}{p+q+n-2} \mathrm{P}(z_1^{p} \overline{z_1}^{q-1})(e_1), \\
\frac{\partial}{\partial z_1} \mathrm{P}(z_1^p \overline{z_1}^q) &= \frac{p(p+n-2)}{p+q+n-2} \mathrm{P}(z_1^{p-1} \overline{z_1}^q), \\
\frac{\partial}{\partial \overline{z_1}} \mathrm{P}(z_1^p \overline{z_1}^q) &= \frac{q(q+n-2)}{p+q+n-2} \mathrm{P}(z_1^{p} \overline{z_1}^{q-1}),
\end{aligned}$$

*ii)* $\mathrm{P}(z_1^p \overline{z_1}^q)$ *is a linear combination of* $z_1^{p-j} \overline{z_1}^{q-j} |z|^{2j}$, $j = 0, \ldots, \min(p,q)$.

*Proof.* Considering $z_1^p \overline{z_1}^q \in Y_{p,q}$ as an element of $Y_{p+q}$ (see Equation (B.3.13)) we can use Lemma 9.1.1 to obtain

$$\mathrm{P}(z_1^p \overline{z_1}^q)(e_1) = \mathrm{P}(z_1 \mathrm{P}(z_1^{p-1} \overline{z_1}^q))(e_1).$$

Now Lemma D.2.2 finishes the proof of the first equation. For the third equation note that $\frac{\partial}{\partial z_1} \mathrm{P}(z_1^p \overline{z_1}^q)$ is an $M$-invariant element of $Y_{p-1,q}$ and therefore a multiple of $\mathrm{P}(z_1^{p-1} \overline{z_1}^q)$ by Equation (B.3.14). We compute this multiple by evaluating at $e_1$. Lemma D.2.2 implies

$$\frac{\partial}{\partial z_1} \mathrm{P}(z_1^p \overline{z_1}^q)(e_1) = p \mathrm{P}(z_1^p \overline{z_1}^q)(e_1) = \frac{p(p+n-2)}{p+q+n-2} \mathrm{P}(z_1^{p-1} \overline{z_1}^q)(e_1).$$

The remaining equations can be proved analogously.

For the second part we use Equation (D.1.2) to get

$$\begin{aligned}
\mathrm{P}(z_1^p \overline{z_1}^q) = \mathrm{P}(z_1 \mathrm{P}(z_1^{p-1} \overline{z_1}^q)) &= z_1 \mathrm{P}(z_1^{p-1} \overline{z_1}^q) - \frac{|z|^2}{p+q+n-2} \frac{\partial}{\partial \overline{z_1}} \mathrm{P}(z_1^{p-1} \overline{z_1}^q) \\
&= z_1 \mathrm{P}(z_1^{p-1} \overline{z_1}^q) - \frac{|z|^2}{p+q+n-2} \frac{q(q+n-2)}{p+q+n-3} \mathrm{P}(z_1^{p-1} \overline{z_1}^{q-1})
\end{aligned}$$

and the result follows by induction. □

**Lemma D.2.4.** *For every $p, q \in \mathbb{N}_0$ we have*

$$\left( z_1 \frac{\partial}{\partial z_1} - \overline{z_1} \frac{\partial}{\partial \overline{z_1}} \right) \mathrm{P}(z_1^p \overline{z_1}^q) = (p-q) \mathrm{P}(z_1^p \overline{z_1}^q).$$

*Proof.* By the proof of Lemma D.2.1 we first obtain that

$$\left(z_1 \frac{\partial}{\partial z_1} - \overline{z_1} \frac{\partial}{\partial \overline{z_1}}\right) = il(iE_{11}),$$

where $l$ denotes the derived left regular representation of $\mathrm{U}(n)$ on $Y_{p,q}$. Since P is $\mathrm{O}(2n)$-equivariant and linear we have

$$il(iE_{11})\mathrm{P}(z_1^p \overline{z_1}^q) = \mathrm{P}(il(iE_{11}) z_1^p \overline{z_1}^q) = \mathrm{P}(p z_1^p \overline{z_1}^q - q z_1^p \overline{z_1}^q). \qquad \square$$

## D.3. Poisson transforms

In this section we define vector-valued Poisson transforms and describe their relationship to the generalized gradients. For each $p, q \in \mathbb{N}_0$ we define the Poisson transform $P_\mu^{Y_{p,q},\mathrm{cpt}} : \mathcal{D}'(K/M) \to C^\infty(G \times_K Y_{p,q})$ by continuous extension of (see Definition 3.3.1)

$$P_\mu^{Y_{p,q},\mathrm{cpt}}(\phi)(g) := \int_K a_I(g^{-1}k)^{-(\mu+\rho)} \tau_{p,q}(k_I(g^{-1}k)) t(\phi(k))\, \mathrm{d}k,$$

where $t \in \mathrm{Hom}_M(\mathbb{C}, Y_{p,q})$ is given by $t(1) := \mathrm{P}(z_1^p \overline{z_1}^q)$. By Equation (3.5) we have

$$P_\mu^{Y_{p,q}} \circ \mathcal{Q}_{\mu-\rho} = P_\mu^{Y_{p,q},\mathrm{cpt}}$$

on $\mathcal{D}'(K/M)$ where $P_\mu^{Y_{p,q}} : H_\mu^{-\infty} \to C^\infty(G \times_K Y_{p,q})$ is given by continuous extension of

$$P_\mu^{Y_{p,q}}(f)(g) := \int_K \tau_{p,q}(k) t(f(gk))\, \mathrm{d}k = F^{-1}(t)(\pi_\mu(g^{-1})f). \tag{D.3.4}$$

The following lemma generalizes Lemma 4.4.1 to $\mathrm{SU}(n,1)$.

**Lemma D.3.1.** *Let $\mu \in \mathfrak{a}^*$ and $f \in H_\mu^{-\infty}$. Then*

$$\begin{aligned}
(\mathbf{d}_{+,\mathrm{hol}} \circ P_\mu^{Y_{p,q}})(f) &= \left(\frac{(\mu+\rho)(H)}{2} + p\right) P_\mu^{Y_{p+1,q}}(f)\ \forall p, q \in \mathbb{N}_0,\\
(\mathbf{d}_{+,\mathrm{ahol}} \circ P_\mu^{Y_{p,q}})(f) &= \left(\frac{(\mu+\rho)(H)}{2} + q\right) P_\mu^{Y_{p,q+1}}(f)\ \forall p, q \in \mathbb{N}_0,\\
(\mathbf{d}_{-,\mathrm{ahol}} \circ P_\mu^{Y_{p,q}})(f) &= \lambda_{p,q}(q)(\mu(H) - \rho(H) - 2(q-1)) P_\mu^{Y_{p,q-1}}(f)\ \forall p \in \mathbb{N}_0,\ q \in \mathbb{N},\\
(\mathbf{d}_{-,\mathrm{hol}} \circ P_\mu^{Y_{p,q}})(f) &= \lambda_{p,q}(p)(\mu(H) - \rho(H) - 2(p-1)) P_\mu^{Y_{p-1,q}}(f)\ \forall p \in \mathbb{N},\ q \in \mathbb{N}_0,\\
(\mathbf{D}^j \circ P_\mu^{Y_{p,q}})(f) &= 0\ \forall p \in \mathbb{N}_0,\ q \in \mathbb{N},\ j \in \{1,2\},
\end{aligned}$$

*where* $\lambda_{p,q}(x) := \frac{x(n+x-2)}{2(p+q+n-2)(p+q+n-1)}$.

*Proof.* Let $\delta_{eM}$ denote the Delta distribution at $eM$ on $K/M$. Then

$$P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM})(g) = a_I(g^{-1})^{-(\mu+\rho)} \tau_{p,q}(k_I(g^{-1})) \mathrm{P}(z_1^p \overline{z_1}^q) \in C^\infty(G \times_K Y_{p,q}).$$

Recall the basis $X_{e_j}, \widetilde{X}_{e_j}, j \in \{1, \ldots, n\}$, from Lemma D.1.2. Since $X_{e_1} = H \in \mathfrak{a}_0$,

$$\begin{aligned}(\nabla P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM}))(e)(H) &= P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM})(\exp tH) \\ &= \frac{d}{dt}\Big|_{t=0} a_I(\exp -tH)^{-(\mu+\rho)}\mathrm{P}(z_1^p\overline{z_1}^q) \\ &= \frac{d}{dt}\Big|_{t=0} e^{t(\mu+\rho)(H)}\mathrm{P}(z_1^p\overline{z_1}^q) = (\mu+\rho)(H)\mathrm{P}(z_1^p\overline{z_1}^q).\end{aligned}$$

Note that for each $j \in \{2, \ldots, n\}$

$$\begin{aligned}\widetilde{X}_{e_1} &= \begin{pmatrix} -i & 0 & i \\ 0 & \mathbf{0}_{n-1} & 0 \\ -i & 0 & i \end{pmatrix} - \mathrm{diag}(-i, \mathbf{0}_{n-1}, i) \in \mathfrak{n}_0 \oplus \mathfrak{k}_0, \\ X_{e_j} &= \begin{pmatrix} 0 & e_{j-1}^T & 0 \\ -e_{j-1} & \mathbf{0}_{n-1} & e_{j-1} \\ 0 & e_{j-1}^T & 0 \end{pmatrix} - \begin{pmatrix} & e_j^T & \\ -e_j & & \\ & & \end{pmatrix} \in \mathfrak{n}_0 \oplus \mathfrak{k}_0, \\ \widetilde{X}_{e_j} &= \begin{pmatrix} 0 & -ie_{j-1}^T & 0 \\ -ie_{j-1} & \mathbf{0}_{n-1} & ie_{j-1} \\ 0 & -ie_{j-1}^T & 0 \end{pmatrix} - \begin{pmatrix} & -ie_j^T & \\ -ie_j & & \\ & & \end{pmatrix} \in \mathfrak{n}_0 \oplus \mathfrak{k}_0.\end{aligned}$$

Let the latter matrices (without the minus sign) be denoted by $\widetilde{Z_1}, Z_j, \widetilde{Z_j} \in \mathfrak{k}_0$ respectively. Since $P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM})$ is constant on $N$ we have

$$(\nabla P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM}))(e)(X_{e_j}) = -(\nabla P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM}))(e)(Z_j) = \tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q)$$

and

$$(\nabla P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM}))(e)(\widetilde{X}_{e_j}) = \tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q).$$

Therefore we deduce that $(\nabla P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM}))(e) \in Y_{p,q} \otimes \mathfrak{p}^*$ is given by

$$\begin{aligned}(\nabla P_\mu^{Y_{p,q},\mathrm{cpt}}(\delta_{eM}))(e) &= (\mu+\rho)(H)\mathrm{P}(z_1^p\overline{z_1}^q) \otimes H^* + \sum_{j=2}^n \tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q) \otimes X_{e_j}^* \\ &\quad + \sum_{j=1}^n \tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q) \otimes \widetilde{X}_{e_j}^*. \qquad \text{(D.3.5)}\end{aligned}$$

By Lemma D.1.2 the projection of the right hand side of (D.3.5) onto $Y_{p,q} \otimes \mathfrak{p}_1^*$ equals

$$\begin{aligned}&(\mu+\rho)(H)\mathrm{P}(z_1^p\overline{z_1}^q) \otimes \Phi^{-1}\left(\frac{z_1}{2}\right) + \sum_{j=2}^n \tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q) \otimes \Phi^{-1}\left(\frac{z_j}{2}\right) \\ &\quad + \sum_{j=1}^n \tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q) \otimes \Phi^{-1}\left(-\frac{iz_j}{2}\right) \in Y_{p,q} \otimes \mathfrak{p}_1^*.\end{aligned}$$

