Two Descriptions of the Quantum Group $U_v\left(A_2^{(2)}\right)$ via Hall Algebras and Derived Equivalences

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Abstract

Let $k = \mathbb{F}_q$, $k \subset K$ be a field extension of degree 4 and

$$\Gamma = \begin{pmatrix} k & 0 \\ K & K \end{pmatrix}.$$

The first new result of this thesis is an explicit description of indecomposable preprojective Γ-modules with endomorphism algebra K, which completes the classification given by D. Baer.

The main focus of this thesis belongs to the structure theory of the quantized enveloping algebra $U_v\left(A_2^{(2)}\right)$ based on the study of the category $\mathrm{mod}(\Gamma)$.

By results of C. Ringel and J. A. Green, the reduced Drinfeld double of the composition algebra of $\mathrm{mod}(\Gamma)$ is isomorphic to $U_v\left(A_2^{(2)}\right)$ (where $v = \sqrt{q}$). This algebra admits another realization $U_v^{Dr}\left(A_2^{(2)}\right)$ due to V. G. Drinfeld. It is known that $\mathrm{mod}(\Gamma)$ is derived equivalent to the category $\mathrm{Coh}(\mathbb{X})$ of coherent sheaves on a certain non-commutative projective hereditary curve $\mathbb{X}$. The main result of this thesis is the description of the composition algebra of $\mathrm{Coh}(\mathbb{X})$ via generators and relations. We identify the reduced Drinfeld double of the composition algebra of $\mathrm{Coh}(\mathbb{X})$ with $U_v^{Dr}\left(A_2^{(2)}\right)$ and use the derived equivalence of $\mathrm{mod}(\Gamma)$ and $\mathrm{Coh}(\mathbb{X})$ to recover the Drinfeld-Beck isomorphism between $U_v\left(A_2^{(2)}\right)$ and $U_v^{Dr}\left(A_2^{(2)}\right)$.

On the way, we detect some (minor) mistakes in the work of T. Akasaka, in which this isomorphism was studied in detail.

As an application of this Hall-theoretical approach, we construct an explicit basis of the positive part $U_v^+\left(A_2^{(2)}\right)$ which is orthogonal with respect to the Drinfeld-Rosso form.

Kurzzusammenfassung

Seien $k = \mathbb{F}_q$, $k \subset K$ eine Körpererweiterung von Grad 4 und

$$\Gamma = \begin{pmatrix} k & 0 \\ K & K \end{pmatrix}.$$

Das erste neue Ergebnis dieser Arbeit ist die explizite Beschreibung der unzerlegbaren präprojektiven Γ-Moduln mit Endomorphismenalgebra K. Dies vervollständigt die Klassifikation von D. Baer.

Der Hauptfokus dieser Dissertation liegt auf der Strukturtheorie der quantisierten einhüllenden Algebra $U_v\left(A_2^{(2)}\right)$ ausgehend von der Kategorie $\mathrm{mod}(\Gamma)$.

Durch Ergebnisse von C. Ringel und J. A. Green ist bekannt, dass das reduzierte Drinfeld-Doppel der Kompositionsalgebra von $\mathrm{mod}(\Gamma)$ isomorph ist zu $U_v\left(A_2^{(2)}\right)$ (wobei $v = \sqrt{q}$). Diese Algebra erlaubt auch eine weitere Realisierung $U_v^{Dr}\left(A_2^{(2)}\right)$ nach V. G. Drinfeld. Außerdem ist bekannt, dass $\mathrm{mod}(\Gamma)$ deriviert äquivalent ist zu der Kategorie $\mathrm{Coh}(\mathbb{X})$ von kohärenten Garben einer bestimmten nicht-kommutativen projektiven erblichen Kurve $\mathbb{X}$. Das Hauptergebnis dieser Arbeit ist die Beschreibung der Kompositionsalgebra von $\mathrm{Coh}(\mathbb{X})$ durch Erzeuger und Relationen. Wir identifizieren das reduzierte Drinfeld-Doppel der Kompositionsalgebra von $\mathrm{Coh}(\mathbb{X})$ mit $U_v^{Dr}\left(A_2^{(2)}\right)$ und benutzen die derivierte Äquivalenz von $\mathrm{mod}(\Gamma)$ und $\mathrm{Coh}(\mathbb{X})$, um den Drinfeld-Beck-Isomorphismus zwischen $U_v\left(A_2^{(2)}\right)$ and $U_v^{Dr}\left(A_2^{(2)}\right)$ zu erhalten.

Währenddessen finden wir kleinere Fehler in der Arbeit von T. Akasaka, wo dieser Isomorphismus genauer studiert wurde.

Als eine Anwendung der Hall-theoretischen Herangehensweise konstruieren wir eine explizite Basis des positiven Teils $U_v^+\left(A_2^{(2)}\right)$, die orthogonal ist bezüglich der Drinfeld-Rosso-Form.

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Contents

0 Introduction and some Preliminary Notions

0.1 Introduction

In this dissertation, our main goal is to use a Hall algebra approach to construct an isomorphism between the two presentations of the quantized enveloping algebra $U_v\left(A_2^{(2)}\right)$ via Drinfeld-Jimbo and $U_v^{Dr}\left(A_2^{(2)}\right)$, the so-called Drinfeld representation. Our approach allows us to construct an orthogonal PBW-basis of the positive part of $U_v\left(A_2^{(2)}\right)$ with respect to the Drinfeld-Rosso form.

Quantum groups have been introduced by Drinfeld and Jimbo. They can be considered as deformations of the universal enveloping algebra of a Lie algebra and form an important class of Hopf algebras which are neither commutative nor cocommutative. They have connections to statistical mechanics, conformal field theory, and knot theory. The variety of areas where quantum groups appear leads to numerous tools that can be used to study them (see e.g. [7], [18], or [24]).

The motivation of the dissertation is coming from the categorical perspective of the two descriptions of the affine quantum group $U_v(\hat{\mathfrak{sl}}_2)$ given by Burban and Schiffmann in their paper [6]. In this work, they considered the Hall algebra of two $\mathbb{F}_q$-linear categories: on one side, the Hall algebras of the category of representations of the Kronecker quiver $\vec{Q}$

$$\bullet \rightrightarrows \bullet$$

and on the other side, the Hall algebra of the category of coherent sheaves of the projective line $\mathbb{P}^1$. In particular, they established the second vertical isomorphism in the following diagram as well as the fact that the isomorphism of the reduced Drinfeld doubles of the Hall algebras restricts to the double composition algebras:

$$\begin{array}{ccc}
U_v(\hat{\mathfrak{sl}}_2) & \xrightarrow[\cong]{\text{Drinfeld-Beck}} & U_v(\mathfrak{Lsl}_2) \\
\downarrow \cong & & \downarrow \cong \\
DC(\mathrm{Rep}_{\mathbb{F}_q}(\vec{Q})) & \xrightarrow{\cong} & DC(\mathrm{Coh}(\mathbb{P}^1)) \\
\cap & & \cap \\
D\mathcal{H}(\mathrm{Rep}_{\mathbb{F}_q}(\vec{Q})) & \xrightarrow[\text{Cramer}]{\cong} & D\mathcal{H}(\mathrm{Coh}(\mathbb{P}^1))
\end{array} \tag{1}$$

In this dissertation, we study another affine case $A_2^{(2)}$, corresponding to the species $\bullet \xrightarrow{(1,4)} \bullet$, to obtain a similar commutative diagram.

The concept of a Hall algebra appeared for the first time in a work by Steinitz [33]. He introduced them in regard to the category of abelian groups to be able to say more about symmetric functions. The second time was by Hall [17] as the algebra of partitions. Only later in the 1980s, it was studied more in detail and the theory has developed since.

Given an abelian $\mathbb{F}_q$-linear category with some conditions, one may construct a $\mathbb{C}$-algebra, called the Hall algebra of the category. The basis of the algebra is given by the isomorphism classes of objects and the multiplication of two of these classes encodes the possible extensions. To be more precise, if we multiply two classes $[M] \cdot [N]$ we sum up over all $[R]$ where there exists a short exact sequence

$$0 \to N \to R \to M \to 0$$

with some prefactors. If the category is hereditary, one can give the Hall algebra a Hopf algebra structure using Green's coproduct [16]. The comultiplication can be seen as splitting $[R]$ via short exact sequences. However, if objects have infinitely many subobjects, the sum over all short exact sequences becomes infinite. In these cases, it is only a topological Hopf algebra [5]. In particular, the algebras on the right-hand side of (1) are topological Hopf algebras.

An important and one of the first examples of a Hall algebra is the so-called classical Hall algebra (see Subsection 1.5). It is the Hall algebra of the category of finite modules over a discrete valuation ring with a finite residue field $\mathbb{F}_q$. In particular, choosing the ring $\mathbb{F}_q[[t]]$, one may equivalently consider the category of nilpotent $\mathbb{F}_q$-representations of the Jordan quiver:

The classical Hall algebra is isomorphic to the ring of symmetric functions studied by Macdonald [25]. However, even though the argument in [25] is given for a commutative ring, we may use the same argument in a non-commutative setting of maximal orders.

Further examples of representations of a quiver or even a species $(Q, \underline{d})$ without loops were considered by Ringel [30], [31] and by Green [16]. They have shown that the Hall algebra in these cases, or more precisely their composition algebra $C(\mathrm{Rep}_{\mathbb{F}_q}(Q))$, correspond to the positive part of a quantum group $U_v(\mathfrak{g}_Q)$ for $v = \sqrt{q}$:

$$C(\mathrm{Rep}_{\mathbb{F}_q}(Q)) \xrightarrow{\cong} U_v^+(\mathfrak{g}_Q).$$

For completeness, in Section 2 we give a brief introduction to the representation theory of species. Afterwards, in Subsection 2.5, we consider the concrete example of a species $\bullet \xrightarrow{(1,4)} \bullet$, which gives rise to the quantized enveloping algebra $U_v\left(A_2^{(2)}\right)$. Instead of the representations of the species $\bullet \xrightarrow{(1,4)} \bullet$, one can equivalently consider modules over the matrix algebra

$$\Gamma = \begin{pmatrix} k & 0 \\ K & K \end{pmatrix}.$$

Here $k = \mathbb{F}_q$ is a finite field, and $k \subset K$ is a field extension of degree 4. It was proven by Dlab and Ringel that Γ is tame and the category $\mathrm{mod}(\Gamma)$ decomposes into a preprojective component, a regular component, and a preinjective component:

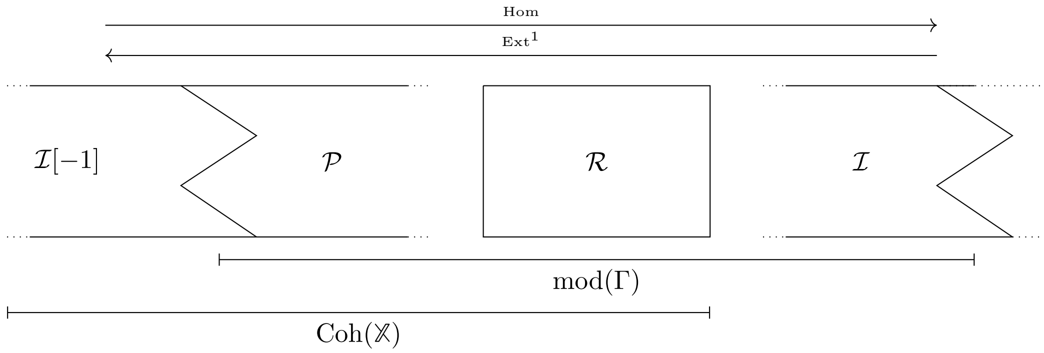
$$\mathrm{mod}(\Gamma) = \mathcal{P} \vee \mathcal{R} \vee \mathcal{I}.$$

Moreover, they have shown that there are two types of indecomposable preprojective objects. In [3], Baer classified the indecomposable preprojective objects with endomorphism ring k. Whereas in this dissertation as the first new result, in Theorem 2.5.1 we explicitly describe the indecomposable preprojective objects with endomorphism ring K. In Subsection 2.6 we complete the picture by describing the isomorphism between the composition algebra of representations of a species and the positive part of the corresponding quantum group due to Ringel and Green.

Until now, the correspondence to quantum groups was to the positive parts. To extend the positive part of a quantum group to the whole quantum group one needs to add the Cartan subalgebra and then apply the reduced Drinfeld double construction [15]. The idea behind the construction is that we have two Hopf algebras A and B with a non-degenerate bilinear Hopf pairing (here $U_v^{\geq 0}(\mathfrak{g})$ and $U_v^{\leq 0}(\mathfrak{g})$) to define a multiplication on $A \otimes B$ such that A and B are both Hopf subalgebras of the tensor product. The final step is the identification of the two copies of the Cartan subalgebras in $U_v^{\geq 0}(\mathfrak{g}) \otimes U_v^{\leq 0}(\mathfrak{g})$ by taking a quotient. The same construction can be performed for Hall algebras, hence we get a presentation of the quantum group as a subalgebra of the reduced Drinfeld double of the corresponding Hall algebra:

$$U_v(\mathfrak{g}_Q) \xrightarrow{\cong} DC(\mathrm{Rep}_{\mathbb{F}_q}(Q)) \subset D\mathcal{H}(\mathrm{Rep}_{\mathbb{F}_q}(Q)).$$

In Section 3 we consider the Hall algebra of another category which is similar to the category of coherent sheaves, hence we suggestively denote it by $\mathrm{Coh}(\mathbb{X})$. The idea of the construction of $\mathrm{Coh}(\mathbb{X})$ is to take the preinjective component which is to the right of the regular component in the Auslander-Reiten quiver of $\mathrm{mod}(\Gamma)$ and glue it to the left of the preprojective component. It is a common method in the theory of exceptional curves (see e.g. [20]).

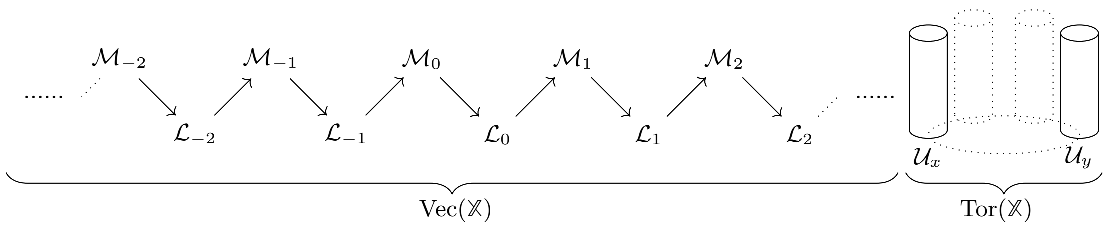


$\mathcal{P}$: prepojective component, $\mathcal{R}$: regular component,
$\mathcal{I}$: preinjective component in $\mathrm{mod}(\Gamma)$

The resulting category $\mathrm{Coh}(\mathbb{X})$ is hereditary, Noetherian, and has a Serre functor τ. Furthermore, similar to the category of coherent sheaves, it decomposes into vector bundles and torsion bundles

$$\mathrm{Coh}(\mathbb{X}) = \mathrm{Vec}(\mathbb{X}) \vee \mathrm{Tor}(\mathbb{X}) = \underbrace{(\mathcal{I}[-1] \vee \mathcal{P})}_{=\mathrm{Vec}(\mathbb{X})} \vee \mathcal{R}.$$

There are indecomposable vector bundles of rank 1, corresponding to the representations with endomorphism ring k, and, contrary to the case of $\mathbb{P}^1$, indecomposable vector bundles of rank 2, corresponding to the representations with endomorphism ring K. Moreover, the Auslander-Reiten quiver of $\mathrm{Coh}(\mathbb{X})$ has the following structure:



More geometrically, $\mathbb{X}$ is the ringed space $(\mathbb{P}^1, \mathcal{A})$, where $\mathcal{A}$ is a certain sheaf of maximal orders (specified by its further properties). Even though $\mathcal{A}_x$ is non-commutative, the modules over $\mathcal{A}_x$ have similar properties to finite modules over a discrete valuation ring. Therefore, the Hall algebra of the torsion part $\mathrm{Tor}(\mathbb{X})$ decomposes into commuting factors, each of which is isomorphic to the ring of symmetric functions analogous to the classical Hall algebra.

By construction, we obtain equivalent bounded derived categories $\mathcal{D}^b(\mathrm{mod}(\Gamma)) \simeq \mathcal{D}^b(\mathrm{Coh}(\mathbb{X}))$ and by a theorem of Cramer (see [9, Theorem 1]), the reduced Drinfeld doubles of the Hall algebras are isomorphic. Using this property and the above-mentioned structure of $\mathrm{Coh}(\mathbb{X})$, we prove our main result, Theorem 4.3.3, in the last section: the presentation of the double composition algebra of $\mathrm{Coh}(\mathbb{X})$ via generators and relations. To be more precise, the double composition algebra $DC(\mathrm{Coh}(\mathbb{X}))$ is isomorphic to the quantum group $U_v^{Dr}\left(A_2^{(2)}\right)$.

To prove Theorem 4.3.3, we introduce the Drinfeld realization of the quantum group of type $A_2^{(2)}$. The definition is given via generators and relations. By an argument due to Damiani [10], we simplify one of the relations, namely by making the indices "constant". Then we prove that the Hall algebra of $\mathrm{Coh}(\mathbb{X})$ has similar relations, which allows us to construct an isomorphism between the double composition algebra of $\mathrm{Coh}(\mathbb{X})$ and the quantum group $U_v^{Dr}\left(A_2^{(2)}\right)$.

Furthermore as an application, in Theorem 4.4.9, we prove that the algebra isomorphism between the reduced Drinfeld doubles of the Hall algebras of $\mathrm{mod}(\Gamma)$ and $\mathrm{Coh}(\mathbb{X})$ restricts to an isomorphism of the double composition algebras. As a result, we obtain a diagram similar to (1):

$$\begin{array}{ccc}
U_v\left(A_2^{(2)}\right) & \xrightarrow[\cong]{} & U_v^{Dr}\left(A_2^{(2)}\right) \\
\downarrow \cong & & \downarrow \cong \\
DC(\mathrm{mod}(\Gamma)) & \xrightarrow{\cong} & DC(\mathrm{Coh}(\mathbb{X})) \\
\cap & & \cap \\
D\mathcal{H}(\mathrm{mod}(\Gamma)) & \xrightarrow{\cong} & D\mathcal{H}(\mathrm{Coh}(\mathbb{X}))
\end{array}$$

The main difficulty compared to [6] is the presence of indecomposable vector bundles of rank 2. They are not needed as generators in the double composition algebra but can be written as q-commutators of line bundles.

Furthermore, the resulting isomorphism between the two realizations of the quantum group

$$U_v\left(A_2^{(2)}\right) \xrightarrow{\cong} U_v^{Dr}\left(A_2^{(2)}\right)$$

has slightly different prefactors than in the paper [1] by Akasaka, leading to the conclusion that [1] contained a mistake.

Our construction allows us on one side to get a better understanding of the isomorphism between the two quantum groups, on the other side one may use this for further applications. For example, the composition algebra side of $\mathrm{Coh}(\mathbb{X})$ has a natural orthogonal basis with respect to Green's form [16], and transferring this to the composition algebra of $\mathrm{mod}(\Gamma)$ we get a PBW-type orthogonal basis of the positive part $U_v^+\left(A_2^{(2)}\right)$ (see Subsection 4.5). This basis may be used to derive an explicit formula for the universal R-matrix of $U_v\left(A_2^{(2)}\right)$.

In general, this approach may be used to get a better insight into more complicated formulas for the quantum groups $U_v\left(A_2^{(2)}\right)$ and $U_v^{Dr}\left(A_2^{(2)}\right)$.

0.2 Quantum Numbers and Notions

In this work we let $\mathbb{N} := \{1, 2, 3, 4, ...\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$ and $k = \mathbb{F}_q$ a finite field.

We put $v := \sqrt{q}$ where $|k| =: q$.

Then we define for $n, r \in \mathbb{N}_0$, $r \leq n$:

$$[n] := \frac{v^n - v^{-n}}{v - v^{-1}} \qquad [n]_+ := \frac{q^n - 1}{q - 1} = q^{\frac{n-1}{2}}[n]$$

$$[n]! := [n] \cdot [n-1] \cdot ... \cdot [1] \qquad [n]_+! := [n]_+ \cdot [n-1]_+ \cdot ... \cdot [1]_+ = q^{\frac{n(n-1)}{4}}[n]!$$

$$\begin{bmatrix} n \\ r \end{bmatrix} := \frac{[n]!}{[r]![n-r]!} \qquad \begin{bmatrix} n \\ r \end{bmatrix}_+ := \frac{[n]_+!}{[r]_+![n-r]_+!} = q^{-\frac{r(n-r)}{2}} \begin{bmatrix} n \\ r \end{bmatrix}$$

Given a set I and natural numbers $d_i \in \mathbb{N}$ for $i \in I$, we denote by $[n]_i$ respectively $\begin{bmatrix} n \\ r \end{bmatrix}_i$ the quantum numbers where we replace the q's by q^{d_i}'s.

Furthermore, if we count the automorphisms of the finite-dimensional vector space k^n, one can easily show that

$$|\operatorname{GL}_n(k)| = (q^n - 1)(q^n - q) \cdot ... \cdot (q^n - q^{n-1}) = q^{\frac{n(n-1)}{2}}(q-1)^n[n]_+!$$

and the number of r-dimensional subspaces is

$$\frac{(q^n - 1)(q^n - q)...(q^n - q^{r-1})}{|\operatorname{GL}_r(k)|} = \frac{[n]_+!}{[n-r]_+![r]_+!} \frac{q^{\frac{r(r-1)}{2}}}{q^{\frac{r(r-1)}{2}}} = \begin{bmatrix} n \\ r \end{bmatrix}_+ .$$

Observe, that the quantum numbers are elements in $\mathbb{Z}[v, v^{-1}]$:

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}} = \begin{cases} v^{n-1} + v^{n-3} + ... + v^{1-n}, & n \in \mathbb{N}, \\ 0, & n = 0. \end{cases}$$

and for $n \geq r + 1$, $n, r \in \mathbb{N}_0$:

$$\begin{bmatrix} n+1 \\ r \end{bmatrix} = v^s \begin{bmatrix} n \\ r \end{bmatrix} + v^{s-r-1} \begin{bmatrix} n \\ r-1 \end{bmatrix} \in \mathbb{Z}[v, v^{-1}],$$

where we set $\begin{bmatrix} n \\ r \end{bmatrix} = 0$ if $r < 0$.

1 Hall Algebras

Since our goal is to classify a specific quantum group via Hall algebras, we first need to introduce and define what a Hall algebra of a category is. Furthermore, we are interested in the reduced Drinfeld double of a Hall algebra which is to the Hall algebra what the whole quantum group is to the positive part $U_v(\mathfrak{n}_+)$. Therefore we start with several definitions and in the last subsection, we explain what is considered to be the classical Hall algebra and its connection to the ring of symmetrical functions.

A good introduction for the topic is [32].

1.1 Definition

First off, we need to make some requirements on the category.

Definition 1.1.1 Let $\mathcal{A}$ be a small abelian category. The category $\mathcal{A}$ is called *finitary*, if for all objects $M,\ N \in \mathrm{Ob}(\mathcal{A})$:

$$|\operatorname{Hom}_{\mathcal{A}}(M,N)| < \infty \qquad \wedge \qquad |\operatorname{Ext}^1_{\mathcal{A}}(M,N)| < \infty.$$

Now let $\mathcal{A}$ be a small finitary abelian hereditary $\mathbb{F}_q$-linear category.

Definition 1.1.2 For objects $M,\ N \in \mathrm{Ob}(\mathcal{A})$ we define the *multiplicative Euler form*:

$$\langle M,N\rangle_m := \sqrt{\frac{|\operatorname{Hom}_{\mathcal{A}}(M,N)|}{|\operatorname{Ext}^1_{\mathcal{A}}(M,N)|}}$$

and the *additive Euler form*:

$$\langle M,N\rangle := \langle M,N\rangle_a := \dim_{\mathbb{F}_q}(\operatorname{Hom}_{\mathcal{A}}(M,N)) - \dim_{\mathbb{F}_q}(\operatorname{Ext}^1_{\mathcal{A}}(M,N))$$

and the corresponding symmetrization:

$$(M,N) := (M,N)_a := \langle M,N\rangle_a + \langle N,M\rangle_a.$$

One can easily see the correspondence between the multiplicative and the additive Euler form, namely:

$$\langle M,N\rangle_m = q^{\frac{1}{2}\langle M,N\rangle_a}.$$

With this setup we may define the main object we are working with:

Definition 1.1.3 Let $\mathcal{X} := \mathrm{Ob}(\mathcal{A})/_{\cong}$ be the set of isomorphism classes of objects in $\mathcal{A}$.

The *Hall algebra* $\mathcal{H}(\mathcal{A})$ is the vector space

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{M\in\mathcal{X}} \mathbb{C}[M]$$

with the following multiplication:

For objects $M,\ N,\ R \in \mathrm{Ob}(\mathcal{A})$ denote by $\mathcal{P}^R_{M,N}$ the set of short exact sequences $0 \to N \to R \to M \to 0$ and set $P^R_{M,N} := |\mathcal{P}^R_{M,N}|$ and $a_M := |\mathrm{Aut}(M)|$. Then we define for $M,\ N \in \mathrm{Ob}(\mathcal{A})$:

$$[M]\cdot[N] = q^{\frac{1}{2}\langle M,N\rangle_a} \sum_{R\in\mathcal{X}} \frac{P^R_{M,N}}{a_M a_N}[R].$$

Remark 1.1.4 If one is considering a category with actual sets plus some additional structure, like $\mathcal{A} = A - \text{mod}$ for a specific algebra A, then we may think of the prefactor above differently: For any three objects $M,\ N,\ R \in \text{Ob}(\mathcal{A})$ it holds:

$$\frac{P^R_{M,N}}{a_M a_N} = |\{L \subset R \mid L \cong N \wedge R/L \cong M\}|,$$

hence we count the number of subobjects L of R where $L \cong N$ and $R/L \cong M$. This follows directly by considering the free action of $\text{Aut}(M) \times \text{Aut}(N)$ on $\mathcal{P}^R_{M,N}$ and the resulting quotient can be identified with the right-hand side above.

One can prove, see [28] or [32]:

Lemma 1.1.5 *The multiplication is associative, i.e.:*
For $L,\ M,\ N \in Ob(\mathcal{A})$:

$$([L] \cdot [M]) \cdot [N] = [L] \cdot ([M] \cdot [N]).$$

Remark 1.1.6 The idea stems from the intuition that in the product $([L] \cdot [M]) \cdot [N]$ as well as in $[L] \cdot ([M] \cdot [N])$ for the prefactor (without the Euler forms) of $[R]$ one counts the number of filtrations

$$0 \subset Q_1 \subset Q_2 \subset R,$$

where $R/Q_2 \cong L$, $Q_2/Q_1 \cong M$ and $Q_1 \cong N$.

Remark 1.1.7 Furthermore, there the multiplication is not just associative but is also unital, to be more specific: [0], the class of the zero object, is neutral:

$$[M] \cdot [0] = [M] = [0] \cdot [M]$$

for all isomorphism classes $[M]$.

Remark 1.1.8 The Hall algebra $\mathcal{H}(\mathcal{A})$ is naturally graded by the Grothendieck group $K_0(\mathcal{A})$ since by definition the multiplication respects the addition in $K_0(\mathcal{A})$.

Now that we have an associative multiplication, we want to extend the Hall algebra by the Grothendieck group. In reference to quantum groups, so far the Hall algebra corresponds to the positive part $U_v(\mathfrak{n}_+)$ and the Grothendieck group to the Cartan subalgebra $U_v(\mathfrak{h})$. Together they generate the Borel subalgebra $U_v(\mathfrak{b}_+)$.

Definition 1.1.9 Let $\mathcal{H}(\mathcal{A})$ be the Hall algebra and $K_0(\mathcal{A})$ the Grothendieck group of the category $\mathcal{A}$.

The *extended Hall algebra* $\tilde{\mathcal{H}}(\mathcal{A})$ is given by the vector space $\tilde{\mathcal{H}}(\mathcal{A}) \otimes_{\mathbb{C}} K_0(\mathcal{A})$ with the subalgebras the Hall algebra $\mathcal{H}(\mathcal{A})$ and the group algebra $\mathbb{C}K_0(\mathcal{A})$ and the relations:

$$\forall \alpha \in K_0(\mathcal{A}),\ [M] \in \mathcal{X}: \qquad K_\alpha [M] K_\alpha^{-1} = q^{\frac{1}{2}(\alpha, M)}[M].$$

In particular, now we have some invertible elements in our algebra. Furthermore, it is (almost) a Hopf algebra, hence we need to say what the comultiplication is and why there is an almost attached to the previous sentence.

1.2 Green's Coproduct

The multiplication of two objects in the Hall algebra $\mathcal{H}(\mathcal{A})$ is a linear combination of the extensions. So it is only natural for the comultiplication to be some linear combination with subobjects and quotients. The problem here is that in general an object does not only have finitely many subobjects. Therefore, we have to either impose a finite subobjects condition on the category, which is the case for example in the category of representations of a species over a finite field, or the comultiplication does not go to $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ but to some completion. For that we proceed as follows:

For $\alpha,\ \beta \in K_0(\mathcal{A})$ we set

$$\begin{aligned} \mathcal{H}(\mathcal{A})[\alpha]\hat{\otimes}\mathcal{H}(\mathcal{A})[\beta] &:= \prod_{\bar{M}=\alpha, \bar{N}=\beta} \mathbb{C}[M] \otimes \mathbb{C}[N], \\ \mathcal{H}(\mathcal{A})\hat{\otimes}\mathcal{H}(\mathcal{A}) &:= \prod_{\alpha,\beta \in K_0(\mathcal{A})} \mathcal{H}(\mathcal{A})[\alpha]\hat{\otimes}\mathcal{H}(\mathcal{A})[\beta]. \end{aligned}$$

Then Green proved the following statements:

Proposition 1.2.1 (Green) *The following defines on the Hall algebra $\mathcal{H}(\mathcal{A})$ the structure of a topological coassociative coproduct:*

For $[R] \in \mathcal{X}$:

$$\Delta([R]) = \sum_{M,N} q^{\frac{1}{2}\langle M,N\rangle} \frac{P^R_{M,N}}{a_R} [M] \otimes [N],$$

with counit $\varepsilon : \mathcal{H}(\mathcal{A}) \to \mathbb{C}$ defined by $\varepsilon([R]) = \delta_{R,0}$.

Theorem 1.2.2 (Green,[16]) *The map $\Delta : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})\hat{\otimes}\mathcal{H}(\mathcal{A})$is a morphism of algebras, i.e. for any $x,\ y \in \mathcal{H}(\mathcal{A})$ we have $\Delta(x \cdot y) = \Delta(x) \cdot \Delta(y)$.*

In particular, we have a sort of topological bialgebra structure. There is also an antipode that gives us a topological Hopf algebra structure, but we do not need it and therefore give only a reference for it. Namely, Xiao [34] constructed a map S for the topological case which naturally works for the categories with the finite subobject condition as well.

We can extend these to the extended Hopf algebra $\tilde{\mathcal{H}}(\mathcal{A})$ by setting for $[R] \in \mathcal{X}$, $\alpha \in K_0(\mathcal{A})$:

$$\tilde{\Delta}([R]K_\alpha) = \sum_{M,N} q^{\frac{1}{2}\langle M,N\rangle} \frac{P^R_{M,N}}{a_R} [M]K_{\bar{N}+\alpha} \otimes [N]K_\alpha,$$

with counit $\varepsilon : \tilde{\mathcal{H}}(\mathcal{A}) \to \mathbb{C}$ defined by $\varepsilon([R]K_\alpha) = \delta_{R,0}$.

1.3 Green's Pairing

In this setup one can define a pairing that respects the previously defined structures, namely:

Proposition 1.3.1 (Green,[16]) *The nondegenerate scalar product $(\ ,\ ) : \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A}) \to \mathbb{C}$ defined by*

$$([M],[N]) = \frac{\delta_{M,N}}{a_M},$$

for $[M],\ [N] \in \mathcal{X}$, is a Hopf pairing, that is for any triple $x,\ y,\ z \in \mathcal{H}(\mathcal{A})$ we have $(xy,z) = (x \otimes y, \Delta(z))$.

Corollary 1.3.2 *The scalar product* $(\ ,\ ) : \tilde{\mathcal{H}}(\mathcal{A}) \otimes \tilde{\mathcal{H}}(\mathcal{A}) \to \mathbb{C}$ *defined by*

$$([M]K_\alpha, [N]K_\beta) = \frac{\delta_{M,N}}{a_M} q^{\frac{1}{2}(\alpha,\beta)},$$

for $[M],\ [N] \in \mathcal{X},\ \alpha,\ \beta \in K_0(\mathcal{A})$*, is a Hopf pairing, i.e. for any triple* $x,\ y,\ z \in \tilde{\mathcal{H}}(\mathcal{A})$ *we have* $(xy, z) = (x \otimes y, \Delta(z))$.

1.4 The Drinfeld Double

Let us make a brief excursion to Drinfeld Doubles. The idea is to double the given algebra where one needs to specify a commutator relation between the two halves. With the idea of quantum groups in mind, the Drinfeld double of the Borel subalgebras is the algebra $U_v(\mathfrak{b}_+) \otimes U_v(\mathfrak{b}_-)$, where one needs to take a quotient to get the quantum group $U_v(\mathfrak{g})$.

For this definition/theorem, let H be a finite-dimensional Hopf algebra and let H^*_{coop} be its dual Hopf algebra with opposite comultiplication.

Theorem 1.4.1 (Drinfeld) *There exists a unique Hopf algebra structure on the vector space* $DH = H \otimes H^*_{coop}$ *such that*

(i) H *and* H^*_{coop} *are Hopf subalgebras of* DH*;*

(ii) For $h \in H$ *and* $h' \in H^*_{coop}$*:* $h \cdot h' = h \otimes h' \in DH$*;*

(iii) The natural pairing $(\ ,\ )$ *on* DH *is a Hopf pairing.*

Moreover, for $h \in H$ *and* $h' \in H^*_{coop}$ *we have*

$$h' \cdot h = \sum (h'_{(1)}, h_{(3)})(S^{-1}(h'_{(3)}), h_{(1)}) h_{(2)} h'_{(2)},$$

where we use Sweedler's notation for the comultiplication of Hopf algebras, i.e. $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ *and* $\Delta^2(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$ *for* $x \in H$*.*

The resulting Hopf algebra is called the Drinfeld double of H*.*

More specifically we use the following version:

Corollary 1.4.2 *Let* H *be a Hopf algebra with a non-degenerate Hopf pairing on* $H \otimes H$*. Let* H^+ *and* H^- *be two copies of* H*. There exists a unique Hopf algebra structure on* $DH = H^+ \otimes H^-$ *such that*

(i) $H^+ \otimes 1$ *respectively* $1 \otimes H^-$ *are Hopf subalgebras isomorphic to* H^+ *respectively* H^-*.*

(ii) For all $a,\ b \in H$*:* $\sum a^-_{(1)} b^+_{(2)} (a_{(2)}, b_{(1)}) = \sum b^+_{(1)} a^-_{(2)} (a_{(1)}, b_{(2)})$*.*

In particular, we may apply it to infinite dimensional topological Hopf algebras with a possibly even degenerate Hopf pairing (see [5]). More specifically we take the Drinfeld double of the extended Hall algebra with respect to Green's pairing.

Since we do not want to double the Grothendieck group in the extended Hall algebra, we take a quotient:

Definition 1.4.3 The *reduced Drinfeld double* $D\mathcal{H}(\mathcal{A})$ of an extended Hall algebra $\tilde{\mathcal{H}}(\mathcal{A})$ is the quotient algebra of the Drinfeld double $D\tilde{\mathcal{H}}(\mathcal{A})$ by the two-sided ideal

$$I = \left\langle K_\alpha^+ \otimes K_\alpha^- - 1^+ \otimes 1^- \mid \alpha \in K_0(\mathcal{A}) \right\rangle .$$

Remark 1.4.4 In this dissertation, we do not consider the Drinfeld double of a non-extended Hall algebra. Therefore $D\mathcal{H}(\mathcal{A})$ always denotes the reduced Drinfeld double.

Furthermore, there is a connection between Hall algebras if the categories are derived equivalent, namely:

The following theorem was proven by Cramer (see [9, Theorem 1]).

Theorem 1.4.5 (Cramer) *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{F}_q$-linear finitary hereditary categories. Assume one of them is Artinian and there is an equivalence of triangulated categories $\mathcal{D}^b(\mathcal{A}) \xrightarrow{\mathfrak{F}} \mathcal{D}^b(\mathcal{B})$. Then there is an algebra isomorphism*

$$\mathfrak{F} : D\mathcal{H}(\mathcal{A}) \longrightarrow D\mathcal{H}(\mathcal{B})$$

uniquely determined by the following properties. For any object $X \in \mathrm{Ob}(\mathcal{A})$ such that $\mathfrak{F}(X) \cong \hat{X}[n]$ with $\hat{X} \in \mathrm{Ob}(\mathcal{B})$ and $n \in \mathbb{Z}$ we have:

$$\mathfrak{F}([X]^\pm) = q^{\frac{n}{2}\langle \bar{X}, \bar{X} \rangle} [\hat{X}]^{\pm\varepsilon(n)} (K_{\hat{X}}^{\pm\varepsilon(n)})^n,$$

where $\varepsilon(n) = (-1)^n$. For $\alpha \in K_0(\mathcal{A})$ we have: $\mathfrak{F}(K_\alpha) = K_{\mathfrak{F}(\alpha)}$.

Later on, we will have derived equivalent categories by construction and can therefore identify the reduced Drinfeld doubles of their Hall algebras via isomorphism.

Now, we have made enough definitions and can start with the first categories to consider. First, we will start with the classical Hall algebra in the next section.

1.5 The Classical Hall Algebra and the Ring of Symmetric Functions

The main reference for this subsection is [25].

The classical Hall algebra is the Hall algebra of finite modules of a discrete valuation ring $\mathfrak{o}$ with a finite residue field k. Let $\mathfrak{p} \subset \mathfrak{o}$ be its maximal ideal. Then we have by definition $k = \mathfrak{o}/\mathfrak{p}$.

Finite modules are $\mathfrak{o}$-modules that have one of the following (equivalent) properties:

- they have a finite composition series;
- they are finitely generated $\mathfrak{o}$-modules M with $\mathfrak{p}^r M = 0$ for some $r \gg 0$;
- or (since we are in the case k finite) they have only a finite number of elements.

This example plays a particular role later on in the Hall algebra of $\mathrm{Coh}(\mathbb{X})$ which we want to study, where $\mathbb{X} = (\mathbb{P}^1, \mathcal{A})$ a ringed space, $\mathcal{A}$ a certain sheaf of hereditary maximal orders. Namely in this case, the torsion part of our category $\mathrm{Tor}(\mathbb{X})$ decomposes into factors $\bigvee_{x \in \mathbb{P}^1} \mathrm{Tor}_x(\mathbb{X})$ where there are no Hom or Ext between each factor. More importantly, the Hall algebra of the torsion part decomposes into factors that commute with each other and each of the factors has a similar structure to finite modules over a discrete valuation ring. The main difference is that the rings are non-commutative. However, that is not a property that we use. In particular, by the same construction, we get an isomorphism between the Hall algebra of a tube $\mathrm{Tor}_x(\mathbb{X})$ and the ring of symmetric functions. We will not go into specifics about why this holds, for more information on maximal orders see [27].

Now let us give the classical argument for the Hall algebra of finite modules of a discrete valuation ring $\mathfrak{o}$ with a finite residue field k.

Example 1.5.1 An important example, which is also talked about in [32], is the case $\mathfrak{o} = k[[t]]$, the power series ring. Then the finite modules are of the form of a finite-dimensional k-vectorspace V equipped with a nilpotent endomorphism $T : V \to V$.

Every finite $\mathfrak{o}$-module decomposes into a direct sum of the form

$$M \cong \bigoplus_{i=1}^{r} \mathfrak{o}/\mathfrak{p}^{\lambda_i} \tag{2}$$

for some $\lambda_i \in \mathbb{N}$. We may assume $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0$. In particular, $\lambda = (\lambda_1, ..., \lambda_r)$ is a partition.

Lemma 1.5.2 *Let M be a finite $\mathfrak{o}$-module and λ a partition such that M decomposes as in (2). We set $\mu_i = \dim_k(\mathfrak{p}^{i-1}M/\mathfrak{p}^i M)$. Then $\mu = (\mu_1, \mu_2, ...)$ is the conjugate partition of λ.*

Proof. For each $j \in \{1, ..., r\}$ let x_j be a generator of the summand $\mathfrak{o}/\mathfrak{p}^{\lambda_j}$ in (2). Let p be a generator of $\mathfrak{p}$. Then $\mathfrak{p}^{i-1}M$ is generated by the $p^{i-1}x_j$ where $\lambda_j \geq i$ (otherwise $p^{i-1}x_j = 0$). In particular we get $\mu_i = \#\{\lambda_j \geq i\}$. Hence μ is the conjugate partition to λ. □

A direct consequence of the Lemma 1.5.2 is that λ is uniquely determined by the module M. Therefore we call λ the type of M. Furthermore, two finite modules are isomorphic if and only if they are of the same type, and to every partition λ there exists a finite $\mathfrak{o}$-module of that type. We have a direct correspondence

$$\mathcal{X} \qquad \overset{1:1}{\longleftrightarrow} \qquad \{\text{partitions}\}$$

where $\mathcal{X}$ is the set of isomorphism classes of finite $\mathfrak{o}$-modules.

