Special values of L-functions

Der Fakultät für
Elektrotechnik, Informatik und Mathematik
der Universität Paderborn
vorgelegte Dissertation
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

von

Martin Barič

Betreuer: Prof. Dr. Fabian Januszewski

This page is intentionally left blank

Abstract

Let $k/\mathbb{Q}$ be a number field, $\wp$ a fixed finite prime place of k and $\mathbb{A}$ the ring of adèles of k. Let further (ρ, σ) be a pair of cuspidal automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}) \times \mathrm{GL}_n(\mathbb{A})$ unramified at $\wp$ and let χ be a quasicharacter of $k^\times \backslash \mathbb{A}^\times$ whose conductor is a power of $\wp$. The automorphic L-function $L(s, \rho \times (\sigma \otimes \chi))$ attached to $\rho \times \sigma$ under the twist of χ has its special value at $s = 1/2$. We will study the $\wp$-adic interpolation of this special value of $L(s, \rho \times (\sigma \otimes \chi))$. We state a conjecture for the value of the $\wp$-adic L-function at χ for general n under the assumption that χ is trivial at $\wp$. We will prove the conjecture in the cases $n = 1, 2$.

Zusammenfassung

Es seien $k/\mathbb{Q}$ ein Zahlkörper, $\wp$ eine feste endliche Primstelle von k und $\mathbb{A}$ der Ring der Adele von k. Es seien ferner (ρ, σ) ein paar von kuspidalen automorphen Darstellungen von $\mathrm{GL}_{n+1}(\mathbb{A}) \times \mathrm{GL}_n(\mathbb{A})$, beide unverzweigt bei $\wp$, und χ ein Quasicharakter von $k^\times \backslash \mathbb{A}^\times$, dessen Führer eine p-Potenz ist. Die automorphe L-Funktion $L(s, \rho \times (\sigma \otimes \chi))$ von $\rho \times \sigma$ unter Twists von χ hat ihren speziellen Wert bei $s = 1/2$. Wir werden die $\wp$-adische Interpolation dieses speziellen Wertes $s = 1/2$ von $L(s, \rho \times (\sigma \otimes \chi))$ untersuchen. Wir werden eine Vermutung über den Wert der $\wp$-adischen L-Funktion für allgemeines n unter der Annahme aufstellen, dass χ trivial bei $\wp$ ist. Diese Vermutung beweisen wir in den Fällen $n = 1, 2$.

Contents

0. Preface

The aim of this writing lies in the $\wp$-adic interpolation of the special value $s = 1/2$ of the L-function $L(s, \rho \times (\sigma \otimes \chi))$ attached to a pair of cuspidal representations (ρ, σ) of $\mathrm{GL}_{n+1}(\mathbb{A}) \times \mathrm{GL}_n(\mathbb{A})$ under the twist of an adelic quasicharacter whose conductor is a power of $\wp$. It readily follows the works of Kazhdan-Mazur-Schmidt ([KMS00]) and Januszewski ([Jan09]). The latter constructs a $\wp$-adic distribution with the interpolation property for χ that is ramified at $\wp$. The author also formulates precise conditions on when the $\wp$-adic distribution is a (pseudo-)measure. We will show that the $\wp$-adic distribution constructed by Januszewski satisfies the relation predicted by Coates in [Coa89] for trivial χ in the case $n = 1, 2$ as well. The crucial part of the construction lies in the modification of $L(s, \rho \times (\sigma \otimes \chi))$ at $\wp$.

Chapter 1 is preliminary in its nature: we define the necessary objects to work with.
Chapter 2 is devoted to the development of both, the automorphic and local representation theory. In addition to general definitions, we will focus on the development of non-Archimedean Whittaker functions right invariant under the Iwahori group J_n. Iwahori-invariant Whittaker functions have been studied by Brubaker-Buciumas-Bump-Gustafsson in [BBBG19] and Brubaker-Bump-Licata in [BBL18]. After the development of the representation theory, Chapter 3 will serve to construct the automorphic L-function attached to $\rho \times \sigma$ by the method of Rankin-Selberg.

Chapter 4 is intended to talk about the connection between motives and cuspidal representations as predicted by Clozel in Conjecture 4.5. of [Clo88]. Coates describes in [Coa89] the construction of a $\wp$-adic measure that interpolates the special values of a motivic L-function. His description serves us as guidance on what we expect to obtain on the automorphic side.
In Chapter 5 we mimic Coates' modification at $\wp$ for motives on the automorphic side. This is the central part of our work. We think that the correct modification of $L(s, \rho \times \sigma)$ at $\wp$ is mirrored in the appearance of a modified local zeta integral of the form

$$\int\limits_{U_n(k_\wp)\backslash \mathrm{GL}_n(k_\wp)} \mathcal{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} h_{n+1} \begin{pmatrix} w_n \pi^{\mathfrak{f}\cdot \mathfrak{t}_n} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(g\pi^{\mathfrak{f}\cdot\mathfrak{t}_n}) \cdot \chi(g) \left|\det(g)\right|^{s-1/2} d^\times g. \qquad (1)$$

Here χ is a character of $\mathrm{GL}_n(k_\wp)$ and the functions $\mathcal{W}, \mathcal{W}'$ are Iwahori-invariant Whittaker functions attached to $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ and are chosen to be eigenvectors of parabolic Hecke operators. The element w_n denotes the long Weyl element of GL_n with respect to the diagonal torus. The matrix h_{n+1} as well as the parameters $\mathfrak{f}$ and $\mathfrak{t}_n$ are further fixed values to match the modification predicted by Coates in [Coa89]. The closed form of this integral is for ramified χ known due to the work of Januszewski; see Section 1.1 in [Jan09]. We conjecture for the case when χ unramified the following:

Conjecture A. If χ is unramified, then the integral (1) has the closed form

$$\mathrm{vol}(J_n, d^\times g) \cdot q^{\frac{(n-1)\cdot n}{2}} \cdot (1 - q^{-1})^{\frac{(n-1)\cdot n}{2}}$$
$$\cdot \left(\delta_{n+1}^{1/2} \otimes \lambda^{w_{n+1}}\right) \left(\begin{pmatrix} \pi^{\mathfrak{t}_n} & \\ & 1 \end{pmatrix}\right)^{\mathfrak{f}} \cdot \left(\delta_n^{1/2} \otimes \mu^{w_n}\right) \left(\pi^{\mathfrak{t}_n}\right)^{\mathfrak{f}}$$

$$\cdot \prod_{i+j>n+1} \left(\frac{q}{q-1} \cdot \frac{1-(\lambda_i \otimes \mu_j)(\pi)^{-1} \cdot q^{s-1}}{1-(\lambda_i \otimes \mu_j)(\pi) \cdot q^{-s}} \right),$$

where $\lambda_i(\pi)$ and $\mu_j(\pi)$ are the Satake parameters of ρ and σ at $\wp$, respectively, and q the cardinality of the residue field of $k_\wp$.

The main result of our work consists in the following:

Theorem A. The Conjecture A holds for $n = 1, 2$.

Finally, in Chapter 6 we recall the construction of the $\wp$-adic distribution of Kazhdan-Mazur-Schmidt ([KMS00]) and Januszewski ([Jan09]), respectively. We modify the L-function $L(s, \rho \times (\sigma \otimes \chi))$ by inserting the modified local zeta integral from Chapter 5. We then show that, under the stated Conjecture, it satisfies the interpolation property for χ unramified at $\wp$ as well, as was predicted by Coates. More precisely, by the work of Januszewski, and assuming the Conjecture A to be true, we have the following result:

Theorem B. If (ρ, σ) are in addition algebraic, regular, ordinary at $\wp$ (this implies that their respective $\wp$-components are spherical) and cohomological, then the $\wp$-adic distribution $\mu_\wp$ on $C_k(\wp^\infty)$ constructed by Januszewski is a measure, that satisfies

$$\begin{aligned}
\int_{C_k(\wp^\infty)} \chi_{\mathbb{A}} d\mu_\wp &= \kappa_{\mathfrak{f}} \cdot P(1/2, \mathcal{W}_\infty, \mathcal{W}'_\infty) \cdot L(\rho \times (\sigma \otimes \chi_{\mathbb{A},1/2})) \cdot \mathrm{vol}(J_n, d^\times g) \\
&\cdot q^{\frac{(n-1)\cdot n}{2}} \cdot (1-q^{-1})^{\frac{(n-1)\cdot n}{2}} \cdot \left(\frac{q}{q-1} \right)^{\frac{n(n+1)}{2}} \\
&\cdot \left(\delta_{n+1}^{1/2} \otimes \lambda^{w_{n+1}} \right) \left(\begin{pmatrix} \pi^{\mathfrak{t}_n} & \\ & 1 \end{pmatrix} \right)^{\mathfrak{f}} \cdot \left(\delta_n^{1/2} \otimes \mu^{w_n} \right) \left(\pi^{\mathfrak{t}_n} \right)^{\mathfrak{f}} \\
&\cdot \prod_{i+j \leq n+1} \left(1 - \frac{(\lambda_i \otimes \mu_j)(\pi)}{q^{1/2}} \right) \cdot \prod_{i+j>n+1} \left(1 - \frac{q^{1/2}}{(\lambda_i \otimes \mu_j)(\pi)} \right).
\end{aligned}$$

Acknowledgments At last, I would like to use the opportunity to express my gratitude: firstly and mostly I would like to thank Prof. Dr. Frank Herrlich for his unconditional support in both personal life and mathematics. I also thank my colleague PD. Dr. Hans Franzen for his friendship, his suggestions and illustrative conversations, specially in geometry. Furthermore, my special thank goes to Assoc. Prof. Dr. Chris Williams. I personally profited a lot from his helpful e-mails and his great expertise in topics of p-adic analysis. I would also like to thank him for accepting to be my examiner. In addition, I would like to thank PD. Dr. Job Kuit for discussions on measure theory and real reductive groups.
My further thanks goes to my colleagues Dr. Aleksandre Maksoud for enlightening talks, to Dr. Ali Suri and Dr. Raphael Müller for the nice working environment at the UPB, to my office colleague Daniel Perniok for his suggestions and corrections on Chapter 2 and to Dr. Stepan Maximov for his suggestions and corrections on this Preface.
Aside from the university environment, I would like to thank my girlfriend Erioneide Ferreira da Silva for her support and for the special person she is in my life. My further

thanks goes to Ana Karla García Peréz for her great friendship and her legacy, and to Ariane Mensato for her constant support.
My last thank is dedicated to my family and further friends who play a role in my life.

0.1. Historical Development and Motivation

The following subsection is intended to give a brief introduction into the historical development of L-functions and their connection to number theory. We will also try to motivate here the aim of this work. It can be skipped by the expert.

0.1.1. Classical approach to L-functions

The Riemann zeta function. The study of *classical* L-functions dates back to Riemann. Riemann introduced in [Rie59] the famous *zeta function* bearing his name

$$\zeta(s) \overset{\text{Df.}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

f This function is a priori defined in the right complex half-plane $\Re(s) > 1$. It can be expanded as the infinite product of *Euler factors*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \tag{2}$$

which was for natural exponents already known to Euler. If we analyze this equality, we find on the left hand side an object from complex analysis, and on the right hand side we see that it encapsulates all the prime numbers, the building blocks of number theory. This was the first established connection between these two areas. The Riemann zeta function is a new tool that can be used to investigate asymptotic behaviour of prime numbers by means of analytic methods.

Riemann could show in two different ways that $\zeta(s)$ extends meromorphically to the whole $\mathbb{C}$ and has a simple pole at $s = 1$. One such argument goes as follows: by a clever mainpulation of the sum $\zeta(s)$, one can extend it meromorphically to $\Re(s) > 0$. Subsequently, the function

$$\Lambda(s) := \pi^{-s/2} \cdot \zeta(s) \cdot \Gamma(s/2),$$

which is called the *complete zeta function*[1], is a meromorphic function on whole $\mathbb{C}$ with simple poles in $\{0, 1\}$, and it satisfies the *functional equation*

$$\Lambda(s) = \Lambda(1-s). \tag{3}$$

Complex functions of the nature of $\zeta(s)$, that

[1]In [Rie59] we find $\Gamma(s/2-1)$. This is not a mistake, just the slight difference in the definition of the Γ-function in times of Riemann, which was shifted by 1 to the right.

- are defined on some right half place,
- have an Euler Product similar to (2),
- satisfy a functional equation similar to (3),
- and (consequently) possess a meromorphic continuation to whole $\mathbb{C}$,

are often referred to as *L-functions*[2]. As such, the Riemann zeta function $\zeta(s)$ is the first of its kind.

Generalizations of $\zeta(s)$. Since its discovery, the Riemann zeta function $\zeta(s)$ went through a series of generalizations:

1. The key to the first generalization consisted in considering the Riemann zeta function $\zeta(s)$ as the L-function corresponding the trivial character $\mathbb{1}$ of the trivial group. Dirichlet introduced the function
$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\Re(s) > 1),$$
where χ is the extension of a finite character of $(\mathbb{Z}/m\mathbb{Z})^{\times}$ to a multiplicative function on $\mathbb{Z}$. This function is called the *Dirichlet L-function* and finite characters of $(\mathbb{Z}/m\mathbb{Z})^{\times}$ for some $m \in \mathbb{N}$ are called *Dirichlet characters*. It possesses the Euler product
$$L(s,\chi) = \prod_{p \text{ prime}} \frac{1}{1-\chi(p)\cdot p^{-s}}$$
and satisfies a function equation similar to (3).
2. Further generalization consisted in understanding the ζ-function as the ζ-function attached to the prime number field $\mathbb{Q}$. This led to the definition of the *Dedekind zeta function*
$$\zeta_k(s) = \sum_{0 \neq \mathfrak{a} \triangleleft \mathcal{O}_k} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} \quad (\Re(s) > 1)$$
where $k/\mathbb{Q}$ is an algebraic number field and $\mathcal{O}_k$ its ring of integers. Here $\mathfrak{a}$ runs over the non-zero ideals of $\mathcal{O}_k$. The Euler product of $\zeta_k(s)$ is
$$\zeta_k(s) = \prod_{\mathfrak{p} \text{ max.ideal}} \frac{1}{(1-\mathfrak{N}(\mathfrak{p})^{-s})}.$$
It also satisfies a function equation similar to (3).

[2]Although a formal definition is still missing.

3. Ultimately, Hecke ([Hec18], [Hec19]) unified these two generalizations as follows: he generalized Dirichlet's characters to so-called *Hecke's Grössencharakter* and studied the functions

$$L_k(s,\chi) = \sum_{\substack{0\neq\mathfrak{a}\subset\mathcal{O}_k \\ (\mathfrak{a},\mathfrak{m})=1}} \frac{\chi(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s} \quad (\Re(s) > 1)$$

where χ is a Hecke's Größencharakter of *modulus* $\mathfrak{m}$. The Euler product of $L_k(s,\chi)$ is given by

$$L_k(s,\chi) = \prod_{\substack{\mathfrak{p} \text{ prime ideal} \\ \mathfrak{p}\notin\mathfrak{m}}} \frac{1}{(1-\chi(\mathfrak{p})\cdot\mathfrak{N}(\mathfrak{p})^{-s})}.$$

The *complete L-function*

$$\Lambda_k(s,\chi) := \gamma(\chi)\cdot\Gamma_k(s,\chi)\cdot A^s\cdot L(s,\chi), \tag{4}$$

where A is a constant and the gamma factor Γ_k depends only on the infinite places of k and is entire in $s\in\mathbb{C}$ for every non-trivial character χ. Furthermore, it satisfies the functional equation

$$\Lambda_k(1-s,\chi^{-1}) = W(\chi^{-1})\cdot\Lambda_k(s,\chi), \tag{5}$$

where $W(\chi)$ is known as the *root number* of χ.

These functions belong to the class of *classical L-functions*.

Remarks.

a) We would like to point out the importance of the study of L-functions in number theory. It is intuitive to use analytic tools in order to investigate the asymptotic behaviour of prime numbers. Indeed, for many statements, there is no elementary proof[3] known or is much harder to derive without analytic tools and often unintuitive. One such prominent example is the Dirichlet's Prime Number Theorem:

   1.) In any arithmetic progression $a+b, a+2b, a+3b$ with a,b coprime, there exist infinitely many prime numbers.

   Dirichlet went even further and showed that the prime numbers are equidistributed with respect to b in the following sense:

   2.) Every prime number (except maybe the prime divisors of b) lies inside the class $a+\left(\mathbb{Z}/b\mathbb{Z}\right)^\times$ with probability $\frac{1}{\varphi(b)}$.

   It took a long time to discover an elementary proof of the Dirichlet's Prime Number Theorem. It was done by Selberg in [Sel49], and he only proved the first statement. The proof itself is tedious and does not reveal anything new.

[3]This is a proof that does not require analytic methods.

b) Hecke also introduced in [Hec37a], [Hec37b] a new type of L-functions. These are attached to modular forms with respect to $\mathrm{SL}_2(\mathbb{Z})$ or its congruences subgroups. These are called Hecke's *modular* L-functions and also belong to the class of classical L-functions.

c) L-functions (or zeta functions) can also be attached to many different objects ; for instance to elliptic curves, or more generally to algebraic varieties. These are known as Hasse-Weil zeta functions.

0.1.2. Adelic approach to L-functions: The paradigm of Tate

L-functions on $\mathrm{GL}_1(\mathbb{A}_k)$. Although Hecke's proof of his functional equation (5) (using theory of generalized theta functions) represented a great achievement so far, it did not lay bare the nature of the gamma factors $\Gamma(s,\chi)$ or of the root number $W(\chi)$. In fact, this was only possible by the work of Tate in the 1950's. Tate, a student of Artin, applied in his doctoral Thesis [Tat50] methods of (abelian) Fourier Theory to the ring of adèles $\mathbb{A}_k$ of a number field $k/\mathbb{Q}$ and, consequently, was able to transfer the classical L-functions to the adelic environment. In this setting, Dirichlet's characters correspond to characters of the idèle class group $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times$ with finite image and in general, Hecke's Grössencharakters for a fixed number field $k/\mathbb{Q}$ correspond to characters of the idèle class group $k^\times \backslash \mathbb{A}_k^\times$ (without the assumption on finite image).

Tate studied *adelic* (or *global*) *zeta functions* of the form

$$Z(s,\Phi,\chi) \overset{\text{Df.}}{=} \int_{\mathbb{A}_k^\times} \Phi(x)\chi(x)\,\|x\|^s\,d^\times x, \tag{6}$$

where Φ is an adelic Schwartz-Bruhat-function, χ is some Hecke's Grössencharakter and $\|\cdot\|$ denotes the adelic norm. These functions are convergent in the right half-plane $\Re(s) > 1$. Any Hecke's Grössencharakter χ is of the form

$$\chi(x) = \prod_\nu \chi_\nu(x_\nu),$$

where ν runs over the prime places of k and the χ_ν are characters of the completions k_ν, unramified at almost all places ν. Since the space of adelic Schwartz-Bruhat-functions $\mathcal{S}(\mathbb{A}_k)$ is by definition the restricted product $\prod_\nu^{\mathcal{S}(\mathcal{O}_\nu)} \mathcal{S}(k_\nu)$, the function Φ is a linear combination of products of the form $\prod_\nu \Phi_\nu$ and hence, Φ can be further assumed to be already such a pure product. In this case, the global zeta function decomposes as an infinite product

$$Z(s,\Phi,\chi) = \prod_\nu Z_\nu(s,\Phi_\nu,\chi_\nu),$$

where Z_ν is the *local zeta function* defined by

$$Z_\nu(s,\Phi_\nu,\chi_\nu) = \int_{k^\times} \Phi_\nu(x_\nu)\chi_\nu(x_\nu)\,|x_\nu|_\nu^s\,d^\times x_\nu. \tag{7}$$

Tate proved that (6), in fact, possesses an analytic continuation to the whole $\mathbb{C}$ with a possible pole and moreover satisfies the functional equation

$$Z(s, \Phi, \chi) = Z(1-s, \hat{\Phi}, \chi^{-1}),$$

where $\hat{\Phi}$ is the Fourier transform of Φ.

Now, the local zeta functions $Z_\nu(s, \Phi_\nu, \chi_\nu)$ for finite primes ν, where χ_ν is unramified, correspond (for a special choice of the Schwartz-Bruhat-functions Φ_ν) to the Euler factors of Hecke's L-functions and thus, the product over the finite places (where χ is not ramified) is in the right half-plane exactly Hecke's classical L-function. The appearance of the Gamma function $\Gamma_k(s, \chi)$ in the functional equation (4) now arises naturally as the product of the local zeta functions at infinity. The root number $W(\chi^{-1})$ comes to existence also in a very natural way; as the product of so-called *local epsilon factors* (modulo a small variation) appearing in the functional equation of the local zeta function $Z_\nu(s, \Phi_\nu, \chi_\nu)$. At last, the global zeta function corresponds to the complete L-function $\Lambda_k(s, \chi)$ modulo some minor corrections factors at the (finite) set of so-called *bad primes* of χ.

L-functions on $\mathrm{GL}_2(\mathbb{A}_k) \times \mathrm{GL}_1(\mathbb{A}_k)$. The adelic approach opened the possibility of further generalization of automorphic L-functions to L-functions over $\mathrm{GL}_n(\mathbb{A}_k)$, since $\mathbb{A}^\times = \mathrm{GL}_1(\mathbb{A}_k)$. The first generalization to $\mathrm{GL}_2(\mathbb{A}_k)$ (or more precisely to $\mathrm{GL}_2(\mathbb{A}_k) \times \mathrm{GL}_1(\mathbb{A}_k)$) was carried out by Jacquet and Langlands in [JL70].

Jacquet and Langlands embedded classical modular/cuspidal forms on the upper half-plane $\mathbb{H} \subset \mathbb{C}$ attached to congruence subgroups of $\mathrm{SL}_2(\mathbb{R})$ naturally into the set of so-called *automorphic/cusp forms* on $\mathrm{GL}_2(\mathbb{A}_k)$ and consequently, were able to transfer Hecke's modular L-functions to automorphic L-functions on $\mathrm{GL}_2(\mathbb{A}_k)$. Their adelic construction of the zeta functions, attached to such automorphic forms, followed techniques of Hecke but was, of course, largely influenced by Tate's work on $\mathrm{GL}_1(\mathbb{A}_k)$.

Roughly speaking, automorphic forms (or more specially, cuspidal forms) on $\mathrm{GL}_n(\mathbb{A}_k)$ are just elements of automorphic representations on $\mathrm{GL}_n(\mathbb{A}_k)$ satisfying certain properties. In this context, Hecke's Grössencharacters are just (cuspidal) automorphic forms on $\mathrm{GL}_1(\mathbb{A}_k)$. This is why we refer to this theory as the theory of automorphic (rather than adelic) L-functions on $\mathrm{GL}_2(\mathbb{A}_k) \times \mathrm{GL}_1(\mathbb{A}_k)$. Following both Hecke and Jacquet-Langlands, one defines the *global zeta function* by

$$Z(s, \varphi, \chi) \overset{\text{Df.}}{=} \int_{k^\times \backslash \mathbb{A}_k^\times} \varphi\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) \chi(x) \, \|x\|^{s-\frac{1}{2}} \, d^\times x, \tag{8}$$

where φ is a cuspidal form coming from an irreducible cuspidal automorphic representation (π, V) of $\mathrm{GL}_2(\mathbb{A}_k)$. The functions defined by (8) are entire on $\mathbb{C}$ and satisfy the functional equation

$$Z(s, \varphi, \chi) = Z(1-s, \widetilde{\varphi}, \chi^{-1}),$$

where $\widetilde{\varphi} \in (\widetilde{\pi}, \widetilde{V})$ (the contragredient representation of (π, V)) is the *dual automorphic form* of φ defined by $\widetilde{\varphi}(g) := \varphi({}^t g^{-1})$.

One may consider now the Fourier expansion of φ, which is, since φ is cuspidal, given for $g \in \mathrm{GL}_2(\mathbb{A}_k)$ by

$$\varphi(g) = \sum_{y \in k^\times} W_\varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} g\right),$$

with absolute and uniform convergence guaranteed on compact subsets of $\mathrm{GL}_2(\mathbb{A}_k)$. The functions W_φ are called global *Whittaker functions*. They are the Fourier coefficients of φ and depend on a fixed additive character on $k \backslash \mathbb{A}_k$. Substituting this in (8), one obtains

$$\begin{aligned} Z(s,\varphi,\chi) &= \int_{k^\times \backslash \mathbb{A}_k^\times} \sum_{y \in k^\times} W_\varphi(\begin{pmatrix} yx & \\ & 1 \end{pmatrix})\chi(x) \|x\|^{s-\frac{1}{2}} \, d^\times x \\ &= \int_{\mathbb{A}_k^\times} W_\varphi(\begin{pmatrix} x & \\ & 1 \end{pmatrix})\chi(x) \|x\|^{s-\frac{1}{2}} \, d^\times x. \end{aligned}$$

One should take care here, because after the unfolding of φ, this expression only converges for $\Re(s) > 1$.

There also exists the notion *local Whittaker functions* for both Archimedean and non-Archimedean places. By the works of Gelfand-Kazhdan ([GK75]) and Shalika ([Sha74]) on the uniqueness of the local Whittaker functions, the global Whittaker function decomposes into a product of local Whittaker functions as

$$W_\varphi(g) = \prod_\nu W_{\varphi_\nu}(g_\nu).$$

We also have similarly a well-known decomposition $\|x\| = \prod_\nu \|x_\nu\|_\nu$ for the adelic norm. At last, by the Flath Decomposition Theorem, [Fla79], Th.3, (π, V) decomposes as a restricted tensor product $\pi \cong \bigotimes'_\nu \pi_\nu$, and thus if φ corresponds to such a pure infinite tensor, then

$$Z(s,\varphi,\chi) = \prod_\nu Z(s,\varphi_\nu,\chi_\nu),$$

where $Z_\nu(s, \mathcal{W}_{\varphi_\nu}, \chi_\nu)$ is the *local zeta function* given by

$$Z(s, \mathcal{W}_{\varphi_\nu}, \chi_\nu) = \int_{k_\nu^\times} W_{\varphi_\nu}(\begin{pmatrix} x_\nu & \\ & 1 \end{pmatrix})\chi_\nu(x_\nu) \|x_\nu\|^{s-\frac{1}{2}} \, d^\times x,$$

a priori only convergent for $\Re(s) > 1$.

L-functions on $\mathrm{GL}_n(\mathbb{A}_k) \times \mathrm{GL}_m(\mathbb{A}_k)$. The work of Jacquet and Langlands on $\mathrm{GL}_2(\mathbb{A}_k) \times \mathrm{GL}_1(\mathbb{A}_k)$ ([JL70]) was further extended by Jacquet in [Jac71] to automorphic L-functions on $\mathrm{GL}_2(\mathbb{A}_k) \times \mathrm{GL}_2(\mathbb{A}_k)$.

The generalization of the theory of automorphic L-functions to $\mathrm{GL}_n(\mathbb{A}_k)$ and to $\mathrm{GL}_n(\mathbb{A}_k) \times \mathrm{GL}_m(\mathbb{A}_k)$ was subsequently achieved in a long series of papers due to Jacquet, Piatetski-Shapiro and Shalika ([PS71], [PS79], [Sha74], [JS76], [JPSS79a], [JPSS81b], [JPSS81a],

[JS81], [JPSS83], [JS85], [JS90]). For a brief introduction to automorphic L-functions see also [Cog00] or [Cog03].

We have already spent some words on zeta functions, but did not really mention automorphic L-functions. So what are these and what are their connections to the zeta integrals? For $n = 1$, these terms are the same. But for general n they differ. We shall cover this in detail in Chapter 3, but to have a vague idea, one can imagine automorphic L-functions as functions generated by all global zeta-integrals when varying over the possible Whittaker functions.

We will study in this work an L-function on $\mathrm{GL}_{n+1}(\mathbb{A}_k) \times \mathrm{GL}_n(\mathbb{A}_k)$.

0.1.3. From geometry to number theory

We briefly mentioned modular forms in the introduction, which are objects from complex analysis and important in number theory. Its geometric counterpart are the elliptic curves. What do we mean by this? By the Eichler-Shiumura construction, if f is a classical newform of weight 2 and level N whose q-expansion has integer coefficients, one can attach to it an elliptic curve E_f of conductor N. This attachment is interpreted by means of number theory: one can define the L-function for both, f and E_f and the attachment $f \mapsto E_f$ satisfies

$$L(f, s) = L(E_f, s).$$

By Taniyama-Shimura, this was actually a conjectural correspondence, i.e. the attachment would go into the other direction as well.
The final proof of this conjecture, today known as the *Modularity Theorem*, was given by Wiles [Wil95] and Taylor-Wiles [TW95] for semi-stable curves and generalized by Breuil-Conrad-Diamond-Taylor [BCDT01] to the *full Modularity Theorem*.
The Modularity Theorem had a nice side-effect. As a corollary, the famous Fermat's Last Theorem was now known to be true: the only possible integer solutions to the equation

$$a^N + b^N = c^N$$

for $N \geq 3$ are only the trivial ones, i.e. those with $abc = 0$.

The Modularity Theorem finds a vast generalization in the Langlands Program. Clozel postulates in Conjecture 4.5 of [Clo88] such a generalization. On the one hand, one has motives, objects from algebraic geometry, that first appeared in 1964 in a letter correspondence between Grothendieck and Serre. On the other hand, one has automorphic (more precisely cuspidal) representations, which are objects studied in representation theory. Their connection is given by means of number theory; indeed one can attach L-functions to motives as well, and if M_Σ is a motive, that is conjecturally attached to an automorphic representation Σ, then their L-functions are the same (modulo a shift).

However, we would like to stress, that the correspondence is in both directions highly conjectural, as one does not really know yet, how to correctly define a motive. There are only very basic examples of this correspondence constructed. We will focus on this part in Chapter 4.

0.1.4. p-adic Interpolation

Recall that a classical L-function, for instance the Riemann zeta function, is an analytic function $\zeta\colon \mathbb{C} \to \mathbb{C}$ (with a simple pole in $s = 1$). It has nice analytic properties and takes rational values at non-positive integers. Since $\mathbb{Z}$ lies in both $\mathbb{C}$ and $\mathbb{Z}_p \subset \mathbb{C}_p$, and is further dense in $\mathbb{Z}_p$, one can question, whether there exists an analytic function

$$\zeta_p\colon \mathbb{Z}_p \to \mathbb{C}_p$$

in the p-adic sense, that 'at least' coincides with ζ at the non-positive integers. We mean such that

$$\zeta(1-n) = (*) \cdot \zeta_p(1-n)$$

at every $n \in \mathbb{N}_0$. We would say in this case, that ζ_p is the *p-adic L-function* attached to ζ, or that it *p-adically interpolates* the values of ζ.
Historically, the first prototype of a p-adic L-function was formally introduced by Kubota and Leopoldt in 1964 (see [KL64]), although its contruction (or rather the continuity of the constructed function) relied on the *Kummer congruences* concerning Bernoulli numbers, which were known for the past 100 years. This function interpolates the *special values* $\{\ldots, -2, -1, 0\}$ of the Riemann zeta function $\zeta(s)$ divided by its Euler factor at p. In other words, the p-adic L-function is a continous function $\zeta_p\colon \mathbb{Z}_p \to \mathbb{C}_p$, such that

$$\zeta_p(1-n) = (1 - p^{-k}) \cdot \zeta(1-n)$$

for all $n \in \mathbb{N}$. And this is the best one can do: There cannot exist a p-adic continuous function interpolating *all* the non-positive values of $\zeta(s)$. Kubota and Leopoldt went a little further and constructed a p-adic L-function, that interpolated simultaneously the non-positive special values of the different Dirichlet L-functions simultaneously. For a nice introduction, see the (yet) unpublished lecture notes of J.R.Jacinto and C.Williams [JW17].

We are interested in $\wp$-adic interpolation of automorphic L-functions. By this analogy, we expect that there must occur a modification the automorphic L-function at $\wp$ as well. Coates-Perrin-Riou ([CPR89]) and Coates ([Coa89]) describe, how one could potentially $\wp$-adically interpolate the L-function attached to a motive and also describes the modification at $\wp$. We shall follow his method: in Chapter 5 we will modify the local zeta integral at $\wp$. Using the description given by Coates ([Coa89]) with the conjectural correspondence between motives and automorphic representations, we will $\wp$-adically interpolatie the L-function of a pair of cuspidal representations $\rho \times \sigma$ of $\mathrm{GL}_{n+1}(\mathbb{A}_k) \times \mathrm{GL}_n(\mathbb{A}_k)$. This is covered in Chapter 6.

1. Preliminaries

1.1. Relevant algebraic structures

We fix a number field $k/\mathbb{Q}$ and denote by $\mathcal{O}_k$ its ring of integers. The letter ν shall be reserved for a variable place of k. This can be either Archimedean (infinite) or non-Archimedean (finite). We will denote by k_ν the completion of k with respect to ν. Every k_ν is a locally compact Hausdorff field when equipped with its standard topology induced by the place ν. If ν is further non-Archimedean, we shall denote by

- $\mathcal{O}_\nu$ its ring of integers,
- $\pi_\nu \in \mathcal{O}_\nu$ a fixed uniformizer of k_ν,
- $\mathfrak{p}_\nu = \pi_\nu \mathcal{O}_\nu$ its maximal ideal,
- $\kappa(\nu) := \mathcal{O}_\nu / \mathfrak{p}_\nu$ its residue field, and by
- q_ν the cardinality of the residue field $\kappa(\nu)$.

Recall also, that for non-Archimedean ν there exists an isomorphism of topological groups

$$k_\nu^\times \cong \mathcal{O}_\nu^\times \times \pi_\nu^{\mathbb{Z}}. \tag{9}$$

This is for example in Chapter II.5., Satz 5.3. of [Neu92].
If ν on the other hand is Archimedean, we shall set

$$\mathcal{O}_\nu^\times := \begin{cases} \mathbb{R}_{>0} & , \nu \text{ real} \\ \mathbb{C}^\times & , \nu \text{ complex} \end{cases}.$$

If ν is now again any place (finite or infinite), we will denote by $|\cdot|_\nu$ the absolute value on k_ν, which is

- for non-Archimedean ν normalized by
$$|\pi_\nu|_\nu := q_\nu^{-1},$$
- for ν real just usual absolute value on $\mathbb{R}$, i.e. $|\cdot|_\nu = |\cdot|_{\mathbb{R}}$,
- and for ν complex the square of the usual complex absolute value on $\mathbb{C}$, i.e. $|\cdot|_\nu = |\cdot|_{\mathbb{C}}^2$.

We denote further by $\mathbb{A} := \mathbb{A}_{\text{fin}} \times \mathbb{A}_\infty$ the ring of *adèles* of k, where $\mathbb{A}_{\text{fin}} = \prod_{\nu<\infty}^{\mathcal{O}_\nu} k_\nu$ is the restricted product of the finite adèles and $\mathbb{A}_\infty = \prod_{\nu|\infty} k_\nu$ denotes the adèles at infinity. The adèles form a locally compact Hausdorff ring. In addition, $\mathbb{A}_{\text{fin}}$ is also totally disconnected. We will interpret k via the diagonal embedding as a topological subring of $\mathbb{A}$. Thus, k is equipped with the discrete topology.
By $\mathbb{A}^\times$ we shall denote the multiplicative group of $\mathbb{A}$ with the initial topology with respect

to the embedding $\mathbb{A}^\times \to \mathbb{A} \times \mathbb{A},\ a \mapsto (a, \frac{1}{a})$. Then $\mathbb{A}^\times$ itself, called the group of *idèles*, becomes a locally compact Hausdorff topological group. We can also embed $k^\times \subset \mathbb{A}^\times$ diagonally, and $k^\times$ inherits the discrete topology as well. Moreover, we have the well-defined *norm map*

$$\|\cdot\|_{\mathbb{A}} : \mathbb{A}^\times \to \mathbb{R}_{>0},\ \|(x_\nu)_\nu\|_{\mathbb{A}} := \prod_\nu |x_\nu|_\nu\,,$$

which is continuous and multiplicative. We will denote by $\mathbb{A}^1$ its kernel. By the well-known Product formula of Artin, $k^\times \subset \mathbb{A}^1$. Moreover, $k^\times \backslash \mathbb{A}^1$ is compact and

$$k^\times \backslash \mathbb{A}^\times \cong \left(k^\times \backslash \mathbb{A}^1\right) \times \mathbb{R}_{>0} \tag{10}$$

as topological groups; see Chapter IV.4., Theorem 6 of [Wei70] for reference.

