Linear programming bounds in classical association schemes

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Lying, thinking
Last night
How to find my soul a home
Where water is not thirsty
And bread loaf is not stone
I came up with one thing
And I don’t believe I’m wrong
That nobody,
But nobody,
Can make it out here alone.

—Maya Angelou, excerpt from *Alone*

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Zusammenfassung

Digitale Kommunikation beruht in hohem Maße auf der Verwendung verschiedener Arten von Codes. Heutzutage wichtige Codes sind Rang-Metrik-Codes und Unterraumcodes – die q-Analoga von binären Codes und binären Codes mit konstantem Gewicht. All diese Codes können als Teilmengen klassischer Assoziationsschemata betrachtet werden. Ein zentrales codierungstheoretisches Problem besteht darin, obere Schranken für die Größe von Codes zu geben. Diese Arbeit untersucht Delsartes mächtiges lineares Optimierungsproblem, dessen Optimum genau eine solche Schranke für Codes in Assoziationsschemata ist. Die linearen Optimierungsprobleme für binäre Codes und binäre Codes mit konstantem Gewicht wurden seit den 1970er Jahren ausgiebig untersucht, aber ihr Optimum ist noch unbekannt. Wir bestimmen auf einheitliche Weise das Optimum des linearen Optimierungsproblems in verschiedenen gewöhnlichen q-Analoga sowie in deren affinen Pendants. Insbesondere werden Schranken und Konstruktionen für Codes in Polarräumen hergeleitet, wobei die Schranken in mehreren Fällen bis auf einen konstanten Faktor optimal sind. Darüber hinaus wird auf der Grundlage dieser Resultate eine fast vollständige Klassifizierung von Steiner-Systemen in Polarräumen gegeben, indem bewiesen wird, dass diese nur in wenigen Spezialfällen existieren könnten.

Abstract

Digital communications relies heavily on the usage of different types of codes. Prominent codes nowadays are rank-metric codes and subspace codes—the q-analogs of binary codes and binary codes with constant weight. All these codes can be viewed as subsets of classical association schemes. A central coding-theoretic problem is to derive upper bounds for the size of codes. This thesis investigates Delsarte's powerful linear program whose optimum is precisely such a bound for codes in association schemes. The linear programs for binary codes and binary constant-weight codes have been extensively studied since the 1970s, but their optimum is still unknown. We determine in a unified way the optimum of the linear program in several ordinary q-analogs as well as in their affine counterparts. In particular, bounds and constructions for codes in polar spaces are established, where the bounds are sharp up to a constant factor in many cases. Moreover, based on these results, an almost complete classification of Steiner systems in polar spaces is provided by showing that they could only exist in some corner cases.

Contents

—Reduction of *Spring 1* (excerpt 47″–56″)
from *Recomposed by Max Richter: Vivaldi - The Four Seasons*

Chapter 1

Introduction

This thesis investigates codes and designs as subsets of association schemes by using a linear programming method developed by Delsarte in the 1970s.

More specifically, we will focus on *rank-metric codes* and *subspace codes*, which gained particular interest in recent years because of their applications in network coding [MS12]—an area on which today's digital communications heavily relies on. Namely, nowadays, large amounts of data are stored or transmitted via a network of intermediate nodes, for example, if one uses video streaming, file distribution, peer-to-peer networking, or distributed storage. In these networks, the data can be disrupted because of noise, for example, by electromagnetic interference in cables when sending data over wires. Codes ensure that the original data can be recovered from the data exposed to noise if not too many errors occurred. In [KK08] and [SKK08], it was proposed to use matrices and finite-dimensional vector spaces over the finite field $\mathbb{F}_q$ for network coding and it turned out that they are very well suited for the task of error-correction in network communications. Rank-metric codes and subspace codes are the q-analogs of the classical codes—the *binary codes* and the *binary constant-weight codes*. We will later see in this thesis that the notion "q-analog" comes from the fact that combinatorics of sets can be seen as the limiting case $q \to 1$ of combinatorics of vector spaces over $\mathbb{F}_q$, where vector spaces are replaced by sets and dimension by cardinality.

In general, one can think of a d-code Y as a finite subset in a metric space such that d is the *minimum distance* that can occur between two distinct elements of Y. Designs on the other hand are subsets of a given space that approximate the whole space in a precise way. Design theory originated in the first half of the 19th century, where the first example of a *t-combinatorial design* was given, which is a collection Y of subsets, having the same cardinality, of a finite set V such that every t-subset of V lies in exactly λ members of Y. For $\lambda = 1$, one obtains a special type of a combinatorial design called *t-Steiner system*. As an example, a 2-Steiner system is given in Figure 1.1, where the set V consists of all the seven depicted points and the lines are the members of the 2-Steiner

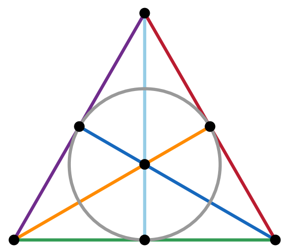


Figure 1.1. A 2-Steiner system.

system.

From an algebraic-combinatorial viewpoint, codes and designs can be seen as dual concepts and an immensely suitable framework to study them is the rich theory of association schemes. As Delsarte and Levenshtein [DL98] phrased it:

> *"In coding theory and related subjects, an association scheme* [...] *should mainly be viewed as a 'structured space' in which objects of interest* (*such as codes, or designs*) *are living."*

More precisely, an *association scheme with n classes* consists of a finite set X together with $n+1$ relations $R_0, R_1, \ldots, R_n$ on $X \times X$ such that the corresponding adjacency matrices $D_0, D_1, \ldots, D_n$ span a commutative matrix algebra over the complex numbers, called *Bose-Mesner algebra*, where D_0 is the identity matrix, the sum of all D_i is the all-ones-matrix, and D_i^T lies in $\{D_0, D_1, \ldots, D_n\}$ for all i. One can show that the Bose-Mesner algebra has a unique basis of primitive idempotent matrices. The coefficients that occur in the change of basis between them and the adjacency matrices are essential for the theory of association schemes and are called *P-* and *Q-numbers*. In particular, the P-numbers are the eigenvalues of the adjacency matrices, which are simultaneously diagonalizable and have exactly $n+1$ maximal common eigenspaces $V_0, V_1, \ldots, V_n$.

We will focus on *classical association schemes*, where the notion "classical" stems from their connection to distance-regular graphs with classical parameters [BCN89, § 6]. They are special in the sense that there exist orderings of the adjacency matrices $D_0, D_1, \ldots, D_n$ and of the eigenspaces $V_0, V_1, \ldots, V_n$ such that the P- and Q-numbers are given as evaluations of orthogonal polynomials. Because of these orderings, we can define a *d-code* as a subset Y of X such that

$$(Y \times Y) \cap R_i = \emptyset \quad \text{for all } i = 1, 2, \ldots, d-1$$

and a *t-design* as a subset Y of X such that its characteristic vector ϕ_Y satisfies

$$\phi_Y \notin V_1 + \cdots + V_t.$$

Particularly, in a classical association scheme, the relations R_i define a metric δ on X by

$$\delta(x,y) = i \quad \text{if and only if} \quad (x,y) \in R_i,$$

so that a d-code in such a classical association scheme is a d-code in the metric space (X, δ).

The objective in coding theory is to construct d-codes with as many elements as possible since we aim for a high information rate during the transmitting and storing process. This gives rise to one of the central coding-theoretic problems.

Problem 1. *What is the maximum cardinality of a d-code in a given metric space?*

So, we would like to derive upper bounds on the cardinality of d-codes and construct codes that reach these upper bounds. For designs on the other hand, we want to give lower bounds on their cardinality since they are some kind of approximation of their whole space leading to the following problem.

Problem 2. *What is the minimum cardinality of a t-design in a classical association scheme?*

One is also interested in the existence of nontrivial designs, especially designs that are extremal in some sense. For example, a t-Steiner system is a t-design in an association scheme, called *Johnson scheme*, and is an optimal $(n-t+1)$-code. So, we also study the following problem.

Problem 3. *Do nontrivial "extremal" t-designs in a classical association scheme exist?*

In this thesis, we will focus on Problem 1 and 3.

A landmark result in Delsarte's PhD thesis [Del73] is the *linear programming method*, which treats the problem of finding the maximum cardinality of a code as an extremum problem for subsets in association schemes. The idea behind the linear program is to first associate an *inner distribution* $(A_0, A_1, \ldots, A_n)$ with a d-code Y given by

$$A_i = \frac{|(Y \times Y) \cap R_i|}{|Y|} \quad \text{for all } i = 0, 1, \ldots, n,$$

so that A_i is basically the average number of pairs from $Y \times Y$ that lie in R_i. It can be readily verified that the inner distribution of a d-code Y satisfies

- $A_0 + A_1 + \cdots + A_n = |Y|$
- $A_0 = 1$
- $A_i \geq 0$ for all $i = 0, 1, \ldots, n$

- $A_i = 0$ for all $i = 1, 2, \ldots, d-1$

Recall that there exist some primitive idempotents $E_0, E_1, \ldots, E_n$ that constitute a second basis for the Bose-Mesner algebra. Hence, the matrices $E_0, E_1, \ldots, E_n$ are positive semidefinite implying that the characteristic vector ϕ_Y of Y satisfies $\phi_Y^T E_k \phi_Y \geq 0$ for all $k = 0, 1, \ldots, n$. It is readily verified that the condition $\phi_Y^T E_k \phi_Y \geq 0$ can be written as

$$\sum_{i=0}^{n} Q_k(i) A_i \geq 0,$$

where $Q_k(i)$ are the Q-numbers that occur when writing E_k as a linear combination of the matrices D_i. Combining all the aforementioned properties of the inner distribution gives Delsarte's linear program of the form

$$\begin{array}{ll} \text{maximize} & A_0 + A_1 + \cdots + A_n \\ \text{subject to} & A_0 = 1 \\ & A_i \geq 0 \quad \text{for all } i = d, d+1, \ldots, n \\ & A_i = 0 \quad \text{for all } i = 1, 2, \ldots, d-1 \\ & \sum\limits_{i=0}^{n} Q_k(i) A_i \geq 0 \quad \text{for all } k = 1, 2, \ldots, n \end{array}$$

The optimum of this linear program gives an upper bound on the cardinality of a d-code in the respective association scheme. The linear program can be solved numerically for a given set of parameters (e.g., number of classes in the association scheme, minimum distance of the code, etc.). However, we will see that the linear program can be also used to obtain analytic bounds and moreover, to characterize the cases, where equality holds.

The most important classical association schemes for classical coding and design theory are the *Hamming scheme* and the *Johnson scheme*, where codes in the first are the q-ary codes and in the latter the binary constant-weight codes. More concretely, the Hamming scheme consists of the set X that contains all n-tuples with entries from $\{1, 2, \ldots, q\}$ and the relations R_i consist of all pairs from $X \times X$ that differ in exactly i positions. For the Johnson scheme, the set X contains all n-subsets of a given v-set and the relations R_i consist of all pairs from $X \times X$ such that their intersection has exactly $n - i$ elements. This thesis will focus on the q-analogs of the Hamming and Johnson scheme that are also classical association schemes. These q-analogs can be further categorized into *ordinary q-analogs* and *affine q-analogs*. The first ones consist of the *q-Johnson scheme $J_q(n, m)$* and six *polar space schemes*: two *Hermitians $^2A_{2n-1}$ and $^2A_{2n}$*, *symplectic C_n*, *hyperbolic D_n*, *parabolic B_n*, and *elliptic $^2D_{n+1}$*. For the polar space schemes, the set X contains all maximal totally isotropic subspaces of a finite vector space equipped with a nondegenerate form and the relations R_i consist of all pairs from $X \times X$ such that the dimension of their intersection is exactly $n - i$, where n is called the *rank* of the polar space and is defined as the dimension of the maximal totally isotropic subspaces. For the q-Johnson scheme $J_q(n,m)$, the set X contains all n-dimensional subspaces of a given $(m+n)$-dimensional vector space over $\mathbb{F}_q$ and the relations R_i consist of all pairs from $X \times X$ such the dimension of their intersection is exactly $n-i$. Moreover, it is well known that the maximal totally isotropic subspaces of the hyperbolic polar space D_n can be partitioned into two systems, called *Latin* and *Greek*, such that each system also gives rise to a classical association scheme denoted by $\frac{1}{2}D_n$ and called *bipartite half of* D_n. The affine q-analogs are the *bilinear forms scheme* $\mathrm{Bil}_q(n,m)$ consisting of all $m \times n$ matrices over $\mathbb{F}_q$, the *alternating bilinear forms scheme* $\mathrm{Alt}_q(m)$ consisting of all alternating $m \times m$ matrices over $\mathbb{F}_q$, and the *Hermitian forms scheme* $\mathrm{Her}_q(n)$ consisting of all Hermitian $n \times n$ matrices over $\mathbb{F}_{q^2}$. The relations R_i of the affine q-analogs contain all pairs from $X \times X$ whose difference has rank i or in the case of $\mathrm{Alt}_q(m)$, it has rank $2i$. All six polar space schemes and the association schemes $J_q(n,m)$, $\frac{1}{2}D_m$, $\mathrm{Bil}_q(n,m)$, $\mathrm{Alt}_q(m)$, and $\mathrm{Her}_q(n)$ have n classes, where $n = \lfloor m/2 \rfloor$ in the case of $\mathrm{Alt}_q(m)$ and $\frac{1}{2}D_m$. The codes in the ordinary q-analogs are the subspace codes and in the affine q-analogs the rank-metric codes. Throughout this thesis, we will see a remarkable resemblance between the ordinary and affine q-analogs.

Known results

Even though at first sight, the idea behind Delsarte's linear program might seem simple, it is a strikingly effective method yielding "good" bounds for different types of codes. For example, the best known asymptotic bound for the cardinality of binary codes and of binary constant-weight codes was derived by applying this linear program in [McE+77]. Moreover, the method was also used to show that a d-code Y with $1 \le d \le n$ in the affine q-analog $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, or $\mathrm{Alt}_q(m)$ satisfies

$$|Y| \le (cb^n)^{n-d+1}, \tag{1.1}$$

where d is required to be odd in the case of $\mathrm{Her}_q(n)$ and the parameters b and c are defined by

$$(b,c) = \begin{cases} (q, q^{m-n}) & \text{for } \mathrm{Bil}_q(n,m) \\ (-q,-1) & \text{for } \mathrm{Her}_q(n) \\ (q^2, 1/q) & \text{for } \mathrm{Alt}_q(m) \text{ if } m \text{ is even} \\ (q^2, q) & \text{for } \mathrm{Alt}_q(m) \text{ if } m \text{ is odd,} \end{cases}$$

see [Del78a], [Sch18], and [DG75]. Except for $\mathrm{Alt}_q(m)$ with even m and odd q, there exist constructions reaching the bound (1.1), which were developed in the three aforementioned papers. So, Problem 1 is answered in $\mathrm{Bil}_q(n,m)$, in

$\mathrm{Her}_q(n)$ for odd d, and in $\mathrm{Alt}_q(m)$ except for even m and odd q. We will see in Chapter 5 that there also exists a bound for even d in $\mathrm{Her}_q(n)$ from [Sch18] that looks slightly more complicated than (1.1). However, there are no known constructions reaching the bound in $\mathrm{Her}_q(n)$ if d is even. For the q-Johnson scheme, a d-code Y with $1 \le d \le n$ in $J_q(n,m)$ satisfies

$$|Y| \le \frac{\begin{bmatrix} m+n \\ n-d+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ n-d+1 \end{bmatrix}_q}. \tag{1.2}$$

This bound was first proved in [WXS03] and a proof based on linear programming was given in [ZJX11]. It was shown in [SKK08] that the bound (1.2) is sharp up to a constant factor by using known optimal rank-metric codes (*MRD codes*) in $\mathrm{Bil}_q(n,m)$, answering Problem 1 for $J_q(n,m)$ asymptotically. In the literature, the bound (1.2) is known under different names, for example, Wang-Xing-Safavi-Naini bound, anticode bound, and packing bound. However, we will see in this thesis that the bounds (1.2) and (1.1) deserve the name *Singleton bound*.[1] Delsarte's linear programming method was also used to derive a bound for codes consisting of symmetric matrices, which is sharp in some cases ([Sch10], [Sch15], [Sch20]). There, the codes correspond to subsets in the *symmetric bilinear forms scheme*, which is a nonclassical association scheme.

Although Problem 1 concerning binary codes and binary constant-weight codes has been studied for decades, a solution for these codes seems still out of reach. In particular, their linear programs have been investigated extensively and it is still unknown what their optima look like. This is different for their q-analogs, for example, since the Singleton bound for codes in the bilinear forms scheme was proved by using the linear program and there exist constructions (MRD codes) reaching this bound [Del78a], the Singleton bound is precisely the optimum of the linear program. Similarly for the other rank-metric codes in the cases, where the bounds obtained in [DG75], [Sch18], and [Sch20] are sharp. In particular, the optima are known for $\mathrm{Alt}_q(m)$ except if m is even and q is odd and for $\mathrm{Her}_q(n)$ if d is odd. To our knowledge, nothing was known about the optimum of the linear program for codes in the q-Johnson scheme.

Codes that are optimal in some sense are often some kind of a design, for example, a d-code in $J_q(n,m)$ of size (1.2) is an $(n-d+1)$-design. The designs in the q-Johnson scheme have a combinatorial interpretation and were introduced in the 1970s (see [Del78b], [Cam74]). They are a q-analog version of a combinatorial design, namely, a *t-design over* $\mathbb{F}_q$ is a collection Y of subspaces, having the same dimension, of a finite-dimensional vector space V

[1] We note that there already exists a different bound called Singleton bound for codes in the q-Johnson scheme [KK08], which can be shown by using a puncturing argument similarly to the proof of the Singleton bound for q-ary codes and rank-metric codes in $\mathrm{Bil}_q(n,m)$.

over $\mathbb{F}_q$ such that every t-dimensional subspace of V lies in exactly λ members of Y. For $\lambda = 1$, one obtains a *t-Steiner system over* $\mathbb{F}_q$ and a d-code Y is of size (1.2) if and only if Y is an $(n-d+1)$-Steiner system over $\mathbb{F}_q$.

It was shown in [Tei87] that nontrivial t-combinatorial designs exist for all t. In [FLV14], a similar result was proved for t-designs over $\mathbb{F}_q$, namely that they exist for all t and q if the size of the ambient vector space is large enough and if the dimension of the members of the design is large enough compared to t. Whereas it was shown in [Kee14] that nontrivial classical Steiner systems are abundant (see also [Glo+16] for a different proof), it is quite sobering that nontrivial Steiner systems over $\mathbb{F}_q$ are so far only known to exist for one set of parameters [Bra+16].

We will examine t-Steiner systems in polar spaces, which are sets Y of maximal totally isotropic subspaces such that every t-dimensional totally isotropic subspace of the polar space lies in exactly one element of Y. The special case of $t = 1$ corresponds to the well-known *spreads* in polar spaces. Although spreads have been heavily studied since the 1960s, their existence question is still not fully resolved (see [HT16, § 7.4] for the current status). For $t = n-1$, it was known that $(n-1)$-Steiner systems cannot exist in ${}^2A_{2n-1}$, B_n, and C_n, and in ${}^2A_{2n}$ and ${}^2D_{n+1}$ if $q = 2$, see [Van11, p. 160]. Moreover, it was known that $(n-1)$-Steiner systems in D_n always exist—namely they are the two systems Latin and Greek. Not much was known about the existence of t-Steiner systems in polar spaces with $1 < t < n-1$. Only recently, it was shown in [Cos+22] that t-Steiner systems cannot exist if $(n,t) = (4,2)$ and $(n,t) = (5,3)$.

New results

To state two main results of this thesis, we extend the definition of the parameters b and c as follows

$$(b,c) = \begin{cases} (q, q^{m-n}) & \text{for } \mathrm{Bil}_q(n,m) \text{ and } J_q(n,m) \\ (-q,-1) & \text{for } \mathrm{Her}_q(n) \text{ and } {}^2A_{2n-1} \\ (q^2, 1/q) & \text{for } \mathrm{Alt}_q(m) \text{ and } \tfrac{1}{2}D_m \text{ if } m \text{ is even} \\ (q^2, q) & \text{for } \mathrm{Alt}_q(m) \text{ and } \tfrac{1}{2}D_m \text{ if } m \text{ is odd.} \end{cases}$$

Our first main result contains the optima of the linear programs for several ordinary and affine q-analogs.

Theorem 1.

(*a*) *Let X be the set of n-spaces in $J_q(n,m)$ or maximal totally isotropic subspaces in ${}^2A_{2n-1}$ or $\frac{1}{2}D_m$, where $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$. Then the optimum* *of the linear program for d-codes with* $1 \le d \le n$ *in* $J_q(n,m)$, ${}^2A_{2n-1}$, *and* $\frac{1}{2}D_m$ *is given by*

$$|X| \prod_{\ell=0}^{d-2} \frac{qb^\ell - 1}{qcb^{n+\ell} - 1}, \tag{1.3}$$

where d is required to be odd in the case of ${}^2A_{2n-1}$.

(*b*) *The optimum of the linear program for d-codes with* $1 \le d \le n$ *in* $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, *and* $\mathrm{Alt}_q(m)$ *is given by*

$$(cb^n)^{n-d+1},$$

where d is required to be odd in the case of $\mathrm{Her}_q(n)$ *and* $n = \lfloor m/2 \rfloor$ *in the case of* $\mathrm{Alt}_q(m)$.

Observe that there is a nice resemblance between the optima for the ordinary and affine q-analogs since we have

$$(cb^n)^{n-d+1} = |X| \prod_{\ell=0}^{d-2} \frac{qb^\ell}{qcb^{n+\ell}},$$

where $|X| = (cb^n)^n$ is the number of matrices in $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$. In the case of $J_q(n,m)$, we have $|X| = \begin{bmatrix} m+n \\ n \end{bmatrix}_q$ and the optimum (1.3) can thus be written as

$$\frac{\begin{bmatrix} m+n \\ n-d+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ n-d+1 \end{bmatrix}_q},$$

which is the known bound (1.2). Therefore, the optima in $J_q(n,m)$, $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ are precisely the Singleton bounds (1.2) and (1.1).

We will moreover obtain the optimum of the linear program for d-codes in $\mathrm{Her}_q(n)$ and ${}^2A_{2n-1}$ if d is even, in D_n if d is even, and in B_n and C_n if d is odd.

The proof of Theorem 1 given in Chapter 5 will rely on unified expressions of the P- and Q-numbers of $J_q(n,m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ as dual q-Hahn polynomials and q-Hahn polynomials, respectively, as well as of the P- and Q-numbers of $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ as affine q-Krawtchouk polynomials.

The bound (1.3) for codes in ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$ will be already derived in Chapter 3. Based on this bound for ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$, we will then obtain bounds for codes in all the remaining polar spaces ${}^2A_{2n}$, B_n, C_n, D_n, and ${}^2D_{n+1}$. We will also use known constructions of codes in the alternating bilinear, Hermitian, and symmetric bilinear forms scheme to construct codes in all polar spaces. Additionally, we will exploit these constructions to show that our obtained bounds are sharp up to a constant factor in several cases and thus, we will answer Problem 1 for codes in polar spaces from an asymptotic viewpoint.

Exploiting our bounds for codes in polar spaces, we will classify nontrivial t-Steiner systems in polar spaces by proving their nonexistence except for some corner cases. This is remarkable since the classical Steiner systems are ubiquitous, whereas basically nothing is known for Steiner systems over $\mathbb{F}_q$. More specifically, our second main result, which will be proved in Chapter 4, is the following.

Theorem 2. *Suppose that a polar space $\mathcal{P}$ of rank n contains a t-Steiner system with $1 < t < n$. Then one of the following holds*

(1) $t = n - 1$ *and* $\mathcal{P} = D_n$*;*

(2) $t = n - 1$ *and* $\mathcal{P} = {}^2A_{2n}$ *or* ${}^2D_{n+1}$ *for* $q \geq 3$*;*

(3) $t = 2$ *and* $\mathcal{P} = {}^2A_{2n}$ *or* ${}^2D_{n+1}$ *for odd* n.

Outline

Chapter 2 will introduce the required preliminaries about association schemes and linear programming. In particular, we will present the classical association schemes in more detail in Section 2.3. Tabular overviews of the classical association schemes can be found in Appendix A.

Every chapter will be preceded with an outline of its structure and a detailed introduction to its contents, especially stating its main results in the case of Chapter 4 and 5. Moreover, Chapter 3, 4, and 5 will be closed with a discussion of open problems arising from the results of the respective chapters or related to the topics considered therein. Finally, we will give a summary of the new results of this thesis in Chapter 6.

A list of all the notation occurring in this thesis can be found on page 155.

Chapter 2

Association schemes

Beautiful things don't ask for attention.
—From the movie *The Secret Life of Walter Mitty*

Association schemes are fundamental in algebraic combinatorics. They give us an algebraic approach to study relations between elements of a finite set. Namely, we can associate a commutative matrix algebra with these relations if they satisfy some "nice" properties. By then using tools from, for example, linear algebra, harmonic analysis, or optimization theory, we can derive combinatorial properties of special subsets, called codes and designs, of this finite set. The theory behind this, and which we will apply in this thesis, goes back to Delsarte's groundbreaking PhD thesis [Del73], where he established a beautiful connection between coding and design theory by unifying both areas in the framework of association schemes. In this setting, codes and designs can be viewed as dual objects. For codes, one is interested in finding large subsets in a metric space such that specified distances cannot occur. Whereas for designs, one wants a small subset of the space such that its characteristic vector cannot lie in some specified eigenspaces associated with the commutative matrix algebra. Delsarte moreover developed a strong method, based on linear programming, to obtain bounds for the size of codes and designs and to characterize the extremal cases, when the bounds are tight. This method will be heavily applied and investigated throughout this thesis.

We start this chapter by giving some background on association schemes in Section 2.1. Afterwards, we will shortly look at some facts from the theory of linear programming that are needed to introduce Delsarte's linear programming method in Section 2.2. The last part of this chapter, Section 2.3, deals with the classical association schemes that we focus on in this thesis.

2.1 Background on association schemes

In this section, we will give an overview of association schemes and present the basic results that will be used throughout this thesis. For proofs and more information, we refer to the books [BI84] and [Ban+21] and especially to Delsarte's thesis [Del73]. Gently introductions to association schemes can also be found in [MS77, § 21] and [LW92].

2.1.1 Definition and examples

Association schemes. In the literature, various definitions of association schemes exist. Here, we will use the following one. An *association scheme* $(X, (R_i))$ *with n classes* is a finite set X with at least two elements together with $n+1$ nonempty relations $R_0, R_1, \dots, R_n$ such that

(A1) all $n+1$ relations R_i partition $X \times X$ and $R_0 = \{(x,x) \mid x \in X\}$;

(A2) for each relation R_i, its transpose $R_i^T = \{(y,x) \mid (x,y) \in R_i\}$ also occurs in $\{R_0, R_1, \dots, R_n\}$;

(A3) for every pair $(x,y) \in R_k$, the number of $z \in X$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant p_{ij}^k depending only on i, j, and k, but not on the particular choice of (x,y);

(A4) $p_{ij}^k = p_{ji}^k$ for all i, j, k.

The numbers p_{ij}^k are called *intersection numbers*. If $R_i = R_i^T$ holds for all i, then the association scheme is called *symmetric* and (A4) holds automatically. Let $R_{i'}$ denote the transposed relation R_i^T of R_i. Then we have $p_{ii'}^0 = p_{i'i}^0$. We write $v_i = p_{ii'}^0$ and call this number the *valency* of the relation R_i. So, for every $x \in X$, we have

$$v_i = |\{y \in X \mid (x,y) \in R_i\}|.$$

The elements of X are called *points* and the set $\{R_0, R_1, \dots, R_n\}$ of relations is henceforth denoted by $\mathcal{R}$.

In a symmetric association scheme, the condition (A3) can be visualized as follows. Associate with each relation R_i a color c_i. Let G be the graph, where the vertices are the points of X and between two vertices $x, y \in X$, there is an edge colored by c_i if (x,y) lies in R_i. Condition (A3) then implies that the number of triangles depicted in Figure 2.1 depends only on the coloring of the edges, but not on the chosen base edge between x and y.

In the following, we look at two examples of association schemes that are the two most important ones for classical coding and design theory—the Hamming scheme and the Johnson scheme. These two will be used as running examples throughout this chapter.

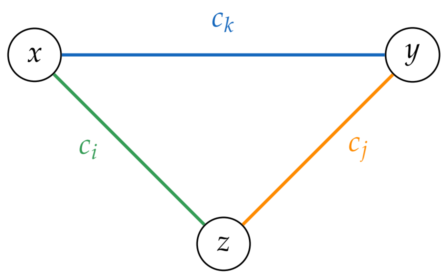


Figure 2.1. Interpretation of the condition (A3) of a symmetric association scheme: exactly p_{ij}^k vertices z form this type of triangle.

Example 2.1.1 (**Hamming scheme**). *Let n and q be positive integers with $q \geq 2$ and let $X = Q^n$, where Q is a set with q elements. The* Hamming distance $d_H(x,y)$ *between two points $x, y \in X$ is the number of positions in which x and y differ. More specifically, the mapping $d_H\colon X \times X \to \mathbb{N}_0$ defined by*

$$d_H(x,y) = |\{i \mid 0 \leq i \leq n,\ x_i \neq y_i\}|$$

is a metric. The relation R_i is given by all pairs of $X \times X$ whose Hamming distance equals i. Then there are $n+1$ relations $R_0, R_1, \ldots, R_n$ that together with X form a symmetric association scheme with n classes, known as the Hamming scheme *and denoted by $H(n,q)$. The valencies are given by*

$$v_i = (q-1)^i \binom{n}{i} \tag{2.1}$$

for all $i = 0, 1, \ldots, n$. For example, $H(2,2)$ and $H(3,2)$ are depicted in Figure 2.2 and 2.3, respectively.

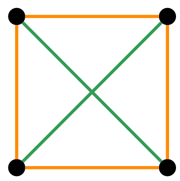


Figure 2.2. Hamming scheme $H(2,2)$ with the relations R_1 and R_2.

Example 2.1.2 (**Johnson scheme**). *Let m and n be positive integers with $m \geq n$. Then the set X of all n-subsets of a given $(m+n)$-set together with*

$$R_i = \{(x,y) \in X \times X \mid |x \cap y| = n - i\} \tag{2.2}$$

for all $i = 0, 1, \ldots, n$ forms a symmetric association scheme with n classes, known as the Johnson scheme, *and is denoted by $J(n,m)$. For all $i = 0, 1, \ldots, n$, we have*

$$v_i = \binom{n}{i}\binom{m}{i}. \tag{2.3}$$

As an example, the Johnson scheme $J(2,2)$ is depicted in Figure 2.4.

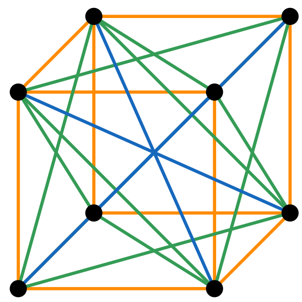


FIGURE 2.3. Hamming scheme $H(3,2)$ with the relations R_1, R_2, and R_3.

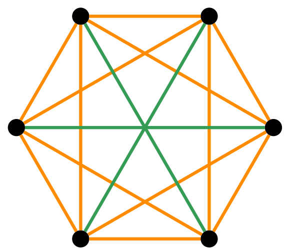


FIGURE 2.4. Johnson scheme $J(2,2)$ with the relations R_1 and R_2.

There is a nice connection between the binary Hamming scheme and the Johnson scheme.

Remark 2.1.3. The Johnson scheme $J(n,m)$ can be embedded into the Hamming scheme $H(m+n,2)$ as follows. Identify an n-set x in $J(n,m)$ with its characteristic vector ϕ_x, defined by $(\phi_x)_a = 1$ if $a \in x$ and $(\phi_x)_a = 0$ otherwise, which lies in $\{0,1\}^{m+n}$. Then, for two n-sets x and y, it holds $n - |x \cap y| = i$ if and only if $d_H(\phi_x, \phi_y) = 2i$. Hence, this gives $(x,y) \in R_i$ in $J(n,m)$ if and only if $d_H(\phi_x, \phi_y) = 2i$.

The following proposition is often very useful to verify that a given finite set X together with some relations forms an association scheme. It can only be applied if the association scheme has a group as an underlying structure such that this group G acts *transitively* on X; that is, for all $x, y \in X$, there exists a $g \in G$ with $gx = y$.

Proposition 2.1.4 ([Ban+21, Example 2.3])**.** *Let G be a finite group and let X be a finite set, where G acts transitively on X. Define an action of G on $X \times X$ by $g(x,y) = (gx, gy)$ for all $x, y \in X$ and $g \in G$. Then $R_0 = \{(x,x) \mid x \in X\}$ is an orbit. If $R_0, R_1, \ldots, R_n$ are all the orbits of $X \times X$ under G, then $(X, \{R_0, R_1, \ldots, R_n\})$ satisfies the conditions (A1)–(A3) of an association scheme.*

We can use this proposition to show that the Hamming scheme and the Johnson scheme are really association schemes.

Example 2.1.5. *Let $(X, \mathcal{R})$ be the Hamming scheme $H(n,q)$. Its underlying group G is the wreath product $S_q \wr S_n$. More concretely, the group G is the semidirect product of $S_q \times \cdots \times S_q$ (n copies of S_q) and S_n with respect to $\varphi\colon S_n \to \mathrm{Aut}(S_q \times \cdots \times S_q)$, where*

$$\varphi(\pi)(\sigma_1, \ldots, \sigma_n) = (\sigma_{\pi(1)}, \ldots, \sigma_{\pi(n)}).$$

Then G acts transitively on X via

$$((\sigma, \pi), x) \mapsto (\sigma_1(x_{\pi(1)}), \ldots, \sigma_n(x_{\pi(n)}))$$

for all $\sigma \in S_q \times \cdots \times S_q$, $\pi \in S_n$, and $x \in X$. The action of G extends to $X \times X$ componentwise and the orbits of this group action are precisely the relations R_i defined in Example 2.1.1. Because $R_i = R_i^T$ for all i, the pair $(X, \{R_0, R_1, \ldots, R_n\})$ is a symmetric association scheme. It is an association scheme of type B_n or C_n *since in the binary case, the group $G = S_2 \wr S_n$ is the hyperoctahedral group, which is a Weyl group of a Chevalley group[2] of type B_n or C_n, see [Sta84, p. 103].*

Example 2.1.6. *Let $(X, \mathcal{R})$ be the Johnson scheme $J(n,m)$. The symmetric group S_{m+n} acts transitively on X by permuting the elements of the $(m+n)$-set. The action of S_{m+n} extends to $X \times X$ componentwise and the orbits of this group action are given by $R_0, R_1, \ldots, R_n$ from* (2.2). *Because of $R_i = R_i^T$ for all i, the pair $(X, \{R_0, R_1, \ldots, R_n\})$ is a symmetric association scheme. It is an association scheme of* type A_{m+n-1} *since the group S_{m+n} is the Weyl group of a Chevalley group of type A_{m+n-1}, see [Sta84, p. 101].*

Strongly regular graphs. If a symmetric association scheme has only two classes, then it is closely related to a *strongly regular graph*. Let $G = (V, E)$ denote a graph with vertex set V and edge set E. If there is an edge between two vertices $x, y \in V$, then write $x \sim y$. A simple graph $G = (V, E)$, which is neither empty nor complete, is called a *strongly regular graph* $\mathrm{srg}(v, k, \theta, \rho)$ if the graph G has v vertices, is k-regular (that is, every vertex has exactly k neighbors), and if the following holds

$$|\{z \in V \mid x \sim z \text{ and } y \sim z\}| = \begin{cases} \theta & \text{if } x \sim y, \\ \rho & \text{if } x \not\sim y. \end{cases}$$

The next proposition shows that symmetric association schemes with two classes are equivalent to strongly regular graphs.

Proposition 2.1.7.

(a) *Let $(X, \{R_0, R_1, R_2\})$ be a symmetric association scheme with two classes. Then the graph $G = (X, R_1)$ is an* $\mathrm{srg}(|X|, p_{11}^0, p_{11}^1, p_{11}^2)$.

[2] The definitions of Weyl and Chevalley groups are out of the scope of this thesis and they are also not required in detail since the type is only used for a categorization of different association schemes.

(*b*) *Let* G *be an* $\operatorname{srg}(v,k,\theta,\rho)$. *Define* $R_0 = \{(x,x) \mid x \in X\}$ *and let* R_1 *be the edge set of* G *and let* R_2 *be the edge set of the complement of* G. *Then the pair* $(X, \{R_0, R_1, R_2\})$ *is a symmetric association scheme with two classes.*

An example of a strongly regular graph can be constructed by using the Hamming scheme $H(2,q)$, for example, see Figure 2.5.

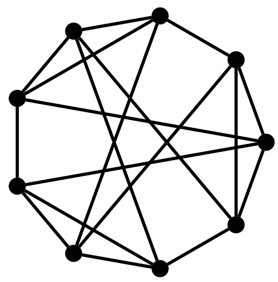


FIGURE 2.5. A strongly regular graph $\operatorname{srg}(9,4,1,2)$ that arises from the Hamming scheme $H(2,3)$.

2.1.2 The Bose-Mesner algebra

We now introduce an algebra associated with an association scheme and use tools from linear algebra to further study association schemes.

Given a relation R_i of an association scheme $(X,\mathcal{R})$ with n classes, let D_i be the *adjacency matrix* of the graph (X,R_i), that is, D_i is an $|X| \times |X|$ matrix with

$$(D_i)_{x,y} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in X$. The conditions (A1)–(A3) are then equivalent to (A1′)–(A3′) with

(A1′) $D_0 + D_1 + \cdots + D_n = J$ and $D_0 = I$ (where J and I are the all-ones-matrix and identity matrix, respectively);

(A2′) for each $i \in \{0,1,\ldots,n\}$, there exists an $i' \in \{0,1,\ldots,n\}$ with $D_i^T = D_{i'}$;

(A3′) for each $i, j \in \{0,1,\ldots,n\}$, there exist nonnegative integers p_{ij}^k such that

$$D_i D_j = \sum_{k=0}^{n} p_{ij}^k D_k.$$

Consider the $\mathbb{C}$-vector space $\mathcal{B}$ spanned by the adjacency matrices $D_0, D_1, \ldots, D_n$ of an association scheme $(X,\mathcal{R})$. This vector space together with the standard matrix multiplication is a commutative matrix algebra, called the *Bose-Mesner algebra*. The following proposition shows that $\mathcal{B}$ has a second basis with some special properties.

Proposition 2.1.8 ([Ban+21, Lemma 2.18]). *Let $\mathcal{B}$ be the Bose-Mesner algebra of an association scheme $(X, \mathcal{R})$ with n classes. Then the algebra $\mathcal{B}$ has a second basis consisting of pairwise orthogonal, idempotent, Hermitian matrices $E_0, E_1, \ldots, E_n$ that satisfy*

$$E_0 + E_1 + \cdots + E_n = I \quad \text{and} \quad \frac{1}{|X|} J \in \{E_0, E_1, \ldots, E_n\}.$$

In what follows, we set $E_0 = \frac{1}{|X|} J$ and write $\mu_k = \operatorname{rank}(E_k)$, which are known as the *multiplicities* of the association scheme. We call the matrices $E_0, E_1, \ldots, E_n$ from Proposition 2.1.8 the *primitive idempotents* of the association scheme.[3]

***P*- and *Q*-numbers.** Since there are two different bases for the Bose-Mesner algebra of an association scheme, we can use a change of basis to define complex numbers $P_i(k)$ and $Q_k(i)$ by

$$D_i = \sum_{k=0}^{n} P_i(k) E_k \quad \text{and} \quad E_k = \frac{1}{|X|} \sum_{k=0}^{n} Q_k(i) D_i.$$

The numbers $P_i(k)$ and $Q_k(i)$ are called *P-numbers* and *Q-numbers* (or *eigenvalues* and *dual eigenvalues*) of the association scheme, respectively. The origin of the second names comes from the fact that the P-numbers are indeed the eigenvalues of the matrices D_i because

$$D_i E_k = \sum_{j=0}^{n} P_i(j) E_j E_k = P_i(k) E_k.$$

Moreover, the column spaces of the matrices $E_0, E_1, \ldots, E_n$ are the maximal common eigenspaces of the matrices $D_0, D_1, \ldots, D_n$. We denote the eigenspaces by V_k, that is, $V_k = E_k \mathbb{C}^X$. We then have the orthogonal direct sum

$$\mathbb{C}^X = V_0 \perp V_1 \perp \cdots \perp V_n$$

with respect to the complex inner product, where V_0 is spanned by the all-ones vector.

Observe that the P- and Q-numbers are real if the association scheme is symmetric.

For the Hamming scheme and the Johnson scheme, the P- and Q-numbers are well known. For example, they were derived in [Del76a] by using semilattices.

Example 2.1.9 ([Del76a, Theorem 11], [Ban+21, Theorem 2.86]). *Consider the Hamming scheme $H(n, q)$. Then there exists a unique ordering of the primitive*

[3] This notion comes from ring theory, where a nonzero idempotent element E in a ring R means that E cannot be written as the sum of two nonzero orthogonal idempotents of R. Indeed, if $E_k = A + B$ for two orthogonal idempotents A and B of the Bose-Mesner algebra, then $E_k A = A$ follows and since A can be written as a linear combination of the matrices $E_0, E_1, \ldots, E_n$, we have $A = E_k$ or $A = 0$.

idempotents $E_0, E_1, \ldots, E_n$ such that the P- and Q-numbers are given by

$$P_i(k) = Q_i(k) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{k}{j} \binom{n-k}{i-j}. \tag{2.4}$$

They can also be written as

$$P_i(k) = Q_i(k) = \sum_{j=0}^{i} (-1)^{i-j} q^j \binom{n-j}{n-i} \binom{n-k}{j},$$

which can be proved by using generating functions, see [MS77, § 5, Theorem 15].

Example 2.1.10 ([Del76a, Theorem 10], [Del78b])**.** *Consider the Johnson scheme $J(n,m)$. Then there exists a unique ordering of the primitive idempotents $E_0, E_1, \ldots, E_n$ such that the P-numbers are given by*

$$P_i(k) = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j}{i-j} \binom{n-k}{j} \binom{m+j-k}{j} \tag{2.5}$$

and the Q-numbers are given by

$$Q_k(i) = \mu_k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{m+n+1-k}{j} \binom{n}{j}^{-1} \binom{m}{j}^{-1} \binom{i}{j}, \tag{2.6}$$

where $\mu_k = \binom{m+n}{k} - \binom{m+n}{k-1}$ are the multiplicities.

Below we summarize some basic facts about the P- and Q-numbers. Writing

$$P = \begin{pmatrix} P_0(0) & P_0(1) & \cdots & P_0(n) \\ P_1(0) & P_1(1) & \cdots & P_1(n) \\ \vdots & \vdots & \ddots & \vdots \\ P_n(0) & P_n(1) & \cdots & P_n(n) \end{pmatrix}$$

and

$$Q = \begin{pmatrix} Q_0(0) & Q_0(1) & \cdots & Q_0(n) \\ Q_1(0) & Q_1(1) & \cdots & Q_1(n) \\ \vdots & \vdots & \ddots & \vdots \\ Q_n(0) & Q_n(1) & \cdots & Q_n(n) \end{pmatrix} \tag{2.7}$$

gives $PQ = QP = |X|I$, which in one case means

$$\frac{1}{|X|} \sum_{k=0}^{n} P_i(k) Q_k(j) = \delta_{ij} \quad \text{for all } i,j = 0,1,\ldots,n. \tag{2.8}$$

The following proposition contains some special values of the P- and Q-numbers.

Proposition 2.1.11 ([Ban+21, Proposition 2.21]). *Let $P_i(k)$ and $Q_k(i)$ be the P- and Q-numbers of an association scheme $(X, \mathcal{R})$ with n classes. Then, for all $i = 0, 1, \ldots, n$, we have*

(*a*) $P_0(i) = Q_0(i) = 1$

(*b*) $P_i(0) = v_i$

(*c*) $Q_i(0) = \mu_i$

The P- and Q-numbers satisfy some orthogonality relations that will be of importance in the next subsection.

Proposition 2.1.12 ([Ban+21, Theorem 2.22]). *Let $P_i(k)$ and $Q_k(i)$ be the P- and Q-numbers of an association scheme $(X, \mathcal{R})$ with n classes. Then we have the two orthogonality relations*

$$\frac{1}{|X|} \sum_{k=0}^{n} \mu_k P_i(k) \overline{P_j(k)} = \delta_{ij} v_i \quad \text{for all } i, j = 0, 1, \ldots, n, \tag{2.9}$$

$$\frac{1}{|X|} \sum_{i=0}^{n} v_i Q_k(i) \overline{Q_j(i)} = \delta_{jk} \mu_k \quad \text{for all } j, k = 0, 1, \ldots, n. \tag{2.10}$$

Moreover, the P- and Q-numbers are connected by

$$\mu_k \overline{P_i(k)} = v_i Q_k(i) \quad \text{for all } i, k = 0, 1, \ldots, n. \tag{2.11}$$

Krein numbers. We remark that the Bose-Mesner algebra of an association scheme with the adjacency matrices $D_0, D_1, \ldots, D_n$ is also an algebra with respect to the Hadamard (entrywise) product $\circ$, where the matrices $D_0, D_1, \ldots, D_n$ form a basis consisting of pairwise orthogonal idempotent elements. Moreover, there exist complex numbers q_{ij}^k, called *Krein numbers*, such that

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{n} q_{ij}^k E_k$$

for all $i, j = 0, 1, \ldots, n$, where $E_0, E_1, \ldots, E_n$ are the primitive idempotents of the association scheme. The Krein numbers satisfy similar properties as the intersection numbers p_{ij}^k. However, no general combinatorial interpretation for them is known. For more details, see [Ban+21, § 2].

2.1.3 P- and Q-polynomial association schemes

In this subsection, we will consider only symmetric association schemes since their P- and Q-numbers are real numbers.

Metric association schemes. Let $(X, \mathcal{R})$ be a symmetric association scheme with n classes and intersection numbers p_{ij}^k, where we assume an ordering of the relations $R_0, R_1, \dots, R_n$. Then $(X, \mathcal{R})$ is called *metric* if $p_{ij}^{i+j} \neq 0$ and

$$p_{ij}^k \neq 0 \implies |i-j| \leq k \leq i+j \quad \text{for all } i, j, k.$$

The motivation for this definition comes from the following proposition.

Proposition 2.1.13 ([Del73, § 5.2])**.** *Let $(X, \mathcal{R})$ be a symmetric association scheme with intersection numbers p_{ij}^k. Define a mapping $\delta\colon X \times X \to \mathbb{N}_0$ by $\delta(x, y) = i$ whenever $(x, y) \in R_i$. Then the following are equivalent:*

(1) δ is a metric.

(2) $(X, \mathcal{R})$ is metric.

Example 2.1.14. *The Hamming scheme $H(n, q)$ is metric and the Hamming distance $d_H(\cdot, \cdot)$ is the corresponding metric.*

Example 2.1.15. *The Johnson scheme $J(n, m)$ is metric: Use the embedding of $J(n, m)$ into $H(m+n, 2)$ from Remark 2.1.3 and identify each set $x \in X$ with its characteristic vector ϕ_x in $\{0, 1\}^{m+n}$. The metric δ is then given by $\delta(x, y) = d_H(\phi_x, \phi_y)/2$.*

***P*-polynomial association scheme.** We will see that metric association schemes have the useful property that the P-numbers are given by polynomials. A symmetric association scheme is called *P-polynomial* with respect to the ordering $R_0, R_1, \dots, R_n$ if there exist polynomials $f_i \in \mathbb{R}[x]$ of degree i and distinct real numbers $y_0, y_1, \dots, y_n$ such that $P_i(k) = f_i(y_k)$ for all $i, k = 0, 1, \dots, n$. The following proposition gives the connection between P-polynomial and metric association schemes.

Proposition 2.1.16 ([Ban+21, Theorem 2.76], [Del73, Theorem 5.6])**.** *Let $(X, \mathcal{R})$ be a symmetric association scheme with the adjacency matrices $D_0, D_1, \dots, D_n$ corresponding to the relations $R_0, R_1, \dots, R_n$. Then the following are equivalent:*

(a) $(X, \mathcal{R})$ is P-polynomial with respect to the ordering $R_0, R_1, \dots, R_n$ of the relations.

(b) $(X, \mathcal{R})$ is metric with respect to the ordering $R_0, R_1, \dots, R_n$.

(c) There exist polynomials $h_i(x) \in \mathbb{R}[x]$ of degree i such that $D_i = h_i(D_1)$.

***Q*-polynomial association scheme.** A definition similar to P-polynomial association scheme can be given in terms of the Q-numbers. A symmetric association scheme is called *Q-polynomial* with respect to the ordering $E_0, E_1, \dots, E_n$ of the primitive idempotents if there exist polynomials $g_k \in \mathbb{R}[x]$ of degree k and distinct real numbers $z_0, z_1, \dots, z_n$ such that $Q_k(i) = g_k(z_k)$ for all $i, k = 0, 1, \dots, n$.

Similar to Proposition 2.1.16, we have the following.[4]

Proposition 2.1.17. *Let $(X, \mathcal{R})$ be a symmetric association scheme with primitive idempotents $E_0, E_1, \ldots, E_n$. Then the following are equivalent:*

(a) *$(X, \mathcal{R})$ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_n$.*

(b) *There exist polynomials $h_k(x) \in \mathbb{R}[x]$ of degree k such that $|X|E_k = h_k(|X|E_1)$, where the Hadamard product is used whenever a matrix is substituted into the polynomial h_k.*

By (2.9), the polynomials f_i associated with a P-polynomial association scheme are orthogonal with respect to the inner product

$$(f, g) = \sum_{k=0}^{n} \mu_k f(y_k) g(y_k) \quad \text{for } f, g \in \mathbb{R}[x]$$

and we have $(f_i, f_j) = |X| v_i \delta_{ij}$. Similarly, by using (2.10), the polynomials g_k associated with a Q-polynomial association scheme are orthogonal with respect to the inner product

$$(f, g) = \sum_{i=0}^{n} v_i f(z_i) g(z_i) \quad \text{for } f, g \in \mathbb{R}[x] \tag{2.12}$$

and we have $(g_j, g_k) = |X| \mu_k \delta_{jk}$. Therefore, for some association schemes, there is a connection to orthogonal polynomials.

Orthogonal polynomials. In what follows, and also in Section 2.3, we will introduce different orthogonal polynomials associated with association schemes. We note that there exist various definitions for the orthogonal polynomials used in this thesis. For the sake of consistency, we will always use the definitions given in [KLS10]. The polynomials will be expressed by hypergeometric functions, which requires the definition of the *Pochhammer symbol* $a^{(i)}$ given by

$$a^{(0)} = 1 \quad \text{and} \quad a^{(i)} = a(a+1)\cdots(a+i-1),$$

where a is a complex number and i is a positive integer.[5] The Pochhammer symbol can be seen as a generalization of the factorial since $i! = 1^{(i)}$. It is also related to the binomial coefficient by

$$\binom{a}{i} = \frac{(-a)^{(i)}}{1^{(i)}}(-1)^i \tag{2.13}$$

[4]There exists a dual version of the term *metric* with respect to the Krein parameters, called *cometric*, see [Del73, Theorem 5.16], for example.