In order to calculate the images under $\mathbf{d}_{+,\mathrm{hol}}$ and $\mathbf{d}_{-,\mathrm{ahol}}$ we therefore have to decompose the polynomial

$$\frac{z_1}{2}(\mu+\rho)(H)\mathrm{P}(z_1^p\overline{z_1}^q) + \sum_{j=2}^n \frac{z_j}{2}\tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q) - \sum_{j=1}^n \frac{iz_j}{2}\tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q) \qquad \text{(D.3.6)}$$

with respect to the decomposition from Equation (D.1.1). Lemma D.2.1 yields

$$\begin{aligned}
&\sum_{j=2}^n \frac{z_j}{2}\tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q) - \frac{iz_j}{2}\tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&= \sum_{j=2}^n \frac{z_j}{2}\left(z_1\frac{\partial}{\partial z_j} + \overline{z_1}\frac{\partial}{\partial \overline{z_j}} - z_j\frac{\partial}{\partial z_1} - \overline{z_j}\frac{\partial}{\partial \overline{z_1}}\right)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&\qquad + \frac{iz_j}{2}i\left(\overline{z_j}\frac{\partial}{\partial \overline{z_1}} - z_j\frac{\partial}{\partial z_1} + \overline{z_1}\frac{\partial}{\partial \overline{z_j}} - z_1\frac{\partial}{\partial z_j}\right)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&= \sum_{j=2}^n \left(z_1 z_j\frac{\partial}{\partial z_j} - z_j\overline{z_j}\frac{\partial}{\partial \overline{z_1}}\right)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&= z_1\left(\sum_{j=1}^n z_j\frac{\partial}{\partial z_j} - z_1\frac{\partial}{\partial z_1}\right)\mathrm{P}(z_1^p\overline{z_1}^q) - (|z|^2 - z_1\overline{z_1})\frac{\partial}{\partial \overline{z_1}}\mathrm{P}(z_1^p\overline{z_1}^q)\\
&= pz_1\mathrm{P}(z_1^p\overline{z_1}^q) - z_1\left(z_1\frac{\partial}{\partial z_1} - \overline{z_1}\frac{\partial}{\partial \overline{z_1}}\right)\mathrm{P}(z_1^p\overline{z_1}^q) - |z|^2\frac{\partial}{\partial \overline{z_1}}\mathrm{P}(z_1^p\overline{z_1}^q).
\end{aligned}$$

Using Lemma D.2.4 this simplifies to

$$qz_1\mathrm{P}(z_1^p\overline{z_1}^q) - |z|^2\frac{\partial}{\partial \overline{z_1}}\mathrm{P}(z_1^p\overline{z_1}^q).$$

Moreover we have by the $K$-equivariance of P and Lemma D.2.1

$$\tau_{p,q}(\widetilde{Z}_1)\mathrm{P}(z_1^p\overline{z_1}^q) = \mathrm{P}(-2i(q-p)z_1^p\overline{z_1}^q) = 2i(p-q)\mathrm{P}(z_1^p\overline{z_1}^q). \qquad \text{(D.3.7)}$$

This proves that the polynomial from Equation (D.3.6) is given by

$$\left(\frac{(\mu+\rho)(H)}{2} + p\right) z_1\mathrm{P}(z_1^p\overline{z_1}^q) - |z|^2\frac{\partial}{\partial \overline{z_1}}\mathrm{P}(z_1^p\overline{z_1}^q).$$

By Equation (D.1.1) and Lemma 9.1.1 we may write the polynomial in the form

$$\begin{aligned}
&\left(\frac{(\mu+\rho)(H)}{2} + p\right)\left(\mathrm{P}(z_1^{p+1}\overline{z_1}^q) + \frac{|z|^2}{p+q+n-1}\frac{\partial}{\partial \overline{z_1}}\mathrm{P}(z_1^p\overline{z_1}^q)\right) - |z|^2\frac{\partial}{\partial \overline{z_1}}\mathrm{P}(z_1^p\overline{z_1}^q)\\
&= \left(\frac{(\mu+\rho)(H)}{2} + p\right)\mathrm{P}(z_1^{p+1}\overline{z_1}^q) + \left(\frac{(\mu+\rho)(H)+2p}{2(p+q+n-1)} - 1\right)|z|^2\frac{\partial}{\partial \overline{z_1}}\mathrm{P}(z_1^p\overline{z_1}^q).
\end{aligned}$$

Using Lemma D.2.3 we infer that this equals

$$\left(\frac{(\mu+\rho)(H)}{2} + p\right)\mathrm{P}(z_1^{p+1}\overline{z_1}^q) + \lambda_{p,q}(q)(\mu(H) - \rho(H) - 2(q-1))|z|^2\mathrm{P}(z_1^p\overline{z_1}^{q-1})$$

since

$$\left(\frac{(\mu+\rho)(H)+2p}{2(p+q+n-1)}-1\right)\frac{q(q+n-2)}{p+q+n-2}=\lambda_{p,q}(q)(\mu(H)-\rho(H)-2(q-1)),$$

where we used $\rho(H)=n$. Since $P_\mu^{Y_{p+1,q},\mathrm{cpt}}(\delta_{eM})(e)=\mathrm{P}(z_1^{p+1}\overline{z_1}^q)$ and $P_\mu^{Y_{p,q-1},\mathrm{cpt}}(\delta_{eM})(e)=\mathrm{P}(z_1^p\overline{z_1}^{q-1})$ the proof of the first and third equation follows from Corollary 3.3.5.

For the second and fourth equation we first describe the projection of the right hand side of Equation (D.3.5) onto $Y_{p,q}\otimes\mathfrak{p}_2^*$. By Lemma D.1.2 it is given by

$$\begin{aligned}(\mu+\rho)(H)\mathrm{P}(z_1^p\overline{z_1}^q)\otimes\Psi^{-1}\left(\frac{\overline{z_1}}{2}\right)+\sum_{j=2}^n\tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q)\otimes\Psi^{-1}\left(\frac{\overline{z_j}}{2}\right)\\+\sum_{j=1}^n\tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q)\otimes\Psi^{-1}\left(\frac{i\overline{z_j}}{2}\right)\in Y_{p,q}\otimes\mathfrak{p}_2^*.\end{aligned}$$

By the definition of $\mathbf{d}_{+,\mathrm{ahol}}$ and $\mathbf{d}_{-,\mathrm{hol}}$ we have to decompose the polynomial

$$\frac{\overline{z_1}}{2}(\mu+\rho)(H)\mathrm{P}(z_1^p\overline{z_1}^q)+\sum_{j=2}^n\frac{\overline{z_j}}{2}\tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q)+\sum_{j=1}^n\frac{i\overline{z_j}}{2}\tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q)\qquad\text{(D.3.8)}$$

with respect to the decomposition from Equation (D.1.1). By Lemma D.2.1 we infer that

$$\begin{aligned}&\sum_{j=2}^n\frac{\overline{z_j}}{2}\tau_{p,q}(Z_j)\mathrm{P}(z_1^p\overline{z_1}^q)+\frac{i\overline{z_j}}{2}\tau_{p,q}(\widetilde{Z}_j)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&=\sum_{j=2}^n\frac{\overline{z_j}}{2}\left(z_1\frac{\partial}{\partial z_j}+\overline{z_1}\frac{\partial}{\partial\overline{z_j}}-z_j\frac{\partial}{\partial z_1}-\overline{z_j}\frac{\partial}{\partial\overline{z_1}}\right)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&\qquad-\frac{i\overline{z_j}}{2}i\left(\overline{z_j}\frac{\partial}{\partial\overline{z_1}}-z_j\frac{\partial}{\partial z_1}+\overline{z_1}\frac{\partial}{\partial\overline{z_j}}-z_1\frac{\partial}{\partial z_j}\right)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&=\sum_{j=2}^n\left(\overline{z_1z_j}\frac{\partial}{\partial\overline{z_j}}-\overline{z_j}z_j\frac{\partial}{\partial z_1}\right)\mathrm{P}(z_1^p\overline{z_1}^q)\\
&=\overline{z_1}\left(\sum_{j=1}^n\overline{z_j}\frac{\partial}{\partial\overline{z_j}}-\overline{z_1}\frac{\partial}{\partial\overline{z_1}}\right)\mathrm{P}(z_1^p\overline{z_1}^q)-(|z|^2-\overline{z_1}z_1)\frac{\partial}{\partial z_1}\mathrm{P}(z_1^p\overline{z_1}^q)\\
&=q\overline{z_1}\mathrm{P}(z_1^p\overline{z_1}^q)+\overline{z_1}\left(z_1\frac{\partial}{\partial z_1}-\overline{z_1}\frac{\partial}{\partial\overline{z_1}}\right)\mathrm{P}(z_1^p\overline{z_1}^q)-|z|^2\frac{\partial}{\partial z_1}\mathrm{P}(z_1^p\overline{z_1}^q)\end{aligned}$$

which simplifies to

$$q\overline{z_1}\mathrm{P}(z_1^p\overline{z_1}^q)-|z|^2\frac{\partial}{\partial z_1}\mathrm{P}(z_1^p\overline{z_1}^q)$$

by Lemma D.2.4. Using Equation (D.3.7) we deduce that the polynomial from Equation (D.3.8) is given by

$$\left(\frac{(\mu+\rho)(H)}{2}+q\right)\overline{z_1}\mathrm{P}(z_1^p\overline{z_1}^q)-|z|^2\frac{\partial}{\partial z_1}\mathrm{P}(z_1^p\overline{z_1}^q).$$

Now we continue as in the previous case to obtain that it equals

$$\left(\frac{(\mu+\rho)(H)}{2} + q\right) \mathrm{P}(z_1^p \overline{z_1}^{q+1}) + \lambda_{p,q}(p)(\mu(H) - \rho(H) - 2(p-1))|z|^2 \mathrm{P}(z_1^{p-1}\overline{z_1}^q).$$

Corollary 3.3.5 finishes the proof of the second and fourth equation.

The last equality follows from the fact that $\mathbf{D}^j$ maps into sections of the bundle associated to the $K$-representation $V^j_{p,q}$ which does not contain the trivial $M$-representation and Corollary 3.3.4. ☐

**Remark D.3.2.** There exist constants $c_{p,q} \in \mathbb{C}$ such that

$$c_{p,q} \cdot P_\mu^{Y_{p,q}}(f)(g) = \mathrm{pr}_{Y_{p,q}}(f(g\bullet))$$

for all $\mu \in \mathfrak{a}^*$, $f \in C^\infty(G)^M$, $g \in G$ and $p, q \in \mathbb{N}_0$, where we extend the Poisson transform to $C^\infty(G)^M$ by the same formula. The constants are given by

$$c_{p,q} = \frac{\dim Y_{p,q}}{\mathrm{P}(z_1^p \overline{z_1}^q)(e_1)}.$$

This leads to the recursion formulas

$$c_{0,0} = 1, \quad c_{p+1,q} = \frac{n+p+q}{p+1} c_{p,q}, \quad c_{p,q+1} = \frac{n+p+q}{q+1} c_{p,q}.$$

*Proof.* The existence and the first expression follow as in Remark 9.2.2 and Lemma 9.2.6. By Remark A.2.3 and Lemma D.2.3 we have

$$\begin{aligned}\frac{c_{p+1,q}}{c_{p,q}} &= \frac{\dim Y_{p+1,q}}{\dim Y_{p,q}} \frac{\mathrm{P}(z_1^p \overline{z_1}^q)(e_1)}{\mathrm{P}(z_1^{p+1}\overline{z_1}^q)(e_1)} \\ &= \frac{\binom{p+n-1}{n-2}}{\binom{p+n-2}{n-2}} \frac{n+p+q}{n+p+q-1} \frac{n+p+q-1}{p+n-1} = \frac{n+p+q}{p+1}\end{aligned}$$

and the second formula is analogous. ☐

## D.4. Fourier characterization

In this section we characterize the spherical principal series representations by relations between their Fourier components (Proposition D.4.8), generalizing Lemma 4.2.1 from the $\mathrm{PSL}(2,\mathbb{R})$ case. We first repeat some notation.