Let λ be the type of a finite $\mathfrak{o}$-module M. Then the length of M is given by $l(M) := |\lambda| = \sum \lambda_i$ which is also the length of a composition series of M. The length is also additive regarding short exact sequences. Namely, for $0 \to M \to M' \to M'' \to 0$ a short exact sequence it holds $l(M') = l(M) + l(M'')$.

For each partition λ, denote by I_λ the isomorphism class of finite $\mathfrak{o}$-modules of type λ.

Remark 1.5.3 Now, let us briefly discuss the Eulerform. Since there is one simple module, namely $\mathfrak{o}/\mathfrak{p} = k$, and for every finite module M there is a composition series of length $l(M)$ with each quotient k, we only need consider the Eulerform $\langle k, k\rangle$.

Furthermore, we know $\mathrm{Hom}(k, k) \cong k$ since k is simple and the only possible extensions are of the form

$$k \longrightarrow k \oplus k \longrightarrow k$$

or

$$k \longrightarrow \mathfrak{o}/\mathfrak{p}^2 \longrightarrow k.$$

This means for the Eulerform we get

$$\langle k, k\rangle = \dim_k \mathrm{Hom}(k, k) - \dim_k \mathrm{Ext}^1(k, k) = 1 - 1 = 0.$$

In particular, this means the Eulerform is equal to zero for any two finite $\mathfrak{o}$-modules M and N:

$$\langle M, N\rangle = 0.$$

Furthermore, there are specific generators we want to consider:

Lemma 1.5.4 *The Hall algebra $\mathcal{H}(\mathfrak{o}-\text{fmod})$ is generated (as a $\mathbb{C}$-algebra) by the elements $I_{(1^r)}$ for $r \geq 1$ and they are algebraically independent over $\mathbb{C}$.*

Before we prove the lemma, let us introduce a partial order on the set of partitions, which will be useful in the proof.

Definition 1.5.5 Given two partitions μ and λ, we write $\mu = (1^{l_1}, 2^{l_2}, ...) \succcurlyeq \lambda = (1^{m_1}, 2^{m_2}, ...)$ if and only if for all $i \geq 1$

$$l_1 + 2l_2 + ... + (i-1)l_{i-1} + i(l_i + l_{i+1} + ...) \leq m_1 + m_2 + ... + (i-1)m_{i-1} + i(m_i + m_{i+1} + ...).$$

Proof. We show that for every partition λ we can write I_λ as a linear combination of monomials of the form $I_{(1^{m_1})}I_{(1^{m_2})}...I_{(1^{m_n})}$.

Let $\lambda = (\lambda_1, ..., \lambda_r) = (1^{l_1}, 2^{l_2}, ..., n^{l_n})$ be a partition. Consider the product

$$X := I_{(1^{l_n})}I_{(1^{l_n+l_{n-1}})}...I_{(1^{l_n+\cdots+l_1})}. \tag{3}$$

Then we have $X = \sum_\mu a_\mu I_\mu$, where a_μ is by definition equal to the number of chains of submodules of M

$$M = M_n \supset M_{n-1} \supset M_{n-2} \supset ... \supset M_0 = 0 \tag{4}$$

with M_i/M_{i-1} is of the type $(1^{l_i+\cdots+l_n})$ for $1 \leq i \leq n$, where M is of type μ.

If there exists such a chain as in (4), then we have $\mathfrak{p}M_i \subset M_{i-1}$ for all $1 \leq i \leq n$ (since each quotient is killed by $\mathfrak{p}$). In particular we get $\mathfrak{p}^i M \subset M_{n-i}$ for $1 \leq i \leq n$. If we write $\mu = (1^{m_1}, 2^{m_2}, ...)$, we get $l(\mathfrak{p}^i M) = m_1 + 2m_2 + ... + i(m_i + m_{i+1} + ...)$ for all i. But also we have $l(M_{n-i}) = l_1 + 2l_2 + ... + i(l_i + ... + l_n)$ by the composition series.

It directly follows

$$m_1 + 2m_2 + ... + i(m_i + m_{i+1} + ...) = l(\mathfrak{p}^i M) \geq l(M_{n-i}) = l_1 + 2l_2 + ... + i(l_i + ... + l_n).$$

Since this holds for all i, we get $\lambda \succcurlyeq \mu$.

One should also note that in the case $\mu = \lambda$ there is just one chain of submodules of the form (4). Therefore we have

$$X \in I_\lambda \oplus \bigoplus_{\mu \preccurlyeq \lambda} \mathbb{C} I_\mu.$$

In particular, if we write the monomials as linear combinations of partitions, this corresponds to a base change in the form of an upper triangular matrix with 1's on the diagonal. Hence the monomials of the form (3) form a basis of the Hall algebra $\mathcal{H}(\mathfrak{o}-\text{fmod})$.

Therefore $\mathcal{H}(\mathfrak{o}-\text{fmod})$ is generated (as a $\mathbb{C}$-algebra) by the elements $I_{(1^r)}$ for $r \geq 1$. □

One may also ask if the Hall algebra $\mathcal{H}(\mathfrak{o}-\text{fmod})$ is commutative or cocommutative. The answer to both is yes. To see that one needs to look at the dual of a module. Let p be a generator of the maximal ideal $\mathfrak{p}$ in $\mathfrak{o}$. For $m \leq n$ the multiplication by p^{n-m} is an injective $\mathfrak{o}$-homomorphism $\mathfrak{o}/\mathfrak{p}^m \to \mathfrak{o}/\mathfrak{p}^n$. We denote by E the direct limit, i.e.

$$E := \lim_{\to} \mathfrak{o}/\mathfrak{p}^n.$$

Definition 1.5.6 Given any finite $\mathfrak{o}$-module M, the *dual* of M is defined as

$$M^\wedge := \mathrm{Hom}_{\mathfrak{o}}(M, E).$$

Lemma 1.5.7 *Given a finite $\mathfrak{o}$-module M, the dual $M^\wedge$ is a finite $\mathfrak{o}$-module isomorphic to M, in particular of the same type as M.*

Proof. First, we note that taking the dual respects direct sums, i.e.

$$(N \oplus M)^\wedge = \mathrm{Hom}_{\mathfrak{o}}(N \oplus M, E) \cong \mathrm{Hom}_{\mathfrak{o}}(N, E) \oplus \mathrm{Hom}_{\mathfrak{o}}(M, E) = N^\wedge \oplus M^\wedge.$$

So let M be indecomposable. Then the dual $M^\wedge$ is also indecomposable and of the same length. But there is only one indecomposable module of a given length up to isomorphism, therefore we have $M \cong M^\wedge$. □

Since E is injective, taking the dual of an exact sequence

$$0 \to N \to M \to R \to 0$$

gives rise to an exact sequence

$$0 \leftarrow N^\wedge \leftarrow M^\wedge \leftarrow R^\wedge \leftarrow 0.$$

Lemma 1.5.8 *The Hall algebra $\mathcal{H}(\mathfrak{o} - \mathrm{fmod})$ is commutative and cocommutative.*

Proof. This follows directly from the above remarks, namely taking the dual of an exact sequence reverses the arrows but does not change the types of the modules. □

Furthermore, let us calculate explicitly some coproducts.

Lemma 1.5.9 *In the Hall algebra $\mathcal{H}(\mathfrak{o} - \mathrm{fmod})$ it holds:*

$$\Delta(I_{(1^n)}) = \sum_{r=0}^{n} q^{-r(n-r)} I_{(1^r)} \otimes I_{(1^{n-r})}.$$

Proof. First we note for a representative M of $I_{(1^n)}$:

$$M \cong (\mathfrak{o}/\mathfrak{p})^{\oplus n} \cong k^n.$$

Furthermore, each submodule of M is of type 1^r for some $r \leq n$ as well as each quotient. Hence we have

$$\Delta(I_{(1^n)}) = \sum_{r=0}^{n} \frac{P^{(1^n)}_{(1^r),(1^{n-r})}}{a_{(1^n)}} I_{(1^r)} \otimes I_{(1^{n-r})}.$$

The number $a_{(1^n)}$ is given by the number of automorphism of k^n, namely

$$a_{(1^n)} = |\,\mathrm{GL}_n(k)| = (q^n - 1)(q^n - q) \cdot \ldots \cdot (q^n - q^{n-1}) = (q-1)^n q^{\frac{n(n-1)}{2}} [n]_+ [n-1]_+ \cdot \ldots \cdot [1]_+.$$

Furthermore $\frac{P^{(1^n)}_{(1^r),(1^{n-r})}}{a_{(1^r)} a_{(1^{n-1})}}$ corresponds to the number of submodules of k^n of dimension r, which is

$$\begin{bmatrix} n \\ r \end{bmatrix}_+ = \frac{[n]_+ [n-1]_+ \ldots . [n-r+1]_+}{[r]_+ [r-1]_+ \ldots [1]_+}.$$

All in all, we get:

$$\begin{aligned}
\frac{P^{(1^n)}_{(1^r),(1^{n-r})}}{a_{(1^n)}} &= a_{(1^r)}a_{(1^{n-r})}\frac{[n]_+[n-1]_+....[n-r+1]_+}{[r]_+[r-1]_+...[1]_+(q-1)^n q^{\frac{n(n-1)}{2}}[n]+[n-1]_+\cdot ...\cdot [1]_+} \\
&= \frac{(q-1)^r(q-1)^{n-r}}{(q-1)^n}\frac{q^{\frac{r(r-1)}{2}}q^{\frac{(n-r)(n-r-1)}{2}}}{q^{\frac{n(n-1)}{2}}} \\
&\quad \frac{[n]_+[n-1]_+....[n-r+1]_+[r]_+[r-1]_+\cdot ...\cdot [1]_+[n-r]_+[n-r-1]_+\cdot ...\cdot [1]_+}{[r]_+[r-1]_+...[1]_+[n]_+[n-1]_+\cdot ...\cdot [1]_+} \\
&= q^{\frac{r^2-r+n^2-2nr-n+r^2+r-n^2+n}{2}} \\
&= q^{r^2-nr} \\
&= q^{-r(n-r)}
\end{aligned}$$

□

Now that we have established some relations and properties of the classical Hall algebra, we would like to identify it with another algebra, namely Macdonald's ring of symmetric functions.

For $n \in \mathbb{N}$ one may consider the ring of symmetric polynomials in n variables

$$\Lambda_n := \mathbb{C}[x_1, ..., x_n]^{\mathfrak{S}_n}.$$

There are projective maps $\Lambda_{n+1} \to \Lambda_n$ for each n by inserting 0 for the variable x_{n+1}. Then taking the projective limit in the category of graded rings leads to

$$\Lambda := \lim_{\leftarrow} \Lambda_n = \text{"}\mathbb{C}[x_1, x_2, ...]^{\mathfrak{S}_\infty}\text{"}.$$

Similar to symmetric polynomials, Λ is generated by elementary symmetric functions: For $r \in \mathbb{N}$ we set

$$e_r = \sum_{i_1<i_2<...<i_r} x_{i_1}x_{i_2}...x_{i_r}$$

and for a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ we define $e_\lambda = e_{\lambda_1}e_{\lambda_2}...e_{\lambda_n}$.

Then Macdonald has proved the following theorem.

Theorem 1.5.10 (Macdonald, [25]) *The set $\{e_\lambda | \lambda$ a partition$\}$ forms a basis of Λ, i.e.*

$$\Lambda \cong \mathbb{C}[e_1, e_2, ...].$$

Furthermore, there is a coproduct on the ring Λ, which was first introduced by Zelevinsky (see [35]): For $n \in \mathbb{N}$ we consider the map $\Delta_n : \Lambda_{2n} \to \Lambda_n \otimes \Lambda_n$ given via the embedding

$$\Lambda_{2n} = \mathbb{C}[x_1, x_2, ..., x_{2n}]^{\mathfrak{S}_{2n}} \hookrightarrow \mathbb{C}[x_1, x_2, ..., x_{2n}]^{\mathfrak{S}_n\times\mathfrak{S}_n} = \mathbb{C}[x_1, ..., x_n]^{\mathfrak{S}_n}\otimes\mathbb{C}[y_1, ..., y_n]^{\mathfrak{S}_n} = \Lambda_n\otimes\Lambda_n.$$

Again, taking the projective limit we get a map $\Delta := \lim \Delta_n : \Lambda \to \Lambda \otimes \Lambda$.

On the generators e_r the coproduct has a rather easy form, namely

$$\Delta(e_r) = \sum_{n=0}^{r} e_{r-n} \otimes e_n,$$

where we set $e_0 = 1$.

Using this setting, we can easily deduce the following theorem:

Theorem 1.5.11 *There is an isomorphism of $\mathbb{C}$-bialgebras*

$$\begin{aligned} \Phi : \mathcal{H}(\mathfrak{o} - \mathrm{fmod}) &\to \Lambda, \\ I_{(1^r)} &\mapsto q^{-\frac{r(r-1)}{2}} e_r. \end{aligned}$$

Proof. Since we have $\mathcal{H}(\mathfrak{o} - \mathrm{fmod}) \cong \mathbb{C}[I_{(1)}, I_{(1^2)}, ...] \cong \mathbb{C}[e_1, e_2, ...] \cong \Lambda$ as $\mathbb{C}$-algebras, there is only left to check the coproduct which is also clear:

$$\begin{array}{ccc} I_{(1^r)} & \xrightarrow{\ \Phi\ } & q^{\frac{-r(r-1)}{2}} e_r \\ \downarrow \Delta & & \downarrow \Delta \\ & & q^{\frac{-r(r-1)}{2}} \sum_{n=0}^{r} e_n \otimes e_{r-n} \\ \sum_{n=0}^{r} q^{-n(r-n)} I_{(1^n)} \otimes I_{(1^{r-n})} & \xrightarrow{\ \Phi \otimes \Phi\ } & \sum_{n=0}^{r} q^{-n(r-n)} q^{\frac{-n(n-1)}{2}} q^{\frac{-(r-n)(r-n-1)}{2}} e_n \otimes e_{r-n} \end{array}$$

Now, just note

$$\begin{aligned} -n(r-n) + \frac{-n(n-1)}{2} + \frac{-(r-n)(r-n-1)}{2} &= \frac{-2n(r-n) - n(n-1) - (r-n)(r-n-1)}{2} \\ &= \frac{-nr + 2n^2 - n^2 + n - r^2 + rn + r + rn - n^2 - n}{2} \\ &= \frac{r^2 - r}{2} \\ &= \frac{-r(r-1)}{2}. \end{aligned}$$

Hence we have $\Delta(\Phi(I_{(1^r)})) = (\Phi \otimes \Phi)(\Delta(I_{(1^r)}))$. □

Furthermore, Macdonald defined a bilinear form on Λ.

First, let us define some other symmetric functions. We introduce these here because later on in Section 3, we define several torsion elements which have an analog on this side with similar properties and the bilinear form will correspond to Green's form. So it should give an idea of how the elements in the Hall algebra of Tor($\mathbb{X}$) will interact.

Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$. We denote

$$x^\alpha := x_1^{\alpha_1} ... x_n^{\alpha_n}.$$

Hence for a partition of length smaller than n, the polynomial

$$m_\lambda = \sum x^\alpha \in \Lambda_n,$$

where we sum over all distinct permutations α of λ, is symmetric. The set $(m_\lambda)_\lambda$ over all partitions of length smaller than n forms a basis of Λ_n. Furthermore, the projection $\Lambda_{n+1} \to \Lambda_n$ sends $m_\lambda(x_1, ..., x_{n+1})$ to $m_\lambda(x_1, ..., x_n)$ if λ has lengths smaller than n and to 0 otherwise. Therefore by taking the projective limit we have a basis of Λ given by the m_λ, the *monomial symmetric functions*, indexed by all partitions.

In this notation, we have $e_r = m_{(1^r)}$.

For $r \geq 0$ we define the rth *complete symmetric function* h_r as

$$h_r := \sum_{|\lambda|=r} m_\lambda,$$

where the sum goes over all partitions $\lambda = (\lambda_1, \lambda_2, ...)$ where $|\lambda| := \sum_i \lambda_i = r$, and for $r \geq 1$ the rth *power sum* as

$$p_r := \sum_i x_i^r = m_{(r)}.$$

Using power series as generating functions, i.e. if we consider $E(t) = \sum_{r\geq 0} e_r t^r$, $H(t) = \sum_{r\geq 0} h_r t^r$ and $P(r) = \sum_{r\geq 1} p_r t^{r-1}$ we have the identities:

$$E(t) = \sum_{r\geq 0} e_r t^r = \prod_{i\geq 1}(1 + x_i t),$$

$$H(t) = \sum_{r\geq 0} h_r t^r = \prod_{i\geq 1}(1 - x_i t)^{-1},$$

and therefore

$$H(t)E(-t) = 1.$$

Furthermore, we get

$$P(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log H(t)$$

resp.

$$P(-t) = \frac{\mathrm{d}}{\mathrm{d}t} \log E(t).$$

Such identities will play a role later on in Section 3. In particular, we specifically use this construction of specific elements to receive similar elements to the power sums p_r.

Furthermore, the bilinear form shall be given by the requirement, that the bases (h_λ) and (m_λ) shall be dual:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

Now, if we set

$$z_\lambda = \prod_{i\geq 1} i^{n_i} \cdot n_i!,$$

for any partition λ, where $n_i = n_i(\lambda)$ is the number of parts of λ equal to i, then we have

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda.$$

In particular, the p_λ form an orthogonal basis.

Example 1.5.12 For $\lambda = (r)$ for some $r \in \mathbb{N}$ we have $z_\lambda = r$, in particular we get

$$\langle p_r, p_r \rangle = r.$$

Later on, the elements of the form p_r will play a key role in Section 3 and 4, or more specifically their analog in the Hall algebra of $\mathrm{Coh}(\mathbb{X})$.

For now, let us turn our attention to another category.

2 Species and their Representations

To talk about Hall algebras, we first need to introduce the categories whose Hall algebras we want to consider. One of these categories is the category of representations of a specific species. Alternatively, it can be considered as a module category of a hereditary algebra.

One may think of species and their representations as a generalization of quivers and their representations. Instead of just considering everything over a fixed field, one takes a bit more general approach by looking at vector spaces over different (skew) fields. To still have linear maps, one takes a modulation which is given via bimodules and we get linear maps for each arrow with tensoring. To make this more precise see the following section.

Their representation theory is rather similar to the one of quivers (see e.g. [12]) and if one is familiar with the latter, the former becomes a rather natural generalization.

Even though this category is not the main object of this thesis, it gives us a clearer picture of what the main category looks like. Namely in the next section, we construct $\mathrm{Coh}(\mathbb{X})$ by taking a specific full subcategory of the derived category $\mathcal{D}^b\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$, therefore by understanding the structure of representations of species, we also understand the structure of $\mathrm{Coh}(\mathbb{X})$.

First, in this section, we give an introduction to the general representation theory of species and then in Subsection 2.5, we go into our specific example which we later on use for our construction of $\mathrm{Coh}(\mathbb{X})$. Lastly, we talk about the connection of Hall algebras of a species and quantum groups due to Ringel and Green.

2.1 Definition

Since we want to consider a specific example of representations of a certain species, we first need to define what a species is. One of the first introductions can be found in [13]. Another reference on the topic of species is [23].

First, we need a specific setup:

Definition 2.1.1 A *weighted graph* $(Q,\ \underline{d})$ is a finite set Q of vertices together with a set $\underline{d} = \{(d_{ij}, d_{ji}) \mid d_{ij} \in \mathbb{N}_0,\ i, j \in Q\}$ of weights, which satisfies the following conditions:

(i) $\forall i \in Q: \quad d_{ii} = 0;$

(ii) $\forall i \in Q\ \exists f_i \in \mathbb{N}: \quad \forall i,\ j \in Q: \quad d_{ij} f_j = d_{ji} f_i.$

If it holds $d_{ij} \neq 0$, we draw an edge between the vertices i and j together with the weight (d_{ij}, d_{ji}), in pictures:

$$i \xrightarrow{\quad (d_{ij}, d_{ji}) \quad} j$$

This way we get an unoriented graph with a finite number of vertices without loops with weights (d_{ij}, d_{ji}) for the edges $\{i,\ j\}$.

A direct consequence of the second condition is that $d_{ij} = 0$ implies directly $d_{ji} = 0$ for vertices $i,\ j \in Q$. In particular, we only draw an edge $\{i,\ j\}$ if $d_{ij} \neq 0$ and $d_{ji} \neq 0$.

Definition 2.1.2 (1) An *orientation* Ω of a weighted graph $(Q,\ \underline{d})$ is an orientation of the edges which are then depicted as arrows.

(2) For a fixed orientation Ω and $i \in Q$ we define a new orientation $s_i\Omega$ by reversing the orientation of the edges which are incident to the vertex i.

(3) An orientation Ω of $(Q, \underline{d})$ is called *admissible* if we can find a total order of the vertices $i_1,\ \ldots,\ i_n$ such that i_1 is a sink in Ω and for all $j \in \{2,\ \ldots, n\}$ the vertex i_j is a sink in $s_{i_{j-1}} \ldots s_{i_1}\Omega$.

In this case, we say $i_1,\ \ldots,\ i_n$ is an *admissible order of sinks.*

Remark 2.1.3 The following statements are equivalent:

1. The orientation Ω of $(Q, \underline{d})$ is admissible.
2. There is no (directed) circle in $(Q, \underline{d})$ with orientation Ω.
3. The relation

$$i \leq j \quad :\Leftrightarrow \quad \text{there is a directed path from } j \text{ to } i \text{ in } (Q, \underline{d}) \text{ with the orientation } \Omega$$

is a partial order on Q.

Definition 2.1.4 Let k be a field and $(Q, \underline{d})$ be a weighted graph. A *k-modulation* $\mathcal{M}$ of $(Q, \underline{d})$ is a family of division algebras $(F_i)_{i\in Q}$ together with an F_i-F_j-bimodule ${}_iM_j$ for all edges $\{i,\ j\}$ such that the following conditions are satisfied:

(i) k lies in the center of all F_i and all F_i are finite-dimensional over k;

(ii) k acts centrally on all ${}_iM_j$;

(iii) The bimodules ${}_jM_i$ and ${}_iM_j^*$ are isomorphic for all edges $\{i,\ j\}$, where ${}_iM_j^* := \operatorname{Hom}_{F_j}({}_iM_j, {}_{F_j}F_{jF_j})$;

(iv) For all edges $\{i,\ j\}$ it holds $\dim({}_iM_j)_{F_j} = d_{ij}$.

Definition 2.1.5 A *k-species* $(\mathcal{M}, \Omega)$ of $(Q, \underline{d})$ is a k-modulation $\mathcal{M}$ of $(Q, \underline{d})$ together with an admissible orientation Ω.

A *representation* $\underline{V} = (V_i, {}_j\varphi_i)_{i,j\in Q}$ is a family of finite-dimensional F_i-right modules V_i for $i \in Q$ and F_j-linear maps

$${}_j\varphi_i : \ V_i \otimes_{F_i} {}_iM_j \to V_j$$

for all oriented edges $i \to j$. If there is no oriented edge $i \to j$ we set ${}_i\varphi_j = 0$.

Given two representations $\underline{V} = (V_i, {}_j\varphi_i)$ and $\underline{W} = (W_i, {}_j\psi_i)$, a *morphism* of representations is a tuple $\underline{f} = (f_i)_{i\in Q} : \underline{V} \to \underline{W}$ of F_i-linear maps $f_i : V_i \to W_i$ such that for all $i,\ j \in Q$ the following diagram commutes:

$$\begin{array}{ccc} V_i \otimes_{F_i} {}_iM_j & \xrightarrow{\ {}_j\varphi_i\ } & V_j \\ {\scriptstyle f_i \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle f_j} \\ W_i \otimes_{F_i} {}_iM_j & \xrightarrow{\ {}_j\psi_i\ } & W_j \end{array}$$

We denote by $\operatorname{Rep}_k(\mathcal{M}, \Omega)$ the category of representations of the k-species $(\mathcal{M}, \Omega)$.

Example 2.1.6 Very important and some of the easiest examples for representations of a species are the representations of a quiver without multiple edges between vertices and loops. The weights are given by $d_{ij} = 1$ if there is an edge from i to j or from j to i and the division algebras are $F_i = k = F_j$ and ${}_iM_j = k$ if $d_{ij} = 1$.

Lemma 2.1.7 *Let V be an F_i-vector space, W_j an F_j-vector space and ${}_iM_j$ an $F_i - F_j$-bimodule, finite-dimensional over F_i and F_j, and ${}_jM_i = {}_iM_j^* := \mathrm{Hom}_{F_j}({}_iM_j, {}_{F_j}F_{jF_j})$ its dual. Let $(m_1, ..., m_d)$ be a F_i-basis of ${}_iM_j$ and $(m^1, ..., m^d)$ the dual basis. Then we have a natural isomorphism of $F_j - F_i$-bimodules*

$$\begin{array}{rcl} \mathrm{Hom}_{F_j}(V_i \otimes_{F_i} {}_iM_j, W_j) & \cong & \mathrm{Hom}_{F_i}(V_i, W_j \otimes_{F_j} {}_jM_i), \\ \varphi & \mapsto & \bar{\varphi}, \\ \bar{\bar{\varphi}} & \leftarrow\!\shortmid & \bar{\varphi}, \end{array}$$

where

$$\bar{\varphi} : V_i \to W_j \otimes_{F_j} {}_jM_i, \qquad x \mapsto \sum_{i=1}^{d} \varphi(x \otimes m_i) \otimes m^i$$

and

$$\bar{\bar{\varphi}} : V_i \otimes_{F_i} {}_iM_j \to W_j, \qquad y \otimes m_z \to y_z \ \ \textit{where} \ \ \bar{\varphi}(y) = \sum_{i=1}^{d} y_i \otimes m^i.$$

In particular, this means that it does not matter on which side we tensor with the bimodules ${}_iM_j$ as long as we are consistent.

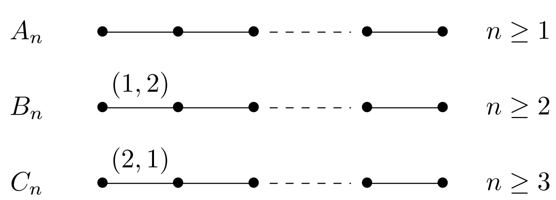
Proof. The natural bijection comes from the adjointness of the functors Hom and $\otimes$:

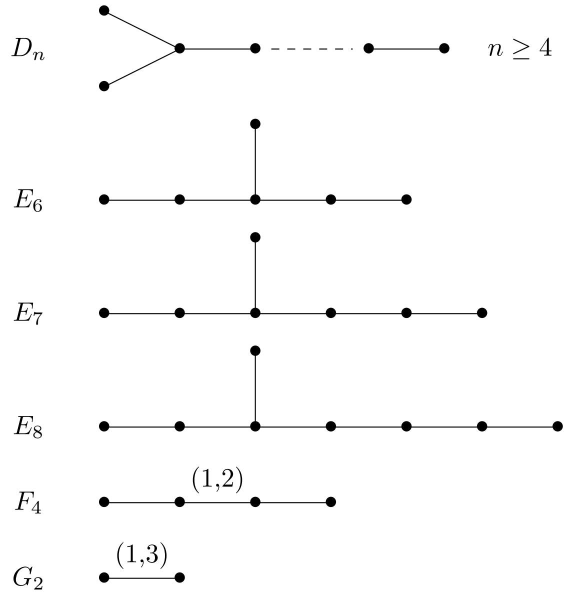
$$\begin{array}{rcl} \mathrm{Hom}_{F_j}(V_i \otimes_{F_i} {}_iM_j, W_j) & \cong & \mathrm{Hom}_{F_i}(V_i, \mathrm{Hom}_{F_j}({}_iM_j, W_j)) \\ & \cong & \mathrm{Hom}_{F_i}(V_i, W_j \otimes_{F_j} {}_jM_i). \end{array}$$

One can check that the above concrete definition of $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ does not depend on the given basis and is therefore natural. □

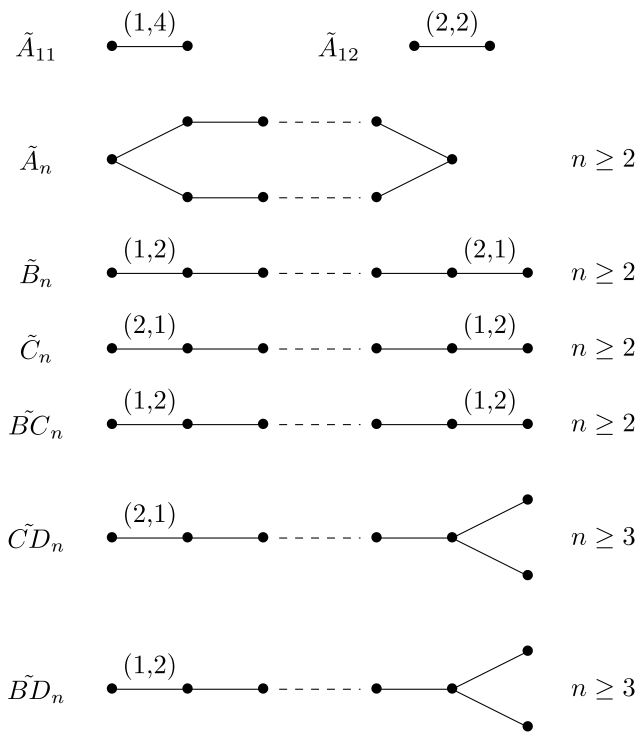
Example 2.1.8 The most studied and understood weighted graphs are the *Dynkin graphs* and *Euclidean graphs*:

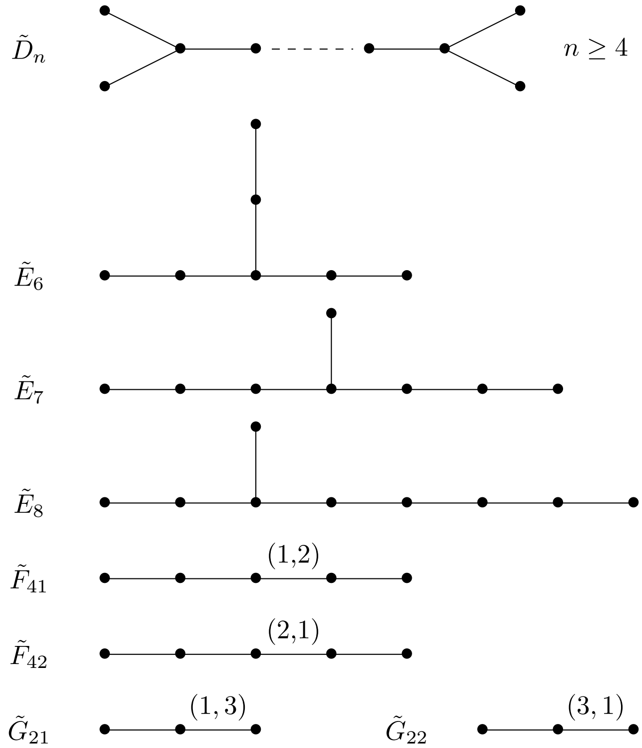
- The indices in the labels for Dynkin graphs coincide with the number of vertices (we omit the weights (1,1)):





- The indices in the labels of the Euclidean graphs coincide with the number of vertices -1. The Euclidean graphs arise from the Dynkin graphs by adding a node at a specific point.





Now let us add some more knowledge about the category of representations of a species (see e.g. [12]). It will not be necessary for the next sections and there are no new results until we go into our concrete example in Subsection 2.5, but it gives a more well-rounded picture of what exactly we are studying and what else can happen. Most of it is very similar to the theory of representations of a quiver.

2.2 The Tensor Algebra

Similar to the path algebra of a quiver one can associate an algebra with a species. Namely an algebra where the module category is equivalent to the category of representations.

Definition 2.2.1 The *tensor algebra* $T(\mathcal{M})$ of a k-species $(\mathcal{M}, \Omega)$ is defined as the k-vector space

$$T(\mathcal{M}) = \bigoplus_{r \geq 0} M^{(r)},$$

where $M^{(0)} := A := \prod_{i \in Q} F_i$, $M^{(1)} := \bigoplus_{(i,j) \in \Omega} {}_iM_j$ also considered as an $A-A$-bimodule where A acts via the projections $A \to F_i$ and then inductively $M^{(r)} := M^{(r-1)} \otimes M^{(1)}$. The multiplication is induced by the canonical isomorphisms $M^{(r)} \otimes M^{(s)} \to M^{(r+s)}$ and distributivity.

Remark 2.2.2 If the orientation Ω of the k-species $(\mathcal{M}, \Omega)$ is admissible, then the tensor algebra $T(\mathcal{M})$ is finite-dimensional.

Example 2.2.3 Consider a field extension $k \subset K$ with $[K : k] = l$ and the k-modulation

$k \xrightarrow{{}_kK_K} K$ of the weighted oriented graph $\bullet \xrightarrow{(1,l)} \bullet$. Then the tensor algebra can be realized as the matrix algebra

$$T = \begin{pmatrix} k & 0 \\ {}_KK_k & K \end{pmatrix}.$$

More explicitly: $M^{(0)} = k \times K$, $M^{(1)} = {}_kK_K$, $M^{(2)} = 0$ and therefore $M^{(r)} = 0$ for $r \geq 2$.

Example 2.2.4 In the case of a quiver, the tensor algebra coincides with the path algebra.

Using this algebra we get the equivalence (see [12]).

Proposition 2.2.5 *Given a k-species $(\mathcal{M}, \Omega)$. Then the category* $\mathrm{Rep}_k(\mathcal{M}, \Omega)$ *of (finite-dimensional) representations of $(\mathcal{M}, \Omega)$ is equivalent to the category* $\mathrm{mod}\,(T(\mathcal{M}))$ *of right $T(\mathcal{M})$-modules of finite length.*

Proof. Here we give the explicit correspondence between objects:

- Let $\underline{V} = (V_i, {}_j\varphi_i)$ be a representation of $(\mathcal{M}, \Omega)$. We define the corresponding $T(\mathcal{M})$-module as the direct sum
$$V = \bigoplus_{i \in Q} V_i,$$
where the A-action is again given via the projections $A \to F_i$ and the M action on V via: For $v_l \in V_l$, $m_{ij} \in {}_iM_j$ we define
$$v_l m_{ij} = \begin{cases} 0, & \text{if } l \neq i, \\ {}_j\varphi_i(v_l \otimes m_{ij}), & \text{if } l = i. \end{cases}$$
The action of $M^{(r)}$ is then defined inductively: For $v_l \in V_l$, $m_{i_1 i_2} \otimes m_{i_2 i_3} \otimes \ldots \otimes m_{i_{r-1} i_r} \otimes m_{i_r i_{r+1}}$ we define
$$v_l(m_{i_1 i_2} \otimes m_{i_2 i_3} \otimes \ldots \otimes m_{i_{r-1} i_r} \otimes m_{i_r i_{r+1}}) = ({}_{i_{r+1}}\varphi_{i_r})((v_l(m_{i_1 i_2} \otimes \ldots \otimes m_{i_{r-1} i_r})) \otimes m_{i_r i_{r+1}}).$$
Furthermore given a morphism $\underline{\psi} : \underline{V} \to \underline{W}$ of representations, the corresponding map $\bigoplus_{i \in Q} \psi_i : V \to W$ is $T(\mathcal{M}, \Omega)$-linear by definition.
- Let V be a $T(\mathcal{M}, \Omega)$-module. Then using the A-action on V we can decompose it as a direct sum
$$V = \bigoplus_{i \in Q} V_i, \qquad \text{where } V_i = VF_i.$$
Then using the M action we get the F_j-linear maps ${}_j\varphi_i : V_i \otimes {}_iM_j \to V_j$ along the arrows $(i, j) \in \Omega$.

□

2.3 Quadratic Forms, Roots and the Weyl Group

Furthermore, we can attach a bilinear form to $\mathbb{Q}^Q$ given a weighted graph $(Q, \underline{d})$: for vectors $\underline{x} = (x_i)_{i \in Q}$, $\underline{y} = (y_i)_{i \in Q} \in \mathbb{Q}^Q$ we define

$$B_Q(\underline{x}, \underline{y}) = \sum_{i \in Q} f_i x_i y_i - \frac{1}{2} \sum_{(i,j) \in Q \times Q} d_{ij} f_j x_i y_j.$$

The corresponding quadratic form $\mathcal{Q}_Q$ is then given by:

$$\mathcal{Q}_Q(\underline{x}) = \sum_{i \in Q} f_i x_i^2 - \sum_{i - j} d_{ij} f_j x_i x_j,$$

where the latter sum is over all edges $\{i, j\}$ in $(Q, \underline{d})$ and we use the fact $d_{ij} f_j = d_{ji} f_i$ for all $i, j \in Q$.

Then we have the following classification (see e.g. [12]).

Proposition 2.3.1 *Given a connected weighted graph $(Q, \underline{d})$ and its corresponding quadratic form $\mathcal{Q}_Q$ it holds:*

1. *$(Q, \underline{d})$ is a Dynkin graph if and only if $\mathcal{Q}_Q$ is positive definite.*
2. *$(Q, \underline{d})$ is an Eucilidean graph if and only if $\mathcal{Q}_Q$ is positive semidefinite.*

This can be proven using two lemmas:

Lemma 2.3.2 *Given a connected weighted graph $(Q, \underline{d})$ which is not a Dynkin graph (see Example 2.1.8) then it contains either a Euclidean graph or one of the following:*

$$\bullet \overset{(d_1,d_2)}{\overline{\qquad\qquad}} \bullet \qquad\qquad \textit{with } d_1 d_2 \geq 5$$

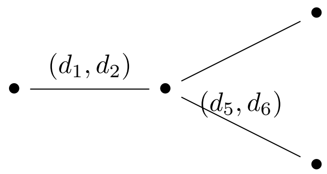
or

$$\bullet \overset{(d_1,d_2)}{\overline{\qquad\qquad}} \bullet \overset{(d_3,d_4)}{\overline{\qquad\qquad}} \bullet \qquad\qquad \textit{with } d_1 d_2 = 3 \textit{ and } d_3 d_4 \in \{2, 3\}.$$

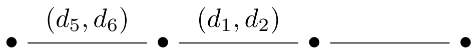
Proof. Since $(Q, \underline{d})$ is not A_1 we may consider an arbitrary edge $\bullet \overset{(d_1,d_2)}{\overline{\qquad\qquad}} \bullet$.

- If $d_1 d_2 \geq 5$ we are done.
- If $d_1 d_2 = 4$ then (d_1, d_2) is either $(1, 4)$, $(4, 1)$ or $(2, 2)$. In the first two cases, the subgraph consisting of just the edge with the incident nodes is the Euclidean graph $\tilde{A}_{11}$, in the latter case the Euclidean graph $\tilde{A}_{12}$.
- If $d_1 d_2 = 3$ then (d_1, d_2) is either $(1, 3)$ or $(3, 1)$. Since $(Q, \underline{d})$ is not G_2 there is another edge $\bullet \overset{(d_1,d_2)}{\overline{\qquad\qquad}} \bullet \overset{(d_3,d_4)}{\overline{\qquad\qquad}} \bullet$. If $d_3 d_4 \geq 2$ we are done ($d_3 d_4 \geq 4$ is done in the first two cases, $d_3 d_4 \in \{2, 3\}$ is the second depicted graph in the lemma). If $d_3 d_4 = 1$ we have $(d_3, d_4) = (1, 1)$ so either this subgraph is $\tilde{G}_{21}$ or $\tilde{G}_{12}$.
- If $d_1 d_2 = 2$ then (d_1, d_2) is either $(1, 2)$ or $(2, 1)$. Since $(Q, \underline{d})$ is not B_2 there is another edge $\bullet \overset{(d_1,d_2)}{\overline{\qquad\qquad}} \bullet \overset{(d_3,d_4)}{\overline{\qquad\qquad}} \bullet$. If $d_3 d_4 \geq 3$ we are in the first three cases. If $d_3 d_4 = 2$ this subgraph is one of the three Euclidean graphs $\tilde{B}_2$, $\tilde{C}_2$ or $\tilde{BC}_2$.

  If $d_3 d_4 = 1$ there is another edge



or



or

Again, if $d_5d_6 \geq 3$ we are in the previous cases, if $d_5d_6 = 2$ we get a Euclidean graph. If $d_5d_6 = 1$ in the former case we have a Euclidean graph of type $\tilde{BD}$ or $\tilde{CD}$, in the latter two cases we can find another edge. ...

This procedure ends since Q is finite.

- If $d_1d_2 = 1$ we can find edges until either we find a Euclidean subgraph of type $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ or find an edge with weights (d_i, d_j) with $d_id_j \geq 2$ which we discussed in the previous cases.

□

Lemma 2.3.3 *Let $(Q, \underline{d})$ be a connected weighted graph with a positive definite or semidefinite quadratic form $\mathcal{Q}_Q$. Then for any proper subgraph $Q' \subset Q$ with corresponding weights, the quadratic form $\mathcal{Q}_{Q'}$ is positive definite.*

Proof. It is clear that for any subgraph the quadratic form is either positive definite or semidefinite (since $\mathcal{Q}_{Q'}(\underline{x}) < 0$ for some $\underline{x} \in \mathbb{Q}^{Q'}$ implies $\mathcal{Q}_Q(\underline{\hat{x}}) = \mathcal{Q}_{Q'}(\underline{x}) < 0$ which is a contradiction to our setting (where $\underline{\hat{x}}$ is $\underline{x} \in \mathbb{Q}^{Q'}$ with $\hat{x}_i = 0$ for $i \notin Q'$)).