We shall reserve the letter R for a commutative locally compact topological Hausdorff ring of characteristic 0, but unless stated otherwise, it will only play the role of $\mathbb{A}$, k, or any of its completions k_ν.

The letter $\wp$ shall stand for a fixed non-Archimedean place of k. Posteriorly, we will define cuspidal representations and an additive character and demand all to be unramified at $\wp$; see Hypothesis 1. We will further denote by $F := k_\wp$ the $\wp$-adic completion, and we shall drop the superscript $(\cdot)_\wp$ from any of its attached objects. In other words, we just write $\mathcal{O}$ for $\mathcal{O}_\wp$, and so on.
Of course most results that we will show for F will also hold for any non-Archimedean k_ν, but with this convention, we would like to emphasize which setting will be needed only for the special places $\wp$ dividing p, and also get rid of unnecessary indexing.

Let us now fix a natural number n. We shall denote by GL_n the general linear group as an affine smooth group scheme (over $\mathbb{Z}$). Unfortunately, we will need to carry the index since we will work with the product $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$. In GL_n we find the following distinguished subgroups:

- $T_n \subset \mathrm{GL}_n$, the maximal split torus of GL_n consisting of diagonal matrices.
- $B_n \subset \mathrm{GL}_n$, the standard Borel subgroup consisting of upper triangular matrices,
- $U_n \subset \mathrm{GL}_n$, the unipotent radical of B_n; these are upper triangular matrices with 1's on the diagonal.
- On occasion, we shall denote by B_n^- and U_n^- the counterpart of B_n and U_n, respectively, consisting of lower triangular matrices.

It is well-known that $B_n(R) = T_n(R)U_n(R)$ and that $U_n(R)$ is a normal subgroup of $B_n(R)$. We will further denote by

- $X^*(T_n) \overset{\text{Df.}}{=} \mathrm{Hom}(T_n, \mathrm{GL}_1)$ the group of algebraic characters of T_n, and dually by
- $X_*(T_n) \overset{\text{Df.}}{=} \mathrm{Hom}(\mathrm{GL}_1, T_n)$ the group of algebraic cocharacters of T_n.

Recall that if α and α' are an algebraic character and an algebraic cocharacter of T_n, respectively, then the map $t \mapsto (\alpha \circ \alpha')(t)$ is a character of GL_1 and hence of the form $t^{\langle \alpha, \alpha' \rangle}$ for some integer $\langle \alpha, \alpha' \rangle$. By Lemma 3.2.11. [Spr98], the induced map

$$\langle \cdot, \cdot \rangle : X^*(T_n) \times X_*(T_n) \to \mathbb{Z}$$

is a perfect pairing. We take $\Delta_n = \{\alpha_{ij}\}_{i \neq j} \subset X^*(T_n) = \mathrm{Hom}(T_n, \mathrm{GL}_1)$ to be the standard root system, which is explicitly given by

$$\alpha_{ij} : T_n \to \mathrm{GL}_1, \ \mathrm{diag}(t_1, \ldots, t_n) \mapsto \frac{t_i}{t_j}.$$

In addition, we fix the Bruhat order in the standard way, i.e. such that $\Delta_n = \Delta_n^+ \sqcup \Delta_n^-$, where

$$\Delta_n^+ = \{\alpha_{ij}\}_{i<j}, \qquad \Delta_n^- = \{\alpha_{ij}\}_{i>j}.$$

If $\alpha \in \Delta_n$, we will denote by s_α its corresponding reflection. We will further denote by $\Sigma_n = \{\alpha_i \mid i = 1, \ldots, n-1\}$ the set of simple roots, where we compactly write $\alpha_i := \alpha_{i,i+1}$. We also simply set $s_i := s_{\alpha_i}$ for the reflections corresponding to simple roots. We will denote by $\Delta_n^\vee$ the dual root system of Δ_n. The coroots are explicitly given by

$$\alpha_{ij}^\vee \colon \mathrm{GL}_1 \to T_n, \ t \mapsto \mathrm{diag}(t_l)_{l=1}^n, \qquad t_l = \begin{cases} t & , i = l, \\ t^{-1} & , j = l, \\ 1 & , \text{otherwise} \end{cases}.$$

If X is any locally compact topological Hausdorff space, specially any of the R-rational points of the algebraic groups mentioned above, we will denote by $C_c(X)$ the set of continuous and compactly supported complex functions on X.

For $m \in \mathbb{N}$ we will denote by 1_m the identity matrix of $\mathrm{GL}_m(R)$ (independent on the choice of R). If further $m < n$, we understand $\mathrm{GL}_m(R) \subset \mathrm{GL}_n(R)$ via the embedding

$$g \mapsto \begin{pmatrix} g & \\ & 1_{n-m} \end{pmatrix}. \tag{11}$$

We shall also use the same identification for any of the subgroups of GL_n mentioned above.

Let us assume again that ν is non-Archimedean. Since T_n is k_ν-split, there is the isomorphism

$$X_*(T_n)(k_\nu) \xrightarrow{\sim} T_n(k_\nu) \big/ (T_n(k_\nu) \cap \mathrm{GL}_n(\mathcal{O}_\nu)) = T_n(k_\nu) \big/ T_n(\mathcal{O}_\nu), \ e \mapsto e(\pi_\nu) T_n(\mathcal{O}_\nu).$$

The latter is by (9) isomorphic to $\mathbb{Z}^n$. Furthermore, our computations will be invariant under $T_n(\mathcal{O}_\nu)$ (so in particular invariant under the choice of the uniformizer π_ν). Thus, if $e = (e_1, \ldots, e_n) \in X_*(T_n)(k_\nu) \cong \mathbb{Z}^n$ is a rational cocharacter of $T_n(k_\nu)$, we will simply write

$$\pi_\nu^e = \begin{pmatrix} \pi_\nu^{e_1} & & \\ & \ddots & \\ & & \pi_\nu^{e_n} \end{pmatrix} \in T_n(k_\nu)$$

for the image of the uniformizer $e(\pi_\nu)$, and drop the modulo $T_n(\mathcal{O}_\nu)$ notation. Finally, we call e *dominant*, iff for every simple (and hence for every positive) root α

$$\langle \alpha, e \rangle \geq 0.$$

In the identification as $e \in \mathbb{Z}^n$, this just means

$$e_1 \geq e_2 \geq \ldots \geq e_n. \tag{12}$$

1.2. Relevant characters

A *quasicharacter* of a locally compact Hausdorff group G is a continuous homomorphism $G \to \mathbb{C}^\times$. Under a *character* of G we mean a quasicharacter with image in $\mathbb{S}^1$. If χ_1, χ_2 are two quasi-characters of G, we denote by $\chi_1 \otimes \chi_2$ its product in the sense that

$$(\chi_1 \otimes \chi_2)(g) := \chi_1(g) \cdot \chi_2(g) \qquad (g \in G).$$

We shall use $\mathbb{1}$ for the trivial character (of any group G).

1.2.1. Additive characters

We fix now a non-trivial additive quasicharacter of the adèles $\psi_\mathbb{A} \colon \mathbb{A} \to \mathbb{C}^\times$, that is trivial on k. Thus, $\psi_\mathbb{A}$ factors through the compact group $k \backslash \mathbb{A}$ and thus, its image is already in $\mathbb{S}^1$. We shall denote by ψ_ν the composition $k_\nu \hookrightarrow \mathbb{A} \xrightarrow{\psi_\mathbb{A}} \mathbb{S}^1$. The character $\psi_\mathbb{A}$ then decomposes as $\psi_\mathbb{A} = \otimes_\nu \psi_\nu$ in the sense that for $a := (x_\nu)_\nu \in \mathbb{A}$ we have

$$\psi_\mathbb{A}(a) = \prod_\nu \psi_\nu(x_\nu),$$

and the product is actually finite, i.e. on almost all places ν we have $\psi_\nu(x_\nu) = 1$. Indeed, almost all non-Archimedean local additive characters ψ_ν satisfy $\mathcal{O}_\nu \subset \mathrm{Kern}(\psi_\nu)$ but $\pi_\nu^{-1}\mathcal{O}_\nu \not\subset \mathrm{Kern}(\psi_\nu)$. Non-Archimedean additive characters with this property are called *unramified* (see Cor.1 of Chapter IV.2. in [Wei70]).
Recall that we assume $\psi_\mathbb{A}$ to be unramified at $\wp$.

An additive character of U_n. In the general case, any additive character

$$\psi \colon R \longrightarrow \mathbb{S}^1,$$

induces the character on $U_n(R)$ given by

$$U_n(R) \longrightarrow \mathbb{C}^\times, \quad \begin{pmatrix} 1 & u_{1,2} & * & \ldots & * \\ & 1 & u_{2,3} & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \ddots & u_{n-1,n} \\ & & & & 1 \end{pmatrix} \mapsto \prod_{i=1}^{n-1} \psi(u_{i,i+1}), \tag{13}$$

which, by abuse of notation, will also be denoted by ψ.

1.2.2. Multiplicative quasicharacters

Let us now consider a multiplicative quasicharacter $\chi_{\mathbb{A},s}\colon \mathbb{A}^\times \to \mathbb{C}^\times$, that is trivial on $k^\times$. Unlike ψ, we shall not fix it but keep it variable. In any case, $\chi_{\mathbb{A},s}$ factors through $k^\times \backslash \mathbb{A}^\times$. This is not compact as in the additive case, but due to (10), $\chi_{\mathbb{A},s}$ is of the form

$$\chi_{\mathbb{A},s} = \chi_{\mathbb{A}} \otimes \|\cdot\|_{\mathbb{A}}^{s-1/2}$$

for some $s \in \mathbb{C}$ (unique in the real part) and some character $\chi_{\mathbb{A}}\colon k^\times \backslash \mathbb{A}^\times \to \mathbb{S}^1$. The choice of the shift $-1/2$ in the exponent of $\|\cdot\|_{\mathbb{A}}$ will be explained later. From now on, we will denote by $\chi_{\nu,s}$ the character at ν induced by $\chi_{\mathbb{A},s}$ (in a similar manner as the additive character ψ_ν was induced by $\psi_{\mathbb{A}}$). Then again,

$$\chi_{\mathbb{A},s} \cong \bigotimes_{\nu} \chi_{\nu,s}, \tag{14}$$

and it is well-known, that in this decomposition, almost all $\chi_{\nu,s}$ are unramified. We shall now recall some basics on the local components of this decomposition. We have the following cases on ν:

ν non-Archimedean. Every non-Archimedean $\chi_{\nu,s}$ itself is due to (9) of the form

$$\chi_{\nu,s}(x_\nu) = (\chi_\nu \times |\cdot|_\nu^{s-1/2})(t_\nu \cdot \pi_\nu^e) := \chi_\nu(t_\nu) \cdot |\pi_\nu^e|_\nu^{s-1/2}$$

for the unique decomposition $x_\nu = t_\nu \cdot \pi_\nu^e \in \mathcal{O}_\nu^\times \cdot \pi_\nu^{\mathbb{Z}}$ and for a unique character $\chi_\nu\colon \mathcal{O}_\nu^\times \to \mathbb{S}^1$.

We define further by $\mathfrak{c}(\chi_\nu)$ the *conductor* of χ_ν. This is the smallest integer $\mathfrak{c}(\chi_\nu) \geq 0$ with $\mathrm{Kern}(\chi_\nu) = 1 + \mathfrak{p}_\nu^{\mathfrak{c}(\chi_\nu)}$ (see [Wei70], VII.3, Definition 7). If $\mathfrak{c}(\chi_\nu) > 0$, we say that $\chi_{\nu,s}$ is *ramified*. In the case $\mathfrak{c}(\chi_\nu) = 0$, we understand $\mathrm{Kern}(\chi_\nu) = 1 + \mathfrak{p}_\nu^0 := \mathcal{O}_\nu^\times$ as usual and say that $\chi_{\nu,s}$ is *unramified*. Alternatively, we might also call χ_ν to be ramified or unramified. If χ_ν is unramified, this just means that χ_ν is trivial. that call $1 + \mathfrak{p}_\nu^{\mathfrak{c}(\chi_\nu)}$ the conductor of χ_ν.

The local characters at infinity are described in Prop.9 of Chapter VII.3 in [Wei70]:

ν real. The local real characters $\chi_{\nu,s}$ are of the form

$$\chi_{\nu,s}(x_\nu) = \mathrm{sgn}(x_\nu)^{\epsilon_\nu} \cdot |x_\nu|_\nu^{s-1/2},$$

for some $\epsilon_\nu \in \{0,1\}$, and we call it *unramified*, if $\epsilon_\nu = 0$, and *ramified* otherwise.

ν complex. The local complex characters $\chi_{\nu,s}$ are of the form

$$\chi_{\nu,s}(x_\nu) = \left(\frac{x_\nu}{|x_\nu|_{\mathbb{C}}}\right)^{l_\nu} \cdot |x_\nu|_\nu^{s-1/2},$$

with a unique $l_\nu \in \mathbb{Z}$. We call $\chi_{\nu,s}$ *unramified*, if $l_\nu = 0$, and *ramified* otherwise. Be aware of the relation $|x_\nu|_\nu = |x_\nu|_{\mathbb{C}}^2$.

Multiplicative quasicharacters of GL_n**.** Since the topological abelianization of $\mathrm{GL}_n(R)$ is[4]

$$(\mathrm{GL}_n(R))^{\mathrm{ab}} = \mathrm{GL}_n(R)\Big/\overline{[\mathrm{GL}_n(R),\mathrm{GL}_n(R)]} = \mathrm{GL}_n(R)\Big/\mathrm{SL}_n(R) \cong R^\times,$$

there is a (1: 1)-correspondence between the quasicharacters of $R^\times$ and quasicharacters of $\mathrm{GL}_n(R)$; more precisely, any quasicharacter of $\mathrm{GL}_n(R)$ is of the form

$$\mathrm{GL}_n(R) \xrightarrow{\det} R^\times \to \mathbb{C}^\times,$$

where the latter is a quasicharacter of $R^\times$. Thus, if χ is a quasicharacter of $R^\times$, we will denote the corresponding quasicharacter of $\mathrm{GL}_n(R)$ by abuse of notation

$$\chi(g) := \chi(\det(g)),$$

where $g \in \mathrm{GL}_n(R)$, and there is no restriction in assuming a quasicharacter of $\mathrm{GL}_n(R)$ to be of this form. By the same principle, the adelic norm extends (uniquely) to the multiplicative map

$$\|\cdot\|_{\mathbb{A}} : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{R}_{>0},\ \|\, g\,\|_{\mathbb{A}} := \|\det(g)\|_{\mathbb{A}}\,.$$

Same of course applies for the absolute value of each local completion k_ν, but we will write

$$\|g_\nu\|_\nu := |\det(g_\nu)|_\nu$$

for $g_\nu \in \mathrm{GL}_n(k_\nu)$.

1.2.3. The local Gauss sum

We focus now on $F = k_\wp$. For $\chi = \chi_\wp$, $\psi = \psi_\wp$ and for $e \in \mathbb{Z}$, we define the e-twisted **local Gauss-sum** by

$$\mathfrak{G}(e,\chi) := \mathfrak{G}_\wp(e,\chi_\wp) := \int_{\mathcal{O}^\times} \chi(t)\cdot\psi(\pi^e t)\, d^\times t.$$

For $e = 0$, this coincides with the definition of [RD91] in Section 7.1. The definition does depend on ψ as well, but ψ is fixed while χ is variable. The local Gauss sum satisfies the following:

Lemma 1.1.

a) If χ is unramified, i.e. $\chi = \mathbb{1}$, then

$$\mathfrak{G}(e,\mathbb{1}) = \begin{cases} 0 & , e \le -2 \\ \frac{1}{1-q} & , e = -1 \\ 1 & , e \ge 0 \end{cases}.$$

In this case, we drop $\mathbb{1}$ from the notation and write just $\mathfrak{G}(e)$.

[4] Warning! This is in general not true, but it does hold for $\mathbb{A}$ or fields of characteristic 0.

b) If χ is ramified (i.e. $\mathfrak{c}(\chi) > 0$) and $e \neq -\mathfrak{c}(\chi)$, we have

$$\mathfrak{G}(e, \chi) = 0.$$

Hence, in this case, we drop e from the notation and write $\mathfrak{G}(\chi) := \mathfrak{G}(-\mathfrak{c}(\chi), \chi)$ instead.

This is proven in the Appendix A.

1.2.4. Modular character of the local Borel subgroup

We continue with $F = k_\wp$. Recall that $B_n(F)$ is for $n \geq 2$ not unimodular. We will denote by δ_n its *modular quasicharacter*. It is defined as follows: if μ is any left invariant Haar measure on $B_n(F)$ and $b \in B_n(F)$, then $d\mu^{(b)}(A) := \mu(b^{-1}Ab) = \mu(Ab)$ is again a left invariant Haar measure and $\delta_n(b) := \frac{\mu(A)}{\mu(Ab)}$, where A is any non-zero Borel set in $B_n(F)$. δ_n is explicitly given by

$$\delta_n(b) = \delta_n(tu) = \prod_{i=1}^{n} |t_i|^{n+1-2i} = |t_1|^{n-1} \cdot |t_1|^{n-3} \cdot \ldots \cdot |t_n|^{1-n}$$

for $b = tu \in T_n(F)U_n(F)$ and thus, is indeed a quasicharacter of $B_n(F)$.

Let us now suppose that we are given $m \leq n$. The following will only be the case for $F = k_\wp$. We interpret $T_m(F) \subset T_n(F)$ by means of the embedding (11). A straightforward computation shows that

$$\delta_n^{1/2}(\pi^e) \cdot \delta_m^{1/2}(\pi^e) = \left(\| \cdot \|^{\left(\frac{n-m}{2}\right)} \otimes \delta_m \right) (\pi^e) \,. \tag{15}$$

This little formula will be useful in Chapter 5.

1.3. The Weyl Group

We still continue with $F = k_\wp$. We will denote by

$$W_n := W_n(\mathrm{GL}_n(F), T_n(F)) := N_{\mathrm{GL}_n(F)}(T_n(F)) \big/ C_{\mathrm{GL}_n(F)}(T_n(F))$$

the Weyl-Group of the pair $(\mathrm{GL}_n(F), T_n(F))$, where $N_{\mathrm{GL}_n(F)}(T_n(F))$ is the normalizer of $T_n(F)$ in $\mathrm{GL}_n(F)$, and $C_{\mathrm{GL}_n(F)}(T_n(F)) = T_n(F)$ its centralizer. It is well-known that W_n is generated by the simple reflections $\{s_i \mid i = 1, \ldots, n-1\}$ and that $W_n \cong S_n$ (the standard symmetric group on n elements). We shall use different representations of W_n, but most frequently we will understand its elements as permutations of $\{1, \ldots, n\}$, or as permutation matrices in $\mathrm{GL}_n(\mathcal{O})$. As mentioned before, there is no danger for our computations in chosing another representative in $\mathrm{GL}_n(\mathcal{O})$. We will consider three diferent actions of the Weyl group:

a) W_n acts naturally (from the left) on the character group $X^*(T_n)(F)$ via

$$(w \cdot \chi)(t) := \chi(w^{-1}tw),$$

b) Via the isomorphism $W_n \cong S_n$, the Weyl group W_n acts naturally (from the left) on $\mathbb{Z}^n \cong X_*(T_n)(F)$ via

$$w \cdot (e_1, \ldots, e_n) := (e_{w^{-1}(1)}, \ldots, e_{w^{-1}(n)}).$$

c) W_n acts (from the right) on the set of $\mathbb{C}$-valued unramified complex-valued characters $\tau\colon T_n(F) \to \mathbb{C}^\times$, i.e. continuous homomorphisms with $\mathrm{Kern}(\tau) = T_n(\mathcal{O})$. For this, we take any representative of w in $\mathrm{GL}_n(\mathcal{O})$, which we will also denote by w, and define

$$(\tau^w)(t) := \tau(wtw^{-1}) = (\tau_1, \ldots, \tau_n)^w(t) = (\tau_{w^{-1}(1)}, \ldots, \tau_{w^{-1}(n)})(t).$$

This is well-defined as τ is unramified.

Furthermore, we denote by

$$w_n := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(\mathcal{O})$$

the long Weyl-element in W_n. This is an involution, which permutes the positive and negative roots, i.e. $w_n \Delta^\pm = \Delta^\mp$.

At last, we will denote by $l(w)$ the *length* of $w \in W_n$ in the usual manner; $l(w)$ is the minimal number r of simple reflections $s_{i_1}, \ldots, s_{i_r}$, such that $w = s_{i_1} \cdot \ldots \cdot s_{i_r}$.

1.4. The Iwahori subgroup

Let us now consider the canonical map

$$\mathcal{O} \to \mathcal{O}/\mathfrak{p} = \mathbb{F}_q,$$

which by functoriality of GL_n induces the canonical map

$$\mathrm{pr}\colon \mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathbb{F}_q).$$

The *Iwahori subgroup* of $\mathrm{GL}_n(\mathcal{O})$ is defined to be

$$J_n := \mathrm{pr}^{-1}(B_n(\mathbb{F}_q)) = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \ldots & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ \mathfrak{p} & \ldots & \mathfrak{p} & \mathcal{O}^\times \end{pmatrix}.$$

Let us take now any Weyl element $w \in W_n$. By Lemma 7 in [Bum87], the length $l(w)$ of w can be expressed as the number of negative roots $\alpha \in \Delta^-$, that satisfy $w\alpha \in \Delta^+$. Thus,

$$q^{l(w)} = [U_n(\mathcal{O})\colon U_n(\mathcal{O}) \cap wJ_nw^{-1}]. \tag{16}$$

1.5. On space decompositions

Recall the well-known decompositions in the $\wp$-adic setting:

- the (already mentioned) Iwasawa-decomposition
$$\mathrm{GL}_n(F) = B_n(F)\,\mathrm{GL}_n(\mathcal{O}), \tag{17}$$
- the Bruhat-decomposition of $\mathrm{GL}_n(\mathcal{O})$ ([Cas80], Chapter 1) as
$$\mathrm{GL}_n(\mathcal{O}) = \coprod_{w \in W_n} J_n w J_n, \tag{18}$$
- and the Iwahori-decomposition
$$J_n = (J_n \cap U_n^-(\mathcal{O}))(J_n \cap T_n(\mathcal{O}))(J_n \cap U_n(\mathcal{O})) = U_n^-(\mathfrak{p}) \cdot T_n(\mathcal{O}) \cdot U_n(\mathcal{O}) \tag{19}$$
where $U_n^-(\mathcal{O}) = w_n U_n(\mathcal{O}) w_n$ is the group of lower-diagonal $(n \times n)$-matrices with 1's on the diagonal.
- More specifically, one can refine the Iwahori-decomposition as follows: if $w \in W_n$, then
$$wJ_nw^{-1} = (wJ_nw^{-1} \cap U_n^-(\mathcal{O})) \cdot T_n(\mathcal{O}) \cdot (wJ_nw^{-1} \cap U_n(\mathcal{O})). \tag{20}$$
It is important to mention, that the factors in both (19) and (20) can be written in any order.

Combining the first three decompositions (17), (18) and (19) together, we obtain the *generalized Bruhat-decomposition* (Chapter I of [Cas80] or alternatively, [MI65] Prop. 2.33.)
$$\mathrm{GL}_n(F) = \coprod_{w \in W_n} B_n(F) w J_n. \tag{21}$$
Furthermore, we can split $B_n(F)$ as $U_n(F)T_n(F)$ and thus, using the fact that $T_n(F) = \pi^{\mathbb{Z}^n} \cdot T_n(\mathcal{O})$ and that $T_n(\mathcal{O}) \subset J_n$, we obtain the finer decomposition
$$\mathrm{GL}_n(F) = \coprod_{\substack{e \in \mathbb{Z}^n \\ w \in W_n}} U_n(F) \cdot \pi^e \cdot w \cdot J_n, \tag{22}$$
to which we will also simply refer to as the generalized Bruhat-decomposition.

1.6. On measures

If Y is any space with a measure dy, and $K \subset Y$ a measurable set, we shall denote by
$$\mathrm{vol}(K, dy)$$
its volume with respect to dy.

Let us now come back to the general case of an arbitrary completion k_ν. We know that

k_ν is *self-dual*; the non-trivial additive character $\psi_\nu : k_\nu \to \mathbb{S}^1$ induced by $\psi_\mathbb{A}$ defines the isomorphism of topological groups

$$(k_\nu, +) \cong \hat{k_\nu},\ x_\nu \mapsto \psi_\nu(x_\nu \cdot -), \tag{23}$$

where $\hat{k_\nu} := \mathrm{Hom}_{\mathrm{Cont}}(k_\nu, \mathbb{S}^1)$ is the character group of k_ν equipped with the compact-open topoology. This is Theorem 3 in Chapter II.5 of [Wei70].

We shall denote by dx_ν the unique additive Haar measure on $(k_\nu, +)$, which is self-dual[5] with respect to the isomorphism (23) induced by ψ_ν. The absolute value $|\cdot|_\nu$ was chosen precisely such that the transformation $x_\nu \mapsto y_\nu x_\nu$ multiplies dx_ν with $|\, y_\nu \,|_\nu$. We shall further denote by $d^\times x_\nu$ the multiplicative Haar measure on $k_\nu^\times$ normalized as follows:

$$d^\times x_\nu = m_\nu \cdot \frac{dx_\nu}{|x_\nu|_\nu} \qquad \text{with} \qquad m_\nu = \begin{cases} 1 & , \nu \text{ Archimedean} \\ \left(1 - \frac{1}{q_\nu}\right)^{-1} & , \nu \text{ non-Archimedean} \end{cases} \tag{24}$$

This fixation has the following advantage: if ψ_ν is unramified at a non-Archimedean place ν, then

$$\mathrm{vol}(\mathcal{O}_\nu, dx_\nu) = 1 = \mathrm{vol}(\mathcal{O}_\nu^\times, d^\times x_\nu). \tag{25}$$

Since we will usually denote by t_ν elements in $\mathcal{O}_\nu^\times$ (or in general $t_\nu = (t_{\nu,1}, \ldots, t_{\nu,n}) \in T_n(\mathcal{O}_\nu)$), we shall use $d^\times t_\nu$ for the restriction of $d^\times x_\nu$ to $\mathcal{O}_\nu^\times$.

For $a := (x_\nu)_\nu \in \mathbb{A}$ we set $da := \prod_\nu dx_\nu$. This implies that da is self-dual as well. For the multiplicative Haar measure on $\mathbb{A}^\times$ we simply set $d^\times a := \prod_\nu d^\times x_\nu$.

The 1-dimensional additive Haar measure on k_ν gives rise to the Haar measure on $U_n(k_\nu)$: For $u_\nu := (u_\nu^{ij})_{i,j}$ we set (symbolically)

$$du_\nu := \prod_{i<j} du_\nu^{ij},$$

where each du_ν^{ij} is the additive measure on k_ν. This automatically implies, that if ν is a finite place where ψ_ν is unramified, then

$$\mathrm{vol}(U_n(\mathcal{O}_\nu), du_\nu) = 1. \tag{26}$$

Let us further fix a Haar measure $d^\times g_\nu$ on $\mathrm{GL}_n(k_\nu)$. We could demand a similar condition as in (24), but we just shall normalize the measure at the non-Archimedean ν where ψ_ν is unramified by

$$\mathrm{vol}(\mathrm{GL}_n(\mathcal{O}_\nu), d^\times g_\nu) = 1. \tag{27}$$

We do not really care about the remaining places.

We will denote by du and $d^\times g$ the Haar measures on $U_n(\mathbb{A})$ and $\mathrm{GL}_n(\mathbb{A})$, respectively, both defined in the same manner as in the 1-dimensional case: for $u := (u_\nu)_\nu \in U_n(\mathbb{A})$ and $g := (g_\nu)_\nu \in \mathrm{GL}_n(\mathbb{A})$ we set

$$du := \prod_\nu du_\nu \quad \text{and} \quad d^\times g := \prod_\nu d^\times g_\nu$$

for their respective measures.

[5] This means, that $\hat{\hat{f}}(x) = f(-x)$ under the Fourier Inversion Formula (which depends on dx_ν), where $f \in L^1(k_\nu) \cap L^2(k_\nu)$.

1.6.1. Remarks on the Iwahori Group

We return now to the case $F = k_\wp$ again.

a) The Iwahori-group J_n sits open inside $\mathrm{GL}_n(\mathcal{O})$ and thus has positive volume w.r.t. the Haar measure $d^\times g$. Hence, the Haar measure $d^\times j$ on J_n coincides up to a positive constant with the restriction of $d^\times g$ on J_n. Although we will write $d^\times j$ instead of $d^\times g$ to emphasize the difference of the underlying group that we are integrating over, informally said, we keep the normalization of $\mathrm{GL}_n(\mathcal{O})$ having volume 1, i.e. we want to have $d^\times j = d^\times g$ on J_n. Now since

$$J_n = \mathrm{pr}^{-1}\left(B_n\left(\mathbb{F}_q\right)\right), \qquad \mathrm{GL}_n(\mathcal{O}) = \mathrm{pr}^{-1}\left(\mathrm{GL}_n\left(\mathbb{F}_q\right)\right),$$

which implies

$$\mathrm{vol}(J_n, d^\times j) = \mathrm{vol}(J_n, d^\times g) = \frac{1}{[\mathrm{GL}_n(\mathcal{O})\colon J_n]} = \frac{1}{[\mathrm{GL}_n\left(\mathbb{F}_q\right) : B_n\left(\mathbb{F}_q\right)]},$$

we derive that

$$\mathrm{vol}(J_n, d^\times j) = \frac{(q-1)^n \cdot q^{\frac{(n-1)n}{2}}}{\prod_{i=0}^{n-1} q^n - q^i}. \tag{28}$$

b) Throughout the text we will encounter the groups

$$U_n^{(w)} := J_n \cap w^{-1} U_n(\mathcal{O}) w, \tag{29}$$

where $w \in W_n$. We will choose the Haar measure $du^{(w)}$ on these as the push-forward of the Haar measure du on $U_n(F)$ via the map

$$U_n(\mathcal{O}) \cap w J_n w^{-1} \longrightarrow J_n \cap w^{-1} U_n(\mathcal{O}) w, \ u \mapsto w^{-1} u w.$$

In particular we have

$$\mathrm{vol}(U_n^{(w)}, du^{(w)}) = \mathrm{vol}(U_n(\mathcal{O}) \cap w J_n w^{-1}, du) = q^{-l(w)} \tag{30}$$

by (16).

1.6.2. On measure decomposition

In general, we will encounter integrals over quotient spaces of the form $U \backslash G$, where G is a locally compact Hausdorff group, $U \subset G$ a closed subgroup, and the respective Haar measures on both G and U have previously been fixed. We shall use the example of $U_n(R) \backslash \mathrm{GL}_n(R)$ for R a local field or $\mathbb{A}$ in order to show, how to choose a suitable measure on the quotient space.

The Haar measures on $U_n(R)$ and $\mathrm{GL}_n(R)$ have been already defined. On $U_n(R) \backslash \mathrm{GL}_n(R)$ we choose the (up to a positive scalar) unique right $\mathrm{GL}_n(R)$-invariant Radon measure, as described in Section 1.5.3 of [Dei14]. In order to distinguish the quotient measure from the Haar measure on $\mathrm{GL}_n(R)$ for now, we will write $d^\times g_U$ for the first one. The measure $d^\times g_U$ is then uniquely given by the condition

$$\int_{\mathrm{GL}_n(R)} f(g)d^\times g = \int_{U_n(R)\backslash \mathrm{GL}_n(R)} \int_{U_n(R)} f(ug_U)du\, d^\times g_U \tag{31}$$

for any $f \in C_c(\mathrm{GL}_n(R))$, and thus, we symbolically write $d^\times g = du\, d^\times g_U$. From now on, we will denote this quotient measure, by abuse of notation, by the same symbol as the Haar measure on $\mathrm{GL}_n(R)$; in this case $d^\times g_U = d^\times g$.

In Chapter 5, we will need to compute an integral over $U_n(F)\Big\backslash \mathrm{GL}_n(F)$ for $F = k_\wp$. Let us thus see now, how to decompose the integral (31) in this case further:
The Iwasawa-Decomposition (17) gives us, due to [Dei14], Prop.1.5.6., the Haar measure decomposition of $d^\times g$ into Haar measures on $B_n(F)$ and $\mathrm{GL}_n(\mathcal{O})$. But the Haar measure on $\mathrm{GL}_n(\mathcal{O})$ is just the restriction of $d^\times g$. We will denote it by $d^\times h$. Then, one has for any $f \in C_c(\mathrm{GL}_n(F))$

$$\int_{\mathrm{GL}_n(F)} f(g)d^\times g = \int_{\mathrm{GL}_n(\mathcal{O})} \int_{B_n(F)} f(bh)d_L b\, d^\times h \tag{32}$$

for a suitable Haar Measure $d_L b$ on $B_n(F)$. The groups $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(\mathcal{O})$ are both unimodular, but $B_n(F)$ is not, and therefore one needs to take $d_L b$ to be the left invariant Haar measure on $B_n(F)$, as pointed out in 4.1. of [Car79]. This decomposition forces $d_L b$ to be normalized, such that

$$\mathrm{vol}(B_n(\mathcal{O}), d_L b) = \int_{B_n(\mathcal{O})} d_L b = \int_{B_n(F)\cap \mathrm{GL}_n(\mathcal{O})} d_L b = 1.$$

Since $d_L b$ is left invariant, *loc.cit.* gives us a further decomposition with respect to $d_L b$ as

$$\int_{B_n(F)} f(b)d_L b = \delta_n^{-1}(t) \cdot \int_{T_n(F)} \int_{U_n(F)} f(ut)du\, d^\times t$$

for a suitable Haar measure $d^\times t$ on $T_n(F)$ and for any $f \in C_c(B_n(F))$. The quasimodular character $\delta_n^{-1}(t)$ is just the Jacobian of the transformation $u \mapsto tut^{-1}$. Since $d_L b$ and du have already been fixed, this implies that the Haar measure $d^\times t$ on $T_n(F)$ is fixed by the normalization $\mathrm{vol}(T_n(\mathcal{O}), d^\times t) = 1$. Thus, equation (32) can be rewritten as

$$\int_{\mathrm{GL}_n(F)} f(g)d^\times g = \delta_n^{-1}(t) \int_{\mathrm{GL}_n(\mathcal{O})} \int_{T_n(F)} \int_{U_n(F)} f(uth)du\, d^\times t\, d^\times h$$

for any $f \in C_c(\mathrm{GL}_n(F))$. Since $T_n(\mathcal{O}) \subset \mathrm{GL}_n(\mathcal{O})$ and $\mathrm{vol}(T_n(\mathcal{O}), d^\times t) = 1$, the latter can be rewritten as

$$\int_{\mathrm{GL}_n(F)} f(g)d^\times g = \sum_{e\in\mathbb{Z}^n} \delta_n^{-1}(\pi^e) \int_{\mathrm{GL}_n(\mathcal{O})} \int_{U_n(F)} f(u\pi^e h)du\, d^\times h.$$

This implies, that for every $f \in C_c(U_n(F)\backslash\mathrm{GL}_n(F))$ one has

$$\int_{U_n(F)\backslash\mathrm{GL}_n(F)} f(g)d^\times g_U = \sum_{e\in\mathbb{Z}^n} \delta_n^{-1}(\pi^e) \int_{\mathrm{GL}_n(\mathcal{O})} f(\pi^e h)d^\times h. \tag{33}$$

We now go one step further: Let us fix a Weyl element $w \in W_n$. By [MI65], Prop.3.2., we know that $\#(J_n w J_n/J_n) = q^{l(w)}$. Thus, the generalized Bruhat decomposition (22) tells us that

$$\int_{U_n(F)\backslash\mathrm{GL}_n(F)} f(g)d^\times g_U = \sum_{e\in\mathbb{Z}^n}\sum_{w\in W_n} \delta_n^{-1}(\pi^e)\cdot q^{l(w)}\cdot \int_{J_n} f(\pi^e wj)d^\times j,$$

where $d^\times j$ is the restriction of $d^\times h = d^\times g_{|\mathrm{GL}_n(\mathcal{O})}$ to J_n as explained in 1.6.1. Furthermore, since f is left $U_n(F)$-invariant, we have

$$f(\pi^e wj) = f(\pi^e w)$$

for all $j \in U_n^{(w)}$. The Quotient Integral Formula 1.5.3. in [Dei14] together with (30) tells us thus, that we can decompose this further as

$$\int_{J_n} f(\pi^e wj)d^\times j = \int_{U_n^{(w)}\backslash J_n}\int_{U_n^{(w)}} f(\pi^e wj_w)du^{(w)}\, d^\times j_w \overset{(30)}{=} q^{-l(w)}\cdot \int_{U_n^{(w)}\backslash J_n} f(\pi^e wj_w)d^\times j_w,$$

where $d^\times j_w$ once again denotes the right J_n-invariant Radon measure on $U_n^{(w)}\backslash J_n$ as given by *loc.cit.*
When there is no danger of ambiguity, we will denote this measure also by $d^\times j$ instead of $d^\times j_w$. This said, if we put everything together, we obtain the following...