**Definition D.4.1.** We embed $C^\infty(G \times_K Y_{p,q})$ into the smooth right $M$-invariant functions $C^\infty(G)^M$ by the map

$$\pi_{Y_{p,q}} : C^\infty(G \times_K Y_{p,q}) \to C^\infty(G)^M, \quad \pi_{Y_{p,q}}(\varphi)(nak) := \varphi(na)(k.e_1),\ n \in N, a \in A, k \in K,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$. Since $\varphi \in C^\infty(G \times_K Y_{p,q})$ we have for each $g = nak \in G = NAK$

$$\pi_{Y_{p,q}}(\varphi)(g) = \varphi(na)(k.e_1) = (\tau_{p,q}(k^{-1})\varphi(na))(e_1) = \varphi(g)(e_1).$$

We further denote by

$$\pi^*_{Y_{p,q}} \colon \mathcal{D}'(G/M) \to \mathcal{D}'(G \times_K Y_{p,q}), \quad \pi^*_{Y_{p,q}}(f)(\varphi) \coloneqq f(\pi_{Y_{q,p}}(\varphi))$$

the transpose of $\pi_{Y_{p,q}}$ defined by duality where $\mathcal{D}'(G \times_K Y_{p,q})$ denotes the dual of $C_c^\infty(G \times_K Y_{q,p})$. We embed smooth sections into distributional sections by

$$\iota_{Y_{p,q}} : C^\infty(G \times_K Y_{p,q}) \to \mathcal{D}'(G \times_K Y_{p,q}), \quad \iota_{Y_{p,q}}(f)(\varphi) \coloneqq \int_G \pi_{Y_{p,q}}(f)(g)\pi_{Y_{q,p}}(f)(g)\,\mathrm{d}gK.$$

Using the same methodology as in the real case, we obtain analogous statements to Lemmas 7.1.2, 9.2.3 and Proposition 7.2.4. In particular we obtain, for each $\mu \in \mathfrak{a}^*$, that $P_\mu^{Y_{p,q},\mathrm{cpt}} = P_\mu^{Y_{p,q}} \circ \mathcal{Q}_{\mu-\rho} = \frac{1}{c_{p,q}}\pi^*_{Y_{p,q}} \circ \mathcal{Q}_{\mu-\rho}$ on $\mathcal{D}'(K/M)$ as in Lemma 9.2.3 iv). This allows the following reformulation of Lemma D.3.1.

**Lemma D.4.2.** *Let $\mu \in \mathfrak{a}^*$ and $f \in H_\mu^{-\infty}$. Then*

$$\begin{aligned}
(\mathbf{d}_{+,\mathrm{hol}} \circ \pi^*_{Y_{p,q}})(f) &= \frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2} + p\right)\pi^*_{Y_{p+1,q}}(f)\,\forall p,q \in \mathbb{N}_0,\\
(\mathbf{d}_{+,\mathrm{ahol}} \circ \pi^*_{Y_{p,q}})(f) &= \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2} + q\right)\pi^*_{Y_{p,q+1}}(f)\,\forall p,q \in \mathbb{N}_0,\\
(\mathbf{d}_{-,\mathrm{ahol}} \circ \pi^*_{Y_{p,q}})(f) &= \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H) - \rho(H) - 2(q-1))\pi^*_{Y_{p,q-1}}(f)\,\forall p \in \mathbb{N}_0,\, q \in \mathbb{N},\\
(\mathbf{d}_{-,\mathrm{hol}} \circ \pi^*_{Y_{p,q}})(f) &= \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H) - \rho(H) - 2(p-1))\pi^*_{Y_{p-1,q}}(f)\,\forall p \in \mathbb{N},\, q \in \mathbb{N}_0,\\
(\mathbf{D}^j \circ \pi^*_{Y_{p,q}})(f) &= 0\ \forall p \in \mathbb{N}_0,\, q \in \mathbb{N},\, j \in \{1,2\},
\end{aligned}$$

*where* $\lambda_{p,q}(x) = \frac{x(n+x-2)}{2(p+q+n-2)(p+q+n-1)}$.

Before proving the Fourier characterization, we first prove two preparatory lemmas, the first of which yields the explicit form of Lemma 7.5.1 in the complex case.

**Lemma D.4.3.** *For each $f \in C^\infty(G \times_K Y_{p,q})$ we have*

$$(\mathbf{d}_{+,\mathrm{hol}} + \mathbf{d}_{-,\mathrm{ahol}} + \mathbf{d}_{-,\mathrm{hol}} + \mathbf{d}_{+,\mathrm{ahol}})f(g) = \sum_{j=1}^n r(X_{e_j})f(g)\frac{z_j + \overline{z_j}}{2} + r(\widetilde{X}_{e_j})f(g)\frac{z_j - \overline{z_j}}{2i}.$$

*Proof.* By definition, the gradient $\nabla$ is given by

$$\nabla f(g) = \sum_{j=1}^n r(X_{e_j})f(g) \otimes X^*_{e_j} + r(\widetilde{X}_{e_j})f(g) \otimes \widetilde{X}^*_{e_j} \in Y_{p,q} \otimes \mathfrak{p}^*.$$

Lemma D.1.2 yields that

$$X^*_{e_j} = \frac{1}{2}(\Phi^{-1}(z_j) + \Psi^{-1}(\overline{z}_j)), \quad \widetilde{X}^*_{e_j} = \frac{1}{2i}(\Phi^{-1}(z_j) - \Psi^{-1}(\overline{z}_j)).$$

By Equation (D.1.1) we obtain for each $\varphi \in Y_{p,q},\ f_1 \in Y_{1,0}$ and $f_2 \in Y_{0,1}$ (as identities on the sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$)

$$\begin{aligned} \varphi f_1 &= (T^{Y_{p,q}}_{Y_{p+1,q}} + T^{Y_{p,q}}_{Y_{p,q-1}})(\varphi \otimes \Phi^{-1}(f_1)), \\ 0 &= (T^{Y_{p,q}}_{Y_{p+1,q}} + T^{Y_{p,q}}_{Y_{p,q-1}})\Big|_{Y_{p,q}\otimes\mathfrak{p}_2^*}, \\ \varphi f_2 &= (T^{Y_{p,q}}_{Y_{p,q+1}} + T^{Y_{p,q}}_{Y_{p-1,q}})(\varphi \otimes \Psi^{-1}(f_2)), \\ 0 &= (T^{Y_{p,q}}_{Y_{p,q+1}} + T^{Y_{p,q}}_{Y_{p-1,q}})\Big|_{Y_{p,q}\otimes\mathfrak{p}_1^*}. \end{aligned}$$

Thus we obtain that $(\mathbf{d}_{+,\mathrm{hol}} + \mathbf{d}_{-,\mathrm{ahol}} + \mathbf{d}_{-,\mathrm{hol}} + \mathbf{d}_{+,\mathrm{ahol}})f(g)$ equals

$$\begin{aligned} &\sum_{j=1}^{n} r(X_{e_j})f(g)\frac{z_j}{2} + r(\widetilde{X}_{e_j})f(g)\frac{1}{2i}z_j + r(X_{e_j})f(g)\frac{\overline{z}_j}{2} - r(\widetilde{X}_{e_j})f(g)\frac{1}{2i}\overline{z}_j \\ =&\sum_{j=1}^{n} r(X_{e_j})f(g)\frac{z_j + \overline{z_j}}{2} + r(\widetilde{X}_{e_j})f(g)\frac{z_j - \overline{z_j}}{2i}. \end{aligned}$$ □

**Lemma D.4.4.** *Let $u_{p,q} \in C^\infty(G \times_K Y_{p,q})$. Then $r(H)\pi_{Y_{p,q}}(u_{p,q})$ equals*

$$\pi_{Y_{p-1,q}}(\mathbf{d}_{-,\mathrm{hol}}u_{p,q}) + \pi_{Y_{p+1,q}}(\mathbf{d}_{+,\mathrm{hol}}u_{p,q}) + \pi_{Y_{p,q-1}}(\mathbf{d}_{-,\mathrm{ahol}}u_{p,q}) + \pi_{Y_{p,q+1}}(\mathbf{d}_{+,\mathrm{ahol}}u_{p,q}).$$

*Proof.* This follows directly from Lemma D.4.3 by evaluating both sides at $e_1$. □

**Lemma D.4.5.** *Let $\mu \in \mathfrak{a}^*$ and $f = \sum_{p,q\in\mathbb{N}_0} f_{p,q} \in \mathcal{D}'(G/M)$ with $f_{p,q} = \pi_{Y_{p,q}}(\pi^*_{Y_{p,q}}(f)) \in \pi_{Y_{p,q}}(C^\infty(G \times_K Y_{p,q}))$ such that the equations from Lemma D.4.2 hold for $f$. Then $r(X)f = (\mu - \rho)(X)f$ for every $X \in \mathfrak{a}_0$, where $r$ denotes the right regular representation of $\mathfrak{a}_0$ on $\mathcal{D}'(G/M)$.*

*Proof.* By definition we have for every $\varphi \in C_c^\infty(G/M)$

$$(r(X)f)(\varphi) = -f(r(X)\varphi) = -\sum_{p,q\in\mathbb{N}_0} f_{p,q}(r(X)\varphi) = \sum_{p,q\in\mathbb{N}_0}(r(X)f_{p,q})(\varphi).$$

By the last equation of Lemma D.4.2 we may apply Lemma D.4.4 to obtain

$$\begin{aligned} r(H)f_{p,q} &= r(H)\pi_{Y_{p,q}}(\pi^*_{Y_{p,q}}(f)) \\ &= \pi_{Y_{p+1,q}}(\mathbf{d}_{+,\mathrm{hol}}\pi^*_{Y_{p,q}}(f)) + \pi_{Y_{p,q+1}}(\mathbf{d}_{+,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f)) \\ &\quad + \pi_{Y_{p,q-1}}(\mathbf{d}_{-,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f)) + \pi_{Y_{p-1,q}}(\mathbf{d}_{-,\mathrm{hol}}\pi^*_{Y_{p,q}}(f)). \end{aligned}$$

The equations from Lemma D.4.2 imply that

$$\begin{aligned} r(H)f_{p,q} = {} & \frac{c_{p,q}}{c_{p+1,q}} \left( \frac{(\mu+\rho)(H)}{2} + p \right) f_{p+1,q} \\ & + \frac{c_{p,q}}{c_{p,q+1}} \left( \frac{(\mu+\rho)(H)}{2} + q \right) f_{p,q+1} \\ & + \frac{c_{p,q}}{c_{p,q-1}} \lambda_{p,q}(q)(\mu(H) - \rho(H) - 2(q-1)) f_{p,q-1} \\ & + \frac{c_{p,q}}{c_{p-1,q}} \lambda_{p,q}(p)(\mu(H) - \rho(H) - 2(p-1)) f_{p-1,q}. \end{aligned}$$

We infer that

$$\begin{aligned} \sum_{p,q=0}^{m} r(H) f_{p,q} = {} & \sum_{p,q=0}^{m-1} a_{p,q} f_{p,q} \\ & + \sum_{p=0}^{m-1} (a_{p,m} - \frac{n+p+m}{m+1} \lambda_{p,m+1}(m+1)(\mu(H) - \rho(H) - 2m)) f_{p,m} \\ & + \sum_{q=0}^{m-1} (a_{m,q} - \frac{n+q+m}{m+1} \lambda_{m+1,q}(m+1)(\mu(H) - \rho(H) - 2m)) f_{m,q} \\ & + (a_{m,m} - 2\frac{n+2m}{m+1} \lambda_{m,m+1}(m+1)(\mu(H) - \rho(H) - 2m)) f_{m,m} \\ & + \sum_{p=0}^{m} \frac{m+1}{n+p+m} \left( \frac{(\mu+\rho)(H)}{2} + m \right) f_{p,m+1} \\ & + \sum_{q=0}^{m} \frac{m+1}{n+q+m} \left( \frac{(\mu+\rho)(H)}{2} + m \right) f_{m+1,q}, \end{aligned} \tag{D.4.9}$$

where $a_{p,q}$ is given by (recall Remark D.3.2 and the definition of $\lambda_{p,q}$)

$$\begin{aligned} a_{p,q} = {} & \frac{c_{p-1,q}}{c_{p,q}} \left( \frac{(\mu+\rho)(H)}{2} + p - 1 \right) + \frac{c_{p,q-1}}{c_{p,q}} \left( \frac{(\mu+\rho)(H)}{2} + q - 1 \right) \\ & + \frac{c_{p,q+1}}{c_{p,q}} \lambda_{p,q+1}(q+1)(\mu(H) - \rho(H) - 2q) \\ & + \frac{c_{p+1,q}}{c_{p,q}} \lambda_{p+1,q}(p+1)(\mu(H) - \rho(H) - 2p) \\ = {} & \frac{p}{n+p+q-1} \left( \frac{(\mu+\rho)(H)}{2} + p - 1 \right) + \frac{q}{n+p+q-1} \left( \frac{(\mu+\rho)(H)}{2} + q - 1 \right) \\ & + \frac{n+p+q}{q+1} \frac{(q+1)(n+q-1)}{2(p+q+n-1)(p+q+n)} (\mu(H) - \rho(H) - 2q) \\ & + \frac{n+p+q}{p+1} \frac{(p+1)(n+p-1)}{2(p+q+n-1)(p+q+n)} (\mu(H) - \rho(H) - 2p) \\ = {} & \frac{p}{n+p+q-1} \left( \frac{(\mu+\rho)(H)}{2} + p - 1 \right) + \frac{q}{n+p+q-1} \left( \frac{(\mu+\rho)(H)}{2} + q - 1 \right) \end{aligned}$$

$$+\frac{n+q-1}{2(p+q+n-1)}(\mu(H)-\rho(H)-2q)$$
$$+\frac{n+p-1}{2(p+q+n-1)}(\mu(H)-\rho(H)-2p)$$
$$=\mu(H)-\rho(H),$$

where we used $\rho(H)=n$ for the last equality.