Now assume there is a proper subgraph Q' where the quadratic form is not positive definite. Without loss of generality, we may consider a minimal proper subgraph Q' with that property.

Let $\underline{y} \in \mathbb{Q}^{Q'} \setminus \{0\}$ with $\mathcal{Q}_{Q'}(\underline{y}) = 0$. Since Q' is a minimal subgraph with positive semidefinite quadratic form, we have $y_i \neq 0$ for all $i \in Q'$. Consider $\underline{x} \in \mathbb{Q}^{Q'}$ with $x_i = |y_i|$. Then it holds

$$\begin{aligned} 0 \leq \mathcal{Q}_{Q'}(\underline{x}) &= \sum_{i \in Q'} f_i x_i^2 - \sum_{i - j} d_{ij} f_j x_i x_j \\ &\leq \sum_{i \in Q'} f_i y_i^2 - \sum_{i - j} d_{ij} f_j y_i y_j \\ &= \mathcal{Q}_{Q'}(\underline{y}) \\ &= 0 \end{aligned}$$

Hence we have $\mathcal{Q}_{Q'}(\underline{x}) = 0$.

Now let $j \in Q \setminus Q'$ with $d_{i_0 j} \neq 0$ for some $i_0 \in Q'$ (this exists since Q' is a proper subgraph and $(Q, \underline{d})$ is connected). Now consider $\underline{z} \in \mathbb{Q}^Q$ with $z_i = x_i$ for $i \in Q'$, $z_j = \frac{1}{2} d_{i_0 j} x_{i_0} > 0$ and $z_i = 0$ else. Then it holds:

$$\begin{aligned}
\mathcal{Q}_Q(\underline{z}) &= \underbrace{\mathcal{Q}_{Q'}(\underline{x})}_{=0} + f_j z_j^2 - \sum_{i-j} d_{ij} f_j z_i z_j \\
&\leq f_j z_j^2 - d_{i_0 j} f_j z_{i_0} z_j \\
&= f_j \frac{1}{4} d_{i_0 j}^2 x_{i_0}^2 - f_j \frac{1}{2} d_{i_0 j}^2 x_{i_0}^2 \\
&< 0
\end{aligned}$$

But this is a contradiction to $\mathcal{Q}_Q$ being positive definite or semidefinite. □

Now to the proof of our proposition:

Proof of Proposition 2.3.1. We prove that a graph with positive definite or semidefinite quadratic form is either a Dynkin graph or a Euclidean graph. The other direction is a direct calculation.

Let $(Q, \underline{d})$ be a connected weighted graph which is not a Dynkin graph with positive definite or semidefinite quadratic form. Then by Lemma 2.3.2 there is either a Euclidean subgraph or one of the exceptional ones. But the quadratic form of the exceptional ones is neither positive definite nor semidefinite. Hence we are in the case with a Euclidean subgraph. But Euclidean graphs have a positive semidefinite quadratic form and by Lemma 2.3.3 proper subgraphs have a positive definite quadratic form. In particular, this means that $(Q, \underline{d})$ is a Euclidean graph itself. □

Now we define what the Weyl group corresponding to a graph is.

Definition 2.3.4 Let $(Q, \underline{d})$ be a weighted graph.

Let $j \in Q$ be a node. We denote by $e_j \in \mathbb{Q}^Q$ the vector with $(e_j)_i = \delta_{ij}$ for $i \in Q$. The (simple) *reflection* $\sigma_j : \mathbb{Q}^Q \to \mathbb{Q}^Q$ is defined as

$$\sigma_j(\underline{x}) = \underline{x} - 2 \frac{B_Q(\underline{x}, e_j)}{B_Q(e_j, e_j)} e_j$$

for any $\underline{x} \in \mathbb{Q}^Q$.

The *Weyl group* W_Q is defined as the group of all linear transformations of $\mathbb{Q}^Q$ generated by the (simple) reflections $(\sigma_j)_{j \in Q}$.

Let $\{1, ..., n\}$ be an admissible order of sinks given by a fixed orientation Ω. Then the corresponding *Coxeter element* $c \in W_Q$ is defined as

$$c = \sigma_n \sigma_{n-1} ... \sigma_1.$$

Remark 2.3.5 One can easily check that the simple reflections σ_j are actual reflections, namely:

$$\begin{aligned}
\sigma_j^2(\underline{x}) &= \sigma_j\left(\underline{x} - 2\frac{B_Q(\underline{x}, e_j)}{B_Q(e_j, e_j)}e_j\right) \\
&= \sigma_j(\underline{x}) - 2\frac{B_Q(\underline{x}, e_j)}{B_Q(e_j, e_j)}\sigma_j(e_j) \\
&= \underline{x} - 2\frac{B_Q(\underline{x}, e_j)}{B_Q(e_j, e_j)}e_j - 2\frac{B_Q(\underline{x}, e_j)}{B_Q(e_j, e_j)}\left(e_j - 2\frac{B_Q(e_j, e_j)}{B_Q(e_j, e_j)}e_j\right) \\
&= \underline{x} - 2\frac{B_Q(\underline{x}, e_j)}{B_Q(e_j, e_j)}e_j + 2\frac{B_Q(\underline{x}, e_j)}{B_Q(e_j, e_j)}e_j \\
&= \underline{x}
\end{aligned}$$

Example 2.3.6 For any $j \in Q$ we have:

$$\sigma_j(e_j) = e_j - 2\frac{B_Q(e_j, e_j)}{B_Q(e_j, e_j)}e_j = -e_j.$$

Remark 2.3.7 Note that the Coxeter element does not depend on the choice of the admissible order for a fixed orientation. Namely, if there are no edges between two nodes i and j, then the simple reflections σ_i and σ_j commute.

Definition 2.3.8

- A vector $\underline{x} \in \mathbb{Q}^Q$ is called *stable* if for all $\omega \in W_Q$:

$$\omega(\underline{x}) = \underline{x}.$$

- We denote by $R_Q := \{\underline{x} \in \mathbb{Q}^Q | \forall \omega \in W_Q : \omega(\underline{x}) = \underline{x}\}$, the *radical subspace* of $\mathbb{Q}^Q$ defined as the set of all stable vectors.
- A vector $\underline{x} \in \mathbb{Q}^Q$ is called a *root* if there exists $j \in Q$ and $\omega \in W_Q$ such that:

$$\underline{x} = \omega(e_j).$$

- A root $\underline{x} \in \mathbb{Q}^Q$ is *positive* (respectively *negative*) if $x_i \geq 0$ for all $i \in Q$ (respectively $x_i \leq 0$ for all $i \in Q$).

One may characterize stable vectors differently:

Lemma 2.3.9 *Given a (connected) weighted graph $(Q, \underline{d})$ and a vector $\underline{x} \in \mathbb{Q}^Q$, the following statements are equivalent:*

1. *$\underline{x}$ is stable;*
2. *$\sigma_k(\underline{x}) = \underline{x}$ for all $k \in Q$;*
3. *$c(\underline{x}) = \underline{x}$ for a Coxeter element c;*
4. *$B_Q(\underline{x}, \underline{y}) = 0$ for all $\underline{y} \in \mathbb{Q}^Q$.*

Proof. The equivalence between 1. and 2. is clear. That 2. implies 3. is also clear. Furthermore note that each reflection σ_j only changes the j^{th} coordinate, so if $c(\underline{x}) = \underline{x}$ then at each reflection σ_j it acts trivial, hence 3. implies 2..

Furthermore if $B_Q(\underline{x}, e_j) = 0$ for each j, this clearly implies 2..

On the other hand by definition $\sigma_k(\underline{x}) = \underline{x}$ implies $B_Q(\underline{x}, e_k) = 0$, and if this holds for all k, then by linearity in the second argument of $B_Q(\underline{x}, -)$ we have 2. implies 4.. ☐

Then one can prove about the radical subspace:

Lemma 2.3.10

- *Given a Dynkin graph $(Q, \underline{d})$, then it holds $R_Q = \{0\}$.*
- *Given a Euclidean graph $(Q, \underline{d})$, then R_Q is one dimensional and has a generator $\underline{\delta}$ with all coordinates natural numbers and at least one coordinate equals one.*

Proof.

- Given a Dynkin graph, then $\mathcal{Q}_Q$ is positive definite, especially for all $\underline{x} \neq 0$ then $\mathcal{Q}_Q(\underline{x}) = B_Q(\underline{x}, \underline{x}) > 0$ and therefore $R_Q = \{0\}$.
- This follows by calculation of R_Q in each case.

☐

Example 2.3.11 In the case $\tilde{A}_{12}$ we have $B_Q((x_1, x_2), (y_1, y_2)) = x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1$.

Hence we have $B_Q((1, 1), (y_1, y_2)) = y_1 + y_2 - y_2 - y_1 = 0$ for all $(y_1, y_2) \in \mathbb{Q}^2$.

In this case it holds $\underline{\delta} = (1, 1)$.

Remark 2.3.12 For $\underline{x} = \omega(e_k)$ a positive root with $\omega \in W_Q$ it holds $\omega\sigma_k(e_k) = \omega(-e_k) = -\omega(e_k) = -\underline{x}$ is a negative root.

Proposition 2.3.13 *Let $(Q, \underline{d})$ be a Dynkin or Euclidean graph. Let $\underline{x} \in \mathbb{Q}^Q$ be a positive root. Then for any $k \in Q$ either $\sigma_k(\underline{x})$ is positive or $\underline{x} = e_k$.*

Instead of showing this, we refer to [12], for most roots this can be viewed as a corollary of a later statement about reflection functors, to be more precise, if we can find an admissible orientation that we may reflect as in $\underline{x} = \omega(e_j)$ with reflection functors, then the statement is clear (see Proposition 2.4.2).

Corollary 2.3.14 *Let $(Q, \underline{d})$ be a Dynkin or Euclidean graph. Then a root $\underline{x} \in \mathbb{Q}^Q$ is either positive or negative.*

Proposition 2.3.15 *Let $(Q, \underline{d})$ be a Dynkin or Euclidean graph. Let $\underline{x} \in \mathbb{Q}^Q$ be a positive root and $c = \sigma_n ... \sigma_1 \in W_Q$ the Coxeter element. Then it holds:*

1. *$c(\underline{x})$ is negative if and only if $\underline{x} = \underline{p}_k := \sigma_1\sigma_2...\sigma_{k-1}(e_k)$ for some $1 \leq k \leq n$;*
2. *$c^{-1}(\underline{x})$ is negative if and only if $\underline{x} = \underline{q}_k := \sigma_n\sigma_{n-1}...\sigma_{k+1}(e_k)$ for some $1 \leq k \leq n$.*

Proof. The statement is a direct consequence of Proposition 2.3.13.

If $c(\underline{x}) = \sigma_n...\sigma_1(\underline{x})$ is negative then choose $k \in \{1, ..., n\}$ minimal such that $\sigma_k...\sigma_1(\underline{x})$ is minimal. By Proposition 2.3.13 we have $\sigma_{k-1}...\sigma_1(\underline{x}) = e_k$, in particular $\underline{x} = \sigma_1\sigma_2....\sigma_{k-1}(e_k)$.

If $\underline{x} = \sigma_1\sigma_2....\sigma_{k-1}(e_k)$, then it holds

$$c(\underline{x}) = \sigma_n ... \sigma_1 \sigma_1 \sigma_2 .... \sigma_{k-1}(e_k) = \sigma_n ... \sigma_k(e_k) = -\sigma_n ... \sigma_{k+1}(e_k).$$

Since $\sigma_n ... \sigma_{k+1}$ does not change the k-th coordinate, by Corollary 2.3.14 $\sigma_n ... \sigma_{k+1}(e_k)$ is positive and therefore $c(\underline{x})$ negative.

The second statement can be proven analogously. □

Furthermore, we can make an actual list of roots for Dynkin and Euclidean graphs. To do this, we first need to define what the defect is.

Recall that the Weyl group W_Q acts trivially on the radical subspace R_Q. Thus each element $\omega \in W_Q$ induces a linear transformation $\bar{\omega}$ on $\mathbb{Q}^Q / R_Q$. These still form a group which we denote by $\bar{W}_Q$.

Proposition 2.3.16 *Given a Dynkin or Euclidean graph $(Q, \underline{d})$, then the group $\bar{W}_Q$ is finite.*

For a proof see [13].

Definition 2.3.17 Given a Euclidean graph $(Q, \underline{d})$ and c a Coxeter element. Let m be the order of $\bar{c}$ in $\bar{W}_Q$. Then we define the *defect* $d_c(\underline{x})$ of $\underline{x} \in \mathbb{Q}^Q$ to be the number for which holds

$$c^m(\underline{x}) = \underline{x} + d_c(\underline{x})\delta.$$

Now we can make a list of roots for Dynkin and Euclidean graphs (see [12]):

Proposition 2.3.18 *1. Let $(Q, \underline{d})$ be a Dynkin graph and $c \in W_Q$ a Coxeter element. For each $k \in \{1, ..., n\}$ let a_k be minimal such that $c^{-a_k - 1}(\underline{p}_k)$ is negative. Then the set*

$$\{c^{-s}\underline{p}_k \mid 1 \leq k \leq n,\ 0 \leq s \leq a_k\}$$

is a list of all positive roots of $(Q, \underline{d})$.

Analogously, for each $k \in \{1, ..., n\}$ let b_k be minimal such that $c^{b_k + 1}(\underline{q}_k)$ is negative. Then the set

$$\{c^s \underline{q}_k \mid 1 \leq k \leq n,\ 0 \leq s \leq b_k\}$$

is a list of all positive roots of $(Q, \underline{d})$.

2. Let $(Q, \underline{d})$ be a Euclidean graph and $c \in W_Q$ a Coxeter element. Then it holds:

(a) The set $\{c^{-s}(\underline{p}_k) \mid 1 \leq k \leq n,\ s \geq 0\}$ is the set of all positive roots of negative defect.

(b) The set $\{c^s(\underline{q}_k) \mid 1 \leq k \leq n,\ s \geq 0\}$ is the set of all positive roots of positive defect.

(c) The set $\{\underline{x}_0 + sg\underline{\delta} \mid s \geq 0,\ \underline{x}_0$ a root of zero defect with $\underline{x}_0 \leq g\underline{\delta}\}$, where $g \in \{1, 2, 3\}$ is a constant for the graph, is the set of all positive roots of zero defect.

Remark 2.3.19 A list for the constant g for the Euclidean graphs can be found in [13].

Example 2.3.20 Consider $1 \xrightarrow{(1,3)} 2$. Then the Coxeter element is $c = \sigma_1 \sigma_2$.

- $\underline{p}_1 = \sigma_2(1,0) = (1,0) - -2\frac{B((1,0),(0,1))}{B((0,1),(0,1))}(0,1) = (1,0) - 2\frac{-\frac{3}{2}}{3}(1,0) = (1,1)$ and $\underline{p}_2 = (0,1)$;
- $\underline{q}_1 = (1,0)$ and $\underline{q}_2 = \sigma_1((0,1)) = (0,1) - 2\frac{B((0,1),(1,0))}{B((1,0),(1,0))}(1,0) = (0,1) - 2\frac{-\frac{3}{2}}{1}(1,0) = (3,1)$;

We have:

$$\begin{aligned}
c^{-1}(\underline{p}_1) &= \sigma_2\sigma_1(1,1) \\
&= \sigma_2\left((1,1) - 2\frac{B((1,1),(1,0))}{B((1,0),(1,0))}(1,0)\right) \\
&= \sigma_2\left((1,1) - 2\frac{-\frac{1}{2}}{1}(1,0)\right) \\
&= \sigma_2(2,1) \\
&= (2,1) - 2\frac{B((2,1),(0,1))}{B((0,1),(0,1))}(0,1) \\
&= (2,1) - 2\frac{0}{3}(0,1) \\
&= (2,1)
\end{aligned}$$

Similarly, we can calculate the positive roots

$$\underline{p}_1 = (1,1) \xmapsto{c^{-1}} (2,1) \xmapsto{c^{-1}} \underline{q}_1 = (1,0)$$

$$\underline{p}_2 = (0,1) \xmapsto{c^{-1}} (3,2) \xmapsto{c^{-1}} \underline{q}_2 = (3,1)$$

Example 2.3.21 Consider $1 \xrightarrow{(1,4)} 2$. Then the Coxeter element is $c = \sigma_1\sigma_2$. Furthermore, we calculate:

- $\underline{p}_1 = \sigma_2(1,0) = (1,0) - -2\frac{B((1,0),(0,1))}{B((0,1),(0,1))}(0,1) = (1,0) - 2\frac{-2}{4}(1,0) = (1,1)$ and $\underline{p}_2 = (0,1)$;
- $\underline{q}_1 = (1,0)$ and $\underline{q}_2 = \sigma_1((0,1)) = (0,1) - 2\frac{B((0,1),(1,0))}{B((1,0),(1,0))}(1,0) = (0,1) - 2\frac{-2}{1}(1,0) = (4,1)$;
- Note: $\underline{\delta} = (2,1)$. It holds:

$$\begin{aligned}
c^{-1}(x,y) &= \sigma_2\sigma_1(x,y) \\
&= \sigma_2\left((x,y) - 2\frac{B((x,y),(1,0))}{B((1,0),(1,0))}(1,0)\right) \\
&= \sigma_2\left((x,y) - 2\frac{x-2y}{1}(1,0)\right) \\
&= \sigma_2(-x+4y, y) \\
&= (-x+4y,y) - 2\frac{B((-x+4y,y),(0,1))}{B((0,1),(0,1))}(0,1) \\
&= (-x+4y,y) - 2\frac{4y+2x-8y}{4}(0,1) \\
&= (-x+4y,y) + (0, 2y-x) \\
&= (-x+4y, -x+3y) \\
&= (x,y) + (-2x+4y, -x+2y) \\
&= (x,y) + (-x+2y)\underline{\delta}
\end{aligned}$$

In particular $\bar{c}$ is trivial in $\bar{W}_Q$.

We also get a formula for the defect in this case:

$$d_c(x,y) = x - 2y$$

Hence we have:

1. The set of all positive roots of negative defect:
$$\{(1+2s, 1+s), (4s, 1+2s) \mid s \geq 0\}.$$
2. The set of all positive roots of positive defect:
$$\{(1+2s, s), (4+4s, 1+2s) \mid s \geq 0\}.$$
3. There are no positive roots of zero defect since $d_c(x,y) = 0$ implies $x = 2y$ and therefore $(x,y) = y(2,1) \in R_Q$. In particular for all $\omega \in W_Q$: $\omega(x,y) = (x,y)$. So by definition, it cannot be a root.

2.4 Reflection Functors

Similar to the definition of reflection functors in the case of representations of quivers, we have reflection functors for representations of species. The idea of reflection functors is to reverse the arrows which end in a specific sink or which begin in a specific source.

The reflection functors have several applications, for one they form an adjoint pair which gives rise to derived equivalences. These may be used to get an automorphism on the reduced Drinfeld double of the Hall algebra. In particular, using the connection between the Hall algebra of representations of a species over a finite field and quantum groups which we will discuss in Subsection 2.6, they give rise to the Lusztig symmetries on the quantum group side (see [26]).

Definition 2.4.1 (i) Let $i_0 \in Q$ be a sink. Then the reflection functor

$$\mathbb{S}^+_{i_0} : \mathrm{Rep}_k(\mathcal{M}, \Omega) \to \mathrm{Rep}_k(\mathcal{M}, \sigma_{i_0}\Omega)$$

is defined as follows:

- On objects:
  Given a representation $\underline{V} = (V_i, {}_j\varphi_i)$, we set
  - $S^+_{i_0}(V_i) := V_i$ for $i \neq i_0$;
  - $S^+_{i_0}(V_{i_0}) := \ker \left(\bigoplus_{j \to i_0} {}_{i_0}\varphi_j \right)$.

  In particular, we have a map $\iota : S^+_{i_0}(V_{i_0}) \to \bigoplus_{j \to i_0} V_j \otimes_{F_j} {}_j M_{i_0}$ and we can set $S^+_{i_0}({}_{i_0}\bar{\varphi}_j) = p_j \circ \iota$.

  We set $S^+_{i_0}({}_j\varphi_i) = {}_j\varphi_i$ for all $i,\ j \neq i_0$.
- On morphisms:
  Given a morphism $\underline{f} = (f_i)_{i \in Q_0} : \underline{V} \to \underline{W}$ between two representations $\underline{V} = (V_i, {}_j\varphi_i)$ and $\underline{W} = (W_i, {}_j\psi_i)$, we set $S^+_{i_0}(f_i) := f_i$ for $i \neq i_0$. In the case $i = i_0$ we use the universal property of the kernel $\mathbb{S}^+_{i_0}(W_{i_0})$, namely there exists a unique morphism $\mathbb{S}^+_{i_0}(f_{i_0}) : \mathbb{S}^+_{i_0}(V_{i_0}) \to \mathbb{S}^+_{i_0}(W_{i_0})$ such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{S}^+_{i_0}(V_{i_0}) & \xrightarrow{\iota_V} & \bigoplus_{j\to i_0} V_j \otimes_{F_j} {}_jM_{i_0} & \longrightarrow & V_{i_0} \\ & & \Big\downarrow \mathbb{S}^+_{i_0}(f_{i_0}) & & \Big\downarrow \bigoplus_{j\to i_0} f_j \otimes \mathrm{id} & & \Big\downarrow f_{i_0} \\ 0 & \longrightarrow & \mathbb{S}^+_{i_0}(W_{i_0}) & \xrightarrow{\iota_W} & \bigoplus_{j\to i_0} W_j \otimes_{F_j} {}_jM_{i_0} & \longrightarrow & W_{i_0} \end{array}$$

(ii) Let $i_0 \in Q$ be a source. Then the reflection functor

$$\mathbb{S}^-_{i_0} : \mathrm{Rep}_k(\mathcal{M}, \Omega) \to \mathrm{Rep}_k(\mathcal{M}, \sigma_{i_0}\Omega)$$

is defined as follows:

- On objects: Given a representation $\underline{V} = (V_i, {}_j\varphi_i)$, we set
  - $S^-_{i_0}(V_i) := V_i$ for $i \neq i_0$;
  - $S^-_{i_0}(V_{i_0}) := \mathrm{coker}\left(\bigoplus_{j\neq i_0} {}_j\bar{\varphi}_{i_0}\right)$.

  In particular, we have a map $\rho : \bigoplus_{j\neq i_0} V_j \otimes_{F_j} {}_jM_{i_0} \to S^-_{i_0}(V_{i_0})$ and we can set $S^-_{i_0}({}_j\varphi_{i_0}) = i_j \circ \rho$ where i_r is the injection $i_r : V_r \otimes_{F_r} {}_rM_{i_0} \to \bigoplus_{j\neq i_0} V_j \otimes_{F_j} {}_jM_{i_0}$ onto the r^{th} summand.
- On morphisms:
  Given a morphism $\underline{f} = (f_i)_{i\in Q_0} : \underline{V} \to \underline{W}$ between two representations $\underline{V} = (V_i, {}_j\varphi_i)$ and $\underline{W} = (W_i, {}_j\psi_i)$, we set $S^-_{i_0}(f_i) := f_i$ for $i \neq i_0$. In the case $i = i_0$ we use the universal property of the cokernel $\mathbb{S}^-_{i_0}(V_{i_0})$, namely there exists a unique morphism $\mathbb{S}^-_{i_0}(f_{i_0}) : \mathbb{S}^-_{i_0}(V_{i_0}) \to \mathbb{S}^-_{i_0}(W_{i_0})$ such that the following diagram commutes:

$$\begin{array}{ccccccc} V_{i_0} & \longrightarrow & \bigoplus_{j\neq i_0} V_j \otimes_{F_j} {}_jM_{i_0} & \xrightarrow{\rho_V} & \mathbb{S}^-_{i_0}(V_{i_0}) & \longrightarrow & 0 \\ \Big\downarrow f_{i_0} & & \Big\downarrow \bigoplus_{j\neq i_0} f_j \otimes \mathrm{id} & & \Big\downarrow \mathbb{S}^-_{i_0}(f_{i_0}) & & \\ W_{i_0} & \longrightarrow & \bigoplus_{j\neq i_0} W_j \otimes_{F_j} {}_jM_{i_0} & \xrightarrow{\rho_W} & \mathbb{S}^-_{i_0}(W_{i_0}) & \longrightarrow & 0 \end{array}$$

Similar to the case of representations of quivers, one can define the Coxeter functor. Namely, given a species $(\mathcal{M}, \Omega)$ with admissible order of sinks $i_1, ..., i_n$, the Coxeter functor is defined by

$$\mathcal{C} := \mathbb{S}^+_{i_n} ... \mathbb{S}^+_{i_2} \mathbb{S}^+_{i_1} : \mathrm{Rep}_k(\mathcal{M}, \Omega) \to \mathrm{Rep}_k(\mathcal{M}, \Omega).$$

Similarly, the 'inverse' Coxeter functor is given by applying the reflection functors $\mathbb{S}^-$

$$\mathcal{C}^- := \mathbb{S}^-_{i_1} ... \mathbb{S}^-_{i_{n-1}} \mathbb{S}^-_{i_n} : \mathrm{Rep}_k(\mathcal{M}, \Omega) \to \mathrm{Rep}_k(\mathcal{M}, \Omega).$$

They are adjoint functors, and give rise to an equivalence not on the whole category, but on the subcategories without projective respectively injective summands (see e.g. [12]).

Furthermore, Brenner and Butler showed (see [4]), if the underlying unoriented graph of our species is a tree (or the oriented graph has no oriented circles), by applying the Auslander-Reiten translation τ to an object we get an object which is isomorphic to the case if we had applied the Coxeter functor. Since we consider only the isomorphism classes in the Hall algebras, instead of the Auslander-Reiten translation τ we may use the Coxeter functor $\mathcal{C}$.

Before we begin to prove some statements, namely in connection to the roots and Weyl groups, we define the dimension vector of a representation.

Given a representation $\underline{V} = (V_i, {}_j\varphi_i)$ of a species $(\mathcal{M}, \Omega)$, we define $\underline{\dim}(\underline{V}) \in \mathbb{N}_0^Q$ to be the vector given by the coordinates

$$\underline{\dim}(\underline{V})_i := \dim(V_i)_{F_i}$$

for $i \in Q$.

Now, let us discuss what happens if we reflect at the same node twice (see [12]):

Lemma 2.4.2 *Let $(\mathcal{M}, \Omega)$ be a species of a connected weighted graph $(Q, \underline{d})$ with admissible orientation Ω. Let $\underline{V} = (V_i, {}_j\varphi_i)$ be a representation of $(\mathcal{M}, \Omega)$.*

1. *Let $1 \in Q$ be a sink. Then it holds*

$$\underline{V} \cong \mathbb{S}_1^- \mathbb{S}_1^+(\underline{V}) \oplus \underline{P},$$

where $\underline{P}$ is isomorphic to finitely many copies of the simple representation S_1 with F_1 at node 1 and 0 else. In particular, if $\underline{V}$ is indecomposable either

$$\underline{V} \cong S_1$$

or

$$\underline{V} \cong \mathbb{S}_1^- \mathbb{S}_1^+(\underline{V}).$$

Furthermore in the latter case we have $\operatorname{End}(\mathbb{S}_1^+(\underline{V})) \cong \operatorname{End}(\underline{V})$, hence $\mathbb{S}_1^+(\underline{V})$ is indecomposable, and $\underline{\dim}(\mathbb{S}_1^+(\underline{V})) = \sigma_1(\underline{\dim}(\underline{V}))$.

2. *Let $n \in Q$ be a source. Then it holds*

$$\underline{V} \cong \mathbb{S}_n^+ \mathbb{S}_n^-(\underline{V}) \oplus \underline{I},$$

where $\underline{I}$ is isomorphic to finitely many copies of the simple representation S_n with F_n at node n and 0 else. In particular, if $\underline{V}$ is indecomposable either

$$\underline{V} \cong S_n$$

or

$$\underline{V} \cong \mathbb{S}_n^+ \mathbb{S}_n^-(\underline{V}).$$

Furthermore in the latter case we have $\operatorname{End}(\mathbb{S}_n^-(\underline{V})) \cong \operatorname{End}(\underline{V})$, hence $\mathbb{S}_n^-(\underline{V})$ is indecomposable, and $\underline{\dim}(\mathbb{S}_n^-(\underline{V})) = \sigma_n(\underline{\dim}(\underline{V}))$.

Proof. We prove the first statement, the second is given by the dual argument.

Consider the diagram of the construction of $\mathbb{S}_1^- \mathbb{S}_1^+(\underline{V})$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathbb{S}_1^+(\underline{V}))_1 & \longrightarrow & \bigoplus_{i\to 1}(V_i \otimes {}_iM_1) & \xrightarrow{\bigoplus{}_i\varphi_1} & V_1 \\ & & & & \downarrow & \overset{\mu_1}{\nearrow} & \\ & & & & (\mathbb{S}_1^-\mathbb{S}_1^+(\underline{V}))_1 & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Since we have $\operatorname{coker}(\ker(\bigoplus{}_i\varphi_1)) = \operatorname{im}(\bigoplus{}_i\varphi_1)$ by definition, there exists a monomorphism $\mu_1 : (\mathbb{S}_1^-\mathbb{S}_1^+(\underline{V}))_1 \to V_1$ such that the diagram commutes.

If $\bigoplus{}_i\varphi_1$ is an epimorphism then μ_1 is an isomorphism. Since the other maps/nodes stay the same, it follows $\underline{V} \cong \mathbb{S}_1^-\mathbb{S}_1^+(\underline{V})$.

If $\bigoplus{}_i\varphi_1$ is not an epimorphism then $\underline{V}$ must be a direct sum of $\mathbb{S}_1^-\mathbb{S}_1^+(\underline{V})$ (which can be embedded via μ_1 at V_1 and id) and copies of S_1. Hence we get $\underline{V} \cong \mathbb{S}_1^-\mathbb{S}_1^+(\underline{V}) \oplus \underline{P}$.

In particular if $\underline{V}$ is indecomposable either $\underline{V} \cong S_1$ (where already $\mathbb{S}_1^+(\underline{V}) = 0$ which only happens if $\underline{V}$ is S_1) or $\underline{V} \cong \mathbb{S}_1^-\mathbb{S}_1^+(\underline{V})$. Now in the latter case, we have $\mathbb{S}_1^+(\underline{V}) \cong \mathbb{S}_1^+\mathbb{S}_1^-\mathbb{S}_1^+(\underline{V})$, in particular, each composition of any two maps is a bijection:

$$\operatorname{End}(\underline{V}) \xrightarrow{\mathbb{S}_1^+} \operatorname{End}(\mathbb{S}_1^+(\underline{V})) \xrightarrow{\mathbb{S}_1^-} \operatorname{End}(\mathbb{S}_1^-\mathbb{S}_1^+(\underline{V})) \xrightarrow{\mathbb{S}_1^+} \operatorname{End}(\mathbb{S}_1^+\mathbb{S}_1^-\mathbb{S}_1^+(\underline{V})).$$

Hence each map is a bijection and we get $\operatorname{End}(\mathbb{S}_1^+(\underline{V})) \cong \operatorname{End}(\underline{V})$.

In particular, if $\underline{V} \ncong S_1$ is indecomposable, so is $\mathbb{S}_1^+(\underline{V})$.

Now to the last statement: the dimension vector of $\mathbb{S}_1^+(\underline{V})$ differs from the dimension vector of $\underline{V}$ only in the first coordinate. So it suffices to check that one. By construction we have a short exact sequence:

$$0 \longrightarrow (\mathbb{S}_1^+(\underline{V}))_1 \longrightarrow \bigoplus_{j\to 1}(V_j \otimes {}_jM_1) \longrightarrow V_1 \longrightarrow 0.$$

(Here we need that $\bigoplus{}_j\varphi_1$ is an epimorphism, which is equivalent to $\underline{V} \cong \mathbb{S}_1^-\mathbb{S}_1^+(\underline{V})$.)

Hence we get for the dimension at the first coordinate (denote v_i for the i^{th} coordinate of the dimension vector of $\underline{V}$):

$$\begin{aligned} \dim((\mathbb{S}_1^+(\underline{V}))_1)_{F_1} &= \sum_{j\to 1} \dim(V_j \otimes {}_jM_1)_{F_1} - \dim(V_1)_{F_1} \\ &= \sum_{j\to 1} v_j \cdot d_{j1} - v_1 \\ &= v_1 - 2(v_1 - \frac{1}{2}\sum_{j\to 1} v_j \cdot d_{j1}) \\ &= v_1 - 2\frac{v_1 f_1 - \frac{1}{2}\sum_{j\to 1} v_j \cdot d_{j1} f_1}{f_1} \\ &= (\sigma_1(\underline{\dim}(\underline{V})))_1 \end{aligned}$$

Hence we have $\underline{\dim}(\mathbb{S}_1^+(\underline{V})) = \sigma_1(\underline{\dim}(\underline{V}))$. □

Given a k-species $(\mathcal{M}, \Omega)$ with admissible orientation Ω with the order $1, ..., n$. Then for $i \in \{1, ..., n\}$ we may define the objects

- $\underline{P}_i := \mathbb{S}_1^- \mathbb{S}_2^- ... \mathbb{S}_{i-1}^-(S_i)$;
- $\underline{Q}_i := \mathbb{S}_n^+ \mathbb{S}_{n-1}^+ ... \mathbb{S}_{i+1}^+(S_i)$,

where S_i is the simple representation in $\text{Rep}_k(\mathcal{M}, s_{i-1}...s_1\Omega)$ (respectively $\text{Rep}_k(\mathcal{M}, s_{i+1}...s_n\Omega)$) with F_i at the node i;

We also get the classification of projective and injective objects (see [12]):

Proposition 2.4.3 *Let $\underline{V} = (V_i, {}_j\varphi_i)$ be an indecomposable representation in $\text{Rep}_k(\mathcal{M}, \Omega)$ and $c \in W_Q$ the Coxeter element corresponding to the admissible order given by Ω. Then the following statements are equivalent:*

(i) $\underline{V}$ *is projective;*

(ii) $\underline{V} \cong \underline{P}_i$ *for some* $1 \leq i \leq n$*;*

(iii) $\mathcal{C}(\underline{V}) = 0$*;*

(iv) $c(\underline{dim}(\underline{V}))$ *is non-positive.*

Similarly, the following are equivalent:

(i) $\underline{V}$ *is injective;*

(ii) $\underline{V} \cong \underline{Q}_i$ *for some* $1 \leq i \leq n$*;*

(iii) $\mathcal{C}^-(\underline{V}) = 0$*;*

(iv) $c^{-1}(\underline{dim}(\underline{V}))$ *is non-positive.*

It can be proven via two lemmas:

Lemma 2.4.4 *The set $\{\underline{P}_i \mid 1 \leq i \leq n\}$ (respectively $\{\underline{Q}_i \mid 1 \leq i \leq n\}$) is the set of all indecomposable projective (respectively injective) objects in $\text{Rep}_k(\mathcal{M}, \Omega)$.*

Proof. We prove the statement about the projective objects, and the statement about injectives works analogously.

Since the set $\{\underline{P}_i \mid 1 \leq i \leq n\}$ consists of n different non-isomorphic elements, it suffices to show that each $\underline{P}_i$ is projective.

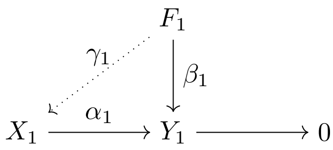
We prove the statement by induction on i.

- For $i = 1$ we have $\underline{P}_1 = S_1$. Consider the diagram

$$\begin{array}{ccccc} & & S_1 & & \\ & & \downarrow{\scriptstyle \underline{\beta}} & & \\ \underline{X} & \xrightarrow{\underline{\alpha}} & \underline{Y} & \longrightarrow & 0 \end{array}$$

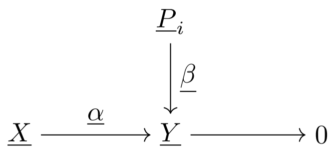
for some epimorphism $\underline{\alpha} : \underline{X} \to \underline{Y}$ and morphism $\underline{\beta} : S_1 \to \underline{Y}$.

Since S_1 is trivial at each node except 1, it suffices to check at node 1. Furthermore, since 1 is a sink, there are no commutativity relations to check concerning the maps between nodes ${}_i\varphi_j$. Hence we consider the diagram

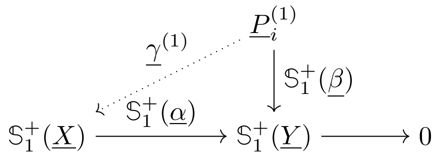


But X_1 and Y_1 are F_1-modules, in particular there exists γ_1 such that the diagram commutes, hence there exists $\underline{\gamma} = (\gamma_i)_i : S_1 \to \underline{X}$ with $\gamma_i = 0$ for $i \neq 1$ such that $\underline{\alpha} \circ \underline{\gamma} = \underline{\beta}$. In particular, S_1 is projective.

- Now let $i > 1$ and assume that $\underline{P}_l$ for $l < i$ is indecomposable and projective for any admissible orientation. In particular $\underline{P}_i^{(1)} := \mathbb{S}_2^- ... \mathbb{S}_{i-1}^-(S_i)$ is indecomposable and projective and $\underline{P}_i = \mathbb{S}_1^-(\underline{P}_i^{(1)})$ by definition. Since $\underline{P}_i^{(1)} \neq S_1$ is indecomposable, $\underline{P}_i \neq S_1$ is indecomposable, too, and we also have $\mathbb{S}_1^+(\underline{P}_i) \cong \underline{P}_i^{(1)}$. Consider the diagram

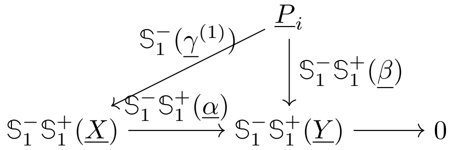


for some epimorphism $\underline{\alpha} : \underline{X} \to \underline{Y}$ and morphism $\underline{\beta} : \underline{P}_i \to \underline{Y}$, and apply the functor $\mathbb{S}_1^+$:



The map $\mathbb{S}_1^+(\underline{\alpha})$ is generally not an epimorphism, but the representation $\underline{P}_i^{(1)}$ has only non-zero components at the nodes $\{2, 3, ..., i\}$, in particular on these nodes $\mathbb{S}_1^+(\underline{\alpha})_j = \alpha_j$, hence it is an epimorphism there and we can apply the assumption that $\underline{P}_i^{(1)}$ is projective. We receive a map $\underline{\gamma}^{(1)}$ as indicated such that the diagram commutes.

Now applying the functor $\mathbb{S}_1^-$ we get the diagram:



By Lemma 2.4.2 we have $\underline{Y} \cong \mathbb{S}_1^-\mathbb{S}_1^+(\underline{Y}) \oplus S_1^{\oplus y}$ for some $y \in \mathbb{N}_0$ and $\underline{X} \cong \mathbb{S}_1^-\mathbb{S}_1^+(\underline{X}) \oplus S_1^{\oplus x}$ for some $x \in \mathbb{N}_0$.

Claim: $\mathrm{im}(\beta) \subset \mathbb{S}_1^-\mathbb{S}_1^+(\underline{Y})$ and $\alpha(\mathbb{S}_1^-\mathbb{S}_1^+(\underline{X})) \subset \mathbb{S}_1^-\mathbb{S}_1^+(\underline{Y})$.

If $\mathrm{im}(\beta) \not\subseteq \mathbb{S}_1^-\mathbb{S}_1^+(\underline{Y})$ then there exists a non-zero map $\underline{P}_i \to S_1$. Since S_1 is simple, it must be an epimorphism but then S_1 must be a direct summand of $\underline{P}_i$ because $S_1 = \underline{P}_1$ is projective. This is a contradiction to $\underline{P}_i$ being indecomposable and by definition non-isomorphic to S_1.

Furthermore $\underline{\alpha}(\mathbb{S}_1^-\mathbb{S}_1^+(\underline{X})) = \mathbb{S}_1^-\mathbb{S}_1^+(\underline{\alpha})(\mathbb{S}_1^-\mathbb{S}_1^+(\underline{X})) \subset \mathbb{S}_1^-\mathbb{S}_1^+(\underline{Y})$ is clear. If the inclusion is proper, and $\underline{\alpha}$ is an epimorphism there exists a non-zero map from S_1 to $\mathbb{S}_1^-\mathbb{S}_1^+(\underline{Y})$ which has no summand isomorphic to S_1 which again is a contradiction.

In particular we may take $\underline{\gamma} = \iota \circ \mathbb{S}_1^-(\underline{\gamma}^{(1)})$, where $\iota : \mathbb{S}_1^-\mathbb{S}_1^+(\underline{X}) \to \underline{X}$ is the inclusion and it holds

$$\underline{\beta} = \underline{\alpha} \circ \underline{\gamma}.$$

Hence $\underline{P}_i$ is projective.

□

Lemma 2.4.5 *Let $(\mathcal{M}, \Omega)$ be a species of a connected weighted graph $(Q, \underline{d})$ with admissible orientation Ω. Let $\underline{V} = (V_i, {}_j\varphi_i)$ be a representation of $(\mathcal{M}, \Omega)$.*

1. *It holds*

$$\underline{V} \cong \mathcal{C}^-\mathcal{C}(\underline{V}) \oplus \underline{P},$$

where $\underline{P}$ is projective. In particular, if $\underline{V}$ is indecomposable either there exists $i \in \{1, ..., n\}$ with

$$\underline{V} \cong \underline{P}_i$$

which is equivalent to $\mathcal{C}(\underline{V}) = 0$ or

$$\underline{V} \cong \mathcal{C}^-\mathcal{C}(\underline{V}).$$

Furthermore in the latter case we have $\mathrm{End}(\mathcal{C}(\underline{V})) \cong \mathrm{End}(\underline{V})$, hence $\mathcal{C}(\underline{V})$ is indecomposable, and $\underline{\dim}(\mathcal{C}(\underline{V})) = c(\underline{\dim}(\underline{V}))$.