[5]The Pochhammer symbol $a^{(i)}$ is commonly denoted by $(a)_i$. However, we will use the latter notation as a a short version of the q-Pochhammer symbol $(a;q)_i$ for a specific q in all upcoming chapters.

for all complex numbers a and all nonnegative integers i. The *hypergeometric function* ${}_rF_s$ is now defined by the series

$$
{}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r)^{(k)}}{(b_1,\ldots,b_s)^{(k)}}\frac{z^k}{k!}, \tag{2.14}
$$

where

$$
(a_1,\ldots,a_r)^{(k)} = a_1^{(k)}\cdots a_r^{(k)}.
$$

The parameters in (2.14) must be chosen such that the denominators in the series are never zero. If one of the parameters a_i in the numerators is a non-positive integer, then the hypergeometric function is a polynomial in z. This is the case for all hypergeometric functions considered in this thesis.

We will now look at the polynomials associated with the Hamming scheme and the Johnson scheme.

Example 2.1.18. *For the Hamming scheme $H(n,q)$, the P-numbers $P_i(k)$ and Q-numbers $Q_i(k)$ can be written by using the* Krawtchouk polynomial *of degree i in x with the parameters n, A given by*

$$
K_i(x;A,n) = {}_2F_1\left(\begin{matrix} -i,-x \\ -n \end{matrix}\,\middle|\, \frac{1}{A}\right) \quad \text{for } i=0,1,\ldots,n,
$$

see [KLS10, § 9.11]. Namely, we have

$$
P_i(k) = Q_i(k) = v_iK_i(k;(q-1)/q,n) = v_i\,{}_2F_1\left(\begin{matrix} -i,-k \\ -n \end{matrix}\,\middle|\, \frac{q}{q-1}\right),
$$

see [CS90, Equation 2.6], for example. Therefore, the Hamming scheme is P-polynomial with respect to the ordering $R_0, R_1, \ldots, R_n$ and Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_n$ that is imposed by (2.4). We call these two orderings the standard orderings. *For the P-polynomial structure, we have $y_k = k$ for all $k = 0,1,\ldots,n$ and in the case of the Q-polynomial structure, we have $z_i = i$ for all $i = 0,1,\ldots,n$.*

Example 2.1.19. *For the Johnson scheme $J(n,m)$, the P-numbers $P_i(k)$ can be written by using a* dual Hahn polynomial *(also called* Eberlein polynomial*) with the parameters n, C, D given by*

$$
E_i(\lambda(x);C,D,n) = {}_3F_2\left(\begin{matrix} -i,-x,x+C+D+1 \\ C+1,-n \end{matrix}\,\middle|\, 1\right) \quad \text{for } i=0,1,\ldots,n,
$$

which is a polynomial of degree i in $\lambda(x) = x(x+C+D+1)$, see [KLS10, § 9.6]. The Q-numbers $Q_k(i)$ can be written by using a Hahn polynomial *with the parameters n, A, B given by*

$$
H_k(x;A,B,n) = {}_3F_2\left(\begin{matrix} -x,-k,k+A+B+1 \\ A+1,-n \end{matrix}\,\middle|\, 1\right) \quad \text{for } k=0,1,\ldots,n,
$$

which is a polynomial of degree k in x, see [*KLS10, Section 9.5*]. *Then, from* (2.5), *we obtain*

$$P_i(k) = v_i E_i(\lambda(k); -m-1, -n-1, n) = v_i\, {}_3F_2\left(\begin{matrix} -i, -k, k-m-n-1 \\ -m, -n \end{matrix}\,\middle|\, 1\right)$$

and

$$Q_k(i) = \mu_k H_k(i; -m-1, -n-1, n) = \mu_k\, {}_3F_2\left(\begin{matrix} -i, -k, k-m-n-1 \\ -m, -n \end{matrix}\,\middle|\, 1\right),$$

see [*Del76b, § 5.2*] *and use* (2.11). *Therefore, the Johnson scheme is P-polynomial with respect to the ordering $R_0, R_1, \ldots, R_n$ and Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_n$. Similarly to the Hamming scheme, we call these two orderings the* standard orderings. *For the P-polynomial structure, we have $y_k = k(k-m-n-1)$ for all $k = 0, 1, \ldots, n$ and in the case of the Q-polynomial structure, we have $z_i = i$ for all $i = 0, 1, \ldots, n$.*

Different orderings. We will now shortly discuss the importance of the orderings required in the definition of P- and Q-polynomial association schemes. We will later see in Section 3.2 why it is useful to study a symmetric association scheme with respect to different orderings if they exist.

It is well known that a P-polynomial association scheme can have at most two different orderings and moreover, it is known which orderings it could have.

Proposition 2.1.20 ([BB80, Theorem 1 and 2]). *A symmetric association scheme, which is not the association scheme of an n-gon*[6], *can only be P-polynomial with respect to at most two orderings of its relations. Moreover, if $(X, \{R_0, R_1, \ldots, R_n\})$ is a P-polynomial association scheme with respect to some ordering $R_0, R_1, \ldots, R_n$, then there are the following four possible patterns of a second P-polynomial ordering:*

(*I*) $R_0, R_2, R_4, R_6, \ldots, R_5, R_3, R_1$

(*II*) $R_0, R_n, R_1, R_{n-1}, R_2, R_{n-2}, R_3, R_{n-3}, \ldots$

(*III*) $R_0, R_n, R_2, R_{n-2}, R_4, R_{n-4}, \ldots, R_{n-5}, R_5, R_{n-3}, R_3, R_{n-1}, R_1$

(*IV*) $R_0, R_{n-1}, R_2, R_{n-3}, R_4, R_{n-5}, \ldots, R_5, R_{n-4}, R_3, R_{n-2}, R_1, R_n$

To determine whether a P-polynomial scheme has a second P-polynomial structure or not, one can use its intersection numbers. For more details, see [Ban+21, § 6.1].

A similar theorem holds for the Q-polynomial structure.

---

[6] This association scheme consists of the point set $X = \{0, 1, \ldots, n-1\}$ together with the relations $R_0, R_1, \ldots, R_{\lfloor n/2 \rfloor}$ given by $(x, y) \in R_i$ if and only if $\min\{|x-y|, n-|x-y|\} = i$. It is P-polynomial with respect to the orderings $R_0, R_\ell, R_{2\ell}, R_{3\ell}, \ldots$ for all ℓ with $\gcd(n, \ell) = 1$, where the indices in the relations are taken modulo n.

Proposition 2.1.21 ([Suz98, Theorem 1], [MW14, Theorem 3]). *A symmetric association scheme, which is not the association scheme of an n-gon*[7], *can only be Q-polynomial with respect to at most two orderings of its primitive idempotents* $E_0, E_1, \dots, E_n$. *Moreover, if an association scheme is Q-polynomial with respect to some ordering* $E_0, E_1, \dots, E_n$ *of its primitive idempotents, then there are the following four possible patterns of a second Q-polynomial ordering:*

(I) $E_0, E_2, E_4, E_6, \dots, E_5, E_3, E_1$

(II) $E_0, E_n, E_1, E_{n-1}, E_2, E_{n-2}, E_3, E_{n-3}, \dots$

(III) $E_0, E_n, E_2, E_{n-2}, E_4, E_{n-4}, \dots, E_{n-5}, E_5, E_{n-3}, E_3, E_{n-1}, E_1$

(IV) $E_0, E_{n-1}, E_2, E_{n-3}, E_4, E_{n-5}, \dots, E_5, E_{n-4}, E_3, E_{n-2}, E_1, E_n$

To determine whether a Q-polynomial scheme has a second Q-polynomial structure or not, one can use its Krein numbers. For more details, see [Ban+21, § 6.1].

We now look at different orderings for the Hamming scheme and the Johnson scheme.

Example 2.1.22. *It is known that the Hamming scheme* $H(n,q)$ *with* $n \geq 3$ *has two P- or Q-polynomial structures if and only if* $q = 2$ *and n is even, see* [*BI84, Remark on p. 259*]. *Thus, assume that* $n \geq 4$ *is even. Then the binary Hamming scheme* $H(n,2)$ *is P-polynomial with respect to the second ordering*

$$R_0, R_{n-1}, R_2, R_{n-3}, R_4, R_{n-5}, \dots, R_1, R_n$$

and Q-polynomial with respect to the second ordering

$$E_0, E_{n-1}, E_2, E_{n-3}, E_4, E_{n-5}, \dots, E_1, E_n.$$

If we use the second P-polynomial structure, then the P-numbers $P'_i(k)$ *are given by*

$$P'_i(k) = \begin{cases} P_i(k) & \text{for even } i \\ P_{n-i}(k) & \text{for odd } i, \end{cases}$$

where $P_i(k)$ *are the P-numbers with respect to the standard ordering, see* [*CS86, Equations (4.14) and (4.15)*]. *From* (2.4), *we obtain* $P'_i(n-k) = (-1)^i P_i(k)$, *which together with* (2.11) *and* (2.1) *gives*

$$P'_i(k) = (-1)^{ki} P_i(k).$$

Similarly, the Q-numbers $Q'_k(i)$ *with respect to the second Q-polynomial ordering are given by*

$$Q'_k(i) = (-1)^{ki} Q_k(i),$$

[7] Let $E_0, E_1, \dots, E_n$ be a Q-polynomial ordering for the association scheme of an n-gon. Then, for all k with $\gcd(n,k) = 1$, it is also Q-polynomial with respect to the ordering $E_0, E_k, E_{2k}, E_{3k}, \dots$, where the indices are taken modulo n.

where $Q_k(i)$ are the Q-numbers with respect to the standard ordering.

Example 2.1.23. *It is known that the Johnson scheme $J(n,m)$ with $n \geq 3$ has two P-polynomial structures if and only if $m = n+1$ or $(n,m) = (3,3)$. Moreover, a second Q-polynomial structure for $J(n,m)$ can only occur if $(n,m) = (3,3)$, see [BI84, Remark on p. 259]. For $m = n+1 \geq 4$, the second P-polynomial ordering for $J(n,n+1)$ is given by*

$$R_0, R_n, R_1, R_{n-1}, R_2, R_{n-2}, \ldots .$$

The eigenvalues $P_i'(k)$ with respect to this ordering are determined by

$$P_i'(k) = \begin{cases} P_{i/2}(k) & \text{for even } i \\ P_{n-(i-1)/2}(k) & \text{for odd } i, \end{cases}$$

where $P_i(k)$ are the eigenvalues with respect to the standard orderings, see [CS86, Table 1].

Bipartite association schemes. Some P-polynomial association schemes contain "subschemes" in the following way. A P-polynomial association scheme $(X,(R_i))$ with n classes, which is not an n-gon (see Footnote 6 on p. 23), is called *bipartite* if

$$D_i D_j = \sum_{k \in \Omega} p_{i,j}^k D_k$$

with $\Omega = \{0, 2, 4, \ldots, \lfloor n/2 \rfloor\}$ holds for all $i, j = 0, 2, 4, \ldots, \lfloor n/2 \rfloor$. This gives an equivalence relation

$$x \sim y \text{ if and only if } (x,y) \in \bigcup_{i \in \Omega} R_i,$$

which partitions X into exactly two equivalence classes X_1 and X_2, called *bipartite halves*. Then the pair $(X_i, (R_{2i})_{0 \leq i \leq \lfloor n/2 \rfloor})$ is also a P-polynomial association scheme, denoted by $\frac{1}{2}X$. If $(X,(R_i))$ is a bipartite association scheme with n classes and P-numbers $P_i(k)$, then the P-numbers $P_i'(k)$ of $\frac{1}{2}X$ are given by $P_i'(k) = P_{2i}(k)$, see [CS86, Theorem 4.1], for example. In Chapter 3 and 5, we will heavily exploit a bipartite half of an association scheme associated with the hyperbolic polar space, which will be introduced in Section 2.3.2. Here, as an example, we will look at the bipartite half of the binary Hamming scheme.

Example 2.1.24. *The binary Hamming scheme $H(n,2)$ is bipartite, where the bipartite halves X_i are given by the set of binary n-tuples with an even/odd number of nonzero entries, see [Ban+21, § 6.4 (iv)]. For example, see Figure 2.6. The relations of $\frac{1}{2}H(n,2)$ are the orbits of $X_i \times X_i$ under the transitive group action of*

$$G = \{g \in S_2 \wr S_n \mid g \text{ has an even number of sign changes}\}.$$

The association scheme $\frac{1}{2}H(n,2)$ *is of* type D_n *since* G *is a Weyl group of a Chevalley group of* type D_n, *see* [*Sta84, p. 104*].

The P-numbers $P_i'(k)$ *of* $\frac{1}{2}H(n,2)$ *are given by*

$$P_i'(k) = v_i' \, {}_2F_1\left(\begin{matrix} -2i, -k \\ -n \end{matrix} \,\middle|\, 2\right)$$

for $i, k = 0, 1, \ldots, \lfloor n/2 \rfloor$, *where* $v_i' = \binom{n}{2i}$ *is the valency of the relation* R_{2i} *of* $H(n,2)$. *This can also be written as*

$$P_i'(k) = v_i' \, {}_3F_2\left(\begin{matrix} -i, -k, k-n \\ -\frac{n}{2}, \frac{-n+1}{2} \end{matrix} \,\middle|\, 1\right)$$

for $i, k = 0, 1, \ldots, \lfloor n/2 \rfloor$, *see* [*CS86, Equation* (*4.13*)]. *Thus,* $P_i'(k)$ *can be expressed by using a Hahn polynomial, namely we have*

$$P_i'(k) = v_{2i} E_i\left(\lambda(k); -\frac{n+1}{2}, -\frac{n+1}{2}, \frac{n}{2}\right) \tag{2.15}$$

with $\lambda(k) = k(k-n)$. *By using* (2.11) *and the multiplicities* $\mu_k' = \binom{n}{k}$ *of* $\frac{1}{2}H(n,2)$, *we obtain the Q-numbers* $Q_k'(i)$ *of* $\frac{1}{2}H(n,2)$ *as follows*

$$Q_k'(i) = \mu_k' \, {}_3F_2\left(\begin{matrix} -i, -k, k-n \\ -\frac{n}{2}, \frac{-n+1}{2} \end{matrix} \,\middle|\, 1\right) = \mu_k' H_k\left(i; -\frac{n+1}{2}, -\frac{n+1}{2}, \frac{n}{2}\right). \tag{2.16}$$

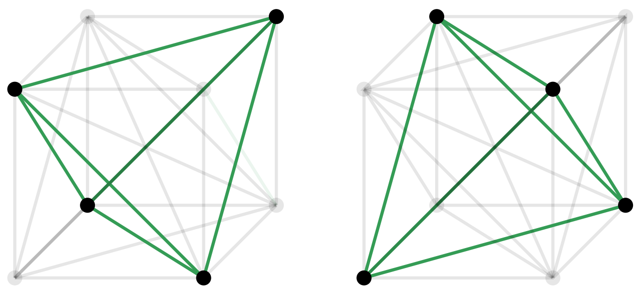


FIGURE 2.6. The bipartite halves of the Hamming scheme $H(3,2)$ with the relation R_1.

2.1.4 Codes and designs in association schemes

The objective of this thesis is to study special subsets of the set X of a (P and Q)-polynomial association scheme—namely, codes and designs. Whereas codes are dense packings in a metric space defined on X, designs are "good" approximations of the whole set X. It will turn out that codes and designs are dual to each other. In this subsection, we will introduce the basic tools to study them and in Section 2.2, we will see how optimization theory can be used to derive bounds for the size of codes.

We start with the definitions of the inner distribution and codes.

Inner distribution and codes. Let $(X, \mathcal{R})$ be an association scheme with n classes. The *inner distribution* of a subset Y of X is the tuple $A = (A_0, A_1, \dots, A_n)$, where

$$A_i = \frac{|(Y \times Y) \cap R_i|}{|Y|}.$$

So, the entry A_i is basically the average number of pairs from $Y \times Y$ that lie in R_i. The motivation for the inner distribution comes from its connection to codes. A *D-code* is a subset Y of X such that $A_i = 0$ for all $i \in \{1, 2, \dots, n\} \setminus D$, where D is a subset of $\{1, \dots, n\}$. If the association scheme $(X, \mathcal{R})$ is P-polynomial, then a subset Y of X is a *d-code* if no pair $(x, y) \in Y \times Y$ lies in one of the relations $R_1, R_2, \dots, R_{d-1}$, that means, Y is a D-code with $D = \{d, d+1, \dots, n\}$. Note that the inner distribution (A_i) of a d-code satisfies

$$A_1 = A_2 = \cdots = A_{d-1} = 0.$$

It turns out that this definition of D-codes coincides with various types of codes introduced in coding theory, as can be seen in the next example.

Example 2.1.25. *The d-codes in the Hamming scheme $H(n, q)$ and in the Johnson scheme $J(n, m)$ are codes defined in coding theory. Namely, we have the following (where we take the standard ordering of the relations in both association schemes).*

(1) *A d-code in $H(n, q)$ is a subset Y of Q^n (where $|Q| = q$) such that*

$$d_H(x, y) \geq d \quad \textit{for all distinct } x, y \in Y.$$

*These codes are the classical q-*ary codes *that have been heavily studied since the 1940s and which have multiple applications in communication and storage systems and are used in, e.g., deep-space communications, solid-state drives, QR codes, Blu-ray discs.*

(2) *A d-code in $J(n, m)$ is a subset Y of n-subsets of an $(m+n)$-set such that*

$$n - |x \cap y| \geq d \quad \textit{for all distinct } x, y \in Y.$$

Identify each n-subset x of the $(m+n)$-set by its characteristic vector ϕ_x in $\mathbb{F}_2^{m+n}$. Then each such vector has exactly n nonzero entries and a d-code Y satisfies

$$d_H(\phi_x, \phi_y) \geq 2d \quad \textit{for all distinct } x, y \in Y.$$

Such codes are called binary constant-weight codes*, which are also used in various digital communication systems, e.g., in barcodes or for frequency hopping in telecommunications.*

Dual distribution. Let $Q_k(i)$ denote the Q-numbers of an association scheme $(X, \mathcal{R})$. The *dual distribution* of a subset Y of X is the tuple $A' = (A'_0, A'_1, \ldots, A'_n)$ defined by

$$A'_k = \sum_{i=0}^{n} Q_k(i) A_i, \tag{2.17}$$

that is $QA = A'$, where Q is the Q-matrix from (2.7) containing the Q-numbers $Q_k(i)$.

The inner and dual distributions have some useful properties summarized in Table 2.1. All properties follow directly from the definitions and sometimes by also using $PQ = |X| I_n$, except for property (f) of the dual distribution, which follows from the expression (e) of A'_k and the fact that the primitive idempotents E_k are positive semidefinite. We emphasize property (f) of the dual distribution since it is essential for several important proofs in the remainder of this thesis.

Proposition 2.1.26 ([Del73, Theorem 3.3]). *All entries of the dual distribution of a subset in an association scheme are nonnegative real numbers.*

Table 2.1. Properties of the inner and dual distributions of a subset Y of X, where $(X, \mathcal{R})$ is an association scheme with the adjacency matrices $D_0, D_1, \ldots, D_n$ and primitive idempotents $E_0, E_1, \ldots, E_n$. Moreover, ϕ_Y denotes the characteristic vector of Y.

	inner distribution $A = (A_i)$	dual distribution $A' = (A'_k)$
(a)	$A_0 = 1$	$A'_0 = \lvert Y \rvert$
(b)	$\sum_{i=0}^{n} A_i = \lvert Y \rvert$	$\sum_{k=0}^{n} A'_k = \lvert X \rvert$
(c)	$A = \frac{1}{\lvert X \rvert} P A'$	$A' = QA$
(d)	$A_i = \frac{1}{\lvert X \rvert} \sum_{k=0}^{n} P_i(k) A'_k$	$A'_k = \sum_{i=0}^{n} Q_k(i) A_i$
(e)	$A_i = \frac{1}{\lvert Y \rvert} \phi_Y^T D_i \phi_Y$	$A'_k = \frac{\lvert X \rvert}{\lvert Y \rvert} \phi_Y^T E_k \phi_Y$
(f)	$A_i \geq 0$	$A'_k \geq 0$

We now look at a dual concept of a code.

Designs. Proposition 2.1.26 motivates the next definition. A subset Y of X is called a *T-design* if $A'_k = 0$ for all $k \in T$, where T is a subset of $\{1, 2, \ldots, n\}$.

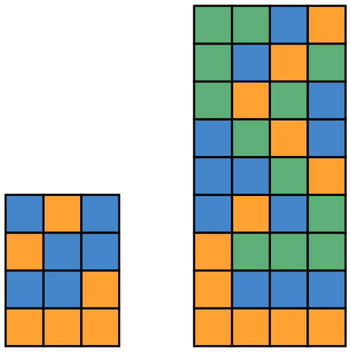


FIGURE 2.7. A 2-$(2, 3, 1)$ orthogonal array (left) and a 2-$(3, 4, 1)$ orthogonal array (right).

If the association scheme is Q-polynomial with the ordered primitive idempotents $E_0, E_1, \ldots, E_n$, then a *t-design* is a subset of X whose dual distribution (A'_k) satisfies

$$A'_1 = A'_2 = \cdots = A'_t = 0.$$

Recall that we have $\mathbb{C}^X = V_0 \perp V_1 \perp \cdots \perp V_n$, where V_k is the column space of E_k. Therefore, the characteristic vector ϕ_Y of a subset Y of X is spanned by the column vectors of the primitive idempotents. In the case of a t-design, we can apply property (e) of A'_k from Table 2.1 to obtain $E_k\phi_Y = 0$ for all $k = 1, 2, \ldots, t$. Thus, the characteristic vector ϕ_Y of a t-design Y satisfies

$$\phi_Y \in V_0 \perp V_{t+1} \perp V_{t+2} \perp \cdots \perp V_n.$$

So, the eigenspaces $V_1, V_2, \ldots, V_t$ are forbidden, whereas for a d-code, the relations $R_1, R_2, \ldots, R_{d-1}$ are forbidden.

Applying a result by Roos [Roo82, Theorem 3.4], one can give a combinatorial interpretation of T-designs in an association scheme that is constructed by using a group as in Proposition 2.1.4. For the Hamming and Johnson scheme, we then obtain the following classification results.

Example 2.1.27. *Designs in the Hamming scheme $H(n, q)$ are related to orthogonal arrays: Take $X = Q^n$ with $|Q| = q \geq 2$ and $1 \leq t \leq n$. Then a subset Y of X is called a t-(q, n, λ)* orthogonal array *if every restriction of Y to any t positions contains every t-tuple from Q^t exactly λ times. For example, a 2-$(2, 3, 1)$ orthogonal array and a 2-$(3, 4, 1)$ orthogonal array can be found in Figure 2.7. We have the following classification result for the Hamming scheme: a T-design with $T = \{1, 2, \ldots, t\}$ for $t \leq n$ in the Hamming scheme $H(n, q)$ is equivalent to a t-(q, n, λ) orthogonal array with $\lambda = |Y|/q^t$, see [Del73, Theorem 4.4] and [Roo82, Theorem 4.1.3].*

Example 2.1.28. *Designs in the Johnson scheme $J(n, m)$ are related to the (classical) combinatorial designs: Let X be the set of n-subsets of an $(m + n)$-set V and let*

$1 \leq t \leq n$. *Then a subset* Y *of* X *is a* t-$(m+n, n, \lambda)$ combinatorial design *if every* t*-subset of* V *lies in exactly* λ *elements of* Y. *For example, a* 2-$(7,3,1)$ *combinatorial design* (*called* Fano plane) *was depicted in Figure 1.1. We have the following classification result for the Johnson scheme: a* T*-design with* $T = \{1, 2, \ldots, t\}$ *and* $t \leq n$ *in the Johnson scheme* $J(n, m)$ *is equivalent to a* t-$(m+n, n, \lambda)$ *combinatorial design with* $\lambda = |Y|\binom{n}{t}/\binom{m+n}{t}$, *see* [*Del73, Theorem 4.7*] *and* [*Roo82, Theorem 4.2.2*].

In some cases, a t-design in a Q-polynomial association scheme can be used to construct a new association scheme.

Proposition 2.1.29 ([Del73, Theorem 5.25])**.** *Let* $(X, \mathcal{R})$ *be a* Q*-polynomial association scheme with* n *classes such that* $Q_k(i) = g_k(z_i)$ *for all* $i, k = 0, 1, \ldots, n$. *Suppose* Y *is a* t*-design in* $(X, \mathcal{R})$ *with* $|Y| \geq 2$ *and that there are exactly* $s+1$ *nonzero entries in the inner distribution of* Y *such that* $t \in \{2s-2, 2s-1, 2s\}$. *Then the pair* $(Y, \mathcal{R}|_Y)$ *becomes a* Q*-polynomial association scheme with* s *classes, where* $\mathcal{R}|_Y$ *contains all relations* R *from* $\mathcal{R}$ *such that* $R \cap (Y \times Y) \neq \emptyset$.

We close this section by looking at an example of Proposition 2.1.29.

Example 2.1.30. *Let* $(X, \mathcal{R})$ *be the binary Hamming scheme* $H(n, 2)$ *and let* Y *be the set of elements in* X *with an even number of nonzero entries. Then* Y *is an* $(n-1)$*-design in* $H(n, 2)$. *Since all Hamming distances in* Y *are even, we have* $s = \lfloor n/2 \rfloor$ *and thus,* $t \in \{2s-1, 2s\}$. *From Proposition 2.1.29, we find that* Y *induces an association scheme with* $\lfloor n/2 \rfloor$ *classes, which is the bipartite half* $\frac{1}{2}H(n, 2)$ *that we already encountered in Example 2.1.24.*

2.2 Linear programming

Besides establishing a connection between association schemes and coding theory—by showing that some codes defined in coding theory are special subsets in association schemes—Delsarte also introduced the powerful linear programming method in [Del73], which treats the problem of finding the maximum size of codes as an extremum problem for subsets in association schemes. Finding upper bounds on the size of codes is one of the main objectives in coding theory since large codes correspond to a high information rate in the transmitting or storing process of data.

This section starts with a small overview on linear programming and afterwards, we will introduce Delsarte's pioneering linear programming method.

2.2.1 Basics from optimization theory

In this subsection, we briefly summarize the most basic facts from the theory of linear programming. For further information, we refer to [Sch86], [Van14], and [GM07].

Primal and dual LP. Let $A \in \mathbb{R}^{n\times s}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}^s$. Then the *primal linear problem* (*primal LP*) is the problem of finding $x \in \mathbb{R}^s$ that maximizes $c^T x$ under the constraints that $x \geq 0$ and $Ax \geq -b$. We write the primal LP as follows

$$\begin{array}{lrcl} \max\limits_{x\in\mathbb{R}^s} & c^T x & & \\ \text{s.t.} & x & \geq & 0 \\ & Ax & \geq & -b \end{array} \tag{2.18}$$

The mapping $x \mapsto c^T x$ for $x \in \mathbb{R}^s$ is called the *objective function*. If a vector $x \in \mathbb{R}^s$ satisfies the constraints of an LP, then x is called a *feasible solution* of that LP. The LP is *bounded* if for all feasible solutions x, the value $c^T x$ of the objective function at x is bounded, otherwise the LP is *unbounded*. If the LP is bounded, then a feasible solution x^* is called *optimal* if $c^T x \leq c^T x^*$ holds for all feasible solutions x. We call the objective function value $c^T x^*$ for an optimal solution x^* the *LP optimum*.

The *dual linear problem* (*dual LP*) is the problem of finding $y \in \mathbb{R}^n$ that minimizes $y^T b$ under the constraints that $y \geq 0$ and $y^T A \leq -c^T$. Similarly to the primal LP, we write

$$\begin{array}{lrcl} \min\limits_{y\in\mathbb{R}^s} & y^T b & & \\ \text{s.t.} & y & \geq & 0 \\ & y^T A & \leq & -c^T \end{array} \tag{2.19}$$

for the dual LP. Hence, the number of variables x_i in the primal LP is equal to the number of inequalities in $y^T A \leq -c^T$ contained in the dual LP. Vice versa, the number of inequalities in $Ax \geq -b$ contained in the primal LP equals the number of variables in the dual LP.

The terms objective function, feasible solution, (un)bounded, and optimal are analogously defined for the dual LP.

Duality theory. There is a beautiful duality theory connecting the primal and dual LP, see [Van14, § 5], for example. Here, we will need the two following duality theorems.

Theorem 2.2.1 (**Weak duality theorem**, [Sch86, Corollary 7.1g])**.** *Every feasible solution of the dual LP gives an upper bound on the LP optimum of the primal LP. More specifically, for every feasible solutions x and y of the primal LP* (2.18) *and dual LP* (2.19)*, respectively, we have*

$$c^T x \leq y^T b.$$

The weak duality theorem implies that the primal LP and dual LP are bounded if both have a feasible solution.

The next theorem gives a connection between the optimal solutions of the primal and dual LP.

Theorem 2.2.2 (**Strong duality theorem**, [Sch86, § 7.9]). *Let x and y be feasible solutions of the primal LP* (2.18) *and dual LP* (2.19)*, respectively. Then both solutions x and y are optimal if and only if their objective function values coincide, that is, $c^T x = y^T b$.*

Another useful property is the *complementary slackness*, which follows directly from the weak and strong duality theorem.

Theorem 2.2.3 (**Complementary slackness**, [Sch86, § 7.9]). *Let x and y be feasible solutions of the primal and dual LP, respectively. Then x and y are optimal if and only if $y^T(b + Ax) = 0$ and $(c^T + y^T A)x = 0$, which means*

(1) *if an entry y_i of y is nonzero, then the i-th inequality in $Ax \geq -b$ is satisfied with equality;*

(2) *if the i-th inequality in $Ax \geq -b$ is not satisfied with equality, then the corresponding entry y_i of y is zero;*

(3) *if an entry x_i of x is nonzero, then the i-th inequality in $y^T A \leq -c^T$ is satisfied with equality;*

(4) *if the i-th inequality in $y^T A \leq -c^T$ is not satisfied with equality, then the corresponding entry x_i of x is zero.*

One can apply *LP solvers* to compute the optimal solution (if it exists) numerically. For example, one can use the package `glpk` in the software GNU Octave [Eat+20], which uses so-called primal and dual simplex methods to solve linear problems.

2.2.2 Delsarte's linear programming method

This subsection is about the powerful linear programming method developed by Delsarte in [Del73], which gives upper bounds for the size of codes in symmetric association schemes. Thus, let $(X, \mathcal{R})$ be a symmetric association scheme with n classes and denote its Q-numbers by $Q_k(i)$ and its multiplicities by μ_k in what follows.

LP for codes. Recall that the inner distribution (A_i) and dual distribution (A'_k) of a D-code Y in X have the following properties (see Table 2.1):

$$\begin{aligned}
\sum_{i=0}^{n} A_i &= |Y| \\
A_0 &= 1 \\
A_i &\geq 0 \quad \text{for all } i = 1,2,\ldots,n \\
A_i &= 0 \quad \text{for all } i \in \{1,2,\ldots,n\} \setminus D \\
A'_k &= \sum_{i=0}^{n} Q_k(i) A_i \geq 0 \quad \text{for all } k = 0,1,\ldots,n
\end{aligned}$$

Since we are interested in upper bounds for the size of a code Y, we want to maximize the sum $\sum_{i=0}^{n} A_i$. This motivates the following two definitions. The *primal LP for D-codes* in $(X,\mathcal{R})$ is given by

$$\begin{array}{llcll}
\max\limits_{x_i\in\mathbb{R}} & \sum\limits_{i=0}^{n} x_i & & & \\
\text{s.t.} & x_0 & = & 1 & \\
& x_i & \geq & 0 & \text{for all } i \in D \\
& x_i & = & 0 & \text{for all } i \in \{1,2,\ldots,n\} \setminus D \\
& \sum\limits_{i=0}^{n} Q_k(i) x_i & \geq & 0 & \text{for all } k = 1,2,\ldots,n
\end{array} \tag{2.20}$$

Henceforth, the LP optimum of (2.20) is denoted by $\mathrm{LP}(D)$ and in the case of a P-polynomial association scheme and $D = \{d, d+1, \ldots, n\}$, it is denoted by $\mathrm{LP}(d)$. (We will presently see that the LP optimum of (2.20) always exists.) Dualizing (2.20) gives the *dual LP for D-codes* in $(X,\mathcal{R})$ of the form

$$\begin{array}{llcll}
\min\limits_{y_i\in\mathbb{R}} & \sum\limits_{k=0}^{n} \mu_k y_k & & & \\
\text{s.t.} & y_0 & = & 1 & \\
& y_k & \geq & 0 & \text{for all } k = 1,2,\ldots,n \\
& \sum\limits_{k=0}^{n} Q_k(i) y_k & \leq & 0 & \text{for all } i \in D,
\end{array} \tag{2.21}$$

where μ_k is the multiplicity.

We then have the following theorem.

Theorem 2.2.4 ([Del73, § 3.2, Theorem 3.8])**.** *Let $(X,\mathcal{R})$ be a symmetric association scheme with n classes and Q-numbers $Q_k(i)$ and let D be a subset of $\{1,2,\ldots,n\}$. Then the linear program* (2.20) *has at least one feasible solution and is bounded. Moreover, if Y is a D-code in X, then its inner distribution is a feasible solution of* (2.20) *and in particular, we have $|Y| \leq \mathrm{LP}(D)$.*

Because of the Weak duality theorem 2.2.1 and Complementary slackness 2.2.3, we obtain the following corollary.

Corollary 2.2.5 ([Del73, Lemma 3.5, § 3.3.2]). *Let $(X, \mathcal{R})$ be a symmetric association scheme with n classes and Q-numbers $Q_k(i)$ and let D be a subset of $\{1, 2, \ldots, n\}$. Then the linear program* (2.21) *has at least one feasible solution and is bounded. Moreover, if* $\mathrm{LP}(D)$ *is the LP optimum of* (2.20), *then the objective function value of every feasible solution y_k of* (2.21) *gives an upper bound on* $\mathrm{LP}(D)$, *that is,*

$$\mathrm{LP}(D) \leq \sum_{k=0}^{n} \mu_k y_k. \tag{2.22}$$

Moreover, if Y is a D-code with dual distribution (A'_k) such that $|Y|$ equals the right-hand side of (2.22), *then $y_k A'_k = 0$ for all $k = 1, 2, \ldots, n$.*

We remark that one can also construct a linear program that minimizes a function under some constraints, whose optimal solution gives lower bounds on the size of designs, see [Del73, § 3.4].

Polynomial LP for codes. The next theorem plays a crucial part in this thesis. It restates Corollary 2.2.5 for Q-polynomial association schemes by using the orthogonal polynomials associated with the Q-polynomial structure.

Theorem 2.2.6 ([Del73, § 4.3]). *Let $(X, \mathcal{R})$ be a Q-polynomial association scheme with n classes, where $Q_k(i) = g_k(z_i)$ for some $g_k \in \mathbb{R}[x]$ of degree k and some real numbers z_i. Let D be a subset of $\{1, 2, \ldots, n\}$. Suppose that $F \in \mathbb{R}[x]$ is a polynomial of degree at most n whose coefficients F_k from the expansion*

$$F = F_0 g_0 + F_1 g_1 + \cdots + F_n g_n$$

satisfy $F_0 = 1$, $F_k \geq 0$ for all $k = 1, 2, \ldots, n$, and $F(z_i) \leq 0$ for all $i \in D$. Then $(F_0, F_1, \ldots, F_n)$ is a feasible solution of the dual LP (2.21) *with objective function value $F(z_0)$. In particular, we have* $\mathrm{LP}(D) \leq F(z_0)$.

The next remark shows how to compute the coefficients F_k for a chosen polynomial F.

Remark 2.2.7. Assume that $(X, \mathcal{R})$ is Q-polynomial with Q-numbers given by $Q_k(i) = g_k(z_i)$ as in Theorem 2.2.6. Let $P_i(k)$, v_i, and μ_k be the P-numbers, the valencies, and the multiplicities of $(X, \mathcal{R})$, respectively. Then, for a polynomial F with $F = F_0 g_0 + F_1 g_1 + \cdots + F_n g_n$, the coefficients F_k can be computed by using the inner product (2.12) and (2.11), namely

$$F_k = \frac{(F, g_k)}{(g_k, g_k)} = \frac{1}{|X| \mu_k} \sum_{i=0}^{n} v_i F(z_i) Q_k(i) = \frac{1}{|X|} \sum_{i=0}^{n} F(z_i) P_i(k). \tag{2.23}$$

We will see throughout this thesis that in the case of a (P and Q)-polynomial association scheme, the polynomial

$$F(z) = c \prod_{i=d}^{n} (z - z_i) \quad \text{for some real constant } c$$

often gives "good" bounds for d-codes in the sense that they are asymptotically optimal. We call this polynomial F the *Singleton polynomial*. The following example explains the origin of this name.

Example 2.2.8 ([Del73, § 4.3.2])**.** *Assume that Y is a d-code in $H(n,q)$. Recall from Example 2.1.18 that we have $z_i = i$ for all $i = 0,1,\dots,n$. We apply Theorem 2.2.6 with the Singleton polynomial*

$$F(z) = q^{n-d+1} \prod_{j=d}^{n} (z-j).$$

We have

$$F(z_i) = F(i) = q^{n-d+1} \frac{\binom{n-i}{n-d+1}}{\binom{n}{d-1}}.$$

The P-numbers satisfy the identity

$$\sum_{i=0}^{n} \binom{n-i}{j} P_i(k) = q^{n-j} \binom{n-k}{n-j} \tag{2.24}$$

for all $j,k = 0,1,\dots,n$, see [Del76a, Theorem 9], for example. Applying this identity, we can compute the coefficients F_k by using (2.23) *and obtain*

$$F_k = \frac{q^{n-d+1}}{q^n \binom{n}{d-1}} \sum_{i=0}^{n} \binom{n-i}{n-d+1} P_i(k) = \frac{\binom{n-k}{d-1}}{\binom{n}{d-1}}.$$

Therefore, the polynomial F satisfies all conditions in Theorem 2.2.6 and we deduce

$$\mathrm{LP}(d) \leq F(z_0) = q^{n-d+1}. \tag{2.25}$$

This is the well-known Singleton bound *from coding theory, see [MS77, Chapter 17 § 4], for example.*

We can moreover say something about the structure of a d-code Y that reaches the bound. Namely, assume that Y has exactly q^{n-d+1} elements. Denote the dual distribution of Y by (A'_k). Then Corollary 2.2.5 implies $F_k A'_k = 0$ for all $k = 1,2,\dots,n$. Since $F_k \neq 0$ if and only if $k \in \{1,2,\dots,n-d+1\}$, the code Y is an $(n-d+1)$-design. More specifically, it is an $(n-d+1)$-$(q,n,1)$ orthogonal array because of Example 2.1.27.

We close this section by also applying Theorem 2.2.6 with the Singleton polynomial to the Johnson scheme.

Example 2.2.9 ([Del73, § 4.3.2])**.** *Assume that Y is a d-code in $J(n,m)$. Recall from Example 2.1.19 that we have $z_i = i$ for all $i = 0,1,\dots,n$. Similarly as in Example 2.2.8, we apply Theorem 2.2.6 together with the Singleton polynomial*

$$F(z) = \frac{\binom{m+n}{n}}{\binom{m+d-1}{d-1}} \prod_{i=d}^{n} (z-i).$$

This gives

$$F(z_i) = \frac{\binom{m+n}{n}\binom{n-i}{n-d+1}}{\binom{m+d-1}{d-1}\binom{n}{d-1}}.$$

The P-numbers satisfy the identity

$$\sum_{i=0}^{n} \binom{n-i}{j} P_i(k) = \binom{n-k}{n-j}\binom{m+n-j-k}{n-j} \tag{2.26}$$

for all $j, k = 0, 1, \ldots, n$, *see* [*Del76a, Theorem 9*], *for example. From* (2.23), *we obtain*

$$F_k = \frac{\binom{n-k}{d-1}\binom{m+d-1-k}{d-1}}{\binom{m+d-1}{d-1}\binom{n}{d-1}}.$$

Hence, the polynomial F *satisfies all conditions in Theorem 2.2.6 and we deduce*

$$\mathrm{LP}(d) \le F(z_0) = \frac{\binom{m+n}{n}}{\binom{m+d-1}{d-1}} = \frac{\binom{m+n}{n-d+1}}{\binom{n}{n-d+1}}. \tag{2.27}$$

Similarly as in Example 2.2.8, we can give a combinatorial interpretation of a d*-code* Y *reaching this bound. Assume that* $|Y|$ *equals the right-hand side of* (2.27) *and let* (A'_k) *be the dual distribution of* Y. *Then Corollary 2.2.5 implies* $F_k A'_k = 0$ *for all* $k = 1, 2, \ldots, n$. *Since* $F_k \neq 0$ *if and only if* $k \in \{1, 2, \ldots, n-d+1\}$, *the code* Y *is an* $(n-d+1)$*-design in the Johnson scheme. More specifically, it is an* $(n-d+1)$*-*$(m+n, n, 1)$ *combinatorial design.*

2.3 Classical association schemes

This section studies several q-analogs of the Hamming scheme and Johnson scheme. These two schemes as well as all the schemes that will be introduced in this section are *classical* association schemes. The notion "classical" stems from their connection to distance-regular graphs with classical parameters [BCN89, Table 6.1]. Moreover, "classical" makes also sense in a broader context since these association schemes are connected to finite classical groups (linear, symplectic, unitary, and orthogonal groups on finite vector spaces), classical forms (bilinear, alternating bilinear, Hermitian, and quadratic forms), and classical orthogonal polynomials (in the sense of [AA85, § 4]: polynomials that are special cases or limiting cases of the q-Racah polynomials or Askey-Wilson polynomials).

The first two subsections deal with the *ordinary q-analogs*—namely, the q-Johnson scheme and six polar space schemes. Afterwards, we will introduce the *affine q-analogs* arising from bilinear forms, alternating bilinear forms, and Hermitian forms.[8] We will end this section by giving a connection between some ordinary q-analogs and affine q-analogs.

[8] We will neglect the classical association scheme arising from quadratic forms on an m-dimensional vector space over $\mathbb{F}_q$ since it was shown in [Ega85] that this scheme has the same parameters, and consequently the same P- and Q-numbers, as the alternating bilinear forms scheme on an $(m+1)$-dimensional vector space over $\mathbb{F}_q$.

A summary of all the classical association schemes occurring in this thesis can be found in Appendix A.

In what follows, we will frequently use the *q-binomial coefficient* defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \frac{q^{n-j+1}-1}{q^j-1},$$

which is a polynomial in q.

2.3.1 q-Johnson scheme

Let m and n be positive integers with $m \geq n$ and let q be a prime power. Choose X to be the set of all n-dimensional[9] subspaces (n-spaces for short) of an $(m+n)$-dimensional vector space over the finite field $\mathbb{F}_q$. We have $|X| = \left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]_q$. The group $\mathrm{GL}_{m+n}(q)$ acts transitively on X and this action extends to $X \times X$ componentwise, where the orbits of the latter group action are given by

$$R_i = \{(U, W) \in X \times X \mid \dim(U \cap W) = n - i\}$$

for all $i = 0, 1, \ldots, n$. By Proposition 2.1.4, $(X, (R_i))$ is an association scheme with n classes, which is known as the *q-Johnson scheme* (also called *Grassmann scheme*) and denoted by $J_q(n, m)$. It is an association scheme of type A_{m+n-1} since $\mathrm{GL}_{m+n}(q)$ is a Chevalley group of type A_{m+n-1}, see [Sta84, p. 110]. We refer to [BI84, § 3.6], [BCN89, § 9.3], and [Ban+21, § 6.4] for more information on this association scheme.

The valencies are

$$v_i = q^{i^2} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ i \end{bmatrix}_q$$

for all $i = 0, 1, \ldots, n$. The P-numbers are given by [Del76a, Theorem 10]

$$P_i(k) = \sum_{j=0}^{i} (-1)^{i-j} \begin{bmatrix} n-j \\ i-j \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q \begin{bmatrix} m+j-k \\ j \end{bmatrix}_q q^{jk+\binom{i-j}{2}} \tag{2.28}$$

and the Q-numbers are given by [Del78b, § 2]

$$Q_k(i) = \mu_k \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} m+n+1-k \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} m \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} i \\ j \end{bmatrix}_q q^{-ij}, \tag{2.29}$$

where the multiplicities are $\mu_k = \left[\begin{smallmatrix} m+n \\ k \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} m+n \\ k-1 \end{smallmatrix}\right]_q$. The q-Johnson scheme is the q-analog of the Johnson scheme; namely by taking the limit $q \to 1$, the P- and Q-numbers as well as the valencies and multiplicities equal those of the Johnson scheme $J(n, m)$, cf. (2.3), (2.5), and (2.6).

[9] We always use algebraic and not projective (geometric) dimension in this thesis.

The P-number $P_i(k)$ from (2.28) is a polynomial of degree i in the variable $[k]_q[m+n+1-k]_q$, where $[n]_q$ is the *q-number* defined by

$$[n]_q = \frac{q^n-1}{q-1}.$$

Therefore, the q-Johnson scheme is P-polynomial with respect to the ordering $R_0, R_1, \ldots, R_n$. Moreover, the Q-number $Q_k(i)$ is a polynomial of degree k in q^{-i}. Thus, the q-Johnson scheme is also Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_n$ of the primitive idempotents that is imposed by (2.29).

We can also write the P- and Q-numbers in a different form by using the q-analogs of the Hahn and dual Hahn polynomials. This requires some definitions. We define the *q-Pochhammer symbol*[10] $(a;q)_n$ by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$$

for a positive integer n and a real number a and the *q-hypergeometric function* ${}_r\phi_s$ by

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} \middle|\, q,z\right) = \sum_{\ell=0}^{\infty} \frac{(a_1;q)_\ell\cdots(a_r;q)_\ell}{(b_1;q)_\ell\cdots(b_s;q)_\ell}\,(-1)^{(1+s-r)\ell}q^{(1+s-r)\binom{\ell}{2}}\,\frac{z^\ell}{(q;q)_\ell}.$$

Observe that the q-binomial coefficient and the q-Pochhammer symbol are connected as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{-n};q)_k}{(q;q)_k}(-1)^k q^{kn-\binom{k}{2}}. \tag{2.30}$$

This can be seen as the q-analog of (2.13). The *dual q-Hahn polynomial*[11] with parameters n, q, C, D is defined by

$$E_i(\mu(x);C,D,n;q) = {}_3\phi_2\left(\begin{matrix} q^{-i},q^{-x},CDq^{x+1} \\ Cq,q^{-n} \end{matrix} \middle|\, q;q\right),$$

which is a polynomial of degree i in $\mu(x) = q^{-x}+CDq^{x+1}$, see [KLS10, § 14.7]. The *q-Hahn polynomial*[12] with parameters n, q, A, B is defined by

$$H_k(q^{-x};A,B,n;q) = {}_3\phi_2\left(\begin{matrix} q^{-x},q^{-k},ABq^{k+1} \\ Aq,q^{-n} \end{matrix} \middle|\, q;q\right),$$

which is a polynomial of degree k in q^{-x}, see [KLS10, § 14.6]. Then the P- and Q-numbers can be written as

$$\begin{aligned} P_i(k) &= v_i E_i(\mu(k);q^{-m-1},q^{-n-1},n;q) \\ &= v_i\,{}_3\phi_2\left(\begin{matrix} q^{-i},q^{-k},q^{k-m-n-1} \\ q^{-m},q^{-n} \end{matrix} \middle|\, q;q\right) \end{aligned} \tag{2.31}$$

[10] This is the q-analog of the Pochhammer symbol in the sense that $(q^a;q)_n/(1-q)^n = a^{(n)}$ for $q \to 1$.

[11] This is indeed the q-analog of the dual Hahn polynomial since $E_i(\mu(x);q^C,q^D,n;q) = E_i(\lambda(x);C,D,n)$ for $q \to 1$.

[12] We have $H_k(q^{-x};q^A,q^B,n;q) = H_k(x;A,B,n)$ for $q \to 1$, which justifies the name q-Hahn polynomial.

and

$$\begin{aligned} Q_k(i) &= \mu_k H_k(q^{-i}; q^{-m-1}, q^{-n-1}, n; q) \\ &= \mu_k\, {}_3\phi_2\left(\begin{matrix} q^{-i}, q^{-k}, q^{k-m-n-1} \\ q^{-m}, q^{-n} \end{matrix}\;\middle|\; q; q\right). \end{aligned} \tag{2.32}$$

see [Del76b, § 5.2] and [Del78b, § 2].

2.3.2 Polar space schemes

Finite classical polar spaces play an important role in finite geometry. In this thesis, we will study codes and special designs in polar spaces. We start by giving some basic facts from finite geometry. For further background, we refer to [Cam92], [Tay92], [BCN89, § 9.4], [Bal15, § 4.2], and [HT16, § 5.1].

Different forms. Let V be a finite-dimensional vector space over $\mathbb{F}_q$ and let σ be a field automorphism of $\mathbb{F}_q$. A *sesquilinear form* on V is a mapping $f\colon V \times V \to \mathbb{F}_q$ that is linear in its first argument and semilinear in its second argument, that is,

$$\begin{aligned} f(av_1 + w_1, v_2) &= af(v_1, v_2) + f(w_1, v_2) \\ f(v_1, av_2 + w_2) &= a^\sigma f(v_1, v_2) + f(v_1, w_2) \end{aligned}$$

for all $v_1, v_2, w_1, w_2 \in V$ and $a \in \mathbb{F}_q$. A sesquilinear form f on V is called *reflexive* if $f(v, w) = 0$ implies $f(w, v) = 0$ for all $v, w \in V$. A reflexive sesquilinear form f is *nondegenerate* if $f(v, w) = 0$ for all $w \in V$ implies $v = 0$, or equivalently if $f(v, w) = 0$ for all $v \in V$ implies $w = 0$. If a mapping $f\colon V \times V \to \mathbb{F}_q$ is linear in both arguments, then f is a *bilinear form*. A sesquilinear form f is *Hermitian* if σ is a nontrivial involution (i.e., $\sigma^2 = 1$) and if $f(v, w) = f(w, v)^\sigma$ for all $v, w \in \mathbb{F}_q$, which requires that q is a square and $\sigma(a) = a^{\sqrt{q}}$ for all $a \in \mathbb{F}_q$. A bilinear form f is *symmetric* if $f(v, w) = f(w, v)$ for all $v, w \in V$ and *alternating* if $f(v, v) = 0$ for all $v \in V$. In the case of an alternating form, the vector space V must have even dimension. A *quadratic form* on V is a mapping $Q\colon V \to \mathbb{F}_q$ such that $Q(av) = a^2 Q(v)$ for all $v \in V$, $a \in \mathbb{F}_q$ and $(v, w) \mapsto Q(v + w) - Q(v) - Q(w)$ is a bilinear form on V. A quadratic form Q on V is *nondegenerate* if $Q(v + w) = Q(w)$ for all $w \in V$ implies $v = 0$.

An *isometry* is a bijective linear mapping φ between two finite-dimensional vector spaces V_1 and V_2 equipped with reflexive sesquilinear forms f_1 and f_2, respectively, such that φ transforms one form into the other, that is, $f_1(v, w) = f_2(\varphi(v), \varphi(w))$ for all $v, w \in V_1$. The same definition holds if the vector spaces are equipped with quadratic forms, where the condition for being an isometry is replaced by $f_1(v) = f_2(\varphi(v))$ for all $v \in V_1$.