Proposition 1.1. For any integrable function f on the quotient $U_n(F)\big\backslash\mathrm{GL}_n(F)$ we have

$$\int_{U_n(F)\backslash\mathrm{GL}_n(F)} f(g)d^\times g_U = \sum_{e\in\mathbb{Z}^n} \delta_n^{-1}(\pi^e)\cdot \sum_{w\in W_n}\int_{U_n^{(w)}\backslash J_n} f(\pi^e wj)d^\times j. \tag{34}$$

Proof. We have seen so far that (34) is an equality that holds for every compactly supported and continuous function with respect to Radon measures. Together with the Riesz Integration Theorem ([Rud87], Prop. 2.14), it follows that (34) indeed holds for every integrable function f on the quotient $U_n(F)\big\backslash\mathrm{GL}_n(F)$. □

1.7. On conjugations

We make a last little remark on conjugations, that will simplify our computations. Let $g = (g_{i,j})_{i,j} \in \mathrm{GL}_n(F)$ be arbitrary. If $w \in W_n$ is a Weyl element, then

$$w^{-1}gw = (g_{w(i),w(j)})_{i,j} = (g_{i,j})_{w^{-1}(i),w^{-1}(j)}. \tag{35}$$

Similarly, for a diagonal element $d := \mathrm{diag}(d_1, \dots, d_n) \in T_n(F) \subset \mathrm{GL}_n(F)$, we have

$$dgd^{-1} = \left(\frac{d_i}{d_j} \cdot g_{i,j} \right)_{i,j}. \qquad (36)$$

Specially, if $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$, we have with the previous identity (36)

$$\pi^e g \pi^{-e} = (\pi^{e_i - e_j} \cdot g_{i,j}). \qquad (37)$$

2. Automorphic Representation Theory

In this chapter we shall recall some basic notions of automorphic representation theory.

2.1. Representations of ℓ-groups

Let us assume for a moment that G is an ℓ-group in the sense of Bernstein-Zelevinsky (see 1.1. in [BZ76]). This means that G is a locally compact, totally disconnected, Hausdorff group.

Example 1. Important examples of ℓ-groups for us are:

a) $\mathrm{GL}_n(k_\nu)$, where ν is any non-Archimedean place of k,

b) $\mathrm{GL}_n(\mathbb{A}_{\mathrm{fin}})$, as restricted products of ℓ-groups are ℓ-groups.

A linear representation (ρ, V) of G is called *smooth*, if

$$\rho \colon G \times V \to V$$

is continuous, when V is equipped with the discrete topology. This is equivalent to

$$\mathrm{Stab}_G(v) \subset G$$

being open for every $v \in V$. Smooth representations possess a 'primary decomposition': Fix any open compact subgroup $K \subset G$. Then (as K-representations),

$$V = \bigoplus_{\alpha \in \widehat{K}} V(\alpha),$$

where $\widehat{K}$ denotes the set of equivalence classes of irreducible finite-dimensional representations of K. Furthermore, we call a smooth representation (ρ, V) *admissible*, if every isotypic component $V(\alpha)$ is of finite dimension. This condition is equivalent to $\dim V^K < \infty$, and is independent of the choice of K.

2.1.1. Hecke algebras

Recall that the representation theory of G is encoded in the Hecke algebra attached to G. To be more precise, let us first define what a Hecke algebra is. If K is any open compact subgroup of an ℓ-group G, we define the *Hecke algebra* attached to the pair (G, K) as

$$\mathcal{H}(G, K) := C_c(K \backslash G / K), \tag{38}$$

with the convolution product given by

$$(f_1 * f_2)(y) := \int_K f_1(x) f_2(x^{-1}y) dx,$$

where dx is any (left invariant) Haar measure on G. To keep computations easier, dx is usually normalized by the condition $\mathrm{vol}(K, dx) = 1$. The convolution integral is well-defined as it is (as a function of x) locally constant and compactly supported. With this product, $\mathcal{H}(G, K)$ becomes an associative and unital $\mathbb{C}$-algebra; see Section 1.3. of [Car79] for more on Hecke algebras. Moreover, there is an equivalence of categories

$$\begin{array}{rcl} \mathrm{Rep}^{\mathrm{sm}}_{\mathbb{C}}(G)^K & \cong & \mathcal{H}(G, K)\text{-mod} \\ (\rho, V) & \mapsto & (\widetilde{\rho}, V) \end{array}$$

between K-invariant complex representations of G (this automatically implies smoothness) and complex vector spaces with a $\mathcal{H}(G, K)$-module structure, where $\widetilde{\rho}$ arises by integral extension as follows:

$$(\widetilde{\rho}(g))\,(v) := \int_G g(x) \cdot \rho(x)(v) dx. \tag{39}$$

Due to this equivalence, we call the elements of a Hecke algebra *Hecke operators*. Moreover, every Hecke operator is a finite $\mathbb{C}$-linear combination of *pure* Hecke operators, which are characteristic functions of some double coset of the form KgK with $g \in G$. We will adopt this notation for the Hecke operators - as finite $\mathbb{C}$-linear combinations of double cosets of the form KgK. Furthemore, any such pure Hecke operator possesses a finite left coset decomposition

$$KgK = \coprod_i g_i K. \tag{40}$$

A priori, the integral in (39) is a Bochner integral, but due to the decomposition (40), the action of the Hecke algebra as in (39) is actually a finite sum: for a pure Hecke operator KgK with decomposition as given in (40), we have

$$(\widetilde{\rho}(g))\,(v) = \sum_i \rho(g_i)(v). \tag{41}$$

2.1.2. Non-Archimedean generic representations

Let us fix a non-Archimedean place ν of k and consider the ℓ-group $\mathrm{GL}_n(k_\nu)$. A very important family of representations of $\mathrm{GL}_n(k_\nu)$ are the *generic* representations: The non-trivial additive character ψ_ν of k_ν induces a generic character on $U_n(k_\nu)$ as explained in (13). We define the *ψ_ν-Whittaker space* to be the smoothly induced space

$$\mathfrak{W}(\psi_\nu) := \mathrm{Ind}_{U_n(k_\nu)}^{\mathrm{GL}_n(k_\nu)}(\psi_\nu).$$

Its elements are called *ψ_ν-Whittaker functions*, but if there is no danger of ambiguity, we refer to these just as Whittaker functions. Hence, a Whittaker function is a function

$$\mathcal{W} \colon \mathrm{GL}_n(k_\nu) \to \mathbb{C},$$

such that

a) for every $u \in U_n(k_\nu), g \in \mathrm{GL}_n(k_\nu)$

$$\mathcal{W}(ug) = \psi_\nu(u) \cdot \mathcal{W}(g), \tag{42}$$

and

b) $\mathcal{W}$ is smooth, i.e. there exists an open compact subgroup $K \subset \mathrm{GL}_n(k_\nu)$, such that

$$\mathcal{W}(gx) = \mathcal{W}(g) \tag{43}$$

for every $g \in \mathrm{GL}_n(k_\nu)$ and every $x \in K$. Whittaker functions with this property are called *K-spherical*. If $K = \mathrm{GL}_n(\mathcal{O}_\nu)$ is the maximal open compact subgroup of $\mathrm{GL}_n(k_\nu)$, we will call the correspondent Whittaker functions just *spherical*.

We call an irreducible admissible representation (ρ_ν, V_ν) of $\mathrm{GL}_n(k_\nu)$ to be *generic*, if there exists an embedding of representations

$$\mathfrak{W}\colon (\rho_\nu, V_\nu) \hookrightarrow \mathfrak{W}(\psi_\nu).$$

By the Local Multiplicity One-Theorem of Gelfand-Kazhdan [GK75], every irreducible admissible representation of $\mathrm{GL}_n(k_\nu)$ appears inside $\mathfrak{W}(\psi_\nu)$ with multiplicity at most one. In this case, the image $\mathfrak{W}(\rho_\nu, \psi_\nu) := \mathfrak{W}\left((\rho_\nu, V_\nu)\right)$ inside $\mathfrak{W}(\psi_\nu)$ is called the *Whittaker-model* of (ρ_ν, V_ν).

Remark 1. In fact, there is a classification of generic representations of $\mathrm{GL}_n(k_\nu)$ due to Zelevinsky, see Theorem 9.7. of [Zel80] and Theorem 9.3 of [PR00], respectively.

2.1.3. Non-Archimedean unramified representations

We continue with $\mathrm{GL}_n(k_\nu)$ for some non-Archimedean place ν of k. The family of unramified representations (at finite places) plays a central role in the study of automorphic representations. An irreducible admissible representation (ρ_ν, V_ν) of $\mathrm{GL}_n(k_\nu)$ is called *unramified*, if $V_\nu^{\mathrm{GL}_n(\mathcal{O}_\nu)} \neq 0$. In this case, $\dim_{\mathbb{C}}\left(V_\nu^{\mathrm{GL}_n(\mathcal{O}_\nu)}\right) = 1$, since $(\mathrm{GL}_n(k_\nu), \mathrm{GL}_n(\mathcal{O}_\nu))$ is a Gelfand-pair. The unramified representations arise as follows:

Consider an *unramified* character of the torus

$$\tau := (\tau_1, \dots, \tau_n)\colon T_n(k_\nu) \to \mathbb{C}^\times.$$

τ being unramified means that $\tau_{|T_n(\mathcal{O})} \equiv 1$. This character inflates to a character of $B_n(k_\nu)$ acting trivially on $U_n(k_\nu)$ and we define

$$\begin{aligned} I(\tau) &= \mathrm{Ind}_{B_n(k_\nu)}^{\mathrm{GL}_n(k_\nu)}(\tau) \\ &= \left\{ f\colon \mathrm{GL}_n(k_\nu) \xrightarrow{\text{loc.cst.}} \mathbb{C} \mid \forall (b,g) \in B_n(k_\nu) \times \mathrm{GL}_n(k_\nu)\colon f(bg) = \left(\delta_n^{1/2} \otimes \tau\right)(b) f(g) \right\}. \end{aligned}$$

Representations of the type $I(\tau)$, i.e. those parabolically induced from a Borel subgroup, are called *principal series representations* of $\mathrm{GL}_n(k_\nu)$. Such a representation has a unique irreducible quotient, which we denote by $Q(\tau)$. By Theorem 9.10 of [PR00], if we assume[6] that $\tau_j \ncong \tau_i \otimes |\cdot|$ for $1 \leq i < j \leq n$, then $Q(\tau)$ is unramified, and on the contrary, any unramified irreducible admissible representation of $\mathrm{GL}_n(k_\nu)$ is isomorphic to such a $Q(\tau)$ with the stated property.

It is due to Theorem 4.2. in [BZ77], that the unramified $I(\tau)$ is irreducible, iff $\tau_i \ncong \tau_j \otimes |\cdot|$ for all (i,j). This is exactly the case, when $I(\tau)$ is generic, see Theorem 9.7. of *loc.cit.* and Theorem 9.3. of [PR00], respectively.

Remark 2. If we relax the condition $\tau_j \ncong \tau_i \otimes |\cdot|$ for $1 \leq i < j \leq n$, then $Q(\tau)$ is K-spherical for $K = J_n \subset \mathrm{GL}_n(\mathcal{O}_\nu)$ the local Iwahori subgroup.

Example 2. a) The trivial representation $\mathbb{1}$ appears as the unique irreducible quotient

$$\mathrm{Ind}_{B_n(k_\nu)}^{\mathrm{GL}_n(k_\nu)}(|\cdot|^{\frac{n-1}{2}}, |\cdot|^{\frac{n-3}{2}}, \ldots, |\cdot|^{-\frac{n-1}{2}}) \to \mathbb{1} \to 0.$$

It is unramified, but not generic.[7]

b) On the other hand, the Steinberg representation St_n is the unique irreducible quotient

$$\mathrm{Ind}_{B_n(k_\nu)}^{\mathrm{GL}_n(k_\nu)}(|\cdot|^{-\frac{n-1}{2}}, |\cdot|^{-\frac{n-3}{2}}, \ldots, |\cdot|^{\frac{n-1}{2}}) \to \mathrm{St}_n \to 0.$$

It is not unramified, but it is generic. It is K-spherical for $K = J_n \subset \mathrm{GL}_n(\mathcal{O}_\nu)$ the Iwahori subgroup as mentioned in Remark 2.

2.1.4. Iwahori-spherical representations

We will only require the following in the case when $F = k_\wp$. Let us consider again an unramified and *regular* character of the torus

$$\tau = (\tau_1, \ldots, \tau_n) \colon T_n(F) \to \mathbb{C}^\times.$$

Regularity means that $\tau^w \ncong \tau$ for all $w \in W_n$. As mentioned previously, the representation $I(\tau)$ is known to be generic and Iwahori-spherical. The intertwiner into its Whittaker model

$$\mathfrak{W}^\tau \colon I(\tau) \xrightarrow{\sim} \mathfrak{W}(\tau, \psi) \subset \mathfrak{W}(\psi),$$

is (up to a constant) explicitly given as follows: we set $\mathfrak{W}^\tau(f) := \mathcal{W}_f^\tau$, where

$$\mathcal{W}_f^\tau(g) \overset{\text{Df.}}{=} \int_{U_n(F)} f(w_n u g) \cdot \overline{\psi(u)} du, \tag{44}$$

whenever it converges. Moreover, if we impose the condition on τ, that $|\tau(\alpha^\vee(\pi))| < 1$ for every simple root α, which means that

$$|\tau_1(\pi)| < |\tau_2(\pi)| < \ldots < |\tau_n(\pi)|, \tag{45}$$

[6] This is the '*does not precede condition*' in [PR00].

[7] Indeed, no one-dimensional representation can be generic.

then the integral (44) converges for every $g \in \mathrm{GL}_n(F)$ by [JS83], Chapter 3, and thus, is a well-defined function $\mathrm{GL}_n(F) \to \mathbb{C}$. Without the condition (45), one would have convergence only for a subset of functions on $I(\tau)$, and would need to extend the intertwiner analytically[8] to whole $I(\tau)$.

Moreover, the image $\mathcal{W}_f^\tau$ for such an $f \in I(\tau)$ is indeed inside $\mathfrak{W}(\psi)$, since for $u' \in U_n(F)$ we have

$$\begin{aligned}\mathcal{W}_f^\tau(u'g) &= \int_{U_n(F)} f(w_n u u' g) \cdot \overline{\psi(u)} du \\ &= \psi(u') \int_{U_n(F)} f(w_n u u' g) \cdot \overline{\psi(u'u)} du \\ &= \psi(u') \mathcal{W}_f^\tau(g).\end{aligned}$$

At last, $\mathcal{W}^\tau$ is non-zero, as will be shown in (49).

The intertwiner $\mathfrak{W}$ should be thought of as the (non-abelian) analogue of the Fourier transform.
As mentioned, $I(\tau)$ is Iwahori-spherical, i.e. $I(\tau)^{J_n} \neq 0$, and, in fact, by [Cas95], Th.6.3.5, its semi-simplification is given by

$$I(\tau)_{\mathrm{ss}}^{J_n} = \sum_{w \in W_n} \delta_n^{1/2} \otimes \tau^w,$$

the sum being direct in this case, since it is equivalent to τ being regular. Recall now the generalized Bruhat decomposition (21). This tells us that in fact, the functions $\{\varphi_w^\tau\}_{w \in W_n}$ given on $\mathrm{GL}_n(F) = \coprod_{w \in W_n} B_n w J_n$ by

$$\varphi_w^\tau(bw'j) = \begin{cases} \left(\delta_n^{1/2} \otimes \tau\right)(b), & w = w' \\ 0, & w \neq w' \end{cases},$$

span $I(\tau)^{J_n}$. These functions induce (via the Fourier transform) the Whittaker functions in $\mathfrak{W}(\tau, \psi)$ given by

$$\mathcal{W}_w^\tau(g) := \mathcal{W}_{\varphi_w^\tau}^\tau(g) = \int_{U_n(F)} \varphi_w^\tau(w_n u g) \cdot \overline{\psi(u)} du. \tag{46}$$

Our primary focus lies on the special Whittaker function $\mathcal{W}_{w_n}^\tau$ corresponding to the large Bruhat cell $B_n w_n J_n$. But in order to evaluate this function, we will need some information about $\mathcal{W}_w^\tau$ for every $w \in W_n$.

2.1.5. Evaluation of Whittaker functions

Langlands' paradigm and spherical Whittaker functions. Langlands conjectured that Whittaker functions on a reductive group G are related to characters of the connected

[8]The group of unramified characters $\tau\colon T_n(F) \to \mathbb{C}^\times$ is naturally isomorphic to $(\mathbb{C}^\times)^n$, which allows us to talk about analytic extension.

Langlands dual[9] group ${}^L G^0$, which in our case $G = \mathrm{GL}_n(k_\nu)$ for ν non-Archimedean is just ${}^L G^0 = \mathrm{GL}_n(\mathbb{C})$. The first in obtaining an explicit formula for spherical[10] Whittaker functions on $\mathrm{GL}_n(k_\nu)$ was Shintani ([Shi76]), followed independently by Kato for spherical Whittaker functions on Chevalley groups ([Kat78]). Casselman and Shalika ([Cas80], [CS80]) further extended the result to spherical Whittaker functions on unramified connected reductive groups using intertwining operators. Indeed, the space of spherical Whittaker functions on $\mathrm{GL}_n(k_\nu)$ is 1-dimensional, i.e. $\mathfrak{W}(\tau, \psi) = \langle \mathcal{W}^0 \rangle$ for some $\mathcal{W}^0$. The Iwasawa decomposition (17) tells us that $\mathcal{W}^0$ is uniquely determined by its values on $T_n(k_\nu) \big/ T_n(\mathcal{O}_\nu) \cong \mathbb{Z}^n$ and Shintani showed that under $\mathcal{W}^0(1_n) = 1$, one has

$$\mathcal{W}^0(\pi^e) = \begin{cases} \delta^{1/2}(\pi^e) \cdot \chi_e(A_\tau), & e \text{ dominant} \\ 0, & \text{otherwise} \end{cases},$$

where χ_e is the character of the irreducible (finite-dimensional) representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight e, and

$$A_\tau = \begin{pmatrix} \tau_1(\pi) & & \\ & \ddots & \\ & & \tau_n(\pi) \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \tag{47}$$

the corresponding *Satake parameter* of τ.

Iwahori Whittaker Functions. We again stick to the special case $F = k_\wp$. Recall that $\psi = \psi_\wp$ is unramified. The evaluation of Iwahori-spherical Whittaker functions was worked out by Brubaker-Bump-Licata in [BBL18] and by Brubaker-Buciumas-Bump-Gustafsson in [BBBG19] relying on previous works of Casselman [Cas80] and Casselman-Shalika [CS80]. We will state some of the results which are of interest to us.

Let us first suppose, that $\mathcal{W}$ is *any* Iwahori-spherical Whittaker function. Due to the right Iwahori-invariance (43) of $\mathcal{W}$ and due to the property (42), it is sufficient to determine the values of $\mathcal{W}(\pi^e w')$ for $e \in \mathbb{Z}^n$ and $w' \in W_n$. We will first investigate, when such a Whittaker function vanishes. In order to distinguish those e and w, for which $\mathcal{W}(\pi^e w') = 0$, we introduce the following notion:

Definition 1. Let $w \in W_n$ be fixed. We say that a cocharacter $e = (e_1, \ldots, e_n) \in X_*(T_n) \cong \mathbb{Z}^n$ is *w-almost dominant*, if for every simple root $\alpha \in \Sigma$

$$\langle \alpha, e \rangle \geq \begin{cases} 0, & w^{-1}\alpha \in \Delta^+ \\ -1, & w^{-1}\alpha \in \Delta^- \end{cases}. \tag{48}$$

Observe that:

- In our setting, the condition $\langle \alpha_i, e \rangle \geq j$ means $e_i - e_{i+1} \geq j$ (for $j \in \{0, -1\}$).

[9] If G is a reductive group attached to the root datum $(X^*, \Phi, X_*, \Phi^\vee)$, the connected Langlands dual group is the complex group $G^\vee(\mathbb{C})$ where $G^\vee$ is the reductive group attached to the dual root datum $(X_*, \Phi^\vee, X^*, \Phi)$.

[10] This means $\mathrm{GL}_n(\mathcal{O}_\nu)$-spherical.

- $w^{-1}\alpha_i \in \Delta^+$ translates as $w^{-1}(i) < w^{-1}(i+1)$.

Example 3. a) The cocharacter $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ being w-almost dominant for $w = 1$ means, e only satisfies the condition $e_1 \geq e_2 \geq \dots \geq e_n$, since the second case in (48) never occurs. Thus, this is just the usual dominance condition (12).

b) For $n = 4$ and the permutation $w = (1\,3\,4\,2) = \begin{pmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{pmatrix}$, we have $w^{-1} = (1\,2\,4\,3)$ and thus the cocharacter $e = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$ is w-almost dominant; since $w^{-1}(2) = 4 > w^{-1}(3) = 1$, we are allowed to have one step 'upwards' in the vector e from $e_2 = 0$ to $e_3 = 1$. But otherwise, it needs to fulfill the non-increasing condition.

c) If $w = w_n$ the long Weyl-element, e is allowed to have single steps 'upwards' on every position. Thus, for example, the vector $e = (0, 1, \dots, n-1)$ is w_n-almost dominant, since the first case in (48) never occurs. This is also the 'most extreme' case that can occur.

The following is Proposition 6 of [BBL18]:

Lemma 2.1. Let $\mathcal{W}$ be an Iwahori-invariant Whittaker function. Let further be $e \in \mathbb{Z}^n$ and $w \in W_n$. Then
$$\mathcal{W}(\pi^e w) = 0,$$
unless e is w-almost dominant.

Proof. Since ψ is unramified, $\psi_{|\mathcal{O}} \equiv 1$, but there is a $t \in \mathcal{O}^\times$, such that $\psi(t \cdot \pi^{-1}) \neq 1$. Pick an $i \in \{1, 2, \dots, n-1\}$ and consider the matrix

$$u = \begin{pmatrix} 1 & & & & \\ & \ddots & t \cdot \pi^{-1} & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in U_n(F)$$

consisting of 1's on its principal diagonal and of $\pi^{-1}t$ at the position $(i, i+1)$ to which in this proof we just refer to as the '*non-zero entry*'. The remaining entries are assumed to be 0. Then, by definition, $\psi(u) \neq 1$ and

$$\psi(u) \cdot \mathcal{W}(\pi^e w) = \mathcal{W}(u \cdot \pi^e w) = \mathcal{W}(\pi^e w \cdot w^{-1}\pi^{-e}u\pi^e w).$$

Consider now the conjugate of u

$$j := w^{-1}\pi^{-e}u\pi^e w.$$

This matrix consists (again) of 1's on its principal diagonal, has 0 everywhere except the non-zero entry, whose value is now $t \cdot \pi^{e_{i+1}-e_i-1}$ by (36), and which by (35) was moved to the position $(w^{-1}(i), w^{-1}(i+1))$. Hence, j is in $U_n(F)$, iff $w^{-1}\alpha_i \in \Delta^+$, and in $U_n^-(F)$ otherwise. Thus,

$$
\begin{aligned}
j \notin J_n \quad &\Longleftrightarrow \quad \begin{cases} \pi^{e_{i+1}-e_i-1} \notin \mathcal{O}, & w^{-1}\alpha_i \in \Delta^+ \\ \pi^{e_{i+1}-e_i-1} \notin \mathfrak{p}, & w^{-1}\alpha_i \in \Delta^- \end{cases} \\
&\Longleftrightarrow \quad \begin{cases} e_{i+1} - e_i - 1 \leq -1, & w^{-1}\alpha_i \in \Delta^+ \\ e_{i+1} - e_i - 1 \leq 0, & w^{-1}\alpha_i \in \Delta^- \end{cases} \\
&\Longleftrightarrow \quad \begin{cases} e_i - e_{i+1} \geq 0, & w^{-1}\alpha_i \in \Delta^+ \\ e_i - e_{i+1} \geq -1, & w^{-1}\alpha_i \in \Delta^- \end{cases} .
\end{aligned}
$$

By varying i, the last condition is exactly the condition on e being w-almost dominant. Otherwise, $j \in J_n$ and

$$\psi(u) \cdot \mathcal{W}(\pi^e w) = \mathcal{W}(u \cdot \pi^e w) = \mathcal{W}(\pi^e w \cdot j) = \mathcal{W}(\pi^e w),$$

which implies $\mathcal{W}(\pi^e w) = 0$ as claimed, since $\psi(u) \neq 1$. □

The case $\mathcal{W}^\tau_{w_n}(\pi^e)$. We start by computing $\mathcal{W}^\tau_{w_n}(\pi^e w)$ in the simple case when $w = 1$.

Lemma 2.2. For $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ and our fixed Whittaker function $\mathcal{W}^\tau_{w_n}$ defined previously we have

$$\mathcal{W}^\tau_{w_n}(\pi^e) = \begin{cases} \left(\delta_n^{1/2} \otimes \tau^{w_n}\right)(\pi^e) & , e_1 \geq e_2 \geq \ldots \geq e_n \\ 0 & , \text{ otherwise} \end{cases} \tag{49}$$

Proof. The case when e is not dominant was covered in Lemma 2.1.
Hence assume now that e is dominant. By (37), $\pi^e U_n(\mathcal{O})\pi^{-e}$ is a subgroup of $U_n(\mathcal{O})$ and thus $\varphi^\tau_{w_n}(b w_n \pi^{-e} u \pi^e)$ vanishes unless $u \in \pi^e U_n(\mathcal{O})\pi^{-e}$. Thus,

$$
\begin{aligned}
\mathcal{W}^\tau_{w_n}(\pi^e) &= \int_{U_n(F)} \varphi^\tau_{w_n}(w_n u \pi^e) \cdot \overline{\psi(u)} du \\
&= \int_{U_n(F)} \varphi^\tau_{w_n}(\pi^{w_n(e)} \cdot w_n \cdot \pi^{-e} u \pi^e) \cdot \overline{\psi(u)} du \\
&= \int_{\pi^e U_n(\mathcal{O})\pi^{-e}} \varphi^\tau_{w_n}(\pi^{w_n(e)} \cdot w_n \cdot u) \cdot \overline{\psi(\pi^e u \pi^{-e})} du \\
&= \varphi^\tau_{w_n}(\pi^{w_n \cdot e}) \cdot \mathrm{vol}(\pi^e U_n(\mathcal{O})\pi^{-e}, du) \\
&= \varphi^\tau_{w_n}(\pi^{w_n \cdot e}) \cdot \delta_n(\pi^e) \\
&= \delta_n^{1/2}(\pi^e) \cdot \tau(\pi^{w_n \cdot e}),
\end{aligned}
$$

where in the last equality we have used that $\delta_n^{1/2}(\pi^{w_n e}) = \delta_n^{-1/2}(\pi^e)$. □

The case $\mathcal{W}^\tau_{w_n w}(\pi^e w)$. Slightly more generally, if we take a Whittaker function of the form $\mathcal{W}^\tau_{w_n w}$, the simplest evaluation turns out to be in the argument $\pi^e w$:

Lemma 2.3. For $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ and $w \in W_n$, we have

$$\mathcal{W}^{\tau}_{w_n w}(\pi^e w) = \begin{cases} q^{-l(w)} \cdot \left(\delta_n^{1/2} \otimes \tau^{w_n}\right)(\pi^e) & , \ e \text{ is } w\text{-almost dominant} \\ 0 & , \ \text{otherwise} \end{cases} . \tag{50}$$

Proof. This is done in Proposition 3.6. of [BBBG19], but since the authors use a slightly different notation and definitions, we shall include the proof for our setting:
As in the case of Lemma 49, we can assume that e is already w-almost dominant. By [Car79], 4.1., the change of variable $u \mapsto \pi^{-e} u \pi^e$ on $U_n(F)$ produces the measure defect by $\delta_n^{-1}(\pi^{-e}) = \delta_n(\pi^e)$:

$$\begin{aligned} \mathcal{W}^{\tau}_{w_n w}(\pi^e w) &= \int_{U_n(F)} \varphi^{\tau}_{w_n w}(w_n u \pi^e w) \cdot \overline{\psi(u)} du \\ &= \delta_n(\pi^e) \cdot \int_{U_n(F)} \varphi^{\tau}_{w_n w}(w_n \pi^e u w) \cdot \overline{\psi(\pi^{-e} u \pi^e)} du \\ &= \delta_n(\pi^e) \cdot \int_{U_n(F)} \varphi^{\tau}_{w_n w}(\pi^{w_n \cdot e} w_n u w) \cdot \overline{\psi(\pi^{-e} u \pi^e)} du. \end{aligned}$$

Let us now suppose that the element $\pi^{w_n \cdot e} w_n u w$ sits inside $B_n w_n w J_n$. Thus, $\pi^{w_n \cdot e} w_n u w = b w_n w j$ for some $b \in B_n(F), j \in J_n$. But this is equivalent to

$$u \in w_n B_n(F) w_n \cdot w J_n w^{-1} = B_n^-(F) \cdot w J_n w^{-1}.$$

By the (generalized) Iwahori decomposition (20), $w J_n w^{-1}$ decomposes as

$$w J_n w^{-1} = (w J_n w^{-1} \cap U_n^-(\mathcal{O})) \cdot T_n(\mathcal{O}) \cdot (w J_n w^{-1} \cap U_n(\mathcal{O})),$$

but since $(w J_n w^{-1} \cap U_n^-(\mathcal{O})) \cdot T_n(\mathcal{O}) \subset B_n^-(F)$, we have that

$$u \in B_n^-(F)(w J_n w^{-1} \cap U_n(\mathcal{O})),$$

and since $B_n^-(F) \cap (w J_n w^{-1} \cap U_n(\mathcal{O})) = \{1\}$, we have indeed that $u \in w J_n w^{-1} \cap U_n(\mathcal{O})$. But since $u \in w J_n w^{-1} \cap U_n(\mathcal{O})$, the fact that e is w-almost dominant tells us that $\overline{\psi(\pi^{-e} u \pi^e)} = 1$. Thus, by (30),

$$\begin{aligned} \mathcal{W}^{\tau}_{w_n w}(\pi^e w) &= \delta_n(\pi^e) \cdot \int_{w J_n w^{-1} \cap U_n(\mathcal{O})} \varphi^{\tau}_{w_n w}(\pi^{w_n \cdot e} w_n u w) \cdot \overline{\psi(\pi^{-e} u \pi^e)} du \\ &= \delta_n(\pi^e) \cdot \left(\delta_n^{1/2} \otimes \tau\right)(\pi^{w_n \cdot e}) \cdot q^{-l(w)} \\ &= q^{-l(w)} \cdot \left(\delta_n^{1/2} \otimes \tau^{w_n}\right)(\pi^e) \end{aligned}$$

as claimed. □

The general case $\mathcal{W}^{\tau}_{w_n}(\pi^e w)$

Example 4. As a starting example, we shall compute $\mathcal{W}^{\tau}_{w_2}(\pi^e w_2)$ in the case $n = 2$: In our case $w_2 = s_1$ is (the unique) simple reflection. We assume first that e is s_1-almost dominant, which means that $e_1 \geq e_2 - 1$. We have

$$\mathcal{W}^{\tau}_{w_2}(\pi^e w_2) = \int_{U_2(F)} \varphi^{\tau}_{w_2}(w_2 u \pi^e w_2) \cdot \overline{\psi(u)} du$$

$$= \int_{U_2(F)} \varphi^\tau_{w_2}(\pi^{w_2\cdot e} \cdot w_2 \cdot \pi^{-e} u \pi^e \cdot w_2) \cdot \overline{\psi(u)} du.$$

Let us now focus on the term $w_2 \cdot \pi^{-e} u \pi^e \cdot w_2$. We have

$$w_2 \cdot \pi^{-e} u \pi^e w_2 = \begin{pmatrix} 1 & \\ u\pi^{-(e_1-e_2)} & 1 \end{pmatrix}.$$

Furthermore, the subset $\{0\} \subset F$ has measure zero, and thus

$$\begin{aligned} \mathcal{W}^\tau_{w_2}(\pi^e w_2) &= \int_F \varphi^\tau_{w_2}(\begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ u\pi^{-(e_1-e_2)} & 1 \end{pmatrix}) \cdot \overline{\psi(u)} du \\ &= \int_{F^\times} \varphi^\tau_{w_2}(\begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ u\pi^{-(e_1-e_2)} & 1 \end{pmatrix}) \cdot \overline{\psi(u)} \cdot (1-q^{-1}) \left|u\right| d^\times u, \end{aligned}$$

where in the last equality we have already performed change of measure from du to $d^\times u$ as in (24). Now it turns out, that the matrix $\begin{pmatrix} 1 & \\ u\pi^{-(e_1-e_2)} & 1 \end{pmatrix}$ for $u \in F^\times$ still lies in the large bruhat cell $B_2(F)w_2U_2(F)$:

$$\begin{pmatrix} 1 & \\ u\pi^{-(e_1-e_2)} & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -u^{-1}\pi^{e_1-e_2} & 1 \\ & u\pi^{-(e_1-e_2)} \end{pmatrix}}_{=:b} \cdot w_2 \cdot \underbrace{\begin{pmatrix} 1 & u^{-1}\pi^{e_1-e_2} \\ & 1 \end{pmatrix}}_{=:j}.$$

Once again using the Iwahori decomposition (19), one can easily see that this lies in $B_2(F)w_2J_2$, iff $j \in U_2(\mathcal{O})$. Thus,

$$\begin{aligned} \mathcal{W}^\tau_{w_2}(\pi^e w_2) &= \sum_{i\in\mathbb{Z}} \int_{\mathcal{O}^\times} \varphi^\tau_{w_2}(\begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_1} \end{pmatrix} \cdot \begin{pmatrix} -t^{-1}\pi^{e_1-e_2-i} & 1 \\ & t\pi^{-(e_1-e_2)+i} \end{pmatrix} \cdot w_2 \\ &\qquad \cdot \begin{pmatrix} 1 & t^{-1}\pi^{e_1-e_2-i} \\ & 1 \end{pmatrix}) \cdot \overline{\psi(t\pi^i)} \cdot (1-q^{-1}) \left|t\pi^i\right| d^\times t \\ &= (1-q^{-1}) \cdot (\delta^{1/2} \otimes \tau)(\pi^e) \cdot \sum_{i=-\infty}^{e_1-e_2} \left(\frac{\tau_2(\pi)}{\tau_1(\pi)}\right)^i \cdot \int_{\mathcal{O}^\times} \overline{\psi(t\pi^i)}\, d^\times t \\ &= (1-q^{-1}) \cdot (\delta^{1/2} \otimes \tau)(\pi^e) \cdot \sum_{i=-\infty}^{e_1-e_2} \mathfrak{G}(i) \cdot \left(\frac{\tau_2(\pi)}{\tau_1(\pi)}\right)^i \\ &\overset{(80)}{=} (1-q^{-1}) \cdot (\delta^{1/2} \otimes \tau)(\pi^e) \cdot \left(\frac{q}{q-1} \cdot \left(\frac{1-q^{-1}\frac{\tau_1(\pi)}{\tau_2(\pi)}}{1-\frac{\tau_2(\pi)}{\tau_1(\pi)}} \right) - \frac{\left(\frac{\tau_2(\pi)}{\tau_1(\pi)}\right)^{e_1-e_2+1}}{1-\frac{\tau_2(\pi)}{\tau_1(\pi)}} \right) \\ &= (\delta^{1/2} \otimes \tau)(\pi^e) \cdot \left(\left(\frac{1-q^{-1}\frac{\tau_1(\pi)}{\tau_2(\pi)}}{1-\frac{\tau_2(\pi)}{\tau_1(\pi)}} \right) - (1-q^{-1}) \frac{\left(\frac{\tau_2(\pi)}{\tau_1(\pi)}\right)^{e_1-e_2+1}}{1-\frac{\tau_2(\pi)}{\tau_1(\pi)}} \right). \end{aligned}$$

Using that $\tau(\pi^e) = \tau^{w_2}(\pi^e) \cdot \left(\frac{\tau_1(\pi)}{\tau_2(\pi)}\right)^{e_1-e_2}$, we shall rewrite this little formula as

$$\mathcal{W}^\tau_{w_2}(\pi^e w_2) = (\delta^{1/2} \otimes \tau^{w_2})(\pi^e) \cdot \left(\frac{1-q^{-1}+q^{-1}\cdot\left(\frac{\tau_1(\pi)}{\tau_2(\pi)}\right)^{e_2-e_1+2} - \left(\frac{\tau_1(\pi)}{\tau_2(\pi)}\right)^{e_2-e_1+1}}{1-\frac{\tau_1(\pi)}{\tau_2(\pi)}} \right). \quad (51)$$

The full description of the formula for $\mathcal{W}^{\tau}_{w_n}(\pi w)$ with $w \in W_n$ arbitrary was worked out by Brubaker, Bump and Licata in [BBL18] by means of intertwiners between principal series representations $I(\tau) \to I(\tau^w)$, based mainly on works of Casselman [Cas80] and Casselman-Shalika [Cas95]. Prop.10 in *loc.cit.* tells us: if s_i is any simple reflection corresponding to a simple root α_i, such that $s_i w < w$, which in our setting $w = w_n$ is always the case, then

$$\mathcal{W}^{\tau}_{s_i w} = \mathcal{I}'_i \mathcal{W}^{\tau}_{w},$$

where $\mathcal{I}'_i$, called a (modified) Demazure-Lusztig operator, applied to $\mathcal{W}^{\tau}_{w}(g)$ is given explicitly as

$$\left(\mathcal{I}'_i \mathcal{W}^{\tau}_{w}\right)(g) := \left(1 - q^{-1}\tau\left(\alpha_i^{\vee}(\pi)\right)\right) \cdot \left(\frac{\mathcal{W}^{\tau}_{w}(g) - \tau\left(\alpha_i^{\vee}(\pi)\right)\mathcal{W}^{\tau^{s_i}}_{w}(g)}{1 - \tau\left(\alpha_i^{\vee}(\pi)\right)}\right) - \mathcal{W}^{\tau}_{w}(g).$$

These operators satisfy the *braid relations* (Prop.12 in *loc.cit.*)

$$\mathcal{I}'_{i+1}\mathcal{I}'_i\mathcal{I}'_{i+1} = \mathcal{I}'_i\mathcal{I}'_{i+1}\mathcal{I}'_i \tag{52}$$

and the *quadratic relations* (Prop.13 in *loc.cit.*)

$$\left(\mathcal{I}'_i\right)^2 = (q^{-1} - 1)\mathcal{I}'_i + q^{-1}. \tag{53}$$

The latter implies that the $\mathcal{I}'_i$'s are invertible with inverse

$$\mathcal{I}_i := \left(\mathcal{I}'_i\right)^{-1} = q\left(\mathcal{I}'_i - (q^{-1} - 1)\right), \tag{54}$$

which is explicitly given by

$$\left(\mathcal{I}_i \mathcal{W}^{\tau}_{w}\right)(g) = q \cdot \left(1 - q^{-1}\tau\left(\alpha_i^{\vee}(\pi)\right)\right) \cdot \left(\frac{\mathcal{W}^{\tau}_{w}(g) - \tau\left(\alpha_i^{\vee}(\pi)\right)\mathcal{W}^{\tau^{s_i}}_{w}(g)}{1 - \tau\left(\alpha_i^{\vee}(\pi)\right)}\right) - \mathcal{W}^{\tau}_{w}(g). \tag{55}$$

If $w = s_{k_n} \cdot \ldots \cdot s_{k_1}$ is now any Weyl element written as reduced product of simple reflections, then

$$\mathcal{W}^{\tau}_{ww_n} = \mathcal{I}'_{k_n} \ldots \mathcal{I}'_{k_1}\mathcal{W}^{\tau}_{w_n}.$$

This is independent of the choice of the reduced product due to the braid relations (52).