We finally claim that the remaining terms in Equation (D.4.9) converge to 0 as $m\to\infty$. Note that every remaining term only involves $f_{p,q}$ with $p+q\geq m$. Moreover, by the orthogonality of the $Y_{p,q}$, we have for each $\varphi\in C_c^\infty(G/M)$ that

$$f_{p,q}(\varphi)=f_{p,q}(\varphi_{p,q})=f(\varphi_{p,q}).$$

As in the proof of Proposition 7.2.4 we see that for each $N>0$ there exists a constant $C$ independent of $m$ such that

$$|f(\varphi_{p,q})|\leq C(1+(p+q)^2)^{-N}\leq C(1+m^2)^{-N} \text{ for } p+q\geq m.$$

Choosing $N$ large enough implies the claim since there are only polynomially many terms and the growth of the coefficients is polynomial. □

**Lemma D.4.6.** *Let $\varphi\in Y_{p,q}$. Then, for $j\in\{2,\dots,n\}$,*

$$\mathrm{P}(z_jf)(e_1)=-\frac{1}{p+q+n-1}\frac{\partial f}{\partial\overline{z}_j}(e_1),\quad \mathrm{P}(\overline{z}_jf)(e_1)=-\frac{1}{p+q+n-1}\frac{\partial f}{\partial z_j}(e_1).$$

*Proof.* This is a direct consequence of Equation (D.1.2) and (D.1.3). □

We now prove the relation between the Fourier recursions and the action of $\mathfrak{n}_0$ to complete the characterization of the spherical principal series by its Fourier components.

**Lemma D.4.7.** *Let $\mu\in\mathfrak{a}^*$ and $f=\sum_{p,q\in\mathbb{N}_0}f_{p,q}\in\mathcal{D}'(G/M)$ with $f_{p,q}\in\pi_{Y_{p,q}}(C^\infty(G\times_K Y_{p,q}))$ such that the equations from Lemma D.4.2 hold for $f$. Then $U_+f=0$ for every $U_+\in C^\infty(G\times_M\mathfrak{n}_0)$.*

*Proof.* For $v\in\mathbb{C}^{n-1}$ let

$$Y_v:=\begin{pmatrix}0 & v^* & 0\\ -v & \mathbf{0}_{n-1} & v\\ 0 & v^* & 0\end{pmatrix}\in\mathfrak{g}_0^\alpha \text{ and } Y_{2\alpha}:=\begin{pmatrix}-i & 0 & i\\ 0 & \mathbf{0}_{n-1} & 0\\ -i & 0 & i\end{pmatrix}\in\mathfrak{g}_0^{2\alpha}.$$

By Appendix B.3.1,

$$Y_{e_1},\dots,Y_{e_{n-1}},Y_{ie_1},\dots,Y_{ie_{n-1}},Y_{2\alpha}$$

is a $\mathbb{R}$-basis of $\mathfrak{n}_0$. We write $U_+\in C^\infty(G\times_M\mathfrak{n}_0)$ as

$$U_+(g)=\sum_{j=1}^{n-1}(\kappa_j^1(g)Y_{e_j}+\kappa_j^2(g)Y_{ie_j})+\kappa(g)Y_{2\alpha},\quad g\in G,$$

for some real-valued smooth functions $\kappa_j^k, \kappa \in C^\infty(G)$. For each $\varphi \in C_c^\infty(G)^M$ we obtain

$$
\begin{aligned}
(U_+ f)(\varphi) &= f(U_+^* \varphi) = \sum_{p,q\in\mathbb{N}_0} f_{p,q}(U_+^*\varphi) = \sum_{p,q\in\mathbb{N}_0} (U_+ f_{p,q})(\varphi) \\
&= \sum_{p,q\in\mathbb{N}_0} \sum_{j=1}^{n-1} ((r(Y_{e_j}) f_{p,q})(\kappa_j^1(g)\varphi) + (r(Y_{ie_j}) f_{p,q})(\kappa_j^2(g)\varphi) + (r(Y_{2\alpha}) f_{p,q})(\kappa(g)\varphi).
\end{aligned}
$$

For $v = (0, v_2, \ldots, v_n) \in \mathbb{C}^n$ we write

$$
Z_v := \begin{pmatrix} & v^* \\ -v & \end{pmatrix} \in \mathfrak{k}_0 \text{ and } Z_{2\alpha} := \operatorname{diag}(-i, 0, \ldots, 0, i) \in \mathfrak{k}_0.
$$

Using the notation from Appendix B.3.1 we obtain for $j \in \{2, \ldots, n\}$

$$
Y_{e_{j-1}} = X_{e_j} + Z_{e_j},\ Y_{ie_{j-1}} = X_{ie_j} + Z_{ie_j},\ Y_{2\alpha} = X_{ie_1} + Z_{2\alpha} \in \mathfrak{p}_0 \oplus \mathfrak{k}_0.
$$

Let us first consider

$$
r(Y_{e_{j-1}}) f_{p,q} = r(X_{e_j}) f_{p,q} + r(Z_{e_j}) f_{p,q}.
$$

The first summand is given by

$$
\begin{aligned}
r(X_{e_j}) f_{p,q}(g) &= r(X_{e_j}) \pi_{Y_{p,q}}(\pi_{Y_{p,q}}^*(f))(g) = \frac{d}{dt}\Big|_{t=0} \pi_{Y_{p,q}}^*(f)(g \exp tX_{e_j})(e_1) \\
&= (\nabla \pi_{Y_{p,q}}^*(f))(g)(X_{e_j})(e_1) \qquad \text{(D.4.10)}
\end{aligned}
$$

for every $g \in G$. We claim that

$$
\begin{aligned}
(\nabla \pi_{Y_{p,q}}^*(f))(g) &= \iota_{Y_{p,q}}^{Y_{p+1,q}}(\mathbf{d}_{+,\mathrm{hol}} \pi_{Y_{p,q}}^*(f)(g)) + \iota_{Y_{p,q}}^{Y_{p-1,q}}(\mathbf{d}_{-,\mathrm{hol}} \pi_{Y_{p,q}}^*(f)(g)) \\
&\quad + \iota_{Y_{p,q}}^{Y_{p,q+1}}(\mathbf{d}_{+,\mathrm{ahol}} \pi_{Y_{p,q}}^*(f)(g)) + \iota_{Y_{p,q}}^{Y_{p,q-1}}(\mathbf{d}_{-,\mathrm{ahol}} \pi_{Y_{p,q}}^*(f)(g)).
\end{aligned}
$$

Indeed, note that both sides are equal after applying some $T_{Y_{a,b}}^{Y_{p,q}}$ by Lemma D.1.3. Thus, using $\mathbf{D}^j \pi_{Y_{p,q}}^*(f) = 0$ for $j \in \{1, 2\}$, the claim follows by using the tensor product decompositions from Propositions A.2.6 and A.2.7. Therefore we may write

$$
\begin{aligned}
(\nabla \pi_{Y_{p,q}}^*(f))(g) &= \iota_{Y_{p,q}}^{Y_{p+1,q}}(\mathbf{d}_{+,\mathrm{hol}} \pi_{Y_{p,q}}^*(f)(g)) + \iota_{Y_{p,q}}^{Y_{p-1,q}}(\mathbf{d}_{-,\mathrm{hol}} \pi_{Y_{p,q}}^*(f)(g)) \\
&\quad + \iota_{Y_{p,q}}^{Y_{p,q+1}}(\mathbf{d}_{+,\mathrm{ahol}} \pi_{Y_{p,q}}^*(f)(g)) + \iota_{Y_{p,q}}^{Y_{p,q-1}}(\mathbf{d}_{-,\mathrm{ahol}} \pi_{Y_{p,q}}^*(f)(g)) \\
&= \frac{c_{p,q}}{c_{p+1,q}} \left( \frac{(\mu+\rho)(H)}{2} + p \right) \iota_{Y_{p,q}}^{Y_{p+1,q}}(\pi_{Y_{p+1,q}}^*(f)(g)) \\
&\quad + \frac{c_{p,q}}{c_{p-1,q}} \lambda_{p,q}(p)(\mu(H) - \rho(H) - 2(p-1)) \iota_{Y_{p,q}}^{Y_{p-1,q}}(\pi_{Y_{p-1,q}}^*(f)(g)) \\
&\quad + \frac{c_{p,q}}{c_{p,q+1}} \left( \frac{(\mu+\rho)(H)}{2} + q \right) \iota_{Y_{p,q}}^{Y_{p,q+1}}(\pi_{Y_{p,q+1}}^*(f)(g)) \\
&\quad + \frac{c_{p,q}}{c_{p,q-1}} \lambda_{p,q}(q)(\mu(H) - \rho(H) - 2(q-1)) \iota_{Y_{p,q}}^{Y_{p,q-1}}(\pi_{Y_{p,q-1}}^*(f)(g)), \qquad \text{(D.4.11)}
\end{aligned}
$$

where we used the equations from Lemma D.4.2. Recall from Lemma D.1.2 that

$$\Phi^{-1}(z_j) = X^*_{e_j} + iX^*_{ie_j} \text{ and } \Psi^{-1}(\overline{z}_j) = X^*_{e_j} - iX^*_{ie_j}.$$

Identifying $Y_{p,q} \otimes \mathfrak{p}^*$ with $\mathrm{Hom}(\mathfrak{p}, Y_{p,q})$ we thus obtain by Lemma D.1.3

$$\begin{aligned}(\nabla\pi^*_{Y_{p,q}}(f))(g)(X_{e_j}) = {} & \frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2}+p\right)\frac{1}{p+1}\frac{\partial\pi^*_{Y_{p+1,q}}(f)(g)}{\partial z_j} \\ & + \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H)-\rho(H)-2(p-1)) \\ & \quad \frac{n+p+q-2}{n+p-2}\mathrm{P}(z_j\pi^*_{Y_{p-1,q}}(f)(g)) \\ & + \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2}+q\right)\frac{1}{q+1}\frac{\partial\pi^*_{Y_{p,q+1}}(f)(g)}{\partial \overline{z}_j} \\ & + \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H)-\rho(H)-2(q-1)) \\ & \quad \frac{n+p+q-2}{n+q-2}\mathrm{P}(\overline{z}_j\pi^*_{Y_{p,q-1}}(f)(g)).\end{aligned}$$

Lemma D.4.6 yields that $r(X_{e_j})f_{p,q}(g) = (\nabla\pi^*_{Y_{p,q}}(f))(g)(X_{e_j})(e_1)$ equals

$$\begin{aligned} & \frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2}+p\right)\frac{1}{p+1}\frac{\partial\pi^*_{Y_{p+1,q}}(f)(g)}{\partial z_j}(e_1) \\ & - \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H)-\rho(H)-2(p-1))\frac{n+p+q-2}{n+p-2}\frac{1}{p+q+n-2}\frac{\partial\pi^*_{Y_{p-1,q}}(f)(g)}{\partial\overline{z}_j}(e_1) \\ & + \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2}+q\right)\frac{1}{q+1}\frac{\partial\pi^*_{Y_{p,q+1}}(f)(g)}{\partial\overline{z}_j}(e_1) \\ & - \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H)-\rho(H)-2(q-1))\frac{n+p+q-2}{n+q-2}\frac{1}{p+q+n-2}\frac{\partial\pi^*_{Y_{p,q-1}}(f)(g)}{\partial z_j}(e_1) \\ = {} & \frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2}+p\right)\frac{1}{p+1}\frac{\partial\pi^*_{Y_{p+1,q}}(f)(g)}{\partial z_j}(e_1) \\ & - \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H)-\rho(H)-2(p-1))\frac{1}{n+p-2}\frac{\partial\pi^*_{Y_{p-1,q}}(f)(g)}{\partial\overline{z}_j}(e_1) \\ & + \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2}+q\right)\frac{1}{q+1}\frac{\partial\pi^*_{Y_{p,q+1}}(f)(g)}{\partial\overline{z}_j}(e_1) \\ & - \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H)-\rho(H)-2(q-1))\frac{1}{n+q-2}\frac{\partial\pi^*_{Y_{p,q-1}}(f)(g)}{\partial z_j}(e_1).\end{aligned}$$