Polar spaces. We can now define polar spaces. A subspace U of V is called *totally isotropic* with respect to a sesquilinear or quadratic form on V if the form vanishes completely on this subspace U, that is, $f(u,v) = 0$ for all $u, v \in U$, or in the case of a quadratic form, $f(u) = 0$ for all $u \in U$. A *finite classical polar space* with respect to a nondegenerate form f consists of all totally isotropic subspaces of V. We will consider only finite classical polar spaces in this thesis and therefore, we will refer to them as *polar spaces*.

The totally isotropic subspaces that are maximal with respect to inclusion in a polar space are called *generators*. It turns out that all generators have the same dimension, which is called the *rank* of the polar space. This comes from the following theorem as can be seen in the corollary thereafter.

Theorem 2.3.1 ([Tay92, Theorem 7.4])**.** *Let V be a finite-dimensional vector space over $\mathbb{F}_q$ equipped with a nondegenerate quadratic, symmetric, alternating, or Hermitian form. Then every isometry between two subspaces of V extends to an isometry of V.*

Corollary 2.3.2 ([Tay92, p. 59])**.** *Every two generators of a polar space have the same dimension.*

Proof. Assume that there are two generators U and W in a polar space defined on a vector space V with $\dim(U) < \dim(W)$. Then there exists a bijective mapping from U to a subspace W' of W, which is an isometry since both spaces U and W are totally isotropic. By Theorem 2.3.1, this isometry extends to an isometry $\varphi\colon V \to V$. Thus, U lies in the totally isotropic space $\varphi^{-1}(W)$, which is a contradiction to U being maximal. □

Since the form f on a vector space V associated with a polar space $\mathcal{P}$ is nondegenerate, we can apply the dimension formula $\dim(U) + \dim(U^\perp) = \dim(V)$ for $U \in \mathcal{P}$, where

$$U^\perp = \{u \in V \mid f(u,v) = 0 \text{ for all } v \in U\},$$

and $U \subseteq U^\perp$ to obtain that the rank of $\mathcal{P}$ is at most $\frac{1}{2}\dim(V)$.

A polar space $\mathcal{P}$ has the parameter e if every $(n-1)$-space in $\mathcal{P}$ lies in exactly $p^{e+1} + 1$ generators.

Types of polar spaces. We will now look at the different types of polar spaces of rank n. Let $V(m,p)$ be an m-dimensional vector space over $\mathbb{F}_p$, where p is a prime power. It is well known that up to a change of the coordinate system, the polar spaces can be described as follows.

- The *Hermitian polar spaces* ${}^2A_{2n-1}$ and ${}^2A_{2n}$. They consist of all totally isotropic subspaces of $V(2n, q^2)$ and $V(2n+1, q^2)$, respectively, where the form associated with $V(m, q^2)$ is Hermitian and given by

  $$x_1 y_1^q + x_2 y_2^q + \cdots + x_m y_m^q.$$

  The group of isometries of ${}^2A_{m-1}$ is the unitary group $\mathrm{U}_m(q^2)$.

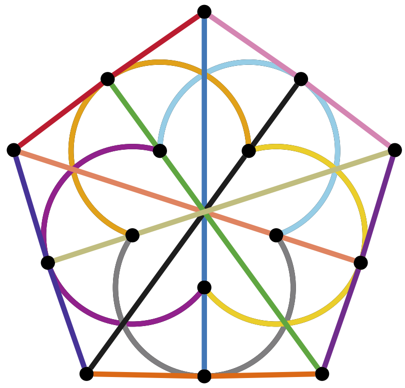


FIGURE 2.8. Symplectic polar space C_2 of rank 2, where the points are the 1-spaces and the lines are the 2-spaces (generators).

- The *symplectic polar space* C_n. It consists of all totally isotropic subspaces of $V(2n, q)$, where the form is alternating and given by

$$x_1y_2 - x_2y_1 + \cdots + x_{2n-1}y_{2n} - x_{2n}y_{2n-1}.$$

The group of isometries is the symplectic group $\mathrm{Sp}_{2n}(q)$. For example, the symplectic polar space C_2 of rank 2 is depicted in Figure 2.8.

- The *hyperbolic polar space* D_n. It consists of all totally isotropic subspaces of $V(2n, q)$, where the form is quadratic and given by

$$x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}.$$

The group of isometries is the orthogonal group $\mathrm{O}^+_{2n}(q)$. For example, the hyperbolic polar space D_2 of rank 2 is depicted in Figure 2.9.

- The *parabolic polar space* B_n. It consists of all totally isotropic subspaces of $V(2n+1, q)$, where the form is quadratic and given by

$$x_1^2 + x_2x_3 + \cdots + x_{2n-2}x_{2n-1} + x_{2n}x_{2n+1}.$$

The group of isometries is the orthogonal group $\mathrm{O}_{2n+1}(q)$.

- The *elliptic polar space* ${}^2D_{n+1}$. It consists of all totally isotropic subspaces of $V(2n+2, q)$, where the form is quadratic and given by

$$g(x_1, x_2) + x_3x_4 + \cdots + x_{2n-1}x_{2n} + x_{2n+1}x_{2n+2}$$

with an irreducible homogeneous polynomial g over $\mathbb{F}_q$ of degree two. The group of isometries is the orthogonal group $\mathrm{O}^-_{2n+2}(q)$.

Table 2.2 contains a summary of all these six polar spaces together with their parameter e.[13] We remark that there are different notations for these

[13] In some literature (for example, [Van11]), one finds the parameters $1/2$, $3/2$, 1, 0, 1, 2 instead of $-1/2$, $1/2$, 0, -1, 0, 1.

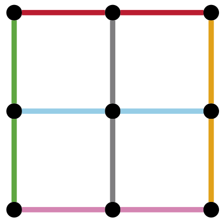


FIGURE 2.9. Hyperbolic polar space D_2 of rank 2, where the points are the 1-spaces and the lines are the 2-spaces (generators).

TABLE 2.2. List of all six finite classical polar spaces.

name	**form**	**notation**	**group**	$\dim(V)$	p	e
Hermitian	Hermitian	${}^2A_{2n-1}$	$\mathrm{U}_{2n}(q^2)$	$2n$	q^2	$-1/2$
Hermitian	Hermitian	${}^2A_{2n}$	$\mathrm{U}_{2n+1}(q^2)$	$2n+1$	q^2	$1/2$
symplectic	alternating	C_n	$\mathrm{Sp}_{2n}(q)$	$2n$	q	0
hyperbolic	quadratic	D_n	$\mathrm{O}^+_{2n}(q)$	$2n$	q	-1
parabolic	quadratic	B_n	$\mathrm{O}_{2n+1}(q)$	$2n+1$	q	0
elliptic	quadratic	${}^2D_{n+1}$	$\mathrm{O}^-_{2n+2}(q)$	$2n+2$	q	1

six polar spaces.[14] Here, we use the notation that comes from identifying the corresponding projective groups of isometries with the Chevalley groups of type ${}^2A_m$, C_n, D_n, B_n, and ${}^2D_{n+1}$, see [Car89, Theorems 11.3.2, 14.5.1, 14.5.2].

There exist various embeddings of the polar spaces into each other.

Remark 2.3.3. Let $\mathcal{P}$ be a polar space consisting of the totally isotropic subspaces of an m-dimensional vector space V over $\mathbb{F}_p$ equipped with a nondegenerate form f. By choosing a hyperplane H of V such that f restricted to this hyperplane is still nondegenerate, we obtain the following embeddings by intersecting the polar space with this hyperplane H:

(a) the Hermitian polar space ${}^2A_{2n-1}$ into the Hermitian polar space ${}^2A_{2n}$;

(b) the hyperbolic polar space D_n into the parabolic polar space B_n;

(c) the parabolic polar space B_n into the elliptic polar space ${}^2D_{n+1}$;

(d) the Hermitian polar space ${}^2A_{2n}$ into the Hermitian polar ${}^2A_{2n+1}$;

(e) the parabolic polar space B_n into the hyperbolic polar space D_{n+1};

(f) the elliptic polar space ${}^2D_{n+1}$ into the parabolic polar space B_{n+1}.

[14] The notation based on the embedding of the polar spaces ${}^2A_m$, C_n, D_n, B_n, and ${}^2D_{n+1}$ into a projective space is given by $H(m,q^2)$, $W(2n-1,q)$, $Q^+(2n-1,q)$, $Q(2n,q)$, and $Q^-(2n+1,q)$, respectively. The last three are sometimes also denoted by $\Omega^+(2n,q)$, $\Omega(2n+1,q)$, and $\Omega^-(2n+2,q)$ in the literature.

For example in (a), (b), and (d), the corresponding hyperplane can be chosen to be $H = \{(x_1, x_2, \dots, x_m) \in V \mid x_1 = 0\}$. Observe that in (a)–(c) the ranks of the polar spaces stay the same, whereas in (d)–(f) they decrease by 1.

If q is even, then the symplectic polar space C_n and the parabolic polar space B_n are isomorphic, see [De 16, Corollary 7.129] for a proof.

Polar space schemes. We will now study the association schemes that arise from polar spaces. Let $\mathcal{P}$ be a polar space of rank n and let G be the corresponding group from Table 2.2. Then G acts transitively on the set X of generators of $\mathcal{P}$. The action of G extends to $X \times X$ componentwise and the orbits of this group action are given by

$$R_i = \{(U, W) \in X \times X \mid \dim(U \cap W) = n - i\} \tag{2.33}$$

for $i = 0, 1, \dots, n$, see [Sta80, Proposition 4.9]. Proposition 2.1.4 implies that $(X, (R_i))$ is an association scheme with n classes. For more information on the polar space schemes, we refer to [BI84, § 3.6], [BCN89, § 9.4], and [Ban+21, § 6.4]. An example of an association scheme arising from a polar space is given in Figure 2.10.

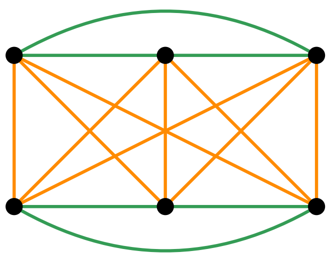


FIGURE 2.10. Association scheme arising from the hyperbolic polar space D_2 with the relations R_1 and R_2.

It is well known that

$$|X| = \prod_{i=1}^{n} (1 + p^{i+e}) \tag{2.34}$$

see [BCN89, Lemma 9.4.1], for example. The P- and Q-numbers of $(X, (R_i))$ are given by

$$P_i(k) = v_i \begin{bmatrix} n \\ k \end{bmatrix}_p^{-1} \sum_{\ell=0}^{i} (-1)^\ell \begin{bmatrix} n-i \\ k-\ell \end{bmatrix}_p \begin{bmatrix} i \\ \ell \end{bmatrix}_p p^{\ell(\ell-i-e-1)} \tag{2.35}$$

and

$$Q_k(i) = \mu_k \begin{bmatrix} n \\ k \end{bmatrix}_p^{-1} \sum_{\ell=0}^{i} (-1)^\ell \begin{bmatrix} n-i \\ k-\ell \end{bmatrix}_p \begin{bmatrix} i \\ \ell \end{bmatrix}_p p^{\ell(\ell-i-e-1)}, \tag{2.36}$$

where

$$v_i = p^{\binom{i+1}{2}+ie} \begin{bmatrix} n \\ i \end{bmatrix}_p \tag{2.37}$$

and

$$\mu_k = p^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{(-p^{e+1};p)_n}{(-p^{e-k+1};p)_{n-k}(-p^{n-k-e-1};p)_k} \tag{2.38}$$

are the valencies and the multiplicities, respectively, see [Sta80, Equation (8.1)], [Sta81, Proposition 2.4].[15,16]

The P-number $P_i(k)$, given in (2.35), is a polynomial of degree i in p^{-k}, and the Q-number $Q_k(i)$, given in (2.36), is a polynomial of degree k in p^{-i}. These polynomials can be written by using *q-Krawtchouk polynomials*[17], which are defined by

$$K_i(q^{-x};A,n;q) = {}_3\phi_2\left(\begin{matrix} q^{-x}, q^{-i}, -Aq^{i+1} \\ 0, q^{-n} \end{matrix} \middle| q;q\right) \quad \text{for } k = 0,1,\dots,n,$$

see [KLS10, § 14.15]. Namely, we have

$$\begin{aligned} P_i(k) &= v_i K_k(p^{-i};p^{-n-e-2},n,;p) \\ &= v_i\, {}_3\phi_2\left(\begin{matrix} p^{-k}, p^{-i}, -p^{-n-e-1+k} \\ 0, p^{-n} \end{matrix} \middle| p;p\right) \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} Q_k(i) &= \mu_k K_k(p^{-i};p^{-n-e-2},n;p) \\ &= \mu_k\, {}_3\phi_2\left(\begin{matrix} p^{-k}, p^{-i}, -p^{-n-e-1+k} \\ 0, p^{-n} \end{matrix} \middle| p;p\right). \end{aligned} \tag{2.40}$$

The association scheme $(X,(R_i))$ is thus P-polynomial with respect to the ordering $R_0, R_1, \dots, R_n$ and Q-polynomial with respect to the ordering $E_0, E_1, \dots, E_n$ of the primitive idempotents that is imposed by (2.33) and (2.39). We call both orderings the *standard orderings*.

Note that the association schemes arising from B_n and C_n have the same parameters (e.g., size of X, eigenvalues), however they are isomorphic only when q is even, see [BCN89, § 9.4.A] or [Ban+21, § 6.4].

$^2A_{2n-1}$ and a second ordering. The association scheme $^2A_{2n-1}$ is Q-polynomial with respect to two different orderings: the standard ordering $E_0, E_1, \dots, E_n$ and $E_0, E_n, E_1, E_{n-1}, E_2, E_{n-2}, \dots$ [CS86]. We continue to use $P_i(k)$ and $Q_k(i)$ to denote the P- and Q-numbers with respect to the standard

[15] It should be noted that p is assumed to be odd in [Sta80] and [Sta81]. However, all parameters of the association scheme as well as $P_i(k)$ and $Q_k(i)$ are polynomials in p. Hence, the expressions for $P_i(k)$ and $Q_k(i)$ hold for all p.

[16] A computation of the P-numbers can also be found in [Van11, Theorem 4.3.6] and [Eis99, Theorem 3.8].

[17] The name comes from the fact that by taking the limit $q \to 1$, a q-Krawtchouk polynomial becomes a Krawtchouk polynomial, more concretely, $K_i(q^{-x};A,n;q) = K_i(x;(A+1)^{-1},n)$ for $q \to 1$. A resemblance can also be seen between (2.4) and (2.35).

ordering and we write $P'_i(k)$ and $Q'_k(i)$ for the P- and Q-numbers with respect to the second ordering. Then $P_i(k)$ is given in (2.39) and $P'_i(k)$ is given by [CS86]

$$\begin{aligned} P'_i(2k) &= P_i(k) && \text{for } i = 0, \ldots, n \text{ and } k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, \\ P'_i(2k+1) &= P_i(n-k) && \text{for } i = 0, \ldots, n \text{ and } k = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned}$$

By applying the quadratic transformation for hypergeometric functions (see [KLS10, (1.13.28)], for example)

$${}_4\phi_3\left(\begin{matrix} A^2, B^2, C, D \\ ABq^{1/2}, -ABq^{1/2}, -CD \end{matrix} \,\middle|\, q; q\right) = {}_4\phi_3\left(\begin{matrix} A^2, B^2, C^2, D^2 \\ A^2B^2q, -CD, -CDq \end{matrix} \,\middle|\, q^2; q^2\right) \tag{2.41}$$

to $P_i(k)$ and $P_i(n-k)$ from (2.39), we obtain

$$P'_i(k) = v_i\, {}_3\phi_2\left(\begin{matrix} (-q)^{-i}, (-q)^{-k}, (-q)^{-2n+k-1} \\ (-q)^{-n}, -(-q)^{-n} \end{matrix} \,\middle|\, -q; -q\right), \tag{2.42}$$

where $v_i = q^{i^2} \left[{n \atop i} \right]_{q^2}$ is the valency of R_i. Using (2.11) gives

$$Q'_k(i) = \mu'_k\, {}_3\phi_2\left(\begin{matrix} (-q)^{-i}, (-q)^{-k}, (-q)^{-2n+k-1} \\ (-q)^{-n}, -(-q)^{-n} \end{matrix} \,\middle|\, -q; -q\right) \tag{2.43}$$

with

$$\mu'_k = \begin{cases} \mu_{k/2} & \text{for even } k \\ \mu_{n-(k-1)/2} & \text{for odd } k, \end{cases}$$

where μ_k denotes the multiplicities of ${}^2A_{2n-1}$ with respect to the standard ordering of $E_0, E_1, \ldots, E_n$. Observe that $P'_i(k)$ and $Q'_k(i)$ can be expressed with a dual q-Hahn polynomial and a q-Hahn polynomial, respectively. Namely, we have

$$\begin{aligned} P'_i(k) &= v_i E_i(\mu(k); (-q)^{-n-1}, (-q)^{-n-1}, n; -q) \\ Q'_k(i) &= \mu'_k H_k((-q)^{-i}; (-q)^{-n-1}, (-q)^{-n-1}, n; -q). \end{aligned}$$

Bipartite halves $\frac{1}{2}D_n$ of D_n. The hyperbolic polar space D_n gives rise to another association scheme, called the *bipartite half of D_n*, in the following way. Let X be the set of generators of D_n and define two generators in X to be equivalent if the dimension of their intersection has the same parity as n. This induces two equivalence classes, X_1 and X_2, which are called *Latin* and *Greek*. See Figure 2.11 for the bipartite halves of D_2. Each pair $(X_i, (R_{2j})_{0 \le j \le \lfloor \frac{n}{2} \rfloor})$ is a P- and Q-polynomial association scheme with $\lfloor n/2 \rfloor$ classes [BCN89, § 9.4.C], denoted by $\frac{1}{2}D_n$. The corresponding group that acts transitively on $\frac{1}{2}D_n$ is the special orthogonal group $\mathrm{SO}^+_{2n}(q)$, see [Sta80, p. 635] and [Tay92, Theorem 11.61], which is a Chevalley group of type D_n. Therefore, the association

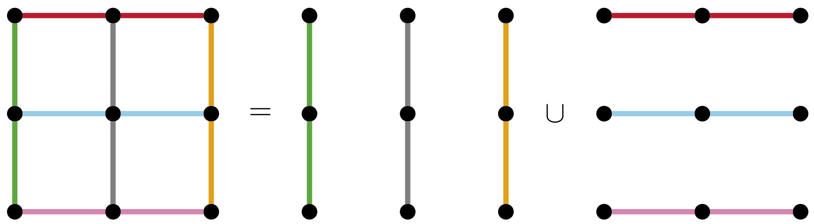


FIGURE 2.11. The bipartite halves of the hyperbolic polar space D_2.

scheme is of type D_n. In what follows, we put $n = \lfloor m/2 \rfloor$ and treat $\frac{1}{2}D_m$. We still denote by $P_i(k)$ and $Q_k(i)$ the P- and Q-numbers of D_m and by $P_i'(k)$ and $Q_k'(i)$ the P- and Q-numbers of $\frac{1}{2}D_m$. From [CS86] we find that

$$P_i'(k) = P_{2i}(k) \quad \text{for } i, k = 0, 1, \dots, n.$$

Applying (2.41) to $P_{2i}(k)$ given in (2.39) implies

$$P_i'(k) = v_{2i}\, {}_3\phi_2 \left(\begin{matrix} q^{-2i}, q^{-2k}, q^{-2m+2k} \\ q^{-m}, q^{-m+1} \end{matrix} \middle| \, q^2; q^2 \right), \tag{2.44}$$

where $v_{2i} = q^{\binom{2i}{2}} \begin{bmatrix} m \\ 2i \end{bmatrix}_q$ is the valency of the relation R_{2i} of D_m. Using (2.11) implies

$$Q_k'(i) = \mu_k\, {}_3\phi_2 \left(\begin{matrix} q^{-2i}, q^{-2k}, q^{-2m+2k} \\ q^{-m}, q^{-m+1} \end{matrix} \middle| \, q^2; q^2 \right), \tag{2.45}$$

where μ_k is the multiplicity of D_m. Observe that similar to ${}^2A_{2n-1}$ with respect to the second ordering, the numbers $P_i'(k)$ and $Q_k'(i)$ can be expressed with a dual q-Hahn polynomial and a q-Hahn polynomial, respectively. Namely, we have

$$P_i'(k) = v_{2i} E_i(\mu(k); q^{-m-1}, q^{-m-1}, m; q^2)$$
$$Q_k'(i) = \mu_k H_k(q^{-2i}; q^{-m-1}, q^{-m-1}, m; q^2).$$

$\frac{1}{2}D_{n+1}$ and B_n/C_n. In the cases of B_n and C_n, one obtains a new association scheme with the classes

$$R_0,\ R_1 \cup R_2,\ R_3 \cup R_4,\ \dots,$$

see [IMU89]. Then the bipartite half $\frac{1}{2}D_{n+1}$ is isomorphic to this new association scheme arising from B_n, which also implies that $\frac{1}{2}D_{n+1}$ has the same parameters as the new association scheme arising from C_n and they are isomorphic if q is even. The first part can be seen as follows. Recall that we can embed B_n into D_{n+1} by intersecting the space D_{n+1} with a nondegenerate hyperplane H. Since $e = 1$ for D_{n+1}, every n-space of D_{n+1} lies in exactly two $(n+1)$-spaces—one from each bipartite half. Because of $\dim(H) = 2n+1$, every n-space contained in H lies in a uniquely determined $(n+1)$-space in $\frac{1}{2}D_{n+1}$. For two n-spaces U_1 and U_2 in H, we have $\dim(U_1 \cap U_2) \in \{n-k, n-k-1\}$ if and only if $\dim(V_1 \cap V_2) = n-k$, where V_1 and V_2 are the corresponding $(n+1)$-spaces containing U_1 or U_2, respectively. See also [BCN89, § 9.4.C].

2.3.3 Affine schemes

It remains to introduce the *affine schemes*, which consist of the matrix groups of type A_{m+n-1}, D_n, and ${}^2A_n$ together with relations defined by the rank metric. It will turn out later that the affine schemes are closely related to the q-Johnson scheme (which is of type A_{m+n-1}), the hyperbolic bipartite half $\frac{1}{2}D_n$, and the association scheme arising from the Hermitian polar space ${}^2A_{2n-1}$ and we will see a remarkable resemblance between these schemes in Subsection 2.3.4 and throughout Chapter 5.

Let m and n be integers with $m \geq n \geq 1$ and let X be the set of $m \times n$ matrices over $\mathbb{F}_q$. The group $G = X \rtimes (\mathrm{GL}_m(q) \times \mathrm{GL}_n(q))$ acts transitively on X via

$$\begin{aligned} G \times X &\to X \\ ((D,(M,N)),A) &\mapsto MAN^{-1} + D. \end{aligned}$$

The action of G extends to $X \times X$ componentwise and the orbits of this group action are given by

$$R_i = \{(A,B) \in X \times X \mid \mathrm{rank}(A-B) = i\}$$

for $i = 0,1,\dots,n$. By Proposition 2.1.4, $(X,(R_i))$ is an association scheme with n classes, which is known as the *bilinear forms scheme* (or *q-Hamming scheme*) since the matrices in $\mathbb{F}_q^{m\times n}$ are in 1-to-1-correspondence with the set of bilinear forms on $\mathbb{F}_q^m \times \mathbb{F}_q^n$. We denote this association scheme by $\mathrm{Bil}_q(n,m)$. Note that $|X| = q^{mn}$. For more information, we refer to [Del78a], [BI84, § 3.6 (II)], [BCN89, § 9.5.A], and [Ban+21, § 6.4.1].

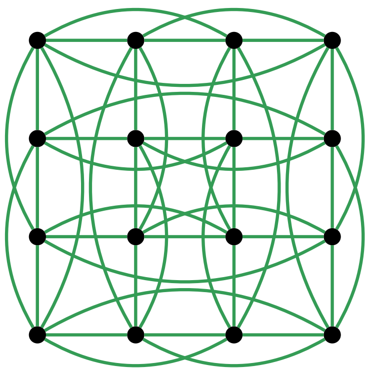


Figure 2.12. Bilinear forms scheme $\mathrm{Bil}_2(2,2)$, where only the relation R_1 is depicted to keep the graph more clearly represented.

An $m \times m$ matrix $A = (a_{ij})$ over $\mathbb{F}_q$ is called *alternating* if it is skew-symmetric with zero main diagonal, that is, $a_{i,i} = 0$ and $a_{i,j} + a_{j,i} = 0$ for all i,j. Every alternating matrix has even rank. Let X now be the set of alternating $m \times m$ matrices. The group $G = X \rtimes \mathrm{GL}_m(q)$ acts transitively on X

via

$$\begin{aligned} G \times X &\to X \\ ((D,M),A) &\mapsto MAM^{-1} + D. \end{aligned}$$

The action of G extends to $X \times X$ componentwise and the orbits of this group action are given by

$$R_i = \{(A,B) \in X \times X \mid \operatorname{rank}(A-B) = 2i\}$$

for $i = 0,1,\dots,n$. By Proposition 2.1.4, $(X,(R_i))$ is an association scheme with $n = \lfloor m/2 \rfloor$ classes. This scheme is known as the *alternating bilinear forms scheme* since a matrix is alternating if and only if the corresponding bilinear form is alternating. We denote this association scheme by $\mathrm{Alt}_q(m)$. Note that $|X| = q^{\binom{m}{2}}$. For more information, we refer to [DG75], [BI84, § 3.6 (II)], [BCN89, § 9.5.B], and [Ban+21, § 6.4.1].

An $n \times n$ matrix A over $\mathbb{F}_{q^2}$ is called *Hermitian* if $A^* = A$, where A^* is obtained from A by conjugation $x \mapsto x^q$ of each entry x of A and transposition. Let X be the set of $n \times n$ Hermitian matrices over $\mathbb{F}_{q^2}$. The group $G = X \rtimes \mathrm{GL}_n(q^2)$ acts transitively on X via

$$\begin{aligned} G \times X &\to X \\ ((D,M),A) &\mapsto MAM^{*} + D. \end{aligned}$$

The action of G extends to $X \times X$ componentwise and the orbits of this group action are given by

$$R_i = \{(A,B) \in X \times X \mid \operatorname{rank}(A-B) = i\}$$

for $i = 0,1,\dots,n$. By Proposition 2.1.4, $(X,(R_i))$ is an association scheme with n classes, which is known as the *Hermitian forms scheme* since a matrix is Hermitian if and only if the corresponding form is Hermitian. We denote it by $\mathrm{Her}_q(n)$. For example, $\mathrm{Her}_2(2)$ is depicted in Figure 2.13. Note that $|X| = q^{n^2}$. For more information, we refer to [Sch18], [BI84, § 3.6 (II)], [BCN89, § 9.5.C], and [Ban+21, § 6.4.1].

In what follows, we write

$$(b,c) = \begin{cases} (q, q^{m-n}) & \text{for } \mathrm{Bil}_q(n,m) \\ (-q,-1) & \text{for } \mathrm{Her}_q(n) \\ (q^2, 1/q) & \text{for } \mathrm{Alt}_q(m) \text{ if } m \text{ is even} \\ (q^2, q) & \text{for } \mathrm{Alt}_q(m) \text{ if } m \text{ is odd.} \end{cases} \tag{2.46}$$

For all three affine schemes $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$, the valencies and multiplicities are given by

$$v_i = \mu_i = b^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_b \prod_{j=0}^{i-1} (cb^{n-j} - 1)$$

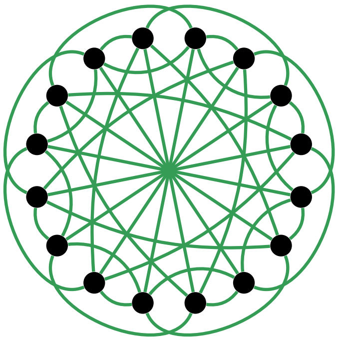


FIGURE 2.13. Hermitian forms scheme $\mathrm{Her}_2(2)$, where the edges represent the relation R_2 and R_1 is neglected for keeping it simple.

and the P- and Q-numbers are given by

$$P_i(k) = Q_i(k) = \sum_{j=0}^{i} (-1)^{i-j} b^{\binom{i-j}{2}} \begin{bmatrix} n-j \\ n-i \end{bmatrix}_b \begin{bmatrix} n-k \\ j \end{bmatrix}_b (cb^n)^j, \tag{2.47}$$

see [Del78a], [DG75], and [Sch18]. Observe that $P_i(k)$ is a polynomial of degree i in b^{-k}. Therefore, these three schemes are P- and Q-polynomial with respect to the ordering $R_0, R_1, \dots, R_n$ of relations and the ordering $E_0, E_1, \dots, E_n$ of the primitive idempotents imposed by (2.47). The corresponding polynomials are related to *affine q-Krawtchouk polynomials* defined by

$$K_i^{\mathrm{aff}}(q^{-x}; B, n; q) = {}_3\phi_2 \left(\begin{matrix} q^{-x}, q^{-i}, 0 \\ Bq, q^{-n} \end{matrix} \,\middle|\, q; q \right),$$

see [KLS10, § 14.16]. Namely, we have

$$\begin{aligned} P_i(k) = Q_i(k) &= v_i K_i^{\mathrm{aff}}(q^{-k}; c^{-1}b^{-n-1}, n; b) \\ &= v_i \, {}_3\phi_2 \left(\begin{matrix} b^{-k}, b^{-i}, 0 \\ c^{-1}b^{-n}, b^{-n} \end{matrix} \,\middle|\, b; b \right). \end{aligned} \tag{2.48}$$

2.3.4 Connection between affine and ordinary q-analogs

The affine schemes $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ are related to the ordinary q-analogs $J_q(n,m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ in the following way. For a vector space V over $\mathbb{F}_p$, let $P_n(V)$ be the set of n-spaces of V. Define the mapping

$$\begin{aligned} v \colon \mathbb{F}_p^{m\times n} &\to P_n(\mathbb{F}_p^{m+n}) \\ A &\mapsto \left\{ \begin{pmatrix} x \\ Ax \end{pmatrix} \mid x \in \mathbb{F}_p^n \right\}. \end{aligned} \tag{2.49}$$

It is well known [BCN89, § 9.5.E] that, after an appropriate choice of the form, $v(A)$ is in ${}^2A_{2n-1}$ if and only if A is Hermitian, and $v(A)$ is in D_n if and only if A is alternating (as before, $p = q^2$ for ${}^2A_{2n-1}$ and $p = q$ otherwise). The mapping v satisfies

$$n - \dim(v(A) \cap v(B)) = \operatorname{rank}(A - B)$$

for all $A, B \in \mathbb{F}_q^{n\times n}$, so in particular v is injective. We therefore obtain the following embeddings:

$$\begin{aligned} \operatorname{Bil}_q(n,m) &\hookrightarrow J_q(n,m) \\ \operatorname{Her}_q(n) &\hookrightarrow {}^2A_{2n-1} \\ \operatorname{Alt}_q(m) &\hookrightarrow \tfrac{1}{2}D_m \end{aligned} \tag{2.50}$$

The P- and Q-numbers of these six association schemes are also quite connected. Recall that we defined the parameters b and c in (2.46), which we used to express the P- and Q-numbers of the affine schemes in a unified way. We can use these parameters b and c for the ordinary q-analogs as well to also write their P- and Q-numbers in a unified way. We thus expand the definition of b and c as follows

$$(b,c) = \begin{cases} (q, q^{m-n}) & \text{for } \operatorname{Bil}_q(n,m) \text{ and } J_q(n,m) \\ (-q,-1) & \text{for } \operatorname{Her}_q(n) \text{ and } {}^2A_{2n-1} \\ (q^2, 1/q) & \text{for } \operatorname{Alt}_q(m) \text{ and } \tfrac{1}{2}D_m \text{ if } m \text{ is even} \\ (q^2, q) & \text{for } \operatorname{Alt}_q(m) \text{ and } \tfrac{1}{2}D_m \text{ if } m \text{ is odd.} \end{cases} \tag{2.51}$$

Observe that because of (2.31), (2.32), (2.44), (2.45), (2.42), and (2.43) the P- and Q-numbers $P_i'(k)$ and $Q_k'(i)$ of $J_q(n,m)$, ${}^2A_{2n-1}$ (with respect to the second Q-polynomial ordering), and $\frac{1}{2}D_m$ are given by dual q-Hahn polynomials and q-Hahn polynomials as follows

$$P_i'(k) = v_i' \, {}_3\phi_2 \left(\begin{matrix} b^{-i}, b^{-k}, q^{-1}c^{-1}b^{-2n+k} \\ b^{-n}, c^{-1}b^{-n} \end{matrix} \,\middle|\, b; b \right) \tag{2.52}$$

and

$$Q_k'(i) = \mu_k' \, {}_3\phi_2 \left(\begin{matrix} b^{-i}, b^{-k}, q^{-1}c^{-1}b^{-2n+k} \\ b^{-n}, c^{-1}b^{-n} \end{matrix} \,\middle|\, b; b \right), \tag{2.53}$$

where the corresponding valencies v_i' and multiplicities μ_k' are stated in Table 2.3.

Note that the parameters of the hypergeometric functions in (2.52) and (2.53) are similar to those in (2.48).

We close this chapter by shortly looking at a nonclassical affine scheme that is connected to the symplectic polar space C_n. Namely, the set of symmetric $n \times n$ matrices over $\mathbb{F}_q$ gives rise to an association scheme, known as the *symmetric bilinear forms scheme* $\operatorname{Sym}_q(n)$, where a relation is indexed by the

Table 2.3. Valencies and multiplicities occurring in (2.52) and (2.53), where μ_k for ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$ is given in (2.38).

	$J_q(n,m)$	${}^2A_{2n-1}$	$\frac{1}{2}D_m$
v'_i	$q^{i^2}\begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} v-n \\ i \end{bmatrix}_q$	$q^{i^2}\begin{bmatrix} n \\ i \end{bmatrix}_{q^2}$	$q^{\binom{2i}{2}}\begin{bmatrix} m \\ 2i \end{bmatrix}_q$
μ'_k	$\begin{bmatrix} m+n \\ k \end{bmatrix}_q - \begin{bmatrix} m+n \\ k-1 \end{bmatrix}_q$	$\mu_{k/2}$ for k even $\mu_{n-(k-1)/2}$ for k odd	μ_k

rank of the difference of any matrix pair from the relation and also by the *type* of this difference if the rank is even, see [Sch15], for example. This association scheme is of *type* C_n [Sta84, p. 117] and it is neither P- nor Q-polynomial since the P- and Q-numbers are not determined from evaluations of orthogonal polynomials. However, they are given by a linear combination of evaluations of affine q-Krawtchouk polynomials, see [Sch15] and [Sch20]. As above, after an appropriate choice of the form, the space $v(A)$ with $A \in \mathbb{F}_q^{n\times n}$ is in C_n if and only if A is symmetric and we additionally have the embedding

$$\mathrm{Sym}_q(n) \hookrightarrow C_n, \tag{2.54}$$

see [BCN89, § 9.5.E].

We note that according to Stanton [Sta84, p. 118] a natural additive matrix group of type B_n or ${}^2A_{2n}$ does not seem to exist.

In Chapter 5, we will encounter more similarities between the eight association schemes occurring in (2.50) and (2.54). Moreover, we will exploit these embeddings to construct codes in the polar space schemes.

Chapter 3

Codes in polar spaces

“Hope” is the thing with feathers -
That perches in the soul -
And sings the tune without the words -
And never stops - at all -

—Emily Dickinson

This chapter studies codes in polar spaces. We will derive upper bounds for the size of codes in all polar spaces and show that most of these bounds are sharp up to a constant factor by giving constructions of codes.

The results of this chapter can also be found in [SW22].

3.1 Introduction

Since the last century codes in the Hamming scheme and Johnson scheme have been heavily exploited in different areas of digital communications. However, today's communications are mostly done via a network of intermediate nodes. This requires new types of codes. It turned out that by using combinatorics of vector spaces over a finite field, one can construct suitable codes for network communications. One then studies rank-metric codes and subspace codes [KK08], which are, for example, codes in the bilinear forms scheme [Del78a], the alternating bilinear forms scheme [DG75], the Hermitian forms scheme [Sch18], and the q-Johnson scheme. Here, we introduce another type of subspace codes; namely, we define a *d-code* in a polar space $\mathcal{P}$ to be a set of generators Y of $\mathcal{P}$ such that $n - \dim U \cap W \geq d$ for all distinct $U, W \in Y$ (the mapping $(U, W) \mapsto n - \dim U \cap W$ agrees with the subspace metric, also known as Grassmann metric, used by coding theorists). This chapter focuses on one of the main coding-theoretic problems for codes in polar spaces:

What is the maximum cardinality of a d-code in a polar space of rank n?

This question will be answered for various polar spaces by showing that the bounds are asymptotically sharp. Namely, our main result in this chapter, Theorem 3.2.1 and Corollary 3.2.4, is a bound on the size of a d-code in a polar space, which is sharp up to a constant factor in many cases. The proof of the bound relies on Delsarte's linear programming method introduced in Section 2.2. In Chapter 4, we will apply the bound obtained here to show the nonexistence of Steiner systems in polar spaces in most cases.

We will start with the derivation of the bound in Section 3.2. Afterwards, we will construct codes in all polar spaces in Section 3.3, and lastly, we will give a list of open problems related to the topic at hand in Section 3.4.

3.2 Bounds

To derive bounds for d-codes in all polar spaces, we begin with bounds for d-codes in the Hermitian polar space ${}^2A_{2n-1}$ and the bipartite half $\frac{1}{2}D_n$ of the hyperbolic polar space D_n. We proceed in this way because by taking the second Q-polynomial ordering for ${}^2A_{2n-1}$ and studying $\frac{1}{2}D_n$ instead of D_n, we can express the resulting Q-numbers $Q_k(i)$ by q-Hahn polynomials of degree k in b^{-i}, see Section 2.3.4. This allows us to treat ${}^2A_{2n-1}$ and $\frac{1}{2}D_n$ in a unified way. We will then use the bounds in ${}^2A_{2n-1}$ and $\frac{1}{2}D_n$ to establish bounds for codes in the remaining polar spaces.

Recall the definition of b and c from (2.51). We write $(x)_i = (x;b)_i$ in what follows.

It will turn out that the bound for ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$ has a similar form as the *Singleton bound*[18] for the q-Johnson scheme $J_q(n,m)$, which is given by[19]

$$|Y| \le \frac{\begin{bmatrix} m+n \\ n-d+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ n-d+1 \end{bmatrix}_q} = \frac{\begin{bmatrix} m+n \\ n \end{bmatrix}_q (q)_{d-1}}{(qcb^n)_{d-1}} \tag{3.1}$$

for d-codes Y in $J_q(n,m)$, see [WXS03, Theorem 5.2], [ZJX11], and [EV11, Theorem 1]. We will give a different proof of (3.1) in the following theorem—which is the main result of this chapter—where we also derive a bound for ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$.

Theorem 3.2.1. *Let X be the set of n-spaces in $J_q(n,m)$ or generators in ${}^2A_{2n-1}$ or $\frac{1}{2}D_m$, where $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$, and let Y be a d-code in X with $1 \le d \le n$. Then we have*

$$|Y| \le \frac{|X|(q)_{d-1}}{(qcb^n)_{d-1}},$$

where d is required to be odd in the case of ${}^2A_{2n-1}$. For even d in ${}^2A_{2n-1}$, we have

$$|Y| \le \frac{|X|(q)_{d-1}}{(qcb^n)_{d-1}} \, \frac{(b^{n-d+2}-1) + q\frac{b^{n+d-2}-1}{qb^{d-2}-1}(b^{n-d+1}-1)}{(b^{n-d+2}-1) + q\frac{b^{n+d-2}-1}{b^{n+d-1}-1}(b^{n-d+1}-1)}.$$

[18] Also known as the *Wang-Xing-Safavi-Naini*, *anticode bound*, or *packing bound*.

[19] Observe the resemblance between (3.1) and the Singleton bound (2.27) for the Johnson scheme.

Moreover, these bounds also hold for d-codes in association schemes with the same P- and Q-numbers as $J_q(n,m)$, ${}^2A_{2n-1}$, or $\frac{1}{2}D_m$.

Before we can prove Theorem 3.2.1, we need the following identity for the Q-numbers. In the case of $J_q(n,m)$, a different proof for this identity can be found in [Del76a].

Lemma 3.2.2. *Let X be the set of n-spaces in $J_q(n,m)$ or generators in ${}^2A_{2n-1}$ or $\frac{1}{2}D_m$, where we put $n = \lfloor m/2 \rfloor$ in the latter case. Let $Q'_k(i)$ be as in (2.53). Then we have*

$$\sum_{k=0}^{n} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} Q'_k(i) = |X| \begin{bmatrix} n-i \\ j \end{bmatrix}_b \tag{3.2}$$

for all $i, j = 0, 1, \ldots, n$.

The proof of Lemma 3.2.2 requires some identities involving the q-Pochhammer symbol. For a real number a and nonnegative integers n, k, we have

$$(a^2;q^2)_k = (a;q)_k(-a;q)_k \tag{3.3}$$

$$(a;q)_{2k} = (a;q^2)_k(aq;q^2)_k \tag{3.4}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k \tag{3.5}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-a)^{-k}q^{\binom{k}{2}-nk+k} \quad \text{for } a \neq 0. \tag{3.6}$$

These identities can be found in [KLS10, § 1.8], for example. We will moreover frequently use the well-known identity

$$\begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} j \\ i \end{bmatrix}_q = \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k-i \\ j-i \end{bmatrix}_q \tag{3.7}$$

without specific reference in the upcoming proof.

Proof of Lemma 3.2.2. Let $P'_i(k)$ and $Q'_k(i)$ be as in (2.52) and (2.53), respectively, for $J_q(n,m)$, $\frac{1}{2}D_m$ and ${}^2A_{2n-1}$. We will show that[20]

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P'_i(k) = b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_q \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}}. \tag{3.8}$$

By multiplying (3.8) with $Q'_k(\ell)$, taking the sum over k, and using (2.8), we obtain the identity in the lemma. It remains to prove (3.8). First, we rewrite

[20] In the case of $J_q(n,m)$, the identity (3.8) can be written as

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_q P'_i(k) = q^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_q \begin{bmatrix} m+n-j-k \\ n-j \end{bmatrix}_q.$$

Observe that this is the q-analog of (2.26).

the valencies v_i', given in Table 2.3, such that we have a similar form for $P_i'(k)$ in both association schemes. For $J_q(n,m)$, we use (2.30) to obtain

$$v_i = q^{i^2} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ i \end{bmatrix}_q = (-1)^i q^{i^2-\binom{i}{2}+mi} \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(q^{-m};q)_i}{(q;q)_i}.$$

For ${}^2A_{2n-1}$, we use (2.30) and (3.3) to obtain

$$\begin{aligned} v_i' = q^{i^2} \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} &= (-1)^i q^{i^2-2\binom{i}{2}+2ni} \frac{((-q)^{-n};-q)_i(-(-q)^{-n};-q)_i}{(-q;-q)_i(q;-q)_i} \\ &= (-1)^i(-q)^{\binom{i}{2}+i+ni} \begin{bmatrix} n \\ i \end{bmatrix}_{-q} \frac{(-(-q)^{-n};-q)_i}{(q;-q)_i}. \end{aligned}$$

For $\frac{1}{2}D_m$, we use (2.30) and (3.4) to obtain

$$v_i' = q^{\binom{2i}{2}} \begin{bmatrix} m \\ 2i \end{bmatrix}_q = q^{2im} \frac{(q^{-m};q^2)_i(q^{-m+1};q^2)_i}{(q^2;q^2)_i(q;q^2)_i}.$$

For even $m = 2n$, we have

$$v_i' = (-1)^i q^{2in+2\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} \frac{(q^{-2n+1};q^2)_i}{(q;q^2)_i}$$

and for odd $m = 2n+1$, we obtain

$$v_i' = (-1)^i q^{2in+2i+2\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} \frac{(q^{-2n-1};q^2)_i}{(q;q^2)_i}.$$

Hence, in all cases, we can write

$$v_i' = (-q)^i c^i b^{\binom{i}{2}+ni} \begin{bmatrix} n \\ i \end{bmatrix}_b \frac{(c^{-1}b^{-n})_i}{(q)_i}.$$

To simplify notation, we set $a = q^{-1}c^{-1}b^{-2n}$. Now, from the expression (2.52) for $P_i'(k)$, we find

$$\begin{aligned} &\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P_i'(k) \\ &\quad = \sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b (-q)^i c^i b^{\binom{i}{2}+ni} \begin{bmatrix} n \\ i \end{bmatrix}_b \frac{(c^{-1}b^{-n})_i}{(q)_i} \, {}_3\phi_2 \left(\begin{matrix} b^{-i}, b^{-k}, ab^k \\ b^{-n}, c^{-1}b^{-n} \end{matrix} \,\middle|\, b; b \right) \\ &\quad = \begin{bmatrix} n \\ j \end{bmatrix}_b \sum_{i,\ell \geq 0} \begin{bmatrix} n-j \\ i \end{bmatrix}_b (-q)^i c^i b^{\binom{i}{2}+ni+\ell} \frac{(c^{-1}b^{-n})_i (b^{-i})_\ell (b^{-k})_\ell (ab^k)_\ell}{(q)_i (b^{-n})_\ell (c^{-1}b^{-n})_\ell (b)_\ell}. \end{aligned}$$

From (2.30) we have

$$\begin{bmatrix} n-j \\ i \end{bmatrix}_b \frac{(b^{-i})_\ell}{(b)_\ell} = (-1)^\ell b^{\binom{\ell}{2}-i\ell} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_b \begin{bmatrix} n-j-\ell \\ i-\ell \end{bmatrix}_b,$$

and therefore

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P_i'(k) = \begin{bmatrix} n \\ j \end{bmatrix}_b \sum_{\ell \geq 0} (-1)^\ell b^{\binom{\ell}{2}+\ell} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_b \frac{(b^{-k})_\ell (ab^k)_\ell}{(b^{-n})_\ell (c^{-1}b^{-n})_\ell} \, S_\ell, \quad (3.9)$$

where

$$S_\ell = \sum_{i\geq 0}(-q)^i c^i b^{\binom{i}{2}+i(n-\ell)} \begin{bmatrix} n-j-\ell \\ i-\ell \end{bmatrix}_b \frac{(c^{-1}b^{-n})_i}{(q)_i}.$$

By interchanging the order of summation and then applying (3.5), we obtain

$$\begin{aligned} S_\ell &= \sum_{i=0}^{n-\ell}(-q)^{i+\ell} c^{i+\ell} b^{\binom{i+\ell}{2}+(i+\ell)(n-\ell)} \begin{bmatrix} n-j-\ell \\ i \end{bmatrix}_b \frac{(c^{-1}b^{-n})_{i+\ell}}{(q)_{i+\ell}} \\ &= \sum_{i=0}^{n-\ell}(-q)^{i+\ell} c^{i+\ell} b^{\binom{i+\ell}{2}+(i+\ell)(n-\ell)} \begin{bmatrix} n-j-\ell \\ i \end{bmatrix}_b \frac{(c^{-1}b^{-n})_\ell (c^{-1}b^{-n+\ell})_i}{(q)_\ell (qb^\ell)_i}. \end{aligned}$$

By using (2.30), this sum becomes

$$\begin{aligned} S_\ell &= (-q)^\ell c^\ell b^{\binom{\ell}{2}-\ell^2+n\ell} \frac{(c^{-1}b^{-n})_\ell}{(q)_\ell} \sum_{i=0}^{n-\ell} (qcb^{2n-j-\ell})^i \frac{(b^{-(n-j-\ell)})_i (c^{-1}b^{-n+\ell})_i}{(b)_i (qb^\ell)_i} \\ &= (-q)^\ell c^\ell b^{\binom{\ell}{2}-\ell^2+n\ell} \frac{(c^{-1}b^{-n})_\ell}{(q)_\ell} \, {}_2\phi_1 \left(\begin{matrix} b^{-(n-j-\ell)}, c^{-1}b^{-n+\ell} \\ qb^\ell \end{matrix} \middle| \, b; qcb^{2n-j-\ell} \right). \end{aligned}$$

The hypergeometric function ${}_2\phi_1$ can be evaluated by using the q-Chu-Vandermonde identity

$${}_2\phi_1 \left(\begin{matrix} b^{-k}, x \\ y \end{matrix} \middle| \, b; \frac{yb^k}{x} \right) = \frac{(x^{-1}y)_k}{(y)_k}$$

(see [KLS10, (1.11.4)], for example), which implies that

$$S_\ell = (-q)^\ell c^\ell b^{\binom{\ell}{2}-\ell^2+n\ell} \frac{(c^{-1}b^{-n})_\ell (qcb^n)_{n-j-\ell}}{(q)_\ell (qb^\ell)_{n-j-\ell}}.$$

Substitute into (3.9) to obtain

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P_i'(k) = \begin{bmatrix} n \\ j \end{bmatrix}_b \sum_{\ell\geq 0} q^\ell c^\ell b^{n\ell} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_b \frac{(b^{-k})_\ell (ab^k)_\ell (qcb^n)_{n-j-\ell}}{(b^{-n})_\ell (q)_\ell (qb^\ell)_{n-j-\ell}}. \tag{3.10}$$

From (3.5) we have

$$(qb^\ell)_{n-j-\ell} = \frac{(q)_{n-j}}{(q)_\ell} \tag{3.11}$$

and from (3.6) we find that

$$(qcb^n)_{n-j-\ell} = \frac{(qcb^n)_{n-j}}{(q^{-1}c^{-1}b^{-2n+1+j})_\ell} (-qcb^n)^{-\ell} b^{\binom{\ell}{2}-(n-j)\ell+\ell}. \tag{3.12}$$

By substituting (3.11) and (3.12) into (3.10) and using (2.30), we have

$$\begin{aligned} &\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P_i'(k) \\ &\quad = \begin{bmatrix} n \\ j \end{bmatrix}_b \sum_{\ell\geq 0} (-1)^\ell b^{\binom{\ell}{2}-(n-j)\ell+\ell} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_b \frac{(b^{-k})_\ell (ab^k)_\ell (qcb^n)_{n-j}}{(b^{-n})_\ell (q)_{n-j} (q^{-1}c^{-1}b^{-2n+1+j})_\ell} \\ &\quad = \begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(qcb^n)_{n-j}}{(q)_{n-j}} \sum_{\ell\geq 0} b^\ell \frac{(b^{-(n-j)})_\ell (b^{-k})_\ell (ab^k)_\ell}{(b)_\ell (b^{-n})_\ell (q^{-1}c^{-1}b^{-2n+1+j})_\ell} \\ &\quad = \begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(qcb^n)_{n-j}}{(q)_{n-j}} \, {}_3\phi_2 \left(\begin{matrix} b^{-(n-j)}, b^{-k}, ab^k \\ b^{-n}, q^{-1}c^{-1}b^{-2n+1+j} \end{matrix} \middle| \, b; b \right). \end{aligned}$$

The hypergeometric function ${}_3\phi_2$ on the right-hand side can be computed via the q-Pfaff-Saalschütz formula

$$ {}_3\phi_2\left(\begin{matrix} b^{-i}, x, y \\ z, xyz^{-1}b^{1-i}\end{matrix}\;\middle|\; b; b\right) = \frac{(x^{-1}z)_i(y^{-1}z)_i}{(z)_i(x^{-1}y^{-1}z)_i} $$

(see [KLS10, (1.11.9)], for example). Note that $qcb^n = a^{-1}b^{-n}$. Therefore, we obtain

$$ \sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P'_i(k) = \begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(b^{-(n-k)})_{n-j}(qcb^{n-k})_{n-j}}{(q)_{n-j}(b^{-n})_{n-j}}. $$

Applying (2.30) to $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_b = \left[\begin{smallmatrix} n \\ n-j \end{smallmatrix}\right]_b$ and using (2.30) one more time leads to the identity (3.8). □

We can now prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Suppose that Y is a d-code in $J_q(n,m)$, ${}^2A_{2n-1}$, or $\frac{1}{2}D_m$. Let $(A_0, A_1, \dots, A_n)$ and $(A'_0, A'_1, \dots, A'_n)$ be the inner and dual distribution of Y, respectively, in terms of the orderings imposed by the P- and Q-numbers given in (2.52) and (2.53). From (2.17) and Lemma 3.2.2, we obtain for all $j = 0, 1, \dots, n$ that

$$ \begin{aligned} &\sum_{k=0}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} A'_k \\ &\quad = \sum_{i=0}^{n} A_i \sum_{k=0}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} Q'_k(i) \\ &\quad = |X| \sum_{i=0}^{n} A_i \begin{bmatrix} n-i \\ j \end{bmatrix}_b . \end{aligned} \tag{3.13} $$

First, assume that d is odd in the case of ${}^2A_{2n-1}$. Since $A_1 = \cdots = A_{d-1} = 0$ and $\left[\begin{smallmatrix} n-i \\ n-d+1 \end{smallmatrix}\right]_b = 0$ for $i \geq d$, we find from (3.13) with $j = n-d+1$ that

$$ \sum_{k=0}^{n-d+1} b^{k(d-1)} \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \frac{(qcb^{n-k})_{d-1}}{(q)_{d-1}} A'_k = |X| \begin{bmatrix} n \\ d-1 \end{bmatrix}_b A_0. $$

Since $A_0 = 1$ and $A'_0 = |Y|$, we obtain

$$ \sum_{k=1}^{n-d+1} b^{k(d-1)} \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \frac{(qcb^{n-k})_{d-1}}{(q)_{d-1}} A'_k = \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \left(|X| - \frac{(qcb^{n})_{d-1}}{(q)_{d-1}} |Y| \right). \tag{3.14} $$

Recall that $A'_k \geq 0$ for all $k \geq 0$. For ${}^2A_{2n-1}$, the sign of $(qcb^{n-k})_{d-1}/(q)_{d-1}$ is $(-1)^{(d-1)(n-k+1)}$ and the sign of $\left[\begin{smallmatrix} n-k \\ d-1 \end{smallmatrix}\right]_b$ is $(-1)^{(d-1)(n-k-d+1)}$. Since d is odd, both signs are thus positive. Hence, all summands on the left-hand side of (3.14) are nonnegative implying

$$ |Y| \leq \frac{|X|(q)_{d-1}}{(qcb^{n})_{d-1}}, $$

as required.