2. *It holds*

$$\underline{V} \cong \mathcal{C}\mathcal{C}^-(\underline{V}) \oplus \underline{Q},$$

where $\underline{Q}$ is injective. In particular, if $\underline{V}$ is indecomposable either there exists $i \in \{1, ..., n\}$ with

$$\underline{V} \cong \underline{Q}_i$$

which ist equivalent to $\mathcal{C}^-(\underline{V}) = 0$ or

$$\underline{V} \cong \mathcal{C}\mathcal{C}^-(\underline{V}).$$

Furthermore in the latter case we have $\mathrm{End}(\mathcal{C}^-(\underline{V})) \cong \mathrm{End}(\underline{V})$, hence $\mathcal{C}^-(\underline{V})$ is indecomposable, and $\underline{\dim}(\mathcal{C}^-(\underline{V})) = c(\underline{\dim}(\underline{V}))$.

Proof. The idea of the proof is to apply Lemma 2.4.2 n times for each reflection:

$$\underline{V} \cong \mathbb{S}_1^- \mathbb{S}_1^+(\underline{V}) \oplus S_1^{\oplus x_1}$$

for some $x_1 \in \mathbb{N}_0$. Applying the statement again to $\mathbb{S}_1^+(\underline{V})$ we get

$$\mathbb{S}_1^+(\underline{V}) \cong \mathbb{S}_2^- \mathbb{S}_2^+ \mathbb{S}_1^+(\underline{V}) \oplus S_2^{\oplus x_2}$$

for some $x_2 \in \mathbb{N}_0$. Hence combining these we get

$$\underline{V} \cong \mathbb{S}_1^- \mathbb{S}_2^- \mathbb{S}_2^+ \mathbb{S}_1^+(\underline{V}) \oplus \underbrace{\mathbb{S}_1^-(S_2)^{\oplus x_2}}_{=\underline{P}_2^{\oplus x_2}} \oplus \underbrace{S_1^{\oplus x_1}}_{=\underline{P}_1^{\oplus x_1}} .$$

Then we can apply the statement again to $\mathbb{S}_2^+ \mathbb{S}_1^+(\underline{V})$ and so on until we end up with

$$\underline{V} \cong \mathcal{C}^- \mathcal{C}(\underline{V}) \oplus \underline{P}_n^{\oplus x_n} \oplus ... \oplus \underline{P}_2^{\oplus x_2} \oplus \underline{P}_1^{\oplus x_1}$$

for some $x_n, ..., x_1 \in \mathbb{N}_0$. Furthermore, $\mathcal{C}(\underline{P}_i) = 0$ for all $i \in \{1, ..., n\}$ is also a direct consequence of Lemma 2.4.2 as well as the proof of the other two statements. □

Proof of Proposition 2.4.3. From Lemma 2.4.4 we get the equivalence of (i) and (ii) and from Lemma 2.4.5 the equivalence of (ii), (iii), and (iv). □

Now, using these functors and our accumulated knowledge about the positive roots for the Euclidean types, we get a bijection between isomorphism classes and roots of non-zero defect (see [12]).

Definition 2.4.6 Let $(\mathcal{M}, \Omega)$ be a k-species of a Euclidean graph $(Q, \underline{d})$. Let $\underline{V} = (V_i, {}_j\varphi_i)$ be a representation of $(\mathcal{M}, \Omega)$. We define the defect $d_c(\underline{V})$ to be the defect of the dimension type $d_c(\underline{\dim}(\underline{V}))$.

Proposition 2.4.7 *Let $(\mathcal{M}, \Omega)$ be a k-species of a Euclidean graph $(Q, \underline{d})$. Then we get a bijection between the isomorphism classes of indecomposable representations of non-zero defect and the positive roots of $(Q, \underline{d})$ of non-zero defect:*

$$\{\textit{indecomposable representations w. non-zero defect}\}_{/\cong} \longrightarrow \{\textit{positive roots w. non-zero defect}\},$$
$$\underline{V} = (V_i, {}_j\varphi_i) \mapsto \underline{\dim}(\underline{V}).$$

2.5 The (1,4)-Case

Now let us look at our main example.

Let $k = \mathbb{F}_q \subset L = k(x) \subset K = k(x,y) = \mathbb{F}_{q^4}$. Hence there exist $c_0 \in k^\times$, a_0, $a_1 \in k$ with $x^2 = c_0$ and $y^2 = a_0 + a_1 x$. Here we consider the species:

$$k \xrightarrow{{}_kK_K} K$$

The projective modules are in this case

$$P_0 = \ k \xrightarrow{(1)} K \quad \text{and} \quad Q_0 = \ 0 \longrightarrow K.$$

By applying j-times the inverse Auslander-Reiten translation to P_0 we get the following prepojective objects (see [3] or [11]):

$$\tau^{-j}(P_0) := P_j \cong \; k^{2j+1} \xrightarrow{(I_{j+1}|A_j)} K^{j+1}$$

where $I_{j+1} \in K^{(j+1)\times(j+1)}$ is the identity matrix and

$$A_j = \begin{pmatrix} x & 0 & \dots & 0 \\ y & x & \ddots & \vdots \\ 0 & y & \ddots & 0 \\ \vdots & \ddots & \ddots & x \\ 0 & \dots & 0 & y \end{pmatrix} \in K^{(j+1)\times j}.$$

Similarly we define $Q_j := \tau^{-j}(Q_0)$.

Theorem 2.5.1 *For $j \in \mathbb{N}$, the representation Q_j is isomorphic to:*

$$k^{4j} \xrightarrow{\left(\begin{array}{ccccc|cccc|cccc|ccc} 1 & 0 & \dots & & 0 & x & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \ddots & & \vdots & y & x & \ddots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & & & 0 & \ddots & \ddots & 0 & & & & & & & \\ & & & 1 & 0 & \vdots & \ddots & y & x & \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 & y & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ \hline 0 & \dots & & \dots & 0 & 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 & x & 0 & 0 \\ \vdots & & & & \vdots & \vdots & & & \vdots & 0 & 1 & \ddots & \vdots & y & \ddots & 0 \\ \vdots & & & & \vdots & \vdots & & & 0 & \vdots & \ddots & \ddots & 0 & 0 & \ddots & x \\ 0 & \dots & & \dots & 0 & 0 & \dots & 0 & xy & 0 & \ddots & 0 & 1 & 0 & 0 & y \end{array}\right)} K^{2j+1}$$

with $\mathrm{End}(Q_j) \cong K$.

As proof, I give here the reasoning on how to arrive at this form. The proof via the endomorphism ring should be manageable but is too much bookkeeping with indices to fully give it here, but one can see that it holds for the first two examples:

Example 2.5.2

- For Q_0 it is clear: $\mathrm{End}(Q_0) \cong K$.
- For Q_1: For $(A, B) \in \mathrm{End}(Q_1)$ holds

$$\begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} A = B \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & xy & 1 \end{pmatrix}$$

Therefore we have

$$\begin{pmatrix} A_{11} + xA_{31} & A_{12} + xA_{32} & A_{13} + xA_{33} & A_{14} + xA_{34} \\ A_{21} + yA_{31} & A_{22} + yA_{32} & A_{23} + yA_{33} & A_{24} + yA_{34} \\ xyA_{31} + A_{41} & xyA_{32} + A_{42} & xyA_{33} + A_{43} & xyA_{34} + A_{44} \end{pmatrix}$$

$$= \begin{pmatrix} B_{11} & B_{12} & xB_{11} + yB_{12} + xyB_{13} & B_{13} \\ B_{21} & B_{22} & xB_{21} + yB_{22} + xyB_{23} & B_{23} \\ B_{31} & B_{32} & xB_{31} + yB_{32} + xyB_{33} & B_{33} \end{pmatrix}$$

In particular, the first two and last columns give us directly the matrix B, and inserting it back into the third column we get the conditions (using $x^2 = c_0 \in k \setminus \{0\}$ and $y^2 = a_0 + a_1 x$ with $a_0, a_1 \in K$):

$$\begin{aligned} A_{13} + xA_{33} &= xB_{11} + yB_{12} + xyB_{13} \\ &= x(A_{11} + xA_{31}) + y(A_{12} + xA_{32}) + xy(A_{14} + xA_{34}) \\ &= c_0 A_{31} + xA_{11} + y(A_{12} + c_0 A_{34}) + xy(A_{32} + A_{14}) \end{aligned}$$

Therefore we have since $A_{ij} \in k$:

$$\begin{aligned} A_{13} &= c_0 A_{31} \\ A_{33} &= A_{11} \\ 0 &= A_{12} + c_0 A_{34} \\ 0 &= A_{32} + A_{14} \end{aligned}$$

Furthermore, we have

$$\begin{aligned} A_{23} + yA_{33} &= xB_{21} + yB_{22} + xyB_{23} \\ &= x(A_{21} + yA_{31}) + y(A_{22} + yA_{32}) + xy(A_{24} + yA_{34}) \\ &= a_0 A_{32} + c_0 a_1 A_{34} + x(A_{21} + a_1 A_{32} + a_0 A_{34}) + yA_{22} + xy(A_{31} + A_{24}) \end{aligned}$$

Hence we get

$$\begin{aligned} A_{23} &= a_0 A_{32} + c_0 a_1 A_{34} \\ 0 &= A_{21} + a_1 A_{32} + a_0 A_{34} \\ A_{33} &= A_{22} \\ 0 &= A_{31} + A_{24} \end{aligned}$$

And lastly

$$\begin{aligned} xyA_{33} + A_{43} &= xB_{31} + yB_{32} + xyB_{33} \\ &= x(xyA_{31} + A_{41}) + y(xyA_{32} + A_{42}) + xy(xyA_{34} + A_{44}) \\ &= c_0 a_1 A_{32} + a_0 c_0 A_{34} + x(A_{41} + a_0 A_{32} + c_0 a_1 A_{34}) + y(c_0 A_{31} + A_{42}) + xyA_{44} \end{aligned}$$

In particular, we have

$$\begin{aligned} A_{43} &= c_0 a_1 A_{32} + a_0 c_0 A_{34} \\ 0 &= A_{41} + a_0 A_{32} + c_0 a_1 A_{34} \\ 0 &= c_0 A_{31} + A_{42} \\ A_{33} &= A_{44} \end{aligned}$$

So we have the form

$$A = \begin{pmatrix} \lambda & \mu & \nu & \gamma \\ a_1\gamma + a_0 c_0^{-1}\mu & \lambda & -a_0\gamma - a_1\mu & -c_0^{-1}\nu \\ c_0^{-1}\nu & -\gamma & \lambda & -c_0^{-1}\mu \\ a_0\gamma + a_1\mu & -\nu & -c_0 a_1\gamma - a_0\mu & \lambda \end{pmatrix}$$

for some $\lambda, \mu, \nu, \gamma \in k$. In particular, the endomorphism ring has dimension 4 over k.

Note, if we calculate A^2 we have as each diagonal entry $\lambda^2 + c_0^{-1}\nu^2 + a_0\gamma^2 + a_0 c_0^{-1}\mu^2 + 2a_1\gamma\mu$.

Whereas if we take the square of $\lambda + xc_0^{-1}\nu + y\gamma + xyc_0^{-1}\mu$ and write it in the basis $(1, x, y, xy)$, we get as the prefactor of 1 the same as on the diagonal of A^2. In particular, it stands to reason that we have an isomorphism between $\mathrm{End}(Q_1)$ and K via sending matrices of the form A to $\lambda + xc_0^{-1}\nu + y\gamma + xyc_0^{-1}\mu$.

Proof of Proposition 2.5.1. This proof is by induction on j.

Therefore, we use the Coxeter functor on Q_j and show that Q_{j+1} is isomorphic to the form above.

Let ψ_j be the map $k^{4j} \overset{\psi_j}{\to} K^{2j+1} \otimes_K {}_K K_k$ of the representation Q_j. First we compute $\mathrm{coker}(\psi_j)$:

It holds:

$$\begin{aligned}\mathrm{im}(\psi_j) \quad = \quad & \langle \underbrace{e_1 \otimes 1, ..., e_{2j+1} \otimes 1}_{2j+1 \text{ generators}}, \underbrace{e_1 \otimes x + e_2 \otimes y, ..., e_{j-1} \otimes x + e_j \otimes y}_{j-1 \text{ generators}}, \\ & e_j \otimes x + e_{j+1} \otimes y + e_{2j+1} \otimes xy, \underbrace{e_{j+2} \otimes x + e_{j+3} \otimes y, ..., e_{2j} \otimes x + e_{2j+1} \otimes y}_{j-1 \text{ generators}} \rangle\end{aligned}$$

So we get $\varphi_j : K^{2j+1} \otimes_K {}_K K_k \to k^{4j+4}$ as $\mathrm{coker}(\psi_j)$, where

$$e_i \otimes 1 \mapsto 0, \qquad e_i \otimes x \mapsto e_{i+1}, \qquad e_i \otimes xy \mapsto e_{2j+3+i}$$

for $1 \leq i \leq 2j+1$ and

$$e_i \otimes y \mapsto -e_i \text{ for } i \in \{1, ..., 2j+1\} \setminus \{j+1, j+2\},$$

$$e_{j+1} \otimes y \mapsto -e_{j+1} - e_{4j+4}, \qquad e_{j+2} \otimes y \mapsto -e_{2j+3}.$$

Hence, we get for $\bar{\varphi}_j : K^{2j+1} \to k^{4j+1} \otimes_{k\ k} K_K$ with corresponding matrix:

$$\left(\begin{array}{cccc|c|c|ccc}
-y &&&&&&&& \\
x & -y &&&&&&& \\
& x & \ddots &&&&&& \\
&& \ddots & -y &&&&& \\
&&& x & -y &&&& \\
&&&& x &&&& \\
\hline
&&&&& x & -y && \\
&&&&&& x & \ddots & \\
&&&&&&& \ddots & -y \\
&&&&&&&& x \\
\hline
&&&&& -y &&& \\
\hline
xy &&&&&&&& \\
& xy &&&&&&& \\
&& \ddots &&&&&& \\
&&& xy &&&&& \\
&&&& xy &&&& \\
&&&&& xy &&& \\
&&&&&& xy && \\
&&&&&&& \ddots & \\
\hline
&&&& -y &&&& xy
\end{array}\right)
\sim
\left(\begin{array}{ccc|c|c|ccc}
-y &&&&&&& \\
x & \ddots &&&&&& \\
& \ddots & -y &&&&& \\
&& x & -y &&&& \\
&&& x &&&& \\
\hline
&&&& -y &&& \\
\hline
&&&& x & -y && \\
&&&&& x & \ddots & \\
&&&&&& \ddots & -y \\
&&&&&&& x \\
\hline
xy &&&&&&& \\
& \ddots &&&&&& \\
&& xy &&&&& \\
&&& xy &&&& \\
&&&& xy &&& \\
&&&&& xy && \\
&&&&&& \ddots & \\
\hline
&&& -y &&&& xy
\end{array}\right)$$

Hence we get for $\psi_{j+1} : k^{4j+1} \otimes_{k\ k} K_K \to K^{2j+3}$ the matrix

$$\left(\begin{array}{cccccc|ccccc|ccccc|cccc}
x &&&&&&&&&&& 1 &&&&&&&& \\
& x &&&&&&&&&& a & 1 &&&&&&& \\
&& x &&&&&&&&&& a & 1 &&&&&& \\
&&& \ddots &&&&&&&&&& \ddots & \ddots &&&&& \\
&&&& x &&&&&&&&&& a & 1 &&&& \\
&&&&& x &&&&&&&&&& a &&&& \\
\hline
&&&&&& x &&&&&&&&&& 1 &&& \\
&&&&&&& x &&&&&&&&& a & 1 && \\
&&&&&&&& \ddots &&&&&&&&& \ddots & \ddots & \\
&&&&&&&&& x &&&&&& -x &&& a & 1 \\
&&&&&&&&&& x &&&&& -1 &&&& a
\end{array}\right)$$

where we denote $a := -xy^{-1} \in K$.

By rearranging the rows and columns, multiplying everything with x^{-1} (after adding $2j+2$nd column to the $3j+5$th), one can get:

$$
\left(\begin{array}{cccccc|ccccc|ccccc|cccc}
1 & & & & & & x^{-1} & & & & & & & & & & & & & \\
& 1 & & & & & y^{-1} & x^{-1} & & & & & & & & & & & & \\
& & 1 & & & & & y^{-1} & x^{-1} & & & & & & & & & & & \\
& & & \ddots & & & & & \ddots & \ddots & & & & & & & & & & \\
& & & & 1 & & & & & y^{-1} & x^{-1} & & & & & & & & & \\
& & & & & 1 & & & & & y^{-1} & & & & & & & & & \\
\hline
& & & & & & & & & & & 1 & & & & & x^{-1} & & & \\
& & & & & & & & & & & & 1 & & & & y^{-1} & x^{-1} & & \\
& & & & & & & & & & & & & \ddots & & & & \ddots & \ddots & \\
& & & & & & & & & & & & & & 1 & & & & y^{-1} & x^{-1} \\
& & & & & & & & & & -x^{-1} & & & & & 1 & & & & y^{-1}
\end{array}\right)
$$

Now one just needs to note that since $x^{-1} = c_0^{-1}x$ and y^{-1} can be written as $d_0y + d_1xy$ for some $d_0,\ d_1 \in k$ (namely since $y^2 = a_0 - a_1x$ implies $y^{-1} = y(a_0 - a_1x)^{-1} = d_0y + d_1xy$ for $(a_0 + a_1x)^{-1} = d_0 + d_1x$) and some more work over k and K, we may get the required form.

Alternatively: One can easily see that the required form is indecomposable, hence it is the unique representation of that dimension vector (see Proposition 2.4.7) up to isomorphism. □

Furthermore, if one considers the representation T_x given by

$$k^2 \xrightarrow{(1\ x)} K,$$

then we have $\tau(T_x) \cong T_x$. As a small reminder: the radical subspace $R_Q \subset \mathbb{Q}^2$, in this case, is generated by $(2,1) =: \delta$ which is also the dimension vector of T_x. Hence it was clear, that the Coxeter functor resp. Auslander-Reiten functor does not change the dimension vector.

Now let us give a short list of some Eulerforms:

$$
\begin{array}{llll}
\langle P_0, P_0\rangle = 1 & \langle P_0, Q_0\rangle = 0 & \langle Q_0, P_0\rangle = 4 & \langle Q_0, Q_0\rangle = 4 \\
\langle P_0, T_x\rangle = 2 & \langle Q_0, T_x\rangle = 4 & \langle T_x, T_x\rangle = 0 & \langle S_1, S_1\rangle = 1
\end{array}
$$

By using the identity $[P_i] = (2i+1)[P_0] - i[Q_0]$ in the Grothendieck group, we get the Eulerforms for $i, j \in \mathbb{N}$:

$$
\begin{aligned}
\langle P_i, P_j\rangle &= (2i+1)(2j+1)\langle P_0, P_0\rangle + ij\langle Q_0, Q_0\rangle \\
&\quad -(2i+1)j\langle P_0, Q_0\rangle - i(2j+1)\langle Q_0, P_0\rangle \\
&= (2i+1)(2j+1) + 4ij - 4i(2j+1) \\
&= 4ij + 2i + 2j + 1 + 4ij - 4i - 8ij \\
&= -2i + 2j + 1
\end{aligned}
$$

And by considering the short exact sequence $0 \longrightarrow P_i \longrightarrow Q_i \longrightarrow P_{i+1} \longrightarrow 0$ we get the identity $[Q_i] = [P_{i+1}] + [P_i]$ in the Grothendieck group and therefore the following:

$$
\begin{aligned}
\langle P_i, Q_j\rangle &= \langle P_i, P_j\rangle + \langle P_i, P_{j+1}\rangle \\
&= -4i + 4j + 2 + 2 \\
&= -4i + 4j + 4
\end{aligned}
$$

and

$$\begin{aligned}\langle Q_i, Q_j\rangle &= \langle P_i, P_j\rangle + \langle P_{i+1}, P_j\rangle + \langle P_{i+1}, P_{j+1}\rangle + \langle P_i, P_{j+1}\rangle \\ &= -8i - 4 + 8j + 4 + 4 \\ &= -8i + 8j + 4\end{aligned}$$

In [11], in the case $\operatorname{char}(k) \neq 2$, one can find an explicit calculation for how the morphisms $\operatorname{Hom}(P_n, P_m)$ look like, or, using the definition via orbit algebras (see [20]), the vector space $\operatorname{Hom}(P_n, P_m)$ is given by the homogeneous part of degree $m-n$ of the k-algebra Π given by the generators X, Y and Z (with $\deg(X) = \deg(Y) = \deg(Z) = 1$) with the relations

$$\begin{aligned}XY - YX &= 0, \\ XZ - ZX &= 0, \\ ZY + YZ + a_1X^2 &= 0, \\ Z^2 + c_0Y^2 + a_0X^2 &= 0.\end{aligned}$$

Lemma 2.5.3 *It holds:*

$$\dim_k \operatorname{Hom}(P_0, P_n) = 2n + 1.$$

Proof. We have:

$$\begin{aligned}\operatorname{Hom}(P_0, P_n) &= \{\omega \in \Pi \mid \omega \text{ homogenous of degree } n\} \\ &= \langle X^n,\ X^{n-1}Y,\ X^{n-1}Z,\ X^{n-2}Y^2,\ X^{n-2}YZ,\ X^{n-3}Y^3, \\ &\qquad X^{n-3}Y^2Z,\ ...,\ XY^{n-2}Z,\ Y^n,\ Y^{n-1}Z\rangle\end{aligned}$$

By considering the exponent of X in the basis, we have in each degree two basis elements with factor X^i except for $i = n$, where we have just one. In particular, we get $\dim \operatorname{Hom}(P_0, P_n) = 2n + 1$. □

2.6 Hall Algebras of Representations of Species and Quantum Groups

Here the goal is to identify the Hall algebra of representations of a species with another algebra, namely the quantum group of a specific Cartan matrix.

This subsection is about a result by Green which generalizes a result by Ringel which is in the case of representations of quivers. Our main reference for this part is a lecture series that was held by Ringel (see [31]).

First, let us introduce what a quantum group is.

Definition 2.6.1 Let I be a finite index set. A matrix $A = (a_{ij})_{i,j\in I} \in \mathbb{Z}^{I\times I}$ is called a *generalized Cartan matrix* if

- $\forall i \in I: \quad a_{ii} = 2;$
- for $i \neq j \in I: \quad a_{ij} \leq 0;$
- $\forall i, j \in I: \quad a_{ij} = 0 \iff a_{ji} = 0.$

If there exists a diagonal matrix $D = \operatorname{diag}(d_i, i \in I)$ with $d_i \in \mathbb{N}$ for all $i \in I$ such that DA is a symmetric matrix, then A is called *symmetrizable*.

Let $q \in \mathbb{C}^\times$ not be a root of unity, $v = \sqrt{q}$. We denote $q_i = q^{d_i}$ (respectively $v_i = v^{d_i}$).

Definition 2.6.2 The *quantum group* $U_v(A)$ associated to a generalized Cartan matrix $A = (a_{ij}) \in \mathbb{Z}^{I\times I}$ is the associative algebra over $\mathbb{C}$ with 1 generated by the set $\{e_i, f_i, K_i^{\pm 1} \mid i \in I\}$ with relations:

(1) $K_i \cdot K_i^{-1} = 1 = K_i^{-1}K_i$ and $K_iK_j = K_jK_i$ for all $i, j \in I$;

(2) $K_je_iK_j^{-1} = v_j^{a_{ji}}e_i$ for all $i, j \in I$;

(3) $K_jf_iK_j^{-1} = v_j^{-a_{ji}}f_i$ for all $i, j \in I$;

(4) $e_if_j - f_je_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}$ for all $i, j \in I$;

(5) $\sum_{k=0}^{1-a_{ij}}(-1)^k \left[{1-a_{ij} \atop k}\right]_i e_i^{1-a_{ij}-k}e_je_i^k = 0$ for all $i \neq j \in I$;

(6) $\sum_{k=0}^{1-a_{ij}}(-1)^k \left[{1-a_{ij} \atop k}\right]_i f_i^{1-a_{ij}-k}f_jf_i^k = 0$ for all $i \neq j \in I$.

Definition 2.6.3 We denote by $U_v^+(A)$ (respectively $U_v^-(A)$) the subalgebra of the quantum group $U_v(A)$ generated by the set $\{e_i \mid i \in I\}$ (respectively $\{f_i \mid i \in I\}$). We denote by $U_v^0(A)$ the subalgebra generated by the set $\{K_i^{\pm 1} \mid i \in I\}$.

Analogously we denote by $U_v^{\geq 0}(A)$ (respectively $U_v^{\leq 0}(A)$) the subalgebra of the quantum group $U_v(A)$ generated by the set $\{e_i, K_i^{\pm 1} \mid i \in I\}$ (respectively $\{f_i, K_i^{\pm 1} \mid i \in I\}$).

Theorem 2.6.4 (Triangular decomposition) *There is an isomorphism of* $\mathbb{C}$*-vector spaces*

$$U_v(A) \cong U_v^-(A) \otimes U_v^0(A) \otimes U_v^+(A)$$

given by the multiplication on the right-hand side.

Furthermore, it is also known that the quantum group is also a reduced Drinfeld double in a sense (see e.g. [19]).

Using this we get the result (due to Drinfeld):

Theorem 2.6.5 *There is an isomorphism of algebras*

$$D\left(U_v^{\geq 0}(A)\right)/(K_i^+K_i^- - 1)_{i\in I} \cong U_v(A).$$

Now that we know what a quantum group associated with a Cartan datum is, let us discuss the Hall algebra of a species. Generally, we are not interested in the whole Hall algebra but in a specific subalgebra.

From now on we fix a species $(\mathcal{M}, \Omega)$ of a weighted graph $(Q, \underline{d})$ over a finite field $k = \mathbb{F}_q$. Then the simple representations are given by

$$S_i = \begin{cases} F_i & \text{at the node } i; \\ 0 & \text{else} \end{cases}$$

for each $i \in Q$.

Definition 2.6.6 The composition algebra $C(\mathrm{Rep}_k(\mathcal{M},\Omega))$ is the subalgebra of $\mathcal{H}(\mathrm{Rep}_k(\mathcal{M},\Omega))$ generated by the set $\{[S_i] \mid i \in Q\}$.

Remark 2.6.7 Consider the product

$$[S_{i_1}] \cdot ... \cdot [S_{i_n}] = q^a \sum_{[M]\in\mathcal{X}} c_M[M]$$

for some $i_1, ..., i_n \in Q$. Then there exist $a \in \frac{1}{2}\mathbb{Z}$ (given by the Euler form) and $c_M \in \mathbb{C}$ for each $[M] \in \mathcal{X}$. Then c_M corresponds to the number of composition series of the form

$$M = L_0 \supset L_1 \supset ... \supset L_n = 0$$

with quotients $L_{j-1}/L_j \cong S_{i_j}$.

Furthermore, we would like to note:

- for each $i \in Q$: $\mathrm{Ext}^1(S_i, S_i) = 0$;
- if there is no arrow $i \to j$ in (Q,Ω) then $\mathrm{Ext}^1(S_j, S_i) = 0$.

In particular, since we do not allow oriented circles, we have either $\mathrm{Ext}^1(S_j, S_i) = 0$ or $\mathrm{Ext}^1(S_i, S_j) = 0$ for each pair $i, j \in Q$.

We set for $i \neq j \in Q$ with $\mathrm{Ext}^1(S_j, S_i) = 0$:

$$\begin{aligned} d_i &= \dim_k \mathrm{End}(S_i), \\ d_j &= \dim_k \mathrm{End}(S_j), \\ e(i,j) &= \dim_k \mathrm{Ext}^1(S_i, S_j), \\ n(i,j) &= \frac{e(i,j)}{d_i} + 1, \\ n(j,i) &= \frac{e(i,j)}{d_j} + 1. \end{aligned}$$

Note: In the case $\mathrm{Ext}^1(S_j, S_i) = 0 = \mathrm{Ext}^1(S_i, S_j)$ we have $e(i,j) = 0 = e(j,i)$ and therefore $n(i,j) = 1 = n(j,i)$.

Theorem 2.6.8 (Ringel, [29]) *Let $i \neq j \in Q$ with $\mathrm{Ext}^1(S_j, S_i) = 0$. Then the following relations hold in the Hall algebra $\mathcal{H}(\mathrm{Rep}_k(\mathcal{M},\Omega))$:*

$$\sum_{t=0}^{n(i,j)} (-1)^t \begin{bmatrix} n(i,j) \\ t \end{bmatrix}_i [S_i]^{n(i,j)-t}[S_j][S_i]^t = 0$$

and

$$\sum_{t=0}^{n(j,i)} (-1)^t \begin{bmatrix} n(j,i) \\ t \end{bmatrix}_j [S_j]^t[S_i][S_j]^{n(j,i)-t} = 0.$$

Proof. We prove the second relation, the first one works similarly.

Let $d := d_j$, $n := n(j,i) = \frac{e(i,j)}{d} + 1$.

Then it holds $e(i,j) = d(n-1)$.

We will prove it in two steps.

1. Step: Calculate the prefactors c_t^M in $X := [S_j]^t[S_i][S_j]^{n-t}$ of each $[M] \in \mathcal{X}$, namely $X = q^{a_t} \sum_{[M]\in\mathcal{X}} c_t^M[M]$.

   Fix $[M] \in \mathcal{X}$ and let M be a representative of the isomorphism class. Then $c_t^M \neq 0$ if and only if there exists a composition series with n factors isomorphic to S_j and one factor isomorphic to S_i.

   Let us consider the case $c_t^M \neq 0$. Let $N \subset M$ be a direct summand of M of minimal length with the composition factor S_i, i.e. $M = N \oplus S_j^m$ for some $m \in \mathbb{N}_0$. The other direct summand follows by using $\operatorname{Ext}^1(S_j, S_j) = 0$.

   Since we also have $\operatorname{Ext}^1(S_j, S_i) = 0$ we get:

   $$\operatorname{rad}(N) \cong S_j^{n-m} \qquad \text{and} \qquad N/\operatorname{rad}(N) \cong S_i.$$

   In particular: $M/\operatorname{rad}(N) \cong S_i \oplus S_j^m$.

   Now let us determine c_t^M, the number of compositions series

   $$M = M_0 \supset M_1 \supset ... \supset M_{n+1} = 0 \tag{5}$$

   with first t factors S_j, then S_i and last $n-t$ factors again S_j.

   If we have $t > m$ then $c_t^M = 0$ since N would not be of minimal length otherwise.

   Hence consider the case $t \leq m$.

   For each composition series (5) it holds: $N \subset M_t$.

   - $M_t/N \subset M/N$ is a submodule of length $n - t - (n - m) = m - t$;
   - $M_{t+1} \subset M_t$ is uniquely determined by $M_t/M_{t+1} \cong S_i$;
   - number of submodules of the form M_t:

     $$\#\{(m-t)\text{-dimensional subspaces of } F_j^m\} = \begin{bmatrix} m \\ m-t \end{bmatrix}_{j+};$$

   - number of composition series of M/M_t:

     $$[t]_{j+}!;$$

   - number of composition series of M_{t+1}:

     $$[n-t]_{j+}!.$$

   Hence in total, we get

   $$c_t^M = \begin{bmatrix} m \\ m-t \end{bmatrix}_{j+} \cdot [t]_{j+}! \cdot [n-t]_{j+}! = \frac{[m]_{j+}![n-t]_{j+}!}{[m-t]_{j+}!}.$$

2. Step: Show $\sum_{t=0}^{n(j,i)} (-1)^t q^{a_t} \begin{bmatrix} n(j,i) \\ t \end{bmatrix}_j c_t^M = 0$ for each $[M] \in \mathcal{X}$.

First, let us determine a_t as well. Using the fact that the Eulerform is determined already on the Grothendieck group we have:

$$\begin{aligned}
2a_t &= \sum_{r=1}^{t-1}\langle S_j^r, S_j\rangle + \langle S_j^t, S_i\rangle + \sum_{r=0}^{n-t-1}\langle S_j^{t+r}\oplus S_i, S_j\rangle \\
&= \sum_{r=1}^{t-1} dr + 0 + \sum_{r=0}^{n-t-1}(d(t+r) - e(i,j)) \\
&= \sum_{r=1}^{t-1} dr + \sum_{r=0}^{n-t-1}(d(t+r) - d(n-1)) \\
&= d\left(\frac{t(t-1)}{2} + \sum_{r=0}^{n-t-1}(r-(n-t-1))\right) \\
&= d\left(\frac{t(t-1)}{2} - \frac{(n-t-1)(n-t)}{2}\right)
\end{aligned}$$

In total we get for each M:

$$\begin{aligned}
\sum_{t=0}^{n(j,i)}(-1)^t q^{a_t}\begin{bmatrix} n(j,i) \\ t\end{bmatrix}_j c_t^M &= \sum_{t=0}^{m}(-1)^t q_j^{\frac{t(t-1)}{4}-\frac{(n-t-1)(n-t)}{4}}\begin{bmatrix} n \\ t\end{bmatrix}_j \frac{[m]_{j+}![n-t]_{j+}!}{[m-t]_{j+}!} \\
&= \sum_{t=0}^{m}(-1)^t q_j^{\frac{t(t-1)}{4}-\frac{(n-t-1)(n-t)}{4}}\frac{[n]_j![m]_{j+}![n-t]_{j+}!}{[t]_j![n-t]_j![m-t]_{j+}!} \\
&= \sum_{t=0}^{m}(-1)^t t[n]_j!\frac{[m]_{j+}![n-t]_{j+}!}{[t]_{j+}![n-t]_j![m-t]_{j+}!} \\
&= [n]_j!\sum_{t=0}^{m}(-1)^t\begin{bmatrix} m \\ t\end{bmatrix}_j \\
&= 0
\end{aligned}$$

□

Hence, if we consider the matrix $A=(a_{ij})$ with $a_{ij}=1-n(i,j)$ for $i\neq j\in Q$ and $a_{ii}=2$, then A is a symmetrizable Cartan matrix with DA symmetric for $D=\operatorname{diag}(d_i, i\in Q)$.

Corollary 2.6.9 *There exists a surjective homomorphism of $\mathbb{C}$-algebras*

$$\begin{aligned}
\pi: U_v^+(A) &\to C(\mathrm{Rep}_k(\mathcal{M},\Omega)), \\
e_i &\mapsto [S_i].
\end{aligned}$$

Theorem 2.6.10 (Green, [16]) *The $\mathbb{C}$-algebra homomorphism $\pi: U_v^+(A)\to C(\mathrm{Rep}_k(\mathcal{M},\Omega))$ is an isomorphism.*

Now, if we apply all this to the (1,4)-case we have studied before we have:

$$\begin{aligned}
&\mathrm{Ext}^1(S_2,S_1)=0, && e(1,2)=\dim_k \mathrm{Ext}^1(S_1,S_2)=4,\\
&d_1=\dim_k \mathrm{End}(S_1)=1, && d_2=\dim_k \mathrm{End}(S_2)=4,\\
&n(1,2)=\frac{4}{1}+1=5, && n(2,1)=\frac{4}{4}+1=2,\\
&a_{12}=1-5=-4, && a_{21}=1-2=-1.
\end{aligned}$$

Hence the corresponding Cartan matrix is

$$A=\begin{pmatrix} 2 & -4\\ -1 & 2\end{pmatrix}.$$

This is the affine Cartan matrix $A_2^{(2)}$.

So there is an isomorphism

$$\begin{aligned}
U_v^+\left(A_2^{(2)}\right) &\longrightarrow C\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right),\\
e_i &\longmapsto [S_i].
\end{aligned}$$

And if we add the Cartan part/Grothendieck group which is given by $P^\vee \cong \mathbb{Z}^2$ and do the reduced Drinfeld double construction, we also get isomorphisms, but note the relations between the positive and negative half:

On one side we have

$$e_if_i-f_ie_i=\frac{K_i-K_i^{-1}}{v_i-v_i^{-1}},$$

but on the Hall algebra side, we get

$$[S_i]^+[S_i]^- - [S_i]^-[S_i]^+ = (K_i^{-1}-K_i)(S_i,S_i) = -\frac{K_i-K_i^{-1}}{q_i-1} = -v_i^{-1}\frac{K_i-K_i^{-1}}{v_i-v_i^{-1}}.$$

Hence we readjust a little the prefactor of $[S_i]^-$:

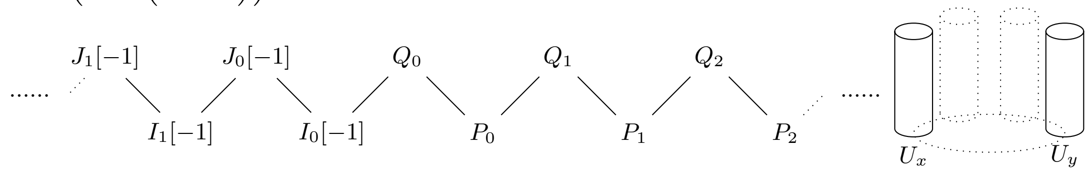
$$\begin{aligned}
U_v\left(A_2^{(2)}\right) &\longrightarrow DC\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right),\\
e_i &\longmapsto [S_i]^+,\\
f_i &\longmapsto -v_i[S_i]^-,\\
K_i &\longmapsto K_i.
\end{aligned}$$

3 The Category $\mathrm{Coh}(\mathbb{X})$ and its Hall Algebra

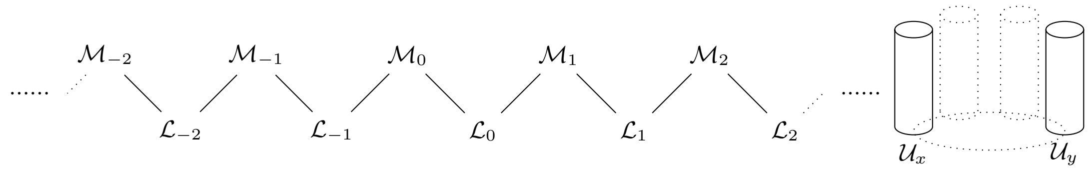
3.1 Introduction of $\mathrm{Coh}(\mathbb{X})$

The other category we want to consider is a category that can be characterized by similar properties to a category of coherent sheaves. The one which is of interest to us can be viewed intuitively as the category of representations of the $\mathbb{F}_q$-species $\bullet \overset{(1,4)}{\longrightarrow} \bullet$ from the Section 2.5, but where we take the preinjective component and glue it to the left of the preprojective one in the Auslander-Reiten quiver. If we consider the bounded derived categories we get the following picture:

$\mathcal{D}^b\left(\mathrm{Rep}_k\left(\bullet \overset{(1,4)}{\longrightarrow} \bullet\right)\right)$:



$\mathcal{D}^b(\mathrm{Coh}(\mathbb{X}))$:



For this introduction, we mainly use [21] as a reference, see also [20].

To be more precise, consider the algebra

$$\Gamma = \begin{pmatrix} k & 0 \\ {}_K K_k & K \end{pmatrix}.$$

We denote by $\mathrm{mod}(\Gamma)$ the finitely generated right Γ-modules. Then by subsection 2.2, we know

$$\mathrm{mod}(\Gamma) \simeq \mathrm{Rep}_k\left(\bullet \overset{(1,4)}{\longrightarrow} \bullet\right).$$

Furthermore the category $\mathrm{mod}(\Gamma)$ may be written as

$$\mathrm{mod}(\Gamma) = \mathrm{mod}_+(\Gamma) \vee \mathrm{mod}_0(\Gamma) \vee \mathrm{mod}_-(\Gamma),$$

where $\mathrm{mod}_+(\Gamma)$ is the full subcategory of the preprojective objects and $\mathrm{mod}_-(\Gamma)$ is the full subcategory of preinjective objects.

Definition 3.1.1 We define the hereditary abelian k-categroy $\mathrm{Coh}(\mathbb{X}) := \mathcal{H}$ as the full subcategory of $\mathcal{D}^b(\mathrm{mod}(\Gamma))$:

$$\mathrm{Coh}(\mathbb{X}) := \mathcal{H} := \underbrace{\mathrm{mod}_-(\Gamma)[-1] \vee \mathrm{mod}_+(\Gamma)}_{=:\mathcal{H}_+} \vee \mathrm{mod}_0(\Gamma).$$

Remark 3.1.2 The category $\mathcal{H}$ is a *homogeneous exceptional curve.* They are characterized by the following properties:

- $\mathcal{H}$ is a connected small abelian k-category with finite-dimensional morphism and extension spaces.
- $\mathcal{H}$ is hereditary, Noetherian, and contains no non-zero projective object.
- $\mathcal{H}$ admits a tilting object.
- For each simple object $S \in \mathrm{Obj}(\mathcal{H})$ we have $\mathrm{Ext}^1(S, S) \neq 0$.

The last condition is the definition that all tubes in $\mathcal{H}_0$ are homogeneous.

Remark 3.1.3 An object is called *tilting* in $\mathcal{H}$ if

1. T has no self-extensions, i.e. $\mathrm{Ext}^1(T, T) = 0$;
2. T generates $\mathcal{H}$, i.e. $\mathrm{Hom}(T, X) = 0 = \mathrm{Ext}^1(T, X)$ implies that X is the zero object.