Let us now investigate the term $r(Z_{e_j})f_{p,q}$. We have

$$r(Z_{e_j})f_{p,q}(g) = r(Z_{e_j})\pi_{Y_{p,q}}(\pi^*_{Y_{p,q}}(f))(g) = \left.\frac{d}{dt}\right|_{t=0}\pi_{Y_{p,q}}(\pi^*_{Y_{p,q}}(f))(g\exp tZ_{e_j}).$$

Since $\pi^*_{Y_{p,q}}(f) \in C^\infty(G \times_K Y_{p,q})$ and $\exp tZ_{e_j} \in K$ we infer

$$\pi_{Y_{p,q}}(\pi^*_{Y_{p,q}}(f))(g \exp tZ_{e_j}) = \pi^*_{Y_{p,q}}(f)(g \exp tZ_{e_j})(e_1) = \tau_{p,q}(\exp -tZ_{e_j})(\pi^*_{Y_{p,q}}(f)(g))(e_1)$$

and conclude that

$$r(Z_{e_j})f_{p,q}(g) = \tau_{p,q}(-Z_{e_j})(\pi^*_{Y_{p,q}}(f)(g))(e_1).$$

Therefore the first equation of Lemma D.2.1 implies that

$$r(Z_{e_j})f_{p,q}(g) = -\left(\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j} + \frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial \overline{z}_j}\right)(e_1).$$

Altogether $r(Y_{e_{j-1}})f_{p,q}(g) = r(X_{e_j})f_{p,q}(g) + r(Z_{e_j})f_{p,q}(g)$ equals

$$\begin{aligned}
&\frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2} + p\right)\frac{1}{p+1}\frac{\partial \pi^*_{Y_{p+1,q}}(f)(g)}{\partial z_j}(e_1) \\
&- \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H) - \rho(H) - 2(p-1))\frac{1}{n+p-2}\frac{\partial \pi^*_{Y_{p-1,q}}(f)(g)}{\partial \overline{z}_j}(e_1) \\
&+ \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2} + q\right)\frac{1}{q+1}\frac{\partial \pi^*_{Y_{p,q+1}}(f)(g)}{\partial \overline{z}_j}(e_1) \\
&- \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H) - \rho(H) - 2(q-1))\frac{1}{n+q-2}\frac{\partial \pi^*_{Y_{p,q-1}}(f)(g)}{\partial z_j}(e_1) \\
&- \left(\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j} + \frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial \overline{z}_j}\right)(e_1).
\end{aligned}$$

Therefore we have

$$\sum_{p,q=0}^{m} r(Y_{e_{j-1}})f_{p,q}(g) = \sum_{p,q=0}^{m-1} a_{p,q}\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j}(e_1) + b_{p,q}\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial \overline{z}_j}(e_1) + \mathrm{Err},$$

where Err consists of terms $\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j}(e_1), \frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial \overline{z}_j}(e_1)$ for $p + q \geq m$ with at most polynomially growing coefficients and

$$\begin{aligned}
a_{p,q} &= \frac{c_{p-1,q}}{c_{p,q}}\left(\frac{(\mu+\rho)(H)}{2} + p - 1\right)\frac{1}{p} - \frac{c_{p,q+1}}{c_{p,q}}\lambda_{p,q+1}(q+1)\frac{\mu(H) - \rho(H) - 2q}{n+q-1} - 1 \\
b_{p,q} &= \frac{c_{p,q-1}}{c_{p,q}}\left(\frac{(\mu+\rho)(H)}{2} + q - 1\right)\frac{1}{q} - \frac{c_{p+1,q}}{c_{p,q}}\lambda_{p+1,q}(p+1)\frac{\mu(H) - \rho(H) - 2p}{n+p-1} - 1.
\end{aligned}$$

We claim that $a_{p,q} = b_{p,q} = 0$. Indeed, Remark D.3.2 and the definition of $\lambda_{p,q}$ imply

$$\begin{aligned}
a_{p,q} = &\frac{p}{n+p+q-1}\left(\frac{(\mu+\rho)(H)}{2} + p - 1\right)\frac{1}{p} \\
&- \frac{n+p+q}{q+1}\lambda_{p,q+1}(q+1)(\mu(H) - \rho(H) - 2q)\frac{1}{n+q-1} - 1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n+p+q-1}\left(\frac{(\mu+\rho)(H)}{2} + p - 1\right) \\
&\quad - \frac{n+p+q}{q+1}\frac{(q+1)(n+q-1)}{2(p+q+n-1)(p+q+n)}(\mu(H)-\rho(H)-2q)\frac{1}{n+q-1} - 1 \\
&= \frac{1}{n+p+q-1}\left(\frac{(\mu+\rho)(H)}{2} + p - 1\right) - \frac{1}{2(p+q+n-1)}(\mu(H)-\rho(H)-2q) - 1 \\
&= 0,
\end{aligned}
$$

where we used $\rho(H) = n$ for the last equality. The proof of $b_{p,q} = 0$ is identical with the roles of $p$ and $q$ interchanged.

Similar to the calculation above we get

$$
\begin{aligned}
\sum_{p,q=0}^{m} r(Y_{ie_{j-1}})f_{p,q}(g) &= \sum_{p,q=0}^{m-1} ia_{p,q}\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j}(e_1) - ib_{p,q}\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial \overline{z}_j}(e_1) + \mathrm{Err}_2 \\
&= \mathrm{Err}_2,
\end{aligned}
$$

where the error term $\mathrm{Err}_2$ has the same properties as Err.

Let us finally compute $r(Y_{2\alpha})f_{p,q} = r(X_{ie_1})f_{p,q} + r(Z_{2\alpha})f_{p,q}$. Similar to Equation (D.4.10) we have

$$
r(X_{ie_1})f_{p,q}(g) = (\nabla \pi^*_{Y_{p,q}}(f))(g)(X_{ie_1})(e_1)
$$

for each $g \in G$. Using Equation (D.4.11), Lemma D.1.3 and Lemma D.1.2 we obtain

$$
\begin{aligned}
(\nabla \pi^*_{Y_{p,q}}(f))(g)(X_{ie_1}) &= \frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2} + p\right)\frac{1}{p+1}\frac{\partial \pi^*_{Y_{p+1,q}}(f)(g)}{\partial z_1} i \\
&\quad + \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H)-\rho(H)-2(p-1)) \\
&\qquad \frac{n+p+q-2}{n+p-2}\mathrm{P}(z_1\pi^*_{Y_{p-1,q}}(f)(g))(-i) \\
&\quad + \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2} + q\right)\frac{1}{q+1}\frac{\partial \pi^*_{Y_{p,q+1}}(f)(g)}{\partial \overline{z}_1}(-i) \\
&\quad + \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H)-\rho(H)-2(q-1)) \\
&\qquad \frac{n+p+q-2}{n+q-2}\mathrm{P}(\overline{z}_1\pi^*_{Y_{p,q-1}}(f)(g))i.
\end{aligned}
$$

By Lemma D.2.2 the evaluation at $e_1$ is given by

$$
\begin{aligned}
r(X_{ie_1})f_{p,q}(g) &= \frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2} + p\right) i\pi^*_{Y_{p+1,q}}(f)(g)(e_1) \\
&\quad + \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H)-\rho(H)-2(p-1))\pi^*_{Y_{p-1,q}}(f)(g)(e_1)(-i) \\
&\quad + \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2} + q\right)\pi^*_{Y_{p,q+1}}(f)(g)(e_1)(-i) \\
&\quad + \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H)-\rho(H)-2(q-1))\pi^*_{Y_{p,q-1}}(f)(g)(e_1)i.
\end{aligned}
$$

Since $r(Z_{2\alpha})f_{p,q}(g) = \tau_{p,q}(-Z_{2\alpha})(\pi^*_{Y_{p,q}}(f)(g))(e_1)$, Lemmas D.2.1 and D.2.2 imply

$$r(Z_{2\alpha})f_{p,q}(g) = 2i\left(\frac{\partial\pi^*_{Y_{p,q}}(f)(g)}{\partial\overline{z}_1} - \frac{\partial\pi^*_{Y_{p,q}}(f)(g)}{\partial z_1}\right)(e_1) = 2i(q-p)\pi^*_{Y_{p,q}}(f)(g)(e_1).$$

Altogether we obtain that $r(Y_{2\alpha})f_{p,q}(g) = r(X_{ie_1})f_{p,q}(g) + r(Z_{2\alpha})f_{p,q}(g)$ equals

$$\begin{aligned} r(Y_{2\alpha})f_{p,q}(g) = {} & \frac{c_{p,q}}{c_{p+1,q}}\left(\frac{(\mu+\rho)(H)}{2}+p\right) i f_{p+1,q}(g) \\ & + \frac{c_{p,q}}{c_{p-1,q}}\lambda_{p,q}(p)(\mu(H)-\rho(H)-2(p-1))f_{p-1,q}(g)(-i) \\ & + \frac{c_{p,q}}{c_{p,q+1}}\left(\frac{(\mu+\rho)(H)}{2}+q\right) f_{p,q+1}(g)(-i) \\ & + \frac{c_{p,q}}{c_{p,q-1}}\lambda_{p,q}(q)(\mu(H)-\rho(H)-2(q-1))f_{p,q-1}(g)i \\ & + 2i(q-p)f_{p,q}(g). \end{aligned}$$

We infer that

$$\sum_{p,q=0}^{m} r(Y_{2\alpha})f_{p,q}(g) = \sum_{p,q=0}^{m-1} \eta_{p,q}f_{p,q}(g) + \mathrm{Err}_3,$$

where $\mathrm{Err}_3$ only consists of terms $f_{p,q}(g)$ for $p+q \geq m$ with at most polynomially growing coefficients and

$$\begin{aligned} (-i)\eta_{p,q} = {} & \frac{c_{p-1,q}}{c_{p,q}}\left(\frac{(\mu+\rho)(H)}{2}+p-1\right) - \frac{c_{p+1,q}}{c_{p,q}}\lambda_{p+1,q}(p+1)(\mu(H)-\rho(H)-2p) \\ & - \frac{c_{p,q-1}}{c_{p,q}}\left(\frac{(\mu+\rho)(H)}{2}+q-1\right) + \frac{c_{p,q+1}}{c_{p,q}}\lambda_{p,q+1}(q+1)(\mu(H)-\rho(H)-2q) \\ & + 2(q-p). \end{aligned}$$