Now, consider ${}^2A_{2n-1}$ for even d. Put

$$\begin{aligned} x_k &= b^{k(d-1)+d-2}\frac{(b^{n-k+1})_{d-1}(b^n)_{d-2}}{(q)_{d-1}(q)_{d-2}}\begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \\ y_k &= b^{k(d-2)+d-1}\frac{(b^{n-k+1})_{d-2}(b^n)_{d-1}}{(q)_{d-2}(q)_{d-1}}\begin{bmatrix} n-k \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b. \end{aligned} \tag{3.15}$$

Use (3.13) with $j = n-d+1$ and $j = n-d+2$ to obtain

$$\begin{aligned} &\sum_{k=0}^{n-d+2}(x_k - y_k)A'_k \\ &= |X|b^{d-2}\frac{(b^n)_{d-2}}{(q)_{d-2}}\left(\begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n \\ d-1 \end{bmatrix}_b + q\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n \\ d-2 \end{bmatrix}_b \frac{b^{n+d-2}-1}{qb^{d-2}-1}\right). \end{aligned} \tag{3.16}$$

Next, we show that the summands on the left-hand side are nonnegative. The sign of $\left[\begin{smallmatrix} m \\ \ell \end{smallmatrix}\right]_b$ is $(-1)^{\ell(m-\ell)}$ and the sign of $(b^m)_\ell/(q)_\ell$ is $(-1)^{m\ell}$. Hence, we have $\operatorname{sign}(x_k) = (-1)^k$ and $\operatorname{sign}(y_k) = -1$, which implies that the left-hand side of (3.16) equals

$$\sum_{k=0}^{n-d+2}((-1)^k|x_k| + |y_k|)A'_k.$$

From

$$\frac{x_k}{y_k} = b^{k-1}\frac{(b^{n-k-d+2}-1)(b^{n-k+d-1}-1)}{(b^{n+d-2}-1)(b^{n-d+1}-1)},$$

we see that $|x_k| \le |y_k|$ for all $k \ge 1$. Therefore, the left-hand side of (3.16) can be bounded from below by $(x_0 - y_0)A'_0$, which is also positive. Since $A'_0 = |Y|$, we thus find from (3.16) that

$$|Y| \le \frac{|X|b^{d-2}\frac{(b^n)_{d-2}}{(q)_{d-2}}\left(\left[\begin{smallmatrix} n-1 \\ d-2 \end{smallmatrix}\right]_b \left[\begin{smallmatrix} n \\ d-1 \end{smallmatrix}\right]_b + q\left[\begin{smallmatrix} n-1 \\ d-1 \end{smallmatrix}\right]_b \left[\begin{smallmatrix} n \\ d-2 \end{smallmatrix}\right]_b \frac{b^{n+d-2}-1}{qb^{d-2}-1}\right)}{\left(b^{d-2}\frac{(b^{n+1})_{d-1}(b^n)_{d-2}}{(q)_{d-1}(q)_{d-2}}\left[\begin{smallmatrix} n \\ d-1 \end{smallmatrix}\right]_b \left[\begin{smallmatrix} n-1 \\ d-2 \end{smallmatrix}\right]_b - b^{d-1}\frac{(b^{n+1})_{d-2}(b^n)_{d-1}}{(q)_{d-2}(q)_{d-1}}\left[\begin{smallmatrix} n \\ d-2 \end{smallmatrix}\right]_b \left[\begin{smallmatrix} n-1 \\ d-1 \end{smallmatrix}\right]_b\right)}.$$

We can now deduce the second inequality of the theorem after elementary manipulations. This completes the proof. □

In what follows, we use Theorem 3.2.1 to obtain bounds for d-codes in the remaining polar spaces ${}^2A_{2n}$, B_n, C_n, D_n, and ${}^2D_{n+1}$. To do so, we write

$$\alpha(n,d) = \left(\prod_{i=1}^{n}(1+q^{2i-1})\right)\left(\prod_{i=1}^{d-1}\frac{1+(-q)^i}{1-(-q)^{n+i}}\right)\varepsilon(n,d),$$

where $\varepsilon(n,d) = 1$ for odd d and

$$\varepsilon(n,d) = \frac{((-q)^{n-d+2}-1) + q\frac{(-q)^{n+d-2}-1}{q(-q)^{d-2}-1}((-q)^{n-d+1}-1)}{((-q)^{n-d+2}-1) + q\frac{(-q)^{n+d-2}-1}{(-q)^{n+d-1}-1}((-q)^{n-d+1}-1)}$$

for even d. Moreover, we write

$$\beta(m,d)=\begin{cases}\left(\prod_{i=1}^{m-1}(1+q^i)\right)\left(\prod_{i=1}^{d-1}\frac{1-q^{2i-1}}{1-q^{m+2i-2}}\right) & \text{for even } m\\ \left(\prod_{i=1}^{m-1}(1+q^i)\right)\left(\prod_{i=1}^{d-1}\frac{1-q^{2i-1}}{1-q^{m+2i-1}}\right) & \text{for odd } m.\end{cases}$$

Observe that, using (2.34), the bounds in Theorem 3.2.1 for ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$ equal $\alpha(n,d)$ and $\beta(m,d)$, respectively.

First, we make the following observation about d-codes in D_n if d is even.

Proposition 3.2.3. *Every d-code in D_n with even d and $2 \le d \le n$ induces a $\frac{d}{2}$-code in $\frac{1}{2}D_n$ of the same size.*

Proof. The strategy of this proof is depicted in Figure 3.1.

Let Y be a d-code in D_n with even d and $2 \le d \le n$. Recall that the set of generators is partitioned into two equivalence classes X_1 and X_2, where two generators lie in the same class if and only if the dimension of their intersection has the same parity as n. For each $y \in Y$, choose an $(n-1)$-space contained in y. Because Y is a d-code with $d > 1$, all these chosen $(n-1)$-spaces are distinct and moreover, the dimension of their intersection is at most $n-d$. Since $e = -1$, each of these $(n-1)$-spaces lies in exactly two generators—one from X_1 and one from X_2. Let $\widehat{Y}$ be the set of all generators in X_1 corresponding to the chosen $(n-1)$-spaces. Then it holds

$$\dim(\widehat{x} \cap \widehat{y}) \le n-d+1$$

for all $\widehat{x}, \widehat{y} \in \widehat{Y}$. However, the case $\dim(\widehat{x} \cap \widehat{y}) = n-d+1$ cannot occur since d is even and $\dim(\widehat{x} \cap \widehat{y})$ must have the same parity as n. Hence, $\widehat{Y} \subseteq X_1$ is a $\frac{d}{2}$-code in $\frac{1}{2}D_n$ with $|Y| = |\widehat{Y}|$, as required. □

We now derive bounds for codes in all polar spaces.

Corollary 3.2.4. *Let $\mathcal{P}$ be a polar space of rank n and let Y be a d-code in $\mathcal{P}$ with $1 \le d \le n$. Put $\delta = \lceil d/2 \rceil$.*

(a) *If $\mathcal{P} = {}^2A_{2n-1}$, then $|Y| \le \alpha(n,d)$.*

(b) *If $\mathcal{P} = {}^2A_{2n}$, then $|Y| \le \alpha(n+1,d)$.*

(c) *If $\mathcal{P} = B_n$ or C_n, then $|Y| \le \beta(n+1,\delta)$.*

(d) *If $\mathcal{P} = D_n$ and d is odd, then $|Y| \le 2\beta(n,\delta)$.*

(e) *If $\mathcal{P} = D_n$ and d is even, then $|Y| \le \beta(n,\delta)$.*

(f) *If $\mathcal{P} = {}^2D_{n+1}$, then $|Y| \le \beta(n+2,\delta)$.*

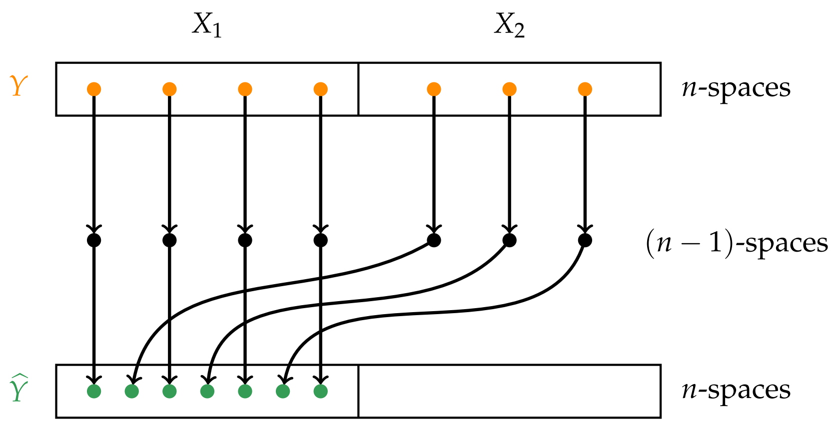


FIGURE 3.1. Idea for the proof of Proposition 3.2.3—a d-code Y in X induces a d-code $\widehat{Y}$ in the equivalence class X_1 of X.

Proof. The bound in (a) follows directly from Theorem 3.2.1 by using (2.34).

A d-code in D_n induces δ-codes in each of the two bipartite halves of D_n, so it is at most twice as large as a δ-code in $\frac{1}{2}D_n$. Theorem 3.2.1 then gives (d). Proposition 3.2.3 immediately implies (e).

Recall from Section 2.3.2 (see p. 46) that in the cases of B_n and C_n, one obtains a new association scheme with the classes

$$R_0, R_1 \cup R_2, R_3 \cup R_4, \dots$$

and this new association scheme has the same P- and Q-numbers as $\frac{1}{2}D_{n+1}$ [IMU89]. Therefore, the size of a d-code in B_n or C_n is at most the upper bound for a δ-code in $\frac{1}{2}D_{n+1}$ given in Theorem 3.2.1, which proves (c).

To establish the remaining cases (b) and (f), note that ${}^2D_{n+1}$ and ${}^2A_{2n}$ arise by intersecting B_{n+1} and ${}^2A_{2n+1}$, respectively, with a hyperplane (see Remark 2.3.3). Hence, ${}^2D_{n+1}$ can be embedded into B_{n+1} and ${}^2A_{2n}$ can be embedded into ${}^2A_{2n+1}$. Note that B_{n+1} and ${}^2A_{2n+1}$ are of rank $n+1$ and each generator in ${}^2D_{n+1}$ or ${}^2A_{2n}$ becomes an n-space in B_{n+1} or ${}^2A_{2n+1}$ under these embeddings. In B_{n+1} and ${}^2A_{2n+1}$, every n-space is contained in exactly $p^{e+1}+1 = q+1$ generators. For each embedded element of Y, we choose one of these $q+1$ generators giving a subset $\widetilde{Y}$ of B_{n+1} or ${}^2A_{2n+1}$. Then $\widetilde{Y}$ is also a d-code and (c) implies (f) and (a) implies (b). $\square$

We close this section by giving the following more useful bounds on $\alpha(n,d)$ and $\beta(n,d)$.

Lemma 3.2.5. *For $1 \le d \le n$, we have*

$$\alpha(n,d) < \begin{cases} \frac{14}{5} q^{n(n-d+1)} & \text{for odd } d \\ \frac{14}{5} q^{n(n-d+2)} & \text{for even } d, \end{cases} \tag{3.17}$$

and

$$\beta(n,d) < \begin{cases} \frac{5}{2}q^{(n-1)(n-2d+2)/2} & \text{for even } n \\ \frac{5}{2}q^{n(n-2d+1)/2} & \text{for odd } n. \end{cases} \tag{3.18}$$

To prove Lemma 3.2.5, we use the identity

$$\frac{x-1}{y-1} \leq \frac{x}{y} \quad \text{for } y \geq x \geq 1 \text{ with } y \neq 1 \tag{3.19}$$

and the following lemma.

Lemma 3.2.6. *Let $n \geq 1$ and $q \geq 2$ be integers. Then we have*

$$\prod_{i=1}^{n}\left(1+\frac{1}{q^i}\right) < \frac{5}{2}, \quad \prod_{i=1}^{n}\left(1+\frac{1}{q^{2i}}\right) < \frac{7}{5} \quad \text{and} \quad \prod_{i=1}^{n}\left(1+\frac{1}{q^{2i-1}}\right) < 2. \tag{3.20}$$

Proof. We use $1+x < \exp(x)$ to obtain

$$\prod_{i=1}^{n}\left(1+\frac{1}{q^i}\right) < \left(1+\frac{1}{q}\right)\exp\left(\frac{1}{q(q-1)}\right) \leq \left(1+\frac{1}{q}\right)\exp\left(\frac{1}{q}\right).$$

Applying $(1+x)\exp(x) < \frac{5}{2}$ for all $x \in [0, \frac{1}{2}]$ leads to the first inequality. Using a similar approach gives us

$$\prod_{i=1}^{n}\left(1+\frac{1}{q^{2i}}\right) < \exp\left(\frac{1}{q^2-1}\right) \leq \exp\left(\frac{1}{3}\right) < \frac{7}{5},$$

and

$$\prod_{i=1}^{n}\left(1+\frac{1}{q^{2i-1}}\right) < \exp\left(\frac{q}{q^2-1}\right) \leq \exp\left(\frac{2}{3}\right) < 2,$$

as required. □

We can now prove Lemma 3.2.5.

Proof of Lemma 3.2.5. For $\beta(n,d)$ and even n, use (3.19) and (3.20) to obtain

$$\begin{aligned}\beta(n,d) &< \left(\prod_{i=1}^{n-1} q^i\left(1+\frac{1}{q^i}\right)\right) q^{(-n+1)(d-1)} \\ &\leq \frac{5}{2}q^{(n-1)(n-2d+2)/2}.\end{aligned}$$

The bound for $\beta(n,d)$ and odd n can be obtained similarly. For $\alpha(n,d)$, we write

$$\alpha(n,d) = \left(\prod_{i=1}^{n}(1+q^{2i-1})\right)\left(\prod_{i=1}^{d-1}\frac{q^i+(-1)^i}{q^{n+i}-(-1)^{n+i}}\right)(-1)^{(n+1)(d-1)}\varepsilon(n,d). \tag{3.21}$$

We have

$$\prod_{i=1}^{d-1} \frac{q^i + (-1)^i}{q^{n+i} - (-1)^{n+i}} = \begin{cases} \prod_{i=1}^{\frac{d-1}{2}} \frac{(q^{2i}+1)(q^{2i-1}-1)}{(q^{n+2i}-(-1)^n)(q^{n+2i-1}+(-1)^n)} & \text{for odd } d \\ \frac{q^{d-1}-1}{q^{n+d-1}+(-1)^n} \prod_{i=1}^{\frac{d-2}{2}} \frac{(q^{2i}+1)(q^{2i-1}-1)}{(q^{n+2i}-(-1)^n)(q^{n+2i-1}+(-1)^n)} & \text{for even } d. \end{cases} \tag{3.22}$$

Using (3.19) and (3.20), we obtain for each $r \geq 1$,

$$\begin{aligned} \prod_{i=1}^{r} \frac{(q^{2i}+1)(q^{2i-1}-1)}{(q^{n+2i}-(-1)^n)(q^{n+2i-1}+(-1)^n)} &\leq \prod_{i=1}^{r} \frac{(q^{2i}+1)(q^{2i-1}-1)}{(q^{n+2i}+1)(q^{n+2i-1}-1)} \\ &\leq \prod_{i=1}^{r} q^{-2n}\left(1+\frac{1}{q^{2i}}\right) \\ &< \frac{7}{5} q^{-2nr}. \end{aligned}$$

Substitute into (3.22) to give

$$\prod_{i=1}^{d-1} \frac{q^i + (-1)^i}{q^{n+i} - (-1)^{n+i}} < \begin{cases} \frac{7}{5} q^{-n(d-1)} & \text{for odd } d \\ \frac{7}{5} q^{-n(d-2)} \frac{q^{d-1}-1}{q^{n+d-1}+(-1)^n} & \text{for even } d. \end{cases} \tag{3.23}$$

From (3.20) we have

$$\prod_{i=1}^{n} (1+q^{2i-1}) = \prod_{i=1}^{n} q^{2i-1}\left(1+\frac{1}{q^{2i-1}}\right) < 2q^{n^2}. \tag{3.24}$$

Substitute (3.23) and (3.24) into (3.21) to obtain

$$\alpha(n,d) < \begin{cases} \frac{14}{5} q^{n(n-d+1)} & \text{for odd } d \\ \frac{14}{5} q^{n(n-d+2)} \frac{q^{d-1}-1}{q^{n+d-1}+(-1)^n} (-1)^{(n+1)(d-1)} \varepsilon(n,d) & \text{for even } d. \end{cases} \tag{3.25}$$

For even d, we have

$$\begin{aligned} &(-1)^{(n+1)(d-1)} \varepsilon(n,d) \\ &\quad = \frac{q^{\frac{q^{n+d-2}-(-1)^n}{q^{d-1}-1}} (q^{n-d+1}+(-1)^n) - (-1)^n (q^{n-d+2}-(-1)^n)}{(q^{n-d+2}-(-1)^n) + q^{\frac{q^{n+d-2}-(-1)^n}{q^{n+d-1}+(-1)^n}} (q^{n-d+1}+(-1)^n)} \\ &\quad = \frac{q^{\frac{(q^{n+d-2}-(-1)^n)(q^{n-d+1}+(-1)^n)}{(q^{d-1}-1)(q^{n-d+2}-(-1)^n)}} - (-1)^n}{q^{\frac{(q^{n+d-2}-(-1)^n)(q^{n-d+1}+(-1)^n)}{(q^{n+d-1}+(-1)^n)(q^{n-d+2}-(-1)^n)}} + 1} \\ &\quad < \frac{q^{n+d-1}+(-1)^n}{q^{d-1}-1}, \end{aligned} \tag{3.26}$$

by using (3.19), so that (3.25) gives the required bound for $\alpha(n,d)$. □

3.3 Constructions

We now discuss the sharpness of the bounds in Corollary 3.2.4 by giving constructions of codes in polar spaces. For a vector space V, let $P_n(V)$ be the set of n-spaces of V. Recall the definition of the injective mapping $v\colon \mathbb{F}_p^{n\times n} \to P_n(\mathbb{F}_p^{2n})$ from (2.49) in Section 2.3.4. We already stated that, after an appropriate choice of the form, $v(A)$ is in ${}^2A_{2n-1}$ if and only if A is Hermitian, $v(A)$ is in D_n if and only if A is alternating, and $v(A)$ is in C_n if and only if A is symmetric (as before, $p = q^2$ for ${}^2A_{2n-1}$ and $p = q$ otherwise). Also recall that the mapping v satisfies

$$n - \dim(v(A) \cap v(B)) = \operatorname{rank}(A - B)$$

for all $A, B \in \mathbb{F}_q^{n\times n}$. Accordingly, to construct codes in polar spaces, we can use different types of codes in $\mathbb{F}_q^{n\times n}$. Such objects were studied in [Sch18], [Sch10], [Sch15], and [DG75] for Hermitian, symmetric, and alternating matrices, and are precisely the codes in affine schemes $\mathrm{Her}_q(n)$, $\mathrm{Sym}_q(n)$, and $\mathrm{Alt}_q(m)$. This implies the following corollary.

Corollary 3.3.1. *The bound in Corollary 3.2.4 for a d-code in a polar space $\mathcal{P}$ of rank n with $1 \le d \le n$ is sharp up to a constant factor if*

(a) $\mathcal{P} = {}^2A_{2n-1}$ *and d is odd;*

(b) $\mathcal{P} = C_n$ *and d is odd;*

(c) $\mathcal{P} = B_n$*, d is odd, and q is even;*

(d) $\mathcal{P} = D_n$ *except possibly when n is even and q is odd.*

Observe that in the case of D_n, the constructed code actually lies in the bipartite half $\frac{1}{2}D_n$ since alternating matrices always have even rank.

Proof of Corollary 3.3.1. From [Sch18] and the injection v, we find that for odd d, there exists a d-code Y in ${}^2A_{2n-1}$ satisfying $|Y| = q^{n(n-d+1)}$. In view of Lemma 3.2.5, this shows that the bound in Corollary 3.2.4 (a) for odd d is sharp up to a constant factor. Likewise from [Sch10] and [Sch15] we find that, for odd d, there exists a d-code Y in C_n satisfying

$$|Y| = \begin{cases} q^{(n+1)(n-d+1)/2} & \text{for even } n \\ q^{n(n-d+2)/2} & \text{for odd } n, \end{cases}$$

showing that the bound in Corollary 3.2.4 (c) for $\mathcal{P} = C_n$ and odd d is sharp up to a constant factor. Since B_n and C_n are isomorphic for even q, the same is true when $\mathcal{P} = B_n$ and q is even. From [DG75] we find that, for even d, there exists a d-code Y in D_n satisfying

$$|Y| = \begin{cases} q^{(n-1)(n-d+2)/2} & \text{for even } n \text{ and even } q \\ q^{n(n-d+1)/2} & \text{for odd } n. \end{cases}$$

Since a d-code is trivially also a $(d-1)$-code, this shows that the bound in Corollary 3.2.4 (d) and (e) is sharp up to a constant factor except possibly when n is even and q is odd. $\square$

We note that in all other cases, one can obtain constructions of d-codes in a similar fashion, showing that the remaining bounds in Corollary 3.2.4 are met up to a small power of q^n.

3.4 Open problems

We close this chapter by looking at some open problems that directly arise from our results or are related to codes in polar spaces.

Problem 3.4.1. *Is there a combinatorial proof for the bounds from Corollary 3.2.4?*

In the case of the q-Johnson scheme, we will see in Chapter 5 that Theorem 2.2.6 with the Singleton polynomial gives the *Singleton bound* (also known as *Wang-Xing-Safavi-Naini bound*, *anticode bound*, or *packing bound*, see [WXS03, Theorem 5.2] and [EV11, Theorem 1 and 2]), which can also be proved combinatorially. Applying the same combinatorial proof for the polar spaces gives the size of an $(n-d+1)$-Steiner system in the polar space as an upper bound for d-codes, see Proposition 5.1.2. However, we will see in Chapter 4 that the bounds from Corollary 3.2.4 are smaller than the size of Steiner systems in most cases. Since we gave only an algebraic proof for the bounds in Corollary 3.2.4, it would be interesting to see whether they can still be proved purely combinatorially. This problem is similar to Problem 5.4.5.

Problem 3.4.2. *Improve the bounds in Corollary 3.2.4 that are not asymptotically optimal, or construct new codes whose sizes reach these bounds up to constant factor.*

Related to Problem 3.4.2 is the following.

Problem 3.4.3. *Can some of the bounds in Corollary 3.2.4 be improved by using semidefinite programming?*

In 2005, Schrijver [Sch05] improved some of Delsarte's linear programming bounds for codes in the binary Hamming scheme $H(n, 2)$ by using semidefinite programming (SDP). Schrijver's method gave new bounds for different types of codes as well (see [Val21] for a survey of this method). In the linear programming (LP), one looks at constraints for pairs of codewords, whereas in the SDP the constraints are for triples, quadruples, etc. of codewords. In the case of the q-Johnson scheme, Schrijver's SDP method did not give new upper bounds for the size of codes so far, which was checked by Ihringer, see his blog post [Ihr18] on this. It would be interesting to see whether SDP yields better bounds than LP for codes in polar spaces.

Problem 3.4.4. *Let $\mathcal{P}$ be a polar space of rank n. What is the maximum number of elements in a subset Y of k-spaces in $\mathcal{P}$ with $1 \leq k < n$ such that $k - \dim(x \cap y) \geq d$ for all distinct $x, y \in Y$?*

Such subsets Y can be seen as a generalization of our definition of a code in a polar space. One possibility to tackle this problem could be to study the nonclassical association scheme arising from the action of the corresponding group from Table 2.2 on the k-spaces (with $k < n$) instead of the generators. This was shown by Stanton [Sta80], who proved that the orbits under this action are indexed by two parameters and who also computed the eigenvalues of this scheme, as did Eisfeld [Eis99].

Problem 3.4.5. *Do 1-perfect codes in B_n and C_n exist if $n = 2^m - 1$ for some integer $m \geq 3$?*

An *e-perfect code* in a polar space $\mathcal{P}$ is a subset of generators such that for every generator x in $\mathcal{P}$, there is a unique generator from Y, whose intersection with x has dimension at least $n - e$. The existence of e-perfect codes was solved in most cases by using Lloyd's theorem [Del73, Theorem 5.7]. More precisely, it was proved by Chihara [Chi87] that nontrivial e-perfect codes do not exist except possibly in B_n or C_n with $e = 1$ and $n = 2^m - 1$ for some integer m (where nontrivial means that the code is neither a singleton nor the full set of generators). A 1-perfect code Y in C_n and B_n is a 3-code with

$$|Y| = \frac{|X|}{v_0 + v_1} = \left(\prod_{i=1}^{n}(1+q^i)\right)\left(\frac{q-1}{q^{n+1}-1}\right),$$

where v_i denotes the valency (2.37). The size of Y is precisely the value of the respective bound in Corollary 3.2.4(c). For $n = 3$, a 1-perfect code in C_3 or B_3 is a so-called *spread*, whose existence question is affirmatively solved in C_3 and still not completely settled in B_3, for example. (See [HT16, § 7.4 and 7.5] for an overview on the existence of spreads.) So far, nothing seems to be known in the case of $n = 2^m - 1$ with $m \geq 3$.

Chapter 4

Steiner systems in polar spaces

> Some things will drop out of the public eye and will go away, but there will always be science, engineering and technology. And there will always, always be mathematics.
>
> —Katherine Johnson

In this chapter, we will give an almost complete classification of Steiner systems in polar spaces by showing that such objects can only exist in some corner cases. This classification result will be proved by using the bounds for codes in polar spaces that were obtained in Chapter 3.

The results of this chapter can also be found in [SW22].

4.1 Introduction

A *t-Steiner system* is a collection Y of n-subsets of a v-set V such that each t-subset of V is contained in exactly one member of Y. The long-standing existence question for t-Steiner systems has been settled recently: it was shown in [Kee14] and [Glo+16] that, for all $t \leq n$ and all sufficiently large v, a t-Steiner system exists, provided that some natural divisibility conditions are satisfied. Observe that a t-Steiner system is a t-$(v, n, 1)$ combinatorial design and by Example 2.1.28 and 2.2.9, a t-design and an optimal $(n - t + 1)$-code in the Johnson scheme $J(n, v - n)$.

It is well known that combinatorics of sets can be regarded as the limiting case $q \to 1$ of combinatorics of vector spaces over $\mathbb{F}_q$. Indeed, following [Cam74] and [Del78b], a *t-Steiner system over* $\mathbb{F}_q$ is a collection Y of n-dimensional subspaces (n-spaces for short) of a v-space V over $\mathbb{F}_q$ such that each t-space of V is contained in exactly one member of Y. It is remarkable that, in the nontrivial case $1 < t < n < v$, Steiner systems over $\mathbb{F}_q$ are only known for a single set of parameters [Bra+16], namely for $(t, n, v) = (2, 3, 13)$ and $q = 2$. Similar to Steiner systems of sets, a t-Steiner system over $\mathbb{F}_q$ is a t-design over $\mathbb{F}_q$ and an optimal $(n-t+1)$-code in the q-Johnson scheme $J_q(n, v-n)$, as we will see in Chapter 5.

We may consider these objects as q-analogs of Steiner systems of type A_{n-1}, as V together with the action of $\mathrm{GL}_n(q)$ is of this type. We study q-analogs of Steiner systems in finite vector spaces of type ${}^2A_{2n-1}$, ${}^2A_{2n}$, B_n, C_n, D_n, and ${}^2D_{n+1}$ (using the notation of [Car89]), which are the polar spaces introduced in Section 2.3. Namely, a *t-Steiner system in a polar space $\mathcal{P}$* is a collection Y of generators in $\mathcal{P}$ such that each totally isotropic t-space of V is contained in exactly one member of Y. These objects are sometimes called *regular systems* or 1-*regular systems* in the literature. Examples of Steiner systems in polar spaces are given in Figure 4.1 and 4.2.

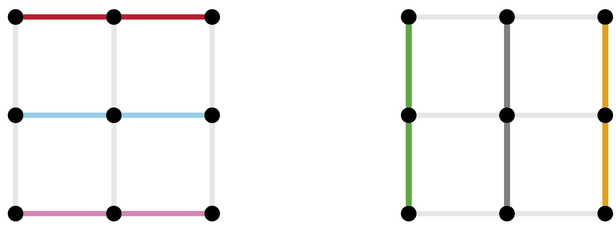


FIGURE 4.1. Two 1-Steiner systems (spreads) in the hyperbolic polar space D_2 from Figure 2.9.

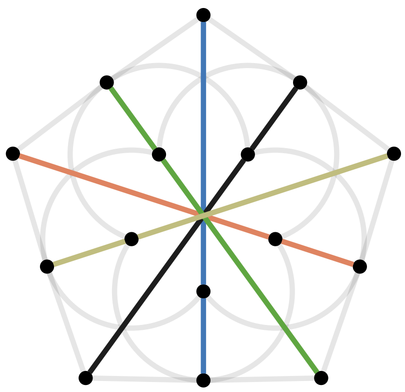


FIGURE 4.2. A 1-Steiner system (spread) in the symplectic polar space C_2 from Figure 2.8.

A 1-Steiner system in a polar space is known as a *spread*, whose existence question has been studied for decades (see [Seg65], [Dye77], [Tha81], [Kan82b], [Kan82a], [Cal+97], for example), but is still not fully resolved (see [HT16, § 7.4] for the current status). The only other known nontrivial t-Steiner systems in polar spaces occur for $t = n-1$ in the hyperbolic polar space D_n and equal one of the two bipartite halves $\frac{1}{2}D_n$ of D_n, see p. 45.

We prove the following classification result.

Theorem 4.1.1. *Suppose that a polar space $\mathcal{P}$ of rank n contains a t-Steiner system with $1 < t < n$. Then one of the following holds*

(1) $t = n - 1$ *and* $\mathcal{P} = D_n$*;*

(2) $t = n - 1$ *and* $\mathcal{P} = {}^2A_{2n}$ *or* ${}^2D_{n+1}$ *for* $q \geq 3$*;*

(3) $t = 2$ *and* $\mathcal{P} = {}^2A_{2n}$ *or* ${}^2D_{n+1}$ *for odd* n*.*

It is unknown whether t-Steiner systems exist in the remaining possibilities and we conjecture the following.

Conjecture 4.1.2. *If a polar space $\mathcal{P}$ of rank n contains a t-Steiner system Y with $1 < t < n$, then $\mathcal{P} = D_n$ and Y is a bipartite half of D_n.*

The special cases $(n, t) = (4, 2)$ and $(n, t) = (5, 3)$ in Theorem 4.1.1 were recently obtained in [Cos+22] and the results in the cases $t = n - 1$ are essentially known (see Case (C1) in Section 4.2). All other cases appear to be new.

An elementary counting argument shows that the size of a t-Steiner system in a polar space necessarily equals the total number of totally isotropic t-spaces divided by the number of t-spaces contained in a generator. Our proof of Theorem 4.1.1 is based on the fact that a set Y of generators in a polar space is a t-Steiner system if and only if Y has the correct size and $\dim U \cap W < t$ for all distinct $U, W \in Y$. Therefore, the intersection of two distinct members of a t-Steiner system can have dimension at most $t - 1$ and so a t-Steiner system is an $(n - t + 1)$-code and its size must satisfy the bounds derived in Chapter 3. However, we will show that in most cases the bound is too small for a t-Steiner system to exist, eventually leading to Theorem 4.1.1. Numerical evidence suggests that in all cases remaining in Theorem 4.1.1, the LP optimum in the corresponding association scheme equals the size of the putative Steiner system. Hence, it seems that entirely new techniques are required to deal with the remaining cases.

After giving a proof of Theorem 4.1.1 in Section 4.2, we will discuss some open problems related to the topic of this chapter in Section 4.3.

4.2 Proof of the classification results

Here, we prove Theorem 4.1.1. The proof is split into the following cases:

(C1) $t = n - 1$ and $\mathcal{P} = {}^2A_{2n}, {}^2D_{n+1}$ for $q = 2$ or $\mathcal{P} = {}^2A_{2n-1}, B_n, C_n$;

(C2) $\mathcal{P} = D_n$ with $1 < t < n - 1$;

(C3) $\mathcal{P} = B_n$ or C_n with $t = 2$ and even n or $2 < t < n - 1$;

(C4) $\mathcal{P} = {}^2D_{n+1}$ with $t \in \{2,3\}$ and odd n or $3 < t < n-1$, but $(n,t) \notin \{(7,4),(8,5)\}$;

(C5) $\mathcal{P} = {}^2A_{2n-1}$ with $1 < t < n-1$;

(C6) $\mathcal{P} = {}^2A_{2n}$ with $t = 2$ and even n, or $2 < t < n-1$ except for $(n,t) = (6,3)$;

(C7) $t = 2$ and $\mathcal{P} = B_n$ or C_n for odd $n > 3$ or $\mathcal{P} = {}^2D_{n+1}$ for even $n > 3$;

(C8) $\mathcal{P} = {}^2D_{n+1}$ with $t = 3$ and even $n > 4$;

(C9) $\mathcal{P} = {}^2D_{n+1}$ with $(n,t) = (7,4)$ or $(8,5)$, or $\mathcal{P} = {}^2A_{2n}$ with $(n,t) = (6,3)$.

The case (C1) is essentially known [Van11, p. 160] and a proof is sketched below for completeness. The cases (C2)–(C6) will follow from Theorem 3.2.1 and Corollary 3.2.4. The cases (C7)–(C9) are some corner cases, which need special treatment.

We begin with a sketch for a proof of (C1).

Proof of (C1). By taking the elements of an $(n-1)$-Steiner system in a polar space of rank n that contain a fixed isotropic 1-space v and taking the quotient by v, one obtains an $(n-2)$-Steiner system in a polar space of the same type but rank $n-1$. This reduces the existence question to 2-Steiner systems in rank 3 or 1-Steiner systems, namely spreads, in rank 2. There are no spreads in B_2 for odd q, ${}^2A_4$ for $q = 2$, and ${}^2A_5$ for all q [HT16, § 7.4] and there are no 2-Steiner systems in ${}^2D_4$ for $q = 2$ [Pan98] and C_3 for all q [Tho96], [CP03]. Since B_n and C_n are isomorphic if q is even, there are also no 2-Steiner systems in B_3 for even q. □

To prove (C2)–(C6), we note that the number of totally isotropic t-spaces in a polar space of rank n is

$$\begin{bmatrix} n \\ t \end{bmatrix}_p \prod_{i=0}^{t-1} (1 + p^{n-i+e}) \tag{4.1}$$

(see [BCN89, Lemma 9.4.1], for example). Since every generator contains exactly $\left[\begin{smallmatrix} n \\ t \end{smallmatrix}\right]_p$ subspaces of dimension t, the size of a t-Steiner system is thus given by

$$\prod_{i=0}^{t-1} (1 + p^{n-i+e}). \tag{4.2}$$

Recall that a t-Steiner system is an $(n-t+1)$-code. Henceforth, we thus write $d = n - t + 1$. Let B denote the corresponding bound of a d-code in Corollary 3.2.4. We denote the size of an $(n-d+1)$-Steiner system by S, hence

$$S = \prod_{i=0}^{n-d} (1 + p^{n-i+e}), \tag{4.3}$$

and in particular

$$S \geq p^{\frac{1}{2}(n-d+1)(n+d+2e)}. \tag{4.4}$$

We set $R = B/S$ and show that $R < 1$.

Proof of $(C2)$. In this case, we assume that $\mathcal{P} = D_n$ and $2 < d < n$. Use Corollary 3.2.4 (d) and (e), (4.4), and (3.18) to obtain

$$R < \begin{cases} \frac{5}{2}q^{\frac{1}{2}(d-2)(d-n)} & \text{for even } n \text{ and even } d \\ 5q^{\frac{1}{2}(d-1)(d-n-1)} & \text{for even } n \text{ and odd } d \\ \frac{5}{2}q^{\frac{1}{2}(d-2)(d-n-1)} & \text{for odd } n \text{ and even } d \\ 5q^{\frac{1}{2}(d-1)(d-n-2)} & \text{for odd } n \text{ and odd } d. \end{cases} \tag{4.5}$$

If n and d have the same parity, then (4.5) implies $R < 1$. If n and d have a different parity, then (4.5) implies $R < 1$, except when $(n, d) = (4, 3)$. In the latter case, Corollary 3.2.4 (d) and (4.3) give

$$R = \frac{2}{1+q^2} < 1.$$

This completes the proof. □

Proof of $(C3)$. In this case, we assume that $\mathcal{P} = B_n$ or C_n and $2 < d < n-1$ or $d = n-1$ is odd. Use Corollary 3.2.4 (c), (4.4), and (3.18) to obtain

$$R < \begin{cases} \frac{5}{2}q^{\frac{1}{2}(d(d-1)-(n+1)(d-2))} & \text{for even } n \text{ and even } d \\ \frac{5}{2}q^{\frac{1}{2}(d(d+1)-(n+1)(d-1))} & \text{for even } n \text{ and odd } d \\ \frac{5}{2}q^{\frac{1}{2}(d(d-1)-n(d-2))} & \text{for odd } n \text{ and even } d \\ \frac{5}{2}q^{\frac{1}{2}(d(d+1)-n(d-1))} & \text{for odd } n \text{ and odd } d. \end{cases}$$

It is the readily verified that $R < 1$, except if (i) $d = 4$ and $n = 6, 7$, or (ii) $d = 3$ and $n = 6, 7$, or (iii) $d = n-2$ is odd, or (iv) $d = n-1$ is odd. For (i) and (ii), Corollary 3.2.4 (c) and (4.3) imply that R equals $(1+q^3)/(1+q^4)$ and $1/(1+q^4)$, respectively, giving $R < 1$ in both cases. For (iii), Corollary 3.2.4 (c) and (4.3) imply that

$$\begin{aligned} R &= \left(\prod_{i=1}^{n-2}(1+q^i)\right)\left(\prod_{i=1}^{\frac{n}{2}-1}\frac{1-q^{2i-1}}{(1-q^{\frac{n}{2}+i})(1+q^{\frac{n}{2}+i})}\right) \\ &= \frac{1}{1+q^{n-1}}\left(\prod_{i=1}^{\frac{n}{2}}(1+q^i)\right)\left(\prod_{i=1}^{\frac{n}{2}-1}\frac{1-q^{2i-1}}{1-q^{\frac{n}{2}+i}}\right) \\ &< \frac{5}{2}\,\frac{q}{1+q^{n-1}} < 1, \end{aligned}$$

by using (3.19), (3.20), and $n \geq 4$. Similarly, for (iv), we deduce

$$R < \frac{5}{2}\,\frac{q}{1+q^{n-2}} < 1,$$

which completes the proof. □

Proof of (C4). In this case, we assume that $\mathcal{P} = {}^2D_{n+1}$ and $2 < d < n-2$ or $d = n-2$ is odd or $d = n-1$ is even, but $(n,d) \notin \{(7,4),(8,4)\}$. Use Corollary 3.2.4 (f), (4.4), and (3.18) to obtain

$$R < \begin{cases} \frac{5}{2}q^{\frac{1}{2}(d(d+1)-(n+1)(d-2))} & \text{for even } n \text{ and even } d \\ \frac{5}{2}q^{\frac{1}{2}(d(d+1)-(n+1)(d-1))} & \text{for even } n \text{ and odd } d \\ \frac{5}{2}q^{\frac{1}{2}(d(d+1)-(n+2)(d-2))} & \text{for odd } n \text{ and even } d \\ \frac{5}{2}q^{\frac{1}{2}(d(d+1)-(n+2)(d-1))} & \text{for odd } n \text{ and odd } d. \end{cases}$$

Then we have $R < 1$, except for (i) $d = 3$ and $n = 5, 6$, or (ii) $d = 4$ and $n = 9, 10$, or (iii) $d = 6$ and $n = 9, 10$. Corollary 3.2.4 (f) and (4.3) imply that, in the respective cases, R equals

$$\frac{1+q^3}{1+q^4}, \qquad \frac{(1+q^3)(1-q^8)}{1-q^{12}}, \qquad \frac{(1-q^8)(1+q^5)}{1-q^{14}}.$$

In all cases, we have $R < 1$, as required. □

Proof of (C5). In this case, we assume that $\mathcal{P} = {}^2A_{2n-1}$ with $2 < d < n$. Use Corollary 3.2.4 (a), (4.4), and (3.17) to obtain

$$R < \begin{cases} \frac{14}{5}q^{(d-1)(d-n-1)} & \text{for odd } d \\ \frac{14}{5}q^{(d-1)(d-n-1)+n} & \text{for even } d. \end{cases}$$

Then we have $R < 1$, except for $(n,d) = (5,4)$. In the latter case, we find from Corollary 3.2.4 (a), (3.21), (4.3), and (3.26) that

$$\begin{aligned} R &< \frac{q^8-1}{q^3-1}\prod_{i=1}^{3}(q^{2i-1}+1)\frac{q^i+(-1)^i}{q^{5+i}+(-1)^i} \\ &= \frac{(q^4-1)(q^5+1)}{(q^7+1)(q^3-1)} \\ &\leq 2q^{-1} \leq 1, \end{aligned}$$

as required. □

Proof of (C6). In this case, we assume that $\mathcal{P} = {}^2A_{2n}$ with $2 < d < n-1$ or $d = n-1$ is odd, where the case $(n,d) = (6,4)$ is excluded. Use Corollary 3.2.4 (b), (4.4), and (3.17) to obtain

$$R < \begin{cases} \frac{14}{5}q^{(d-1)(d-n-2)+2d-1} & \text{for odd } d \\ \frac{14}{5}q^{d(d-n-1)+2n+2} & \text{for even } d. \end{cases} \tag{4.6}$$

For odd d, it follows $R < 1$, except when $(n,d) = (4,3)$. In the latter case, we find from Corollary 3.2.4 (b) and (4.3) that

$$R = \frac{(q^4-1)(q^5+1)}{(q^3-1)(q^7+1)} < \frac{q^2+q^{-3}}{q^3-1} < 1.$$

If d is even, then (4.6) implies $R < 1$, except when $(n, d) = (8, 6)$ (recall that we excluded $(n, d) = (6, 4)$). In this case, we find from Corollary 3.2.4 (b), (3.21), and (4.3) that

$$R = \left(\prod_{i=1}^{6}(1 + q^{2i-1})\right)\left(\prod_{i=1}^{4}\frac{q^i + (-1)^i}{q^{9+i} + (-1)^i}\right)\frac{q^5 - 1}{q^{14} - 1}\,\varepsilon(9, 6)$$

with

$$\begin{aligned}\varepsilon(9,6) &= \frac{q^5 + 1 + q\frac{q^{13}+1}{q^5-1}(q^4 - 1)}{q^5 + 1 + q\frac{q^{13}+1}{q^{14}-1}(q^4 - 1)} \\ &= \frac{(q^{14} - 1)}{(q^5 - 1)}\,\frac{(q^{18} - q^{14} + q^{10} + q^5 - q - 1)}{(q^{19} + q^{18} - q - 1)} \\ &< \frac{(q^{14} - 1)}{(q^5 - 1)}\,\frac{1}{q}.\end{aligned}$$

This gives

$$\begin{aligned}R &< \frac{1}{q}\left(\prod_{i=1}^{6}(1 + q^{2i-1})\right)\left(\prod_{i=1}^{4}\frac{q^i + (-1)^i}{q^{9+i} + (-1)^i}\right) \\ &= \frac{1}{q}\,\frac{(q^8 - 1)(q^7 + 1)(q^9 + 1)}{(q^5 - 1)(q^6 + 1)(q^{13} - 1)} \\ &< q^{-3}\,\frac{q^7 + 1}{q^5 - 1} < 1,\end{aligned}$$

by using (3.19), which completes the proof. □

Now, it remains to prove the corner cases (C7)–(C9). This is done by showing that the dual distribution of the Steiner system has a negative entry, which contradicts Proposition 2.1.26. In what follows, all inner and dual distributions (in particular those in ${}^2A_{2n-1}$) are determined with respect to the standard orderings imposed by (2.33) and (2.35). We require the following result on the inner and dual distributions of t-Steiner systems.

Proposition 4.2.1. *Let X be the set of generators in a polar space of rank n and suppose that Y is a t-Steiner system in X with $1 \le t \le n$. Let (A_i) and (A'_k) be the inner distribution and dual distribution of Y, respectively, in terms of the standard orderings imposed by* (2.33) *and* (2.35)*. Then we have*

$$A_{n-i} = \sum_{j=i}^{t-1}(-1)^{j-i}p^{\binom{j-i}{2}}\begin{bmatrix} j \\ i \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \left(\prod_{\ell=j}^{t-1}(1 + p^{n-\ell+e}) - 1\right)$$

for all $i = 0, 1, \dots, n-1$ and $A'_1 = A'_2 = \cdots = A'_t = 0$.

To prove Proposition 4.2.1, we use the following counterpart of Lemma 3.2.2 for the Q-numbers of the association scheme of polar spaces.

Lemma 4.2.2. *Let X be the set of generators in a polar space of rank n and let $Q_k(i)$ be the corresponding Q-numbers given by* (2.36)*. Then we have*

$$\sum_{k=0}^{n} p^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_p \prod_{\ell=1}^{n-j} (1+p^{\ell-k+e}) Q_k(i) = |X| \begin{bmatrix} n-i \\ j \end{bmatrix}_p$$

for all $i, j = 0, 1, \dots, n$.

Proof. We will frequently use the identity (3.7) without specific reference in this proof.

Let $P_i(k)$ and $Q_k(i)$ be as in (2.35) and (2.36), respectively. We will prove

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_p P_i(k) = p^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_p \prod_{\ell=1}^{n-j} (1+p^{\ell-k+e}). \tag{4.7}$$

By multiplying (4.7) with $Q_k(\ell)$, taking the sum over k, and using (2.8), we obtain the identity in the lemma. It remains to prove (4.7). For all $i, j = 0, 1, \dots, n$, we have

$$\begin{aligned} &\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_p P_i(k) \\ &\quad = \sum_{i=0}^{n} \sum_{\ell=0}^{i} (-1)^{\ell} \begin{bmatrix} n \\ k \end{bmatrix}_p^{-1} \begin{bmatrix} n-i \\ j \end{bmatrix}_p \begin{bmatrix} n-i \\ k-\ell \end{bmatrix}_p \begin{bmatrix} n \\ i \end{bmatrix}_p \begin{bmatrix} i \\ \ell \end{bmatrix}_p p^{\ell(\ell-i-e-1)+\binom{i+1}{2}+ie}, \end{aligned}$$

which becomes

$$\begin{aligned} &\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_p P_i(k) \\ &\quad = \sum_{i=0}^{n} \sum_{\ell=0}^{i} (-1)^{\ell} \begin{bmatrix} n \\ k \end{bmatrix}_p^{-1} \begin{bmatrix} n-i \\ j \end{bmatrix}_p \begin{bmatrix} n \\ \ell \end{bmatrix}_p \begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}_p \begin{bmatrix} n-k \\ i-\ell \end{bmatrix}_p p^{\ell(\ell-i-e-1)+\binom{i+1}{2}+ie}. \end{aligned}$$

Interchanging the order of summation by putting $m = i - \ell$ gives us

$$\begin{aligned} &\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_p P_i(k) \\ &\quad = \sum_{m=0}^{n-k} \left(\sum_{\ell=0}^{k} (-1)^{\ell} p^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_p \begin{bmatrix} n-m-\ell \\ j \end{bmatrix}_p \right) \begin{bmatrix} n-k \\ m \end{bmatrix}_p p^{\binom{m}{2}+m(e+1)}. \end{aligned} \tag{4.8}$$

To evaluate the inner sum, we use the q-Chu-Vandermonde identity

$$\begin{bmatrix} x+y \\ z \end{bmatrix}_p = \sum_{i=0}^{x} p^{i(y-z+i)} \begin{bmatrix} x \\ i \end{bmatrix}_p \begin{bmatrix} y \\ z-i \end{bmatrix}_p, \tag{4.9}$$

where x, y, z are integers, see [GJ83, § 2, 2.6.3(c)]. Applying the q-binomial inversion formula

$$\sum_{j=i}^{k} (-1)^{j-i} p^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_p \begin{bmatrix} k \\ j \end{bmatrix}_p = \delta_{ik} \tag{4.10}$$

for nonnegative integers i, k to (4.9), which can be deduced from the q-binomial theorem

$$\sum_{i=0}^{k} p^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_p z^i = \prod_{i=0}^{k-1} (1 + zp^i) \tag{4.11}$$

for example, reveals that

$$\sum_{\ell=0}^{x} (-1)^\ell p^{\binom{\ell}{2}} \begin{bmatrix} x \\ \ell \end{bmatrix}_p \begin{bmatrix} x-\ell+y \\ z \end{bmatrix}_p = p^{x(y-z+x)} \begin{bmatrix} y \\ z-x \end{bmatrix}_p.$$

Put $x = k$, $y = n-k-m$, and $z = j$ to obtain

$$\sum_{\ell=0}^{k} (-1)^\ell p^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_p \begin{bmatrix} n-m-\ell \\ j \end{bmatrix}_p = p^{k(n-m-j)} \begin{bmatrix} n-k-m \\ j-k \end{bmatrix}_p.$$

Substitute into (4.8) to give

$$\begin{aligned} \sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_p P_i(k) &= \sum_{m=0}^{n-k} p^{k(n-m-j)} \begin{bmatrix} n-k-m \\ j-k \end{bmatrix}_p \begin{bmatrix} n-k \\ m \end{bmatrix}_p p^{\binom{m}{2}+m(e+1)} \\ &= \left(\sum_{m=0}^{n-j} \begin{bmatrix} n-j \\ m \end{bmatrix}_p p^{\binom{m}{2}+m(e+1)-km} \right) p^{k(n-j)} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_p. \end{aligned} \tag{4.12}$$

Applying the q-binomial theorem (4.11) to the sum on the right-hand side of (4.12) leads to the identity (4.7). □

We can now prove Proposition 4.2.1.