Example 5. Let us now see how this applies to our case $g = \pi^e s_i$, where e is s_i-almost dominant. Using the fact that in the GL_n-case we have $w_n s_i = s_{w_n(i)} w_n = s_{n-i} w_n$, as well as (55) together with (50), we have

$$\begin{aligned}
\mathcal{W}^{\tau}_{w_n}\left(\pi^e s_i\right) &= \left(\mathcal{I}_{w_n(i)}\mathcal{W}^{\tau}_{s_{w_n(i)}w_n}\right)\left(\pi^e s_i\right) \\
&= \left(\mathcal{I}_{w_n(i)}\mathcal{W}^{\tau}_{w_n s_i}\right)\left(\pi^e s_i\right) \\
&= q \cdot \left(1 - q^{-1}\tau\left(\alpha_{n-i}^{\vee}(\pi)\right)\right) \cdot \left(\frac{\mathcal{W}^{\tau}_{w_n s_i}(\pi^e s_i) - \tau\left(\alpha_{n-i}^{\vee}(\pi)\right)\mathcal{W}^{\tau s_{n-1}}_{w_n s_i}(\pi^e s_i)}{1 - \tau\left(\alpha_{n-i}^{\vee}(\pi)\right)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad -\mathcal{W}^{\tau}_{w_n s_i}(\pi^e s_i)\\
&= \left(1-q^{-1}\tau\left(\alpha^{\vee}_{n-i}(\pi)\right)\right)\cdot\left(\frac{\left(\delta_n^{1/2}\otimes\tau^{w_n}\right)(\pi^e)-\tau\left(\alpha^{\vee}_{n-i}(\pi)\right)\left(\delta_n^{1/2}\otimes\tau^{s_{n-i}w_n}\right)(\pi^e)}{1-\tau\left(\alpha^{\vee}_{n-i}(\pi)\right)}\right)\\
&\quad -q^{-1}\cdot\left(\delta_n^{1/2}\otimes\tau^{w_n}\right)(\pi^e)\\
&= \left(\delta_n^{1/2}\otimes\tau^{w_n}\right)(\pi^e)\left(\left(1-q^{-1}\frac{\tau_{n-i}(\pi)}{\tau_{n-i+1}(\pi)}\right)\left(\frac{1-\left(\frac{\tau_{n-i}(\pi)}{\tau_{n-i+1}(\pi)}\right)^{e_i-e_{i+1}+1}}{1-\frac{\tau_{n-i}(\pi)}{\tau_{n-i+1}(\pi)}}\right)-q^{-1}\right).
\end{aligned}
$$

For $n=2$ and $i=1$ we easily recover the special case (51).

We will also use the following identity in the computation for the case $n=2$ in Chapter 5:

Lemma 2.4. If $e,f\in\mathbb{Z}^n$ with $s_i\bullet f=f$, then

$$
\mathcal{W}^{\tau}_{w_n}(\pi^{e+f}s_i)=\left(\delta_n^{1/2}\otimes\tau^{w_n}\right)(\pi^f)\cdot\mathcal{W}^{\tau}_{w_n}(\pi^e s_i). \tag{56}
$$

2.2. $(\mathfrak{g},K)$-modules

Let G be a real reductive Lie group. We shall denote by $\mathfrak{g}$ its Lie algebra, by $K\subset G$ a fixed maximal compact subgroup and by $\mathfrak{k}$ the Lie algebra of K.

Example 6. Important examples of $(\mathfrak{g},K)$-modules for us arise as follows:

a) $\mathrm{GL}_n(k_\nu)$, where ν is any Archimedean place of k, has the structure of a real reductive Lie group. For its maximal compact subgroup we then usually take $K\in\{O(n),U(n)\}$ depending on whether ν is real or complex.

b) $\mathrm{GL}_n(\mathbb{A}_\infty)\cong\prod_{\nu|\infty}\mathrm{GL}_n(k_\nu)$ has the structure of a real reductive Lie group as well.

A $(\mathfrak{g},K)$-*module* in the sense of Lepowsky is a $\mathbb{C}$-vector space V, that is simultaneously

- a Lie algebra representation of $\mathfrak{g}$,
- and a linear representation of K (we disregard for instance the topology on K),

such that

1. for all $v\in V, k\in K, X\in\mathfrak{g}$:

$$
k\cdot(X\cdot v)=\mathrm{Ad}(k)(X)\cdot(k\cdot v),
$$

2. V is K-finite, i.e. for any $v_0\in V$, the subspace

$$
K\cdot v_0:=\langle x\cdot v_0\,|\,x\in K\rangle
$$

is of finite dimension, such that the action of K on any $K\cdot v_0$ is continuous,

3. for all $v \in V$ and $Y \in \mathfrak{k}$,

$$\left(\frac{d}{dt}\exp(tY)\cdot v\right)_{|t=0} = Y \cdot v.$$

We want to stress that a $(\mathfrak{g}, K)$-module is a purely algebraic concept, and therefore we do not speak about 'smooth' $(\mathfrak{g}, K)$-modules. But in analogy with the non-Archimedean case, we can understand the second condition as '*smoothness*'; indeed, it implies, that we have an algebraic direct sum decomposition

$$V = \bigoplus_{\alpha \in \widehat{K}} V(\alpha),$$

see for instance Lemma 3.3.3. in [Wal88]. Here, $\widehat{K}$, as in the non-Archimedean case, denotes the set of equivalence classes of irreducible finite-dimensional representations of K. In analogy with the non-Archimedean case, we will call a $(\mathfrak{g}, K)$-module V *admissible*, if every $V(\alpha)$ is of finite dimension.

Remark 3. Another reason for the 'smoothness' arises via the Casselman-Wallach's *Globalization Theorem* (see [Cas89] as well as the recent paper [KB14]): given a $(\mathfrak{g}, K)$-module V, there exists a (unique) smooth representation (of moderate growth) W of G, whose subspace of K-finite vectors W_K is isomorphic to V as a $(\mathfrak{g}, K)$-module[11].

2.2.1. Archimedean Generic Representations

There is a similar notion of generic Archimedean representations for $(\mathfrak{g}, K)$-modules, see Chapter 8 of [JPSS79b]. However, we shall not be interested in this case for now. The only fact important to us is the Multiplicity One-Theorem of Shalika [Sha74], Theorem 3.1., which is the analogue statement of Gelfand-Kajdan, but for Archimedean places, and the fact that we can talk about Archimedean Whittaker functions.

2.3. Automorphic representations

Automorphic representations should be thought of as spaces of automorphic forms. We thus start by recalling the notion of an automorphic form, see Chapter 3.3. in Bump's [Bum96]. Recall that $\mathrm{GL}_n(\mathbb{A})$ possesses the standard maximal compact subgroup $K_{\max} := K_{\mathrm{fin}} K_\infty$, where

$$K_{\mathrm{fin}} := \prod_{\nu<\infty} \mathrm{GL}_n(\mathcal{O}_\nu) \quad \text{and} \quad K_\infty := \prod_{\nu \text{ real}} O(n) \times \prod_{\nu \text{ complex}} U(n).$$

Definition 2. a) A function $\varphi\colon \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ is called an *automorphic form* on $\mathrm{GL}_n(\mathbb{A})$ with *central character* $\omega\colon k^\times \backslash \mathbb{A}^\times \to \mathbb{S}^1$, if the following hold:

- $\varphi(\gamma g) = \varphi(g)$ for $\gamma \in \mathrm{GL}_n(k), g \in \mathrm{GL}_n(\mathbb{A})$,

[11] More precisely, the Casselman-Wallach functor defines an equivalence of the corresponding categories.

- $\varphi(zg) = \omega(z)\varphi(g)$ for $z \in \mathbb{A}^\times$, center of $\mathrm{GL}_n(\mathbb{A})$,
- φ is *smooth* of *moderate growth*,
- φ is $K_{\max}$-*finite*,
- φ is $\mathcal{Z}$-*finite*[12],

If ω is character of $k^\times \backslash \mathbb{A}^\times$, we shall denote by $\mathcal{A}(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$ the set of automorphic forms with central character ω.

b) Moreover, we call an automorphic form φ on $\mathrm{GL}_n(\mathbb{A})$ to be *cuspidal*, if

$$\int_{n \in N(k)\backslash N(\mathbb{A})} \varphi(ng)dn = 0, \tag{57}$$

where $N \subset \mathrm{GL}_n$ is any (standard maximal) unipotent subgroup. The subspace of cuspidal forms with central character ω will be denoted by $\mathcal{A}_0(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$.

Recall that $\mathrm{GL}_n(\mathbb{A}_\infty)$ has the structure of a real (reductive) Lie group. We shall denote by $\mathfrak{g}_\infty$ its Lie algebra. Now the space of automorphic forms $\mathcal{A}(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$ with a central character ω is not necessarily a representation of $\mathrm{GL}_n(\mathbb{A})$, where $\mathrm{GL}_n(\mathbb{A})$ acts via right translation in the argument $g \cdot \varphi(x) := \varphi(xg)$, but it is a

- smooth representation of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{fin}})$,
- and a $(\mathfrak{g}_\infty, K_\infty)$-module,

such that the two actions commute. By abuse of notation, we will write ρ for any of these actions.

Definition 3. 1. An *automorphic representation* (ρ, V) is any irreducible subquotient V of $\mathcal{A}(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$ for some central character ω.

2. An *admissible representation* is an automorphic representation (ρ, V), if it is admissible for both parts: as representation of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{fin}})$ and as $(\mathfrak{g}_\infty, K_\infty)$-module.

3. At last, a *cuspidal representation* (ρ, V) of $\mathrm{GL}_n(\mathbb{A})$ is any (automorphic) admissible representation, that is realizable as subquotient of some $\mathcal{A}_0(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$.

Thus, for us, automorphic representations are always irreducible. Suppose now, we are given an (irreducible) admissible (not necessarily cuspidal) representation (ρ, V) of $\mathrm{GL}_n(\mathbb{A}_k)$. Then, by the Flath-Decomposition Theorem 2 and Theorem 3 in [Fla79], ρ decomposes as a restricted tensor product into local components

$$\rho \cong {\bigotimes_{\nu<\infty}}' \rho_\nu \otimes \bigotimes_{\nu\mid\infty} \rho_\nu,$$

where ρ_ν is

[12] $\mathcal{Z}$ stands for the center of the universal enveloping algebra of the complexification of the infinity part Lie algebra $\mathfrak{g}_{\infty,\mathbb{C}}$.

- an irreducible admissible representation of $\mathrm{GL}_n(k_\nu)$ for finite ν,
- and an irreducible admissible $(\mathfrak{g}_\nu, K_\nu)$-module at $\nu|\infty$.

Furthermore, by *loc.cit.*, this decomposition is K-compatible in the following sense: if $K = \prod_\nu K_{\nu<\infty} \subset \mathrm{GL}_n(\mathbb{A}_{\mathrm{fin}})$ is any open compact subgroup, then

$$\left({\bigotimes_{\nu<\infty}}' \rho_\nu\right)^K \cong {\bigotimes_{\nu<\infty}}' \rho_\nu^{K_\nu}$$

But due to the topology on $\mathrm{GL}_n(\mathbb{A})$, we have $K_\nu = \mathrm{GL}_n(\mathcal{O}_\nu)$ on almost all finite places ν. In addition, due to the K-finiteness of ρ itself and the structure of K, we thus have on almost all finite places $\dim_{\mathbb{C}}(\rho_\nu^{\mathrm{GL}_n(\mathcal{O}_\nu)}) = 1$. In other words, we have an irreducible unramified representation of $\mathrm{GL}_n(k_\nu)$ at almost all finite places ν of k.

2.3.1. Remarks on Cuspidal Representations

Cuspidal representations have nice properties:

- One important future of cuspidal representations is that they are *generic*[13]. Genericity for automorphic representations is defined analogously as in the local cases (see [PS79]). As a consequence of the local Multiplicity One-Theorems, one has the global Multiplicity One-Theorem (very first Theorem of *loc.cit.*), and indeed, if $\mathcal{W} = \mathcal{W}_{\varphi,\psi_{\mathbb{A}}}$ is a global Whittaker function attached to some cuspidal form $\varphi \in (\rho, V)$ and the global additive character $\psi_{\mathbb{A}}$, such that φ corresponds under the Flath Decomposition to a pure tensor of the form $\varphi = \otimes_\nu \varphi_\nu$, then every constituent ρ_ν is generic and
$$\mathcal{W}(g) = \prod_\nu \mathcal{W}_{\varphi_\nu,\psi_\nu}(g_\nu), \tag{58}$$
where $\mathcal{W}_{\varphi_\nu,\psi_\nu}$ are local Whittaker functions attached to the local component φ_ν realized in the Whittaker space $\mathfrak{W}(\psi_\nu)$ and $g = (g_\nu)_\nu \in \mathrm{GL}_n(\mathbb{A})$. As we have seen above, we have the irreducible principal series representation $\rho_\nu = I(\tau_\nu)$ for an unramified character τ_ν of $T_n(k_\nu)$ on almost all finite places ν.
- A consequence of genericity is that one can attach an L-function to a cuspidal representation and the L-function has desirable properties (functional equation, meromorphic continuation). Indeed, we shall explain in the next chapter, how to attach an L-function to a pair of cuspidal representations by means of Rankin-Selberg convolutions.

[13]At least over $\mathrm{GL}_n(\mathbb{A})$. This may be in general not true, as there exist for example cuspidal non-generic representations for the spin groups; the holomorphic Siegel modular forms do not have a Whittaker model.

3. Automorphic L-Functions

Let us fix a pair of (irreducible) cuspidal representations (ρ, σ) of $\mathrm{GL}_{n+1}(\mathbb{A}_k) \times \mathrm{GL}_n(\mathbb{A}_k)$. As we have seen in Chapter 2, both representations decompose as restricted tensor products into local components

$$\rho \cong \bigotimes\nolimits_\nu' \rho_\nu, \qquad \sigma \cong \bigotimes\nolimits_\nu' \sigma_\nu.$$

Let us set

$$S_\infty := \{\nu \mid \nu \text{ arch.prime place of } k\}$$

and

$$S_{\mathrm{ram}} := \{\nu \mid \nu \text{ non-arch.prime place of } k, \text{ s.t. } \rho_\nu, \sigma_\nu \text{ or } \psi_\nu \text{ ramify}\}.$$

Both sets are finite and so is

$$S := S_\infty \cup S_{\mathrm{ram}}.$$

We already mentioned that we want $\psi_{\mathbb{A}}$ to be unramified at $\wp$. From now on we impose the stronger condition:

Hypothesis 1. $\wp \notin S$.

For now, we assume that $\chi_{\mathbb{A},s}$ is unramified at every place, i.e. $\chi_{\mathbb{A},s} = \|\cdot\|^{s-1/2}$. The global (complete) L-function attached to the pair (ρ, σ) under the twist of $\chi_{\mathbb{A},s}$ is defined formally as the Euler product of local L-functions

$$L(\rho \times (\sigma \otimes \chi_{\mathbb{A},s})) := \prod_\nu L(\rho_\nu \times (\sigma_\nu \otimes \chi_{\nu,s}))$$

with ν ranging over all prime places of k. The local L-functions are described below. We would like to point out, that in general, for cuspidal representations of $\mathrm{GL}_n \times \mathrm{GL}_m$ one uses the shift $\|\cdot\|^{s-\frac{n-m}{2}}$, which explains our choice of the exponent in this case.

Recall that ρ and σ are generic for any place ν (finite or infinite). We define the *local zeta integral* (attached to a pair of local Whittaker functions of $\mathfrak{W}(\rho_\nu, \psi_\nu) \times \mathfrak{W}(\sigma_\nu, \overline{\psi_\nu})$) by

$$Z(\mathcal{W}_\nu, \mathcal{W}'_\nu, \chi_{\nu,s}) = \int\limits_{U_n(k_\nu)\backslash \mathrm{GL}_n(k_\nu)} \mathcal{W}_\nu(\begin{pmatrix} g_\nu & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'_\nu(g_\nu) \cdot \chi_{\nu,s}(g_\nu)\, d^\times g_\nu. \tag{59}$$

We shall now explain the Euler factors (these are the local L-functions) at every place ν and their connection to the local zeta integrals:

- ν non-archimedean:
  Here we follow Chapter 2 of [JPSS83]. For every pair of Whittaker functions $(\mathcal{W}_\nu, \mathcal{W}'_\nu) \in \mathfrak{W}(\rho_\nu, \psi_\nu) \times \mathfrak{W}(\sigma_\nu, \overline{\psi_\nu})$, the local Zeta integral is a rational function in q_ν^{-s} and thus converges absolutely for sufficiently large $\Re(s)$. Moreover, the family of these integrals

$$\left\{Z(\mathcal{W}_\nu, \mathcal{W}'_\nu, \chi_{\nu,s}) \mid (\mathcal{W}_\nu, \mathcal{W}'_\nu) \in \mathfrak{W}(\rho_\nu, \psi_\nu) \times \mathfrak{W}(\sigma_\nu, \overline{\psi_\nu})\right\}$$

span a (principal) fractional ideal of $\mathbb{C}[q_\nu^{\pm s}]$ inside its fraction field $\mathbb{C}(q_\nu^{-s})$, which is independent of ψ_ν and any generator is of the form $\frac{1}{P(q_\nu^{-s})}$ for some $P(X) \in \mathbb{C}[X]$ with $P(0) \neq 0$. We define

$$L(\rho_\nu \times (\sigma_\nu \otimes \chi_{\nu,s}))$$

to be the generator P normalized by $P(0) = 1$. This is all included in Theorem 2.7 of *loc.cit.*

Furthermore, if $\nu \notin S$, i.e. both representations ρ_ν and σ_ν, as well as ψ_ν are unramified, then $\rho_\nu \cong I(\lambda_\nu), \sigma_\nu \cong I(\mu_\nu)$, and the corresponding local L-function is actually of the form (59) for the unique pair of spherical Whittaker functions $(\mathcal{W}_\nu, \mathcal{W}'_\nu)$ normalized by the condition $\mathcal{W}_\nu(1) = 1 = \mathcal{W}'_\nu(1)$. In this case,

$$L(\rho_\nu \times (\sigma_\nu \otimes \chi_{\nu,s})) = \frac{1}{\det(1 - A_{\lambda_\nu} \otimes A_{\mu_\nu} \cdot q^{-s})},$$

where A_{λ_ν} and A_{μ_ν} are the respective Satake parameters of $I(\lambda_\nu)$ and $I(\mu_\nu)$ as defined in 47. The proof of this can be found in Chapter 7 in the lecture notes of Cogdell [Cog03]. In general, the local non-archimedean L-factor is only a finite sum of such local non-archimedean zeta integrals.

Remark 4. If $\nu \notin S$, but χ_ν is ramified at ν, then

$$L(\rho_\nu \times (\sigma_\nu \otimes \chi_{\nu,s})) = 1$$

by definition.

- ν archimedean: In this case, the local zeta integrals converge for sufficiently large $\Re(s)$ and can be extended meromorphically to the whole plane $\mathbb{C}$. Moreover, by the local archimedean Langlands-correspondence, there corresponds to every (pair of) local component(s) $\rho_\nu \times \sigma_\nu$ a pair of $((n+1), n)$-dimensional semi-simple representations $\left(\rho_\nu^{\mathrm{WR}}, \sigma_\nu^{\mathrm{WR}}\right)$ of the Weil-group W_{k_ν} (for a reference, see Theorem 2 and Theorem 5 of [Kna79]), and one defines

  $$L(\rho_\nu \times (\sigma_\nu \otimes \chi_{\nu,s}) := L(s, \rho_\nu^{\mathrm{WR}} \otimes \sigma_\nu^{\mathrm{WR}})$$

  in the sense of Artin (we will always have $\chi_\nu = \mathbb{1}$ at infinite places, so $\chi_{\nu,s} = \|\cdot\|^{s-1/2}$). Moreover, every archimedean local zeta integral $Z(\mathcal{W}_\nu, \mathcal{W}'_\nu, \chi_{\nu,s})$ at ν is a holomorphic multiple of $L(\rho_\nu \times (\sigma_\nu \otimes \chi_{\nu,s})$, and this L-factor is essentially just a product of gamma factors.

To sum up, there exists a holomorphic function $P(s, \mathcal{W}_\infty, \mathcal{W}'_\infty)$ in s depending only on the pair of families of Whittaker functions at infinity $\mathcal{W}_\infty = (\mathcal{W}_\nu)_{\nu|\infty}$ and $\mathcal{W}'_\infty = (\mathcal{W}'_\nu)_{\nu|\infty}$, such that the global L-function satisfies

$$P(s, \mathcal{W}_\infty, \mathcal{W}'_\infty) \cdot L(\rho \times (\sigma \otimes \chi_{\mathbb{A},s})) = \sum_\iota \prod_\nu Z(\mathcal{W}_\nu^{(\iota)}, \mathcal{W}_\nu'^{(\iota)}, \chi_{\nu,s})$$

for some finite number of pairs of families of local Whittaker functions $(\mathcal{W}^{(\iota)}, \mathcal{W}'^{(\iota)}) := \left(\mathcal{W}_\nu^{(\iota)}, \mathcal{W}_\nu'^{(\iota)}\right)$. Here, for every $\nu \notin S$ we choose $\mathcal{W}_\nu^{(\iota)}$ and $\mathcal{W}_\nu'^{(\iota)}$ to be the normalized spherical vector, respectively. Thus, if $(\varphi_\iota, \varphi_\iota')$ are the cuspidal forms associated to $(\mathcal{W}^{(\iota)}, \mathcal{W}'^{(\iota)})$, then together with (58), we have the integral representation

$$P(s, \mathcal{W}_\infty, \mathcal{W}'_\infty) \cdot L(\rho \times (\sigma \otimes \chi_{\mathbb{A},s})) = \sum_\iota \int_{\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \varphi_\iota(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) \cdot \varphi_\iota(g) \cdot \chi_{\mathbb{A},s}(g) d^\times g. \quad (60)$$

4. Motives

By Clozel, Conjecture 4.5. in [Clo88], one expects a correspondence between cuspidal representations on the automorphic side and motives on the geometric side. More precisely, if Σ is a algebraic automorphic (and isobaric) cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_k)$ with purity weight[14] w and values in some number field $E/\mathbb{Q}$, then there should exist attached to Σ an irreducible n-dimensional pure motive M_Σ of weight w, defined over k with coefficients in some number field extension E'/E, such that (up to a shift), their L-functions coincide. To be more specific, such that

$$L_\nu\left(\Sigma, s+\frac{1-n}{2}\right) = L_\nu(M_\Sigma, s)$$

holds. One expects this correspondence to be functorial by means of their L-functions. As an example of functoriality which is of importance for us, let us take our pair of cuspidal representations ($\rho \cong \otimes'\rho_\nu, \sigma \cong \otimes'\sigma_\nu$). If M_ρ and M_σ are their conjectural motives, one expects that

$$L_\nu\left(\rho_\nu \times \sigma_\nu, s+\frac{1-n(n+1)}{2}\right) = L_\nu(M_\rho \otimes M_\sigma, s)$$

at every place ν in k, where the left hand side is the local Euler factor at ν as described in Chapter 3. But let us start from the beginning. What is a (pure) motive?

Fix for a moment a projective non-singular variety X over $\mathbb{Q}$ and an integer $m \geq 0$. The theory of algebraic and analytic geometry provides X with three different cohomology groups:

- The singular or Betti cohomology $H^m_B(X(\mathbb{C}))$ with rational coefficients of the complex manifold $X(\mathbb{C})$.
- The de Rham cohomology $H^m_{dR}(X)$ of the algebraic variety X.
- For every prime ℓ the ℓ-adic cohomology $H^m_\ell(X)$ of the algebraic variety X over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ with coefficients in $\mathbb{Q}_\ell$.

As it turns out, these cohomology groups have a wide interplay with each other. For our number-theoretic needs, we just abstract these properties and adopt it as the definition in a naive sense. Hence, we define a motive by its 'realization'. A definition of such realizations over $\mathbb{Q}$ can be found in [Coa89], Section 3, as well as in [CPR89], Section 2. Deligne defines (realizations of) motives over a general number field with coefficients in possibly another number field in [Del79]. Nevertheless, we shall also follow the unpublished paper [Pan90] of Panchishkin.

[14] We did not define it here, nor will we do so, since we do not really need it, but it is a technical assumption. For a definition see for example [Clo88].

Definition 4. A *pure motive* M (or just motive) over the number field $k/\mathbb{Q}$ with coefficients in (possibly another) number field $E/\mathbb{Q}$ is a collection of the form

$$M := \Big((H_{B,\beta}(M))_{\beta: k \hookrightarrow \mathbb{C}}\,, \quad H_{dR}(M), \quad (H_\ell(M))_{\ell \text{ fin.prime in } E} \Big)\,,$$

together with two constants

$$d := d(M) \in \mathbb{N}_0, \text{ and } w := w(M) \in \mathbb{Z}$$

called the *dimension* and the *weight* of M, respectively. Here, β runs over the different field embeddings $k \hookrightarrow \mathbb{C}$ and ℓ over the different finite prime places of E. The collection consists of free modules of the same rank d as follows:

- each $H_{B,\beta}(M)$ is a d-dimensional vector space over E,
- $H_{dR}(M)$ is a free module of rank d over the ring $(E \otimes_{\mathbb{Q}} k)$,
- each $H_\ell(M)$ is a d-dimensional vector space over E_ℓ,

that satisfy the following properties:

i) Each $H_{B,\beta}(M)$, where $\beta \colon k \hookrightarrow \mathbb{R}$ is a real embedding, admits an E-rational involution

$$\rho_{B,\beta} \colon H_{B,\beta}(M) \to H_{B,\beta}(M).$$

ii) For each embedding $\beta \colon k \hookrightarrow \mathbb{C}$ there is a *Hodge-Decomposition*

$$H_{B,\beta}(M) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=w} H_\beta^{p,q}(M) \tag{61}$$

into $(E \otimes \mathbb{C})$-modules. Futhermore, we demand $(\rho_B \otimes 1_{\mathbb{C}})(H_\beta^{i,j}(M)) = H_\beta^{j,i}(M)$ for any pair (i,j) with $i + j = w$, provided β is real.

iii) There is a decreasing filtration $\{F^m H_{dR}(M)\}_{m \in \mathbb{Z}}$ of (not necessarily free) $(E \otimes_{\mathbb{Q}} k)$-modules on $H_{dR}(M)$, i.e.

$$H_{dR}(M) = \bigcup_{m \in \mathbb{Z}} F^m H_{dR}(M) \supseteq \ldots \supseteq F^n H_{dR}(M) \supseteq F^{n+1} H_{dR}(M) \supseteq \ldots .$$

iv) There is a continuous group action of the absolute galois group $\mathrm{Gal}(\overline{k}/k)$ of k on each $H_\ell(M)$, such that system of ℓ-adic representations

$$r_\ell \colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}\left(H_\ell(M)\right)$$

is *compatible*.

v) For each finite prime ℓ of E and each embedding $\beta \colon k \hookrightarrow \mathbb{C}$ there is a *comparison isomorphism* of E_ℓ-vector spaces

$$\psi_{\ell,\beta} \colon H_\ell(M) \xrightarrow{\sim} H_{B,\beta}(M) \otimes_E E_\ell.$$

vi) For each embedding $\beta: k \hookrightarrow \mathbb{C}$ there is a *comparison isomorphism* (at infinity) of $(E \otimes_{\mathbb{Q}} \mathbb{C})$-modules

$$\psi_{\infty,\beta}: H_{dR}(M) \otimes_{k,\beta} \mathbb{C} \xrightarrow{\sim} H_{B,\beta}(M) \otimes_{\mathbb{Q}} \mathbb{C},$$ [15]

such that for each $m \in \mathbb{Z}$ and each embedding $\beta: k \hookrightarrow \mathbb{C}$ we have

$$\psi_{\infty,\beta}^{-1}\left(\bigoplus_{i \geq m} H_{\beta}^{i,j}(M)\right) = F^m H_{dR}(M) \otimes_{k,\beta} \mathbb{C}.$$

4.1. Complex L-functions attached to motives

Let us now suppose that M is a motive. One can attach to M the L-function $L(M, s)$ defined as follows:

Let us fix some non-Archimedean prime place ν of k. Let us further denote by D_ν the (*absolute*) decomposition subgroup of some prime $\overline{\nu}$ in $\overline{k}$ lying above ν and I_ν its inertia subgroup. Then we have a short exact sequence of groups

$$1 \to I_\nu \to D_\nu \to \operatorname{Gal}\left(\overline{\kappa(\nu)}/\kappa(\nu)\right) \to 1.$$

The *arithemetic Frobenius* Frob_ν is the element of ${D_\nu}\big/{I_\nu}$ that acts on the algebraic closure $\overline{\kappa(\nu)}$ by the automorphism $x \mapsto x^{q_\nu}$. If ℓ is any non-Archimedean place coprime to ν (in the sense that ℓ and ν do not lie over the same prime of $\mathbb{Q}$), we define the polynomial in X

$$L_\nu^{(1)}(M, X) := \frac{1}{\det\left(1 - r_\ell\left(\mathrm{Frob}_\nu^{-1}\right)_{|H_\ell(M)^{I_\nu}} \cdot X\right)}.$$

Observe that we did not use ℓ on the left hand side. We justify this by imposing yet another standard hypothesis on M:

Hypothesis 2. $L_\nu^{(1)}(M, X)$ is a rational polynomial with coefficients in E and independent of the choice of non-Archimedean ℓ coprime to ν.

Under this hypothesis, if $\gamma: E \hookrightarrow \mathbb{C}$ is any field embedding, we can consider $L_\nu^{(\gamma)}(M, X)$ by acting via γ on each coefficient of $L_\nu^{(1)}(M, X)$.
With this, we define

$$L_\nu(M, s) := \left(L_\nu^{(\gamma)}\left(M, q_\nu^{-s}\right)\right)_\gamma \in \prod_\gamma \mathbb{C}_\gamma(q_\nu^s) = (E \otimes_{\mathbb{Q}} \mathbb{C})(q_\nu^s),$$

where γ runs through the embeddings $E \hookrightarrow \mathbb{C}$. Finally, we define the corresponding L-function via

$$L(M, s) := \prod_{\nu < \infty} L_\nu(M, s).$$

[15]The subscript in the tensorproduct $\otimes_{k,\beta}$ means a tensorproduct $\otimes_k$ twisted by β as follows: for any $x \in k$ one has $x \otimes - = - \otimes \beta(x)$.

Remark 5. a) One can extend $L(M,s)$ to the *complete* L-function $\Lambda(M,s)$ defined as

$$\Lambda(M,s) = L_\infty(M,s) \cdot L(M,s),$$

where $L_\infty(M,s)$ is the Γ-factor (at infinity). The complete L-function should conjecturally satisfy a functional equation 'as usual' in the language of L-functions (meromorphic extension and functional equation). The infinity part is defined for $k = \mathbb{Q}$ in [Coa89]. In the case of a general number field $k/\mathbb{Q}$, the restriction $\operatorname{Res}_{k/\mathbb{Q}} M$ is a pure motive over $\mathbb{Q}$ (with coefficients in E), whose ℓ-adic representations are obtained by induction from $\operatorname{Gal}(\overline{k}/k)$ to $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It satisfies the identity

$$L_\nu(M,s) = L_\nu(\operatorname{Res}_{k/\mathbb{Q}} M, s).$$

at every place ν (finite or infinite).

b) We would like to remark that we can also restrict the coefficient field E, by regarding the realization of M as vector spaces over a smaller subfield. In other words, we have two different 'restriction functors'.

4.2. p-adic L-functions attached to motives

Let us restrict now to the case $k = E = \mathbb{Q}$. Coates and Perrin-Riou defined in Section 4 of [CPR89] the notion of a p-ordinary motive. For sake of completeness, we shall give the definition here. We fix a prime p and as before let $\overline{p}$ to be any prime in $\overline{\mathbb{Q}}$ with $\overline{p}|p$. Moreover we denote by

$$\phi_p \colon D_p = \operatorname{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$$

the cyclotomic character at p.

Definition 5. We say M is *ordinary*[16] at p, if the following two conditions are satisfied:

i) I_p acts trivially on $H_\ell(M)$ for any prime $\ell \neq p$.

ii) There exists a D_p-stable filtration of $\mathbb{Q}_p$-subspaces of $H_p(M)$

$$H_p(M) = W_0(M) \supsetneq W_1(M) \supsetneq \ldots \supsetneq W_t(M) = 0$$

with some $t \in \mathbb{N}$, such that I_p acts on every quotient $W_{i-1}(M)/W_i(M)$ by some power of ϕ_p, call it $\phi_p^{-e_i(M)}$ for $1 \leq i \leq t$, and such that these integers satisfy the domination condition

$$e_1(M) \geq \ldots \geq e_t(M).$$

With acting by some power $-e_i(M)$ of ϕ_p, we mean that for $g \in I_p$ and $w \in W_i(M)$,

$$g \cdot (w + W_i(M)) = \phi_p^{-e_i(M)}(g) \cdot w + W_i(M).$$

[16]By modern standards we would call it *ordinary* and *unramified at p*.