The objects in $\mathcal{H}_+$ are called vector bundles and any line bundle (i.e. object of rank 1, e.g. P_0) may play the role of the structure sheaf L. Consider another bundle $\bar{L}$ such that there exists an irreducible morphism $L \to \bar{L}$, in our example $L = P_0$ take $\bar{L} = Q_1$. Then ${}_K K_k = \mathrm{Hom}(L, \bar{L})$. Furthermore $T = L \oplus \bar{L}$ is a tilting bundle such that $\Gamma = \mathrm{End}(T)$.

From now on we denote by $\mathcal{L}_0$ the structure sheaf, $\tau^i(\mathcal{L}_0) =: \mathcal{L}_i$, by $\mathcal{M}_1$ the corresponding $\bar{L}$ and so on.

There are other approaches to defining the category. A more concrete construction via orbit algebras is given as follows (see [20], [21]):

Definition 3.1.4 Let L be a fixed line bundle and σ a positive automorphism (i.e. $\deg(\sigma L) > \deg(L) = 0$). Then the *orbit algebra* R is given by the vector space

$$R = \Pi(L, \sigma) := \bigoplus_{n \geq 0} \mathrm{Hom}_\Gamma(L, \sigma^n L)$$

and the multiplication for $f \in \mathrm{Hom}(L, \sigma^n L)$ and $g \in \mathrm{Hom}(L, \sigma^m L)$

$$f \star g := \sigma^m(f) \otimes g.$$

Then we consider the quotient category

$$\mathcal{H} := \mathrm{mod}(R)/\,\mathrm{mod}_0(R),$$

namely the Serre quotient category of finitely generated R-modules by finite length R-modules.

Remark 3.1.5 In particular, using the terminology of [2], the category $\mathcal{H}$ is a noetherian projective scheme. Hence in our case, we from now on denote it by Coh($\mathbb{X}$).

In correspondence to the (1,4)-species we considered before, we have:

Theorem 3.1.6 (Kussin, [20]) *Let k be a field. Consider the tower of (commutative) fields*

$$k \subset k(x) \subset k(x, y) = K$$

such that $x^2 = c_0$ and $y^2 = a_0 + a_1 x$ for some $c_0,\ a_0,\ a_1 \in k$. Let M be the tame bimodule $M = {}_k K_K$.

1. *The orbit algebra $R = \Pi(L, \sigma_x)$ is the k-algebra on the three generators X, Y and Z with relations*

$$\begin{array}{rcl} XY - YX &=& 0, \\ XZ - ZX &=& 0, \\ ZY + YZ + a_1X^2 &=& 0, \\ Z^2 + c_0Y^2 - a_0X^2 &=& 0. \end{array}$$

   If $\mathrm{char}(k) = 2$ *and* $a_1 = 0$ *then R is commutative, otherwise its center is given by* $k[X, Y^2]$.

2. *The function field $k(\mathbb{X})$ is isomorphic to the quotient division ring of*

$$k\langle U, V\rangle/(VU + UV + a_1, V^2 + c_0U^2 - a_0).$$

   If $\mathrm{char}(k) = 2$ *and* $a_1 = 0$ *then $k(\mathbb{X})$ is commutative, otherwise its center is* $k(U^2)$.

Remark 3.1.7 Note, that this is the $(1, 4)$-case we have studied since by construction we have the same orbit algebra already mentioned at the end of Section 2.5. However, we use for calculations for the most part just our knowledge about the representations of the species and transfer it via the concrete construction in Definition 3.1.1 instead of this one.

Now to a direct consequence, namely the Grothendieck groups $\mathrm{K}_0(\mathrm{Coh}(\mathbb{X}))$ and $\mathrm{K}_0\left(\mathrm{Rep}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$ are isomorphic. Both are isomorphic to $\mathbb{Z}^2$, and one can for example consider the automorphism:

$$\begin{array}{ccc} \mathrm{K}_0(\mathrm{Coh}(\mathbb{X})) & \xrightarrow{\begin{pmatrix}\mathrm{rk}\\ \deg\end{pmatrix}} & \mathbb{Z}^2 \\ \Big\downarrow = & \begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix} \Big\Uparrow \Big\Downarrow \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix} & \\ \mathrm{K}_0\left(\mathrm{Rep}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right) & \xrightarrow[\underline{\dim}]{} & \mathbb{Z}^2 \end{array}$$

In particular, we have the basis elements:

$$\begin{array}{ccc} \overline{\mathcal{L}_0} & \longmapsto & \begin{pmatrix}1\\0\end{pmatrix} \\ \big\downarrow & & \big\downarrow \\ \overline{P_0} & \longmapsto & \begin{pmatrix}1\\1\end{pmatrix} \end{array} \qquad \begin{array}{ccc} \overline{\mathcal{T}_x} & \longmapsto & \begin{pmatrix}0\\1\end{pmatrix} \\ \big\downarrow & & \big\downarrow \\ \overline{T_x} & \longmapsto & \begin{pmatrix}2\\1\end{pmatrix} \end{array}$$

Then the (additive) Euler form is given by: For $\mathcal{F},\ \mathcal{G} \in \mathrm{Ob}(\mathrm{Coh}(\mathbb{X}))$:

$$\langle \mathcal{F}, \mathcal{G}\rangle = \mathrm{rk}(\mathcal{F})\,\mathrm{rk}(\mathcal{G}) + 2\,\mathrm{rk}(\mathcal{F})\deg(\mathcal{G}) - 2\deg(\mathcal{F})\,\mathrm{rk}(\mathcal{G})$$

or written as bilinear form

$$(\mathrm{rk}\,\mathcal{F}, \deg\mathcal{F})\begin{pmatrix}1 & 2\\ -2 & 0\end{pmatrix}\begin{pmatrix}\mathrm{rk}\,\mathcal{G}\\ \deg\mathcal{G}\end{pmatrix}.$$

Remark 3.1.8 For any $\alpha = (a,b) \in K_0(\mathrm{Coh}(\mathbb{X}))$ it holds:

$$\begin{aligned} ((0,1),(a,b)) &= \langle (0,1),(a,b)\rangle + \langle (a,b),(0,1)\rangle \\ &= 0\cdot a + 2\cdot 0\cdot b - 2\cdot 1\cdot a + a\cdot 0 + 2\cdot a\cdot 1 - 2\cdot b\cdot 0 \\ &= 0 \end{aligned}$$

In particular, in $\tilde{\mathcal{H}}(\mathrm{Coh}(\mathbb{X}))$ the element $C := K_{(0,1)}$ is central.

Also note, by the above identification, $(0,1) \in K_0(\mathrm{Coh}(\mathbb{X}))$ corresponds to the element $\delta = (2,1) \in \mathrm{K}_0\left(\mathrm{Rep}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$, the generator of the radical subspace of $\mathbb{Q}^Q = \mathbb{Q}^2$.

As mentioned above, the category $\mathrm{Coh}(\mathbb{X})$ decomposes into vector bundles $\mathcal{H}_+ =: \mathrm{Vec}(\mathbb{X})$ and torsion bundles $\mathrm{Tor}(\mathbb{X})$. As one might expect of torsion bundles, they decompose further into subcategories that have no homomorphisms or extensions between them:

$$\mathrm{Tor}(\mathbb{X}) = \bigvee_{x\in\mathbb{P}^1_k} \mathrm{Tor}_x(\mathbb{X}) = \bigvee_{d\in\mathbb{N}} \bigvee_{x\in\mathbb{P}^1_k[d]} \mathrm{Tor}_x(\mathbb{X}),$$

where $\mathbb{P}^1_k[d]$ are the points of degree d.

To each point x, there is a unique simple object S_x in $\mathrm{Tor}_x(\mathbb{X}) =: \mathcal{U}_x$. The difference to coherent sheaves of $\mathbb{P}^1_k$ is given by two points (see [22, Example 10.8]):

- a point we denote by π with $\deg(\pi) = 1$ and $\mathrm{End}(S_\pi) \cong \mathbb{F}_{q^2}$,
- a point we denote by ω with $\deg(\omega) = 2$ and $\mathrm{End}(S_\omega) \cong \mathbb{F}_{q^4}$.

For all other points $x \in \mathbb{P}^1_k \setminus \{\pi,\omega\}$ where $d = \deg(x)$ it holds $\mathrm{End}(S_x) \cong \mathbb{F}_{q^d}$.

Furthermore, in this setup, the Hall algebras $\mathcal{H}(\mathcal{U}_x)$ of each tube is isomorphic as a $\mathbb{C}$-algebra to the ring of symmetric functions Λ over the field $\mathrm{End}(S_x) \cong \mathbb{F}_{q^d}$ studied in the Subsection 1.5 as a Hall algebra of a discrete valuation ring.

With no homomorphisms and extensions between the tubes $\mathcal{U}_x$ it also follows that the Hall algebra of all torsion bundles $\mathcal{H}(\mathrm{Tor}(\mathbb{X}))$ is commutative.

Now let us start with some more concrete calculations.

3.2 Some Calculations in $D\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$

For the computations always keep in mind the corresponding elements in the bounded derived category $\mathcal{D}^b\left(\mathrm{Rep}_k\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$, e.g. $P_n \mapsto \mathcal{L}_n$, $Q_n \mapsto \mathcal{M}_n$ for $n \in \mathbb{N}_0$.

Example 3.2.1 It holds:

$$\begin{aligned} [\mathcal{L}_0][\mathcal{L}_{-1}] &= q^{\frac{1+2(-1)}{2}}\left(q^{2+1}[\mathcal{L}_0 \oplus \mathcal{L}_{-1}] + \frac{q^4-1}{q-1}[\mathcal{M}_0]\right) \\ &= q^{\frac{5}{2}}[\mathcal{L}_0 \oplus \mathcal{L}_{-1}] + q^{-\frac{1}{2}}\frac{q^4-1}{q-1}[\mathcal{M}_0] \end{aligned}$$

and

$$\begin{aligned} [\mathcal{L}_{-1}][\mathcal{L}_0] &= q^{\frac{1+2\cdot 1}{2}}[\mathcal{L}_0 \oplus \mathcal{L}_{-1}] \\ &= q^{\frac{3}{2}}[\mathcal{L}_0 \oplus \mathcal{L}_{-1}]. \end{aligned}$$

Hence we can write $[\mathcal{M}_0]$ using the q-commutator of rank 1 line bundles:

$$\begin{aligned}[\mathcal{M}_0] &= q^{\frac{1}{2}}\frac{q-1}{q^4-1}([\mathcal{L}_0][\mathcal{L}_{-1}] - q^{\frac{5}{2}}[\mathcal{L}_0 \oplus \mathcal{L}_{-1}]) \\ &= q^{\frac{1}{2}}\frac{q-1}{q^4-1}([\mathcal{L}_0][\mathcal{L}_{-1}] - q^{\frac{5}{2}}q^{-\frac{3}{2}}[\mathcal{L}_{-1}][\mathcal{L}_0]) \\ &= q^{\frac{1}{2}}\frac{q-1}{q^4-1}\left[[\mathcal{L}_0],[\mathcal{L}_{-1}]\right]_q .\end{aligned}$$

Furthermore, we have

$$[\mathcal{M}_0][\mathcal{L}_{-1}] = q^0 \cdot q^4[\mathcal{M}_0 \oplus \mathcal{L}_{-1}] \qquad \text{and} \qquad [\mathcal{L}_{-1}][\mathcal{M}_0] = q^2[\mathcal{M}_0 \oplus \mathcal{L}_{-1}].$$

In particular, we get the commutator relation:

$$\Big[\left[[\mathcal{L}_0],[\mathcal{L}_{-1}]\right]_q, [\mathcal{L}_{-1}]\Big]_{q^2} = 0.$$

Now let us introduce some more general elements. The definition and properties are analogs to elements which can be found in [6] and [32] for Hall algebras of other categories.

Definition 3.2.2 We define the elements $\mathbb{1}_{(r,d)}$ for $(r,d) \in \mathbb{Z}^2$ as:

$$\mathbb{1}_{(r,d)} := \sum_{\substack{[\mathcal{F}]:\\ \overline{\mathcal{F}}=(r,d)}} [\mathcal{F}].$$

Then we can make several calculations straight away. It holds:

$$\mathbb{1}_{(1,s)} = \sum_{l\geq 0} q^{-l}[\mathcal{L}_{s-l}]\mathbb{1}_{(0,l)}$$

In particular:

$$[\mathcal{L}_s] = \mathbb{1}_{(1,s)} - \sum_{l\geq 1} q^{-l}[\mathcal{L}_{s-l}]\mathbb{1}_{(0,l)}$$

$$\begin{aligned}\Rightarrow \qquad [\mathcal{L}_0] &= \mathbb{1}_{(1,0)} - \sum_{l\geq 1} q^{-l}[\mathcal{L}_{-l}]\mathbb{1}_{(0,l)} \\ &= \mathbb{1}_{(1,0)} - \sum_{l_1\geq 1} q^{-l_1}\left(\mathbb{1}_{(1,-l_1)} - \sum_{l_2\geq 1} q^{-l_2}[\mathcal{L}_{-l_1-l_2}]\mathbb{1}_{(0,l_2)}\right)\mathbb{1}_{(0,l_1)} \\ &= \mathbb{1}_{(1,0)} - \left(\sum_{l_1\geq 1} q^{-l_1}\mathbb{1}_{(1,-l_1)}\right) + \sum_{l_1,\ l_2\geq 1} q^{-l_1-l_2}[\mathcal{L}_{-l_1-l_2}]\mathbb{1}_{(0,l_2)}\mathbb{1}_{(0,l_1)} \\ &= \mathbb{1}_{(1,0)} - \left(\sum_{l_1\geq 1} q^{-l_1}\mathbb{1}_{(1,-l_1)}\right) \\ &\quad + \sum_{l_1,\ l_2\geq 1} q^{-l_1-l_2}\left(\mathbb{1}_{(1,-l_1-l_2)} - \sum_{l_3\geq 1} q^{-l_3}[\mathcal{L}_{-l_1-l_2-l_3}]\mathbb{1}_{(0,l_3)}\right)\mathbb{1}_{(0,l_2)}\mathbb{1}_{(0,l_1)} \\ &= \ldots \\ &= \sum_{n\geq 0} q^{-n}\mathbb{1}_{(1,-n)}\sum_{r\geq 0}(-1)^r \sum_{l_1+\ldots+l_r=n} \mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}\end{aligned}$$

This leads us to the following definition:

Definition 3.2.3 We define the elements:

$$\chi_n = \sum_{r\geq 1}(-1)^r \sum_{l_1+\ldots+l_r=n} \mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)},$$

and the power sum series:

$$\mathbb{1}(s) = 1 + \sum_{l\geq 1}\mathbb{1}_{(0,l)}s^l$$

and

$$\chi(s) = 1 + \sum_{l\geq 1}\chi_l s^l.$$

Using these, we can rewrite the above formula as follows:

$$[\mathcal{L}_0] = \sum_{n\geq 0} q^{-n}\mathbb{1}_{(1,-n)}\chi_n \tag{6}$$

and it holds:

$$\begin{aligned}
\mathbb{1}(s)\chi(s) &= \left(1+\sum_{l\geq 1}\mathbb{1}_{(0,l)}s^l\right)\left(1+\sum_{n\geq 1}\chi_n s^n\right)\\
&= \left(1+\sum_{l\geq 1}\mathbb{1}_{(0,l)}s^l\right)\left(1+\sum_{n\geq 1}\sum_{r\geq 1}(-1)^r\sum_{l_1+\ldots+l_r=n}\mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}s^n\right)\\
&= 1+\sum_{l\geq 1}\mathbb{1}_{(0,l)}s^l+\sum_{n\geq 1}\sum_{r\geq 1}(-1)^r\sum_{l_1+\ldots+l_r=n}\mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}s^n\\
&\quad+\sum_{l,n\geq 1}\sum_{r\geq 1}(-1)^r\sum_{l_1+\ldots+l_r=n}\mathbb{1}_{(0,l)}\mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}s^{l+n}\\
&= 1+\underbrace{\sum_{l\geq 1}\mathbb{1}_{(0,l)}s^l-\sum_{n\geq 1}\mathbb{1}_{(0,n)}s^n}_{=0}+\sum_{n\geq 1}\sum_{r\geq 2}(-1)^r\sum_{l_1+\ldots+l_r=n}\mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}s^n\\
&\quad+\sum_{m\geq 2}\sum_{r\geq 2}(-1)^{r-1}\sum_{l_1+\ldots+l_r=m}\mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}s^m\\
&= 1+\sum_{n\geq 2}\sum_{r\geq 2}(-1)^r\sum_{l_1+\ldots+l_r=n}\mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}s^n-\sum_{m\geq 2}\sum_{r\geq 2}(-1)^r\sum_{l_1+\ldots+l_r=m}\mathbb{1}_{(0,l_1)}\ldots\mathbb{1}_{(0,l_r)}s^m\\
&= 1
\end{aligned}$$

We are also interested in the coproduct but refer to the explicit proof somewhere else which works analogously to this case.

Lemma 3.2.4 *It holds:*

$$\tilde{\Delta}(\mathbb{1}_{(r,d)}) = \sum_{\alpha+\beta=(r,d)} q^{-\frac{1}{2}\langle\alpha,\beta\rangle}\mathbb{1}_\alpha K_\beta\otimes\mathbb{1}_\beta$$

Proof. [32] Schiffmann, Lectures on Hall algebras, Lemma 1.7 □

Now let us go back to consider the coproduct on the (not extended) Hall algebra $\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$.

Corollary 3.2.5 *It holds:*

$$\Delta(\mathbb{1}(s)) = \mathbb{1}(s) \otimes \mathbb{1}(s)$$

Proof.

$$\begin{aligned}
\Delta(\mathbb{1}(s)) &= \Delta\left(1 + \sum_{l\geq 1} \mathbb{1}_{(0,l)} s^l\right) \\
&= \Delta(1) + \sum_{l\geq 1} s^l \Delta(\mathbb{1}_{(0,l)}) \\
&= 1 \otimes 1 + \sum_{l\geq 1} s^l \Delta(\mathbb{1}_{(0,l)}) \\
&= 1 \otimes 1 + \sum_{l\geq 1} \sum_{a+b=l} q^{-\langle (0,a),(0,b)\rangle} (s^a \mathbb{1}_{(0,a)}) \otimes (s^b \mathbb{1}_{(0,b)}) \\
&= \left(\sum_{a\geq 0} \mathbb{1}_{(0,a)} s^a\right) \otimes \left(\sum_{b\geq 0} \mathbb{1}_{(0,b)} s^b\right) \\
&= \mathbb{1}(s) \otimes \mathbb{1}(s)
\end{aligned}$$

□

Since $\mathbb{1}(s)\chi(s) = 1$, it follows $\Delta(\chi_n) = \sum_{r=0}^{n} \chi_r \otimes \chi_{n-r}$. In particular, in the setting of the extended Hall algebra $\tilde{\mathcal{H}}(\mathrm{Coh}(\mathbb{X}))$ it holds for $n \in \mathbb{N}$:

$$\tilde{\Delta}(\chi_n) = \sum_{i=0}^{n} \chi_i K_{(0,n-i)} \otimes \chi_{n-i}.$$

We can use these coproducts now to calculate the comultiplication of line bundles:

Proposition 3.2.6 *It holds:*

$$\tilde{\Delta}([\mathcal{L}_0]) = [\mathcal{L}_0] \otimes 1 + \sum_{l\geq 0} \theta_l K_{(1,-l)} \otimes [\mathcal{L}_{-l}],$$

where $\theta_l = \sum_{i=0}^{l} q^{l-2i} \mathbb{1}_{(0,l-i)} \chi_i$.

Proof. Since $\mathcal{L}_0$ is a line bundle, the only subobjects are other line bundles $\mathcal{L}_{-l}$ for $l \in \mathbb{N}$ or the zero object.

In particular, it follows for the comultiplication:

$$\tilde{\Delta}([\mathcal{L}_0]) = \tilde{\Delta}_{(1,0),(0,0)}([\mathcal{L}_0]) + \sum_{l\geq 0} \tilde{\Delta}_{(0,l),(1,-l)}([\mathcal{L}_0])$$

By equation (6) and using that Δ is an algebra homomorphism we get:

$$\begin{aligned}
\tilde{\Delta}([\mathcal{L}_0]) &= \tilde{\Delta}\left(\sum_{n\geq 0} q^{-n}\mathbb{1}_{(1,-n)}\chi_n\right)\\
&= \sum_{n\geq 0} q^{-n}\tilde{\Delta}(\mathbb{1}_{(1,-n)})\tilde{\Delta}(\chi_n)\\
&= \sum_{n\geq 0} q^{-n}\left(\sum_{\substack{\alpha,\ \beta:\\ \alpha+\beta=(1,-n)}} q^{-\frac{1}{2}\langle\alpha,\beta\rangle}\mathbb{1}_\alpha K_\beta \otimes \mathbb{1}_\beta\right)\left(\sum_{i=0}^{n}\chi_i K_{(0,n-i)}\otimes\chi_{n-i}\right)\\
&= \sum_{n\geq 0}\sum_{\substack{\alpha,\ \beta:\\ \alpha+\beta=(1,-n)}}\sum_{i=0}^{n} q^{-n-\frac{1}{2}\langle\alpha,\beta\rangle}q^{\frac{1}{2}(\beta,(0,i))}\mathbb{1}_\alpha\chi_i K_{\beta+(0,n-i)}\otimes\mathbb{1}_\beta\chi_{n-i}
\end{aligned}$$

In particular for $\tilde{\Delta}_{(0,l),(1,-l)}([\mathcal{L}_0])$ the indices α and β are for fixed n and i: $\beta = (1,-l-n+i)$ and $\alpha = (0, l-i)$, therefore we get $\langle\alpha,\beta\rangle = -2(l-i)$ and $(\beta,(0,i)) = 0$.

$$\begin{aligned}
\tilde{\Delta}_{(0,l),(1,-l)}([\mathcal{L}_0]) &= \sum_{n\geq 0}\sum_{i=0}^{n} q^{-n+l-i}\mathbb{1}_{(0,l-i)}\chi_i K_{(1,-l)}\otimes\mathbb{1}_{(1,-l-n+i)}\chi_{n-i}\\
&= \sum_{i=0}^{l}\sum_{n\geq i} q^{-n+l-i}\mathbb{1}_{(0,l-i)}\chi_i K_{(1,-l)}\otimes\mathbb{1}_{(1,-l-n+i)}\chi_{n-i}\\
&= \sum_{i=0}^{l} q^{l-i}\mathbb{1}_{(0,l-i)}\chi_i K_{(1,-l)}\otimes\sum_{n\geq i} q^{-n}\mathbb{1}_{(1,-l-n+i)}\chi_{n-i}\\
&\overset{k=n-i}{=} \sum_{i=0}^{l} q^{l-i}\mathbb{1}_{(0,l-i)}\chi_i K_{(1,-l)}\otimes\sum_{k\geq 0} q^{-k-i}\mathbb{1}_{(1,-l-k)}\chi_k\\
&\overset{(6)}{=} \underbrace{\sum_{i=0}^{l} q^{l-2i}\mathbb{1}_{(0,l-i)}\chi_i}_{=:\theta_l} K_{(1,-l)}\otimes[\mathcal{L}_{-l}]
\end{aligned}$$

In total we get the formula:

$$\begin{aligned}
\tilde{\Delta}([\mathcal{L}_0]) &= \tilde{\Delta}_{(1,0),(0,0)}([\mathcal{L}_0]) + \sum_{l\geq 0}\tilde{\Delta}_{(0,l),(1,-l)}([\mathcal{L}_0])\\
&= [\mathcal{L}_0]\otimes 1 + \sum_{l\geq 0}\theta_l K_{(1,-l)}\otimes[\mathcal{L}_{-l}]
\end{aligned}$$

□

Lemma 3.2.7 *Let $d \in \mathbb{N}$ and $n \in \mathbb{Z}$. It holds:*

$$\mathbb{1}_{d\delta}[\mathcal{L}_n] = \sum_{s=0}^{d} q^{-s}[2s+1]_+[\mathcal{L}_{n+s}]\mathbb{1}_{(d-s)\delta}$$

Proof. There are $\frac{q^{2s+1}-1}{q-1}$ injective maps $\mathcal{L}_n \hookrightarrow \mathcal{L}_{n+s}$ up to a scalar by Lemma 2.5.3. Alltogether we have $\frac{q^{2s+1}-1}{q-1}q^{2(d-s)}$ injective maps $\mathcal{L}_n \hookrightarrow \mathcal{L}_{n+s} \oplus \mathcal{T}_{(0,d-s)}$ for a torsion bundle $\mathcal{T}_{(0,d-s)}$ of degree $d-s$. Hence we get:

$$\begin{aligned}\mathbb{1}_{d\delta}[\mathcal{L}_n] &= \sum_{s=0}^{d} q^{\frac{1}{2}\langle(0,d),(1,n)\rangle}\frac{q^{2s+1}-1}{q-1}q^{2(d-s)}\sum_{\bar{\mathcal{T}}=(0,d-s)}[\mathcal{L}_{n+s}\oplus\mathcal{T}]\\ &= \sum_{s=0}^{d}\underbrace{q^{\frac{1}{2}(\langle(0,d),(1,n)\rangle-\langle(1,n+s),(0,d-s)\rangle)}}_{=q^{-2d+s}}\frac{q^{2s+1}-1}{q-1}q^{2(d-s)}[\mathcal{L}_{n+s}]\mathbb{1}_{(d-s)\delta}\\ &= \sum_{s=0}^{d} q^{-s}[2s+1]_+[\mathcal{L}_{n+s}]\mathbb{1}_{(d-s)\delta}.\end{aligned}$$

□

3.3 The Elements T_r

Now, let us briefly focus on the torsion part. Recall the last paragraphs in Subsection 3.1:

Since there are no extensions and homomorphisms between different tubes $\mathcal{U}_x$ we have

$$\mathrm{Tor}(\mathbb{X}) = \bigvee_{x\in\mathbb{P}^1_k}\mathcal{U}_x,$$

where $\mathrm{Tor}(\mathbb{X})$ is the full subcategory with all objects of rank 0.

Following the terminology in [22], the curve $\mathbb{X}$ is a curve of genus zero with ramification sequence $(2^1, 2^2)$ (see [22, Example 10.8]). This means that the $\mathcal{U}_x$ correspond to the skyscraper sheaves in $\mathrm{coh}(\mathbb{P}^1)$ except in two points. One is given by the simple regular representation

$$S_\pi = (k^2 \otimes K \xrightarrow{(1,x)} K)$$

where $k \subset k(x) =: L \subset k(x,y) =: K$ are field extensions with $[L : k] = [K : L] = 2$. Furthermore, we have $\mathrm{End}(S_\pi) \cong L$ which is unique. In this case we have $q_\pi := |\,\mathrm{End}(S_\pi)| = q^2$ but $\deg(\pi) = d_\pi = 1$. The other point which differs has $q_\omega = q^4$ and $\deg(\omega) = d_\omega = 2$.

Furthermore, the Hall algebra of each tube $\mathcal{U}_x$ is isomorphic to the ring of symmetric functions Λ_{q_x}, where $q_x = |k_x|$, as a discrete valuation ring with residue field k_x. Let $\psi_x : \Lambda_{q_x} \to \mathcal{H}(\mathcal{U}_x)$ be that isomorphism and d_x the degree of x.

Definition 3.3.1 We define the elements $T_{x,k}$ as the image of the power sums p_r up to a factor, namely for $x \in \mathbb{P}^1$, $k \in \mathbb{N}$ we define

$$T_{x,k} := \begin{cases} \frac{[2k]}{k} d_x \psi_x\left(p_{\frac{k}{d_x}}\right), & \text{if } d_x | k,\\ 0, & \text{else.}\end{cases}$$

The elements T_k are then given by the sum

$$T_k := \sum_{x\in\mathbb{P}^1} T_{x,k}.$$

Remark 3.3.2 From Subsection 1.5 we know the elementary symmetric function e_r corresponds to the indecomposable of degree r. The power sums p_r correspond to the elements $T_{x,r}$ up to a factor. Similarly, since the complete symmetric functions h_r are by definition the sum of all monomials of rank r, the corresponding elements in the Hall algebra are all torsion bundles of a given rank, namely

$$\mathbb{1}_{x,(0,k)} := \sum_{\substack{[\mathcal{F}]:\ \mathcal{F}\in \mathrm{Ob}(\mathcal{U}_x): \\ \bar{\mathcal{F}}=(0,k)}} [F].$$

Lemma 3.3.3 *The elements T_k can also be expressed via a power sum series:*

$$\exp\left(\sum_{k\geq 1} \frac{T_k}{[2k]} s^k\right) = \mathbb{1}(s).$$

Proof. Since the Hall algebra of Tor($\mathbb{X}$) is commutative and by definition of T_k as a sum, we have

$$\exp\left(\sum_{k\geq 1} \frac{T_k}{[2k]} s^k\right) = \prod_{x\in\mathbb{P}^1} \exp\left(\sum_{k\geq 1} \frac{T_{x,k}}{[2k]} s^k\right).$$

We may also split $\mathbb{1}(s)$ into factors of the subalgebras $\mathcal{H}(\mathcal{U}_x)$, i.e. let

$$\mathbb{1}_{x,(0,k)} := \sum_{\substack{[\mathcal{F}]:\ \mathcal{F}\in \mathrm{Ob}(\mathcal{U}_x): \\ \bar{\mathcal{F}}=(0,k)}} [F],$$

then by the commutativity and the fact there are no extensions between tubes, we get

$$\mathbb{1}(s) = \prod_{x\in\mathbb{P}^1} \left(1 + \sum_{r\geq 1} \mathbb{1}_{x,(0,r)} s^r\right).$$

In particular, it is enough to show the identity in each $\mathcal{H}(\mathcal{U}_x)$, namely

$$\exp\left(\sum_{k\geq 1} \frac{T_{x,k}}{[2k]} s^k\right) = 1 + \sum_{k\geq 1} \mathbb{1}_{x,(0,k)} s^k.$$

But this follows by the identity in the ring of symmetric functions:

$$\begin{aligned}
\psi_x^{-1}\left(\exp\left(\sum_{k\geq 1}\frac{T_{x,k}}{[2k]}s^k\right)\right) &= \exp\left(\sum_{k\geq 1}\frac{\psi_x^{-1}(T_{x,k})}{[2k]}s^k\right)\\
&= \exp\left(\sum_{\substack{l\geq 1,\\ d_x|l}}\frac{p_{\frac{l}{d_x}}\cdot d_x}{l}s^{\frac{l}{d_x}}\right)\\
&= 1+\sum_{k\geq 1}h_k s^k\\
&= 1+\sum_{k\geq 1}\psi_x^{-1}(\mathbb{1}_{x,(0,k)})s^k\\
&= \psi_x^{-1}\left(1+\sum_{k\geq 1}\mathbb{1}_{x,(0,k)}s^k\right)
\end{aligned}$$

This proves the Lemma. □

Remark 3.3.4 By rewriting the formula we get

$$\sum_{k\geq 1}\frac{T_k}{[2k]}s^k = \log(\mathbb{1}(s)),$$

and therefore for $k \in \mathbb{N}$:

$$T_k = [2k]\sum_{r=1}^{k}\sum_{\substack{i_1,\ldots,i_r\in\mathbb{N}:\\ i_1+\ldots+i_r=k}}\frac{(-1)^{r+1}}{r}\mathbb{1}_{i_1\delta}\ldots\mathbb{1}_{i_r\delta}.$$

The first three can be easily written down:

$$\begin{aligned}
T_1 &= [2]\mathbb{1}_\delta\\
T_2 &= [4]\left(\mathbb{1}_{2\delta}-\frac{1}{2}\mathbb{1}_\delta^2\right)\\
T_3 &= [6]\left(\mathbb{1}_{3\delta}-\mathbb{1}_{2\delta}\mathbb{1}_\delta+\frac{1}{3}\mathbb{1}_\delta^3\right)
\end{aligned}$$

Remark 3.3.5 By construction we have that the T_r are primitive, i.e. for $r \in \mathbb{N}$

$$\tilde{\Delta}(T_r) = T_r\otimes 1 + C^r\otimes T_r.$$

Remark 3.3.6 The θ_n can also be defined via the generating series:

$$\theta(s) := 1+\sum_{n=1}^{\infty}\theta_n s^n = \exp\left((q^{\frac{1}{2}}-q^{-\frac{1}{2}})\sum_{r=1}^{\infty}T_r s^r\right).$$

Namely, using the definition of the T_k we have:

$$\mathbb{1}(s) = \exp\left(\sum_{k\geq 1}\frac{T_k}{[2k]}s^k\right).$$

Furthermore, using $\chi(s)\mathbb{1}(s) = 1$, we have

$$\chi(s) = \exp\left(-\sum_{k\geq 1} \frac{T_k}{[2k]} s^k \right).$$

Hence we have

$$\begin{aligned}
\theta(s) := 1 + \sum_{n=1}^{\infty} \theta_n s^n &= 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{n} q^{n-2i} \mathbb{1}_{(0,n-i)} \chi_i s^n \\
&= \mathbb{1}(qs)\chi(q^{-1}s) \\
&= \exp\left(\sum_{k\geq 1} \frac{T_k}{[2k]} q^k s^k \right) \exp\left(-\sum_{k\geq 1} \frac{T_k}{[2k]} q^{-k} s^k \right) \\
&= \exp\left(\sum_{k\geq 1} \frac{q^k - q^{-k}}{[2k]} T_k s^k \right) \\
&= \exp\left((q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{k\geq 1} T_k s^k \right)
\end{aligned}$$

From now on we denote by T_k for integers $k < 0$ the elements $T^-_{-k} \in D\tilde{\mathcal{H}}(\mathrm{Coh}(\mathbb{X}))$.

Using that the comultiplication of the line bundles takes the same form as the ones in the category $\mathrm{Coh}(\mathbb{P}^1)$ (see e.g. [32]), one gets the same formula for the reduced Drinfeld double for the commutator $[[\mathcal{L}_n]^+, [\mathcal{L}_m]^-]$. Namely:

Corollary 3.3.7 [6] *For $n, m \in \mathbb{Z}$ it holds:*

$$[[\mathcal{L}_n]^+, [\mathcal{L}_m]^-] = -q^{-\frac{1}{2}} \frac{K_{(1,m)}\theta^+_{n-m} - K_{(-1,-n)}\theta^-_{n-m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

where $\sum_{k=0}^{\infty} \theta^{\pm}_{\pm k} u^k = \exp\left(\pm(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{l=1}^{\infty} T_{\pm l} u^l \right)$.

Proof. We use the formula of Corollary 1.4.2: $\sum a^-_{(1)} b^+_{(2)} (a_{(2)}, b_{(1)}) = \sum b^+_{(1)} a^-_{(2)} (a_{(1)}, b_{(2)})$

We have

$$\tilde{\Delta}([\mathcal{L}_n]^+) = [\mathcal{L}_n]^+ \otimes 1 + \sum_{l\geq 0} \theta^+_l K_{(1,n-l)} \otimes [\mathcal{L}_{n-l}]^+$$

and

$$\tilde{\Delta}([\mathcal{L}_m]^-) = [\mathcal{L}_m]^- \otimes 1 + \sum_{j\geq 0} \theta^-_l K^{-1}_{(1,m-j)} \otimes [\mathcal{L}_{m-j}]^-.$$

Hence inserting in the formula yields on the left-hand side:

$$\begin{aligned}
&[\mathcal{L}_m]^- (1 \cdot \underbrace{(1, [\mathcal{L}_n])}_{=0} + \sum_{l\geq 0} [\mathcal{L}_{n-l}]^+ \cdot \underbrace{(1, \theta_l K_{(1,n-l)}))}_{=\delta_{l,0}} \\
&+ \sum_{j\geq 0} \theta^-_l K^{-1}_{(1,m-j)} (1 \cdot \underbrace{([\mathcal{L}_{m-j}], [\mathcal{L}_n])}_{=\delta_{m-j,n} \frac{1}{q-1}} + \sum_{l\geq 0} [\mathcal{L}_{n-l}]^+ \underbrace{([\mathcal{L}_{m-j}], \theta_l K_{(1,n-l)}))}_{=0} \\
= \; &[\mathcal{L}_m]^- [\mathcal{L}_n]^+ + \theta^-_{m-n} K^{-1}_{(1,n)} \frac{1}{q-1}
\end{aligned}$$

and on the right-hand side:

$$
\begin{aligned}
& [\mathcal{L}_n]^+(1\cdot \underbrace{(1,[\mathcal{L}_m])}_{=0} + \sum_{j\geq 0}[\mathcal{L}_{m-j}]^- \cdot \underbrace{(1,\theta_j K^{-1}_{(1,m-j)}))}_{=\delta_{j,0}} \\
& + \sum_{l\geq 0}\theta_l^+ K_{(1,n-l)}(1\cdot \underbrace{([\mathcal{L}_{n-l}],[\mathcal{L}_m])}_{=\delta_{n-l,m}\frac{1}{q-1}} + \sum_{j\geq 0}[\mathcal{L}_{m-j}]^- \underbrace{([\mathcal{L}_{n-l}],\theta_j K^{-1}_{(1,m-j)}))}_{=0} \\
= \quad & [\mathcal{L}_n]^+[\mathcal{L}_m]^- + \theta^+_{m-n}K_{(1,m)}\frac{1}{q-1}
\end{aligned}
$$

Comparing both sides we get:

$$
\begin{aligned}
[[\mathcal{L}_n]^+,[\mathcal{L}_m]^-] &= [\mathcal{L}_n]^+[\mathcal{L}_m]^- - [\mathcal{L}_m]^-[\mathcal{L}_n]^+ \\
&= \theta^-_{m-n}K^{-1}_{(1,n)}\frac{1}{q-1} - \theta^+_{m-n}K_{(1,m)}\frac{1}{q-1} \\
&= -q^{-\frac{1}{2}}\frac{K_{(1,m)}\theta^+_{n-m} - K_{(-1,-n)}\theta^-_{n-m}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}
\end{aligned}
$$

□

Now, let us consider some more the elements T_r. Since there are no homomorphisms and extensions between the tubes $\mathcal{U}_x$ we have that

$$
T_r = \sum_{x\in\mathbb{P}^1} T_r^x = \sum_{d|r}\sum_{\substack{x\in\mathbb{P}^1\\ d_x=d}} T_r^x, \tag{7}
$$

where each $T_r^x \in \mathcal{H}(\mathcal{U}_x)$ and d_x is the degree of the point $x\in\mathbb{P}^1$. But as mentioned before, each $\mathcal{H}(\mathcal{U}_x)$ is isomorphic to the ring of symmetric functions Λ_{q_x}. Using our definition of T_r we have a direct correspondence to power sums p_r. In particular, we can use this to calculate the bilinear forms.

Remark 3.3.8 In [32] one can find also the case for general curves and the analogous definition for the $\tilde{T}_r$ respectively $\tilde{T}_r^x = \frac{[r]}{r}\deg(x)\tilde{\Phi}_x(p_{\frac{r}{\deg(x)}})$ and their bilinear forms

$$
(\tilde{T}_r^x,\tilde{T}_r^x) = \frac{[r]^2\deg(x)}{r\left(q_x^{\frac{r}{\deg(x)}}-1\right)}
$$

Remark 3.3.9 In the case of coherent sheaves of the projective line $\mathrm{Coh}(\mathbb{P}^1)$, the elements T_r have a slightly different normalization, namely, they are given by the generating series

$$
\sum_{k\geq 1}\frac{T_k^{\mathbb{P}^1}}{[k]}s^k = \log(\mathbb{1}(s)).
$$

And the bilinear form is given by (see [6])

$$
(T_r^{\mathbb{P}^1},T_s^{\mathbb{P}^1}) = \delta_{r,s}\frac{[2r]}{r(v-v^{-1})}.
$$

(Note: in [6] they switch v and v^{-1}.)

In particular, using the result in $\mathrm{Coh}(\mathbb{P}^1)$ and the knowledge of how it differs, namely in only two specific points $x \in \mathbb{P}^1$, we can calculate the bilinear form in our case:

Lemma 3.3.10 *It holds for all $r,\ s \in \mathbb{N}$:*

$$(T_r, T_s) = \delta_{rs} \frac{[2r]}{r} \frac{q^r + (-1)^{r+1} + q^{-r}}{v - v^{-1}}.$$

Proof. First, using (7) and the orthogonality we have

$$(T_r, T_r) = \sum_{d|r} \sum_{\substack{x \in \mathbb{P}^1 \\ d_x = d}} (T_r^x, T_r^x).$$

We consider two cases, namely r odd and r even.