Proof of Proposition 4.2.1. From (2.17) and Lemma 4.2.2, we find that, for all $j \geq 0$,

$$\sum_{k=0}^{j} A_k' \, p^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_p \prod_{\ell=1}^{n-j} (1 + p^{\ell-k+e}) = |X| \sum_{i=0}^{n} A_i \begin{bmatrix} n-i \\ j \end{bmatrix}_p. \tag{4.13}$$

Since Y is an $(n-t+1)$-code, we have $A_0 = 1$ and $A_1 = \cdots = A_{n-t} = 0$ and therefore obtain, by setting $j = t$ in (4.13),

$$\sum_{k=0}^{t} A_k' \, p^{k(n-t)} \begin{bmatrix} n-k \\ n-t \end{bmatrix}_p \prod_{\ell=1}^{n-t} (1 + p^{\ell-k+e}) = |X| \begin{bmatrix} n \\ t \end{bmatrix}_p.$$

From $A_0' = |Y|$, we find that

$$\sum_{k=1}^{t} A_k' \, p^{k(n-t)} \begin{bmatrix} n-k \\ n-t \end{bmatrix}_p \prod_{i=1}^{n-t} (1 + p^{\ell-k+e}) = \begin{bmatrix} n \\ t \end{bmatrix}_p \left(|X| - |Y| \prod_{\ell=1}^{n-t} (1 + p^{\ell+e}) \right).$$

From the expression (2.34) for $|X|$ and the expression (4.2) for $|Y|$, we see that the right-hand side is zero. Since $A_k' \geq 0$ by Proposition 2.1.26, we conclude $A_1' = A_2' = \cdots = A_t' = 0$. Moreover, (4.13) simplifies to

$$\begin{bmatrix} n \\ j \end{bmatrix}_p \prod_{\ell=1}^{n-j} (1 + p^{\ell+e}) |Y| = |X| \left(\begin{bmatrix} n \\ j \end{bmatrix}_p + \sum_{i=n-t+1}^{n} A_i \begin{bmatrix} n-i \\ j \end{bmatrix}_p \right)$$

for all $j = 0, 1, \dots, t-1$. Using (2.34) and the expression (4.2) for $|Y|$ again, we obtain

$$\sum_{i=0}^{t-1} A_{n-i} \begin{bmatrix} i \\ j \end{bmatrix}_p = \begin{bmatrix} n \\ j \end{bmatrix}_p \left(\prod_{\ell=j}^{t-1} (1 + p^{n-\ell+e}) - 1 \right).$$

Applying the q-binomial inversion formula (4.10) gives the desired expression for A_{n-i}. □

We now prove (C7)–(C9). Henceforth, we denote by (A_i) and (A'_k) the inner and dual distribution, respectively, of a putative t-Steiner system Y.

Proof of (C7). We now assume that $t = 2$ and $\mathcal{P} = B_n$ or C_n for odd $n > 3$ or $\mathcal{P} = {}^2D_{n+1}$ for even $n > 3$. We will show that $A'_{n-1} < 0$ in the first case and $A'_n < 0$ in the second case. By (2.17) and (2.11), we have

$$\frac{A'_k}{\mu_k} = 1 + \frac{P_{n-1}(k)}{v_{n-1}} A_{n-1} + \frac{P_n(k)}{v_n} A_n.$$

By Proposition 4.2.1, we have

$$A_{n-1} = q^{n-1+e} \begin{bmatrix} n \\ 1 \end{bmatrix}_q$$

and

$$A_n = (q^{n+e} + 1)(q^{n-1+e} + 1) - \begin{bmatrix} n \\ 1 \end{bmatrix}_q q^{n-1+e} - 1.$$

From (2.35) and (2.37), we find for B_n and C_n that

$$\frac{P_{n-1}(n-1)}{v_{n-1}} = \begin{bmatrix} n \\ 1 \end{bmatrix}_q^{-1} \left(q^{-n+1} - q^{-2n+4} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \right)$$

and

$$\frac{P_n(n-1)}{v_n} = q^{-2n+2},$$

and for ${}^2D_{n+1}$ that

$$\frac{P_{n-1}(n)}{v_{n-1}} = -q^{-2n+2} \quad \text{and} \quad \frac{P_n(n)}{v_n} = q^{-2n}.$$

Here, we have crucially used the assumed parity of n. For B_n and C_n, we then obtain

$$\frac{A'_{n-1}}{\mu_{n-1}} = 2 - \frac{q^n - 1}{(q-1)q^{n-1}} - \frac{1}{q^{2n-2}} - \frac{q^{n-1} - 1}{(q-1)q^{n-3}} + \frac{(q^n+1)(q^{n-1}+1)}{q^{2n-2}}.$$

For $n > 3$, we have

$$\begin{aligned} 2 - \frac{q^n - 1}{(q-1)q^{n-1}} - \frac{1}{q^{2n-2}} &= \frac{q^{2n-1} - 2q^{2n-2} + q^{n-1} - q + 1}{(q-1)q^{2n-2}} \\ &< \frac{q^{2n-1} - 2q^{n+1} + q^{n-1} - q + 1}{(q-1)q^{2n-2}} \\ &= \frac{q^{n-1} - 1}{(q-1)q^{n-3}} - \frac{(q^n+1)(q^{n-1}+1)}{q^{2n-2}} \end{aligned}$$

and therefore $A'_{n-1} < 0$ if $\mathcal{P} = B_n$ or C_n, which completes the proof in the first case. For ${}^2D_{n+1}$, we obtain

$$\frac{A'_n}{\mu_n} = 1 - \frac{q^n - 1}{(q-1)q^n} - \frac{1}{q^{2n}} - \frac{(q^n-1)q^2}{(q-1)q^n} + \frac{(1+q^{n+1})(1+q^n)}{q^{2n}}.$$

For $n > 2$, we have

$$\begin{aligned} 1 - \frac{q^n - 1}{(q-1)q^n} - \frac{1}{q^{2n}} &= \frac{q^{2n+1} - 2q^{2n} + q^n - q + 1}{(q-1)q^{2n}} \\ &< \frac{q^{2n+1} - 2q^{n+2} + q^n - q + 1}{(q-1)q^{2n}} \\ &= \frac{(q^n-1)q^2}{(q-1)q^n} - \frac{(1+q^{n+1})(1+q^n)}{q^{2n}}, \end{aligned}$$

and therefore $A'_n < 0$ in the case $\mathcal{P} = {}^2D_{n+1}$. This completes the proof. □

Proof of (C8). We now assume $\mathcal{P} = {}^2D_{n+1}$ for $t = 3$ and even $n > 4$. As in (C7) we compute

$$\frac{A'_{n-1}(q-1)^2(q+1)}{2\mu_{n-1}} = -q(q+1)(1-q^{2-n})(1-q^{4-n}) + q^{5-3n}(1+q^{-2}),$$

from which it is readily verified that $A'_{n-1} < 0$, as required. □

Proof of (C9). As in (C7) we compute the following. For $\mathcal{P} = {}^2D_8$ and $t = 4$, we have

$$\frac{A'_6}{\mu_6} = -2q^{-5}(q+1)^2(q^2+1)(q^3+q+1) < 0,$$

for ${}^2D_9$ and $t = 5$, we have

$$\frac{A'_7}{\mu_7} = -2q^{-5}(q+1)^4(q^2-q+1)(q^2+1)^2 < 0,$$

and for ${}^2A_{12}$ and $t = 3$, we have

$$\frac{A'_5}{\mu_5} = -q^{-7}(q+1)^3(q^2-q+1)(q^4-q^3+q^2+1) < 0.$$

In all cases, we obtain the required nonexistence of t-Steiner systems. □

This completes the proof of Theorem 4.1.1.

4.3 Open problems

We close this chapter by discussing some open problems that are related to Steiner systems in polar spaces.

Problem 4.3.1. *Prove Conjecture 4.1.2.*

This conjecture is true if there are no t-Steiner systems in ${}^2A_{2n}$ and ${}^2D_{n+1}$ in the case $t = n - 1$ and $q \geq 3$, and in the case $t = 2$ and odd n. For the first case, recall from the proof of (C1) in Section 4.2 that an $(n-1)$-Steiner system in a polar space of rank n induces a 2-Steiner system in a polar space of the same type of rank 3, or a 1-Steiner system, hence a spread, in rank 2. Thus, showing the nonexistence of spreads in ${}^2A_4$ and ${}^2D_3$ for $q \geq 3$ could prove the conjecture. However, for ${}^2D_3$ and all $q \geq 2$, there always exists a spread, whereas nothing is known for ${}^2A_4$ with $q \geq 3$. Similarly, for the second case, a 2-Steiner system in odd rank n induces a spread in even rank $n - 1$. The existence question for spreads in a general rank in ${}^2D_{n+1}$ and ${}^2A_{2n}$ is also not completely settled yet, see [HT16, § 7.4 and 7.5]. The case ${}^2A_{2n}$ with $q \geq 3$ is the most important open case concerning spreads in polar spaces since only the special case ${}^2A_4$ with $q = 2$ has been solved so far.

Problem 4.3.1 is also related to the existence of strongly regular graphs with specific parameters. Namely, a putative 2-Steiner system Y in rank 3 is a 2-design and a 2-code and, by Proposition 2.1.29, it induces a symmetric association scheme with two classes and hence a strongly regular graph.

Problem 4.3.2. *Do nontrivial t-designs in polar spaces exist for all $t \geq 2$?*

A t-(v, n, λ) design over $\mathbb{F}_q$ is a collection Y of n-subspaces of $\mathbb{F}_q^v$ such that every t-subspace of $\mathbb{F}_q^v$ lies in exactly λ members of Y. These are precisely the t-designs in the q-Johnson scheme. Using a probabilistic argument, it was shown in [FLV14] that t-designs over $\mathbb{F}_q$ exist for all t and q if $n > 12(t + 1)$ and v is sufficiently large. The definition of a design over $\mathbb{F}_q$ can be extended to polar spaces: a *t-(v, n, λ) design in a polar space* $\mathcal{P}$ of rank n is a collection of generators of $\mathcal{P}$ such that every t-space of $\mathcal{P}$ lies in exactly λ members of Y. Nontrivial designs in polar spaces were recently found by computer constructions in [Lan20] and by Kiermaier, Schmidt, Wassermann, which was announced in [Was]. See also [Cos+22] for more information on designs in polar spaces. However, a general existence result for designs in polar spaces is presently not known.

Problem 4.3.3. *Let $\mathcal{P}$ be a polar space of rank n, and let k, t be integers with $1 \leq t < k < n$. Do collections Y of k-spaces in $\mathcal{P}$ exist such that every t-space in $\mathcal{P}$ lies in exactly one member of Y?*

Such collections Y can be seen as a generalization of a t-Steiner system in a polar space. This problem is related to Problem 3.4.4

Problem 4.3.4. *Do nontrivial t-Steiner systems over $\mathbb{F}_q$ exist for all $t \geq 2$?*

Recall from Section 4.1 that so far, nontrivial t-Steiner systems over $\mathbb{F}_q$, which are designs in the q-Johnson scheme, are only known for a single set of parameters. In the case of t-Steiner systems in the Johnson scheme, the existence question was open for over 150 years and has been only settled recently in [Kee14] and [Glo+16]. It would be interesting to see whether the methods from these papers could be modified for the q-analog case. However, this is certainly a very challenging problem and the methods applied in these two papers are way out of scope of this thesis.

CHAPTER 5

Optimal solution of the linear program

I can see that, without being excited,
mathematics can look pointless and cold.

—Maryam Mirzakhani

This chapter focuses on Delsarte's linear program for codes in various classical association schemes. We will derive the LP optimum in the bilinear forms scheme, Hermitian forms scheme, and alternating bilinear forms scheme as well as in several ordinary q-analogs. This is done by using the duality theory of linear programming.

5.1 Introduction

It is well known that Delsarte's linear programming method, which was introduced in Section 2.2, yields asymptotically optimal bounds in many association schemes. In particular, the best known asymptotic bound for codes in the binary Hamming scheme $H(n,2)$ and in the Johnson scheme $J(n,m)$ arises from Delsarte's linear program (2.20), see [McE+77]. However, even though the linear program for d-codes in $H(n,q)$ and $J(n,m)$ has been studied since the 1970s no explicit expression for the LP optimum is known so far except for the special case that $q \geq \max\{d, n-d+2\}$ for $H(n,q)$, see [Del73, § 4.3.2]. For the bilinear forms scheme, Hermitian forms scheme, and alternating bilinear forms scheme, Delsarte's linear program was used to derive bounds that are sharp in most cases implying that in these respective cases, the bounds are precisely the LP optima, see [Del78a], [DG75], and [Sch18]. For the ordinary q-analogs, nothing was known about the LP optimum so far. Here, we will give an explicit expression for the LP optimum for codes in several classical association schemes. We will see that these LP optima are sharp up to a constant factor in most cases.

The main result of this chapter is the following theorem. Recall the definition of the parameters b and c from (2.51).

Theorem 5.1.1.

(a) *Let X be the set of n-spaces in $J_q(n,m)$ or generators in ${}^2A_{2n-1}$ or $\frac{1}{2}D_m$, where $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$. Then the LP optimum for d-codes with $1 \le d \le n$ in $J_q(n,m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ is given by*

$$\mathrm{LP}(d) = \frac{|X|(q)_{d-1}}{(qcb^n)_{d-1}}, \tag{5.1}$$

where d is required to be odd in the case of ${}^2A_{2n-1}$. For even d with $2 \le d \le n$, the LP optimum for d-codes in ${}^2A_{2n-1}$ is given by

$$\mathrm{LP}(d) = \frac{|X|(q)_{d-1}}{(qcb^n)_{d-1}} \, \frac{(b^{n-d+2}-1) + q\frac{b^{n+d-2}-1}{qb^{d-2}-1}(b^{n-d+1}-1)}{(b^{n-d+2}-1) + q\frac{b^{n+d-2}-1}{b^{n+d-1}-1}(b^{n-d+1}-1)}. \tag{5.2}$$

Moreover, both LP optima (5.1) *and* (5.2) *also hold for association schemes with the same P- and Q-numbers as $J_q(n,m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$.*

(b) *The LP optimum for d-codes with $1 \le d \le n$ in $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ is given by*

$$\mathrm{LP}(d) = (cb^n)^{n-d+1}, \tag{5.3}$$

where d is required to be odd in the case of $\mathrm{Her}_q(n)$ and $n = \lfloor m/2 \rfloor$ in the case of $\mathrm{Alt}_q(m)$. For even d with $2 \le d \le n$, the LP optimum for d-codes in $\mathrm{Her}_q(n)$ is given by

$$\mathrm{LP}(d) = (cb^n)^{n-d+1} \, \frac{(b^{n-d+2}-1) + b^n(b^{n-d+1}-1)}{b^{n-d+2} - b^{n-d+1}}. \tag{5.4}$$

Observe that the LP optima in Theorem 5.1.1(a) are exactly the bounds we obtained in Chapter 3. In particular, we want to emphasize that for $J_q(n,m)$, the LP optimum is the well-known Singleton bound

$$\mathrm{LP}(d) = \frac{\begin{bmatrix} m+n \\ n-d+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ n-d+1 \end{bmatrix}_q},$$

see p. 54 for more information on this bound. Moreover, it was known that (5.3) and (5.4) are upper bounds for the size of d-codes in $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$. Namely, they are also the Singleton bounds, which were derived by using Delsarte's linear program in [Del78a], [DG75], and [Sch18]. Therefore, it seems reasonable to use a similar approach for the six polar space schemes to obtain some kind of a Singleton bound. This is done in Proposition 5.1.2, where we derive an upper bound for d-codes in all six polar space schemes with respect to the standard ordering by using a Singleton polynomial together with Theorem 2.2.6. Moreover, we also give a purely combinatorial proof of Proposition 5.1.2. It will turn out that the bound therein is precisely the number of elements in a Steiner system. Since we showed in Chapter 4 that Steiner systems cannot exist in most cases, the bound of Proposition 5.1.2 is weaker than the one in Corollary 3.2.4 in most cases, motivating our approach to study ${}^2A_{2n-1}$ with respect to the second ordering and $\frac{1}{2}D_m$.

Proposition 5.1.2. *Let $\mathcal{P}$ be a polar space of rank n with parameter e and let Y be a d-code in $\mathcal{P}$ with $1 \le d \le n$. Then we have*

$$|Y| \le \prod_{i=0}^{n-d} (1 + p^{n-i+e}).$$

Algebraic proof of Proposition 5.1.2. Let X be the set of generators in $\mathcal{P}$, let $P_i(k)$ be the P-numbers of the corresponding association scheme given in (2.35) and consider the standard ordering of the Q-polynomial structure. We apply Theorem 2.2.6 with the Singleton polynomial

$$F(z) = \begin{bmatrix} n \\ d-1 \end{bmatrix}_p \prod_{j=d}^{n} p^j \frac{z - z_j}{p^j - 1},$$

where $z_j = p^{-j}$. Then we have $F(z_i) = 0$ for all $i = d, d+1, \dots, n$ and we also obtain

$$F(z_i) = \begin{bmatrix} n \\ d-1 \end{bmatrix}_p \prod_{j=d}^{n} p^j \frac{p^{-i} - p^{-j}}{p^j - 1} = \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_p$$

for all $i = 0, 1, \dots, n$. By using (2.23) and (4.7), we have

$$F_k = \frac{1}{|X|} \sum_{i=0}^{n} \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_p P_i(k) = \frac{1}{|X|} p^{k(d-1)} \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_p \prod_{i=1}^{d-1} (1 + p^{\ell-k+e}).$$

Hence, it holds $F_k \ge 0$ for all $k = 0, 1, \dots, n$. The polynomial F/F_0 thus satisfies all conditions of Theorem 2.2.6 giving the desired bound by using (2.34). □

Combinatorial proof of Proposition 5.1.2. Every element of the d-code Y contains exactly $\left[\begin{smallmatrix} n \\ n-d+1 \end{smallmatrix}\right]_p$ totally isotropic subspaces of dimension $n-d+1$. A given $(n-d+1)$-space in $\mathcal{P}$ is contained in at most one element of Y since otherwise there would exist distinct $x, y \in Y$ such that $n - \dim(x \cap y) \le d-1$, contradicting to Y being a d-code. By (4.1), the number of $(n-d+1)$-spaces in $\mathcal{P}$ is

$$\begin{bmatrix} n \\ n-d+1 \end{bmatrix}_p \prod_{i=0}^{n-d} (1 + p^{n-i+e}).$$

Therefore, we have

$$|Y| \le \frac{\left[\begin{smallmatrix} n \\ n-d+1 \end{smallmatrix}\right]_p \prod_{i=0}^{n-d} (1 + p^{n-i+e})}{\left[\begin{smallmatrix} n \\ n-d+1 \end{smallmatrix}\right]_p} = \prod_{i=0}^{n-d} (1 + p^{n-i+e}),$$

as stated. □

Theorem 5.1.1 also gives the LP optimum in B_n, C_n, and D_n in the following way. First, recall from Section 2.3.2 (see p. 46) that B_n and C_n induce new association schemes with the classes

$$R_0, R_1 \cup R_2, R_3 \cup R_4, \ldots,$$

which have the same P- and Q-numbers as $\frac{1}{2}D_{n+1}$. Therefore, without loss of generality for odd d, we can study the linear program for $\frac{d+1}{2}$-codes in $\frac{1}{2}D_{n+1}$ instead of the linear program for d-codes in B_n and C_n. Second for even d, by Proposition 3.2.3, we can require without loss of generality that the inner distribution $(A_0, A_1, \ldots, A_n)$ of a d-code in D_n satisfies $A_i = 0$ for all odd i. Thus, without loss of generality for even d, we can add this constraint to the linear program (2.20) for d-codes in D_n and can study the linear program for $\frac{d}{2}$-codes in $\frac{1}{2}D_n$ instead. These observations imply the following corollary from Theorem 5.1.1.

Corollary 5.1.3.

(a) *Let X be the set of generators in B_n or C_n. Assume that d is odd with $1 \le d \le n$. Then the LP optimum for d-codes in C_n and B_n is given by*

$$\mathrm{LP}(d) = \begin{cases} |X| \prod_{i=1}^{\frac{d-1}{2}} \dfrac{1-q^{2i-1}}{1-q^{n+2i-1}} & \text{for odd } n \\ |X| \prod_{i=1}^{\frac{d-1}{2}} \dfrac{1-q^{2i-1}}{1-q^{n+2i}} & \text{for even } n. \end{cases} \tag{5.5}$$

(b) *Let X be the set of generators in D_n. Assume that d is even with $2 \le d \le n$. Then the LP optimum for d-codes in D_n is given by*

$$\mathrm{LP}(d) = \begin{cases} \dfrac{|X|}{2} \prod_{i=1}^{\frac{d}{2}-1} \dfrac{1-q^{2i-1}}{1-q^{n+2i-1}} & \text{for odd } n \\ \dfrac{|X|}{2} \prod_{i=1}^{\frac{d}{2}-1} \dfrac{1-q^{2i-1}}{1-q^{n+2i-2}} & \text{for even } n. \end{cases}$$

We pose the following conjecture that the bound from Corollary 3.2.4(d) for D_n is precisely the LP optimum if n is odd. This was checked with a computer for many small values of q, n, and d.

Conjecture 5.1.4. *Let X be the set of generators in D_n. Assume that n and d are odd integers with $1 \le d \le n$. Then the LP optimum for d-codes in D_n is given by*

$$\mathrm{LP}(d) = |X| \prod_{i=1}^{\frac{d-1}{2}} \frac{1-q^{2i-1}}{1-q^{n+2i-1}}.$$

Observe that the LP optima in Corollary 5.1.3 and in Conjecture 5.1.4 are exactly the bounds we obtained in Chapter 3.

We will see in the next remark that there is a nice similarity between the LP optimum in the ordinary q-analogs and their affine counterparts.

Remark 5.1.5. The LP optimum (5.3) can also be written as

$$\mathrm{LP}(d) = \frac{|X|}{(cb^n)^{d-1}},$$

where $|X|$ is the number of matrices in the corresponding association scheme $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ with

$$|X| = (cb^n)^n. \tag{5.6}$$

We can then write (5.1) and (5.3) as

$$\mathrm{LP}(d) = |X| \prod_{\ell=0}^{d-2} \frac{qb^\ell - 1}{qcb^{n+\ell} - 1} \tag{5.7}$$

$$\mathrm{LP}(d) = |X| \prod_{\ell=0}^{d-2} \frac{qb^\ell}{qcb^{n+\ell}},$$

respectively. If we neglect the ones in the fractions of the right-hand side in (5.7), then both LP optima have the same form. This resemblance also occurs in ${}^2A_{2n-1}$ and $\mathrm{Her}_q(n)$ for even d by neglecting some ones in the long fraction appearing in (5.2) in a similar way.

In [Sch10] and [Sch15], it was shown that a d-code Y in the association scheme $\mathrm{Sym}_q(n)$ of symmetric $n \times n$ matrices over $\mathbb{F}_q$ with odd d, which means that $\mathrm{rank}(x - y) \geq d$ for all distinct $x, y \in Y$, satisfies

$$|Y| \leq \begin{cases} |X| q^{-n(d-1)/2} & \text{for odd } n \\ |X| q^{-(n+1)(d-1)/2} & \text{for even } n \end{cases}$$

with $|X| = q^{n(n+1)/2}$. Since these bounds are sharp and were proved by using Delsarte's linear programming method, they are precisely the LP optimum for d-codes in $\mathrm{Sym}_q(n)$ if d is odd. Observe that we again have a nice resemblance to the LP optimum in C_n by neglecting the minus ones in the fractions occurring in (5.5).

Since it was known that (5.3) and (5.4) are upper bounds for the size of d-codes in $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ that were derived by using Delsarte's linear program in [Del78a], [DG75], and [Sch18] and moreover, sharp constructions were given in the aforementioned papers except for $\mathrm{Alt}_q(m)$ if m is even and q is odd and for $\mathrm{Her}_q(n)$ if d is even, the LP optimum (5.3) was basically known, except in the latter cases. Here, we derive the LP optimum for $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ without using the known constructions. In particular, we obtain as a new result the LP optimum in the case of $\mathrm{Alt}_q(m)$ with even m and odd q and in the case of $\mathrm{Her}_q(n)$ with even d. We note that for the latter case, it was conjectured in [Sch18] that (5.4) is precisely the LP optimum.

We already saw in Section 3.3 that the LP optimum (5.1) is reached up to a constant factor for ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$ except possibly when m is even and q is odd in the case of $\frac{1}{2}D_m$. It is well known that the Singleton bound—and thus the LP optimum—for $J_q(n,m)$ is also reached up to a constant factor, see [SKK08, § IV.A.]. Moreover, a d-code Y in $J_q(n,m)$ is of size (5.1) if and only if Y is an $(n-d+1)$-Steiner system over $\mathbb{F}_q$.

Proof strategy for Theorem 5.1.1. The proof of Theorem 5.1.1 is split into two parts. First, in Section 5.2, we will give a feasible solution of the dual LP (2.21) whose objective function value coincides with (5.1), (5.2), (5.3), or (5.4). Second, in Section 5.3, we will compute the inner distribution of a d-code whose size is precisely the stated LP optimum in Theorem 5.1.1 and show that the inner distribution is a feasible solution of the primal LP (2.20). The Strong duality theorem 2.2.2 then proves Theorem 5.1.1.

At the end of this chapter, we will discuss some related open problems in Section 5.4.

We close this section by stating some preliminaries that are needed for the proof of Theorem 5.1.1.

Identities for the P- and Q-numbers. Crucial for the proofs in Section 5.2 and 5.3 are the following identities for the P-numbers in the studied classical association schemes.

- Recall from (3.8) that the P-numbers (2.52) of $J_q(n,m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ satisfy

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P'_i(k) = b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} \tag{5.8}$$

  for all $j,k = 0,1,\dots,n$, where $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$.

- The P-numbers (2.47) of $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ satisfy

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P_i(k) = \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b (cb^n)^{n-j} \tag{5.9}$$

  for all $j,k = 0,1,\dots,n$, where $n = \lfloor m/2 \rfloor$ in the case of $\mathrm{Alt}_q(m)$. This follows by using the P-numbers (2.47) and the q-binomial inversion formula of the form

$$\sum_{j=i}^{k} (-1)^{k-j} b^{\binom{k-j}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \begin{bmatrix} k \\ j \end{bmatrix}_b = \delta_{ik}. \tag{5.10}$$

The identities (5.8) and (5.9) can be also written as

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P_i'(k) = \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \prod_{\ell=0}^{n-j-1} b^k \frac{qcb^{n-k+\ell}-1}{qb^\ell - 1}$$
$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b P_i(k) = \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \prod_{\ell=0}^{n-j-1} \frac{qcb^{n+\ell}}{qb^\ell}.$$

We have a similar resemblance between these identities as for the corresponding LP optima: by neglecting the ones in the fractions of the first identity, the terms on the right-hand side of both identities become the same.

Finally, for the sake of convenience, we rewrite the inequality (3.19) as

$$\frac{y-1}{x-1} \geq \frac{y}{x} \quad \text{for } y \geq x > 1. \tag{5.11}$$

5.2 Feasible solution of the dual LP

In this section, we will apply Theorem 2.2.6 together with a suitable Singleton polynomial or variations of it to construct a feasible solution of the dual LP (2.21) for the ordinary q-analogs $J_q(n,m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ as well as for the affine q-analogs $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$. We start with the latter three. Recall from (2.47) that $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ are Q-polynomial with $z_i = b^{-i}$.

Proposition 5.2.1. *There exists a feasible solution of the dual LP* (2.21) *for d-codes in* $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, *and* $\mathrm{Alt}_q(m)$ *with objective function value* (5.3) *for* $1 \leq d \leq n$, *where d is required to be odd in the case of* $\mathrm{Her}_q(n)$ *and* $n = \lfloor m/2 \rfloor$ *in the case of* $\mathrm{Alt}_q(m)$.

Proof. Let $P_i(k)$ be given in (2.47). Take the Singleton polynomial

$$F(z) = (cb^n)^{n-d+1} \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \prod_{j=d}^{n} b^j \frac{z - z_j}{b^j - 1}$$

with $z_j = b^{-j}$. Then we have $F(z_i) = 0$ for all $i = d, d+1, \ldots, n$. Moreover, we obtain

$$F(z_i) = (cb^n)^{n-d+1} \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \prod_{j=d}^{n} b^j \frac{b^{-i} - b^{-j}}{b^j - 1} = (cb^n)^{n-d+1} \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b$$

for all $i = 0, 1, \ldots, n$. Together with (2.23), (5.6), and (5.9), we have

$$F_k = \frac{(cb^n)^{n-d+1}}{(cb^n)^n} \sum_{i=0}^{n} \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b P_i(k) = \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b.$$

Observe that in the case of $\mathrm{Her}_q(n)$, the sign of $\left[\begin{smallmatrix} n-k \\ d-1 \end{smallmatrix}\right]_b$ is $(-1)^{(d-1)(n-k+d-1)} = 1$ since d is odd. We thus obtain $F_k \geq 0$ for all $k = 0, 1, \ldots, n$. Therefore, the polynomial F/F_0 satisfies all conditions of Theorem 2.2.6 with

$$\frac{F(z_0)}{F_0} = (cb^n)^{n-d+1},$$

which implies the stated result. $\square$

We now look at $\mathrm{Her}_q(n)$ with even d and proceed similarly as in Proposition 5.2.1, but we take a linear combination of two Singleton polynomials instead of just one Singleton polynomial.

Proposition 5.2.2. *Let n and d be integers with $2 \le d \le n$ and even d. Then there exists a feasible solution of the dual LP* (2.21) *for d-codes in* $\mathrm{Her}_q(n)$ *with objective function value* (5.4).

Proof. Take

$$F(z) = \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \beta_1(z) - \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \beta_2(z)$$

with

$$\beta_1(z) = q^{n(n-d+1)} \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \prod_{j=d}^{n} b^j \frac{z - z_j}{b^j - 1}$$

$$\beta_2(z) = (-1)^{n+1} q^{n(n-d+2)} \begin{bmatrix} n \\ d-2 \end{bmatrix}_b \prod_{j=d-1}^{n} b^j \frac{z - z_j}{b^j - 1},$$

where $z_j = b^{-j}$. We then have $F(z_i) = 0$ for all $i = d, d+1, \dots, n$ and also obtain

$$\beta_1(z_i) = q^{n(n-d+1)} \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b$$

$$\beta_2(z_i) = (-1)^{n+1} q^{n(n-d+2)} \begin{bmatrix} n-i \\ n-d+2 \end{bmatrix}_b$$

for all $i = 0, 1, \dots, n$. This gives

$$F(z_i) = q^{n(n-d+1)} \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b + (-1)^n q^{n(n-d+2)} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-i \\ n-d+2 \end{bmatrix}_b$$

for all $i = 0, 1, \dots, n$. From (2.23), (5.6), and (5.9), we find

$$F_k = (-1)^{n+1} \left(\begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b - \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-2 \end{bmatrix}_b \right).$$

The sign of $\left[\begin{smallmatrix} n-1 \\ d-2 \end{smallmatrix}\right]_b \left[\begin{smallmatrix} n-k \\ d-1 \end{smallmatrix}\right]_b$ and $\left[\begin{smallmatrix} n-1 \\ d-1 \end{smallmatrix}\right]_b \left[\begin{smallmatrix} n-k \\ d-2 \end{smallmatrix}\right]_b$ is $(-1)^{n-k+1}$ and $(-1)^n$, respectively, which gives

$$F_k = (-1)^k \left| \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \right| + \left| \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-2 \end{bmatrix}_b \right|.$$

For even k, we immediately have $F_k \geq 0$. For $k = 1$, we obtain $F_1 = 0$ and for all odd $k \geq 3$, we have

$$\left| \frac{\left[{n-1 \atop d-1}\right]_b \left[{n-k \atop d-2}\right]_b}{\left[{n-1 \atop d-2}\right]_b \left[{n-k \atop d-1}\right]_b} \right| = \left| \frac{(-q)^{n-d+1} - 1}{(-q)^{n-k-d+2} - 1} \right| \geq \frac{q^{n-d+1} - 1}{q^{n-k-d+2} + 1} \geq 1.$$

Hence, we obtain $F_k \geq 0$ for all $k = 0, 1, \dots, n$. Therefore, the polynomial F/F_0 satisfies all conditions of Theorem 2.2.6 with

$$\frac{F(z_0)}{F_0} = \frac{q^{n(n-d+1)} \left(\left[{n-1 \atop d-2}\right]_b \left[{n \atop d-1}\right]_b + b^n \left[{n-1 \atop d-1}\right]_b \left[{n \atop d-2}\right]_b \right)}{(-1)^{n+1} \left(\left[{n-1 \atop d-2}\right]_b - \left[{n-1 \atop d-1}\right]_b \left[{n \atop d-2}\right]_b \right)}.$$

After some elementary manipulations, this gives the stated objective function value and thus proves the proposition. □

We will now look at the ordinary q-analogs $J_q(n, m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ and again use Theorem 2.2.6 with a Singleton polynomial. Similarly to $\mathrm{Her}_q(n)$, we will distinguish between even and odd d in the case of ${}^2A_{2n-1}$. Recall from (2.53) that the association schemes $J_q(n, m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ are Q-polynomial with $z_i = b^{-i}$, where we take the second ordering for ${}^2A_{2n-1}$.

Proposition 5.2.3. *Let X be the set of n-spaces in $J_q(n, m)$ or generators in ${}^2A_{2n-1}$ or $\frac{1}{2}D_m$, where $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$. Then there exists a feasible solution of the dual LP* (2.21) *for d-codes in $J_q(n, m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ with objective function value* (5.1) *for $1 \leq d \leq n$, where d is required to be odd in the case of ${}^2A_{2n-1}$.*

Proof. Let $P'_i(k)$ be given in (2.52). Take the Singleton polynomial

$$F(z) = \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \prod_{j=d}^{n} b^j \frac{z - z_j}{b^j - 1}$$

with $z_j = b^{-j}$. This gives $F(z_i) = 0$ for all $i = d, d+1, \dots, n$ and

$$F(z_i) = \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \prod_{j=d}^{n} b^j \frac{b^{-i} - b^{-j}}{b^j - 1} = \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b$$

for all $i = 0, 1, \dots, n$. From (2.23) and (3.8), we find that

$$F_k = \frac{1}{|X|} \sum_{i=0}^{n} \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b P'_i(k) = \frac{1}{|X|} b^{k(d-1)} \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \frac{(qcb^{n-k})_{d-1}}{(q)_{d-1}}.$$

For $J_q(n, m)$ (since $m \geq k$) and $\frac{1}{2}D_m$, we see that

$$\frac{(qcb^{n-k})_{d-1}}{(q)_{d-1}} \geq 0$$

and thus, we have $F_k \geq 0$ for all $k = 0, 1, \dots, n$. For ${}^2A_{2n-1}$, the sign of

$$\frac{(qcb^{n-k})_{d-1}}{(q)_{d-1}} = \frac{((-q)^{n-k+1}; -q)_{d-1}}{(q; -q)_{d-1}}$$

is $(-1)^{(n-k+1)(d-1)}$ and $\left[\begin{smallmatrix} n-k \\ d-1 \end{smallmatrix}\right]_b$ has the sign $(-1)^{(d-1)(n-k-d+1)}$. Since d is odd for ${}^2A_{2n-1}$, we also obtain $F_k \geq 0$ for all $k = 0, 1, \ldots, n$. Therefore, the polynomial F/F_0 satisfies all conditions of Theorem 2.2.6 with

$$\frac{F(z_0)}{F_0} = \frac{|X|(q)_{d-1}}{(qcb^n)_{d-1}}.$$

This concludes the proof. □

It remains to look at ${}^2A_{2n-1}$ with even d, where we take a linear combination of two Singleton polynomials like in the proof of Proposition 5.2.2 for $\mathrm{Her}_q(n)$ with even d.

Proposition 5.2.4. *Let X be the set of generators in ${}^2A_{2n-1}$ and let d be an even integer with $2 \leq d \leq n$. Then there exists a feasible solution of the dual LP* (2.21) *for d-codes in ${}^2A_{2n-1}$ with objective function value* (5.2).

Proof. Take

$$F(z) = \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \beta_1(z) - \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \beta_2(z)$$

with

$$\beta_1(z) = \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \prod_{j=d}^{n} b^j \, \frac{z - z_j}{b^j - 1}$$

$$\beta_2(z) = b \, \frac{(b^{n+d-2} - 1)}{(qb^{d-2} - 1)} \begin{bmatrix} n \\ d-2 \end{bmatrix}_b \prod_{j=d-1}^{n} b^j \, \frac{z - z_j}{b^j - 1},$$

where $z_j = b^{-j}$. We then obtain $F(z_i) = 0$ for all $i = d, d+1, \ldots, n$. Moreover, we have

$$\beta_1(z_i) = \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b$$

$$\beta_2(z_i) = b \, \frac{(b^{n+d-2} - 1)}{(qb^{d-2} - 1)} \begin{bmatrix} n-i \\ n-d+2 \end{bmatrix}_b$$

for all $i = 0, 1, \ldots, n$. Therefore, we find

$$F(z_i) = \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-i \\ n-d+1 \end{bmatrix}_b - b \, \frac{(b^{n+d-2} - 1)}{(qb^{d-2} - 1)} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-i \\ n-d+2 \end{bmatrix}_b$$

for all $i = 0, 1, \ldots, n$. Using (2.23) and (3.8) gives

$$F_k = \frac{1}{|X|} \Bigg(b^{k(d-1)} \frac{(b^{n-k+1})_{d-1}}{(q)_{d-1}} \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b - b^{k(d-2)+1} \frac{(b^{n+d-2} - 1)}{(qb^{d-2} - 1)} \, \frac{(b^{n-k+1})_{d-2}}{(q)_{d-2}} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-2 \end{bmatrix}_b \Bigg).$$

Since the sign of $\begin{bmatrix} m \\ \ell \end{bmatrix}_b$ and $(b^m)_\ell/(q)_\ell$ is $(-1)^{\ell(m-\ell)}$ and $(-1)^{m\ell}$, respectively, we obtain

$$F_k = \frac{1}{|X|}\left((-1)^k \left| b^{k(d-1)} \frac{(b^{n-k+1})_{d-1}}{(q)_{d-1}} \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \right| \right.$$
$$\left. + \left| b^{k(d-2)+1} \frac{(b^{n+d-2}-1)}{(qb^{d-2}-1)} \frac{(b^{n-k+1})_{d-2}}{(q)_{d-2}} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-2 \end{bmatrix}_b \right| \right).$$

We have $F_k \geq 0$ for all even k. Assume that k is odd. This gives

$$\left| \frac{b^{k(d-2)+1} \frac{(b^{n+d-2}-1)}{(qb^{d-2}-1)} \frac{(b^{n-k+1})_{d-2}}{(q)_{d-2}} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-2 \end{bmatrix}_b}{b^{k(d-1)} \frac{(b^{n-k+1})_{d-1}}{(q)_{d-1}} \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b} \right|$$
$$= \left| \frac{b^{-k+1}(b^{n+d-2}-1)(b^{n-d+1}-1)}{(b^{n-k+d-1}-1)(b^{n-k-d+2}-1)} \right|.$$

For $k = 1$, this becomes 1 and for $k \geq 3$, it can be bounded from below by

$$q^{-k+1} \frac{(q^{n+d-2}+1)(q^{n-d+1}-1)}{(q^{n-k+d-1}-1)(q^{n-k-d+2}+1)} \geq 1.$$

Hence, we also have $F_k \geq 0$ for all odd k. Thus, the polynomial F/F_0 satisfies all conditions of Theorem 2.2.6 with

$$\frac{F(z_0)}{F_0} = \frac{\begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n \\ d-1 \end{bmatrix}_b - b\, \frac{(b^{n+d-2}-1)}{(qb^{d-2}-1)} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n \\ d-2 \end{bmatrix}_b}{\frac{1}{|X|}\left(\frac{(b^{n+1})_{d-1}}{(q)_{d-1}} \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n \\ d-1 \end{bmatrix}_b - b\, \frac{(b^{n+d-2}-1)}{(qb^{d-2}-1)} \frac{(b^{n+1})_{d-2}}{(q)_{d-2}} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n \\ d-2 \end{bmatrix}_b \right)}.$$

After some elementary manipulations, this gives the stated objective function value and thus proves the proposition. □

5.3 Feasible solution of the primal LP

The goal of this section is to show the existence of a feasible solution of the primal LP (2.20) whose objective function value equals the stated LP optimum in Theorem 5.1.1 for the affine q-analogs $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, $\mathrm{Alt}_q(m)$, and for the ordinary q-analogs $J_q(n,m)$, ${}^2A_{2n-1}$, $\frac{1}{2}D_m$. The strategy is to first compute the inner and dual distribution of a code whose size equals the respective stated LP optimum in Theorem 5.1.1. Afterwards, we will show that these distributions are nonnegative and therefore, the inner distribution (A_i) is a feasible solution of the primal LP (2.20) such that its objective function value—the sum of the entries A_i—is precisely the stated LP optimum in Theorem 5.1.1.

This is done in Section 5.3.1 and 5.3.2 for the affine and ordinary q-analogs, respectively, where d is required to be odd for $\mathrm{Her}_q(n)$ and ${}^2A_{2n-1}$. The case d even in the latter two association schemes is treated separately in Section 5.3.3 and 5.3.4.

5.3.1 Affine q-analogs

In this subsection, we will prove the existence of a feasible solution of the primal LP (2.20) for $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$, where d is required to be odd in the case of $\mathrm{Her}_q(n)$. In all these three cases, the inner distribution of a code whose size equals (5.3) was computed in [Del78a], [DG75], and [Sch18]. Moreover, it was shown therein that a d-code of size (5.3) is an $(n-d+1)$-design. Here, we use these results to determine the corresponding dual distributions and show that both distributions are nonnegative implying that the inner distribution is a feasible solution of the primal LP (2.20). It actually suffices to look at $\mathrm{Alt}_q(m)$ with even m and odd q since for all the other cases, there are known constructions of codes in $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, and $\mathrm{Alt}_q(m)$ whose sizes equal the respective stated LP optimum (5.3), see [Del78a], [DG75], and [Sch18]. Nevertheless, we will give a proof for the nonnegativity of the inner and dual distributions without using the known constructions. The case of $\mathrm{Her}_q(n)$ with even d is handled in Section 5.3.3.

The main result of this subsection is the following proposition.

Proposition 5.3.1. *There exists a feasible solution of the primal LP* (2.20) *for d-codes with* $1 \le d \le n$ *in* $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, *and* $\mathrm{Alt}_q(m)$ *with objective function value* (5.3), *where d is required to be odd in the case of* $\mathrm{Her}_q(n)$ *and* $n = \lfloor m/2 \rfloor$ *in the case of* $\mathrm{Alt}_q(m)$.

Observe that Proposition 5.3.1 and 5.2.1 together with the Strong duality theorem 2.2.2 imply the first part of Theorem 5.1.1(b).

To prove Proposition 5.3.1, we first derive the dual distribution of a code of size (5.3) by using the inner distribution that was computed in [Del78a], [DG75], and [Sch18].

Proposition 5.3.2. *Let Y be a d-code with* $1 \le d \le n$ *in* $\mathrm{Bil}_q(n,m)$, $\mathrm{Her}_q(n)$, *or* $\mathrm{Alt}_q(m)$ *of size* (5.3), *where d is required to be odd in the case of* $\mathrm{Her}_q(n)$ *and* $n = \lfloor m/2 \rfloor$ *in the case of* $\mathrm{Alt}_q(m)$. *Then the inner distribution* (A_i) *of Y satisfies*

$$A_{n-i} = \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \begin{bmatrix} n \\ j \end{bmatrix}_b ((cb^n)^{n-d+1-j} - 1)$$

for all $i = 0, 1, \dots, n-1$, *and the dual distribution* (A'_k) *of Y satisfies*

$$A'_{n-i} = (cb^n)^{n-d+1} \sum_{j=i}^{d-2} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \begin{bmatrix} n \\ j \end{bmatrix}_b ((cb^n)^{d-1-j} - 1)$$

for all $i = 0, 1, \dots, n-1$. *In particular, Y is an* $(n-d+1)$*-design.*

Proof. Let (A_i) and (A'_k) be the inner and dual distribution of Y, respectively. Then (A_i) was determined in [Del78a, Theorem 5.6], [DG75, Theorem 4], and [Sch18, Theorem 3] and moreover, it was shown that Y is an $(n-d+1)$-design. It remains to compute (A'_k). By using (2.17), $Q_k(i) = P_k(i)$, and (5.9), we obtain

$$\begin{aligned}\sum_{k=0}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b A'_k &= \sum_{i=0}^{n} A_i \sum_{k=0}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b P_k(i) \\ &= (cb^n)^j \sum_{i=0}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b A_i\end{aligned}$$

for all $j = 0, 1, \dots, n$. We have $A_1 = A_2 = \cdots = A_{d-1} = 0$ and

$$A'_1 = A'_2 = \cdots = A'_{n-d+1} = 0.$$

Because of $A_0 = 1$, $A'_0 = |Y|$, and $\left[\begin{smallmatrix} n-i \\ j \end{smallmatrix}\right]_b = 0$ if $i \geq d$ and $j \geq n-d+2$, we obtain

$$\sum_{k=n-d+2}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b A'_k = \begin{bmatrix} n \\ j \end{bmatrix}_b ((cb^n)^j - (cb^n)^{n-d+1})$$

for all $j = n-d+2, \dots, n$. Interchanging the order of summation gives

$$\sum_{k=j}^{d-2} \begin{bmatrix} k \\ j \end{bmatrix}_b A'_{n-k} = (cb^n)^{n-d+1} \begin{bmatrix} n \\ j \end{bmatrix}_b ((cb^n)^{d-1-j} - 1)$$

for all $j = 0, 1, \dots, d-2$. Applying the q-binomial inversion formula (4.10) implies the desired expression of A'_{n-k}. □

We need the following lemma to show that both distributions (A_i) and (A'_k) are nonnegative.

Lemma 5.3.3. *Let q be an integer with $q \geq 2$ and $b = -q$.*

(*a*) *For all nonnegative integers n, i, j with $n - i \geq j + 2$, we have*

$$\frac{\left|\left[\begin{smallmatrix} n-i \\ j \end{smallmatrix}\right]_b\right|}{\left|\left[\begin{smallmatrix} n-i \\ j+2 \end{smallmatrix}\right]_b\right|} \geq q^{-2n+4j+2i+2}. \tag{5.12}$$

(*b*) *Let n and i be nonnegative integers. If $n - i \geq 1$, then we have*

$$\left|\begin{bmatrix} n-i \\ 1 \end{bmatrix}_b\right| \leq q^{n-i-1}. \tag{5.13}$$

If $n - i \geq 2$, then we have

$$\left|\begin{bmatrix} n-i \\ 2 \end{bmatrix}_b\right| \leq \frac{1}{3} q^{2n-2i-2}. \tag{5.14}$$

Proof. (a) We have

$$\frac{\left|\left[\begin{smallmatrix} n-i \\ j \end{smallmatrix}\right]_b\right|}{\left|\left[\begin{smallmatrix} n-i \\ j+2 \end{smallmatrix}\right]_b\right|} = \left|\frac{((-q)^{j+2}-1)((-q)^{j+1}-1)}{((-q)^{n-i-j}-1)((-q)^{n-i-j-1}-1)}\right|$$
$$\geq \frac{(q^{j+2}+1)\,(q^{j+1}-1)}{(q^{n-i-j}-1)\,(q^{n-i-j-1}+1)}.$$

This gives

$$\frac{\left|\left[\begin{smallmatrix} n-i \\ j \end{smallmatrix}\right]_b\right|}{\left|\left[\begin{smallmatrix} n-i \\ j+2 \end{smallmatrix}\right]_b\right|} \geq q^{-2n+4j+2i+2},$$

as stated.

(b) For $n-i \geq 1$, by using (5.11), we have

$$\left|\begin{bmatrix} n-i \\ 1 \end{bmatrix}_b\right| = \frac{|(-q)^{n-i}-1|}{q+1} \leq \frac{q^{n-i}+1}{q+1} \leq q^{n-i-1}.$$

For $n-i \geq 2$, by again using (5.11), we obtain

$$\left|\begin{bmatrix} n-i \\ 2 \end{bmatrix}_b\right| \leq \frac{(q^{n-i}-1)(q^{n-i-1}+1)}{(q+1)(q^2-1)} \leq \frac{1}{3}q^{2n-2i-2},$$

as wanted.

□

We can now prove Proposition 5.3.1.

Proof of Proposition 5.3.1. Let $1 \leq d \leq n$. For $d = 1$, the set of all matrices in the respective affine scheme is a 1-code and thus, there exists a feasible solution of the primal LP with the required objective function value.

Assume now that $d \geq 2$. Let (A_i) and (A'_k) be given in Proposition 5.3.2. We will show that all entries of (A_i) and (A'_k) are nonnegative, which implies that (A_i) is a feasible solution of the primal LP (2.20). First, we rewrite A_{n-i} by applying (3.7) and interchanging the order of summation and obtain

$$A_{n-i} = \begin{bmatrix} n \\ i \end{bmatrix}_b \sum_{j=0}^{n-d-i} (-1)^j b^{\binom{j}{2}} \begin{bmatrix} n-i \\ j \end{bmatrix}_b ((cb^n)^{n-d+1-j-i}-1)$$

for all $i = 0, 1, \ldots, n-d$.