The first condition of ordinarity at p implies that the (inverse of the) Euler factor at p has exactly degree d, let us say

$$L_p(M, X) = \frac{1}{(1 - \alpha_1 X) \cdot \ldots \cdot (1 - \alpha_d X)}$$

with some $\alpha_i \in \overline{\mathbb{Q}_p} \backslash \{0\}$. We assume that the α_i are ordered in such a way that

$$|\alpha_1|_p \leq \ldots \leq |\alpha_w|_p \,, \tag{62}$$

where $|\cdot|_p$ denotes here (by abuse of notation) the natural extension of the absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ to the decomposition field $\mathbb{Q}_p(\alpha_1, \ldots, \alpha_w)$. Conjecture 4.2 of *loc.cit.* states the connection between the integers $e_i(M)$ and the $|\alpha_j|_p$.

Remark 6. Due to Clozel's conjectural correspondence, we expect to have the notion of p-ordinarity for our cuspidal representations as well. We will come back to p-ordinarity for cuspidal representations in Chapter 6.

4.3. Tensor product of motives

There are several operations on motives. We already introduced the two different restrictions, and briefly mentioned the tensor product. Let us write down what we precisely mean by the latter: given two pure motives M and M' (both defined over k with coefficients in E) of dimensions $d(M)$ and $d(M')$, respectively, we define $M \otimes M'$ 'tensor-wise' in every realization. More precisely;

- its Betti-realization is for every embedding $\beta\colon k \hookrightarrow \mathbb{C}$ given as

$$H_{B,\beta}(M \otimes M') := H_{B,\beta}(M) \otimes_E H_{B,\beta}(M')$$

- its de Rham realization is just

$$H_{dR}(M \otimes M') := H_{dR}(M) \otimes_E H_{dR}(M')$$

- its ℓ-adic realization is given in a similar manner as

$$H_\ell(M \otimes M') := H_\ell(M) \otimes_{E_\ell} H_\ell(M')$$

  at every prime place ℓ.

Its dimension is obviously $d(M \otimes M') = d(M) \cdot d(M')$. In a similar manner, we obtain the corresponding de Rham filtration, the Hodge decomposition, as well as the corresponding comparison morphisms and each E-rational involution just by 'point-wise' tensoring.

Moreover, let us suppose M and M' are both ordinary at p: since I_p acts trivially on both $H_\ell(M)$ and $H_\ell(M')$, it does so on their tensor product $H_\ell(M \otimes M')$ for $\ell \neq p$. Moreover, we have p-adic cohomology filtrations

$$H_p(M) \;\; = \;\; W_0(M) \supsetneq W_1(M) \supsetneq \ldots \supsetneq W_t(M) = 0$$

$$H_p(M') = W_0(M') \supsetneq W_1(M') \supsetneq \ldots \supsetneq W_u(M') = 0,$$

as in the definition of p-ordinarity, with

$$e_1(M) \geq \ldots \geq e_t(M),$$
$$e_1(M) \geq \ldots \geq e_s(M).$$

We impose now the following

Hypothesis 3. For any $m \in \{2, 3, \ldots, t+u\}$ and any $i, j \geq 1$ with $i + j = m$, we have that

$$e_m(M \otimes M') := e_i(M) + e_j(M') \tag{63}$$

is constant (i.e. invariant of the choice of i, j).

Under this hypothesis, we can construct a p-adic cohomology filtration of $H_p(M \otimes M')$ according to p-ordinarity as follows:

$$\begin{aligned} H_p(M \otimes M') &= W_0(M) \otimes W_0(M') \supsetneq W_0(M) \otimes W_1(M') + W_1(M) \otimes W_0(M') \supsetneq \ldots \\ &\supsetneq \sum_{i+j=m} W_i(M) \otimes W_j(M') \supsetneq \ldots \supsetneq W_{t-1}(M) \otimes W_{u-1}(M) \supsetneq 0, \end{aligned}$$

and we have

$$e_2(M \otimes M') \geq e_3(M \otimes M') \geq \ldots \geq e_{t+u}(M \otimes M').$$

At last let us explain why we need p-ordinarity. Assume that M is a p-ordinary motive. Coates predicts in [Coa89] the existence of a unique p-adic (pseudo-)measure, that interpolates the special values of the motivic L-function of M. For a more precise statement, see Conjecture 6 of his paper. In Chapter 6 we will assume ρ and σ to be $\wp$-ordinary, and thus, under an invariance condition as in (63) but on the automorphic side, we will $\wp$-adically interpolate the special value $s = 1/2$ of $L(\rho \times \sigma, s)$ as described by Coates.

5. Modification at $\wp$

Given a motive M over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$, and a prime number p, Coates describes in Chapter 5 of [Coa89], how the Euler factor $L_p(M,s)$ at p should be modified in order to p-adically interpolate the values of $L(M,s)$. We mimic this modification on the automorphic side over the number field k at our fixed non-Archimedean place $\wp$:

5.1. The modified setting

Recall that we denoted by $F = k_\wp$ the corresponding local field at $\wp$. In order to interpolate the L-function attached to $\rho \times \sigma$ at $\wp$, we shall modify the Euler factor at $\wp$ as follows:

Fix a natural number $\mathfrak{f} = \mathfrak{f}_\wp \geq \max\{\mathfrak{c}(\chi), 1\}$. The number $\mathfrak{f}$ plays the role of the congruence level in the $\wp$-adic interpolation. By our choice,

$$1 + \pi^{\mathfrak{f}} \cdot \mathcal{O} \subset \mathrm{Kern}(\chi).$$

Moreover, let

$$\mathfrak{t}_n := \begin{pmatrix} n \\ n-1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}^n, \qquad h_{n+1} := \begin{pmatrix} 1 & & & & 1 \\ & 1 & & & 1 \\ & & \ddots & & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \in U_{n+1}(F).$$

Let us fix a pair of unramified and regular $\mathbb{C}$-valued characters (λ, μ) of $T_{n+1}(F) \times T_n(F)$ that satisfy the condition (45) and such that $I(\lambda), I(\mu)$ are principal series representations. Let us further take the pair $(\mathcal{W}, \mathcal{W}') := (\mathcal{W}^\lambda_{w_{n+1}}, \mathcal{W}^\mu_{w_n})$ of (Iwahori-spherical) (ψ, ψ^{-1})-Whittaker functions on $\mathrm{GL}_{n+1}(F) \times \mathrm{GL}_n(F)$, each one supported on the large Bruhat cell, as defined in (46). We define the *modified local zeta integral*

$$\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) = \int\limits_{U_n(F)\backslash \mathrm{GL}_n(F)} \mathcal{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} h_{n+1} \begin{pmatrix} w_n \pi^{\mathfrak{f}\cdot \mathfrak{t}_n} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(g\pi^{\mathfrak{f}\cdot\mathfrak{t}_n}) \cdot \chi_s(g)\, d^\times g, \quad (64)$$

where $s \in \mathbb{C}$. This integral converges for $\Re(s) \gg 0$. Recall that $\chi_s = \chi \times \|\cdot\|^{s-\frac{1}{2}}$ for some multiplicative character χ on $\mathrm{GL}_n(\mathcal{O})$.

5.2. The case $n = 1$

We start by the computation of $\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s)$ for $\mathrm{GL}_2 \times \mathrm{GL}_1$. In this case we have $\mu = (\mu_1)$ so we just denote it by μ. We further have $U_1(F) = \{1\} = W_1$, and $\mathrm{GL}_1(F) = F^\times$ is abelian. Also, $\delta_1 = 1$. Moreover, $J_1 = \mathcal{O}^\times$, and since $\mathfrak{t}_1 = 1$, we obtain

$$\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) \quad = \quad \int\limits_{F^\times} \mathcal{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} \pi^{\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(g\pi^{\mathfrak{f}}) \cdot \chi_s(g)\, d^\times g$$

$$
\begin{aligned}
&= \int_{F^\times} \mathcal{W}(\begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} \begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} \pi^{\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(g\pi^{\mathfrak{f}}) \cdot \chi_s(g)\, d^\times g \\
&= \int_{F^\times} \psi(g) \cdot \mathcal{W}(\begin{pmatrix} g \cdot \pi^{\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(g \cdot \pi^{\mathfrak{f}}) \cdot \chi_s(g)\, d^\times g \\
&= \sum_{e\in\mathbb{Z}} \int_{\mathcal{O}^\times} \psi(\pi^e t) \cdot \mathcal{W}(\begin{pmatrix} \pi^e t \cdot \pi^{\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e t \cdot \pi^{\mathfrak{f}}) \cdot \chi(t) \cdot |\pi^e|^{s-\frac{1}{2}}\, d^\times t \\
&= \sum_{e\in\mathbb{Z}} q^{-e(s-\frac{1}{2})} \cdot \mathcal{W}(\begin{pmatrix} \pi^{e+\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^{e+\mathfrak{f}}) \cdot \int_{\mathcal{O}^\times} \chi(t) \cdot \psi(\pi^e t) d^\times t \\
&= \sum_{e\in\mathbb{Z}} q^{-e(s-\frac{1}{2})} \cdot \mathcal{W}(\begin{pmatrix} \pi^{e+\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^{e+\mathfrak{f}}) \cdot \mathfrak{G}(e,\chi) \\
&\overset{(49)}{=} \sum_{e\in\mathbb{Z}} \mathfrak{G}(e,\chi) \cdot \left(\delta_2^{1/2} \otimes \lambda^{w_2}\right)(\begin{pmatrix} \pi^{e+\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot (\delta_1^{1/2} \otimes \mu_1)(\pi)^{e+\mathfrak{f}} \cdot q^{-e(s-\frac{1}{2})} \\
&= \delta_2^{1/2}(\begin{pmatrix} \pi^{\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot (\lambda_2 \otimes \mu)(\pi)^{\mathfrak{f}} \cdot \sum_{e\in\mathbb{Z}} \mathfrak{G}(e,\chi) \cdot \left((\lambda_2 \otimes \mu_1)(\pi) \cdot q^{-s}\right)^e.
\end{aligned}
$$

5.2.1. χ ramified

If χ is ramified, all factors from the local zeta integral vanish by Lemma 1.1.b) unless $e = -\mathfrak{c}(\chi)$, and hence

$$
\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) = q^{-\mathfrak{f}/2} \cdot (\lambda_2 \otimes \mu)(\pi)^{\mathfrak{f}} \cdot \mathfrak{G}(\chi) \cdot \left((\lambda_2 \otimes \mu)(\pi) \cdot q^{-s}\right)^{-\mathfrak{c}(\chi)}
$$

for all $s \in \mathbb{C}$.

5.2.2. χ unramified

If χ_s is unramified, our local zeta integral results to be

$$
\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) \quad = \quad q^{-\mathfrak{f}/2} \cdot (\lambda_2 \otimes \mu)(\pi)^{\mathfrak{f}} \cdot \left(\frac{q}{q-1} \cdot \frac{1 - (\lambda_2 \otimes \mu)(\pi)^{-1} \cdot q^{s-1}}{1 - (\lambda_2 \otimes \mu)(\pi) \cdot q^{-s}}\right)
$$

by virtue of (81) for all $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large. This expression then meromorphically extends $\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s)$ to the whole $\mathbb{C}$.

5.3. A general recursion

How should one approach the resolution for a general $n \in \mathbb{N}$? Recall the matrix

$$
h_{n+1} = \begin{pmatrix} 1 & & & 1 \\ & \ddots & & \vdots \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \in U_{n+1}(F)
$$

sitting inside $\mathcal{W}$. If we set

$$b_n := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in F^n,$$

one can interpret h_{n+1} as the element $(1_n, b_n)$ in the affine linear group $\mathrm{GL}_n(F) \ltimes F^n$. Thus, for $g = (g, 0) \in \mathrm{GL}_n(F) \ltimes F^n$, one has

$$(g, 0) \cdot (1_n, b_n) = (g, gb_n) = (1_n, g \cdot b_n) \cdot (g, 0)$$

which means

$$\begin{pmatrix} g & \\ & 1 \end{pmatrix} h_{n+1} = \begin{pmatrix} 1_n & g \cdot b_n \\ & 1 \end{pmatrix} \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

Thus,

$$\mathcal{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} h_{n+1} *) = \prod_{i=1}^{n} \psi(g_{ni}) \cdot \mathcal{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} *). \tag{65}$$

If we define for a vector $a := (a_1, \ldots, a_n) \in \mathbb{N}_0$

$$\psi_a(g) := \prod_{i=1}^{n} \psi(g_{ni}\pi^{a_i}), \tag{66}$$

the equation (65) can be restated as

$$\mathcal{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} h_{n+1} *) = \psi_{0_n}(g) \cdot \mathcal{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} *). \tag{67}$$

Thus,

$$\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) = \int\limits_{U_n(F)\backslash \mathrm{GL}_n(F)} \psi_{0_n}(g)\, \mathcal{W}(\begin{pmatrix} g w_n \pi^{\mathfrak{f}\cdot\mathfrak{t}_n} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(g\pi^{\mathfrak{f}\cdot\mathfrak{t}_n}) \cdot \left(\chi \times \|\cdot\|^{s-\frac{1}{2}}\right)(g)\, d^\times g.$$

By (34), this can be rewritten as

$$\begin{aligned} \widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) \;&=\; \sum_{e\in\mathbb{Z}^n} \left(\delta_n^{-1} \otimes \|\cdot\|^{s-\frac{1}{2}}\right)(\pi^e) \cdot \sum_{w\in W_n} \chi(w) \\ &\quad \cdot \int\limits_{U_n^{(w)}\backslash J_n} \mathcal{W}(\begin{pmatrix} \pi^e w j \cdot w_n \pi^{\mathfrak{f}\cdot\mathfrak{t}_n} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e w j \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_n}) \cdot \psi_{0_n}(\pi^e w j)\chi(j)\, d^\times j. \end{aligned}$$

Let us set

$$I_n^{(a)}(w, e) := \int\limits_{U_n^{(w)}\backslash J_n} \mathcal{W}(\begin{pmatrix} \pi^e w j \cdot w_n \pi^{\mathfrak{f}\cdot\mathfrak{t}_n} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e w j \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_n}) \cdot \psi_a(\pi^e w j)\chi(j)\, d^\times j \tag{68}$$

for $a \in \mathbb{N}_0^n$. Then

$$\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) = \sum_{\substack{e\in\mathbb{Z}^n \\ w\in W_n}} \left(\delta_n^{-1} \otimes \|\cdot\|^{s-\frac{1}{2}}\right)(\pi^e) \cdot \chi(w) \cdot I_n^{(0_n)}(w, e). \tag{69}$$

Decomposition of $U_n^{(w)}\backslash J_n$. The space $U_n^{(w)}\backslash J_n$ can be interpreted as an upper-triangular matrix group conjugated by an Weyl element (such that below the diagonal appears $\mathfrak{p}$ instead of $\mathcal{O}$): It has exactly one 'full' row in the sense that every other row has at least one 0 inside:

$$U_n^{(w)}\backslash J_n \cong \begin{pmatrix} \mathcal{O}^\times & * & * & * & * & * & * & * \\ * & \ddots & * & * & * & * & * & * \\ * & * & \ddots & * & * & * & * & * \\ * & * & * & \mathcal{O}^\times & * & * & * & * \\ \mathfrak{p} & \dots & \dots & \mathfrak{p} & \mathcal{O}^\times & \mathcal{O} & \dots & \mathcal{O} \\ * & * & * & * & * & \mathcal{O}^\times & * & * \\ * & * & * & * & * & * & \ddots & * \\ * & * & * & * & * & * & * & \mathcal{O}^\times \end{pmatrix}.$$

The highlighted row is the $w^{-1}(n)$-th row. Due to the appearance of $\psi_a(\pi^e wj)$ inside $I_n^{(a)}(w,e)$, which is just a product running over the $w^{-1}(n)$-th row of j, it makes sense to integrate $I_n^{(a)}(w,e)$ row-wise: starting by its $w^{-1}(n)$-th row.

A further observation we can make is that the subset of $U_n^{(w)}\backslash J_n$, which has at least one 0 in its $w^{-1}(n)$-th row, is a nullset w.r.t. $d^\times j$. We shall now decompose $U_n^{(w)}\backslash J_n$ such that we can perform the integration row-wise:

- We denote by $(U_n^{(w)}\backslash J_n)'$ the subset of $U_n^{(w)}\backslash J_n$ that results by fixing its $w^{-1}(n)$-th row to be $(0_{w^{-1}(n)-1}, 1, 0_{n-w^{-1}(n)})$. Thus, it is of the form

$$\left(U_n^{(w)}\backslash J_n\right)' \cong \begin{pmatrix} \mathcal{O}^\times & * & * & * & * & * & * & * \\ * & \ddots & * & * & * & * & * & * \\ * & * & \ddots & * & * & * & * & * \\ * & * & * & \mathcal{O}^\times & * & * & * & * \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\ * & * & * & * & * & \mathcal{O}^\times & * & * \\ * & * & * & * & * & * & \ddots & * \\ * & * & * & * & * & * & * & \mathcal{O}^\times \end{pmatrix}.$$

- We define

$$\mathcal{A}_w^{(n)} := \mathbb{N}^{w^{-1}(n)-1} \times \{0\} \times \mathbb{N}_0^{n-w^{-1}(n)}.$$

  and for $a \in \mathcal{A}_w^{(n)}$ set

$$r_a := \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \pi^{a_1} & \dots & \dots & \pi^{a_{w^{-1}(n)-1}} & 1 & \pi^{a_{w^{-1}(n)+1}} & \dots & \pi^{a_n} \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

With this definitions we have the following decomposition:

$$U_n^{(w)}\backslash J_n = \coprod_{a\in\mathcal{A}_w^{(n)}} (U_n^{(w)}\backslash J_n)' \cdot r_a \cdot T_n(\mathcal{O}) \sqcup \{\text{nullset}\} \tag{70}$$

Thus, inside the integration over $j \in U_n^{(w)}\backslash J_n$, we can assume j to be of the form

$$j = j' \cdot \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ \pi^{a_1} & \dots & \dots & \pi^{a_{w^{-1}(n)-1}} & 1 & \pi^{a_{w^{-1}(n)+1}} & \dots & \pi^{a_n} \\ & & & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 \end{pmatrix} \cdot t \in (U_n^{(w)}\backslash J_n)' \cdot r_a \cdot T_n(\mathcal{O})$$

for some $a \in \mathcal{A}_w^{(n)}$.

Measure decomposition. By the decomposition (70), we have for any $f \in C_c(U_n^{(w)}\backslash J_n)$

$$\int_{U_n^{(w)}\backslash J_n} f(j)\, d^\times j = \sum_{a\in\mathcal{A}_w^{(n)}} (*) \cdot \int_{(U_n^{(w)}\backslash J_n)'} \int_{T_n(\mathcal{O})} f(j' r_a \cdot t)\, d^\times t\, d^\times j'$$

where $d^\times j'$ is some Radon measure on $(U_n^{(w)}\backslash J_n)'$. In order to determine $d^\times j'$ we do the following:

If we apply conjugation by w^{-1} to the refined Iwahori decomposition (20), we have the decomposition

$$J_n = U_n^{(w)} \cdot U_n^{(w_n w)} \cdot T_n(\mathcal{O}).$$

Together with the Product Integral Theorem in [Dei14], prop.1.5.6, we obtain that for any $f \in C_c(U_n^{(w)}\backslash J_n)$, one has

$$\int_{U_n^{(w)}\backslash J_n} f(j) d^\times j = c(w) \cdot \int_{U_n^{(w_n w)}} \int_{T_n(\mathcal{O})} f(u^{(w_n w)} t)\, d^\times t\, du^{(w_n w)}$$

where $c(w) > 0$ is a suitable constant. This is due to the fact, that now all the appearing measures have already been fixed. If we choose f to be the characteristic function on $U_n^{(w)}\backslash J_n$, the constant results to be the quotient

$$c(w) = \frac{\text{vol}(J_n, d^\times j)}{\text{vol}(U_n^{(w)}, du^{(w)})} \cdot \frac{1}{\text{vol}(U_n^{(w_n w)}, du^{(w_n w)})} \cdot \frac{1}{\text{vol}(T_n(\mathcal{O}), d^\times t)} = \frac{\text{vol}(J_n, d^\times j)}{q^{-\frac{(n-1)n}{2}}},$$

which is w-invariant, and thus we set $c := c(w)$.
Observe also, that the order of $u^{(w_n w)}$ and t in the argument is not relevant since all the groups are compact and hence unimodular. We furthermore know the structure of these measures:

- The measure on $T_n(\mathcal{O})$ is just the n-fold product measure of the multiplicative measure on $\mathcal{O}^\times$, i.e. $d^\times t = d^\times t_1 \cdot \ldots \cdot d^\times t_n$.
- Since the measure on $U_n(\mathcal{O})$ is given by $du = \prod_{i<j} du_{ij}$, the du_{ij} being the additive measure on $\mathcal{O}$, we also know the form of the pullback measure $du^{(w_n w)}$ on $U_n^{(w_n w)}$: it also a product of additive measures on $\mathcal{O}$.
- And of course, the measure $d^\times j$ on $U_n^{(w)} \backslash J_n$ is just the product of these two (up to the constant c).

Thus, we define now $d^\times j'$ on $(U_n^{(w)} \backslash J_n)'$ as the product measure of the $d^\times t$ and $du^{(w_n w)}$ but leaving out all the measures that were corresponding to elements lying in the $w^{-1}(n)$-th row of j. In order to fill it up to the measure $d^\times j$, we thus need to add an $(n-1)$-fold additive measure on $\mathcal{O}$ and 1 multiplicative measure on $\mathcal{O}^\times$ (corresponding to the $w^{-1}(n)$-th row of j). But due to (70), we convert the $n-1$ additive measures on $\mathcal{O}$ into its multiplicative counterparts and thus obtain by (24) informally

$$d^\times j = c \cdot \|\pi^a\| \left(1 - q^{-1}\right)^{n-1} d^\times t \, d^\times j',$$

This should be understood as follows; one has

$$\int\limits_{U_n^{(w)} \backslash J_n} f(j) \, d^\times j = c \cdot \left(1 - q^{-1}\right)^{n-1} \cdot \sum_{a \in \mathcal{A}_w} \|\pi^a\| \cdot \int\limits_{(U_n^{(w)} \backslash J_n)'} \int\limits_{T_n(\mathcal{O})} f(j' r_a \cdot t) \, d^\times t \, d^\times j' \tag{71}$$

for any $f \in C_c(U_n^{(w)} \backslash J_n)$.
Now observe that r_a can be decomposed into a 'left part' $r_a^{(L)}$ and a 'right part' $r_a^{(R)}$ as

$$r_a = \underbrace{\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \pi^{a_1} & \ldots & \ldots & \pi^{a_{w^{-1}(n)-1}} & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix}}_{=: r_a^{(L)}} \underbrace{\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & \pi^{a_{w^{-1}(n)+1}} & \ldots & \pi^{a_n} \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}}_{=: r_a^{(R)}},$$

and that $r_a^{(L)} r_a^{(R)} = r_a^{(R)} r_a^{(L)}$. Thus, we obtain:

Lemma 5.1. For any $w \in W_n$ and $e \in \mathbb{Z}^n$ we have

$$\begin{aligned} I_n^{(0_n)}(w, e) = q^{\frac{n(n-1)}{2}} \cdot \operatorname{vol}(J_n, d^\times j) \cdot \left(1 - q^{-1}\right)^{n-1} \cdot \sum_{a \in \mathcal{A}_w} \|\pi^a\| \cdot \prod_{i=1}^{n} \mathfrak{G}(e_n + a_i, \chi) \\ \cdot \int\limits_{(U_n^{(w)} \backslash J_n)'} \mathcal{W}(\begin{pmatrix} \pi^e w j' \cdot r_a^{(L)} \cdot w_n \pi^{\mathfrak{f} \cdot t_n} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e w j' \cdot r_a^{(R)} \pi^{\mathfrak{f} \cdot t_n}) \cdot \chi(j') \, d^\times j'. \end{aligned} \tag{72}$$

Proof. From (71), a straight-forward computation shows

$$\begin{aligned}
I_n^{(0_n)}(w,e) &= c\cdot\left(1-q^{-1}\right)^{n-1}\cdot\sum_{a\in\mathcal{A}_w}\|\pi^a\|\cdot\int_{\left(U_n^{(w)}\backslash J_n\right)'}\int_{T_n(\mathcal{O})}\mathcal{W}(\begin{pmatrix}\pi^e w(j'r_a t)\cdot w_n\pi^{\mathfrak{f}\cdot\mathfrak{t}_n} & \\ & 1\end{pmatrix}) \\
&\qquad\cdot\mathcal{W}'(\pi^e w(j'r_a t)\pi^{\mathfrak{f}\cdot\mathfrak{t}_n})\cdot\psi_{0_n}\left(\pi^e w(j'r_a t)\right)\cdot\chi\left(j'r_a t\right)d^\times t\,d^\times j' \\
&= c\cdot\left(1-q^{-1}\right)^{n-1}\cdot\sum_{a\in\mathcal{A}_w}\|\pi^a\|\cdot\int_{(U_n^{(w)}\backslash J_n)'}\mathcal{W}(\begin{pmatrix}\pi^e wj'\cdot r_a^{(L)}\cdot w_n\pi^{\mathfrak{f}\cdot\mathfrak{t}_n} & \\ & 1\end{pmatrix})\cdot \\
&\qquad\cdot\mathcal{W}'(\pi^e wj'\cdot r_a^{(R)}\pi^{\mathfrak{f}\cdot\mathfrak{t}_n})\cdot\int_{T_n(\mathcal{O})}\psi_{0_n}\left(\pi^e w(j'r_a t)\right)\cdot\chi\left(t\right)d^\times t\,d^\times j' \\
&= c\cdot\left(1-q^{-1}\right)^{n-1}\cdot\sum_{a\in\mathcal{A}_w}\|\pi^a\|\cdot\int_{(U_n^{(w)}\backslash J_n)'}\mathcal{W}(\begin{pmatrix}\pi^e wj'\cdot r_a^{(L)}\cdot w_n\pi^{\mathfrak{f}\cdot\mathfrak{t}_n} & \\ & 1\end{pmatrix}) \\
&\qquad\cdot\chi\left(j'\right)\cdot\mathcal{W}'(\pi^e wj'\cdot r_a^{(R)}\pi^{\mathfrak{f}\cdot\mathfrak{t}_n})\cdot\left(\prod_{i=1}^n\int_{\mathcal{O}^\times}\psi(\pi^{e_n+a_i}t_i)\chi\left(t_i\right)d^\times t_i\right)d^\times j' \\
&= c\cdot\left(1-q^{-1}\right)^{n-1}\cdot\sum_{a\in\mathcal{A}_w}\|\pi^a\|\cdot\prod_{i=1}^n\mathfrak{G}(e_n+a_i,\chi) \\
&\qquad\cdot\int_{(U_n^{(w)}\backslash J_n)'}\mathcal{W}(\begin{pmatrix}\pi^e wj'\cdot r_a^{(L)}\cdot w_n\pi^{\mathfrak{f}\cdot\mathfrak{t}_n} & \\ & 1\end{pmatrix})\cdot\mathcal{W}'(\pi^e wj'\cdot r_a^{(R)}\pi^{\mathfrak{f}\cdot\mathfrak{t}_n})\cdot\chi\left(j'\right)d^\times j'.
\end{aligned}$$

The statement follows using $c = q^{\frac{n(n-1)}{2}}\cdot\mathrm{vol}(J_n, d^\times j)$. □

Remark 7. At last, we would need to extract the $r_a^{(L)}$ and $r_a^{(R)}$ from $\mathcal{W}$ and $\mathcal{W}'$, respectively.
If we would perform the general extraction, we would need additional notation, but we would eventually decrease the integration and get a recursive formula of the form

$$I_n^{(0_n)}(w,e) = \sum_{(*)}(*)\cdot I_{n-1}^{(*)}(*,*),$$

where in $I_{n-1}^{(*)}(*,*)$ appears the integration over $(U_n^{(w)}\backslash J_n)'$. We would perform the same strategy but now on the $w^{-1}(n-1)$-th row of $(U_n^{(w)}\backslash J_n)'$.
We would eventually end up with a sum of the form

$$I_n^{(0_n)}(w,e) = \sum_{(*)}(*)\cdot I_1^{(*)}(*,*),$$

where $I_1^{(*)}(*,*)$ consists of a product of the form $\mathcal{W}(*)\cdot\mathcal{W}'(*)\cdot\chi(*)$. Thus, the problem of computing the modified local zeta integral $\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s)$ is of combinatorial nature. Unfortunately, no identity, that would help us to handle the formula, is at the present known to us, nor could we think of any other strategy for the evaluation of such a sum.

Nevertheless, if we believe Langlands paradigm (see Section 2.1.5), then it should be possible to interpret the Whittaker functions by means of representation theory, and thus there might exist some new Cauchy-type identity that would permit us to obtain a closed formula for $\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s)$.

We shall now compute by hand the case $n = 2$. This shall underline the combinatorial problem we face, as well as give an insight on the general recursion in this special case.

5.4. The case $n = 2$

Let us assume that $n = 2$. Then $W_2 = \{1, w_2\}$ and $\mathrm{vol}(J_2, d^\times j) = \frac{1}{q+1}$. By (68) and (69) we have

$$\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) \;=\; \sum_{e \in \mathbb{Z}^2} \left(\| \cdot \|^{s-1/2} \otimes \delta^{-1} \right)(\pi^e) \cdot \sum_{w \in W_2} \chi(w) \cdot I_2^{(0_2)}(w, e),$$

where

$$I_2^{(0_2)}(w, e) = \int_{U_2^{(w)} \backslash J_2} \psi_{0_2}(\pi^e w j) \cdot \mathcal{W}(\begin{pmatrix} \pi^e w j \cdot w_2 \pi^{\mathfrak{f} \cdot \mathfrak{t}_2} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e w j \cdot \pi^{\mathfrak{f} \cdot \mathfrak{t}_2}) \cdot \chi(j) \, d^\times j.$$

A priori we take $s \in \mathbb{C}$ with $\Re(s) \gg 0$. **The expressions $I_2^{(0_2)}(w, e)$:**
We now compute the expression $I_2^{(0_2)}(w, e)$ for each $w \in W_2$:

<u>$w = \mathrm{id}$:</u> In this case $U_2^{(w)} = U_2^{(\mathrm{id})} = U_2(\mathcal{O})$ and $\mathcal{A}_{\mathrm{id}}^{(2)} = \mathbb{N} \times \{0\}$ and hence, the decomposition (70) is simply given by

$$\begin{aligned} U_2^{(\mathrm{id})} \Big\backslash J_2 \;&\cong\; \begin{pmatrix} \mathcal{O}^\times & \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix} \\ &= \coprod_{a \in \mathcal{A}_{\mathrm{id}}^{(2)}}^{\infty} \begin{pmatrix} \mathcal{O}^\times & \\ & 1 \end{pmatrix} \cdot r_a \cdot \begin{pmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{pmatrix} \sqcup \begin{pmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{pmatrix} \\ &= \coprod_{a=1}^{\infty} \underbrace{\begin{pmatrix} \mathcal{O}^\times & \\ & 1 \end{pmatrix}}_{\cong \left(U_2^{(\mathrm{id})} \big\backslash J_2 \right)'} \cdot \begin{pmatrix} 1 & \\ \pi^a & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{pmatrix} \sqcup \begin{pmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{pmatrix}. \end{aligned}$$

Moreover, $\begin{pmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{pmatrix} = T_2(\mathcal{O})$ has measure 0 w.r.t. $d^\times j$ on $U_2^{(w)} \big\backslash J_2$ and hence we can restrict the integration to the first set. The matrix r_a decomposes as

$$r_a = \begin{pmatrix} 1 & \\ \pi^a & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \\ \pi^a & 1 \end{pmatrix}}_{= r_a^{(L)}} \cdot \underbrace{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}_{= r_a^{(R)}}.$$

At last, the measure $d^\times j'$ on $\left(U_2^{(\mathrm{id})}\backslash J_2\right)' \cong \begin{pmatrix} \mathcal{O}^\times & \\ & 1 \end{pmatrix}$ is just the multiplicative (Haar) measure on $\mathcal{O}^\times$ normalized by (25). We write $j' = \begin{pmatrix} t & \\ & 1 \end{pmatrix}$ with $t \in \mathcal{O}^\times$ and thus, by (72),

$$\begin{aligned} I_2^{(0_2)}(\mathrm{id}, e) &= q \cdot \frac{1}{q+1} \cdot \left(1 - q^{-1}\right) \cdot \sum_{a=(a_1,a_2)\in\mathbb{N}\times\{0\}} \|\pi^a\| \cdot \prod_{i=1}^{2} \mathfrak{G}(e_2 + a_i, \chi) \\ &\quad \cdot \int_{(U_2^{(w)}\backslash J_2)'} \mathcal{W}(\begin{pmatrix} \pi^e j' \cdot r_a^{(L)} \cdot w_2 \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e j' \cdot r_a^{(R)} \pi^{\mathfrak{f}\cdot\mathfrak{t}_2}) \\ &\quad \cdot \chi(j')\, d^\times j' \\ &= \frac{q-1}{q+1} \cdot \mathfrak{G}(e_2, \chi) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}} \end{pmatrix}) \cdot \sum_{a=1}^{\infty} q^{-a} \cdot \mathfrak{G}(e_2 + a, \chi) \\ &\quad \cdot \int_{\mathcal{O}^\times} \mathcal{W}(\begin{pmatrix} \pi^e \begin{pmatrix} t & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \pi^a & 1 \end{pmatrix} \cdot w_2 \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2} & \\ & 1 \end{pmatrix}) \cdot \chi(t)\, d^\times t. \end{aligned}$$

We now proceed to extract the matrix $r_a^{(L)}$ from $\mathcal{W}(\cdot)$. If $a \in \{1, \ldots, \mathfrak{f} - 1\}$, then

$$\begin{aligned} &\mathcal{W}(\begin{pmatrix} \pi^e \begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \pi^a & 1 \end{pmatrix} \cdot w_2 \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2} & \\ & 1 \end{pmatrix}) \\ = \;& \mathcal{W}(\begin{pmatrix} \pi^e \begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \pi^{-a} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\pi^{-a} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \pi^a & 1 \end{pmatrix} \cdot w_2 \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2} & \\ & 1 \end{pmatrix}) \\ = \;& \psi(\pi^{e_1-e_2-a} t) \cdot \mathcal{W}(\begin{pmatrix} \pi^e \begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} -\pi^{-a} & \\ 1 & \pi^a \end{pmatrix} \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2} & \\ & 1 \end{pmatrix}) \\ = \;& \psi(\pi^{e_1-e_2-a} t) \cdot \mathcal{W}(\begin{pmatrix} \pi^e \begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} \pi^{2\mathfrak{f}-a} & \\ & \pi^{\mathfrak{f}+a} \end{pmatrix} \begin{pmatrix} -1 & \\ \pi^{\mathfrak{f}-a} & 1 \end{pmatrix} & \\ & 1 \end{pmatrix}) \\ = \;& \psi(\pi^{e_1-e_2-a} t) \cdot \mathcal{W}(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}-a} & & \\ & \pi^{e_2+\mathfrak{f}+a} & \\ & & 1 \end{pmatrix}). \end{aligned}$$

Otherwise if $a \geq \mathfrak{f}$, then we simply have

$$\mathcal{W}(\begin{pmatrix} \pi^e \begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \pi^a & 1 \end{pmatrix} \cdot w_2 \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2} & \\ & 1 \end{pmatrix}) = \mathcal{W}(\begin{pmatrix} \begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}} \end{pmatrix} \cdot w_2 & \\ & 1 \end{pmatrix}).$$