We claim that $\eta_{p,q} = 0$. Indeed, by Remark D.3.2 and the definition of $\lambda_{p,q}$ we obtain

$$\begin{aligned} (-i)\eta_{p,q} = {} & \frac{p}{n+p+q-1}\left(\frac{(\mu+\rho)(H)}{2}+p-1\right) - \frac{n+p+q}{p+1}\lambda_{p+1,q}(p+1)(\mu(H)-\rho(H)-2p) \\ & - \frac{q}{n+p+q-1}\left(\frac{(\mu+\rho)(H)}{2}+q-1\right) \\ & + \frac{n+p+q}{q+1}\lambda_{p,q+1}(q+1)(\mu(H)-\rho(H)-2q) + 2(q-p) \\ = {} & \frac{p}{n+p+q-1}\left(\frac{(\mu+\rho)(H)}{2}+p-1\right) - \frac{n+p-1}{2(p+q+n-1)}(\mu(H)-\rho(H)-2p) \\ & - \frac{q}{n+p+q-1}\left(\frac{(\mu+\rho)(H)}{2}+q-1\right) \\ & + \frac{n+q-1}{2(p+q+n-1)}(\mu(H)-\rho(H)-2q) + 2(q-p) \\ = {} & 0, \end{aligned}$$

where we used $\rho(H) = n$ for the last equality. Let us finally prove that the error terms converge to zero when tested against some test function $\varphi \in C_c^\infty(G)^M$. We first prove that

$$\int_G \frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j}(e_1)\varphi(g)\,\mathrm{d}g \to 0 \tag{D.4.12}$$

for $p, q \to \infty$. Lemma D.2.1 yields

$$(z_1\frac{\partial}{\partial z_j} - \overline{z_j}\frac{\partial}{\partial \overline{z_1}})\pi^*_{Y_{p,q}}(f)(g) = -\frac{1}{2}(\tau_{p,q}(E_{j1} - E_{1j}, 0) - i\tau_{p,q}(iE_{j1} + iE_{1j}, 0))\pi^*_{Y_{p,q}}(f)(g)$$

and thus, by Proposition 2.4.4 ii) and iii), $\frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j}(e_1)$ equals

$$-\frac{1}{2}\langle(\tau_{p,q}(E_{j1} - E_{1j}, 0) - i\tau_{p,q}(iE_{j1} + iE_{1j}, 0))\pi^*_{Y_{p,q}}(f)(g), \phi_{Y_{p,q}}\rangle \dim Y_{p,q}.$$

Thus,

$$\begin{aligned}
&\int_G \frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j}(e_1)\varphi(g)\,\mathrm{d}g \\
&= -\frac{\dim Y_{p,q}}{2}\langle(\int_G \tau_{p,q}(E_{j1} - E_{1j}, 0) - i\tau_{p,q}(iE_{j1} + iE_{1j}, 0))\pi^*_{Y_{p,q}}(f)(g)\varphi(g)\,\mathrm{d}g, \phi_{Y_{p,q}}\rangle.
\end{aligned}$$

Taking absolute values on both sides and using the Cauchy-Schwarz inequality we infer

$$\begin{aligned}
&|\int_G \frac{\partial \pi^*_{Y_{p,q}}(f)(g)}{\partial z_j}(e_1)\varphi(g)\,\mathrm{d}g| \\
&\leq \frac{\dim Y_{p,q}}{2}|\langle\int_G \tau_{p,q}(E_{j1} - E_{1j}, 0)\pi^*_{Y_{p,q}}(f)(g)\varphi(g)\,\mathrm{d}g, \phi_{Y_{p,q}}\rangle| \\
&\quad + \frac{\dim Y_{p,q}}{2}|\langle\int_G \tau_{p,q}(iE_{j1} + iE_{1j}, 0)\pi^*_{Y_{p,q}}(f)(g)\varphi(g)\,\mathrm{d}g, \phi_{Y_{p,q}}\rangle| \\
&= \frac{\dim Y_{p,q}}{2}|\langle\int_G r(E_{1j} - E_{j1}, 0)\pi^*_{Y_{p,q}}(f)(g)\varphi(g)\,\mathrm{d}g, \phi_{Y_{p,q}}\rangle| \\
&\quad + \frac{\dim Y_{p,q}}{2}|\langle\int_G r(-iE_{j1} - iE_{1j}, 0)\pi^*_{Y_{p,q}}(f)(g)\varphi(g)\,\mathrm{d}g, \phi_{Y_{p,q}}\rangle| \\
&= \frac{\dim Y_{p,q}}{2}|\langle\int_G \pi^*_{Y_{p,q}}(f)(g)r(E_{j1} - E_{1j}, 0)\varphi(g)\,\mathrm{d}g, \phi_{Y_{p,q}}\rangle| \\
&\quad + \frac{\dim Y_{p,q}}{2}|\langle\int_G \pi^*_{Y_{p,q}}(f)(g)r(iE_{j1} + iE_{1j}, 0)\varphi(g)\,\mathrm{d}g, \phi_{Y_{p,q}}\rangle| \\
&= \frac{\dim Y_{p,q}}{2}\|\int_G \pi^*_{Y_{p,q}}(f)(g)r(E_{j1} - E_{1j}, 0)\varphi(g)\,\mathrm{d}g\|_{L^2(K)}\|\phi_{Y_{p,q}}\|_{L^2(K)} \\
&\quad + \frac{\dim Y_{p,q}}{2}\|\int_G \pi^*_{Y_{p,q}}(f)(g)r(iE_{j1} + iE_{1j}, 0)\varphi(g)\,\mathrm{d}g\|_{L^2(K)}\|\phi_{Y_{p,q}}\|_{L^2(K)} \\
&= \frac{\sqrt{\dim Y_{p,q}}}{2}\|\int_G \pi^*_{Y_{p,q}}(f)(g)r(E_{j1} - E_{1j}, 0)\varphi(g)\,\mathrm{d}g\|_{L^2(K)}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sqrt{\dim Y_{p,q}}}{2} \| \int_G \pi^*_{Y_{p,q}}(f)(g) r(iE_{j1} + iE_{1j}, 0)\varphi(g)\,\mathrm{d}g \|_{L^2(K)} \\
\le{}& \frac{\sqrt{\dim Y_{p,q}}}{2} \sup_{k\in K} | \int_G \pi^*_{Y_{p,q}}(f)(g)(k.e_1) r(E_{j1} - E_{1j}, 0)\varphi(g)\,\mathrm{d}g | \\
&+ \frac{\sqrt{\dim Y_{p,q}}}{2} \sup_{k\in K} | \int_G \pi^*_{Y_{p,q}}(f)(g)(k.e_1) r(iE_{j1} + iE_{1j}, 0)\varphi(g)\,\mathrm{d}g | \\
\le{}& \frac{\sqrt{\dim Y_{p,q}}}{2} \sup_{k\in K} | \int_G \pi^*_{Y_{p,q}}(f)(g)(e_1) r(E_{j1} - E_{1j}, 0)\varphi(gk^{-1})\,\mathrm{d}g | \\
&+ \frac{\sqrt{\dim Y_{p,q}}}{2} \sup_{k\in K} | \int_G \pi^*_{Y_{p,q}}(f)(g)(e_1) r(iE_{j1} + iE_{1j}, 0)\varphi(gk^{-1})\,\mathrm{d}g |.
\end{aligned}
$$

Now the functions $r(E_{j1} - E_{1j}, 0)\varphi(\bullet\, k^{-1})$ and $r(iE_{j1} + iE_{1j}, 0)\varphi(\bullet\, k^{-1})$ are again smooth and compactly supported and by Remark 7.2.2 and the polynomial growth of $\dim Y_{p,q}$ (see Remark A.2.3) and $\|\pi_{Y_{p,q}}(\pi^*_{Y_{p,q}}(f))\|_{L^2(G)}$ we obtain (D.4.12). The convergence of the remaining error terms can be treated in an analogous manner. □

**Proposition D.4.8.** *Let $\mu \in \mathfrak{a}^*$ and $f_{p,q} \in \pi_{Y_{p,q}}(C^\infty(G \times_K Y_{p,q}))$ be such that $f = \sum_{p,q\in\mathbb{N}_0} f_{p,q} \in \mathcal{D}'(G/M)$. Then $f \in H^{-\infty}_\mu$ if and only if the equations from Lemma D.4.2 hold for $f$.*

*Proof.* The if-part follows from Lemma D.4.5 and Lemma D.4.7. The only-if-part is given by Lemma D.4.2. □

Let us finally mention that, using the embeddings $\pi_{Y_{p,q}}$ from Definition D.4.1, the generalized gradients $\mathbf{d}_{+,\mathrm{(a)hol}}$ and $-\mathbf{d}_{-,\mathrm{(a)hol}}$ are adjoint with respect to the $L^2$-inner product.

**Lemma D.4.9.** *For $u_{p,q} \in C^\infty_c(G \times_K Y_{p,q})$ we have*

$$
\begin{aligned}
\langle \pi_{Y_{p,q}}(u_{p,q}), \pi_{Y_{p,q}}(\mathbf{d}_{+,\mathrm{hol}} u_{p-1,q})\rangle_{L^2(G)} &= -\langle \pi_{Y_{p-1,q}}(\mathbf{d}_{-,\mathrm{hol}} u_{p,q}), \pi_{Y_{p-1,q}}(u_{p-1,q})\rangle_{L^2(G)} \\
\langle \pi_{Y_{p,q}}(u_{p,q}), \pi_{Y_{p,q}}(\mathbf{d}_{+,\mathrm{ahol}} u_{p,q-1})\rangle_{L^2(G)} &= -\langle \pi_{Y_{p,q-1}}(\mathbf{d}_{-,\mathrm{ahol}} u_{p,q}), \pi_{Y_{p,q-1}}(u_{p,q-1})\rangle_{L^2(G)}
\end{aligned}
$$

*for each $u_{a,b} \in C^\infty_c(G \times_K Y_{a,b})$ whenever no index is negative.*

*Proof.* The orthogonality of the $Y_{p,q}$ implies that

$$\langle \pi_{Y_{p,q}}(u_{p,q}), \pi_{Y_{p',q'}}(u_{p',q'})\rangle_{L^2(G)} = 0$$

for $(p,q) \neq (p',q') \in \mathbb{N}_0^2$ by splitting the integral into $G/K$ and $K$. Using Lemma D.4.4 we thus obtain

$$\langle \pi_{Y_{p,q}}(u_{p,q}), \pi_{Y_{p,q}}(\mathbf{d}_{+,\mathrm{hol}} u_{p-1,q})\rangle_{L^2(G)} = \langle \pi_{Y_{p,q}}(u_{p,q}), r(H)\pi_{Y_{p-1,q}}(u_{p-1,q})\rangle_{L^2(G)}.$$

The right invariance of the Haar measure on $G$ implies that

$$\langle \pi_{Y_{p,q}}(u_{p,q}), r(H)\pi_{Y_{p-1,q}}(u_{p-1,q})\rangle_{L^2(G)} = -\langle r(H)\pi_{Y_{p,q}}(u_{p,q}), \pi_{Y_{p-1,q}}(u_{p-1,q})\rangle_{L^2(G)}.$$

Another application of Lemma D.4.4 and the orthogonality gives

$$-\langle r(H)\pi_{Y_{p,q}}(u_{p,q}), \pi_{Y_{p-1,q}}(u_{p-1,q})\rangle_{L^2(G)} = -\langle \pi_{Y_{p-1,q}}(\mathbf{d}_{-,\mathrm{hol}} u_{p,q}), \pi_{Y_{p-1,q}}(u_{p-1,q})\rangle_{L^2(G)}.$$

This proves the first equation and the second is proven analogously. □

## D.5. Spectral correspondence

Without using Theorem 6.1.1, we can still determine constraints on the location of $\Gamma$-invariant elements using only the mapping properties of the scalar Poisson transform and the relations between the Fourier coefficients from Lemma D.3.1.