We start with $\mathrm{Bil}_q(n,m)$ and $\mathrm{Alt}_q(m)$. Observe that it suffices to show that the inner distribution is nonnegative because by taking $n-d+2$ instead of d, the dual distribution becomes a positive multiple of the inner distribution. For $n = d$, we immediately have $A_n \geq 0$. Assume now that $2 \leq d < n$. Write

$$a_{i,j} = b^{\binom{j}{2}} \begin{bmatrix} n-i \\ j \end{bmatrix}_b ((cb^n)^{n-d+1-j-i}-1)$$

for all $i = 0, 1, \ldots, n-d$ and $j = 0, 1, \ldots, n-d-i$. Take $i \in \{0, 1, \ldots, n-d\}$. For all $j = 0, 1, \ldots, n-d-i-1$, we have

$$\begin{aligned}\frac{a_{i,j}}{a_{i,j+1}} &= \frac{b^{\binom{j}{2}}\left[\begin{smallmatrix}n-i\\ j\end{smallmatrix}\right]_b((cb^n)^{n-d+1-j-i}-1)}{b^{\binom{j+1}{2}}\left[\begin{smallmatrix}n-i\\ j+1\end{smallmatrix}\right]_b((cb^n)^{n-d-j-i}-1)}\\ &= b^{-j}\frac{(b^{j+1}-1)}{(b^{n-i-j}-1)}\frac{((cb^n)^{n-d+1-j-i}-1)}{((cb^n)^{n-d-j-i}-1)}.\end{aligned}$$

From (5.11) we find

$$\frac{a_{i,j}}{a_{i,j+1}} > cb^{i+j} \geq 1$$

for all $i, j \geq 0$ except for $\mathrm{Alt}_q(m)$ with even m and $(i,j) = (0,0)$, where we have

$$\frac{a_{0,0}}{a_{0,1}} = \frac{(q^2-1)(q^{(2n-1)(n-d+1)}-1)}{(q^{2n}-1)(q^{(2n-1)(n-d)}-1)} \geq \frac{q^2-1}{q} > 1.$$

This completes the proof for $\mathrm{Bil}_q(n,m)$ and $\mathrm{Alt}_q(m)$.

Now, consider $\mathrm{Her}_q(n)$ with odd $d \geq 3$. Write

$$a_{i,j} = (-1)^j b^{\binom{j}{2}}\begin{bmatrix}n\\ i\end{bmatrix}_b\begin{bmatrix}n-i\\ j\end{bmatrix}_b((cb^n)^{m-j-i}-1)$$

with $m \in \{n-d+1, d-1\}$ for all $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, m-i-1$. The sign of $a_{i,j}$ is $(-1)^{\binom{j}{2}+ij+j}$ since $(-1)^{(n+1)m} = 1$. Therefore, we have $a_{i,2j} \geq 0$ if $j \geq 0$ is even, and $a_{i,2j+1} \geq 0$ if $j + i \equiv 1 \pmod 2$. Thus, in what follows, we look at $a_{i,2j} = |a_{i,2j}|$ and $a_{i,2j+1} = |a_{i,2j+1}|$ in the respective cases. For $n = d$, we immediately have $A_n \geq 0$. Henceforth, assume that $3 \leq d < n$. We will show that

$$\begin{aligned} a_{i,0} &\geq |a_{i,2}| \text{ for all odd } i \geq 1\\ a_{i,0} &\geq |a_{i,1}| + |a_{i,2}| \text{ for all even } i \geq 0,\\ a_{i,2j} &\geq |a_{i,2j+2}| \text{ for all } i \geq 0 \text{ and all even } j \geq 2\\ a_{i,2j+1} &\geq |a_{i,2j+3}| \text{ for all } i, j \geq 0 \text{ with } i + j \equiv 1 \pmod 2.\end{aligned} \tag{5.15}$$

Observe that the nonnegativity of the inner and dual distribution follows by showing that $(a_{i,j})$ satisfies (5.15).

Take $i \in \{0, 1, \ldots, m-1\}$. For all $j = 0, 1, \ldots, m-i-3$, by using (5.12), we have

$$\frac{|a_{i,j}|}{|a_{i,j+2}|} = \left|\frac{b^{\binom{j}{2}}\left[\begin{smallmatrix}n-i\\ j\end{smallmatrix}\right]_b((cb^n)^{m-j-i}-1)}{b^{\binom{j+2}{2}}\left[\begin{smallmatrix}n-i\\ j+2\end{smallmatrix}\right]_b((cb^n)^{m-j-i-2}-1)}\right| \geq q^{2j+2i+1}\frac{1-q^{-n(m-j-i)}}{1+q^{-n(m-j-i-2)}}.$$

Since $n \geq 4$ and $3 \leq m-j-i \leq m$, we obtain

$$\frac{|a_{i,j}|}{|a_{i,j+2}|} < \frac{47}{50}q^{2j+2i+1} > 1$$

for all i, j. It remains to show that

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} \geq 1$$

if i is even. Let i be an even number with $0 \leq i \leq m-2$. We have

$$\begin{aligned} \frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} &= \frac{|(cb^n)^{m-i} - 1|}{\left|\left[{n-i \atop 1}\right]_b((cb^n)^{m-i-1} - 1)\right| + q\left|\left[{n-i \atop 2}\right]_b((cb^n)^{m-i-2} - 1)\right|} \\ &\geq \frac{q^{n(m-i)} - 1}{q^{n-i-1}(q^{n(m-i-1)} + 1) + \frac{1}{3}q^{2n-2i-1}(q^{n(m-i-2)} + 1)}, \end{aligned}$$

by using (5.13) and (5.14). This becomes

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} \geq \frac{1 - q^{-n(m-i)}}{q^{-i-1}(1 + q^{-n(m-i-1)}) + \frac{1}{3}q^{-2i-1}(1 + q^{-n(m-i-2)})}.$$

For $n \geq 4$, we obtain

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} > 1$$

if $i \leq m-4$, and if $i = m-2$ (where $a_{i,2}$ cannot occur), we have

$$\frac{|a_{i,0}|}{|a_{i,1}|} > 1.$$

This completes the proof. □

5.3.2 Ordinary q-analogs

The goal of this subsection is to prove the following proposition.

Proposition 5.3.4. *Let X be the set of n-spaces in $J_q(n,m)$ or generators in ${}^2A_{2n-1}$ or $\frac{1}{2}D_m$, where $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$. Then there exists a feasible solution of the primal LP* (2.20) *for d-codes in $J_q(n,m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ with objective function value* (5.1) *for $1 \leq d \leq n$, where d is required to be odd in the case of ${}^2A_{2n-1}$.*

Observe that Proposition 5.3.4 and 5.2.3 together with the Strong duality theorem 2.2.2 imply the first part of Theorem 5.1.1(a).

To prove Proposition 5.3.4, we start with the derivation of the inner distribution of a code whose size equals the stated LP optimum (5.1).

Proposition 5.3.5. *Assume that Y is a d-code with $1 \leq d \leq n$ in $J_q(n,m)$, ${}^2A_{2n-1}$, or $\frac{1}{2}D_m$ of size* (5.1), *where d is required to be odd in the case of ${}^2A_{2n-1}$ and $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$. Then Y is an $(n-d+1)$-design, where the ordering of the dual distribution is imposed by* (2.53)*—in particular, the second ordering is taken for ${}^2A_{2n-1}$. Moreover, the inner distribution (A_i) of Y satisfies*

$$A_{n-i} = \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \begin{bmatrix} n \\ j \end{bmatrix}_b \left(\frac{(qcb^n)_{n-j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{n-j}} - 1 \right) \tag{5.16}$$

for all $i = 0, 1, \ldots, n-1$.

Before we prove this proposition, we rephrase the design property obtained therein in the case of ${}^2A_{2n-1}$ with respect to the standard ordering. Namely, if Y is a d-code in ${}^2A_{2n-1}$ for odd d whose size equals (5.1), then its dual distribution with respect to the standard ordering satisfies

$$A'_i = A'_{n-i+1} = 0 \quad \text{for all } i = 1, 2, \dots, \tfrac{n-d+1}{2}$$

if $n - d + 1$ is even, and

$$A'_i = A'_{n-i+1} = 0,\ A'_{n-\frac{n-d}{2}} = 0 \quad \text{for all } i = 1, 2, \dots, \tfrac{n-d}{2}$$

if $n - d + 1$ is odd.

Proof of Proposition 5.3.5. Let X be the set of n-spaces in $J_q(n, m)$ or generators in ${}^2A_{2n-1}$ or $\frac{1}{2}D_m$. Then we have

$$|Y| = \frac{|X|(q)_{d-1}}{(qcb^n)_{d-1}}. \tag{5.17}$$

Let (A_i) and (A'_k) denote the inner and dual distribution of Y, respectively, in terms of the orderings imposed by (2.52) and (2.53). Recall the formulae (3.13) and (3.14) from the proof of Theorem 3.2.1, namely we have

$$\sum_{k=0}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} A'_k = |X| \sum_{i=0}^{n} A_i \begin{bmatrix} n-i \\ j \end{bmatrix}_b \tag{5.18}$$

for all $j = 0, 1, \dots, n$ and in particular,

$$\sum_{k=1}^{n-d+1} b^{k(d-1)} \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \frac{(qcb^{n-k})_{d-1}}{(q)_{d-1}} A'_k = \begin{bmatrix} n \\ d-1 \end{bmatrix}_b \left(|X| - \frac{(qcb^n)_{d-1}}{(q)_{d-1}} |Y| \right), \tag{5.19}$$

where all coefficients of A'_k on the left-hand side of (5.19) are nonnegative. Observe that the bracket on the right-hand side of (5.19) equals zero because of (5.17). Therefore, we have $A'_1 = A'_2 = \cdots = A'_{n-d+1} = 0$, which means that Y is an $(n - d + 1)$-design.

By using $A'_0 = |Y|$ and (5.17), we then find from (5.18) that the inner distribution is determined by the equations

$$\begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(qcb^n)_{n-j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{n-j}} = \begin{bmatrix} n \\ j \end{bmatrix}_b + \sum_{i=d}^{n} \begin{bmatrix} n-i \\ j \end{bmatrix}_b A_i$$

for all $j = 0, 1, \dots, n - d$. This can be rewritten as

$$\sum_{k=0}^{n-d} \begin{bmatrix} k \\ j \end{bmatrix}_q A_{n-k} = \begin{bmatrix} n \\ j \end{bmatrix}_q \left(\frac{(qcb^n)_{n-j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{n-j}} - 1 \right) \tag{5.20}$$

for all $j = 0, 1, \dots, n - d$. Applying the q-binomial inversion formula (4.10) gives the desired expression of the inner distribution. □

The derivation of the dual distribution requires the following lemma. To simplify the notation, we set $a = q^{-1}c^{-1}b^{-2n}$.

Lemma 5.3.6. *Let C and Q be the $n \times n$ matrices defined by $Q = (Q'_k(i))_{k,i}$ and $C = (c_{ji})_{j,i}$ with $Q'_k(i)$ as in (2.53) and $c_{j,i} = \left[\begin{smallmatrix} n-i \\ j \end{smallmatrix}\right]_b$ for all $i, j, k = 0, 1, \ldots, n$. Then C is invertible and the product QC^{-1} is given by*

$$(QC^{-1})_{k,j} = \mu'_k a^k b^{k^2+\binom{j}{2}}(-bc)^j \frac{(b^{-k})_j (ab^k)_j (qb^{n-k})_k}{(b^{-n})_j (q^{-1}b^{1-n})_j (c^{-1}b^{-n})_k} \tag{5.21}$$

for all $k, j = 0, 1, \ldots, n$ with μ'_k given in Table 2.3.

To prove Lemma 5.3.6, we need an extension of the q-Pochhammer symbol for negative subscripts as follows

$$(a;q)_{-k} = \prod_{i=1}^{k}(1 - aq^{-i})^{-1}$$

for $k = 1, 2, 3, \ldots$ if $a \neq q, q^2, \ldots, q^k$.

Proof of Lemma 5.3.6. The inverse of C is given by

$$(C^{-1})_{i,j} = (-1)^{i+j-n} b^{\binom{i+j-n}{2}} \begin{bmatrix} j \\ n-i \end{bmatrix}_b \tag{5.22}$$

for all $i, j = 0, 1, \ldots, n$ since the q-binomial inversion formula (5.10) implies

$$\sum_{i=0}^{n} \begin{bmatrix} n-i \\ k \end{bmatrix}_b (-1)^{i+j-n} b^{\binom{i+j-n}{2}} \begin{bmatrix} j \\ n-i \end{bmatrix}_b = \delta_{k,j}.$$

Let $k, j \in \{0, 1, \ldots, n\}$. By substituting (5.22) and (2.53), we have

$$\begin{aligned}(QC^{-1})_{k,j} &= \sum_{i=0}^{n} Q'_k(i)(C^{-1})_{i,j} \\ &= \sum_{i=0}^{n} \mu'_k \, {}_3\phi_2\left(\begin{matrix} b^{-i}, b^{-k}, ab^k \\ b^{-n}, c^{-1}b^{-n} \end{matrix} \middle| b; b\right) (-1)^{i+j-n} b^{\binom{i+j-n}{2}} \begin{bmatrix} j \\ n-i \end{bmatrix}_b.\end{aligned}$$

Use the definition of the q-hypergeometric function and (2.30) to obtain

$$\begin{aligned}(QC^{-1})_{k,j} &= \mu'_k \sum_{i,\ell \geq 0} \frac{(b^{-i})_\ell (b^{-k})_\ell (ab^k)_\ell}{(b^{-n})_\ell (c^{-1}b^{-n})_\ell (b)_\ell} (-1)^{i+j-n} b^{\binom{i+j-n}{2}+\ell} \begin{bmatrix} j \\ n-i \end{bmatrix}_b \\ &= \mu'_k \sum_{\ell \geq 0} \frac{(b^{-k})_\ell (ab^k)_\ell}{(b^{-n})_\ell (c^{-1}b^{-n})_\ell} (-1)^\ell b^{\binom{\ell}{2}+\ell} S_\ell, \end{aligned} \tag{5.23}$$

where

$$S_\ell = \sum_{i=0}^{n} (-1)^{i+j-n} b^{\binom{i+j-n}{2}-i\ell} \begin{bmatrix} i \\ \ell \end{bmatrix}_b \begin{bmatrix} j \\ n-i \end{bmatrix}_b.$$

Interchanging the order of summation gives

$$S_\ell = b^{-n\ell} \sum_{i=0}^{n} (-1)^{j-i} b^{\binom{j-i}{2}+i\ell} \begin{bmatrix} n-i \\ \ell \end{bmatrix}_b \begin{bmatrix} j \\ i \end{bmatrix}_b.$$

To compute this sum, we use the q-Chu-Vandermonde identity

$$\begin{bmatrix} x+y \\ z \end{bmatrix}_b = \sum_{i=0}^{x} b^{i(y-z+i)} \begin{bmatrix} x \\ i \end{bmatrix}_b \begin{bmatrix} y \\ z-i \end{bmatrix}_b \tag{5.24}$$

(see [GJ83, § 2, 2.6.3(c)], for example). Applying the q-inversion formula (5.10) to (5.24) reveals that

$$\sum_{k=0}^{x} (-1)^k b^{\binom{k}{2}} \begin{bmatrix} x \\ k \end{bmatrix}_b \begin{bmatrix} x-k+y \\ z \end{bmatrix}_b = b^{x(y-z+x)} \begin{bmatrix} y \\ z-x \end{bmatrix}_b .$$

Put $x = i$, $y = n - i$, and $z = n - \ell$ to obtain

$$\sum_{k=0}^{i} (-1)^k b^{\binom{k}{2}} \begin{bmatrix} i \\ k \end{bmatrix}_b \begin{bmatrix} n-k \\ n-\ell \end{bmatrix}_b = b^{i\ell} \begin{bmatrix} n-i \\ \ell \end{bmatrix}_b ,$$

which, by again applying the q-binomial inversion formula (5.10), gives

$$\sum_{i=0}^{j} (-1)^{j-i} b^{\binom{j-i}{2}+i\ell} \begin{bmatrix} n-i \\ \ell \end{bmatrix}_b \begin{bmatrix} j \\ i \end{bmatrix}_b = (-1)^j b^{\binom{j}{2}} \begin{bmatrix} n-j \\ n-\ell \end{bmatrix}_b .$$

Therefore, we have

$$S_\ell = (-1)^j b^{\binom{j}{2}-n\ell} \begin{bmatrix} n-j \\ n-\ell \end{bmatrix}_b .$$

Substitute into (5.23) to obtain

$$(QC^{-1})_{k,j} = \mu'_k (-1)^j b^{\binom{j}{2}} \sum_{\ell \ge 0} \frac{(b^{-k})_\ell (ab^k)_\ell}{(b^{-n})_\ell (c^{-1}b^{-n})_\ell} (-1)^\ell b^{\binom{\ell}{2}-(n-1)\ell} \begin{bmatrix} n-j \\ n-\ell \end{bmatrix}_b .$$

Interchanging the order of summation gives

$$(QC^{-1})_{k,j} = \mu'_k b^{j^2-nj} \sum_{\ell=0}^{n} \frac{(b^{-k})_{\ell+j} (ab^k)_{\ell+j}}{(b^{-n})_{\ell+j} (c^{-1}b^{-n})_{\ell+j}} (-1)^\ell b^{\binom{\ell}{2}-(n-j-1)\ell} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_b .$$

By (2.30), this becomes

$$(QC^{-1})_{k,j} = \mu'_k b^{j^2-nj} \sum_{\ell=0}^{n} \frac{(b^{-k})_{\ell+j} (ab^k)_{\ell+j} (b^{-(n-j)})_\ell}{(b^{-n})_{\ell+j} (c^{-1}b^{-n})_{\ell+j} (b)_\ell} b^\ell .$$

Applying (3.5) gives

$$\begin{aligned} (QC^{-1})_{k,j} &= \mu'_k b^{j^2-nj} \frac{(b^{-k})_j (ab^k)_j}{(b^{-n})_j (c^{-1}b^{-n})_j} \sum_{\ell=0}^{n} \frac{(b^{-k+j})_\ell (ab^{k+j})_\ell}{(c^{-1}b^{-n+j})_\ell (b)_\ell} b^\ell \\ &= \mu'_k b^{j^2-nj} \frac{(b^{-k})_j (ab^k)_j}{(b^{-n})_j (c^{-1}b^{-n})_j} \, {}_2\phi_1 \left(\begin{matrix} b^{-(k-j)}, ab^{k+j} \\ c^{-1}b^{-n+j} \end{matrix} \,\middle|\, b; b \right) . \end{aligned}$$

Using the q-Chu-Vandermonde identity of the form

$${}_2\phi_1 \left(\begin{matrix} b^{-i}, x \\ y \end{matrix} \,\middle|\, b; b \right) = \frac{(x^{-1}y)_i}{(y)_i} \, x^i$$

(see [KLS10, Eq. (1.11.5)], for example) implies

$$(QC^{-1})_{k,j} = \mu'_k a^{k-j} b^{k^2-nj} \frac{(b^{-k})_j (ab^k)_j (a^{-1}c^{-1}b^{-n-k})_{k-j}}{(b^{-n})_j (c^{-1}b^{-n})_j (c^{-1}b^{-n+j})_{k-j}}.$$

From (3.5) we have $(c^{-1}b^{-n+j})_{k-j} = (c^{-1}b^{-n})_k/(c^{-1}b^{-n})_j$ and from (3.6) we find

$$(a^{-1}c^{-1}b^{-n-k})_{k-j} = \frac{(a^{-1}c^{-1}b^{-n-k})_k}{(acb^{n+1})_j} (-a^{-1}c^{-1}b^{-n-k})^{-j} b^{\binom{j}{2}-kj+j}.$$

Substituting these and using $a = q^{-1}c^{-1}b^{-2n}$ give the required value of $(QC^{-1})_{k,j}$. □

We are now in position to derive the dual distribution.

Proposition 5.3.7. *Let Y be a d-code with $1 \le d \le n$ in $J_q(n,m)$, ${}^2A_{2n-1}$, or $\frac{1}{2}D_m$ of size* (5.1)*, where d is required to be odd in the case of ${}^2A_{2n-1}$ and $n = \lfloor m/2 \rfloor$ in the case of $\frac{1}{2}D_m$. Let (A'_k) be the dual distribution of Y in terms of the ordering imposed by* (2.53)*—in particular, the second ordering is taken for ${}^2A_{2n-1}$. Then we have*

$$A'_{n-k} = c_k \sum_{j=0}^{d-2-k} (-1)^j b^{\binom{n-k-j}{2}} \frac{(qb^k)_j}{(qcb^{2k+1})_j} \begin{bmatrix} n-k \\ j \end{bmatrix}_b \left(1 - \frac{(qb^{k+j})_{d-k-j-1}}{(qcb^{n+k+j})_{d-k-j-1}}\right) \tag{5.25}$$

for all $k = 0,1,\dots,n-1$, where

$$c_k = \mu'_{n-k}(-q^{-1}b^{-n+1})^{n-k} \frac{(qb^k)_{n-k}(ab^{n-k})_{n-k}}{(c^{-1}b^{-n})_{n-k}(q^{-1}b^{1-n})_{n-k}}$$

with μ'_{n-k} given in Table 2.3.

Proof. Similarly to the derivation of the inner distribution, we solve a system of linear equations. Because of (5.18) and $A_0 = 1$, $A_1 = \cdots = A_{d-1} = 0$, we have

$$\sum_{k=0}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} A'_k = |X| \left(\begin{bmatrix} n \\ j \end{bmatrix}_b + \sum_{i=d}^{n} A_i \begin{bmatrix} n-i \\ j \end{bmatrix}_b \right)$$

for all $j = 0,1,\dots,n$. Since $\left[\begin{smallmatrix} n-i \\ j \end{smallmatrix}\right]_b = 0$ if $i \ge d$ and $j \ge n-d+1$, we see that

$$\sum_{k=0}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} A'_k = |X| \begin{bmatrix} n \\ j \end{bmatrix}_b \tag{5.26}$$

for all $j = n-d+1,\dots,n$. Recall from Proposition 5.3.5 that Y is an $(n-d+1)$-design. Therefore, we have $A'_1 = A'_2 = \cdots = A'_{n-d+1} = 0$. Because of $A'_0 = |Y|$, we obtain

$$\sum_{k=n-d+2}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}} A'_k = |X| \begin{bmatrix} n \\ j \end{bmatrix}_b \left(1 - \frac{(qcb^n)_{n-j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{n-j}}\right) \tag{5.27}$$

for all $j = n-d+2,\dots,n$. In the case of the inner distribution, we simply applied the q-binomial inversion formula to solve the system (5.20). Here, we need a slightly different approach. Recall that the Q-numbers are determined by (3.2). Define the $n \times n$ matrices $Q = (Q'_k(i))_{k,i}$, $B = (b_{jk})_{j,k}$, and $C = (c_{ji})_{j,i}$ by

$$b_{jk} = b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(qcb^{n-k})_{n-j}}{(q)_{n-j}}, \quad c_{ji} = \begin{bmatrix} n-i \\ j \end{bmatrix}_b$$

for all $i,j,k = 0,1,\dots,n$. Then we can write (3.2) as $BQ = |X|C$, which implies that the inverse of B is determined by $B^{-1} = \frac{1}{|X|}QC^{-1}$. Multiplication of (5.27) with B^{-1} gives

$$A'_k = \sum_{j=n-d+2}^{n} (QC^{-1})_{k,j} \begin{bmatrix} n \\ j \end{bmatrix}_b \left(1 - \frac{(qcb^n)_{n-j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{n-j}}\right)$$

for all $k = n-d+2,\dots,n$. Substituting (5.21) and using (2.30) imply

$$A'_k = \mu'_k a^k b^{k^2} \frac{(qb^{n-k})_k}{(c^{-1}b^{-n})_k} \sum_{j=n-d+2}^{n} (b^{n+1}c)^j \frac{(b^{-k})_j (ab^k)_j}{(b)_j (q^{-1}b^{1-n})_j} \times \left(1 - \frac{(qcb^n)_{n-j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{n-j}}\right).$$

Using (2.30) and interchanging the order of summation give

$$A'_k = \mu'_k a^k \frac{(qb^{n-k})_k}{(c^{-1}b^{-n})_k} \sum_{j=0}^{k-n+d-2} (-b^{n+1}c)^{k-j} b^{\binom{k-j}{2}+kj} \frac{(ab^k)_{k-j}}{(q^{-1}b^{1-n})_{k-j}} \begin{bmatrix} k \\ j \end{bmatrix}_b \times \left(1 - \frac{(qcb^n)_{n-k+j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{n-k+j}}\right).$$

Apply (3.6) and $a = q^{-1}c^{-1}b^{-2n}$ to obtain

$$A'_{n-k} = c_k \sum_{j=0}^{d-2-k} (-1)^j b^{\binom{n-k-j}{2}} \frac{(qb^k)_j}{(qcb^{2k+1})_j} \begin{bmatrix} n-k \\ j \end{bmatrix}_b \left(1 - \frac{(qcb^n)_{k+j}(q)_{d-1}}{(qcb^n)_{d-1}(q)_{k+j}}\right)$$

with c_k as stated in the proposition. The desired expression of A'_{n-k} now follows by using (3.5). $\square$

It remains to show that both distributions (A_i) and (A'_k) are nonnegative. This requires the following lemma.

Lemma 5.3.8. *Let n and q be integers with $n \geq 1$ and $q \geq 2$. Then we have*

$$\prod_{i=1}^{n} \left(1 - \frac{1}{q^i}\right) \geq \frac{1}{4}. \tag{5.28}$$

Proof. Use $1 - x \geq 4^{-x}$ for all $x \in [0, \frac{1}{2}]$ to obtain

$$\prod_{i=1}^{n} \left(1 - \frac{1}{q^i}\right) \geq \prod_{i=1}^{n} \left(1 - \frac{1}{2^i}\right) \geq \prod_{i=1}^{n} 4^{-1/2^i} \geq 4^{-\sum_{i=1}^{\infty} 2^{-i}} = \frac{1}{4},$$

as stated. $\square$

We start with the nonnegativity of the inner distribution.

Proposition 5.3.9. *For* $1 < d \le n$*, all entries of the inner distribution* (A_i) *given in* (5.16) *are nonnegative.*

Proof. Let (A_i) be given in (5.16). By using (3.5) and (3.7), and interchanging the order of summation, we have

$$A_{n-i} = \sum_{j=0}^{n-d-i} (-1)^j b^{\binom{j}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_b \begin{bmatrix} n-i \\ j \end{bmatrix}_b \left(\frac{(qcb^{n+d-1})_{n-i-j-d+1}}{(qb^{d-1})_{n-i-j-d+1}} - 1 \right)$$

for all $i = 0, 1, \dots, n-d$. Set

$$a_{i,j} = (-1)^j b^{\binom{j}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_b \begin{bmatrix} n-i \\ j \end{bmatrix}_b \left(\frac{(qcb^{n+d-1})_{n-i-j-d+1}}{(qb^{d-1})_{n-i-j-d+1}} - 1 \right)$$

for all $i = 0, 1, \dots, n-d$ and $j = 0, 1, \dots, n-d-i$.

We begin with $J_q(n,m)$ and $\frac{1}{2}D_m$. Observe that $A_n \ge 0$ if $n = d$. Assume now that $n > d$. We will show that the sequence $(|a_{i,j}|)_j$ is decreasing for all $i = 0, 1, \dots, n-d$, which implies $A_{n-i} \ge 0$. Take $i \in \{0, 1, \dots, n-d\}$. For all $j = 0, 1, \dots, n-d-i-1$, we have

$$\begin{aligned}
\frac{|a_{i,j}|}{|a_{i,j+1}|} &= \frac{b^{\binom{j}{2}} \begin{bmatrix} n-i \\ j \end{bmatrix}_b \left(\frac{(qcb^{n+d-1})_{n-i-j-d+1}}{(qb^{d-1})_{n-i-j-d+1}} - 1 \right)}{b^{\binom{j+1}{2}} \begin{bmatrix} n-i \\ j+1 \end{bmatrix}_b \left(\frac{(qcb^{n+d-1})_{n-i-j-d}}{(qb^{d-1})_{n-i-j-d}} - 1 \right)} \\
&= b^{-j} \, \frac{(b^{j+1}-1)}{(b^{n-i-j}-1)} \, \frac{\left(\frac{(qcb^{2n-i-j-1}-1)}{(qb^{n-i-j-1}-1)} \, \frac{(qcb^{n+d-1})_{n-i-j-d}}{(qb^{d-1})_{n-i-j-d}} - 1 \right)}{\left(\frac{(qcb^{n+d-1})_{n-i-j-d}}{(qb^{d-1})_{n-i-j-d}} - 1 \right)}.
\end{aligned}$$

Since $(qcb^{2n-i-j-1}-1)/(qb^{n-i-j-1}-1) > 1$, we can apply (5.11) to obtain

$$\frac{|a_{i,j}|}{|a_{i,j+1}|} > b^{-j} \, \frac{(b^{j+1}-1)(qcb^{2n-i-j-1}-1)}{(b^{n-i-j}-1)(qb^{n-i-j-1}-1)}.$$

Then again from (5.11) we find

$$\frac{|a_{i,j}|}{|a_{i,j+1}|} > cb^{i+j+1} \left(1 - \frac{1}{b^{j+1}} \right) \ge cb \left(1 - \frac{1}{b} \right) \ge 1$$

for all $i, j \ge 0$, as required.

Now, consider ${}^2A_{2n-1}$ with odd $d \ge 3$. Write

$$a'_{i,j} = (-1)^j b^{\binom{j}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_b \begin{bmatrix} n-i \\ j \end{bmatrix}_b \quad \text{and} \quad \varepsilon_{i,j} = \frac{(b^{n+d})_{n-i-j-d+1}}{(-b^d)_{n-i-j-d+1}},$$

so that $a_{i,j} = a'_{i,j}(\varepsilon_{i,j} - 1)$. Observe that the signs of $a'_{i,j}$ and $\varepsilon_{i,j}$ are $(-1)^{\binom{j}{2}+(i+j)(n-i)}$ and $(-1)^{(n+1)(i+j)}$, respectively, which implies

$$\operatorname{sign}(a_{i,j}) = (-1)^{\binom{j}{2}+ij+j}.$$

For all $i \geq 0$, we have $a_{i,2j} \geq 0$ for all even $j \geq 0$ and $a_{i,2j+1} \geq 0$ for all j with $j + i \equiv 1 \pmod 2$. Hence in what follows, we look at $a_{i,2j} = |a_{i,2j}|$ and $a_{i,2j+1} = |a_{i,2j+1}|$ in the respective cases. If $n = d$, then we immediately obtain $A_n \geq 0$. Assume now that $n > d$. We will show that $a_{i,j}$ satisfies (5.15). Observe that this will prove the nonnegativity of the inner distribution.

Take $i \in \{0, 1, \dots, n-d\}$. For all $j = 0, 1, \dots, n-d-i-2$, use (5.12) to obtain

$$\frac{|a'_{i,j}|}{|a'_{i,j+2}|} = \frac{q^{\binom{j}{2}} \left|\left[{n-i \atop j}\right]_b\right|}{q^{\binom{j+2}{2}} \left|\left[{n-i \atop j+2}\right]_b\right|} \geq q^{-2n+2j+2i+1}. \tag{5.29}$$

For all $j = 0, 1, \dots, n-d-i$, we have

$$\begin{aligned} |\varepsilon_{i,j}| &= \prod_{\ell=0}^{n-j-i-d} \frac{q^{n+d+\ell} - (-1)^{n+d+\ell}}{q^{d+\ell} - (-1)^{\ell}} \\ &\geq \prod_{\ell=0}^{n-j-i-d} \frac{q^{n+d+\ell} - 1}{q^{d+\ell} + 1} \\ &\geq q^{(n-2)(n-j-i-d+1)} \geq 4 \end{aligned}$$

because of $i + j \leq n - d$ and $3 \leq d < n$. Since $\varepsilon_{i,j}$ and $\varepsilon_{i,j+2}$ have the same sign, we thus either have

$$\frac{|\varepsilon_{i,j} - 1|}{|\varepsilon_{i,j+2} - 1|} = \frac{|\varepsilon_{i,j}| + 1}{|\varepsilon_{i,j+2}| + 1} \quad \text{or} \quad \frac{|\varepsilon_{i,j} - 1|}{|\varepsilon_{i,j+2} - 1|} = \frac{\varepsilon_{i,j} - 1}{\varepsilon_{i,j+2} - 1}.$$

Because of $|\varepsilon_{i,j}| \geq |\varepsilon_{i,j+2}|$, we find in both cases that

$$\frac{|\varepsilon_{i,j} - 1|}{|\varepsilon_{i,j+2} - 1|} \geq \frac{|\varepsilon_{i,j}|}{2|\varepsilon_{i,j+2}|} = \frac{1}{2} \frac{(q^{2n-j-i} - (-1)^{j-i})(q^{2n-j-i-1} - (-1)^{j-i-1})}{(q^{n-j-i} - (-1)^{n-j-i-1})(q^{n-j-i-1} - (-1)^{n-j-i})},$$

from which we obtain

$$\frac{|\varepsilon_{i,j} - 1|}{|\varepsilon_{i,j+2} - 1|} \geq \frac{1}{2} \frac{(q^{2n-j-i} + 1)(q^{2n-j-i-1} - 1)}{(q^{n-j-i} - 1)(q^{n-j-i-1} + 1)} \geq \frac{1}{2} q^{2n-1}$$

by using (5.11). Combining this with (5.29) gives

$$\frac{|a_{i,j}|}{|a_{i,j+2}|} \geq \frac{1}{2} q^{2j+2i} \geq 2$$

for all $(i, j) \neq (0, 0)$.

It remains to prove that

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} \geq 1$$

for all even i. Let i be an even number in $\{0, 1, \dots, n-d-1\}$. Using (5.13) and (5.14) gives

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} = \frac{(\varepsilon_{i,0} - 1)}{\left|\left[{n-i \atop 1}\right]_b\right| |\varepsilon_{i,1} - 1| + q \left|\left[{n-i \atop 2}\right]_b\right| (\varepsilon_{i,2} - 1)} \geq \frac{\left(1 - \frac{1}{\varepsilon_{i,0}}\right)}{D}$$

with
$$D = q^{n-i-1}\left(\left|\frac{\varepsilon_{i,1}}{\varepsilon_{i,0}}\right| + \frac{1}{\varepsilon_{i,0}}\right) + \frac{1}{3}q^{2n-2i-1}\left(\frac{\varepsilon_{i,2}}{\varepsilon_{i,0}} - \frac{1}{\varepsilon_{i,0}}\right).$$
We have
$$\left|\frac{\varepsilon_{i,1}}{\varepsilon_{i,0}}\right| = \left|\frac{b^{n-i}-1}{b^{2n-i}-1}\right| \le \frac{q^{n-i}+1}{q^{2n-i}-1}$$
and
$$\frac{\varepsilon_{i,2}}{\varepsilon_{i,0}} = \frac{(b^{n-i}-1)(b^{n-i-1}-1)}{(b^{2n-i}-1)(b^{2n-i-1}-1)} \le q^{-n+1}\frac{(q^{n-i-1}+1)}{(q^{2n-i}-1)}.$$
We thus obtain
$$D \le q^{-i-1}\frac{\left(1+\frac{1}{q^{n-i}}\right)}{\left(1-\frac{1}{q^{2n-i}}\right)} + \frac{1}{3}q^{-2i-1}\frac{\left(1+\frac{1}{q^{n-i-1}}\right)}{\left(1-\frac{1}{q^{2n-i}}\right)} - \frac{1}{\varepsilon_{i,0}}\left(\frac{1}{3}q^{2n-2i-1} - q^{n-i-1}\right).$$
From $2n-i \ge 7$, $n-i \ge 3$, and $i \ge 0$, we see that
$$q^{-i-1}\frac{\left(1+\frac{1}{q^{n-i}}\right)}{\left(1-\frac{1}{q^{2n-i}}\right)} + \frac{1}{3}q^{-2i-1}\frac{\left(1+\frac{1}{q^{n-i-1}}\right)}{\left(1-\frac{1}{q^{2n-i}}\right)} \le \frac{296}{381} < 1.$$
Since $n-i \ge 3$, we also have
$$\frac{1}{3}q^{2n-2i-1} - q^{n-i-1} > 1.$$
This gives
$$D < 1 - \frac{1}{\varepsilon_{i,0}}$$
and thus,
$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} > 1$$
for all $i \le n-d-2$, and in particular,
$$\frac{|a_{i,0}|}{|a_{i,1}|} > 1$$
for $i = n-d-1$ since $a_{i,2}$ does not exist in this case. This completes the proof. □

Using a similar approach as in the preceding proof, we will now show that the dual distribution is also nonnegative.

Proposition 5.3.10. *For $1 < d \le n$, all entries of the dual distribution (A'_k) given in (5.25) are nonnegative.*

Proof. Let (A'_k) be given in (5.25). Write

$$b_{k,j} = c_k(-1)^j b^{\binom{n-k-j}{2}} \frac{(qb^k)_j}{(qcb^{2k+1})_j} \begin{bmatrix} n-k \\ j \end{bmatrix}_b \left(1 - \frac{(qb^{k+j})_{d-k-j-1}}{(qcb^{n+k+j})_{d-k-j-1}}\right)$$

for all $k = 0, 1, \ldots, d-2$ and $j = 0, 1, \ldots, d-2-k$.

We start with $J_q(n,m)$ and $\frac{1}{2}D_m$. Observe that for all k, the factor c_k in A'_k is nonnegative. We will show that the sequence $(|b_{k,j}|)_j$ is decreasing for all $k = 0, 1, \ldots, d-2$ implying $A'_{n-k} \geq 0$. Take $k \in \{0, 1, \ldots, d-2\}$. For all $j = 0, 1, \ldots, d-3-k$, we have

$$\begin{aligned}
\frac{|b_{k,j}|}{|b_{k,j+1}|} &= \frac{b^{\binom{n-k-j}{2}} \frac{(qb^k)_j}{(qcb^{2k+1})_j} \left[\begin{smallmatrix} n-k \\ j \end{smallmatrix}\right]_b \left(1 - \frac{(qb^{k+j})_{d-k-j-1}}{(qcb^{n+k+j})_{d-k-j-1}}\right)}{b^{\binom{n-k-j-1}{2}} \frac{(qb^k)_{j+1}}{(qcb^{2k+1})_{j+1}} \left[\begin{smallmatrix} n-k \\ j+1 \end{smallmatrix}\right]_b \left(1 - \frac{(qb^{k+j+1})_{d-k-j-2}}{(qcb^{n+k+j+1})_{d-k-j-2}}\right)} \\
&= b^{n-k-j-1} \frac{(qcb^{2k+j+1}-1)(b^{j+1}-1)}{(qb^{k+j}-1)(b^{n-k-j}-1)} \frac{\left(1 - \frac{(qb^{k+j})_{d-k-j-1}}{(qcb^{n+k+j})_{d-k-j-1}}\right)}{\left(1 - \frac{(qcb^{n+k+j}-1)(qb^{k+j})_{d-k-j-1}}{(qb^{k+j}-1)(qcb^{n+k+j})_{d-k-j-1}}\right)}.
\end{aligned}$$

Since $(qcb^{n+k+j}-1)/(qb^{k+j}-1) > 1$, we obtain

$$\frac{|b_{k,j}|}{|b_{k,j+1}|} \geq b^{n-k-j-1} \frac{(qcb^{2k+j+1}-1)(b^{j+1}-1)}{(qb^{k+j}-1)(b^{n-k-j}-1)}. \tag{5.30}$$

By using (5.11), we have

$$\frac{|b_{k,j}|}{|b_{k,j+1}|} \geq cb^{k+j} \geq 1$$

for all $k, j \geq 0$ except for $\frac{1}{2}D_m$ with even m and $k = j = 0$, in which case (5.30) gives

$$\frac{|b_{0,0}|}{|b_{0,1}|} \geq \frac{q^{2n-2}(q+1)(q^2-1)}{(q^{2n}-1)} \geq \frac{q^2-1}{q} > 1,$$

as required.

Now, consider ${}^2A_{2n-1}$ with odd $d \geq 3$. The sign of c_k is $(-1)^{\binom{n-k}{2}}$. For $0 \leq k+j \leq d-2$ and $3 \leq d \leq n$, we obtain

$$|\delta_{k,j}| \leq \prod_{\ell=0}^{d-k-j-2} \frac{q^{k+j+\ell+2}}{q^{n+k+j+\ell}} = q^{(n-2)(k+j+1-d)} \tag{5.31}$$

and therefore,

$$|\delta_{k,j}| \leq \frac{1}{q^{n-2}} < \frac{1}{2}. \tag{5.32}$$

Hence, the entry $b_{k,j}$ has the sign $(-1)^{\binom{j}{2}+kj+j}$. Thus, we have $b_{k,2j} = |b_{k,2j}|$ if j is even and $b_{k,2j+1} = |b_{k,2j+1}|$ if $k+j \equiv 1 \pmod 2$ for all k, j. Similarly to the inner distribution, we will show that $(b_{k,j})$ satisfies (5.15). Observe that this will prove the nonnegativity of (A'_k). Set

$$b'_{k,j} = b^{\binom{n-k-j}{2}} \frac{(qb^k)_j}{(b^{2k+2})_j} \begin{bmatrix} n-k \\ j \end{bmatrix}_b \quad \text{and} \quad \delta_{k,j} = \frac{(qb^{k+j})_{d-k-j-1}}{(b^{n+k+j+1})_{d-k-j-1}},$$

so that $b_{k,j} = c_k(-1)^j b'_{k,j}(1-\delta_{k,j})$. Take $k \in \{0,1,\dots,d-2\}$. For all $j = 0,1,\dots,d-4-k$, we have

$$\begin{aligned}\frac{|b'_{k,j}|}{|b'_{k,j+2}|} &= \frac{q^{\binom{n-k-j}{2}}\left|\frac{(qb^k)_j}{(b^{2k+2})_j}\left[\begin{smallmatrix} n-k \\ j \end{smallmatrix}\right]_b\right|}{q^{\binom{n-k-j-2}{2}}\left|\frac{(qb^k)_{j+2}}{(b^{2k+2})_{j+2}}\left[\begin{smallmatrix} n-k \\ j+2 \end{smallmatrix}\right]_b\right|} \\ &= q^{2(n-k-j)-3}\frac{\left|(b^{2k+j+2}-1)(b^{2k+j+3}-1)\left[\begin{smallmatrix} n-k \\ j \end{smallmatrix}\right]_b\right|}{\left|(qb^{k+j}-1)(qb^{k+j+1}-1)\left[\begin{smallmatrix} n-k \\ j+2 \end{smallmatrix}\right]_b\right|}.\end{aligned}$$

Using (5.12) and (5.11) gives

$$\frac{|b'_{k,j}|}{|b'_{k,j+2}|} \geq q^{2(n-k-j)-3}\frac{(q^{2k+j+2}-1)(q^{2k+j+3}+1)}{(q^{k+j+1}+1)(q^{k+j+2}-1)}q^{-2n+4j+2k+2} \geq q^{2k+2j}.$$

From (5.32) we see that $\frac{1}{2} < 1-\delta_{k,j} < \frac{3}{2}$, which gives

$$\frac{|1-\delta_{k,j}|}{|1-\delta_{k,\ell}|} > \frac{1}{3}$$

for all $j,\ell = 0,1,\dots,d-2-k$. Hence, we find

$$\frac{|b_{k,j}|}{|b_{k,j+2}|} \geq \frac{1}{3}q^{2k+2j} > 1$$

for all $(k,j) \neq (0,0)$, as required.

It remains to show that

$$\frac{|b_{k,0}|}{|b_{k,1}|+|b_{k,2}|} \geq 1$$

for all even k. Let k be an even number in $\{0,1,\dots,d-3\}$. We have

$$\begin{aligned}&\frac{|b_{k,0}|}{|b_{k,1}|+|b_{k,2}|} \\ &\quad = \frac{\left|b^{\binom{n-k}{2}}(1-\delta_{k,0})\right|}{\left|b^{\binom{n-k-1}{2}}\frac{(qb^k)_1}{(b^{2k+2})_1}\left[\begin{smallmatrix} n-k \\ 1 \end{smallmatrix}\right]_b(1-\delta_{k,1})\right| + \left|b^{\binom{n-k-2}{2}}\frac{(qb^k)_2}{(b^{2k+2})_2}\left[\begin{smallmatrix} n-k \\ 2 \end{smallmatrix}\right]_b(1-\delta_{k,2})\right|},\end{aligned}$$

which becomes

$$\frac{|b_{k,0}|}{|b_{k,1}|+|b_{k,2}|} = \frac{|1-\delta_{k,0}|}{\frac{q^{-n+k+1}}{(q^{k+1}+1)}\left|\left[\begin{smallmatrix} n-k \\ 1 \end{smallmatrix}\right]_b\right|\,|1-\delta_{k,1}| + \frac{q^{-2n+2k+3}(q^{k+2}+1)}{(q^{k+1}+1)(q^{2k+3}+1)}\left|\left[\begin{smallmatrix} n-k \\ 2 \end{smallmatrix}\right]_b\right|\,|1-\delta_{k,2}|}.$$

Applying (5.13) and (5.14) gives

$$\begin{aligned}\frac{|b_{k,0}|}{|b_{k,1}|+|b_{k,2}|} &\geq \frac{(q^{k+1}+1)|1-\delta_{k,0}|}{|1-\delta_{k,1}|+\frac{1}{3}q^{-k}\left(1+\frac{1}{q^{k+2}}\right)|1-\delta_{k,2}|} \\ &\geq \frac{3|1-\delta_{k,0}|}{|1-\delta_{k,1}|+\frac{1}{3}\left(1+\frac{1}{4}\right)|1-\delta_{k,2}|}.\end{aligned}$$

From (5.31) and $k \leq d-3$, we find

$$\begin{aligned}|\delta_{k,0}| &\leq q^{(n-2)(k+1-d)} \leq \frac{1}{q^{2n-4}} \leq \frac{1}{4} \\ |\delta_{k,1}| &\leq q^{(n-2)(k+2-d)} \leq \frac{1}{q^{n-2}} \leq \frac{1}{2} \\ |\delta_{k,2}| &\leq q^{(n-2)(k+3-d)} \leq \frac{1}{q^{2n-4}} \leq \frac{1}{4},\end{aligned}$$

where the latter follows from $k \leq d-5$ since otherwise $j=2$ cannot occur. This implies

$$\frac{|b_{k,0}|}{|b_{k,1}|+|b_{k,2}|} \geq \frac{3\left(1-\frac{1}{4}\right)}{1+\frac{1}{2}+\frac{1}{3}\left(1+\frac{1}{4}\right)^2} > 1$$

for $k \leq d-5$ and

$$\frac{|b_{k,0}|}{|b_{k,1}|} > 1$$

for $k = d-3$ since $b_{k,2}$ does not exist in this case. This completes the proof. □

We can now prove the existence of a feasible solution of the primal LP with the required objective function value.

Proof of Proposition 5.3.4. For $d=1$, take $Y = X$ and thus, there exists a feasible solution of the primal LP (2.20) with the required objective function value. For $d > 1$, combine Proposition 5.3.9 and 5.3.10. □

5.3.3 The Hermitian matrices and even d

The following proposition is the main result of this subsection.

Proposition 5.3.11. *There exists a feasible solution of the primal LP* (2.20) *for d-codes in* $\mathrm{Her}_q(n)$ *with objective function value* (5.4) *for all even d with* $2 \leq d \leq n$.

Observe that Proposition 5.3.11 and 5.2.2 together with the Strong duality theorem 2.2.2 imply the second part of Theorem 5.1.1(b).

As in the previous sections, we will compute the inner and dual distribution of a d-code with even d whose size equals (5.4). Afterwards, we will show that these distributions are nonnegative implying the existence of a feasible solution of the primal LP.

We start with the computation of the inner distribution.

Proposition 5.3.12. *Let n and d be integers with $2 \leq d \leq n$ and even d. Assume that Y is a d-code in $\mathrm{Her}_q(n)$ of size (5.4). Let (A_i) and (A'_k) be the inner and dual distribution of Y, respectively. Then we have*

$$A_{n-i} = \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \begin{bmatrix} n \\ j \end{bmatrix}_b \times \left((-1)^j \frac{|Y|}{b^{nj}} - 1 - \frac{b^j - 1}{b^{n-d+1} - 1} \left((-1)^j \frac{|Y|}{b^{nj}} + (-1)^{n+j} b^{n(n-d+1-j)} \right) \right)$$

for all $i = 0, 1, \ldots, n-1$. Moreover, we have $A'_2 = \cdots = A'_{n-d+2} = 0$ and

$$A'_1 = \frac{b^n - 1}{b^{n-d+1} - 1} \left((-1)^{n+1} q^{n(n-d+1)} - |Y| \right). \tag{5.33}$$

Proof. By using (2.47), (2.17) and (5.9), we obtain for all $j = 0, 1, \ldots, n$

$$\sum_{k=0}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b A'_k = (-1)^j b^{nj} \sum_{i=0}^{n} A_i \begin{bmatrix} n-i \\ j \end{bmatrix}_b. \tag{5.34}$$

Put

$$x_k = \begin{bmatrix} n-k \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \quad \text{and} \quad y_k = \begin{bmatrix} n-k \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b.$$

Using (5.34) with $j = n-d+1$ and $j = n-d+2$ gives

$$(-1)^{n+1} \sum_{k=0}^{n-d+2} (x_k - y_k)\, A'_k = q^{n(n-d+1)} \left(\begin{bmatrix} n \\ d-1 \end{bmatrix}_b \begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b + b^n \begin{bmatrix} n \\ d-2 \end{bmatrix}_b \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \right).$$

It was shown in [Sch18, Proof of Theorem 2] that the coefficients of A'_k on the left-hand side are nonnegative. Since Y is of size (5.4), we have

$$\sum_{k=1}^{n-d+2} (x_k - y_k) A'_k = 0.$$

Because of $x_1 = y_1$ and $x_k \neq y_k$ for all $k = 2, 3, \ldots, n-d+2$, we obtain

$$A'_2 = A'_3 = \cdots = A'_{n-d+2} = 0.$$

From (5.34) we thus find

$$\begin{bmatrix} n \\ j \end{bmatrix}_b |Y| + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_b A'_1 = (-1)^j b^{nj} \left(\begin{bmatrix} n \\ j \end{bmatrix}_b + \sum_{k=0}^{n-d} A_{n-k} \begin{bmatrix} k \\ j \end{bmatrix}_b \right)$$

for all $j = 0, 1, \ldots, n-d$. Applying the q-binomial inversion formula (4.10) gives

$$A_{n-i} = \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \left(\begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(-1)^j |Y|}{b^{nj}} + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_b \frac{(-1)^j A'_1}{b^{nj}} - \begin{bmatrix} n \\ j \end{bmatrix}_b \right) \tag{5.35}$$

for all $i = 0, 1, \ldots, n-d$. It remains to compute A'_1. Set $j = n-d+1$ in (5.34) to obtain

$$\begin{bmatrix} n \\ d-1 \end{bmatrix}_b |Y| + \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b A'_1 = (-1)^{n+1} q^{n(n-d+1)} \begin{bmatrix} n \\ d-1 \end{bmatrix}_b.$$

This gives

$$A'_1 = \frac{b^n - 1}{b^{n-d+1} - 1} \left((-1)^{n+1} q^{n(n-d+1)} - |Y| \right).$$

By substituting this into (5.35) and doing some elementary manipulations, we have

$$\begin{aligned} A_{n-i} = \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \Bigg(\begin{bmatrix} n \\ j \end{bmatrix}_b \left((-1)^j \frac{|Y|}{b^{nj}} - 1 \right) \\ - \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_b \frac{b^n - 1}{b^{n-d+1} - 1} \left((-1)^j \frac{|Y|}{b^{nj}} + (-1)^{n+j} b^{n(n-d+1-j)} \right) \Bigg) \end{aligned}$$

for all $i = 0, 1, \ldots, n-d$. After some elementary manipulations, the desired expression of the inner distribution follows. □

We now derive the dual distribution.