- Case r odd: In this case, we only have one exceptional point to consider, namely the point we denoted by π, where $d_\pi = 1$ but $q_\pi = q^2$.

$$\begin{aligned}
(T_r, T_r) &= (T_r^{\mathbb{P}^1}, T_r^{\mathbb{P}^1}) \cdot \frac{[2r]^2}{[r]^2} - \underbrace{\frac{[2r]^2}{r}\left(\frac{1}{q^r - 1} - \frac{1}{q^{2r} - 1}\right)}_{\text{difference given by } \pi} \\
&= \frac{[2r]}{r(v - v^{-1})} \cdot \frac{[2r]^2}{[r]^2} - \frac{[2r]^2}{r}\left(\frac{q^r + 1}{q^{2r} - 1} - \frac{1}{q^{2r} - 1}\right) \\
&= \frac{[2r]^2}{[r]^2 r}\left(\frac{[2r]}{(v - v^{-1})} - \frac{[r]^2 q^r}{q^{2r} - 1}\right) \\
&= \frac{[2r]^2}{[r]^2 r}\left(\frac{v^{2r} - v^{-2r}}{(v - v^{-1})^2} - \frac{(v^r - v^{-r})^2 v^{2r}}{(v - v^{-1})^2 (v^{4r} - 1)}\right) \\
&= \frac{[2r]^2}{[r]^2 r}\left(\frac{(v^{2r} - v^{-2r})^2}{(v^{2r} - v^{-2r})(v - v^{-1})^2} - \frac{(v^r - v^{-r})^2}{(v - v^{-1})^2 (v^{2r} - v^{-2r})}\right) \\
&= \frac{[2r]^2 (v^r - v^{-r})^2 ((v^r + v^{-r})^2 - 1)}{[r]^2 r (v - v^{-1})^2 (v^{2r} - v^{-2r})} \\
&= \frac{[2r]^2 (v^{2r} + 2 + v^{-2r} - 1)}{r(v^{2r} - v^{-2r})} \\
&= \frac{[2r](v^{2r} + 1 + v^{-2r})}{r(v - v^{-1})}
\end{aligned}$$

- Case r even: In this case, there exists another point of degree 2, namely the point ω with

$q_\omega = q^4$ instead of $q_\omega = q^2$. Hence we have

$$\begin{aligned}
(T_r, T_r) &= \frac{[2r](v^{2r}+1+v^{-2r})}{r(v-v^{-1})} - \frac{2\cdot[2r]^2}{r}\left(\frac{1}{q^r-1} - \frac{1}{q^{2r}-1}\right) \\
&= \frac{[2r]}{r}\left(\frac{(v^{2r}+1+v^{-2r})}{v-v^{-1}} - \frac{2\cdot(v^{2r}-v^{-2r})\cdot q^r}{(v-v^{-1})(q^{2r}-1)}\right) \\
&= \frac{[2r]}{r}\left(\frac{(v^{2r}+1+v^{-2r})}{v-v^{-1}} - \frac{2\cdot(v^{2r}-v^{-2r})}{(v-v^{-1})(v^{2r}-v^{-2r})}\right) \\
&= \frac{[2r]}{r}\cdot\frac{(v^{2r}+1+v^{-2r}-2)}{v-v^{-1}} \\
&= \frac{[2r](v^{2r}-1+v^{-2r})}{r(v-v^{-1})}
\end{aligned}$$

□

Similar using the bilinear form for the T_r with itself we can calculate this for θ_r with the T_r:

Corollary 3.3.11 *It holds for all $r \in \mathbb{N}$:*

$$(\theta_r, T_r) = \frac{[2r]}{r}(q^r + (-1)^{r+1} + q^{-r})$$

Proof. Considering the generating series (see Remark 3.3.6)

$$\theta(s) = \exp\left((q^{\frac{1}{2}} - q^{-\frac{1}{2}})\sum_{k\geq 1} T_k s^k\right)$$

and the fact $(T_s, T_r) = 0$ for $r \neq s$ we get:

$$\begin{aligned}
(\theta_r, T_r) &= ((v-v^{-1})T_r, T_r) \\
&= \frac{[2r]}{r}(q^r + (-1)^{r+1} + q^{-r}).
\end{aligned}$$

□

Furthermore, we may use the bilinear pairing to calculate the relations of T_r in $D\tilde{\mathcal{H}}(\mathrm{Coh}(\mathbb{X}))$:

Corollary 3.3.12 *For $k,\ l \in \mathbb{Z}\setminus\{0\}$ it holds in $D\tilde{\mathcal{H}}(\mathrm{Coh}(\mathbb{X}))$:*

$$[T_k, T_l] = \delta_{k+l,0}\frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})\frac{C^k - C^{-k}}{v - v^{-1}}.$$

Proof. First, we note, that if both indices are positive or both negative the commutator is zero since the torsion part of the Hall algebra commutes. Hence it remains to check the relation for $k > 0$ and $l < 0$ (and vice versa).

Let $k > 0$ and $l < 0$. We use the relation in Corollary 1.4.2 and the fact that the T_r are (quasi) primitive:

$$T_l T_k(1, C^k) + C^k(T_k, T_{-l}) \quad = \quad T_k T_l(C^l, 1) + C^l(T_{-l}, T_k)$$

Hence we get

$$T_lT_k - T_kT_l = (T_k, T_{-l}) \cdot (C^l - C^k).$$

Inserting gives:

$$[T_k, T_l] = \delta_{k+l,0}\frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})\frac{C^k - C^{-k}}{v - v^{-1}}.$$

□

We are also able to calculate the commutator of the T_r with the line bundles $[\mathcal{L}_n]$:

Proposition 3.3.13 *Let $n \in \mathbb{Z}$, $k \in \mathbb{N}$. Then it holds:*

$$[T_k, [\mathcal{L}_n]] = \frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})[\mathcal{L}_{n+k}].$$

Proof. By applying Lemma 3.2.7 we get the following:

$$\begin{aligned}
T_k[\mathcal{L}_n] &= [2k]\sum_{r=1}^{k}\sum_{\substack{i_1,\dots,i_r\in\mathbb{N}:\\ i_1+\dots+i_r=k}}\frac{(-1)^{r+1}}{r}\mathbb{1}_{i_1\delta}\dots\mathbb{1}_{i_r\delta}[\mathcal{L}_n]\\
&= [2k]\sum_{r=1}^{k}\sum_{\substack{i_1,\dots,i_r\in\mathbb{N}:\\ i_1+\dots+i_r=k}}\frac{(-1)^{r+1}}{r}\mathbb{1}_{i_1\delta}\dots\mathbb{1}_{i_{r-1}\delta}\sum_{s_r=0}^{i_r}q^{-s_r}[2s_r+1]_+[\mathcal{L}_{n+s_r}]\mathbb{1}_{(i_r-s_r)\delta}\\
&= [2k]\sum_{r=1}^{k}\sum_{\substack{i_1,\dots,i_r\in\mathbb{N}:\\ i_1+\dots+i_r=k}}\frac{(-1)^{r+1}}{r}\sum_{s_1=0}^{i_1}\dots\sum_{s_r=0}^{i_r}q^{-(s_1+\dots+s_r)}\left(\prod_{j=1}^{r}[2s_j+1]_+\right)[\mathcal{L}_{n+s_1+\dots+s_r}]\prod_{j=1}^{r}\mathbb{1}_{(i_j-s_j)\delta}
\end{aligned}$$

Notice that this is a telescoping sum. Only the two cases $s_1 = s_2 = \dots = s_r = 0$ and $(s_1 = i_1 \wedge s_2 = i_2 \wedge \dots \wedge s_r = i_r)$ remain. In particular, we get:

$$\begin{aligned}
T_k[\mathcal{L}_n] &= [2k]\sum_{r=1}^{k}\sum_{\substack{i_1,\dots,i_r\in\mathbb{N}:\\ i_1+\dots+i_r=k}}\frac{(-1)^{r+1}}{r}\left([\mathcal{L}_n]\mathbb{1}_{i_1\delta}\dots\mathbb{1}_{i_r\delta} + q^{-k}\left(\prod_{j=1}^{r}[2i_j+1]_+\right)[\mathcal{L}_{n+k}]\right)\\
&= [\mathcal{L}_n]T_k + \underbrace{[2k]q^{-k}\sum_{r=1}^{k}\sum_{\substack{i_1,\dots,i_r\in\mathbb{N}:\\ i_1+\dots+i_r=k}}\frac{(-1)^{r+1}}{r}\left(\prod_{j=1}^{r}[2i_j+1]_+\right)}_{=:\Phi_k}[\mathcal{L}_{n+k}]
\end{aligned}$$

It remains to be shown $\Phi_k = \frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})$. Here our previous calculations of the bilinear forms and the coproduct come to play (see Proposition 3.2.6 and Corollary 3.3.11):

Note that $[\mathcal{L}_n]T_k$ always has torsion, therefore pairing the commutator with $[\mathcal{L}_{n+k}]$ we get:

$$
\begin{aligned}
\Phi_k([\mathcal{L}_{n+k}],[\mathcal{L}_{n+k}]) &= (T_k[\mathcal{L}_n],[\mathcal{L}_{n+k}]) \\
&= (T_k \otimes [\mathcal{L}_n], \tilde{\Delta}([\mathcal{L}_{n+k}])) \\
&= \underbrace{\left(T_k \otimes [\mathcal{L}_n], [\mathcal{L}_{n+k}] \otimes 1 + \sum_{l \geq 0} \theta_l K_{(1,n+k-l)} \otimes [\mathcal{L}_{n+k-l}] \right)}_{\text{only the summand } l=k \text{ is non-zero in the pairing}} \\
&= (T_k, \theta_k K_{(1,n)})([\mathcal{L}_n],[\mathcal{L}_n]) \\
&= \frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})([\mathcal{L}_n],[\mathcal{L}_n])
\end{aligned}
$$

Now just using the fact $([\mathcal{L}_{n+k}],[\mathcal{L}_{n+k}]) = ([\mathcal{L}_n],[\mathcal{L}_n]) = \frac{1}{q-1}$, we are done. □

Remark 3.3.14 By considering partitions of $k \in \mathbb{N}$ we get:

	Partition	#	
$k=1$	1	1	$[3]_+ = 1+q+q^2$
$k=2$	2	1	$[5]_+ = 1+q+q^2+q^3+q^4$
	$1+1$	1	$[3]_+^2 = 1+2q+3q^2+2q^3+q^4$
$k=3$	3	1	$[7]_+ = 1+q+q^2+q^3+q^4+q^5+q^6$
	$2+1$	2	$[5]_+[3]_+ = 1+2q+3q^2+3q^3+3q^4+2q^5+q^6$
	$1+1+1$	1	$[3]_+^3 = 1+3q+6q^2+7q^3+6q^4+3q^5+q^6$
$k=4$	4	1	$[9]_+ = 1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8$
	$3+1$	2	$[7]_+[3]_+ = 1+2q+3q^2+3q^3+3q^4+3q^5+3q^6+2q^7+q^8$
	$2+2$	1	$[5]_+^2 = 1+2q+3q^2+4q^3+5q^4+4q^5+3q^6+2q^7+q^8$
	$2+1+1$	3	$[5]_+[3]_+^2 = 1+3q+6q^2+8q^3+9q^4+8q^5+6q^6+3q^7+q^8$
	$4\cdot 1$	1	$[3]_+^4 = 1+4q+10q^2+16q^3+19q^4+16q^5+10q^6+4q^7+q^8$
$k=5$	5	1	$[11]_+ = 1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}$
	$4+1$	2	$[9]_+[3]_+ = 1+2q+3q^2+3q^3+3q^4+3q^5+3q^6+3q^7+3q^8+2q^9+q^{10}$
	$3+2$	2	$[7]_+[5]_+ = 1+2q+3q^2+4q^3+5q^4+5q^5+5q^6+4q^7+3q^8+2q^9+q^{10}$
	$3+1+1$	3	$[7]_+[3]_+^2 = 1+3q+6q^2+8q^3+9q^4+9q^5+9q^6+8q^7+6q^8+3q^2+q^{10}$
	$2+2+1$	3	$[5]_+^2[3]_+ = 1+3q+6q^2+9q^3+12q^4+13q^5+12q^6+9q^7+6q^8+3q^9+q^{10}$
	$2+3\cdot 1$	4	$[5]_+[3]_+^3 = 1+4q+10q^2+17q^3+23q^4+25q^5+23q^6+17q^7+10q^8+4q^9+q^{10}$
	$5\cdot 1$	1	$[3]_+^5 = 1+5q+15q^2+30q^3+45q^4+51q^5+45q^6+30q^7+15q^8+5q^9+q^{10}$

So by direct calculation, one may check the first few values of Φ_k in the first formula:

$$\begin{aligned}
\Phi_1 &= [2]q^{-1}\frac{(-1)^2}{1}[3]_+ \\
&= [2](q^{-1}+1+q) \\
\Phi_2 &= [4]q^{-2}\left(\frac{(-1)^2}{1}[5]_+ + \frac{(-1)^3}{2}[3]_+^2\right) \\
&= [4]q^{-2}(1+q+q^2+q^3+q^4-\frac{1}{2}(1+2q+3q^2+2q^3+q^4)) \\
&= \frac{[4]}{2}q^{-2}(1+0-q^2+0+q^4) \\
&= \frac{[4]}{2}(q^{-2}-1+q^2) \\
\Phi_3 &= [6]q^{-3}\left(\frac{(-1)^2}{1}[7]_+ + 2\frac{(-1)^3}{2}[5]_+[3]_+ + \frac{(-1)^4}{3}[3]_+^3\right) \\
&= [6]q^{-3}(1+q+q^2+q^3+q^4+q^5+q^6-(1+2q+3q^2+3q^3+3q^4+2q^5+q^6) \\
&\qquad +\frac{1}{3}(1+3q+6q^2+7q^3+6q^4+3q^5+q^6)) \\
&= \frac{[6]}{3}q^{-3}(1+0+0+q^3+0+0+q^6) \\
&= \frac{[6]}{3}(q^{-3}+1+q^3)
\end{aligned}$$

Furthermore we get the relation in the reduced Drinfeld double $D\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$, where again we denote $T_k := T^-_{-k}$ for $k < 0$ and $T_k := T^+_k$ for $k > 0$:

Corollary 3.3.15 *Let $n \in \mathbb{Z}$, $k \in \mathbb{Z} \setminus \{0\}$. Then it holds:*

$$\left[T_k, [\mathcal{L}_n]^\pm\right] = \pm\frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})[\mathcal{L}_{n+k}]^\pm.$$

Proof.

- For $k > 0$:

  By Proposition 3.3.13 we have

  $$\left[T_k, [\mathcal{L}_n]^+\right] = \frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})[\mathcal{L}_{n+k}]^+.$$

  By the definition of the Drinfeld double and the coproduct of T_r (see Remark 3.3.5) we can see since $[\mathcal{L}_n]^-$ has the opposite coproduct:

  $$\left[[\mathcal{L}_n]^-, T_k\right] = \frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})[\mathcal{L}_{n+k}]^-$$

  Hence we have

  $$\left[T_k, [\mathcal{L}_n]^\pm\right] = \pm\frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})[\mathcal{L}_{n+k}]^\pm.$$

- For $k < 0$:

By definition of T_k for $k < 0$ we have by Proposition 3.3.13

$$\big[T_k, [\mathcal{L}_n]^-\big] = -\frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})[\mathcal{L}_{n+k}]^-.$$

Also by the previous case, we get

$$\big[T_k, [\mathcal{L}_n]^\pm\big] = \pm\frac{[2k]}{k}(q^k + (-1)^{k+1} + q^{-k})[\mathcal{L}_{n+k}]^\pm.$$

□

3.4 The Product of Line Bundles

Now, as a last calculation in this section, let us go back to vector bundles, or to be more accurate, the line bundles $\mathcal{L}_n$.

Proposition 3.4.1 *Let $n,\ m \in \mathbb{Z}$. Then it holds:*

$$[\mathcal{L}_n][\mathcal{L}_m] = \begin{cases} q^{\frac{1+2(m-n)}{2}}[\mathcal{L}_m \oplus \mathcal{L}_n], & \text{if } n < m, \\ q^{\frac{1}{2}}(q+1)[\mathcal{L}_n^{\oplus 2}], & \text{if } n = m, \\ q^{\frac{1+2(m-n)}{2}}\Big(q^{2(n-m)+1}[\mathcal{L}_n \oplus \mathcal{L}_m] + \sum_{a=1}^{\lfloor\frac{n-m-1}{2}\rfloor}(q-1)^{-1}\gamma_a^{(n-m-2a)}[\mathcal{L}_{n-a} \oplus \mathcal{L}_{m+a}] & \\ \qquad + I_{2\mathbb{Z}}(n-m)(q-1)^{-1}\alpha_{\frac{n-m}{2}}[\mathcal{L}_{\frac{n-m}{2}} \oplus \mathcal{L}_{\frac{n-m}{2}}] & \\ \qquad + I_{2\mathbb{Z}+1}(n-m)(q-1)^{-1}\beta_{\frac{n-m-1}{2}}[\mathcal{M}_{\frac{n+m+1}{2}}]\Big), & \text{if } n > m, \end{cases}$$

where $I_{2\mathbb{Z}}$ respectively $I_{2\mathbb{Z}+1}$ is the indicator function of $2\mathbb{Z}$ respecitvely $2\mathbb{Z}+1$, and $\alpha_\bullet$, $\gamma_\bullet$ and $\beta_\bullet$ are given by the series

- $A(w) = \sum_{l=0}^{\infty}\alpha_l w^l = (q^2-1)\frac{1+q^2w}{1+qw}\frac{1-q^2w}{1-q^4w}$
- $B(w) = \sum_{l=0}^{\infty}\beta_l w^l = (q^4-1)\frac{1-q^2w}{1-q^4w}\frac{1}{1+qw}$
- $C^{(r)}(w) = \sum_{l=0}^{\infty}\gamma_l^{(r)} w^l = q^{2r+1}(q-1)\frac{(1-w)(1+q^3w)}{(1+qw)(1-q^4w)}$

For the proof, we first start with the following lemma:

Lemma 3.4.2 *Let $n,\ m \in \mathbb{Z}$ and $\mathcal{F} \in \mathrm{Obj}(\mathrm{Coh}(\mathbb{X}))$, such that there exists a short exact sequence*

$$0 \longrightarrow \mathcal{L}_m \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{L}_n \longrightarrow 0.$$

Then there are only the following cases:

(i) If $n < m$, then $\mathcal{F} \cong \mathcal{L}_m \oplus \mathcal{L}_n$.

(ii) If $n \geq m$ and $n+m$ even, then there exists $0 \leq a \leq \frac{n-m}{2}$ such that $\mathcal{F} \cong \mathcal{L}_{n-a} \oplus \mathcal{L}_{m+a}$.

(iii) If $n \geq m$ and $n+m$ odd, then there exists $0 \leq a \leq \lfloor\frac{n-m}{2}\rfloor$ such that $\mathcal{F} \cong \mathcal{L}_{n-a} \oplus \mathcal{L}_{m+a}$ OR $\mathcal{F} \cong \mathcal{M}_{\frac{n+m+1}{2}}$.

Proof. First off, we note that $\mathcal{F}$ is torsion free because there are no homomorphisms from a torsion bundle to a line bundle, in this case $\mathcal{L}_n$.

Since the Euler form is additive, we know $\overline{\mathcal{F}} = (2, n+m) = (1, n) + (1, m) = \overline{\mathcal{L}}_n + \overline{\mathcal{L}}_m$. In particular, since $\mathcal{F}$ is torsion free, $\mathcal{F}$ decomposes into $\mathcal{L}_r \oplus \mathcal{L}_s$ for some r and s or $\mathcal{F}$ is isomorphic to $\mathcal{M}_t$ for some t. In the latter case, we know $(2, 2t-1) = \overline{\mathcal{M}}_t = \overline{\mathcal{F}} = (2, n+m)$. Hence we get $t = \frac{n+m+1}{2} \in \mathbb{Z}$ and $n+m$ must be odd.

In the first case, the short exact sequence either splits, therefore $r = n$ and $s = m$ or vice versa, or if we write $f = i \oplus j$ and $g = k \oplus l$, then $i \neq 0 \neq j$ and $k \neq 0 \neq l$. Therefore we have $m \leq \min(r, s)$ and $n \geq \max(r, s)$ because otherwise there is no nonzero morphism in one of the four cases. In particular, it follows $m \leq n$, and again by degree considerations, we get $r + s = n + m$. □

Example 3.4.3 Using the power series $A(w)$, $B(w)$ and $C(w)$, we get:

- For $A(w)$:

$$\begin{aligned} A(w) &= (q^2-1)\frac{1+q^2w}{1+qw}\frac{1-q^2w}{1-q^4w} \\ &= (q^2-1)(1-q^4w^2)(1-qw+q^2w^2-q^3w^3+...)(1+q^4w+q^8w^2+q^{12}w^3+...) \\ &= (q^2-1)(1+(q^4-q)w+(q^8-q^5-q^4+q^2)w^2+(q^{12}-q^3+q^5-q^8-q^9+q^6)w^3+...) \end{aligned}$$

  Hence we get:

  - $\alpha_0 = q^2 - 1$,
  - $\alpha_1 = (q^2-1)(q^4-q)$,
  - $\alpha_2 = (q^2-1)(q^8-q^5-q^4+q^2)$,
  - $\alpha_3 = (q^2-1)(q^{12}-q^9-q^8+q^6+q^5-q^3)$.

- For $B(w)$:

$$\begin{aligned} B(w) &= (q^4-1)\frac{1-q^2w}{1-q^4w}\frac{1}{1+qw} \\ &= (q^4-1)(1-q^2w)(1+q^4w+q^8w^2+q^{12}w^3+...)(1-qw+q^2w^2-q^3w^3+...) \\ &= (q^4-1)(1+(q^4-q^2-q)w+(q^8+q^2-q^5+q^3-q^6)w^2+...) \end{aligned}$$

  Hence we get:

  - $\beta_0 = q^4 - 1$,
  - $\beta_1 = (q^4-1)(q^4-q^2-q)$,
  - $\beta_2 = (q^4-1)(q^8-q^6-q^5+q^3+q^2)$.

- For $C^{(r)}(w)$:

$$\begin{aligned} C^{(r)}(w) &= q^{2r+1}(q-1)\frac{(1-w)(1+q^3w)}{(1+qw)(1-q^4w)} \\ &= q^{2r+1}(q-1)(1-w)(1+q^3w)(1-qw+q^2w^2-q^3w^3+...)(1+q^4w+q^8w^2+q^{12}w^3+...) \\ &= q^{2r+1}(q-1)(1+(-1+q^3-q+q^4)w+(q^2+q^8-q^3+q-q^4-q^4+q^7-q^5)w^2+...) \end{aligned}$$

  Hence we get:

- $\gamma_0^{(r)} = q^{2r+1}(q-1)$,
- $\gamma_1^{(r)} = q^{2r+1}(q-1)(q^4+q^3-q-1)$,
- $\gamma_2^{(r)} = q^{2r+1}(q-1)(q^8+q^7-q^5-2q^4-q^3+q^2+q)$.

Proof of Proposition 3.4.1. By the previous Lemma 3.4.2 for the cases $n \leq m$ there is only one possible extension, namely $[\mathcal{L}_n \oplus \mathcal{L}_m]$.

In the case $n < m$ there is only one possible way to embed $\mathcal{L}_m$ into the direct sum $\mathcal{L}_n \oplus \mathcal{L}_m$, hence there we only have the prefactor given by the Eulerform $\langle \mathcal{L}_n, \mathcal{L}_m \rangle = 1 + 2(m-n)$.

In the case $n = m$ there are the possibilities

$$\mathcal{L}_n \xrightarrow{(1,\alpha)} \mathcal{L}_n^{\oplus 2} \longrightarrow \mathcal{L}_n \qquad \text{for some } \alpha \in k$$

and

$$\mathcal{L}_n \xrightarrow{(0,1)} \mathcal{L}_n^{\oplus 2} \longrightarrow \mathcal{L}_n$$

up to automorphism. Hence the prefactor is given by the product of $(q+1)$ and the factor corresponding to the Eulerform $\langle \mathcal{L}_n, \mathcal{L}_n \rangle = 1$.

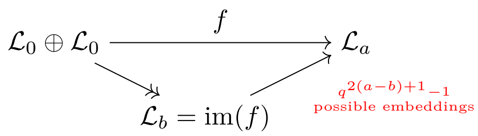
In the case $n < m$ one summand is just the direct sum $[\mathcal{L}_n \oplus \mathcal{L}_m]$ where the prefactor is given by $|\operatorname{Hom}(\mathcal{L}_m, \mathcal{L}_n)| = q^{2(n-m)+1}$ (see Lemma 2.5.3). For the other cases by Lemma 3.4.2 we calculate the prefactors as follows.

We do α_a, β_a and γ_a case by case:

- α: The number α_a is given by $\#\{\mathcal{L}_0 \oplus \mathcal{L}_0 \twoheadrightarrow \mathcal{L}_a\} = \frac{\alpha_a}{q-1}$. Let us consider the morphisms $\mathcal{L}_0 \oplus \mathcal{L}_0 \to \mathcal{L}_a$. It holds:

$$\#\operatorname{Hom}(\mathcal{L}_0 \oplus \mathcal{L}_0, \mathcal{L}_a) = q^{2a+2}$$

The image of a non-zero morphism in $\operatorname{Hom}(\mathcal{L}_0 \oplus \mathcal{L}_0, \mathcal{L}_a)$ is isomorphic to a line bundle $\mathcal{L}_b$ with $0 \leq b \leq a$.



We get the formula

$$\#\{\varphi : \mathcal{L}_0 \oplus \mathcal{L}_0 \to \mathcal{L}_a \mid \operatorname{im}(\varphi) \cong \mathcal{L}_b\} = \alpha_b \cdot \underbrace{\frac{q^{2(a-b)+1}-1}{q-1}}_{=:\delta_{a-b}}.$$

In particular, for all morphisms we get:

$$\#\operatorname{Hom}(\mathcal{L}_0 \oplus \mathcal{L}_0, \mathcal{L}_a) \setminus \{0\} = \sum_{b=0}^{n} \alpha_b \cdot \delta_{a-b}. \tag{8}$$

Define the power sum series:

$$A(w) = \sum_{k=0}^{\infty} \alpha_k w^k, \qquad D(w) := \sum_{k=0}^{\infty} \delta_k w^k \qquad \text{and} \qquad E(w) := \sum_{k=0}^{\infty} (q^{4k+2}-1) w^k.$$

By the equation (8) we get the identity:

$$E(w) = A(w) \cdot D(w).$$

We may rewrite the powersum series $E(w)$ and $D(w)$:

$$\begin{aligned}
E(w) &= \sum_{k=0}^{\infty}(q^{4k+2}-1)w^k \\
&= \sum_{k=0}^{\infty} q^{4k+2}w^k - \sum_{k=0}^{\infty} w^k \\
&= \sum_{k=0}^{\infty} q^2(q^4w)^k - \sum_{k=0}^{\infty} w^k \\
&= q^2\frac{1}{1-q^4w} - \frac{1}{1-w} \\
&= \frac{q^2(1-w)-(1-q^4w)}{(1-q^4w)(1-w)} \\
&= \frac{q^2-q^2w-1+q^4w}{(1-q^4w)(1-w)} \\
&= \frac{(q^2-1)(1+q^2w)}{(1-q^4w)(1-w)}
\end{aligned}$$

$$\begin{aligned}
D(w) &= \frac{1}{q-1}\sum_{k=0}^{\infty}\delta_k w^k \\
&= \frac{1}{q-1}\sum_{k=0}^{\infty}(q^{2k+1}-1)w^k \\
&= \frac{1}{q-1}\left(\sum_{k=0}^{\infty} q(q^2w)^k - \sum_{k=0}^{\infty} w^k\right) \\
&= \frac{1}{q-1}\left(\frac{q}{1-q^2w} - \frac{1}{1-w}\right) \\
&= \frac{1}{q-1}\frac{q(1-w)-(1-q^2w)}{(1-q^2w)(1-w)} \\
&= \frac{1}{q-1}\frac{q-qw-1+q^2w}{(1-q^2w)(1-w)} \\
&= \frac{1}{q-1}\frac{(q-1)(1+qw)}{(1-q^2w)(1-w)} \\
&= \frac{1+qw}{(1-q^2w)(1-w)}
\end{aligned}$$

Hence we get:

$$A(w) = \frac{E(w)}{D(w)} = \frac{(q^2-1)(1+q^2w)}{(1-q^4w)(1-w)}\frac{(1-q^2w)(1-w)}{1+qw} = (q^2-1)\frac{1+q^2w}{1+qw}\frac{1-q^2w}{1-q^4w}$$

- β: Similar to α_a, the constant β_a is given by $\#\{\mathcal{M}_0 \twoheadrightarrow \mathcal{L}_a\} = \frac{\beta_a}{q-1}$. Considering all morphisms, we have

$$\#\operatorname{Hom}(\mathcal{M}_0, \mathcal{L}_a) = q^{4(a+1)}.$$

The image of a non-zero morphism $\mathcal{M}_0 \to \mathcal{L}_a$ is isomorphic to a line bundle $\mathcal{L}_b$ with $0 \leq b \leq a$. It holds:

$$\#\{\varphi : \mathcal{M}_0 \to \mathcal{L}_a \mid \operatorname{im}(\varphi) \cong \mathcal{L}_b\} = \beta_b \cdot \underbrace{\frac{q^{2(a-b)+1} - 1}{q-1}}_{=\delta_{a-b}}$$

Using the powersum series $D(w)$, $B(w) := \sum_{k=0}^{\infty} \beta_k w^k$ and $F(w) := \sum_{k=0}^{\infty} (q^{4(k+1)} - 1)w^k$ we get the identity:

$$F(w) = B(w)D(w).$$

We rewrite $F(w)$:

$$\begin{aligned} F(w) &= \sum_{k=0}^{\infty} (q^{4(k+1)} - 1)w^k \\ &= \sum_{k=0}^{\infty} q^4 (q^4 w)^k - \sum_{k=0}^{\infty} w^k \\ &= \frac{q^4}{1 - q^4 w} - \frac{1}{1-w} \\ &= \frac{q^4(1-w) - (1 - q^4 w)}{(1-q^4 w)(1-w)} \\ &= \frac{q^4 - q^4 w - 1 + q^4 w}{(1-q^4 w)(1-w)} \\ &= \frac{q^4 - 1}{(1-q^4 w)(1-w)} \end{aligned}$$

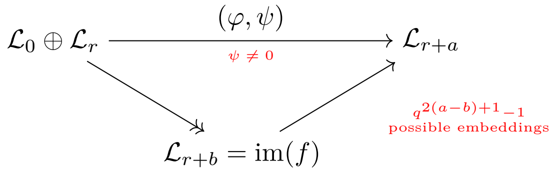
Hence we get:

$$B(w) = \frac{F(w)}{D(w)} = \frac{q^4 - 1}{(1-q^4 w)(1-w)} \frac{(1-q^2 w)(1-w)}{1+qw} = (q^4 - 1)\frac{(1-q^2 w)}{(1-q^4 w)(1+qw)}$$

- γ: The constant $\gamma_a^{(r)}$ is given by $\#\{\mathcal{L}_0 \oplus \mathcal{L}_r \twoheadrightarrow \mathcal{L}_{r+a}\} = \frac{\gamma_a^{(r)}}{q-1}$. Considering all morphisms we have:

$$\#\operatorname{Hom}(\mathcal{L}_0 \oplus \mathcal{L}_r, \mathcal{L}_{r+a}) = q^{2(r+a)+1} q^{2a+1} = q^{2r+4a+2}.$$

We consider morphisms with image $\mathcal{L}_{r+b}$.



So we have the formula for the number of all morphisms $(\varphi, \psi) : \mathcal{L}_0 \oplus \mathcal{L}_r \to \mathcal{L}_{r+a}$ with $\psi \neq 0$:

$$\sum_{k=0}^{a} \gamma_k \delta_{a-k} = q^{2r+1} q^{2a}(q^{2a+1} - 1).$$

So using the powersum series $D(w)$, $C^{(r)}(w) = \sum_{k=0}^{\infty} \gamma_k^{(r)} w^k$, and

$$G^{(r)}(w) = q^{2r+1} \sum_{k=0}^{\infty} q^{2k}(q^{2k+1} - 1) w^k$$

we get the identity:

$$G^{(r)}(w) = C^{(r)}(w) D(w).$$

Rewriting $G^{(r)}(w)$ we get:

$$\begin{aligned}
G^{(r)}(w) &= q^{2r+1} \sum_{k=0}^{\infty} q^{2k}(q^{2k+1} - 1) w^k \\
&= q^{2r+1} \left(\sum_{k=0}^{\infty} q(q^4 w)^k - \sum_{k=0}^{\infty} (q^2 w)^k \right) \\
&= q^{2r+1} \left(\frac{q}{1 - q^4 w} - \frac{1}{1 - q^2 w} \right) \\
&= q^{2r+1} \frac{q(1 - q^2 w) - (1 - q^4 w)}{(1 - q^4 w)(1 - q^2 w)} \\
&= q^{2r+1} \frac{q - q^3 w - 1 + q^4 w}{(1 - q^4 w)(1 - q^2 w)} \\
&= q^{2r+1} \frac{(q - 1)(q^3 w + 1)}{(1 - q^4 w)(1 - q^2 w)}
\end{aligned}$$

Hence we get:

$$\begin{aligned}
C^{(r)}(w) &= \frac{G^{(r)}(w)}{D(w)} \\
&= q^{2r+1} \frac{(q - 1)(q^3 w + 1)}{(1 - q^4 w)(1 - q^2 w)} \frac{(1 - q^2 w)(1 - w)}{1 + qw} \\
&= q^{2r+1}(q - 1) \frac{(q^3 w + 1)(1 - w)}{(1 + qw)(1 - q^4 w)}.
\end{aligned}$$

□

Now let us introduce the Drinfeld representation of the quantum group/quantum loop algebra to show which other relations we should discuss.

4 The Drinfeld Realization $U_v^{Dr}\left(A_2^{(2)}\right)$

4.1 Definition of $U_v^{Dr}\left(A_2^{(2)}\right)$ and its Relations

For this subsection, we follow [10]. The goal is to define the quantum group $U_v^{Dr}\left(A_2^{(2)}\right)$ and give the argument by I. Damiani of how we may simplify the relations without changing the algebra.

To consider the relations, we define several algebras with generators, where some of the relations hold.

Let $v = \sqrt{q} \in \mathbb{C}$.

This definition can also be found in e.g. [8].

Definition 4.1.1 We denote by

- $U_v^{Dr}\left(A_2^{(2)}\right) = U_v(L(\mathfrak{sl}_3)^\tau)$ the associative $\mathbb{C}$-algebra with generators $(G) = \{X_k^{\pm},\ H_l,\ K^{\pm 1},\ C^{\pm 1} \mid k \in \mathbb{Z},\ l \in \mathbb{Z} \setminus \{0\}\}$ and the following relations:

(CU) C is central;

(CK) $CC^{-1} = 1 = C^{-1}C$, $KK^{-1} = 1 = K^{-1}K$;

$(KX^{\pm})$ $KX_k^{\pm}K^{-1} = v^{\pm 2}X_k^{\pm}$ for $k \in \mathbb{Z}$;

(KH) $KH_k = H_kK$ for $k \in \mathbb{Z} \setminus \{0\}$;

(XX) $[X_k^+, X_l^-] = \frac{C^{-l}K\psi_{k+l}^+ - C^{-k}K^{-1}\psi_{k+l}^-}{v-v^{-1}}$ for $k,\ l \in \mathbb{Z}$,
where $\sum_{k=0}^{\infty} \psi_{\pm k}^{\pm} u^k = \exp\left(\pm(v-v^{-1})\sum_{l=1}^{\infty} H_{\pm l}u^l\right)$;

$(HX^{\pm})$ $[H_k, X_l^{\pm}] = \pm \underbrace{\frac{1}{k}[2k]_v(v^{2k} + v^{-2k} + (-1)^{k+1})}_{=:b_k} X_{k+l}^{\pm}$ for $k \in \mathbb{Z} \setminus \{0\},\ l \in \mathbb{Z}$;

(HH) $[H_k, H_l] = \delta_{k+l,0}b_k\frac{C^k - C^{-k}}{v-v^{-1}}$ for $k,\ l \in \mathbb{Z} \setminus \{0\}$;

$(X2^{\pm})$ $X_{k+2}^{\pm}X_l^{\pm} + (v^{\mp 2} - v^{\pm 4})X_{k+1}^{\pm}X_{l+1}^{\pm} - v^{\pm 2}X_k^{\pm}X_{l+2}^{\pm} =$
$v^{\pm 2}X_l^{\pm}X_{k+2}^{\pm} + (v^{\pm 4} - v^{\mp 2})X_{l+1}^{\pm}X_{k+1}^{\pm} - X_{l+2}^{\pm}X_k^{\pm}$ for $k,\ l \in \mathbb{Z}$;

$(X3^{-,\pm})$ $\mathrm{Sym}(v^3X_{k\mp 1}^{\pm}X_l^{\pm}X_m^{\pm} - (v+v^{-1})X_k^{\pm}X_{l\mp 1}^{\pm}X_m^{\pm} + v^{-3}X_k^{\pm}X_l^{\pm}X_{m\mp 1}^{\pm}) = 0$ for $k,\ l,\ m \in \mathbb{Z}$;

$(X3^{+,\pm})$ $\mathrm{Sym}(v^{-3}X_{k\pm 1}^{\pm}X_l^{\pm}X_m^{\pm} - (v+v^{-1})X_k^{\pm}X_{l\pm 1}^{\pm}X_m^{\pm} + v^3X_k^{\pm}X_l^{\pm}X_{m\pm 1}^{\pm}) = 0$ for $k,\ l,\ m \in \mathbb{Z}$.

- $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$ the $\mathbb{C}$-algebra generated by the generators (G) with the relations (CU), (CK), $(KX^{\pm})$, (XX) and $(HX^{\pm})$.

- $\bar{U}_v^{Dr}\left(A_2^{(2)}\right)$ the $\mathbb{C}$-algebra generated by $(\bar{G}) = \{C^{\pm 1},\ K^{\pm 1},\ X_k^{\pm} \mid k \in \mathbb{Z}\}$ and relations (CU) and (CK).

The above definition for $U_v^{Dr}\left(A_2^{(2)}\right)$ is the standard you can find in most papers. Furthermore, in the original definition of Drinfeld [14] only $(X3^{-,-})$ and $(X3^{+,+})$ appear. The relations $(X3^{-,+})$ and $(X3^{+,-})$ are introduced for symmetry in [8] as consequences of other relations.

In particular, it suffices to use only the two relations $(X3^{-,-})$ and $(X3^{+,+})$ without the other two.

Remark 4.1.2 As shown in [10], we can reformulate the relations with commutators:

$$\begin{aligned}
& \mathrm{Sym}(v^{-3}X^\pm_{k\pm1}X^\pm_l X^\pm_m - (v+v^{-1})X^\pm_k X^\pm_{l\pm1}X^\pm_m + v^3 X^\pm_k X^\pm_l X^\pm_{m\pm1}) \\
= \; & \mathrm{Sym}(v^{-3}(X^\pm_{k\pm1}X^\pm_l X^\pm_m - v^4 X^\pm_k X^\pm_{l\pm1}X^\pm_m - v^2 X^\pm_k X^\pm_{l\pm1}X^\pm_m + v^6 X^\pm_k X^\pm_l X^\pm_{m\pm1})) \\
= \; & \mathrm{Sym}(v^{-3}(X^\pm_{k\pm1}X^\pm_l X^\pm_m - v^2 X^\pm_l X^\pm_{k\pm1}X^\pm_m - v^4 X^\pm_m X^\pm_{k\pm1}X^\pm_l + v^6 X^\pm_m X^\pm_l X^\pm_{k\pm1})) \\
= \; & \mathrm{Sym}(v^{-3}([X^\pm_{k\pm1}, X^\pm_l]_{v^2} X^\pm_m - v^4 X^\pm_m [X^\pm_{k\pm1} X^\pm_l]_{v^2})) \\
= \; & \mathrm{Sym}(v^{-3}[[X^\pm_{k\pm1}, X^\pm_l]_{v^2}, X^\pm_m]_{v^4})
\end{aligned}$$

Hence the relation $(X3^{+,\pm})$ is equivalent to

$$\mathrm{Sym}([[X^\pm_{k\pm1}, X^\pm_l]_{v^2}, X^\pm_m]_{v^4}) = 0 \qquad \text{for all } m,k,l \in \mathbb{Z}.$$

Similarly the relation $(X3^{-,\pm})$ is equivalent to

$$\mathrm{Sym}([[X^\pm_{k\mp1}, X^\pm_l]_{v^{-2}}, X^\pm_m]_{v^{-4}}) = 0 \qquad \text{for all } m,k,l \in \mathbb{Z}.$$

Remark 4.1.3 In the relation (XX) one should note that $\psi_k^+ = 0$ for $k < 0$, $\psi_k^- = 0$ for $k > 0$ and $\psi_0^+ = \psi_0^- = 1$. In particular, only in the case $k + l = 0$ are there two terms in the numerator of the relation for $[X_k^+, X_l^-]$. Furthermore, using this relation one can write the generators H_r's with the other generators.

Remark 4.1.4 We have:

1. the algebra $U_v^{Dr}\left(A_2^{(2)}\right)$ is a quotient of $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$;
2. the algebra $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$ is a quotient of $\bar{U}_v^{Dr}\left(A_2^{(2)}\right)$ by the previous remark (see [10, Section 6]).

Let (R) be a set of relations in an algebra U given via a set Z depending on parameters $r \in \mathbb{Z}^l$ and $s \in \mathbb{Z}^{\tilde{l}}$ for some $l, \tilde{l} \in \mathbb{N}_0$, i.e. for $\zeta \in Z$ we have relations of the form $S_\zeta(r,s) = 0$.

Then we define the ideals:

$$\mathcal{I}(R) = (S_\zeta(r,s) \mid \zeta \in Z, r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}}),$$

$$\mathcal{I}_c(R) = (S_\zeta(r\mathbb{1}, s) \mid \zeta \in Z, r \in \mathbb{Z}, s \in \mathbb{Z}^{\tilde{l}}),$$

$$\mathcal{I}_0(R) = (S_\zeta(\underline{0}) \mid \zeta \in Z),$$

where $\underline{0} \in \mathbb{Z}^{l+\tilde{l}}$.

By definition we have $\mathcal{I}_0(R) \subset \mathcal{I}_c(R) \subset \mathcal{I}(R)$.