**Proposition D.5.1.** *Let $\mu \coloneqq -\rho - 2k\alpha \in \mathfrak{a}^*$, for some $k \in \mathbb{N}_0$, be an exceptional spectral parameter (see Lemma 5.2.4). Then*

$$P_\mu^{Y_{p,q}}(f) = 0 \quad \forall (p,q) \in \{0,\dots,k\}^2,\ f \in {}^\Gamma H_\mu^{-\infty}.$$

*Proof.* Note first that the scalar Poisson transform $P_\mu^{Y_{0,0}}$ maps $H_\mu^{-\infty}$ into

$$\{f \in C^\infty(\mathbb{C}\mathbf{H}^n) \colon \Delta f = (\rho(H)^2 - \mu(H)^2) f\},$$

where $\Delta$ denotes the Laplacian. Thus, ${}^\Gamma H_\mu^{-\infty}$ is mapped into

$$\{f \in C^\infty(\Gamma\backslash\mathbb{C}\mathbf{H}^n) \colon \Delta f = (\rho(H)^2 - \mu(H)^2) f\}, \tag{D.5.13}$$

by identifying ${}^\Gamma C^\infty(\mathbb{C}\mathbf{H}^n) \cong C^\infty(\Gamma\backslash\mathbb{C}\mathbf{H}^n)$. Since $\Gamma\backslash\mathbb{C}\mathbf{H}^n$ is a smooth compact Riemannian manifold the positivity of the Laplacian and

$$\rho(H)^2 - \mu(H)^2 = -4k\rho(H) - 4k^2 < 0 \text{ if } k \neq 0$$

imply that the space in Equation (D.5.13) is the zero space if $k \neq 0$ and $P_\mu^{Y_{0,0}}\big|_{{}^\Gamma H_\mu^{-\infty}} = 0$ in this case. Let us prove the same equality for $k = 0$, i.e. $\mu = -\rho$. In this case we have $\rho(H)^2 - \mu(H)^2 = 0$ so that $P_\mu^{Y_{0,0}}(f)$ is constant as a harmonic function on a compact Riemannian manifold. The fourth equation of Lemma D.3.1 for $p = 1$, $q = 0$ shows that

$$(\mathbf{d}_{-,\mathrm{hol}} \circ P_\mu^{Y_{1,0}})(f) = -P_\mu^{Y_{0,0}}(f)$$

and we obtain by Lemma D.4.9 that

$$\begin{aligned}
\|\pi_{Y_{0,0}}(P_\mu^{Y_{0,0}}(f))\|_{L^2(G)}^2 &= -\langle \pi_{Y_{0,0}}(\mathbf{d}_{-,\mathrm{hol}} P_\mu^{Y_{1,0}}(f)), \pi_{Y_{0,0}}(P_\mu^{Y_{0,0}}(f))\rangle_{L^2(G)} \\
&= \langle \pi_{Y_{1,0}}(P_\mu^{Y_{1,0}}(f)), \pi_{Y_{1,0}}(\mathbf{d}_{+,\mathrm{hol}} P_\mu^{Y_{0,0}}(f))\rangle_{L^2(G)} \\
&= 0,
\end{aligned}$$

where we used $\mathbf{d}_{+,\mathrm{hol}} P_\mu^{Y_{0,0}}(f) = 0$ since $P_\mu^{Y_{0,0}}(f)$ is constant and $\mathbf{d}_{+,\mathrm{hol}}$ is a differential operator. Thus, $\pi_{Y_{0,0}}(P_\mu^{Y_{0,0}}(f)) = 0$ and, since $\pi_{Y_{0,0}}$ is injective, $P_\mu^{Y_{0,0}}(f) = 0$.

Finally the first two equations of Lemma D.3.1 read

$$\begin{aligned}
(\mathbf{d}_{+,\mathrm{hol}} \circ P_\mu^{Y_{p,q}})(f) &= (p-k) P_\mu^{Y_{p+1,q}}(f)\, \forall p,q \in \mathbb{N}_0, \\
(\mathbf{d}_{+,\mathrm{ahol}} \circ P_\mu^{Y_{p,q}})(f) &= (q-k) P_\mu^{Y_{p,q+1}}(f)\, \forall p,q \in \mathbb{N}_0,
\end{aligned}$$

and the proposition follows by recursion. □

We now turn to the spectral correspondence. To make the description clearer, we introduce the following rescaled versions of the generalized gradients.

**Notation D.5.2.** For each $p, q \in \mathbb{N}_0$ we define the generalized gradients

$$\widetilde{\mathbf{d}}_{+,\mathrm{hol}} := 2(p+q+n-1)(p+q+n)\mathbf{d}_{+,\mathrm{hol}}, \qquad \widetilde{\mathbf{d}}_{-,\mathrm{hol}} := \frac{1}{p(n+p-2)}\mathbf{d}_{-,\mathrm{hol}},$$
$$\widetilde{\mathbf{d}}_{+,\mathrm{ahol}} := 2(p+q+n-1)(p+q+n)\mathbf{d}_{+,\mathrm{ahol}}, \qquad \widetilde{\mathbf{d}}_{-,\mathrm{ahol}} := \frac{1}{q(n+q-2)}\mathbf{d}_{-,\mathrm{ahol}}$$

on $C^\infty(G \times_K Y_{p,q})$. Moreover, we define the constants $\xi^1_{p,q}$, $\xi^2_{p,q}$, $\xi^3_{p,q}$ and $\xi^4_{p,q}$ by

$$\xi^1_p := \frac{1}{2}(\mu(H)^2 - (\rho(H) + 2(p-1))^2),$$
$$\xi^2_p := \frac{1}{2}(\mu(H)^2 - (\rho(H) + 2p)^2),$$
$$\xi^3_q := \frac{1}{2}(\mu(H)^2 - (\rho(H) + 2q)^2),$$
$$\xi^4_q := \frac{1}{2}(\mu(H)^2 - (\rho(H) + 2(q-1))^2).$$

**Lemma D.5.3.** *Let $\mu \in \mathfrak{a}^*$ and $f \in H^{-\infty}_\mu$. Then*

$$\widetilde{\mathbf{d}}_{+,\mathrm{hol}}\widetilde{\mathbf{d}}_{-,\mathrm{hol}}\pi^*_{Y_{p,q}}(f) = \xi^1_p \pi^*_{Y_{p,q}}(f)$$
$$\widetilde{\mathbf{d}}_{-,\mathrm{hol}}\widetilde{\mathbf{d}}_{+,\mathrm{hol}}\pi^*_{Y_{p,q}}(f) = \xi^2_p \pi^*_{Y_{p,q}}(f)$$
$$\widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\widetilde{\mathbf{d}}_{+,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f) = \xi^3_q \pi^*_{Y_{p,q}}(f)$$
$$\widetilde{\mathbf{d}}_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f) = \xi^4_q \pi^*_{Y_{p,q}}(f)$$
$$\widetilde{\mathbf{d}}_{+,\mathrm{hol}}\widetilde{\mathbf{d}}_{+,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f) = \widetilde{\mathbf{d}}_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}_{+,\mathrm{hol}}\pi^*_{Y_{p,q}}(f)$$
$$\widetilde{\mathbf{d}}_{-,\mathrm{hol}}\widetilde{\mathbf{d}}_{+,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f) = \widetilde{\mathbf{d}}_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}_{-,\mathrm{hol}}\pi^*_{Y_{p,q}}(f)$$
$$\widetilde{\mathbf{d}}_{+,\mathrm{hol}}\widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f) = \widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\widetilde{\mathbf{d}}_{+,\mathrm{hol}}\pi^*_{Y_{p,q}}(f)$$
$$\widetilde{\mathbf{d}}_{-,\mathrm{hol}}\widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\pi^*_{Y_{p,q}}(f) = \widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\widetilde{\mathbf{d}}_{-,\mathrm{hol}}\pi^*_{Y_{p,q}}(f).$$

*Proof.* This follows from Lemma D.4.2. □

**Theorem D.5.4** (Quantum-classical correspondence for exceptional spectral parameters)**.** *Let $\mu := -\rho - 2k\alpha \in \mathfrak{a}^*$, for some $k \in \mathbb{N}_0$, be an exceptional spectral parameter (see Lemma 5.2.4). Recall the irreducible subrepresentation $I_k$ of $H_\mu$ from Lemma B.3.2. By Proposition 5.2.6, the minimal $K$-type of $I_k$ is given by $Y_{k+1,k+1}$. Then the map*

$$P^{Y_{k+1,k+1}}_\mu\Big|_{{}^\Gamma H^{-\infty}_\mu} : {}^\Gamma H^{-\infty}_\mu \to \mathcal{H}_\mathbb{C},$$

*where $\mathcal{H}_\mathbb{C}$ denotes the space*

$$\{u \in {}^\Gamma C^\infty(G \times_K Y_{k+1,k+1}) \mid \textit{properties } i) - vii)\},$$

*is an isomorphism, where the properties are given by* *i)* $\widetilde{\mathbf{d}}_{+,\mathrm{hol}}\widetilde{\mathbf{d}}^q_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}^p_{+,\mathrm{hol}}u = \widetilde{\mathbf{d}}^q_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}^{p+1}_{+,\mathrm{hol}}u \quad \forall (p,q) \in \mathbb{N}_0 \times \mathbb{N}$,

*ii)* $\widetilde{\mathbf{d}}_{-,\mathrm{hol}}\widetilde{\mathbf{d}}^q_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}^p_{+,\mathrm{hol}}u = \xi^2_{k+p}\widetilde{\mathbf{d}}^q_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}^{p-1}_{+,\mathrm{hol}}u \quad \forall (p,q) \in \mathbb{N} \times \mathbb{N}_0$,

*iii)* $\widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\widetilde{\mathbf{d}}^q_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}^p_{+,\mathrm{hol}}u = \xi^3_{k+q}\widetilde{\mathbf{d}}^{q-1}_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}^p_{+,\mathrm{hol}}u \quad \forall (p,q) \in \mathbb{N}_0 \times \mathbb{N}$,

*iv)* $\widetilde{\mathbf{d}}_{-,\mathrm{ahol}}\widetilde{\mathbf{d}}^p_{+,\mathrm{hol}}u = \widetilde{\mathbf{d}}^p_{+,\mathrm{hol}}\widetilde{\mathbf{d}}_{-,\mathrm{ahol}}u \quad \forall p \in \mathbb{N}_0$,

*v)* $\mathbf{D}^j\widetilde{\mathbf{d}}^q_{+,\mathrm{ahol}}\widetilde{\mathbf{d}}^p_{+,\mathrm{hol}}u = 0 \quad \forall (p,q) \in \mathbb{N}_0^2,\ q \in \{1,2\}$,

*vi)* $\mathbf{d}_{-,\mathrm{ahol}}u = 0$,

*vii)* $\mathbf{d}_{-,\mathrm{hol}}u = 0$.

*Moreover, we have*

$$\mathcal{H}_{\mathbb{C}} \cong {}^{\Gamma}I_k^{-\infty},$$

*where* ${}^{\Gamma}I_k^{-\infty}$ *denotes the* $\Gamma$*-invariant distribution vectors in* $I_k$*.*

*Proof.* The proof is separated into the following steps:

i) $\mathrm{im}\left(P_\mu^{Y_{k+1,k+1}}\Big|_{{}^{\Gamma}H_\mu^{-\infty}}\right) \subseteq \mathcal{H}_{\mathbb{C}}$,

ii) $P_\mu^{Y_{k+1,k+1}}\Big|_{{}^{\Gamma}H_\mu^{-\infty}}$ is injective, and

iii) $\mathrm{im}\left(P_\mu^{Y_{k+1,k+1}}\Big|_{{}^{\Gamma}H_\mu^{-\infty}}\right) = \mathcal{H}_{\mathbb{C}}$.

i) The properties $i) - v)$ follow from Lemma D.5.3 respectively Lemma D.4.2. For property $vi)$ note that

$$\mathbf{d}_{-,\mathrm{ahol}}P_\mu^{Y_{k+1,k+1}}(f)$$

is a multiple of $P_\mu^{Y_{k+1,k}}(f)$ by Lemma D.4.2 and Proposition D.5.1 implies that $P_\mu^{Y_{k+1,k}}(f)$ is zero. Property $vii)$ follows from an analogous argument.

ii) The injectivity follows from Proposition 5.1.3 since $\mathbf{Irr}(\mu) = \{I_k\}$ by Lemma B.3.2 and $\mathrm{mult}_K(Y_{k+1,k+1}, I_k) = 1 \neq 0$.

iii) For the surjectivity let $u \in \mathcal{H}_{\mathbb{C}}$. Recall the constants $c_{p,q} \in \mathbb{C} \setminus \{0\}$ from Remark D.3.2. Define $u_{p,q} \coloneqq 0$ for $(p,q) \in \mathbb{N}_0 \times \{0,\dots,k\} \cup \{0,\dots,k\} \times \mathbb{N}_0$ and $u_{k+1,k+1} \coloneqq c_{k+1,k+1} \cdot u$. Fitting the equations from Lemma D.4.2 we recursively define

$$u_{k+1+\ell,k+1} \coloneqq \frac{c_{k+1+\ell,k+1}}{c_{k+\ell,k+1}}\frac{1}{\ell}\mathbf{d}_{+,\mathrm{hol}}u_{k+\ell,k+1}$$

for $\ell \in \mathbb{N}$ and

$$u_{p,k+1+\ell} \coloneqq \frac{c_{p,k+1+\ell}}{c_{p,k+\ell}}\frac{1}{\ell}\mathbf{d}_{+,\mathrm{ahol}}u_{p,k+\ell}$$

for $\ell \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq k+1}$. Let $f$ denote the formal sum

$$f \coloneqq \sum_{p,q \in \mathbb{N}_0} \pi_{Y_{p,q}}(u_{p,q}). \tag{D.5.14}$$

We claim that $f$ defines a distribution and that the equations from Lemma D.4.2 hold for $f$. Then Proposition D.4.8 implies that $f \in H_\mu^{-\infty}$ and by construction (recall $P_\mu^{Y_{p,q}} = \frac{1}{c_{p,q}} \pi_{Y_{p,q}}^*$ from Section D.4) we have

$$P_\mu^{Y_{k+1,k+1}}(f) = \frac{1}{c_{k+1,k+1}} u_{k+1,k+1} = u,$$

finishing the proof of iii) (note that $f$ is $\Gamma$-invariant since $u$ is $\Gamma$-invariant and the generalized gradients are $G$-equivariant).