Proposition 5.3.13. *Let n and d be integers with $2 \le d \le n$ and even d. Assume that Y is a d-code in $\mathrm{Her}_q(n)$ of size (5.4). Then the dual distribution (A'_k) of Y satisfies*

$$A'_1 = \frac{b^n - 1}{b^{n-d+1} - 1} \left((-1)^{n+1} q^{n(n-d+1)} - |Y| \right)$$

and for all $k = 0, 1, \ldots, n-2$,

$$A'_{n-k} = \sum_{j=0}^{d-k-3} (-1)^{n-k} b^{\binom{j}{2} + n(n-j-k)} \begin{bmatrix} n \\ k \end{bmatrix}_b \begin{bmatrix} n-k \\ j \end{bmatrix}_b (1 - \delta_{k,j}),$$

where

$$\delta_{k,j} = (-1)^{n-j-k} \frac{|Y|}{b^{n(n-j-k)}} - (-1)^{n-j-k} \frac{(b^{n-j-k} - 1)}{(b^{n-d+1} - 1)} \frac{(|Y| + (-1)^n q^{n(n-d+1)})}{b^{n(n-j-k)}}$$

for all $j = 0, 1, \ldots, d-k-3$.

Proof. Let (A_i) and (A'_k) denote the inner and dual distribution of Y, respectively. Recall from Proposition 5.3.12 that $A'_1, A'_2, \ldots, A'_{n-d+2}$ were already determined with $A'_2 = \cdots = A'_{n-d+2} = 0$. Since Y is a d-code, we also have $A_1 = \cdots = A_{d-1} = 0$. Combine this with (5.34) to obtain

$$\sum_{k=n-d+3}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b A'_k = \begin{bmatrix} n \\ j \end{bmatrix}_b ((-1)^j b^{nj} - |Y|) - \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_b A'_1$$

for all $j = n-d+3, \ldots, n$. Changing the index j and interchanging the order of summation give

$$\sum_{k=j}^{d-3} \begin{bmatrix} k \\ j \end{bmatrix}_b A'_{n-k} = \begin{bmatrix} n \\ j \end{bmatrix}_b \left((-1)^{n-j} b^{n(n-j)} - |Y|\right) - \begin{bmatrix} n-1 \\ j \end{bmatrix}_b A'_1$$

for all $j = 0, 1, \ldots, d-3$. By applying the q-binomial inversion formula (4.10), we have

$$A'_{n-k} = \sum_{j=k}^{d-3} (-1)^{j-k} b^{\binom{j-k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_b \left(\begin{bmatrix} n \\ j \end{bmatrix}_b \left((-1)^{n-j} b^{n(n-j)} - |Y|\right) - \begin{bmatrix} n-1 \\ j \end{bmatrix}_b A'_1 \right)$$

for all $k = 0, 1, \ldots, d-3$. Using (5.33) and doing some manipulations give

$$\begin{aligned} A'_{n-k} &= \sum_{j=k}^{d-3} (-1)^{j-k} b^{\binom{j-k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_b \begin{bmatrix} n \\ j \end{bmatrix}_b \\ &\quad \times \left((-1)^{n-j} b^{n(n-j)} - |Y| - \frac{b^{n-j}-1}{b^{n-d+1}-1} \left((-1)^{n+1} q^{n(n-d+1)} - |Y| \right) \right). \end{aligned}$$

Interchanging the order of summation implies the stated expression of the dual distribution. □

To show that the inner and dual distribution given in Proposition 5.3.12 and 5.3.13, respectively, are nonnegative, we write

$$\gamma(n,d) = (-b^n)^{n-d+1} \delta(n,d)$$

with

$$\delta(n,d) = \frac{(b^{n-d+2}-1) + b^n(b^{n-d+1}-1)}{b^{n-d+2} - b^{n-d+1}},$$

where d is assumed to be even. Observe that (5.4) equals $\gamma(n,d)$. We first need some bounds on $\gamma(n,d)$.

Lemma 5.3.14. *Let n and d be integers, where $2 \leq d \leq n$ and d is even. Then we have*

$$\frac{1}{3} q^{n(n-d+2)-1} \leq \gamma(n,d) \leq \frac{1}{2} q^{n(n-d+2)}. \tag{5.36}$$

Proof. For odd n, we have

$$\gamma(n,d) = q^{n(n-d+1)} \, \frac{q^{n-d+2} + 1 + q^n(q^{n-d+1}-1)}{q^{n-d+2} + q^{n-d+1}}, \tag{5.37}$$

and for even n,

$$\gamma(n,d) = q^{n(n-d+1)} \, \frac{-q^{n-d+2} + 1 + q^n(q^{n-d+1}+1)}{q^{n-d+2} + q^{n-d+1}}. \tag{5.38}$$

Hence, we obtain

$$\begin{aligned}\gamma(n,d) &\leq q^{n(n-d+1)}\ \frac{q^{2n-d+1}+q^n-q^{n-d+2}+1}{q^{n-d+1}(q+1)}\\ &\leq \frac{1}{3}q^{n(n-d+1)}\ \frac{q^{2n-d+1}+q^n}{q^{n-d+1}}.\end{aligned}$$

This gives

$$\gamma(n,d) \leq \frac{1}{3}q^{n(n-d+2)}\left(1+\frac{1}{q^{n-d+1}}\right) \leq \frac{1}{2}q^{n(n-d+2)}.$$

From (5.37) and (5.38), we see that

$$\gamma(n,d) \geq q^{n(n-d+1)}\ \frac{q^{2n-d+1}-q^n+q^{n-d+2}+1}{q^{n-d+1}(q+1)}.$$

We have

$$\frac{q^{2n-d+1}-q^n+q^{n-d+2}+1}{q^{n-d+1}(q+1)} \geq \frac{q^{2n-d+1}-q^n}{q^{n-d+1}(q+1)} = q^{n-1}\frac{\left(1-\frac{1}{q^{n-d+1}}\right)}{\left(1+\frac{1}{q}\right)} \geq \frac{1}{3}q^{n-1}.$$

Thus, we obtain

$$\gamma(n,d) \geq \frac{1}{3}q^{n(n-d+2)-1},$$

which completes the proof. □

To prove the nonnegativity of the inner distribution (A_i) given in Proposition 5.3.12, we rewrite it by doing some elementary manipulations and interchanging the order of summation and obtain

$$A_{n-i} = \sum_{j=0}^{n-d-i} a_{i,j} \tag{5.39}$$

with $a_{i,j} = a'_{i,j}(1-\varepsilon_{i,j})$, where

$$a'_{i,j} = (-1)^i b^{\binom{j}{2}-n(j+i)} \begin{bmatrix} n \\ i \end{bmatrix}_b \begin{bmatrix} n-i \\ j \end{bmatrix}_b |Y| \tag{5.40}$$

and

$$\varepsilon_{i,j} = (-1)^{j+i}\frac{b^{n(j+i)}}{|Y|} + \frac{b^{j+i}-1}{b^{n-d+1}-1}\left(1+(-1)^n\frac{q^{n(n-d+1)}}{|Y|}\right)$$

for all $i = 0,1,\ldots,n-d$ and $j = 0,1,\ldots,n-d-i$.

We start by deriving bounds on $1-\varepsilon_{i,j}$.

Lemma 5.3.15. *Let n and d be integers, where $2 \leq d < n$ and d is even. For $0 \leq i+j \leq n-d$, we have*

$$1-\varepsilon_{i,j} \geq \begin{cases} \frac{29}{32} & \text{for } n \text{ and } i+j \text{ odd} \\ \frac{125}{128} & \text{for } n \text{ and } i+j \text{ even} \\ \frac{186}{256} & \text{for odd } n \text{ and even } i+j \\ \frac{31}{64} & \text{for even } n \text{ and odd } i+j \end{cases}$$

and

$$1-\varepsilon_{i,j} \leq \begin{cases} 2 & \text{for } n \text{ and } i+j \text{ odd} \\ \frac{27}{16} & \text{for } n \text{ and } i+j \text{ even} \\ 1 & \text{for odd } n \text{ and even } i+j \\ \frac{2051}{2048} & \text{for even } n \text{ and odd } i+j. \end{cases}$$

Moreover, for odd $n \geq 3$, we have

$$1-\varepsilon_{0,0} \geq \frac{253}{256}.$$

Proof. We have to look at four different cases depending on the parity of n and $i+j$.

Assume that $n \geq 3$ and $i+j$ are odd, hence $1 \leq i+j \leq n-d$. We have

$$1-\varepsilon_{i,j} = 1 - \frac{q^{n(j+i)}}{|Y|} + \frac{q^{j+i}+1}{q^{n-d+1}-1}\left(1 - \frac{q^{n(n-d+1)}}{|Y|}\right). \tag{5.41}$$

By using the lower bound from (5.36), we obtain

$$1-\varepsilon_{i,j} \geq 1 - \frac{3}{q^{n(n-d-j-i+2)-1}} \geq \frac{29}{32}.$$

Moreover, applying the lower bound from (5.36) to (5.41) gives

$$1-\varepsilon_{i,j} \leq 1 + \frac{q^{j+i}+1}{q^{n-d+1}-1} \leq 1 + \frac{q^{n-d}+1}{q^{n-d+1}-1} \leq 2,$$

where the last inequality holds since $n-d \geq 1$.

Assume that $n \geq 4$ and $i+j$ are even, thus $0 \leq i+j \leq n-d$. We have

$$1-\varepsilon_{i,j} = 1 - \frac{q^{n(j+i)}}{|Y|} + \frac{q^{j+i}-1}{q^{n-d+1}+1}\left(1 + \frac{q^{n(n-d+1)}}{|Y|}\right). \tag{5.42}$$

Because of the lower bound from (5.36), we obtain

$$1-\varepsilon_{i,j} \geq 1 - \frac{3}{q^{n(n-d-j-i+2)-1}} \geq \frac{125}{128}.$$

Applying the lower bound from (5.36) to (5.42) gives

$$1-\varepsilon_{i,j} \leq 1 + \frac{q^{j+i}-1}{q^{n-d+1}+1}\left(1 + \frac{3}{q^{n-1}}\right) \leq \frac{27}{16}.$$

Assume that $n \geq 3$ is odd and $i+j$ is even, thus $0 \leq i+j \leq n-d-1$. Then we have

$$1-\varepsilon_{i,j} = 1 - \frac{q^{n(j+i)}}{|Y|} - \frac{q^{j+i}-1}{q^{n-d+1}-1}\left(1 - \frac{q^{n(n-d+1)}}{|Y|}\right).$$

The bounds from (5.36) imply

$$1-\varepsilon_{i,j} \geq 1-\frac{3}{q^{n(n-d-j-i+2)-1}}-\frac{q^{j+i}-1}{q^{n-d+1}-1}.$$

Applying (3.19) and $i+j \leq n-d-1$ gives

$$1-\varepsilon_{i,j} \geq 1-\frac{3}{q^{3n-1}}-\frac{1}{q^2} \geq \frac{186}{256}.$$

Moreover, we obtain

$$1-\varepsilon_{0,0} = 1-\frac{1}{|Y|} \geq 1-\frac{3}{q^{n(n-d+2)-1}} \geq \frac{253}{256}.$$

Because of the lower bound from (5.36), we also have

$$1-\varepsilon_{i,j} \leq 1.$$

Assume that $n \geq 4$ is even and $i+j$ is odd, hence $1 \leq i+j \leq n-d-1$. Then we have

$$1-\varepsilon_{i,j} = 1+\frac{q^{n(j+i)}}{|Y|}-\frac{q^{j+i}+1}{q^{n-d+1}+1}\left(1+\frac{q^{n(n-d+1)}}{|Y|}\right).$$

Using the lower bound from (5.36) gives

$$\begin{aligned}
1-\varepsilon_{i,j} &\geq 1-\frac{q^{j+i}+1}{q^{n-d+1}+1}\left(1+\frac{3}{q^{n-1}}\right)\\
&\geq 1-\frac{1}{q^2}\left(1+\frac{1}{q^{n-d-1}}\right)\left(1+\frac{3}{q^{n-1}}\right)\\
&\geq \frac{31}{64}.
\end{aligned}$$

We also obtain

$$1-\varepsilon_{i,j} \leq 1+\frac{3}{q^{n(n-d-j-i+2)-1}} \leq 1+\frac{3}{q^{3n-1}} < \frac{2051}{2048}.$$

This finishes the proof. □

We can now prove the nonnegativity of the inner distribution.

Proposition 5.3.16. *For $2 \leq d \leq n$, all entries of the inner distribution (A_i) given in Proposition 5.3.12 are nonnegative.*

Proof. Let (A_i) be given in (5.39).

First, assume that $n = d$. Then n is even and we only have to show that $A_n \geq 0$. We have

$$A_n = |Y|-1 \geq \frac{1}{3}q^{2n-1} \geq 0.$$

So, we henceforth assume that $2 \leq d < n$. Since $1-\varepsilon_{i,j} > 0$ for all i,j by Lemma 5.3.15, the sign of $a_{i,j}$ is $(-1)^{\binom{j}{2}+ij+j}$. Hence, for all $i = 0,1,\ldots,n-d$, we have $\operatorname{sign}(a_{i,2j}) = 1$ if $j \geq 0$ is even and $\operatorname{sign}(a_{i,2j+1}) = 1$ if $j + i \equiv 1 \pmod 2$. In all other cases, we have $\operatorname{sign}(a_{i,j}) = -1$. We will show that $a_{i,j}$ satisfies (5.15). Observe that this will prove the nonnegativity of the inner distribution.

Take $i \in \{0, 1, \ldots, n-d\}$. For all $j = 0, 1, \ldots, n-d-i-2$, by using (5.12), we obtain

$$\frac{|a'_{i,j}|}{|a'_{i,j+2}|} = \frac{q^{\binom{j}{2}-n(j+i)}\left|\begin{bmatrix} n-i \\ j \end{bmatrix}_b\right|}{q^{\binom{j+2}{2}-n(j+i+2)}\left|\begin{bmatrix} n-i \\ j+2 \end{bmatrix}_b\right|} \geq q^{2j+2i+1}. \tag{5.43}$$

Assume that $n \geq 4$. By using Lemma 5.3.15, we deduce

$$\frac{1-\varepsilon_{i,j}}{1-\varepsilon_{i,j+2}} \geq \begin{cases} \frac{29}{64} & \text{for } n \text{ and } i+j \text{ odd} \\ \frac{125}{216} & \text{for } n \text{ and } i+j \text{ even} \\ \frac{186}{256} & \text{for odd } n \text{ and even } i+j \\ \frac{992}{2051} & \text{for even } n \text{ and odd } i+j \end{cases}$$

for all i, j with $0 \leq i + j \leq n - d$. We thus obtain

$$\frac{|a_{i,j}|}{|a_{i,j+2}|} = \frac{|a'_{i,j}|(1-\varepsilon_{i,j})}{|a'_{i,j+2}|(1-\varepsilon_{i,j+2})} \geq q^{2j+2i+1}\frac{1-\varepsilon_{i,j}}{1-\varepsilon_{i,j+2}} \geq 1$$

for all i, j with $i + j \geq 0$.

It remains to show that

$$\frac{|a_{i,0}|}{|a_{i,1}|+|a_{i,2}|} \geq 1$$

for all even $n \geq 3$ and $i \geq 0$. Thus, let $i \geq 0$ be even. Then, by using (5.13) and (5.14), we have

$$\begin{aligned} \frac{|a_{i,0}|}{|a_{i,1}|+|a_{i,2}|} &= \frac{1-\varepsilon_{i,0}}{q^{-n}\left|\begin{bmatrix} n-i \\ 1 \end{bmatrix}_b\right|(1-\varepsilon_{i,1}) + q^{1-2n}\left|\begin{bmatrix} n-i \\ 2 \end{bmatrix}_b\right|(1-\varepsilon_{i,2})} \\ &\geq \frac{1-\varepsilon_{i,0}}{q^{-i-1}(1-\varepsilon_{i,1}) + \frac{1}{3}q^{-2i-1}(1-\varepsilon_{i,2})}. \end{aligned}$$

Together with Lemma 5.3.15, this gives

$$\frac{|a_{i,0}|}{|a_{i,1}|+|a_{i,2}|} \geq 1$$

for all $n \geq 3$ and all even $i \geq 0$ except for odd $n \geq 3$ and $i = 0$. In the latter case, by using (5.13), (5.14), and Lemma 5.3.15, we obtain

$$\begin{aligned} \frac{|a_{0,0}|}{|a_{0,1}|+|a_{0,2}|} &= \frac{1-\varepsilon_{0,0}}{q^{-n}\left|\begin{bmatrix} n \\ 1 \end{bmatrix}_b\right|(1-\varepsilon_{0,1}) + q^{1-2n}\left|\begin{bmatrix} n \\ 2 \end{bmatrix}_b\right|(1-\varepsilon_{0,2})} \\ &\geq \frac{1-\varepsilon_{0,0}}{\frac{(1+q^{-n})}{(q+1)}(1-\varepsilon_{0,1}) + \frac{(1+q^{-n})}{(q+1)(q^2-1)}(1-\varepsilon_{0,2})} \\ &\geq 1. \end{aligned}$$

This completes the proof. □

It remains to show the nonnegativity of the dual distribution.

Proposition 5.3.17. *For $2 \leq d \leq n$, all entries of the dual distribution (A'_k) given in Proposition 5.3.13 are nonnegative.*

Proof. Let (A'_k) be given in Proposition 5.3.13.

First, we have

$$\begin{aligned} A'_1 &= \frac{b^n - 1}{b^{n-d+1} - 1}\left((-1)^{n+1} q^{n(n-d+1)} - |Y|\right) \\ &= \frac{q^n - (-1)^n}{q^{n-d+1} + (-1)^n}\left(|Y| + (-1)^n q^{n(n-d+1)}\right). \end{aligned}$$

From Lemma 5.3.14, we find $A'_1 \geq 0$. In particular, this proves the proposition for $d = 2$. Henceforth, we assume $4 \leq d \leq n$. We write

$$A'_{n-k} = \sum_{j=0}^{d-k-3} b_{k,j}$$

with $b_{k,j} = b'_{k,j}(1 - \delta_{k,j})$, where $\delta_{k,j}$ as in Proposition 5.3.13 and

$$b'_{k,j} = (-1)^{n-k} b^{\binom{j}{2} + n(n-j-k)} \begin{bmatrix} n \\ k \end{bmatrix}_b \begin{bmatrix} n-k \\ j \end{bmatrix}_b$$

for all $k = 0, 1, \ldots, d-3$ and $j = 0, 1, \ldots, d-k-3$.

First, we look at $\delta_{k,j}$. For $0 \leq k + j \leq d - 3$, applying the upper bound from (5.36) gives

$$\begin{aligned} |\delta_{k,j}| &\leq \frac{|Y|}{q^{n(n-j-k)}} + \frac{(q^{n-j-k} + 1)(|Y| + q^{n(n-d+1)})}{q^{n(n-j-k)}(q^{n-d+1} - 1)} \\ &\leq \frac{q^{n(n-d+2)}}{2q^{n(n-j-k)}} + \frac{(q^{n-j-k} + 1)(\frac{1}{2} q^{n(n-d+2)} + q^{n(n-d+1)})}{q^{n(n-j-k)}(q^{n-d+1} - 1)}. \end{aligned}$$

Using $k + j \leq d - 3$ and doing some elementary manipulations imply

$$|\delta_{k,j}| \leq \frac{1}{2q^n} + \left(\frac{1}{q^{d-3}} + \frac{1}{q^n}\right)\left(\frac{1}{2} + \frac{1}{q^n}\right) \leq \frac{89}{256}$$

for all $n \geq 4$. This gives

$$\frac{1 - |\delta_{k,j}|}{1 + |\delta_{k,\ell}|} \geq \frac{167}{345} \tag{5.44}$$

for all $k, j, \ell \geq 0$ with $k + j \leq d - 3$ and $k + \ell \leq d - 3$. In particular, we have $1 - \delta_{k,j} > 0$ and hence, the sign of $b_{k,j}$ is $(-1)^{\binom{j}{2} + kj + j}$. Thus, for all $k = 0, 1, \ldots, d-3$, we have $\operatorname{sign}(b_{k,2j}) = 1$ if j is even and $\operatorname{sign}(b_{k,2j+1}) = 1$ if $j + k \equiv 1 \pmod 2$. In the other cases, we have $\operatorname{sign}(b_{k,j}) = -1$.

We will now show that $b_{k,j}$ has the same properties as $a_{k,j}$ in (5.15). Observe that this will prove the nonnegativity of the dual distribution.

We have $b'_{k,j} = \frac{q^{n^2}}{|Y|} a'_{k,j}$ for $a'_{k,j}$ as in (5.40). Because of (5.43), we conclude

$$\frac{|b_{k,j}|}{|b_{k,j+2}|} \geq \frac{|a'_{k,j}|}{|a'_{k,j+2}|} \frac{(1 - |\delta_{k,j}|)}{(1 + |\delta_{k,j+2}|)} \geq \frac{167}{345} q^{2j+2k+1} > 1$$

for all $k, j \geq 0$ with $(k, j) \neq (0, 0)$.

It remains to show that

$$\frac{|b_{k,0}|}{|b_{k,1}| + |b_{k,2}|} \geq 1$$

for all even $k \geq 0$. Let $k \geq 0$ be even. From (5.44), (5.13), and (5.14), we have

$$\begin{aligned} \frac{|b_{k,0}|}{|b_{k,1}| + |b_{k,2}|} &\geq \frac{167}{345} \frac{1}{\left(q^{-n} |\left[{n-k \atop 1}\right]_b| + q^{1-2n} |\left[{n-k \atop 2}\right]_b|\right)} \\ &\geq \frac{167}{345} \frac{1}{(q^{-k-1} + \frac{1}{3} q^{-2k-1})}. \end{aligned}$$

This gives

$$\frac{|b_{k,0}|}{|b_{k,1}| + |b_{k,2}|} \geq 1$$

for all $k \geq 2$. Similarly to the estimation of $|\delta_{k,j}|$, we obtain

$$|\delta_{0,0}| \leq \frac{1}{2q^{n(d-2)}} + \left(\frac{1}{q^{n(d-3)}} + \frac{1}{q^{n(d-2)}}\right) \left(\frac{1}{2} + \frac{1}{q^n}\right) \leq \frac{161}{4096}$$

for all n, d with $4 \leq d \leq n$. Hence, we have

$$\frac{|b_{0,0}|}{|b_{0,1}| + |b_{0,2}|} \geq \frac{1}{(q^{-1} + \frac{1}{3} q^{-1})} \frac{(1 - \frac{161}{4096})}{(1 + \frac{89}{256})} \geq 1.$$

This completes the proof. □

Combining Proposition 5.3.16 and 5.3.17 proves Proposition 5.3.11.

5.3.4 The Hermitian polar space ${}^2A_{2n-1}$ and even d

The goal of this subsection is to prove the following proposition.

Proposition 5.3.18. *Let X be the set of generators in ${}^2A_{2n-1}$ and let d be an even integer with $2 \leq d \leq n$. Then there exists a feasible solution of the primal LP* (2.20) *for d-codes in ${}^2A_{2n-1}$ with objective function value* (5.2)*.*

Observe that Proposition 5.3.18 and 5.2.4 together with the Strong duality theorem 2.2.2 imply the second part of Theorem 5.1.1(a).

We will use the same approach as for odd d and compute the inner and dual distribution of a d-code in ${}^2A_{2n-1}$ of size (5.2). However, the expressions for the inner and dual distribution are slightly more complicated if d is even.

This comes from the fact that for even d, the first entry of the dual distribution is nonzero whereas for odd d, we obtain an $(n-d+1)$-design.

We start by computing the inner distribution. To do so, we write the LP optimum (5.2) as

$$\mathrm{LP}(d) = |X| \frac{(q)_{d-1}}{(b^{n+1})_{d-1}} \varepsilon(n,d),$$

where d is assumed to be even and

$$\varepsilon(n,d) = \frac{((-q)^{n-d+2}-1) + q\frac{(-q)^{n+d-2}-1}{q(-q)^{d-2}-1}((-q)^{n-d+1}-1)}{((-q)^{n-d+2}-1) + q\frac{(-q)^{n+d-2}-1}{(-q)^{n+d-1}-1}((-q)^{n-d+1}-1)}. \tag{5.45}$$

Thus, a code Y of size (5.2) satisfies

$$|Y| = |X| \frac{(q)_{d-1}}{(b^{n+1})_{d-1}} \varepsilon(n,d). \tag{5.46}$$

We can now derive the inner distribution.

Proposition 5.3.19. *Let X be the set of generators in ${}^2A_{2n-1}$ and let d be an even integer with $2 \le d \le n$. Assume that Y is a d-code in ${}^2A_{2n-1}$ of size* (5.2)*. Let (A_i) and (A'_k) be the inner and dual distribution of Y, respectively, in terms of the orderings imposed by* (2.52) *and* (2.53)*. Then we have*

$$\begin{aligned} A_{n-i} = \frac{|Y|}{|X|} \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(b^{n+1})_{n-j}}{(q)_{n-j}} \\ \times \Bigg(1 - (1-\varepsilon(n,d)^{-1}) b^{n-j-d+1} \frac{(b^{n+d-1}-1)(b^j-1)}{(b^{n-d+1}-1)(b^{2n-j}-1)} \\ - \frac{(qb^{d-1})_{n-j-d+1}}{(b^{n+d})_{n-j-d+1}} \varepsilon(n,d)^{-1} \Bigg) \end{aligned}$$

for all $i = 0, 1, \dots, n-1$. Moreover, we have $A'_2 = \dots = A'_{n-d+2} = 0$ and

$$A'_1 = |X| b^{-d+1} \frac{(q)_{d-1}}{(b^n)_{d-1}} \frac{(b^n-1)}{(b^{n-d+1}-1)} (1-\varepsilon(n,d)). \tag{5.47}$$

Proof. Recall the formulae (3.13) and (3.16) from the proof of Theorem 3.2.1, namely we have

$$\sum_{k=0}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(b^{n-k+1})_{n-j}}{(q)_{n-j}} A'_k = |X| \sum_{i=0}^{n} A_i \begin{bmatrix} n-i \\ j \end{bmatrix}_b \tag{5.48}$$

for all $j = 0, 1, \dots, n$ and

$$\begin{aligned} &\sum_{k=0}^{n-d+2} (x_k - y_k) A'_k \\ &= |X| b^{d-2} \frac{(b^n)_{d-2}}{(q)_{d-2}} \left(\begin{bmatrix} n-1 \\ d-2 \end{bmatrix}_b \begin{bmatrix} n \\ d-1 \end{bmatrix}_b + q \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \begin{bmatrix} n \\ d-2 \end{bmatrix}_b \frac{b^{n+d-2}-1}{qb^{d-2}-1} \right) \end{aligned} \tag{5.49}$$

with x_k and y_k as in (3.15), where all coefficients of A'_k on the left-hand side of (5.49) are nonnegative. Using $A'_0 = |Y|$ and doing some elementary manipulations give

$$\sum_{k=1}^{n-d+2} (x_k - y_k) A'_k = 0.$$

Since $x_1 = y_1$ and $x_k \neq y_k$ for all $k \geq 2$, we obtain

$$A'_2 = A'_3 = \cdots = A'_{n-d+2} = 0.$$

We therefore find from (5.48) that

$$\begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(b^{n+1})_{n-j}}{(q)_{n-j}} |Y| + b^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_b \frac{(b^n)_{n-j}}{(q)_{n-j}} A'_1 = |X| \left(\begin{bmatrix} n \\ j \end{bmatrix}_b + \sum_{k=0}^{n-d} A_{n-k} \begin{bmatrix} k \\ j \end{bmatrix}_b \right)$$

for all $j = 0, 1, \ldots, n-d$. Applying the q-binomial inversion formula (4.10) gives

$$\begin{aligned} A_{n-i} = {} & \frac{1}{|X|} \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \\ & \times \left(\begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(b^{n+1})_{n-j}}{(q)_{n-j}} |Y| + b^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_b \frac{(b^n)_{n-j}}{(q)_{n-j}} A'_1 - |X| \begin{bmatrix} n \\ j \end{bmatrix}_b \right) \end{aligned} \tag{5.50}$$

for all $i = 0, 1, \ldots, n-d$. It remains to compute A'_1. Set $j = n-d+1$ in (5.48) to obtain

$$\begin{bmatrix} n \\ d-1 \end{bmatrix}_b \frac{(b^{n+1})_{d-1}}{(q)_{d-1}} |Y| + b^{d-1} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b \frac{(b^n)_{d-1}}{(q)_{d-1}} A'_1 = |X| \begin{bmatrix} n \\ d-1 \end{bmatrix}_b.$$

This implies

$$A'_1 = b^{-d+1} \frac{(q)_{d-1}}{(b^n)_{d-1}} \frac{\begin{bmatrix} n \\ d-1 \end{bmatrix}_b}{\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_b} \left(|X| - \frac{(b^{n+1})_{d-1}}{(q)_{d-1}} |Y| \right).$$

Using (5.46) gives

$$A'_1 = |X| b^{-d+1} \frac{(q)_{d-1}}{(b^n)_{d-1}} \frac{(b^n - 1)}{(b^{n-d+1} - 1)} (1 - \varepsilon(n,d)).$$

By substituting this into (5.50) and doing some elementary manipulations, we obtain

$$\begin{aligned} A_{n-i} = \frac{|Y|}{|X|} \sum_{j=i}^{n-d} (-1)^{j-i} b^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_b \begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(b^{n+1})_{n-j}}{(q)_{n-j}} \times \left(1 - \frac{|X|}{|Y|} \frac{(q)_{n-j}}{(b^{n+1})_{n-j}} \right. \\ \left. + \frac{|X|}{|Y|} (1 - \varepsilon(n,d)) b^{n-j-d+1} \frac{(b^j - 1)(b^n - 1)}{(b^{n-d+1} - 1)(b^{2n-j} - 1)} \frac{(q)_{d-1}}{(b^n)_{d-1}} \right) \end{aligned}$$

Using (5.46) and applying (3.5) give the desired expression of A_{n-i}. □

We can now compute the dual distribution.

Proposition 5.3.20. *Let X be the set of generators in ${}^2A_{2n-1}$ and let d be an even integer with $2 \le d \le n$. Assume that Y is a d-code in ${}^2A_{2n-1}$ of size* (5.2). *Let (A'_k) be the dual distribution of Y in terms of the second ordering imposed by* (2.53). *Then we have*

$$A'_1 = |X| b^{-d+1} \frac{(q)_{d-1}}{(b^n)_{d-1}} \frac{(b^n-1)}{(b^{n-d+1}-1)} (1-\varepsilon(n,d)),$$

and for all $k = 0, 1, \dots, n-2$,

$$A'_{n-k} = c_k \sum_{j=0}^{d-k-3} b^{\binom{j}{2}-nj} \frac{(b^{-n-1-k})_{n-k-j}}{(-b^{-n})_{n-k-j}} \begin{bmatrix} n-k \\ j \end{bmatrix}_b (1-\delta_{k,j}), \tag{5.51}$$

where $c_k = \mu'_{n-k} b^{\binom{n-k}{2}-n(n-k)} \frac{(qb^k)_{n-k}}{(-b^{-n})_{n-k}}$ with μ'_{n-k} given in Table 2.3 and

$$\begin{aligned} \delta_{k,j} = \varepsilon(n,d) &\frac{(qb^{k+j})_{d-k-j-1}}{(b^{n+k+j+1})_{d-k-j-1}} \\ &+ (1-\varepsilon(n,d)) b^{k+j-d+1} \frac{(b^{n-k-j}-1)}{(b^{n-d+1}-1)} \frac{(qb^{k+j})_{d-k-j-1}}{(b^{n+k+j})_{d-k-j-1}} \end{aligned} \tag{5.52}$$

for all $j = 0, 1, \dots, d-k-3$. In particular, we have $A'_2 = A'_3 = \cdots = A'_{n-d+2} = 0$.

Proof. First, observe that $A'_1, A'_2, \dots, A'_{n-d+2}$ were already determined in Proposition 5.3.19 with $A'_2 = A'_3 = \cdots = A'_{n-d+2} = 0$. To obtain A'_{n-k} for $k = 0, 1, \dots d-3$, we proceed similarly as in the case of odd d and solve a system of linear equations by using the inverse matrix from Lemma 5.3.6. As in (5.26), we have

$$\sum_{k=0}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(b^{n-k+1})_{n-j}}{(q)_{n-j}} A'_k = |X| \begin{bmatrix} n \\ j \end{bmatrix}_b$$

for all $j = n-d+1, \dots, n$. Use $A'_2 = A'_3 = \cdots = A'_{n-d+2} = 0$ to obtain

$$\begin{aligned} \sum_{k=n-d+3}^{j} &b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(b^{n-k+1})_{n-j}}{(q)_{n-j}} A'_k \\ &= |X| \begin{bmatrix} n \\ j \end{bmatrix}_b - \begin{bmatrix} n \\ j \end{bmatrix}_b \frac{(b^{n+1})_{n-j}}{(q)_{n-j}} |Y| - b^{n-j} \begin{bmatrix} n-1 \\ n-j \end{bmatrix}_b \frac{(b^n)_{n-j}}{(q)_{n-j}} A'_1 \end{aligned} \tag{5.53}$$

for all $j = n-d+3, \dots, n$. By using (5.46) and (5.47), the right-hand side of (5.53) can be written as

$$\begin{aligned} |X| \begin{bmatrix} n \\ j \end{bmatrix}_b \Bigg(1 - \varepsilon(n,d) &\frac{(b^{n+1})_{n-j}(q)_{d-1}}{(b^{n+1})_{d-1}(q)_{n-j}} \\ &- (1-\varepsilon(n,d)) b^{n-j-d+1} \frac{(b^j-1)}{(b^{n-d+1}-1)} \frac{(b^n)_{n-j}(q)_{d-1}}{(b^n)_{d-1}(q)_{n-j}} \Bigg). \end{aligned}$$

By applying (3.5), we have

$$\begin{aligned}
&\sum_{k=n-d+3}^{j} b^{k(n-j)} \begin{bmatrix} n-k \\ n-j \end{bmatrix}_b \frac{(b^{n-k+1})_{n-j}}{(q)_{n-j}} A'_k \\
&= |X| \begin{bmatrix} n \\ j \end{bmatrix}_b \Bigg(1 - \varepsilon(n,d) \frac{(qb^{n-j})_{d-n+j-1}}{(b^{2n-j+1})_{d-n+j-1}} \\
&\qquad\qquad - (1-\varepsilon(n,d)) b^{n-j-d+1} \frac{(b^j - 1)}{(b^{n-d+1}-1)} \frac{(qb^{n-j})_{d-n+j-1}}{(b^{2n-j})_{d-n+j-1}} \Bigg).
\end{aligned}$$

Similarly as in the case of odd d, we multiply the matrix QC^{-1} from Lemma 5.3.6, which gives

$$\begin{aligned}
A'_k = \mu'_k b^{-(2n+1)k+k^2} &\frac{(qb^{n-k})_k}{(-b^{-n})_k} \sum_{j=n-d+3}^{n} b^{\binom{j}{2}+j} \frac{(b^{-k})_j (b^{-2n-1+k})_j}{(b^{-n})_j (-b^{-n})_j} \begin{bmatrix} n \\ j \end{bmatrix}_b \\
&\times \Bigg(1 - \varepsilon(n,d) \frac{(qb^{n-j})_{d-n+j-1}}{(b^{2n-j+1})_{d-n+j-1}} \\
&\qquad - (1-\varepsilon(n,d)) b^{n-j-d+1} \frac{(b^j - 1)}{(b^{n-d+1}-1)} \frac{(qb^{n-j})_{d-n+j-1}}{(b^{2n-j})_{d-n+j-1}} \Bigg).
\end{aligned}$$

Apply (2.30) to obtain

$$\begin{aligned}
A'_{n-k} = \mu'_{n-k} b^{-(n-k)(n+k+1)} &\frac{(qb^{k})_{n-k}}{(-b^{-n})_{n-k}} \sum_{j=n-d+3}^{n-k} b^{\binom{j}{2}+j+kj} \frac{(b^{-n-1-k})_j}{(-b^{-n})_j} \begin{bmatrix} n-k \\ j \end{bmatrix}_b \\
&\times \Bigg(1 - \varepsilon(n,d) \frac{(qb^{n-j})_{d-n+j-1}}{(b^{2n-j+1})_{d-n+j-1}} \\
&\qquad - (1-\varepsilon(n,d)) b^{n-j-d+1} \frac{(b^j - 1)}{(b^{n-d+1}-1)} \frac{(qb^{n-j})_{d-n+j-1}}{(b^{2n-j})_{d-n+j-1}} \Bigg).
\end{aligned}$$

Changing the order of summation and doing some elementary manipulations give the stated expression of A'_{n-k}. □

Proving the nonnegativity of both distributions requires the following bounds on $\varepsilon(n,d)$.

Lemma 5.3.21. *Let n and d be integers, where $2 \le d \le n$ and d is even. Then we have*

$$\varepsilon(n,d) > \begin{cases} -\dfrac{q^{n+d-1}+1}{q^{d-1}-1} & \text{for even } n \\[2ex] \frac{1}{2} q^{d-2}(q^{n-d+1}-1) & \text{for odd } n \end{cases} \tag{5.54}$$

and

$$\varepsilon(n,d) < \begin{cases} -\dfrac{q^{n}}{q+1} & \text{for even } n \\[2ex] \dfrac{q^{n+d-1}-1}{q^{d-1}-1} & \text{for odd } n. \end{cases} \tag{5.55}$$

Proof. Since d is even, we find from (5.45) that

$$(-1)^{n+1}\varepsilon(n,d) = \frac{-q^{n-d+2} + (-1)^n + (-1)^n q\,\frac{q^{n+d-2}-(-1)^n}{q^{d-1}-1}(q^{n-d+1} + (-1)^n)}{(-1)^n q^{n-d+2} - 1 + (-1)^n q\,\frac{q^{n+d-2}-(-1)^n}{q^{n+d-1}+(-1)^n}(q^{n-d+1} + (-1)^n)}.$$

In (3.26), it was shown that

$$(-1)^{n+1}\varepsilon(n,d) < \frac{q^{n+d-1} + (-1)^n}{q^{d-1} - 1},$$

which implies the stated lower and upper bound for even n and odd n, respectively. Assume that n is even. Then we have

$$(-1)^{n+1}\varepsilon(n,d) = \frac{-q^{n-d+2} + 1 + q\frac{q^{n+d-2}-1}{q^{d-1}-1}(q^{n-d+1} + 1)}{q^{n-d+2} - 1 + q\frac{q^{n+d-2}-1}{q^{n+d-1}+1}(q^{n-d+1} + 1)}.$$

Using (5.11) gives

$$\frac{q^{n+d-2} - 1}{q^{d-1} - 1} \geq q^{n-1}$$

and we also have

$$\frac{q^{n+d-2} - 1}{q^{n+d-1} + 1} \leq \frac{1}{q}.$$

We thus obtain

$$\begin{aligned}(-1)^{n+1}\varepsilon(n,d) &\geq \frac{-q^{n-d+2} + 1 + q^{2n-d+1} + q^n}{q^{n-d+2} + q^{n-d+1}} \\ &> \frac{q^{2n-d+1}}{q^{n-d+2} + q^{n-d+1}} \\ &= \frac{q^n}{q+1},\end{aligned}$$

as stated. For odd n, we have

$$(-1)^{n+1}\varepsilon(n,d) = \frac{q^{n-d+2} + 1 + q\frac{q^{n+d-2}+1}{q^{d-1}-1}(q^{n-d+1} - 1)}{q^{n-d+2} + 1 + q\frac{q^{n+d-2}+1}{q^{n+d-1}-1}(q^{n-d+1} - 1)}.$$

Because of

$$\frac{q^{n+d-2} + 1}{q^{d-1} - 1} \geq q^{n-1} \quad \text{and} \quad \frac{q^{n+d-2} + 1}{q^{n+d-1} - 1} \leq 1,$$

we obtain

$$(-1)^{n+1}\varepsilon(n,d) \geq \frac{q^{n-d+2} + 1 + q^n(q^{n-d+1} - 1)}{q^{n-d+2} + 1 + q(q^{n-d+1} - 1)}.$$

Observe that it holds

$$q^{n-d+2} + 1 > \frac{1}{2}(-q^n + q^{n-1} + q^{d-1} - q^{d-2})$$

since $-q^n + q^{n-1} + q^{d-1} \leq 0$. This can be used to show, by elementary manipulations, that

$$q^{n-d+2} + 1 + q^n(q^{n-d+1} - 1) > \frac{1}{2}q^{d-2}(q^{n-d+1} - 1)\left(q^{n-d+2} + 1 + q(q^{n-d+1} - 1)\right),$$

which implies

$$(-1)^{n+1}\varepsilon(n,d) > \frac{1}{2}q^{d-2}(q^{n-d+1} - 1),$$

as required. □

To prove the nonnegativity of the inner distribution given in Proposition 5.3.19, we rewrite it as

$$A_{n-i} = \frac{|Y|}{|X|}\sum_{j=0}^{n-d-i} a_{i,j} \tag{5.56}$$

with $a_{i,j} = a'_{i,j}(1 - \varepsilon_{i,j})$, where

$$a'_{i,j} = (-1)^j b^{\binom{j}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_b \begin{bmatrix} n-i \\ j \end{bmatrix}_b \frac{(b^{n+1})_{n-j-i}}{(q)_{n-j-i}}$$

and

$$\varepsilon_{i,j} = (1 - \varepsilon(n,d)^{-1})b^{n-j-i-d+1}\frac{(b^{n+d-1} - 1)(b^{j+i} - 1)}{(b^{n-d+1} - 1)(b^{2n-j-i} - 1)} + \frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}\varepsilon(n,d)^{-1}$$

for all $i = 0, 1, \ldots, n-d$ and $j = 0, 1, \ldots, n-d-i$.

We first give lower and upper bounds on $\varepsilon_{i,j}$.

Lemma 5.3.22. *Let n and d be integers with even d such that $2 \leq d < n$ and $n \geq 4$. For all integers i, j with $0 \leq i + j \leq n - d$, we have*

$$1 - \varepsilon_{i,j} > \begin{cases} \frac{31}{32} & \text{for } n \text{ and } i+j \text{ odd} \\ \frac{61}{64} & \text{for } n \text{ and } i+j \text{ even} \\ \frac{191}{256} & \text{for odd } n \text{ and even } i+j \\ \frac{109}{128} & \text{for even } n \text{ and odd } i+j \end{cases}$$

and

$$1 - \varepsilon_{i,j} < \begin{cases} 2 & \text{for } n \text{ and } i+j \text{ odd} \\ \frac{51}{32} & \text{for } n \text{ and } i+j \text{ even} \\ 1 & \text{for odd } n \text{ and even } i+j \\ \frac{259}{256} & \text{for even } n \text{ and odd } i+j. \end{cases}$$

Moreover, we have

$$1 - \varepsilon_{0,0} > \frac{255}{256}$$

for odd n.

Proof. From (5.54) and (5.55), we find

$$-\frac{q+1}{q^n} < \varepsilon(n,d)^{-1} < -\frac{q^{d-1}-1}{q^{n+d-1}+1} \quad \text{for even } n \tag{5.57}$$

and

$$\frac{q^{d-1}-1}{q^{n+d-1}-1} < \varepsilon(n,d)^{-1} < 2\,\frac{q^{-d+2}}{q^{n-d+1}-1} \quad \text{for odd } n. \tag{5.58}$$

Note that the sign of $(qb^{d-1})_{n-j-i-d+1}/(b^{n+d})_{n-j-i-d+1}$ is $(-1)^{(n+1)(n-j-i+1)}$. Moreover, we have

$$\begin{aligned}\left|\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}\right| &= \left|\prod_{\ell=0}^{n-j-i-d} \frac{qb^{d-1+\ell}-1}{b^{n+d+\ell}-1}\right| \\ &\le \prod_{\ell=0}^{n-j-i-d} \frac{q^{d+\ell}+1}{q^{n+d+\ell}-1} \\ &\le q^{-(n-2)(n-j-i-d+1)} \end{aligned}\tag{5.59}$$

for all i, j with $0 \le i + j \le n - d$.

We have to look at four different cases depending on the parity of n and $i + j$.

Assume that $n \ge 5$ and $i + j$ are odd. We have

$$\begin{aligned}1 - \varepsilon_{i,j} = 1 + (1 - \varepsilon(n,d)^{-1})q^{n-j-i-d+1}\,&\frac{(q^{n+d-1}-1)(q^{j+i}+1)}{(q^{n-d+1}-1)(q^{2n-j-i}+1)} \\ &- \frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}\varepsilon(n,d)^{-1}.\end{aligned}\tag{5.60}$$

Use that the second summand is nonnegative and

$$\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}} \ge 0$$

together with the upper bound from (5.58) to obtain

$$\begin{aligned}1 - \varepsilon_{i,j} &> 1 - \frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}\,\frac{2q^{-d+2}}{(q^{n-d+1}-1)} \\ &> 1 - \frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}} 2q^{-n+2}.\end{aligned}$$

Because of (5.59), we have

$$1 - \varepsilon_{i,j} > 1 - \frac{2}{q^{(n-2)(n-j-i-d+2)}}.$$

Since $n \geq 5$ and $i + j \leq n - d$, we deduce

$$1 - \varepsilon_{i,j} > \frac{31}{32}.$$

From (5.60) we also obtain

$$1 - \varepsilon_{i,j} < 1 + (1 - \varepsilon(n,d)^{-1}) q^{n-j-i-d+1} \frac{(q^{n+d-1} - 1)(q^{j+i} + 1)}{(q^{n-d+1} - 1)(q^{2n-j-i} + 1)}.$$

Using the lower bound from (5.58) gives

$$\begin{aligned} 1 - \varepsilon_{i,j} &< 1 + q^{n-j-i} \frac{(q^n - 1)(q^{j+i} + 1)}{(q^{n-d+1} - 1)(q^{2n-j-i} + 1)} \\ &< 1 + \frac{q^{j+i} + 1}{q^{n-d+1} - 1}. \end{aligned}$$

Because of $i + j \leq n - d$ and $n - d \geq 1$, we have

$$1 - \varepsilon_{i,j} < 1 + \frac{q^{n-d} + 1}{q^{n-d+1} - 1} \leq 2,$$

as required.

Assume now that n and $j + i$ are even. We have

$$\begin{aligned} 1 - \varepsilon_{i,j} = 1 + (1 - \varepsilon(n,d)^{-1}) q^{n-j-i-d+1} &\frac{(q^{n+d-1} + 1)(q^{j+i} - 1)}{(q^{n-d+1} + 1)(q^{2n-j-i} - 1)} \\ &- \frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}} \varepsilon(n,d)^{-1}. \end{aligned} \tag{5.61}$$

The second summand is nonnegative, whereas $\varepsilon(n,d)^{-1}$ and

$$\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}$$

are negative. Therefore, we obtain

$$1 - \varepsilon_{i,j} \geq 1 - \left| \frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}} \right| |\varepsilon(n,d)^{-1}|.$$

Using (5.59) and the lower bound from (5.57) gives

$$1 - \varepsilon_{i,j} > 1 - \frac{q + 1}{q^{n+(n-2)(n-j-i-d+1)}} \geq \frac{61}{64},$$

where the last inequality follows from $j + i \leq n - d$ and $n \geq 4$. From (5.61) we also obtain

$$1 - \varepsilon_{i,j} < 1 + (1 - \varepsilon(n,d)^{-1}) q^{n-j-i-d+1} \frac{(q^{n+d-1} + 1)(q^{j+i} - 1)}{(q^{n-d+1} + 1)(q^{2n-j-i} - 1)}.$$

Using the lower bound from (5.57) gives

$$1 - \varepsilon_{i,j} < 1 + q^{-j-i-d+1} \frac{(q^n + q + 1)(q^{n+d-1} + 1)(q^{j+i} - 1)}{(q^{n-d+1} + 1)(q^{2n-j-i} - 1)}.$$

By (3.19) and (5.11), we have

$$1-\varepsilon_{i,j}<1+\frac{q^n+q+1}{q^{2n-d-j-i+1}}.$$

Since $i+j\le n-d$ and $n\ge 4$, we obtain

$$1-\varepsilon_{i,j}<1+\frac{1}{q}+\frac{1}{q^n}+\frac{1}{q^{n+1}}\le\frac{51}{32}.$$

Assume that $n\ge 5$ is odd and $i+j$ is even, which implies $i+j\le n-d-1$. We have

$$\begin{aligned}1-\varepsilon_{i,j}=1-(1-\varepsilon(n,d)^{-1})q^{n-j-i-d+1}&\frac{(q^{n+d-1}-1)(q^{j+i}-1)}{(q^{n-d+1}-1)(q^{2n-j-i}-1)}\\&-\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}\,\varepsilon(n,d)^{-1}.\end{aligned}\tag{5.62}$$

Since $(qb^{d-1})_{n-j-i-d+1}/(b^{n+d})_{n-j-i-d+1}\ge 0$, we use the bounds from (5.58) to obtain

$$\begin{aligned}1-\varepsilon_{i,j}>1-q^{n-j-i}&\frac{(q^n-1)(q^{j+i}-1)}{(q^{n-d+1}-1)(q^{2n-j-i}-1)}\\&-2\,\frac{q^{-d+2}}{(q^{n-d+1}-1)}\,\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}.\end{aligned}\tag{5.63}$$

From (3.19) we find

$$\begin{aligned}\frac{q^n-1}{q^{2n-j-i}-1}&\le q^{-n+j+i}\\ \frac{q^{j+i}-1}{q^{n-d+1}-1}&\le q^{-n+j+i+d-1},\end{aligned}$$

which implies

$$1-\varepsilon_{i,j}>1-\frac{1}{q^{n-j-i-d+1}}-2\,\frac{q^{-d+2}}{(q^{n-d+1}-1)}\,\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}.$$

Using (5.59) gives

$$1-\varepsilon_{i,j}>1-\frac{1}{q^{n-j-i-d+1}}-\frac{2}{q^{(n-2)(n-j-i-d+2)}}.$$

Because of $n\ge 5$ and $j+i\le n-d-1$, we have

$$1-\varepsilon_{i,j}>1-\frac{1}{4}-\frac{1}{2^8}=\frac{191}{256}.$$

In particular, from (5.63) we find similarly

$$1-\varepsilon_{0,0}>1-\frac{1}{2^8}=\frac{255}{256}.$$

By applying $1-\varepsilon(n,d)^{-1}>0$ and

$$\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}\,\varepsilon(n,d)^{-1}>0$$

to (5.62), we obtain $1-\varepsilon_{i,j}<1$.