Our goal is to show that we can reduce $(X3^{\pm,\pm})$ to $(X3_c^{\pm,\pm})$ where we only consider $k = l = m \in \mathbb{Z}$.

Lemma 4.1.5 *In $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$ consider, for $r = (r_1, r_2, r_3) \in \mathbb{Z}^3$,*

$$S^\pm_{X3}(r_1,r_2,r_3) := \sum_{\sigma \in \mathfrak{S}_3} \sigma.(v^{-3}X^\pm_{r_1\pm1}X^\pm_{r_2}X^\pm_{r_3} - (v+v^{-1})X^\pm_{r_1}X^\pm_{r_2\pm1}X^\pm_{r_3} + v^3 X^\pm_{r_1}X^\pm_{r_2}X^\pm_{r_3\pm1})$$

and let $p \in \mathbb{Z} \setminus 0$. It holds:

$$[H_p, S^{\pm}_{X3}(r)] = \pm b_p \sum_{u=1}^{3} S^{\pm}_{X3}(r + pe_u),$$

where $e_u \in \mathbb{Z}^3$ is the vector consisting of 0s with one 1 at the u^{th} spot.

Proof. The lemma follows by simple calculation and applying $(HX^{\pm})$ three times:

Let $p \in \mathbb{Z} \setminus \{0\}$, $r_1, r_2, r_3 \in \mathbb{Z}$. It holds:

$$\begin{aligned} H_p X^{\pm}_{r_1} X^{\pm}_{r_2} X^{\pm}_{r_3} &= \pm b_p X^{\pm}_{r_1+p} X^{\pm}_{r_2} X^{\pm}_{r_3} + X^{\pm}_{r_1} H_p X^{\pm}_{r_2} X^{\pm}_{r_3} \\ &= \pm b_p X^{\pm}_{r_1+p} X^{\pm}_{r_2} X^{\pm}_{r_3} \pm b_p X^{\pm}_{r_1} X^{\pm}_{r_2+p} X^{\pm}_{r_3} + X^{\pm}_{r_1} X^{\pm}_{r_2} H_p X^{\pm}_{r_3} \\ &= \pm b_p X^{\pm}_{r_1+p} X^{\pm}_{r_2} X^{\pm}_{r_3} \pm b_p X^{\pm}_{r_1} X^{\pm}_{r_2+p} X^{\pm}_{r_3} \pm b_p X^{\pm}_{r_1} X^{\pm}_{r_2} X^{\pm}_{r_3+p} + X^{\pm}_{r_1} X^{\pm}_{r_2} X^{\pm}_{r_3} H_p \end{aligned}$$

Hence by definition of $S^{\pm}_{X3}(r)$ we get

$$[H_p, S^{\pm}_{X3}(r)] = \pm b_p \sum_{u=1}^{3} S^{\pm}_{X3}(r + pe_u).$$

□

Definition 4.1.6 Define the algebra homomorphism $\tilde{t} : \tilde{U}_v^{Dr}\left(A_2^{(2)}\right) \to \tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$ on the generators:

$$C^{\pm 1} \mapsto C^{\pm 1},\ K^{\pm 1} \mapsto (KC^{-1})^{\pm 1},\ X^{\pm}_r \mapsto X^{\pm}_{r \mp 1},\ H_r \mapsto H_r.$$

Remark 4.1.7 It is easy to check, that $\tilde{t}$ is a well-defined automorphism. The least obvious relation is (XX):

$$\begin{aligned} \tilde{t}([X^+_k, X^-_l]) &= [X^+_{k-1}, X^-_{l+1}] \\ &= \frac{C^{-l-1} K \psi^+_{k+l} - C^{-k+1} K^{-1} \psi^-_{k+l}}{v - v^{-1}}. \end{aligned}$$

By definition of $\psi^{\pm}_r$ we have $\tilde{t}(\psi^{\pm}_r) = \psi^{\pm}_r$ since they are given by terms in H_r's. Hence we get:

$$\begin{aligned} \frac{C^{-l-1} K \psi^+_{k+l} - C^{-k+1} K^{-1} \psi^-_{k+l}}{v - v^{-1}} &= \frac{C^{-l} \tilde{t}(K) \psi^+_{k+l} - C^{-k} \tilde{t}(K^{-1}) \psi^-_{k+l}}{v - v^{-1}} \\ &= \tilde{t}\left(\frac{C^{-l} K \psi^+_{k+l} - C^{-k} K^{-1} \psi^-_{k+l}}{v - v^{-1}}\right). \end{aligned}$$

Furthermore, we note $\tilde{t}(S^{\pm}_{X3}(r_1, r_2, r_3)) = S^{\pm}_{X3}(r_1 \mp 1, r_2 \mp 1, r_3 \mp 1)$. In particular the ideals $\mathcal{I}(X3^{+,\pm})$ and $\mathcal{I}_c(X3^{+,\pm})$ in $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$ are $\tilde{t}$-stable.

Analogously, we get that the ideals $\mathcal{I}(X3^{-,\pm})$ and $\mathcal{I}_c(X3^{-,\pm})$ in $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$ are $\tilde{t}$-stable.

To prove $\mathcal{I}(X3^{+,\pm}) = \mathcal{I}_c(X3^{+,\pm})$ and $\mathcal{I}(X3^{-,\pm}) = \mathcal{I}_c(X3^{-,\pm})$ in $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$, we first make a general statement about algebras:

Lemma 4.1.8 *Let U be an algebra over a field of characteristic 0. Let t be an automorphism of U, $Y \subset U$ a subset and consider elements z_m, $N_y(r) \in U$ for $m \in \mathbb{N}, y \in Y, r \in \mathbb{Z}^l$ with $l \in \mathbb{N}$ fixed such that:*

(i) $\forall y \in U, r \in \mathbb{Z}^l: \ t(N_y(r)) = N_y(r + \mathbb{1})$;

(ii) $\forall m \in \mathbb{N}: \ [z_m, Y] \subset Y$;

(iii) $\forall m \in \mathbb{N}, y \in U, r \in \mathbb{Z}^l: \ [z_m, N_y(r)] = N_{[z_m,y]}(r) + \sum_{u=1}^{l} N_y(r + me_u)$.

For $y \in U$ define $S_y(r) = \sum_{\sigma \in \mathfrak{S}_l} N_y(\sigma(r))$. Let Y be a subset of U.

Then: If $N_y(0) = 0$ for all $y \in Y$ then $S_y(r) = 0$ for all $y \in Y, r \in \mathbb{Z}^l$.

Proof. First we note that $S_y(\sigma(r)) = S_y(r)$ by definition. Hence it is enough to consider $r_1 \leq r_2 \leq ... \leq r_l$. Furthermore, the properties (i) and (iii) also hold for S_y, e.g.

$$t(S_y(r)) = t\left(\sum_{\sigma \in \mathfrak{S}_l} N_y(\sigma(r))\right) = \sum_{\sigma \in \mathfrak{S}_l} t(N_y(\sigma(r))) = \sum_{\sigma \in \mathfrak{S}_l} N_y(\sigma(r + \mathbb{1})) = S_y(r + \mathbb{1}).$$

Hence by applying the automorphism t^{-r_1} to $S_y(r)$, we can reduce the claim to the case $0 = r_1 \leq r_2 \leq ... \leq r_l$.

Define $v := \max\{u \in \{1, ..., l\} \mid r_u = 0\}$. If $v = l$ we have $S_y(r) = S_y(0) = l!N_y(0) = 0$ by definition of S_y and hypothesis. If $v < l$ then for $a := r_{v+1} \neq 0$:

$$\max\{u \in \{1, ..., l\} \mid (r - ae_{v+1})_u = 0\} = v + 1$$

and

$$\forall b > v + 1: \ \max\{u \in \{1, ..., l\} \mid (r - ae_{v+1} + ae_b)_u = 0\} = v + 1.$$

By induction on v, we may assume $S_y(r - ae_{v+1}) = 0 = S_y(r - ae_{v+1} + ae_b)$ for $l \geq b > v + 1$ and for all $y \in Y$.

Using the commutator with z_a we get:

$$\begin{aligned} 0 = [z_a, S_y(r - ae_{v+1})] &= \underbrace{S_{[z_a,y]}(r - ae_{v+1})}_{=0} + \sum_{u=1}^{l} S_y(r - ae_{v+1} + ae_u) \\ &= \sum_{u=1}^{v+1} \underbrace{S_y(r - ae_{v+1} + ae_u)}_{=S_y(r) \text{ by symmetry}} + \sum_{u=v+2}^{l} \underbrace{S_y(r - ae_{v+1} + ae_u)}_{=0} \\ &= (v + 1)S_y(r) \end{aligned}$$

Hence, by induction, we get the claim for all $y \in Y, r \in \mathbb{Z}^l$: $S_y(r) = 0$. □

Proposition 4.1.9 *In $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$ we have $\mathcal{I}(X3^{+,+}) = \mathcal{I}_c(X3^{+,+})$ and $\mathcal{I}(X3^{-,-}) = \mathcal{I}_c(X3^{-,-})$.*

Proof. Consider the following data:

- $U := \tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$;
- $Y := \{0, 1\}$;

- $t := \tilde{t}^{-1}$;
- $z_m := \frac{1}{b_m} H_m$ for $m \in \mathbb{N}$;
- $N_y(r) := y(v^{-3} X^+_{r_1+1} X^+_{r_2} X^+_{r_3} - (v+v^{-1}) X^+_{r_1} X^+_{r_2+1} X^+_{r_3} + v^3 X^+_{r_1} X^+_{r_2} X^+_{r_3+1})$ for $y \in Y$.

These satisfy the conditions of Lemma 4.1.8.

In $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)/\mathcal{I}_c(X3^+)$ we have $N_1(0) = \frac{1}{6} S^+_{X3}(0) = 0$ and therefore by applying Lemma 4.1.8 we have $S^+_{X3}(r) = 0$ for all $r \in \mathbb{Z}^3$. In particular we have $\mathcal{I}(X3^+) = 0$. Therefore we get $\mathcal{I}(X3^+) \subset \mathcal{I}_c(X3^+)$ in $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$.

Similarly, we get $\mathcal{I}(X3^-) \subset \mathcal{I}_c(X3^-)$ in $\tilde{U}_v^{Dr}\left(A_2^{(2)}\right)$. □

In total this means, in the definition of $U_v^{Dr}\left(A_2^{(2)}\right)$ we may substitute the relations $(X3^{\pm,\pm})$ with $(X3_c^{+,+})$ and $(X3_c^{-,-})$, where

$(X3_c^{+,+})$ $[[X^+_{r+1}, X^+_r]_{v^2}, X^+_r]_{v^4} = 0$ for all $r \in \mathbb{Z}$;

$(X3_c^{-,-})$ $[[X^-_{r+1}, X^-_r]_{v^{-2}}, X^-_r]_{v^{-4}} = 0$ for all $r \in \mathbb{Z}$.

4.2 Relations in the Hall Algebra

Now, to the main part of this thesis: We want to find elements in $D\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$ with the same relations, such that we have a subalgebra isomorphic to $U_v^{Dr}\left(A_2^{(2)}\right)$. We already know that $C = K_{(0,1)}$ is central. Let us consider the following subalgebra:

Definition 4.2.1 The *double composition algebra* $DC(\mathrm{Coh}(\mathbb{X}))$ is defined as the $\mathbb{C}$-subalgebra of $D\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$ generated by the following elements $\left\{[\mathcal{L}_n]^\pm, T_k, C^{\pm 1}, K^{\pm 1} \mid n \in \mathbb{Z}, k \in \mathbb{Z} \setminus \{0\}\right\}$, where $K = K_{(1,0)}$.

Now, let us prove some more relations that are similar to the one of the quantum group $U_v^{Dr}\left(A_2^{(2)}\right)$.

Proposition 4.2.2 *In* $\tilde{\mathcal{H}}(\mathrm{Coh}(\mathbb{X}))$ *it holds for all* $k,\ l \in \mathbb{Z}$*:*

$$[\mathcal{L}_{k+2}][\mathcal{L}_l] + (q^{-1} - q^2)[\mathcal{L}_{k+1}][\mathcal{L}_{l+1}] - q[\mathcal{L}_k][\mathcal{L}_{l+2}] = q[\mathcal{L}_l][\mathcal{L}_{k+2}] + (q^2 - q^{-1})[\mathcal{L}_{l+1}][\mathcal{L}_{k+1}] - [\mathcal{L}_{l+2}][\mathcal{L}_k]$$

Here we use the following lemma:

Lemma 4.2.3 *We have the following identities for the power sums in Proposition 3.4.1:*

(i) $A(w) - \alpha_2 w^2 - \alpha_1 q - \alpha_0 + (q - q^{-1})(A(w) - \alpha_1 w - \alpha_0)w - q^5(A(w) - \alpha_0)w^2 = 0$

(ii) $B(w) - \beta_2 w^2 - \beta_1 q - \beta_0 + (q - q^{-1})(B(w) - \beta_1 w - \beta_0)w - q^5(B(w) - \beta_0)w^2 = 0$

(iii) $C^{(l)}(w) - \gamma_2^{(l)} w^2 - \gamma_1^{(l)} q - \gamma_0^{(l)} + (q - q^{-1})(C^{(l)}(w) - \gamma_1^{(l)} w - \gamma_0^{(l)})w - q^5(C^{(l)}(w) - \gamma_0^{(l)})w^2 = 0$

Proof. Direct calculations. □

Proof of Proposition 4.2.2. We reformulate the equation to:

$$\begin{aligned}([\mathcal{L}_{k+2}][\mathcal{L}_l] + [\mathcal{L}_{l+2}][\mathcal{L}_k]) + (q^{-1} - q^2)([\mathcal{L}_{k+1}][\mathcal{L}_{l+1}] + [\mathcal{L}_{l+1}][\mathcal{L}_{k+1}]) \\ -q([\mathcal{L}_k][\mathcal{L}_{l+2}] + [\mathcal{L}_l][\mathcal{L}_{k+2}]) \quad = \quad 0\end{aligned}$$

We need to consider the cases $k = l$, $k = l+1$, $k = l+2$ and $k > l+2$. Because of the symmetry in the equation the cases $k < l$ follow.

Due to the Auslander-Reiten translation, it is enough to show the equation for $l = 0$. Applying τ alternatively τ^- often enough, we get the equation for an arbitrary l.

- Case $k = l = 0$:

$$\begin{aligned} & [\mathcal{L}_2][\mathcal{L}_0] + [\mathcal{L}_2][\mathcal{L}_0] + (q^{-1} - q^2)([\mathcal{L}_1][\mathcal{L}_1]^+ + [\mathcal{L}_1][\mathcal{L}_1]) - q([\mathcal{L}_0][\mathcal{L}_2] + [\mathcal{L}_0][\mathcal{L}_2]) \\ = \quad & 2 \cdot (q^{\frac{1+2\cdot(-2)}{2}}(q^{2\cdot 2+1}[\mathcal{L}_2 \oplus \mathcal{L}_0] + (q^2-1)\frac{q^4-q}{q-1}[\mathcal{L}_1^{\oplus 2}]) \\ & +(q^{-1} - q^2)q^{\frac{1}{2}}(q+1)[\mathcal{L}_1^{\oplus 2}] - q \cdot q^{\frac{1+2\cdot 2}{2}}[\mathcal{L}_2 \oplus \mathcal{L}_0]) \\ = \quad & 2 \cdot (\underbrace{(q^{-\frac{3}{2}} \cdot q^5 - q \cdot q^{\frac{5}{2}})}_{=0}[\mathcal{L}_2 \oplus \mathcal{L}_0] + \underbrace{(q^{-\frac{3}{2}}(q^2-1)q(q^2+q+1) + (q^{-1}-q^2)q^{\frac{1}{2}}(q+1))}_{=0}[\mathcal{L}_1^{\oplus 2}]) \\ = \quad & 0 \end{aligned}$$

- Case $k = l+1 = 1$:

$$\begin{aligned} & [\mathcal{L}_3][\mathcal{L}_0] + [\mathcal{L}_2][\mathcal{L}_1] + (q^{-1} - q^2)([\mathcal{L}_2][\mathcal{L}_1] + [\mathcal{L}_1][\mathcal{L}_2]) - q([\mathcal{L}_1][\mathcal{L}_2] + [\mathcal{L}_0][\mathcal{L}_3]) \\ = \quad & q^{\frac{1+2\cdot(-3)}{2}}(q^{2\cdot 3+1}[\mathcal{L}_0 \oplus \mathcal{L}_3] + q^{2+1}(q^4+q^3-q-1)[\mathcal{L}_1 \oplus \mathcal{L}_2] + \frac{q^4-1}{q-1}(q^4-q^2-q)[\mathcal{M}_2]) \\ & +q^{\frac{1+2\cdot(-1)}{2}}(q^{2\cdot 1+1}[\mathcal{L}_1 \oplus \mathcal{L}_2] + \frac{q^4-1}{q-1}[\mathcal{M}_2]) \\ & +(q^{-1}-q^2)(q^{\frac{1+2\cdot(-1)}{2}}(q^{2\cdot 1+1}[\mathcal{L}_1 \oplus \mathcal{L}_2] + \frac{q^4-1}{q-1}[\mathcal{M}_2]) + q^{\frac{1+2\cdot 1}{2}}[\mathcal{L}_1 \oplus \mathcal{L}_2]) \\ & -q(q^{\frac{1+2\cdot 1}{2}}[\mathcal{L}_1 \oplus \mathcal{L}_2] + q^{\frac{1+2\cdot 3}{2}}[\mathcal{L}_0 \oplus \mathcal{L}_3]) \\ = \quad & (q^{-\frac{5}{2}} \cdot q^7 - q \cdot q^{\frac{7}{2}})[\mathcal{L}_0 \oplus \mathcal{L}_3] \\ & +(q^{-\frac{5}{2}} \cdot q^{2+1}(q^4+q^3-q-1) + q^{-\frac{1}{2}} \cdot q^3 + (q^{-1}-q^2)(q^{-\frac{1}{2}} \cdot q^3 + q^{\frac{3}{2}}) - q \cdot q^{\frac{3}{2}})[\mathcal{L}_1 \oplus \mathcal{L}_2] \\ & +(q^{-\frac{5}{2}} \cdot \frac{q^4-1}{q-1}(q^4-q^2-q) + q^{-\frac{3}{2}}\frac{q^4-1}{q-1} + (q^{-1}-q^2)q^{-\frac{3}{2}}\frac{q^4-1}{q-1})[\mathcal{M}_2] \\ = \quad & 0 \end{aligned}$$

- Case $k = l+2 = 2$: Analogously.

- Case $k > 2,\ l = 0$:

$$
\begin{aligned}
& ([\mathcal{L}_{k+2}][\mathcal{L}_0] + [\mathcal{L}_2][\mathcal{L}_k]) + (q^{-1} - q^2)([\mathcal{L}_{k+1}][\mathcal{L}_1] + [\mathcal{L}_1][\mathcal{L}_{k+1}]) \\
& -q([\mathcal{L}_k]^+[\mathcal{L}_2]^+ + [\mathcal{L}_0]^+[\mathcal{L}_{k+2}]^+) \\
= \; & q^{\frac{1+2(-k-2)}{2}}(q^{2(k+2)+1}[\mathcal{L}_{k+2} \oplus \mathcal{L}_0] + \sum_{a=1}^{\lfloor \frac{k+1}{2} \rfloor} (q-1)^{-1}\gamma_a^{(k+2-2a)}[\mathcal{L}_{k+2-a} \oplus \mathcal{L}_a] \\
& \quad + I_{2\mathbb{Z}}(k)(q-1)^{-1}\alpha_{\frac{k+2}{2}}[\mathcal{L}_{\frac{k+2}{2}} \oplus \mathcal{L}_{\frac{k+2}{2}}] + I_{2\mathbb{Z}+1}(k)(q-1)^{-1}\beta_{\frac{k+1}{2}}[\mathcal{M}_{\frac{k+3}{2}}]) \\
& + q^{\frac{1+2(k-2)}{2}}[\mathcal{L}_k \oplus \mathcal{L}_2] \\
& + (q^{-1} - q^2)q^{\frac{1+2(-k)}{2}}(q^{2k+1}[\mathcal{L}_{k+1} \oplus \mathcal{L}_1] + \sum_{a=1}^{\lfloor \frac{k-1}{2} \rfloor} (q-1)^{-1}\gamma_a^{(k-2a)}[\mathcal{L}_{k-a} \oplus \mathcal{L}_a] \\
& \quad + I_{2\mathbb{Z}}(k)(q-1)^{-1}\alpha_{\frac{k}{2}}[\mathcal{L}_{\frac{k+2}{2}} \oplus \mathcal{L}_{\frac{k+2}{2}}]^+ + I_{2\mathbb{Z}+1}(k)(q-1)^{-1}\beta_{\frac{k-1}{2}}[\mathcal{M}_{\frac{k+3}{2}}]) \\
& + (q^{-1} - q^2)q^{\frac{1+2k}{2}}[\mathcal{L}_{k+1} \oplus \mathcal{L}_1] \\
& - q \cdot q^{\frac{1+2(-k+2)}{2}}(q^{2(k-2)+1}[\mathcal{L}_k \oplus \mathcal{L}_2] + \sum_{a=1}^{\lfloor \frac{k-3}{2} \rfloor} (q-1)^{-1}\gamma_a^{(k-2-2a)}[\mathcal{L}_{k-2-a} \oplus \mathcal{L}_{2+a}] \\
& \quad + I_{2\mathbb{Z}}(k)(q-1)^{-1}\alpha_{\frac{k-2}{2}}[\mathcal{L}_{\frac{k+2}{2}} \oplus \mathcal{L}_{\frac{k+2}{2}}] + I_{2\mathbb{Z}+1}(k)(q-1)^{-1}\beta_{\frac{k-3}{2}}[\mathcal{M}_{\frac{k+3}{2}}]) \\
& - q \cdot q^{\frac{1+2(k+2)}{2}}[\mathcal{L}_{k+2} \oplus \mathcal{L}_0] \\
= \; & \left(q^{-k-\frac{3}{2}} \cdot q^{2k+5} - q^{k+\frac{7}{2}}\right)[\mathcal{L}_{k+2} \oplus \mathcal{L}_0] \\
& + \left(q^{-k-\frac{3}{2}}\frac{\gamma_1^{(k)}}{q-1} + (q^{-1} - q^2)q^{k+\frac{3}{2}} + (q^{-1} - q^2)q^{k+\frac{1}{2}}\right)[\mathcal{L}_{k+1} \oplus \mathcal{L}_1] \\
& + \left(q^{-k-\frac{3}{2}}\frac{\gamma_2^{(k-2)}}{q-1} + q^{k-\frac{3}{2}} + (q^{-1} - q^2)q^{-k+\frac{1}{2}}\frac{\gamma_1^{(k-2)}}{q-1} - q^{k+\frac{1}{2}}\right)[\mathcal{L}_k \oplus \mathcal{L}_2] \\
& + \sum_{a=1}^{\lfloor \frac{k-3}{2} \rfloor}\left(q^{-k-\frac{3}{2}}\frac{\gamma_{2+a}^{(k-2+2a)}}{q-1} + (q^{-1} - q^2)q^{-k+\frac{1}{2}}\frac{\gamma_{a+1}^{(k-2-2a)}}{q-1} - q \cdot q^{-k+\frac{5}{2}}\frac{\gamma_a^{(k-2-2a)}}{q-1}\right)[\mathcal{L}_{k-a} \oplus \mathcal{L}_{a+2}] \\
& + I_{2\mathbb{Z}}(k)\left(q^{-k-\frac{3}{2}}\frac{\alpha_{\frac{k+2}{2}}}{q-1} + (q^{-1} - q^2)q^{-k+\frac{1}{2}}\frac{\alpha_{\frac{k}{2}}}{q-1} - q \cdot q^{-k+\frac{5}{2}}\frac{\alpha_{\frac{n-2}{2}}}{q-1}\right)[\mathcal{L}_{\frac{k+2}{2}} \oplus \mathcal{L}_{\frac{k+2}{2}}] \\
& + I_{2\mathbb{Z}+1}(k)\left(q^{-k-\frac{3}{2}}\frac{\beta_{\frac{k+1}{2}}}{q-1} + (q^{-1} - q^2)q^{-k+\frac{1}{2}}\frac{\beta_{\frac{k-1}{2}}}{q-1} - q \cdot q^{-k+\frac{5}{2}}\frac{\beta_{\frac{k-3}{2}}}{q-1}\right)[\mathcal{M}_{\frac{k+3}{2}}]
\end{aligned}
$$

So we need to check, that the factors are zero.

The first three can be checked easily with Example 3.4.3:

- $q^{-k-\frac{3}{2}} \cdot q^{2k+5} - q^{k+\frac{7}{2}} = 0$
- $q^{-k-\frac{3}{2}}\frac{\gamma_1^{(k)}}{q-1} + (q^{-1} - q^2)q^{k+\frac{3}{2}} + (q^{-1} - q^2)q^{k+\frac{1}{2}}$
  $= q^{-k-\frac{3}{2}}q^{2k+1}(q^4 + q^3 - q - 1) + (q^{-1} - q^2)q^{k+\frac{3}{2}} + (q^{-1} - q^2)q^{k+\frac{1}{2}}$
  $= q^k(q^{\frac{7}{2}} + q^{\frac{5}{2}} - q^{\frac{1}{2}} - q^{-\frac{1}{2}} + q^{\frac{1}{2}} - q^{\frac{7}{2}} + q^{-\frac{1}{2}} - q^{\frac{5}{2}})$
  $= 0$

$$
\begin{aligned}
&- q^{-k-\frac{3}{2}} \frac{\gamma_2^{(k-2)}}{q-1} + q^{k-\frac{3}{2}} + (q^{-1}-q^2)q^{-k+\frac{1}{2}} \frac{\gamma_1^{(k-2)}}{q-1} - q^{k+\frac{1}{2}} \\
&= q^k(q^{-\frac{3}{2}}q^{-3}(q^8+q^7-q^5-2q^4-q^3+q^2+q)+q^{-\frac{3}{2}}+(q^{-1}-q^2)q^{\frac{1}{2}}q^{-3}(q^4+q^3-q-1)-q^{\frac{1}{2}}) \\
&= q^k(q^{\frac{7}{2}}+q^{\frac{5}{2}}-q^{\frac{1}{2}}-2q^{-\frac{1}{2}}-q^{-\frac{3}{2}}+q^{-\frac{5}{2}}+q^{-\frac{7}{2}}+q^{\frac{1}{2}}+q^{-\frac{1}{2}}-q^{-\frac{5}{2}}-q^{-\frac{7}{2}}-q^{\frac{7}{2}}-q^{\frac{5}{2}} \\
&\qquad +q^{\frac{1}{2}}+q^{-\frac{1}{2}}-q^{\frac{1}{2}}) \\
&= 0
\end{aligned}
$$

For the other three, we need to apply Lemma 4.2.3.

To prove $q^{-k-\frac{3}{2}} \frac{\gamma_{2+a}^{(k-2+2a)}}{q-1} + (q^{-1}+q^2)q^{-k+\frac{1}{2}} \frac{\gamma_{a+1}^{(k-2-2a)}}{q-1} - q \cdot q^{-k+\frac{5}{2}} \frac{\gamma_a^{(k-2-2a)}}{q-1} = 0$ for all $a > 0$, $k > 2$, it is equivalent to show:

$$
q^{-\frac{3}{2}} \frac{\gamma_{2+a}^{(r)}}{q-1} + (q^{-1}+q^2)q^{\frac{1}{2}} \frac{\gamma_{a+1}^{(r)}}{q-1} - q \cdot q^{\frac{5}{2}} \frac{\gamma_a^{(r)}}{q-1} = 0 \qquad \text{for all } r \in \mathbb{N},\ a > 0.
$$

Multiplying the left side with w^{a+2} and adding up over all a, this is equivalent to:

$$
\begin{aligned}
0 &= \sum_{a=1}^{\infty} \left(q^{-\frac{3}{2}} \frac{\gamma_{2+a}^{(r)}}{q-1} + (q^{-1}-q^2)q^{\frac{1}{2}} \frac{\gamma_{a+1}^{(r)}}{q-1} - q \cdot q^{\frac{5}{2}} \frac{\gamma_a^{(r)}}{q-1} \right) w^{a+2} \\
&= q^{-\frac{3}{2}} \left(\sum_{a=1}^{\infty} \frac{\gamma_{2+a}^{(r)}}{q-1} w^{a+2} \right) + (q^{\frac{1}{2}} - q^{\frac{5}{2}}) \left(\sum_{a=1}^{\infty} \frac{\gamma_{a+1}^{(r)}}{q-1} w^{a+1} \right) w - q^{\frac{7}{2}} \left(\sum_{a=1}^{\infty} \frac{\gamma_a^{(r)}}{q-1} w^a \right) w^2 \\
&= q^{-\frac{3}{2}}(C^{(r)}(w) - \gamma_2^{(r)}w^2 - \gamma_1^{(r)}q - \gamma_0^{(r)} + (q-q^{-1})(C^{(r)}(w) - \gamma_1^{(r)}w - \gamma_0^{(r)})w \\
&\qquad -q^5(C^{(r)}(w) - \gamma_0^{(r)})w^2)
\end{aligned}
$$

This is the identity (iii) in Lemma 4.2.3.

Similarly, one can show the other two identities.

□

Hence we get the following result:

Proposition 4.2.4 *There is a surjective* $\mathbb{C}$*-algebra morphism* $\Phi : U_v^{Dr}\left(A_2^{(2)}\right) \to DC(\mathrm{Coh}(\mathbb{X}))$, *given on the generators by:*

$$
X_n^+ \mapsto [\mathcal{L}_n]^+,\ X_n^- \mapsto -v[\mathcal{L}_{-n}]^-,\ H_r \mapsto T_r,\ K \mapsto K,\ C \mapsto C \qquad \textit{for } n \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\}.
$$

Proof. Since we are defining Φ on the generators, we need to show that the images of the generators satisfy the relations of $U_v^{Dr}\left(A_2^{(2)}\right)$. Then the map Φ is well-defined. By using the fact that the images also generate $DC(\mathrm{Coh}(\mathbb{X}))$ as an algebra by definition, the surjectivity immediately follows.

We have already proven most of the relations:

(CU) Remark 3.1.8;

(CK) by definition;

$(KX^{\pm})$ follows from the definition of the extended Hall algebra (1.1.9)/reduced Drinfeld double of the Hall algebra $D\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$ (1.4.1):

$$K[\mathcal{L}_n]^{\pm}K = q^{\pm\frac{1}{2}((1,0),(1,n))}[\mathcal{L}_n]^{\pm} = q^{\pm 1}[\mathcal{L}_n]^{\pm} \qquad \text{for all } n \in \mathbb{Z};$$

(KH) follows as above from the definitions:

$$KT_rK = q^{\pm\frac{1}{2}((1,0),(0,r))}T_r = T_r \qquad \text{for all } r \in \mathbb{Z}\setminus\{0\};$$

(XX) Consider Corollary 3.3.7: For $n, m \in \mathbb{Z}$ it holds:

$$[[\mathcal{L}_n]^+, [\mathcal{L}_m]^-] = -q^{-\frac{1}{2}}\frac{K_{(1,m)}\theta^+_{n-m} - K_{(-1,-n)}\theta^-_{n-m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

In particular, we get

$$\begin{aligned}\Phi([X_n^+, X_m^-]) &= -v[[\mathcal{L}_n]^+, [\mathcal{L}_{-m}]^-] \\ &= \frac{K_{(1,-m)}\theta^+_{n+m} - K_{(-1,-n)}\theta^-_{n+m}}{v - v^{-1}} \\ &= \Phi\left(\frac{C^{-m}K\psi^+_{n+m} - C^{-n}K^{-1}\psi^-_{n+m}}{v - v^{-1}}\right);\end{aligned}$$

$(HX^{\pm})$ Proposition 3.3.13 and Corollary 3.3.15;

(HH) follows from Corollary 3.3.12;

$(X2^{\pm})$ Proposition 4.2.2 for the positive half, note for the negative half for $k, l \in \mathbb{Z}$:

$$\begin{aligned}&\Phi\left(X^-_{k+2}X^-_l + X^-_{l+2}X^-_k + (q-q^{-2})(X^-_{k+1}X^-_{l+1} + X^-_{l+1}X^-_{k+1}) - q^{-1}(X^-_kX^-_{l+2} + X^-_lX^-_{k+2})\right)\\
&= q\left([\mathcal{L}_{-k-2}]^-[\mathcal{L}_{-l}]^- + [\mathcal{L}_{-l-2}]^-[\mathcal{L}_{-k}]^-\right.\\
&\qquad +(q-q^{-2})([\mathcal{L}_{-k-1}]^-[\mathcal{L}_{-l-1}]^- + [\mathcal{L}_{-l-1}]^-[\mathcal{L}_{-k-1}]^-)\\
&\qquad \left.-q^{-1}([\mathcal{L}_{-k}]^-[\mathcal{L}_{-l-2}]^- + [\mathcal{L}^-_{-l}[\mathcal{L}_{-k-2}]^-)\right)\\
&= q([\mathcal{L}_{-k-2}]^-[\mathcal{L}_{-l}]^- + [\mathcal{L}_{-l-2}]^-[\mathcal{L}_{-k}]^-)\\
&\qquad +(q^2-q^{-1})([\mathcal{L}_{-k-1}]^-[\mathcal{L}_{-l-1}]^- + [\mathcal{L}_{-l-1}]^-[\mathcal{L}_{-k-1}]^-)\\
&\qquad -([\mathcal{L}_{-k}]^-[\mathcal{L}_{-l-2}]^- + [\mathcal{L}^-_{-l}[\mathcal{L}_{-k-2}]^-)\\
&= -\left([\mathcal{L}_{-k}]^-[\mathcal{L}_{-l-2}]^- + [\mathcal{L}_{-l}]^-[\mathcal{L}_{-k-2}]^-\right.\\
&\qquad +(q^{-1}-q^2)([\mathcal{L}_{-k-1}]^-[\mathcal{L}_{-l-1}]^- + [\mathcal{L}_{-l-1}]^-[\mathcal{L}_{-k-1}]^-)\\
&\qquad \left.-q([\mathcal{L}_{-k-2}]^-[\mathcal{L}_{-l}]^- + [\mathcal{L}_{-l-2}]^-[\mathcal{L}_{-k}]^-)\right)\\
&\overset{4.2.2}{=} 0;\end{aligned}$$

$(X3^{-,\pm})$ follows from $(X3_c^{+,+})$ and $(X3_c^{-,-})$;

$(X3^{+,\pm})$ follows from $(X3_c^{+,+})$ and $(X3_c^{-,-})$. By Remark 4.1.2 we write $(X3_c^{+,+})$ as

$$[[X^+_{r+1}, X^+_r]_{v^2}, X^+_r]_{v^4} = 0 \qquad \text{for all } r \in \mathbb{Z}.$$

By using the Auslander-Reiten translation τ and its inverse τ^-, this follows from our calculation in Example 3.2.1 where the relation $(X3_c^{+,+})$ is proven for $r = -1$. Similarly one gets $(X3_c^{-,-})$, namely for the two halves we have the same relation in the composition algebra but the map Φ also switches the indices hence we have

$$\begin{aligned}\Phi([[X_{r+1}^-, X_r^-]_{v^{-2}}, X_r^-]_{v^{-4}}) &= v^3[[[\mathcal{L}_{-r-1}]^-, [\mathcal{L}_{-r}]^-]_{v^{-2}}, [\mathcal{L}_{-r}]^-]_{v^{-4}} \\ &= p(v)[[\mathcal{M}_{-r-1}]^-, [\mathcal{L}_{-r}]^-]_{v^{-4}} \\ &= p(v)(v^4[\mathcal{M}_{-r-1}\oplus\mathcal{L}_{-r}]^- - v^{-4}q^4[\mathcal{M}_{-r-1}\oplus\mathcal{L}_{-r}]^-) \\ &= 0\end{aligned}$$

for some function p.

□

4.3 The Isomorphism between $U_v^{Dr}\left(A_2^{(2)}\right)$ and $DC(\mathrm{Coh}(\mathbb{X}))$

From the Proposition 4.2.4 in the previous section, we already have a surjective map

$$\Phi : U_v^{Dr}\left(A_2^{(2)}\right) \twoheadrightarrow DC(\mathrm{Coh}(\mathbb{X})).$$

Hence it remains to be shown that this map is also injective.

Remark 4.3.1 Using the known basis of the Hall algebra $\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$ and our knowledge of the Auslander-Reiten quiver we get the following basis of the double composition algebra $DC(\mathrm{Coh}(\mathbb{X}))$: A basis element in $\mathcal{H}(\mathrm{Coh}(\mathbb{X}))$ is of the form

$$[\underbrace{\mathcal{L}_i^{\oplus n_1}\oplus\mathcal{M}_{i+1}^{\oplus m_1}\oplus\mathcal{L}_{i+1}^{\oplus n_2}\oplus ...\oplus\mathcal{M}_j^{\oplus m_{j-i}}}_{\text{vector bundle part}}\oplus\underbrace{\mathcal{T}_{r,z}^{\oplus p_{r,z}}\oplus ...\oplus\mathcal{T}_{s,y}^{\oplus p_{s,y}}}_{\text{torsion part}}]$$

where $n_1, ..., n_{j-i}, m_1, ..., m_{j-i}, p_{r,z}, ..., r, ..., s \in \mathbb{N}_0$, $i, j \in \mathbb{Z}$, $y, ..., z \in \mathbb{X}$, written as a direct sum of indecomposables.

In $DC(\mathrm{Coh}(\mathbb{X}))$ we get the indecomposable rank 2 vector bundles by using the q-commutator. However, we do not get all of the torsion modules, but only "direct sums of the T_r for $r \in \mathbb{Z}\backslash\{0\}$", where we set $T_{-r} = T_r^-$ for $r \in \mathbb{N}$. This follows from the fact that Φ in Proposition 4.2.4 is surjective.

Since there are only trivial extensions $M \to M\oplus N \to N$ if M is to the right of N in the Auslander-Reiten quiver, we get as a basis written as products in the generators of the double composition algebra elements of the form:

$$\begin{gathered}[\mathcal{L}_i]^{+n_1}\cdot[[\mathcal{L}_{i+1}]^+, [\mathcal{L}_i]^+]_q^{b_1}\cdot[\mathcal{L}_{i+1}]^{+n_2}\cdot ...\cdot[[\mathcal{L}_{j+1}]^+, [\mathcal{L}_j]^+]_q^{b_{j-i}}\cdot \\ T_{r_1}^{t_1}\cdot ...\cdot T_{r_s}^{t_s}\cdot K^k\cdot C^c\cdot \\ [[\mathcal{L}_x]^-, [\mathcal{L}_{x-1}]^-]_{q^{-1}}^{d_1}\cdot[\mathcal{L}_{x-1}]^{-m_1}\cdot ...\cdot[\mathcal{L}_y]^{-m_{x-y}}\end{gathered} \tag{9}$$

for $i, j, x, y \in \mathbb{Z}$, $n_1, ..., n_{j-i}, m_1, ..., m_{x-y}, b_1, ..., d_1, ..., d_{x-y} \in \mathbb{N}_0$ with $n_1 + b_1 \neq 0 \neq n_{j-i} + b_{j-i}$, $r_1 < r_2 < ... < r_s \in \mathbb{Z}$, $t_1, ..., t_s \in \mathbb{N}, k, c \in \mathbb{Z}$.

Lemma 4.3.2 *Any element in $U_v^{Dr}\left(A_2^{(2)}\right)$ may be written as a $\mathbb{C}$-linear combination of elements of the form*

$$X_i^{+l_i}[X_{i+1}^+, X_i^+]_q^{b_i}X_{i+1}^{+l_{i+1}}[X_{i+2}^+, X_{i+1}^+]_q^{b_{i+1}}...X_j^{+l_j}\cdot H_{g_1}^{h_1}...H_{g_r}^{h_r}\cdot K^kC^c$$

$$\cdot X_n^{-m_n}[X_{n-1}^-, X_n^-]_q^{a_n} X_{n-1}^{-m_{n-1}}[X_{n-2}^-, X_{n-1}^-]_q^{a_{n-1}} ... X_o^{-m_o}, \tag{10}$$

where $g_1 < g_2 < g_3 < ... < g_r$*,* $i < j$*,* $n > o$*,* $l_i, b_i, ..., b_j, l_j, h_1, ..., h_r, m_n, a_n, ..., a_o, m_o \in \mathbb{N}_0$*,* $k, c \in \mathbb{Z}$*.*

Proof. The basic idea is, given a word in the generators, one can use the relations of $U_v^{Dr}\left(A_2^{(2)}\right)$ to write it as a linear combination of elements of the above form.

In more detail:

Every element in $U_v^{Dr}\left(A_2^{(2)}\right)$ is a linear combination of words in the generators. Hence we show that any word can be written as a linear combination of elements of the above form instead of taking a general element. We do this by following these steps:

- We pull all letters X_n^- to the right:
  - Since C is central, we can pull X_n^- past C-factors.
  - Using (KX^-) we can pull X_n^- past K-factors and get factors of v.

    $$X_n^- \cdot K^r = v^{2r} K^r \cdot X_n^- \qquad \text{for } r \in \mathbb{Z}.$$

  - Using (XX) we can pull X_n^- past X_m^+ by generating additionally new summands without X_n^- and X_m^+ but H_r's.

    $$X_n^- X_m^+ = X_m^+ X_n^- + \frac{C^{-n}K^{-1}\psi_{n+m}^- - C^{-m}K\psi_{n+m}^+}{v - v^{-1}},$$

    where the $\psi^\pm$'s are given by the generating series $\sum_{k=0}^\infty \psi_{\pm k}^\pm u^k = \exp\left(\pm(v - v^{-1})\sum_{l=1}^\infty H_{\pm l} u^l\right)$.
  - Using (HX^-) we can pull X_n^- past H_r by generating an additional summand without X_n^- and H_r but with X_{n+r}^-.

    $$X_n^- H_r = H_r X_n^- - \frac{1}{r}[2r]_v(v^{2r} + v^{-2r} + (-1)^{r+1})X_{n+r}^-.$$

  Hence we can write any word in the generators as a linear combination of words where all X_n^--factors are on the very right.