Plugging this back into the equation of $I_2^{(0_2)}(\mathrm{id}, e)$, we obtain

$$
\begin{aligned}
I_2^{(0_2)}(\mathrm{id}, e) &= \frac{q-1}{q+1}\mathfrak{G}(e_2,\chi)\cdot \mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix}) \\
&\quad \cdot \Bigg(\sum_{a=1}^{\mathfrak{f}-1} \mathfrak{G}(e_2+a,\chi)\cdot \mathcal{W}(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}-a} & & \\ & \pi^{e_2+\mathfrak{f}+a} & \\ & & 1\end{pmatrix})\cdot q^{-a} \\
&\quad \cdot \underbrace{\int_{T_1(\mathcal{O})} \psi(\pi^{e_1-e_2-a}\,t)\cdot\chi(t)d^\times t}_{=\mathfrak{G}(e_1-e_2-a,\chi)} \\
&\quad + \sum_{a=\mathfrak{f}}^{\infty} \mathfrak{G}(e_2+a,\chi)\cdot \mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & \pi^{e_2+2\mathfrak{f}} & \\ & & 1\end{pmatrix}\cdot w_2)\cdot q^{-a} \\
&\quad \cdot \underbrace{\int_{t\in T_1(\mathcal{O})} \chi(t)\, d^\times t}_{=\mathfrak{G}(0;\chi)} \Bigg) \\
&= \frac{q-1}{q+1}\cdot \mathfrak{G}(e_2,\chi)\cdot \mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix}) \\
&\quad \cdot \Bigg(\sum_{a=1}^{\mathfrak{f}-1} \mathfrak{G}(e_2+a,\chi)\cdot \mathfrak{G}(e_1-e_2-a,\chi)\cdot q^{-a} \\
&\quad \cdot \mathcal{W}(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}-a} & & \\ & \pi^{e_2+\mathfrak{f}+a} & \\ & & 1\end{pmatrix}) \\
&\quad + \mathfrak{G}(0;\chi)\cdot \mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & \pi^{e_2+2\mathfrak{f}} & \\ & & 1\end{pmatrix}\cdot w_2)\cdot \sum_{a=\mathfrak{f}}^{\infty} \mathfrak{G}(e_2+a,\chi)\cdot q^{-a} \Bigg).
\end{aligned}
$$

<u>$w = w_2$:</u> We will use exactly the same strategy as in the $w = \mathrm{id}$ case. In this case $U_2^{(w_2)} = U_2^-(\mathfrak{p})$, as well as $\mathcal{A}_{\mathrm{id}}^{(2)} = \{0\}\times\mathbb{N}_0$ and hence, the decomposition (70) is here

$$
U_2^{(w)}\Big\backslash J_2 \cong \begin{pmatrix}\mathcal{O}^\times & \mathcal{O} \\ & \mathcal{O}^\times\end{pmatrix} = \coprod_{a=0} \underbrace{\begin{pmatrix}1 & \\ & \mathcal{O}^\times\end{pmatrix}}_{\cong \left(U_2^{(w_2)}\Big\backslash J_2\right)'} \begin{pmatrix}1 & \pi^a \\ & 1\end{pmatrix}\begin{pmatrix}\mathcal{O}^\times & \\ & \mathcal{O}^\times\end{pmatrix} \sqcup \begin{pmatrix}\mathcal{O}^\times & \\ & \mathcal{O}^\times\end{pmatrix}.
$$

Moreover, the r_a decomposes as

$$
r_a = \begin{pmatrix}1 & \pi^a \\ & 1\end{pmatrix} = \underbrace{\begin{pmatrix}1 & \\ & 1\end{pmatrix}}_{=r_a^{(L)}}\cdot\underbrace{\begin{pmatrix}1 & \pi^a \\ & 1\end{pmatrix}}_{=r_a^{(R)}}.
$$

Again, we only need to integrate over the first set. We write $j' = \begin{pmatrix} 1 & \\ & t \end{pmatrix}$ and with (72) we have

$$I_2^{(0_2)}(w_2, e) = q \cdot \frac{1}{q+1} \cdot \left(1 - q^{-1}\right) \cdot \sum_{a=(a_1,a_2)\in\{0\}\times\mathbb{N}_0} \|\pi^a\| \cdot \prod_{i=1}^{2} \mathfrak{G}(e_2 + a_i, \chi)$$
$$\cdot \int_{(U_2^{(w)}\backslash J_2)'} \mathcal{W}(\begin{pmatrix} \pi^e w_2 j' \cdot r_a^{(L)} \cdot w_2 \pi^{\mathfrak{f}\cdot\mathfrak{t}_2} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e w_2 j' \cdot r_a^{(R)} \pi^{\mathfrak{f}\cdot\mathfrak{t}_2})$$
$$\cdot \chi(j')\, d^\times j'$$
$$= \frac{q-1}{q+1} \cdot \mathfrak{G}(e_2, \chi) \cdot \mathcal{W}(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1 \end{pmatrix}) \cdot \sum_{a=0}^{\infty} q^{-a} \cdot \mathfrak{G}(e_2 + a, \chi)$$
$$\cdot \int_{\mathcal{O}^\times} \mathcal{W}'(\pi^e \cdot w_2 \cdot \begin{pmatrix} 1 & \\ & t \end{pmatrix} \cdot \begin{pmatrix} 1 & \pi^a \\ & 1 \end{pmatrix} \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2}) \cdot \chi(t)\, d^\times t.$$

Now, again, if $a \in \{0, \ldots, \mathfrak{f}-1\}$, then

$$\begin{aligned}
& \mathcal{W}'(\pi^e \cdot w_2 \cdot \begin{pmatrix} 1 & \\ & t \end{pmatrix} \cdot \begin{pmatrix} 1 & \pi^a \\ & 1 \end{pmatrix} \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2}) \\
= \;& \mathcal{W}'(\pi^e w_2 \begin{pmatrix} 1 & \\ & t \end{pmatrix} \begin{pmatrix} 1 & \\ \pi^{-a} & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -\pi^{-a} & 1 \end{pmatrix} \begin{pmatrix} 1 & \pi^a \\ & 1 \end{pmatrix} \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2}) \\
= \;& \psi^{-1}(\pi^{e_1-e_2-a}\, t) \cdot \mathcal{W}'(\pi^e w_2 \begin{pmatrix} 1 & \\ & t \end{pmatrix} \begin{pmatrix} 1 & \pi^a \\ -\pi^{-a} & \end{pmatrix} \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2}) \\
= \;& \psi^{-1}(\pi^{e_1-e_2-a}\, t) \cdot \mathcal{W}'(\pi^e \begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} -\pi^{-a} & \\ 1 & \pi^a \end{pmatrix} \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2}) \\
= \;& \psi^{-1}(\pi^{e_1-e_2-a}\, t) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}-a} & \\ & \pi^{e_2+\mathfrak{f}+a} \end{pmatrix}).
\end{aligned}$$

Otherwise if $a \geq \mathfrak{f}$, then we simply have

$$\mathcal{W}'(\pi^e w_2 \begin{pmatrix} 1 & \\ & t \end{pmatrix} \begin{pmatrix} 1 & \pi^a \\ & 1 \end{pmatrix} \cdot \pi^{\mathfrak{f}\cdot\mathfrak{t}_2}) = \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}} \end{pmatrix} \cdot w_2).$$

Plugging this back into $I_2^{(0_2)}(w_2, e)$, we obtain

$$\begin{aligned}
I_2^{(0_2)}(w_2, e) \;=\; & \frac{q-1}{q+1} \cdot \mathfrak{G}(e_2, \chi) \cdot \mathcal{W}(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1 \end{pmatrix}) \\
& \cdot \Big(\sum_{a=0}^{\mathfrak{f}-1} \mathfrak{G}(e_2 + a, \chi) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}-a} & \\ & \pi^{e_2+\mathfrak{f}+a} \end{pmatrix}) \cdot q^{-a}
\end{aligned}$$

$$\cdot \int_{\mathcal{O}^\times} \psi^{-1}(\pi^{e_1-e_2-a}\, t) \cdot \chi\,(t)\, d^\times t$$

$$+ \sum_{a=\mathfrak{f}}^{\infty} q^{-a} \cdot \mathfrak{G}(e_2+a,\chi) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}} \end{pmatrix} \cdot w_2) \cdot \int_{\mathcal{O}^\times} \chi\,(t)\, d^\times t\Bigg).$$

The resulting integrals are now again (local) Gauss sums but with respect to ψ^{-1}. But since ψ is unitary, using the fact that $\psi^{-1}(u) = \psi(-u)$ for any $u \in F$, we can multiply the local Gauss sums by $\chi(-1) = \chi(w_2)$ in order to convert them to Gauss sums with respect to ψ. Obviously $\chi(w_2)^2 = 1$, and thus,

$$\begin{aligned} I_2^{(0_2)}(w_2,e) \quad = \quad & \chi(w_2) \cdot \frac{q-1}{q+1} \cdot \mathfrak{G}(e_2,\chi) \cdot \mathcal{W}(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1 \end{pmatrix}) \\ & \cdot\Bigg(\mathfrak{G}(e_2,\chi) \cdot \mathfrak{G}(e_1-e_2,\chi) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}} \end{pmatrix}) \\ & + \sum_{a=1}^{\mathfrak{f}-1} \mathfrak{G}(e_2+a,\chi) \cdot \mathfrak{G}(e_1-e_2-a,\chi) \\ & \qquad \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}-a} & \\ & \pi^{e_2+\mathfrak{f}+a} \end{pmatrix}) \cdot q^{-a} \\ & \quad + \mathfrak{G}(0;\chi) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}} \end{pmatrix} \cdot w_2) \cdot \sum_{a=\mathfrak{f}}^{\infty} \mathfrak{G}(e_2+a,\chi) \cdot q^{-a}\Bigg). \end{aligned}$$

Observe that we have split the case $a = 0$ from the sum $\sum_{a=0}^{\mathfrak{f}-1}$.

Back to the Zeta Integral:
We now return to the zeta integral. With what we now computed we have

$$\begin{aligned} & \widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s) \\ = \; & \sum_{e\in\mathbb{Z}^2} \left(\|\cdot\|^{s-1/2} \otimes \delta_2^{-1}\right)(\pi^e) \cdot \sum_{w\in W_2} \chi(w) \cdot I_2^{(0_2)}(w,e) \\ = \; & \sum_{e\in\mathbb{Z}^2} \left(\|\cdot\|^{s-1/2} \otimes \delta_2^{-1}\right)(\pi^e) \\ & \cdot\Bigg[\chi(\mathrm{id}) \cdot \frac{q-1}{q+1} \cdot \mathfrak{G}(e_2,\chi) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}} \end{pmatrix}) \\ & \quad \cdot \Bigg(\sum_{a=1}^{\mathfrak{f}-1} \mathfrak{G}(e_2+a,\chi) \cdot \mathfrak{G}(e_1-e_2-a,\chi) \cdot q^{-a} \\ & \qquad \cdot \mathcal{W}(\begin{pmatrix} \pi^{e_1+2\mathfrak{f}-a} & & \\ & \pi^{e_2+\mathfrak{f}+a} & \\ & & 1 \end{pmatrix}) \\ & \quad + \mathfrak{G}(0;\chi) \cdot \mathcal{W}(\begin{pmatrix} \begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}} \end{pmatrix} \cdot w_2 & \\ & 1 \end{pmatrix}) \cdot \sum_{a=\mathfrak{f}}^{\infty} \mathfrak{G}(e_2+a,\chi) \cdot q^{-a} \Bigg) \end{aligned}$$

$$+\chi(w_2)\cdot\chi(w_2)\cdot\frac{q-1}{q+1}\cdot\mathfrak{G}(e_2,\chi)\cdot\mathcal{W}(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix})$$

$$\cdot\Bigg(\mathfrak{G}(e_2,\chi)\cdot\mathfrak{G}(e_1-e_2,\chi)\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix})$$

$$+\sum_{a=1}^{\mathfrak{f}-1}\mathfrak{G}(e_2+a,\chi)\cdot\mathfrak{G}(e_1-e_2-a,\chi)$$

$$\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}-a} & \\ & \pi^{e_2+\mathfrak{f}+a}\end{pmatrix})\cdot q^{-a}$$

$$+\mathfrak{G}(0;\chi)\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}}\end{pmatrix}\cdot w_2)\cdot\sum_{a=\mathfrak{f}}^{\infty}\mathfrak{G}(e_2+a,\chi)\cdot q^{-a}\Bigg)\Bigg]$$

$$=\frac{q-1}{q+1}\cdot\sum_{e\in\mathbb{Z}^2}\left(\|\cdot\|^{s-1/2}\otimes\delta^{-1}\right)(\pi^e)\cdot\mathfrak{G}(e_2,\chi)$$

$$\cdot\Bigg[\mathfrak{G}(e_2,\chi)\cdot\mathfrak{G}(e_1-e_2,\chi)\cdot\mathcal{W}(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix})$$

$$+\sum_{a=1}^{\mathfrak{f}-1}\mathfrak{G}(e_2+a,\chi)\cdot\mathfrak{G}(e_1-e_2-a,\chi)$$

$$\cdot\Bigg(\mathcal{W}(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}-a} & & \\ & \pi^{e_2+\mathfrak{f}+a} & \\ & & 1\end{pmatrix})\mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix})$$

$$+\mathcal{W}(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}-a} & \\ & \pi^{e_2+\mathfrak{f}+a}\end{pmatrix})\Bigg)\cdot q^{-a}$$

$$+\mathfrak{G}(0,\chi)\cdot\Bigg(\mathcal{W}(\begin{pmatrix}\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}}\end{pmatrix}w_2 & \\ & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix})$$

$$+\mathcal{W}(\begin{pmatrix}\pi^{e_1+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}}\end{pmatrix}w_2)\Bigg)$$

$$\cdot\sum_{a=\mathfrak{f}}^{\infty}\mathfrak{G}(e_2+a,\chi)\cdot q^{-a}\Bigg].$$

Performing a change of variable on $e=(e_1,e_2)\in\mathbb{Z}^2$ by $e_1\mapsto e_1+e_2$, we obtain that

$$\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s)$$

$$=\frac{q-1}{q+1}\cdot\sum_{e\in\mathbb{Z}^2}\left(\|\cdot\|^{s-1/2}\otimes\delta_2^{-1}\right)(\begin{pmatrix}\pi^{e_1+e_2} & \\ & \pi^{e_2}\end{pmatrix})\cdot\mathfrak{G}(e_2,\chi)$$

$$
\begin{aligned}
&\cdot\Big[\mathfrak{G}(e_2,\chi)\cdot\mathfrak{G}(e_1,\chi)\cdot\mathcal{W}(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}}&&\\&\pi^{e_2+\mathfrak{f}}&\\&&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}}&\\&\pi^{e_2+\mathfrak{f}}\end{pmatrix})\\
&+\sum_{a=1}^{\mathfrak{f}-1}\mathfrak{G}(e_2+a,\chi)\cdot\mathfrak{G}(e_1-a,\chi)\\
&\qquad\cdot\Bigg(\mathcal{W}(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}-a}&&\\&\pi^{e_2+\mathfrak{f}+a}&\\&&1\end{pmatrix})\,\mathcal{W}'(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}}&\\&\pi^{e_2+\mathfrak{f}}\end{pmatrix})\\
&\qquad\quad+\mathcal{W}(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}}&&\\&\pi^{e_2+\mathfrak{f}}&\\&&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}-a}&\\&\pi^{e_2+\mathfrak{f}+a}\end{pmatrix})\Bigg)\cdot q^{-a}\\
&\mathfrak{G}(0,\chi)\cdot\Bigg(\mathcal{W}(\begin{pmatrix}\begin{pmatrix}\pi^{e_1+e_2+\mathfrak{f}}&\\&\pi^{e_2+2\mathfrak{f}}\end{pmatrix}w_2&\\&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}}&\\&\pi^{e_2+\mathfrak{f}}\end{pmatrix})\\
&\qquad+\mathcal{W}(\begin{pmatrix}\pi^{e_1+e_2+2\mathfrak{f}}&&\\&\pi^{e_2+\mathfrak{f}}&\\&&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+e_2+\mathfrak{f}}&\\&\pi^{e_2+2\mathfrak{f}}\end{pmatrix}w_2)\Bigg)\\
&\qquad\cdot\sum_{a=\mathfrak{f}}^{\infty}\mathfrak{G}(e_2+a,\chi)\cdot q^{-a}\Big].
\end{aligned}
$$

5.4.1. χ ramified

If χ is ramified, the Gauss sums vanish unless $e_2=e_1=-\mathfrak{c}(\chi)$ by Lemma 1.1.b), and what remains is

$$
\begin{aligned}
\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s) &= \frac{q-1}{q+1}\cdot\left(\|\cdot\|^{s-1/2}\otimes\delta_2^{-1}\right)(\begin{pmatrix}\pi^{-2\mathfrak{c}(\chi)}&\\&\pi^{-\mathfrak{c}(\chi)}\end{pmatrix})\cdot\mathfrak{G}(\chi)^3\\
&\qquad\cdot\mathcal{W}(\begin{pmatrix}\pi^{2(\mathfrak{f}-\mathfrak{c}(\chi))}&&\\&\pi^{\mathfrak{f}-\mathfrak{c}(\chi)}&\\&&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{2(\mathfrak{f}-\mathfrak{c}(\chi))}&\\&\pi^{\mathfrak{f}-\mathfrak{c}(\chi)}\end{pmatrix}).
\end{aligned}
$$

Using (49) together with (15), we obtain

$$
\begin{aligned}
\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s) &= \frac{q-1}{q+1}\cdot\mathfrak{G}(\chi)^3\cdot\left(\delta_3^{1/2}\otimes\lambda^{w_3}\right)(\begin{pmatrix}\pi^{\mathfrak{t}_2}&\\&1\end{pmatrix})^{\mathfrak{f}}\cdot\left(\delta_2^{1/2}\otimes\mu^{w_2}\right)(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\\
&\qquad\cdot\left(\prod_{i+j>3}(\lambda_i\otimes\mu_j)(\pi)\cdot q^{-s}\right)^{-\mathfrak{c}(\chi)}
\end{aligned}
$$

We will explain in section 5.5, why we put the result in this form.

5.4.2. χ unramified

If χ is unramified, the zeta-integral vanishes unless $-1 \le e_1, e_2$ due to 1.1.a). Recall that we omit now χ in the notation of the Gauss sums $\mathfrak{G}(\cdot)$. Thus, again using (49) together with (15), as well as Example 5 and Lemma 56, $\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s)$ equals

$$
\begin{aligned}
&\frac{q-1}{q+1} \cdot \sum_{\substack{e_1=-1\\ e_2=-1}}^{\infty} \left(\|\cdot\|^{s-1/2} \otimes \delta_2^{-1}\right)\left(\begin{pmatrix} \pi^{e_1+e_2} & \\ & \pi^{e_2}\end{pmatrix}\right) \cdot \mathfrak{G}(e_2) \\
&\cdot \Bigg[\mathfrak{G}(e_2) \cdot \mathfrak{G}(e_1) \cdot \mathcal{W}\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix}\right) \cdot \mathcal{W}'\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix}\right) \\
&+ \sum_{a=1}^{\mathfrak{f}-1} \mathfrak{G}(e_2+a) \cdot \mathfrak{G}(e_1-a) \\
&\qquad \cdot \Bigg(\mathcal{W}\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}-a} & & \\ & \pi^{e_2+\mathfrak{f}+a} & \\ & & 1\end{pmatrix}\right) \mathcal{W}'\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix}\right) \\
&\qquad + \mathcal{W}\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix}\right) \cdot \mathcal{W}'\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}-a} & \\ & \pi^{e_2+\mathfrak{f}+a}\end{pmatrix}\right)\Bigg) \cdot q^{-a} \\
&+ \Bigg(\mathcal{W}\left(\begin{pmatrix} \begin{pmatrix} \pi^{e_1+e_2+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}}\end{pmatrix} w_2 & \\ & 1\end{pmatrix}\right) \cdot \mathcal{W}'\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix}\right) \\
&\qquad + \mathcal{W}\left(\begin{pmatrix} \pi^{e_1+e_2+2\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix}\right) \cdot \mathcal{W}'\left(\begin{pmatrix} \pi^{e_1+e_2+\mathfrak{f}} & \\ & \pi^{e_2+2\mathfrak{f}}\end{pmatrix} w_2\right)\Bigg) \cdot \frac{q^{-\mathfrak{f}}}{1-q^{-1}}\Bigg] \\
= \; &\frac{q-1}{q+1} \cdot (\delta_3^{1/2} \otimes \lambda^{w_3})\left(\begin{pmatrix} \pi^{\mathfrak{f}} & & \\ & \pi^{\mathfrak{f}} & \\ & & 1\end{pmatrix}\right) \cdot (\delta_2^{1/2} \otimes \mu^{w_2})\left(\begin{pmatrix} \pi^{\mathfrak{f}} & \\ & \pi^{\mathfrak{f}}\end{pmatrix}\right) \\
&\cdot \sum_{e_2=-1}^{\infty} \left\| \begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix} \right\|^{s} \cdot \mathfrak{G}(e_2) \cdot \lambda^{w_3}\left(\begin{pmatrix} \pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix}\right) \cdot \mu^{w_2}\left(\begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right) \\
&\cdot \sum_{e_1=-1}^{\infty} \left(\|\cdot\|^{s-1/2} \otimes \delta_2^{-1}\right)\left(\begin{pmatrix} \pi^{e_1} & \\ & 1\end{pmatrix}\right) \\
&\cdot \Bigg[\mathfrak{G}(e_2) \cdot \mathfrak{G}(e_1) \cdot \mathcal{W}\left(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix}\right) \cdot \mathcal{W}'\left(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix}\right) \\
&+ \sum_{a=1}^{\mathfrak{f}-1} \mathfrak{G}(e_1-a) \cdot \Bigg(\mathcal{W}\left(\begin{pmatrix} \pi^{e_1+\mathfrak{f}-a} & & \\ & \pi^{a} & \\ & & 1\end{pmatrix}\right) \cdot \mathcal{W}'\left(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix}\right)
\end{aligned}
$$

$$+\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}-a} & \\ & \pi^a\end{pmatrix})\Big)\, q^{-a}$$

$$+\Big(\mathcal{W}(\begin{pmatrix}\begin{pmatrix}\pi^{e_1} & \\ & \pi^{\mathfrak{f}}\end{pmatrix}\cdot w_2 & \\ & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix})$$

$$+\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1} & \\ & \pi^{\mathfrak{f}}\end{pmatrix}\cdot w_2)\Big)\cdot\frac{q^{-\mathfrak{f}}}{1-q^{-1}}\Big].$$

In the last equation we have extracted the factor $\begin{pmatrix}\pi^{e_2+\mathfrak{f}} & & \\ & \pi^{e_2+\mathfrak{f}} & \\ & & 1\end{pmatrix}$ from $\mathcal{W}(*)$ and the factor $\begin{pmatrix}\pi^{e_2+\mathfrak{f}} & \\ & \pi^{e_2+\mathfrak{f}}\end{pmatrix}$ from $\mathcal{W}'(*)$. This is possible due to (55). Now observe that the last summand inside the $[\,\cdot\,]$-brackets disappear unless $e_1 \geq \mathfrak{f}-1$, which is equivalent to the fact that $\begin{pmatrix}\pi^{e_1} & \pi^{\mathfrak{f}} & 1\end{pmatrix}$ is w_2-almost dominant, and thus we further make a distinction on whether $e_1 \leq \mathfrak{f}-2$. Thus,

$$\begin{aligned}
&\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s)\\
=\ &\frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{f}} & & \\ & \pi^{\mathfrak{f}} & \\ & & 1\end{pmatrix})\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\begin{pmatrix}\pi^{\mathfrak{f}} & \\ & \pi^{\mathfrak{f}}\end{pmatrix})\\
&\cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right\|^s\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix})\\
&\cdot\Big[\sum_{e_1=-1}^{\mathfrak{f}-2}\big(\|\cdot\|^{s-1/2}\otimes\delta_2^{-1}\big)(\begin{pmatrix}\pi^{e_1} & \\ & 1\end{pmatrix})\\
&\quad\cdot\Big(\mathfrak{G}(e_2)\cdot\mathfrak{G}(e_1)\cdot\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix})\\
&\qquad+\sum_{a=1}^{e_1+1}\mathfrak{G}(e_1-a)\cdot\Big(\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}-a} & & \\ & \pi^a & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix})\\
&\qquad\quad+\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}-a} & \\ & \pi^a\end{pmatrix})\Big)\cdot q^{-a}\Big)\\
&+\sum_{e_1=\mathfrak{f}-1}^{\infty}\big(\|\cdot\|^{s-1/2}\otimes\delta_2^{-1}\big)(\begin{pmatrix}\pi^{e_1} & \\ & 1\end{pmatrix})
\end{aligned}$$

$$\cdot \Bigg(\mathfrak{G}(e_2) \cdot \mathfrak{G}(e_1) \cdot \mathcal{W}(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1 \end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1 \end{pmatrix})$$

$$+ \sum_{a=1}^{\mathfrak{f}-1} \Bigg(\mathcal{W}(\begin{pmatrix} \pi^{e_1+\mathfrak{f}-a} & & \\ & \pi^a & \\ & & 1 \end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1 \end{pmatrix})$$

$$+ \mathcal{W}(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1 \end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}-a} & \\ & \pi^a \end{pmatrix}) \Bigg) \cdot q^{-a}$$

$$+ \Bigg(\mathcal{W}(\begin{pmatrix} \begin{pmatrix} \pi^{e_1} & \\ & \pi^{\mathfrak{f}} \end{pmatrix} \cdot w_2 & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1 \end{pmatrix})$$

$$+ \mathcal{W}(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1 \end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1} & \\ & \pi^{\mathfrak{f}} \end{pmatrix} \cdot w_2) \Bigg) \cdot \frac{q^{-\mathfrak{f}}}{1-q^{-1}} \Bigg) \Bigg] .$$

We shall keep tracking the term $\mathfrak{G}(e_1)$ in the sum when $e_1 \in \{\mathfrak{f}-1, \mathfrak{f}, \mathfrak{f}+1, \ldots\}$, even though its value is 1.

Evaluation of the inner factors:
Observe now that we are in the situation where none of the factors is zero anymore (i.e. no Gauss sum and no Whittaker function) and thus, we can substitute the Whittaker functions for their respective values. Let us first focus on the interior sum-term of $\sum_{a=1}^{e_1+1}$ and $\sum_{a=1}^{\mathfrak{f}-1}$, respectively. Observe that the first term is 0 for $e_1 = -1$. Otherwise we have

$$\sum_{a=1}^{e_1+1} \mathfrak{G}(e_1-a) \cdot \Bigg(\mathcal{W}(\begin{pmatrix} \pi^{e_1+\mathfrak{f}-a} & & \\ & \pi^a & \\ & & 1 \end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1 \end{pmatrix})$$

$$+ \mathcal{W}(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1 \end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix} \pi^{e_1+\mathfrak{f}-a} & \\ & \pi^a \end{pmatrix}) \Bigg) q^{-a}$$

$$= \quad (\delta_3^{1/2} \otimes \lambda^{w_3})(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1 \end{pmatrix}) \cdot (\delta_2^{1/2} \otimes \mu^{w_2})(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1 \end{pmatrix}) \cdot \sum_{a=1}^{e_1+1} \mathfrak{G}(e_1-a)$$

$$\cdot \Bigg((\delta_3^{1/2} \otimes \lambda^{w_3})(\begin{pmatrix} \pi^{-a} & & \\ & \pi^a & \\ & & 1 \end{pmatrix}) + (\delta_2^{1/2} \otimes \mu^{w_2})(\begin{pmatrix} \pi^{-a} & \\ & \pi^a \end{pmatrix}) \Bigg) q^{-a}$$

$$= \quad (\delta_3^{1/2} \otimes \lambda^{w_3})(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1 \end{pmatrix}) \cdot (\delta_2^{1/2} \otimes \mu^{w_2})(\begin{pmatrix} \pi^{e_1+\mathfrak{f}} & \\ & 1 \end{pmatrix})$$

$$\sum_{a=1}^{e_1+1} \mathfrak{G}(e_1-a)\cdot\left(\lambda^{w_3}(\begin{pmatrix}\pi^{-1}&&\\&\pi&\\&&1\end{pmatrix})^a+\mu^{w_2}(\begin{pmatrix}\pi^{-1}&\\&\pi\end{pmatrix})^a\right)$$

$$\overset{a\mapsto e_1-a}{=} (\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&&\\&1&\\&&1\end{pmatrix})\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&\\&1\end{pmatrix})$$

$$\sum_{a=-1}^{e_1-1}\mathfrak{G}(a)\cdot\left(\left(\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{a-e_1}+\left(\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{a-e_1}\right)$$

$$= (\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&&\\&1&\\&&1\end{pmatrix})\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&\\&1\end{pmatrix})$$

$$\left[\left(\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{-e_1}\cdot\left(\left(\frac{q}{q-1}\right)\cdot\left(\frac{1-\left(q\cdot\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{-1}}{1-\frac{\lambda_3(\pi)}{\lambda_2(\pi)}}\right)-\frac{\left(\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{e_1}}{1-\frac{\lambda_3(\pi)}{\lambda_2(\pi)}}\right)+\right.$$

$$\left.\left(\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{-e_1}\cdot\left(\left(\frac{q}{q-1}\right)\cdot\left(\frac{1-\left(q\cdot\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{-1}}{1-\frac{\mu_2(\pi)}{\mu_1(\pi)}}\right)-\frac{\left(\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{e_1}}{1-\frac{\mu_2(\pi)}{\mu_1(\pi)}}\right)+\right].$$

We will denote by

$$F(\lambda,e_1):=\frac{1}{1-\frac{\lambda_3(\pi)}{\lambda_2(\pi)}}\cdot\left(\left(\frac{q}{q-1}\right)\cdot\left(1-\left(q\cdot\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{-1}\right)\cdot\left(\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{-e_1}-1\right)$$

the first factor in the brackets, and by

$$F'(\mu,e_1):=\frac{1}{1-\frac{\mu_2(\pi)}{\mu_1(\pi)}}\cdot\left(\left(\frac{q}{q-1}\right)\cdot\left(1-\left(q\cdot\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{-1}\right)\cdot\left(\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{-e_1}-1\right),$$

the second one. Thus, if $e_1\geq 0$, the first interior sum can be written as

$$\sum_{a=1}^{e_1+1}\mathfrak{G}(e_1-a)\cdot\left(\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}-a}&&\\&\pi^a&\\&&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&\\&1\end{pmatrix})\right.$$

$$\left.+\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&&\\&1&\\&&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}-a}&\\&\pi^a\end{pmatrix})\right)q^{-a}$$

$$= \left(\delta_2\otimes\|\cdot\|^{1/2}\right)(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&\\&1\end{pmatrix})\cdot(\lambda_3\otimes\mu_2)(\pi)^{e_1+\mathfrak{f}}\cdot(F(\lambda,e_1)+F'(\mu,e_1)).$$

Analogously for the second interior sum, we have

$$\sum_{a=1}^{\mathfrak{f}-1}\left(\mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}-a}&&\\&\pi^a&\\&&1\end{pmatrix})\cdot\mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}}&\\&1\end{pmatrix})\right.$$

$$+ \mathcal{W}(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix}) \cdot \mathcal{W}'(\begin{pmatrix}\pi^{e_1+\mathfrak{f}-a} & \\ & \pi^a\end{pmatrix})\Bigg) q^{-a}\Bigg)$$

$$= (\delta_3^{1/2} \otimes \lambda^{w_3})(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix}) \cdot (\delta_2^{1/2} \otimes \mu^{w_2})(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix})$$

$$\cdot \left(\frac{\frac{\lambda_2(\pi)}{\lambda_3(\pi)} - \left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}}{1 - \frac{\lambda_2(\pi)}{\lambda_3(\pi)}} + \frac{\frac{\mu_1(\pi)}{\mu_2(\pi)} - \left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}}{1 - \frac{\mu_1(\pi)}{\mu_2(\pi)}} \right)$$

$$= \left(\delta_2 \otimes \|\cdot\|^{1/2}\right)(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix}) \cdot (\lambda_3 \otimes \mu_2)(\pi)^{e_1+\mathfrak{f}} \cdot (G(\lambda) + G'(\mu)),$$

where we have set

$$G(\lambda) := \frac{\frac{\lambda_2(\pi)}{\lambda_3(\pi)} - \left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}}{1 - \frac{\lambda_2(\pi)}{\lambda_3(\pi)}}, \qquad G'(\mu) := \frac{\frac{\mu_1(\pi)}{\mu_2(\pi)} - \left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}}{1 - \frac{\mu_1(\pi)}{\mu_2(\pi)}}.$$

Evaluation of Whittaker Functions with w_2 in its argument:
By the example 5, since $w_2 = s_1$,

$$\mathcal{W}(\begin{pmatrix}\begin{pmatrix}\pi^{e_1} & \\ & \pi^{\mathfrak{f}}\end{pmatrix} \cdot w_2 & \\ & 1\end{pmatrix}) = \left(\delta_3^{1/2} \otimes \lambda^{w_3}\right)(\begin{pmatrix}\pi^{e_1} & & \\ & \pi^{\mathfrak{f}} & \\ & & 1\end{pmatrix})$$

$$\cdot \left(\left(1 - q^{-1} \cdot \frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right) \left(\frac{1 - \left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{e_1+1-\mathfrak{f}}}{1 - \frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) - q^{-1} \right)$$

and

$$\mathcal{W}'(\begin{pmatrix}\pi^{e_1} & \\ & \pi^{\mathfrak{f}}\end{pmatrix} \cdot w_2) = \left(\delta_2^{1/2} \otimes \mu^{w_2}\right)(\begin{pmatrix}\pi^{e_1} & \\ & \pi^{\mathfrak{f}}\end{pmatrix})$$

$$\cdot \left(\left(1 - q^{-1} \cdot \frac{\mu_1(\pi)}{\mu_2(\pi)}\right) \left(\frac{1 - \left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{e_1+1-\mathfrak{f}}}{1 - \frac{\mu_1(\pi)}{\mu_2(\pi)}} \right) - q^{-1} \right).$$

If we set

$$H(\lambda, e_1) := \left(1 - q^{-1} \cdot \frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right) \left(\frac{1 - \left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{e_1+1-\mathfrak{f}}}{1 - \frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) - q^{-1}$$

and

$$H'(\mu, e_1) := \left(1 - q^{-1} \cdot \frac{\mu_1(\pi)}{\mu_2(\pi)}\right) \left(\frac{1 - \left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{e_1+1-\mathfrak{f}}}{1 - \frac{\mu_1(\pi)}{\mu_2(\pi)}} \right) - q^{-1},$$

then

$$\mathcal{W}(\left(\begin{pmatrix}\pi^{e_1} & \\ & \pi^{\mathfrak{f}}\end{pmatrix}\cdot w_2 \quad \begin{matrix} \\ 1\end{matrix}\right)) = \left(\delta_3^{1/2}\otimes\lambda^{w_3}\right)(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix})\cdot\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}\cdot q^{\mathfrak{f}}\cdot H(\lambda,e_1),$$

and similarly

$$\mathcal{W}'(\begin{pmatrix}\pi^{e_1} & \\ & \pi^{\mathfrak{f}}\end{pmatrix}\cdot w_2) = \left(\delta_2^{1/2}\otimes\mu^{w_2}\right)(\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & \\ & 1\end{pmatrix})\cdot\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}\cdot q^{\mathfrak{f}}\cdot H'(\mu,e_1).$$

Plugging back:

If we plug everything back into $\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s)$, and this time extract $\begin{pmatrix}\pi^{e_1+\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix}$ from $\mathcal{W}(*)$ and $\begin{pmatrix}\pi^{e_2+\mathfrak{f}} & \\ & 1\end{pmatrix}$ from $\mathcal{W}'(*)$, we get

$$\begin{aligned}
& \widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s) \\
= \; & \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{f}} & & \\ & \pi^{\mathfrak{f}} & \\ & & 1\end{pmatrix})\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\begin{pmatrix}\pi^{\mathfrak{f}} & \\ & \pi^{\mathfrak{f}}\end{pmatrix}) \\
& \cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right\|^s\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}) \\
& \cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{f}} & & \\ & 1 & \\ & & 1\end{pmatrix})\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\begin{pmatrix}\pi^{\mathfrak{f}} & \\ & 1\end{pmatrix}) \\
& \cdot\Bigg[\sum_{e_1=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_1} & \\ & 1\end{pmatrix}\right\|^s\cdot\mathfrak{G}(e_2)\cdot\mathfrak{G}(e_1)\cdot(\lambda_3\otimes\mu_2)(\pi)^{e_1} \\
& +\sum_{e_1=0}^{\mathfrak{f}-2}\left\|\begin{pmatrix}\pi^{e_1} & \\ & 1\end{pmatrix}\right\|^s\cdot(\lambda_3\otimes\mu_2)(\pi)^{e_1}\cdot(F(\lambda,e_1)+F'(\mu,e_1)) \\
& +\sum_{e_1=\mathfrak{f}-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_1} & \\ & 1\end{pmatrix}\right\|^s\cdot(\lambda_3\otimes\mu_2)(\pi)^{e_1}\cdot(G(\lambda)+G'(\mu,e_1)) \\
& +\sum_{e_1=\mathfrak{f}-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_1} & \\ & 1\end{pmatrix}\right\|^s\cdot(\lambda_3\otimes\mu_2)(\pi)^{e_1}\cdot\frac{1}{1-q^{-1}} \\
& \qquad\cdot\left(\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}H(\lambda,e_1)+\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}\cdot H'(\mu,e_1)\right)\Bigg] \\
\overset{81}{=} \; & \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}
\end{aligned}$$

$$\cdot \sum_{e_2=-1}^{\infty} \left\| \begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2} \end{pmatrix} \right\|^s \cdot \mathfrak{G}(e_2) \cdot \lambda^{w_3}(\begin{pmatrix} \pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1 \end{pmatrix}) \cdot \mu^{w_2}(\begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2} \end{pmatrix})$$

$$\cdot \Bigg[\mathfrak{G}(e_2) \cdot \frac{q}{q-1} \cdot \left(\frac{1-(\lambda_3 \otimes \mu_2)(\pi)^{-1} \cdot q^{s-1}}{1-(\lambda_3 \otimes \mu_2)(\pi) \cdot q^{-s}} \right)$$

$$+ \sum_{e_1=0}^{\mathfrak{f}-2} \left\| \begin{pmatrix} \pi^{e_1} & \\ & 1 \end{pmatrix} \right\|^s \cdot (\lambda_3 \otimes \mu_2)(\pi)^{e_1} \cdot (F(\lambda, e_1) + F'(\mu, e_1))$$

$$+ \sum_{e_1=\mathfrak{f}-1}^{\infty} \left\| \begin{pmatrix} \pi^{e_1} & \\ & 1 \end{pmatrix} \right\|^s \cdot (\lambda_3 \otimes \mu_2)(\pi)^{e_1} \cdot (G(\lambda) + G'(\mu, e_1))$$

$$+ \sum_{e_1=\mathfrak{f}-1}^{\infty} \left\| \begin{pmatrix} \pi^{e_1} & \\ & 1 \end{pmatrix} \right\|^s \cdot (\lambda_3 \otimes \mu_2)(\pi)^{e_1} \cdot$$

$$\cdot \left(\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)} \right)^{\mathfrak{f}} H(\lambda, e_1) + \left(\frac{\mu_1(\pi)}{\mu_2(\pi)} \right)^{\mathfrak{f}} \cdot H'(\mu, e_1) \right) \Bigg].$$

Evaluation of further interior sums:
Let us from now on write

$$r_{ij} := (\lambda_i \otimes \mu_j)(\pi) \cdot q^{-s}.$$

For the sums with the expressions F and F' we have

$$\sum_{e_1=0}^{\mathfrak{f}-2} \left\| \begin{pmatrix} \pi^{e_1} & \\ & 1 \end{pmatrix} \right\|^s \cdot (\lambda_3 \otimes \mu_2)(\pi)^{e_1} \cdot F(\lambda, e_1)$$

$$= \frac{1}{1-\frac{\lambda_3(\pi)}{\lambda_2(\pi)}} \cdot \left(\left[\left(\frac{q}{q-1} \right) \cdot \left(1 - \left(q \cdot \frac{\lambda_3(\pi)}{\lambda_2(\pi)} \right)^{-1} \right) \cdot \sum_{e_1=0}^{\mathfrak{f}-2} r_{22}^{e_1} \right] - \sum_{e_1=0}^{\mathfrak{f}-2} r_{32}^{e_1} \right)$$

$$= \frac{1}{1-\frac{\lambda_3(\pi)}{\lambda_2(\pi)}} \cdot \left(\left[\left(\frac{q}{q-1} \right) \cdot \left(1 - \left(q \cdot \frac{\lambda_3(\pi)}{\lambda_2(\pi)} \right)^{-1} \right) \cdot \frac{1-r_{22}^{\mathfrak{f}-1}}{1-r_{22}} \right] - \frac{1-r_{32}^{\mathfrak{f}-1}}{1-r_{32}} \right),$$

and

$$\sum_{e_1=0}^{\mathfrak{f}-2} \left\| \begin{pmatrix} \pi^{e_1} & \\ & 1 \end{pmatrix} \right\|^s \cdot (\lambda_3 \otimes \mu_2)(\pi)^{e_1} \cdot F'(\mu, e_1)$$

$$= \frac{1}{1-\frac{\mu_2(\pi)}{\mu_1(\pi)}} \cdot \left(\left[\left(\frac{q}{q-1} \right) \cdot \left(1 - \left(q \cdot \frac{\mu_2(\pi)}{\mu_1(\pi)} \right)^{-1} \right) \cdot \frac{1-r_{31}^{\mathfrak{f}-1}}{1-r_{31}} \right] - \frac{1-r_{32}^{\mathfrak{f}-1}}{1-r_{32}} \right),$$

respectively. For the sums with the expressions H and H' we have

$$\sum_{e_1=\mathfrak{f}-1}^{\infty} \left\| \begin{pmatrix} \pi^{e_1} & \\ & 1 \end{pmatrix} \right\|^s \cdot (\lambda_3 \otimes \mu_2)(\pi)^{e_1} H(\lambda, e_1)$$

$$= \left(\left(\frac{1-q^{-1} \cdot \frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) - q^{-1} \right) \cdot \sum_{e_1=\mathfrak{f}-1}^{\infty} r_{32}^{e_1} - \left(\frac{\lambda_3(\pi)}{\lambda_2(\pi)} \right)^{\mathfrak{f}-1} \cdot \left(\frac{1-q^{-1} \cdot \frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) \cdot \sum_{e_1=\mathfrak{f}-1}^{\infty} r_{22}^{e_1}$$

$$
\begin{aligned}
&= \left(\left(\frac{1-q^{-1}\cdot\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}\right)-q^{-1}\right)\cdot\left(\frac{r_{32}^{\mathfrak{f}-1}}{1-r_{32}}\right)-\left(\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{\mathfrak{f}-1}\cdot\left(\frac{1-q^{-1}\cdot\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}\right)\cdot\left(\frac{r_{22}^{\mathfrak{f}-1}}{1-r_{22}}\right)\\
&= \left(\frac{1-q^{-1}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}\right)\cdot\left(\frac{r_{32}^{\mathfrak{f}-1}}{1-r_{32}}\right)-\left(\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{\mathfrak{f}-1}\cdot\left(\frac{1-q^{-1}\cdot\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}\right)\cdot\left(\frac{r_{22}^{\mathfrak{f}-1}}{1-r_{22}}\right)\\
&= \frac{r_{32}^{\mathfrak{f}-1}}{\left(1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)}\cdot\left(\frac{1-q^{-1}}{1-r_{32}}-\frac{1-q^{-1}\cdot\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-r_{22}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{e_1=\mathfrak{f}-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_1} & \\ & 1\end{pmatrix}\right\|^s\cdot(\lambda_3\otimes\mu_2)(\pi)^{e_1}\,H'(\mu,e_1)\\
&= \left(\left(\frac{1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)-q^{-1}\right)\cdot\sum_{e_1=\mathfrak{f}-1}^{\infty}r_{32}^{e_1}-\left(\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{\mathfrak{f}-1}\cdot\left(\frac{1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)\cdot\sum_{e_1=\mathfrak{f}-1}^{\infty}r_{31}^{e_1}\\
&= \left(\left(\frac{1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)-q^{-1}\right)\cdot\left(\frac{r_{32}^{\mathfrak{f}-1}}{1-r_{32}}\right)-\left(\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{\mathfrak{f}-1}\cdot\left(\frac{1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)\cdot\left(\frac{r_{31}^{\mathfrak{f}-1}}{1-r_{31}}\right)\\
&= \left(\frac{1-q^{-1}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)\cdot\left(\frac{r_{32}^{\mathfrak{f}-1}}{1-r_{32}}\right)-\left(\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{\mathfrak{f}-1}\cdot\left(\frac{1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)\cdot\left(\frac{r_{31}^{\mathfrak{f}-1}}{1-r_{31}}\right)\\
&= \frac{r_{32}^{\mathfrak{f}-1}}{\left(1-\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)}\cdot\left(\frac{1-q^{-1}}{1-r_{32}}-\frac{1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-r_{31}}\right),
\end{aligned}
$$

respectively.

Putting back II:

If we plug everything back once again, using that $\frac{1}{1-y^{-1}} = -\frac{y}{1-y}$ which we will use for $y\in\{\frac{\mu_2(\pi)}{\mu_1(\pi)},\frac{\lambda_3(\pi)}{\lambda_2(\pi)},q\}$, the zeta integral becomes

$$
\begin{aligned}
&\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s)\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\\
&\quad\cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right\|^s\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix})\\
&\quad\cdot\Bigg[\mathfrak{G}(e_2)\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)+\left(\frac{r_{32}^{\mathfrak{f}-1}}{1-r_{32}}\right)\cdot\left(\frac{\frac{\lambda_2(\pi)}{\lambda_3(\pi)}-\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}+\frac{\frac{\mu_1(\pi)}{\mu_2(\pi)}-\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)+\\
&\qquad+\frac{1}{1-\frac{\lambda_3(\pi)}{\lambda_2(\pi)}}\cdot\left(\left[\left(\frac{q}{q-1}\right)\cdot\left(1-\left(q\cdot\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{-1}\right)\cdot\frac{1-r_{22}^{\mathfrak{f}-1}}{1-r_{22}}\right]-\frac{1-r_{32}^{\mathfrak{f}-1}}{1-r_{32}}\right)+
\end{aligned}
$$

$$+\frac{1}{1-\frac{\mu_2(\pi)}{\mu_1(\pi)}}\cdot\left(\left[\left(\frac{q}{q-1}\right)\cdot\left(1-\left(q\cdot\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{-1}\right)\cdot\frac{1-r_{31}^{\mathfrak{f}-1}}{1-r_{31}}\right]-\frac{1-r_{32}^{\mathfrak{f}-1}}{1-r_{32}}\right)+$$

$$+\frac{1}{1-q^{-1}}\cdot\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}\cdot\frac{r_{32}^{\mathfrak{f}-1}}{\left(1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)}\cdot\left(\frac{1-q^{-1}}{1-r_{32}}-\frac{1-q^{-1}\cdot\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-r_{22}}\right)$$

$$+\frac{1}{1-q^{-1}}\cdot\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}\cdot\frac{r_{32}^{\mathfrak{f}-1}}{\left(1-\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)}\cdot\left(\frac{1-q^{-1}}{1-r_{32}}-\frac{1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-r_{31}}\right)\Bigg]$$

$$=\frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2}&\\&1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}$$

$$\cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2}&\\&\pi^{e_2}\end{pmatrix}\right\|^{s}\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2}&&\\&\pi^{e_2}&\\&&1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2}&\\&\pi^{e_2}\end{pmatrix})$$

$$\Bigg[\mathfrak{G}(e_2)\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)$$

$$+\left(\frac{1}{1-r_{32}}\right)\cdot\left(\left(\frac{1}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}\right)\cdot\left(r_{32}^{\mathfrak{f}-1}\cdot\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}-\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}\right)+\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\cdot\left(1-r_{32}^{\mathfrak{f}-1}\right)\right.\right.$$

$$\left.+r_{32}^{\mathfrak{f}-1}\cdot\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}\right)+$$

$$+\left(\frac{1}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)\cdot\left(r_{32}^{\mathfrak{f}-1}\cdot\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}-\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}\right)+\frac{\mu_1(\pi)}{\mu_2(\pi)}\cdot\left(1-r_{32}^{\mathfrak{f}-1}\right)\right.$$

$$\left.\left.+r_{32}^{\mathfrak{f}-1}\cdot\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}\right)\right)$$

$$+\left(\frac{1}{1-r_{31}}\right)\cdot\left(\frac{1}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}}\right)\cdot\left(-\frac{1}{1-q^{-1}}\cdot\left(\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)^{\mathfrak{f}}\cdot r_{32}^{\mathfrak{f}-1}\cdot\left(1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)+\right.$$

$$\left.-\frac{\mu_1(\pi)}{\mu_2(\pi)}\cdot\left(\frac{q}{q-1}\right)\cdot\left(1-\left(q\cdot\frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{-1}\right)\cdot(1-r_{31}^{\mathfrak{f}-1})\right)$$

$$+\left(\frac{1}{1-r_{22}}\right)\cdot\left(\frac{1}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}\right)\cdot\left(-\frac{1}{1-q^{-1}}\cdot\left(\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\right)^{\mathfrak{f}}\cdot r_{32}^{\mathfrak{f}-1}\cdot\left(1-q^{-1}\cdot\frac{\mu_1(\pi)}{\mu_2(\pi)}\right)+\right.$$

$$\left.-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\cdot\left(\frac{q}{q-1}\right)\cdot\left(1-\left(q\cdot\frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{-1}\right)\cdot(1-r_{22}^{\mathfrak{f}-1})\right)\Bigg]$$

$$=\frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2}&\\&1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}$$

$$\cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2}&\\&\pi^{e_2}\end{pmatrix}\right\|^{s}\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2}&&\\&\pi^{e_2}&\\&&1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2}&\\&\pi^{e_2}\end{pmatrix})$$

$$\cdot \Bigg[\mathfrak{G}(e_2) \cdot \frac{q}{q-1} \cdot \left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)$$

$$+ \left(\frac{1}{1-r_{32}}\right) \cdot \left(\left(\frac{\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) + \left(\frac{\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}} \right) \right)$$

$$- \left(\frac{1}{1-r_{31}}\right) \cdot \left(\frac{\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}} \right) \cdot \left(\frac{1-\left(q \cdot \frac{\mu_2(\pi)}{\mu_1(\pi)}\right)^{-1}}{1-q^{-1}} \right)$$

$$- \left(\frac{1}{1-r_{22}}\right) \cdot \left(\frac{\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) \cdot \left(\frac{1-\left(q \cdot \frac{\lambda_3(\pi)}{\lambda_2(\pi)}\right)^{-1}}{1-q^{-1}} \right)\Bigg]$$

$$= \frac{q-1}{q+1} \cdot (\delta_3^{1/2} \otimes \lambda^{w_3})(\begin{pmatrix} \pi^{\mathfrak{t}_2} & \\ & 1 \end{pmatrix})^{\mathfrak{f}} \cdot (\delta_2^{1/2} \otimes \mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}$$

$$\cdot \sum_{e_2=-1}^{\infty} \left\| \begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2} \end{pmatrix} \right\|^s \cdot \mathfrak{G}(e_2) \cdot \lambda^{w_3}(\begin{pmatrix} \pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1 \end{pmatrix}) \cdot \mu^{w_2}(\begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2} \end{pmatrix})$$

$$\cdot \Bigg[\mathfrak{G}(e_2) \cdot \frac{q}{q-1} \cdot \left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right) +$$

$$+ \frac{q}{q-1} \cdot \left(\frac{\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) \cdot \left(\frac{1}{1-r_{32}}\right) \cdot \left(\frac{1}{1-r_{22}}\right)$$

$$\cdot \left(1 - r_{22} - q^{-1} + q^{-1}r_{22} - 1 + r_{32} + q^{-1} \cdot \frac{\lambda_2(\pi)}{\lambda_3(\pi)} - q^{-1}r_{22}\right)$$

$$+ \frac{q}{q-1} \cdot \left(\frac{\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}} \right) \cdot \left(\frac{1}{1-r_{32}}\right) \cdot \left(\frac{1}{1-r_{31}}\right)$$

$$\cdot \left(1 - r_{31} - q^{-1} + q^{-1}r_{31} - 1 + r_{32} + q^{-1} \cdot \frac{\lambda_2(\pi)}{\lambda_3(\pi)} - q^{-1}r_{31}\right)\Bigg]$$

$$= \frac{q-1}{q+1} \cdot (\delta_3^{1/2} \otimes \lambda^{w_3})(\begin{pmatrix} \pi^{\mathfrak{t}_2} & \\ & 1 \end{pmatrix})^{\mathfrak{f}} \cdot (\delta_2^{1/2} \otimes \mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}$$

$$\cdot \sum_{e_2=-1}^{\infty} \left\| \begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2} \end{pmatrix} \right\|^s \cdot \mathfrak{G}(e_2) \cdot \lambda^{w_3}(\begin{pmatrix} \pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1 \end{pmatrix}) \cdot \mu^{w_2}(\begin{pmatrix} \pi^{e_2} & \\ & \pi^{e_2} \end{pmatrix})$$

$$\cdot \Bigg[\mathfrak{G}(e_2) \cdot \frac{q}{q-1} \cdot \left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right) +$$

$$+ \frac{q}{q-1} \cdot \left(\frac{\frac{\lambda_2(\pi)}{\lambda_3(\pi)}}{1-\frac{\lambda_2(\pi)}{\lambda_3(\pi)}} \right) \cdot \left(\frac{1}{1-r_{32}}\right) \cdot \left(\frac{1}{1-r_{22}}\right) \cdot (r_{32}-r_{22}) \cdot \left(1-(qr_{32})^{-1}\right)$$

$$+ \frac{q}{q-1} \cdot \left(\frac{\frac{\mu_1(\pi)}{\mu_2(\pi)}}{1-\frac{\mu_1(\pi)}{\mu_2(\pi)}} \right) \cdot \left(\frac{1}{1-r_{32}}\right) \cdot \left(\frac{1}{1-r_{31}}\right) \cdot (r_{32}-r_{31}) \cdot \left(1-(qr_{32})^{-1}\right)\Bigg]$$

$$
\begin{aligned}
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right\|^{s}\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix})\\
&\quad\cdot\left[\mathfrak{G}(e_2)+\frac{\lambda_2(\pi)}{\lambda_3(\pi)}\cdot r_{32}\cdot\left(\frac{1}{1-r_{22}}\right)+\frac{\mu_1(\pi)}{\mu_2(\pi)}\cdot r_{32}\cdot\left(\frac{1}{1-r_{31}}\right)\right]\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right\|^{s}\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix})\\
&\quad\cdot\left[\mathfrak{G}(e_2)+\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)(r_{22}\cdot(1-r_{31})+r_{31}\cdot(1-r_{22})\right]\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\sum_{e_2=-1}^{\infty}\left\|\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right\|^{s}\cdot\mathfrak{G}(e_2)\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix})\\
&\quad\cdot\left[\mathfrak{G}(e_2)+\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)(r_{22}+r_{31}-2\cdot r_{22}\cdot r_{31})\right].
\end{aligned}
$$

One can observe that the term $\left\|\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix}\right\|^{s}\cdot\lambda^{w_3}(\begin{pmatrix}\pi^{e_2} & & \\ & \pi^{e_2} & \\ & & 1\end{pmatrix})\cdot\mu^{w_2}(\begin{pmatrix}\pi^{e_2} & \\ & \pi^{e_2}\end{pmatrix})$ is just $(r_{31}\cdot r_{22})^{e_2}$. Thus, using (80), we finally obtain that

$$
\begin{aligned}
\tilde{Z}(\mathcal{W},\mathcal{W}',\chi_s) &= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\sum_{e_2=-1}^{\infty}\mathfrak{G}(e_2)\cdot(r_{31}\cdot r_{22})^{e_2}\\
&\quad\quad\cdot\left[\mathfrak{G}(e_2)+\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)(r_{22}+r_{31}-2\cdot r_{22}\cdot r_{31})\right]\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\left[\frac{(r_{31}\cdot r_{22})^{-1}-1+(1-q)^2}{(1-q)^2(1-r_{31}\cdot r_{22})}\right.\\
&\quad+\frac{(r_{31}\cdot r_{22})^{-1}-1+(1-q)^1}{(1-q)^1(1-r_{31}\cdot r_{22})}\cdot\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)\\
&\quad\quad\left.\cdot(r_{22}+r_{31}-2\cdot r_{22}\cdot r_{31})\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\left(\frac{1}{(1-q)^2}\right)\cdot\left(\frac{1}{1-r_{31}\cdot r_{22}}\right)\cdot\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)\cdot\\
&\quad\cdot\Big[\left((r_{31}\cdot r_{22})^{-1}+q^2-2q\right)(1-r_{22})(1-r_{31})\\
&\quad+((r_{31}\cdot r_{22})^{-1}-q)(1-q)(r_{22}+r_{31}-2\cdot r_{22}\cdot r_{31})\Big]\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\left(\frac{1}{(1-q)^2}\right)\cdot\left(\frac{1}{1-r_{31}\cdot r_{22}}\right)\cdot\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)\cdot\\
&\quad\cdot\Big[\left((r_{31}\cdot r_{22})^{-1}+q^2-2q\right)(1-r_{22}-r_{31}+r_{22}\cdot r_{31})\\
&\quad+((r_{31}\cdot r_{22})^{-1}-q)\\
&\qquad\cdot(r_{22}+r_{31}-2\cdot r_{22}\cdot r_{31}-qr_{22}-qr_{31}+2q\cdot r_{22}\cdot r_{31})]\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\left(\frac{1}{(1-q)^2}\right)\cdot\left(\frac{1}{1-r_{31}\cdot r_{22}}\right)\cdot\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)\cdot\\
&\quad\cdot\Big[((r_{31}\cdot r_{22})^{-1}-r_{31}^{-1}-r_{22}^{-1}+1\\
&\qquad\qquad+q^2-q^2r_{22}-q^2r_{31}+q^2\cdot r_{22}\cdot r_{31}\\
&\qquad\qquad-2q+2qr_{22}+2qr_{31}-2qr_{22}\cdot r_{31})\\
&\quad+(r_{31}^{-1}+r_{22}^{-1}-2-qr_{31}^{-1}-qr_{22}^{-1}+2q\\
&\qquad-qr_{22}-qr_{31}+2q\cdot r_{22}\cdot r_{31}+q^2r_{22}+q^2r_{31}-2q^2\cdot r_{22}\cdot r_{31})\Big]\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\left(\frac{1}{(1-q)^2}\right)\cdot\left(\frac{1}{1-r_{31}\cdot r_{22}}\right)\cdot\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)\cdot\\
&\quad\cdot\Big[((r_{31}\cdot r_{22})^{-1}-qr_{31}^{-1}-qr_{22}^{-1}+q^2-1+qr_{31}+qr_{22}-q^2\cdot r_{22}\cdot r_{31})\Big]\\
&= \frac{q-1}{q+1}\cdot(\delta_3^{1/2}\otimes\lambda^{w_3})(\begin{pmatrix}\pi^{\mathfrak{t}_2} & \\ & 1\end{pmatrix})^{\mathfrak{f}}\cdot(\delta_2^{1/2}\otimes\mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}\cdot\frac{q}{q-1}\cdot\left(\frac{1-(qr_{32})^{-1}}{1-r_{32}}\right)\\
&\quad\cdot\left(\frac{1}{(1-q)^2}\right)\cdot\left(\frac{1}{1-r_{31}\cdot r_{22}}\right)\cdot\left(\frac{1}{1-r_{22}}\right)\left(\frac{1}{1-r_{31}}\right)\cdot\\
&\quad\cdot q^2\cdot(1-r_{22}\cdot r_{31})\cdot\left(1-(qr_{22})^{-1}\right)\cdot\left(1-(qr_{31})^{-1}\right).
\end{aligned}
$$

Thus,

$$
\widetilde{Z}(\mathcal{W},\mathcal{W}',\chi_s) \;=\; \mathrm{vol}(J_2)\cdot q^{\frac{(2-1)\cdot 2}{2}}\cdot(1-q^{-1})^{\frac{(2-1)\cdot 2}{2}}
$$

$$\cdot (\delta_3^{1/2} \otimes \lambda^{w_3})(\begin{pmatrix} \pi^{\mathfrak{t}_2} & \\ & 1 \end{pmatrix})^{\mathfrak{f}} \cdot (\delta_2^{1/2} \otimes \mu^{w_2})(\pi^{\mathfrak{t}_2})^{\mathfrak{f}}$$
$$\cdot \prod_{i+j>2+1} \left(\frac{q}{q-1} \cdot \frac{1-(\lambda_j \otimes \mu_j)(\pi)^{-1} \cdot q^{s-1}}{1-(\lambda_j \otimes \mu_j)(\pi) \cdot q^{-s}} \right).$$

As in the case when χ is ramified, we will explain in Section 5.5, why we put the result in this form.

5.5. The case n arbitrary

Let us for a moment assume that ρ and σ are ordinary at $\wp$, see Subsection 6.3. Then all $\lambda_i(\pi)$ and $\mu_j(\pi)$ lie in a number field E/k. Under a fixed embedding of E into $\overline{\mathbb{Q}_p}$ with $\wp|p$ in $\mathbb{Q}$, we can further assume that exactly half of the Satake parameters satisfy $|(\lambda_i \otimes \mu_j)(\pi)|_\wp < q^{-1/2}$, namely those with $i+j > n+1$. These correspond to the different α in Lemma 7 of [Coa89]. The remaining Satake parameteres for $i+j \leq n+1$ satisfy $|(\lambda_i \otimes \mu_j)(\pi)|_\wp > q^{-1/2}$.

5.5.1. χ ramified

Let us supose that χ is ramified. This case has been worked out by Januszewski in Chapter 1 of [Jan09]. If one takes a look at the formula (72)

$$I_n^{(0_n)}(w,e) = q^{\frac{(n-1)n}{2}} \cdot \mathrm{vol}(J_n, d^\times j) \cdot \left(1-q^{-1}\right)^{n-1} \cdot \sum_{a \in \mathcal{A}_w} \|\pi^a\| \cdot \prod_{i=1}^{n} \mathfrak{G}(e_n + a_i, \chi)$$
$$\cdot \int_{(U_n^{(w)} \backslash J_n)'} \mathcal{W}(\begin{pmatrix} \pi^e w j' \cdot r_a^{(L)} \cdot w_n \pi^{\mathfrak{f} \cdot \mathfrak{t}_n} & \\ & 1 \end{pmatrix}) \cdot \mathcal{W}'(\pi^e w j' \cdot r_a^{(R)} \pi^{\mathfrak{f} \cdot \mathfrak{t}_n}) \cdot \chi(j') \, d^\times j',$$

Lemma 1.1t, 2b) forces only those $w \in W_n$ and those $a \in \mathcal{A}_w$ to survive, for which the product $\prod_{i=1}^n \mathfrak{G}(e_n + a_i, \chi)$ does not vanish. But since $a_{w^{-1}(n)} = 0$, this forces $a = (0, \ldots, 0)$, which can only occur for $w^{-1}(n) = 1$ and for $e_n = -\mathfrak{c}(\chi)$. After the extraction of r_a, the pattern would repeat: one would inductively generate in every step k a factor of $\mathfrak{G}(\chi)^{n+1-k}$, and force the condition $w^{-1}(n+1-k) = k$ as well as $e_{n+1-k} = -k \cdot \mathfrak{c}(\chi)$. In other words, there is no combinatorics involved; the $I_n^{(0_n)}(w,e)$ as well as the whole modified integral is just one factor. The only factor that survives is

$$\begin{aligned} \widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) &= \mathrm{vol}(J_n, d^\times g) \cdot q^{\frac{(n-1)n}{2}} \cdot (1-q^{-1})^{\frac{(n-1)\cdot n}{2}} \cdot \mathfrak{G}(\chi)^{\frac{n(n+1)}{2}} \\ &\quad \cdot \left(\delta_{n+1}^{1/2} \otimes \lambda^{w_{n+1}}\right) \left(\begin{pmatrix} \pi^{\mathfrak{t}_n} & \\ & 1 \end{pmatrix}\right)^{\mathfrak{f}} \cdot \left(\delta_n^{1/2} \otimes \mu^{w_n}\right) \left(\pi^{\mathfrak{t}_n}\right)^{\mathfrak{f}} \\ &\quad \cdot \left(\prod_{i+j>n+1} (\lambda_i \otimes \mu_j)(\pi) \cdot q^{-s} \right)^{-\mathfrak{c}(\chi)}. \end{aligned}$$

In this case it is easy to see that the zeta integral is entire (holomorphic for all $s \in \mathbb{C}$). As pointed out in Remark 4, in this case $L(\rho_\wp \times (\sigma_\wp \otimes \chi_{\wp,s}) = 1$. Thus, $\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s)$ coincides (up to a non-zero constant in $E^\times$) with the prediction of Coates stated in Lemma 7 of [Coa89].

5.5.2. χ unramified

If χ is unramified, the value of the local zeta integral remains unknown for general n. Nevertheless, based on the results for $n = 1, 2$, based on its structure in the ramified case together with Coates prediction in Lemma 7 of *loc.cit.*, we conjecture the following:

Conjecture 1. If χ is unramified, then

$$\begin{aligned}\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s) &= \mathrm{vol}(J_n, d^\times g) \cdot q^{\frac{(n-1)\cdot n}{2}} \cdot (1 - q^{-1})^{\frac{(n-1)\cdot n}{2}} \\ &\quad \cdot \left(\delta_{n+1}^{1/2} \otimes \lambda^{w_{n+1}}\right) \left(\begin{pmatrix} \pi^{\mathfrak{t}_n} & \\ & 1 \end{pmatrix}\right)^{\mathfrak{f}} \cdot \left(\delta_n^{1/2} \otimes \mu^{w_n}\right) \left(\pi^{\mathfrak{t}_n}\right)^{\mathfrak{f}} \\ &\quad \cdot \prod_{i+j>n+1} \left(\frac{q}{q-1} \cdot \frac{1 - (\lambda_i \otimes \mu_j)(\pi)^{-1} \cdot q^{s-1}}{1 - (\lambda_i \otimes \mu_j)(\pi) \cdot q^{-s}}\right).\end{aligned}$$

As we could observe in the computations for $n = 1, 2$, we also expect that a priori the integral converges only for $\Re(s) \gg 0$, and the conjectured result is actually the meromorphic continuation of the local zeta integral to whole $\mathbb{C}$ with the obvious poles. In this case we have further

$$L(\rho_\wp \times (\sigma_\wp \otimes \chi_{\wp,s})) = \prod_{i,j} \frac{1}{(1 - \lambda_i(\pi) \otimes \mu_j \cdot q^{-s})}.$$

and the quotient

$$\frac{\widetilde{Z}(\mathcal{W}, \mathcal{W}', \chi_s)}{L(\rho_\wp \times (\sigma_\wp \otimes \chi_{\wp,s}))}$$

coincides with the statement of Lemma 7 of [Coa89] for motives, modulo a non-zero constant in $E^\times$.

6. p-adic L-function attached to $GL_{n+1} \times GL_n$

6.1. The modified automorphic L-function

Recall the equation (60) involving the global L-function $L(\rho \times (\sigma \otimes \chi_{\mathbb{A},s}))$. We modify the integral expression as follows: set $h_{\mathbb{A},n+1} := (h_{\nu,n+1})_\nu \in \mathrm{GL}_{n+1}(\mathbb{A})$ and $h_{\mathbb{A},n} := (h_{\nu,n})_\nu \in \mathrm{GL}_n(\mathbb{A})$ where

$$h_{\nu,n+1} = \begin{cases} h_{n+1} \cdot \begin{pmatrix} w_n \pi^{\mathfrak{f}_\wp \cdot \mathfrak{t}_n} & \\ & 1 \end{pmatrix}, & \nu = \wp, \\ 1_{n+1}, & \text{otherwise} \end{cases}$$

and

$$h_{\nu,n} = \begin{cases} \pi^{\mathfrak{f}_\wp \cdot \mathfrak{t}_n}, & \nu = \wp \\ 1_n, & \text{otherwise} \end{cases},$$

respectively, according to the setting in 5.1. We further modify the cuspidal forms φ_i and φ_i' as follows: we interchange its component at $\wp$ by the factor that corresponds to the local Iwahori-invariant Whittaker functions as described in *loc.cit.*, and call this new cuspidal forms $\tilde{\varphi}_i$ and $\tilde{\varphi}_i'$ respectively. We obtain a modified version[17] of (60) as

$$\begin{aligned} \sum_\iota \int_{\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} & \tilde{\varphi}_\iota(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot h_{\mathbb{A},n+1}) \cdot \tilde{\varphi}_\iota'(g \cdot h_{\mathbb{A},n}) \\ & \cdot \chi_{\mathbb{A},s}(g) d^\times g = P(s, \mathcal{W}_\infty, \mathcal{W}_\infty') \cdot L(\rho \times (\sigma \otimes \chi_{\mathbb{A},s})) \cdot \frac{\widetilde{Z}(\mathcal{W}_\wp, \mathcal{W}_\wp', \chi_{\wp,s})}{L(\rho_\wp \times (\sigma_\wp \otimes \chi_{\wp,s}))} \end{aligned} \tag{73}$$

From now on, we shall follow Chapter 4 of [KMS00] and Chapter 2 of [Jan09].

6.2. The spherical and parabolic Hecke algebra

Since $\wp \notin S$, the Hecke algebra of interest at $\wp$ is the *spherical* Hecke algebra

$$\mathcal{H} := \mathcal{H}(\mathrm{GL}_n(F), \mathrm{GL}_n(\mathcal{O})) = C_c(\mathrm{GL}_n(\mathcal{O})\backslash \mathrm{GL}_n(F)/\mathrm{GL}_n(\mathcal{O})).$$

Moreover, this Hecke algebra is commutative as we have the *Satake isomorphism* of $\mathbb{C}$-algebras

$$\mathcal{H} \xrightarrow{\sim} \mathbb{C}[X_*(T)(F)]^{W_n},$$

see Theorem 4.1. of Cartier. We define in $\mathcal{H}$ the *standard* Hecke operators

$$\mathcal{T}_i := \mathrm{GL}_n(\mathcal{O}) \begin{pmatrix} 1_i & \\ & \pi \cdot 1_{n-i} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}) \tag{74}$$

for $i \in \{0, 1, \ldots, n\}$.

Let us now set $\mathcal{H}' := \mathcal{H}(B_n(F), B_n(\mathcal{O}))$. This Hecke algebra is called *parabolic*. It is

[17] Kazhdan-Mazur-Schmidt in Chapter 3.2 of [KMS00] and Januszewski in Chapter 1.2 of [Jan09] call this (under a mild modification) a *generalized global Birch Lemma*.

easily seen to satisfy the properties of the Lemma on Embeddings[18] of [Gri92], and thus, we have a natural ring embedding

$$\mathcal{H} \hookrightarrow \mathcal{H}', \; f \mapsto f_{|B_n(F)}.$$

Moreover, by Theorem 2 of *loc.cit.*, the *Hecke polynomial*

$$H(X) := \sum_{i=0}^{n} (-1)^i \cdot q^{\frac{(i-1)i}{2}} \cdot \mathcal{T}_i \cdot X^{n-i} \in \mathcal{H}[X]$$

decomposes over $\mathcal{H}'$ into linear factors as

$$H(X) = \prod_{j=1}^{n} (X - \mathcal{U}_j) = (X - \mathcal{U}_1) \cdot \ldots \cdot (X - \mathcal{U}_n) \in \mathcal{H}'[X],$$

where

$$\mathcal{U}_j := B_n(\mathcal{O}) \begin{pmatrix} 1_{j-1} & & \\ & \pi & \\ & & 1_{n-j} \end{pmatrix} B_n(\mathcal{O}) \in \mathcal{H}' \tag{75}$$

for $j \in \{1, \ldots, n\}$. One should be careful as $\mathcal{H}'$ itself is not commutative, and neither are the operators $\{\mathcal{U}_j\}_{j=1}^n$. However, if we now define for $j \in \{0, \ldots, n\}$

$$\mathcal{V}_j := q^{-\frac{(j-1)j}{2}} \cdot \mathcal{U}_1 \ldots \mathcal{U}_j \in \mathcal{H}', \tag{76}$$

then there operators commute pairwise by [Gri92]. In addition, there is another interesting fact about these operators.

Recall from Linear Algebra, that a well-known feature of pairwise commuting (linear) operators of a vector space (over $\mathbb{C}$) is that they possess a simultaneous eigenvector. A prototype example in number theory is the family of classical Hecke operators for modular forms.