Assume now that n is even and $i+j$ is odd, which implies $n-d\geq 2$ and $i+j\geq 1$. We have

$$\begin{aligned}1-\varepsilon_{i,j}=1-(1-\varepsilon(n,d)^{-1})q^{n-j-i-d+1}\frac{(q^{n+d-1}+1)(q^{j+i}+1)}{(q^{n-d+1}+1)(q^{2n+j+i}+1)}\\-\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}\,\varepsilon(n,d)^{-1}.\end{aligned}\tag{5.64}$$

Since $\varepsilon(n,d)^{-1}<0$ and $(qb^{d-1})_{n-j-i-d+1}/(b^{n+d})_{n-j-i-d+1}\geq 0$, the third summand is nonnegative. Together with the lower bound from (5.57) we obtain

$$1-\varepsilon_{i,j}>1-q^{-j-i-d+1}\frac{(q^n+q+1)(q^{n+d-1}+1)(q^{j+i}+1)}{(q^{n-d+1}+1)(q^{2n+j+i}+1)}.$$

By (5.11) we have

$$\frac{q^{n+d-1}+1}{q^{n-d+1}+1}<q^{2d-2},$$

which gives us

$$\begin{aligned}1-\varepsilon_{i,j}&>1-\frac{q^n+q+1}{q^{2n+j+i-d}}\\&=1-\frac{1}{q^{n+j+i-d}}-\frac{1}{q^{2n+j+i-d-1}}-\frac{1}{q^{2n+j+i-d}}.\end{aligned}$$

Because of $n-d\geq 2$, $n\geq 4$, and $i+j\geq 1$, we deduce

$$1-\varepsilon_{i,j}>\frac{109}{128}.$$

The second summand in (5.64) is nonpositive since $\varepsilon(n,d)^{-1}<0$. Together with the lower bound from (5.57), we obtain

$$1-\varepsilon_{i,j}<1+\frac{(q+1)}{q^n}\,\frac{(qb^{d-1})_{n-j-i-d+1}}{(b^{n+d})_{n-j-i-d+1}}.$$

Using (5.59) and $i+j\leq n-d-1$ as well as $n\geq 4$ gives

$$1-\varepsilon_{i,j}<1+\frac{(q+1)}{q^n}q^{-(n-2)(n-j-i-d+1)}\leq 1+\frac{q+1}{q^{3n-4}}\leq\frac{259}{256},$$

which completes the proof. □

We now show that the inner distribution is nonnegative.

Proposition 5.3.23. *For $2\leq d\leq n$, all entries of the inner distribution (A_i) given in Proposition 5.3.19 are nonnegative.*

Proof. Let (A_i) be given in (5.56).

First, assume that $d = n$. Then n is even and we only have to show that $A_n \geq 0$. By using (5.46), we have

$$\begin{aligned}
A_n &= \frac{|Y|}{|X|}\,\frac{(b^{n+1})_n}{(q)_n}\,\left(1 - \frac{(qb^{n-1})_1}{(b^{2n})_1}\varepsilon(n,n)^{-1}\right)\\
&= \frac{(q)_{n-1}(b^{n+1})_n}{(b^{n+1})_{n-1}(q)_n}\,\varepsilon(n,n) - \frac{(q)_{n-1}(b^{n+1})_n(qb^{n-1})_1}{(b^{n+1})_{n-1}(q)_n(b^{2n})_1}\\
&= -(q^n+1)\varepsilon(n,n) - 1.
\end{aligned}$$

From (5.54) we obtain

$$A_n > \frac{q^n(q^n+1)}{q+1} - 1 > 0.$$

Assume now that $2 \leq d < n$. Since $1 - \varepsilon_{i,j} > 0$ for all i, j by Lemma 5.3.22 and the sign of $(b^{n+1})_{n-j-i}/(q)_{n-j-i}$ is $(-1)^{(n+1)(n-j-i)}$, the sign of $a_{i,j}$ is $(-1)^{\binom{j}{2}+ij+j}$. Hence for all $i = 0, 1, \ldots, n-d$, we have $\operatorname{sign}(a_{i,2j}) = 1$ if $j \geq 0$ is even and $\operatorname{sign}(a_{i,2j+1}) = 1$ if $j + i \equiv 1 \pmod 2$. In all other cases, we have $\operatorname{sign}(a_{i,j}) = -1$. Similarly as in the case of odd d, we will show that $(a_{i,j})$ satisfies (5.15). Observe that this will prove the nonnegativity of the inner distribution.

Take $i \in \{0, 1, \ldots, n-d\}$. For all $j = 0, 1, \ldots, n-d-i-2$, by using (5.12), we obtain

$$\begin{aligned}
\frac{|a'_{i,j}|}{|a'_{i,j+2}|} &= \frac{q^{\binom{j}{2}}\left|\begin{bmatrix} n-i \\ j \end{bmatrix}_b \frac{(b^{n+1})_{n-j-i}}{(q)_{n-j-i}}\right|}{q^{\binom{j+2}{2}}\left|\begin{bmatrix} n-i \\ j+2 \end{bmatrix}_b \frac{(b^{n+1})_{n-j-i-2}}{(q)_{n-j-i-2}}\right|}\\
&\geq q^{2j+2i-2n+1}\,\frac{|(b^{2n-j-i-1}-1)(b^{2n-j-i}-1)|}{|(qb^{n-j-i-2}-1)(qb^{n-j-i-1}-1)|}.
\end{aligned}$$

Applying (5.11) gives

$$\frac{|a'_{i,j}|}{|a'_{i,j+2}|} \geq q^{2j+2i-2n+1}\,\frac{(q^{2n-j-i-1}-1)(q^{2n-j-i}+1)}{(q^{n-j-i-1}+1)(q^{n-j-i}-1)} > q^{2j+2i}.$$

Assume that $n \geq 4$. By using Lemma 5.3.22, we deduce

$$\frac{1-\varepsilon_{i,j}}{1-\varepsilon_{i,j+2}} > \begin{cases} \frac{31}{64} & \text{for } n \text{ and } i+j \text{ odd}\\ \frac{61}{102} & \text{for } n \text{ and } i+j \text{ even}\\ \frac{191}{256} & \text{for odd } n \text{ and even } i+j\\ \frac{218}{259} & \text{for even } n \text{ and odd } i+j \end{cases}$$

for all i, j with $0 \leq i + j \leq n - d - 2$. For all i, j with $1 \leq i + j \leq n - d - 2$, we thus obtain

$$\frac{|a_{i,j}|}{|a_{i,j+2}|} = \frac{|a'_{i,j}|(1-\varepsilon_{i,j})}{|a'_{i,j+2}|(1-\varepsilon_{i,j+2})} > q^{2i+2j}\,\frac{1-\varepsilon_{i,j}}{1-\varepsilon_{i,j+2}} > 1,$$

as wanted. In the case of $n \geq 4$, it remains to show that

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} \geq 1$$

for all even $i \geq 0$. Thus, let $i \geq 0$ be even. We have

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} = \frac{\left|\frac{(b^{n+1})_{n-i}}{(q)_{n-i}}\right| (1 - \varepsilon_{i,0})}{\left|\begin{bmatrix} n-i \\ 1 \end{bmatrix}_b \frac{(b^{n+1})_{n-i-1}}{(q)_{n-i-1}}\right| (1 - \varepsilon_{i,1}) + q \left|\begin{bmatrix} n-i \\ 2 \end{bmatrix}_b \frac{(b^{n+1})_{n-i-2}}{(q)_{n-i-2}}\right| (1 - \varepsilon_{i,2})}.$$

It holds that

$$\left|\frac{(b^{n+1})_{n-i-1}}{(q)_{n-i-1}}\right| = \frac{(q^{n-i} + (-1)^n)}{(q^{2n-i} - 1)} \left|\frac{(b^{n+1})_{n-i}}{(q)_{n-i}}\right|$$

and

$$\left|\frac{(b^{n+1})_{n-i-2}}{(q)_{n-i-2}}\right| = \frac{(q^{n-i} + (-1)^n)(q^{n-i-1} - (-1)^n)}{(q^{2n-i} - 1)(q^{2n-i-1} + 1)} \left|\frac{(b^{n+1})_{n-i}}{(q)_{n-i}}\right|.$$

Combining this with

$$\left|\begin{bmatrix} n-i \\ 1 \end{bmatrix}_b\right| = \frac{q^{n-i} - (-1)^n}{q+1}$$

$$\left|\begin{bmatrix} n-i \\ 2 \end{bmatrix}_b\right| = \frac{(q^{n-i} - (-1)^n)(q^{n-i-1} + (-1)^n)}{(q+1)(q^2-1)}$$

gives

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} = \frac{1 - \varepsilon_{i,0}}{\frac{(q^{2n-2i}-1)}{(q+1)(q^{2n-i}-1)}(1 - \varepsilon_{i,1}) + \frac{q(q^{2n-2i}-1)(q^{2n-2i-2}-1)}{(q+1)(q^2-1)(q^{2n-i}-1)(q^{2n-i-1}+1)}(1 - \varepsilon_{i,2})}.$$

Because of (3.19) and $i \geq 0$, we obtain

$$\begin{aligned} \frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} &\geq \frac{1 - \varepsilon_{i,0}}{\frac{q^{-i}}{(q+1)}(1 - \varepsilon_{i,1}) + \frac{q^{-2i}}{(q+1)(q^2-1)}(1 - \varepsilon_{i,2})} \\ &> \frac{1 - \varepsilon_{i,0}}{\frac{1}{3}(1 - \varepsilon_{i,1}) + \frac{1}{18}(1 - \varepsilon_{i,2})}. \end{aligned}$$

Using the lower and upper bounds on $1 - \varepsilon_{i,j}$ from Lemma 5.3.22 gives

$$\frac{|a_{i,0}|}{|a_{i,1}| + |a_{i,2}|} > 1$$

for all $n \geq 4$ and even $i \geq 0$, as required.

It remains to look at $n = 3$ and $d = 2$. Because of the sign of $a_{1,0}$, we immediately have $A_2 = \frac{|Y|}{|X|} a_{1,0} \geq 0$. For the entry A_3, we only need to show $|a_{0,0}| / |a_{0,1}| \geq 1$. We have

$$\frac{|a'_{0,0}|}{|a'_{0,1}|} = \frac{\left|\frac{(q^4)_3}{(q)_3}\right|}{\left|\begin{bmatrix} 3 \\ 1 \end{bmatrix}_b \frac{(q^4)_2}{(q)_2}\right|} = q + 1 \geq 3.$$

We also require bounds on $1 - \varepsilon_{0,0}$ and $1 - \varepsilon_{0,1}$. Use the upper bound from (5.58) to obtain

$$\begin{aligned}
1 - \varepsilon_{0,0} &= 1 - \frac{(q^2+1)}{(q^5+1)(q^3+1)}\,\varepsilon(3,2)^{-1} \\
&\geq 1 - \frac{2(q^2+1)}{(q^5+1)(q^3+1)(q^2-1)} \\
&\geq 1 - \frac{2}{q^6} \\
&\geq \frac{31}{32}.
\end{aligned}$$

Applying the lower bound from (5.58) gives

$$\begin{aligned}
1 - \varepsilon_{0,1} &= 1 + (1 - \varepsilon(3,2)^{-1})\,q\,\frac{(q^4-1)(q+1)}{(q^2-1)(q^5+1)} - \frac{(q^2+1)}{(q^5+1)}\,\varepsilon(3,2)^{-1} \\
&\leq 1 + \left(1 - \frac{q-1}{q^4-1}\right) q\,\frac{(q^2+1)(q+1)}{(q^5+1)} \\
&= 1 + \frac{q^2(q^3-1)}{(q-1)(q^5+1)} \\
&\leq 2.
\end{aligned}$$

In summary, we conclude

$$\frac{|a_{0,0}|}{|a_{0,1}|} = \frac{|a'_{0,0}|}{|a'_{0,1}|}\,\frac{(1-\varepsilon_{0,0})}{(1-\varepsilon_{0,1})} > 1.$$

This completes the proof. □

It remains to show that the dual distribution is nonnegative.

Proposition 5.3.24. *For $2 \leq d \leq n$, all entries of the dual distribution (A'_k) given in Proposition 5.3.20 are nonnegative.*

Proof. Let (A'_k) be given in Proposition 5.3.20. We first show that $A'_1 \geq 0$ for $2 \leq d \leq n$. The sign of $(q)_{d-1}/(b^n)_{d-1}$ is $(-1)^n$ and from (5.47), we thus find

$$A'_1 = |X| q^{-d+1} \left| \frac{(q)_{d-1}}{(b^n)_{d-1}} \right| \frac{(q^n - (-1)^n)}{(q^{n-d+1} + (-1)^n)} ((-1)^n - (-1)^n \varepsilon(n,d)).$$

From Lemma 5.3.21, we see that $\varepsilon(n,d) \geq 1$ for odd n and $\varepsilon(n,d) < 0$ for even n. Thus, we have $(-1)^n - (-1)^n\varepsilon(n,d) \geq 0$ implying $A'_1 \geq 0$, as required. Observe that we can now consider $d \geq 4$ since for $d = 2$, we only need to show $A'_1 \geq 0$.

Set $b_{k,j} = b'_{k,j}(1 - \delta_{k,j})$ with $\delta_{k,j}$ as in (5.52) and

$$b'_{k,j} = b^{\binom{j}{2} - nj} \frac{(b^{-n-1-k})_{n-k-j}}{(-b^{-n})_{n-k-j}} \begin{bmatrix} n-k \\ j \end{bmatrix}_b$$

for all $k = 0, 1, \ldots, d-3$ and $j = 0, 1, \ldots, d-k-3$, so that

$$A'_{n-k} = c_k \sum_{j=0}^{d-k-3} b_{k,j}$$

as in (5.51). Since the sign of $(qb^k)_{n-k}/(-b^{-n})_{n-k}$ is $(-1)^{\binom{n-k}{2}+(n-k)(k+1)}$, we obtain $c_k \geq 0$. Hence, we need to show that $\sum_{j=0}^{d-k-3} b_{k,j} \geq 0$. Observe that this will follow from proving that $(b_{k,j})$ satisfies (5.15). The proof is split into some intermediate results, starting with bounds for $|\delta_{k,j}|$.

Claim 1. *For all $q \geq 2$ and $n \geq 8$, we have*

$$|\delta_{k,j}| \leq \begin{cases} 0.0032 & \text{if } 0 \leq k+j \leq d-4, \\ 0.05 & \text{if } k+j = d-3. \end{cases}$$

For all $q \geq 4$ and $n \geq 4$, we have

$$|\delta_{k,j}| \leq \begin{cases} 0.014 & \text{if } 0 \leq k+j \leq d-4, \\ 0.095 & \text{if } k+j = d-3. \end{cases}$$

Proof of Claim 1. For $0 \leq k+j \leq d-3$, we have

$$\begin{aligned} |\delta_{k,j}| \leq |\varepsilon(n,d)| &\left| \frac{(qb^{k+j})_{d-k-j-1}}{(b^{n+k+j+1})_{d-k-j-1}} \right| \\ &+ (1 + |\varepsilon(n,d)|) q^{k+j-d+1} \frac{(q^{n-k-j}+1)}{(q^{n-d+1}-1)} \left| \frac{(qb^{k+j})_{d-k-j-1}}{(b^{n+k+j})_{d-k-j-1}} \right|. \end{aligned}$$

Using $|\varepsilon(n,d)| \leq (q^{n+d-1}+1)/(q^{d-1}-1)$ from Lemma 5.3.21 gives

$$\begin{aligned} |\delta_{k,j}| \leq \frac{(q^{n+d-1}+1)}{(q^{d-1}-1)} &\left| \frac{(qb^{k+j})_{d-k-j-1}}{(b^{n+k+j+1})_{d-k-j-1}} \right| \\ &+ q^{k+j} \frac{(q^n+1)(q^{n-k-j}+1)}{(q^{d-1}-1)(q^{n-d+1}-1)} \left| \frac{(qb^{k+j})_{d-k-j-1}}{(b^{n+k+j})_{d-k-j-1}} \right|. \end{aligned} \tag{5.65}$$

For $m \in \{n, n+1\}$, by using (3.20) and (5.28), we find

$$\left| \frac{(qb^{k+j})_{d-k-j-1}}{(b^{m+k+j})_{d-k-j-1}} \right| \leq \prod_{\ell=0}^{d-k-j-2} \frac{q^{k+j+1+\ell}+1}{q^{m+k+j+\ell}-1} \leq 10q^{-(m-1)(d-k-j-1)}.$$

This gives us

$$\begin{aligned} |\delta_{k,j}| \leq 10q^{-n(d-k-j-1)} &\frac{q^{n+d-1}+1}{q^{d-1}-1} \\ &+ 10q^{k+j-(n-1)(d-k-j-1)} \frac{(q^n+1)(q^{n-k-j}+1)}{(q^{d-1}-1)(q^{n-d+1}-1)}, \end{aligned}$$

which becomes

$$\begin{aligned}|\delta_{k,j}| \le 10q^{-n(d-k-j-2)}\,&\frac{1+q^{-n-d+1}}{1-q^{-d+1}}\\&+10q^{-(n-1)(d-k-j-2)+1}\frac{(1+q^{-n})(1+q^{-n+k+j})}{(1-q^{-d+1})(1-q^{-n+d-1})}.\end{aligned}$$

For all $n \ge 8$, $q \ge 2$, and $0 \le k+j \le d-4$, we obtain $|\delta_{k,j}| \le 0.0032$. For all $n \ge 4$, $q \ge 4$, and $0 \le k+j \le d-4$, we obtain $|\delta_{k,j}| \le 0.014$. In the case of $k+j = d-3$, we find from (5.65) that

$$\begin{aligned}|\delta_{k,j}| &\le \frac{q^{d-2}+1}{q^{n+d-2}-1}+q^{d-3}\frac{(q^n+1)(q^{n-d+3}+1)(q^{d-2}+1)}{(q^{n-d+1}-1)(q^{n+d-3}-1)(q^{n+d-2}+1)}\\&\le q^{-n}\,\frac{1+q^{-d+2}}{1-q^{-n-d+2}}+q^{-n+2}\frac{(1+q^{-n})(1+q^{-n+d-3})(1+q^{-d+2})}{(1-q^{-n+d-1})(1-q^{-n-d+3})}\end{aligned}$$

For all $n \ge 8$ and $q \ge 2$, we obtain $|\delta_{k,j}| \le 0.05$, and for all $n \ge 4$ and $q \ge 4$, we obtain $|\delta_{k,j}| \le 0.095$. This completes the proof of Claim 1.

Claim 2. *For all $q \ge 2$, $n \ge 4$, $k = 0, 1, \ldots, d-3$, and $j = 0, 1, \ldots, d-k-3$, we have*

$$\frac{|b'_{k,j}|}{|b'_{k,j+2}|} \ge q^{2j+2k+1}\,\frac{1-q^{-2k-j-2}}{1+q^{-k-j-1}}.$$

Proof of Claim 2. Using the definition of the q-Pochhammer symbol, we see that

$$\frac{|b'_{k,j}|}{|b'_{k,j+2}|} = \left|\frac{b^{\binom{j}{2}-nj}\left[{n-k \atop j}\right]_b(1-b^{-2k-j-3})(1-b^{-2k-j-2})}{b^{\binom{j+2}{2}-n(j+2)}\left[{n-k \atop j+2}\right]_b(1+b^{-k-j-2})(1+b^{-k-j-1})}\right|.$$

Applying (5.12) gives

$$\frac{|b'_{k,j}|}{|b'_{k,j+2}|} \ge q^{2j+2k+1}\frac{(1+q^{-2k-j-3})(1-q^{-2k-j-2})}{(1-q^{-k-j-2})(1+q^{-k-j-1})} \ge q^{2j+2k+1}\,\frac{1-q^{-2k-j-2}}{1+q^{-k-j-1}},$$

as wanted.

Claim 3. *For all $q \ge 2$ and $n \ge 8$, we have $|b_{k,0}|/|b_{k,1}| \ge 2$ for all $k \ge 1$. For all $q \ge 4$ and $n \ge 4$, we have $|b_{k,0}|/|b_{k,1}| \ge 2$ for all $k \ge 0$.*

Proof of Claim 3. By using the definition of the q-Pochhammer symbol, we have

$$\frac{|b'_{k,0}|}{|b'_{k,1}|} = \left|\frac{1-b^{-2k-2}}{b^{-n}(1+b^{-k-1})\left[{n-k \atop 1}\right]_b}\right| = \frac{(1-q^{-2k-2})(q+1)}{q^{-n}(1+q^{-k-1})(q^{n-k}+1)}.$$

For $q \ge 2$, $n \ge 8$, and $k \ge 1$, we have

$$\frac{|b'_{k,0}|}{|b'_{k,1}|} \ge \frac{(1-q^{-4})(q+1)}{q^{-n}(1+q^{-2})(q^{n-1}+1)} \ge \frac{128}{43}.$$

For $q \geq 4$, $n \geq 4$, and $k \geq 0$, we obtain

$$\frac{|b'_{k,0}|}{|b'_{k,1}|} \geq \frac{(1-q^{-2})(q+1)}{q^{-n}(1+q^{-2})(q^n+1)} \geq \frac{960}{257}.$$

Together with Claim 1, this proves Claim 3.

Claim 4. *The sign of $b_{k,j}$ is $(-1)^{\binom{j}{2}+kj+j}$ for all $j,k \geq 0$ if $q \geq 2$ and $n \geq 8$, or if $q \geq 4$ and $n \geq 4$.*

Proof of Claim 4. This follows from Claim 1, which gives $1-\delta_{k,j} \geq 0$, and from

$$\frac{(b^{-n-1-k})_{n-k-j}}{(-b^{-n})_{n-k-j}} \geq 0.$$

We now combine the results from the previous claims to show that $b_{k,j}$ satisfies (5.15). Assume that $q \geq 2$ and $n \geq 8$, or that $q \geq 4$ and $n \geq 4$. Because of Claim 4, we then have

$$\frac{b_{k,2j}}{|b_{k,2j+2}|} = \frac{|b_{k,2j}|}{|b_{k,2j+2}|} \quad \text{for all } k \geq 0 \text{ and even } j \geq 0$$

and

$$\frac{b_{k,2j+1}}{|b_{k,2j+3}|} = \frac{|b_{k,2j+1}|}{|b_{k,2j+3}|} \quad \text{for all } k,j \geq 0 \text{ with } k+j \equiv 1 \pmod 2.$$

We can thus look at $|b_{k,j}|/|b_{k,j+2}|$. Claim 1 and 2 imply

$$\frac{|b_{k,j}|}{|b_{k,j+2}|} \geq \frac{|b'_{k,j}|}{|b'_{k,j+2}|}\,\frac{(1-|\delta_{k,j}|)}{(1+|\delta_{k,j+2}|)} \geq 2$$

for all $k,j \geq 0$ if $q \geq 4$, $n \geq 4$, or for all $k,j \geq 0$ with $k+j \neq 0$ if $q \geq 2$, $n \geq 8$, which also gives

$$\frac{|b_{k,0}|}{|b_{k,1}|+|b_{k,2}|} \geq 1$$

in the respective cases by using Claim 3.

It remains to look at $q \geq 2$, $n \geq 8$, and $k = j = 0$, that is

$$\frac{|b_{0,0}|}{|b_{0,1}|+|b_{0,2}|}.$$

Using the definition of the q-Pochhammer symbol, we see that

$$\begin{aligned}\frac{|b_{0,0}|}{|b_{0,1}|+|b_{0,2}|} &= \frac{1-\delta_{0,0}}{q^{-n}\frac{(1-q^{-1})}{(1-q^{-2})}\left|\begin{bmatrix}n\\1\end{bmatrix}_b\right|(1-\delta_{0,1}) + q^{1-2n}\frac{(1-q^{-1})(1+q^{-2})}{(1-q^{-2})(1+q^{-3})}\left|\begin{bmatrix}n\\2\end{bmatrix}_b\right|(1-\delta_{0,2})} \\ &\geq \frac{1-\delta_{0,0}}{q^{-n}\frac{1}{(1-q^{-2})}\left|\begin{bmatrix}n\\1\end{bmatrix}_b\right|(1-\delta_{0,1}) + q^{1-2n}\frac{(1+q^{-2})}{(1-q^{-2})}\left|\begin{bmatrix}n\\2\end{bmatrix}_b\right|(1-\delta_{0,2})}.\end{aligned}$$

By applying

$$\left| \begin{bmatrix} n \\ 1 \end{bmatrix}_b \right| \leq \frac{q^n+1}{q+1} \quad \text{and} \quad \left| \begin{bmatrix} n \\ 2 \end{bmatrix}_b \right| \leq \frac{(q^n-1)(q^{n-1}+1)}{(q+1)(q^2-1)},$$

we find

$$\frac{|b_{0,0}|}{|b_{0,1}|+|b_{0,2}|} \geq \frac{1-\delta_{0,0}}{\frac{(1+q^{-n})}{(1-q^{-2})(q+1)}(1-\delta_{0,1})+\frac{(1+q^{-2})(1+q^{-n+1})}{(1-q^{-2})(q+1)(q^2-1)}(1-\delta_{0,2})}.$$

By using $q \geq 2$ and Claim 1, this becomes

$$\frac{|b_{0,0}|}{|b_{0,1}|+|b_{0,2}|} \geq \frac{1-|\delta_{0,0}|}{\frac{257}{576}(1+|\delta_{0,1}|)+\frac{215}{1152}(1+|\delta_{2,2}|)} \geq 1,$$

as wanted.

For the remaining cases with $q \in \{2,3\}$, $n \in \{4,5,6,7\}$, and $d \in \{4,\dots,n\}$, the dual distribution is nonnegative as well, which was checked with a computer. This concludes the proof. □

Combining Proposition 5.3.23 and 5.3.24 proves Proposition 5.3.18.

5.4 Open problems

We close this chapter by giving a list of open problems related to the LP optimum in classical association schemes.

Problem 5.4.1. *Prove Conjecture 5.1.4 concerning the LP optimum in D_n for odd d and odd n.*

Problem 5.4.2. *Determine the LP optimum for d-codes in a polar space $\mathcal{P}$ of rank n if*

(*a*) $\mathcal{P} = B_n$ *or* C_n *and d is even*

(*b*) $\mathcal{P} = {}^2D_{n+1}$

(*c*) $\mathcal{P} = {}^2A_{2n}$

Numerical computations of the LP optimum for small values of q, n, and d suggest that the LP optimum is strictly smaller than the bound of Corollary 3.2.4 in the cases (a)–(c).

Problem 5.4.3. *Determine the LP optimum for d-codes in the symmetric bilinear forms scheme* $\mathrm{Sym}_q(n)$ *if q is odd and n and d are even or if q is even and d is even.*

This problem goes back to a conjecture given in [Sch15, p. 647]. A d-code Y in $\mathrm{Sym}_q(n)$ is a set of symmetric $n \times n$ matrices over $\mathbb{F}_q$ such that $\mathrm{rank}(x-y) \geq d$ for all distinct $x, y \in Y$. In [Sch15], the case q odd was studied and sharp bounds were proved for d-codes with odd d, which were obtained by using linear programming and are thus exactly the LP optimum. Moreover, it was conjectured in [Sch15] that the LP optimum for even n and even d is given by

$$q^{(n+1)(n-d+2)/2}\,\frac{1+q^{-n+d-1}}{q+1}.$$

For odd n and even d, a bound was also determined in [Sch15], but it is unknown so far what the LP optimum might look like. In [Sch10], the case q even was treated and sharp bounds were obtained for odd d by using linear programming as well. Thus, the LP optimum is also known for odd d and even q. But the LP optimum for even d and even q is unknown.

Problem 5.4.4. *Determine the LP optimum in the Hamming scheme $H(n,q)$ and the Johnson scheme $J(n,m)$.*

It was stated in [Del73, p. 55] that the LP optimum in $H(n,q)$ is precisely the Singleton bound (2.25) if $q \geq \max\{d, n-d+2\}$. Otherwise, it is still unknown what the LP optimum in $H(n,q)$ looks like. The case $q=2$ for the Hamming scheme is the most important and challenging problem concerning the LP optimum in classical association schemes. Here, one also looks at the asymptotic behavior of the maximal number $A(n,d)$ of elements in a d-code in $H(n,2)$, when $n \to \infty$ and $0 \leq d/n \leq 1$. A comparison of this asymptotic behavior for various bounds in the binary Hamming scheme is given in Figure 5.1, where the Gilbert-Varshamov bound (GV bound) is a lower bound and the others are all upper bounds. In Figure 5.1, we used a weaker version of the McEliece-Rodemich-Rumsey-Welch bound (MRRW bound), which equals its stronger version for $\delta > 0.273$ and we note that for $\delta \leq 0.273$, the Elias bound is not as good as the stronger version of the MRRW bound [McE+77], see also [MS77, § 17] or [Lin99, § 5] for an overview of the bounds occurring in Figure 5.1. It is known that the asymptotic version of the LP optimum strictly exceeds the GV bound and is at least as large as the arithmetic mean of the weaker MRRW bound and the GV bound, see the gray area in Figure 5.1. It was conjectured in [Sam01] that this arithmetic mean of the weaker MRRW bound and the GV bound is precisely the LP optimum. However, it was pointed out in [BJ01] that this conjecture might be false and that the stronger MRRW bound might be the asymptotic LP optimum.

Similarly, for the Johnson scheme, there exists a GV bound and an MRRW bound for binary constant-weight codes and the asymptotic version of the LP optimum is as large as the arithmetic mean of the MRRW bound and the GV bound [Sam01].

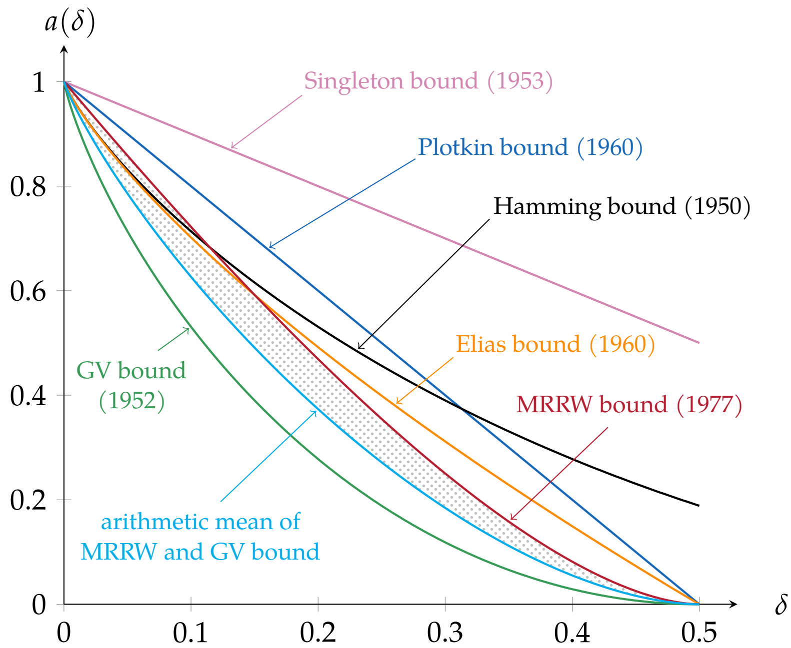
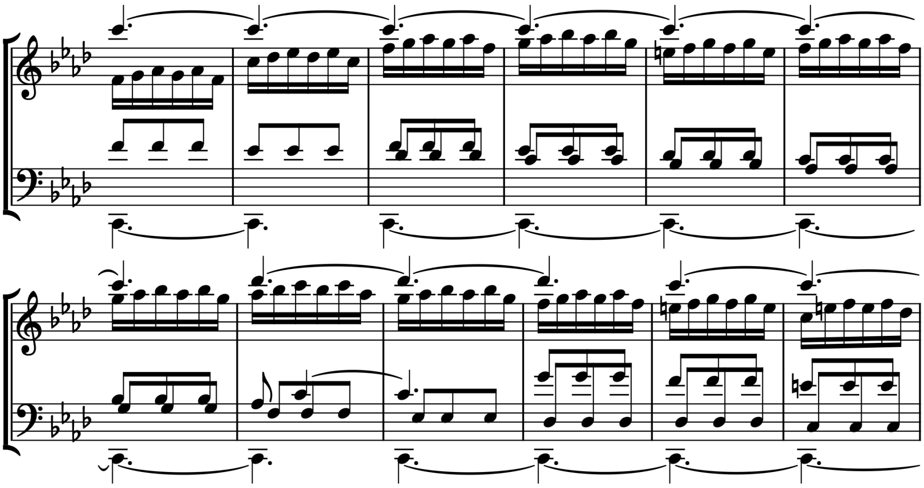


FIGURE 5.1. A comparison of the asymptotic bounds for codes in the binary Hamming scheme, where $a(\delta) = \limsup_{n\to\infty} n^{-1} \log_2 A(n, \lceil \delta n \rceil)$ for $0 \le \delta \le 1/2$.

Problem 5.4.5. *Is there a combinatorial proof for the Singleton bound for* $\mathrm{Her}_q(n)$ *and* $\mathrm{Alt}_q(m)$*?*

In the case of $\mathrm{Bil}_q(n, m)$, there exists a combinatorial proof for the Singleton bound (1.1). Presently, it is unknown whether the Singleton bound for $\mathrm{Her}_q(n)$ and $\mathrm{Alt}_q(m)$ can be proved purely combinatorially. This problem is similar to Problem 3.4.1.



—Reduction of *Winter 3* (excerpt 40″–50″)
from *Recomposed by Max Richter: Vivaldi - The Four Seasons*

Chapter 6

Summary

The most notable results of this thesis are briefly summarized below in a simplified version. (See Table A.1 for the notation of the association schemes.) Open problems related to the investigated topics can be found in Section 3.4, 4.3, and 5.4, which pose future research directions.

(R1) In Theorem 3.2.1 on p. 54, a bound for codes in $J_q(n, m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ was derived in a unified way, where in the case of $J_q(n, m)$, it is the well-known bound from [WXS03], showing that there exists a bound of a similar form for ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$ as for $J_q(n, m)$. The proof of Theorem 3.2.1 relies on the fact that the P- and Q-numbers of $J_q(n, m)$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$ can be written in a similar way as dual q-Hahn polynomials and q-Hahn polynomials, where we used the second ordering of the primitive idempotents in the case of ${}^2A_{2n-1}$.

(R2) In Corollary 3.2.4 on p. 60, we applied the bounds for codes in ${}^2A_{2n-1}$ and $\frac{1}{2}D_n$ from (R1) to derive bounds for codes in all the remaining polar spaces B_n, C_n, D_n, ${}^2A_{2n}$, and ${}^2D_{n+1}$.

(R3) In Section 3.3, we constructed codes in all polar spaces by "lifting" known codes in the alternating bilinear, Hermitian, and symmetric bilinear forms schemes. Moreover, we proved that the constructed codes reach the respective bound obtained in (R1) and (R2) up to a constant factor for ${}^2A_{2n-1}$ with odd d; C_n with odd d; B_n with odd d and even q; and D_n except possibly for even n and odd q; whereas in the remaining cases for all polar spaces, the bounds are met up to a small power of q^n.

(R4) In Proposition 4.2.1, we computed the inner distribution of Steiner systems in polar spaces.

(R5) In Theorem 4.1.1 on p. 68, we provided an almost complete classification of nontrivial Steiner systems in polar spaces; namely for $1 < t < n$, if a t-Steiner system in a polar space of rank n exists, then we are in D_n

with $t = n - 1$ (and in this case, it is one of the bipartite halves $\frac{1}{2}D_n$) or possibly in ${}^2A_{2n}$ or ${}^2D_{n+1}$ with $t = n - 1$ and $q \geq 3$, or in ${}^2A_{2n}$ or ${}^2D_{n+1}$ with odd n and $q \geq 2$. We moreover posed Conjecture 4.1.2 on p. 69 stating that t-Steiner systems with $1 < t < n$ can only exist in D_n for $t = n - 1$.

(R6) Theorem 5.1.1 on p. 82 contains the optimum of the linear program for codes in $J_q(n, m)$, $\frac{1}{2}D_m$, and ${}^2A_{2n-1}$ as well as for their affine counterparts $\mathrm{Bil}_q(n, m)$, $\mathrm{Alt}_q(m)$, and $\mathrm{Her}_q(n)$. In the last three cases, the optimum was basically known except for $\mathrm{Alt}_q(m)$ with even m and odd q and for $\mathrm{Her}_q(n)$ with even d. We gave a proof for all parameters. It is remarkable that the optimum in $J_q(n, m)$, $\frac{1}{2}D_m$, and ${}^2A_{2n-1}$ has a similar form as the optimum in their affine counterpart. Moreover, we derived the optimum for d-codes in D_n if d is even and in B_n and C_n if d is odd. The optimum for $J_q(n, m)$ is precisely the bound from [WXS03] and the optima for $\frac{1}{2}D_m$, ${}^2A_{2n-1}$, D_n, B_n, and C_n are precisely the bounds obtained in (R1) and (R2). Additionally, we posed Conjecture 5.1.4 stating the optimum in D_n if d and n are odd.

Appendix A

List of classical association schemes

This chapter contains a tabular overview of the studied classical association schemes.

Table A.1. Summary of the studied classical association schemes divided into three categories: (a) basic, (b) ordinary q-analog, and (c) affine q-analog, together with their type and group.

	association scheme	type	**group**
(a)	Johnson scheme $J(n,m)$	A_{m+n-1}	S_{m+n}
	Hamming scheme $H(n,q)$	B_n or C_n	$S_q^n \rtimes S_n$
	bipartite half $\frac{1}{2}H(n,2)$ of $H(n,2)$	D_n	$S_2^{n-1} \rtimes S_n$
(b)	q-Johnson scheme $J_q(n,m)$	A_{m+n-1}	$\mathrm{GL}_{m+n}(q)$
	parabolic polar space scheme	B_n	$\mathrm{SO}_{2n+1}(q)$
	symplectic polar space scheme	C_n	$\mathrm{Sp}_{2n}(q)$
	elliptic polar space scheme	${}^2D_{n+1}$	$\mathrm{SO}^-_{2n+2}(q)$
	Hermitian polar space schemes	${}^2A_m$	$\mathrm{SU}_m(q^2)$
	hyperbolic polar space scheme	D_n	$\mathrm{O}^+_{2n}(q)$
	bipartite half $\frac{1}{2}D_n$ of D_n	D_n	$\mathrm{SO}^+_{2n}(q)$
(c)	bilinear forms scheme $\mathrm{Bil}_q(n,m)$	A_{m+n-1}	$X \rtimes (\mathrm{GL}_m(q) \times \mathrm{GL}_n(q))$
	Hermitian forms scheme $\mathrm{Her}_q(n)$	${}^2A_m$	$X \rtimes \mathrm{GL}_n(q^2)$
	alternating bilinear forms scheme $\mathrm{Alt}_q(m)$	D_n	$X \rtimes \mathrm{GL}_m(q)$

TABLE A.2. Classical association schemes and their orthogonal polynomials $f_i(z)$ of degree i evaluated at y_k such that $P_i(k) = v_i f_i(y_k)$.

association scheme	**polynomial**	$f_i(y_k)$	v_i
$H(n,q)$	Krawtchouk	${}_2F_1\left(\begin{matrix}-i,-k\\-n\end{matrix}\,\middle\|\,\frac{q}{q-1}\right)$	$(q-1)^i\binom{n}{i}$
$J(n,m)$	dual Hahn	${}_3F_2\left(\begin{matrix}-i,-k,k-m-n-1\\-m,-n\end{matrix}\,\middle\|\,1\right)$	$\binom{n}{i}\binom{m}{i}$
$\frac{1}{2}H(n,2)$	dual Hahn	${}_3F_2\left(\begin{matrix}-i,-k,k-n\\-\frac{n}{2},\frac{-n+1}{2}\end{matrix}\,\middle\|\,1\right)$	$\binom{n}{2i}$
$B_n, C_n, D_n,$ ${}^2D_{n+1}, {}^2A_{2n-1}, {}^2A_{2n}$	q-Krawtchouk	${}_3\phi_2\left(\begin{matrix}p^{-k},p^{-i},-p^{-n-e-1+k}\\0,p^{-n}\end{matrix}\,\middle\|\,p;p\right)$	$p^{\binom{i+1}{2}+ie}\begin{bmatrix}n\\i\end{bmatrix}_p$
$J_q(n,m)$	dual q-Hahn	${}_3\phi_2\left(\begin{matrix}q^{-i},q^{-k},q^{k-m-n-1}\\q^{-m},q^{-n}\end{matrix}\,\middle\|\,q;q\right)$	$q^{i^2}\begin{bmatrix}n\\i\end{bmatrix}_q\begin{bmatrix}m\\i\end{bmatrix}_q$

$\frac{1}{2}D_m$	dual q-Hahn	${}_3\phi_2\left(\begin{matrix} q^{-2i}, q^{-2k}, q^{-2m+2k} \\ q^{-m}, q^{-m+1} \end{matrix} \middle\vert q^2; q^2\right)$	$q^{\binom{2i}{2}}\begin{bmatrix} m \\ 2i \end{bmatrix}_q$
${}^2A_{2n-1}$ (2nd ordering)	dual q-Hahn	${}_3\phi_2\left(\begin{matrix} (-q)^{-i}, (-q)^{-k}, (-q)^{-2n+k-1} \\ (-q)^{-n}, -(-q)^{-n} \end{matrix} \middle\vert -q; -q\right)$	$q^{i^2}\begin{bmatrix} n \\ i \end{bmatrix}_{q^2}$
$\mathrm{Bil}_q(n, m)$	affine q-Krawtchouk	${}_3\phi_2\left(\begin{matrix} q^{-k}, q^{-i}, 0 \\ q^{-m}, q^{-n} \end{matrix} \middle\vert q; q\right)$	$q^{\binom{i}{2}}\begin{bmatrix} n \\ i \end{bmatrix}_q \prod_{j=0}^{i-1}(q^{m-j}-1)$
$\mathrm{Alt}_q(m)$	affine q-Krawtchouk	${}_3\phi_2\left(\begin{matrix} q^{-2k}, q^{-2i}, 0 \\ q^{-m}, q^{-m+1} \end{matrix} \middle\vert q^2; q^2\right)$	$q^{i^2-i}\begin{bmatrix} \lfloor m/2 \rfloor \\ i \end{bmatrix}_{q^2} \prod_{j=0}^{i-1}(q^{2\lceil m/2 \rceil-2j-1}-1)$
$\mathrm{Her}_q(n)$	affine q-Krawtchouk	${}_3\phi_2\left(\begin{matrix} (-q)^{-k}, (-q)^{-i}, 0 \\ (-q)^{-n}, -(-q)^{-n} \end{matrix} \middle\vert -q; -q\right)$	$(-q)^{\binom{i}{2}}\begin{bmatrix} n \\ i \end{bmatrix}_{-q} \prod_{j=0}^{i-1}(-(-q)^{n-j}-1)$

Table A.4. Classical association schemes and their orthogonal polynomials $g_k(z)$ of degree k evaluated at z_k such that $Q_k(i) = \mu_k g_k(z_i)$.

association scheme	**polynomial**	$g_k(z_i)$	μ_k
$H(n,q)$	Krawtchouk	${}_2F_1\left(\begin{matrix}-i,-k\\ -n\end{matrix}\,\middle\vert\,\frac{q}{q-1}\right)$	$(q-1)^k\binom{n}{k}$
$J(n,m)$	Hahn	${}_3F_2\left(\begin{matrix}-i,-k,k-v-1\\ -v+n,-n\end{matrix}\,\middle\vert\,1\right)$	$\binom{m+n}{k}-\binom{m+n}{k-1}$
$\frac{1}{2}H(n,2)$	Hahn	${}_3F_2\left(\begin{matrix}-i,-k,k-n\\ -\frac{n}{2},\frac{-n+1}{2}\end{matrix}\,\middle\vert\,1\right)$	$\binom{n}{k}$
$B_n, C_n, D_n,$ ${}^2D_{n+1}, {}^2A_{2n-1}, {}^2A_{2n}$	q-Krawtchouk	${}_3\phi_2\left(\begin{matrix}p^{-k},p^{-i},-p^{-n-e-1+k}\\ 0,p^{-n}\end{matrix}\,\middle\vert\,p;p\right)$	$p^{k(k-n)}\begin{bmatrix}n\\ k\end{bmatrix}_p\dfrac{(-p^{e+1};p)_n}{(-p^{e-k+1};p)_{n-k}(-p^{n-k-e-1};p)_k}$
$J_q(n,m)$	q-Hahn	${}_3\phi_2\left(\begin{matrix}q^{-i},q^{-k},q^{k-m-n-1}\\ q^{-m},q^{-n}\end{matrix}\,\middle\vert\,q;q\right)$	$\begin{bmatrix}m+n\\ k\end{bmatrix}_q-\begin{bmatrix}m+n\\ k-1\end{bmatrix}_q$

$\frac{1}{2}D_m$	q-Hahn	${}_3\phi_2\left(\begin{matrix} q^{-2i}, q^{-2k}, q^{-2m+2k} \\ q^{-m}, q^{-m+1} \end{matrix}\middle\vert\, q^2; q^2\right)$	$q^{k(k-m)}\begin{bmatrix} m \\ k \end{bmatrix}_q \dfrac{(-1;q)_m}{(-q^{-k};q)_{m-k}(-q^{m-k};q)_k}$
${}^2A_{2n-1}$ (2nd ordering)	q-Hahn	${}_3\phi_2\left(\begin{matrix} (-q)^{-i}, (-q)^{-k}, (-q)^{-2n+k-1} \\ (-q)^{-n}, -(-q)^{-n} \end{matrix}\middle\vert\, -q; -q\right)$	$\mu_{k/2}$ or $\mu_{n-(k-1)/2}$ from the natural ordering
$\mathrm{Bil}_q(n,m)$	affine q-Krawtchouk	${}_3\phi_2\left(\begin{matrix} q^{-k}, q^{-i}, 0 \\ q^{-m}, q^{-n} \end{matrix}\middle\vert\, q; q\right)$	$q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1}(q^{m-j}-1)$
$\mathrm{Alt}_q(m)$	affine q-Krawtchouk	${}_3\phi_2\left(\begin{matrix} q^{-2k}, q^{-2i}, 0 \\ q^{-m}, q^{-m+1} \end{matrix}\middle\vert\, q^2; q^2\right)$	$q^{k^2-k}\begin{bmatrix} \lfloor m/2 \rfloor \\ k \end{bmatrix}_{q^2} \prod_{j=0}^{k-1}(q^{2\lceil m/2 \rceil-2j-1}-1)$
$\mathrm{Her}_q(n)$	affine q-Krawtchouk	${}_3\phi_2\left(\begin{matrix} (-q)^{-k}, (-q)^{-i}, 0 \\ (-q)^{-n}, -(-q)^{-n} \end{matrix}\middle\vert\, -q; -q\right)$	$(-q)^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{-q} \prod_{j=0}^{k-1}(-(-q)^{n-j}-1)$

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$\circ$	Hadamard product	
$a^{(i)}$	Pochhammer symbol	21
$(a;q)_n$	q-Pochhammer symbol	38
$(x)_i$	q-Pochhammer symbol $(x;b)_i$	54
$\binom{n}{k}$	binomial coefficient	
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	q-binomial coefficient	37
$[n]_q$	q-number	38
$A = (A_0, A_1, \ldots, A_n)$	inner distribution of a subset of an association scheme	27
${}^2A_{2n-1}$	Hermitian polar space of even dimension	40
${}^2A_{2n}$	Hermitian polar space of odd dimension	40
$A' = (A'_0, A'_1, \ldots, A'_n)$	dual distribution of a subset of an association scheme	28
$\alpha(n,d)$	bound for d-codes in ${}^2A_{2n-1}$	59
$\mathrm{Alt}_q(m)$	alternating forms scheme	48
A^T	transpose of a matrix A	
$\mathrm{Aut}(G)$	automorphism group of a group G	
b	parameter used for $\mathrm{Bil}_q(n,m)$, $\mathrm{Alt}_q(m)$, $\mathrm{Her}_q(n)$, $J_q(n,m)$, $\frac{1}{2}D_m$, and ${}^2A_{2n-1}$	50
$\mathcal{B}$	Bose-Mesner algebra of an association scheme	16
$\beta(m,d)$	bound for d-codes in $\frac{1}{2}D_m$	60
$\mathrm{Bil}_q(n,m)$	bilinear forms scheme (also known as q-Hamming scheme)	47
B_n	parabolic polar space	41
c	parameter used for $\mathrm{Bil}_q(n,m)$, $\mathrm{Alt}_q(m)$, $\mathrm{Her}_q(n)$, $J_q(n,m)$, $\frac{1}{2}D_m$, and ${}^2A_{2n-1}$	50
$\mathbb{C}$	the set of complex numbers	
C_n	symplectic polar space	41
$\delta_{x,y}$	Kronecker delta	
d_H	Hamming metric	13

D_i	adjacency matrix of the relation R_i of an association scheme	16
D_n	hyperbolic polar space	41
${}^2D_{n+1}$	elliptic polar space	41
$\frac{1}{2}D_n$	bipartite half of the hyperbolic polar space D_n	45
e	parameter associated to a polar space	40
$E_i(\mu(x); C, D, n; q)$	dual q-Hahn polynomial of degree i in $\mu(x)$	38
$E_i(\lambda(x); C, D, n)$	dual Hahn polynomial of degree i in $\lambda(x)$	22
$\varepsilon(n, d)$	factor occuring in $\alpha(n, d)$	59
$\mathbb{F}_q$	the finite field with q elements	
${}_rF_s$	hypergeometric function	22
$\mathrm{GL}_n(q)$	the general linear group	
$H_k(q^{-x}; A, B, n; q)$	q-Hahn polynomial of degree k in q^{-x}	38
$H_k(x; A, B, n)$	Hahn polynomial of degree k in x	22
$\frac{1}{2}H(n, 2)$	bipartite half of the Hamming scheme $H(n, 2)$	25
$H(n, q)$	Hamming scheme	13
$\mathrm{Her}_q(n)$	Hermitian forms scheme	48
I	identity matrix	16
J	all-ones-matrix	16
$J(n, m)$	Johnson scheme	13
$J_q(n, m)$	q-Johnson scheme	37
$K^{m\times n}$	the ring of $m \times n$ matrices over a field K	
$K^{X\times Y}$	the ring of $\lvert X\rvert \times \lvert Y\rvert$ matrices over a field K, where the rows and columns are indexed by elements of X and Y	
$K_i(x; A, n)$	Krawtchouk polynomial of degree i in x	22
$K_i(q^{-x}; A, n; q)$	q-Krawtchouk polynomial of degree i in q^{-x}	44
$K_i^{\mathrm{aff}}(q^{-x}; B, n; q)$	affine q-Krawtchouk polynomial of degree i in q^{-x}	49
$\mathrm{LP}(D)$	LP optimum of the LP for a D-code	33
$\mathrm{LP}(d)$	LP optimum of the LP for a d-code	33
μ_k	multiplicities of an association scheme	17
$\mathbb{N}$	the set of positive integers	

$\mathbb{N}_0$	the set of nonnegative integers	
$\mathrm{O}_{2n+1}(q)$	group of $(2n+1) \times (2n+1)$ orthogonal matrices over $\mathbb{F}_q$	41
$\mathrm{O}^-_{2n+2}(q)$	group of $(2n+2) \times (2n+2)$ orthogonal matrices over $\mathbb{F}_q$ with sign $-$	41
$\mathrm{O}^+_{2n}(q)$	group of $2n \times 2n$ orthogonal matrices over $\mathbb{F}_q$ with sign $+$	41
P	matrix containing the P-numbers $P_i(k)$	18
ϕ_Y	characteristic vector of a set Y	14
${}_r\phi_s$	q-hypergeometric function	38
p^k_{ij}	intersection numbers of an association scheme	12
$P_i(k)$	P-numbers of an association scheme	17
Q	matrix containing the Q-numbers $Q_k(i)$	18
q^k_{ij}	Krein numbers of an association scheme	19
$Q_k(i)$	Q-numbers of an association scheme	17
$\mathbb{R}$	the set of real numbers	
$\mathcal{R}$	the set of relations of an association scheme	12
S_n	symmetric group on n elements	
$\mathrm{Sp}_{2n}(q)$	group of $2n \times 2n$ symplectic matrices over $\mathbb{F}_q$	41
$\mathrm{Sym}_q(n)$	symmetric bilinear forms scheme	50
$\mathrm{U}_m(q^2)$	group of unitary $m \times m$ matrices over $\mathbb{F}_{q^2}$	40

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