Let us finally prove the claim. First, a straightforward calculation using the definition of the $u_{p,q}$ and Notation D.5.2 yields that

$$u_{p,q} = c_{p,q} \frac{1}{m_1! m_2!} \widetilde{\mathbf{d}}_{+,\mathrm{ahol}}^{m_2} \widetilde{\mathbf{d}}_{+,\mathrm{hol}}^{m_1} u \cdot \frac{1}{2^{m_1+m_2}(p+q+n-1)(2k+n+1)\prod_{\ell=2k+2}^{p+q-2}(\ell+n)^2},$$

where $m_1 \coloneqq p-(k+1)$ and $m_2 \coloneqq q-(k+1)$. Using this formula it is now direct to see that the properties $i) - vii)$ ensure that the equations from Lemma D.4.2 hold for each $u_{p,q}$. For a proof of the fact that $f$ defines a distribution, we refer the reader to Lemma 8.2.2. □

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# List of Symbols

| Symbol | Description | Page |
|---|---|---|
| $a_I(g)$ | $A$-component of the Iwasawa decomposition of $g \in G$ | 7 |
| $\mathcal{A}'_\tau(\sigma)$ | refined exceptional set of [Olb94] | 46 |
| $a^\mu$ | short notation for $e^{\mu(\log a)}$ | 7 |
| $B_\pm$ | initial resp. end point map | 13 |
| $C^\infty(G \times_\Upsilon V)$ | (isomorphic to) smooth sections of $G \times_\Upsilon V$ | 7 |
| $\mathbf{D}$ | generalized gradient belonging to $\mathrm{pr}_{V_\ell}$ | 107 |
| $\mathbf{D}^j$ | generalized gradient belonging to $\mathrm{pr}_{V^j_{k,\ell}}$ | 188 |
| $\mathbf{d}_-$ | generalized gradient belonging to $T_{\ell-1}$ | 107 |
| $\mathbf{d}_{-,\mathrm{ahol}}$ | generalized gradient belonging to $T^{p,q,-,\mathrm{ahol}}$ | 188 |
| $\mathbf{d}_{-,\mathrm{hol}}$ | generalized gradient belonging to $T^{p,q,-,\mathrm{hol}}$ | 188 |
| $\mathbf{d}_+$ | generalized gradient belonging to $T^{\ell+1}$ | 107 |
| $\mathbf{d}_{+,\mathrm{ahol}}$ | generalized gradient belonging to $T^{p,q,+,\mathrm{ahol}}$ | 188 |
| $\mathbf{d}_{+,\mathrm{hol}}$ | generalized gradient belonging to $T^{p,q,+,\mathrm{hol}}$ | 188 |
| $\mathcal{D}'(G \times_K Y)$ | dual of $C_c^\infty(G \times_K \tilde{Y})$ | 67 |
| $\mathrm{d}_V^Y$ | generalized gradient given by $T_V^Y \circ \nabla$ | 78 |
| $E_0, E_u, E_s$ | Anosov splitting of the tangent bundle | 8 |
| $E_{\sigma,\mu}$ | space of joint eigensections | 22 |
| $F$ | Frobenius isomorphism | 23 |
| $\mathfrak{g}$ | complexification of $\mathfrak{g}_0$ | 7 |
| $\mathfrak{g}_0$ | Lie algebra of $G$ | 7 |
| $G \times_\Upsilon V$ | associated vector bundle | 7 |
| $H$ | element in $\mathfrak{a}_0$ defined by $\alpha(H) = 1$ | 9 |
| $\mathbf{H}^n$ | hyperbolic space of real dimension $n$ | 8 |
| $H_{\sigma,\mu}$ | principal series representation | 11 |
| $H^{\mathrm{cpt}}_{\sigma,\mu}$ | compact picture of principal series representation | 12 |
| $H_\mu$ | spherical principal series representation | 12 |
| $H^{\mathrm{opp}}_{\sigma,\mu}$ | opposite principal series representation | 12 |
| $H(g)$ | $\mathfrak{a}_0$-component $\log a_I(g)$ of $g$ | 12 |
| $H_-(g)$ | $\mathfrak{a}_0$-component $\log a_I^-(g)$ of $g$ | 12 |
| $\mathcal{H}^{\pm\infty}$ | smooth resp. distributional globalization of $(\pi, \mathcal{H})$ | 16 |
| $\mathbf{I}$ | isomorphism between $\mathfrak{p}$ and $\mathfrak{p}^*$ | 53 |
| $\hat{K}$ | set of equivalence classes of irreducible unitary representation of $K$ | 7 |
| $k_I(g)$ | $K$-component of the Iwasawa decomposition of $g \in G$ | 7 |
| $\hat{K}_M$ | $M$-spherical representations in $\hat{K}$ | 18 |

| Symbol | Description | Page |
|---|---|---|
| $k_\varphi$ | specific element of PSO(2) | 28 |
| $\mathcal{K}$ | canonical line bundle on $\mathbf{H}^2$ | 28 |
| $\mathcal{K}_\Gamma$ | canonical line bundle on $\Gamma\backslash\mathbf{H}^2$ | 27 |
| $\ell$ | derived left regular representation on $C^\infty(K/M)$ | 18 |
| $L$ | left regular representation on $L^2(K/M)$ | 18 |
| $(L_m, \mathbb{C}_m)$ | representation of PSO(2) | 28 |
| $M$ | centralizer of $A$ in $K$ | 7 |
| $\mathbf{M}$ | smooth compact Riemannian locally symmetric space of rank 1 defined by $\Gamma\backslash G/K$ | 8 |
| $\mathcal{M}$ | unit sphere bundle of $\mathbf{M}$, isomorphic to $\Gamma\backslash G/M$ | 8 |
| $n_I(g)$ | $N$-component of the Iwasawa decomposition of $g \in G$ | 7 |
| P | projection onto the harmonic part of a homogeneous polynomial | 160 |
| $P_\mu^{Y_\ell}$ | specific Poisson transform associated with $Y_\ell$ | 113 |
| $P_\mu^{Y_\ell,\mathrm{cpt}}$ | specific Poisson transform associated with $Y_\ell$, compact realization | 112 |
| $P_\mu^{Y_{p,q}}$ | specific Poisson transform associated with $Y_{p,q}$ | 196 |
| $P_\mu^{Y_{p,q},\mathrm{cpt}}$ | specific Poisson transform associated with $Y_{p,q}$, compact realization | 196 |
| $P_\mu^\tau$ | vector-valued Poisson transform associated with $\tau$ | 44 |
| $\mathcal{Q}_{\mu,\pm}$ | initial resp. end point transform | 14 |
| $\mathrm{Res}_X(\lambda)$ | Ruelle resonant states | 9 |
| $\mathrm{Res}_X^0(\lambda)$ | first band Ruelle resonant states | 9 |
| $\mathrm{Res}_{X^*}(\lambda)$ | Ruelle co-resonant states | 9 |
| $\mathrm{Res}_{X^*}^0(\lambda)$ | first band Ruelle co-resonant states | 9 |
| $\mathcal{R}_\pm(\mu)$ | resonant states on the cover $S\mathbf{H}^n = G/M$ | 10 |
| $\mathbb{S}^{n-1}$ | unit sphere in $\mathbb{R}^n$ | 160 |
| $S\mathbf{H}^n$ | unit sphere bundle of $\mathbf{H}^n$, isomorphic to $G/M$ | 8 |
| $\mathrm{Sp}(m)$ | symplectic group | 174 |
| $T_{\ell-1}$ | element of $\mathrm{Hom}_K(Y_\ell \otimes \mathfrak{p}^*, Y_{\ell-1})$ | 108 |
| $T^{\ell+1}$ | element of $\mathrm{Hom}_K(Y_\ell \otimes \mathfrak{p}^*, Y_{\ell+1})$ | 108 |
| $T_V^Y$ | specific element of $\mathrm{Hom}_K(Y \otimes \mathfrak{p}^*, V)$ | 77 |
| $\mathcal{U}$ | universal enveloping algebra | 7 |
| $V_{a,b}$ | irreducible $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$-representation | 175 |
| $W$ | Weyl group of $(\mathfrak{g}_0, \mathfrak{a}_0)$ | 7 |
| $w_-$ | longest Weyl group element | 12 |
| $\mathrm{WF}(u)$ | wave-front set of a distribution $u$ | 9 |
| $Y_\ell$ | spherical harmonics of degree $\ell$ | 160 |
| $\xi_\ell$ | a constant | 117 |
| $\alpha$ | unique simple positive restricted root of $(\mathfrak{g}_0, \mathfrak{a}_0)$ | 9 |
| $\Delta_\mathfrak{k}$ | root system of $\mathfrak{k}$ | 45 |
| $\Delta_\mathfrak{k}^+$ | positive roots of $\Delta_\mathfrak{k}$ | 45 |
| $\eta_\pm$ | raising/lowering operator | 33 |

| **Symbol** | **Description** | **Page** |
|---|---|---|
| $\iota_{G/M}$ | embedding of $C^\infty(G/M)$ into $\mathcal{D}'(G/M)$ | 67 |
| $\iota_Y$ | embedding of $C^\infty(G \times_K Y)$ into $\mathcal{D}'(G \times_K Y)$ | 67 |
| $\iota_Y^V$ | right inverse of $T_V^Y$ | 78 |
| $\kappa$ | Killing form of $\mathfrak{g}$ | 7 |
| $\lambda_\ell$ | a constant | 113 |
| $\omega$ | matrix coefficient map from $\mathfrak{p}$ to $C^\infty(K/M)$ | 77 |
| $\Phi_\pm$ | $K$-invariant, smooth elements of $H_\pm^{\alpha+\rho}$ | 14 |
| $\pi_\Gamma$ | quotient map from $S\mathbf{H}^n$ to $S\mathbf{M}$ | 10 |
| $\pi_\ell^*$ | embedding of $C^\infty(G \times_K Y_\ell)$ into $C^\infty(G)^M$ | 115 |
| $\pi_m^*$ | embedding of $C_c^\infty(\mathbf{M}, \mathcal{K}_\Gamma^m)$ into $C_c^\infty(S\mathbf{M})$ | 27 |
| $\tilde{\pi}_m^*$ | embedding of $C_c^\infty(\mathbf{H}^2, \mathcal{K}^m)$ into $C_c^\infty(S\mathbf{H}^2)$ | 29 |
| $\pi_{m*}$ | pullback of $\pi_m^*$ | 27 |
| $\pi_{p,q}^*$ | embedding of $C^\infty(G \times_K Y_{p,q})$ into $C^\infty(G)^M$ | 200 |
| $\pi_Y$ | embedding of $\mathcal{D}'(G \times_K Y)$ into $\mathcal{D}'(G/M)$ | 67, 69 |
| $\pi_Y^*$ | pullback of $\pi_{\tilde{Y}}$ | 68 |
| $\rho$ | half-sum of the positive restricted roots with multiplicities | 7 |
| $\rho_c$ | half-sum of the positive roots $\Delta_{\mathfrak{k}}^+$ | 46 |
| $\Sigma$ | restricted roots | 7 |
| $\Sigma^+$ | positive restricted roots | 7 |
| $\theta$ | Cartan involution on $\mathfrak{g}_0$ and $G$ | 7 |