  So we now consider only words in the form AB where A is a word in the generators X_i^+'s, H_r's, $K^{\pm 1}$ and $C^{\pm 1}$ and B is a word only in the generators X_n^-'s.
- We pull all letters with X_i^+ to the left:

  So we want to order the word A. Similar to the X_n^- case, we use the relations:
  - Since C is central, we can pull X_i^+ past C-factors.
  - Using (KX^+) we can pull X_i^+ past K-factors and get factors of v.

    $$X_i^+ \cdot K^r = v^{-2r} K^r \cdot X_i^+ \qquad \text{for } r \in \mathbb{Z}.$$

  - Using (HX^+) we can pull X_i^+ past H_r by generating an additional summand without X_i^+ and H_r but with X_{i+r}^+.

    $$X_i^+ H_r = H_r X_i^+ + \frac{1}{r}[2r]_v(v^{2r} + v^{-2r} + (-1)^{r+1})X_{i+r}^+.$$

Hence any element can be written as a linear combination of words of the form $CA'B$ where C is a word in X_i^+'s, A' is a word in the generators H_r's, $K^{\pm 1}$ and $C^{\pm 1}$ and B is a word only in the generators X_n^-'s.

- Now we want to bring more order to each part of the word.
  - The first part, the word in X_i^+'s:
    The idea is to use the relations $(X2^+)$ and $(X3^{\pm,+})$ to get the required form.
    Using $(X2^+)$ we can reduce the distance between the indices, i.e. $X_k^+X_l^+$ can be written as:
    $$X_k^+X_l^+ = (v^4-v^{-2})X_{k-1}^+X_{l+1}^+ + v^2X_{k-2}^+X_{l+2}^+ + v^2X_l^+X_k^+ + (v^4-v^{-2})X_{l+1}^+X_{k-1}^+ - X_{l+2}^+X_{k-2}^+$$
    If $k > l+2$ then the indices are closer together and some even in the right order where the left index is smaller than the right one in a product.
    In the case $k = l+2$ we have by pulling the last summand to the left-hand side and dividing by 2:
    $$X_k^+X_{k-2}^+ = (v^4-v^{-2})X_{k-1}^{+2} + v^2X_{k-2}^+X_k^+$$
    Hence both products are in the required form.
    Now we need to consider the case $X_{k+1}^{+n}X_k^{+m}$ with $n, m \in \mathbb{N}$. Using the relation $(X2^+)$ we would not gain anything. But we can use $(X3^{\pm,+})$ to pull at least some to the right side, worst case we get something of the form $(X_{k+1}^+X_k^+)^a$.
    We do induction on $n+m$:
    1. Case $n+m=2$: We get directly
       $$X_{k+1}^+X_k^+ = v^{-2}X_k^+X_{k+1}^+ - v^{-2}[X_k^+, X_{k+1}^+]_{v^2},$$
       which is the required form.
    2. Case $n+m>2$:
       * $m=1,\ n>1$:
         $$\begin{aligned} X_{k+1}^{+n}X_k^+ &= X_{k+1}^{+n-2}X_{k+1}^+X_{k+1}^+X_k^+ \\ &\overset{(X3^{-,+})}{=} X_{k+1}^{+n-2}(-v^6X_k^+X_{k+1}^+X_{k+1}^+ + (v^4+v^2)X_{k+1}^+X_k^+X_{k+1}^+) \\ &= -v^6\underbrace{X_{k+1}^{+n-2}X_k^+}_{=:\star 1}X_{k+1}^+X_{k+1}^+ + (v^4+v^2)\underbrace{X_{k+1}^{+n-1}X_k^+}_{=:\star 2}X_{k+1}^+ \end{aligned}$$
         $\star 1$ and $\star 2$ can be written by induction in the required form, hence this whole section can be written in such a way.
       * $m>1,\ n\geq 1$:
         $$\begin{aligned} X_{k+1}^{+n}X_k^{+m} &= X_{k+1}^{+n-1}X_{k+1}^+X_k^+X_k^+X_k^{+m-2} \\ &\overset{(X3^{+,+})}{=} X_{k+1}^{+n-1}((v^4+v^2)X_k^+X_{k+1}^+X_k^+ - v^6X_k^+X_k^+X_{k+1}^+)X_k^{+m-2} \\ &= \ldots \qquad \text{'apply } (X3^{+,+})\ (m-1)\text{-times'} \\ &= X_{k+1}^{+n-1}(\lambda X_k^{+m-1}X_{k+1}^+X_k^+ + \mu X_k^{+m}X_{k+1}^+) \\ &= \lambda\underbrace{X_{k+1}^{+n-1}X_k^{+m-1}X_{k+1}^+X_k^+}_{=:\star 3} + \mu\underbrace{X_{k+1}^{+n-1}X_k^{+m}}_{=:\star 4}X_{k+1}^+ \end{aligned}$$

for some $\lambda, \mu \in \mathbb{C}$.
On $\star 4$ we can use the induction hypothesis. For $\star 3$ either $n = 1$ and we can apply the first case on the two right factors $X_{k+1}^+ X_k^+$ and are done.
Also note:

$$\begin{aligned} X_{k+1}^+[X_k^+, X_{k+1}^+]_{v^2} &= X_{k+1}^+ X_k^+ X_{k+1}^+ - v^2 X_{k+1}^{+2} X_k^+ \\ &= X_{k+1}^+ X_k^+ X_{k+1}^+ + v^8 X_k^+ X_{k+1}^{+2} - (v^6 + v^4) X_{k+1}^+ X_k^+ X_{k+1}^+ \\ &= (1 - v^6 - v^4) X_{k+1}^+ X_k^+ X_{k+1}^+ + v^8 X_k^+ X_{k+1}^{+2} \end{aligned} \tag{11}$$

The first summand can be written as claimed by applying the first case on the first two factors.
In particular using the induction hypothesis on $X_{k+1}^{+n-1} X_k^{+m-1}$ in $\star 3$ we are considering linear combinations of

$$\begin{aligned} X_k^{+a}[X_k^+, X_{k+1}^+]_{v^2}^b X_{k+1}^{+c} X_{k+1}^+ X_k^+ &= -v^{-2} X_k^{+a}[X_k^+, X_{k+1}^+]_{v^2}^b X_{k+1}^{+c}[X_k^+, X_{k+1}^+]_{v^2} \\ &\quad + v^{-2} X_k^{+a}[X_k^+, X_{k+1}^+]^b X_{k+1}^{+c} X_k^+ X_{k+1}^+, \end{aligned}$$

where $a + b = m - 1$, $b + c = n - 1$. The first summand can be reordered by equation (11) and the last one by using the induction hypothesis on all but the last factor X_{k+1}^+.

In total, we can write any word in X_i^+'s as a linear combination of words of the form

$$X_{i_1}^{+n_1}[X_{i_2}^+, X_{i_2+1}^+]_{v^2}^{n_2} X_{i_3}^{+n_3} \dots [X_{i_r}^+, X_{i_r+1}^+]_{v^2}^{n_r},$$

where $i_1 \leq i_2 < i_3 \leq i_4 < \dots \leq i_r$ and $n_1, n_2, \dots, n_r \in \mathbb{N}_0$.

– The word in X_i^-'s works analogously to the X_i^+'s case due to the symmetry of the relations.
Hence we can write any word in X_i^-'s as a linear combination of words of the form

$$X_{i_1}^{-n_1}[X_{i_2}^-, X_{i_2-1}^-]_{v^{-2}}^{n_2} X_{i_3}^{-n_3} \dots [X_{i_r}^-, X_{i_r-1}^-]_{v^{-2}}^{n_r},$$

where $i_1 \geq i_2 > i_3 \geq i_4 > \dots \geq i_r$ and $n_1, n_2, \dots, n_r \in \mathbb{N}_0$.

– Since the H_r commute among themselves and with K and C, we may order the middle word in any way. In particular we can write any word in the generators H_r's, K and C in the form:

$$H_{g_1}^{n_1} H_{g_2}^{n_2} \dots H_{g_r}^{n_r} K^k C^c,$$

where $g_1 < g_2 < \dots . < g_r$, for some $r \in \mathbb{N}_0$, $n_1, \dots, n_r \in \mathbb{N}$, $k, c \in \mathbb{Z}$.

All in all, we may order any expression in the generators as a linear combination of elements of the form (10). Hence we are done. □

Now, this finally allows us to formulate and prove the main theorem.

Theorem 4.3.3 *There is $\mathbb{C}$-algebra isomorphism $\Phi : U_v^{Dr}\left(A_2^{(2)}\right) \to DC(\mathrm{Coh}(\mathbb{X}))$, given on the generators by:*

$$X_n^+ \mapsto [\mathcal{L}_n]+,\ X_n^- \mapsto -v[\mathcal{L}_{-n}]^-,\ H_r \mapsto T_r,\ K \mapsto K,\ C \mapsto C \qquad \textit{for } n \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\}.$$

Proof. From Proposition 4.2.4 we already have a surjection $\Phi : U_v^{Dr}\left(A_2^{(2)}\right) \to DC(\mathrm{Coh}(\mathbb{X}))$ sending the generators to generators. It remains the injectivity.

Let $f \in \ker(\Phi)$. By the above Lemma 4.3.2 we may write f as a linear combination of elements of the form:

$$X_i^{+l_i}[X_{i+1}^+, X_i^+]_q^{b_i} X_{i+1}^{+l_{i+1}}[X_{i+2}^+, X_{i+1}^+]_q^{b_{i+1}} ... X_j^{+l_j} \cdot H_{g_1}^{h_1} ... H_{g_r}^{h_r} \cdot K^k C^c$$

$$\cdot X_n^{-m_n}[X_{n-1}^-, X_n^-]_q^{a_i} X_{n-1}^{-m_{n-1}}[X_{n-2}^-, X_{n-1}^-]_q^{a_{n-1}} ... X_o^{-m_o},$$

But we already know that the corresponding images are a basis in $DC(\mathrm{Coh}(\mathbb{X}))$. In particular, if $\Phi(f) = \sum \lambda ... = 0$ all pre-factors must be 0. Hence $f = 0$ and therefore $\ker(\Phi) = \{0\}$. In particular, Φ is injective.

□

4.4 Passing to $\mathbb{Q}(\tilde{v})$-Algebras

In most cases, quantum groups are defined as algebras over $\mathbb{C}(\tilde{v})$ or even $R = \mathbb{Z}[\tilde{v}, \tilde{v}^{-1}]$, where $\tilde{q}$ respectively $\tilde{v}$ is an indeterminate instead of $q \in \mathbb{C}^\times \setminus \{1\}$ respectively $v \in \mathbb{C}^\times \setminus \{\pm 1\}$. Then one may go from there via tensoring to the definition we have used. Now recall the Definition 2.6.2.

Definition 4.4.1

- The *generalized quantum group* $\tilde{U}_{\tilde{v}}(A)$ associated to a generalized Cartan matrix $A \in \mathbb{Z}^{I \times I}$ is the associative algebra over $\mathbb{Q}(\tilde{v})$ with 1 generated by the set $\{e_i, f_i, K_i^{\pm 1} \mid i \in I\}$ with relations:

  (1) $K_i \cdot K_i^{-1} = 1 = K_i^{-1} K_i$ and $K_i K_j = K_j K_i$ for all $i, j \in I$;

  (2) $K_j e_i K_j^{-1} = \tilde{v}_j^{a_{ji}} e_i$ for all $i, j \in I$;

  (3) $K_j f_i K_j^{-1} = \tilde{v}_j^{-a_{ji}} f_i$ for all $i, j \in I$;

  (4) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{\tilde{v}_i - \tilde{v}_i^{-1}}$ for all $i, j \in I$;

  (5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix} 1-a_{ij} \\ k \end{smallmatrix}\right]_i e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for all $i \neq j \in I$;

  (6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix} 1-a_{ij} \\ k \end{smallmatrix}\right]_i f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for all $i \neq j \in I$.

  where now the quantum numbers are with the parameter $\tilde{q} := \tilde{v}^2$ (respectively $\tilde{v}$) instead of q respectively v.

- Analogously, the *generealized Drinfeld representation* $\tilde{U}_{\tilde{v}}^{Dr}\left(A_2^{(2)}\right)$ the associative $\mathbb{Q}(\tilde{v})$-algebra with generators $(G) = \{X_k^{\pm},\ H_l,\ K^{\pm 1},\ C^{\pm 1} \mid k \in \mathbb{Z},\ l \in \mathbb{Z} \setminus \{0\}\}$ and the relations of Definition 4.1.1 where we replace v by $\tilde{v}$.

Remark 4.4.2 For $v \in \mathbb{C}^\times$ not a root of unity, one may get back to our original definitions by considering $\mathbb{C}$ as a $R := \mathbb{Z}[\tilde{v}, \tilde{v}^{-1}]$-module where $\tilde{v}$ acts via multiplication with v. Then it holds

$$\tilde{U}_{\tilde{v}}(A) \otimes_R \mathbb{C} \cong U_v(A) \qquad \text{and} \qquad \tilde{U}_{\tilde{v}}^{Dr}\left(A_2^{(2)}\right) \otimes_R \mathbb{C} \cong U_v^{Dr}\left(A_2^{(2)}\right)$$

as $\mathbb{C}$-algebras.

Similarly, we want to consider generalized double composition algebras. They are defined as follows:

Let $\mathcal{P} = \{p^t \mid t \in \mathbb{N}, p \text{ prime}\} \subset \mathbb{N}$. Denote by $D\mathcal{H}_q(\mathrm{Coh}(\mathbb{X}))$ respectively $D\mathcal{H}\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$ the reduced Drinfeld doubles of the $\mathbb{F}_q$-linear categories as before for $q \in \mathcal{P}$.

Definition 4.4.3 We define the generalized (reduced) Hall algebras by

$$D\mathcal{H}_{gen}(\mathrm{Coh}(\mathbb{X})) := \prod_{q\in\mathcal{P}} D\mathcal{H}_q(\mathrm{Coh}(\mathbb{X}))$$

and

$$D\mathcal{H}_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right) := \prod_{q\in\mathcal{P}} D\mathcal{H}\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right).$$

Let us consider the following elements in $D\mathcal{H}_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$:

- $\tilde{v} := (v_q)_{q\in\mathcal{P}}$, where $v_q \in D\mathcal{H}\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$ is given by the (positive) root of q;
- $\tilde{S}_i^{\pm} = ([S_i^q]^{\pm})_{q\in\mathcal{P}}$, where $[S_i^q]$ is the class of the simple object S_i in $\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ for $i \in \{1,2\}$;
- $\tilde{K}_\alpha = (K_{\alpha,q})_{q\in\mathcal{P}}$, where $K_{\alpha,q}$ is the element of the group algebra $\mathbb{C}\left(K_0\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)\right)$ corresponding to $\alpha \in K_0$.

Then $D\mathcal{H}_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ can be considered as a $\mathbb{Q}(\tilde{v})$-algebra.

Similarly, consider the following elements in $D\mathcal{H}_{gen}(\mathrm{Coh}(\mathbb{X}))$:

- $\tilde{v} := (v_q)_{q\in\mathcal{P}}$, where $v_q \in D\mathcal{H}_q(\mathrm{Coh}(\mathbb{X}))$ is given by the root of q;
- $\tilde{\mathcal{L}}_n^{\pm} := ([\mathcal{L}_n^q]^{\pm})_{q\in\mathcal{P}}$, where $[\mathcal{L}_n^q]$ is the class of the line bundle $\mathcal{L}_n$ in $\mathrm{Coh}(\mathbb{X})$ (over the field $\mathbb{F}_q$);
- $\tilde{K} := (K_q)_{q\in\mathcal{P}}$, where K_q is the element $K \in D\mathcal{H}_q(\mathrm{Coh}(\mathbb{X}))$;
- $\tilde{C} := (C_q)_{q\in\mathcal{P}}$, where C_q is the element $C \in D\mathcal{H}_q(\mathrm{Coh}(\mathbb{X}))$;

Then $D\mathcal{H}_{gen}(\mathrm{Coh}(\mathbb{X}))$ can be considered as a $\mathbb{Q}(\tilde{v})$-algebra.

In general, for Y an object, denote by $\tilde{Y}$ the element in the generalized Hall algebra where each component is the class of Y in the respective category (if Y can be considered as an element in each F_q-linear category).

Definition 4.4.4

- The *generalized double composition algebra* $DC_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ is defined as the $\mathbb{Q}(\tilde{v})$-subalgebra of the generalized Hall algebra $D\mathcal{H}_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ generated by the elements $\{\tilde{S}_i^{\pm}, K_\alpha \mid i \in \{1,2\}, \alpha \in K_0\}$.

- The *generalized double composition algebra* $DC_{gen}(\mathrm{Coh}(\mathbb{X}))$ is defined as the $\mathbb{Q}(\tilde{v})$-subalgebra of the generalized Hall algebra $D\mathcal{H}_{gen}(\mathrm{Coh}(\mathbb{X}))$ generated by the elements $\{\tilde{\mathcal{L}}_n^{\pm}, \tilde{K}, \tilde{C} \mid n \in \mathbb{Z}\}$.

Then by a result of Green (see [16], [31]) we have

$$\begin{array}{rcl} DC_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right) & \xrightarrow{\cong} & \tilde{U}_{\tilde{q}}\left(A_2^{(2)}\right), \\ \tilde{S}_1^+ & \mapsto & e_1, \\ \tilde{S}_2^+ & \mapsto & e_2, \\ \tilde{S}_1^- & \mapsto & -\tilde{v}^{-1} f_1, \\ \tilde{S}_2^- & \mapsto & -\tilde{v}^{-4} f_2, \\ K_1 & \mapsto & K_1, \\ K_2 & \mapsto & K_2 \end{array}$$

as $\mathbb{Q}(\tilde{v})$-algebras.

Similarly, by Theorem 4.3.3 and its proof, we get an isomorphism

$$\begin{array}{rcl} \tilde{U}_{\tilde{v}}^{Dr}\left(A_2^{(2)}\right) & \xrightarrow{\cong} & DC_{gen}(\mathrm{Coh}(\mathbb{X})), \\ X_i^+ & \mapsto & \tilde{\mathcal{L}}_i^+, \\ X_i^- & \mapsto & -\tilde{v}\tilde{\mathcal{L}}_{-i}^-, \\ K & \mapsto & \tilde{K}, \\ C & \mapsto & \tilde{C} \end{array}$$

as $\mathbb{Q}(\tilde{v})$-algebras.

Furthermore, since by the Theorem of Cramer (Theorem 1.4.5 or see [9]) the factors $D\mathcal{H}_q(\mathrm{Coh}(\mathbb{X}))$ and $D\mathcal{H}\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$ are isomorphic for each $q \in \mathcal{P}$, their generic forms $D\mathcal{H}_{gen}(\mathrm{Coh}(\mathbb{X}))$ and $D\mathcal{H}_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ are isomorphic as well as $\mathbb{Q}(\tilde{v})$-algebras.

Hence as a last goal, it remains to check that the generalized double composition algebras are also isomorphic.

For this we note for the isomorphism $\mathcal{F} : D\mathcal{H}_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right) \to D\mathcal{H}_{gen}(\mathrm{Coh}(\mathbb{X}))$ it holds:

$$\begin{array}{rcl} \tilde{S}_1^+ & \mapsto & \tilde{v}^{\langle \mathcal{L}_{-1}, \mathcal{L}_{-1} \rangle} \tilde{\mathcal{L}}_{-1}^- \tilde{K}^{-1} \tilde{C} = \tilde{v} \tilde{\mathcal{L}}_{-1}^- \tilde{K}^{-1} \tilde{C}, \\ \tilde{S}_2^+ & \mapsto & \tilde{\mathcal{M}}_0^+ = \tilde{v} \dfrac{1}{[4]_+} \left[\tilde{\mathcal{L}}_0^+, \tilde{\mathcal{L}}_{-1}^+\right]_{\tilde{v}^2}, \\ \tilde{S}_1^- & \mapsto & \tilde{v} \tilde{\mathcal{L}}_{-1}^+ \tilde{K} \tilde{C}^{-1}, \\ \tilde{S}_2^- & \mapsto & \tilde{v} \dfrac{1}{[4]_+} \left[\tilde{\mathcal{L}}_0^-, \tilde{\mathcal{L}}_{-1}^-\right]_{\tilde{v}^2}, \\ \tilde{K}_1 & \mapsto & \tilde{K}^2 \tilde{C}^{-1}, \\ \tilde{K}_2 & \mapsto & \tilde{K}\tilde{C}. \end{array}$$

In particular, the images of the generators of the generalized double composition algebra $DC_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ is contained in $DC_{gen}(\mathrm{Coh}(\mathbb{X}))$. For the other direction, we need the following statement:

Proposition 4.4.5 (Reinecke, [28]) *Let $(\mathcal{M},\Omega)$ be an $\mathbb{F}_q$-species of a Dynkin or Euclidean graph.*

The derived reflection functors $R\mathbb{S}_*^+$ and $L\mathbb{S}_*^-$ induce an isomorphism on the reduced Drinfeld doubles $D\mathcal{H}(\mathrm{Rep}_{\mathbb{F}_q}(\mathcal{M},\Omega))$ and $D\mathcal{H}(\mathrm{Rep}_{\mathbb{F}_q}(\mathcal{M},s_*\Omega))$, which may be restricted to an isomorphism of the double composition algebras $DC(\mathrm{Rep}_{\mathbb{F}_q}(\mathcal{M},\Omega))$ and $DC(\mathrm{Rep}_{\mathbb{F}_q}(\mathcal{M},s_*\Omega))$.

More specifically, there are formulas for the image of the generators, analogous to the Lusztig symmetries.

For the proof see [28, Theorem 3.29].

Using this, we immediately get the following corollaries for our case $k \xrightarrow{{}_kK_K} K$:

Corollary 4.4.6 *For $n \in \mathbb{N}_0$ it holds $[P_n]^\pm, [I_n]^\pm \in DC\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$.*

Proof. It holds: $[I_0]^\pm = [S_1]^\pm \in DC\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$ is clear by definition.

For P_0 it holds:

$$\begin{aligned}[S_1][S_2] &= v^{\langle S_1,S_2\rangle}([S_1\oplus S_2] + \frac{q^4-1}{q-1}[P_0]) \\ &= v^{-4}(\tilde{v}[S_2][S_1] + \frac{q^4-1}{q-1}[P_0])\end{aligned}$$

Hence $[P_0]$ can be written as an expression in $[S_1]$ and $[S_2]$.

For $n > 0$, we apply Proposition 4.4.5 $2n$-times by considering the Coxeter functors. The Coxeter functor $\mathcal{C} = \mathbb{S}_1^+\mathbb{S}_2^+$ induces by the same argument an automorphism of $DC\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$.

In particular, for $n \in \mathbb{N}_0$ it holds

$$[P_n]^\pm = [\mathcal{C}^n(P_0)]^\pm \in DC\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$$

and

$$[I_n]^\pm = [\mathcal{C}^{-n}(I_0)]^\pm \in DC\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right).$$

□

Corollary 4.4.7 *For $n \in \mathbb{N}_0$ it holds $\tilde{P}_n^\pm, \tilde{I}_n^\pm \in DC_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$.*

Corollary 4.4.8 *The generators of $DC_{gen}(\mathrm{Coh}(\mathbb{X}))$ are contained in the image $\mathcal{F}\left(DC_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$.*

Theorem 4.4.9 *The generalized double composition algebras $DC_{gen}(\mathrm{Coh}(\mathbb{X}))$ and $DC_{gen}\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ are isomorphic via $\mathcal{F}$.*

Hence if we combine the discussed morphisms, we get an isomorphism $\tilde{U}_{\tilde{q}}\left(A_2^{(2)}\right) \to \tilde{U}_{\tilde{v}}^{Dr}\left(A_2^{(2)}\right)$:

$$\begin{array}{rcl} e_1 & \mapsto & -X_{-1}^{-}K^{-1}C, \\ e_2 & \mapsto & \tilde{v}\dfrac{1}{[4]_+}\left[X_0^+, X_{-1}^+\right]_{\tilde{v}^2}, \\ f_1 & \mapsto & -\tilde{v}^2 X_{-1}^{+} K C^{-1}, \\ f_2 & \mapsto & -\tilde{v}^5\dfrac{1}{[4]_+}\left[X_1^-, X_0^-\right]_{\tilde{v}^{-2}}, \\ K_1 & \mapsto & K^{-1}C, \\ K_2 & \mapsto & K^2C^{-1}. \end{array}$$

Note, that this is not the same isomorphism that can be found in the literature, e.g. in [1] by Akasaka where it has the form

$$\begin{array}{rcl} e_1 & \mapsto & X_0^+, \\ e_2 & \mapsto & \tilde{v}^{-2}\dfrac{1}{[4]_+}CK^{-2}\left[X_0^-, X_1^-\right]_{\tilde{v}^2}, \\ f_1 & \mapsto & X_0^-, \\ f_2 & \mapsto & \tilde{v}^2\dfrac{1}{[4]_+}C^{-1}\left[X_{-1}^+, X_0^+\right]_{\tilde{v}^{-2}}K^2, \\ K_1 & \mapsto & K, \\ K_2 & \mapsto & CK^{-2}, \\ c^{\frac{1}{2}} & \mapsto & C^{\frac{1}{2}}, \end{array}$$

where there is an additional generator $c^{\frac{1}{2}}$, or in [8] by Chari and Pressley

$$\begin{array}{rcl} e_1 & \mapsto & X_0^+, \\ e_2 & \mapsto & K^{-2}\left[X_0^+, X_{-1}^+\right]_{\tilde{v}^2}, \\ f_1 & \mapsto & X_0^-, \\ f_2 & \mapsto & \dfrac{1}{[4]^2}\left[X_0^-, X_{-1}^-\right]_{\tilde{v}^{-2}}K^2, \\ K_1 & \mapsto & K, \\ K_2 & \mapsto & K^{-2}, \end{array}$$

where in the latter one there is no generator C.

But there are certainly similarities in the given forms, in particular, one may note that in our case the generator e_1 is sent to the negative part of the Drinfeld double whereas in the isomorphisms stated by Akasaka and Chari and Pressley the generator e_2 is sent to the negative part, which is a result of considering the k-species $\bullet \xrightarrow{(1,4)} \bullet$ instead of $\bullet \xleftarrow{(1,4)} \bullet$ which is derived equivalent via the derived reflection functors, namely the derived equivalence

$$\mathcal{D}^b\left(\mathrm{Rep}\left(\bullet \xleftarrow{(1,4)} \bullet\right)\right) \to \mathcal{D}^b(\mathrm{Coh}(\mathbb{X}))$$

has the form on the simple objects

$$S_1 \mapsto \mathcal{L}_0 \qquad \text{and} \qquad S_2 \mapsto \mathcal{M}_0[1].$$

Hence by the known isomorphisms and the theorem of Cramer, we get

$$\begin{aligned}
e_1 &\mapsto X_0^+, \\
e_2 &\mapsto \tilde{v}^3 \frac{1}{[4]_+} \left[X_0^-, X_1^-\right]_{\tilde{v}^2} CK^{-2}, \\
f_1 &\mapsto X_0^-, \\
f_2 &\mapsto \tilde{v}^8 \frac{1}{[4]_+} C^{-1} \left[X_0^+, X_{-1}^+\right]_{\tilde{v}^{-2}} K^2, \\
K_1 &\mapsto K, \\
K_2 &\mapsto CK^{-2}.
\end{aligned}$$

This is (almost) the isomorphism which can be found in [1], namely the two prefactors of the images of e_2 and f_2 differ in the exponent of $\tilde{v}$ but the rest is the same. One should note though, that Akasaka uses a slightly different definition. To be specific: instead of the form of the relations (HX) and (XX) which we use here (see Definition 4.1.1), the exponents of the C are different. They use the relation

$$[H_k, X_n^\pm] = \pm \frac{1}{k}[2k] b_k C^{\mp \frac{|k|}{2}} X_{n+k}^\pm$$

and

$$[X_k^+, X_l^-] = \frac{C^{\frac{k-l}{2}} K \psi_{k+l}^+ - C^{\frac{l-k}{2}} K^{-1} \psi_{k+l}^-}{v - v^{-1}}.$$

However, this small discrepancy may be fixed as follows:

Instead of using the generating series as in Lemma 3.3.3

$$\exp\left(\sum_{k\geq 1} \frac{T_k}{[2k]} s^k\right) = 1 + \sum_{r\geq 1} \mathbb{1}_{(0,r)} s^r$$

and the generating series in Remark 3.3.6

$$1 + \sum_{n=1}^{\infty} \theta_n s^n = \exp\left((v - v^{-1}) \sum_{r=1}^{\infty} T_r s^r\right),$$

we replace them with

$$\exp\left(\sum_{k\geq 1} \underbrace{T_k C^{-\frac{k}{2}}}_{=:\tilde{T}_k} \frac{1}{[2k]} s^k\right) = 1 + \sum_{r\geq 1} \mathbb{1}_{(0,r)} C^{-\frac{r}{2}} s^r$$

and

$$1 + \sum_{n=1}^{\infty} \underbrace{\theta_n C^{-\frac{n}{2}}}_{=:\tilde{\theta}_n} s^n = \exp\left((v - v^{-1}) \sum_{r=1}^{\infty} \underbrace{T_r C^{-\frac{r}{2}}}_{=:\tilde{T}_r} s^r\right).$$

With these replacements we still have the same pairing for $r > 0$

$$(\tilde{\theta}_r, \tilde{T}_r) = (\theta_r C^{-\frac{r}{2}}, T_r C^{-\frac{r}{2}}) = (\theta_r, T_r) \overset{3.3.11}{=} \frac{[2r]}{r}(q^r + (-1)^{r+1} + q^{-r})$$

but slightly different coproducts:

- For $n \in \mathbb{Z}$:

$$\begin{aligned}\tilde{\Delta}([\mathcal{L}_n]) &= [\mathcal{L}_n] \otimes 1 + \sum_{l \geq 0} \theta_l C^{n-l} K \otimes [\mathcal{L}_{n-l}] \\ &= [\mathcal{L}_n] \otimes 1 + \sum_{l \geq 0} \tilde{\theta}_l C^{n-\frac{l}{2}} K \otimes [\mathcal{L}_{n-l}]\end{aligned}$$

- For $n \in \mathbb{N}$:

$$\begin{aligned}\tilde{\Delta}(\tilde{T}_n) &= \tilde{\Delta}(T_n) \cdot \tilde{\Delta}(C^{-\frac{n}{2}}) \\ &= \tilde{T}_n \otimes C^{-\frac{n}{2}} + C^{\frac{n}{2}} \otimes \tilde{T}_n.\end{aligned}$$

In particular in turn, they yield the relations for $k \in \mathbb{N}$, $n, m \in \mathbb{Z}$

$$[\tilde{T}_k, [\mathcal{L}_n]] = \frac{1}{k}[2k] b_k C^{\mp \frac{k}{2}} [\mathcal{L}_{n+k}]$$

and

$$[[\mathcal{L}_n]^+, [\mathcal{L}_m]^-] = \frac{C^{\frac{m-n}{2}} K \tilde{\theta}^+_{n-m} - C^{\frac{n-m}{2}} K^{-1} \tilde{\theta}^-_{n-m}}{v - v^{-1}}.$$

Similarly, the other relations may be deduced. Hence via an analogous argument and replacing the generators T_r by $\tilde{T}_r$ etc. we get the relations as in [1].

4.5 An Orthogonal PBW-Basis of $U_v^+\left(A_2^{(2)}\right)$

As a last small application, we now discuss how one may compute the universal R-matrix for $U_v\left(A_2^{(2)}\right)$. By definition, it is an element in the completion $U_v\left(A_2^{(2)}\right) \hat{\otimes} U_v\left(A_2^{(2)}\right)$. The computation may be done by constructing an orthogonal basis, namely if we consider the Drinfeld double $D(A)$ of an algebra A via a non-degenerate form and one has an orthonormal basis $(a_i)_{i \in I}$ of A, then the universal R-matrix may be expressed as

$$R = \sum_{i \in I} (a_i \otimes 1) \otimes (1 \otimes a_i) \in D(A) \hat{\otimes} D(A).$$

One of the properties is that they solve the quantum Yang-Baxter equation

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}.$$

Considering our constructed isomorphisms, Green's form is easily calculated on the Hall algebra side $C\left(\mathrm{Rep}_{\mathbb{F}_q}\left(\bullet \xrightarrow{(1,4)} \bullet\right)\right)$. The hardest part is to determine the contribution of regular representations, but this easily follows from our knowledge of the torsion part in $DC(\mathrm{Coh}(\mathbb{X}))$. To be more precise, given the Auslander-Reiten quiver (recall the beginning of Section 3.1), we can directly write down an orthogonal basis with elements of the form

$$[Q_0^{\oplus q_0}][P_0^{\oplus p_0}][Q_1^{\oplus q_1}][P_1^{\oplus p_1}] \cdot \ldots \cdot R_1^{t_1} R_2^{t_2} \ldots \cdot [I_1^{\oplus i_1}][J_0^{\oplus j_0}][I_0^{\oplus i_0}] =: Z_{\underline{q},\underline{p},\underline{t},\underline{i},\underline{j}} \tag{12}$$

in the positive part, where $q_l, p_l, t_r, i_l, j_l \in \mathbb{N}_0$ for $l \in \mathbb{N}_0$ and $r \in \mathbb{N}$ and almost all are zero, and the factors R_r are the pre-image of the T_r in the Hall algebra of $\mathrm{Coh}(\mathbb{X})$ of the isomorphism Φ. Furthermore, with our knowledge of the bilinear form regarding the elements T_r, one can easily deduce the norms to compute the universal R-matrix in $DC\left(\bullet \xrightarrow{(1,4)} \bullet\right)$ and $U_v\left(A_2^{(2)}\right)$. Only the fact that monomials in T_r's form an orthogonal basis has not been discussed, though it will be proven more in detail.

Elements of the form (12) correspond to the elements

$$[Q_0^{\oplus q_0} \oplus P_0^{\oplus p_0} \oplus Q_1^{\oplus q_1} \oplus P_1^{\oplus p_1} \oplus ...\text{"}R_1^{t_1}R_2^{t_2}\text{"}... \oplus I_1^{\oplus i_1} \oplus J_0^{\oplus j_0} \oplus I_0^{\oplus i_0}]$$

up to a pre-factor by construction, namely using the knowledge given by the Auslander-Reiten quiver, there are no other extensions that could appear as a summand in the product. Furthermore, it is a concrete description of the decomposition into indecomposable summands. In particular, if one takes two such products and if even one of the factors differs, Green's form of the two vanishes.

It remains to compute the norms. First, one may consider the norms on the $C(\text{Coh}(\mathbb{X}))$-side of the T_r. The torsion part is generated by the T_r in the sense

$$\Lambda := C(\text{Tor}(\mathbb{X})) = \mathbb{C}[T_1, T_2, ...],$$

which is also $\mathbb{N}_0$-graded with $\deg(T_r) = r$. Furthermore, each generator is primitive (in the unextended version)

$$\Delta(T_r) = T_r \otimes 1 + 1 \otimes T_r$$

for $r \in \mathbb{N}$. If we consider elements of the form

$$T_{\underline{m}} = T_1^{m_1} T_2^{m_2} T_3^{m_3} \cdot ...,$$

where $\underline{m} = (m_1, m_2, ...) \in \mathbb{N}_0^{\mathbb{N}}$ with almost all entries 0, it holds for Green's form:

Lemma 4.5.1 *For $\underline{m}, \underline{n} \in \mathbb{N}_0^{\mathbb{N}}$ with almost all entries 0 it holds*

- *$(T_{\underline{m}})_{\underline{m}}$ is an orthogonal basis of Λ;*
- *$(T_{\underline{m}}, T_{\underline{n}}) = \delta_{\underline{m},\underline{n}} \cdot \prod_{j\geq 1}(m_j! \cdot b_j^{m_j})$, where $b_j = \frac{[2j]}{j}\frac{q^j+(-1)^{j+1}+q^{-j}}{v-v^{-1}}$ as before.*

Proof. An important fact that we use here is that Green's form is a Hopf pairing.

- Claim: $(T_r^m, T_r^m) = m! \cdot b_r^m$ for $m, r \in \mathbb{N}$.

  This can be done by induction. $(T_r, T_r) = b_r$ was proven in Lemma 3.3.10.

  Now for the case $m > 1$:

$$\begin{aligned}(T_r^m, T_r^m) &= (\Delta(T_r)^m, T_r^{m-1} \otimes T_r) \\ &= ((T_r \otimes 1 + 1 \otimes T_r)^m, T_r^{m-1} \otimes T_r) \\ &= m \cdot (T_r^{m-1}, T_r^{m-1})(T_r, T_r) \\ &\overset{\text{ind.}}{=} m! \cdot b_r^m\end{aligned}$$

  where we use that in $(T_r \otimes 1 + 1 \otimes T_r)^m$ only the summand $mT_r^{m-1} \otimes T_r$ yields a non-zero pairing by considering the degree of each term.

  Also by degree considerations, we have $(T_r^m, T_r^n) = 0$ for $n \neq m$.

- Furthermore, if we pair T_r with $T_{\underline{m}}$ with $m_j \neq 0$ for some $j \neq r$, we get using the m^{th} coproduct

$$\Delta^{(m)}(T_r) = T_r \otimes \underbrace{1 \otimes 1 \otimes ... \otimes 1}_{m \text{ times}} + 1 \otimes T_r \otimes 1 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes T_r$$

where $m := \sum_j m_j < \infty$:

$$\begin{aligned}(T_r, T_{\underline{m}}) &= (\Delta^{(m)}(T_r), T_1^{\otimes m_1} \otimes T_2^{\otimes m_2} \otimes ...) \\ &= ... + ...(1, T_j)... + ...(T_r, T_j)... + ...(1, T_j)... + ... \\ &= 0,\end{aligned}$$

since each summand has at least one factor 0. Similarly, it follows $(T_r^a, T_{\underline{m}}) = 0$ with $m_r = 0$, since $\Delta^{(a)}(T_{\underline{m}})$ can be written using expressions in T_j with $j \neq r$ and 1's.

- By an analogous argument, if say $\underline{m} = (m_1, m_2, ..., m_l)$ and $\underline{n} = (n_1, ..., n_l)$ with $n_l \neq 0$ and $m_l = 0$:

$$\begin{aligned}(T_{\underline{m}}, T_{\underline{n}}) &= (T_{\underline{m}}, T_1^{n_1} \cdot ... \cdot T_{l-1}^{n_{l-1}} \cdot T_l^{n_l}) \\ &= (\Delta(T_{\underline{m}}), T_1^{n_1} \cdot ... \cdot T_{l-1}^{n_{l-1}} \otimes T_l^{n_l}) \\ &= \sum (T', T_1^{n_1} \cdot ... \cdot T_{l-1}^{n_{l-1}})(T'', T_l^{n_l}) \\ &= 0,\end{aligned}$$

since T'' is an expression in $T_1, ..., T_{l-1}$ and by the previous argument, the pairing $(T'', T_l^{n_l})$ vanishes.

- Now, consider $\underline{n} = (n_1, ..., n_l)$ with $n_l \neq 0$ and $a \in \mathbb{N}$. The pairing $(T_l^a, T_{\underline{n}})$ is only unequal zero if $a \cdot l = \sum_{j=1}^{l} j n_j$, therefore unless $T_{\underline{n}} = T_l^a$ which was discussed in the first case, we may assume $0 \neq n_l < a$.

$$\begin{aligned}(T_l^a, T_{\underline{n}}) &= (T_l^a, T_1^{n_1} \cdot ... \cdot T_{l-1}^{n_{l-1}} \cdot T_l^{n_l}) \\ &= (\Delta(T_l)^a, \underbrace{T_1^{n_1} \cdot ... \cdot T_{l-1}^{n_{l-1}}}_{\text{orthgonal to } T_l^b} \otimes T_l^{n_l}) \\ &= (T', T_1^{n_1} \cdot ... \cdot T_{l-1}^{n_{l-1}})(T_l^{n_l}, T_l^{n_l}) \\ &= 0,\end{aligned}$$

since T' is an expression in T_l and 1, and there exists $j < l$ with $n_j \neq 0$.

- Now, by the last case and analogous step-by-step calculations we get $(T_{\underline{m}}, T_{\underline{n}}) \neq 0$ if and only if $\underline{m} = \underline{n}$. Hence the monomials form an orthogonal basis.

  Furthermore, the second to last line in the calculation yields

$$\begin{aligned}(T_{\underline{m}}, T_{\underline{n}}) &= \delta_{\underline{m},\underline{n}} \cdot \prod_{j \geq 1} (T_j^{m_j}, T_j^{m_j}) \\ &= \delta_{\underline{m},\underline{n}} \cdot \prod_{j \geq 1} (m_j! \cdot b_j^{m_j}).\end{aligned}$$

□

To recap, therefore elements of the form (12) are orthogonal and by construction form an orthogonal basis.

As mentioned before, one may construct the universal R-matrix for $U_v\left(A_2^{(2)}\right)$ with a more accessible understanding of the formulas, or even some further applications.

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