The natural $\mathcal{H}'$-module, where the different $\mathcal{V}_j$ operate is the $\mathbb{C}$-vector space $\mathcal{M}' := C_c(B_n(F)/B_n(\mathcal{O}))$. The $\mathcal{H}'$-module structure on $\mathcal{M}'$ is as described in (41). In this case, it is explicitly given by

$$\bullet \colon \begin{cases} \mathcal{H}' \times \mathcal{M}' & \to & \mathcal{M}', \\ \left(\coprod_i g_i B_n(\mathcal{O}), \, \eta \right) & \mapsto & \sum_i \eta(- \cdot g_i). \end{cases} \tag{77}$$

We now state a preparatory Lemma, which is used to find a simultaneous eigenvector of the different $\mathcal{V}_j$:

Lemma 6.1. Let $z \in \mathbb{C}$ be arbitrary. Then we have the decomposition

$$q^{-\frac{(n-1)n}{2}} \cdot \prod_{j=0}^{n-1} \mathcal{V}_j \cdot H(z) = \prod_{j=1}^{n} \left(z \cdot q^{1-j} \mathcal{V}_{j-1} - \mathcal{V}_j \right). \tag{78}$$

[18]Gritsenko calls it Lemma on Imbeddings.

We now state a criterion on how to find a simultaneous eigenvector of the different $\mathcal{V}_j$:

Lemma 6.2. Let $\lambda := (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{C}^{n-1}$ and $\theta \in \mathcal{M}$ satisfying

$$H(\lambda_j) \bullet \theta = 0$$

for any $j \in \{1, \dots, n-1\}$. Then the function

$$\theta_\lambda := \left(\prod_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\lambda_i \cdot q^{1-j} \mathcal{V}_{j-1} - \mathcal{V}_j \right) \right) \bullet \theta$$

is a simultaneous eigenvector of $\mathcal{V}_j$ with corresponding eigenvalue

$$c_j := q^{-\frac{(j-1)j}{2}} \cdot \prod_{i=1}^{j} \lambda_i.$$

where $j = 1, \dots, n-1$.

Proof. This is Lemma 2.2.3 of [Jan09]. □

6.3. The $\wp$-adic distribution

At this point, we will assume that ρ and σ are both *ordinary* at $\wp$. By this, we mean that

- both $\rho_\wp$ and $\sigma_\wp$ are defined over a number field. Thus, there is a (common) number field $E/\mathbb{Q}$, such that the zeroes

$$\lambda_1(\pi), \dots, \lambda_n(\pi), \lambda_{n+1}(\pi), \quad \text{and} \quad \mu_1(\pi), \dots, \mu_n(\pi)$$

  of the corresponding Hecke polynomials of $\rho_\wp$ and $\sigma_\wp$, respectively, are all in E, and

- with respect to a fixed embedding $E \subset \overline{E} = \overline{\mathbb{Q}} \subset \overline{E}_\wp$ we have

$$|\lambda_1(\pi)|_\wp = 1,\ |\lambda_2(\pi)|_\wp = q^{-1}, \dots,\ |\lambda_n(\pi)|_\wp = q^{-(n-1)},\ |\lambda_{n+1}(\pi)|_\wp = q^{-n}$$

  and

$$|\mu_1(\pi)|_\wp = 1,\ |\mu_2(\pi)|_\wp = q^{-1}, \dots,\ |\mu_n(\pi)|_\wp = q^{-(n-1)}.$$

We do not really need to demand $|\lambda_{n+1}(\pi)|_\wp = q^{-n}$ and $|\mu_n(\pi)|_\wp = q^{-(n-1)}$, but under this condition, the powers of q satisfy the condition 62. If we assume that M_ρ and M_σ are the motives conjecturally attached to ρ and σ, respectively, then the powers $e_i(M_\rho)$ and $e_j(M_\sigma)$ of the cyclotomic character as in the definition 5, are exactly the q-powers of $|\lambda_i(\pi)|_\wp$ and $|\mu_j(\pi)|_\wp$, respectively. We can thus assume that M_ρ and M_σ are ordinary at $\wp$. Since M_ρ and M_σ further satisfy the Hypothesis 3, their tensor product $M_\rho \otimes M_\sigma$, provided it exists, would be ordinary at $\wp$ as well.

Remark 8. Observe that exactly half of the mixed roots satisfy

$$|(\lambda_i \otimes \mu_j)(\pi)|_\wp < q^{-1/2} \cdot (*)$$

up to a shift $(*)$, namely exactly those with $i + j > n + 1$. This is the same condition we mentioned in section 5.5 and Coates stated in Lemma 7 of [Coa89], respectively.

Kazhdan-Mazur-Schmidt construct in Chapter 4 of [KMS00] a p-adic distribution, that interpolates the special value $s = \frac{1}{2}$ of the modified L-function over $\mathbb{Q}$, under the assumptions that $\chi_{\mathbb{A}}$ is trivial everywhere except p, and the conductors of all $\chi_{\mathbb{A}}, \chi_{\mathbb{A}}^2, \ldots, \chi_{\mathbb{A}}^{n-1}$ are the same power of p. Januszewski generalizes this construction in Section 2.3 of [Jan09] to a number field $k/\mathbb{Q}$, where $\chi_{\mathbb{A}}$ is unramified everywhere except at $\wp$. He also shows that the interpolation condition holds without the conductor assumption on the powers of $\chi_{\mathbb{A}}$. The construction of the distribution is as follows: (we understand that $\delta_0 = 1$)

Let us set

$$\kappa_\lambda := \delta_n(\pi^{t_n}) \cdot \prod_{i=1}^{n} \lambda_i(\pi)^{(n+1)-i}, \qquad \kappa_\mu := \delta_{n-1}(\pi^{t_{n-1}}) \cdot \prod_{i=1}^{n-1} \mu_i(\pi)^{n-i}.$$

Then both κ_λ and κ_μ are $\wp$-adic units. We further set

$$\kappa_{\mathfrak{f}} := \left(\frac{\delta_{n+1}(\pi^{t_{n+1}}) \cdot \delta_n(\pi^{t_n})}{\kappa_\lambda \cdot \kappa_\mu} \right)^{\mathfrak{f}}.$$

We set now

$$C_k(\wp^\infty) := k^\times \Big\backslash \mathbb{A}^\times \Big/ \prod_{\nu \neq \wp} \mathcal{O}_\nu^\times \cong \varprojlim_{\mathfrak{f}} k^\times \Big\backslash \mathbb{A}^\times \Big/ (1 + \pi^{\mathfrak{f}}) \cdot \prod_{\nu \neq \wp} \mathcal{O}_\nu^\times$$

to be the ray class group of level ∞ at $\wp$. Consider for an $a \in \mathbb{A}^\times$ the set

$$\Theta(a) := k^\times \Big\backslash k^\times \cdot a \cdot \prod_{\nu} \mathcal{O}_\nu^\times \Big/ \prod_{\nu \neq \wp} \mathcal{O}_\nu^\times \cong \varprojlim_{\mathfrak{f}} k^\times \Big\backslash k^\times \cdot a \cdot \prod_{\nu} \mathcal{O}_\nu^\times \Big/ (1 + \pi^{\mathfrak{f}}) \cdot \prod_{\nu \neq \wp} \mathcal{O}_\nu^\times.$$

This is a compact open subset of $C_k(\wp^\infty)$. As the double quotient $k^\times \Big\backslash \mathbb{A}^\times \Big/ \prod_{\nu} \mathcal{O}_\nu^\times$ is naturally isomorphic to the ideal class group of k, which has say h elements, there exist idèles $a_1, \ldots, a_h \in \mathbb{A}^\times$ such that

$$C_k(\wp^\infty) = \Theta(a_1) \sqcup \ldots \sqcup \Theta(a_h).$$

We can assume w.l.o.g. assume that all a_i are 1 at the place $\wp$. Let us now decompose the space $\mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A}^\times)$. Let us set for the $a_i \in \mathbb{A}^\times$

$$C_{i,\mathfrak{f}} := \det{}^{-1} \left(k^\times \Big\backslash k^\times \cdot a_i \cdot (1 + \pi^{\mathfrak{f}}) \prod_{\nu} \mathcal{O}_{\nu \neq \wp}^\times \right) \subset \mathrm{GL}_n(k) \Big\backslash \mathrm{GL}_n(\mathbb{A}).$$

For $x \in \mathcal{O}_\wp^\times$ we set further $\epsilon_x := (\epsilon_{x,\nu})_\nu \in \mathrm{GL}_n(\mathbb{A})$ with

$$\epsilon_{x,\nu} := \begin{cases} \begin{pmatrix} x & \\ & 1_{n-1} \end{pmatrix} & , \nu = \wp \\ 1_n & , \text{otherwise} \end{cases}.$$

Then

$$\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}) = \coprod_{i=1}^{h}\coprod_{x} C_{i,\mathfrak{f}}\cdot\epsilon_x, \tag{79}$$

where x runs over a representative system of $\mathcal{O}_\wp^\times/(1+\pi^\mathfrak{f})$. For the different idèles $a_i \in \mathbb{A}^\times$, Januszewski defines in Section 2.3 of [Jan09] the $\wp$-adic distribution on $\Theta(a_i)$ as

$$\mu_{a_i}(x+\wp^\mathfrak{f}) := \kappa_\mathfrak{f}\cdot\sum_\iota\int_{C_{i,\mathfrak{f}}}\tilde{\varphi}_\iota(\begin{pmatrix} g\epsilon_x & \\ & 1\end{pmatrix}\cdot h_{\mathbb{A},n+1})\cdot\tilde{\varphi}'_\iota(g\cdot\epsilon_x\cdot h_{\mathbb{A},n})d^\times g$$

and shows that it is indeed a $\wp$-adic distribution under the assumption that we modify the local Whittaker functions at $\wp$ of both $\tilde{\varphi}'_\iota$ and $\tilde{\varphi}_\iota$ according to Lemma 6.2. But our choice of Iwahori Whittaker functions is already a simultaneous eigenvector of the different $\mathcal{V}_j$'s (see Corollary 5.5. of [Har98] and Proposition 1.3. of [Jan18]), and thus a scalar multiple of the modification as stated in Lemma 6.2, so there is no need for modification in our case.

The different $\mu_{a_1},\dots,\mu_{a_h}$ sum up to a $\wp$-adic distribution $\mu_\wp$ on $C_k(\wp^\infty)$ and for the trivial adelic character $\chi_\mathbb{A} = \mathbb{1}$ it satisfies

$$\begin{aligned}
\int_{C_k(\wp^\infty)}\chi_\mathbb{A}d\mu_\wp &= \sum_{i=1}^{h}\int_{\Theta(a_i)}d\mu_{a_i} = \sum_{i=1}^{h}\sum_{x\in\mathcal{O}_\wp^\times/(1+\pi^\mathfrak{f})}\mu_{a_i}(x+\pi^\mathfrak{f})\\
&= \kappa_\mathfrak{f}\cdot\sum_{i=1}^{h}\sum_{x\in\mathcal{O}_\wp^\times/(1+\pi^\mathfrak{f})}\sum_\iota\int_{C_{i,\mathfrak{f}}}\tilde{\varphi}_\iota(\begin{pmatrix} g & \\ & 1\end{pmatrix}\cdot\epsilon_x\cdot h_{\mathbb{A},n+1})\cdot\tilde{\varphi}'_\iota(g\cdot\epsilon_x\cdot h_{\mathbb{A},n})d^\times g\\
&\overset{(79)}{=} \kappa_\mathfrak{f}\cdot\sum_\iota\int_{\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\tilde{\varphi}_\iota(\begin{pmatrix} g & \\ & 1\end{pmatrix}\cdot h_{\mathbb{A},n+1})\cdot\tilde{\varphi}'_\iota(g\cdot h_{\mathbb{A},n})d^\times g\\
&\overset{(73)}{=} \kappa_\mathfrak{f}\cdot P(1/2,\mathcal{W}_\infty,\mathcal{W}'_\infty)\cdot L(\rho\times(\sigma\otimes\chi_{\mathbb{A},1/2}))\cdot\frac{\widetilde{Z}(\mathcal{W}_\wp,\mathcal{W}'_\wp,\chi_{\wp,1/2})}{L(\rho_\wp\times(\sigma_\wp\otimes\chi_{\wp,1/2}))}\\
&= \kappa_\mathfrak{f}\cdot P(1/2,\mathcal{W}_\infty,\mathcal{W}'_\infty)\cdot L(\rho\times(\sigma\otimes\chi_{\mathbb{A},1/2}))\cdot\mathrm{vol}(J_n,d^\times g)\\
&\quad\cdot q^{\frac{(n-1)\cdot n}{2}}\cdot(1-q^{-1})^{\frac{(n-1)\cdot n}{2}}\cdot\left(\frac{q}{q-1}\right)^{\frac{n(n+1)}{2}}\\
&\quad\cdot\left(\delta_{n+1}^{1/2}\otimes\lambda^{w_{n+1}}\right)\left(\begin{pmatrix}\pi^{\mathfrak{t}_n} & \\ & 1\end{pmatrix}\right)^\mathfrak{f}\cdot\left(\delta_n^{1/2}\otimes\mu^{w_n}\right)\left(\pi^{\mathfrak{t}_n}\right)^\mathfrak{f}\\
&\quad\cdot\prod_{i+j\leq n+1}\left(1-\frac{(\lambda_i\otimes\mu_j)(\pi)}{q^{1/2}}\right)\cdot\prod_{i+j>n+1}\left(1-\frac{q^{1/2}}{(\lambda_i\otimes\mu_j)(\pi)}\right).
\end{aligned}$$

Januszewski [Jan09] treats the question when the $\wp$-adic distribution is a measure. By his work,

Theorem 6.1. If (ρ,σ) are in algebraic, regular, automorphic cuspidal representations of $\mathrm{GL}_{n+1}(\mathbb{A}_k)\times\mathrm{GL}_n(\mathbb{A}_k)$, that are ordinary at $\wp$ (this implies that their respective $\wp$-components are spherical) and cohomological[19], then the $\wp$-adic distribution $\mu_\wp$ on $C_k(\wp^\infty)$

[19]For a precise definition of cohomological representation, see [Jan09].

constructed by Januszewski is a measure, that satisfies for $\chi_{\mathbb{A}}$ unramified at $\wp$

$$\begin{aligned}\int_{C_k(\wp^\infty)} \chi_{\mathbb{A}} d\mu_\wp &= \kappa_{\mathfrak{f}} \cdot P(1/2, \mathcal{W}_\infty, \mathcal{W}'_\infty) \cdot L(\rho \times (\sigma \otimes \chi_{\mathbb{A},1/2})) \cdot \mathrm{vol}(J_n, d^\times g) \\ &\quad \cdot q^{\frac{(n-1)\cdot n}{2}} \cdot (1-q^{-1})^{\frac{(n-1)\cdot n}{2}} \cdot \left(\frac{q}{q-1}\right)^{\frac{n(n+1)}{2}} \\ &\quad \cdot \left(\delta_{n+1}^{1/2} \otimes \lambda^{w_{n+1}}\right) \left(\begin{pmatrix} \pi^{\mathfrak{t}_n} & \\ & 1 \end{pmatrix}\right)^{\mathfrak{f}} \cdot \left(\delta_n^{1/2} \otimes \mu^{w_n}\right) \left(\pi^{\mathfrak{t}_n}\right)^{\mathfrak{f}} \\ &\quad \cdot \prod_{i+j \leq n+1} \left(1 - \frac{(\lambda_i \otimes \mu_j)(\pi)}{q^{1/2}}\right) \cdot \prod_{i+j > n+1} \left(1 - \frac{q^{1/2}}{(\lambda_i \otimes \mu_j)(\pi)}\right).\end{aligned}$$

A. Local Gauss sums

For $\chi = \chi_\wp$, $\psi = \psi_\wp$ (recall that ψ is unramified at $\wp$) as introduced in Section 1.2, and for $e \in \mathbb{Z}$, we defined the e-twisted local Gauss-sum by

$$\mathfrak{G}(e,\chi) = \int_{\mathcal{O}^\times} \chi(t) \cdot \psi(\pi^e t)\, d^\times t.$$

It satisfies the following:

Lemma A.1.

a) Is χ unramified, i.e. $\chi = \mathbb{1}$, we have

$$\mathfrak{G}(e,\mathbb{1}) = \begin{cases} 0 & , e \leq -2 \\ \frac{1}{1-q} & , e = -1 \\ 1 & , e \geq 0 \end{cases}.$$

In this case, we drop $\mathbb{1}$ from the notation and write just $\mathfrak{G}(e)$.

b) Is χ ramified, i.e. $\mathfrak{c}(\chi) > 0$ and $e \neq -\mathfrak{c}(\chi)$, we have

$$\mathfrak{G}(e,\chi) = 0.$$

Hence, in this case, we drop e from the notation and write $\mathfrak{G}(\chi) := \mathfrak{G}(-\mathfrak{c}(\chi),\chi)$ instead.

Proof. a) Since $\mathcal{O}^\times = \mathcal{O}\backslash\pi\mathcal{O}$ and using the measure comparision given in (24), we obtain

$$\begin{aligned} \mathfrak{G}(e,\mathbb{1}) &= \int_{\mathcal{O}^\times} \mathbb{1}(t)\psi(\pi^e t) d^\times t \\ &= \int_{\mathcal{O}^\times} \psi(\pi^e u) \frac{1}{1-q^{-1}} \cdot \frac{du}{|u|} \\ &= \frac{1}{1-q^{-1}} \left(\int_{\mathcal{O}} \psi(\pi^e u) du - \int_{\mathfrak{p}} \psi(\pi^e u) du \right) \\ &= \frac{1}{1-q^{-1}} \left(\int_{\mathcal{O}} \psi(\pi^e u) du - \int_{\mathcal{O}} \psi(\pi^e (\pi u)) d(\pi u) \right) \\ &= \frac{1}{1-q^{-1}} \left(\int_{\mathcal{O}} \psi(\pi^e u) du - q^{-1} \int_{\mathcal{O}} \psi(\pi^{e+1} u) du \right). \end{aligned}$$

Hence:

- For $e \leq -2$ we have

$$\mathfrak{G}(e,\mathbb{1}) = \frac{1}{1-q^{-1}} \left(\int_{\mathcal{O}} \psi(\pi^e u) du - q^{-1} \int_{\mathcal{O}} \psi(\pi^{e+1} u) du \right) = 0,$$

  since both $\psi(\pi^e \cdot -)$ and $\psi(\pi^{e+1} \cdot -)$ are non-trivial on $\mathcal{O}$.

- In the case $e = -1$ we have

$$\mathfrak{G}(e, \mathbb{1}) = \frac{1}{1-q^{-1}} \left(\underbrace{\int_{\mathcal{O}} \psi(\pi^{-1}u)du}_{=0} - q^{-1} \underbrace{\int_{\mathcal{O}} \psi(u)du}_{=1} \right) = -\frac{1}{1-q^{-1}} q^{-1} = \frac{1}{1-q},$$

since $\psi(\pi^{-1} \cdot -)$ is non-trivial on $\mathcal{O}$, but $\psi(-)$ is trivial on $\mathcal{O}$.

- At last for $e \geq 0$ we get

$$\mathfrak{G}(e, \mathbb{1}) = \frac{1}{1-q^{-1}} \left(\int_{\mathcal{O}} \psi(\pi^e u)du - q^{-1} \int_{\mathcal{O}} \psi(\pi^{e+1}u)du \right) = \frac{1}{1-q^{-1}}(1-q^{-1}) = 1.$$

Here, both $\psi(\pi^e \cdot -)$ and $\psi(\pi^{e+1} \cdot -)$ are trivial on $\mathcal{O}$.

b) Now let χ be ramified.

- First we consider the case $e < -\mathfrak{c}(\chi)$. Here we have $1+\mathfrak{p}^{\mathfrak{c}(\chi)} \subsetneq \mathcal{O}^\times$. Moreover since $\mathcal{O}^\times$ and $1 + \mathfrak{p}^{\mathfrak{c}(\chi)}$ are unimodular, by the quotient integral formula 1.5.3 in [Dei14] there exists a $\mathcal{O}^\times$-invariant Radon measure $d^\times h$ on the quotient space $\mathcal{O}^\times / 1 + \mathfrak{p}^{\mathfrak{c}(\chi)}$, such that we can split the integration of the full space $\mathcal{O}^\times$ into the product integration of the subspace $1+\mathfrak{p}^{\mathfrak{c}(\chi)}$ and the quotient space $\mathcal{O}^\times / 1 + \mathfrak{p}^{\mathfrak{c}(\chi)}$, and hence

$$\mathfrak{G}(e, \chi) = \int_{\mathcal{O}^\times} \chi(t)\psi(\pi^e t)d^\times t = \int_{\mathcal{O}^\times/(1+\mathfrak{p}^{\mathfrak{c}(\chi)})} \int_{1+\mathfrak{p}^{\mathfrak{c}(\chi)}} \chi(xy)\psi(\pi^e xy)\, d^\times x\, d^\times y.$$

But since χ is trivial on $1 + \mathfrak{p}^{\mathfrak{c}(\chi)}$, we can rewrite the interior integral as

$$\int_{1+\mathfrak{p}^{\mathfrak{c}(\chi)}} \chi(xy)\psi(\pi^e xy)d^\times x = \chi(y) \int_{1+\mathfrak{p}^{\mathfrak{c}(\chi)}} \psi(\pi^e xy)d^\times x.$$

Moreover since $\mathfrak{c}(\chi) < -e$, and hence the character $\psi(\pi^e y \cdot -)$ is non-trivial on $1 + \mathfrak{p}^{\mathfrak{c}(\chi)}$, we obtain

$$\int_{1+\mathfrak{p}^{\mathfrak{c}(\chi)}} \psi(\pi^e xy)d^\times x = 0,$$

since we can again split the integration over $1+\mathfrak{p}^{\mathfrak{c}(\chi)}$ by the subspace $1+\mathfrak{p}^{-e} \subsetneq 1+\mathfrak{p}^{\mathfrak{c}(\chi)}$, and use the same argument, that $\psi(\pi^e y \cdot -)$ is non-trivial on the finite group $1 + \mathfrak{p}^{\mathfrak{c}(\chi)} / 1 + \mathfrak{p}^{-e}$, and the integration over this (discrete) finite group happens to be the same as over the additive group $\mathfrak{p}^{\mathfrak{c}(\chi)} / \mathfrak{p}^{-e}$.

- Now we consider the case $e > -\mathfrak{c}(\chi)$. If we suppose that $e \geq 0$, then obviously

$$\mathfrak{G}(e, \chi) = \int_{\mathcal{O}^\times} \chi(t)\psi(\pi^e t)d^\times t = \int_{\mathcal{O}^\times} \chi(t)d^\times t = 0.$$

Otherwise if $0 < -e < \mathfrak{c}(\chi)$, we consider the closed subgroup

$$\mathcal{O}^\times \supseteq 1 + \mathfrak{p}^{-e},$$

As previously, we can express the Gauss sum as follows:

$$\mathfrak{G}(e,\chi) = \int_{\mathcal{O}^\times} \chi(t)\psi(\pi^e t) d^\times t = \int_{\mathcal{O}^\times/(1+\mathfrak{p}^{\mathfrak{c}(\chi)})} \int_{1+\mathfrak{p}^{-e}} \chi(xy)\psi(\pi^e xy)\, d^\times x\, d^\times y.$$

By looking at the interior integral, we see that $x \in 1+\mathfrak{p}^{-e}$ means $x = 1 + x'$ for some $x' \in \mathfrak{p}^{-e}$, but for those x' we have $\psi(\pi^k x' y) = 1$, and thus

$$\int_{1+\mathfrak{p}^{-e}} \chi(xy)\psi(\pi^e xy) d^\times x = \chi(y)\psi(\pi^e y) \int_{1+\mathfrak{p}^{-e}} \chi(x) d^\times x = 0,$$

since χ is again non-trivial on $1 + \mathfrak{p}^{-e}$ because of the assumption $-e < \mathfrak{c}(\chi)$.

□

Remark 9. Let us fix an $r \in \mathbb{N}$. If $z \in \mathbb{C}\backslash\{1,0\}$, $N \geq -1$ an integer and if χ is unramified, then

$$\sum_{e=-\infty}^{N} \mathfrak{G}(e)^r \cdot z^e = \frac{z^{-1} - 1 + (1-q)^r}{(1-q)^r(1-z)} - \frac{z^{N+1}}{1-z}, \tag{80}$$

since with help of the previous lemma 1.1

$$\begin{aligned} \sum_{e=-\infty}^{N} \mathfrak{G}(e)^r \cdot z^e &= \frac{1}{1-z} \sum_{e=-1}^{N} \mathfrak{G}(e)^r \cdot z^e (1-z) \\ &= \frac{\mathfrak{G}(-1)^r (z^{-1} - 1) + 1 - z^{N+1}}{1-z} \\ &= \frac{z^{-1} - 1 + (1-q)^r}{(1-q)^r(1-z)} - \frac{z^{N+1}}{1-z} \end{aligned}$$

Specially in the case when $|z| < 1$ and $r = 1$, taking $N \to \infty$ this simplifies further as

$$\sum_{e\in\mathbb{Z}} \mathfrak{G}(e) \cdot z^e = \frac{q}{q-1} \cdot \frac{1 - (qz)^{-1}}{1-z}. \tag{81}$$

We will need this little computation in Chapter 5.

References

[BBBG19] Brubaker, B. ; Buciumas, V. ; Bump, D. ; Gustafsson, H.P.A.: Colored Vertex Models and Iwahori Whittaker Functions. In: *https://arxiv.org/abs/1906.04140#* (2019)

[BBL18] Brubaker, Ben ; Bump, Daniel ; Licata, Anthony: Whittaker Functions and Demazure Operators. In: *https://arxiv.org/abs/1111.4230* (2018)

[BCDT01] Breuil, C. ; Conrad, B. ; Diamond, F. ; Taylor, R.: On the modularity of elliptic curves over $\mathbb{Q}$: Wild 3-adic exercises. In: *Journal of the American Mathematical Society, Vol. 14, No. 4* (2001), S. 843–939

[Bum87] Bump, Daniel: Hecke algebras *(unpublished lecture notes)*. In: *http://sporadic.stanford.edu/bump/math263/hecke.pdf* (1987)

[Bum96] Bump, Daniel: *Automorphic Forms and Representations*. Cambridge University Press, 1996

[BZ76] Bernstein., I.N. ; Zelevinskii, A.V.: Representation Theory of $\mathrm{GL}(n, F)$ where F is a local non-Archimedean Field. In: *Russian Math. Surveys 31:3* (1976), S. 1–68

[BZ77] Bernstein, I.N. ; Zelevinskii, A.V.: Induced Representations of reductive $\mathfrak{p}$-adic groups I. In: *Annales scientifiques de l'É.N.S,* 4^{e} *série, tome 10, n*^{0} *4* (1977), S. 441–472

[Car79] Cartier, P.: Automorphic Forms, Representations and L-Functions: 1.7. Representations of $\mathfrak{p}$-adic groups: A survey. In: *Proceedings of Symposia in Pure Mathematics, Volume 33, Part 1: the Corvallis readings* (1979), S. 111–155

[Cas80] Casselman, W.: The unramified principal series for p-adic groups, I. The spherical function. In: *COMPOSITIO MATHEMATICA, Vol. 40, nr.3* (1980), S. 387–406

[Cas89] Casselman, W.: Canonical extensions of Harish-Chandra modules to representations of G. In: *Canadian Journal of Mathematics, n.41(3)* (1989), S. 385–438

[Cas95] Casselman, W.: Introduction to the Theory of admissible Representations of p-adic reductive Groups ('draft'). (1995)

[Clo88] Clozel, L.: Automorphic Forms, Shimura Varieties, and L-Functions: Motifs et Formes Automorphes: Applications du Principe de Fonctorialité. In: *PERSPECTIVES IN MATHEMATICS, Vol.10* (1988), S. 77–159

[Coa89] Coates, John: On p-adic L-functions. In: *Séminaire Bourbaki, Astérisque, tome 177-178, exp. n*^{o} *701* (1989), S. 33–59

[Cog00] Cogdell, J.W.: Notes on L-functions for GL_n. In: *(unpublished lecture notes)* (2000)

[Cog03] Cogdell, J.W.: Lectures on L-functions, Converse Theorems, and Functoriality for GL_n. In: *(unpublished lecture notes)* (2003)

[CPR89] Coates, J. ; Perrin-Riou, B.: On p-adic L-functions attached to motives over $\mathbb{Q}$. In: *Advanced Studies in Pure Mathematics 17* (1989), S. 23–54

[CS80] Casselman, W. ; Shalika, J.: The unramified principal series for p-adic groups II. In: *COMPOSITIO MATHEMATICA, Vol. 41* (1980), S. 207–231

[Dei14] Deitmar, Anton: *Principles of Harmonic Analysis.* Springer, 2014

[Del79] Deligne, P.: Valeurs de Fonctions L et Périods d'Intégrales. In: *Proceedings of Symposia in Pure Mathematics, Volume 33, Part 2: the Corvallis readings* (1979), S. 313–346

[Fla79] Flath, D.: Automorphic Forms, Representations and L-Functions: 2.1. Decomposition of Representations into Tensor Products. In: *Proceedings of Symposia in Pure Mathematics, Volume 33, Part 1: the Corvallis readings* (1979), S. 179–183

[GK75] Gelfand, I.M. ; Kazhdan, D.A.: Representations of $\mathrm{GL}(n, K)$ where K is a local field. In: *Lie Groups and Their Representations* (1975), S. 95–118

[Gri92] Gritsenko, V.A.: Parabolic extensions of the Hecke ring of the general linear group II. In: *Journal of Soviet Mathematics 62* (1992), S. 2869–2882

[Har98] Haruzo, Hida: Automorphic induction and Leopoldt type conjectures for $GL(n)^*$. In: *Asian Journal of Mathematics 2* (1998), S. 667–710

[Hec18] Hecke, E.: Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen, Erste Mitteilung. In: *Mathematische Zeitschrift, Band* 1 (1918), S. 357–376

[Hec19] Hecke, E.: Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen, Zweite Mitteilung. In: *Mathematische Werke, Band* 6 (1919), S. 11–51

[Hec37a] Hecke, E.: Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I. In: *Mathematische Annalen, Band 114* (1937), S. 1–28

[Hec37b] Hecke, E.: Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung II. In: *Mathematische Annalen, Band 114* (1937), S. 316—-351

[Jac71] Jacquet, H.: *Automorphic Forms on GL(2) II.* Springer Verlag, 1971

[Jan09] Januszewski, F.: p-adische Rankin-Selberg-Faltungen, Ph.D. thesis. (2009)

[Jan18] Januszewski, Fabian: Non-abelian p-adic Rankin-Selberg L-Functions and non-vanishing of central L-values. In: *https://arxiv.org/abs/1708.02616* (2018)

[JL70] Jacquet, H. ; Langlands, R.: *Automorphic Forms on GL(2).* Springer Verlag, 1970

[JPSS79a] Jacquet, H. ; Piatetski-Shapiro, I.I. ; Shalika, J.: Automorphic Forms on GL(3). In: *Ann.Math 109, Nr.1* (1979), S. 169–212

[JPSS79b] Jacquet, H. ; Piatetski-Shapiro, I.I. ; Shalika, J.: Automorphic Forms on GL(3) II. In: *Ann.Math Vol.109, Nr.2* (1979), S. 213–258

[JPSS81a] Jacquet, H. ; Piatetski-Shapiro, I.I. ; Shalika, J.: Conducteur des représentations du groupe linéaire. In: *Math. Ann., 256* (1981), S. 199–214

[JPSS81b] Jacquet, H. ; Piatetski-Shapiro, I.I. ; Shalika, J.: Relevement cubique non normal. In: *C. R. Acad. Sci. Paris, Ser. I. Math., 292* (1981)

[JPSS83] Jacquet, H. ; Piatetski-Shapiro, I.I. ; Shalika, J.: Rankin–Selberg convolutions. In: *Amer. Journal of Math., 105* (1983), S. 367–464

[JS76] Jacquet, H. ; Shalika, J.: A non-vanishing theorem for zeta functions of GL_n. In: *Inventiones mathematicae 38* (1976), S. 1–16

[JS81] Jacquet, H. ; Shalika, J.: On Euler products and the classification of automorphic representations I. In: *Amer. Journal of Math., 103* (1981), S. 499–558

[JS83] Jacquet, H. ; Shalika, J.: The Whittaker Models of Induced Representations. In: *Pacific Journal of Mathematics, Vol. 109, No. 1* (1983), S. 107–120

[JS85] Jacquet, H. ; Shalika, J.: A lemma on highly ramified ϱ-factors. In: *Math. Ann., 271* (1985), S. 319–322

[JS90] Jacquet, H. ; Shalika, J.: Rankin–Selberg convolutions: Archimedean theory. In: *Weizmann Science Press, Jerusalem, 1990* (1990), S. 125–207

[JW17] Jacinto, J.R. ; Williams, C.: An introduction to p-adic L-functions, *(unpublished lecture notes).* (2017)

[Kat78] Kato, S i.: On an explicit formula for class one Whittaker functions on classical groups over $\mathcal{P}$-adic groups, *(preprint).* (1978)

[KB14] Krötz, B. ; Bernstein, J.: Smooth Fr echet globalizations of Harish-Chandra modules. In: *Israel Journal of Mathematics, n.199(1)* (2014), S. 45–111

[KL64] Kubota, T. ; Leopoldt, H.W.: Eine p-adische Theorie der Zetawerte. In: *Journal für die reine und angewandte Mathematik* (1964), S. 238–339

[KMS00] Kazhdan, D. ; Mazur, B. ; Schmidt, C.G.: Relative modular symbols and Rankin-Selberg convolutions. In: *Journal für die reine und angewandte Mathematik* (2000), S. 97—141

[Kna79] Knapp, A.W.: Local Langlands Correspondence: The Archimedean Case. In: *Proceedings of Symposia in Pure Mathematics, Volume 33, Part 2: the Corvallis readings* (1979), S. 313–346

[MI65] Matsumoto, Hideya ; Iwahori, Nagayoshi: On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. In: *Publications mathématiques de l'I.H.É.S* (1965), S. 5–48

[Neu92] Neukirch, Jürgen: *Algebraische Zahlentheorie*. Springer, 1992

[Pan90] Panchishkin, A.A.: Convolutions of Hilbert modular forms, motives, and p-adic L-functions. In: *MPIM Preprint Series 1990 (43)* (1990)

[PR00] Prasad, D. ; Raghuram, A.: Representation Theory of GL(n) over Non-Archimedean Local Fields. In: *(Lectures at School on Automorphic Forms on GL(n))* (2000)

[PS71] Piatetski-Shapiro, I.I.: Euler Subgroups, in Lie Groups and Their Representations. In: *John Wiley & Sons, New York–Toronto* (1971)

[PS79] Piatetski-Shapiro, I.I.: Multiplicity one theorems. In: *Proceedings of Symposia in Pure Mathematics, Volume 33, Part 1: the Corvallis readings* (1979)

[RD91] Ramakrishnan D., Valenza R.: *Fourier Analysis on Number Fields*. Springer, 1991

[Rie59] Riemann, B.: Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. (1859)

[Rud87] Rudin, W.: Real and Complex Analysis (third edition). In: *McGraw-Hill Book Company* (1987)

[Sel49] Selberg, A.: An Elementary Proof of Dirichlet's Theorem About Primes in an Arithmetic Progression. In: *Annals of Mathematics, Vol. 50, No.2* (1949), S. 297–304

[Sha74] Shalika, J.: The multiplicity one theorem for GL(n). In: *Annals of Mathematics 100* (1974), S. 273–330

[Shi76] Shintani, T.: On an Explicit Formula for Class-1 Whittaker Functions on GL_n. In: *Proc. Japan Acad., 52* (1976)

[Spr98] Springer, T.A.: Linear Algebraic Groups. In: *Birkhäuser* (1998)

[Tat50] Tate, J.: Fourier analysis in number fields, and Hecke's zeta-functions, Ph.D. thesis. In: *Princeton University, Princeton NJ* (1950)

[TW95] Taylor, R. ; Wiles, A.: Ring-Theoretic Properties of Certain Hecke Algebras. In: *Annals of Mathematics Second Series, Vol.141, No. 3* (1995), S. 553–572

[Wal88] Wallach, N.R.: *Real Reductive Groups I.* Academic Press (Pure and applied Mathematics 132), San Diego, 1988

[Wei70] Weil, André: *Basic Number Theory.* Springer, 1970

[Wil95] Wiles, A.: Modular Elliptic Curves and Fermat's Last Theorem. In: *Annals of Mathematics Second Series, Vol.141, No. 3* (1995), S. 443–551

[Zel80] Zelevinskii, A.V.: Induced Representations of reductive $\mathfrak{p}$-adic groups II;. In: *Annales scientifiques de l'É.N.S,* 4^e *série, tome 13,* n^0 *2* (1980), S. 165–210